

MATH597

Analysis II

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\stackrel{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition we're using to deduce that particular equality.

Notice that there are some proofs is **intended** left as assignments, and for completeness, I put them in [Appendix A](#), use it in your **own risks**! You'll lose the chance to practice and really understand the materials.

Additionally, we'll use [\[FF99\]](#) as our main text, while using [\[Tao13\]](#) and [\[Ax19\]](#) as supplementary references.

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Lecture 1: σ -algebra

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1 Measure

Example. Before we start, we first see some examples.

1. Let $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) := \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which is the *power set* of X . We see that

$$\#X = n \implies \#\mathcal{P}(X) = 2^n$$

for $n < \infty$.

2. If $n = \infty$, say $X = \mathbb{N}$, then

$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while \mathbb{N} is a countable set. We can see this as follows. Consider

$$\phi: \mathcal{P}(\mathbb{N}) \rightarrow [0, 1], \quad A \mapsto 0.a_1a_2a_3\dots \text{(base 2)},$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example, A can be $A = \{2, 3, 6, \dots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have

$$\#\mathcal{P}(\mathbb{N}) \geq \#[0, 1].$$

But since $[0, 1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X . Hence, we are intriguing to define a map μ such that

$$\mu: \mathcal{P}(X) \rightarrow [0, \infty].$$

Example. We first see some examples.

1. Let $X = \{0, 1, 2\}$. Then we want to define $\mu: \mathcal{P}(X) \rightarrow [0, \infty]$, we can have

- $\mu(A) = \#A$. Then we have
 - $\mu(\{0, 1\}) = 2$
 - $\mu(\{0\}) = 1$
- $\mu(A) = \sum_{i \in A} 2^i$. Then we have
 - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$

2. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu: \mathcal{P}(\mathbb{N}) \rightarrow [0, \infty]$, we can have

- $\mu(A) = \#A$. Then we have
 - $\mu(\{2, 3, 4, 5, \dots\}) = \infty = \mu(\{\text{even numbers}\})$
- $\mu(A) = e^{-1} \sum_{i \in A} \frac{1}{i!}$. Then we have
 - $\mu(\{0, 2, 4, 6, \dots\}) = e^{-1} (1 + \frac{1}{2!} + \frac{1}{3!} + \dots)$
- $\mu(A) = \sum_{i \in A} a_i$

3. Let $X = \mathbb{R}$. Then we want to define $\mu: \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$, we can have

- $\mu(A) = \#A$
- $\mu((a, b)) = b - a$.

Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

Answer. No!

- $\mu((a, b)) = e^b - e^a$.

Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

Answer. No!

We immediately see the problems. To extend our native measure method into \mathbb{R} is hard and will cause something counter-intuitive!¹ Hence, rather than define measurement on *all* subsets in the power set of X , we only focus on *some* subsets. In other words, we want to define

$$\mu: \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \rightarrow [0, \infty].$$

¹https://en.wikipedia.org/wiki/Banach-Tarski_paradox

1.1 σ -algebras

We start from the definition of the most fundamental element in measure theory.

Definition 1.1 (σ -algebra). Let X be a set. A collection \mathcal{A} of subsets of X , i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a σ -algebra on X if

- $\emptyset \in \mathcal{A}$.
- \mathcal{A} is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- \mathcal{A} is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\emptyset}_{\in \mathcal{A}}$ and \mathcal{A} is closed under complement.
- $\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c$, namely \mathcal{A} is closed under countable intersections.
- $A_1 \cup A_2 \cup \dots \cup A_n = A_1 \cup A_2 \cup \dots \cup A_n \cup \emptyset \cup \emptyset \cup \dots$, hence \mathcal{A} is closed under finite unions and intersections.

An immediate definition can be given. We now define so-called *Borel set*.

Definition 1.2 (Borel set). Given a topological space X , a *Borel set* is any set in X that can be formed from open sets through the operations of countable union, countable intersection and relative complement.

Lecture 2: Measure

07 Jan. 11:00

Example. Again, we first see some examples.

1. Let $\mathcal{A} = \mathcal{P}(X)$, which is the power σ -algebra.
2. Let $\mathcal{A} = \{\emptyset, X\}$, which is a trivial σ -algebra.
3. Let $B \subset X$, $B \neq \emptyset$, $B \neq X$. Then we see that $\mathcal{A} = \{\emptyset, B, B^c, X\}$ is a σ -algebra.

Lemma 1.1. Let \mathcal{A}_α , $\alpha \in I$, be a family of σ -algebra on X . Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is a σ -algebra on X .

Remark. Notice that I may be an uncountable intersection.

Proof. A simple proof can be made as follows. Firstly, $\emptyset \in \mathcal{A}_\alpha$ for every α clearly. Moreover, closure under complement and countable unions for every \mathcal{A}_α implies the same must be true for $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$. Hence, $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ -algebra. ■

The above allows us to give the following definition.

Definition 1.3 (Generation of σ -algebra). Given $\mathcal{E} \subset \mathcal{P}(X)$, where \mathcal{E} is not necessarily a σ -algebra. Let $\langle \mathcal{E} \rangle$ be the intersection of all σ -algebras on X containing \mathcal{E} , then we call $\langle \mathcal{E} \rangle$ the σ -algebra generated by \mathcal{E} .

Remark. Clearly, $\langle \mathcal{E} \rangle$ is the smallest σ -algebra containing \mathcal{E} , and it is unique. To check the uniqueness, we suppose there are two different $\langle \mathcal{E} \rangle_1$ and $\langle \mathcal{E} \rangle_2$ generated from \mathcal{E} . It's easy to show

$$\langle \mathcal{E} \rangle_1 \subseteq \langle \mathcal{E} \rangle_2,$$

and by symmetry, they are equal.

Example. We see that $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle$.

Lemma 1.2. We have

1. Given \mathcal{A} a σ -algebra, $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \mathcal{A}$
2. $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$

Proof. We'll see that after proving the first claim, the second follows smoothly.

1. The first claim is trivial, since we know that $\langle \mathcal{E} \rangle$ is the smallest σ -algebra containing \mathcal{E} , then if $\mathcal{E} \subset \mathcal{A}$, we clearly have $\langle \mathcal{E} \rangle \subset \mathcal{A}$ by the definition.
2. The second claim is also easy. From the first claim and the definition, we have

$$\mathcal{E} \subset \mathcal{F} \subset \langle \mathcal{F} \rangle \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle.$$

■

At this point, we haven't put any specific structure on X . Now we try to describe those spaces with good structure, which will give the space some nice properties.

Definition 1.4 (Borel σ -algebra). For a topological space X , the *Borel σ -algebra on X* , denoted as $\mathcal{B}(X)$, is the σ -algebra generated by the collection of all open sets in X .

Example. We see that $\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a, b) \mid a < b; a, b \in \mathbb{R}\}$.

- $\mathcal{E}_2 = \{[a, b] \mid a < b; a, b \in \mathbb{R}\}$ since $[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.
- $\mathcal{E}_3 = ((a, b] \mid a < b; a, b \in \mathbb{R})$ since $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$.
- $\mathcal{E}_4 = ([a, b) \mid a < b; a, b \in \mathbb{R})$ since $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b)$.
- $\mathcal{E}_5 = ((a, \infty) \mid a \in \mathbb{R})$ since $(a, \infty) = \bigcup_{n=1}^{\infty} (a, a + n)$.
- $\mathcal{E}_6 = ([a, \infty) \mid a \in \mathbb{R})$ since $[a, \infty) = \bigcup_{n=1}^{\infty} [a, a + n)$.
- $\mathcal{E}_7 = ((-\infty, b) \mid b \in \mathbb{R})$ since $(-\infty, b) = \bigcup_{n=1}^{\infty} (b - n, b)$.
- $\mathcal{E}_8 = ((-\infty, b] \mid b \in \mathbb{R})$ since $(-\infty, b] = \bigcup_{n=1}^{\infty} (b - n, b]$.

Proposition 1.1. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \dots, 8$.

Proof. Firstly, we see that $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \implies \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$ by Lemma 1.2. Secondly, by definition, $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$ where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R}\}.$$

It's enough to show $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$ since if so, $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$, and clearly $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$, then we will have $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$. Let $O \subset \mathbb{R}$ be an open set, i.e., $O \in \mathcal{E}$. We claim that every open set in \mathbb{R} is a countable union of disjoint open intervals.²

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

where I_j open interval with the form of $(a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$.

For example, \mathcal{E}_1 is trivially true, and

$$(a, b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{[a + \frac{1}{n}, b - \frac{1}{n}]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for \mathcal{E}_2 and

$$(a, \infty) = \bigcup_{k=1}^{\infty} (a, a + k)$$

shows the case for \mathcal{E}_5 . It's now straightforward to check open intervals are in $\langle \mathcal{E}_i \rangle$ for every i . ■

²<https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals>

Now, to put a structure on a space, we define the following.

Definition 1.5 (Measurable space, \mathcal{A} -measurable set). A *measurable space* or *Borel space* is a tuple of a set X and a σ -algebra \mathcal{A} on X , denoted by (X, \mathcal{A}) .

Definition 1.6 (Measurable set). Given a measurable space (X, \mathcal{A}) , every $E \in \mathcal{A}$ is a so-called \mathcal{A} -measurable set.

1.2 Measures

With the definition of measurable space, we now can refine our measure function μ as follows.

Definition 1.7 (Measure, Measure space). Given a measurable space on (X, \mathcal{A}) , a *measure* is a function μ such that

$$\mu: \mathcal{A} \rightarrow [0, \infty]$$

with

1. $\mu(\emptyset) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_1, A_2, \dots \in \mathcal{A}$ are **disjoint**. We call this *Countable additivity*.

We denote (X, \mathcal{A}, μ) a *measure space*.

Notation. We denote $[0, \infty] := [0, \infty) \cup \{\infty\}$.

Remark. The motivation of why we only want *countable additivity* but not uncountable additivity can be seen by the following example. We'll consider the most intuitive *measure* on $\mathbb{R}, \mathcal{B}(\mathbb{R})$.

Since we have

$$(0, 1] = (1/2, 1] \cup (1/4, 1/2] \cup (1/8, 1/4] \cup \dots$$

and also

$$(0, 1] = \bigcup_{x \in (0, 1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2}, 1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4}, \frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8}, \frac{1}{4}])} + \dots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0, 1]} 0$$

since $\mu(x) = 0$ for $x \in \mathbb{R}$, which is clearly not what we want.

Example. We see some examples.

1. For any (X, \mathcal{A}) , we let $\mu(A) := \#A$. This is called *counting measure*.
2. Let $x_0 \in X$. For any (X, \mathcal{A}) , the *Dirac-Delta measure* at x_0 is

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A. \end{cases}$$

3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$,

$$\mu(A) = \sum_{i \in A} a_i,$$

where $a_1, a_2, \dots \in [0, \infty)$.

Lecture 3: Construct a Measure

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Note. If $A, B \in \mathcal{A}$ and $A \subset B$, then

$$\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

Theorem 1.1. Given (X, \mathcal{A}, μ) be a *measure space*.

1. Monotonicity.

$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

2. Countable subadditivity.

$$A_1, A_2, \dots \in \mathcal{A} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

3. Continuity from below/ monotone convergence theorem (MCT) for sets.

$$\begin{cases} A_1, A_2, \dots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \dots \end{cases} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

4. Continuity from above.

$$\begin{cases} A_1, A_2, \dots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \dots \\ \mu(A_1) < \infty \end{cases} \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Proof. We prove this theorem one by one.

1. Since $A \subset B$, hence we have

$$\mu(B) = \mu\left(\underbrace{(B \setminus A) \cup A}_{\text{disjoint}}\right) \stackrel{!}{=} \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A).$$

2. This should be trivial from [countable additivity](#) with the fact that $\mu(A) \geq 0$ for all A .

DIY!

3. Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$, then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

are a disjoint union and $B_i \in \mathcal{A}$, hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i).$$

With $\mu\left(\bigcup_{i=1}^n B_i\right) = \mu(A_n)$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n B_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

4. Let $E_i = A_1 \setminus A_i \implies E_i \in \mathcal{A}$, $E_1 \subset E_2 \subset \dots$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right),$$

which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i\right) \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$$

since $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu(A_1) < \infty$. Then from [continuity from below](#), we further have

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(E_n) = \mu(A_1) - \lim_{n \rightarrow \infty} (\mu(A_1) - \mu(A_n)).$$

From [monotonicity](#), we see that $\mu(A_n) \leq \mu(A_1) < \infty$, hence we can split the limit and further get

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu(A_1) + \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \mu(A_n).$$

■

Example. Given $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting measure})$. Then we see

- $A_n = \{n, n+1, n+2, \dots\} \implies \mu(A_n) = \infty$
- $A_1 \supset A_2 \supset A_3 \supset \dots$
- $\bigcap_{i=1}^{\infty} A_i = \emptyset \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = 0$

Remark. We see that in this case, since $\mu(A_1) \not\leq \infty$, hence [continuity from above](#) doesn't hold.

We now try to characterize some properties of a measure space.

Definition 1.8 (μ -null, μ -subnull, Complete measure space). Given (X, \mathcal{A}, μ) ,

- $A \subset X$ is a μ -null set if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- $A \subset X$ is a μ -subnull set if $\exists \mu$ -null set B such that $A \subset B$.
- (X, \mathcal{A}, μ) is a complete measure space if every μ -subnull set is \mathcal{A} -measurable.

Note. We see that for a μ -subnull set, it's not necessary \mathcal{A} -measurable.

There are some useful terminologies we'll use later relating to μ -null.

Definition 1.9 (Almost everywhere). Given (X, \mathcal{A}, μ) , a statement $P(x)$, $x \in X$ holds μ -almost everywhere (a.e.) if the set

$$\{x \in X : P(x) \text{ does not hold}\}$$

is μ -null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

Definition 1.10 (Finite measure). , Given (X, \mathcal{A}, μ)

- μ is a finite measure if $\mu(X) < \infty$.
- μ is a σ -finite measure if $X = \bigcup_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.

Exercise. Every [measure space](#) can be [completed](#). Namely, we can always find a bigger σ -algebra to [complete](#) the space.

1.3 Outer Measures

We start by giving a definition.

Definition 1.11 (Outer measure). An *outer measure* on X is a map

$$\mu^*: \mathcal{P}(X) \rightarrow [0, \infty]$$

such that

- $\mu^*(\emptyset) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$
- (countable subadditivity) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ for every $A_i \subset X$.

Example. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

is an **outer measure** due to the **Proposition 1.2** we're going to show.

Remark. We see that an **outer measure** need not be a **measure**. Check **Definition 1.7**.

Proposition 1.2. Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\emptyset, X \in \mathcal{E}$. Let

$$\rho: \mathcal{E} \rightarrow [0, \infty]$$

such that $\rho(\emptyset) = 0$. Then

$$\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

is an **outer measure** on X .

Note. Recall the Tonelli's Theorem³ for series:

If $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Specifically, in [Tao13] Theorem 0.0.2.

Lecture 4: Carathéodory extension Theorem

12 Jan. 11:00

As previously seen. We now prove the **Proposition 1.2**.

Proof. We need to prove

³https://en.wikipedia.org/wiki/Fubini%27s_theorem

- μ^* is well-defined. i.e., inf is taken over a non-empty set. This is trivial since $X \in \mathcal{E}$ and $X \supset A$ for any $A \in \mathcal{E}$.
- $\mu^*(\emptyset) = 0$. Since $\emptyset \in \mathcal{E}$ and

$$\mu^*(\emptyset) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \emptyset \right\} = 0$$

since $\rho(\emptyset) = 0$ for all i and further, by Squeeze Theorem⁴, we see that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(\emptyset) = 0.$$

- $A \subset B \implies \mu^*(A) \leq \mu^*(B)$. We simply show this by contradiction. Suppose $A \subset B$ and $\mu^*(A) > \mu^*(B)$, then by definition of μ^* , we have

$$\begin{aligned} \mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\} \\ &> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B \right\} = \mu^*(B). \end{aligned}$$

Now, let $B = (B \setminus A) \cup A$, then we have

$$\begin{aligned} \mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\} \\ &> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B \setminus A) \cup A \right\} = \mu^*(B). \end{aligned}$$

Now, since $B \setminus A \supseteq \emptyset$, then this inequality can't hold, hence a contradiction⁵.

- Countable subadditivity. Let $A_1, A_2, \dots \in X$. If one of $\mu^*(A_n) = \infty$, then result holds. So we may assume $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Now, fix any $\epsilon > 0$, we will show that

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon.$$

For each $n \in \mathbb{N}$, $\exists E_{n,1}, E_{n,2}, \dots \in \mathcal{E}$ such that

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n$$

and

$$\mu^*(A_n) + \frac{\epsilon}{2^n} \geq \sum_{k=1}^{\infty} \rho(E_{n,k}).$$

Then we see that

$$\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k) \in \mathbb{N}^2} E_{k,n},$$

⁴https://en.wikipedia.org/wiki/Squeeze_theorem

⁵This is an important trick!!

which implies

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{(n,k) \in \mathbb{N}^2} \rho(E_{k,n}) \stackrel{!}{=} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{k,n}) \leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right)$$

from the inequality just derived. Now, since the last term is just

$$\sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon,$$

hence we finally have

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon$$

for arbitrarily small fixed $\epsilon > 0$, hence the subadditivity is proved. ■

Definition 1.12 (Carathéodory measurable). Let μ^* be an **outer measure** on X . We say $A \subset X$ is *Carathéodory measurable* (*C-measurable*) with respect to μ^* if

$$\forall E \subset X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Lemma 1.3. Let μ^* be an **outer measure** on X . Suppose B_1, \dots, B_N are disjoint **C-measurable** sets. Then,

$$\forall E \subset X, \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \right) \right) = \sum_{i=1}^N \mu^*(E \cap B_i).$$

Proof. Since we have

$$\begin{aligned} \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \right) \right) &= \mu^*(E' \cap B_1) + \mu^*(E' \setminus B_1) \text{ }^6 \\ &= \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \cap B_1 \right) \right) + \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \right) \cap B_1^c \right) \\ &= \mu^*(E \cap B_1) + \mu^* \left(E \cap \left(\bigcup_{i=2}^N B_i \right) \right) \end{aligned}$$

where the equality comes from the fact that B_1 is **C-measurable** and disjoint from $B_i, i \neq 1$. Then, we simply iterate this argument and have the result. ■

Remark. This implies that if we restrict an **outer measure** on a **C-measurable** set, then it becomes finite additive.

⁶Here, $E' := E \cap \left(\bigcup_{i=1}^N B_i \right)$ for the simplicity of notation.

Theorem 1.2 (Carathéodory extension Theorem). Let μ^* be an outer measure on X . Let \mathcal{A} be the collection of C-measurable sets (with respect to μ^*). Then,

1. \mathcal{A} is a σ -algebra on X .
2. $\mu = \mu^*|_{\mathcal{A}}$ is a measure on (X, \mathcal{A}) .
3. (X, \mathcal{A}, μ) is a complete measure space.

Proof. We divide the proof in several steps.

1. We show \mathcal{A} is a σ -algebra by showing

- (a) $\emptyset \in \mathcal{A}$. To show this, we simply check that \emptyset is C-measurable. We see that

$$\forall_{E \subset X} \mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E \setminus \emptyset) = \mu^*(E),$$

which just shows $\emptyset \in \mathcal{A}$.

- (b) \mathcal{A} closed under complements. This is equivalent to say that if A is C-measurable, so is A^c . We see that if A is C-measurable, then for every $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Observing that $E \cap A = E \setminus A^c$ and $E \setminus A = E \cap A^c$, hence

$$\mu^*(E) = \mu^*(E \setminus A^c) + \mu^*(E \cap A^c).$$

We immediately see that above implies $A^c \in \mathcal{A}$.

- (c) \mathcal{A} closed under countable unions.

Note. To show \mathcal{A} closed under countable unions, we show that \mathcal{A} is closed under:

finite unions $\xRightarrow{\text{then}}$ countable disjoint unions $\xRightarrow{\text{then}}$ countable unions.

- We show \mathcal{A} is closed under finite unions.

Claim. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

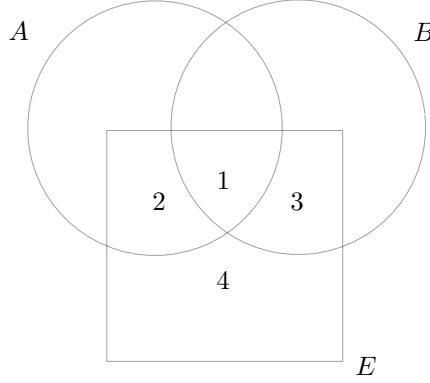
Fix $E \subset X$ arbitrary. We need to show that

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)),$$

i.e.,

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

given $A, B \in \mathcal{A}$.



- Since A is **C-measurable**,
 - * $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$
 - * $\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$
- Since B is **C-measurable**,
 - * $\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$

Hence, we have

$$\begin{aligned} \mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4). \end{aligned}$$

- We show \mathcal{A} is closed under countable disjoint unions.

Let $A_1, A_2, \dots \in \mathcal{A}$ and disjoint. Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^{\infty} A_i\right),$$

hence we only need to show another way around.

Fix $N \in \mathbb{N}$, we have $\bigcup_{n=1}^N A_n \in \mathcal{A}$ since **N is finite**, and

$$\begin{aligned} \mu^*(E) &= \mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right) + \mu^*\left(E \setminus \left(\bigcup_{n=1}^N A_n\right)\right) \\ &\geq \underbrace{\sum_{n=1}^N \mu^*(E \cap A_n)}_{=\mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right)} + \underbrace{\mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right)}_{\leq \mu^*\left(E \setminus \left(\bigcup_{n=1}^N A_n\right)\right)} \\ &\stackrel{!}{=} \mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right) + \mu^*\left(E \setminus \left(\bigcup_{n=1}^N A_n\right)\right) \end{aligned}$$

Now, take $N \rightarrow \infty$ then we are done.

- We show \mathcal{A} is closed under countable unions.

DIY

The proof will be *continued*...

Lecture 5: Hahn-Kolmogorov Theorem

14 Jan. 11:00

Firstly, we see a stronger version of [Lemma 1.3](#) we have seen before.

Lemma 1.4. Let μ^* be an [outer measure](#) on X . Suppose B_1, B_2, \dots are disjoint [C-measurable](#) sets. Then,

$$\forall E \subset X, \mu^* \left(E \cap \left(\bigcup_{i=1}^{\infty} B_i \right) \right) = \sum_{i=1}^{\infty} \mu^* (E \cap B_i).$$

Proof.

$$\sum_{n=1}^{\infty} \mu^* (E \cap B_i) \geq \mu^* \left(E \cap \bigcup_{n=1}^{\infty} B_n \right) \geq \mu^* \left(E \cap \left(\bigcup_{n=1}^N B_n \right) \right) \stackrel{!}{=} \sum_{n=1}^N \mu^* (E \cap B_n).$$

Now, we just take $N \rightarrow \infty$ (or note that $N \in \mathbb{N}$ is arbitrary, we then get the result according to [Squeeze Theorem](#)⁷). ■

Let's continue the proof of [Theorem 1.2](#).

2. Since from [Definition 1.7](#), we need to show

- $\mu(\emptyset) = 0$. This means that we need to show $\mu^*|_{\mathcal{A}}(\emptyset) = 0$. Since $\emptyset \in \mathcal{A}$ and μ^* is an [outer measure](#), hence from the [property](#) of [outer measure](#), it clearly holds.
- [Countable additivity](#) of μ^* on \mathcal{A} follows from the [Lemma 1.4](#) with $E = X$

3. The proof is given in [Theorem A.1](#). ■

1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets \mathcal{E} and any ρ such that it assigns [measure](#) on \mathcal{E} , then it induces an [outer measure](#) by [Proposition 1.2](#), finally [complete](#) the [outer measure](#) by [Theorem 1.2](#).

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.2}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to [σ-algebra](#) we are working on till now.

⁷https://en.wikipedia.org/wiki/Squeeze_theorem

Definition 1.13 (Algebra). Let X be a set. A collection \mathcal{A} of subsets of X , i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on X* if

- $\emptyset \in \mathcal{A}$.
- \mathcal{A} is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- \mathcal{A} is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^n A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an [algebra](#) and a [\$\sigma\$ -algebra](#) is whether they are closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an [outer measure](#).

Definition 1.14 (Pre-measure). Let \mathcal{A}_0 be an [algebra](#) on X . We say

$$\mu_0: \mathcal{A}_0 \rightarrow [0, \infty]$$

is a *pre-measure* if

1. $\mu_0(\emptyset) = 0$
2. (finite additivity) $\mu_0\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu_0(A_i)$ if $A_1, \dots, A_n \in \mathcal{A}_0$ are disjoint.
3. (countable additivity within the [algebra](#)) If $A \in \mathcal{A}_0$ and $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, disjoint, then

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 1.5. (1) + (3) \implies (2) in [Definition 1.14](#).

Proof. It's easy to see that since μ_0 is monotone. ■

Theorem 1.3 (Hahn-Kolmogorov Theorem). Let μ_0 be a [pre-measure](#) on [algebra](#) \mathcal{A}_0 on X . Let μ^* be the [outer measure](#) induced by (\mathcal{A}_0, μ_0) in [Proposition 1.2](#). Let \mathcal{A} and μ be the [Carathéodory \$\sigma\$ -algebra](#) and [measure](#) for μ^* , then (\mathcal{A}, μ) extends (\mathcal{A}_0, μ_0) . i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts.

- We first show $\mathcal{A} \supset \mathcal{A}_0$. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is [C-measurable](#), i.e.,

$$\forall E \subset X \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We first fix an $E \subset X$. From [countable subadditivity](#) of μ^* , we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

Fix $\epsilon > 0$. By the [Proposition 1.2](#) of μ^* , $\exists B_1, B_2, \dots \in \mathcal{A}_0$, $\bigcup_{n=1}^{\infty} B_n \supset E$ such that

$$\mu^*(E) + \epsilon \geq \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left(\underbrace{\mu_0(B_n \cap A)}_{\in \mathcal{A}_0} + \underbrace{\mu_0(B_n \cap A^c)}_{\in \mathcal{A}_0} \right)$$

by the [finite additivity](#) of μ_0 . Note that

$$\left\{ \begin{array}{l} \bigcup_{n=1}^{\infty} (B_n \cap A) \supset E \cap A \\ \bigcup_{n=1}^{\infty} (B_n \cap A^c) \subset E \cap A^c \end{array} \right. \implies \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

since

$$\mu^*(E \cap A) \leq \mu^* \left(\bigcup_{n=1}^{\infty} (B_n \cap A) \right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A)$$

and

$$\mu^*(E \cap A^c) \leq \mu^* \left(\bigcup_{n=1}^{\infty} (B_n \cap A^c) \right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A^c).$$

We then see that for any $\epsilon > 0$, the inequality

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

holds, hence so does

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

which implies $\mathcal{A} \supset \mathcal{A}_0$.

The proof will be [continued](#)...

Lecture 6: Hahn-Kolmogorov Theorem and Extension.

18 Jan. 11:00

Let's continue the proof of [Theorem 1.3](#).

- Let $A \in \mathcal{A}_0$, we want to show that

$$\mu(A) = \mu_0(A).$$

– Firstly, let

$$B_i = \begin{cases} A, & \text{if } i = 1 \\ \emptyset, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,$$

hence $\bigcup_{i=1}^{\infty} B_i = A$, then we see that

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu_0(B_i) = \mu_0(A)$$

from the [definition](#) of μ^* and [countable additivity within the algebra](#) of μ_0 .

– Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A \cap B_1 \in \mathcal{A}_0$, $C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right) \in \mathcal{A}_0$ for $i \geq 2$ since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

are disjoint countable unions, by [countable additivity within the algebra](#), we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i) \implies \mu_0(A) \leq \mu^*(A)$$

by taking the infimum from the [definition](#) of μ^* .

Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every $A \in \mathcal{A}_0$, which implies

$$\mu(A) = \mu_0(A)$$

for every $A \in \mathcal{A}_0$ from [Theorem 1.2](#), where we extend μ^* to μ respect to \mathcal{A}_0 . ■

Definition 1.15 (HK extension). (\mathcal{A}, μ) obtained from [Theorem 1.3](#) is the *Hahn-Kolmogorov extensions* of (\mathcal{A}_0, μ_0) .

We can show the uniqueness of [HK extension](#).

Theorem 1.4 (Uniqueness of HK extension). Let \mathcal{A}_0 be an [algebra](#) on X , μ_0 be a [pre-measure](#) on \mathcal{A}_0 . Let (\mathcal{A}, μ) be the [HK extension](#) of (\mathcal{A}_0, μ_0) . Let (\mathcal{A}', μ') be another extension of (\mathcal{A}_0, μ_0) . Then if μ_0 is [σ-finite](#), $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Note. Notice that $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$ since they both extend \mathcal{A}_0 .

Proof. Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing the arbitrary cover of A and using the [definition](#) of μ^* .

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) < \infty$, and fix $\epsilon > 0$. Then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup_{i=1}^{\infty} B_i \supset A$ such that

$$\mu(A) + \epsilon = \mu^*(A) + \epsilon \geq \sum_{i=1}^{\infty} \mu_0(B_i) \stackrel{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

This implies that

$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$

where the first equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

$$\mu(B) = \lim_{N \rightarrow \infty} \mu\left(\bigcup_{i=1}^N B_i\right) \stackrel{8}{=} \lim_{N \rightarrow \infty} \mu'\left(\bigcup_{i=1}^N B_i\right) = \mu'(B),$$

hence,

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \stackrel{9}{\leq} \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon$$

for arbitrary ϵ , so we conclude $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is [σ-finite](#), so we know $X = \bigcup_{n=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ such that

$$\mu_0(X_n) < \infty.$$

Replacing X_n by $X_1 \cup \dots \cup X_n \in \mathcal{A}_0$, we may assume that

$$X_1 \subset X_2 \subset \dots$$

Then,

$$\forall_{n \in \mathbb{N}} \mu(A \cap X_n) < \infty \stackrel{!}{\implies} \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

From the continuity of [measure](#), we then have

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A \cap X_n) \leq \lim_{n \rightarrow \infty} \mu'(A \cap X_n) = \mu'(A).$$

■

⁸ $\mu = \mu'$ on \mathcal{A}_0 .

⁹From the first part.

Corollary 1.1. Let μ_0 be a [pre-measure](#) on [algebra](#) \mathcal{A}_0 on X . Suppose μ_0 is [\$\sigma\$ -finite](#), then

$\exists!$ [measure](#) μ on $\langle \mathcal{A}_0 \rangle$ that extends \mathcal{A}_0 .

Furthermore,

- The completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the [HK extension](#) of (\mathcal{A}_0, μ_0) .

-

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \forall_{i \in \mathbb{N}} \bigcup_{i=1}^{\infty} B_i \supset A \right\}$$

for all $A \in \overline{\langle \mathcal{A}_0 \rangle}$.

Lecture 7: Borel Measures

21 Jan. 11:00

1.5 Borel Measures on \mathbb{R}

We first introduce so-called *distribution function*.

Definition 1.16 (Distribution function). An [increasing](#)^a function

$$F: \mathbb{R} \rightarrow \mathbb{R}$$

and [right-continuous](#). F is then a *distribution function*.

^aHere, increasing means $F(x) \leq F(y)$ for $x < y$.

Example. Here are some examples of right-continuous functions.

1. $F(x) = x$.

2. $F(x) = e^x$.

3. Define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

4. Let $\mathbb{Q} := \{r_1, r_2, \dots\}$. Define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$

and

$$F(x) := \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$

Then F is a distribution function (hence right-continuous). This is shown in [Lemma A.1](#).

Note. If F is increasing, and

$$F(\infty) := \lim_{x \nearrow \infty} F(x), \quad F(-\infty) := \lim_{x \searrow -\infty} F(x)$$

exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.¹⁰

Now, we can define a *Borel measure* on $(X, \mathcal{B}(\mathbb{R}))$.

Definition 1.17 (Borel measure). A *Borel measure* is any **measure** μ defined on the **σ -algebra** of **Borel sets**.

Definition 1.18 (Locally finite). Let X be a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Note. Some authors will require a **Borel measure** equipped with the **locally finite** property. But formally, this is not so common.

Lemma 1.6. Let μ be a **locally finite Borel measure** on \mathbb{R} , then

$$F_\mu(x) = \begin{cases} \mu((0, x]), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu((x, 0]), & \text{if } x < 0 \end{cases}$$

is a **distribution function**.

Proof. To show F_μ is increasing, consider $x < y$ such that

$$F_\mu(x) \leq F_\mu(y)$$

by considering

- $x > 0$: Then $F_\mu(x) = \mu((0, x])$ and

$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$

- $x = 0$: Then $F_\mu(x) = 0$ and

$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$

since $y > 0$.

- $x < 0$: Follows the same argument with $x > 0$.

¹⁰There are distributions [FF99] Ch9., but these are different from distribution functions.

Now, we need to show F_μ is right-continuous. Firstly, assume that $x \geq 0$, then we see that

$$F_\mu(x) = \mu((0, x]) = \mu((0, x^+])$$

from the fact that a measure is right-continuous.¹¹ Now, if $x \leq 0$, the same argument follows since multiplying -1 will not change the fact that a [measure](#) is continuous. ■

Definition 1.19 (Half intervals). We call

$$\emptyset, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

half-intervals.

Lemma 1.7. Let \mathcal{H} be the collection of finite disjoint unions of [half-intervals](#). Then, \mathcal{H} is an [algebra](#) on \mathbb{R} .

Proof. We see that

- $\emptyset \in \mathcal{H}$. Clearly.
- To show \mathcal{H} is closed under complements, we have
 - $\emptyset^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$.
 - $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$.¹²
 - $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$.
 - $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$.
 - $(-\infty, \infty)^c = \emptyset \in \mathcal{H}$.
- \mathcal{H} is closed under finite unions, clearly.

■

¹¹Actually, a measure is always continuous.

¹²Since it's a two disjoint union of half intervals.

Proposition 1.3 (Distribution function defines a pre-measure). Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be a [distribution function](#). For a [half interval](#) I , define

$$\ell(I) := \ell_F(I) = \begin{cases} 0, & \text{if } I = \emptyset; \\ F(b) - F(a), & \text{if } I = (a, b]; \\ F(\infty) - F(a), & \text{if } I = (a, \infty]; \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b]; \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 := \mu_{0,F}$ as

$$\mu_{0,F}: \mathcal{H} \rightarrow [0, \infty]$$

by

$$\mu_0(A) = \sum_{k=1}^N \ell(I_k) \text{ if } A = \bigcup_{k=1}^N I_k,$$

where A is a finite disjoint union of [half intervals](#) I_1, \dots, I_N . Then, μ_0 is a [pre-measure](#) on \mathcal{H} .

Proof. We see that

1. μ_0 is well-defined.
2. $\mu_0(\emptyset) = 0$.
3. μ_0 is finite additive.
4. μ_0 is [countable additivity within \$\mathcal{H}\$](#) .

Suppose $A \in \mathcal{H}$ where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.¹³

Focus on the case $I = (a, b]$. Let

$$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

which is a disjoint union. Then we only need to check

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

- Since $(a, b] \supset \bigcup_{n=1}^N (a_n, b_n]$ for any fixed $N \in \mathbb{N}$, hence

$$\forall_{N \in \mathbb{N}} F(b) - F(a) \geq \sum_{n=1}^N (F(b_n) - F(a_n)).$$

¹³Since \mathcal{H} is only a collection of *finite* disjoint [half intervals](#), hence after considering $A = I$, we can apply the same argument iteratively and stop in finite steps. Formally, we can consider $H \in \mathcal{H}$, $H = \bigcup_{i=1}^{\infty} A^i$, where A^i being a [half interval](#). Then by the above argument, we have $A^i = I^i$ and so on.

By letting $N \rightarrow \infty$, we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

- Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

$$F(a') - F(a) < \epsilon.$$

For each $n \in \mathbb{N}$, $\exists b'_n > b_n$ such that

$$F(b'_n) - F(b_n) < \frac{\epsilon}{2^n}.$$

Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b'_n),$$

hence

$$\exists_{N \in \mathbb{N}} [a', b] \subset \bigcup_{n=1}^N (a_n, b'_n),^{14}$$

which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^N F(b'_n) - F(a_n).$$

Finally, we see that

$$\begin{aligned} F(b) - F(a) &\leq F(b) - F(a') + \epsilon \\ &\leq \sum_{n=1}^{\infty} (F(b'_n) - F(a_n)) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\epsilon}{2^n} \right) + \epsilon \\ &= \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) + 2\epsilon \end{aligned}$$

for any fixed $\epsilon > 0$, hence

$$F(b) - F(a) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

Combine these two inequalities, we have

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

as we desired.

¹⁴This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.



Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8: Lebesgue-Stieltjes Measure on \mathbb{R}

24 Jan. 11:00

To classify all **measures**, we now see this last theorem to complete the task.

Theorem 1.5 (Locally finite Borel measures on \mathbb{R}). We have

1. $F: \mathbb{R} \rightarrow \mathbb{R}$ a **distribution function**, then there exists a **unique locally finite Borel measure** μ_F on \mathbb{R} satisfying

$$\mu_F((a, b]) = F(b) - F(a)$$

for every $a < b$.

2. Suppose $F, G: \mathbb{R} \rightarrow \mathbb{R}$ are **distribution functions**. Then,

$$\mu_F = \mu_G$$

on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof.



HW.

Remark. **Theorem 1.5** simply states that given a **distribution function**, if we restrict our attention on **locally finite measures** on \mathbb{R} following our usual convention, then it defines the **measure** on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

1.6 Lebesgue-Stieltjes Measure on \mathbb{R}

We see that

$$F \text{ distribution function} \xRightarrow{!} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 1.20 (Lebesgue-Stieltjes measure). Given a **distribution function** F , we say μ_F on \mathcal{A}_{μ_F} is called the *Lebesgue-Stieltjes measure* corresponding to F .

Definition 1.21 (Lebesgue measure, Lebesgue σ -algebra). From **Definition 1.20**, if $F(x) = x$, then the induced $(\mathcal{A}_{\mu_F}, \mu_F)$ is denoted as (\mathcal{L}, m) , where \mathcal{L} is called *Lebesgue σ -algebra*, and m is called *Lebesgue measure*.

Remark. Recall that \mathcal{L} is induced by [Theorem 1.2](#), namely given m , for all $A \subset \mathbb{R}$, we have

$$\mathcal{L} := \left\{ A \subset \mathbb{R} \mid \forall_{E \subset \mathbb{R}} m(A) = m(A \cap E) + m(A \setminus E) \right\}$$

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).^{15}$$

Example. We first see some examples.

1. $\mu_F((a, b]) = F(b) - F(a)$. Then

- $\mu_F(\{a\}) = F(a) - F(a^-)$
- $\mu_F([a, b]) = F(b) - F(a^-)$
- $\mu_F((a, b)) = F(b^-) - F(a)$

This is so-called *discrete measure*.

2. We define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Then

- $\mu_F(\{0\}) = 1$
- $\mu_F(\mathbb{R}) = 1$
- $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. This is easy to see since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, hence

$$\begin{aligned} \mu_F(\mathbb{R} \setminus \{0\}) &= \mu_F((-\infty, 0) \cup (0, \infty)) \\ &= \underbrace{\mu_F((-\infty, 0))}_{0-0^{16}} + \underbrace{\mu_F((0, \infty))}_{1-1^{17}} = 0. \end{aligned}$$

We call that μ_F is the *Dirac measure* at 0.

3. Denote $\mathbb{Q} = \{r_1, r_2, \dots\}$, and we define

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n. \end{cases}$$

Then

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.

¹⁵Some text will use $x-$ and $x+$ instead of x^- and x^+ , respectively.

¹⁶It follows from $F(0^-) - F(-\infty) = 0 - 0 = 0$.

¹⁷It follows from $F(\infty) - F(0) = 1 - 1 = 0$.

- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$.

This is shown in [Lemma A.2](#).

4. If F is continuous at a , then $\mu_F(\{a\}) = 0$.
5. $F(x) = x$, then recall that we denote $\mu_F := m$, and we have
 - $m((a, b]) = m((a, b)) = m([a, b]) = b - a$.
6. $F(x) = e^x$
 - $\mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$.

Example (Middle thirds Cantor set). Let $C := \bigcap_{n=1}^{\infty} K_n$, where we have

$$\begin{aligned} K_0 &:= [0, 1] \\ K_1 &:= K_0 \setminus \left(\frac{1}{3}, \frac{2}{3} \right) \\ K_2 &:= K_1 \setminus \left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \\ &\vdots \\ K_n &:= K_{n-1} \setminus \bigcup_{k=1}^{3^{n-1}} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right). \end{aligned}$$

We see that C is uncountable and with $m(C) = 0$. And observe that $x \in C$ if and only if $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for some $a_n \in \{0, 2\}$. Hence, we can instead formulate K_n by

$$K_n = \bigcup_{\substack{a_i \in \{0, 2\} \\ 1 \leq i \leq n}} \left[\sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n} \right].$$

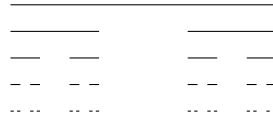


Figure 1: The top line corresponds to K_0 , and then K_1 , etc.

The proof of $m(C) = 0$ is given in [Lemma A.3](#).

1.6.1 Cantor Function

Consider F as follows. We define a function F to be 0 to the left of 0, and 1 to the right of 1. Then, define F to be $\frac{1}{2}$ on $(\frac{1}{3}, \frac{2}{3})$, $\frac{1}{4}$ on $(\frac{1}{9}, \frac{2}{9})$, $\frac{3}{4}$ on $(\frac{7}{9}, \frac{8}{9})$ and so on. This is so-called *Cantor Function*. We can show F is continuous and increasing, which makes F a distribution function. Also, we see that the measure this F induced is called *Cantor measure*.



Figure 2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

Cantor Measure μ_F		Lebesgue Measure m
$\mu_F(\mathbb{R} \setminus C) = 0$		$m(\mathbb{R} \setminus C) = \infty > 0$
$\mu_F(C) = 1$	\iff	$m(C) = 0$
$\mu_F(\{a\}) = 0$		$m(\{a\}) = 0$

Remark. μ_F and m are said to be **singular** to each other.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 1.8. Let μ be **Lebesgue-Stieltjes measure** on \mathbb{R} . Then we have

$$\begin{aligned} \mu(A) &\stackrel{!}{=} \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

for every $A \in \mathcal{A}_\mu$

Proof. The second equality follows from the **continuity of the measure**. ■

Remark. This is similar to

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - 1/n], \quad (a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n].$$

Lecture 9: Properties of Lebesgue-Stieltjes measure

26 Jan. 11:00

As previously seen. Let $X \subset [0, \infty]$. Recall that

- Finite supremum.

$$\alpha = \sup X < \infty \iff \begin{cases} \forall_{x \in X} \alpha \geq x \\ \forall_{\epsilon > 0} \exists_{x \in X} x + \epsilon \geq \alpha. \end{cases}$$

- Infinite supremum.

$$\alpha = \sup X = \infty \iff \forall_{L > 0} \exists_{x \in X} x \geq L.$$

This should be useful latter on.

Theorem 1.6 (Regularity). Let μ be [Lebesgue-Stieltjes measure](#). Then, for all $A \in \mathcal{A}_\mu$,

1. (outer regularity) $\mu(A) = \inf\{\mu(O) \mid O \supset A, O \text{ is open}\}$
2. (inner regularity) $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$

Proof. We check them separately.

1. DIY
2. Let $s := \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$, then by [monotonicity](#), we have $\mu(A) \geq s$. To show the other direction, we consider

- A is a bounded set.

Then $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, \overline{A} is also bounded $\implies \mu(\overline{A}) < \infty$. Fix $\epsilon > 0$, then by [outer regularity](#), there exists an open $O \supset \overline{A} \setminus A$, and $\mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \epsilon$. Let $K := \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$, we

show that

$$\mu(K) \geq \mu(A) - \epsilon.$$

DIY

- A is an unbounded set with $\mu(A) < \infty$.

Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \dots$, then

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) < \infty.$$

- A is an unbounded set with $\mu(A) = \infty$.

We can show that

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) = \infty.$$

Fix $L > 0$, then $\exists N$ such that $\mu(A_N) \geq L$.

■

Definition 1.22 (G_δ -set, F_σ -set). Let X be a topological space. Then

- A G_δ -set is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open.
- A F_σ -set is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.7. Let μ be a Lebesgue-Stieltjes measure. Then *TFAE*^a:

1. $A \in \mathcal{A}_\mu$
2. $A = G \setminus M$, G is a G_δ -set, M is a μ -null set.
3. $A = F \setminus N$, F is a F_σ -set, N is a μ -null set.

^aTFAE: The following are equivalent.

Proof. We see that (2.) \implies (1.) and (3.) \implies (1.) are clear.

- (1.) \implies (3.)

– Assume $\mu(A) < \infty$. From the inner regularity, we have

$$\forall n \in \mathbb{N} \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

Let $F = \bigcup_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ -null.

Check!

– Assume $\mu(A) = \infty$. Let $A = \bigcup_{k \in \mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

$$\forall k \in \mathbb{Z} \ A_k = F_k \cup N_k, \quad A = \underbrace{\left(\bigcup_k F_k \right)}_{F_\sigma\text{-set}} \cup \underbrace{\left(\bigcup_k N_k \right)}_{\mu\text{-null}}.$$

- (1.) \implies (2.)

We see that

$$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

■

Proposition 1.4. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0 \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using **outer regularity** and the fact that every open set is $\bigcup_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. ■

DIY

We now see some properties of **Lebesgue measure**.

Theorem 1.8. Let $A \in \mathcal{L}$, then we have $A + s \in \mathcal{L}$, $rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,

$$m(A + s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. ■

DIY

Example. We now see some examples.

1. Let $\mathbb{Q} = \{r_i\}_{i=1}^{\infty}$ which is dense in \mathbb{R} . Let $\epsilon > 0$, and

$$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i} \right).$$

We see that O is open and dense¹⁸ in \mathbb{R} . But we see

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$

Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$

2. There exists uncountable set A with $m(A) = 0$.
3. There exists A with $m(A) > 0$ but A contains no non-empty open intervals.
4. There exists $A \notin \mathcal{L}$. e.g. Vitali set.¹⁹
5. There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

Lecture 10: Integration

26 Jan. 11:00

2 Integration

2.1 Measurable Function

We start with a definition.

Definition 2.1 (Measurable function). Suppose (X, \mathcal{A}) , (Y, \mathcal{B}) are **measurable spaces**. Then we say $f: X \rightarrow Y$ is $(\mathcal{A}, \mathcal{B})$ -*measurable* if

$$\forall_{B \in \mathcal{B}} f^{-1}(B) \in \mathcal{A}.$$

Remark. If \mathcal{A} and \mathcal{B} are given, we'll sometimes say f is **measurable** if it'll not cause any confusions.

¹⁸https://en.wikipedia.org/wiki/Dense_set

¹⁹https://en.wikipedia.org/wiki/Vitali_set

Lemma 2.1. Given two measurable spaces (X, \mathcal{A}) and (Y, \mathcal{B}) , and suppose $\mathcal{B} = \langle \mathcal{E} \rangle$ for some $\mathcal{E} \subset Y$. Then,

$$f: X \rightarrow Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall_{E \in \mathcal{E}} f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part (\implies) is clear. On the other direction, we consider the following. Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $\mathcal{E} \subset \mathcal{D}$ by assumption
- \mathcal{D} is a σ -algebra

Check!

hence, we see that $\langle \mathcal{E} \rangle = \mathcal{B} \subset \mathcal{D}$ from Lemma 1.2. The result then follows from the definition of $(\mathcal{A}, \mathcal{B})$ -measurable. ■

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup_{\alpha} E_{\alpha}\right) = \bigcup_{\alpha} f^{-1}(E_{\alpha})$

Definition 2.2 (\mathcal{A} -measurable). Let (X, \mathcal{A}) be a measurable space. Then,

$$\left. \begin{array}{l} f: X \rightarrow \mathbb{R} \\ f: X \rightarrow \overline{\mathbb{R}} \\ f: X \rightarrow \mathbb{C} \end{array} \right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{ \begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f: X \rightarrow \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{array} \right.$$

Notation. Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$
- $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$.
- $\Re f$ is the real part of f , while $\Im f$ is the imaginary part of f .

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \implies$ Every function is \mathcal{A} -measurable.
- $\mathcal{A} = \{\emptyset, X\} \implies$ The only \mathcal{A} -measurable functions are constant functions.

Definition 2.3 (Lebesgue measurable). A Lebesgue measurable function f is a measurable function

$$f: (\mathbb{R}, \mathcal{L}) \rightarrow (\mathbb{C}, \mathcal{B}(\mathbb{C})).$$

Lemma 2.2. Given $f: X \rightarrow \mathbb{R}$, *TFAE*.

1. f is \mathcal{A} -measurable
2. $\forall a \in \mathbb{R}, f^{-1}((a, \infty)) \in \mathcal{A}$
3. $\forall a \in \mathbb{R}, f^{-1}([a, \infty)) \in \mathcal{A}$
4. $\forall a \in \mathbb{R}, f^{-1}((-\infty, a)) \in \mathcal{A}$
5. $\forall a \in \mathbb{R}, f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from Lemma 2.1 we just saw. ■

Remark (Operations preserve \mathcal{A} -measurability). Given $f, g: X \rightarrow \mathbb{R}$ and is \mathcal{A} -measurable, then

1. $\phi: \mathbb{R} \rightarrow \mathbb{R}$, \mathcal{A} -measurable²⁰, then

$$\phi \circ f: X \rightarrow \mathbb{R}$$

is \mathcal{A} -measurable.

2. $-f, 3f, f^2, |f|$ are all \mathcal{A} -measurable, and $\frac{1}{f}$ is \mathcal{A} -measurable if $f(x) \neq 0, \forall x \in X$.
3. $f + g$ is \mathcal{A} -measurable. We see this from

$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} (f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty)))$$

with Lemma 2.2.

4. $f \cdot g$ is \mathcal{A} -measurable. We see this from

$$f(x)g(x) = \frac{1}{2} ((f(x) + g(x))^2 - f(x)^2 - g(x)^2).$$

5. We see that

$$(f \vee g)(x) := \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) := \min\{f(x), g(x)\}$$

are \mathcal{A} -measurable.

6. Let $f_n: X \rightarrow \overline{\mathbb{R}}$ be \mathcal{A} -measurable. Then

$$\sup_{n \in \mathbb{N}} f_n, \inf_{n \in \mathbb{N}} f_n, \limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n$$

are \mathcal{A} -measurable.

Proof. Consider $\sup_{n \in \mathbb{N}} f_n =: g$, then

$$g^{-1}((a, \infty]) = \bigcup_{n \in \mathbb{N}} f_n^{-1}((a, \infty])$$

for $\sup_n f_n(x) = g(x) > a$. A similar argument can prove the case of $\inf_{n \in \mathbb{N}} f_n$. check

And notice that $\limsup_{n \rightarrow \infty} f_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n$, then the similar argument also proves this case. ■

7. If $\lim_{n \rightarrow \infty} f_n(x)$ converges for every $x \in X$, then f is \mathcal{A} -measurable.

8. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous

$\implies f$ is Borel measurable

$\implies f$ is Lebesgue measurable

since the preimage of an open set of a continuous function is open, then we consider $f^{-1}((a, \infty))$.

Definition 2.4 (Support). The *support* of function $f: X \rightarrow \overline{\mathbb{R}}$ is

$$\text{supp } f := \{x \in X \mid f(x) \neq 0\}.$$

Definition 2.5 (Positive and Negative part). For $f: X \rightarrow \overline{\mathbb{R}}$, let $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$,^a where we call f^+ the *positive part* of f while f^- the *negative part* of f .

^ai.e., $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$

Remark. If $\text{supp } f^+ \cap \text{supp } f^- = \emptyset$ and $f(x) = f^+(x) - f^-(x)$, then

f is \mathcal{A} -measurable $\iff f^+, f^-$ are \mathcal{A} -measurable.

Definition 2.6 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator) function* of E is

$$\chi_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is \mathcal{A} -measurable $\iff E \in \mathcal{A}$.

Definition 2.7 (Simple function). Let (X, \mathcal{A}) be a measurable space. Then a *simple function* $\phi: X \rightarrow \mathbb{C}$ that is \mathcal{A} -measurable and takes only finitely many values.

Remark. We see that if

$$\phi(X) = \{c_1, \dots, c_N\},$$

²⁰In this case, we also call it *Borel measurable*.

then

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \implies \phi = \sum_{i=1}^N \underbrace{c_i}_{\neq \pm\infty} \mathbb{1}_{\underbrace{E_i}_{\in \mathcal{A}}}.$$

Lecture 11: Integration of nonnegative functions

31 Jan. 11:00

As previously seen. For a [simple function](#) ϕ , c_i can actually be in \mathbb{C} .

Theorem 2.1. Given a [measurable space](#) (X, \mathcal{A}) and let $f: X \rightarrow [0, \infty]$, the followings are equivalent.

1. f is a [mathcal{A}-measurable](#) function.
2. There exists [simple functions](#) $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$ such that

$$\forall_{x \in X} \lim_{n \rightarrow \infty} \phi_n(x) = f(x)$$

i.e., f is a [pointwise upward](#) limit of [simple functions](#).

Proof. We'll prove both directions.

- It's clear that (2.) \implies (1.) from the fact that $f(x) = \sup_n \phi_n(x)$ and [the remark](#).
- We want to show that (1.) \implies (2.). Assume f is [mathcal{A}-measurable](#), and fix $n \in \mathbb{N}$.

Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. Also, for $0 \leq k \leq 2^{2^n} - 1$, $E_{n,k} = f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}]) \in \mathcal{A}$.

Then, define ϕ_n be

$$\phi_n = \sum_{k=0}^{2^{2^n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n},$$

we have

- $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$ for every $x \in X$
- $\forall x \in X \setminus F_n$, we have $0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$

Furthermore, we see that

$$F_1 \supset F_2 \supset \dots, \quad \bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\}),$$

then

- $x \in f^{-1}([0, \infty]) = X \setminus \bigcap_{n=1}^{\infty} F_n \implies \lim_{n \rightarrow \infty} \phi_n(x) = f(x)$
- $x \in f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} F_n \implies f_n(x) \geq 2^n \implies \lim_{n \rightarrow \infty} \phi_n(x) = \infty = f(x)$

■

Corollary 2.1. If f is bounded on a set $A \subset \mathbb{R}$, i.e., $\exists L > 0$ such that

$$\forall_{x \in A} |f(x)| \leq L,$$

then there exists a sequence of [simple functions](#) $\{\phi_n\}$ such that $\phi_n \rightarrow f$ [uniformly](#) on A .

Proof.

■

DIY

Corollary 2.2. If $f: X \rightarrow \mathbb{C}$ is a [measurable function](#) if and only if there exists [simple functions](#) $\phi_n: X \rightarrow \mathbb{C}$ such that

$$0 \leq |\phi_1(x)| \leq |\phi_2(x)| \leq \dots \leq |f(x)|$$

with

$$\forall_{x \in X} \lim_{n \rightarrow \infty} \phi_n(x) = f(x).$$

Proof.

■

DIY

2.2 Integration of Nonnegative Functions

We start with our first definition about integral.

Definition 2.8 (Integration of nonnegative function). Let (X, \mathcal{A}, μ) be a [measure space](#), and $\phi: X \rightarrow [0, \infty]$ such that

$$\phi = \sum_{i=1}^N c_i \mathbb{1}_{E_i}$$

be a [simple function](#). Define

$$\int \phi = \int \phi \, d\mu = \int_X \phi \, d\mu = \sum_{i=1}^N c_i \mu(E_i).$$

Furthermore, for $A \in \mathcal{A}$,

$$\int_A \phi = \int \phi \, d\mu = \int \phi \mathbb{1}_A \, d\mu.$$

Note. Note that

- In the expression $\sum_{i=1}^N c_i \mu(E_i)$, we're using the convention $0 \cdot \infty = 0$.

- The function $\phi \mathbb{1}_A$ is also a **simple function** since both ϕ and $\mathbb{1}_A$ are **simple function**.

Proposition 2.1. Suppose we have $\phi, \psi \geq 0$ be two **simple functions**. Then,

- **Definition 2.8** is well-defined.
- $\int c\phi = c \int \phi$ for $c \in [0, \infty)$.
- $\int \phi + \psi = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x)$ for all $x \implies \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi d\mu$ is a **measure** on (X, \mathcal{A}) .

Proof.



DIY

Definition 2.9 (Generalization of Integration of nonnegative function). Given (X, \mathcal{A}, μ) with $f: X \rightarrow [0, \infty]$ be **\mathcal{A} -measurable**. Define

$$\int f = \int f d\mu = \sup \left\{ \int \phi : 0 \leq \phi \leq f \text{ such that } \phi \text{ is simple} \right\}.$$

Note. Note that

- If f is a **simple function**, the **Definition 2.8** and **Definition 2.9** of $\int f$ are the same.
- $\int cf = c \int f$ for $c \in [0, \infty)$.
- If $f \geq g \geq 0 \implies \int f \geq \int g$.
- But $\int f + g = \int f + \int g$ is not trivial.

Theorem 2.2 (Monotone Convergence Theorem (MCT)). Given (X, \mathcal{A}, μ) be a **measure space**. Then if

- $f_n: X \rightarrow [0, \infty]$ be **\mathcal{A} -measurable** for every $n \in \mathbb{N}$;
- $0 \leq f_1(x) \leq f_2(x) \leq \dots$ for every $x \in X$;
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for every $x \in X$,

we have

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

Proof. Note that if $\lim_{n \rightarrow \infty} \int f_n$ exists, then it's equal to $\sup_n \int f_n$.

Then

- $f_n \leq f \implies \int f_n \leq \int f \implies \lim_{n \rightarrow \infty} \int f_n \leq \int f$.

- Fix a **simple function** $0 \leq \phi \leq f$, then it's enough to show $\lim_{n \rightarrow \infty} \int f_n \geq \int \phi$.

We first fix $\alpha = (0, 1)$, then it's also enough to show

$$\lim_{n \rightarrow \infty} \int f_n \geq \alpha \int \phi.$$

Let $A_n := \{x \in X \mid f_n(x) \geq \alpha \phi(x)\}$, then since f_n is **measurable**,

- $A_n \in \mathcal{A}$
- $A_1 \subset A_2 \subset A_3 \subset \dots$
- $\bigcup_{n=1}^{\infty} A_n = X$

Check!

We then have

$$\int f_n \geq \int f_n \mathbb{1}_{A_n} \geq \int \alpha \phi \mathbb{1}_{A_n} = \alpha \int_{A_n} \phi = \alpha \nu(A_n)$$

where $\nu(A) = \int_A \phi$ is a **measure**. This implies

$$\lim_{n \rightarrow \infty} \int f_n \geq \alpha \lim_{n \rightarrow \infty} \nu(A_n) \stackrel{21}{=} \alpha \nu(X) = \alpha \int \phi.$$

■

Corollary 2.3 (Linearity of nonnegative integral). Let $f, g \geq 0$ be **measurable**, then

$$\int f + g = \int f + \int g.$$

Proof. There exists **simple functions** ϕ_n and ψ_n such that

- $0 \leq \phi_1 \leq \phi_2 \leq \dots$ and $\phi_n \rightarrow f$ **pointwise**
- $0 \leq \psi_1 \leq \psi_2 \leq \dots$ and $\psi_n \rightarrow g$ **pointwise**

Then,

$$\int (f + g) \stackrel{!}{=} \lim_{n \rightarrow \infty} \int (\phi_n + \psi_n) = \lim_{n \rightarrow \infty} \int \phi_n + \int \psi_n \stackrel{!}{=} \int f + \int g.$$

■

Lecture 12: Fatou's Lemma

2 Feb. 11:00

We start with a useful corollary.

²¹This follows from the **continuity of measure from below**

Corollary 2.4 (Tonelli's theorem for nonnegative series and integrals). Given $g_n \geq 0$ for every $n \in \mathbb{N}$ and let g_n be measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Remark. Recall that we have seen [two series case](#) before. We'll later see two integrals cases.

Proof. Let $f_N := \sum_{n=1}^N g_n$ such that $\lim_{N \rightarrow \infty} f_N = \sum_{n=1}^{\infty} g_n =: f$, then since $g_n \geq 0$, we have $0 \leq f_1 \leq f_2 \leq \dots$ with

$$\lim_{N \rightarrow \infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

By [Theorem 2.2](#), we have

$$\lim_{N \rightarrow \infty} \underbrace{\int \sum_{n=1}^N g_n}_{f_N} = \underbrace{\int \sum_{n=1}^{\infty} g_n}_f.$$

Now, since the terms in the limit on the left-hand side is just a finite sum, by [Corollary 2.3](#), we have

$$\underbrace{\lim_{N \rightarrow \infty} \sum_{n=1}^N \int g_n}_{\sum_{n=1}^{\infty} \int g_n} = \lim_{N \rightarrow \infty} \int \sum_{n=1}^N g_n = \int \sum_{n=1}^{\infty} g_n,$$

hence

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

■

Theorem 2.3 (Fatou's Lemma). Suppose $f_n \geq 0$ and measurable, then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Remark. Recall that

$$\liminf_{n \rightarrow \infty} f_n := \lim_{k \rightarrow \infty} \inf_{n \geq k} f_n = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f_n$$

and

$$\exists \lim_{n \rightarrow \infty} a_n \iff \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

Proof. Let $g_k = \inf_{n \geq k} f_n$, then g_k is measurable and $0 \leq g_1 \leq g_2 \leq \dots$. Now, from Theorem 2.2, we have

$$\int \lim_{k \rightarrow \infty} g_k = \lim_{k \rightarrow \infty} \int g_k.$$

Notice that the left-hand side is just $\int \liminf_{n \rightarrow \infty} f_n$, while the right-hand side is just $\lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n$, i.e.,

$$\int \liminf_{n \rightarrow \infty} f_n = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n.$$

We see that we want to take the inf outside the integral on the right-hand side. Observe that

$$\forall_{m \geq k} \inf_{n \geq k} f_n \leq f_m \implies \forall_{m \geq k} \int \inf_{n \geq k} f_n \leq \int f_m \implies \int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m.$$

Then, we have

$$\int \liminf_{n \rightarrow \infty} f_n = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n \leq \lim_{k \rightarrow \infty} \inf_{m \geq k} \int f_m = \liminf_{m \rightarrow \infty} \int f_m.$$

■

Example. Given $(\mathbb{R}, \mathcal{L}, m)$.

1. **Escape to horizontal infinity.** Let $f_n := \mathbb{1}_{(n, n+1)}$. We immediately see that

- $f_n \rightarrow 0$ pointwise
- $\int f_n = 1$ for every n
- $\int f = 0$

From Theorem 2.3, we have a strict inequality

$$0 = \int \liminf_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} \int f_n = 1.$$

2. **Escape to width infinity.** Let $f_n := \frac{1}{n} \mathbb{1}_{(0, n)}$.
3. **Escape to vertical infinity.** Let $f_n := n \mathbb{1}_{(0, \frac{1}{n})}$.

Lemma 2.3 (Markov's inequality). Let $f \geq 0$ be measurable. Then

$$\forall_{c \in (0, \infty)} \mu(\{x \mid f(x) \geq c\}) \leq \frac{1}{c} \int f.$$

Proof. Denote $\{x \mid f(x) \geq c\} =: E$, then

$$f(x) \geq c \mathbb{1}_E(x) \implies \int f \geq c \int \mathbb{1}_E = c \cdot \mu(E).$$

■

Remark. Notice that $E = f^{-1}([c, \infty])$, hence E is measurable.

Proposition 2.2. Let $f \geq 0$ be measurable. Then,

$$\int f = 0 \iff f = 0 \text{ a.e..}$$

i.e.,

$$\int f \, d\mu = 0 \iff \mu(A) = 0$$

where $A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$.

Proof. Firstly, assume that $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^N c_i \mathbb{1}_{E_i}$$

where E_i are disjoint and $c_i \in (0, \infty)$. Then,

$$\begin{aligned} \int \phi &= \sum_{i=1}^N c_i \mu(E_i) = 0 \\ \iff \mu(E_1) &= \dots = \mu(E_N) = 0 \\ \iff \mu(A) &= 0, \quad A = \bigcup_{i=1}^N E_i. \end{aligned}$$

Now, assume that f is a general function where $f \geq 0$ is the only constraint.

1. Assume $\mu(A) = 0$ (i.e., $f = 0$ a.e.). Let $0 \leq \phi \leq f$, where ϕ is simple. Then

$$\forall_{x \in A^c} \phi(x) = 0$$

since $f(x) = 0, \forall x \in A^c$. This implies that $\phi = 0$ a.e. since $\mu(A) = 0$, so $\int \phi = 0$. We then have

$$\int f = 0$$

from Definition 2.9.

2. Assume $\int f = 0$. Let $A_n = f^{-1}([\frac{1}{n}, \infty])$. Then we see that

- $A_1 \subset A_2 \subset \dots$
- $\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} [\frac{1}{n}, \infty]\right) = f^{-1}((0, \infty)) = A$.

We then have

$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \stackrel{!}{\leq} n \int f = 0,$$

which further implies

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n) = 0$$

from the [continuity of measure from below](#).

■

Corollary 2.5. If $f, g \geq 0$ are both [measurable](#) and $f = g$ [a.e.](#), then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \neq g(x)\}$ ²². Then by assumption, $\mu(A) = 0$, hence

$$f \mathbb{1}_A = 0 \text{ [a.e.](#), } g \mathbb{1}_A = 0 \text{ [a.e.](#)..}$$

This further implies that

$$\begin{aligned} \int f &= \int f(\mathbb{1}_A + \mathbb{1}_{A^c}) \\ &\stackrel{!}{=} \int f \mathbb{1}_A + \int f \mathbb{1}_{A^c} \\ &= \int f \mathbb{1}_{A^c} = \int g \mathbb{1}_{A^c} \\ &= \int g \mathbb{1}_{A^c} + \int g \mathbb{1}_A = \int g. \end{aligned}$$

■

Corollary 2.6. Let $f_n \geq 0$ be [measurable](#). Then

1.
$$\left. \begin{array}{l} 0 \leq f_1 \leq f_2 \leq \dots \leq f \text{ [a.e.](#) } \\ \lim_{n \rightarrow \infty} f_n = f \text{ [a.e.](#) } \end{array} \right\} \implies \lim_{n \rightarrow \infty} \int f_n = \int f.$$
2.
$$\lim_{n \rightarrow \infty} f_n = f \text{ [a.e.](#) } \implies \int f \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Proof.

■

DIY

Remark. Almost all the theorems we've proved can be replaced by theorems dealing with [almost everywhere](#) condition.

Lecture 13: Integration of Complex Functions

4 Feb. 11:00

2.3 Integration of Complex Functions

As usual, we start with a definition.

²² A is [measurable](#) indeed.

Definition 2.10 (Integrable). Let (X, \mathcal{A}, μ) be a **measure space** and let $f: X \rightarrow \overline{\mathbb{R}}$ and $g: X \rightarrow \mathbb{C}$ be **measurable**.^a

Then f, g are called *integrable* if $\int |f| < \infty$ and $\int |g| < \infty$, and we define

$$\int f = \int f^+ - \int f^-, \quad \int g = \int \Re g + i \int \Im g.$$

Furthermore, for $f: X \rightarrow \overline{\mathbb{R}}$, we define

$$\int f = \begin{cases} \infty, & \text{if } \int f^+ = \infty, \int f^- < \infty; \\ -\infty, & \text{if } \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

^aRecall that for a complex-valued function like g , this means that both $\Re g$ and $\Im g$ are **measurable**.

We now see a lemma.

Lemma 2.4. Let $f, g: X \rightarrow \overline{\mathbb{R}}$ or \mathbb{C} **integrable**. Assume that $f(x) + g(x)$ is well-defined for all $x \in X$.^a

Then we have

1. $f + g, cf$ for all $c \in \mathbb{C}$ are **integrable**.
2. $\int f + g = \int f + \int g$. This is not trivial since $(f + g)^+ \neq f^+ + g^+$.
3. $|\int f| \leq \int |f|$.

^aThat is, we never see $\infty + (-\infty)$ or $(-\infty) + \infty$.

Proof. Check [FF99] page 53. ■

Lemma 2.5. Let (X, \mathcal{A}, μ) be a **measure space** and let f be an **integrable** function on X . Then

1. f is finite **a.e.** i.e., $\{x \in X \mid |f(x)| = \infty\}$ is a **null set**.
2. The set $\{x \in X \mid f(x) \neq 0\}$ is **σ -finite**.

Proof. _____ ■

HW 5
Q8 by
Lemma 2.3

Proposition 2.3. Let (X, \mathcal{A}, μ) be a **measure space**, then

1. If h is **integrable** on X , then

$$\forall_{E \in \mathcal{A}} \int_E h = 0 \iff \int |h| = 0 \iff h = 0 \text{ a.e.}$$

2. If f, g are **integrable** on X , then

$$\forall_{E \in \mathcal{A}} \int_E f = \int_E g \iff f = g \text{ a.e.}$$

Proof. We prove this one by one.

1. We see that the second equivalence is done in **Proposition 2.2**, hence we prove the first equivalence only. Since we have

$$\int |h| = 0 \implies \left| \int_E h \right| \leq \int_E |h| \leq \int |h| = 0,$$

which shows one implication. Now assume that $\int_E h = 0$ for all $E \in \mathcal{A}$, then we can write h as

$$h = u + iv = (u^+ - u^-) + i(v^+ - v^-).$$

Let $B := \{x \in X \mid u^+(x) > 0\}$, then by assumption, we have

$$0 = \int_B h = \Re \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+,$$

hence $u^+ = 0$ **almost everywhere**. Similarly, we have u^-, v^+, v^- are all zero **almost everywhere**. This gives us that h is zero **almost everywhere** as desired.

2. _____

DIY

■

Theorem 2.4 (Dominated Convergence Theorem). Let (X, \mathcal{A}, μ) be a **measure space**, and

- Let f_n be **integrable** on X .
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ **almost everywhere**.
- There is a $g: X \rightarrow [0, \infty]$ such that g is **integrable** and

$$\forall_{n \in \mathbb{N}} |f_n(x)| \leq g(x) \text{ a.e.}$$

Then we have

$$\lim_{n \rightarrow \infty} \int f_n = \int f = \int \lim_{n \rightarrow \infty} f_n.$$

Proof. Let F be the countable union of [null set](#) on which the three conditions may fail. Then we see that after modifying the definition of f_n, f and g on F , we may assume that all three conditions hold everywhere since modifying on a [null set](#) does not change the integral.

We now consider the \mathbb{R} -valued case only. Note that the second and the third conditions imply that f is [integrable](#) since $|f| \leq g(x)$. We then see that $g + f_n \geq 0$ and $g - f_n \geq 0$ because $-g \leq f_n \leq g$. From [Theorem 2.3](#), we have

 Check
 \mathbb{C} -valued
 case

$$\int g + f \leq \liminf_{n \rightarrow \infty} \int g + f_n, \quad \int g - f \leq \liminf_{n \rightarrow \infty} \int g - f_n.$$

From the [linearity of integral](#), we have

$$\int g + \int f \leq \int g + \liminf_{n \rightarrow \infty} \int f_n, \quad \int g - \int f \leq \int g - \liminf_{n \rightarrow \infty} \int f_n.$$

Now, since $\int g < \infty$, we can cancel it, which gives

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n, \quad -\int f \leq \liminf_{n \rightarrow \infty} \int -f_n = -\limsup_{n \rightarrow \infty} \int f_n,$$

which implies

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n \leq \limsup_{n \rightarrow \infty} \int f_n \leq \int f.$$

This shows that the limit exists, and the desired result indeed holds. ■

Corollary 2.7 (Tonelli's theorem for series and integrals). Suppose f_n are [integrable](#) functions such that

$$\sum_{n=1}^{\infty} \int |f_n| < \infty,$$

then we have

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

Proof. Take $G(x)$ to be

$$G(x) := \sum_{n=1}^{\infty} |f_n(x)|,$$

then we see

$$G(x) \geq |F_N(x)|$$

where

$$F_N(x) := \sum_{n=1}^N f_n(x).$$

By [Corollary 2.4](#), we have

$$\int G(x) = \sum_{n=1}^{\infty} \int |f_n(x)| < \infty.$$

Lastly, from [Theorem 2.4](#), the result follows. ■

Remark. Compare to [Corollary 2.4](#), we see that we further generalize the result!

Lecture 14: L^1 Space

7 Feb. 11:00

2.4 L^1 Space

We now introduce another space called L^p spaces, which are function spaces defined using a natural generalization of the [p-norm](#) for finite-dimensional vector spaces. We sometimes call it Lebesgue spaces also.

Before we start, we need to define a *norm*.

Definition 2.11 (Seminorm). Let V be a vector space over field \mathbb{R} or \mathbb{C} . A *seminorm* on V is

$$\|\cdot\| : V \rightarrow [0, \infty)$$

such that

- $\|cv\| = |c| \|v\|$ for every $v \in V$ and every scalar c .
- $\|v + w\| \leq \|v\| + \|w\|$ for every $v, w \in V$.

Definition 2.12 (Norm). A *norm* is a [seminorm](#) with

- $\|v\| = 0 \iff v = 0$.

Lemma 2.6. A [normed](#) vector space is a metric space with metric

$$\rho(v, w) = \|v - w\|.$$

Proof.



DIY

Example (p -norm). $V = \mathbb{R}^d$ with

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^d |x_i|^p \right)^{1/p}, & \text{if } p \in [0, \infty); \\ \max_{1 \leq i \leq d} |x_i|, & \text{if } p = \infty \end{cases}$$

is a [normed](#) vector space. The unit ball

$$\{x \in \mathbb{R}^d \mid \|x\|_p \leq 1\}$$

for different p has the following figures.



Remark. All $\|\cdot\|_p$ norms induce the same topology. i.e., if U is open in p -norm, it is open in p' -norm as well.

Note. Recall that we say f is **integrable** means

$$\int |f| < \infty,$$

and if $f = g$ **a.e.**, then

$$\int f = \int g$$

Definition 2.13 (L^1 Space). Given (X, \mathcal{A}, μ) ,

$$f \in L^1(X, \mathcal{A}, \mu) (= L^1(X, \mu) = L^1(X) = L^1(\mu))$$

means that f is an **integrable** function on X .

Lemma 2.7. $L^1(X, \mathcal{A}, \mu)$ is a vector space with **seminorm**

$$\|f\|_1 = \int |f|.$$

Proof.

■

Check this
is indeed a
seminorm.

Definition 2.14 (L^1 Space with equivalence class). Define $f \sim g$ if $f = g$ **a.e.**, then

$$L^1(X, \mathcal{A}, \mu) / \sim = L^1(X, \mathcal{A}, \mu),$$

i.e., we simply denote the collection of equivalence classes by itself.^a

^aBy some abusing of notation of L^1 .

Remark. We have

- With Definition 2.14, $L^1(X, \mathcal{A}, \mu)$ is a **normed** vector space.
- We say that the L^1 -metric $\rho(f, g)$ is simply

$$\rho(f, g) = \int |f - g|.$$

2.4.1 Dense Subsets of L^1

Note. Recall the definition of a *dense set*²³.

Definition 2.15 (Step function). A *step function* on \mathbb{R} is

$$\psi = \sum_{i=1}^N c_i \mathbb{1}_{I_i},$$

where I_i is an interval.

Theorem 2.5. We have the following.

1. {integrable simple functions} is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1 -metric).
2. $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$, where μ is a Lebesgue-Stieltjes-measure. Then {integrable simple functions} is dense in $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu)$.
3. $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

Notation. We denote the collection of continuous functions with compact support by $C_c(\mathbb{R})$.

Proof. We prove this one by one.

1. Since there exists simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|$, where $\phi_n \rightarrow f$ pointwise. Then by Theorem 2.4, we have

$$\lim_{n \rightarrow \infty} \int \underbrace{|f_n - f|}_{\leq |\phi_n| + |f| \leq 2|f|} = 0$$

where $2|f|$ is in L^1 .

2. Let $\mathbb{1}_E$ approximate by $\sum_{i=1}^{\infty} c_i \mathbb{1}_{I_i}$. From Theorem 1.6 for Lebesgue-Stieltjes-measure,

$$\forall \epsilon' > 0 \exists I = \bigcup_{i=1}^N I_i \text{ such that } \mu(E \triangle I) \leq \epsilon'.$$

²³https://en.wikipedia.org/wiki/Dense_set

3. To approximate $\mathbb{1}_{(a,b)}$, we simply consider function $g \in C_c(\mathbb{R})$ such that

$$\int |\mathbb{1}_{(a,b)} - g| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

■

Lecture 15: Riemann Integral

9 Feb. 11:00

2.5 Riemann Integrability

We are now working in $(\mathbb{R}, \mathcal{L}, m)$. Let's first revisit the definition of Riemann Integral. Let P be a partition of $[a, b]$ as

$$P = \{a = t_0 < t_1 < \dots < t_k = b\}.$$

Then the *lower Riemann sum* of f using P is equal to L_P , which is defined as

$$L_P = \sum_{i=1}^K \left(\inf_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1}),$$

and the *upper Riemann sum* of f using P is equal to U_P , which is defined as

$$U_P = \sum_{i=1}^K \left(\sup_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1}).$$

Then we call

- *Lower Riemann integral* of $f = \underline{I} = \sup_P L_P$
- *Upper Riemann integral* of $f = \bar{I} = \inf_P U_P$

Definition 2.16 (Riemann (Darboux) integrable). A bounded function $f: [a, b] \rightarrow \mathbb{R}$ is called *Riemann (Darboux) integrable* if

$$\underline{I} = \bar{I}$$

If so, then $\underline{I} = \bar{I} = \int_a^b f(x) dx$.

Note. We see that

- If $P \subset P'$, then

$$L_P \leq L_{P'}, \quad U_{P'} \leq U_P.$$

- Recall that continuous functions on $[a, b]$ are [Riemann integrable](#) on $[a, b]$.

Theorem 2.6. Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function. Then

1. If f is Riemann integrable, then f is Lebesgue measurable, thus Lebesgue integrable. Further,

$$\int_a^b f(x) dx = \int_{[a,b]} f dm.$$

2. If f is Riemann integrable $\iff f$ is continuous Lebesgue a.e.^a

^aHere, we mean that the set where f is discontinuous is a null set under Lebesgue measure.

Proof. There exists $P_1 \subset P_2 \subset \dots$ such that $L_{P_n} \nearrow \underline{I}$ and $U_{P_n} \searrow \bar{I}$.²⁴ Now, define simple (step) functions

$$\begin{aligned} \bullet \phi_n &= \sum_{i=1}^K \left(\inf_{[t_{i-1}, t_i]} f \right) \mathbb{1}_{(t_{i-1}, t_i]} \\ \bullet \psi_n &= \sum_{i=1}^K \left(\sup_{[t_{i-1}, t_i]} f \right) \mathbb{1}_{(t_{i-1}, t_i]} \end{aligned}$$

if $P_n = \{a = t_0 < t_1 < \dots < t_K\}$. Let $\phi := \sup_n \phi_n$ and $\psi := \inf_n \psi_n$. We then see that ϕ, ψ are Lebesgue (Borel) measurable function.

Note. Note that

$$\begin{aligned} \bullet \exists M > 0 \text{ such that } \forall_{n \in \mathbb{N}} |\phi_n|, |\psi_n| &\leq M \mathbb{1}_{[a,b]} \\ \bullet \int \phi_n dm &= L_{P_n}, \int \psi_n dm = U_{P_n} \end{aligned}$$

By [Theorem 2.4](#) and the fact that $M \mathbb{1}_{[a,b]} \in L^1(\mathbb{R}, \mathcal{L}, m)$, we have

$$\underline{I} = \lim_{n \rightarrow \infty} \int \phi_n dm = \int \phi dm, \quad \bar{I} = \int \psi dm.$$

Thus,

$$\begin{aligned} f \text{ is Riemann integrable} &\iff \int \phi = \int \psi \\ &\iff \int (\psi - \phi) = 0 \\ &\iff \psi = \phi \text{ Lebesgue a.e.} \end{aligned}$$

■

2.6 Modes of Convergence

As we should already see, there are different *modes* of convergence. Let's formalize them.

²⁴Here, we took refinements of P_n if needed.

Definition 2.17 (Pointwise, Uniformly convergence). Let

$$f_n, f: X \rightarrow \mathbb{C},$$

and $S \subset X$. Then we say

- $f_n \rightarrow f$ *pointwise* on S if

$$\forall_{x \in S} \forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{n \geq N} |f_n(x) - f(x)| < \epsilon.$$

- $f_n \rightarrow f$ *uniformly* on S if

$$\forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{x \in S} \forall_{n \geq N} |f_n(x) - f(x)| < \epsilon.$$

Remark. We see that we can replace $\forall \epsilon > 0$ by $\forall k \in \mathbb{N}$ with ϵ changing to $\frac{1}{k}$.

Lemma 2.8. Let $B_{n,k}$ be

$$B_{n,k} := \left\{ x \in X \mid |f_n(x) - f(x)| < \frac{1}{k} \right\}.$$

Then

1. $f_n \rightarrow f$ *pointwise* on S if and only if

$$S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$

2. $f_n \rightarrow f$ *uniformly* on S if and only if $\exists N_1, N_2, \dots \in \mathbb{N}$ such that

$$S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Proof. This essentially follows from [Definition 2.17](#). ■

Definition 2.18 (Converges a.e., Converges in L^1). Let (X, \mathcal{A}, μ) be a *measure space*. Assuming that f_n, f are *measurable function*, then

1. $f_n \rightarrow f$ *almost everywhere* means

$$\exists \text{ null set } E \text{ such that } f_n \rightarrow f \text{ pointwise on } E^c.$$

2. $f_n \rightarrow f$ *in L^1* means

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0.$$

Example. Given $(\mathbb{R}, \mathcal{L}, m)$ and let $f = 0$. We see the followings.

1. $f_n = \mathbb{1}_{(n, n+1)}$
2. $f_n = \frac{1}{n} \mathbb{1}_{(0, n)}$
3. $f_n = n \mathbb{1}_{(0, \frac{1}{n})}$
4. **Typewriter functions.**



Lecture 16: Modes of Convergence

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Let's start with a proposition.

Proposition 2.4 (Fast L^1 convergence leads to a.e. convergence). Let (X, \mathcal{A}, μ) be a **measure space**, and f_n, f are all **measurable** functions on X . Then

$$\sum_{n=1}^{\infty} \|f_n - f\|_1 < \infty \implies f_n \rightarrow f \text{ a.e.}$$

Proof. Let

$$E := \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c = \{x \in X \mid f_n(x) \not\rightarrow f(x)\}.$$

By Lemma 2.3, we see that

$$\forall_k \forall_N \mu(B_{n,k}^c) \leq k \int |f_n - f| \implies \forall_k \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) \leq \sum_{n=N}^{\infty} k \|f_n - f\|_1 \rightarrow 0$$

as $N \rightarrow \infty$. Now, by continuity of measure from above,

$$\forall_k \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = \lim_{N \rightarrow \infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \mu(E) = 0$$

since $f_n \rightarrow f$ pointwise on E^c . ■

Corollary 2.8. Given $\{f_n\}_n$ such that $f_n \rightarrow f$ in L^1 , there exists a subsequence $\{f_{n_j}\}_{n_j}$ where $f_{n_j} \rightarrow f$ a.e.

Proof. Since

$$\forall_{j \in \mathbb{N}} \forall_{n_j \in \mathbb{N}} \|f_{n_j} - f\|_1 \leq \frac{1}{j^2}.$$

Then,

$$\sum_{j=1}^{\infty} \|f_{n_j} - f\|_1 < \infty.$$

From Proposition 2.4, we have the desired result. ■

Definition 2.19 (Converge in measure). Let f_n, f be measurable functions on (X, \mathcal{A}, μ) . Then $f_n \rightarrow f$ in measure means

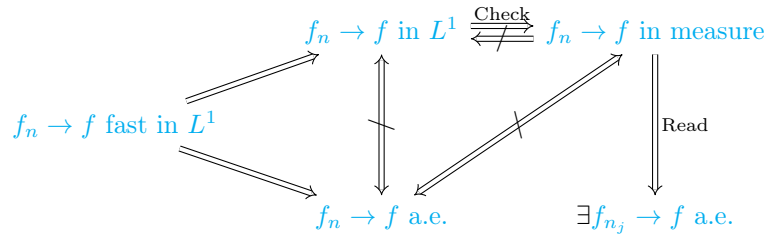
$$\forall_{\epsilon > 0} \lim_{n \rightarrow \infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

Example. Let $f_n = n \mathbb{1}_{(0, \frac{1}{n})}$ and $f = 0$. We see that

$$\forall \epsilon > 0 \quad \{x \in X \mid |f_n(x) - f(x)| > \epsilon\} = \left(0, \frac{1}{n}\right),$$

$f_n \rightarrow 0$ in measure. (Recall that $f_n \not\rightarrow 0$ in L^1)

Remark. We see that



Definition 2.20 (Uniformly a.e., Almost uniformly). Let f_n, f be measurable functions on (X, \mathcal{A}, μ) .

1. $f_n \rightarrow f$ *uniformly almost everywhere* means \exists null set F such that $f_n \rightarrow f$ *uniformly* on F^c .
2. $f_n \rightarrow f$ *almost uniformly* means $\forall \epsilon > 0 \exists F \in \mathcal{A}$ such that $\mu(F) < \epsilon$, $f_n \rightarrow f$ *uniformly* on F^c .

Lemma 2.9. We have

$$f_n \rightarrow f \text{ uniformly on } S \iff \exists N_1, N_2, \dots \in \mathbb{N} \ S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Theorem 2.7 (Egorov's Theorem). Let f_n, f be measurable functions on (X, \mathcal{A}, μ) . Suppose $\mu(X) < \infty$, then

$$f_n \rightarrow f \text{ a.e.} \iff f_n \rightarrow f \text{ almost uniformly.}$$

Proof. We prove two directions.

• \Leftarrow

DIY

• \Rightarrow Fix $\epsilon > 0$. We see that

$$\begin{aligned} f_n \rightarrow f \text{ a.e.} &\implies \mu \left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0 \\ &\implies \forall_k \mu \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0. \end{aligned}$$

From *continuity of measure from above* and $\mu(X) < \infty$, we further have

$$\forall_k \lim_{N \rightarrow \infty} \mu \left(\bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0 \implies \forall_k \exists_{N_k \in \mathbb{N}} \mu \left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c \right) < \frac{\epsilon}{2^k}.$$

Now, let

$$F := \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c,$$

we see that $\mu(F) < \epsilon$, hence $f_n \rightarrow f$ *uniformly*.

■

3 Product Measure

3.1 Product σ -algebra

Before we start, we see the setup.

- Product space.

$$X = \prod_{\alpha \in I} X_{\alpha}$$

where $x = (x_{\alpha})_{\alpha \in I} \in X$.

- Coordinate map.

$$\pi_{\alpha}: X \rightarrow X_{\alpha}.$$

Now we see the formal definition.

Definition 3.1 (Product σ -algebra). Let $(X_{\alpha}, \mathcal{A}_{\alpha})$ be a measurable space for all $\alpha \in I$. Then a product σ -algebra on $X = \prod_{\alpha \in I} X_{\alpha}$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_{\alpha} = \left\langle \bigcup_{\alpha \in I} \pi_{\alpha}^{-1}(\mathcal{A}_{\alpha}) \right\rangle,$$

where

$$\pi_{\alpha}^{-1}(\mathcal{A}_{\alpha}) = \{\pi_{\alpha}^{-1}(E) \mid E \in \mathcal{A}_{\alpha}\}.$$

Notation. We denote $I = \{1, \dots, d\} \implies X = \prod_{i=1}^d X_i, x = (x_1, \dots, x_d)$. Also,

$$\bigotimes_{i=1}^d \mathcal{A}_i = \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_d.$$

Lemma 3.1. If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \forall_i E_i \in \mathcal{A}_i \right\} \right\rangle.$$

Proof. If $E_i \in \mathcal{A}_i$, then $\pi_i^{-1}(E_i) = \prod_{j=1}^{\infty} E_j$, where $E_j = X$ for $j \neq i$. On the other hand, since

$$\prod_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} \pi_i^{-1}(E_i),$$

from Lemma 1.2, the result follows. ■

Lecture 17: Product Measure

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We now see a lemma.

Lemma 3.2. Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle$ for every $\alpha \in I$. Then

1. $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$
2. $\bigotimes_\alpha \mathcal{A}_\alpha = \langle \bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$
3. If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \forall_i E_i \in \mathcal{E}_i \right\} \right\rangle$$

Proof. We prove this one by one.

1. Note that for $f: Y \rightarrow Z$, and \mathcal{B} be a σ -algebra on Z , then $f^{-1}(\mathcal{B})$ is also a σ -algebra.²⁵ Hence, π_α^{-1} is a σ -algebra on X , i.e.,

$$\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \stackrel{!}{\implies} \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha).$$

To show the other direction, let \mathcal{M} being

$$\mathcal{M} = \{B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle\}.$$

We now check

- \mathcal{M} is a σ -algebra.
- $\mathcal{E}_\alpha \subset \mathcal{M}$. This is true by definition of \mathcal{M} .

Thus, $\langle \mathcal{E}_\alpha \rangle = \mathcal{A}_\alpha \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha$, $E \in \mathcal{M}$, implying

$$\pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle,$$

i.e., $\mathcal{A}_\alpha \subset \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

2.

3.

Check
(Easy)!

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■

²⁵Since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(B)^c = f^{-1}(B^c)$, and $\bigcup_n f^{-1}(B_n) = f^{-1}(\bigcup_n B_n)$.

Theorem 3.1. Suppose X_1, \dots, X_d are metric spaces. Let $X = \prod_{i=1}^d X_i$ with product metric defined as

$$\rho(x, y) = \sum_{i=1}^d \rho_i(x_i, y_i).$$

Then,

1. $\bigotimes_{i=1}^d \mathcal{B}(X_i) \subset \mathcal{B}(X)$
2. If in addition, each X_i has a countable dense subset,

$$\bigoplus_{i=1}^d \mathcal{B}(X_i) = \mathcal{B}(X).$$

Proof.



DIY

Remark. We see that

- $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \dots \otimes \mathcal{B}(\mathbb{R})$
- let $f = u + iv: X \rightarrow \mathbb{C}$, and \mathcal{A} be a σ -algebra on X . Then

$$\forall_{E \in \mathcal{B}(\mathbb{R})} u^{-1}(E), v^{-1}(E) \in \mathcal{A} \iff f^{-1}(F) \in \mathcal{A}, \forall F \in \mathcal{B}(\mathbb{C})$$

with $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Definition 3.2 (x -section, y -section). Let X, Y be two sets. Then

1. For $E \subset X \times Y$,

$$E_x = \{y \in Y \mid (x, y) \in E\}, \quad E^y = \{x \in X \mid (x, y) \in E\}.$$

2. For $f: X \times Y \rightarrow \mathbb{C}$, define

$$f_x: Y \rightarrow \mathbb{C}, \quad f^y: X \rightarrow \mathbb{C}$$

by

$$f_x(y) = f(x, y) = f^y(x).$$

Example. We see that

$$(\mathbb{1}_E)_x = \mathbb{1}_{E_x}$$

and

$$(\mathbb{1}_E)^y = \mathbb{1}_{E^y}.$$

Proposition 3.1. Given two measurable spaces (X, \mathcal{A}) and (Y, \mathcal{B}) , then

1. If $E \in \mathcal{A} \otimes \mathcal{B}$, then

$$\forall_{x \in X} \forall_{y \in Y} E_x \in \mathcal{B}, E^y \in \mathcal{A}.$$

2. If $f: X \times Y \rightarrow \mathbb{C}$ is $\mathcal{A} \otimes \mathcal{B}$ -measurable, then

$$\forall_{x \in X} \forall_{y \in Y} f_x \text{ is } \mathcal{B}\text{-measurable, } f^y \text{ is } \mathcal{A}\text{-measurable.}$$

Proof. We prove this one by one.

1. Let $\mathcal{F} := \left\{ E \subset X \times Y \mid \forall_{x \in X} \forall_{y \in Y} E_x \in \mathcal{B}, E^y \in \mathcal{A} \right\}$, then

- \mathcal{F} is a σ -algebra.

$$- \emptyset \in \mathcal{F}.$$

$$- (E^c)_x = E_x^c.$$

$$- \left(\bigcup_{j=1}^{\infty} E_j \right)_x = \bigcup_{j=1}^{\infty} (E_j)_x.$$

And the same is true for y .

- Let $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$, which is again easy to show from definition.

Hence, we see that $\langle \mathcal{R}_0 \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{F}$.

2. Since

$$(f_x)^{-1}(B) = (f^{-1}(B))_x$$

and

$$(f^y)^{-1}(B) = (f^{-1}(B))^y,$$

the result follows from 1. ■

3.2 Product Measures

We start with the definition.

Definition 3.3 (Rectangle). Given two measurable spaces, a (measurable) rectangle is $R = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, we let

$$\mathcal{R}_0 := \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$

and

$$\mathcal{R} := \left\{ \bigcup_{i=1}^N R_i \mid N \in \mathbb{N}, R_1, \dots, R_N \text{ disjoint rectangles} \right\}.$$

Note. Whenever we're talking about [rectangle](#), they're always [measurable](#).

Lemma 3.3. \mathcal{R} is an [algebra](#), and

$$\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}.$$

Proof. Simply observe that

$$(A \times B)^c = (A^c \times Y) \cup (A \times B^c)$$



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Lecture 18: Monotone Class

16 Feb. 11:00

Let's start with a theorem.

Theorem 3.2. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be [measure spaces](#). Then

1. There is a [measure](#) $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

for every $A \in \mathcal{A}, B \in \mathcal{B}$.

2. If μ, ν are [σ-finite](#), then $\mu \times \nu$ is unique.

Proof. We prove this one by one.

1. Define $\mu: \mathcal{R} \rightarrow [0, \infty]$ by $\mu(A \times B) = \mu(A)\nu(B)$, and extending linearly, we have

$$\pi(A \times B) = \mu(A)\nu(B),$$

hence

$$\pi\left(\prod_{i=1}^N A_i \times B_i\right) = \sum_{i=1}^n \pi(A_i \times B_i).$$

We claim that π is a [pre-measure](#). To show this, it's enough to check that $\pi(A \times B) = \sum_{n=1}^{\infty} \pi(A_n \times B_n)$ if $A \times B = \prod_n A_n \times B_n$. Since $A_n \times B_n$ are disjoint, so

$$\mathbb{1}_{A \times B}(x, y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n \times B_n}(x, y).$$

Thus,

$$\mathbb{1}_A(x)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y).$$

Integrating with respect to x , and applying [Proposition 1.3](#), we have

$$\int_X \mathbb{1}_A(x)\mathbb{1}_B(y) d\mu(x) = \sum_{n=1}^{\infty} \int_X \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y) d\mu(x),$$

which implies

$$\mu(A)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mu(A_n)\mathbb{1}_{B_n}(y)$$

for every y . We can then integrate again with respect to y and apply [Proposition 1.3](#), we have

$$\int_Y \mu(A)\mathbb{1}_B(y) d\nu(y) = \sum_{n=1}^{\infty} \int_Y \mu(A_n)\mathbb{1}_{B_n}(y) d\nu(y),$$

which gives us

$$\mu(A)\nu(B) = \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n).$$

Hence, we see that μ is indeed a [pre-measure](#), so [Theorem 1.3](#) gives $\mu \times \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on \mathcal{R} .

2. If μ, ν are [\$\sigma\$ -finite](#), then π is [\$\sigma\$ -finite](#) on \mathcal{R} , then [Theorem 1.4](#) applies. Moreover, we have that

$$(\mu \times \nu)(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \mid E \subset \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\}.$$

■

3.3 Monotone Class Lemma

Let's start with a definition.

Definition 3.4 (Monotone Class). If X is a set, and $C \subset \mathcal{P}(X)$, we say that C is a *monotone class* on X if

- C is closed under countable increasing unions.
- C is closed under countable decreasing intersections.

Example. We see that

1. Every [\$\sigma\$ -algebra](#) is a [monotone class](#).
2. If C_α are (arbitrarily many) [monotone classes](#) on a set X , then $\bigcap_{\alpha} C_\alpha$ is a [monotone class](#). Furthermore, if $\mathcal{E} \subset \mathcal{P}(X)$, there is a unique smallest [monotone class](#) containing \mathcal{E} , denoted by $\langle \mathcal{E} \rangle$, which follows the same idea as in [Definition 1.3](#).

Theorem 3.3 (Monotone Class Lemma). Suppose \mathcal{A}_0 is an [algebra](#) on X . Then $\langle \mathcal{A}_0 \rangle^a$ is the [monotone class](#) generated by \mathcal{A}_0 .

^a $\langle \mathcal{A}_0 \rangle$ is the [\$\sigma\$ -algebra](#) generated by \mathcal{A}_0 by [Definition 1.3](#).

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$ and let \mathcal{C} be the monotone class generated by \mathcal{A}_0 . Since \mathcal{A} is a σ -algebra, it's a monotone class. Note that it contains \mathcal{A}_0 , hence $\mathcal{A} \supset \mathcal{C}$.

To show $\mathcal{C} \supset \mathcal{A}$, it's enough to show that \mathcal{C} is a σ -algebra. We check that

1. $\emptyset \in \mathcal{A}_0 \subseteq \mathcal{C}$.
2. Let $\mathcal{C}' = \{E \subset X \mid E^c \in \mathcal{C}\}$.
 - \mathcal{C}' is a monotone class.
 - $\mathcal{A}_0 \subset \mathcal{C}'$ because if $E \in \mathcal{A}_0$, then $E^c \in \mathcal{A}_0$, so $E^c \in \mathcal{C}$, thus $E \in \mathcal{C}'$.

We see that $\mathcal{C}' \subset \mathcal{C}'$, so \mathcal{C} is closed under complements.

3. For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.
 - $\mathcal{D}(E) \subset \mathcal{C}$.
 - $\mathcal{D}(E)$ is a monotone class.
 - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. We see this by picking $F \in \mathcal{A}_0$, then $E \cup F \in \mathcal{A}_0 \subset \mathcal{C}$.

Hence, $C = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.

4. Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\}$. That is $\mathcal{D} = \{E \in \mathcal{C} \mid E \cup F, \forall F \in \mathcal{C}\}$. Then we have
 - $\mathcal{A}_0 \subset \mathcal{D}$ by 3.
 - \mathcal{D} is a monotone class.
 - $\mathcal{D} \subset \mathcal{C}$ by definition.

Thus, $\mathcal{D} = \mathcal{C}$, so if $E, F \in \mathcal{C}$, then $E \cup F \in \mathcal{C}$. This implies that \mathcal{C} is closed under finite unions.

5. Now to show that \mathcal{C} is closed under countable unions, let $E_1, E_2, \dots \in \mathcal{C}$. We may then define

$$F_N = \bigcup_{n=1}^N E_n \in \mathcal{C}.$$

Then we see that $F_1 \subset F_2 \subset \dots$, hence $\bigcup_N F_N \in \mathcal{C}$. But this simply implies

$$\bigcup_N F_N = \bigcup_n E_n,$$

so we're done. ■

Lecture 19: Fubini-Tonelli's Theorem

18 Feb. 11:00

As previously seen. If $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A} \forall x \in X, \forall y \in Y$. Note that the reverse is not true.

3.4 Fubini-Tonelli Theorem

We start with a theorem.

Theorem 3.4 (Tonelli's theorem for characteristic functions). Given (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be σ -finite measure space. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$, then

1. $\alpha(x) := \nu(E_x): X \rightarrow [0, \infty]$ is a \mathcal{A} -measurable function.
2. $\beta(y) := \mu(E^y): Y \rightarrow [0, \infty]$ is a \mathcal{B} -measurable function.
3. $(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$.

Proof. We prove this one by one.

1. Assume that μ, ν are finite measure. Let

$$C := \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{Conditions 1., 2., 3., hold}\}.$$

It's enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset C$. We further observe that from the Theorem 3.3 and the fact that \mathcal{R} is an algebra, it's also enough to show that

- $\mathcal{R} \subset C$.
- C is a monotone class.

From condition 1.,

$$\alpha(x) = \nu((A \times B)_x) = \begin{cases} \nu(B), & \text{if } x \in A; \\ 0, & \text{if } x \notin A \end{cases} = \nu(B) \mathbb{1}_A.$$

And from condition 2.,

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

and

$$\int_X \nu((A \times B)_x) d\mu(x) = \nu(B)\mu(A).$$

Let $E_n \in C$, $E_1 \subset E_2 \subset \dots$. We need to show $E = \bigcup_{n=1}^{\infty} E_n \in C$. We now see that

$$\begin{aligned} E_x &= \bigcup_{n=1}^{\infty} (E_n)_x, (E_1)_x \subset (E_2)_x \subset \dots \\ \implies \alpha(x) &= \nu(E_x) \stackrel{!}{=} \lim_{n \rightarrow \infty} \nu((E_n)_x) \quad \forall x \in X. \end{aligned}$$

This implies that 1. is proved.

For 3., we see that

$$\begin{aligned} (\mu \times \nu)(E) &\stackrel{!}{=} \lim_{n \rightarrow \infty} (\mu \times \nu)(E_n) \\ &= \lim_{n \rightarrow \infty} \int_X \nu((E_n)_x) d\mu(x) \\ &\stackrel{!}{=} \int_X \nu(E_x) d\mu(x). \end{aligned}$$

Now let $F_n \in C$, $F_1 \supset F_2 \supset \dots$. We need to show that $F = \bigcap_{n=1}^{\infty} F_n \in C$. Instead of using [Theorem 2.2](#), we now want to use [Theorem 2.4](#), which is applicable since $\mu(X), \nu(Y) < \infty$ by assumption. Then assume that μ, ν are σ -finite, then

$$X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n), \begin{cases} X_1 \subset X_2 \subset \dots, & \mu(X_k) < \infty \\ Y_1 \subset Y_2 \subset \dots, & \nu(Y_k) < \infty. \end{cases}$$

DIY

Theorem 3.5 (Fubini-Tonelli's Theorem). Given two σ -finite measure space $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$, we have the following two versions.

(Tonelli) If $f: X \times Y \rightarrow [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$ -measurable, then

1. $g(x) := \int_Y f(x, y) d\nu(y)$, $X \rightarrow [0, \infty]$ is a \mathcal{A} -measurable function.
2. $h(y) := \int_X f(x, y) d\mu(x)$, $Y \rightarrow [0, \infty]$ is a \mathcal{B} -measurable function.
3. We have

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y).$$

(Fubini) If $f \in L^1(X \times Y, \mu \times \nu)$, then

1. $f_x \in L^1(Y, \nu)$ for μ -a.e. x , and $g(x) \in L^1(X, \mu)$ defined μ -a.e.
2. $f_y \in L^1(X, \mu)$ for ν -a.e. y , and $h(y) \in L^1(Y, \nu)$ defined ν -a.e.
3. The iterated integral formulas hold. Namely, we have

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y).$$

Proof. Read [FF99].

Lecture 20: Lebesgue Measure on \mathbb{R}^d

21 Feb. 11:00

3.5 Lebesgue Measure on \mathbb{R}^d

Example. We first see some examples.

1. $(\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m)$ is not complete.
 - Let $A \in \mathcal{L}$, $A \neq \emptyset$, $m(A) = 0$.
 - Let $B \subset [0, 1]$, $B \notin \mathcal{L}$ (Vital set for example).
 - Let $E = A \times B$, $F = A \times [0, 1]$.

We see that $E \subset F$, $F \in \mathcal{L} \otimes \mathcal{L}$, $(m \times m)(F) = m(A)m([0, 1]) = 0$, i.e., F is a **null** set. But E is **not** $\mathcal{L} \otimes \mathcal{L}$ -measurable-function since otherwise, its sections are all **measurable**.

Definition 3.5. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the *completion* of

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \dots \times m),$$

which is same as the *completion* of

$$(\mathbb{R}^d, \mathcal{L} \otimes \dots \otimes \mathcal{L}, m \times \dots \times m).$$

Remark. We see that

$$\mathcal{L}^d \supsetneq \mathcal{L} \otimes \dots \otimes \mathcal{L} = \left\langle \left\{ \prod_{i=1}^d E_i \mid E_i \in \mathcal{L} \right\} \right\rangle.$$

Definition 3.6 (General rectangle). A *rectangle* in \mathbb{R}^d is $R = \prod_{i=1}^d E_i$ where $E_i \in \mathcal{B}(\mathbb{R})$.

Definition 3.7. We let

$$m^d(E) := \inf \left\{ \sum_{k=1}^{\infty} m^d(R_k) \mid E \subset \bigcup_{k=1}^{\infty} R_k, R_k \text{ is rectangles} \right\}.$$

Theorem 3.6. Let $E \subset \mathcal{L}^d$. Then

1. $m^d(E) = \inf \{m^d(O) \mid \text{open } O \supset E\} = \sup \{m^d(K) \mid \text{compact } K \subset E\}$.
2. $E = A_1 \cup N_1 = A_2 \setminus N_2$, where A_1 is F_σ , A_2 is G_δ , and N_i are **null**.
3. If $m^d(E) < \infty$, $\forall \epsilon > 0$, $\exists R_1, \dots, R_m$ **rectangles** whose sides are intervals such that

$$m^d \left(E \triangle \left(\bigcup_{i=1}^m R_i \right) \right) < \epsilon.$$

Proof. Similar to $d = 1$ case. ■

Theorem 3.7. **Integrable step functions** and $C_c(\mathbb{R}^d)$, the collection of continuous functions, are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$

Proof. See [FF99]. ■

Theorem 3.8. Lebesgue measure in \mathbb{R}^d is translation-invariant.

Proof. See [FF99]. ■

Theorem 3.9 (Effect of linear transformation on Lebesgue measure). If $T \in \text{GL}(\mathbb{R}^d)$, $e \in \mathcal{L}^d$, then $T(E)$ is measurable and

$$m(T(E)) = |\det T| \cdot m(E).$$

Proof. See [FF99]. ■

4 Differentiation on Euclidean Space

As previously seen. Given $f: [a, b] \rightarrow \mathbb{R}$, there are two versions of **fundamental theorem of calculus**:

1.

$$\int_a^b f'(x) dx = f(b) - f(a).$$

2.

$$\frac{d}{dx} \int_a^x f(t) dt = f(x),$$

which follows from

$$\lim_{r \rightarrow 0^+} \frac{1}{r} \int_x^{x+r} f(t) dt = f(x) = \lim_{r \rightarrow 0^+} \frac{1}{r} \int_{x-r}^x f(t) dt.$$

Remark. We see that

$$\lim_{r \rightarrow 0^+} \frac{1}{r} \int_x^{x+r} (f(t) - f(x)) dt = 0 = \lim_{r \rightarrow 0^+} \frac{1}{r} \int_{x-r}^x (f(t) - f(x)) dt,$$

where we have

$$f(x) = \frac{1}{r} \int_x^{x+r} f(t) dt.$$

This generalized to $f: \mathbb{R}^d \rightarrow \mathbb{R}$, namely

$$\lim_{r \rightarrow 0^+} \frac{1}{\text{vol}(B(x, r))} \int_{B(x, r)} (f(t) - f(x)) \underbrace{dt}_{\mathbb{R}^d} \stackrel{?}{=} 0.$$

4.1 Hardy-Littlewood Maximal Function

We first see our notation.

Notation. Given a(n) (open) ball in \mathbb{R}^d , $B = B(a, r)$, denote $cB = B(a, cr)$ for $c > 0$.

Lemma 4.1 (Vitali-type covering lemma). Let B_1, \dots, B_k be a finite collection of open balls in \mathbb{R}^d . Then there exists a sub-collection B'_1, \dots, B'_m of disjoint open balls such that

$$\bigcup_{i=1}^m (3B'_i) \supset \bigcup_{i=1}^k B_i.$$

Proof. Greedy Algorithm. ■

Lecture 21: Hardy-Littlewood Maximal Function and Inequality

25 Feb. 11:00

Notation. We let

$$\int_E f \, dm = \int_E f(x) \, dx.$$

The problem we're working on is

$$\frac{1}{m(B(w, r))} \int_{B(w, r)} f(y) \, dy \xrightarrow[r \rightarrow 0]{?} f(x).$$

Definition 4.1 (Locally integrable). Given $f: \mathbb{R}^d \rightarrow \mathbb{C}$ be [Lebesgue measurable](#) function. Then we say f is *locally integrable* if for every compact $K \subset \mathbb{R}^d$,

$$\int_K |f| \, dm < \infty.$$

We write $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Definition 4.2 (Hardy-Littlewood maximal function). Given $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the *Hardy-Littlewood maximal function* for f is defined as

$$Hf(x) := \sup \{A_r(x) \mid r > 0\},$$

where

$$A_r(x) := \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y)| \, dy.$$

Note. We note that $A_r(\cdot)$ means *averaging function*.

Lemma 4.2. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then

1. $A_r(x)$ is jointly continuous for $(x, r) \in \mathbb{R}^d \times (0, \infty)$.
2. $Hf(x)$ is [Borel measurable](#).

Proof. We outline the proof.

1. Let $(x, r) \rightarrow (x^*, r^*) \implies A_r(x) \rightarrow A_{r^*}(x^*)$. Let (x_n, r_n) be any sequence which converges to x^*, r^* , then we consider $\lim_{n \rightarrow \infty} A_{r_n}(x_n)$ and we can calculate

$$\int \underbrace{|f(y)| \mathbb{1}_{B(x_n, r_n)}(y)}_{:=h_n(y)},$$

then we apply [Theorem 2.4](#) to h_n .

2. Observe that

$$(Hf)^{-1}(\underbrace{(a, \infty)}_{\text{open}}) = \bigcup_{r>0} A_r^{-1}((a, \infty))$$

is open, since $A_r^{-1}((a, \infty))$ is open from the 1. Note that the equality comes from the fact that $Hf = \sup_r A_r$. ■

Theorem 4.1 (Hardy-Littlewood maximal inequality). There exists $C_d > 0$ such that for every $f \in L^1(\mathbb{R}^d)$,

$$\forall_{\alpha>0} m(\{x \in \mathbb{R}^d \mid Hf(x) > \alpha\}) \leq \frac{C_d}{\alpha} \int |f(x)| \, dx.$$

Proof. We first fix $f \in L^1$ and $\alpha > 0$. We define

$$E := \{x \mid Hf(x) > \alpha\},$$

which is a [Borel measurable set](#) by [Lemma 4.2](#). Then

$$x \in E \implies \exists_{r_x>0} A_{r_x}(x) > \alpha \implies m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x, r_x)} |f(y)| \, dy.$$

From [inner regularity](#), we have

$$m(E) = \sup \{m(K) \mid \text{compact } K \subset E\}.$$

Let $K \subset E$ be compact, then

$$K \subset \bigcup_{x \in K} B(x, r_x) \xrightarrow{K \text{ compact}} K \subset \bigcup_{i=1}^N B_i \xrightarrow{!} K \subset \bigcup_{i=1}^m \{3B'_j\}.$$

From here, we further have

$$m(K) \leq \sum_{i=1}^m m(3B'_j) = 3^d \sum_{j=1}^m m(B'_j) \leq \frac{3^d}{\alpha} \sum_{j=1}^m \int_{B'_j} |f(y)| \, dy.$$

Now, since B'_1, \dots, B'_m are disjoint, hence we finally have

$$m(K) \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \, dy. \quad \blacksquare$$

Lecture 22: Lebesgue Differentiation Theorem

07 Mar. 11:00

We should compare the [Hardy-Littlewood maximal inequality](#) to [Markov's inequality](#). Namely, there exists $C_d > 0$ (can take 3^d) such that for all $f \in L^1(\mathbb{R}^d)$, $\alpha > 0$, we have

$$\begin{cases} m(\{x \mid Hf(x) > \alpha\}) \leq \frac{C_d}{\alpha} \int |f|; \\ m(\{x \mid |f(x)| > \alpha\}) \leq \frac{1}{\alpha} \int |f|. \end{cases}$$

4.2 Lebesgue Differentiation Theorem

We start with a theorem!

Theorem 4.2 (Lebesgue Differentiation Theorem). Let $f \in L^1$, then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dy = 0$$

for [a.e.](#) x .

Proof. The result holds for $f \in C_c(\mathbb{R}^d)$, namely for those continuous functions with **compact support**. This is because for any $\epsilon > 0$, if r is small and $|f(y) - f(x)| < \epsilon$, then

$$\frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dy < \epsilon.$$

Now, let $f \in L^1(\mathbb{R}^d)$ and fix $\epsilon > 0$. By density, there exists $g \in C_c(\mathbb{R}^d)$ with $\|f - g\|_1 < \epsilon$. We then have

$$\begin{aligned} \int_{B(x, r)} |f(y) - f(x)| \, dy &\leq \int_{B(x, r)} |f(y) - g(y)| \, dy \\ &\quad + \int_{B(x, r)} |g(y) - g(x)| \, dy \\ &\quad + \int_{B(x, r)} |g(x) - f(x)| \, dy. \end{aligned}$$

Divide all of these by $m(B(x, r))$, and take $\limsup_{r \rightarrow \infty}$, we need to understand the error terms, namely

$$\frac{1}{m(B(x, r))} \int_{B(x, r)} |f(x) - g(x)| \, dy = |g(x) - f(x)|$$

and

$$\frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - g(y)| \, dy \leq (H(f - g))(x).$$

We define

$$Q(x) := \limsup_{r \rightarrow \infty} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dy.$$

We want to show $m(\{x \in X \mid Q(x) > 0\}) = 0$. Let $E_\alpha = \{x \in X \mid Q(x) > \alpha\}$. It is enough to show $m(E_\alpha) = 0$ for all $\alpha > 0$ because $\{x \in X \mid Q(x) > 0\} = \bigcup_n E_{\frac{1}{n}}$. We know by the above that

$$Q(x) \leq (H(f - g))(x) + 0 + |g(x) - f(x)|.$$

Therefore,

$$E_\alpha \subset \{x \in X \mid (H(f - g))(x) > \alpha/2\} \cup \{x \in X \mid |g(x) - f(x)| > \alpha/2\}.$$

By the [Hardy-Littlewood maximal inequality](#) and [Markov's inequality](#), we have

$$\begin{cases} m(\{x \mid (H(f - g))(x) > \alpha/2\}) \leq \frac{2C_d}{\alpha} \int |f - g|; \\ m(\{x \mid |g(x) - f(x)| > \alpha/2\}) \leq \frac{2}{\alpha} \int |f - g|. \end{cases}$$

Thus,

$$0 \leq m(E_\alpha) \leq \frac{2C_d}{\alpha} \|f - g\|_1 + \frac{2}{\alpha} \|f - g\|_1 \leq \frac{2(C_d + 1)}{\alpha} \epsilon.$$

Taking $\epsilon \rightarrow 0$, $m(E_\alpha)$ does not depend on ϵ and g , hence $m(E_\alpha) = 0$. ■

Corollary 4.1. [Theorem 4.2](#) also holds for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Proof. Using the fact that m^d is σ -finite, and apply [Theorem 4.2](#). Specifically, partition \mathbb{R}^d into countably many compact sets K_i and apply [Theorem 4.2](#) to $f \mathbb{1}_{K_i}$ for all i . ■

Corollary 4.2. For $f \in L^1_{\text{loc}}$, we have

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) \, dy = f(x)$$

for [a.e.](#) x .

Proof. Use that

$$f(x) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f(x) \, dy$$

and the triangle inequality. ■

Definition 4.3 (Lebesgue point). Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the point $x \in \mathbb{R}^d$ is called a *Lebesgue point* of f if

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dy = 0.$$

Remark. [Corollary 4.1](#) tells us that almost all points in \mathbb{R}^d in \mathbb{R}^d are [Lebesgue points](#) for f .

DIY

Definition 4.4 (Shrink nicely). We say that $\{E_r\}_{r>0}$ *shrinks nicely* to x as $r \rightarrow 0$ if $E_r \subset B(x, r)$ and

$$\exists_{c>0} c \cdot m(B(x, r)) \leq m(E_r).$$

Corollary 4.3. Suppose E_r shrink nicely to 0, and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, and x is a Lebesgue point. Then

$$\begin{cases} \lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r+x} |f(y) - f(x)| \, dy = 0; \\ \lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r+x} |f(y)| \, dy = f(x). \end{cases}$$

Corollary 4.4. If $f \in L^1_{\text{loc}}(\mathbb{R})$, then $F(x) = \int_0^x f(y) \, dy$ is differentiable and $F'(x) = f(x)$ almost everywhere.

Lecture 23: Metric, normed and L^p Spaces

09 Mar. 11:00

5 Normed Vector Space

5.1 Metric Spaces and Normed Spaces

We have seen the definition of a norm before, now we formally introduce the concept of *metric*.

Definition 5.1 (Metric). Let Y be a set, a function $\rho: Y \times Y \rightarrow [0, \infty)$ is a *metric* on Y if

- $\rho(x, y) = \rho(y, x)$ for all $x, y \in Y$.
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in Y$.
- $\rho(x, y) = 0$ if and only if $x = y$.

Note. The followings make sense in a metric space.

1. Open/closed balls.
2. Open/closed sets.
3. Convergence sequences ($x_n \rightarrow x$ with respect to ρ if and only if $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$).
4. Continuous functions.

Example. We have the following metric spaces.

1. \mathbb{Q} with $\rho(x, y) = |x - y|$.

2. \mathbb{R} with $\rho(x, y) = |x - y|$.
3. \mathbb{R}_+ with $\rho(x, y) = |\ln(y/x)|$.
4. \mathbb{R}^d with

$$\rho_p(x, y) = \left(\sum_{i=1}^d |x_i - y_i|^p \right)^{1/p}$$

and

$$\rho_\infty(x, y) = \max_{1 \leq i \leq d} |x_i - y_i|.$$

These all give the same open sets, hence they are topologically equivalent.

5. $C([0, 1])$ with

$$\rho_p(f, g) = \left(\int_0^1 |f - g|^p \right)^{1/p}$$

and

$$\rho_\infty(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|.$$

6. Let (X, \mathcal{A}, μ) be a **measure space** with $\mu(X) < \infty$. Let Y be the set of **measurable functions** on X , then

$$\rho(f, g) = \int \min\{|f(x) - g(x)|, 1\} \, d\mu(x)$$

is a **metric** and $f_n \rightarrow f$ in ρ if and only if $f_n \rightarrow f$ in **measure**.

Let V be a vector space over scalar field $K = \mathbb{R}$ or $K = \mathbb{C}$.

As previously seen (Metric induced by a norm). Recall the definition of **seminorm** and **norm**. We see that a **norm** induces a metric

$$\rho(v, w) := \|v - w\|,$$

and we have

$$v_n \rightarrow v \iff \lim_{n \rightarrow \infty} \|v_n - v\| = 0.$$

Example. We first see some common examples of **normed** vector space.

1. $L^1(X, \mathcal{A}, \mu)$ with $\|f\|_1 := \int |f| \, d\mu$.
2. $C([0, 1])$ with $\|f\|_1 := \int_0^1 |f(x)| \, dx$, $\|f\|_\infty := \max_{0 \leq x \leq 1} |f(x)|$.
3. For \mathbb{R}^d and $0 < p < \infty$, we have

$$\|x\|_p := \left(\sum_{i=1}^d |x_i|^p \right)^{1/p}, \quad \|x\|_\infty := \max_{1 \leq i \leq d} |x_i|.$$

5.2 L^p Space

It turns out that we can generalize L^1 into L^p .

Definition 5.2 (L^p space). Given a **measure space** (X, \mathcal{A}, μ) and a **measurable function** f and p such that $0 < p < \infty$, we define a **seminorm** $\|\cdot\|_p$ such that

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p},$$

which induces the so-called L^p space $L^p(X, \mathcal{A}, \mu)$, where

$$L^p(X, \mathcal{A}, \mu) := \left\{ f \mid \|f\|_p < \infty \right\}.$$

Remark. Note that $\|\cdot\|_p$ is only a **seminorm**. But if we identity functions which are equal **almost everywhere**, then it's indeed a **norm**.

Example. $(\mathbb{R}, \mathcal{L}, m)$ has $f(x) = x^{-\alpha} \mathbb{1}_{(1, \infty)}(x) \in L^p$ if and only if $\alpha p > 1$. In contrast, $g(x) = x^{-\beta} \mathbb{1}_{(0, 1)}(x) \in L^p$ if and only if $\beta p < 1$.

Similar to **Definition 5.2**, we have the following.

Definition 5.3 (ℓ^p space). If $(X, \mathcal{P}(X), \nu)$ is equipped with the **counting measure**, then we say it's an ℓ^p space such that

$$\ell^p(X) := L^p(X, \mathcal{P}(X), \nu).$$

Remark. We are interested in $\ell^p(\mathbb{N})$ in particular. We have

$$\ell^p := \ell^p(\mathbb{N}) = \left\{ a = (a_1, a_2, \dots) \mid \|a\|_p = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} < \infty \right\}.$$

Lemma 5.1. $L^p(X, \mathcal{A}, \nu)$ is a vector space for all $p \in (0, \infty)$.

Proof. We verify the following.

- $c \cdot f \in L^p(X, \mathcal{A}, \mu)$ for $c \in \mathbb{R}$. Indeed, since

$$\|cf\|_p = \left(\int |cf|^p d\mu \right)^{1/p} = |c| \|f\|_p < \infty \iff \|f\|_p < \infty,$$

which implies $c \cdot f \in L^p(X, \mathcal{A}, \mu)$.

- $f + g \in L^p(X, \mathcal{A}, \mu)$. Indeed, since for any real numbers α, β , we have

$$(\alpha + \beta)^p \leq (2 \cdot \max\{|\alpha|, |\beta|\})^p = 2^p \cdot \max\{|\alpha|^p, |\beta|^p\} \leq 2^p (|\alpha|^p + |\beta|^p),$$

which implies that for $f, g \in L^p(X, \mathcal{A}, \mu)$, we have

$$\|f + g\|_p < \infty \iff \|f + g\|_p^p = \int |f + g|^p d\mu \leq 2^p \int (|f|^p + |g|^p) < \infty.$$

This further implies

$$\|f + g\|_p < \infty \iff \|f\|_p, \|g\|_p < \infty,$$

which is what we want. ■

We see that in the above derivation, it doesn't give us the triangle inequality, namely

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

hence we need some new results.

Theorem 5.1 (Hölder's inequality). Let $1 < p < \infty$, and let $q := p/(p-1)$ so that $1/p + 1/q = 1$. Then we have

$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. We prove this in steps.

1. Note that

$$t \leq \frac{t^p}{p} + 1 - \frac{1}{p} = \frac{t^p}{p} + \frac{1}{q}$$

for all $t \geq 0$. Hence, by taking $F(t) := t - t^p/p$ and $t \geq 0$, we see that the maximum of F implies the above inequality.

2. Young's Inequality.²⁶ We have

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

for $\alpha, \beta > 0$. This follows by taking $t := \alpha/\beta^{q-1}$ in the first inequality we obtained.

3. Without loss of generality, we can assume that $0 < \|f\|_p, \|g\|_q < \infty$. Now, consider $F(x) = f(x)/\|f\|_p$, $G(x) = g(x)/\|g\|_q$. We know that $\|F\|_p = 1 = \|G\|_q$. Then by Young's Inequality, we have

$$\int |F(x)G(x)| d\mu \leq \int \frac{|F(x)|^p}{p} + \frac{|G(x)|^q}{q} \implies \frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

which implies our desired result. ■

²⁶https://en.wikipedia.org/wiki/Young's_inequality_for_products

Example. For $p = q = 2$, $X = \{1, \dots, d\}$ with μ being the [counting measure](#), then for any $x, y \in \mathbb{R}^d$, we have

$$\sum_{i=1}^d |x_i y_i| \leq \sqrt{\sum_{i=1}^d x_i^2} \sqrt{\sum_{i=1}^d y_i^2}$$

We now see how we can obtain the desired triangle inequality.

Theorem 5.2 (Minkowski's Inequality). Let $1 \leq p < \infty$, then for $f, g \in L^p$,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. For $p = 1$, it's easy since it's just triangle inequality. Now, we assume that $1 < p < \infty$, and we may assume also that $\|f + g\| \neq 0$ without loss of generality. Then

$$\begin{aligned} \int |f(x) + g(x)|^p &\leq \int |f(x) + g(x)|^{p-1} (|f(x)| + |g(x)|) \\ &\leq \left(\int |f + g|^{(p-1)q} \right)^{1/q} \left[\left(\int |f|^p \right)^{1/p} + \left(\int |g|^p \right)^{1/p} \right] \\ &\leq \left(\int |f + g|^p \right)^{1/q} (\|f\|_p + \|g\|_p). \end{aligned}$$

We then see that

$$\underbrace{(|f(x) + g(x)|^p)^{1-1/q}}_{(|f(x)+g(x)|^p)^{1/p}} \leq \|f\|_p + \|g\|_p,$$

which is just $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. ■

Lecture 24: Embedding L^p Space

11 Mar. 11:00

Definition 5.4 (Essential supremum). For a [measurable function](#) f on (X, \mathcal{A}, μ) , we define

$$\begin{aligned} S &:= \{\alpha \geq 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\} \\ &= \{\alpha \geq 0 \mid |f(x)| \leq \alpha \text{ a.e.}\}. \end{aligned}$$

Then, we say that the *essential supremum* of f , denoted as $\|f\|_\infty$, is defined as

$$\|f\|_\infty := \begin{cases} \inf S, & \text{if } S \neq \emptyset; \\ \infty, & \text{if } S = \emptyset. \end{cases}$$

Definition 5.5 (L^∞ space). Let $L^\infty(X, \mathcal{A}, \mu)$ be

$$L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\}.$$

Definition 5.6 (ℓ^∞ space). We let ℓ^∞ be defined as

$$\ell^\infty = L^\infty(\mathcal{N}, \mathcal{P}(\mathcal{N}), \nu),$$

where ν is the [counting measure](#).

Example. Consider $(\mathbb{R}, \mathcal{L}, m)$. Then

$$f(x) = \frac{1}{x} \mathbb{1}_{(0, \infty)}(x) \notin L^\infty;$$

$$g(x) = x \mathbb{1}_{\mathbb{Q}}(x) + \frac{1}{1+x^2} \in L^\infty.$$

If f is continuous on $(\mathbb{R}, \mathcal{L}, m)$, then $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$. For $a \in \ell^\infty$, we have $\|a\|_\infty = \sup_{i \in \mathbb{N}} |a_i|$, and sequences in ℓ^∞ are exactly the bounded sequences.

Lemma 5.2. We have the following.

1. Suppose $f \in L^\infty(X, \mathcal{A}, \mu)$. Then,

$$\begin{cases} \mu(\{x \mid |f(x)| > \alpha\}) = 0, & \text{if } \alpha \geq \|f\|_\infty; \\ \mu(\{x \mid |f(x)| > \alpha\}) > 0, & \text{if } \alpha < \|f\|_\infty. \end{cases}$$

2. $|f(x)| \leq \|f\|_\infty$ [almost everywhere](#).
3. $f \in L^\infty$ if and only if there exists a bounded [measurable function](#) g such that $f = g$ [almost everywhere](#).

Proof. .



DIY

Theorem 5.3. We have the following.

1. $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.
2. $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
3. $f_n \rightarrow f$ in L^∞ if and only if $f_n \rightarrow f$ [uniformly almost everywhere](#).

Remark. The motivation for 1. is that

$$\frac{1}{1} + \frac{1}{\infty} = 1,$$

and we want to have the similar result as in [Theorem 5.1](#).

Proof. . We'll do one implication in 3. Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$. Let $A = \bigcup_n A_n$, we see that $\mu(A) = 0$ as well.



DIY

For $x \in A^c$ and for every n , we have

$$|f_n(x) - f(x)| \leq \|f_n - f\|_\infty.$$

Given $\epsilon > 0$, there is an N so that

$$\|f_n - f\| < \epsilon$$

for all $n \geq N$. But then for all $x \in A^c$, $|f_n(x) - f(x)| < \epsilon$ as well. ■

Proposition 5.1. We have the following.

1. For $1 \leq p < \infty$, the collection of **simple functions** with finite measure **support** is dense in $L^p(X, \mathcal{A}, \mu)$.
2. For $1 \leq p < \infty$, the collection of **step functions** with finite measure **support** is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$, so is $C_c(\mathbb{R})$.
3. For $p = \infty$, the collection of **simple functions** is dense in $L^\infty(X, \mathcal{A}, \mu)$.

Remark. Note that $C_c(\mathbb{R})$ is **not** dense in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

Proof. .

DIY

5.3 Embedding Properties of L^p Spaces

Definition 5.7 (Equivalent norm). Two **norms** $\|\cdot\|, \|\cdot\|'$ on V are *equivalent* if there exists $c_1, c_2 > 0$, such that

$$c_1 \|v\| \leq \|v\|' \leq c_2 \|v\|$$

for all $v \in V$.

Note. We see that

1. These **norms** gives the same topological properties (open sets, closed sets, convergence, etc.).
2. **Definition 5.7** is an equivalence relation on **norms**.

Example. For \mathbb{R}^d we have the **norms** $\|\cdot\|_p$ for $1 \leq p \leq \infty$. All of these are equivalent. We see that for $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{1/p} \leq (d \|x\|_\infty^p)^{1/p} = d^{1/p} \|x\|_\infty.$$

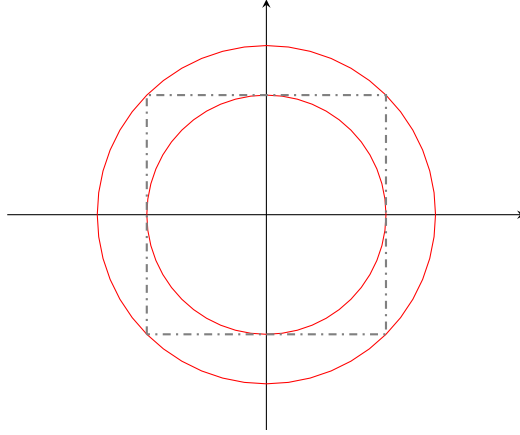
Also,

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{1/p} \geq (\|x\|_\infty^p)^{1/p} = \|x\|_\infty.$$

Thus, $\|\cdot\|_p$ is equivalent to $\|\cdot\|_\infty$ for every $1 \leq p < \infty$, and transitivity gives that they are all equivalent.

Another way of thinking of this, by assuming $v \neq 0$, and scaling by some t , we may assume v lies on the unit circle in one of the **norms**. Then we are squeezing

a unit circle in $\|\cdot\|'$ between two circles of radius c_1, c_2 in $\|\cdot\|$. In picture, we have to show that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent, we have



since the circles in $\|\cdot\|_\infty$ are squares.

Example. For $1 \leq p, q \leq \infty$, we have $L^p(\mathbb{R}, m)$ -norm and $L^q(\mathbb{R}, m)$ -norm are not equivalent, even worse, we have that

$$L^p(\mathbb{R}, m) \not\subseteq L^1(\mathbb{R}, m), \quad L^p(\mathbb{R}, m) \not\supseteq L^1(\mathbb{R}, m).$$

Lecture 25: Banach Spaces

14 Mar. 11:00

Proposition 5.2. Suppose $\mu(X) < \infty$, then for every $0 < p < q \leq \infty$, $L^q \subseteq L^p$.

Proof. Suppose $q < \infty$, then

$$\int |f|^p \leq \left(\int (|f|^p)^{q/p} \right)^{p/q} \left(\int 1^{q/(q-p)} \right)^{1-p/q} = \left(\int |f|^q \right)^{p/q} \mu(X)^{1-p/q} < \infty$$

where we split $\int |f|^p$ into $\int |f|^p \cdot 1$. From Hölder's inequality with $q/p > 1$, we have

$$\|f\|_p \leq \|f\|_q \mu(X)^{1/p-1/q} < \infty.$$

The case that $q = \infty$ is left as an exercise. ■

DIY

Proposition 5.3. If $0 < p < q \leq \infty$, then $\ell^p \subseteq \ell^q$.

Proof. When $q = \infty$, we have

$$\|a\|_\infty^p = \left(\sup_i |a_i| \right)^p = \sup_i |a_i|^p \leq \sum_{i=1}^{\infty} |a_i|^p.$$

Thus $\|a\|_\infty \leq \|a\|_p$.

When $q < \infty$, we see that

$$\sum_{i=1}^{\infty} |a_i|^q = \sum_{i=1}^{\infty} |a_i|^p \cdot |a_i|^{q-p} \leq \|a\|_{\infty}^{q-p} \sum_{i=1}^{\infty} |a_i|^p \leq \|a\|_p^{q-p} \cdot \|a\|_p^p = \|a\|_p^q.$$

Therefore,

$$\|a\|_q \leq \|a\|_p.$$

■

Proposition 5.4. For all $0 < p < q < r \leq \infty$, $L^p \cap L^r \subseteq L^q$.

Proof.

■

DIY

5.4 Banach Spaces

Let's start with a definition.

Definition 5.8 (Cauchy sequence). Let Y, ρ be a **metric** space. We call x_n a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\rho(x_n, x_m) < \epsilon$.

Note. Convergent sequence are **Cauchy**.

Definition 5.9 (Complete). A **metric** space (Y, ρ) is called *complete* if every **Cauchy sequence** in Y converges.

Example. We first see some examples.

1. We see that \mathbb{Q} with $\rho(x, y) = |x - y|$ is **not complete**, but \mathbb{R} with the same **metric** is **complete**.
2. $C([0, 1])$ with $\rho(f, g) = \|f - g\|_{\infty}$ is **complete**, but with $\rho(f, g) = \int |f - g|$ is not.

Definition 5.10 (Banach space). A *Banach space* is a **complete normed** vector space.

Remark. Namely, a vector space equipped with a **norm** whose **metric induced by the norm** is **complete**.

Theorem 5.4. Let $(V, \|\cdot\|)$ be a **normed** space. Then,

V is **complete** \iff every absolutely convergent series is convergent.

i.e., if $\sum_{i=1}^{\infty} \|v_i\| < \infty$, then $\left\{ \sum_{i=1}^N v_i \right\}_{N \in \mathbb{N}}$ converges to some $s \in V$.

Before we prove [Theorem 5.4](#), we first see one of the result based on this theorem.²⁷

Theorem 5.5 (Riesz-Fischer theorem). For every $1 \leq p \leq \infty$, we have $L^p(X, \mathcal{A}, \mu)$ is [complete](#), hence a [Banach space](#).

Proof. We prove this in steps.

1. We first prove this for $1 \leq p < \infty$. Suppose $f_n \in L^p$ and $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

We need to show that there is an $F \in L^p$ such that $\left\| \sum_{n=1}^N f_n - F \right\|_p \rightarrow 0$ as $N \rightarrow \infty$. i.e., we need to show

- (a) $\sum_{n=1}^{\infty} f_n(x)$ is convergent [a.e.](#) In fact, we can show $\int \sum_{n=1}^{\infty} |f_n(x)| < \infty$.

Let $G(x) = \sum_{n=1}^{\infty} |f_n(x)| = \sup_N \sum_{n=1}^N |f_n(x)|$, $G: X \rightarrow [0, \infty]$. Also, let $G_N(x) = \sum_{n=1}^N |f_n(x)|$. Then, we have

$$0 \leq G_1 \leq G_2 \leq \dots \leq G,$$

and $G_N \rightarrow G$. Furthermore,

$$0 \leq G_1^p \leq G_2^p \leq \dots \leq G^p,$$

and $G_N^p \rightarrow G^p$. From [monotone convergence theorem](#),

$$\int G^p = \lim_{N \rightarrow \infty} \int G_N^p.$$

From [Minkowski inequality](#), we further have

$$\|G_N\|_p \leq \sum_{n=1}^N \|f_n\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p := B < \infty.$$

Thus,

$$\int G(x)^p = \lim_{N \rightarrow \infty} \int G_N^p = \lim_{N \rightarrow \infty} \|G_N\|_p^p \leq B^p < \infty.$$

We see that G is finite [a.e.](#) as desired. This implies that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ [a.e.](#), so $\sum_{n=1}^{\infty} f_n(x)$ converges [a.e.](#) Now, we simply let

$$F(x) = \begin{cases} \sum_{n=1}^{\infty} f_n(x), & \text{if it converges;} \\ 0, & \text{otherwise.} \end{cases}$$

²⁷The proof can be found in [here](#).

- (b) $F \in L^p$, where $F(x) := \sum_{n=1}^{\infty} f_n(x)$ **a.e.** and say is zero elsewhere.

This is clear since

$$|F(x)| \leq G(x) \implies \int |F|^p \leq \int G^p < \infty,$$

hence $F \in L^p$.

- (c) $\left\| \sum_{n=1}^N f_n - F \right\|_p \rightarrow 0$ as $N \rightarrow \infty$.

We now see that

$$\left| \sum_{n=1}^N f_n(x) - F(x) \right|^p \leq \left(\sum_{n=1}^{\infty} |f_n(x)| + |F(x)| \right)^p \leq (2G(x))^p.$$

Since $2G \in L^p$, so $2G^p \in L^1$. Thus, by **dominated convergence theorem**, we have

$$\lim_{N \rightarrow \infty} \int \left| \sum_{n=1}^N f_n(x) - F(x) \right|^p dx = 0.$$

This implies

$$\left\| \sum_{n=1}^N f_n - F \right\|_p \rightarrow 0$$

as $N \rightarrow \infty$.

2. The case that $1 \leq p \leq \infty$ is left as an exercise.

DIY

■

Lecture 26: Bounded Linear Transformations

16 Mar. 11:00

We now prove **Theorem 5.4**, completing the proof of **Theorem 5.5** since the latter relies on this result.

Proof. We prove it by proving two directions.

(\implies) Suppose V is **complete**, and fix an absolutely convergent series $\sum_n v_n$. Define $s_N = \sum_{n=1}^N v_n$. It suffices to show the partial sums are a **Cauchy Sequence**.

Fix $\epsilon > 0$, then because $\sum_{n=1}^{\infty} \|v_n\| < \infty$, there is a $K \in \mathbb{N}$ so that

$$\sum_{n=K}^{\infty} \|v_n\| < \epsilon.$$

Now let $M > N > K$, we see that

$$\|s_M - s_N\| = \left\| \sum_{n=N+1}^M v_n \right\| \leq \sum_{n=N+1}^M \|v_n\| \leq \sum_{n=N}^{\infty} \|v_n\| < \epsilon,$$

so this is **Cauchy**.

(\Leftarrow) Now suppose $v_n, n \in \mathbb{N}$ is a **Cauchy sequence**. For all $j \in \mathbb{N}$, there exists an $N_j \in \mathbb{N}$ such that

$$\|v_n - v_m\| < \frac{1}{2^j}$$

for all $n, m \geq N_j$. Without loss of generality, we may assume $N_1 < N_2 < \dots$

Let $w_1 = v_{N_1}$, $w_j = v_{N_j} - v_{N_{j-1}}$ for $j \geq 2$. Therefore,

$$\sum_{j=1}^{\infty} \|w_j\| \leq \|v_{N_1}\| + \sum_{j=2}^{\infty} \frac{1}{2^{j-1}} < \infty.$$

Thus, $\sum_{j=1}^k w_j \rightarrow s \in V$ as $k \rightarrow \infty$. But by telescoping, we have

$$v_{N_k} = \sum_{j=1}^k w_j \rightarrow s.$$

Now we claim that since v_n is **Cauchy**, so $v_n \rightarrow s$.

Explicitly, take $\epsilon > 0$, and let k be large enough so that $\|v_{N_k} - s\| < \epsilon$ and $1/2^k < \epsilon$. Then if $n > N_k$ then

$$\|v_n - s\| \leq \|v_n - v_{N_k}\| + \|v_{N_k} - s\| < \epsilon + \epsilon = 2\epsilon.$$

Thus, $v_n \rightarrow s$. ■

5.5 Bounded Linear Transformations

Definition 5.11 (Bounded linear transformation). Given two **normed** vector spaces $(V, \|\cdot\|)$, $(W, \|\cdot\|')$, a linear map $T: V \rightarrow W$ is called a *bounded map* if there exists $c \geq 0$ such that

$$\|Tv\|' \leq c\|v\|$$

for all $v \in V$.

Proposition 5.5. Suppose $T: (V, \|\cdot\|) \rightarrow (W, \|\cdot\|')$ is a linear map. Then the followings are equivalent.

1. T is continuous.
2. T is continuous at 0.
3. T is a **bounded map**.

Proof. 1. \implies 2. is clear. For 2. \implies 3., take $\epsilon = 1$, then there exists a $\delta > 0$ such that $\|Tu\|' < 1$ if $\|u\| < \delta$.

Now take an arbitrary $\|v\| \in V$, $v \neq 0$. Let $u = \frac{\delta}{2\|v\|}v$. Then $\|u\| < \delta$. Therefore,

$$\|Tu\|' < 1 \implies \frac{\delta}{2\|v\|} \|Tv\|' < 1 \implies \|Tv\|' < \frac{2}{\delta} \|v\|.$$

Then $2/\delta$ is our constant.

For 3. \implies 1., fix $v_0 \in V$. Then for some constant c

$$\|Tv - Tv_0\|' = \|T(v - v_0)\|' \leq c \|v - v_0\|.$$

Thus, T is continuous, as when $v \rightarrow v_0$ the right-hand side goes to zero, and so $Tv \rightarrow Tv_0$. ■

Example. Let's see some examples.

1. We can look at

$$\begin{aligned} T: \ell^1 &\rightarrow \ell^1 \\ (a_1, a_2, \dots) &\mapsto (a_2, a_3, \dots). \end{aligned}$$

Then clearly $\|Ta\|_1 \leq \|a\|_1$, so T is a **bounded linear transformation**.

2. We can also look at $S: (C([-1, 1]), \|\cdot\|_1) \rightarrow \mathbb{C}$, where $Sf = f(0)$. S is not a **bounded linear transformation**, because we can make

$$\begin{cases} \|Sf\| &= |f(0)| = n \\ \|f\|_1 &= 1 \end{cases}$$

for every $n \in \mathbb{N}$ (take f 's graph to be a skinny triangle shooting up to n at 0).

3. But $U: (C([-1, 1]), \|\cdot\|_\infty) \rightarrow \mathbb{C}$ defined by $Uf = f(0)$ is a **bounded linear transformation**, because $|f(0)| \leq \|f\|_\infty$.
4. Let A be an $n \times m$ matrix. Then $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ defined by $v \mapsto Av$ is a **bounded linear transformation**.

Explicitly this is

$$(Tv)_i = (Av)_i = \sum_{j=1}^m A_{ij}v_j.$$

5. Let $K(x, y)$ be a continuous function on $[0, 1] \times [0, 1]$. We'll define

$$T: (C[0, 1], \|\cdot\|_\infty) \rightarrow (C[0, 1], \|\cdot\|_\infty)$$

by

$$(Tf)(x) = \int_0^1 K(x, y)f(y) \, dy.$$

This is an analogue of matrix multiplication (K is like a continuous matrix). This is a **bounded linear transformation**.

6. Let us look at $T: L^1(\mathbb{R}) \rightarrow (C(\mathbb{R}), \|\cdot\|_\infty)$ defined by

$$(Tf)(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx$$

that is the Fourier transform of f .

7. $T: (C^\infty[0, 1], \|\cdot\|_\infty) \rightarrow (C^\infty[0, 1], \|\cdot\|_\infty)$. Define

$$(Tf)(x) = f'(x).$$

This is not a **bounded linear transformation**. In contrast, S , defined on the same spaces

$$(Sf)(x) = \int_0^x f(t) dt$$

is bounded.

Definition 5.12 (Operator norm). Let $L(V, W)$ be defined as a vector space such that

$$L(V, W) := \{T: V \rightarrow W \mid T \text{ is a bounded linear transformation}\}.$$

Then for $T \in L(V, W)$, the *operator norm* of T is

$$\begin{aligned} \|T\| &:= \inf\{c \geq 0 \mid \|Tv\|'' \leq c\|v\|' \text{ for all } v \in V\} \\ &= \sup\left\{\frac{\|Tv\|''}{\|v\|'} \mid v \neq 0, v \in V\right\} \\ &= \sup\{\|Tv\|'' \mid \|v\|' = 1, v \in V\}. \end{aligned}$$

Lemma 5.3. We have that

1. The **three definitions** of $\|T\|$ above are all equal.
2. $(L(V, W), \|\cdot\|)$ is indeed a **normed** space.

Proof.



DIY

Lecture 27: Dual Space

18 Mar. 11:00

As previously seen. From **Definition 5.12**, we have that

$$\|Tv\|'' \leq \|T\| \|v\|'.$$

Remark. Notice that this **Definition 5.12** is only for **bounded linear transformation**.

Theorem 5.6. If W is **complete**, then $L(V, W)$ is **complete**.

Proof. Suppose T_n is a **Cauchy sequence** in $L(V, W)$. Fix $v \in V$ and let $w_n := T_n v \in W$, we then have

$$\|w_n - w_m\| = \|T_n v - T_m v\| = \|(T_n - T_m)v\| \leq \underbrace{\|T_n - T_m\|}_{\rightarrow 0} \underbrace{\|v\|}_{\text{fixed value}}.$$

Thus, w_n is **Cauchy**, so it converges since W is **complete**. We call its unique limit Tv . This makes $T: V \rightarrow W$ into a function. We must show it is a **bounded linear transformation** and that $\|T_n - T\| \rightarrow 0$.



DIY

5.6 Dual of L^p Spaces

Example. Let $w \in \mathbb{R}^d$, and denote the inner product between w and $v \in \mathbb{R}^d$ by

$$v \cdot w := \langle v, w \rangle.$$

Then we can consider

$$\max\{v \cdot w \mid \|v\|_2 = 1\} = \|w\|_2.$$

If $w \in \mathbb{C}^d$, this is similar since

$$\max\{|v \cdot w| \mid \|v\|_2 = 1\} = \|w\|_2.$$

These maximums are achieved by $v = \frac{\bar{w}}{\|w\|}$ if $w \neq 0$.

Proposition 5.6. Let $1/p + 1/q = 1$ with $1 \leq q < \infty$. For every $g \in L^q$,

$$\|g\|_q = \sup \left\{ \left| \int fg \right| \mid \|f\|_p = 1 \right\}.$$

Suppose μ is σ -finite. Then the result also holds for $q = \infty$, $p = 1$.

As previously seen. For $\alpha \in \mathbb{C}$, $\text{sgn}(\alpha) := e^{i\theta}$ where $\alpha = |\alpha| e^{i\theta}$.

Proof. By Hölder's inequality, we know that

$$\left| \int fg \right| \leq \int |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q = \|g\|_q.$$

Thus, the supremum is less or equal to $\|g\|_q$.

1. Let

$$f(x) = \frac{|g(x)|^{q-1} \cdot \overline{\text{sgn}(g(x))}}{\|g\|_q^{q-1}}$$

Then $\int |f|^p = 1$, and $\int fg = \|g\|_q$.

2. The case that μ is σ -finite and $q = \infty$, $p = 1$ can be shown.

Check

DIY

■

Remark. One could use the above to prove Minkowski's inequality (as it only uses Hölder's inequality).

Definition 5.13 (Dual space). For a normed space $(V, \|\cdot\|)$, its dual space is $V^* = L(V, \mathbb{R})$ or $V^* = L(V, \mathbb{C})$.

Remark. Namely, the dual space of V contains bounded linear transformations with codomain being the scalar field.

Definition 5.14 (Linear functional). Given a **normed** space $(V, \|\cdot\|)$, $\ell \in V^*$ is called a *linear functional* on V . i.e.,

- $\ell: V \rightarrow \mathbb{R}$ (or \mathbb{C}).
- ℓ is linear.
- There exists a $c \geq 0$ such that $|\ell(v)| = c \|v\|$.

Note. V^* is always a **Banach space** (even if V is not **complete**).

Corollary 5.1. We have the followings.

1. Let $1/p + 1/q = 1, 1 \leq q < \infty$. For $g \in L^q$ define $\ell_g \in L^p \rightarrow \mathbb{C}$ by

$$\ell_g(f) = \int fg.$$

Then $\ell_g \in (L^p)^*$. Furthermore, $\|\ell_g\| = \|g\|_q$.

2. If μ is **σ -finite** then this also holds for $q = \infty, p = 1$.

Proof. ℓ_g is clearly linear in f because the integral is linear. Then **Proposition 5.6** gives in both 1. and 2. that

$$\|g\|_q = \sup\{|\ell_g(f)| \mid \|g\|_p = 1\} = \|\ell_g\|$$

and so ℓ_g is a **bounded linear transformation** with the desired properties. ■

Theorem 5.7. We have the followings.

1. Let $1/p + 1/q = 1, 1 \leq q < \infty$. The map $T: L^q \rightarrow (L^p)^*$ given by $Tg = \ell_g$ is an isometric^a linear isomorphism.

This means that

- T is a **bounded linear transformation**.
- T is bijective.
- T is **norm-preserving**.

2. If μ is **σ -finite** then this also holds for $q = \infty, p = 1$.

^aA map T is called isometric if for a given g , $\|Tg\| = \|g\|$.

Note. Even if μ is **σ -finite** we might not have $L^1 \cong (L^\infty)^*$.

Also note that $L^2 \cong (L^2)^*$, and for all $1 < p < \infty$ we have $(L^p)^{**} \cong L^p$.

Proof. We have already proved this is isometric in **Corollary 5.1**, it is clearly linear, and isometry implies injectivity.

We will prove that it is surjective later. ■

Fix!!!

Lecture 28: Signed Measure

21 Mar. 11:00

6 Signed and Complex Measures

As previously seen. Suppose $f: X \rightarrow [0, \infty]$ is a measurable function on (X, \mathcal{A}, μ) .

We can define $\nu(E) = \int_E f \, d\mu$ for $E \in \mathcal{A}$, and ν is a measure on (X, \mathcal{A}) .

This gives a map from the set of non-negative measurable functions on X to measures on X . This is injective if we identify functions which are equal almost everywhere. But it is not necessarily surjective. We can then think of measures as a generalization of functions.

For an example, think of a Dirac-Delta measure on \mathbb{R} . This is not the Lebesgue integral of any non-negative measurable function.

What if instead we took $f: X \rightarrow \mathbb{R}, \bar{\mathbb{R}}$ or \mathbb{C} . We could take the same construction to get $\nu(E) = \int_E f \, d\mu$, but this is no longer a measure as it can take $\mathbb{R}, \bar{\mathbb{R}}$ or \mathbb{C} values.

6.1 Signed Measures

Definition 6.1. Let (X, \mathcal{A}) be a measurable space. A signed measure is $\nu: \mathcal{A} \rightarrow [-\infty, \infty]$ or $\nu: \mathcal{A} \rightarrow (-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$.
- If $A_1, A_2, \dots \in \mathcal{A}$ are disjoint then

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$$

where the series on the right-hand side converges absolutely if

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \in (-\infty, \infty).$$

This means the series does not depend on rearrangement.

Example. Consider

1. ν is a positive measure (i.e., measure), then ν is a signed measure.
2. If we have positive measures μ_1, μ_2 such that either $\mu_1(X) < \infty$ or $\mu_2(X) < \infty$, then $\nu = \mu_1 - \mu_2$ is a signed measure.
3. If $f: X \rightarrow \bar{\mathbb{R}}$ on a measure space (X, \mathcal{A}, μ) such that $\int_X f^+ \, d\mu < \infty$ or $\int_X f^- \, d\mu < \infty$, we can define

$$\nu(E) = \int_E f \, d\mu$$

and this will be a signed measure.

Note. The following weird things happen with [signed measures](#).

1. $A \subseteq B$ does not imply $\nu(A) \leq \nu(B)$, as $\nu(B) = \nu(A) + \nu(B \setminus A)$, and $\nu(B \setminus A)$ may be negative.
2. If $A \subseteq B$ and $\nu(A) = \infty$, then $\nu(B) = \infty$, because $\nu(B \setminus A) \in (-\infty, \infty]$.
3. Similarly, if $A \subseteq B$ and $\nu(A) = -\infty$ then $\nu(B) = -\infty$.

Lemma 6.1. If ν is a [signed measure](#) on (X, \mathcal{A}) , then we have

1. Continuity from below. If $E_n \in \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \dots$ then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{N \rightarrow \infty} \nu(E_N).$$

2. Continuity from above. If $E_n \in \mathcal{A}$, $E_1 \supseteq E_2 \supseteq \dots$, and $-\infty < \nu(E_1) < \infty$ then

$$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{N \rightarrow \infty} \nu(E_N).$$

Proof. Read [FF99]. ■

Definition 6.2 (Positive, negative, null set for a signed measure).

Let ν be a [signed measure](#) on (X, \mathcal{A}) . Let $E \in \mathcal{A}$, then we say that

1. E is *positive* for ν if for all $F \subseteq E$, $\nu(F) \geq 0$.
2. E is *negative* for ν if for all $F \subseteq E$, $\nu(F) \leq 0$.
3. E is *null* for ν if for all $F \subseteq E$, $\nu(F) = 0$.

Note. We see that

1. If E is a [positive set](#), $F \subseteq E$, then $\nu(F) \leq \nu(E)$.
2. If E is a [negative set](#), $F \subseteq E$, then $\nu(F) \geq \nu(E)$.

Lemma 6.2. Let ν be a [signed measure](#) on (X, \mathcal{A}) , then

1. If E is [positive](#), $G \subseteq E$ is [measurable](#), then G is [positive](#).
2. If E is [negative](#), $G \subseteq E$ is [measurable](#), then G is [negative](#).
3. If E is [null](#), $G \subseteq E$ is [measurable](#), then G is [positive](#).
4. E_1, E_2, \dots are [positive](#) sets, then $\bigcup_{i=1}^{\infty} E_i$ is [positive](#).

Proof. ■

DIY

Lemma 6.3. Suppose that ν is a signed measure with $\nu: \mathcal{A} \rightarrow [-\infty, \infty)$. Suppose $E \in \mathcal{A}$ and $0 < \nu(E) < \infty$, then there exists a measurable $A \subseteq E$ such A is a positive set and $\nu(A) > 0$.

Proof. If E is positive, we're done. Otherwise, there exist measurable subsets with negative measure. Let $n_1 \in \mathbb{N}$ be the least such n_1 such that there exists $E_1 \subseteq E$ with $\nu(E_1) < -1/n_1$.

If $E \setminus E_1$ is positive, we're done. Else we can inductively define n_2, n_3, \dots as well as E_2, E_3, \dots

Explicitly, if $E \setminus \bigcup_{i=1}^{k-1} E_i$ is not positive, let n_k be the least such that there exists $E_k \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ with $\nu(E_k) < -1/n_k$.

Note then that if $n_k \geq 2$, for all $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ we have that $\nu(B) \geq -\frac{1}{n_k-1}$.

Now let $A = E \setminus \bigcup_{i=1}^{\infty} E_i$. Since $E = A \cup (\bigcup_i E_i)$ we have by countable additivity that

$$0 < \nu(E) = \nu(A) + \sum_{k=1}^{\infty} \nu(E_k) < \nu(A).$$

Furthermore, $\nu(E), \nu(A)$ are both in $(0, \infty)$, and we see that

$$0 < \nu(E) \leq \nu(A) - \sum_{k=1}^{\infty} \frac{1}{n_k}.$$

Therefore, the sum on the right-hand side must converge, meaning that $1/n_k \rightarrow 0$ as $k \rightarrow \infty$. That is $\lim_{k \rightarrow \infty} n_k = \infty$.

Now if $B \subseteq A$, then $B \subseteq E \setminus \bigcup_{i=1}^{\infty} E_i$. Therefore, $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$. By the note above, for large enough k such that $n_k \geq 2$ we have

$$\nu(B) \geq \frac{-1}{n_k - 1},$$

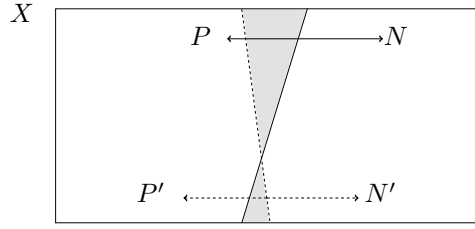
then taking $k \rightarrow \infty$ we have $\nu(B) \geq 0$, and so A is a positive set as desired. ■

Theorem 6.1 (Hahn decomposition theorem). If ν is a signed measure on (X, \mathcal{A}) , then there exist $P, N \in \mathcal{A}$ such that

$$P \cap N = \emptyset, \quad P \cup N = X,$$

where P is positive for ν , N is negative for ν .

Furthermore, if P', N' are another such pair, then $P \Delta P' (= N \Delta N')$ is null for ν .



Lecture 29: Hahn and Jordan Decomposition Theorem

23 Mar. 11:00

We now prove Theorem 6.1.

Proof. We first show the uniqueness. We see that $P \setminus P' \subseteq P, P \setminus P' \subseteq N'$. Thus, $P \setminus P' \subseteq P \cap N'$ is both positive and negative, hence $P \setminus P'$ is null.

Similarly, for $P' \setminus P$, and then their union $P \Delta P'$ is null as well.

To show the existence, without loss of generality suppose $\nu: \mathcal{A} \rightarrow [-\infty, \infty)$. If not, consider $-\nu$. Let

$$s := \sup\{\nu(E) \mid E \in \mathcal{A} \text{ is a positive set}\},$$

which is a nonempty supremum because \emptyset is positive. Then there exist P_1, P_2, \dots positive sets such that $\lim_{n \rightarrow \infty} \nu(P_n) = s$.

Then we have that $P = \bigcup_n P_n$ is positive by Lemma 6.2. We then have $\nu(P) \leq s$ and $\nu(P) = \nu(P_n) + \nu(P \setminus P_n) \geq \nu(P_n)$. Thus,

$$\nu(P) \geq \lim_{n \rightarrow \infty} \nu(P_n) = s.$$

Hence, $\nu(P) = s$ and the supremum is in fact a max. We then know that $s = \nu(P) < \infty$ because ν does not attain the value infinity.

Now let $N = X \setminus P$. We claim that N is negative. If not then there exists a measurable $E \subseteq N$ with $\nu(E) > 0$. By assumption, $\nu(E) < \infty$. Then $0 < \nu(E) < \infty$, so by Lemma 6.3 there exists a measurable $A \subseteq E$ such that A is positive and $\nu(A) > 0$.

But we then know that

$$\nu(P \cup A) = \nu(P) + \nu(A) > \nu(P)$$

which is a contradiction since $P \cup A$ is a positive set, and $\nu(P)$ is maximal.

Therefore, N is **negative**, and the theorem holds. ■

Definition 6.3 (Singular). If μ, ν are **signed measures** on (X, \mathcal{A}) , then we say μ and ν are *singular to each other*, denoted as $\mu \perp \nu$, if there exists $E, F \in \mathcal{A}$ such that $E \cap F = \emptyset, E \cup F = X$, F is **null** for μ , E is **null** for ν .

Example. Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

1. The **Lebesgue measure** m .
2. The **Cantor measure** μ_C defined by the **Cantor function**.
3. The **discrete measure** $\mu_D = \delta_1 + 2\delta_{-1}$.

We then see that

1. Take $E = \mathbb{R} \setminus \{-1, 1\}, F = \{1, -1\}$ to see that $m \perp \mu_D$.
2. Take $E = \mathbb{R} \setminus K$ and $F = K$ where K is the **Cantor set** to see that $m \perp \mu_C$.
3. We can also see that $\mu_C \perp \mu_D$.

Theorem 6.2 (Jordan decomposition theorem). Let ν be a **signed measure** on (X, \mathcal{A}) . Then there exists unique **positive measures** ν^+, ν^- on (X, \mathcal{A}) such that for all $E \in \mathcal{A}$ we have

$$\nu(E) = \nu^+(E) - \nu^-(E)$$

and

$$\nu^+ \perp \nu^-.$$

Proof. For existence, we take $\nu^+(E) := \nu(E \cap P), \nu^-(E) := -\nu(E \cap N)$. ■

DIY for uniqueness

Lecture 30: Absolutely Continuous Measures

25 Mar. 11:00

Example. For an example of **Theorem 6.2**, let (X, \mathcal{A}, μ) be a **measure space**, $f: X \rightarrow \mathbb{R}$, and $\nu(E) = \int_E f \, d\mu$. Then

$$\nu^+(E) = \int_E f^+ \, d\mu \qquad \nu^-(E) = \int_E f^- \, d\mu.$$

Definition 6.4 (Total variation). Let ν be a **signed measure** on (X, \mathcal{A}) . The *total variation measure* of ν is $|\nu| := \nu^+ + \nu^-$.

Remark. This is a **positive measure** on X .

Example. In the above **example**, $|\nu|(E) = \int_E |f| \, d\mu$.

Lemma 6.4. We have the following

1. $|\nu(E)| \leq |\nu|(E)$.
2. E is ν -null if and only if E is $|\nu|$ -null.
3. If κ is another signed measure, then $\kappa \perp \nu$ if and only if $\kappa \perp |\nu|$ if and only if $\kappa \perp \nu^+$ and $\kappa \perp \nu^-$.

Proof.

DIY

Definition 6.5 (Finite signed measure). A signed measure ν is *finite* if $|\nu|$ is a finite measure, and similarly for σ -finite.

This holds if and only if ν^+, ν^- are both finite (resp. σ -finite) measures.

6.2 Absolutely Continuous Measures

Definition 6.6 (Absolutely continuous). Let μ be a positive measure, ν be a signed measure, both on (X, \mathcal{A}) . We say that ν is *absolutely continuous with respect to μ* , denoted as $\nu \ll \mu$, provided that for all $E \in \mathcal{A}$, $\mu(E) = 0$ implies $\nu(E) = 0$.

Remark. This is equivalent to every μ -null set being ν -null.

Example. If (X, \mathcal{A}, μ) , $f: X \rightarrow \overline{\mathbb{R}}$, $\nu(E) = \int_E f d\mu$, then $\nu \ll \mu$.

Notation. $d\nu = f d\mu$ means ν is a signed measure defined by

$$\nu(E) = \int_E f d\mu.$$

Lemma 6.5. If μ is a positive measure, ν is a signed measure on (X, \mathcal{A}) , then

1. $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.
2. $\nu \ll \mu$ and $\nu \perp \mu$ implies $\nu = 0$.

Proof.

DIY 1.

For 2., write $X = A \cup B$, $A \cap B = \emptyset$, A μ -null, B ν -null. Then

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap A).$$

Then $E \cap A \subseteq A$, so $\nu(E \cap A) = 0$. By absolute continuity, $\nu(E \cap A) = 0$, thus $\nu(E) = 0$. ■

Theorem 6.3 (Radon-Nikodym theorem). Suppose μ is a σ -finite positive measure, ν is a σ -finite signed measure, and suppose $\nu \ll \mu$. Then there exists $f: X \rightarrow \overline{\mathbb{R}}$ such that $d\nu = f d\mu$, in other words $\nu(E) = \int_E f d\mu$. If g is another such function with $d\nu = g d\mu$ then $f = g$ μ -a.e..

Proof. Next class, we'll prove a more general form called **Lebesgue Radon Nikodym theorem**. ■

Definition 6.7 (Radon-Nikodym derivative). Suppose $\nu \ll \mu$. The Radon-Nikodym derivative of ν with respect to μ is a function

$$\frac{d\nu}{d\mu}: X \rightarrow \overline{\mathbb{R}}$$

such that

$$\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu$$

for all $E \in \mathcal{A}$.

Remark. i.e. we have $d\nu = \frac{d\nu}{d\mu} d\mu$.

Note. By **Theorem 6.3**, such a function exists and is unique up to equivalence μ -a.e. in the σ -finite case.

Example. Say $F(x) = e^{2x}: \mathbb{R} \rightarrow \mathbb{R}$. This is continuous and strictly increasing, so we may define a **Lebesgue-Stieltjes measure** μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

This is defined to be the unique **locally finite** measure satisfying $\mu_F([a, b]) = F(b) - F(a) = e^{2b} - e^{2a}$. Then one can check that

$$\mu_F(E) = \int_E 2e^{2x} dx$$

by uniqueness and the classical **fundamental theorem of calculus**, since the right-hand side is a **locally finite** Borel measure, and $\kappa([a, b]) = e^{2b} - e^{2a}$, thus $\mu_F = \kappa$.

Therefore, $\mu_F \ll m$ and $\frac{d\mu_F}{dm} = 2e^{2x} = \frac{dF}{dx}$.

Example. Let $C(x): \mathbb{R} \rightarrow \mathbb{R}$ be the **Cantor function**. Then $C'(x) = 0$ outside the **Cantor set**. But we don't always have

$$\mu_C(E) \neq \int_E 0 dx.$$

So the candidate derivative is 0, but this fails. In particular,

$$C(b) - C(a) \neq \int_a^b C'(x) dx.$$

In fact, $\mu_C \not\ll m$ because $\mu_C \perp m$ and $\mu_C \neq 0$.

Thus, the existence of a derivative [almost everywhere](#) and continuity is not enough to guarantee a version of the [fundamental theorem of calculus](#) holds.

Lecture 31: Lebesgue-Radon-Nikodym Theorem

28 Mar. 11:00

Lemma 6.6. Let μ, ν be [finite positive measures](#) on (X, \mathcal{A}) . Then either

1. $\nu \perp \mu$.
2. There exists an $\epsilon > 0$, an $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a [positive set for the measure](#) $\nu - \epsilon\mu$.
i.e., for all $G \subseteq F$, $\nu(G) \geq \epsilon\mu(G)$.

Proof. Let $\kappa_n = \nu - (1/n)\mu$. By [Theorem 6.1](#) we have $X = P_n \cup N_n$ for P_n [positive](#) for κ_n , N_n [negative](#) for κ_n .

Let $P = \bigcup_n P_n$, $N = \bigcap_n N_n = X \setminus P$, then $X = P \cup N$.

We see that for any N we have $\kappa_n(N) \leq 0$ because $N \subseteq N_n$. Thus,

$$0 \leq \nu(N) \leq \frac{1}{n}\mu(N).$$

This implies $\nu(N) = 0$. Because ν is [positive](#) for any $N' \subseteq N$ we have $0 \leq \nu(N') \leq \nu(N)$, and thus $\nu(N') = 0$. This shows N is [null](#) for ν .

If $\mu(P) = 0$, then $\nu \perp \mu$.

If $\mu(P) \neq 0$, then we have $\mu(P_n) > 0$ for some n .

With $F = P_n$ and $\epsilon = 1/n$, then F is a [positive set](#) for $\kappa_n = \nu - (1/n)\mu$ as desired. ■

Theorem 6.4 (Lebesgue-Radon-Nikodym theorem). Let μ be a [σ-finite positive measure](#), ν a [σ-finite signed measure](#) on (X, \mathcal{A}) .

Then there are unique λ, ρ [σ-finite signed measures](#) on (X, \mathcal{A}) such that $\lambda \perp \mu$, $\rho \ll \mu$, $\nu = \lambda + \rho$.

Furthermore, there exists a [measurable function](#) $f: X \rightarrow \overline{\mathbb{R}}$ such that $d\rho = f d\mu$ (that is for all $E \in \mathcal{A}$, $\rho(E) = \int_E f d\mu$).

And if there is another g such that $d\rho = g d\mu$, then $f = g$, [μ-a.e.](#)

Notationally, we may write $d\nu = d\lambda + f d\mu$, where $d\lambda$ and $d\mu$ are [singular](#) to each other.

Proof. We prove it step by step.

1. Assume μ, ν are [finite positive measures](#). Let

$$\begin{aligned} \mathcal{F} &= \left\{ g: X \rightarrow [0, \infty] \mid \int_E g d\mu \leq \nu(E), \forall E \in \mathcal{A} \right\} \\ &= \{ g: X \rightarrow [0, \infty] \mid d\nu - g d\mu \text{ is a positive measure} \}. \end{aligned}$$

This set is nonempty since $g = 0 \in \mathcal{F}$. Let $s = \sup\{\int_X g d\mu \mid g \in \mathcal{F}\}$.

Claim. There is an $f \in \mathcal{F}$ such that $s = \int_X f \, d\mu$.

If $g, h \in \mathcal{F}$, we can define $u(x) = \max\{g(x), h(x)\}$. Then $u \in \mathcal{F}$. Why? Well let $A = \{x \mid g(x) \geq h(x)\}$. Then

$$\begin{aligned} \int_E u \, d\mu &= \int_{E \cap A} g \, d\mu + \int_{E \cap A^c} h \, d\mu \\ &\leq \nu(E \cap A) + \nu(E \cap A^c) = \nu(E). \end{aligned}$$

There exist measurable functions $g_1, g_2, \dots \in \mathcal{F}$ such that

$$\lim_{n \rightarrow \infty} \int_X g_n \, d\mu = s.$$

We can replace g_2 by $\max(g_1, g_2)$, g_3 by $\max(g_1, g_2, g_3)$, so that we may assume $0 \leq g_1 \leq g_2 \leq \dots$

Then we still know that $\lim_{n \rightarrow \infty} \int_X g_n \, d\mu = s$, as all the relevant integrals are bounded above by s . Now let $f(x) = \sup_n g_n(x) = \lim_{n \rightarrow \infty} g_n(x)$. By Monotone convergence theorem,

$$\int_E f \, d\mu = \lim_{n \rightarrow \infty} \int_E g_n \, d\mu \leq \nu(E).$$

Thus, $f \in \mathcal{F}$. When $E = X$ we get $\int_X f \, d\mu = s$ as desired.

Let $\rho(E) := \int_E f \, d\mu$. We of course have $\rho \ll \mu$. And also we know

$$0 \leq \rho(X) = \int_X f \, d\mu \leq \nu(X) < \infty.$$

Thus, ρ is a finite positive measure. We can define $\lambda(E) := \nu(E) - \rho(E)$. Then

$$\lambda(E) = \nu(E) - \int_E f \, d\mu \geq 0$$

because $f \in \mathcal{F}$. Thus, λ is also a positive measure, and $\lambda(X) \leq \nu(X) < \infty$. It remains to show the following.

Claim. $\lambda \perp \mu$.

Suppose not, by Lemma 6.6, there exists $\epsilon > 0$, $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for $\lambda - \epsilon\mu$.

Then this says that $d\lambda - \epsilon \mathbb{1}_F d\mu$ is a positive measure, that is $d\nu - f \, d\mu - \epsilon \mathbb{1}_F d\mu$ is a positive measure. This will break maximality of f .

Explicitly, let $g(x) = f(x) + \epsilon \mathbb{1}_F(x)$. Then for all $E \in \mathcal{A}$ we have

$$\begin{aligned} \int_E g \, d\mu &= \int_E f \, d\mu + \epsilon \mu(E \cap F) \\ &= \nu(E) - \lambda(E) + \epsilon \mu(E \cap F) \\ &\leq \nu(E) - \lambda(E \cap F) + \epsilon \mu(E \cap F) \leq \nu(E) \end{aligned}$$

since $\lambda(E \cap F) - \epsilon \mu(E \cap F) \geq 0$. Thus, $g \in \mathcal{F}$. We then see that

$$s \geq \int_X g \, d\mu = \int_X f \, d\mu + \int_X \epsilon \mathbb{1}_F \, d\mu = s + \epsilon \mu(F) > s.$$

This is a contradiction! Perfect!

There are now technical things, such as extending to σ -finite measures and uniqueness. These are relatively easy compared to this part. ■

Lecture 32: Lebesgue Differentiation Theorem for Regular Borel Measures on \mathbb{R}^d

30 Mar. 11:00

Example (Lebesgue-Radon-Nikodym). Let $\mu = m$, $\nu = \mu_F$ (Lebesgue-Stieltjes measure for F). We'll define $F(x)$ by

$$F(x) = \begin{cases} e^{3x}, & \text{if } x \leq 0; \\ 1, & \text{if } 0 < x < 1; \\ 5, & \text{if } x \geq 1. \end{cases}$$

Then we will have that

$$\mu_F(E) = \int_{E \cap \mathbb{R}_{<0}} 3e^{3x} \, dx + 4\delta_1(E).$$

It is enough to check on $(-\infty, x]$ because these are locally finite Borel measures on \mathbb{R} .

Then we have $\mu_F = d\rho + d\lambda = f \, dm + d\lambda$ where $f = 1_{\mathbb{R}_{<0}} 3e^{3x}$ and $\lambda = 4\delta_1$, $\lambda \perp m$.

Read: Theorem 3.5, Proposition 3.9, Corollary 3.10 of section 3.2 of [FF99].

Skip: Complex measures (section 3.3) of [FF99].

As previously seen. If $\nu = \nu^+ - \nu^-$, we defined the total variation $|\nu| = \nu^+ + \nu^-$.

Then we have $|\nu(E)| \leq |\nu|(E)$.

6.3 Lebesgue Differentiation Theorem for Regular Borel Measures on \mathbb{R}^d

See page 99 of [FF99].

Definition 6.8 (Regular). A Borel signed measure ν on \mathbb{R}^d is called *regular* if

1. (compact finite) $|\nu|(K) < \infty$ for all compact K .
2. (outer regularity) We have *outer regularity*

$$|\nu|(E) = \inf\{|\nu|(U) \mid \text{open } U \supseteq E\}$$

for every Borel set E .

Example. Any Lebesgue-Stieltjes measure on \mathbb{R} has this property (see Theorem 1.6), so is the difference between two of them (at least if one of them is finite).

The Lebesgue measure on \mathbb{R}^d is regular.

Note. From compact finiteness, if ν is regular then ν is σ -finite. Also, if $d\nu = f dm$ is regular, then

$$|\nu|(K) = \int_K |f| dm < \infty$$

GMM result= 'pixel-error': 32.02873544435314 for all compact K . Thus, $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Lemma 6.7. $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ if and only if $d\nu = f dm$ is regular.

Proof.



DIY

As previously seen. Recall the Lebesgue differentiation theorem, here we had that if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ implies that for Lebesgue almost every x ,

$$\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x)$$

for any E_r shrinks nicely to x (think of $B_r(x)$).

Corollary 6.1. Let ρ be a regular signed Borel measure on \mathbb{R}^d . Suppose $\rho \ll m$. Then $d\rho = f dm$ for some $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. So then for Lebesgue almost every x we have

$$\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x).$$

Writing this nicely, using established notation, this is

$$\lim_{r \rightarrow 0} \frac{\rho(E_r)}{m(E_r)} = \frac{d\rho}{dm}(x)$$

for every E_r shrinks nicely to x .

Proposition 6.1. Let λ be a **regular positive Borel measure** on \mathbb{R}^d . Suppose $\lambda \perp m$.

For Lebesgue **almost every** x , we have

$$\lim_{r \rightarrow 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for every E_r **shrinking to x nicely** (equivalently, **shrinking to 0 nicely**).

Proof. It is enough to consider $E_r = B(x, r)$. We wish to prove that

$$G := \left\{ x \mid \limsup_{r \rightarrow 0} \frac{\lambda(E_r)}{m(E_r)} \neq 0 \right\} = \bigcup_{n=1}^{\infty} G_n$$

where

$$G_n := \left\{ x \mid \limsup_{r \rightarrow 0} \frac{\lambda(E_r)}{m(E_r)} > \frac{1}{n} \right\}.$$

It is enough to show $m(G_n) = 0$ for all n .

$\lambda \perp m$, so we know $\mathbb{R}^d = A \cup B$ disjoint, $\lambda(A) = 0$, $m(B) = 0$. Thus, it suffices to show $m(G_n \cap A) = 0$.

Fix $\epsilon > 0$, since λ is **regular**, there exists an open set $U \supseteq A$ such that $\lambda(U) \leq \lambda(A) + \epsilon = \epsilon$.

For every $x \in G_n \cap A$, there is an $r_x > 0$ such that $\lambda(B(x, r_x))/m(B(x, r_x)) > 1/n$ and $B(x, r_x) \subseteq U$.

Let $K \subseteq G_n \cap A$, compact. Then $K \subseteq \bigcup_{x \in K} B(x, r_x)$. By compactness, we can take a finite sub-cover, and then use **Lemma 4.1** to find B_1, B_2, \dots, B_N disjoint each of type $B(x, r_x)$ such that $K \subseteq \bigcup_i 3B_i$.

Therefore,

$$m(K) \leq 3^d \sum_{i=1}^N m(B_i) \leq 3^d n \sum_{i=1}^N \lambda(B_i) = 3^d n \lambda\left(\bigcup_i B_i\right) \leq 3^d n \lambda(U) = 3^d n \epsilon.$$

By inner regularity, $m(G_n \cap A) \leq 3^d n \epsilon$ for any $\epsilon > 0$. Taking $\epsilon \rightarrow 0$ yields $m(G_n \cap A) = 0$, so then $m(G_n) = 0$ as desired. ■

Lecture 33: Monotone Differentiation Theorem

1 Apr. 11:00

From last time we have that if $\rho \ll m$ is **regular** then

$$\lim_{r \rightarrow 0} \frac{\rho(E_r)}{m(E_r)} = \frac{d\rho}{dm}(x)$$

for **Lebesgue almost every** x , where E_r **shrinks nicely** to x . Likewise, if $\lambda \perp m$ **regular (positive measure)** then

$$\lim_{r \rightarrow 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for **Lebesgue almost every** x , where E_r **shrinks nicely** to x .

Theorem 6.5 (Lebesgue differentiation theorem for regular measures). Let ν be a regular Borel signed measure on \mathbb{R}^d . Then $d\nu = d\lambda + f dm$, $\lambda \perp m$ by Theorem 6.4.

Then for Lebesgue almost every x ,

$$\lim_{r \rightarrow 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every E_r shrinks nicely to x .

Proof. It must be checked that ν regular implies $\lambda, f dm$ are regular. ■

Check!

6.4 Monotone Differentiation Theorem

Definition 6.9. For $F: \mathbb{R} \rightarrow \mathbb{R}$ that is monotonically increasing, denote

$$F(x^+) = \lim_{y \rightarrow x^+} F(y), \quad F(x^-) = \lim_{y \rightarrow x^-} F(y).$$

These exist and are

$$\inf_{y > x} F(y), \quad \sup_{y < x} F(y).$$

So they always exist (being bounded below/above respectively by $F(x)$).

Lemma 6.8. If F is monotonically increasing, then

$$D = \{x \mid F \text{ is discontinuous at } x\}$$

is a countable set.

Proof. $x \in D$ if and only if $F(x^+) > F(x^-)$. For each $x \in D$, let $I_x = (F(x^-), F(x^+))$, not empty.

Also, if $x, y \in D$, $x \neq y$, then I_x, I_y are disjoint. Say if $x < y$ then

$$F(x^-) < F(x^+) \leq F(x) \leq F(y) \leq F(y^-) < F(y^+).$$

Taking a rational number in each interval gives an injective map $D \rightarrow \mathbb{Q}$, so D is countable. ■

Theorem 6.6 (Monotone Differentiation Theorem). Let F be increasing. Then

- F is differentiable Lebesgue almost everywhere.
- $G(x) = F(x^+)$ (which is right-continuous) is differentiable almost everywhere.
- $G' = F'$ almost everywhere

Proof. Start with G , which is increasing and right-continuous on \mathbb{R} . There is then a **Lebesgue-Stieltjes measure** μ_G on \mathbb{R} . Thus, it is **regular** on \mathbb{R} . We see

$$\frac{G(x+h) - G(x)}{h} = \begin{cases} \frac{\mu_G((x, x+h])}{m((x, x+h])}, & \text{if } h > 0; \\ \frac{\mu_G((x+h, x])}{m((x+h, x])}, & \text{if } h < 0. \end{cases}$$

These both **shrink nicely** to x . By **Theorem 6.5** (since these **shrink nicely**), we know then that these both converge for **Lebesgue almost every** x to some common limit $f(x)$. Hence, G' exists **Lebesgue almost everywhere**.

Define $H(x) = G(x) - F(x) \geq 0$. We see that

$$\{x \mid H(x) > 0\} \subseteq \{x \mid F \text{ is discontinuous at } x\}$$

This is then countable by **Lemma 6.8**, and we can write $\{x \mid H(x) > 0\} = \{x_n\}$. Then let

$$\mu := \sum_n H(x_n) \delta_{x_n}.$$

This is a **Borel measure**, so we check if it is **locally finite**. That is we check

$$\mu((-N, N)) = \sum_{-N < x_n < N} H(x_n) \leq G(N) - F(-N) < \infty$$

checking the inequality just consists of seeing that the intervals $(F(x_n), G(x_n))$ are disjoint and a subset of $(F(-N), G(N))$ so

$$\sum_{-N < x_n < N} H(x_n) = \mu \left(\bigcup_n (F(x_n), G(x_n)) \right) \leq \mu((F(-N), G(N))).$$

Thus, μ is a **Lebesgue-Stieltjes measure** on \mathbb{R} , so it is **regular**.

Remark. Special to \mathbb{R} , we have

$$\begin{aligned} \text{that locally finite Borel} &\implies \text{Lebesgue-Stieltjes} \\ &\implies \text{regular} \\ &\implies \text{outer regularity.} \end{aligned}$$

Then we have that

$$\left| \frac{H(x+h) - H(x)}{h} \right| \leq \frac{H(x+h) + H(x)}{|h|} \leq \frac{\mu((x-2h, x+2h))}{|h|}.$$

This goes to 0 for **Lebesgue almost every** x by **Theorem 6.5** and that $\mu \perp m$. Check!

Thus, H is differentiable **almost everywhere** and $H' = 0$ **almost everywhere**. Thus, F is differentiable **almost everywhere** and $F' = G'$ **almost everywhere**. ■

Proposition 6.2. Suppose F is an increasing function. Then F' exists almost everywhere and is measurable, then

$$\int_a^b F'(x) \, dx \leq F(b) - F(a).$$

Example. Take $F(x)$ to be 0 on $x \leq 0$, 1 on $x > 0$. Then $F'(x) = 0$ almost everywhere. So

$$\int_{-1}^1 F'(x) \, dx = 0 < 1 = F(1) - F(-1).$$

Even if F is continuous we might not have equality. Take $F(x)$ to be the Cantor function. Then $F'(x) = 0$ almost everywhere, but

$$\int_0^1 F'(x) \, dx = 0 < 1 = F(1) - F(0).$$

Appendix

A Additional Proofs

A.1 Measure

This section gives all additional proofs in [Section 1](#).

Theorem A.1 ([Theorem 1.2 3.](#)). Under the setup of [Theorem 1.2](#), (X, \mathcal{A}, μ) is a [complete measure space](#).

Proof. We see this in two parts.

1. **Claim:** If $A \subset X$ satisfies $\mu^*(A) = 0$, then A is [Carathéodory measurable](#) with respect to μ^* .

Proof. If $A \subset X$ and $\mu^*(A) = 0$, where μ^* is an outer measure on X , we'll show that A is [Carathéodory measurable](#) with respect to μ^* .

Equivalently, we want to show that for any $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Firstly, noting that $(E \cap A) \subset A$, and by [monotonicity](#) of μ^* , we see that

$$\mu^*(E \cap A) \leq \mu^*(A) = 0,$$

and since $\mu^* \geq 0$, hence $\mu^*(E \cap A) = 0$. Now, we only need to show that

$$\mu^*(E) = \mu^*(E \setminus A).$$

Since $E \setminus A = E \cap A^c$, and hence we have $E \cap A^c \subset E$, so

$$\mu^*(E) \geq \mu^*(E \setminus A).$$

To show another direction, we note that

$$\mu^*(E) \leq \mu^*(E \cup A) = \mu^*((E \setminus A) \cup A) \leq \mu^*(E \setminus A),$$

hence we conclude that A is [Carathéodory measurable](#) with respect to μ^* if $\mu^*(A) = 0$. ■

2. **Claim:** If A is [μ-subnull](#), then $A \in \mathcal{A}$.

Proof. Let \mathcal{A} denotes the [Carathéodory σ-algebra](#), and $\mu := \mu^*|_{\mathcal{A}}$. We want to show if A is [μ-subnull](#), then $A \in \mathcal{A}$.

Firstly, if A is [μ-subnull](#), then there exists a $B \in \mathcal{A}$ such that $A \subset B$ and $\mu(B) = 0$. But since from the [monotonicity](#) of μ^* , we further have

$$0 = \mu(B) = \mu^*(B) \geq \mu^*(A),$$

hence $\mu^*(A) = 0$.

From the first claim, we immediately see that A is [Carathéodory measurable](#) with respect to μ^* , which implies A is in [Carathéodory σ-algebra](#), hence $A \in \mathcal{A}$. ■

We see that the second claim directly proves that (X, \mathcal{A}, μ) is a [complete measure space](#). ■

Lemma A.1. The function F defined in [this example](#) is a [distribution function](#)

Proof. We define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n \end{cases}$$

where $\{r_1, r_2, \dots\} = \mathbb{Q}$, and

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} = \sum_{n; r_n \leq x} \frac{1}{2^n}$$

is both increasing and right-continuous.

- Increasing. Consider $x < y$. We see that

$$F(x) = \sum_{n; r_n \leq x} \frac{1}{2^n} \leq \sum_{n; r_n \leq y} \frac{1}{2^n} = F(y)$$

clearly.^{[28](#)}

- Right-continuous. We want to show $F(x^+) = F(x)$. Let $x^+(\epsilon) := x + \epsilon$ with $\epsilon > 0$, we'll show that

$$\lim_{\epsilon \rightarrow 0} F(x^+(\epsilon)) = \lim_{\epsilon \rightarrow 0} F(x + \epsilon) = F(x).$$

Firstly, we have

$$F(x^+(\epsilon)) - F(x) = \sum_{n; r_n \leq x+\epsilon} \frac{1}{2^n} - \sum_{n; r_n \leq x} \frac{1}{2^n} = \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n},$$

and we want to show

$$\lim_{\epsilon \rightarrow 0} F(x^+(\epsilon)) - F(x) = \lim_{\epsilon \rightarrow 0} \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} = 0.$$

Before we show how we choose ϵ ,^{[30](#)} we see that

$$\sum_{n=k}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

²⁸This is trivial since we're always going to sum more strictly positive terms in $F(y)$ than in $F(x)$.

²⁹The strict is crucial to show the result, since if $x = r_k$ for some fixed k , then we can't make the summation arbitrarily small.

³⁰To be precise, how ϵ depends on r_n .

Now, we observe that

$$\sum_{n; x < r_n \leq x + \epsilon} \frac{1}{2^n} \leq \sum_{n = \arg \min_k x < r_k \leq x + \epsilon}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

With this observation, it should be fairly easy to see that we can choose ϵ based on how small we want to make 2^{1-k} be,³¹ and we indeed see that

$$\lim_{k \rightarrow \infty} 2^{1-k} = 0,$$

which implies that F is right-continuous by squeeze theorem. ■

Lemma A.2. The function F defined in [this example](#) satisfies

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$

given in [this example](#).

Proof. We prove them one by one. And notice that F is indeed a distribution function as we proved in [Lemma A.1](#).

1. To show $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$, we first note that

$$\{r\} = \bigcap_{a-1 \leq x < r} (x, r].$$

Then, we see that

$$\mu_F(\{r\}) = \mu_F \left(\bigcap_{a-1 \leq x < a} (x, r] \right),$$

where each $(x, r] \in \mathcal{A}$ and $(x, r] \supset (y, r]$ whenever $r - 1 \leq x \leq y < r$. Notice that we implicitly assign the order of the index by the order of x . Then, we see that $\mu_F(r - 1, r] < \infty$.³² Then, from [continuity from above](#), we see that

$$\mu_F(\{r\}) = \lim_{i \rightarrow \infty} \mu_F((x_i, r]),$$

where we again implicitly assign an order to x as the usual order on \mathbb{R} by given index i . It's then clear that as $i \rightarrow \infty$, $x_i \rightarrow r$. From the definition of F , we see that

$$F((x_i, r]) = F(r) - F(x_i) = \sum_{n; r_n \leq r} \frac{1}{2^n} - \sum_{n; r_n \leq x_i} \frac{1}{2^n} = \sum_{n; x_i < r_n \leq r} \frac{1}{2^n}.$$

³¹We're referring to $\epsilon - \delta$ proof approach.

³²This will be $\mu(A_1)$ in the condition of [continuity from above](#). Furthermore, since \mathbb{Q} is countable, hence $F(x) < \infty$ is promised.

It's then clear that since $r \in \mathbb{Q}$, there exists an i' such that $r_{i'} = r$. Then, we immediately see that no matter how close $x_i \rightarrow r$, this sum is at least

$$\frac{1}{2^{i'}}$$

for a fixed i' . Hence, we conclude that $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$.

2. Now, we show $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$. Firstly, we claim that

$$\mu_F(\mathbb{Q}) = 1$$

and

$$\mu_F(\mathbb{R}) = 1$$

as well. Since $\mu_F(\mathbb{Q}) = 1$ is clear, we note that the second one essentially follows from the fact that we can write

$$\mathbb{R} = \lim_{N \rightarrow \infty} \bigcup_{i=1}^N (a - i, a + i]$$

for any $a \in \mathbb{R}$, say 0. From [continuity from below](#), we have

$$\mu_F\left(\bigcup_{i=1}^{\infty} (-i, +i]\right) = \lim_{n \rightarrow \infty} \mu_F((-n, n]) = \sum_{n; r_n \in \mathbb{Q}} \frac{1}{2^n} = 1.$$

Given the above, from countable additivity of μ_F , we have

$$\mu_F(\mathbb{R} \setminus \mathbb{Q}) + \underbrace{\mu_F(\mathbb{Q})}_1 = \underbrace{\mu_F(\mathbb{R})}_1 \implies \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$$

as we desired. ■

Lemma A.3 (Cantor set has measure 0). Let C denotes the [middle thirds Cantor set](#), then the [Lebesgue measure](#) of C is 0. i.e.,

$$m(C) = 0.$$

Proof. Since we're removing $\frac{1}{3}$ of the whole interval at each n , we see that the measure of those removing parts, denoted by r , is

$$m(r) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 1.$$

Then, by [countable additivity](#) of m , we see that

$$m(C) = m([0, 1]) - m(r) = 1 - 1 = 0. \quad \blacksquare$$

A.2 Integration

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