MATH635
Riemannian Geometry

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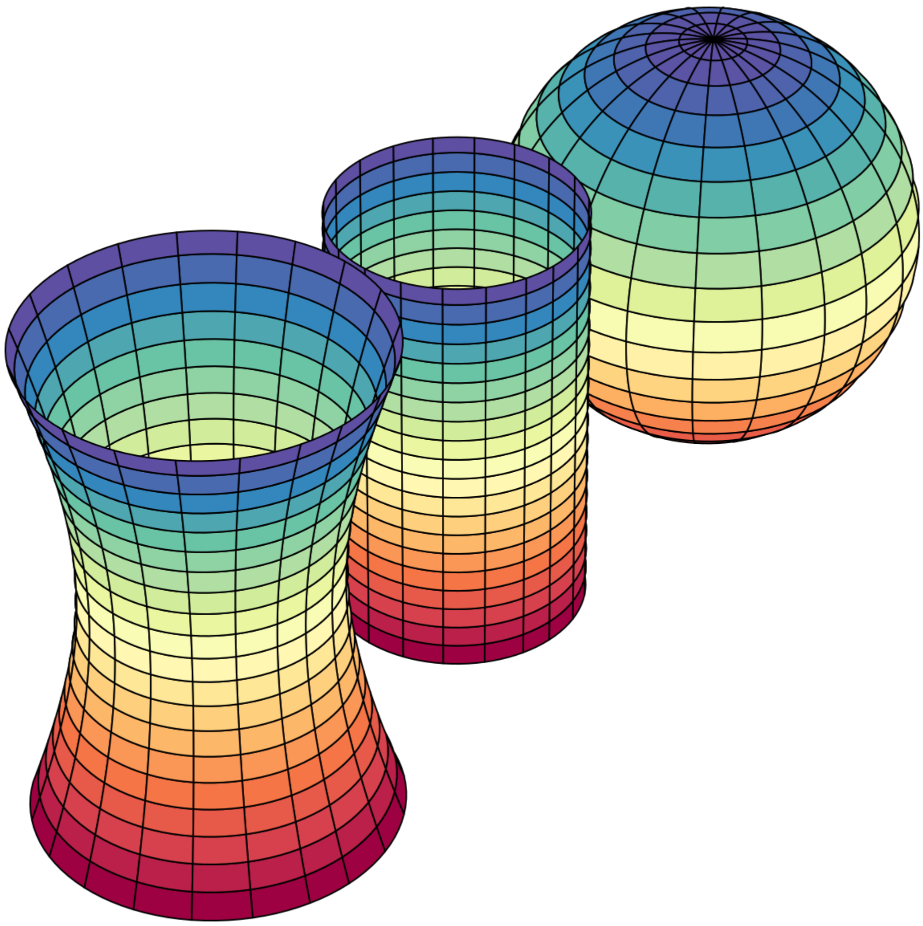
Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by Lydia Bieri at University of Michigan. Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [FC13] as our reference. Apart from this, I also found [Sch15] very useful.

A noticeable different is that we introduce geodesics differently from do Carmo [FC13], where we set the solution of the variations of energy to define a geodesic first, and then draw connection to the "curve with zero acceleration" after introduce the covariant derivative; however, do Carmo [FC13] first introduce covariant derivative and then return the variation view point much later.



This course is taken in Winter 2023, and the date on the cover page is the last updated time.

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Chapter 1

Smooth Manifolds

Lecture 1: A Foray to Smooth Manifolds

5 Jan. 13:00

1.1 Topological Manifolds

Let's start with a common definition.

Definition 1.1.1 (Topological manifold). A *topological manifold* $\mathcal{M}$ of dimension n is a Hausdorff and second-countable (topological) space such that each point $p \in \mathcal{M}$ has a neighborhood U homeomorphic via $\varphi\colon U \to U'$ to an open subset $U' \subseteq \mathbb{R}^n$.

Definition 1.1.2 (Local coordinate map). For every $p \in \mathcal{M}$, the corresponding homeomorphism φ is called the *local coordinate map*.

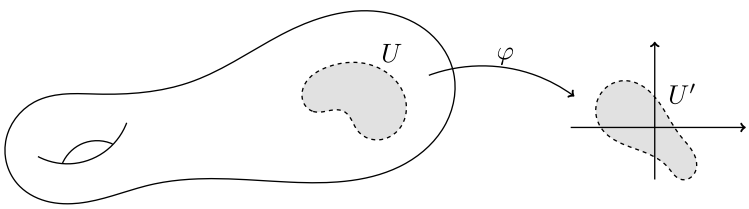
Definition 1.1.3 (Local coordinate). The pull-back $(x^1, \dots, x^n)$ of the local coordinate map φ from $\mathbb{R}^n$ is called the *local coordinates* on U, given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

Definition 1.1.4 (Coordinate chart). The pair (U, φ) is called a *(coordinate) chart* on $\mathcal{M}$.

Remark. The reason why we want the space to be Hausdorff and second-countable is because we can then have partition of unity.

In other words, a topological manifold can be thought of as a space such that it looks like $\mathbb{R}^n$ locally.



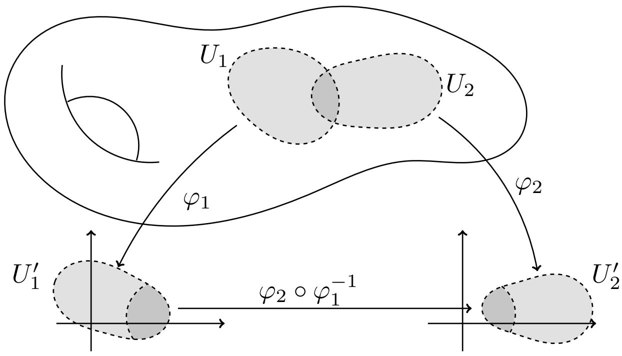
Definition 1.1.5 (Atlas). An *atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ for a manifold $\mathcal{M}$ is a collection of charts such that $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$ are an open covering of $\mathcal{M}$, i.e., $\mathcal{M} = \bigcup_\alpha U_\alpha$.

In other words, for all $p \in \mathcal{M}$, there exists a neighborhood $U \subseteq \mathcal{M}$ and homeomorphism $h\colon U \to U' \subseteq \mathbb{R}^n$ open.

Definition 1.1.6 (Locally finite)**.** An atlas is said to be *locally finite* if each point $p \in \mathcal{M}$ is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as $\mathbb{R}^n$, there's no clear way to make sense of differentiability of a manifold. But thankfully, we now have an explicit relation to the ambient space $\mathbb{R}^n$ via φ_α. To formalize, let $\mathcal{A}$ be an atlas for a manifold $\mathcal{M}$, and assume that $(U_1, \varphi_1), (U_2, \varphi_2)$ are 2 elements of $\mathcal{A}$. Then clearly, the map $\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a homeomorphism between 2 open sets of Euclidean spaces since both φ_1 and φ_2 are homeomorphisms. Due to this map's importance, it has its own name.

Definition 1.1.7 (Coordinate transition)**.** The map $\varphi_2 \circ \varphi_1^{-1}$ is called the *coordinate transition* of $\mathcal{A}$ for the pair of charts $(U_1, \varphi_1), (U_2, \varphi_2)$.



1.2 Differentiable Manifolds

Notice that the coordinate transitions are from $\mathbb{R}^n$ to $\mathbb{R}^n$; hence differentiability makes sense now, which induces the following.

Definition 1.2.1 (Differentiable atlas)**.** The atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is *differentiable* if all transitions are differentiable.

Remark. Here, the differentiability depends on the content. Sometimes, we may want it to be C^∞, and sometimes may be C^k for some finite k. On the other hand, smooth always refers to C^∞. We'll use them interchangeably if it's clear which case we're referring to.

Definition 1.2.2 (Equivalence atlas)**.** Two atlases $\mathcal{U}, \mathcal{V}$ of a manifold are equivalent if for every $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$,

$$\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

Notation (Smoothly compatible)**.** Two charts (U, φ) and (V, ψ) are *smoothly compatible* if either $U \cap V = \varnothing$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

This suggests the following.

Definition 1.2.3 (Smooth structure)**.** A *smooth structure* on $\mathcal{M}$ is an equivalence class $\mathcal{U}$ of coordinate atlas with the property that all transition functions are diffeomorphisms.

Remark. We can also use the *maximal* differentiable atlas to be our differentiable structure.

Definition 1.2.4 (Smooth manifold)**.** A *smooth manifold* is a manifold $\mathcal{M}$ with a smooth structure.

In this way, we can do calculus on smooth manifolds! Furthermore, it now makes sense to say that a function $f\colon \mathcal{M} \to \mathbb{R}$ is differentiable (or C^∞) by considering differentiability of $f \circ \varphi^{-1}$ around p. This leads to the following.

Notation. The collection of smooth functions on smooth manifold $\mathcal{M}$ is denoted by $C^\infty(\mathcal{M}, \mathbb{R})$, or $C^k(\mathcal{M}, \mathbb{R})$.[a]

[a] We will formalize this later in Definition 1.2.8.

Remark. The class $C^\infty(\mathcal{M}, \mathbb{R})$ consists of functions with property is well-defined.

Proof. Let $\mathcal{A}$ be any given atlas from equivalence class that defines the smooth structure, and as we have shown, if $(U, \varphi) \in \mathcal{A}$, then $f \circ \varphi^{-1}$ is smooth on $\mathbb{R}^n$. This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.2.2 requirement that defines the equivalent manifolds. ⊛

1.2.1 Orientation

Another essential property of a manifold is its orientability.

Definition. Consider an atlas $\mathcal{A}$ for a differentiable manifold $\mathcal{M}$.

Definition 1.2.5 (Oriented)**.** $\mathcal{A}$ is *oriented* if all transitions have positive functional determinant.

Definition 1.2.6 (Orientable)**.** $\mathcal{M}$ is *orientable* if $\mathcal{A}$ is an oriented atlas.

Motivated by the above definitions, we see that we can actually use an atlas to define an orientation.

Definition 1.2.7 (Orientation)**.** Let $\mathcal{M}$ be an orientable manifold. Then a oriented differentiable structure is called an *orientation* of $\mathcal{M}$.

If $\mathcal{M}$ possesses an orientation, we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with Definition 1.2.5.

Remark. Two differentiable structures obeying Definition 1.2.5 determine the same orientation if the union again satisfying Definition 1.2.5.

Remark. If $\mathcal{M}$ is orientable and connected, then there exists exactly 2 distinct orientations on $\mathcal{M}$.

Now, we can see some examples of smooth manifolds.

Example (Sphere)**.** The sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by

$$S^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\right\}.$$

Consider $U_i^+ = \{x \in S^n \mid x_i > 0\}$, $U_i^- = \{x \in S^n \mid x_i < 0\}$ for $i = 1, \dots, n+1$, and $h_i^\pm\colon U_i^\pm \to \mathbb{R}^n$ such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, .., x_{n+1}).$$

Note that the minimum charts needed to cover S^n is 2.

Example. Let $\mathcal{M} = U \subseteq \mathbb{R}^n$, then $\{(U, \varphi)\}$ is a smooth structure with $\varphi = \mathbb{1}$.

Example. Open sets of C^∞-manifolds are C^∞-manifolds.

Example (General linear group)**.** $\mathrm{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, open.

Example (Real projective space)**.** $\mathbb{RP}^n = S^n \,/ \sim$ where $x \sim -x$ with $\pi \colon S^n \to \mathbb{RP}^n$, $x \mapsto [x]$.

Proof. π is a homeomorphism on each U_i^+ for $i = 1, \dots, n+1$, with

$$\left\{ \left(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1} \right), i = 1, \dots, n+1 \right\}$$

is a C^∞-atlas for $\mathbb{R}P^n$. ⊛

Note. Observe that $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} \,/ \sim$.

Lecture 2: Maps Between Smooth Manifolds

1.2.2 Smooth Maps

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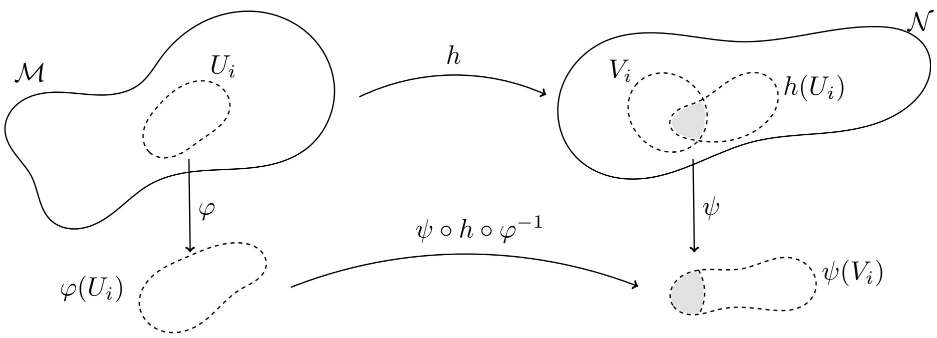
We can now consider the maps between manifolds, specifically, the smooth manifolds.

Definition 1.2.8 (Smooth function)**.** Let $\mathcal{M}, \mathcal{N}$ be two smooth manifolds, and let $\mathcal{U}$ be locally finite atlas from the equivalence class that gives the smooth structure on $\mathcal{M}$, and let $\mathcal{V}$ be the corresponding for $\mathcal{N}$. A map $h \colon \mathcal{M} \to \mathcal{N}$ is said to be *smooth* if each map in the collection

$$\left\{ \psi \circ h \circ \varphi^{-1} \colon h(U) \cap V \neq \varnothing \right\}$$

where $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$ is C^∞-differentiable as a map from one Euclidean space to another.



Remark. Equivalence relation guarantees that Definition 1.2.8 depends only on the smooth structure of $\mathcal{M}, \mathcal{N}$, but not on the chosen representative coordinate atlas.

Definition. Consider two smooth manifolds $\mathcal{M}, \mathcal{N}$ and a smooth homeomorphism $h \colon \mathcal{M} \to \mathcal{N}$ with smooth inverse.

Definition 1.2.9 (Diffeomorphic)**.** The two manifolds $\mathcal{M}, \mathcal{N}$ are said to be *diffeomorphic*.

Definition 1.2.10 (Diffeomorphism)**.** The map h is said to be a *diffeomorphism*.

Let $\mathcal{M}_1, \mathcal{M}_2$ be two smooth manifolds, and let $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a diffeomorphism. Then

(a) $\mathcal{M}_1$ is orientable if and only if $\mathcal{M}_2$ is orientable.

(b) If in addition, $\mathcal{M}_1$ and $\mathcal{M}_2$ are both connected and oriented, then φ induces an orientation on $\mathcal{M}_2$ that may or may not coincide with the initial orientation of $\mathcal{M}_2$.

If the induced orientation coincides, then we say φ preserves the orientation, otherwise φ reverses the orientation.

1.2.3 Grassmannian Manifold

Before proceeding, let's consider an interesting smooth manifold.

Definition 1.2.11 (Grassmannian manifold)**.** Given $m, n \in \mathbb{N}$, the so-called *Grassmannian manifold* $G(n, m)$ is the set of all n-dimensional subspaces of $\mathbb{R}^{n+m}$.

Note. $G(1, m)$ is just $\mathbb{R}P^m$, and $G(0, m)$, $G(n, 0)$ are one-point sets.

As we will soon see, $G(n, m)$ has the smooth structure of an mn-dimensional manifold.

Intuition. We obtain the structure by exhibiting an atlas whose transitions are diffeomorphisms.

Firstly, we give $G(n, m)$ a suitable topology, i.e., the metric topology. Let $\Pi \in G(n, m)$, and let $\mathcal{L}(\Pi, \Pi^\perp)$ denote the mn-dimensional space of linear maps from Π to $\Pi^\perp$. Define the map

$$\varphi_\Pi\colon \mathcal{L}(\Pi, \Pi^\perp) \to G(n, m), \qquad \varphi_\Pi(\alpha) = (\mathbb{1}_\Pi \oplus \alpha)(\Pi)$$

where $\mathbb{1}_\Pi \oplus \alpha$ is regarded as a map $\Pi \to \Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.[1] Clearly, φ_Π is injective, and thus, $\big(\mathcal{L}(\Pi, \Pi^\perp), \varphi_\Pi\big)$ is an mn-dimensional chart of $G(n, m)$.

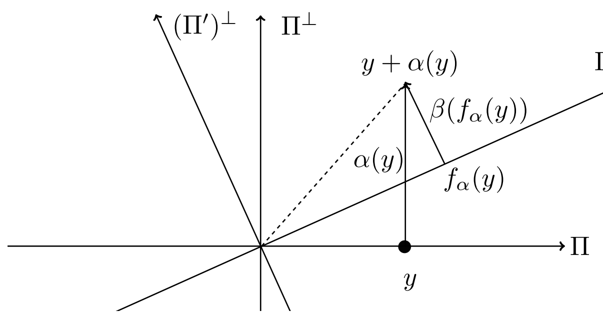
Remark. The images $\varphi_\Pi\big(\mathcal{L}(\Pi, \Pi^\perp)\big)$ cover $G(n, m)$.

Example. $\Pi = \varphi_\Pi(0) \in \varphi_\Pi\big(\mathcal{L}(\Pi, \Pi^\perp)\big)$.

We can now prove that these charts are mutually compatible. Let $\Pi, \Pi' \in G(n, m)$, and let P, P' be orthogonal projections from $\mathbb{R}^{n+m}$ onto Π, Π' respectively. Firstly,

$$F = \varphi_{\Pi'}^{-1}\varphi_\Pi\colon \varphi_\Pi^{-1}\big(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))\big) \to \varphi_{\Pi'}^{-1}\big(\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))\big)$$

is smooth.



Consider $\alpha \in \mathcal{L}(\Pi, \Pi^\perp)$, and $\beta \in \mathcal{L}(\Pi', (\Pi')^\perp)$, then for α, β, the equality $F(\alpha) = \beta$ means that $\varphi_\Pi(\alpha) = \varphi_{\Pi'}(\beta)$. Let $f_\alpha\colon \Pi \to \Pi'$ be defined by

$$f_\alpha = P' \circ (\mathbb{1}_\Pi \oplus \alpha).$$

We need to check

[1] In other words, $\varphi_\Pi(\alpha)$ is the graph of α in $\Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.

(a) f_α is invertible, and

(b) $\forall y \in \Pi$, $y + \alpha(y) = f_\alpha(y) + \beta(f_\alpha(y))$.

Note. The condition $\det f_\alpha \neq 0$ gives an exact description of the subset $\varphi_{\Pi^{-1}}\left(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))\right)$ of $\mathcal{L}(\Pi, \Pi^\perp)$, which is therefore open.

For β, it is $\left(\mathbb{1}_{\Pi'} \oplus \beta\right) \circ f_\alpha = \mathbb{1}_\Pi \oplus \alpha$, and hence

$$\beta = F(\alpha) = (\mathbb{1}_\Pi \oplus \alpha) \circ f_\alpha^{-1} - \mathbb{1}_{\Pi'}.$$

It follows by the construction that the image of β is contained in $(\Pi')^\perp$.

Remark. We obtain an infinite atlas for $G(n, m)$ with charts labeled by $\Pi \in G(n, m)$. But it suffices to consider only $\binom{n+m}{n}$ charts corresponding to subspaces Π spanned with n coordinate axes.

1.2.4 Other Manifold Properties

We now introduce two notions.

Definition 1.2.12 (Closed manifold)**.** A manifold is *closed* if it is compact and without boundary.

Definition 1.2.13 (Open manifold)**.** A manifold is *open* if it has only non-compact components without boundary.

Lemma 1.2.1. If $\mathcal{M}$ can be covered by two coordinate neighborhoods V_1, V_2 such that $V_1 \cap V_2$ is connected, then $\mathcal{M}$ is orientable.

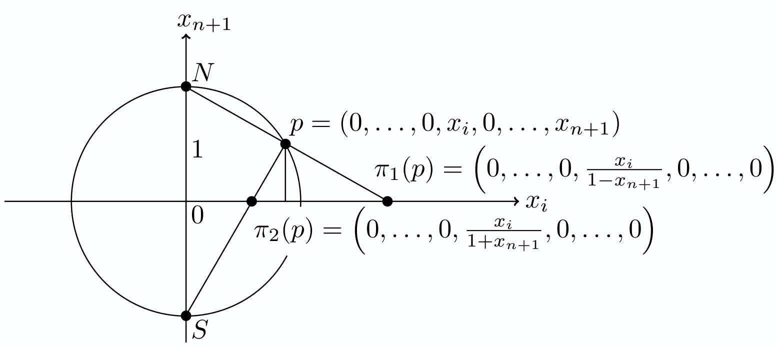
Proof. The determinant of the differential of the coordinate change $\neq 0$, so it does not change sign in $V_1 \cap V_2$. If it's negative at a single point, it's enough to change the sign of one of the coordinates to make it positive at that point, hence on $V_1 \cap V_2$. ∎

Example. Let $S^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\right\} \subseteq \mathbb{R}^{n+1}$ is orientable.

Proof. Let $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$, consider given $p = (0, \ldots, 0, x_i, 0, \ldots, x_{n+1})$, then $\pi_1 \colon S^n \setminus \{N\} \to \mathbb{R}^n$ given by

$$\pi_1(p) = \left(0, \ldots, 0, \frac{x_i}{1 - x_{n+1}}, 0, \ldots, 0\right)$$

to be the stereographic projection from the North Pole N.



More generally, it takes $p(x_1, \ldots, x_{n+1}) \in S^n - \{N\}$ into the intersection at the hyperplane

$x_{n+1} = 0$ with the line passing through p ad N. In this way, we have

$$\pi_1(x_1, \ldots, x_n) = \left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right),$$

hence $\pi_1 \colon S^n \setminus \{N\} \to \mathbb{R}^n$ is differentiable, and is injective. Similarly, $\pi_2 \colon S^n \setminus \{S\} \to \mathbb{R}^n$ for S can also be defined and everything holds similarly. We see that these two parametrizations $(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})$ cover S^n. The change of coordinate is given by

$$y_j = \frac{x_j}{1-x_{n+1}} \leftrightarrow y_j' = \frac{x_j}{1+x_{n+1}}, \ (y_1, \ldots, y_n) \in \mathbb{R}^n, \ j = 1, \ldots, n,$$

where

$$y_j' = \frac{y_j}{\sum_{i=1}^n y_i^2}.$$

This implies that $\left\{(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})\right\}$ is a differentiable structure for S^n. Now, consider

$$\pi_1^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^n) = S^n \setminus \{N, S\},$$

which is connected, hence S^n is orientable, and the above structure gives an orientation of S^n. ⊛

Lecture 3: Tangent Spaces and Bundles

Let's look at two more examples about orientation. 12 Jan. 13:00

Example. Let $A \colon S^n \to S^n$ be the antipodal map given by $A(p) = -p$ for $p \in \mathbb{R}^{n+1}$. It's easy to see that A is differentiable with $A^2 = \mathbb{1}$. Furthermore, A is diffeomorphism of $S^n \subseteq \mathbb{R}^{n+1}$. We see that

- if n is even, A reverses the orientation;
- if n is odd, A preserves the orientation.

Example. $G(k, n)$ is orientable if and only if n is even or $n = 1$.

Finally, we introduce the notion of complex manifolds.

Definition 1.2.14 (Complex manifold). A *complex manifold* $\mathcal{M}$ of complex dimension d ($\dim_{\mathbb{C}} \mathcal{M} = d$) is a differentiable manifold of (real) dimension $2d$ ($\dim_{\mathbb{R}} \mathcal{M} = 2d$) whose charts take values in open subsets of $\mathbb{C}^d$ with holomorphic chart transitions.

As previously seen. The chart transitions $z_\beta \circ z_\alpha^{-1} \colon z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ is holomorphic if $\partial z_\beta^j / \partial \overline{z_\alpha^k} = 0$ for all j, k where

$$\frac{\partial}{\partial \overline{z^k}} = \frac{1}{2}\left(\frac{\partial}{\partial \overline{x^k}} + i \frac{\partial}{\partial \overline{y^k}}\right).$$

We're not going to spend more time on complex manifolds in this course, however, one important thing to note is the following.

Remark. Complex Grassmannians $G_{\mathbb{C}}(k, n)$ are all orientable. More generally, complex manifolds are always orientable because holomorphic maps always have positive functional determinant.

1.2.5 Partition of Unity

So far, we have defined functions between manifolds in a local way. However, to get a global definition from the local definitions, it's not that straightforward. One way to do this is by the so-called partition of unity.

Definition 1.2.15 (Partition of unity)**.** Let $\mathcal{M}$ be a differentiable manifold, and let $(U_\alpha)_{\alpha\in\mathcal{A}}$ be an open covering of $\mathcal{M}$. Then a *partition of unity* is a locally finite refinement $(V_\beta)_{\beta\in\mathcal{B}}$ of (U_α) and C^∞-functions $\varphi_\beta\colon \mathcal{M}\to\mathbb{R}$ with

(a) $\operatorname{supp}(\varphi_\beta)\subseteq V_\beta$ for all $\beta\in\mathcal{B}$;

(b) $0\leq\varphi_\beta(x)\leq 1$ for all $x\in\mathcal{M}$, $\beta\in\mathcal{B}$;

(c) $\sum_{\beta\in\mathcal{B}}\varphi_\beta = 1$ for all $x\in\mathcal{M}$.[a]

[a] There are only finitely many non-vanishing summands of each point, since only finitely many φ_β are non-zero of any given point as the covering (V_β) is locally finite.

Intuition. Essentially, a partition of unity gives us a way to weight all the "local functions" with domains (i.e., the local coordinates) containing a particular point. By patching these weighted function values together, we get a final, consistent function value of that point.

We state, without proof, of an important lemma about the partition of unity.

Lemma 1.2.2 (Partition of unity)**.** Let $\mathcal{M}$ be a differentiable manifold, and let $(U_\alpha)_{\alpha\in\mathcal{A}}$ be an open covering of $\mathcal{M}$. Then there exists a partition of unity subordinate to (U_α),

Remark. Lemma 1.2.2 is only possible by requiring $\mathcal{M}$ being Hausdorff and second-countable.

1.3 Tangent and Cotangent Spaces

1.3.1 Tangent Spaces in Euclidean Spaces

To discuss the concept of calculus between manifolds formally, we start with our discussion in Euclidean spaces, where we naturally have the coordinates for every point.

Definition. Let $\mathcal{M}$ be a Euclidean manifold of dimension d, $x=(x^1,\dots,x^d)$ be Euclidean coordinates of $\mathbb{R}^d$, and $x_0\in\Omega\subseteq\mathbb{R}^d$ where Ω is open.

Definition 1.3.1 (Tangent space of Euclidean space)**.** The *tangent space* $T_{x_0}\Omega$ *of* Ω *at* x_0 is the vector space $\{x_0\}\times E$[a] spanned by the basis $(\partial/\partial x^1,\dots,\partial/\partial x^d)$.

[a] E is a d-dimensional Euclidean space.

Definition 1.3.2 (Tangent vector of Euclidean space)**.** The elements in the tangent space of Euclidean spaces is called *tangent vectors*.

Before proceeding, we introduce a shorthand notation.

Notation (Einstein notation)**.** The *Einstein notation* abbreviates the summation $\sum_i v^i x_i$ as $v^i x_i$, where we implicitly sum over the upper and lower index.

Definition 1.3.3 (Differential of Euclidean space)**.** If $\Omega,\Omega'\subseteq\mathbb{R}^d$ are open, and $f\colon\Omega\to\Omega'$ is differentiable, then the *differential* $\mathrm{d}f(x_0)$ for $x_0\in\Omega$ is the induced linear map between tangent spaces

$$\mathrm{d}f(x_0)\colon T_{x_0}\Omega\to T_{f(x_0)}\Omega',\quad v=v^i\frac{\partial}{\partial x^i}\mapsto v^i\frac{\partial f^j}{\partial x^i}(x_0)\frac{\partial}{\partial f^j}.$$

Definition 1.3.4 (Tangent bundle of Euclidean space). The *tangent bundle* is defined as $T\Omega := \coprod_{x\in\Omega} T_x\Omega \cong \Omega \times E \cong \Omega \times \mathbb{R}^d$, which is an open subset of $\mathbb{R}^d \times \mathbb{R}^d$.

Note (Total space). $T\Omega$ is also called the *total space*.

Remark. Given a tangent bundle $T\Omega$, we define π to be the projection $\pi\colon T\Omega \to \Omega$ given by $\pi(x, v) = x$. This makes $T\Omega$ naturally a differentiable manifold.

With the notion of tangent bundle, given $f\colon \Omega \to \Omega'$, we can also define $\mathrm{d}f\colon T\Omega \to T\Omega'$ as

$$\left(x, v^i \frac{\partial}{\partial x^i}\right) \mapsto \left(f(x), v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}\right).$$

Notation. We often write $\mathrm{d}f(x)(v)$ instead of $\mathrm{d}f(x, v)$ to coincide with the notation of differential.

In particular, for $v = v^i \partial/\partial x^i$, we have

$$\mathrm{d}f(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R},$$

and we write $v(f)(x)$ for $\mathrm{d}f(x)(v)$.

1.3.2 Tangent Spaces in Manifolds

We now try to formally define the tangent space on a smooth manifold. A natural idea is the following.

Intuition. Let $\mathcal{M}^d$ be a differentiable manifold with a chart $x\colon U \to \Omega \subseteq \mathbb{R}^d$ and $p \in U \subseteq \mathcal{M}$ where U is open. The *tangent space* $T_p\mathcal{M}$ of $\mathcal{M}$ at p *should be* represented in the chart x by $T_{x(p)}x(U)$.

To see that the above are well-defined, i.e., $T_p\mathcal{M}$ are independent of the choice of charts, let $x'\colon U' \to \mathbb{R}^d$ to be another chart with $p \in U' \subseteq \mathcal{M}$ where U' is also open. Denote $\Omega := x(U)$, and $\Omega' := x'(U')$, then the transition map

$$x' \circ x^{-1}\colon x(U \cap U') \to x'(U \cap U')$$

induces a vector space isomorphism

$$L := \mathrm{d}(x' \circ x^{-1})(x(p))\colon T_{x(p)}\Omega \to T_{x'(p)}\Omega',$$

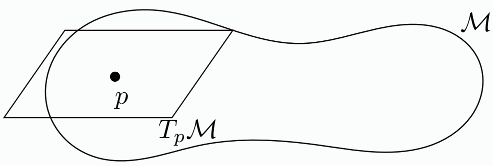
such that $v \in T_{x(p)}\Omega$ and $L(v) \in T_{x'(p)}\Omega'$ represent the same tangent vector in $T_p\mathcal{M}$.

Remark. A tangent vector in $T_p\mathcal{M}$ is given by the family of the coordinate representations.

Now, we want to define the similar notion of differential of Euclidean spaces. Let consider a simple case first, where we let $f\colon \mathcal{M} \to \mathbb{R}$ to be a differentiable function, and assume that the tangent vector $w \in T_p\mathcal{M}$ is represented by $v \in T_{x(p)}x(U)$.

Intuition. We want to define $\mathrm{d}f(p)$ as a linear map from $T_p\mathcal{M} \to \mathbb{R}$. In chart x, let $w \in T_p\mathcal{M}$ be given as $v = v^i \partial/\partial x^i \in T_{x(p)}x(U)$. Say that $\mathrm{d}f(p)(w)$ in this chart represented by

$$\mathrm{d}(f \circ x^{-1})(x(p))(v).$$



Remark. $T_p\mathcal{M}$ is a vector space of dimension d isomorphic to $\mathbb{R}^d$, where the isomorphism depends on choice of chart.

Intuition. Pull functions on $\mathcal{M}$ back by a chart to an open subset of $\mathbb{R}^d$, differentiate there.

In order to obtain a tangent space which does not depend on charts, we need to have transformation behavior under change of charts. Let $F\colon \mathcal{M}^d \to \mathcal{N}^c$ be a differentiable map where $\mathcal{M}, \mathcal{N}$ are smooth manifolds. Then we want to represent $\mathrm{d}F$ in local charts $x\colon U \subseteq \mathcal{M} \to \mathbb{R}^d, y\colon V \subseteq \mathcal{N} \to \mathbb{R}^c$ by $\mathrm{d}(y \circ F \circ x^{-1})$. The local coordinates on U is given by $(x^1, \dots, x^d)$, and on V is $(F^1, \dots, F^c)$ such that

$$F(x) = \left(F^1(x^1, \dots, x^d), \dots, F^c(x^1, \dots, x^d)\right).$$

Then, $\mathrm{d}F$ induces a linear map $\mathrm{d}F\colon T_p\mathcal{M} \to T_{F(x)}\mathcal{N}$ which in our coordinate representation is given by the matrix

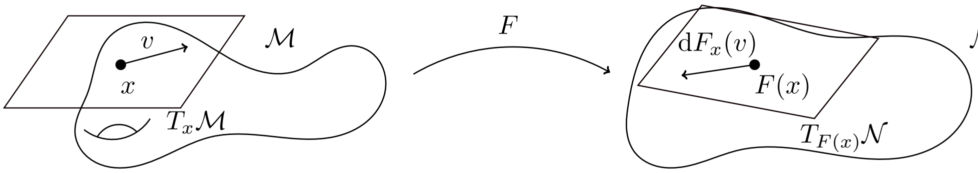
$$\left(\frac{\partial F^\alpha}{\partial x^i}\right)_{\substack{\alpha=1,\dots,c\\ i=1,\dots,d}},$$

and a change of charts is then just the base change at tangent spaces: if

$$\begin{aligned}(x^1, \dots, x^d) &\mapsto (\xi^1, \dots, \xi^d)\\ (F^1, \dots, F^c) &\mapsto (\phi^1, \dots, \phi^c)\end{aligned}$$

are coordinate changes, then $\mathrm{d}F$ represented in the new coordinates is given by

$$\left(\frac{\partial \phi^\beta}{\partial \xi^j}\right) = \left(\frac{\partial \phi^\beta}{\partial F^\alpha}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial x^i}{\partial \xi^j}\right).$$



Lecture 4: Tangent Bundles, Vector Fields, and Submanifolds

Definition. Let $\mathcal{M}^d$ be a differentiable manifold with a chart $x\colon U \to \Omega \subseteq \mathbb{R}^d$ and $p \in U \subseteq \mathcal{M}$ where U is open. On $\left\{(x, v) \mid v \in T_{x(p)}\Omega\right\}$, we define an equivalence relation by $(x, v) \sim (y, w)$ if and only if $w = \mathrm{d}(y \circ x^{-1})v$. 17 Jan. 13:00

Definition 1.3.5 (Tangent space)**.** The space of equivalence classes is called the *tangent space* $T_p\mathcal{M}$ at point p to $\mathcal{M}$.

Definition 1.3.6 (Tangent vector)**.** The elements in the tangent space is called *tangent vectors*.

Remark. $T_p\mathcal{M}$ naturally caries the structure of a vector space.

Now, $T\mathcal{M}$ is defined as

$$T\mathcal{M} \coloneqq \coprod_{p\in\mathcal{M}} T_p\mathcal{M}.$$

Recall the projection $\pi\colon T\mathcal{M} \to \mathcal{M}$ with $\pi(V) = p$ for $V \in T_p\mathcal{M}$. Then we can define the following.

Definition 1.3.7 (Derivation)**.** If $x\colon U \to \mathbb{R}^d$ be a chart for $\mathcal{M}$, and let $TU = \coprod_{p\in U} T_pU$. Then we define the *derivation* $\mathrm{d}x\colon TU \to Tx(U) \coloneqq \coprod_{p\in x(U)} T_p\mathcal{M}$ by $w \mapsto \mathrm{d}x(\pi(w))(w) \in T_{x(\pi(w))}x(U)$.

The transition maps

$$\mathrm{d}x' \circ (\mathrm{d}x)^{-1} = \mathrm{d}(x' \circ x^{-1})$$

are differentiable. π is local represented by $x \circ \pi \circ \mathrm{d}x^{-1}$ maps $(x_0, v) \in Tx(U)$ to x_0.

Definition 1.3.8 (Tangent bundle). The triple $(T\mathcal{M}, \pi, \mathcal{M})$ is called the *tangent bundle* of $\mathcal{M}$.

Definition 1.3.9 (Total space). $T\mathcal{M}$ is called the *total space* of the tangent bundle.

We can choose the courses (the initial) topology for total space $T\mathcal{M}$ such that π is continuous. Furthermore, we can construct a C^∞-atlas $\mathcal{A}_{T\mathcal{M}}$ on $T\mathcal{M}$ from the C^∞-atlas $\mathcal{A}$ of $\mathcal{M}$. Specifically, consider $\mathcal{A}_{T\mathcal{M}} \coloneqq \{(TU, \xi_x) \mid (U, x) \in \mathcal{A}\}$ where $\xi_x \colon TU \to \mathbb{R}^{2\cdot d}$ such that

$$x \mapsto \left((x^1 \circ \pi)(x), \dots, (x^d \circ \pi)(x), (\mathrm{d}x^1)_{\pi(x)}(X), \dots, (\mathrm{d}x^d)_{\pi(x)}(X)\right).$$

Intuition. We know that $X = X^i_x (\partial/\partial x^i)_{\pi(x)}$, and we might tempt to write X^i as the last d components. But we write it in the above way is because

$$(\mathrm{d}x^j)_{\pi(x)}(X) = (\mathrm{d}x^j)_{\pi(x)}\left(X^i_x \left(\frac{\partial}{\partial x^i}\right)_{\pi(x)}\right) = X^i_x \delta^j_i = X^j_x.$$

Note. We can check that ξ_x^{-1} exists, and it's also smooth, hence $T\mathcal{M}$ has a natural topology and a C^∞-atlas making it a $2\dim\mathcal{M}$-dimensional smooth manifold.

1.3.3 Cotangent Spaces

Another important objects is the cotangent spaces.

Definition. Let $\mathcal{M}^d$ be a differentiable manifold, and $T_p\mathcal{M}$ be the tangent space at p to $\mathcal{M}$.

Definition 1.3.10 (Cotangent space). The *cotangent space* $T^*_p\mathcal{M}$ to $\mathcal{M}$ is the dual of $T_p\mathcal{M}$, i.e., $T^*_p\mathcal{M} = (T_p\mathcal{M})^*$.

Definition 1.3.11 (Cotangent vector). The elements in the cotangent space is called *cotangent vectors*.

Remark. $T^*_p\mathcal{M}$ is the space of 1-forms on $T_p\mathcal{M}$.

Notation (Covariant vector). The cotangent vectors are also called *covariant vectors*.

Notation (Contravariant vector). The tangent vectors are also called *contravariant vectors*.

Similarly, we can define the projection $\pi \colon T^*\mathcal{M} \to \mathcal{M}$ with $\pi(\omega) = p$ for $\omega \in T^*_p\mathcal{M}$, and we have the following.

Definition 1.3.12 (Cotangent bundle). The triple $(T^*\mathcal{M}, \pi, \mathcal{M})$ is called the *cotangent bundle* of $\mathcal{M}$.

1.4 Vector Fields and Brackets

1.4.1 Vector Fields

We now introduce the notion of vector field.

Definition 1.4.1 (Vector field). A *(tangent) vector field* X on a differentiable manifold $\mathcal{M}$ is a correspondence associating to each point $p \in \mathcal{M}$ a vector $X(p) \in T_p\mathcal{M}$, i.e., $X\colon \mathcal{M} \to T\mathcal{M}$.

Remark. Naturally, we say that the field X is differentiable if the map X is differentiable.

Considering a local chart $x\colon U \subseteq \mathbb{R}^n \to \mathcal{M}$, we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i},$$

where $a_i\colon U \to \mathbb{R}$ are functions on U for $i = 1, \dots, n$, and $\{\partial/\partial x_i\}_i$ is the basis associated to x.

Remark. X is differentiable if and only if a_i are differentiable for some (and, therefore, for any) x.

It's convenient to think of a vector field as a mapping $X\colon \mathcal{D} \to \mathcal{F}$ from the set $\mathcal{D}$ of differentiable functions on $\mathcal{M}$ to the set $\mathcal{F}$ of the functions on $\mathcal{M}$, defined by

$$(Xf)(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p),$$

where f is implicitly denoting the expression of f in the chart x.

Intuition. This idea of a vector as a directional derivative is precisely what was used to define the notion of tangent vector.

Remark. Xf does not depend on the choice of x.

Remark. X is differentiable if and only if $X\colon \mathcal{D} \to \mathcal{D}$, i.e., $Xf \in \mathcal{D}$ for all $f \in \mathcal{D}$.

Observe that if $\varphi\colon \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, $v \in T_p\mathcal{M}$ and f differentiable function in a neighborhood of $\varphi(p)$, we have

$$(\mathrm{d}\varphi(v)f)\varphi(p) = v(f \circ \varphi)(p)$$

since by letting $\alpha\colon (-\epsilon, \epsilon) \to \mathcal{M}$ be a differentiable curve with $\alpha'(0) = v$, $\alpha(0) = p$,[2] then

$$(\mathrm{d}\varphi(v)f)\varphi(p) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi \circ \alpha)\bigg|_{t=0} = v(f \circ \varphi)(p).$$

1.4.2 Brackets

By viewing X as an operator on $\mathcal{D}$, we can consider the iterates of X, i.e, given differentiable fields X and Y and $f\colon M \to \mathbb{R}$ being a differentiable function, consider $X(Yf)$ and $Y(Xf)$.

Note. In general, $X(Yf)$ (and hence $Y(Xf)$) is not a field.

Proof. It involves derivatives of order higher than one. ⊛

But we have the following.

Lemma 1.4.1. Let X, Y be differentiable vector fields on a smooth manifold $\mathcal{M}$. Then there exists a unique vector field Z such that for all $f \in \mathcal{D}$, $Zf = (XY - YX)f$.

Proof. See do Carmo [FC13, §0 Lemma 5.2]. ∎

This Z is called the bracket.

[2] This is the way do Carmo [FC13] used to define tangent vectors.

Definition 1.4.2 (Bracket). Given two differentiable vector fields X, Y on a smooth manifold $\mathcal{M}$, the *bracket* of X and Y is defined by

$$[X, Y] \coloneqq XY - YX.$$

Clearly, $[X, Y]$ is differentiable.

Proposition 1.4.1. If X, Y and Z are differentiable vector fields on $\mathcal{M}$, $a, b \in \mathbb{R}$, f, g are differentiable functions, then we have the following.

(a) $[X, Y] = -[Y, X]$ (*anti-commutativity*),

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (*linearity*),

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (*Jacobi identity*),

(d) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

Proof. See do Cargo [FC13, §0 Proposition 5.3]. ■

Example. $[\partial/\partial x^i, \partial/\partial x^j] = 0$ for $i = j$.

1.5 Submanifolds, Immersions, and Embeddings

We now study the relation between manifolds.

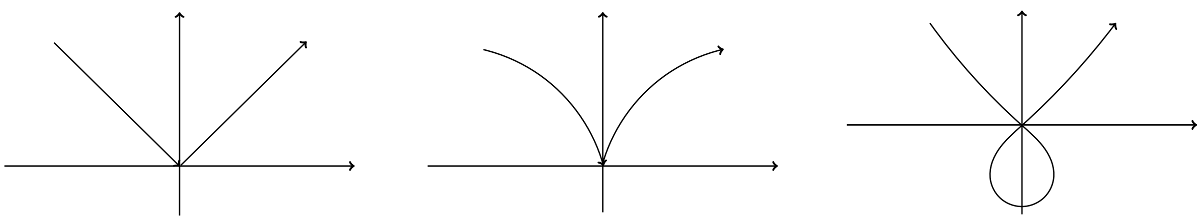
Definition 1.5.1 (Immersion). Let $\mathcal{M}^m, \mathcal{N}^n$ be smooth manifolds. A differentiable mapping $\varphi\colon \mathcal{M} \to \mathcal{N}$ is an *immersion* if

$$\mathrm{d}\varphi_p\colon T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$$

is injective for every $p \in \mathcal{M}$.

Definition 1.5.2 (Embedding). An immersion $\varphi\colon \mathcal{M} \to \mathcal{N}$ is an *embedding* if it is also a homeomorphism onto $\varphi(\mathcal{M}) \subseteq \mathcal{N}$, with $\varphi(\mathcal{M})$ having the subspace topology induced from $\mathcal{N}$.

Definition 1.5.3 (Submanifold). If the inclusion $\iota\colon \mathcal{M} \hookrightarrow \mathcal{N}$ between two manifolds is an embedding, then $\mathcal{M}$ is a *submanifold* of $\mathcal{N}$.



(a) Non-differentiable curve. (b) Non-immersion curve. (c) Non-embedding curve.

Figure 1.1: Three simple examples

Lemma 1.5.1. Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ to be an immersion and $x \in \mathcal{M}$.[a] Then there exists a neighborhood U of x and a chart (V, y) on $\mathcal{N}$ with $f(x) \in V$ such that $f|_U$ is a differentiable embedding and $y^{m+1}(p) = \cdots = y^n(p) = 0$ for all $p \in f(U \cap V)$.

[a] Hence, $n \geq m$.

Proof. In the local coordinates $(z^1, \ldots, z^n)$ on $\mathcal{N}$, and $(x^1, \ldots, x^m)$ on $\mathcal{M}$, without loss of generality,[a] let

$$\left(\frac{\partial z^\alpha(f(x))}{\partial x^i}\right)_{i,\alpha=1,\ldots,m}$$

be non-singular. Consider

$$F(z, x) \coloneqq \left(z^1 - f^1(x), \ldots, z^n - f^n(x)\right),$$

which has maximal rank in $x^1, \ldots, x^m, z^{m+1}, \ldots, z^n$. By the implicit function theorem, locally, there exists a map $\varphi\colon U \to \mathbb{R}^n$ such that

$$(z^1, \ldots, z^m) \mapsto (\varphi^1(z^1, \ldots, z^m), \ldots, \varphi^n(z^1, \ldots, z^m)) = x$$

such that $F(z, x) = 0$, i.e.,

$$\varphi^i(z^1, \ldots, z^m) = \begin{cases} x^i, & \text{if } i = 1, \ldots, m; \\ z^i, & \text{if } i = m+1, \ldots, n, \end{cases}$$

for which

$$\left(\frac{\partial \varphi^i}{\partial z^\alpha}\right)_{\alpha,i=1,\ldots,m}$$

has maximal rank. Now, we choose a new coordinate

$$\begin{aligned}(y^1, \ldots, y^n) = \big(&\varphi^1(z^1, \ldots, z^m), \ldots, \varphi^m(z^1, \ldots, z^m), \\ &z^{m+1} - \varphi^{m+1}(z^1, \ldots, z^m), \ldots, z^n - \varphi^n(z^1, \ldots, z^m)\big).\end{aligned}$$

Then, we have $z = f(x) \Leftrightarrow F(z, x) = 0$, i.e., $(y^1, \ldots, y^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$, proving the result. ■

[a] Since $\mathrm{d}f(x)$ is injective.

Lemma 1.5.2. Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ be a differentiable map such that $m \geq n$ with $p \in \mathcal{N}$. Let $\mathrm{d}f(x)$ has rank n for all $x \in \mathcal{M}$ with $f(x) = p$. Then $f^{-1}(p)$ is the union of differentiable submanifolds of $\mathcal{M}$ of dimension $m - n$.

Remark. Let $\mathcal{N}^n$ be a smooth manifold, and let $1 \leq m \leq n$. Then an arbitrary subset $\mathcal{M} \subseteq \mathcal{N}$ has the structure of differentiable submanifold of $\mathcal{N}$ of dimension m if and only if for all $p \in \mathcal{M}$, there exists a smooth chart (U, φ) of $\mathcal{N}$ such that $p \in U$, $\varphi(p) = 0$, $\varphi(U)$ is open, and

$$\varphi(U \cap \mathcal{M}) = (-\epsilon, +\epsilon)^n \times \{0\}^{n-m},$$

where $(-\epsilon, +\epsilon)^n$ is the cube. Noticeably, the C^∞-manifold structure of $\mathcal{M}$ is uniquely determined.

Remark. Let $\mathcal{M} \subseteq \mathcal{N}$ be a differentiable submanifold of $\mathcal{N}$, and let $\iota\colon \mathcal{M} \hookrightarrow \mathcal{N}$ be the inclusion. Then, for $p \in \mathcal{M}$, $T_p\mathcal{M}$ can be considered as subspace of $T_p\mathcal{N}$, namely as the image of $\mathrm{d}\iota(T_p\mathcal{M})$.

Lemma 1.5.3. Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ be a differentiable map such that $m \geq n$ with $p \in \mathcal{N}$. Let $\mathrm{d}f(x)$ has rank n for all $x \in \mathcal{M}$ with $f(x) = p$. For the submanifold $X = f^{-1}(p)$ and for $q \in X$, it is true that

$$T_q X = \ker \mathrm{d}f(q) \subseteq T_q\mathcal{M}.$$

Chapter 2

Riemannian Manifolds

Lecture 5: Riemannian Manifolds

In this chapter, we start our discussion on Riemannian manifolds. 19 Jan. 13:00

2.1 Riemannian Metrics

We start by defining the Riemannian metric.

Definition 2.1.1 (Riemannian metric). A *Riemannian metric* g on a differentiable manifold $\mathcal{M}$ is given by a scalar product I on each $T_p\mathcal{M}$ which depends smoothly on the base point p.

Definition 2.1.2 (Riemannian manifold). A *Riemannian manifold* $(\mathcal{M}, g)$ is a smooth manifold $\mathcal{M}$ equipped with a Riemannian metric g.

Let $x = (x^1, \ldots, x^d)$ be the local coordinates. In these, a metric is represented by a positive definite symmetric matrix $\left(g_{ij}(x)\right)_{i,j=1,\ldots,d}$, i.e., $g_{ij} = g_{ji}$, and $g_{ij}\xi^i\xi^j > 0$ for all $\xi = (\xi^1, \ldots, \xi^d) \neq 0$ with coefficients smoothly depending on x.

2.1.1 Transformation Behavior

We now see that the smoothness does not depend on coordinates, i.e., the smooth dependence on the base point (as required in Definition 2.1.1) can be represented in the local coordinates. Given 2 tangent vectors $v, w \in T_p\mathcal{M}$ with coordinate representations $(v^1, \ldots, v^d), (w^1, \ldots, w^d)$ given by x such that $v = v^i\frac{\partial}{\partial x^i}$ and $w = w^i\frac{\partial}{\partial x^i}$, their product is

$$\langle v, w\rangle \coloneqq g_{ij}(x(p))v^iw^j.$$

In particular,

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle = g_{ij}.$$

Remark. The length of v is given as $\|v\| \coloneqq \langle v, v\rangle^{1/2}$.

Let $y = f(x)$ define different local coordinates. In these, v, w are given as

$$(\widetilde{v}^1, \ldots, \widetilde{v}^d), (\widetilde{w}^1, \ldots, \widetilde{w}^d)$$

with $\widetilde{v}^j = v^i\frac{\partial f^j}{\partial x^i}$ and $\widetilde{w}^j = w^i\frac{\partial f^j}{\partial x^i}$. Denote the metric in new coordinates y by $h_{k\ell}(y)$, then we have

$$h_{k\ell}(f(x))\widetilde{v}^k\widetilde{w}^\ell = \langle v, w\rangle = g_{ij}(x)v^iw^j.$$

Plug everything in, we have

$$h_{k\ell}(f(x))\frac{\partial f^k}{\partial x^i}\frac{\partial f^\ell}{\partial x^j}v^iw^j = g_{ij}(x)v^iw^j.$$

We see that this holds for any tangent vectors v, w, therefore,

$$h_{k\ell}(f(x))\frac{\partial f^k}{\partial x^i}\frac{\partial f^\ell}{\partial x^j} = g_{ij}(x),$$

which is the transformation behavior under coordinates changes.

Remark. This shows that the smoothness does not depend on the choice of coordinates!

Example. Consider the Euclidean space Ω, then given $v, w \in T_p\Omega$, we have

$$\langle v, w\rangle = \delta_{ij}v^i w^j = v^i w_i.$$

Theorem 2.1.1. Every differentiable manifold can be equipped with a Riemannian metric.

Proof. From Lemma 1.2.2, there exists a differentiable partition of unity $\{f_\alpha\}$ of $\mathcal{M}$ subordinate to a covering $\{V_\alpha\}$ of $\mathcal{M}$. Consider the induced metric $\langle\cdot,\cdot\rangle^\alpha$ of the system of local coordinates on each V_α. Then, for every $p \in M$, a Riemannian metric $\langle\cdot,\cdot\rangle_p$ can be defined naturally as

$$\langle u, v\rangle_p = \sum_\alpha f_\alpha(p)\,\langle u, v\rangle_p^\alpha$$

for all $u, v \in T_pM$. Given the fact that $\{f_\alpha\}$ is the partition of unity, we know that

(a) $f_\alpha \geq 0$, and $f_\alpha = 0$ on $\overline{V}_\alpha^c$,

(b) $\sum_\alpha f_\alpha(p) = 1$ for all p on $\mathcal{M}$,

it's then immediate that the defined is indeed a Riemannian metric. ■

2.1.2 Isometry

After introducing any type of mathematical structure, we must introduce a notion of when two objects are the same, hence we now characterize g.

Definition 2.1.3 (Isometry). A diffeomorphism $h\colon \mathcal{M} \to \mathcal{N}$ is an *isometry* between two Riemannian manifolds if it preserves the Riemannian metric, i.e., for $p \in \mathcal{M}$, $v, w \in T_p\mathcal{M}$,

$$\langle v, w\rangle_{\mathcal{M}} = \langle \mathrm{d}h(v), \mathrm{d}h(w)\rangle_{\mathcal{N}}.$$

Definition 2.1.4 (Local isometry). A diffeomorphism $h\colon \mathcal{M} \to \mathcal{N}$ is a *local isometry* between two Riemannian manifolds if for every $p \in \mathcal{M}$, there exists a neighborhood U such that $h|_U : U \to h(U)\colon \mathcal{M} \to \mathcal{N}$ is an isometry and $h(U) \subseteq \mathcal{N}$ is open.

If's common to say that a Riemannian manifold $\mathcal{M}$ is locally isometric to a Riemannian manifold $\mathcal{N}$ if for every $p \in \mathcal{M}$, there exists a neighborhood U of p in $\mathcal{M}$ and a local isometry $f\colon U \to f(U) \subseteq \mathcal{N}$.

Example (Euclidean space). The *Euclidean space of dimension* n $\mathcal{M} = \mathbb{R}^n$ with $\partial/\partial x_i$ identified with $e_i = (0, \ldots, 1, \ldots, 0)$ is with the metric

$$\langle e_i, e_j\rangle = \delta_{ij}.$$

The Riemannian geometry of this space is metric Euclidean geometry.

Example (Lie group). See Appendix A.2 for reference.

2.2 Geodesics

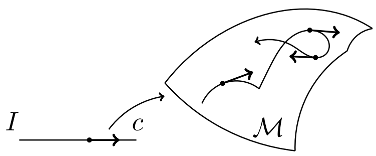
This is the first focus on the study of Riemannian geometry, i.e., the geodesics. The up-shot is that a geodesic minimizes the arc length for points *sufficiently close* (in a sense to be made precise); in addition, if a curve minimizes arc length between any two of its points, it is a geodesic.

2.2.1 Vector Fields along Curves

We are now going to show how a Riemannian metric can be used to calculate the length of a curve.

Definition 2.2.1 (Curve)**.** A (parametrized) *curve* is a differentiable mapping $c\colon I \subseteq \mathbb{R} \to \mathcal{M}$ to a smooth manifold $\mathcal{M}$.

Note. A parametrized curve can admit self-intersections as well as corners.



Definition 2.2.2 (Vector field along a curve)**.** A *(smooth) vector field X along a curve $c\colon I \subseteq \mathbb{R} \to \mathcal{M}$* on a smooth manifold $\mathcal{M}$ is defined as $X\colon I \to T\mathcal{M}$ such that $X(t) \in T_{c(t)}\mathcal{M}$ for all $t \in I$.

Notation. The set of smooth vector fields along c is denoted as $\chi_c(\mathcal{M})$.

Note. To say V is differentiable means that for any differentiable function f on $\mathcal{M}$, the function $t \mapsto V(t)f$ is a differentiable function on I.

Example (Velocity field)**.** The vector field along c $\mathrm{d}c/\mathrm{d}t \coloneqq \mathrm{d}c(\mathrm{d}/\mathrm{d}t)$ is called the *velocity field* or *tangent vector field*.

Remark. A vector field along c can't necessarily be extended to a vector field on an open set of $\mathcal{M}$.

Notation (Segment)**.** The restriction of a curve c to a closed interval $[a, b] \subseteq I$ is called a *segment*.

2.2.2 Lengths and Energies

We're interested in the following two quantities.

Definition. Let $\gamma\colon [a, b] \to \mathcal{M}$ be a curve on a Riemannian manifold $(\mathcal{M}, g)$.

Definition 2.2.3 (Length)**.** The *length* of γ is defined as

$$L(\gamma) \coloneqq \int_a^b \left\lVert \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \right\rVert \,\mathrm{d}t.$$

Definition 2.2.4 (Energy)**.** The *energy* of γ is defined as

$$E(\gamma) \coloneqq \frac{1}{2} \int_a^b \left\lVert \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \right\rVert^2 \,\mathrm{d}t.$$

We now want to compute $L(\gamma)$, $E(\gamma)$ in local coordinates. Let the local coordinates be

$$(x^1(\gamma(t)), \dots, x^d(\gamma(t))),$$

we write

$$\dot{x}^i(t) = \frac{\mathrm{d}}{\mathrm{d}t}(x^i(\gamma(t))).$$

Then, we have

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)}\,\mathrm{d}t, \quad E(\gamma) = \frac{1}{2}\int_a^b g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)\,\mathrm{d}t.$$

Definition 2.2.5 (Distance). Given a Riemannian manifold $(\mathcal{M}, g)$, the *distance* between 2 points $p, q \in \mathcal{M}$ is defined as

$$d(p,q) \coloneqq \inf\left\{L(\gamma) \mid \gamma\colon [a,b] \to \mathcal{M} \text{ piecewise curve with } \gamma(a) = p, \gamma(b) = q\right\}.$$

Note. Any 2 points $p, q \in \mathcal{M}$ can be connected by a piecewise curve, hence $d(p,q)$ always exists.

Corollary 2.2.1. The topology of $\mathcal{M}$ induced by the distance function d coincides with the original manifold topology of $\mathcal{M}$.

Lemma 2.2.1. If $\gamma\colon [a,b] \to \mathcal{M}$ is a curve, and $\psi\colon [\alpha,\beta] \to [a,b]$ is a reparametrization, then $L(\gamma \circ \psi) = L(\gamma)$.

Proof. This can be proved by computation, and the take-away is that the length functional is invariant under parameter changes. ∎

2.2.3 Geodesic Equations as Euler-Lagrange Equations

We want to find a curve which minimizes the length between sufficiently close two points. It turns out that instead of working with length directly, we should work with energy instead.

Notation. Let's first fix some common notations.

(a) $\left(g^{ij}\right)_{i,j=1,\dots,d} = (g_{ij})^{-1}_{i,j=1,\dots,d}$.[a]

(b) $g_{j\ell,k} \coloneqq \frac{\partial}{\partial x^k} g_{j\ell}$.

[a] Technically, g^{-1} is not an inverse: g is a $(0,2)$-tensor field, while g^{-1} is a $(2,0)$-tensor field.

Note. In the above notations, we have $g^{i\ell}g_{\ell j} = \delta^i_j$.

And the following is particularly important.

Notation (Christoffel symbol). The *Christoffel symbol* is defined for all i as

$$\Gamma^i_{jk} \coloneqq \frac{1}{2}g^{i\ell}\left(g_{j\ell,k} + g_{k\ell,j} - g_{jk,\ell}\right).$$

Remark. The notion of Γ is a bit cryptic at first, and we will come back to this after. Now, just treat it as a calculation tool.

The up-shot is that the Euler-Lagrange equations for the energy E has a nice form, and the solution of which has exactly the properties we want, hence we define it as geodesics.

Proposition 2.2.1. The Euler-Lagrange equations for the energy E are

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0 \text{ for } i = 1, \dots, d. \tag{2.1}$$

Proof. The Euler-Lagrange equations of a functional[a]

$$I(x) = \int_a^b f(t, x(t), \dot{x}(t))\,\mathrm{d}t$$

are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0$$

for $i = 1, \dots, d$. Just by plugging in, we obtain for E, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ik}(x(t))\dot{x}^k(t) + g_{ji}(x(t))\dot{x}^j(t)\right) - g_{jk,i}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$$

for $i = 1, \dots, d$. Hence,

$$g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j + g_{ik,\ell}\dot{x}^\ell\dot{x}^k + g_{ji,\ell}\dot{x}^\ell\dot{x}^j - g_{jk,i}\dot{x}^\ell\dot{x}^j = 0$$

Rename some indices and use $g_{ij} = g_{ji}$, we have that

$$2g_{\ell m}\ddot{x}^m + \left(g_{k\ell,j} + g_{j\ell,k} - g_{jk,\ell}\right)\dot{x}^j\dot{x}^k = 0$$

for $\ell = 1, \dots, d$. Hence, we have

$$g^{i\ell}g_{\ell m}\ddot{x}^m + \frac{1}{2}g^{i\ell}\left(g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell}\right)\dot{x}^j\dot{x}^k = 0$$

for $i = 1, \dots, d$. Finally, observe that $g^{i\ell}g_{\ell m} = \delta_{im}$, i.e., $g^{i\ell}g_{\ell m}\ddot{x}^m = \ddot{x}^i$, hence the claim follows. ■

[a] The Lagrangian is $\mathcal{L} = \frac{1}{2}g_{jk}\dot{x}^j\dot{x}^k$.

Finally, we define the geodesics as the solution of Equation 2.1.

Definition 2.2.6 (Geodesic)**.** A curve $\gamma\colon [a, b] \to \mathcal{M}$ that obeys Equation 2.1 is called a *geodesic*.

Intuition. Geodesic is the critical points of energy.[a]

[a] In fact, we can also start from length and get the same thing, which might be more natural.

2.2.4 Variation of Energies

We now discuss why geodesic is well-defined, i.e., we want to show that Equation 2.1 has a unique solution. We solve this via the variational principal, and we first define the action functional.

Definition 2.2.7 (Action functional)**.** Let $\mathcal{L}$ be the Lagrangian, then the *action functional*

$$I[w(\cdot)] := \int_0^t \mathcal{L}(\dot{w}(s), w(s))\,\mathrm{d}s$$

is defined for functions $w(\cdot) = (w^1(\cdot), \dots w^n(\cdot))$ of the admissible class

$$\mathcal{A} = \left\{w(\cdot) \in C^2([0, t]; \mathbb{R}^n) \mid w(0) = y, w(t) = x\right\}.$$

Example. Both length and energy are action functionals.

From the calculus of variation, we can find a curve $x(\cdot) \in \mathcal{A}$ such that $I[x(\cdot)] = \min_{w(\cdot)\in\mathcal{A}} I[w(\cdot)]$.

Theorem 2.2.1 (Euler-Lagrangian equations)**.** The solution $x(\cdot)$ from $I[x(\cdot)] = \min_{w(\cdot)\in\mathcal{A}} I[w(\cdot)]$ solves the system of Euler-Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(D_{\dot{x}}\mathcal{L}(\dot{x}(s), x(s)) + D_x\mathcal{L}(\dot{x}(s), x(s))\right) = 0$$

for $0 \leq s \leq t$.

Lecture 6: Geodesics and the Exponential Map

Now, we draw some relations between length and energy and see why starting from energy makes sense. 24 Jan. 13:00

Proposition 2.2.2. For all curves $\gamma\colon [a, b] \to \mathcal{M}$,

$$\mathcal{L}(\gamma)^2 \leq 2(b-a)E(\gamma)$$

with equality if and only if $\|\mathrm{d}\gamma/\mathrm{d}t\|$ is a constant.

Proof. From Hölder's inequality,

$$\int_a^b \left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\| \,\mathrm{d}t \leq (b-a)^{1/2}\left(\int_a^b \left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\|^2 \,\mathrm{d}t\right)^{1/2}$$

with equality if and only if $\|\mathrm{d}\gamma/\mathrm{d}t\|$ is a constant. ∎

Example. Let

$$\mathcal{L}(q, x) = \frac{1}{2}m|q|^2 - V(x)$$

with $m > 0, q = \dot{x}$, the Euler-Lagrangian equations is given by $m\ddot{x}(s) = F(x(s))$ for $F \coloneqq -\mathrm{D}V$.

Since regular curves can be parametrized by arc length with unit speed $\|\mathrm{d}\gamma/\mathrm{d}t\| = \|\dot{\gamma}\| \equiv 1$, the following is natural.

Lemma 2.2.2. Each geodesic is parametrized proportionally to the arc length, i.e., $\|\dot{\gamma}\|$ is a constant.

Proof. For a solution $x(t)$ of $\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$ (i.e., the geodesic), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\dot{x}, \dot{x}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)\right) = 0.$$

∎

Remark. This is one of the advantages of working with the energyrather than the length.

Since the length and the energy functionals are invariants under parameter changes, it's enough to look at curves parametrized by arc length.

Theorem 2.2.2. Let $\mathcal{M}$ be a Riemannian manifold, $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$. Then there exists an $\epsilon > 0$ and a unique geodesic such that $c\colon [0, \epsilon] \to \mathcal{M}$ with $c(0) = p$ and $\dot{c}(0) = v$. In addition, c smoothly depend on p, v.

Proof. Since Equation 2.1 is a system of second order ODE, by Picard-Lindelöf theorem, we have local existence and uniqueness of the solution with prescribed initial values and derivative such that the solution depends smoothly on p, v. ∎

If $x(t)$ is the solution of Equation 2.1, then $x(\lambda t)$ is also a solution for any constant $\lambda \in \mathbb{R}$. Denote geodesic from Theorem 2.2.2 by c_v, then

$$c_v(t) = c_{\lambda v}(t/\lambda)$$

for $\lambda > 0$, $t \in [0, \epsilon]$, and hence $c_{\lambda v}$ defined on $[0, \epsilon/\lambda]$.

Remark. Since c_v depends smoothly on v, the set $\{v \in T_p\mathcal{M} \mid \|v\| = 1\}$ is compact, hence there exists $\epsilon_0 > 0$ such that for $\|v\| = 1$, c_v defined at least on $[0, \epsilon_0]$, implying that for all $w \in T_p\mathcal{M}$ with $\|w\| \leq \epsilon_0$, c_w is defined at least on $[0, 1]$.

2.2.5 Exponential Maps and Normal Coordinates

The above discussion permits us to introduce the concept of the exponential map in the following manner.

Definition 2.2.8 (Exponential map)**.** Let $(\mathcal{M}, g)$ be a Riemannian manifold, $p \in \mathcal{M}$, and $V_p := \{v \in T_p\mathcal{M} \mid c_v \text{ defined on } [0, 1]\}$. The *exponential map of* $\mathcal{M}$ *at* p, $\exp_p \colon V_p \to \mathcal{M}$, is defined as $v \mapsto c_v(1)$.

Clearly, exp is differentiable, and we shall utilize the restriction of exp to an open subset of the tangent space $T_q\mathcal{M}$, i.e., we define

$$\exp_p \colon B(0, \epsilon) \subseteq T_p\mathcal{M} \to \mathcal{M},$$

where $B(0, \epsilon)$ is an open ball with center at the origin 0 of $T_p\mathcal{M}$ of radius ϵ. It's easy to see that $\exp_p$ is differentiable and that $\exp_p(0) = p$.

Intuition. Geometrically, $\exp_p(v)$ is a point of $\mathcal{M}$ obtained by going out the length equal to $|v|$, starting from p, along a geodesic which passes through p with velocity equal to $v/|v|$.

Proposition 2.2.3. The exponential map $\exp_p$ maps a neighborhood of $0 \in T_p\mathcal{M}$ diffeomorphically onto a neighborhood of $p \in \mathcal{M}$.

Proof. We see that

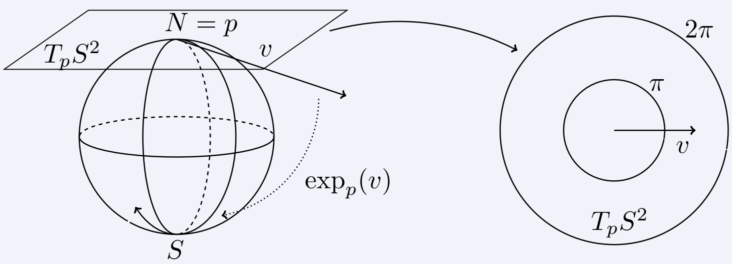
$$\mathrm{d}(\exp_p)_0(v) = \frac{\mathrm{d}}{\mathrm{d}t} \exp_p(tv)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} c_{tv}(1)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} c_v(t)\bigg|_{t=0} = v,$$

i.e., $\mathrm{d}(\exp_p)_0$ is the identity of $T_q\mathcal{M}$. By the inverse function theorem, $\exp_p$ is a local diffeomorphism on a neighborhood of 0. ■

Example. Let $\mathcal{M} = \mathbb{R}^n$, then the exponential map is the identity.[a]

[a] With the usual identification of $T_p\mathbb{R}^n$ at p with $\mathbb{R}^n$.

Example. Let $\mathcal{M} = S^2$.



Now we know that $\exp_p \colon B(0, \epsilon) \subseteq T_p\mathcal{M} \to \mathcal{M}$ maps diffeomorphically onto its image, we then define the following.

Definition 2.2.9 (Normal coordinate)**.** Given an exponential map $\exp_p \colon B(0, \epsilon) \to \mathcal{M}$, let $(e_1, \ldots, e_n)$ be the orthonormal basis of $T_p\mathcal{M}$. Then the associated local coordinates are the *normal coordinates*.

In this case, given $p \in \mathcal{M}^n$, $0 \in \mathbb{R}^n$, for all i, j, k,[1]

$$g_{ij}(p) = \delta_{ij}, \quad \Gamma^k_{ij}(p) = 0, \quad g_{ij,k} = 0.$$

Intuition. The first derivative vanishes, so locally, the manifold looks Euclidean.

Note. [FC13] introduces everything above using $T\mathcal{M}$ instead of $T_p\mathcal{M}$.

2.3 Hopf-Rinow Theorem

With all the tools we have developed, we now want to characterize the minimizing property of geodesics.

2.3.1 Riemannian Polar Coordinates

A particular useful tool is the Riemannian polar coordinates, which is introduced as follows.

Theorem 2.3.1. For all $p \in \mathcal{M}$, there exists $\rho > 0$ such that the Riemannian polar coordinates may be introduced on $B(p, \rho) = \{q \in \mathcal{M} \mid d(p, q) \leq \rho\}$. For any such ρ and $q \in \partial B(p, \rho)$, there exists a unique geodesic of shortest length ($= \rho$) from p to q. In the polar coordinates, this geodesic is given by the straight line $x(t) = (t, \varphi_0)$, $0 \leq t \leq \rho$, with q represented by coordinates (ρ, φ_0), $\varphi_0 \in S^{d-1}$.

Proof. Take an arbitrary curve from p to q, namely $c(t) = \big(r(t), \varphi(t)\big)$, $0 \leq t \leq T$, which does not have to be entirely be contained in $B(p, \rho)$. Let t_0 be defined as

$$t_0 \coloneqq \inf \{t \leq T \mid d(x(t), p) \geq \rho\}.$$

Then $t_0 \leq T$ such that $c|_{[0,t_0]}$ lies entirely in $B(p, \rho)$. We want to show that

(a) $L\left(c|_{[0,t_0]}\right) \geq \rho$, and

(b) $L\left(c|_{[0,t_0]}\right) = \rho$ only for a straight line in the polar coordinates,

where

$$L\left(c|_{[0,t_0]}\right) \coloneqq \int_0^{t_0} \sqrt{g_{ij}(c(t))\dot{c}^i\dot{c}^j}\,\mathrm{d}t.$$

Observe that $g_{r\varphi} = 0$, with $g_{\varphi\varphi}$ being positive definite and $g_{rr} \equiv 1$, we have

$$L\left(c|_{[0,t_0]}\right) \geq \int_0^{t_0} \sqrt{g_{rr}(c(t))\dot{r}\dot{r}}\,\mathrm{d}t = \int_0^{t_0} |\dot{r}|\,\mathrm{d}t \geq \int_0^{t_0} \dot{r}\,\mathrm{d}t = r(t_0) = \rho.$$

∎

Corollary 2.3.1. Let $\mathcal{M}$ be a compact Riemannian manifold. Then there exists $\rho_0 > 0$ such that

(a) for any $p \in \mathcal{M}$, Riemannian polar coordinates may be introduced on $B(p, \rho_0)$;

(b) for any $p, q \in \mathcal{M}$ with $d(p, q) \leq \rho_0$, they can be connected by precisely one geodesic or shortest length which depends continuously on p and q.

Lecture 7: Hopf-Rinow Theorem

2.3.2 Hopf-Rinow Theorem

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By using Corollary 2.3.1, we have shown the following in the homework.

[1] Note that this only holds at p. We will come back to this when we formally introduce the linear connection.

Lemma 2.3.1. Let $(\mathcal{M}, g)$ be a compact Riemannian manifold. Then for all $p \in \mathcal{M}$, the exponential map $\exp_p$ is defined on all of $T_p\mathcal{M}$ and any geodesic may be extended indefinitely in each direction.

Then we use Lemma 2.3.1 to show the following.

Theorem 2.3.2. Let $(\mathcal{M}, g)$ be a compact Riemannian manifold.

(a) For any 2 points $p, q \in \mathcal{M}$, there exists a geodesic in every homotopy class of curves from p to q. Moreover, we can choose a shortest curve as the geodesic in the homotopy class.

(b) Every homotopy class of closed curves in $\mathcal{M}$ contains a curve that is shortest and geodesic.

We now want to generalize it. However, this is not true in the most general setting, and we need one more requirement.

Definition 2.3.1 (Geodesically complete)**.** A Riemannian manifold $(\mathcal{M}, g)$ is *geodesically complete* if for all $p \in \mathcal{M}$, $\exp_p$ is defined on all of $T_p\mathcal{M}$.

In other words, a Riemannian manifold $\mathcal{M}$ is geodesically complete if any geodesic $c(t)$ with $c(0) = p$ can be extended for all $t \in \mathbb{R}$. Then, we have the following.

Theorem 2.3.3 (Hopf-Rinow theorem)**.** Let $(\mathcal{M}, g)$ be a Riemannian manifold, then the following statements are equivalent.

(a) $\mathcal{M}$ is complete as a metric space.[a]

(b) The closed and bounded subsets of $\mathcal{M}$ are compact.

(c) There exists $p \in \mathcal{M}$ such that $\exp_p$ is defined on all $T_p\mathcal{M}$.

(d) $\mathcal{M}$ is geodesically complete.

Furthermore, (d) (and hence (a), (b), and (c)) implies

(e) for two points $p, q \in \mathcal{M}$ can be joined by a minimizing geodesic, i.e., geodesic of the shortest distance $d(p, q)$.

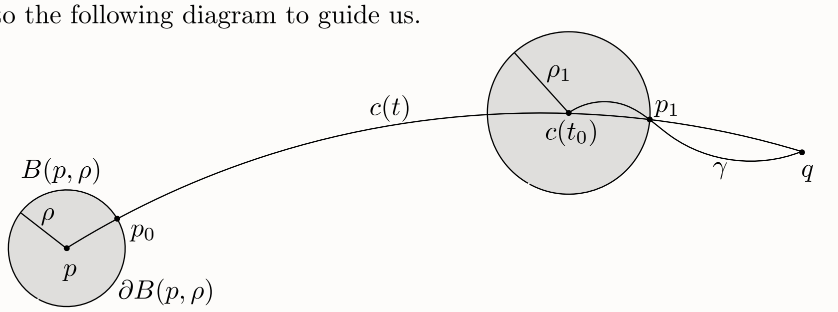
[a]Hence, equivalently, complete as a topological space w.r.t. the underlying topology.

Proof. We start by proving (d) implies (e). Let $\mathcal{M}$ be geodesically complete, and let $r := d(p, q)$, and let ρ be as in Corollary 2.3.1. Let $p_0 \in \partial B(p, \rho)$ be a point where the continuous functional $d(q, \cdot)$ attains its minimum on the compact set $\partial B(p, \rho)$. Then, for some $V \in T_p\mathcal{M}$, $p_0 = \exp_p \rho V$.

Consider the geodesic $c(t) = \exp_p tV$, by showing $c(r) = q$, $c|_{[0,r]}$ will be the shortest geodesic from p to q. We start by defining

$$I := \{t \in [0, r] \mid d(c(t), q) = r - t\},$$

and referring to the following diagram to guide us.



Now, we want to show that $I = [0, r]$, which will follow from showing that I is open.

Note. I is not empty since by definition it contains 0 and r. Further, I is closed by continuity.

Let $t_0 \in I$, and let $\rho_1 > 0$ be the radius as in the corollary, without loss of generality, $\rho_1 < r - t_0$. Let $p_1 \in \partial B(c(t_0), \rho_1)$ be the point where the continuous functional $d(q, \cdot)$ attains its minimum on the compact set $\partial B(c(t_0), \rho_1)$. By the triangle inequality,

$$d(p, q) \leq d(p, p_1) + d(p_1, q).$$

For every curve γ from $c(t_0)$ to q, there exists $\gamma(t) \in \partial B(c(t_0), \rho_1)$, hence

$$L(\gamma) \geq d(c(t_0), \gamma(t)) + d(\gamma(t), q) = \rho_1 + d(p_1, q) \Rightarrow d(q, c(t_0)) \geq \rho_1 + d(p_1, q)$$

where we use $d(c(t_0), \gamma(t)) = \rho_1$. But from the triangle inequality, we actually have

$$d(q, c(t_0)) = \rho_1 + d(p_1, q) \Leftrightarrow d(p_1, q) = \underbrace{d(q, c(t_0))}_{r - t_0} - \rho_1,$$

hence $d(p_1, p) \geq r - (r - t_0 - \rho_1) = t_0 + \rho_1$, i.e., this is a minimizing curve!

On the other hand, there exists a curve from p to p_1 of length $t_1 + \rho_1$ since it's composed by the portion from p to $c(t_0)$ along $c(t)$ and the portion being the geodesic from $c(t_0)$ to p_1 of length ρ_1. Then, by Theorem 2.3.2, this curve is a geodesic curve. Finally, from the uniqueness of geodesic with the given extra data, this geodesic coincides with c. Hence, $p_1 = c(t_0 + \rho_1)$. With $d(p_1, q) = r - t_0 - \rho_1$,

$$d(c(t_0 + \rho_1), q) = d(p_1, q) = r - t_0 - \rho = r - (t_0 + \rho_1),$$

so $t_0 + \rho_1 \in I$, implying that I is open, i.e., $I = [0, r]$, so $c(r) = q$ follows.[a]

[a] For a detailed proof, see [FC13, Corollary 3.9].

Lecture 8: Injectivity Radius and Vector Bundles

Let's finish the proof of Hopf-Rinow theorem. 31 Jan. 13:00

Proof of Theorem 2.3.3 (Continued). We see that (d) implies (e), hence we only need to show that (a), (b), (c), and (d) are equivalent.

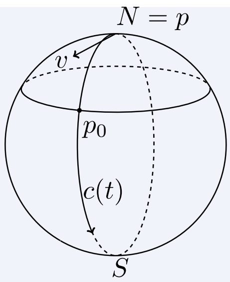
- (d) $\Rightarrow$ (c): It is trivial.
- (c) $\Rightarrow$ (b): Let $K \subseteq \mathcal{M}$ be closed and bounded. As K bounded, $K \subseteq B(p, r)$ for some $r > 0$. Then any point in $B(p, r)$ can be joined with p by geodesic of length $\leq r$, and $B(p, r)$ is the image of the compact ball in $T_p\mathcal{M}$ of radius r under continuous map $\exp_p$, hence $B(p, r)$ is compact. As K closed and $K \subseteq B(p, r)$, K is compact.
- (b) $\Rightarrow$ (a): Let $(p_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be a Cauchy sequence, so it's bounded, and by (b), its closure is compact. It contains a convergent subsequence, so it converges, i.e., $\mathcal{M}$ is complete.
- (a) $\Rightarrow$ (d): Let c be a geodesic in $\mathcal{M}$, parametrized by arc length defined on a maximal interval I. Since I s non-empty, and we can show that I is both open and closed.

■

Remark. It's worth mentioning that we do have uniqueness after choosing p_0,[a] so the non-uniqueness really comes from the initial choice of p_0.

[a] In other words, after choosing p_0, everything is fixed.

Example. Consider S^2, after fixing p_0, $c(t_0)$ is extended uniquely.



2.3.3 Injectivity Radius

One might wonder, though we have Lemma 2.3.1, how far can $\exp_p$ extends while maintaining injectivity?

Definition 2.3.2 (Injectivity radius). Let $\mathcal{M}$ be a Riemannian manifold, and $p \in \mathcal{M}$. The *injectivity radius* $i(p)$ of p is

$$i(p) := \sup \left\{ \rho > 0 \mid \exp_p \text{ defined on } B(0, \rho) \subseteq T_p\mathcal{M} \text{ and injective} \right\}.$$

Similarly, the *injectivity radius* $i(\mathcal{M})$ of $\mathcal{M}$ is defined as $i(\mathcal{M}) := \inf_{p \in \mathcal{M}} i(p)$.

Example (Sphere). $i(S^n) = \pi$.

Example (Torus). $i(T^n) = 1/2$.

Remark. Any manifold carries a complete Riemannian metric. If $(\mathcal{M}, g_1)$ is not complete, we can find g_2 such that $(\mathcal{M}, g_2)$ is complete.

Example (Hyperbolic half-plane). The half-plane $P = \left\{ (x, y) \in \mathbb{R}^2 \mid y > 0 \right\}$ equipped with metric induced by the Euclidean metric on $\mathbb{R}^2$, which is not complete.

However, it becomes complete when equipped with the following metric

$$\frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2).$$

In fact, P with the above metric is called the *hyperbolic half-plane* $\mathbb{H}^2$, and we can extend it to $\mathbb{H}^n$.

Another question we may ask is the following.

Problem. Is the converse of Hopf-Rinow theorem true? I.e., can we show that (e) implies (d)?

Answer. No! Any 2 points in the open half-sphere can be joint by a unique minimal geodesic, but this manifold is not geodesically complete. ⊛

Example. The injectivity radius of H^n is ∞.

Remark. Given a compact $\mathcal{M}$, the injectivity radius is always > 0 by continuity argument.

Now, given a complete but not compact $\mathcal{M}$, the injectivity radius can be 0.

Example. Take the quotient of the Poincaré half-plane by the translations

$$(x, y) \mapsto (x + n, y), \quad n \in \mathbb{Z}.$$

We then obtain a complete Riemannian manifold $\mathcal{M}$ with $i(\mathcal{M}) = 0$.

Note. Finding lower bounds for $i(\mathcal{M})$ introduces curvature estimates.

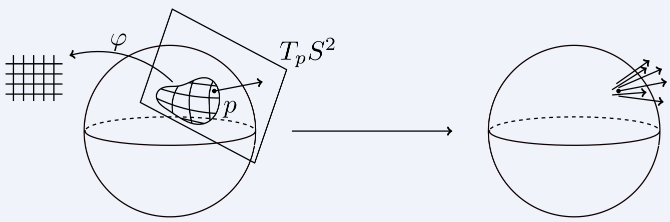
2.4 Vector Bundles and Tensor Fields

We now introduce the theory of bundles, which allows us to introduce vector fields, and hence tensor fields. Noticeably, nearly every structure we can put on a Riemannian manifold will be in the form of tensor fields, which is why we care about them.

As a motivating example, recall the tangent bundle[2] $(T\mathcal{M}, \pi, \mathcal{M})$, which captures the idea of "for every $p \in \mathcal{M}$, we associate a space $T_p\mathcal{M}$".

Intuition. This helps us construct tangent vector fields since a tangent vector field X of $\mathcal{M}$ is defined by associating p to a tangent vector $X(p)$ in the associated tangent space $T_p\mathcal{M}$.

Example. The tangent vector field assigns every $p \in S^2$ a "point" in the associated tangent space.



2.4.1 Bundles

The above example of tangent bundle generalizes quite easily for defining a general bundle.

Definition 2.4.1 (Bundle). A *bundle* is a tuple $(E, \pi, \mathcal{M})$ consists of the total space E, the base space $\mathcal{M}$, and the bundle projection $\pi\colon E \to \mathcal{M}$.

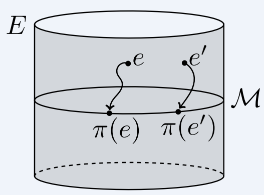
Definition 2.4.2 (Total space). The differentiable manifold E is called the *total space*.

Definition 2.4.3 (Base space). The differentiable manifold $\mathcal{M}$ is called the *base space*.

Definition 2.4.4 (Bundle projection). The (differentiable) continuous surjection $\pi\colon E \to \mathcal{M}$ is called the *bundle projection*.

Example. A tangent bundle $(T\mathcal{M}, \pi, \mathcal{M})$ is a bundle.

Example. Let E be a cylinder, $\mathcal{M}$ be a circle.



As we can see, the number of possible π is enormous, as long as it's surjective and smooth.

[2]Where just use the name "bundle" and don't know what it is. We'll see now!

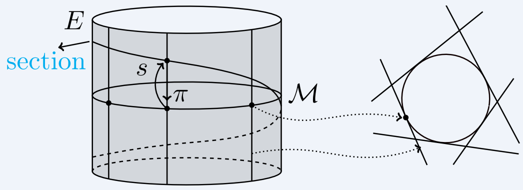
Notation. Sometimes, we will just denote a bundle as $E \xrightarrow{\pi} \mathcal{M}$, or even more compactly, just π since it captures all the data.

Definition 2.4.5 (Fiber)**.** Given a bundle $(E, \pi, \mathcal{M})$, the *fiber* over $p \in \mathcal{M}$ under π is the preimage of a $\{p\}$, i.e., $\pi^{-1}(\{p\})$.

Definition 2.4.6 (Section)**.** A *section* of a bundle $(E, \pi, \mathcal{M})$ is a differentiable map $s\colon \mathcal{M} \to E$ such that $\pi \circ s = \mathrm{id}_{\mathcal{M}}$.

Remark. We see that a section s encodes lots of information of a bundle, since s includes $E, \mathcal{M}$, and the condition deal with π.

Example. Again let E be a cylinder, $\mathcal{M}$ be a circle. This time, we choose π to be the trivial one.



We see that in this way, this bundle really captures all the tangent spaces structure of a circle!

2.4.2 Vector Bundles

Then, we're interested in the so-called vector bundle.

Definition 2.4.7 (Vector bundle)**.** A (differentiable) *vector bundle* of rank n is a bundle $(E, \pi, \mathcal{M})$ such that each fiber $E_x := \pi^{-1}(x)$ of $x \in \mathcal{M}$ carries a structure of an n-dimensional (real) vector space, and local triviality condition holds.

> **Definition 2.4.8** (Local trivialization)**.** For all $x \in \mathcal{M}$, the *local trivialization* (U, φ) consists a neighborhood U and diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that for all $y \in U$,
>
> $$\varphi_y := \varphi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^n$$
>
> is a vector space isomorphism.

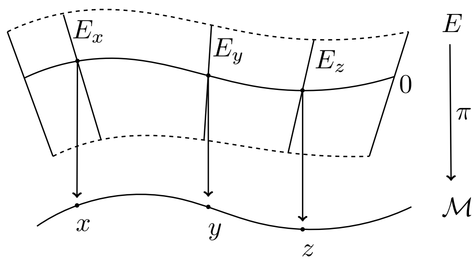


Figure 2.1: An illustration of vector bundle $(E, \pi, \mathcal{M})$.

Definition 2.4.9 (Trivial)**.** A vector bundle is *trivial* if it's isomorphic to $\mathcal{M} \times \mathbb{R}^n$.[a]

[a] n is the rank of the vector bundle.

Intuition. Local trivialization shows that *locally* π looks like the projection of $U \times \mathbb{R}^n$ on U.

Definition 2.4.10 (Bundle chart)**.** The pair (φ, U) is the *bundle chart* in local trivialization.

Remark. From Definition 2.4.7, vector bundle is locally, but not necessarily globally a product of base space and the fiber.

Intuition. We may look at a vector bundle as a family of vector spaces, all isomorphic to a fixed $\mathbb{R}^n$, "parametrized" (locally trivially) by a manifold.

2.4.3 Vector Fields

We can now introduce the notion of vector field in terms of section.

Definition 2.4.11 (Vector field)**.** A (smooth) *vector field* X is a smooth section of a vector bundle.

Note. We see that a smooth tangent vector field is indeed a smooth vector field with the bundle being the tangent bundle.

Notation. Since we will nearly always be talking about tangent vector fields, we will abuse the notation a bit and just simply call it vector fields. But always keep in mind that more broadly, a vector field should be a section of a vector bundle, not always $T\mathcal{M}$.

Lecture 9: Tensors and Connections

2.4.4 Tensor Fields

2 Feb. 13:00

We can introduce the notion of "tensor fields" in a brute-fore way[3] by following a similar path of how we define vector field, i.e., we first define some bundle called tensor bundle, and then the tensor field is just a smooth section of which. But first, it might be beneficial to see how does a tensor look like.

Definition 2.4.12 (Tensor)**.** Let V be a vector space of dimension $m < \infty$, and the dual space V^*. The vector space of the *r-times contravariant and s-times covariant tensors over V*, denoted as $T^r_s(V)$, is defined as

$$T^r_s(V) = \{T \colon \underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \to \mathbb{R}\} = (V^*)^{\otimes r} \otimes V^{\otimes s}.$$

Intuition. Just as vector field, we're trying to assign a vector on every point $p \in \mathcal{M}$. Here, we're trying to assign a tensor on every point $p \in \mathcal{M}$, which is just an element in $(V_p^*)^{\otimes r} \otimes V_p^{\otimes s}$.

As one might imagine, since a tensor is an element in $(V_p^*)^{\otimes r} \otimes V_p^{\otimes s}$, the corresponding tensor bundle is defined as follows.

Definition 2.4.13 (Tensor bundle)**.** A *(r, s)-tensor bundle* $T^r_s\mathcal{M} = \bigcup_{p\in\mathcal{M}} T^r_s(T_p\mathcal{M}) = T^r_s(T\mathcal{M})$ on $\mathcal{M}$ is a fiber bundle where the fiber is the tensor product of s tangent spaces and r cotangent spaces.

Remark. It's clear that one can also define tensor bundle in terms of a general vector bundle V on $\mathcal{M}$ instead of the tangent bundle $T\mathcal{M}$ specifically.

[3]See Appendix A.1 for another view point.

So in a tensor bundle, the fiber is a vector space and the tensor bundle is a special kind of vector bundle.[4] Finally, the tensor field is defined as follows.

Definition 2.4.14 (Tensor field). A (r, s)-*tensor field* is a section of a (r, s)-tensor bundle.

A convenient notation is the following.

Notation. Let $\mathcal{M}^n$ be a smooth manifold and $\pi\colon E \to \mathcal{M}$ a smooth vector bundle, then the set of sections is denoted as

$$\Gamma(E) \coloneqq \{s \in C^\infty(\mathcal{M}, E) \mid \pi \circ s = \mathrm{id}_{\mathcal{M}}\}.$$

Then, we see the following.

Example. Consider the vector bundle $(T\mathcal{M}, \pi, \mathcal{M})$, then $\Gamma(T\mathcal{M}) \coloneqq \{\text{vector fields on } \mathcal{M}\}$.

Example. A (r, s)-tensor field on $\mathcal{M}$ is then equivalently defined as an element in $\Gamma(T^r_s\mathcal{M})$.

Notation. For $s \in \mathbb{N}$, let $\Lambda^s(V^*) \coloneqq \left\{A \in T^0_s(V) \mid A \text{ skew-symmetric}\right\}$.

Example. $\Gamma(\Lambda_s\mathcal{M}) \coloneqq \{s\text{-forms on } \mathcal{M}\}$ with $\Lambda_s\mathcal{M} = \Lambda^s\left(\bigcup_{p\in\mathcal{M}} T^*_p\mathcal{M}\right)$.

Example. A Riemannian metric g on $\mathcal{M}$ is a $(0, 2)$-tensor field, i.e., $g \in \Gamma(T^0_2(\mathcal{M}))$ for all $p \in \mathcal{M}$.

Proof. Since $g_p\colon T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$, so by regarding p as the argument of the map g, $g\colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$. ⊛

Note. It's in fact unnecessary to have such a general Definition 2.4.14 on a Riemannian manifold.

Proof. Since given a Riemannian metric g, it associates to each $X \in \Gamma(T\mathcal{M})$ a unique $\omega \in \Gamma(T^*\mathcal{M})$ given by $\omega(Y) = g(X, Y)$ for all $X, Y \in \Gamma(T\mathcal{M})$. ⊛

2.5 Other Metrics

Finally, we conclude this chapter by introducing some other metrics a manifold can equip with.

Definition 2.5.1 (Pseudo-Riemannian metric). A *pseudo-Riemannian metric* on a differentiable manifold $\mathcal{M}$ is a $(0, 2)$-tensor field $g \in \Gamma(T^0_2(\mathcal{M}))$ for all $p \in \mathcal{M}$ with

(a) $g(X, Y) = g(Y, X)$ for all $X, Y \in T\mathcal{M}$;

(b) for all $p \in \mathcal{M}$, g_p is non-degenerate bilinear form on $T_p\mathcal{M}$, i.e., $g_p(X, Y) = 0$ for all $X, Y \in T_p\mathcal{M}$ if and only if $Y = 0$.

Note. A pseudo Riemannian metric is a Riemannian metric if it's positive definite at every $p \in \mathcal{M}$.

Definition 2.5.2 (Lorentzian metric). A *Lorentzian metric* g is a continuous assignment of a non-degenerate[a] quadratic form g_p of index 1[b] in $T_p\mathcal{M}$ for all $p \in \mathcal{M}$.

[a] $g_p(X, Y) = 0$ for all $Y \in T_p\mathcal{M}$ implies $X = 0$.
[b] It means that the maximal dimension of a subspace of $T_p\mathcal{M}$ on which g_p is negative definite is 1.

[4] There are vector bundles which are not tensor bundles.

An equivalent definition is the following.

Definition 2.5.3 (Lorentzian). A quadratic form g_p in $T_p\mathcal{M}$ is *Lorentzian* if there exists a vector $V \in T_p\mathcal{M}$ such that $g_p(V,V) < 0$ while setting $\Sigma_V = \{X \mid g_p(X,V) = 0\}$ such that $g_p|_{\Sigma_V}$ [a] is positive definite.

[a] The g_p-orthogonal complement of V.

Example (Minkowski space). The Minkowski space on $\mathbb{R}^4$ is the prototypical example from physics (flat spacetime). Namely, the metric is given by the quadratic form

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with the coordinates being (t, x, y, z).

Chapter 3

Connections and Curvatures

So far, we saw that a vector field X can be used to provide a directional derivative since it gives us a tangent vector at each point smoothly. Now, we will introduce a new symbol ∇ where for $f \in C^\infty(\mathcal{M})$,

$$\nabla_X f \coloneqq Xf.$$

Problem. Does this notation overkill? We already know that $Xf = (\mathrm{d}f)(X)$!

Answer. No! While $\nabla, X \colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, while $\mathrm{d}f \colon \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$, we can generalize ∇_X to act from vector fields to vector fields! The insight is that if X can be extended naturally (without providing any extra structures), then we certainly won't bother introducing a new symbol. However, as you might guess, to let ∇ doing this, we do need to provide extra structures, and ∇ stands exactly for these extra structures! ⊛

In some sense, this new notions ∇ allows us to "connect" tangent spaces, which allows us to make sense of "curvatures" and other geometric property of a Riemannian manifold.

3.1 Levi-Civita Connections

We start by talking about linear connections, and then realize that after specifying a Riemannian metric g, with an additional (technical) assumption, a unique linear connection, defined as Levi-Civita connections, exists for any Riemannian manifold. In other words, specifying g is the same as specifying the "shape of the space." We'll make sense of all these on the way.

3.1.1 Affine Connections

We first formulate a *wish list* of properties which the ∇_X should have. Any remaining freedom in choosing ∇ will need to be provided as additional structures beyond the structures on $\mathcal{M}$ we already have.

Definition 3.1.1 (Linear connection)**.** A *linear connection* (or *affine connection*) on a smooth manifold $\mathcal{M}$ is a bilinear map

$$\nabla \colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M}),$$

which is denoted by $\nabla(X, Y) = \nabla_X Y$ and which satisfies

(a) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$;

(b) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;

(c) $\nabla_X fY = f\nabla_X Y + X(f)Y$;

for all vector fields $X, Y, Z \in \Gamma(T\mathcal{M})$ and $f, g \in C^\infty(\mathcal{M})$.

Remark. Definition 3.1.1 (c) shows that this is actually a local notion as we will see.

Note. There's a similar notation called covariant derivative, denoted by D, satisfies similar properties as a linear connection. Hence, we often write D and ∇ interchangeably.[a]

[a] ∇ is more general than D; however, we treat them as the same as suggested by Proposition 3.4.1.

Now, one might be wondering that, after fixing these rules we want, how much freedom is left? To see this, let's first do some calculations...

3.1.2 Connection Coefficients

Choose a system of coordinates $(x_1, \ldots, x_n)$ at $p \in \mathcal{M}$, we can write $X = X^i \frac{\partial}{\partial x_i}, Y = Y^j \frac{\partial}{\partial x_j}$, then

$$\nabla_X Y = \nabla_{X^i \frac{\partial}{\partial x_i}} \left(Y^j \frac{\partial}{\partial x_j} \right) = X^i Y^j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + X^i \frac{\partial}{\partial x_i}(Y^j) \frac{\partial}{\partial x_j}.$$

Now, we see that $\nabla_{\partial / \partial x_i} \frac{\partial}{\partial x_j}$ is another vector field, hence can again write

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} =: \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

in terms of the basis with a new set of coefficients Γ.

Notation (Connection coefficient). The coefficients Γ_{ij}^k are called the *connection coefficients*.

Intuition. Γ are the *corrections* to an ordinary derivative on a "curved" manifold w.r.t. ∇.

It's tempting to say that the connection coefficients are the same as Christoffel symbols since we're using the same symbols. Indeed, they are the same if ∇ is chosen to be the Levi-Civita connection.[1]

Note. It's clear that Γ_{ij}^k are differentiable and charts-dependent and hence ∇ is local.

Finally, in a particular domain U, we have

$$\nabla_X Y = \left(X^i Y^j \Gamma_{ij}^k + X(Y^k) \right) \frac{\partial}{\partial x_k} \Rightarrow (\nabla_X Y)^k = X(Y^k) + \Gamma_{ij}^k X^i Y^j,$$

meaning that we have $(\dim \mathcal{M})^3$ many Γ's (freedom) when choosing Γ_{ij}^k with Definition 3.1.1.

Remark. One might ask what about other tensor fields? Fortunately, the same set of Γ's fix the action of ∇ on any tensor fields.

Proof. The key observation is that if we define $\nabla_{\frac{\partial}{\partial x^j}}(\mathrm{d}x^i) =: \Sigma_{jk}^i \mathrm{d}x^k$, then

$$\nabla_{\frac{\partial}{\partial x^j}} \left(\mathrm{d}x^i \left(\frac{\partial}{\partial x^k} \right) \right) = \begin{cases} \frac{\partial}{\partial x^j}(\delta_k^i) = 0; \\ \left(\nabla_{\frac{\partial}{\partial x^j}} \mathrm{d}x^i \right) \frac{\partial}{\partial x^k} + \mathrm{d}x^i \underbrace{\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \right)}_{\Gamma_{jk}^\ell \frac{\partial}{\partial x^\ell}}, \end{cases}$$

and since $\mathrm{d}x^i \frac{\partial}{\partial x^\ell} = \delta_\ell^i$, the above leads to

$$\left(\nabla_{\frac{\partial}{\partial x^j}} \mathrm{d}x^i \right) \frac{\partial}{\partial x^k} = -\mathrm{d}x^i \left(\Gamma_{jk}^\ell \frac{\partial}{\partial x^\ell} \right) \Rightarrow \left(\nabla_{\frac{\partial}{\partial x^j}} \mathrm{d}x^i \right)_k = -\Gamma_{jk}^i.$$

⊛

In summary, we have

$$\begin{cases} (\nabla_X Y)^k = X(Y^k) + \Gamma_{ij}^k X^i Y^j, & \text{if } Y \text{ is a vector field;} \\ (\nabla_X \omega)_k = X(\omega_k) - \Gamma_{ik}^j X^i \omega_j, & \text{if } \omega \text{ is a co-vector field.} \end{cases}$$

[1] As we will soon see, it means the torsion free and Riemannian connection. See this for a more detailed explanation.

3.1.3 Levi-Civita Connections

The basic insight is that, after choosing a particular connection,[2] the space is basically fixed: i.e., the *shape*, or "curvature", of the space is determined by the choice of ∇! We now formalize this idea. A particularly natural notion related to "curvature" is the torsion, defined as follows.

Definition 3.1.2 (Torsion). The *torsion* T of a linear connection ∇ is the $(1,2)$-tensor field

$$T(\omega, X, Y) \coloneqq \omega\left(\nabla_X Y - \nabla_Y X - [X, Y]\right).$$

Notation. We usually write this as $T(X, Y)$ by neglecting ω.

Remark. T is actually C^∞-linear in each entry,[a] hence a tensor field.

[a]See Appendix A.1.

Proof. Since $T(f\cdot\omega, X, Y) = f\cdot\omega(\dots) = fT(\omega, X, Y)$ and $T(\omega+\psi, X, Y) = \dots = T(\omega, X, Y) + T(\psi, X, Y)$, and also

$$\begin{aligned} T(\omega, fX, Y) &= \omega\left(\nabla_{fX} Y - \nabla_Y(fX) - [fX, Y]\right) \\ &= \omega(f\nabla_X Y - (Yf)X - f\nabla_Y X - f[X,Y] + (Yf)X) = f\cdot T(\omega, X, Y) \end{aligned}$$

since

$$([fX, Y])g = f\cdot X(Yg) - Y(f\cdot Xg) = f\cdot X(Yg) - (Yf)(Xg) - f\cdot Y(Xg) = (f\cdot[X,Y] - (Yf)X)\,g.$$

Finally, we claim that the additivity at X holds, with $T(\omega, X, Y) = -T(\omega, Y, X)$, we're done. ⊛

Intuition. Definition 3.1.2 makes sense (in such a form) since this will make T actually a tensor field. For example, without the Lie bracket term, we don't have the linearity at X (hence Y).

Definition 3.1.3 (Torsion-free). A linear connection ∇ is *torsion-free* if $T = 0$.

Notation (symmetric). A torsion-free ∇ is sometimes said to be *symmetric*.

In a chart,

$$T^i_{jk} \coloneqq T\left(\mathrm{d}x^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \Gamma^i_{jk} - \Gamma^i_{kj} = 2\Gamma^i_{[jk]},$$

hence if $T = 0$, we can interchange the lower two indexes of Γ^k_{ij}, i.e., $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Definition 3.1.4 (Riemannian). Let ∇ be a linear connection and g be a Riemannian metric on $\mathcal{M}$. Then ∇ is *Riemannian* (or *metric*) if for all $X, Y, Z \in \Gamma(T\mathcal{M})$,[a]

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

[a]We view $g(X, Y) \in C^\infty(\mathcal{M})$ as suggested by Appendix A.1.

Notation (Compatible). A Riemannian ∇ is sometimes said to be *compatible*.

Remark. Equivalently, Definition 3.1.4 can be formulated as $\nabla g = 0$.

We are now able to state the fundamental theorem of this section.

[2]Remember that we have freedom to choose Γ's.

Theorem 3.1.1 (Levi-Civita). On each Riemannian manifold $(\mathcal{M}, g)$, there exists a unique Riemannian, torsion-free connection ∇ on $T\mathcal{M}$ determined by the *Koszul formula*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle). \tag{3.1}$$

Proof sketch. Firstly, we show that every Riemannian and torsion-free connection satisfies Koszul formula, which implies uniqueness. For existence, we verify that the unique map $\nabla\colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ given by Koszul formula is Riemannian and torsion-free.[a] ■

[a] For a detail proof, see [FC13, §2 Theorem 3.6].

Note. I rearrange the Koszul formula to make it easier to memorize.

Finally, we define the following.

Definition 3.1.5 (Levi-Civita connection). The *Levi-Civita connection* is the unique linear connection ∇ defined by the Koszul formula.

Remark. This means, given a Riemannian metric g, with the condition of torsion-free, the shape of the space is also fixed since there's a unique linear connection ∇ such that $T = \nabla g = 0$.

Lecture 10: Curvatures and Flow of Vector Fields

3.2 Riemannian Curvatures

7 Feb. 13:00

Given all these definitions, we can now introduce the notion of "curvatures." Consider the following.[3]

Definition 3.2.1 (Riemannian curvature). The *Riemannian curvature* R of a Levi-Civita connection ∇ is the $(1, 3)$-tensor field[a]

$$R(\omega, Z, X, Y) \coloneqq \omega\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right).$$

[a] R is indeed C^∞-linear in each entry (see Appendix A.1) although we omit the proof here.

Notation. We usually write this as $R(X, Y)Z$ by emphasizing Z and neglecting ω.

Example (Euclidean space). If $\mathcal{M} = \mathbb{R}^n$ (with the "flat" ∇), $R(X, Y)Z = 0$ for all $X, Y, Z \in \Gamma(T\mathbb{R}^n)$.

Proof. Since given $Z = (z_1, \dots, z_n)$ with the components from natural coordinates of $\mathbb{R}^n$, $\nabla_X Z = (Xz_1, \dots, Xz_n)$, then $\nabla_Y \nabla_X Z = (Y X z_1, \dots, Y X z_n)$, hence $R(X, Y)Z = 0$. ⊛

Hence, we see the following.

Intuition. $R(X, Y)Z$ is trying to measure how much $\mathcal{M}$ deviates from being Euclidean.

Another way to look at this is the following.

Intuition. Consider a system of coordinates $\{x_i\}$ around $p \in \mathcal{M}$. Since $[\partial/\partial x_i, \partial/\partial x_j] = 0$,

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = (\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}})\frac{\partial}{\partial x_k},$$

i.e., $R(X, Y)Z$ is trying to measure the non-commutativity of the covariant derivative.

[3] In do Carmo [FC13], the corresponding definition of Definition 3.2.1 differs by a sign.

3.2.1 Local Expressions

It;s convenient to express things in a local coordinates. Consider a chart (U, x) at $p \in \mathcal{M}$ and let $\partial/\partial x_i = X_i$. We define R_{ijk}^{ℓ} as[4]

$$R_{ijk}^{\ell} X_\ell \coloneqq R(X_i, X_j) X_k.$$

If $X = u^i X_i, Y = v^j X_j, Z = w^k X_k$, from the linearity of R,

$$R(X, Y)Z = R_{ijk}^{\ell} u^i v^j w^k X_\ell.$$

Then the above intuition can be rewritten as follows.

Remark (Algebraic significant of Riemannian curvature). Since

$$(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) = R(\cdot, Z, X, Y) + \nabla_{[X,Y]} Z,$$

by letting $\nabla_i \coloneqq \nabla_{\frac{\partial}{\partial x^i}}, \nabla_j \coloneqq \nabla_{\frac{\partial}{\partial x^j}}$, in a chart (U, x), we have

$$(\nabla_i \nabla_j Z)^k - (\nabla_j \nabla_i Z)^k = R_{\ell ij}^k Z^\ell + \nabla_{\underbrace{\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]}_{=0}} Z = R_{\ell ij}^k Z^\ell,$$

i.e., the components of R contains all the information of how ∇_i and ∇_j fail to commute.

We can also express R_{ijk}^{ℓ} in terms of Γ_{ij}^k by observing

$$R(X_i, X_j) X_k = \nabla_{X_i} \nabla_{X_j} X_k - \nabla_{X_j} \nabla_{X_i} X_k = \nabla_{X_i} (\Gamma_{jk}^{\ell} X_\ell) - \nabla_{X_j} (\Gamma_{ik}^{\ell} X_\ell),$$

hence,

$$R_{ijk}^s = \Gamma_{jk}^{\ell} \Gamma_{i\ell}^s - \Gamma_{ik}^{\ell} \Gamma_{j\ell}^s + \Gamma_{jk,i}^s - \Gamma_{ik,j}^s.$$

Lastly, we write

$$\langle R(X_i, X_j) X_k, X_\ell \rangle = R_{ijk}^s g_{\ell s} \eqqcolon R_{ijk\ell}.$$

3.2.2 Identities

There are many important identities related to R, and we should see some of them.

Note. Although the above interpretations and intuitions are more or less formal, we should first get used to the formal properties of R and postpone a more geometric interpretation of curvature later.

The following two are due to Bianchi (both are proved in homework 2).

Proposition 3.2.1 (First Bianchi identity). Given the Riemannian curvature tensor R, for all vector fields X, Y, Z,

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0;$$

or equivalently, $R_{k\ell ij} + R_{kij\ell} + R_{kj\ell i} = 0$.

Proof. See also do Carmo [FC13, Proposition 2.4]. ■

Proposition 3.2.2 (Second Bianchi identity). Given the Riemannian curvature tensor R,

$$\frac{\partial}{\partial x^h} R_{k\ell ij} + \frac{\partial}{\partial x^k} R_{\ell hij} + \frac{\partial}{\partial x^\ell} R_{hkij} = 0;$$

or equivalently, $\nabla_{[\alpha} R_{\beta\gamma]\delta\epsilon} \coloneqq \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0.$[a]

[a]This notation is a bit cryptic: see Ricci calculus.

Moreover, we can also talk about exchanging two indices.

[4]This is how we define connection coefficients, i.e., R_{ijk}^{ℓ} are components of R in (U, x).

Proposition 3.2.3. Given the Riemannian curvature tensor R,

(a) $R(X,Y)Z = -R(Y,X)Z$, i.e., $R_{k\ell ij} = -R_{k\ell ji}$;

(b) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$, i.e., $R_{k\ell ij} = -R_{\ell kij}$;

(c) $\langle R(X,Y)Z, W\rangle = -\langle R(Y,X)Z, W\rangle$, i.e., $R_{k\ell ij} = -R_{\ell kji}$;

(d) $\langle R(X,Y)Z, W\rangle = -\langle R(Z,W)X, Y\rangle$, i.e., $R_{k\ell ij} = R_{ij\ell k}$.

Proof. See also do Carmo [FC13, Proposition 2.5]. ■

3.2.3 Other Curvatures

There are other notions of curvature, but they all depend on the Riemannian curvature, and appearing to be some sorts of "average" of R. We have already seen the first one.

Definition 3.2.2 (Riemannian-Christoffel curvature)**.** The *Riemannian-Christoffel curvature* is defined by

$$R_{k\ell ij} \coloneqq g_{km} R^m_{\ell ij} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k}\right\rangle.$$

Definition 3.2.3 (Ricci curvature)**.** The *Ricci curvature* is defined by $R_{ab} = g^{cm} R_{camb} = R^m_{\ amb}$.

Definition 3.2.4 (Ricci scalar curvature)**.** The *(Ricci) scalar curvature* is defined by $R = g^{ab} R_{ab}$.

Note. For a more formal treatment, see do Carmo [FC13, §4.4].[a]

[a]Notice that the order in do Carmo [FC13] is a bit different: it introduces sectional curvature first.

3.3 Flows of Vector Fields

Let $\mathcal{M}$ be a smooth manifold, and X a vector field on $\mathcal{M}$. Then X defines a first order differential equation[5]

$$\dot{c} = X(c).$$

And this ODE has a solution, as guaranteed by Proposition 3.3.1.

Proposition 3.3.1. For all $p \in \mathcal{M}^d$, there exists an open interval $I = I_p \subseteq \mathbb{R}$ with $0 \in I_p$ such that a smooth curve $c\colon I_p \to \mathcal{M}$ solves

$$\begin{cases} \dfrac{\mathrm{d}c(t)}{\mathrm{d}t} = X(c(t)), & t \in I; \\ c(0) = p. \end{cases}$$

Further, the solution depends smoothly on the initial data (i.e., p).[a]

[a]This directly follows from ODE theory.

Proof. For all $p \in \mathcal{M}$, we want to find an open interval $I = I_p$ around $0 \in \mathbb{R}$ and a solution of the following ODE for $c\colon I \to \mathcal{M}$:

$$\begin{cases} \dfrac{\mathrm{d}c(t)}{\mathrm{d}t} = X(c(t)), & t \in I; \\ c(0) = p. \end{cases}$$

[5]If $\dim \mathcal{M} > 1$, it is a system of first order differential equations.

We can check in local coordinates that this is a system of ODE. In such coordinates, let $c(t)$ be given by $c(t) = \left(c^1(t), c^2(t), \dots, c^d(t)\right)$. Let $X =: X^i \partial/\partial x^i$, then the above system becomes

$$\frac{\mathrm{d}c^i(t)}{\mathrm{d}t} = X^i\left(c(t)\right), \quad i = 1, \dots, d.$$

From the Picard-Lindelöf theorem, with the initial data $c(0) = p$, there is a unique solution. ■

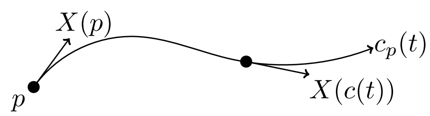
Proposition 3.3.2. For all $p \in \mathcal{M}$, there exists an open neighborhood U of p and an open interval I_p with $0 \in I_p$ such that for all $q \in U$, the curve c_q with

$$\dot{c}_q(t) = X(c_q(t)), \quad c_q(0) = q$$

is defined on I and the map $c \colon I \times U \to \mathcal{M}$, $(t, q) \mapsto c_q(t)$ is smooth.

Proposition 3.3.2 suggests the following definition.

Definition 3.3.1 (Local flow)**.** The map $c_q(t) \colon I \times U \to \mathcal{M}$, $(t, q) \mapsto c_q(t)$ from Proposition 3.3.2 is called the *local flow* of the vector field X.



Definition 3.3.2 (Integral curve)**.** The local flow $c_q(t)$ is called the *integral curve* of X through q.

3.3.1 Local 1-Parameter Groups

Now, given a local flow $c_q(t)$ of a vector field X, by fixing t, we can vary q and see the following.

Theorem 3.3.1. Let $\varphi_t(q) \coloneqq c_q(t)$ such that $\varphi_t \circ \varphi_s(q) = \varphi_{t+s}(q)$ for $s, t, (t + s) \in I_q$. If φ_t is defined on $U \subseteq \mathcal{M}$, it maps U diffeomorphically onto its image.

We see that φ_t defines a family of diffeomorphism around p, which gives the following.

Definition 3.3.3 (Local 1-parameter group)**.** A family $(\varphi_t)_{t \in I}$ of diffeomorphism from $\mathcal{M}$ to $\mathcal{M}$ satisfying Theorem 3.3.1 is called a *local* 1*-parameter group* of diffeomorphisms.

In general, a local 1-parameter group needs not be extendible to a group because the maximum interval $I = I_q$ in Definition 3.3.3 need not be all of $\mathbb{R}$.

Example. Let $\mathcal{M} = \mathbb{R}$, $X(t) = \tau^2 \mathrm{d}/\mathrm{d}\tau$. Then the solution of $\dot{c}(t) = c^2(t)$ is not defined over all $\mathbb{R}$.

To get the whole group structure, consider the following.

Theorem 3.3.2. Let X be a vector field on a smooth manifold $\mathcal{M}$ with a compact support. Then the corresponding local flow is defined for every $q \in \mathcal{M}$ and $t \in \mathbb{R}$, and the local 1-parameter group becomes a group of diffeomorphisms.

Proof. By using $\operatorname{supp}(X) \subseteq K$, K compact, we can cover K by a finite covering, then using Proposition 3.3.2, we're done. ■

This leads to the following.

Corollary 3.3.1. On a compact differentiable manifold $\mathcal{M}$, any vector field generates a local 1-parameter group.

Lecture 11: Geodesic & Cogeodesic Flows and Parallel Transport

3.3.2 Geodesic and Cogeodesic Flows

9 Feb. 13:00

This section is weird... Need fix

A particularly interesting flow is the cogeodesic flow: let's first transform Equation 2.1 (which is a second order ODE) into a first order system on the cotangent bundle $T^*\mathcal{M}$, and locally trivialize $T^*\mathcal{M}$ by chart $T^*\mathcal{M}|_U \cong U \times \mathbb{R}^d$ with coordinates $(x^1, \ldots, x^d, p_1, \ldots, p_d)$. Now, set

$$H(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j, \tag{3.2}$$

Theorem 3.3.3. Equation 2.1 is equivalent to the system on $T^*\mathcal{M}$:

$$\begin{cases} \dot{x}^i = \dfrac{\partial H}{\partial p_i} g^{ij}(x) p_j; \\ \dot{p}_i = -\dfrac{\partial H}{\partial x^i} = -\dfrac{1}{2} g^{jk}_{,i}(x) p_j p_k. \end{cases} \tag{3.3}$$

Proof. This is just computation (recall that $g^{ik} g_{kj} = \delta^i_j$). ■

Definition 3.3.4 (Cogeodesic flow)**.** The *cogeodesic flow* is the local flow determined by Equation 3.3.

Definition 3.3.5 (Geodesic flow)**.** The geodesic flow on $T\mathcal{M}$ is obtained from the cogeodesic flow by the first equation in Equation 3.3.

Intuition. The geodesic is the projection of the integral curve of the geodesic flow onto $\mathcal{M}$.

Note (Hamiltonian flow)**.** The cogeodesic flow is a *Hamiltonian flow* for the Hamiltonian H.

Proof. By Equation 3.3, along the integral curves,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = H_{x^i}\dot{x}^i + H_{p_i}\dot{p}^i = -\dot{p}_i x \dot{x}^i + \dot{x}^i \dot{p}_i = 0.$$

⊛

Observe that the cogeodesic flow maps $E_\lambda \coloneqq \{(x,p) \in T^*\mathcal{M} \mid H(x,p) = \lambda\}$[6] onto itself for all $\lambda \geq 0$, and if $\mathcal{M}$ is compact, then all E_λ are compact, then all geodesic flows are defined on all E_λ for all λ.

3.4 Covariant Derivatives and Parallelism

An important concept related to curvatures is "parallelism," which needs a formal introduction of covariant derivatives. As a motivating example, the following is an equivalent definition of geodesic.

Example (Autoparallel)**.** The geodesic c satisfies $\nabla_{\dot{c}}\dot{c} = 0$. This is called autoparallel.

Proof. In the local coordinates, we have $\dot{c} = \dot{c}^i \partial/\partial x^i$, and note that

$$\nabla_{\dot{c}}\dot{c} = \dot{c}^i \nabla_{\frac{\partial}{\partial x^i}} \dot{c}^j \frac{\partial}{\partial x^j} = \dot{c}^i \dot{c}^j \Gamma^k_{ij} \frac{\partial}{\partial x^k} + \ddot{c}^k \frac{\partial}{\partial x^k} = \left(\ddot{c}^k + \Gamma^k_{ij}\dot{c}^i\dot{c}^j\right) \frac{\partial}{\partial x^k} = 0 \tag{3.4}$$

since a geodesic is the solution of Equation 2.1. ⊛

Intuition. A geodesic is a curve with "zero acceleration".

To understand what $\nabla_{\dot{c}}\dot{c}$ is doing beyond just calculation, we need to understand parallel transports.

[6] $\mathcal{M} = \bigcup_{\lambda \geq 0} PE_\lambda$ for P being the projection.

3.4.1 Covariant Derivatives

We can now finally define covariant derivative formally.

As previously seen. Let $X = X^i \frac{\partial}{\partial x_i}$, $V = V^k \frac{\partial}{\partial x_k}$, and let D be the Levi-Civita connection. Then

$$\mathrm{D}_V X = \mathrm{D}_V \left(X^i \frac{\partial}{\partial x_i} \right) = V(X^i) \frac{\partial}{\partial x_i} + X^i \mathrm{D}_V \frac{\partial}{\partial x_i} = V(X^i) \frac{\partial}{\partial x_i} + V^k X^i \Gamma_{ki}^j \frac{\partial}{\partial x_j}.$$

Proposition 3.4.1 (Covariant derivative)**.** Let $(\mathcal{M}, g)$ be a Riemannian manifold, D the Levi-Civita connection, and c a smooth curve in $\mathcal{M}$ with the set of smooth vector fields along c $\mathcal{X}_c(\mathcal{M})$. Then there exists a unique operator $\mathrm{D}/\mathrm{d}t$ defined as the vector space of vector fields along c satisfying

(i) (a) $\frac{\mathrm{D}}{\mathrm{d}t}(fY)(t) = f'(t)Y(t) + f(t)\frac{\mathrm{D}}{\mathrm{d}t}Y(t)$ for all $f \in C^\infty(I)$ and $Y \in \mathcal{X}_c(\mathcal{M})$;

(b) $\frac{\mathrm{D}}{\mathrm{d}t}(V + W) = \frac{\mathrm{D}V}{\mathrm{d}t} + \frac{\mathrm{D}W}{\mathrm{d}t}$ for all $V, W \in \mathcal{X}_c(\mathcal{M})$;

(ii) if there exists a neighborhood of in I such that Y is the restriction to c of a vector field X defined on a neighborhood of $c(t_0)$ in $\mathcal{M}$, then $\frac{\mathrm{D}}{\mathrm{d}t}Y(t_0) = (\mathrm{D}_{c(t_0)}X)_{c(t_0)}$.

Proof. Consider defining such an operator $\mathrm{D}/\mathrm{d}t$ as

$$\frac{\mathrm{D}}{\mathrm{d}t}\left(Y^i(t) \frac{\partial}{\partial x_i} \right) = \frac{\mathrm{d}V^i}{\mathrm{d}t}\frac{\partial}{\partial x_i} + \dot{c} Y^i \Gamma_{ji}^k(c(t)) \frac{\partial}{\partial x_k},$$

where $\dot{c} = \dot{c}^k \frac{\partial}{\partial x_k}$. This shows (i) (a) and (b) hold. Next, to show (ii), let x be a smooth vector field in $\mathcal{M}$. Then the induced vector field along c is given by $Y(t) = X_{c(t)}$, i.e., in terms of the coordinate basis, we have $Y(t) = Y^i(t)\frac{\partial}{\partial x_i}$, $X_x = X^i(x)\frac{\partial}{\partial x_i}$, and $Y^i(t) = X^i(c(t))$. Then,

$$\begin{aligned}
\mathrm{D}_i X = \mathrm{D}_i \left(X^i \frac{\partial}{\partial x_i} \right) &= \dot{c}(X^i)\frac{\partial}{\partial x_i} + X^i \mathrm{D}_i \frac{\partial}{\partial x_i} = X^i \dot{c}^k \underbrace{\mathrm{D}_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}}_{\Gamma_{ki}^\ell \frac{\partial}{\partial x_\ell}} \\
&= \partial_t (X^i \circ c) \frac{\partial}{\partial x_i} + \dot{c}^k X^i \Gamma_{ki}^\ell \frac{\partial}{\partial x_\ell} = \partial_t (X^i \circ c)\frac{\partial}{\partial x_i} + \dot{c}^k Y^i \Gamma_{ki}^\ell \frac{\partial}{\partial x_\ell} = \frac{\mathrm{D}}{\mathrm{d}t} Y.
\end{aligned}$$

■

Note. From Proposition 3.4.1, $\mathrm{D}/\mathrm{d}t$ is what we want, and note how it depends on c.

Definition 3.4.1 (Covariant derivative)**.** The *covariant derivative* of V along c is the vector field $\mathrm{D}V/\mathrm{d}t$.

Problem 3.4.1. Why not just define $\mathrm{D}Y/\mathrm{d}t$ by (ii)?

Answer. A vector field Y along c may not always be extended to a neighborhood of c in $\mathcal{M}$. But, in local coordinates, Y is always a linear combination of vector fields along c since

$$Y(t) = \sum_{i=1}^n Y^i(t) \left(\frac{\partial}{\partial x^i} \right)_{c(t)},$$

i.e., it can be extended. ⊛

Proposition 3.4.1 shows that the choice of a linear connection on $\mathcal{M}$ leads to a bona fide (satisfying (a) and (b)) derivative of vector fields along curves.

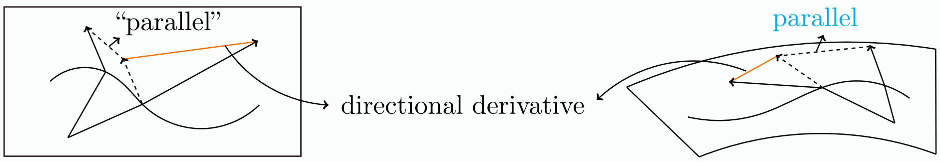
Remark. The notion of connection furnishes a manner of differentiating vectors along curves.

3.4.2 Parallel Transports

Finally, we introduce the notion of parallel.

Definition 3.4.2 (Parallel)**.** A vector field X on $\mathcal{M}$ along a curve c is *parallel* (or *parallelly transported*) along c if $\mathrm{D}X/\mathrm{d}t = 0$ for all $t \in I$.

Intuition. In the (flat) Euclidean space, we know what is "parallel," and hence we can define the directional derivative. But now the logic is reversed: we first define what is parallel in a curved space, and then we can make sense of directional derivative in a curved space!

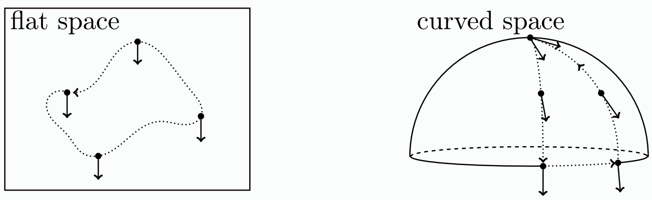


Given the definition of a parallel vector fields along curves, we can talk about parallel transport.

Definition 3.4.3 (Parallel transport)**.** The *parallel transport* from $c(0)$ to $c(t)$ along the curve c in a Riemannian manifold $(\mathcal{M}, g)$ is the linear map $P_i \colon T_{c(0)}\mathcal{M} \to T_{c(t)}\mathcal{M}$ associating $v \in T_{c(0)}\mathcal{M}$ with $X_v(i) \in T_{c(i)}\mathcal{M}$ with X_v being the parallel vector field along c such that $X_v(0) = v$.

It's clear that how we can extend Definition 3.4.3 for a piece-wise smooth curve.

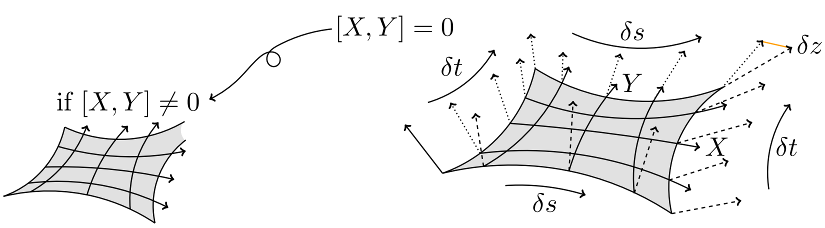
Intuition. When the space is flat, keeping the "arrow" (which defines a vector field) in one direction and moving around won't produce any changes, while when the space is curved, it will.



We make a surprising remark on the relation between Riemannian curvature and parallel transport.

Remark (Geometric significant of Riemannian curvature)**.** The idea is that for a manifold with torsion free ∇, if we parallel transporting along two paths on an infinitesimal patch (which induces X, Y) such that $[X, Y] = 0$, we can detect curvature in terms of δz, where[a]

$$(\delta z)^i = \cdots = R^i_{jk\ell} X^k Y^\ell Z^j \cdot \delta s \delta t + O(\delta s^2 \delta t, \delta s \delta t^2).$$



We will come back to this later.

[a]This is a deep theorem! In the ..., we use $T \equiv 0$.

Proposition 3.4.2. The parallel transport exists, uniquely.

Proof. do Carmo [FC13, Proposition 2.6] ∎

Proposition 3.4.3. Let $(\mathcal{M}, g)$ be a Riemannian manifold. The parallel transport defines for all t an isometry from $T_{c(0)}\mathcal{M}$ onto $T_{c(t)}\mathcal{M}$; more generally, if X, Y are vector fields along c, then

$$\frac{\mathrm{d}}{\mathrm{d}t} g(x(t), y(t)) = g\left(\frac{\mathrm{D}X(t)}{\mathrm{d}t}, Y(t)\right) + g\left(X(t), \frac{\mathrm{D}Y(t)}{\mathrm{d}t}\right).$$

Proof. See do Carmo [FC13, Proposition 3.2] ∎

3.4.3 Autoparallel Curves

Now we can formally introduce the notion of autoparallel.

Definition 3.4.4 (Autoparallel)**.** Let ∇ be a connection on $T\mathcal{M}$ of a differentiable manifold $\mathcal{M}$. A curve $c\colon I \to \mathcal{M}$ is called *autoparallel* (or *geodesic*) w.r.t. ∇ if

$$\nabla_{\dot{c}}\dot{c} = 0.$$

Intuition. An autoparallel curve is the *straightest line* (hence geodesic) in the space w.r.t. ∇!

Remark (Physical interpretation)**.** One can start from introducing ∇, considering $\nabla_{\dot{c}}\dot{c} \coloneqq 0$ (which is just Equation 2.1), and realize that we don't need to consider gravity as a force, rather a "curvature of spacetime," in order to make sense of Newton's first law, i.e., mass without forces will undergo a autoparallel curve.

Example (Euclidean plane)**.** Let $U = \mathbb{R}^2$, $x = \mathrm{id}_{\mathbb{R}^2}$, $\Gamma^i_{jk} = 0$, then $\ddot{c}^k = 0$ in Equation 3.4. Hence,

$$c^k(t) = a^k t + b^k \text{ for } a^k, b^k \in \mathbb{R}^d.$$

Example (Round sphere)**.** The geodesics on a "round sphere" are the great circles.

Proof. Consider a "unit round sphere" $\mathcal{M} = S^2$ with spherical coordinates $x(p) = (r, \theta, \varphi)$ such that $r = 1$, $\theta \in (0, \pi)$, and $\varphi \in 0, 2\pi$. The "roundness" is given by ∇_{round} where we specify (at one point)

$$\Gamma^1_{22} \coloneqq -\sin\theta\cos\theta, \quad \Gamma^2_{21} = \Gamma^2_{12} \coloneqq \cot\theta,$$

where we let $x^1(p) = \theta(p), x^2(p) = \varphi(p)$. The autoparallel equation tells us

$$\begin{cases} \ddot{\theta} + \Gamma^1_{22}\dot{\varphi}\dot{\varphi} = 0; \\ \ddot{\varphi} + 2\Gamma^2_{12}\dot{\theta}\dot{\varphi} = 0; \end{cases} \Leftrightarrow \begin{cases} \ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\varphi}\dot{\varphi} = 0; \\ \ddot{\varphi} + 2\cot(\theta)\dot{\theta}\dot{\varphi} = 0. \end{cases}$$

Then, we see that $\theta(t) = \pi/2$, $\varphi(t) = \omega t + \varphi_0$ is a solution.[a] Hence, we conclude that if we run at a constant speed around the great circle of S^2, it'll be autoparallel, hence a geodesic. ⊛

[a] Note that $\theta(t) = \pi/2$, $\varphi(t) = \omega t^2 + \varphi_0$ is not a solution.

Similarly, given any ∇ on a space, we can find the straightest curve on which.

Lecture 12: Tangent and Cotangent Bundles

3.5 More on Tangent and Cotangent Bundles

14 Feb. 13:00

Let $f\colon \mathcal{M} \to \mathcal{N}$ be a differentiable map between two differentiable manifolds, until now, we have only talked about how to transform tangent vectors or 1-form via f. Implicitly, these are just pullback (f^*) and pushforward (f_*), as we now define formally.

Definition. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a smooth map between two smooth manifolds and $p \in \mathcal{M}$.

Definition 3.5.1 (Pushforward)**.** The *pushforward* is the linear map $f_* \coloneqq \mathrm{d}f_p\colon T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$.

Definition 3.5.2 (Pullback)**.** The *pullback* is the linear map $f^*\colon T^*_{f(p)}\mathcal{N} \to T^*_p\mathcal{M}$ where

$$(f^*\omega)(X) = \omega(f_* X)$$

for $\omega \in T^*_{f(p)}\mathcal{N}$ and $X \in T_p\mathcal{M}$.

In all, the following diagram commutes:

$$\begin{tikzcd}
T^*_p\mathcal{M} \arrow[d, "\pi"'] & T^*_p\mathcal{N} \arrow[l, "f^*"'] \arrow[d, "\pi"] \\
\mathcal{M} \arrow[r, "f"'] & \mathcal{N}
\end{tikzcd}
\qquad
\begin{tikzcd}
T_p\mathcal{M} \arrow[r, "f_*"] \arrow[d, "\pi"'] & T_p\mathcal{N} \arrow[d, "\pi"] \\
\mathcal{M} \arrow[r, "f"'] & \mathcal{N}
\end{tikzcd}$$

3.5.1 Pullbacks and Pushforwards on Bundles

Now, consider a vector bundle $(E, \pi, \mathcal{N})$ over $\mathcal{N}$, we want to use f to "pull back" the vector bundle, i.e., construct a vector bundle, denote as f^*E, for which the fiber over $x \in \mathcal{M}$ is $E_{f(x)}$.

Definition 3.5.3 (Pullback bnudle)**.** The *pullback bundle* f^*E is the vector bundle over $\mathcal{M}$ with the bundle charts $(\varphi \circ f, f^{-1}(U))$ if (φ, U) is the bundle charts of E.

Similarly, we can "push forward" a vector bundle $(E, \pi, \mathcal{M})$ over $\mathcal{M}$ via f in the same fashion.

Definition 3.5.4 (Pushforward bnudle)**.** The *pushforward bundle* f_*E is the vector bundle over $\mathcal{N}$ with the bundle charts $(\varphi \circ f^{-1}, f(U))$ if (φ, U) is the bundle charts of E.

Note. In Definition 3.5.4, it only makes sense if $\mathcal{M} \hookrightarrow \mathcal{N}$.

Definition 3.5.5 (Bundle homomorphism)**.** Consider 2 vector bundles $(E_1, \pi_1, \mathcal{M}), (E_2, \pi_2, \mathcal{M})$ over $\mathcal{M}$, and let the differentiable map $f\colon E_1 \to E_2$ be fiber preserving, i.e., $\pi_2 \circ f = \pi_1$. If the fiber maps $f_x\colon E_{1,x} \to E_{2,x}$ is linear,[a] then f is called a *bundle homomorphism*.

[a] I.e., vector homomorphisms.

Definition 3.5.6 (Subbundle)**.** Let $(E, \pi, \mathcal{M})$ of rank n be a vector bundle. Let $E^1 \subseteq E$, and assume that for all $x \in \mathcal{M}$, there exists a bundle chart (φ, U) for $x \in U$ and

$$\varphi(\pi^{-1}(U) \cap E^1) = U \times \mathbb{R}^m \subseteq U \times \mathbb{R}^n$$

for $m \leq n$. Then the *subbundle* of E of rank m is the vector bundle $(E^1, \pi|_{E^1}, \mathcal{M})$.

Example. Consider $f\colon \mathcal{M} \hookrightarrow \mathcal{N}$ where $g_{\mathcal{N}}$ is a metric on $\mathcal{N}$. Then, $g_{\mathcal{N}}$ induces a metric $g_{\mathcal{M}}$ on $\mathcal{M}$ by f since we can define

$$g_{\mathcal{M}}(X, Y) \coloneqq g_{\mathcal{N}}(f_*(X), f_*(Y)).$$

3.5.2 Pullbacks and Pushforwards of Vector Fields

Now, we consider to "pull back" or "push forward" a vector field, i.e., a section of a bundle.

Definition 3.5.7 (Pushforward). Let $\psi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism between smooth manifolds, and let X be a vector field on $\mathcal{M}$. Then the *pushforward* vector field $Y = \psi_* X = \mathrm{d}\psi X$ on $\mathcal{N}$ is

$$Y(p) = \mathrm{d}\psi(X(\psi^{-1}(p))).$$

Definition 3.5.8 (Pullback). Let $\psi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism between smooth manifolds, and let Y be a vector field on $\mathcal{N}$. Then the *pullback* vector field $X = \psi^* Y$ on $\mathcal{M}$ is just $X(p) = Y_{\psi(p)}$.

Note. We let ψ be a diffeomorphism just for convenient.

Lemma 3.5.1. For every differentiable function $f\colon \mathcal{N} \to \mathbb{R}$, $(\psi_* X)(f)(p) = X(f \circ \psi)(\psi^{-1} p)$.

Lemma 3.5.2. Let X be a vector field on $\mathcal{M}$ and $\psi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. If the local 1-parameter group $(\varphi_t)_{t\in I}$ generated by X, then the local 1-parameter group generated by $\psi_* X$ is $\psi \circ \varphi_t \circ \psi^{-1}$.

3.5.3 Induced Bundle Metrics

Let $(\mathcal{M}, g)$ be a Riemannian manifold, then g induces the *bundle metrics* on all vector bundles over $\mathcal{M}$: for $T^*\mathcal{M}$, it is given by

$$g(\omega, \eta) \coloneqq g^{ij} \omega_i \eta_i$$

for $\omega = \omega_i \mathrm{d}x^i, \eta = \eta_i \mathrm{d}x^i$. Hence, we can talk about the identification between $T\mathcal{M}$ and $T^*\mathcal{M}$ through g:

$$\begin{gathered} V = V^i \tfrac{\partial}{\partial x^i} \in T\mathcal{M} \\ \Updownarrow \\ \omega = \omega_j \mathrm{d}x^j \in T^*\mathcal{M} \end{gathered}$$

with $\omega_j = g_{ij} V^i$ (or $V^i = g^{ij}\omega_j$) such that

(a) $g(X, Y) = g_{ij} X^i Y^j$ for $X, Y \in T\mathcal{M}$;

(b) $g(\omega, \eta) = g^{ij}\omega_i \eta_j$ for $\omega, \eta \in T^*\mathcal{M}$.

Thus, for $V \in T_x\mathcal{M}$, there corresponds a 1-form $\omega \in T_x^*\mathcal{M}$ via the metric $\omega(Y) \coloneqq g(V, Y)$ for all Y, and we further have $\|\omega\| = \|V\|$.

We can also consider the coordinate transformation behavior. Let $(e_i)_{i=1,\dots,d}$ be a basis of $T_x\mathcal{M}$ and $(\omega^j)_{j=1,\dots,d}$ the dual basis of $T_x^*\mathcal{M}$, i.e., $w^j(e_i) = \delta_i^j$. Given $V = V^i e_i \in T_x\mathcal{M}$, $\eta = \eta_j \omega^j \in T_x^*\mathcal{M}$, we then have $\eta(V) = \eta_i V^i$. Now, consider bases $(e_i), (\omega^j)$ in the local coordinates, i.e., $e_i = \partial/\partial x^i$ and $\omega^j = \mathrm{d}x^j$. Let f be a local coordinates change, then V and η transformed as

$$f_*(V) \coloneqq V^i \frac{\partial f^\alpha}{\partial x^i} \frac{\partial}{\partial f^\alpha}, \qquad f^*(\eta) \coloneqq \eta_j \frac{\partial x^j}{\partial f^\beta} \mathrm{d}f^\beta$$

correspondingly, and we see that

$$f^*(\eta)(f_*(V)) = \eta_j \frac{\partial x^j}{\partial f^\alpha} V^i \frac{\partial f^\alpha}{\partial x^i} = \eta_i V^i = \eta(V).$$

Intuition. The above means that

- the tangent vectors transform with the functional matrix of coordinates change;
- the cotangent vectors transform with the transposed inverse of the above matrix.

To compute the coordinates change $y \mapsto x(y)$ for $\omega = \omega_i \mathrm{d}x^i$, $\eta = \eta_i \mathrm{d}x^i$ with $\langle \omega, \eta \rangle = g^{ij}\omega_i\eta_j$, we have

$$\omega_i \mathrm{d}x^i = \omega_i \frac{\partial x^i}{\partial y^\alpha} \mathrm{d}y^\alpha =: \widetilde{w}_\alpha \mathrm{d}y^\alpha.$$

As previously seen. g^{ij} is transformed as

$$h^{\alpha\beta} = g^{ij} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}.$$

Then, we see that $h^{\alpha\beta}\widetilde{w}_\alpha \widetilde{\eta}_\beta = g^{ij}\omega_i\eta_j$ and $\|\omega(x)\| = \sup\{\omega(x)(V) \mid V \in T_x\mathcal{M}, \|v\| = 1\}$.

Remark. If we consider $T\mathcal{M} \otimes T\mathcal{M}$, then metric is

$$\langle V \otimes Y, \xi \otimes \eta \rangle = g_{ij} V^i Y^i g_{k\ell} \omega^k \eta^\ell.$$

As previously seen (Lie derivative)**.** Consider a vector field X with a local 1-parameter group $(\psi_t)_{t\in I}$ and a tensor field S on $\mathcal{M}$. The Lie derivative of S in the direction of X is defined as

$$\mathcal{L}_X S \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^* S)\Big|_{t=0}.$$

Lecture 13: Sectional Curvatures and Space Forms

Let $X = X^i \partial/\partial x^i$ be a vector field. Then consider $(\psi_t)_* X(\psi_t(X))$ to get a curve X_t in $T_x\mathcal{M}$ for $t \in I$. 16 Feb. 13:00
By differentiate that curve, i.e.,

$$(\psi_t)_* \frac{\partial}{\partial x^i}(\psi_t(x)) = \frac{\partial \psi_t^k}{\partial x^i} \frac{\partial}{\partial x^k}.$$

Note. For $\varphi\colon \mathcal{M} \to \mathcal{N} \coloneqq \mathcal{M}$ and X and $\varphi(x)$ are in the same coordinate neighborhood,

$$\varphi_* \frac{\partial}{\partial x^i} = \frac{\partial \varphi^k}{\partial x^i} \frac{\partial}{\partial \varphi^k}$$

since $\frac{\partial}{\partial \varphi^k} = \frac{\partial}{\partial x^k}$.

On the other hand, let $\omega = \omega_i \mathrm{d}x^i$ be a 1-form, then we have

$$(\psi_t^*)(\omega)(x) = \omega_i(\psi_t(x)) \frac{\partial \psi_t^i}{\partial x^k} \mathrm{d}x^k,$$

which is a curve in $T_x^*\mathcal{M}$.

Note. For $\varphi\colon \mathcal{M} \to \mathcal{N}$ (φ need not be a diffeomorphism) with for the 1-form $\omega = \omega_i \mathrm{d}x^i$ on $\mathcal{N}$,

$$\varphi^*\omega = \omega_i(\varphi(x)) \frac{\partial z^i}{\partial x^k} \mathrm{d}x^k.$$

Let $\varphi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism, Y be a vector field on $\mathcal{N}$. Then, set

$$\varphi^* Y \coloneqq (\varphi^{-1})_* Y,$$

and for other contravariant tensors, φ^* can be defined in an analogous way.

Example. For a vector field X and a local 1-parameter group $(\psi_t)_{t\in I}$, it is $(\psi_t^* X) = (\psi_t)_* X$.

3.6 Sectional Curvatures

Beyond Riemannian curvature and other "averaging" variations of which, the following one is in particular interesting and is the one considered by Riemann.

Definition 3.6.1 (Sectional curvature)**.** The *sectional curvature* of the plane Σ spanned by the (linearly independent) tangent vectors $X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i} \in T_x\mathcal{M}$ of a Riemannian manifold $(\mathcal{M}, g)$ is

$$K(\sigma) \coloneqq K(X \wedge Y) = \frac{g(R(X,Y)Y, X)}{|X \wedge Y|^2}$$

where $|X \wedge Y|^2 = g(X,X)g(Y,Y) - g(X,Y)^2$.[a]

[a]Given a vector space V and $x, y \in V$, $|x \wedge y| \coloneqq \sqrt{|x|^2|y|^2 - \langle x, y\rangle^2}$ represents the area of the two-dimensional parallelogram spanned by x, y.

Note. Definition 3.6.1 is well-defined since $K(\sigma)$ is invariant under different bases of σ.

Remark. Sectional curvature determines the whole Riemannian curvature.

Proof. Given $g(R(X,Y)Z,W)$, we can express this entirely by K [FC13, Lemma 3.3]. ⊛

Remark (Gauss curvature)**.** For $\dim \mathcal{M} = 2$, $R_{ijk\ell} = K(g_{ik}g_{j\ell} - g_{ij}g_{k\ell})$ since $T_x\mathcal{M}$ contains only one plane, i.e., $T_x\mathcal{M}$ itself. In this case, K is called the *Gauss curvature*.

In particular, the space form considers the space with constant sectional curvature.

Definition 3.6.2 (Space form)**.** A Riemannian manifold $(\mathcal{M}, g)$ is a *space form* if $K(X \wedge Y)$ is a constant for all linearly independent tangent vectors $X, Y \in T_p\mathcal{M}$ for all $p \in \mathcal{M}$.

Definition 3.6.3 (Spherical)**.** A space form is called *spherical* if $K > 0$.

Definition 3.6.4 (Flat)**.** A space form is called *flat* if $K = 0$.

Definition 3.6.5 (Hyperbolic)**.** A space form is called *hyperbolic* if $K > 0$.

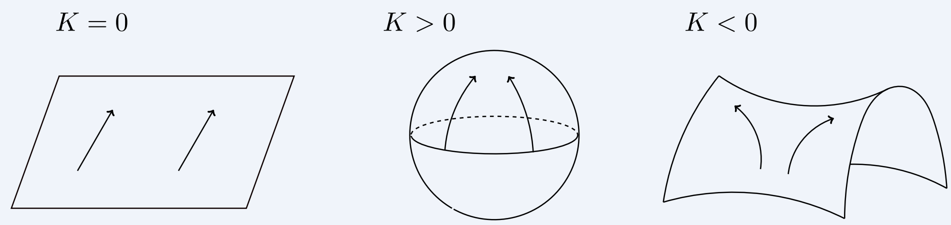
Generalize Definition 3.6.2 a bit, we have the so-called Einstein manifolds.

Definition 3.6.6 (Einstein manifold)**.** A Riemannian manifold $(\mathcal{M}, g)$ is called an *Einstein manifold* if $R_{ik} = cg_{ik}$ for a constant c.[a]

[a]Which does not depend on the choice of local coordinates.

Remark. Every space form is an Einstein manifold.

Example. $\mathbb{R}^n$ is flat, S^n is spherical, and $\mathbb{H}^n$ is hyperbolic. And all are Einstein manifolds.



Definition 3.6.7 (Flat). A connection ∇ on $T\mathcal{M}$ is *flat* if each point in $\mathcal{M}$ has a neighborhood U with local coordinates for which all the coordinate vector fields $\partial/\partial x^i$ are parallel, i.e., $\nabla \partial/\partial x^i = 0$.

Theorem 3.6.1. A connection ∇ on $T\mathcal{M}$ is flat if and only if its curvature and torsion vanish identically.

Proof. Flat connection implies $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$, hence all $\Gamma_{ij}^k = 0$, so T, R vanish. Conversely, find the local coordinates such that $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ for all i, j and use Frobenius theorem. ∎

Example. The following are flat manifolds with their usual shape, i.e., connections.

- $\mathbb{R}^n$.
- Torus T^2.
- Every 1-dimensional Riemannian manifold.
- Products of flat manifolds.
- Tori.

Theorem 3.6.2 (Schur theorem). Let $(\mathcal{M}, g)$ be a Riemannian manifold with $\dim \mathcal{M} \geq 3$.

(a) If the sectional curvature of $\mathcal{M}$ is constant at each point, i.e., $K(X \wedge Y) = f(x)$ for $X, Y \in T_x\mathcal{M}$, then $f(x)$ is a constant on $\mathcal{M}$, hence $\mathcal{M}$ is a space form.

(b) If the Ricci curvature is a constant at each point, i.e., $R_{ik} = c(x) g_{ik}$, then $c(x)$ is a constant, hence $\mathcal{M}$ is an Einstein manifold.

Remark. Schur theorem says that the isotropy[a] of a Riemannian manifold implies the homogeneity.[b] Hence, a point-wise property implies a global one!

[a] I.e., the property that at each point, all directions are geometrically indistinguishable.
[b] I.e., all points are geometrically indistinguishable.

3.7 More on Covariant Derivatives

To end this chapter, we revisit covariant derivative. But this time, we generalize it from vector field to tensor field, i.e., we will show that it's also possible to covariantly differentiate tensors. The motivation is that given a 1-form ω, and vector fields X, Y, we have

$$X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y),$$

and for arbitrary tensors S, T, we similarly have

$$\nabla_X (S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T.$$

Consider the following.[7]

Definition 3.7.1 (Covariant diffential). Let T be a $(0, s)$-tensor. The *covariant differential* ∇T of T is a $(0, s+1)$-tensor given by

$$\nabla T(Y_1, \ldots, Y_s, Z) = Z(T(Y_1, \ldots, Y_s)) - T(\nabla_Z Y_1, \ldots, Y_s) - \cdots - T(Y_1, \ldots, Y_{s-1}, \nabla_Z Y_s).$$

Definition 3.7.2 (Covariant derivative). For each $Z \in \Gamma(T\mathcal{M})$, the *covariant derivative* $\nabla_Z T$ of T relative to Z is a $(0, s)$-tensor given by

$$\nabla_Z T(Y_1, \ldots, Y_s) = \nabla T(Y_1, \ldots, Y_s, Z).$$

We primarily focus on covariant tensor, however, we also have the following.

[7] Definition 3.7.2 is natural by considering a certain *frame* [FC13, §4.5].

Remark. For T a (p, q)-tensor,

$$\begin{aligned}(\nabla_Y T)(\alpha_1, \ldots, \alpha_q, X_1, \ldots, X_p) = Y(T(\alpha_1, \ldots, \alpha_q, X_1, \ldots, X_p)) \\ - \sum_{i=1}^{q} T(\alpha_1, \ldots, \nabla_Y \alpha_i, \ldots, \alpha_q, X_1, \ldots, X_p) \\ - \sum_{i=1}^{p} T(\alpha_1, \ldots, \alpha_q, X_1, \ldots, \nabla_Y X_i, \ldots, X_p).\end{aligned}$$

Example. Consider the metric tensor $g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$, then $\nabla_X g = 0$ for all vector fields X.

Proof. For all $X, Y, Z \in \Gamma(T\mathcal{M})$,

$$\nabla g(X, Y, Z) = Z\langle X, Y\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle = 0$$

since ∇ is Riemannian. ⊛

It's convenient to use the following identification.

Notation. Let $X \in \Gamma(T\mathcal{M})$ and identify X with the tensor that associates to $Y \in \Gamma(T\mathcal{M})$ the function $\langle X, Y\rangle$.

Intuition. Consider the covariant derivative of the tensor X relative to $Z \in \Gamma(T\mathcal{M})$, which is such that for all $Y \in \Gamma(T\mathcal{M})$,

$$\nabla_Z X(Y) = \nabla X(Y, Z) = Z(X(Y)) - X(\nabla_Z Y) = Z\langle X, Y\rangle - \langle X, \nabla_Z Y\rangle = \langle \nabla_Z X, Y\rangle.$$

This shows that the tensor $\nabla_Z X$ can be identified with the vector field $\nabla_Z X$ as well by our new notation!

Remark. This justifies the notation adopted, and shows that the Definition 3.7.2 is a generalization of Definition 3.4.1.

Chapter 4

Isometric Immersions

Consider $f\colon \mathcal{M} \to \widetilde{\mathcal{M}}$ be a differentiable immersion of a manifold $\mathcal{M}^n$into a Riemannian manifold $\widetilde{\mathcal{M}}^k$ for $k = n + m$. The Riemannian metric of $\widetilde{\mathcal{M}}$ induces, naturally, a Riemannian metric on $\mathcal{M}$: if $v_1, v_2 \in T_p\mathcal{M}$, we let

$$\langle v_1, v_2 \rangle \coloneqq \langle \mathrm{d}f_p(v_1), \mathrm{d}f_p(v_2) \rangle.$$

This makes f an isometric immersion of $\mathcal{M}$ into $\widetilde{\mathcal{M}}$, and we want to study the relationship between the geometry of $\mathcal{M}$ and that of $\widetilde{\mathcal{M}}$.

While do Carmo [FC13] directly discusses the second fundamental form, we start by introducing the Riemannian covering map, which has a strong connection to the second fundamental form and furnishes a broader view of the theory of isometric immersions.

4.1 Riemannian Covering Maps

Let's first review the basic notion in algebraic topology.

Definition 4.1.1 (Covering map)**.** Let $\mathcal{M}, \widetilde{\mathcal{M}}$ be two manifolds. A map $p\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ is a *covering map* if

(a) p is smooth and surjective;

(b) for all $m \in \mathcal{M}$, there exists a neighborhood U at m in $\mathcal{M}$ with $p^{-1}(U) = \coprod_{i\in I} U_i$ with $p\colon U_i \to U$ being a diffeomorphism and U_i are disjoint open subsets of $\widetilde{\mathcal{M}}$.

Notation (Covering space)**.** $\widetilde{\mathcal{M}}$ in Definition 4.1.1 is called the *covering space*.

Notation (Universal covering space)**.** A covering space is *universal* if it's simply connected.

By introducing local isometry, we have the so-called Riemannian covering map.

Definition 4.1.2 (Riemannian covering map)**.** Let $(\mathcal{M}, g), (\mathcal{N}, h)$ be Riemannian manifolds. A map $p\colon \mathcal{N} \to \mathcal{M}$ is a *Riemannian covering map* if p is a smooth covering map and is a local isometry.

4.1.1 Induced Riemannian Covering Maps

Given a covering map, from a Riemannian metric g on the covering space, we obtain a induced Riemannian metric on the base space and a Riemannian covering map.

Proposition 4.1.1. Let $p\colon \mathcal{N} \to \mathcal{M}$ be a smooth covering map. For every Riemannian metric g on $\mathcal{M}$, there exists a unique Riemannian metric h on $\mathcal{N}$ such that p is a Riemannian covering map.

Note. The converse of Proposition 4.1.1 is generally not true.

Let's first see some examples.

Example. Every space covers itself trivially.

Example. $\mathbb{R}$ is the universal covering space of S^1.

Example. $U(n)$ has universal covers $U(n) \times \mathbb{R}$.

Example. S^n is a double cover for $\mathbb{R}P^n$ and is universal for $n > 1$.

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Proposition 4.1.2. Let $(\mathcal{N}, h)$ be a Riemannian manifold and G be a free and proper group of isometries of $(\mathcal{N}, h)$. Then, there exists a unique Riemannian metric g on the quotient manifold $\mathcal{M} = \mathcal{N} \,/\, G$ such that the connected projection $p\colon \mathcal{N} \to \mathcal{M}$ is a Riemannian covering map.

Proof. Let $n, n' \in \mathcal{N}$ such that $n, n' \in p^{-1}(m)$ for $m \in \mathcal{M}$. Hence, there exists an isometry $f \in G$ such that $f(n) = n'$. Also, $p \circ f = p$, and p is a local diffeomorphism, so we can define a scalar product g_m on $T_m\mathcal{M}$: for all $u, v \in Tvm\mathcal{M}$,

$$g_m(u, v) = h_n\big((T_n p)^{-1} u, (T_n p)^{-1} v\big)$$

for $n \in p^{-1}(m)$. This does not depend on the choice of $n \in p^{-1}(m)$ since $(T_n p)^{-1} = T_n f \circ (T_n p)^{-1}$ and $T_n f$ is an isometry of the Euclidean vector spaces $T_n\mathcal{N}$ and $T_{n'}\mathcal{N}$. It can be shown that g is smooth. Thus, we have constructed a metric g on $\mathcal{M}$ such that p is a Riemannian covering map, which is unique. ∎

4.1.2 Totally Geodesic

A particular interesting condition is the following.

Definition 4.1.3 (Totally geodesic)**.** A submanifold $\mathcal{M}$ of $(\widetilde{\mathcal{M}}, \widetilde{g})$ is called *totally geodesic* if for all $m \in \mathcal{M}$ and $v \in T_m\mathcal{M}$, the geodesic c of $(\widetilde{M}, g)$ with $c(0) = m$ and $c'(0) = v$ is contained fully in $\mathcal{M}$.

Proposition 4.1.3. Let $p\colon (\mathcal{N}, h) \to (\mathcal{M}, g)$ be a Riemannian covering map. The geodesic of $(\mathcal{M}, g)$ are the projections of the geodesic in $(\mathcal{N}, h)$; and the geodesic of $(\mathcal{N}, h)$ are the liftings of those in $(\mathcal{M}, g)$.

Proof. Since p is a local isometry, if γ is a geodesic of $\mathcal{N}$, then $c = p \circ \gamma$ is also a geodesic of $\mathcal{M}$. From the uniqueness theorem for geodesics shows that these are indeed the only geodesics on $\mathcal{M}$. Conversely, if $p \circ \gamma$ is a geodesic in $\mathcal{M}$, then γ is a geodesic in $\mathcal{N}$. ∎

Example. In Euclidean spaces, the totally geodesic submanifold are affine linear subspaces and their open subsets.

Example. Each closed geodesic in Riemannian manifolds defines a 1-dimensional compact totally geodesic submanifold.

Example. The totally geodesic submanifolds of $S^n \subseteq \mathbb{R}^{n+1}$ are the intersections of S^n with linear subspaces of $\mathbb{R}^{n+1}$.

Example. In general, Riemannian manifolds do not have any totally geodesic submanifolds of dimensional > 1.

Remark. We will see that $\mathcal{M}$ is totally geodesic in $\widetilde{M}$ if and only if all the second fundamental forms vanish identically.

4.2 The Second Fundamental Form

Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ be an immersion between two Riemannian manifolds. We already know that a metric on $\mathcal{N}$ induces a metric on $\mathcal{M}$ naturally by the immersion (inclusion). We now ask: given the Levi-Civita connection $\nabla^{\mathcal{N}}$ of $\mathcal{N}$, how to get $\nabla^{\mathcal{M}}$ of $\mathcal{M}$?

Note. In the following discussion, we consider $\mathcal{M}^m \subseteq \mathcal{N}^n$, i.e., we simply consider the case of inclusion. However, everything works out nicely by identifying $\mathcal{M}$ with the image of $f(\mathcal{M})$ in $\mathcal{N}$.

4.2.1 The Immersion-Induced Levi-Civita Connection

This immersion-induced Levi-Civita connection is given by the central object $(\nabla^{\mathcal{N}}_X Y)^{\top}$, where $\top\colon T_x\mathcal{N} \to T_x\mathcal{M}$ for $x \in \mathcal{M}$ is the orthogonal projection.[1] The formal guarantee is given by Theorem 4.2.1.

Theorem 4.2.1. For $X, Y \in \Gamma(T\mathcal{M})$, $\nabla^{\mathcal{M}}_X Y = \left(\nabla^{\mathcal{N}}_X Y\right)^{\top}$.

Proof. Firstly, we have to make sure that the right-hand side is defined. This can be done by extending vector fields X, Y locally to a neighborhood of $\mathcal{M}$ in $\mathcal{N}$. We do this in the local coordinates around $x \in \mathcal{M}$ locally mapping $\mathcal{M}$ to $\mathbb{R}^m \subseteq \mathbb{R}^n$. Specifically, the extension of $X = \xi^i(x)\partial/\partial x^i$ is

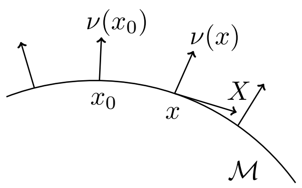
$$\widetilde{X}(x^1, \ldots, x^n) = \sum_{i=1}^{m} \xi^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}.$$

Then $\langle \widetilde{X}, \widetilde{Y}\rangle(x) = \langle X, Y\rangle(x)$ and $[\widetilde{X}, \widetilde{Y}](x) = [X, Y](x)$. From Levi-Civita theorem, the Koszul formula holds for both $\mathcal{N}$ and $\mathcal{M}$, hence

- $(\nabla^{\mathcal{N}}_X Y)^{\top}$ does not depend on the choice of extensions: follows from the fact that the representation of $\nabla^{\mathcal{N}}$ is done by Γ;
- $(\nabla^{\mathcal{N}}_X Y)^{\top}$ defines a torsion-free connection on $\mathcal{M}$: as $\nabla^{\mathcal{N}}_X Y - \nabla^{\mathcal{N}}_Y X - [X, Y]$ vanishes, also the tangential (to $\mathcal{M}$) part has to vanish.

■

Let $\nu(x)$ be a vector field in a neighborhood of $x_0 \in \mathcal{M} \subseteq \mathcal{N}$ that is orthogonal to $\mathcal{M}$, i.e., $\langle \nu(x), X\rangle = 0$ for all $X \in T_x\mathcal{M}$.



[1] We note this one last time: this makes sense since we can identify $T_x\mathcal{M} \subseteq T_x\mathcal{N}$ by the immersion $f\colon \mathcal{M} \to \mathcal{N}$.

Notation. Let $T_x\mathcal{M}^\perp$ be the orthogonal complement of $T_x\mathcal{M}$ in $T_x\mathcal{N}$.

With this notation, we see that $\langle \nu(x), X\rangle = 0$ for all $X \in T_x\mathcal{M}$ means $\nu(x) \in T_x\mathcal{M}^\perp$.

Notation (Normal bundle)**.** The *normal bundle* $T\mathcal{M}^\perp$ of $\mathcal{M}$ in $\mathcal{N}$ is the bundle with fiber $T_x\mathcal{M}^\perp$ of $x \in \mathcal{M}$.

Lemma 4.2.1. $(\nabla^{\mathcal{N}}_X \nu)^\top(x)$ only depends on $\nu(x)$.

Proof. For a real-valued function f on a neighborhood of x, we have

$$(\nabla^{\mathcal{N}}_X f\nu)^\top(x) = \big(X(f)(x)\nu(x)\big)^\top + f(x)(\nabla^{\mathcal{N}}_X \nu)^\top(x) = f(x)(\nabla^{\mathcal{N}}_X \nu)^\top(x)$$

as $\big(X(f)(x)\nu(x)\big)^\top = 0$ with $\nu(x) \in T_x\mathcal{M}^\top$. ∎

4.2.2 The Second Fundamental Form

With the notations we have developed, we define the following.

Definition 4.2.1 (Second fundamental tensor)**.** The *second fundamental tensor* $S\colon T_x\mathcal{M} \times T_x\mathcal{M}^\perp \to T_x\mathcal{M}$ of $\mathcal{M}$ at point $x \in \mathcal{M}$ is defined by

$$S(X, \nu) = (\nabla^{\mathcal{N}}_X \nu)^\top.$$

Lemma 4.2.2. For $X, Y \in T_x\mathcal{M}$, $\ell_\nu(X, Y) \coloneqq \langle S(X,\nu), Y\rangle$ is symmetric in X, Y.

Proof. We see that

$$\begin{aligned}
\ell_\nu(X,Y) &= \langle (\nabla^{\mathcal{N}}_X \nu)^\top, Y\rangle \\
&= \langle \nabla^{\mathcal{N}}_X \nu, Y\rangle && (Y \in T_x\mathcal{M}) \\
&= -\langle \nu, \nabla^{\mathcal{N}}_X Y\rangle && (\nabla^{\mathcal{N}} \text{ is metric and } \langle \nu, Y\rangle = 0) \\
&= -\langle \nu, \nabla^{\mathcal{N}}_Y X + [X,Y]\rangle && (\nabla^{\mathcal{N}} \text{ is torsion-free}) \\
&= -\langle \nu, \nabla^{\mathcal{N}}_Y X\rangle - \langle \nu, [X,Y]\rangle \\
&= -\langle \nu, \nabla^{\mathcal{N}}_Y X\rangle && (\nu \in T_x\mathcal{M}^\perp, [X,Y] \in T_x\mathcal{M}) \\
&= \langle \nabla^{\mathcal{N}}_Y \nu, X\rangle && (\nabla^{\mathcal{N}} \text{ is metric}) \\
&= \langle (\nabla^{\mathcal{N}}_Y \nu)^\top, X\rangle && (X \in T_x\mathcal{M}) \\
&= \ell_\nu(Y, X).
\end{aligned}$$

∎

Definition 4.2.2 (Second fundamental form)**.** The *second fundamental form* $\ell_\nu(\cdot,\cdot)$ of $\mathcal{M}$ w.r.t. $\mathcal{N}$ is defined as $\ell_\nu(X,Y) \coloneqq \langle S(X,\nu), Y\rangle$.

Note. do Carmo [FC13] defines the second fundamental form as $\ell_\nu(X, X)$.

Note (First fundamental form)**.** The *first fundamental form* is the metric applied to $X, Y \in T_x\mathcal{M}$, i.e., $\langle X, Y\rangle$.

Now, fix a normal field ν, and let $S_\nu(X) \coloneqq S(X,\nu)$, then $S_\nu\colon T_x\mathcal{M} \to T_x\mathcal{M}$ is self-adjoint w.r.t. the metric $\langle\cdot,\cdot\rangle$ by Lemma 4.2.2.

4.2.3 Curvatures and Second Fundamental Forms

Due to the time, we can only talk about the definition of the following. For a detailed discussion, see [FC13, §6 Example 2.4 – Example 2.8].

Definition. Assume that $\langle \nu, \nu \rangle \equiv 1$, i.e., ν is the unit normal field, then S_ν has m real eigenvalues.

Definition 4.2.3 (Principal curvature)**.** The eigenvalues are called *principal curvatures* of $\mathcal{M}$ in direction ν.

Definition 4.2.4 (Principal curvature vector)**.** The corresponding eigenvectors are called *principal curvature vectors* of $\mathcal{M}$ in direction ν.

Definition 4.2.5 (Mean curvature)**.** The *mean curvature* of $\mathcal{M}$ in direction ν is defined by

$$H_\nu \coloneqq \frac{1}{m} \operatorname{Tr} S_\nu.$$

Definition 4.2.6 (Gauss-Kronecker curvature)**.** The *Gauss-Kronecker curvature* of $\mathcal{M}$ in direction ν is defined by

$$K_\nu \coloneqq \det S_\nu.$$

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4.2.4 Totally Geodesic and Second Fundamental Form

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Let $\dim \mathcal{N} = m+1$, $\dim \mathcal{M} = m$, then for all $x \in \mathcal{M}$, there are exactly 2 normal vectors $\nu \in T_x\mathcal{M}^\perp$ with $\langle \nu, \nu \rangle \equiv 1$, i.e., $\nabla^{\mathcal{N}}_X \nu$ always tangential to $\mathcal{M}$. Now, we fix locally such a normal field and drop the subscript ν in the following discussion.

Note. If we choose an opposite normal field, then ℓ, S, and mean curvature will change their sign. However, for even m, the Gauss-Kronecker curvature does not depend on the choice of the direction of ν.

Intuition. $\nabla^{\mathcal{N}}_X \nu$ measures the "tilting velocity" with which ν is tilted relative to a fixed parallel vector field in $\mathcal{N}$, when on $\mathcal{M}$ in direction X.

Theorem 4.2.2. Given $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$, then $\mathcal{M}$ is totally geodesic in $\widetilde{\mathcal{M}}$ if and only if all second fundamental form of $\mathcal{M}$ vanish identically.

Proof. Let $c\colon I \to \mathcal{M}$ be a geodesic in $\mathcal{M}$, i.e., $\nabla^{\mathcal{M}}_{\dot{c}} \dot{c} = 0$. By Theorem 4.2.1, $\nabla^{\mathcal{M}}_{\dot{c}} \dot{c} = (\nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c})^\top = 0$, implying c is a geodesic in $\widetilde{\mathcal{M}}$ if and only if $(\nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c})^\top = 0$, i.e., $\langle \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c}, \nu \rangle = 0$ for all $\nu \in T\mathcal{M}^\perp$. Notice that

- $\langle \dot{c}, \nu \rangle = 0$, and hence
- $\dot{c}\langle \dot{c}, \nu \rangle = \langle \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c}, \nu \rangle + \langle \dot{c}, \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \nu \rangle = 0$.

In all, we have $0 = \langle \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c}, \nu \rangle = -\langle \dot{c}, \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \nu \rangle = -\ell_\nu(\dot{c}, \dot{c})$, proving the theorem. ■

Note. Theorem 4.2.2 also holds for Lorentzian manifolds $(\widetilde{\mathcal{M}}, \widetilde{g})$.

Example (Initial value problem for Einstein equations)**.** Given a $(\widetilde{\mathcal{M}}^4, \widetilde{g})$ a Lorentzian manifolds satisfying Einstein equations, and a $(\mathcal{M}^3, g)$ non-degenerate Riemannian manifold. If the second fundamental form of $\mathcal{M}^3$ in $\widetilde{\mathcal{M}}^4$ vanishes identically, then $\mathcal{M}^3$ is totally geodesic.[a]

[a] This is just a special case of Theorem 4.2.2; in general, it does not vanish.

Theorem 4.2.2 allows us to get what is probably the best geometric interpretation of sectional curvature. Let $\mathcal{M}$ be a Riemannian manifold and let $p \in \mathcal{M}$. Let $B \subseteq T_p\mathcal{M}$ be an open ball in $T_p\mathcal{M}$ on which $\exp_p$ is a diffeomorphism, and let $\sigma \subseteq T_p\mathcal{M}$ be a subspace of dimension 2. Then, $\exp_p(\sigma \cap B) = S$ is a submanifold of dimension 2 of $\mathcal{M}$ passing through p.

Intuition. S is the surface formed by "small" geodesics that start from p and are tangent to σ at p.

Note. By Theorem 4.2.2, S is geodesic at p, hence the second fundamental forms of the inclusion $\iota\colon S \subseteq \mathcal{M}$ vanish at p.

As a submanifold of $\mathcal{M}$, S has an induced Riemannian metric whose Gauss curvature at p will be denoted by K_S. It follows from the Gauss formula [FC13, §6 Theorem 2.5][2] that

$$K_S(p) = K(p, \sigma),$$

i.e., the sectional curvature $K(p, \sigma)$ is the Gauss curvature, at p, of a small surface formed by geodesics of $\mathcal{M}$ that start from p and are tangent to σ.

Remark. This was exactly the way in which Riemann defined sectional curvature.

4.3 The Fundamental Equations

Given an isometric immersion $f\colon \mathcal{M}^m \to \mathcal{N}^n$ with $n = m + k$, at each $p \in \mathcal{M}$, we have

$$T_p\mathcal{N} = T_p\mathcal{M} \oplus (T_p\mathcal{M})^\perp,$$

which varies differentiably with p.

Intuition. Locally, the portion of the tangent bundle $T\mathcal{N}$ which sits over $\mathcal{M}$ can be decomposed into the direct sum of the *tangent bundle* $T\mathcal{M}$ and the *normal bundle* $T\mathcal{M}^\perp$.

Everything about immersions occurs as if the geometry decomposes into two geometries: the geometry of the tangent bundle and the geometry of the normal bundle, and these geometries are related by the second fundamental form of the immersions.

Notation. Greek indices $(\alpha, \beta, \dots)$ occurring twice are summed from 1 to k for $X, Y, Z, W \in T_x\mathcal{M}$.

Theorem 4.3.1 (Gauss' equations)**.** Let $\mathcal{N}$ be a Riemannian manifold with $\dim \mathcal{N} = n$, and let $\mathcal{M} \subseteq \mathcal{N}$ be a submanifold with $\dim \mathcal{M} = m$. Let $k = n - m$, and $x \in \mathcal{M}$, $\nu_1, \dots, \nu_k$ be an orthonormal basis of $(T_x\mathcal{M})^\perp$, $S_\alpha := S_{\nu_\alpha}$, $\ell_\alpha := \ell_{\nu_\alpha}$, $\alpha = 1, \dots, k$. Then,

$$R^{\mathcal{M}}(X, Y)Z - \left(R^{\mathcal{N}}(X, Y)Z\right)^\top = \ell_\alpha(Y, Z)S_\alpha(X) - \ell_\alpha(X, Z)S_\alpha(Y).$$

Thus, we also have

$$\left\langle R^{\mathcal{M}}(X, Y)Z, W\right\rangle - \left\langle R^{\mathcal{N}}(X, Y)Z, W\right\rangle = \ell_\alpha(Y, Z)\ell_\alpha(X, W) - \ell_\alpha(X, Z)\ell_\alpha(Y, W).$$

Proof. We can extend X, Y, Z, W, ad $\nu, \dots, \nu_k$ to vector fields inn $T_{\mathcal{M}}$ and $T\mathcal{M}^\perp$, respectively. Let

[2] Which is just a special case of Gauss' equations.

ν_α be orthonormal, then

$$\nabla^{\mathcal{N}}_Y Z = (\nabla^{\mathcal{N}}_Y Z)^\top = (\nabla^{\mathcal{N}}_X Z)^\perp = \nabla^{\mathcal{M}}_Y Z + \langle \nu_\alpha, \nabla^{\mathcal{N}}_Y Z \rangle \nu_\alpha$$

as ν_α form orthonormal basis of $T\mathcal{M}^\perp$. Hence,

$$\nabla^{\mathcal{N}}_X \nabla^{\mathcal{N}}_Y Z = \nabla^{\mathcal{N}}_X \nabla^{\mathcal{M}}_Y Z + X(\langle \nu_\alpha, \nabla^{\mathcal{N}}_Y Z \rangle)\nu_\alpha + \langle \nu_\alpha, \nabla^{\mathcal{N}}_Y Z \rangle \nabla^{\mathcal{N}}_X \nu_\alpha.$$

Then,

$$(\nabla^{\mathcal{N}}_X \nabla^{\mathcal{N}}_Y Z)^\top = \nabla^{\mathcal{M}}_X \nabla^{\mathcal{M}}_Y Z + \underbrace{\langle \nu_\alpha, \nabla^{\mathcal{N}}_Y Z \rangle}_{-\ell_\alpha(Y,Z)} \underbrace{(\nabla^{\mathcal{N}}_X \nu_\alpha)^\top}_{S_\alpha(X)} = \nabla^{\mathcal{M}}_X \nabla^{\mathcal{M}}_Y Z - \ell_\alpha(Y,Z) S_\alpha(X).$$

Analogously, we have

$$(\nabla^{\mathcal{N}}_Y \nabla^{\mathcal{N}}_X Z)^\top = \nabla^{\mathcal{M}}_Y \nabla^{\mathcal{M}}_X Z - \ell_\alpha(X,Z) S_\alpha(Y),$$

and also, we have

$$(\nabla^{\mathcal{N}}_{[X,Y]} Z)^\top = \nabla^{\mathcal{M}}_{[X,Y]} Z.$$

By collecting terms, we have

$$\begin{aligned} &(\nabla^{\mathcal{N}}_X \nabla^{\mathcal{N}}_Y Z)^\top - (\nabla^{\mathcal{N}}_Y \nabla^{\mathcal{N}}_X Z)^\top - (\nabla^{\mathcal{N}}_{[X,Y]} Z)^\top \\ &\qquad = \nabla^{\mathcal{M}}_X \nabla^{\mathcal{M}}_Y Z - \nabla^{\mathcal{M}}_Y \nabla^{\mathcal{M}}_X Z - \nabla^{\mathcal{M}}_{[X,Y]} Z - \ell_\alpha(Y,Z) S_\alpha(X) + \ell_\alpha(X,Z) S_\alpha(Y), \end{aligned}$$

equivalently,

$$R^{\mathcal{M}}(X,Y)Z - (R^{\mathcal{N}}(X,Y)Z)^\top = \ell_\alpha(Y,Z) S_\alpha(X) - \ell_\alpha(X,Z) S_\alpha(Y).$$

■

Theorem 4.3.1 tells us that for a surface $\mathcal{M}$ in $\mathbb{R}^3$, the Gauss-Kronecker curvature coincides with the Riemannian curvature of $\mathcal{M}$, which is independent of the embedding. Therefore, Gauss-Kronecker curvature does not depend on embeddings of $\mathcal{M}$ into $\mathbb{R}^3$.

Remark (Codazzi equations). Let $\mathcal{M}^m \subseteq \mathcal{N}^{m+1}$ where N is unit normal on $\mathcal{M}$. Then, the *Codazzi equations* is defined as

$$\langle R(X,Y)e_j, N \rangle = (\nabla^{\mathcal{M}}_X \ell)(Y, e_j) - (\nabla^{\mathcal{M}}_Y \ell)(X, e_j) = X^k Y^i \nabla^{\mathcal{M}}_k \ell_{ij} - Y^k X^i \nabla^{\mathcal{M}}_k \ell_{ij}, \tag{4.1}$$

i.e., $\langle R(X,Y)Z, N \rangle = (\nabla^{\mathcal{M}}_X \ell)(Y,Z) - (\nabla^{\mathcal{M}}_Y \ell)(X,Z)$.

The Codazzi equations, together with Gauss' equations, form the fundamental equations of the local theory of isometric immersions.

Chapter 5

Jacobi Fields

Lecture 16: Jacobi Field

In this chapter, we derive a first relation between the two basic concepts introduced, i.e., geodesics and curvatures. This is done by introducing Jacobi field: vector fields along geodesics, defined by means of differential equations naturally from exponential map. Moreover, Jacobi fields allow us to obtain a simple characterization of the singularities of the exponential map. 7 Mar. 13:00

Intuition. The upshot is, as we will see, the curvature $K(p, \sigma)$, $\sigma \subseteq T_p\mathcal{M}$, determines how fast the geodesics, that start from p and are tangent to σ, spread apart. This is described by Jacobi field.

5.1 Jacobi Fields

As mentioned, we want to consider neighboring geodesics under a vector field along which, and study how do they move. Their behaviors are essentially governed by curvature.

Definition 5.1.1 (Jacobi field)**.** Let $\mathcal{M}$ be a d-dimensional Riemannian manifold. Let $c\colon I \to \mathcal{M}$ be a geodesic. A vector field X along c is called a *Jacobi field* if it satisfies the *Jacobi equation*

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X + R(X, \dot{c})\dot{c} = 0. \tag{5.1}$$

Notation. We write $\dot{X} \coloneqq \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X$ and $\ddot{X} \coloneqq \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X$.

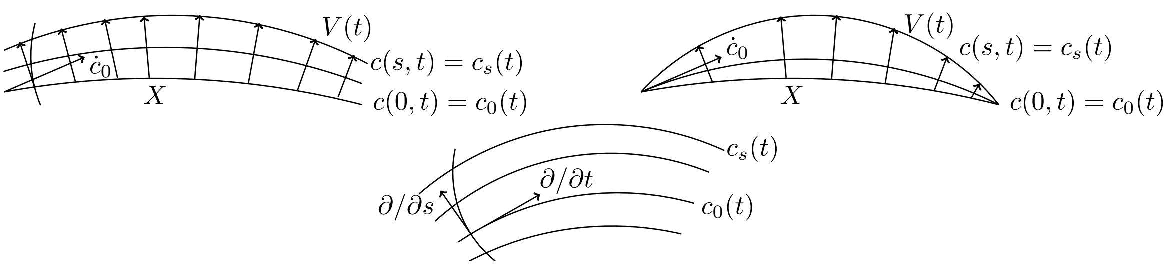
Using new notations, the Jacobi equation is rewritten as

$$\ddot{X} + R(X, \dot{c})\dot{c} = 0.$$

To understand Jacobi field, we first recall the variation.

As previously seen (Variation)**.** For some $\epsilon > 0$, the *variation* of a smooth curve $c\colon [a, b] \to \mathcal{M}$ is a differentiable map $F\colon [a, b] \times (-\epsilon, \epsilon) \to \mathcal{M}$ such that $F(t, 0) = c(t)$ for $t \in [a, b]$ with $s \in (-\epsilon, \epsilon)$.

Essentially, a Jacobi field studies the variation of geodesics: we can label geodesics c as



Notation (Proper variation). A *proper variation* is a variation where the endpoints are fixed, i.e., $F(a,s) = c(a)$ and $F(b,s) = c(b)$ for all $s \in (-\epsilon, \epsilon)$.

Note. We might either fix the endpoints or left them open, i.e., we can consider both proper and non-proper cases.

Intuition. The Jacobi equation can be viewed as the linearization of the geodesic equation.

Formally, we define the following.

Definition 5.1.2 (Geodesic variation). Let $\mathcal{M}$ be a (semi-)Riemannian manifold. A variation of curves $c\colon I \times (-\epsilon, \epsilon) \to \mathcal{M}$ is called a *geodesic variation* if for all $s \in (-\epsilon, \epsilon)$, the curve $t \mapsto c_s(t) \coloneqq c(t,s)$ is a geodesic.

Notation. We set $c_s(t) = c(t,s) = F(t,s)$, and

- $\dot{c}(t,s) \coloneqq \frac{\partial}{\partial t} c(t,s)$, i.e., $\mathrm{d}F(\partial/\partial t)c(t,s)$;
- $c'(t,s) \coloneqq \frac{\partial}{\partial s} c(t,s)$, i.e., $\mathrm{d}F(\partial/\partial s)c(t,s)$.

5.2 Variations of Length and Energy

Recall the following.

As previously seen. Given a variation of a geodesic $c_s(t)$, The energy for c_s is defined as

$$E(s) \coloneqq \frac{1}{2} \int_a^b \left\langle \frac{\partial c(t,s)}{\partial t}, \frac{\partial c(t,s)}{\partial t} \right\rangle \mathrm{d}t,$$

and the length for c_s is defined as

$$L(s) \coloneqq \int_a^b \left\langle \frac{\partial c(t,s)}{\partial t}, \frac{\partial c(t,s)}{\partial t} \right\rangle^{1/2} \mathrm{d}t.$$

And we want to compute

- the first variations $E'(0)$ and $L'(0)$, i.e., the first derivatives;
- for $c = c_0$ geodesic, compute the second variations $E''(0)$ and $L''(0)$, i.e., the second derivatives.

5.2.1 First Variations

Let's consider the first variations, i.e., $E'(0)$ and $L'(0)$.

Lemma 5.2.1. If $L(s)$, $E(s)$ are differentiable w.r.t. s, then

$$L'(0) = \int_a^b \left(\frac{\frac{\partial}{\partial t} \langle c', \dot{c} \rangle}{\langle \dot{c}, \dot{c} \rangle^{1/2}} - \frac{\left\langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c} \right\rangle}{\langle \dot{c}, \dot{c} \rangle^{1/2}} \right) \mathrm{d}t,$$

and

$$E'(0) = \langle c'(b,0), \dot{c}(b,0) \rangle - \langle c'(a,0), \dot{c}(a,0) \rangle - \int_a^b \left\langle \frac{\partial c}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial t}(t,s) \right\rangle \mathrm{d}t.$$

Proof. We have already proved this in different notations. ■

Note. If $c = c_0$ is parametrized proportionally to the arc-length, i.e., $\|\dot{c}(t,0)\|$ is a constant. Then $L'(0)$ becomes

$$L'(0) = \frac{1}{\langle \dot{c}, \dot{c} \rangle^{1/2}} \left(\langle c', \dot{c} \rangle \big|_{t=a,s=0}^{t=b,s=0} - \int_a^b \left\langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c} \right\rangle \mathrm{d}t \right).$$

If we consider the fixed endpoints case (i.e., proper variation), we observe that E and L are stationary if and only if

$$\nabla_{\frac{\partial}{\partial t}} \dot{c}(t,0) = 0,$$

i.e., when c is a geodesic.

5.2.2 Second Variations

Now, let $c = c_0$ be a geodesic. Then we compute the second derivatives w.r.t. s of E and L at $s = 0$.

Theorem 5.2.1. Let $c\colon [a,b] \to \mathcal{M}$ be a geodesic. Then

$$E''(0) = \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} c'(t,0), \nabla_{\frac{\partial}{\partial t}} c'(t,0) \right\rangle \mathrm{d}t - \int_a^b \langle R(\dot{c}, c')c', \dot{c} \rangle \, \mathrm{d}t \bigg|_{s=0} + \left\langle \nabla_{\frac{\partial}{\partial s}} c', \dot{c} \right\rangle \bigg|_{t=a,s=0}^{t=b,s=0}.$$

By letting $c'^{\perp} := c' - \left\langle \frac{\dot{c}}{\|\dot{c}\|}, c' \right\rangle \frac{\dot{c}}{\|\dot{c}\|}$,[a] we have

$$L''(0) = \frac{1}{\|\dot{c}\|} \left(\int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} c'^{\perp}, \nabla_{\frac{\partial}{\partial t}} c'^{\perp} \right\rangle \mathrm{d}t - \int_a^b \langle R(\dot{c}, c'^{\perp})c'^{\perp}, \dot{c} \rangle \, \mathrm{d}t + \left\langle \nabla_{\frac{\partial}{\partial s}} c', \dot{c} \right\rangle \bigg|_{t=a}^{t=b} \right) \Bigg|_{s=0}.$$

[a] I.e., the component of c' orthogonal to $\dot{c}$.

Remark. By keeping the endpoints fixed, if the sectional curvature of $\mathcal{M}$ is non-positive, then the Riemannian curvature in $E''(0)$ and $L''(0)$ are non-negative. This implies $E''(0) > 0$, then $E(c_s) > E(c_0)$ for small $|s|$.

Corollary 5.2.1. On a manifold with non-positive sectional curvature, the geodesics with fixed endpoints are always locally minimizing.

5.3 Index Form

5.3.1 Pullback Connections

Let $\mathcal{M}$ be a Riemannian manifold of dimension d, and $\mathcal{H}$ be a differentiable manifold.[1] Let $f\colon \mathcal{H} \to \mathcal{M}$, smooth, and f may not be injective. We ask the following question.

Problem 5.3.1. What is the tangent space of $f(\mathcal{H})$ of point $p \in f(\mathcal{H})$?

We see that even if f is an immersion, since it can be non-injective, there maybe issues.

Example. Let $p = f(x) = f(y)$ for $x \neq y$. For f being an immersion, we may restrict f to a sufficiently small neighborhood U, V at x, y, respectively, such that $f(U), f(V)$ have well-defined tangent spaces at p. Then, in a double point (e.g., p) of $f(\mathcal{H})$, the tangent space can be specified by specifying the preimage (x or y).

Formally, consider $f^*(T\mathcal{M})$, the tangent bundle $T\mathcal{M}$ pullback by f.

[1] Often times, $\mathcal{H}$ is an interval $I \subseteq \mathbb{R}$ or a square $I \times I \subseteq \mathbb{R}^2$.

Note. The fiber over $x \in \mathcal{H}$ is $T_{f(x)}\mathcal{M}$.

Then, we can introduce a connection $f^*(\nabla)$ on $f^*(T\mathcal{M})$: let $X \in T_x\mathcal{H}$, Y a section of $f^*(T\mathcal{M})$. Set

$$(f^*\nabla)_X Y \coloneqq \nabla_{\mathrm{d}f(X)} Y,$$

where $f^*(T\mathcal{M})_x$ is identified with $T_{f(x)}\mathcal{M}$ with ∇ for $f^*\nabla$.

Note. For $\nabla_{\mathrm{d}f(X)} Y$ to be well-defined, we need to extend Y to a neighborhood of $f(\mathcal{H})$.[a]

[a]Hence, it does not depend on the choice of extension.

Notation. Write ∇ for $f^*\nabla$ in what follows.

5.3.2 Index Form

Now, let $f = c\colon I \to \mathcal{M}$ (often, c is a geodesic), i.e., we consider vector field along c.[2] Specifically, let X be a vector field along c where c is a geodesic. Then, there exists a geodesic variation

$$c\colon [a,b] \times (-\epsilon, \epsilon) \to \mathcal{M}$$

of $c(t)$ with $\frac{\partial c}{\partial s}\big|_{s=0} = X$. Consider the second variation of energy: inspired from Theorem 5.2.1, we write

$$I(X,X) \coloneqq \int_a^b \Big(\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} X \rangle - \langle R(\dot{c}, X)X, \dot{c} \rangle \Big) \,\mathrm{d}t,$$

i.e., $I(X,X) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} E(0)$ if $X(a) = X(b) = 0$. Moreover, instead of considering a 1-parameter variation, we can also consider a 2-parameter variation on X and $Y \coloneqq \frac{\partial c}{\partial t}$. In this case, we propose the following.

Definition 5.3.1 (Index form). The *index form* of a geodesic c on $X = \frac{\partial c}{\partial s}\big|_{s=0}$ and $Y = \frac{\partial c}{\partial t}$ is

$$I(X,Y) \coloneqq \int_a^b \Big(\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} Y \rangle - \langle R(\dot{c}, X)Y, \dot{c} \rangle \Big) \,\mathrm{d}t.$$

Note. We see that $I(X,Y)$ is a bilinear and symmetric in X, Y.

As previously seen. Recall the Jacobi equation, i.e., $\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X + R(X, \dot{c})\dot{c} = 0$.

Proposition 5.3.1 (Jacobi field). A vector field X along a geodesic $c\colon [a,b] \to \mathcal{M}$ is a Jacobi-field if and only if the index form of c satisfies $I(X,Y) = 0$ for all vector fields Y along c with $Y(a) = Y(b) = 0$.

Proof. Observe that

$$\begin{aligned} I(X,Y) &= \int_a^b \Big(\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} Y \rangle - \langle R(\dot{c}, X)Y, \dot{c} \rangle \Big) \,\mathrm{d}t \\ &= \int_a^b \Big(\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} Y \rangle - \langle R(X, \dot{c})\dot{c}, Y \rangle \Big) \,\mathrm{d}t = \int_a^b \Big(\langle -\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} X, Y \rangle - \langle R(X, \dot{c})\dot{c}, Y \rangle \Big) \,\mathrm{d}t, \end{aligned} \tag{5.2}$$

where the second inequality follows from the fact that ∇ is Riemannian, hence

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \langle \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X, Y \rangle = \langle \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X, Y \rangle + \langle \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X, \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} Y \rangle,$$

[2]In deed, a section of $f^*(T\mathcal{M})$ is a *vector field along* f in general even for $f\colon I^2 \to \mathcal{M}$.

with $Y(a) = 0 = Y(b)$,

$$\int_a^b \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \langle \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X, Y\rangle \,\mathrm{d}t = \langle \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X, Y\rangle \Big|_a^b = 0,$$

so

$$\int_a^b \langle \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X, Y\rangle \,\mathrm{d}t = -\int_a^b \langle \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} x, \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} Y\rangle \,\mathrm{d}t.$$

We see that the right-hand side of Equation 5.2 vanishes for every Y if and only if

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X + R(X, \dot{c})\dot{c} = 0,$$

which is just the Jacobi equation, so the result follows. ∎

Intuition. Proposition 5.3.1 is really where the Jacobi equation comes from.

Remark. do Carmo [FC13] introduce Jacobi equation slightly differently, but it's basically the same.

Lecture 17: Jacobi Fields and General Relativity

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Lemma 5.3.1. A vector field along a geodesic $c\colon [a, b] \to \mathcal{M}$ is a Jacobi field if and only if it is a critical point of $I(X, X)$ w.r.t. all variations with fixed endpoints, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}s} I(X + sY, X + sY)\Big|_{s=0} = 0$$

for every vector field along c with $Y(a) = Y(b) = 0$.

Proof. We just use the proof of Proposition 5.3.1 with the fact that

$$\frac{\mathrm{d}}{\mathrm{d}s} I(X + sY, X + sY)\Big|_{s=0} = 2\int_a^b \left(-\langle \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X, Y\rangle - \langle R(X, \dot{c})\dot{c}, Y\rangle\right) \,\mathrm{d}t.$$

∎

Remark. Lemma 5.3.1 tells us that the Jacobi equation is the Euler-Lagrange equations for $I(X) := I(X, X)$.

5.3.3 Existence and Uniqueness of Jacobi Fields

Given the initial data, how can we characterize the Jacobi equation on a Riemannian manifold $(\mathcal{M}, g)$ with $\dim \mathcal{M} = d$? Firstly, we know that the Jacobi equation is a system of d linear second order ODE.

Theorem 5.3.1. Let $c\colon [a, b] \to \mathcal{M}$ be a geodesic. For all $v, w \in T_{c(a)}\mathcal{M}$, there exists a unique Jacobi field X along c with $X(a) = v$, $\dot{X}(a) = w$.

Proof. Let $\{v_i\}_{i=1}^d$ be an orthonormal basis of $T_{c(a)}\mathcal{M}$. Let $\{X_i\}_{i=1}^d$ be parallel vector field along c with $X_i(a) = v_i$ for $i = 1, \ldots, d$. Then for all $t \in [a, b]$, $X_1(t), \ldots, X_d(t)$ is an orthonormal basis of $T_{c(t)}\mathcal{M}$. Choose arbitrary vector field X along c as $X = \xi^i X_i$, i.e., $\xi^i(t) = \langle X(t), X_i(t)\rangle$. As vector fields X_i's are parallel, we have

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X = \frac{\mathrm{d}\xi^i}{\mathrm{d}t} X_i + \xi_i \underbrace{\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X_i}_{0} = \frac{\mathrm{d}\xi^i}{\mathrm{d}t} X_i \Rightarrow \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} X = \frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2} X_i.$$

To write the Jacobi equation in these coordinates, we first write the curvature as

$$R(X,\dot{c})\dot{c} = \xi^i \rho_i^k X_k,$$

where we let $\rho_i^k \coloneqq \langle R(X_i,\dot{c})\dot{c}, X_k\rangle$,[a] i.e., $R(X_i,\dot{c})\dot{c} = \rho_i^k X_k$. Then, the Jacobi equation becomes

$$\left(\frac{\mathrm{d}^2\xi^k}{\mathrm{d}t^2} + \xi^i\rho_i^k\right) X_k = 0 \Rightarrow \frac{\mathrm{d}^2\xi^k(t)}{\mathrm{d}t^2} + \xi^i(t)\rho_i^k(t) = 0, \quad k = 1,\dots,d$$

since $\{X_i\}$ is a orthonormal basis. Then, by the linear algebra and ODE theory, we have existence and uniqueness. ■

[a] ρ_i^k is sometimes referred to *rotation*.

Let's see some examples of Jacobi fields.

Example ($\mathbb{R}^n$). Since the geodesics are "straight lines", consider the Jacobi field X along straight line c with $X(a) = v, \dot{X}(a) = w$. Let $V(t), W(t)$ be parallel vector fields along c with $V(a) = v, W(a) = w$, by linearizing, we have

$$X(t) = V(t) + (t-a)W(t).$$

Example ($S^n \subseteq \mathbb{R}^{n+1}$). Let $c\colon [0,T] \to S^n$ be a geodesic with $\|\dot{c}\| = 1$, and $v, w \in T_{c(0)}S^n$, V, W parallel vector fields along c with $V(0) = v, W(0) = w$. Also, assume that $\langle v, \dot{c}(0)\rangle = 0 = \langle w, \dot{c}(0)\rangle$, then the Jacobi field X is

$$X(t) = V(t)\cos t + W(t)\sin t.$$

Proof. We see that

$$\dot{X}(t) = -V(t)\sin t + W(t)\cos t,$$

and

$$\ddot{X}(t) = -V(t)\cos t - W(t)\sin t.$$

By using the Riemannian curvature on S^n, we have

$$R(X,\dot{c})\dot{c} = \underbrace{\langle \dot{c},\dot{c}\rangle}_{1} X - \underbrace{\langle X,\dot{c}\rangle}_{0} \dot{c} = X.$$

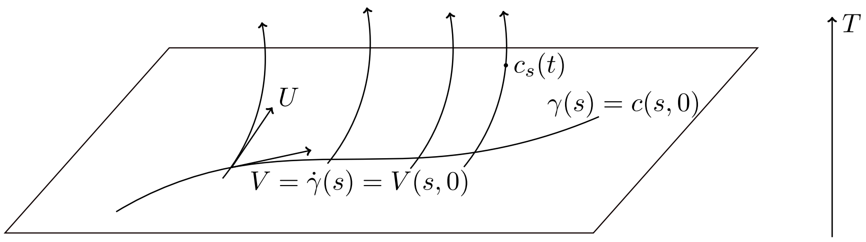
Then $\ddot{X} + R(X,\dot{c})\dot{c} = 0$. ⊛

Remark. We can also consider $S^n_\rho \subseteq \mathbb{R}^{n+1}$ with $\|\dot{c}\| = 1$ and play the above game, i.e., by letting

$$X(t) = V(t)\cos\frac{t}{\rho} + W(t)\sin\frac{t}{\rho}.$$

5.3.4 Application of General Relativity

We take a quick detour to see a huge break through in general relativity related to Jacobi field. Consider the universe as a $(\mathcal{M}^4, g)$ a Lorentzian manifold,



Here, we have $[\partial/\partial s, \partial/\partial t] = [U,V] = 0$. Hence, the Jacobi equation is now

$$\nabla_U^2 V + R(V,U,U) = 0.$$

For given U, the right-hand side defines of each $p \in \mathcal{M}$ a linear map

$$N \mapsto R(N, U)U$$

for N unit normal of subspace of $T_p\mathcal{M}$ perpendicular to U.[3] Hence, locally,

- the gravitational field g, the "fields strengths" Γ can be transformed away;
- variation of gravitational fields strengths can be described by Riemannian curvature tensor, hence cannot be transformed away.

All these imply that the Jacobi equation with Riemannian curvature tensor can describe the relative accelerations (or field forces) of nearby geodesics.

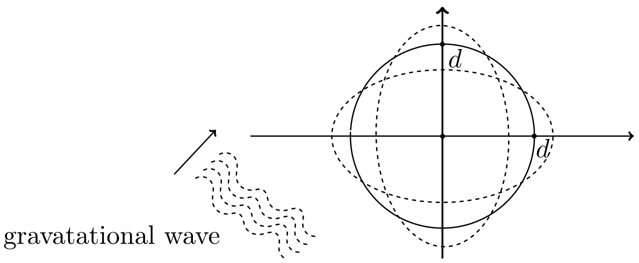


Figure 5.1: LIGO [Abb+16], $\frac{\Delta\lambda}{\lambda} \approx 10^{-21}$.

5.4 Jacobi Fields and Geodesics

Consider a Jacobi field transversal along c, then we can split the Jacobi field into

- tangential component: do not depend on geometry of $\mathcal{M}$, hence no information about $\mathcal{M}$;
- normal component: very useful!

Specifically, consider $X = X^\top + X^\perp$, we have the following.

Lemma 5.4.1. Let $c\colon [a, b] \to \mathcal{M}$ be a geodesic, and $\lambda, \mu \in \mathbb{R}$. Then, the Jacobi field X along c with $X(a) = \lambda\dot{c}(a)$, $\dot{X}(a) = \mu\dot{c}(a)$ is given by $X(t) = (\lambda + (t - a)\mu)\dot{c}(t)$.

Lecture 18: Jacobi Fields and Geodesics

5.4.1 Jacobi Fields and the Linearization of Geodesic Equations

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As previously seen. Recall that Equation 5.2 is linear, hence the sum of solutions is a solution.

Theorem 5.4.1. Consider a geodesic $c\colon [0, 1] \to \mathcal{M}$, $t \mapsto c(t)$, and the geodesic variation $c\colon [0, 1] \times (-\epsilon, \epsilon) \to \mathcal{M}$ of c.[a] Then $X(t) \coloneqq \frac{\partial}{\partial s} c(t, s)\big|_{s=0}$ is a Jacobi field along $c(t) = c_0(t)$. Conversely, every Jacobi field along $c(t)$ can be obtained in this way, i.e., by variation of geodesics.

[a] I.e., for all curves $c(\cdot, s) =: c_s(\cdot)$ are geodesics.

Proof. The forward direction is straightforward: since $c(t, s)$ for a fixed s is a geodesic, hence

[3] This is often called the *field force operator*.

$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} c(t, s) = 0$ for all s, implying $\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} c(t, s) = 0$. Then,

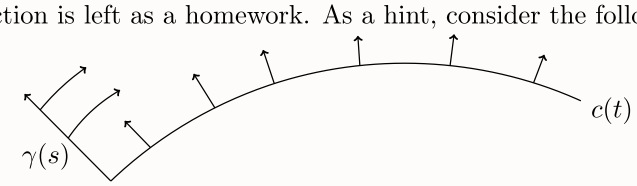
$$\begin{aligned} 0 &= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} c(t, s) \\ &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} c(t, s) + \left(-\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} + \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \right) \frac{\partial}{\partial t} c(t, s) \\ &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} c(t, s) + R\left(\frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \right) \frac{\partial c}{\partial t}, \end{aligned}$$

since $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$, and hence also $\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} = \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}$. Plugging in the definition of X, we have

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} X + R\left(X, \frac{\partial c}{\partial t} \right) \frac{\partial c}{\partial t} = 0,$$

i.e., X is a Jacobi field.

The converse direction is left as a homework. As a hint, consider the following:



Then, let

$$c(t, s) = \exp_{\gamma(s)} (t(\dot{c}(0) + s \cdot V))$$

for some V. Once we have this, we just let $X(t) = \frac{\partial}{\partial s} c(t, s) \big|_{s=0}$. ■

Remark. This confirms the intuition: Jacobi equation is the linearization of the geodesic equation!

Theorem 5.4.1 gives us a clear picture of how Jacobi field arise from the geodesic variation.

5.4.2 Killing Fields

To proceed, we need a new definition called killing field.

As previously seen. Recall that

$$\begin{aligned} (\mathcal{L}_X S)(Y_1, \ldots, Y_p) &= X(S(Y_1, \ldots, Y_p)) - \sum_{i=1}^{p} S(Y_1, \ldots, [X, Y_i], \ldots, Y_p) \\ &= (\nabla_X S)(Y_1, \ldots, Y_p) + \sum_{i=1}^{p} S(Y_i, \ldots, \nabla_{Y_i} X, \ldots, Y_p) \end{aligned}$$

since ∇ is torsion-free, we have $\nabla_X Y_i - \nabla_{Y_i} X = [X, Y_i]$.

Definition 5.4.1 (Killing field). Consider a Riemannian manifold $(\mathcal{M}, g)$, and $g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$. Then a vector field X such that

$$\mathcal{L}_X g = 0$$

is called a *killing field* (or *infinitesimal isometry*).

Here are two basic facts about killing fields.

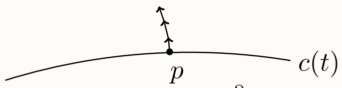
Lemma 5.4.2. A vector field X on $(\mathcal{M}, g)$ is a killing field if and only if the local 1-parameter group generated by X consisted of local isometries.

Lemma 5.4.3. The killing fields of a Riemannian manifold constitute a Lie algebra.

Theorem 5.4.1 implies the following.

Corollary 5.4.1. Every killing field X on $\mathcal{M}$ is a Jacobi field along any geodesic in $\mathcal{M}$.

Proof idea. Since we have a killing field X, we use it to construct $\Phi_s \colon \mathcal{M} \to \mathcal{M}$, which is an isometry since X is a killing field.

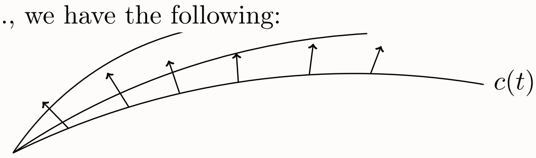


The idea is to consider $c(t,s) = \Phi_s \circ c(t)$, and let $X = \frac{\partial}{\partial s} c(t,s)$. By Theorem 5.4.1, we're done. ■

Corollary 5.4.2. Let $c \colon [0,T] \to \mathcal{M}$ be a geodesic with $p = c(0)$, i.e., $c(t) = \exp_p(t\dot{c}(0))$. For $W \in T_p\mathcal{M}$, the Jacobi field x along c with $X(0) = 0$, $\dot{X}(0) = W$, is given as

$$X(t) = \mathrm{D}(\exp_p)\big|_{(t\dot{c}(0))}(tW).$$

Proof. This is a direct consequence of Theorem 5.4.1, since now $X(0) = 0$, we don't need to worry about constructing $\gamma(s)$, i.e., we have the following:



Now, we consider $c(t,s) = \exp_p(t(\dot{c}(0) + s \cdot W))$, hence

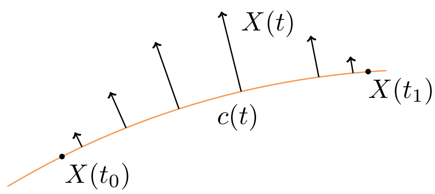
$$\frac{\partial}{\partial s} c(t,s) = \frac{\partial}{\partial s} \exp_p(t\dot{c} + s \cdot W)\bigg|_{s=0}.$$

To have $\mathrm{D}(\exp_p)\big|_V (W)$, we construct a Jacobi field W such that $X(0) = 0$, $\dot{X}(0) = W$. ■

Remark. Thus, derivative of exp can be computed from Jacobi field along radial geodesics.

5.5 Conjugate Points

Consider the following, where $c \colon I \to \mathcal{M}$ is a geodesic, and $X(t)$ is a Jacobi field with $X(t_0) = X(t_1) = 0$ such that $t_0 \neq t_1 \in I$.



Note. Notice that X is always normal to c.

To characterize this scenario, we define the following.

Definition 5.5.1 (Conjugate point)**.** Let $c \colon I \to \mathcal{M}$ be a geodesic. For $t_0, t_1 \in I$ with $t_0 \neq t_1$, $c(t_0)$ and $c(t_1)$ are called *conjugate* along c if there exists a Jacobi field $X(t)$ along c which does not vanish identically but satisfies $X(t_0) = 0 = X(t_1)$.

Note. We see that $\langle X(t), \dot{c}(t)\rangle = 0$ for all t.

Proof. Since $\nabla_{\partial t}\langle X(t), \dot{c}(t)\rangle = \langle \dot{X}, \dot{c}\rangle$, so

$$\nabla_{\partial t}\nabla_{\partial t}\langle X(t), \dot{c}(t)\rangle = \langle \ddot{X}, \dot{c}\rangle = -\langle R(X, \dot{c})\dot{c}, \dot{c}\rangle = 0.$$

This is a linear function, and if two endpoints are both 0, everything is 0. ⊛

Note. If $t_0, t_1 \in I$, $t_0 \neq t_1$ are not conjugate along c, then for $V \in T_{c(t_0)}\mathcal{M}, W \in T_{c(t_1)}\mathcal{M}$, there exists a unique Jacobi field $Y(t)$ along c such that $Y(t_0) = V, Y(t_1) = W$.

Proof. Let $\mathcal{J}_c$ be the vector space of Jacobi fields along c. Construct the linear map

$$A\colon \mathcal{J}_c \to T_{c(t_0)}\mathcal{M} \times T_{c(t_1)}\mathcal{M}, \quad Y \mapsto (Y(t_0), Y(t_1)).$$

Since $\mathcal{J}_c$ is a vector space with $\dim \mathcal{J}_c = 2n$, and the target space is also with dimension $2n$, and because $t_0 \neq t_1$ are not conjugate, $\ker A = \{0\}$, i.e., A is injective, hence A is bijective as the domain and the range of A have the same dimension. ⊛

Example. Any antipodal points of S^n are conjugate points.

Example. $\mathbb{R}^n$ with flat metric doesn't have conjugate points.

Example. Riemannian manifolds with non-positive sectional curvature has no conjugate points.

5.5.1 Length-Minimizing Geodesics

We can formalize the above examples by the following.

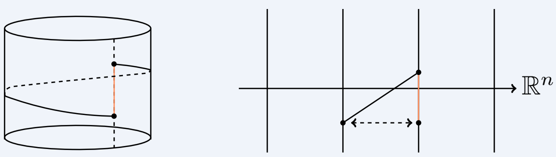
Theorem 5.5.1. Let $c\colon [a, b] \to \mathcal{M}$ be a geodesic.

(a) If there does not exist a point conjugate to $c(a)$ along $c(t)$, then there exists $\epsilon > 0$ such that for all piecewise smooth curve $g\colon [a, b] \to \mathcal{M}$ with $g(a) = c(a), g(b) = c(b)$ and $d(g(t), c(t)) < \epsilon$ for all $t \in [a, b]$, we have $L(c) \leq L(g)$, and the equality holds when if and only if g is a reparametrization of c.

(b) If there is $\tau \in (a, b)$ such that $c(a)$ and $c(\tau)$ are conjugate points along c, then there exists a proper variation $c(t, s)\colon [a, b] \times (-\epsilon, \epsilon) \to \mathcal{M}$ such that $L(c_s) < L(c)$ for $s \in (-\epsilon, \epsilon) \setminus \{0\}$.

Theorem 5.5.1 (a) implies that if there are no conjugate points, a geodesic is length-minimizing w.r.t. *sufficiently close curves*. As we have seen multiple times, this is not global.

Example (Cylinder)**.** Consider the cylinder, where we identify every (integer-multiple) line of $\mathbb{R}^n$ below.



There are two geodesics, but one is strictly longer.

Example (Torus)**.** Consider geodesics on a "flat" torus which winds around more than once on the torus. Then, even without conjugate points, it's not length-minimizing globally.

To prove Theorem 5.5.1, we need the following.

Corollary 5.5.1. Let $p \in \mathcal{M}$ and $V \in T_p\mathcal{M}$ is contained in the domain of definition of $\exp_p$. Let $c(t) = \exp_p(tV)$, and $\gamma\colon [0, 1] \to T_p\mathcal{M}$ be a piecewise smooth curve contained in the domain of $\exp_p$

with $\gamma(0) = 0, \gamma(1) = V$. Then

$$\|v\| = L\left(\exp_p(tV)\big|_{t\in[0,1]}\right) \leq L\left(\exp_p \circ\gamma(t)\right)$$

and the equality holds if and only if γ differs from the curve tV, $t \in [0,1]$ only by reparametrization.

Proof hint. We directly estimate

$$L(\exp\circ\gamma) = \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}t}\exp\circ\gamma\right| \,\mathrm{d}t = \int_0^1 |\mathrm{D}\exp\circ\gamma| \,\mathrm{d}t.$$

■

Lecture 19: Length-Minimizing Geodesics and Conjugacy

Now, let's prove Theorem 5.5.1. 14 Mar. 13:00

Proof of Theorem 5.5.1. We prove them one by one.

(a) We want to show that if there's no conjugate point, then for all curves as in (a), there exists a curve γ as in Corollary 5.5.1. Without loss of generality, let $a = 0$, $b = 1$, and we set $V := \dot{c}(0)$. Then, we know that since there are no conjugate points along c, $\exp_p$ of maximal rank along any radial curve tV, $0 \leq t \leq 1$. By the inverse function theorem, for all t, $\exp_p$ is a diffeomorphism in a neighborhood of tV.

Now, cover $\{tV \mid 0 \leq t \leq 1\}$ by finitely many such neighborhoods $\{\Omega_i\}_{i=1}^k$, and let $U_i = \exp_p \Omega_i$. Assume that $tV \in \Omega_i$, for $t_{i-1} \leq t \leq t_i$ (with $t_0 = 0, t_k = 1$). Let $\epsilon > 0$ sufficiently small. Then, for all curve $g\colon [0,1] \to \mathcal{M}$ satisfying the assumption, $g([t_{i-1}, t_i]) \subseteq U_i$.

Claim. For all g satisfying $g([t_{i-1}, t_i]) \subseteq U_i$, there exists a curve $\gamma \subseteq T_p\mathcal{M}$ such that $\exp_p \gamma = g$ with $\gamma(0) = 0, \gamma(1)gV$.

Proof. Put $\gamma(t) = (\exp_p|_{\Omega_i})^{-1}(g(t))$ for $t_{i-1} \leq t \leq t_i$, so γ satisfies Corollary 5.5.1. ⊛

(b) Without loss of generality, let $a = 0, b = 1$. Let X be a non-trivial Jacobi field along c with $X(0) = 0 = X(\tau)$. We have $\dot{X}(\tau) \neq 0$, as otherwise $X \equiv 0$ by the uniqueness. Let $Z(t)$ be an arbitrary vector field X along c with $Z(0) = 0 = Z(1)$, $Z(\tau) = -\dot{X}(\tau)$. Let $\eta > 0$, set

$$Y_\eta(t) = \begin{cases} Y_\eta^1(t) = X(t) + \eta Z(t), & \text{if } 0 \leq t \leq \tau; \\ Y_\eta^2(t) = \eta Z(t), & \text{if } \tau \leq t \leq 1, \end{cases}$$

and we let $Z^1 := Z|_{[0,\tau]}, Z^2 := Z|_{[\tau,1]}$. Now, since

$$I(Y_\eta^1, Y_\eta^1) = \left\langle \dot{X}(\tau), 2\eta Z(\tau)\right\rangle + \eta^2 I(Z^1, Z^1) - 2\eta\|\dot{X}(\tau)\|^2 + \eta^2 I(Z^1, Z^1),$$

with

$$I(Y_\eta^2, Y_\eta^2) = \eta^2 I(Z^2, Z^2),$$

with

$$I(Y_\eta, Y_\eta) = I(Y_\eta^1, Y_\eta^1) + I(Y_\eta^2, Y_\eta^2) = -2\eta\|\dot{X}(\tau)\|^2 + \eta^2 I(Z, Z)$$

for sufficiently small $\eta > 0$. Now, consider the variation $c(t,s) := \exp_{c(t)} sY_\eta(t)$, we have $L'(0) = 0$[a] and

$$L''(0) = I(Y_\eta, Y_\eta) < 0.$$

By the Taylor theorem, this is a minimum, i.e., $L(c_s) < L(c)$.

■

[a]Note that $L(s) := L(c_s)$, $L(0) = L(c)$.

Remark. Given a geodesic γ from q to p, q is conjugates to p along γ if there exists a non-trivial Jacobi field along γ vanishing at p and q.

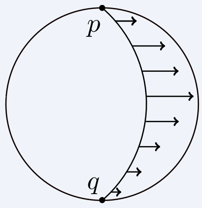
We finally note that there's one concept that's related to our discussion.

Definition 5.5.2 (Order)**.** The *order* (or multiplicity) of conjugacy is the dimension of the space of Jacobi fields vanishing at two conjugate points.

Given $\dim \mathcal{M} = n$, by the existence and uniqueness theorem for Jacobi fields, we see that

- there is an n-dimensional space of Jacobi fields vanishing at $p \in \mathcal{M}$;
- there is an at most $(n-1)$-dimensional space of Jacobi fields vanishing at $p, q \in \mathcal{M}$, as tangential Jacobi fields vanishes at most at one point.

Example (S_r^n)**.** On S_r^n and p, q antipodal points on S_r^n, there is a Jacobi filed vanishing at p and q for all parallel normal vector field along γ, thus p, q conjugate to order exactly $(n-1)$.



5.5.2 Characterization via Exponential Maps and Index Forms

We now characterize the conjugate points by exponential map and the index form: firstly, they are precisely the images of singularities of the exponential map.

Proposition 5.5.1. Let $p \in \mathcal{M}$, $V \in T_p\mathcal{M}$, $q = \exp V$. Then $\exp_p$ is a local diffeomorphism in a neighborhood of V if and only if q does not conjugate to p along geodesic $\gamma(t) = \exp_p tV$, $t \in [0,1]$.

For simplicity, let's develop some shorthand notations.

Notation. Let $c\colon [a,b] \to \mathcal{M}$ be a curve. Denote

- ν_c: the space of vector field X along c, i.e., $\nu_c = \Gamma(c^*(T\mathcal{M}))$;
- $\mathring{\nu}_c$: the space of vector field X along c with $V(a) = V(b) = 0$.

Another characterization is related to index form.

Lemma 5.5.1. Let $c\colon [a,b] \to \mathcal{M}$ be a geodesic. Then there is no pair of conjugate points along c if and only if the index form I of c is strictly positive definite on $\mathring{\nu}_c$

Proof. Assume that c has no conjugate points, then Theorem 5.5.1 (a) implies that $I(X,X) \geq 0$ for all $X \in \mathring{\nu}_c$ because otherwise $c(t,s) := \exp_{c(t)} sX(t)$ would be locally length-decreasing.

If $I(Y,Y) = 0$ for some $Y \in \mathring{\nu}_c$, then by $I(X,X) \geq 0$, for all $Z \in \mathring{\nu}_c$ and $\lambda \in \mathbb{R}$,

$$0 \leq I(Y - \lambda Z, Y - \lambda Z) = 0 - 2\lambda I(Y,Z) + \lambda^2 I(Z,Z).$$

This inequality holds only if $I(Y,Z) = 0$ for all $Z \in \mathring{\nu}_c$, implying Y is a Jacobi field form Proposition 5.3.1. As there are no conjugate points along c, $Y = 0$, i.e., I is strictly positive definite.

Lecture 20: Sobolev Spaces and Cut Locus

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Proof of Lemma 5.5.1 (Continue). For the backward direction, assume that for $t_0, t_1 \in [a, b]$ (without loss of generality, let $t_0 < t_1$) such that $c(t_0), c(t_1)$ are two conjugate points along c. Then, there exists a non-trivial Jacobi field X along c such that $X(t_0) = 0 = X(t_1)$. Now, consider

$$Y(t) = \begin{cases} 0, & \text{if } a \leq t \leq t_0; \\ X(t), & \text{if } t_0 \leq t \leq t_1; \\ 0, & \text{if } t_1 \leq t \leq b; \end{cases} \Rightarrow J(Y, Y) = 0,$$

hence I is not positive definite, a contradiction. ■

5.6 The Cut Locus

We end this chapter by developing one more notion for later. We first take a detour to Sobolev spaces.

5.6.1 Sobolev Spaces

On $\mathring{\nu}_c$, we introduce the norm

$$\|X\| := \left(\int_a^b \left(\langle \dot{X}, \dot{X} \rangle + \langle X, X \rangle \right) \mathrm{d}t \right)^{1/2},$$

and denote $\mathring{H}_c$ the completion of $\mathring{\nu}_c$ w.r.t. $\|\cdot\|$.

Definition 5.6.1 (Schwartz space). A *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ is defined as

$$\mathcal{S}(\mathbb{R}^d) := \left\{ u \in C^\infty(\mathbb{R}^d) \mid \forall \alpha, \beta \in \mathbb{N}^d \ \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta u(x)| < \infty \right\}.$$

Definition 5.6.2 (Tempered distribution). A *tempered distribution* is a continuous linear functional f on $\mathcal{S}(\mathbb{R}^d)$, i.e., $f \colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$.

Notation. The space of tempered distributions is denoted as $\mathcal{S}^1(\mathbb{R}^d)$.

Definition 5.6.3 (Locally integrable). Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $f \colon \Omega \to \mathbb{C}$ be Lebesgue measurable. Then the *locally integrable* (or locally summable) space is defined as

$$L^1_{\mathrm{loc}}(\Omega) := \left\{ f \colon \Omega \to \mathbb{C} \text{ measurable} \mid f|_K \in L^1(K) \ \forall \text{ compact } K \subseteq \Omega \right\}.$$

Definition 5.6.4 (Weak derivative). Let $U \subseteq \mathbb{R}^n$ be open, and $u, v \in L^1_{\mathrm{loc}}(U)$. Let α be a multi-index. Then v is the α^{th}*-weak derivative* of u, denoted as $D^\alpha u = v$ provided

$$\int_U u \cdot D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_U v \varphi \, \mathrm{d}x$$

for all test functions $\varphi \in C_c^\infty(U)$.

Notation. $C_c^\infty(U)$ is the space of smooth functions with compact support defined on U.

Notation. Here, $D^\alpha \varphi$ means

$$D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Note. We can write $D^\alpha u$ since it's unique (up to measure 0).

Remark. If the weak derivative exists, then it's unique up to a set of measure zero.

Definition 5.6.5 (Sobolev space)**.** Fix $1 \leq p \leq \infty$, and let k be a non-negative integer. The *Sobolev space* $W^{k,p}(U)$ consists of all locally integrable functions $u\colon U \to \mathbb{R}$ for all α with $|\alpha| \leq k$ such that $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Remark. If $p = 2$, $H^k(U) \coloneqq W^{k,2}(U)$ for $k = 0, 1, \ldots$ is a Hilbert space.

Example. $H^0(U) = L^2(U)$.

On $W^{k,p}(U)$, we introduce the norm

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p \,\mathrm{d}x\right)^{1/p}, & \text{if } 1 \leq p < \infty; \\ \displaystyle\sum_{|\alpha| \leq k} \operatorname*{ess\,sup}_U |D^\alpha u|, & \text{if } p = \infty. \end{cases}$$

Notation. Denote the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$ as $W_0^{k,p}(U)$.

Thus, $u \in W_0^{k,p}(U)$ if and only if there exists functions $u_n \in C_c^\infty(U)$ such that $u_n \to u$ in $W^{k,p}(U)$.

Remark. Lastly, the upshot is that $u \in W_0^{k,p}(U)$ if $u \in W^{k,p}(U)$ such that "$D^\alpha u = 0$ on ∂U" for all $|\alpha| \leq k - 1$, more precisely, use traces.

5.6.2 The Index

Let $\{V_i\}_{i=1}^d$ for $d = \dim \mathcal{M}$ be an orthonormal basis of parallel vector fields. Now, write $X = \xi^i V_i$, so $\dot{X}_i = \dot{\xi}^i V_i$, hence

$$\|X\| = \left(\int_a^b \left(\dot{\xi}^i \dot{\xi}^j + \xi^i \xi^j\right) \,\mathrm{d}t\right)^{1/2}.$$

Then, $\mathring{H}_c^1$ can be identified with Sobolev space $\mathring{H}^{1,2}(I, \mathbb{R}^d)$. Next, consider I (the index form) of c as quadratic form on $\mathring{H}_c^1$, i.e., $I\colon \mathring{H}_c^1 \times \mathring{H}_c^1 \to \mathbb{R}$ with

$$I(X, Y) = \int_a^b \left(\langle \dot{X}, \dot{Y}\rangle - \langle R(\dot{c}, X)Y, \dot{c}\rangle\right) \,\mathrm{d}t,$$

and we define the following.

Definition 5.6.6 (Index)**.** The *index* of c, $\operatorname{Ind}(c)$, is the dimension of the largest subspace of $\mathring{H}_c^1$, on which I is negative definite.

Definition 5.6.7 (Extended index)**.** The *extended index* of c, $\operatorname{Ind}_0(c)$, is the dimension of the largest subspace of $\mathring{H}_c^1$, on which I is negative semi-definite.

Definition 5.6.8 (Nullity)**.** The *nullity* is defined as $N(c) \coloneqq \operatorname{Ind}_\circ - \operatorname{Ind}(c)$.

Notation. For $t \in (a, b]$, let $\mathcal{J}_c^t$ be the space of Jacobi fields x along c with $X(a) = 0 = X(t)$.

Lemma 5.6.1. $\operatorname{Ind}(c)$ and $N(c)$ are always finite.

Proof. See [FC13] (by contradiction). ∎

Lemma 5.6.2. $\dim \mathcal{J}_c^b = N(c)$.

5.6.3 The Cut Locus

Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold. Let $p \in \mathcal{M}$, and denote $d(\cdot) := d(p, \cdot)$. Then we have seen that there is a normal neighborhood where geodesics are minimizing and d is smooth away from p. For all $v \in S^{n-1}$, we can find a geodesic $c_v(t) = \exp_t(tv)$. Let $R(v) := \sup \left\{T \mid c_v|_{[0,T]} \text{ minimizing}\right\}$.

Note. If $t < R(v)$, then $d(p, c_v(t)) = t$; moreover, if $R(v) = \infty$, c_v is minimizing.

We can then define the following.

Definition 5.6.9 (Cut locus)**.** The *cut locus* of p is defined as

$$C(p) := \left\{c_v(R(v)) \mid v \in S^{n-1} \text{ such that } R(v) < \infty\right\}.$$

Definition 5.6.10 (Cut point)**.** Consider a geodesic c with $d(c(0), c(t))d(p, c(t)) = t$ on $t \in [0, t_0]$ for t_0 being the last point this holds. Then we say $c(t_0)$ is a *cut point* of p along c.

Intuition. The cut locus $C(p)$ of p is the union of the cut points of p along all geodesics starting from p.

Lecture 21: Morse Index Theorem

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As previously seen. Fix $p \in \mathcal{M}$, let $q \in C(p)$ a cut point. Then there exists a geodesic c such that

(a) c minimizing up to and including q, and

(b) c uniquely minimizing up to but not including q.

Thus, c is not minimizing after that point.

Let's first see the following under the above setting.

Proposition 5.6.1. At each cut point $q \in C(p)$, q is either a conjugate point or there exists two minimizing geodesics connecting p, q.

Proof. See [FC13]. ∎

Chapter 6

Morse Index, Rauch Comparison, Sphere Theorems, and More

Now, we have everything to prove three important theorems: the Morse index theorem, Rauch comparison theorem, and the sphere theorem.

Intuition. In short,

- Morse index theorem relates the number (with multiplicities) of conjugate points on a geodesic segment to the index.
- Rauch comparison theorem is one of the basic facts in Riemannian geometry. Intuitively, it expresses the plausible fact that as the curvature grows, lengths shorten.
- Sphere theorem is one of the most beautiful theorems of global differential geometry, which says that under some mild curvature bounds, the space is homeomorphic to a sphere.

In what follows, we prove each theorem one by one.

Note. After proving the Morse index theorem, we detour to study the Morse function and Morse homology before going to the Rauch comparison theorem.

Note. After prove the sphere theorem, we prove another important uniformization theorem, which is worth noting here.

Let's start by proving the Morse index theorem.

6.1 Morse Index Theorem

In this section, we study the Morse index theorem, which gives information about conjugate points via index form.

As previously seen. The index form $I(X, Y)$, index $\operatorname{Ind}(c)$, and also Ind_0, and Lemma 5.6.1.

Let $c\colon [0, T] \to \mathcal{M}$ be a geodesic. Then the index $\operatorname{Ind}(c)$ on the space $\mathring{\nu}_c$ is finite and equals the number of points $c(t)$ conjugate to $c(0)$ for $t \in (0, T)$, counted with multiplicities.

6.1.1 The Conjugate Locus

Before proving the Morse index theorem, let's see one last definition.

Definition 6.1.1 (Conjugate locus). Let $(\mathcal{M}, g)$ be a Riemannian manifold. The set of (first) conjugate points of point $p \in \mathcal{M}$ for all geodesics starting at p is called the *conjugate locus* of p.

Proposition 6.1.1. Let $(\mathcal{M}, g)$ be a complete Riemannian manifold. Let $c\colon [0, \infty) \to \mathcal{M}$ be a normalized geodesic with $c(0) = p$. Assume that $c(t_0)$ is the cut point at $p = c(0)$ along c. Then, either $c(t_0)$ is the first conjugate point of $c(0)$ along c or there exists another geodesic $\sigma \neq c$ from p to $c(t_0)$ such that $\ell(\sigma) = \ell(c)$. Conversely, if either the above are true, then there exists $t_0 \in (0, t_0]$ such that $c(t_1)$ is the cut point of p along c.

Proof. See [FC13]. ∎

6.1.2 Morse Index Theorem

Consider the following.

Theorem 6.1.1 (Morse index theorem). Let $c\colon [a, b] \to \mathcal{M}$ be a geodesic. Then, there are at most finitely many points conjugate to $c(a)$ along c, and

$$\operatorname{Ind}(c) = \sum_{t\in(a,b)} \dim \mathcal{J}_c^t, \quad \operatorname{Ind}_0(c) = \sum_{t\in(a,b]} \dim \mathcal{J}_c^t.$$

Proof. For all $t_i \in (a, b]$, for which $c(t_i)$ conjugate to $c(a)$, there exists a Jacobi field X_i along with $X_i(a) = 0 = X_i(t_i)$. Set

$$Y_i(t) := \begin{cases} X_i(t), & \text{if } a \leq t \leq t_i; \\ 0, & \text{otherwise,} \end{cases}$$

we have that $Y_i(t)$ are linearly independent such that $I(Y_i, Y_i) = 0$ for all i. This implies that the number of conjugate points is at most $\operatorname{Ind}_0(c)$, which is finite from Lemma 5.6.1.

For $\tau \in (a, b]$, set

$$\varphi(\tau) := \operatorname{Ind}\left(c|_{[a,\tau]}\right), \quad \varphi_0(\tau) := \operatorname{Ind}_0\left(c|_{[a,\tau]}\right).$$

Claim. $\varphi(\tau)$ is left-continuous.

Proof. For $\tau \in (a, b]$, let I_τ be the index form of $c|_{[a,\tau]}$, and let X be a vector field along $c|_{[a,\tau]}$ satisfy $I_\tau(X, X) < 0$ and $\|X\| = 1$.

Let $\widetilde{X}$ be vector field defined by $\widetilde{X}(t) := X(\tau t/\sigma)$ on $[a, \sigma]$. Then,

$$\int_0^\sigma \langle \dot{\widetilde{X}}(t), \dot{\widetilde{X}}(t)\rangle \,\mathrm{d}t = \int_0^\sigma \left(\frac{\tau}{\sigma}\right)^2 \langle \dot{X}(\tau t/\sigma), \dot{X}(\tau t/\sigma),\rangle \,\mathrm{d}t = \frac{\tau}{\sigma}\int_0^\tau \langle \dot{X}(s), \dot{X}(s)\rangle \,\mathrm{d}s,$$

implying

$$\int_0^\sigma \langle \dot{\widetilde{X}}(t), \dot{\widetilde{X}}(t)\rangle \,\mathrm{d}t \to \int_0^\tau \langle \dot{X}(t), \dot{X}(t)\rangle \,\mathrm{d}t$$

for $\sigma \to \tau$. Also, we have $\|X\| = 1$, and X is continuous,[a] we see that $\widetilde{X}$ converges point-wise to X as $\sigma \to \tau$, hence

$$\int_0^\sigma \langle R(\dot{c}, \widetilde{X})\widetilde{X}, \dot{c}\rangle \,\mathrm{d}t \to \int_0^\tau \langle R(\dot{c}, X)X, \dot{c}\rangle \,\mathrm{d}t$$

as $\sigma \to \tau$, hence $I_\sigma(\widetilde{X}, \widetilde{X}) \to I_\tau(X, X)$ as $\sigma \to \tau$. Notice that the above also implies $I_\sigma(\widetilde{X}, \widetilde{X}) < 0$ if σ is sufficiently close to τ.

Finally, for all orthonormal basis of a space on which I_τ is negative definite, we may also find a basis of some space on which I_σ is negative definite if σ is sufficiently close to τ. As φ is monotonically increasing, we have left-continuity. ⊛

[a] This is from something called Sobolev theorem.

Claim. $\varphi_0(\tau)$ is right-continuous.

Proof. Let $(\tau_n)_{n\in\mathbb{N}} \subseteq (a,b]$ converge to $\tau \in (a,b]$ for all $n \in \mathbb{N}$, let X_n be a vector field along $c|_{[0,\tau_n]}$ with $\|X\| = 1$ and $I_{\tau_n}(X_n, X_n) \leq 0$. After selecting a subsequence, X_n converges weakly in Sobolev space $H^{1,2}$ topology to some vector field X along $c|_{[a,\tau]}$. Then, we just check every ingredient of index form (see [FC13]). ⊛

Finally, let $a < t_1 < t_2 < \cdots < t_k \leq b$ be the points $c(t_i)$ conjugate to $c(a)$. Then, $\varphi_0(t) - \varphi(t) = 0$ for $t_i \in (a,b]$. Then,

$$\sum_{t\in(a,b]} \dim \mathcal{J}_c^t = \sum_{t\in(a,b]} (\varphi_0(t) - \varphi(t)) = \sum_{i=1}^k (\varphi_0(t_i) - \varphi(t_i)).$$

Since φ is left-continuous, and φ_0 is right-continuous, hence we have

$$\varphi_0(t_i) = \varphi(t_{i+1})$$

for $i = 1, \ldots, k-1$, we finally have

$$\sum_{i=1}^k (\varphi_0(t_i) - \varphi(t_i)) = \varphi_0(t_k) - \varphi(t_1).$$

From φ being left-continuous again, $\varphi(t_1) = 0$. Finally, again, from the continuity properties of φ, φ_0, they can "jump" only at the points τ where $\varphi_0(\tau) \neq \varphi(\tau)$, i.e., at the conjugate points. In particular, φ_0 is constant on $[t_k, b]$ hence, $\varphi_0(t_k) = \varphi_0(b)$, i.e.,

$$\varphi_0(b) = \sum_{t\in(a,b]} \dim \mathcal{J}_c^t.$$

∎

Intuition. The "jump" only happens at conjugate points.

Lecture 22: Bonnet-Mayers Theorem and Morse Functions

6.1.3 Bonnet-Mayers Theorem

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Definition 6.1.2 (Diameter). The *diameter* of a manifold $\mathcal{M}$ is defined as

$$\operatorname{diam}(\mathcal{M}) \coloneqq \sup_{p,q\in\mathcal{M}} d(p,q).$$

Theorem 6.1.2 (Bonnet-Mayers theorem). Let $(\mathcal{M}^n, g)$ be a complete[a] Riemannian manifold and Ricci curvature $\geq \lambda > 0$, i.e.,

$$\operatorname{Ric}(X,X) \geq \lambda \langle X, X\rangle$$

for all $X \in T\mathcal{M}$. Then the diameter of $\mathcal{M}$ is less than $\pi\sqrt{(n-1)/\lambda}$. In particular, $\mathcal{M}$ is compact and has finite fundamental group $\pi_1(\mathcal{M})$.

[a] I.e., closed and any two points can be joined by a minimizing geodesic.

Proof. For all $\rho < \operatorname{diam}(\mathcal{M})$, there exists $p, q \in \mathcal{M}$ with $d(p,q) = \rho$. As $\mathcal{M}$ complete, there exists a shortest geodesic arc $c\colon [0,\rho] \to \mathcal{M}$ with $c(0) = p$ and $c(\rho) = q$. Now, let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $T_p\mathcal{M}$, such that $e_1 = \dot{c}(0)$. Now, consider a parallel orthonormal basis $\{\dot{c}(t), X_1(t), \ldots, X_n(t)\}$ along c. Furthermore, consider $Y_i(t) \coloneqq (\sin \pi t/\rho) X_i(t)$ with $i = 2, \ldots, n$.

Then,

$$I(Y_i, Y_i) = \int_0^\rho -\langle \ddot{Y}_i, Y_i \rangle - \langle R(Y_i, \dot{c})\dot{c}, Y_i \rangle \,\mathrm{d}t = \int_0^\rho \sin^2 \frac{\pi t}{\rho} \left(\frac{\pi^2}{\rho^2} - \langle R(X_i, \dot{c})\dot{c}, X_i \rangle \right) \mathrm{d}t.$$

Since c is the shortest curve connecting p, q, it follows that there are no conjugate points between p, q. Hence, $I(Y_i, Y_i) \geq 0$ for all i, so

$$0 \leq \sum_{i=2}^n I(Y_i, Y_i) = \int_0^\rho \sin^2 \frac{\pi t}{\rho} \left(\frac{\pi^2}{\rho^2}(n-1) - R(\dot{c}, \dot{c}) \right) \mathrm{d}t \leq \left(\frac{\pi^2}{\rho^2}(n-1) - \lambda \right) \int_0^\rho \sin^2 \frac{\pi t}{\rho} \,\mathrm{d}x,$$

implying

$$0 \leq \frac{1}{2}\rho \left(\frac{\pi^2(n-1)}{\rho^2} - \lambda \right) \Rightarrow \rho^2 \leq \frac{\pi^2(n-1)}{\lambda} \Rightarrow \rho \leq \pi\sqrt{\frac{n-1}{\lambda}}.$$

Since this is true for all $\rho < \operatorname{diam}(\mathcal{M})$, hence we see that $\operatorname{diam}(\mathcal{M}) \leq \pi\sqrt{(n-1)/\lambda}$.

Furthermore, the universal cover of $\mathcal{M}$ satisfies the same assumptions as Ricci curvature, by computation, we have finite $\pi_1(\mathcal{M})$. ∎

Remark. We choose $Y_i(t) = \sin(\pi t/\rho) X_i(t)$ is just because it satisfies the needed condition, and makes the computation works out nicely.

Intuition. Bonnet-Mayers theorem says that if $\mathcal{M}$ has Ricci curvature not less than the one of S_r^n, then $\operatorname{diam}(\mathcal{M})$ is at most the one of S_r^n.

Consider the hyperbolic space $\mathbb{H}^n$ in $\mathbb{R}^{n+1}$ where we define

$$\langle x, x \rangle \coloneqq -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2$$

for $x = (x^0, \ldots, x^1)$. Then

$$\mathbb{H}^n \coloneqq \left\{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -1, x^0 > 0 \right\}.$$

Also, consider the half-space of $\mathbb{R}^n$ such that

$$H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

with metric on $\mathbb{H}^n$, we have

$$g_{ij}(x_1, \ldots, x_n) = \frac{\delta_{ij}}{x_n^2}.$$

Then, we see that we have a constant sectional curvature of -1.

6.2 Morse Theory and Flow Homology

We detour to study Morse functions, and some related topics. In what follows, we focus on critical points of functions.

6.2.1 Morse Functions

Let $(\mathcal{M}, g)$ be a complete Riemannian manifold. Let $f\colon \mathcal{M} \to \mathbb{R}$ be a smooth[1] function. Then

$$\mathrm{d}f(x) = 0$$

means that x is a critical point of f.

Definition 6.2.1 (Non-degenerate)**.** A critical point a of f is *non-degenerate* if the Hessian of f is non-singular at a.

[1] It's typically enough to ask for $f \in C^3(\mathcal{M}, \mathbb{R})$.

Definition 6.2.2 (Morse index). The *index* $\mu(p)$ of non-degenerate critical point p of f is the dimension of the largest subspace of $T_p\mathcal{M}$ on which the Hessian is negative definite.

Intuition. That is, the number of directions in which f decreases.

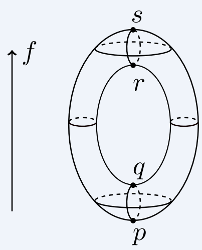
Note. The degeneracy and index are independent of coordinate choice.

Now, we define the critical set of f as

$$C(f) \coloneqq \{x \in \mathcal{M} \mid \mathrm{d}f(x) = 0\}.$$

Definition 6.2.3 (Morse function). A *Morse function* f is a function as introduced such that all critical points are non-degenerate.

Example. Consider f is the height function, which is a Morse function such that the index of s is 2, r is 1, q is 1, and p is 0 by looking at the decreasing directions.



Now, define $M^a = f^{-1}(-\infty, a]$, then we see that

(a) Pass p: M^a for $0 < a < f(q)$ is a disk, which is homotopy equivalent to a point, i.e., 0-cell.

(b) Pass q: M^a for $f(q) < a < f(r)$ is a cylinder, where we attach a 1-cell.

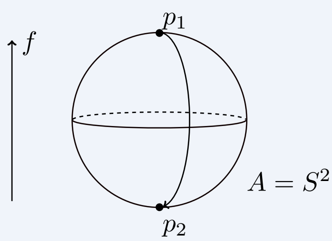
(c) Pass r: M^a for $f(r) < a < f(s)$ is a torus with disk removed.

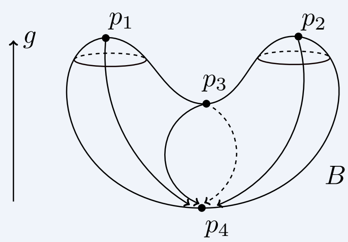
(d) Pass s: M^a for $a > f(s)$ is a torus.

Lecture 23: Morse Theory and Flow Homology

Throughout this lecture, let $\mathcal{M}$ be a compact Riemannian manifold, also, we'll keep mentioning the following example. 30 Mar. 13:00

Example. Consider the following.





Then,

- f: $\mu_f(p_1) = 2$, $\mu_f(p_2) = 0$.
- g: $\mu_g(p_1) = \mu_g(p_2) = 2$, $\mu_g(p_3) = 1$, and $\mu_g(p_4) = 0$.

Additionally, we can study some "invariants" about spaces, such as the Euler characteristic[2] of $\mathcal{M}$, which is "defined" as

$$\chi(\mathcal{M}) = \sum_{p\colon \text{critical point of } f} (-1)^{\mu(p)} \mu(p).$$

Example. $\chi(A) = \chi(S^2) = 2$, and $\chi(B) = 4 - 1 = 3$.

To make χ formal, we need to consider the homology as we did in algebraic topology, i.e., we need to define the "boundary map" and "complexes".

Intuition. Intuitively, our complexes should be a vector space built on top of critical points; on the other hand, for two critical points p, q such that $\mu(p) - \mu(q) = 1$, we want to count the trajectories from p to q modulo 2, i.e., the boundary map ∂ should be somehow defined as

$$\partial p \coloneqq \sum_{\substack{p \text{ critical point of } f \\ \mu(q) = \mu(p) - 1}} (\#\{\text{flow lines from } p \text{ to } q\} \bmod 2) \cdot q.$$

6.2.2 Morse Complex

We study so-called Morse complex, and define the so-called *flow homology*. We start by defining the so-called negative gradient flow.

Definition 6.2.4 (Negative gradient flow)**.** The *negative gradient flow* of f on $\mathcal{M}$ is defined as the solution $\phi\colon \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ of

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(x,t) = -\operatorname{grad}(f(\phi(x,t))); \\ \phi(x,0) = x & \text{for } x \in \mathcal{M}. \end{cases}$$

Note. We will simply call negative gradient flow a *flow* for simplicity.

Remark. From the Picard-Lindelöf theorem, local existence of the flow is guaranteed. Moreover, if we have "very good" conditions, such a flow may even exist globally.

More generally, the Euler characteristic is an example of a flow $\phi\colon \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ such that

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(x,t) &= -V(f(\phi(x,t))); \\ \phi(x,0) &= x \end{cases}$$

with some vector field V on $\mathcal{M}$.

Note (Autonomous)**.** We have $V(\phi(x,t))$, not $V(\phi(x,t),t)$, i.e., it doesn't explicitly depend on t.

Remark. The flow satisfies group property, i.e., $\phi(x, t_1 + t_2) = \phi(\phi(x, t_1), t_2)$ for all $t_1, t_2 \in \mathbb{R}$.

- Moreover, for all $x \in \mathcal{M}$, the flow line or orbit $\gamma_x \coloneqq \{\phi(x,t) \mid t \in \mathbb{R}\}$ through point x is flow-invariant, i.e., for all $y \in \gamma_x$, $t \in \mathbb{R}$, we have $\phi(y,t) \in \gamma_x$.
- Finally, for all $t \in \mathbb{R}$, $\phi(\cdot, t)\colon \mathcal{M} \to \mathcal{M}$ is a diffeomorphism of $\mathcal{M}$ onto its image.

Naturally, we can consider the following two kinds of points of $\mathcal{M}$.

[2]We will make it formal.

Definition 6.2.5 (Stable manifold)**.** The *stable manifold* at x_0 of the flow ϕ are defined as

$$W^s(x_0) \coloneqq \left\{ y \in \mathcal{M} \mid \lim_{t\to\infty} \phi(y,t) = x_0 \right\}.$$

Definition 6.2.6 (Unstable manifold)**.** The *unstable manifold* at x_0 of the flow ϕ are defined as

$$W^u(x_0) \coloneqq \left\{ y \in \mathcal{M} \mid \lim_{t\to-\infty} \phi(y,t) = x_0 \right\}.$$

That is to say, we should focus on the dimension of $W^u(p)$.

Intuition. For $t \to \pm\infty$, each flow line $x(t)$ defined as $x\colon \mathbb{R} \to \mathcal{M}$ with $\dot{x}(t) = -\operatorname{grad} f(x(t))$ for all $t \in \mathbb{R}$ converges to critical point, i.e.,

$$p = x(-\infty), \quad p = x(+\infty) \Rightarrow W^u(p) \text{ the all flow lines } x(t) \text{ with } x(-\infty) = p.$$

As previously seen. For a Morse function f, the set of critical points $C(f) \coloneqq \{x \in \mathcal{M} \mid \mathrm{d}f(x) = 0\}$

Notation. Denote the set of critical points of f of index k as $\mathrm{Crit}_k(f)$.

Definition 6.2.7 (Morse complex)**.** Define the vector space over $\mathbb{Z} / 2\mathbb{Z}$ as

$$C_k(f, \mathbb{Z}_2) = C_k(f) \coloneqq \left\{ \sum_{a \in \mathrm{Crit}_k(f)} m_a a \mid m_a \in \mathbb{Z}/2\mathbb{Z} \right\}.$$

Definition 6.2.8 (Boundary operator)**.** The *boundary operator* $\partial_k\colon C_k(f) \to C_{k-1}(f)$ by specifying its behavior on the basis elements. Given a critical point $a \in C_k(f)$, ∂_k sends a to a linear combination of points in $\mathrm{Crit}_{k-1}(f)$ defined as

$$\partial_k(a) = \sum_{b \in \mathrm{Crit}_{k-1}(f)} m(a,b) b$$

with $m(a,b) \in \mathbb{Z} / 2\mathbb{Z}$ being the number mod2 of trajectories from a to b.[a]

[a] We can check that $\partial \circ \partial = 0$.

6.2.3 Morse Homology

With all the notions we have established, we have the following naturally.

Definition 6.2.9 (Morse homology group)**.** The *Morse homology group* is defined as

$$H_k(\mathcal{M}, f, \mathbb{Z}_2) \coloneqq {\ker \partial \text{ on } C_k(f)} \Big/ {\operatorname{Im} \partial \text{ from } C_{k+1}(f)}.$$

Remark. The image of ∂ from $C_{k+1}(f, \mathbb{Z}_2)$ is always contained in the kernel of ∂ on $C_k(f, \mathbb{Z}_2)$.

Definition 6.2.10 (Betti number)**.** The *Betti number* is defined as $b_k \coloneqq \dim_{\mathbb{Z}_2} H_k(\mathcal{M}, f, \mathbb{Z}_2)$.

Definition 6.2.11 (Euler characteristic). The *Euler characteristic* of $\mathcal{M}$ is defined as

$$\chi(\mathcal{M}) = \sum_i (-1)^i b^i.$$

Let's now calculate all these on our examples $A = S^2$ and B.

Example. Revisit the example for f, we have

- $C_2(f) = \mathbb{Z} / 2\mathbb{Z}[p_1]$;
- $C_0(f) = \mathbb{Z} / 2\mathbb{Z}[p_2]$;
- $C_1(f) = 0$.

Also, we see that the chain complexes are

- $C_2 = \mathbb{Z}_2[p_1]$;
- $C_0 = \mathbb{Z}_2[p_2]$;
- $C_1 = 0$;

For kernels,

- $\ker \partial_2 = \{p_1\}$;
- $\ker \partial_0 = \{p_2\}$;
- and since ∂_1 is trivial, so all images are trivial.

Finally, we calculate the homology groups as

- $H_2(A, f, \mathbb{Z}_2) = \mathbb{Z}_2$;
- $H_1(A, f, \mathbb{Z}_2) = 0$;
- $H_0(A, f, \mathbb{Z}_2) = \mathbb{Z}_2$.

Example. Revisit the example for g, we have

- $C_2(g) = \mathbb{Z} / 2\mathbb{Z}[p_1]$;
- $C_1(g) = \mathbb{Z} / 2\mathbb{Z}[p_3]$;
- $C_0(g) = \mathbb{Z} / 2\mathbb{Z}[p_4]$;
- $C_k(g) = 0$ for $k \geq 3$.

Also, we see that the chain complexes are

- $\partial_2(p_1) = p_3 = \partial_2(p_2)$, and $\partial_2(p_1 + p_2) = 2p_3 = 0$;
- $\partial_1(p_3) = 2p_4 = 0$.

So the kernels are

- $\ker \partial_1 = \operatorname{Im} \partial_2 = \mathbb{Z}_2[p_3]$;
- $\ker \partial_2 = \mathbb{Z}_2[p_1 + p_2]$;
- $\ker \partial_0 = \mathbb{Z}_2[p_4]$,

and all other images and kernels are trivial. Finally, we calculate the homology groups as

- $H_2(B, g, \mathbb{Z}_2) = \ker \partial_2 = \mathbb{Z}_2$;
- $H_1(B, g, \mathbb{Z}_2) = \ker \partial_1 \,/\, \operatorname{Im} \partial_2 = 0$;
- $H_0(B, g, \mathbb{Z}_2) = \mathbb{Z}_2$.

Lecture 24: Introduction to the Rauch Comparison Theorem

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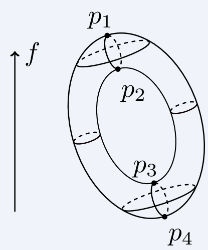
Example (Tilted torus). Consider the tilted torus



We see that

- $C_2 = \mathbb{Z}_2[p_1]$;
- $C_1 = \mathbb{Z}_2[p_2] \oplus \mathbb{Z}_2[p_3]$;
- $C_0 = \mathbb{Z}_2[p_1]$.

Moreover, the chain complex is

$$0 \xrightarrow{\partial_3} \mathbb{Z}[p_1] \xrightarrow{\partial_2} \mathbb{Z}_2[p_2] + \mathbb{Z}_2[p_3] \xrightarrow{\partial_1} \mathbb{Z}_2[p_4] \xrightarrow{\partial_0} 0$$

6.3 The Rauch Comparison Theorem

In this section, our goal is to compare Riemannian manifolds $(\mathcal{M}, g)$ with other Riemannian manifolds of constant curvatures model spaces, e.g., S^n, $\mathbb{R}^n$, and $\mathbb{H}^n$.

Notation (Model space). The set of *model spaces* is denoted as $\mathcal{M}_m \in \{S^n, \mathbb{R}^n, \mathbb{H}^n\}$.

6.3.1 Preliminary Estimations

Let $c(t)$ be a geodesic with $\|\dot{c}\| = 1$, $v \in T_{c(0)}\mathcal{M}$. Furthermore, let $\mathcal{J}(t)$ be the Jacobi field along $c(t)$ with $\mathcal{J}(0) = 0$ and $\dot{\mathcal{J}}(0) = v$ given by

$$\begin{cases} (\sin t)v, & \text{for } S^n; \\ tv, & \text{for } \mathbb{R}^n; \\ (\sinh t)v, & \text{for } \mathbb{H}^n. \end{cases}$$

Now, consider $(\mathcal{M}, g)$ such that $\lambda \leq \kappa \leq \mu$ with $\lambda \leq 0$ and $\mu \geq 0$.

Notation. For $\rho \in \mathbb{R}$,

$$c_\rho(t) = \begin{cases} \cos(\sqrt{\rho} t), & \text{if } \rho > 0; \\ 1, & \text{if } \rho = 0; \\ \cosh\left(\sqrt{-\rho} t\right), & \text{if } \rho < 0, \end{cases}$$

and also,

$$s_\rho(t) = \begin{cases} \frac{1}{\sqrt{\rho}}\sin(\sqrt{\rho}t), & \text{if } \rho > 0; \\ t, & \text{if } \rho = 0; \\ \frac{1}{\sqrt{-\rho}}\sinh\left(\sqrt{-\rho}t\right), & \text{if } \rho < 0, \end{cases}$$

These are solutions of Jacobi equations for constant sectional curvature ρ, i.e.,

$$\ddot{f}(t) + \rho f(t) = 0$$

with corresponding initial values $f(0) = 0$, $\dot{f}(0) = 1$, respectively, $f(0) = 1$, $\dot{f}(0) = 0$.

Theorem 6.3.1. Assume $\kappa \leq \mu$ and $\|\dot{c}\| \equiv 1$, and assume either $\mu \geq 0$ or $\mathcal{J}^{\tan} \equiv 0$. Let $f_\mu := |\mathcal{J}(0)|c_\mu + |\mathcal{J}|'(0)s_\mu$ solve

$$\ddot{f} + \mu f = 0$$

with $f(0) = |\mathcal{J}(0)|$ and $\dot{f}(0) = |\mathcal{J}|'(0)$. If $f_\mu(t) > 0$ for $0 < t < \tau$, then the following holds.

(a) $\langle \mathcal{J}, \dot{\mathcal{J}}\rangle f_\mu \geq \langle \mathcal{J}, \mathcal{J}\rangle \dot{f}_\mu$ on $[0, \tau]$.

(b) $1 \leq \frac{|\mathcal{J}(t_0)|}{f_\mu(t_1)} \leq \frac{|\mathcal{J}(t_2)|}{f_\mu(t_2)}$ if $0 < t_1 \leq t_2 < \tau$.

(c) $|\mathcal{J}(0)|c_\mu(t) + |\mathcal{J}|'(0)s_\mu(t) \leq |\mathcal{J}(t)|$ for $0 \leq t \leq \tau$.

Proof. Firstly, we have that

$$|\mathcal{J}|' = \frac{\langle \mathcal{J}, \dot{\mathcal{J}}\rangle}{|\mathcal{J}|}, \quad |\mathcal{J}|'' = \frac{\langle \dot{\mathcal{J}}, \dot{\mathcal{J}}\rangle}{|\mathcal{J}|} + \frac{\langle \mathcal{J}, \ddot{\mathcal{J}}\rangle}{|\mathcal{J}|} - \frac{\langle \mathcal{J}, \dot{\mathcal{J}}\rangle^2}{|\mathcal{J}|^3},$$

so

$$|\mathcal{J}|'' + \mu|\mathcal{J}| = \frac{1}{|\mathcal{J}|}\left(-\langle R(\mathcal{J}, \dot{c})\dot{c}, \mathcal{J}\rangle + \mu\langle \mathcal{J}, \mathcal{J}\rangle\right) + \frac{1}{|\mathcal{J}|^3}\left(|\dot{\mathcal{J}}|^2|\mathcal{J}|^2 - \langle \mathcal{J}, \dot{\mathcal{J}}\rangle^2\right) \geq 0$$

since $\kappa \leq \mu$ for $0 < t < \tau$, provided $\mathcal{J}$ has no zeros on $(0, \tau)$. Moreover,

$$\left(|\mathcal{J}|' f_\mu - |\mathcal{J}|\dot{f}_\mu\right)' = |\mathcal{J}|'' f_\mu - |\mathcal{J}|\ddot{f}_\mu \geq 0$$

since $\ddot{f}_\mu + \mu f_\mu = 0$ for $f_\mu(t) \geq 0$. Also, we have $|\mathcal{J}|(0) = f_\mu(0)$, $|\mathcal{J}|'(0) = \dot{f}_\mu(0)$, implying

$$|\mathcal{J}|' f_\mu - |\mathcal{J}|\dot{f}_\mu \geq 0,$$

which proves the first claim.

Furthermore,

$$\left(\frac{|\mathcal{J}|}{f_\mu}\right)' = \frac{1}{f_\mu^2}\left(|\mathcal{J}|' f_\mu - |\mathcal{J}|\dot{f}_\mu\right) \geq 0,$$

then since first zero of $\mathcal{J}$ cannot occur before the first zero of f_μ, the second claim is proved. The last claim follows directly from this. ■

Remark. $f_\mu(t) > 0$ for $0 < t < \tau$ is necessary.

Proof. Take $S^{n(\mu-\epsilon)}$ with $\mathcal{J}(0) = 0$. We see that $f_\mu(t)$ has a zero at $t = \pi/\sqrt{\mu}$ and $\mathcal{J}(t)$ has one at $t = \pi/\sqrt{\mu-\epsilon}$. For small $\epsilon > 0$ and any t, only slightly longer than $\pi/\sqrt{\mu-\epsilon}$, we have $\frac{|\mathcal{J}(t)|}{f(t)} < 1$. ⊛

Corollary 6.3.1. Suppose that $\kappa \leq \mu$, $c_\mu \geq 0$ on $(0, \tau)$, and $\mu \geq 0$ or $\mathcal{J}^{\tan} \equiv 0$. Let $\|\dot{c}\| \equiv 1$,

$\mathcal{J}(0) = 0$, $|R| < \Lambda$ with R being the curvature tensor. Then,

$$|\mathcal{J}(t) - t\dot{\mathcal{J}}(t)| \leq |\mathcal{J}(\tau)|\frac{1}{2}\Lambda t^2.$$

Theorem 6.3.2. Assume that $\lambda \leq \kappa \leq \mu$, and either $\lambda \leq 0$ or $\mathcal{J}^{\tan} \equiv 0$, $\|\dot{c}\| \equiv 1$, and $\mathcal{J}(0)$, $\dot{\mathcal{J}}(0)$ be linearly dependent. Finally, assume $s_{(\lambda+\mu)/2} > 0$ on $(0, \tau)$. Then, for $0 \leq t \leq \tau$,

$$|\mathcal{J}(t)| \leq |\mathcal{J}(0)| c_\lambda(t) + |\mathcal{J}|'(0) s_\lambda(t).$$

Proof idea. Let $\rho \in \mathbb{R}$, $\eta \coloneqq \max(\mu - \rho, \rho - \lambda)$. Let A be a vector field along c with $\ddot{A} + \rho A = 0$, $A(0) = \mathcal{J}(0)$, and $\dot{A}(0) = \dot{\mathcal{J}}(0)$.

Let $a\colon I \to \mathbb{R}$ being a solution of $\ddot{a} + (\rho - \eta)a = \eta|A|$, $a(0) = \dot{a}(0) = 0$, and $b\colon I \to \mathbb{R}$ solving $\ddot{b} + \rho b = \eta|\mathcal{J}|$, $b(0) = \dot{b}(0) = 0$. ∎

Lecture 25: Rauch Comparison Theorems and Sphere Theorem

6.3.2 Rauch Comparison Theorem

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We're now ready to provide the general statement of Rauch.

Theorem 6.3.3 (Rauch comparison theorem)**.** Let $(\mathcal{M}^m, g)$, $(\overline{\mathcal{M}}^m, \overline{g})$ be Riemannian manifolds and $\gamma\colon [0, a] \to \mathcal{M}$, $\overline{\gamma}\colon [0, a] \to \overline{\mathcal{M}}$ be normalized geodesics with $\gamma(0) = p$, $\overline{\gamma}(0) = \overline{p}$. Let $X, \overline{X}$ be Jacobi fields along $\gamma, \overline{\gamma}$, respectively such that $X(0) = \overline{X} = 0$, $|\nabla_{\dot{\gamma}(0)} X| = |\overline{\nabla}_{\dot{\overline{\gamma}}(0)} \overline{X}|$, and $\langle \dot{\gamma}(0), \nabla_{\dot{\gamma}(0)} X \rangle = \langle \dot{\overline{\gamma}}(0), \overline{\nabla}_{\dot{\overline{\gamma}}(0)} \overline{X} \rangle$. Furthermore, assume that

(a) γ has no conjugate points on $[0, a]$;

(b) sectional curvatures $K, \overline{K}$ of $\mathcal{M}, \overline{\mathcal{M}}$ satisfy $\overline{K} \leq K$ for all 2-planes containing $\dot{\gamma}, \dot{\overline{\gamma}}$.

Then, $\overline{\gamma}$ has no conjugate points on $[0, a]$, and for all $t \in [0, a]$,

$$|X(t)| \leq |\overline{X}(t)|.$$

Proof idea. To prove this, we first see a lemma.

Lemma 6.3.1. Let $(\mathcal{M}^m, g)$, $(\overline{\mathcal{M}}^m, \overline{g})$ be Riemannian manifolds and $\gamma\colon [0, a] \to \mathcal{M}$, $\overline{\gamma}\colon [0, a] \to \overline{\mathcal{M}}$ be normalized geodesics with $\gamma(0) = p$, $\overline{\gamma}(0) = \overline{p}$. Let $X, \overline{X}$ be Jacobi fields along $\gamma, \overline{\gamma}$, respectively such that $X(0) = \overline{X} = 0$. Furthermore, assume that

(a) γ has no conjugate points on $[0, a]$;

(b) sectional curvatures $K, \overline{K}$ of $\mathcal{M}, \overline{\mathcal{M}}$ satisfy $\overline{K} \leq K$ for all 2-planes containing $\dot{\gamma}, \dot{\overline{\gamma}}$.

Finally, assume that $|X(a)| = |\overline{X}(a)|$. Then, $I(X, X) \leq I(\overline{X}, \overline{X})$.

Proof idea. We first choose an orthonormal frame in $(\mathcal{M}, g)$ and $(\overline{\mathcal{M}}, \overline{g})$ with $e_1 = \dot{\gamma}$ and $\overline{e}_1 = \dot{\overline{\gamma}}$, and $e_2 = X(a)/|X(a)| \neq 0$, etc. Consider $X(t) = X^i(t) e_i(t)$ and the same for $\overline{X}$. Then, the second variation of the energy shows $I(X, X) \leq I(\overline{X}, \overline{X})$. ∎

Now, consider normal components of $X, \overline{X}$ only, and we can show that

$$\lim_{t \to 0} \frac{|\overline{X}(t)|^2}{|X(t)|^2} \eqqcolon \lim_{t \to 0} \frac{\overline{u}(t)}{u(t)} = 1,$$

thus to prove $|X| \leq |\overline{X}|$, it's enough to show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{|\overline{X}(t)|^2}{|X(t)|^2} \geq 0,$$

equivalently, $\dot{\bar{u}} - \bar{u}\dot{u} \geq 0$. Then, since γ has no conjugate points, we have $u(t) > 0$. Let $c \in [0, a]$ be the greatest number such that $\bar{u}(t) > 0$ on $(0, c)$. Then, for all $b \in (0, c)$, define

$$X_b(t) = \frac{X(t)}{|X(b)|}, \quad \overline{X}_b(t) = \frac{\overline{X}(t)}{|\overline{X}(b)|}.$$

From Lemma 6.3.1 to $I(X_b, X_b), I(\overline{X}_b, \overline{X}_b)$, then we're done. ■

Corollary 6.3.2. Let $(\mathcal{M}, g)$ be a complete and simply-connected Riemannian manifold with non-positive sectional curvature, and $\triangle ABC$ is a geodesic triangle in $\mathcal{M}$, then

(a) $|AB|^2 + |AB|^2 - 2|AB||AC| \cos\angle A \leq |BC|^2$;

(b) $\angle A + \angle B + \angle C \leq \pi$.

Corollary 6.3.3. Suppose that sectional curvature of $(\mathcal{M}, g)$ satisfies

$$0 < C_1 \leq K \leq C_2$$

for some constants C_1, C_2. Let γ be any geodesic in $\mathcal{M}$. Then, the distance d between any two conjugate points of γ satisfies

$$\frac{\pi}{\sqrt{C_2}} \leq d \leq \frac{\pi}{\sqrt{C_1}}.$$

Corollary 6.3.4. Let $(\mathcal{M}, g)$ be compact Riemannian manifold where the sectional curvature K satisfies $K \leq C$ for some constant C. Then, either the injectivity radius

$$i(\mathcal{M}, g) \geq \pi/\sqrt{C},$$

or there exists a closed geodesic γ in $\mathcal{M}$ whose length is minimal among all closed geodesics such that

$$i(\mathcal{M}, g) \geq \frac{1}{2}L(\gamma).$$

6.4 The Sphere Theorem

In this section, we want to prove the following.

Theorem 6.4.1 (Sphere theorem)**.** Let $\mathcal{M}^n$ be a compact and simply-connected Riemannian manifold with sectional curvature K such that

$$0 < hK_{\max} < K \leq K_{\max}.$$

Then if $h = 1/4$, then $\mathcal{M}$ is homeomorphic to a sphere S^n.

Notation (Pinching number)**.** h in the sphere theorem is called the *pinching number* of $\mathcal{M}$.

Remark. Another version of the sphere theorem is to assume $0 < h < K \leq 1$ by scaling.

To prove this, Borger [Ber60], Klingenberg [Kli61] used Rauch comparison theorem with Morse index theorem in the 1960s.

6.4.1 Gauss-Bonnet Theorem and Theorem by Hamilton

To understand the sphere theorem, we should consider $n = 2, 3$. In this case, it suffices to assume $h \geq 0$, i.e., for a compact and simply-connected Riemannian manifold $\mathcal{M}^n$ with $n = 2, 3$ such that it has positive sectional curvature, then $\mathcal{M}^n$ is homeomorphic to S^n.

Note. For $n = 2$, it follows from the Gauss-Bonnet theorem, and for $n = 3$, it follows from a theorem by R. Hamilton.

Theorem 6.4.2 (Gauss-Bonnet theorem). Let $\mathcal{M}$ be a compact connected 2-dimensional Riemannian manifold $\mathcal{M}$ with Gauss curvature K. Then, its characteristic is given by

$$\chi(\mathcal{M}) = \frac{1}{2\pi} \int_{\mathcal{M}} K \, \mathrm{d}\mu_{\mathcal{M}}.$$

The Gauss-Bonnet theorem generalizes the so-called Gauss-Bonnet formula.

Note (Gauss-Bonnet formula). Let γ be a curved polygon on an oriented Riemannian 2-manifold $(\mathcal{M}, g)$ such that γ is positive oriented as the boundary of an open set Ω with compact closure. Then,

$$\int_{\Omega} K \, \mathrm{d}A + \int_{\gamma} k_N \, \mathrm{d}s + \sum_i \epsilon_i = 2\pi,$$

where $k_N(t) = \langle \mathrm{D}_t \dot{\gamma}(t), N(t) \rangle$,[a] and ϵ_i are the exterior angles.

[a] $N(t)$ is the normal vector field.

To understand all these, we need the following concept.

Definition 6.4.1 (Smooth triangulation). For $\mathcal{M}$ smooth, compact 2-manifold, a *smooth triangulation* of $\mathcal{M}$ is a finite collection of curved triangles such that

- the union of the closed regions $\overline{\Omega}_i$ bounded by the triangles is actually $\mathcal{M}$;
- the intersection of any pair (if not empty) is either a single vertex of each or a single edge of each.

Theorem 6.4.3 (Radó [Rad25]). Every compact topological 2-manifold has a triangulation.

Note. Let $\mathcal{M}$ be a triangulated 2-manifold. Then, the Euler characteristic is

$$\chi(\mathcal{M}) = N_v - N_e + N_f.$$

This implies that

$$\int_{\mathcal{M}} K \, \mathrm{d}A = 2\pi\chi(\mathcal{M}).$$

Then, we can start proving Gauss-Bonnet theorem.

Proo of Theorem 6.4.2. Let $\{\Omega_i\}_{i=1}^N$ denote the faces of triangulation, and for all i, let $\{\gamma_{ij} \mid j = 1, 2, 3\}$ be the edges of Ω_i and $\{\theta_{ij} \mid j = 1, 2, 3\}$ be its interior angles.

As each exterior angle is π minus the interior angle, by applying the Gauss-Bonnet formula to each triangle and sum over i, we have

$$\begin{aligned}
&\sum_{i=1}^{N_f} \int_{\Omega_i} K \, \mathrm{d}A + \sum_{i=1}^{N_f} \sum_{j=1}^{3} \int_{\gamma_{ij}} k_N \, \mathrm{d}s + \sum_{i=1}^{N_f} \sum_{j=1}^{3} (\pi - \theta_{ij}) = \sum_{i=1}^{N_f} 2\pi \\
\Leftrightarrow &\int_{\mathcal{M}} K \, \mathrm{d}\mu_{\mathcal{M}} + 0 + 3\pi N_f - \sum_{i=1}^{N_f} \sum_{j=1}^{3} \theta_{ij} = 2\pi N_f
\end{aligned}$$

where the second term vanishes since each edge appears twice but with opposite sign. Since degrees at each vertex adds up to 2π, we have

$$\int_{\mathcal{M}} K \,\mathrm{d}A = 2\pi N_v - \pi N_f.$$

As each edge is in exactly 2 triangles and each triangle has 3 edges, we see that $2N_e = 3N_f$,

$$\int_{\mathcal{M}} K \,\mathrm{d}A = 2\pi N_v - 2\pi N_e + 2\pi N_f = 2\pi\chi(\mathcal{M}).$$

■

Theorem 6.4.4 (Hamilton). Let $\mathcal{M}$ be a compact and simply-connected 3-dimensional Riemannian manifold $\mathcal{M}$ with strictly positive Ricci curvature. Then, $\mathcal{M}$ is diffeomorphic to S^3.

Lecture 26: Toward Proving the Sphere Theorem

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Corollary 6.4.1. Let $\mathcal{M}$ be a compact 2-dimensional Riemannian manifold, and K be the Gauss curvature.

(a) If $\mathcal{M}$ is homeomorphic to the sphere or the projective plane, then $K > 0$ somewhere.

(b) If $\mathcal{M}$ is homeomorphic to torus or Klein bottle, then either $K = 0$ or K takes on both positive and negative values.

(c) If $\mathcal{M}$ is any other compact surfaces, then $K < 0$ somewhere.

Corollary 6.4.2. Let $\mathcal{M}$ be a compact 2-dimensional Riemannian manifold, and K be the Gauss curvature.

(a) If $K > 0$, then $\mathcal{M}$ is homeomorphic to sphere or projective plane, and $\pi_i(\mathcal{M})$ is finite.

(b) If $K \leq 0$, then $\pi_1(\mathcal{M})$ is infinite and $\mathcal{M}$ has genus at least 1.

6.4.2 Beyond 2-Dimension

To go beyond 2-dimension, some consider the so-called phaffian.

Definition 6.4.2 (Pfaffian). Let $\mathcal{P}$ be the map from $(0,4)$-tensors to $\mathbb{R}$ with the domain carries symmetries as Riemannian curvature.

Theorem 6.4.5. On any oriented vector space, there exists a basis independent functions $\mathcal{P}$ such that for all compact, even-dimension Riemannian manifold $\mathcal{M}$,

$$\int_{\mathcal{M}} \mathcal{P}(R) \,\mathrm{d}V = \frac{1}{2}\operatorname{vol}(S^n)\chi(\mathcal{M}).$$

Note. This is too much information swallowed...

Another approach is to consider the following.

Notation. Let $p \in \mathcal{M}$, then $d_p \colon \mathcal{M} \to \mathbb{R}$ such that $d_p(q) = \operatorname{dist}(p, q)$.

We see that d_p is Lipschitz continuous and smooth on $\mathcal{M} \setminus (\{p\} \cup \operatorname{Cut}(p)) =: \mathcal{M}_p$. At nay $q \in \mathcal{M}_p$, the gradient ∇d_p is the tangent vector at q of the unique normal minimizing geodesic from p to q. In particular, $\lvert \nabla d_p \rvert = 1$ at most points of $\mathcal{M}$. Now, we want to compare distance functions on different manifolds. This requires comparing the Hessian.

As previously seen (Hessian)**.** For all smooth function f on $\mathcal{M}$, its Hessian $\nabla^2 f$ is defined as

$$\nabla^2 f(X,Y) = \langle \nabla_X \nabla f, Y\rangle.$$

The Hessian of f is symmetric, and we can write $\Delta f = \operatorname{Tr}(\nabla^2 f)$.

Notation. Let $K^+ := \max_{\sigma\subseteq T_p\mathcal{M}} K(\sigma)$ and $K^- := \min_{\sigma\subseteq T_p\mathcal{M}} K(\sigma)$.

Theorem 6.4.6 (Hessian comparsion theorem)**.** Let $(\mathcal{M}, g)$, $(\widetilde{\mathcal{M}}, \widetilde{g})$ be complete Riemannian manifolds, and $\gamma\colon [0,b]\to\mathcal{M}$ and $\widetilde{\gamma}\colon [0,b]\to\widetilde{\mathcal{M}}$ be minimizing normal geodesics in $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, respectively, such that

$$\widetilde{K}^+(t) \leq K^-(t)$$

for all $t\in[0,b]$. Denote $q=\gamma(a)$, $\widetilde{q}=\widetilde{\gamma}(a)$ for $a\leq b$. Suppose $X_q\in T_q\mathcal{M}$, $\widetilde{X}\in T_{\widetilde{q}}\widetilde{\mathcal{M}}$ satisfy

$$\langle X_q, \dot{\gamma}(a)\rangle = \langle X_{\widetilde{q}}, \dot{\widetilde{\gamma}}(a)\rangle$$

and $|X_q| = |\widetilde{X}_{\widetilde{q}}|$. Then,

$$\nabla^2 d_p(X_q, X_q) \leq \widetilde{\nabla}^2 \widetilde{d}_{\widetilde{p}}(\widetilde{X}_{\widetilde{q}}, \widetilde{X}_{\widetilde{q}}).$$

6.4.3 Toponogor Theorem

We now state the main tools we need in order to prove the sphere theorem.

Definition. Let $(\mathcal{M}, g)$ be a complete Riemannian manifold.

Definition 6.4.3 (Geodesic triangle)**.** A *geodesic triangle* $\triangle ABC$ consists of 3 points $A,B,C\in\mathcal{M}$ and 3 minimizing geodesics (sides) $\gamma_{AB},\gamma_{BC},\gamma_{CA}$ joining each 2 of them.

Definition 6.4.4 (Generalized geodesic triangle)**.** A *generalized geodesic triangle* $\triangle ABC$ consists of 3 points $A,B,C\in\mathcal{M}$ and 2 minimizing geodesics γ_{AB},γ_{AC} and 1 geodesic γ_{BC} of length $L(\gamma_{BC})\leq L(\gamma_{AB})+L(\gamma_{AC})$, joining each 2 of them.

Definition 6.4.5 (Geodesic hinge)**.** A *geodesic hinge* $\angle BAC$ consists of a point $A\in\mathcal{M}$ and 2 minimizing geodesics γ_{AB},γ_{AC} emanating from A with endpoints B,C.

Definition 6.4.6 (Generalized geodesic hinge)**.** A *generalized geodesic hinge* $\angle BAC$ consists of a point $A\in\mathcal{M}$ and 2 geodesics γ_{AB},γ_{AC} emanating from A with endpoints B,C, with only one is minimizing.

For all $k\in\mathbb{R}$, denote $\mathcal{M}_k^n$ the n-dimensional space form of constant curvature k, i.e.,

$$\mathcal{M}_k^n = S^n(k) \text{ or } \mathbb{R}^n \text{ or } \mathbb{H}^n(k).$$

Lemma 6.4.1. Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold with sectional curvature $K\geq k$.

(a) For all generalized geodesic hinge $\angle BAC$ in $\mathcal{M}$, there exists a geodesic hinge $\angle\widetilde{B}\widetilde{A}\widetilde{C}$ in $\mathcal{M}_k^n$ with the same angle and corresponding sides are with the same length as $\angle BAC$.

(b) For all generalized geodesic triangle $\triangle ABC$ in $\mathcal{M}$, there exists a geodesic triangle $\triangle\widetilde{A}\widetilde{B}\widetilde{C}$ in $\mathcal{M}_k^n$ whose corresponding sides have the same length as $\triangle ABC$.

Theorem 6.4.7 (Toponogor theorem)**.** Let $(\mathcal{M}, g)$ be a complete Riemannian manifold with sectional curvature $K \geq k$.

(a) Let $\angle BAC$ be a geodesic hinge in $\mathcal{M}$ and $\angle \widetilde{B}\widetilde{A}\widetilde{C}$ in $\mathcal{M}^n_k$. Then, $\operatorname{dist}(B, C) = \operatorname{dist}(\widetilde{B}, \widetilde{C})$.

(b) Let $\triangle ABC$ be a geodesic triangle in $\mathcal{M}$, $\triangle \widetilde{A}\widetilde{B}\widetilde{C}$ in $\mathcal{M}^n_k$. Then, the 3 angles in $\triangle ABC$ are greater than the corresponding angles in $\triangle \widetilde{A}\widetilde{B}\widetilde{C}$.

Theorem 6.4.8 (Klingenberg)**.** Let $(\mathcal{M}, g)$ be a complete, simply-connected Riemannian manifold with sectional curvature $1/4 < K \geq 1$. Then,

$$\operatorname{Inj}(\mathcal{M}, g) \geq \pi.$$

6.4.4 Proof of the Sphere Theorem

Now, we can prove the sphere theorem. Let's first restate it (after scaling) for our reference.

Theorem 6.4.9 ((Scaled) Sphere theorem)**.** Let $\mathcal{M}^n$ be a compact and simply-connected Riemannian manifold with sectional curvature K such that

$$\frac{1}{4} < K \leq 1.$$

Then $\mathcal{M}$ is homeomorphic to a sphere S^n.

Proof. By Bonnet-Mayers theorem, we know that $\mathcal{M}$ is compact, hence there exists $k > 1/4$ such that $k \leq K \leq 1$. By the Klingenberg theorem,

$$\ell = \operatorname{diam}(\mathcal{M}, g) \geq \operatorname{Inj}(\mathcal{M}, g) \geq \pi > \frac{\pi}{2\sqrt{k}}.$$

Take $p, q \in \mathcal{M}$ such that $\operatorname{dist}(p, q) = \operatorname{diam}(\mathcal{M}, g)$. Let $q_0 \in \mathcal{M}$ such that $\ell_1 = \operatorname{dist}(p, q_0) > \pi/2\sqrt{k}$, and γ_1 be a minimizing normal geodesic connecting $p = \gamma_1(0)$ and $q_0 = \gamma_1(\ell_1)$. Then, consider the following.

Lemma 6.4.2. Let $(\mathcal{M}, g)$ be a compact Riemannian manifold where there exists $p, q \in \mathcal{M}$ such that $\operatorname{dist}(p, q) = \operatorname{diam}(\mathcal{M}, g)$. Then, for all $X_p \in T_p\mathcal{M}$, there exists a minimizing geodesic γ connecting $p = \gamma(0)$ to q such that

$$\langle \dot{\gamma}(0), X_p \rangle \geq 0.$$

From this, there exists a minimizing normal geodesic γ_2 connecting $p = \gamma_2(0)$ to $q = \gamma_2(\ell)$ such that $\langle \dot{\gamma}_1(0), \dot{\gamma}_2(0) \rangle \geq 0$, i.e., the angle α between $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ is no more than $\pi/2$. According Toponogor theorem, by looking at the geodesic hinge $\angle q_1 pq$, $\operatorname{dist}(q_1, q) \leq \operatorname{dist}(\widetilde{q}_1, \widetilde{q})$ for a comparison geodesic hinge $\angle \widetilde{q}_1 \widetilde{p} \widetilde{q}$ in $\mathcal{M}^n_k = S^n(1/\sqrt{k})$. Now, due to the cosine law for $S^n(1/\sqrt{k})$,

$$\cos\left(\sqrt{k} \cdot \operatorname{dist}(q_{,1}, q)\right) \geq \cos\left(\sqrt{k} \cdot \operatorname{dist}(\widetilde{q}_1, \widetilde{q})\right) = \cos\sqrt{k}\ell \cdot \cos\sqrt{k}\ell_1 + \sin\sqrt{k}\ell \cdot \sin\sqrt{k}\ell_1 \cos\alpha \geq \cdots > 0.$$

Lecture 27: Uniformization Theorem

Before continue the proof of Theorem 6.4.9, we need one more tool. 13 Apr. 13:00

Theorem 6.4.10 (Brown's theorem)**.** Let $\mathcal{M}$ be a smooth and compact manifold. If $\mathcal{M} = U_1 \cap U_2$ with U_1, U_2 open subsets in $\mathcal{M}$ homeomorphic to $\mathbb{R}^n$, then $\mathcal{M}$ is homeomorphic to S^n.

We can then finish the proof.

Proof of Theorem 6.4.9 (Continue). We have shown that

$$\overline{B_{\frac{\pi}{2\sqrt{k}}}(p)} \cup \overline{B_{\frac{\pi}{2\sqrt{k}}}(q)} = \mathcal{M}.$$

Denote r to be

$$r := \frac{1}{2}\left(\operatorname{Inj}(\mathcal{M}, g) + \frac{\pi}{2\sqrt{k}}\right) > \frac{\pi}{2\sqrt{k}},$$

then $\mathcal{M} = B_r(p) \cup B_r(q)$. Moreover, since $r < \operatorname{Inj}(\mathcal{M}, g)$, both $B_r(p)$ and $B_r(q)$ are homeomorphic to $\mathbb{R}^n$. By Brown's theorem, the result follows. ■

6.4.5 The Family of the Sphere Theorem

The sphere theorem doesn't hold for $1/4 \leq K \leq 1$.[3]

Example. The complex projective spaces $\mathbb{CP}^n$ are also compact, simply connected Riemannian manifold such that $1/4 \leq K \leq 1$. But they are not homeomorphic to S^{2m}.

However, this is "almost" the only counterexample in the following sense.

Theorem 6.4.11. Let $\mathcal{M}^n$ be a compact and simply-connected Riemannian manifold.

(a) If m is even, then there exists $\epsilon(m) > 0$ such that $1/4 - \epsilon(m) \leq K \leq 1$, then $\mathcal{M}$ is either homeomorphic to S^n or diffeomorphic to either $\mathbb{CP}^{m/2}$, $\mathbb{HP}^{m/4}$, or $\mathbb{C}_a\mathbb{P}^2$ [Ber83].

(b) If m is odd, then there exists $\epsilon > 0$ such that if $1/4 - \epsilon \leq K \leq 1$, then $\mathcal{M}$ homeomorphic to S^n [AM96].

Looking back,

- Rauch in 1951 proved the sphere theorem for $3/4 < K \leq 1$ [Rau51];
- Klingenberg in 1959 proved the sphere theorem for $0.55 < K \leq 1$ [Kli59];
- Berger in 1960 proved the sphere theorem for $1/4 < K \leq 1$ when m is even [Ber60];
- Klingenberg in 1961 proved the sphere theorem [Kli61].

Now, what if we want diffeomorphism? "Exotic spheres" exists: manifolds that are homeomorphic to sphere but not diffeomorphic.

Example (J. Milnor)**.** If $n = 7$, we can construct as S^3-bundles over S^4.

Consider $m = 2$, from the Gauss-Bonnet theorem, $\mathcal{M}$ is diffeomorphic to S^2 since

$$0 < \int_{\mathcal{M}} K \,\mathrm{d}A = 2\pi\chi(\mathcal{M}),$$

with the fact that S^2 is the only such object. As for $m = 3$, by the Hamilton's proof, if $(\mathcal{M}, g)$ is a 3-dimension compact Riemannian manifold with Ricci curvature > 0 then $(\mathcal{M}, g)$ is diffeomorphic to S^3 using Ricci flow, e.g., [BS08].

[3] Where we originally have $1/4 < K \leq 1$.

Chapter 7

Epilogue

In the end of this long journey, we wrap up this course by showing some results and directions that can be further explored based on what we have learned.

7.1 Uniformization Theorem

Recall the Gauss-Bonnet theorem, where we let $(\mathcal{M}, g)$ be a compact, 2-dimension, oriented Riemannian manifold without boundary. And let K be the Gauss curvature, γ be the genus of $\mathcal{M}$, then,

$$\int_{\mathcal{M}} K \, \mathrm{d}\mu_g = 2\pi\chi.$$

In particular, there are three cases:

(a) $\gamma = 1$, then $\chi = 0$;

(b) $\gamma \geq 2$, then χ is negative integer;

(c) $\gamma = 0$, then $\chi = 2$.

Now, we want to generalize it. To do this, we need the following notion.

Definition 7.1.1 (Conformal). A metric g is *conformal* to another metric $\widetilde{g}$ if there exists a positive smooth function Ω on $\mathcal{M}$ such that

$$\widetilde{g} = \Omega^2 \circ g$$

has constant Gauss curvature.

Note (Constant rescaling). If we take $\Omega = c$ for some constant, then $\widetilde{K} = c^{-2} K$.

Now we state the uniformization theorem.

Theorem 7.1.1 (Uniformization theorem). If g is conformal to a metric of constant Gauss curvature, then

$$\widetilde{K} = \begin{cases} 0, & \text{if } \gamma = 1; \\ -1, & \text{if } \gamma \geq 2; \\ 1, & \text{if } \gamma = 0. \end{cases}$$

First, we check how Gauss curvature transforms under conformal transformations. Consider an n-dimensional Riemannian manifold $(\mathcal{M}, g)$ and $\widetilde{g}_{ij} = \Omega^2 g_{ij}$, then

$$\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \Omega^{-1}(\delta^k_i \partial_j \Omega + \delta^k_j \partial_i \Omega - g^{k\ell} g_{ij} \partial_\ell \Omega).$$

Moreover, we can also compute how do the Riemannian curvature, Ricci curvature, and also sectional curvature transform.

In particular, for $n = 2$, set $\Omega = e^u$ for some smooth function u, then the Gauss curvature K is transformed as

$$\widetilde{K} = e^{-2u}(K - \Delta_g u).$$

Now, we want to find some smooth solutions u of

$$\Delta_g u + \widetilde{K} e^{2u} = K$$

when $\gamma = 0$, $\chi = 2$, we have

$$\Delta_g u + e^{2u} = K.$$

Note. The maximum principle does not work.

7.1.1 Proof of the Uniformization Theorem

We now start to prove the uniformization theorem. We follow 5 steps to prove the theorem.

1. Find points $p, o \in \mathcal{M}$ such that $\mathrm{dist}(p, o) = \mathrm{diam}(\mathcal{M})$.
2. Find a function w on $\mathcal{M} \setminus \{p\}$ such that $\widetilde{g} = e^{2w} g$ on $\mathcal{M} \setminus \{p\}$ makes $(\mathcal{M} \setminus \{p\}, \widetilde{g})$ isometric to a plane, i.e., $\widetilde{K} = 0$. From here, we get $\Delta_g w - K = 0$ on $\mathcal{M} \setminus \{p\}$. So naturally, we define w to be a solution of

$$\Delta_g w - K = -4\pi\delta_p$$

on $\mathcal{M}$, where δ_p is the delta Dirac distribution at p.
3. Let $\widetilde{d}_o$ be the distance function on $(\mathcal{M} \setminus \{p\}, \widetilde{g})$ from o. Set

$$e^v = \frac{1}{1 + \widetilde{d}_o^2/4},$$

then metric $\widetilde{\widetilde{g}} = 2^{2v}\widetilde{g}$ has Gauss curvature $\widetilde{\widetilde{K}} = 1$. Namely, $(\mathcal{M} \setminus \{p\}, \widetilde{\widetilde{g}})$ will be isometric to a standard sphere minus the North Pole N.
4. Set $u = w + v$, then we show that this function extends continuously to the point p, i.e., $\widetilde{\widetilde{g}} = e^{2u} g$ is a metric on $\mathcal{M}$ such that $(\mathcal{M}, \widetilde{\widetilde{g}})$ is isometric to a standard sphere.
5. Show that u is smooth on $\mathcal{M}$.[1]

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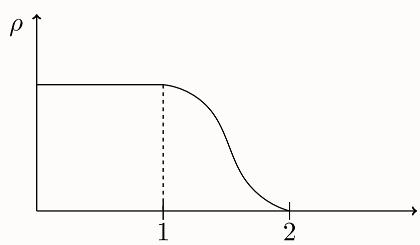
18 Apr. 13:00

Proof of Theorem 7.1.1. We now start our proof following the 5 steps.

1. This step is trivial.
2. Use the exponential map. Let r_p be the injectivity radius (relative to $(\mathcal{M}, g)$) of $\exp_p$. Choose $\epsilon > 0$ such that $2\epsilon < r_p$. Let ρ be a C^∞ non-increasing function on $[0, \infty)$ such that $\rho = 1$ on $[0, 1]$ and $\rho = 0$ on $[2, \infty)$.



[1] $w \to \infty$ while $v \to -\infty$ as we approach p. To show $u = w + v$, we need to analyze the blow-behavior for w and v.

Define the cur-off function for $q \in \mathcal{M}$ such that

$$\eta(q) = \begin{cases} \rho(d_p/\epsilon), & \text{if } q \in B_{2\epsilon}(p); \\ 0, & \text{otherwise.} \end{cases}$$

Namely, $\eta = 1$ on $B_\epsilon(p)$ and $\eta = 0$ on $\mathcal{M} \setminus B_{2\epsilon}(p)$. Define on $\mathcal{M}$ the function

$$w_o = \begin{cases} -2\eta \log d_p, & \text{in } B_{2\epsilon}(p); \\ 0, & \text{on } \mathcal{M} \setminus B_{2\epsilon}(p). \end{cases}$$

In $B_\epsilon(p)$, $w_0 = -2\log d_p$. As $2\epsilon < r_p$, $\exp_p$ is a diffeomorphism of the ball of radius 2ϵ with center 0 in $T_p\mathcal{M}$ onto $B_{2\epsilon}(p)$ in $\mathcal{M}$. Consider choosing a polar normal coordinate (r, θ) in $R_{2\epsilon}(p)$ such that $\mathrm{d}\rho = r$ and $g = \mathrm{d}r^2 + R^2(r,\theta)\mathrm{d}\theta^2$ such that

$$\int_0^\pi R(r,\theta)\,\mathrm{d}\theta = L(r)$$

be the perimeter of the geodesic circles. Then $L(r)/r \to 2\pi$ as $r \to 0$ by local euclidicity at p.

> **Notation** (Geodesic curavture). The *geodesic curvature* of circles κ is defined as $\kappa = \frac{1}{R}\frac{\partial R}{\partial r}$.

We have

$$\frac{\partial \kappa}{\partial r} = -\kappa^2 - K.$$

Now, express the Laplace operator Δ in polar coordinates.

> **Remark.** In arbitrary coordinates, we have
>
> $$\Delta_g = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^a} \cdot \sqrt{\det g}(g^{-1})^{ab}\frac{\partial}{\partial x^b}.$$

Then in polar coordinates,

$$\Delta_g = \frac{1}{R}\frac{\partial}{\partial r} \cdot R\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} \cdot \frac{1}{R}\frac{\partial}{\partial \theta},$$

and in $B_{\epsilon(p)}$, we have $\omega_0 = -2\log r$ and

$$\Delta_g \omega_0 = -\frac{2}{R}\frac{\partial R}{\partial r}\left(\frac{R}{r}\right) = -\frac{2\lambda}{r}$$

with $\lambda = \frac{1}{R}\frac{\partial R}{\partial r} - \frac{1}{r} = \kappa - \frac{1}{r}$. Hence,

$$\underbrace{\frac{\partial \kappa}{\partial r} + \kappa^2}_{-K} = \frac{\partial \lambda}{\partial r} + \frac{2\lambda}{r} + \lambda^2.$$

Set $\mu = r^2\lambda$, it becomes

$$\frac{\partial \mu}{\partial r} + \frac{\mu^2}{r^2} = -r^2 K.$$

Since as $r \to 0$, $\mu \to 0$, hence along each ray, the integral equation $\mu(r,\theta)$ is

$$\mu(r,\theta) = -\int_0^r \frac{\mu(r',\theta)^2}{r'^2} + r'^2 K(r',\theta)\,\mathrm{d}r',$$

and we have $\lambda(r,\theta) = O(r)$. Moreover, $\lambda/r \to -K_p/3$ as $r \to 0$. It follows that $\Delta_g\omega_0$ is bounded and $\Delta_g\omega_0 \to 2K_p/3$ as approaching p.

Now, set $\omega = \omega_0 + \omega_1$, then ω_1 has to satisfy

$$\Delta_g \omega_1 = \Delta_g \omega - \Delta_g \omega_0 = K - \Delta_g \omega_0$$

on $\mathcal{M} \backslash p$. Let $f = K - \Delta_g \omega_0$ be a function on $\mathcal{M}$. We can show that f extends to a continuous function on $\mathcal{M}$.

Claim. There is a solution w_1 of $\Delta_g w_1 = f$ unique up to an additive constant, provided that

$$\int_{\mathcal{M}} f \,\mathrm{d}\mu_g = 0.$$

Proof. To prove this, we integrate f on $\mathcal{M} \setminus B_\delta(P)$ with $0 < \delta \leq \epsilon$, i.e.,

$$-\int_{\mathcal{M} \backslash B_{\epsilon(p)}} \Delta_g w_0 \,\mathrm{d}\mu_g = \int_{\partial B_\delta(p)} \nabla_N w_0 \,\mathrm{d}s,$$

where $\mathrm{d}s$ is the element of arc length of $\partial B_\delta(p)$. In $\overline{B}_\delta(p)$ we have, in polar coordinates, $w_0 = -2\log r$ and $\nabla_N = \partial/\partial r$, so $\nabla_N w_0 = -2/r$. Moreover, it is $\mathrm{d}s = R\,\mathrm{d}\theta$. So, we have

$$\int_{\partial B_\delta(p)} \nabla_N w_0 \,\mathrm{d}s = -\frac{2}{\delta}\int_0^{2\pi} R(\delta, \theta)\,\mathrm{d}\theta \to -4\pi \text{ as } \delta \to 0.$$

On the other hand,

$$\lim_{\delta \to 0} \int_{\mathcal{M} \backslash B_\delta(p)} K \,\mathrm{d}\mu_g = \int_{\mathcal{M}} K \,\mathrm{d}\mu_g = 4\pi$$

by Gauss-Bonnet. We conclude that indeed $\int_{\mathcal{M}} f \,\mathrm{d}\mu_g = 0$. ⊛

So, the equation is solvable for w_1. In fact, we can show that w_1 is bounded on $\mathcal{M}$ and is in fact continuous.[a]

3. Step 3 is also trivial.
4. Now, it is $u = w + v$ where $w = w_0 + w_1$, and w_1 is bounded on $\mathcal{M}$, while $w_0 = -2\eta \log d_p$ and d_p is the g-distance from p. On the other hand,

$$e^v = \frac{1}{1 + \widetilde{d}_o^2/4},$$

where $\widetilde{d}_o$ is the $\widetilde{g}$-distance from o. Hence,

$$v = -2\log \widetilde{d}_o + O(1).$$

5. It follows that u is bounded on $\mathcal{M}$ if and only if in $B_{\epsilon(p)}$ (relative to g)$d_p \cdot \widetilde{d}_o$ is bounded above and below by positive constants.[b]

■

[a] For, $f = \Delta_g w_1$ being bounded, in particular $f \in L^2(\mathcal{M})$ implies $w_1 \in H_2(\mathcal{M})$, hence w_1 is bounded.
[b] For detail, see the note.

7.1.2 Yamabe Problem

From the proof of uniformization theorem, the following problem arises.

Problem 7.1.1 (Yamabe poroblem). Given a compact Riemannian manifold $\mathcal{M}, g$ of dimension $n \geq 3$. Find a metric $\widetilde{g}$ conformal to g such that the scalar curvature of $\widetilde{g}$ is constant.

If $\mathcal{M}$ has no boundary, then Aubin [Aub76b; Aub76a], Schoen [Sch84], Trudinger [Tru68] solves it.

With boundary, Escober [Esc92]. If we write $g = u^{\frac{4}{n-2}} g_0$, the scalar curvature R_g is

$$R_g = u^{-\frac{u+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_g u \right). \tag{7.1}$$

g has constant scalar curvature c if and only if u is a solution of the *Yamabe equation*

$$\frac{4(n-1)}{n-2} \Delta_{g_0} u - R_{g_0} u + c u^{\frac{n+2}{n-2}} = 0. \tag{7.2}$$

To solve this, consider the variational approach, where we have the following.

Definition 7.1.2 (Einstein-Hilbert action)**.** The *Einstein-Hilbert action* $\mathcal{E}(g)$ is defined as

$$\mathcal{E}(g) = \frac{\int_{\mathcal{M}} R_g \,\mathrm{d}\,\mathrm{vol}_g}{\mathrm{vol}(\mathcal{M}, g)^{\frac{n-2}{n}}}.$$

Definition 7.1.3 (Einstein metric)**.** A metric g is called an *Einstein metric* if $\mathrm{Ric}(g) = c \cdot g$ for some constant c.

Remark. A metric g is a critical point of $\mathcal{E}$ if and only if g is an Einstein metric.

Given any positive function u, consider the *Yamabe functional*

$$\mathcal{E}_{g_0}(u) = \mathcal{E}(u^{\frac{4}{n-2}} g_0).$$

Equation 7.1 implies

$$\mathcal{E}_{g_0}(u) = \frac{\int_{\mathcal{M}} \left(\frac{4(n-1)}{n-2} \lvert \mathrm{d} u_{g_0} \rvert^2 + R_{g_0} u^2 \right) \mathrm{d}\,\mathrm{vol}_{g_0}}{\left(\int_{\mathcal{M}} u^{\frac{2n}{n-2}} \,\mathrm{d}\,\mathrm{vol}_{g_0} \right)^{\frac{n-2}{n}}}.$$

u is a critical point if and only if u satisfies the Yamabe equation.

7.2 Lorentzian Manifolds and General Relativity

Consider

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

Remark. We have $g(X, X) = -(X^0)^2 + \sum_{i=1}^{3} (X_i)^2$.

Definition 7.2.1 (Arc length)**.** The *arc length* of causal curve γ between 2 points corresponding to parameter values $\lambda = a$ and $\lambda = b$ is

$$L[\gamma](a, b) = \int_a^b \sqrt{-g(\dot{\gamma}(\lambda), \dot{\gamma}(\lambda))} \,\mathrm{d}\lambda.$$

If $q \in \mathcal{J}^+(p)$, define temporal distance q from p as $\tau(q, p) = \sup L[\gamma]$ over all future-directed casual curves (p, q).

As previously seen. In the Riemannian manifold case, we have Hopf-Rinow theorem.

The analogous for Lorentzian manifolds is for maximization, i.e., it holds if spacetime admits Cauchy hypersurfaces when the supremum is achieved, and metric C', maximizing curve is a causal geodesic.

7.3 Ricci Flow

The basic idea of Ricci flow is to consider the metric g is changing over time, i.e., the shape of the manifold changes w.r.t. $g(t)$, described by an O.D.E. related to the Ricci curvature as

$$\frac{\partial g(t)}{\partial t} = -2\operatorname{Ric}(g).$$

Appendix

Appendix A

Additional Notes

A.1 The $C^\infty(\mathcal{M})$-Module Viewpoint of Tensor Fields

To start this section, we need some primarily tools.

Definition A.1.1 (Left module)**.** Suppose R is a ring with 1. A *left R-module M* consists of an Abelian group $(M, +)$ and an operation $\cdot\colon R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$,

(a) $r \cdot (x + y) = r \cdot x + r \cdot y$;

(b) $(r + s) \cdot x = r \cdot x + s \cdot x$;

(c) $(rs) \cdot x = r \cdot (s \cdot x)$;

(d) $1 \cdot x = x$.

Note. A *right R-module M* can also be defined similarly by consider $\cdot\colon M \times R \to M$.

Definition A.1.2 (Module)**.** If R is commutative, then the left and right R-module M are the same, and we call M a *module*.

Intuition. We're basically relaxing the notion of $\mathbb{F}$-vector space, but this time, the field $\mathbb{F}$ is replaced by a ring R.

Remark. The most noticeable difference between a module and a vector space is that a module usually don't have a basis.

The reason why we introduce the notion of module is because of the following: we can understand tensor field better in the following way. Observe that $\Gamma(T\mathcal{M}) = \{X\colon \text{vector fields on } \mathcal{M}\}$ is actually a $C^\infty(\mathcal{M})$-module:

Claim. $\Gamma(T\mathcal{M})$ carries a natural $C^\infty(\mathcal{M})$-module structure.

Proof. Firstly, observe that $C^\infty(\mathcal{M}) = \big((C^\infty(\mathcal{M}), +, \cdot)\big)$ is not a field but a ring.[a] Then, naturally, the $C^\infty(\mathcal{M})$-module $(\Gamma(T\mathcal{M}), \oplus, \odot)$ where

- $\oplus$: $(X \oplus \widetilde{X})(f) \coloneqq (Xf) + \widetilde{X}(f)$;
- $\odot$: $(g \odot X)(f) \coloneqq g \cdot X(f)$,

for $X, \widetilde{X} \in \Gamma(T\mathcal{M})$, $g, f \in C^\infty(\mathcal{M})$. ⊛

[a] Since given $f \in C^\infty(\mathcal{M})$, we might not have f^{-1}.

Notation. Notice that given a vector field $X\colon \mathcal{M} \to T\mathcal{M}$ with $p \mapsto X(p)$, we let

$$Xf\colon \mathcal{M} \to \mathbb{R}, \quad p \mapsto X(p)f.$$

This makes sense since we can't always do things globally, e.g., Hairy ball theorem. Specifically, we can't choose a basis $X_1, \ldots, X_d \in \Gamma(T\mathcal{M})$ for our vector field globally as we already know. Similarly, we can define $\Gamma(T^*\mathcal{M})$, i.e., the set of "convector field"[1] is again a $C^\infty(\mathcal{M})$-module.

Example. Given $\omega \in \Gamma(T^*\mathcal{M})$ and $X \in \Gamma(T\mathcal{M})$, ω acts on X to yield smooth functions by point-wise evaluation, i.e., we define

$$(\omega(X))\,(p) := \omega(p)(X(p)).$$

Then, the action of ω on X is a $C^\infty(\mathcal{M})$-linear map since

$$(\omega(fX))(p) = f(p)\omega(p)(X(p)) = (f\omega)(p)(X(p)) = (f\omega(X))(p)$$

for $f \in C^\infty(\mathcal{M})$. This suggests that we should not regard ω just as a section of $T^*\mathcal{M}$, but also a linear mapping of $X \in \Gamma(T\mathcal{M})$ into $C^\infty(\mathcal{M})$.

Then, in this view point, we have the following.

Definition A.1.3 (Tensor field*). A (r,s)-*tensor field* T on a smooth manifold $\mathcal{M}$ is a $C^\infty(\mathcal{M})$ multilinear map

$$T\colon \underbrace{\Gamma(T^*\mathcal{M}) \times \cdots \times \Gamma(T^*\mathcal{M})}_{r} \times \underbrace{\Gamma(T\mathcal{M}) \times \ldots \Gamma(T\mathcal{M})}_{s} \to C^\infty(\mathcal{M}).$$

Comparing to Definition 2.4.14, this definition is more general!

Example. The linear connection ∇ $(X,Y) \mapsto \nabla_X Y$ does not define a tensor field.

Proof. Since ∇ is only $\mathbb{R}$-linear in Y. ⊛

A.2 Lie Groups and Lie Algebra

A.2.1 Lie Groups

Lie groups are an important topic to study for Riemannian geometry, hence we now introduce it.

Definition A.2.1 (Lie group). A *Lie group* is a group G with a differentiable structure such that the mapping $G \times G \to G$ given by $(x,y) \to xy^{-1}$, $x, y \in G$, is differentiable.

Definition (Transformation). Let G be a Lie group.

Definition A.2.2 (Left transformation). The *translations from the left* $L_x\colon G \to G$ is defined as $L_x(y) = xy$.

Definition A.2.3 (Right transformation). The *translations from the right* $R_x\colon G \to G$ is defined as $R_x(y) = yx$.

Remark. Both L_x and R_x are diffeomorphisms.

In the following discussion, let G be a Lie group. Turns out that G admits some nice properties on left invariant vector fields.

[1] We won't define it formally, but it's defined similarly.

Definition (Invariant of Riemannian metric). Let g be a Riemannian metric on G.

Definition A.2.4 (Left invariant). g is *left invariant* if

$$\langle u, v\rangle_y = \langle \mathrm{d}(L_x)_y u, \mathrm{d}(L_x)_y v\rangle_{L_x(y)}$$

for all $x, y \in G$, $u, v \in T_y G$, i.e., L_x is an isometry.

Definition A.2.5 (Right invariant). g is *right invariant* if

$$\langle u, v\rangle_y = \langle \mathrm{d}(R_x)_y u, \mathrm{d}(R_x)_y v\rangle_{R_x(y)}$$

for all $x, y \in G$, $u, v \in T_y G$, i.e., R_x is an isometry.

Definition A.2.6 (Bi-invariant). g is *bi-invariant* if it's both right and left invariant.

Definition (Invariant of vector field). Let X be a vector field on G.

Definition A.2.7 (Left invariant). X is *left invariant* if $\mathrm{d}L_x X = X$ for all $x \in G$.

Definition A.2.8 (Right invariant). X is *right invariant* if $\mathrm{d}R_x X = X$ for all $x \in G$.

Definition A.2.9 (Bi-invariant). X is *bi-invariant* if it's both right and left invariant.

As we mentioned, the left invariant vector fields are completely determined by their values at a single point of G, which allows us to introduce an additional structure on the tangent space to the neutral element $e \in G$ in the following manner.

To each vector $X_e \in T_e G$, we associate the left invariant X defined by

$$X_a \coloneqq \mathrm{d}L_a X_e, \quad a \in G.$$

A.2.2 Lie Algebras

Let X, Y be left invariant vector fields on G. Since for each $x \in G$ and for any differentiable function f on G,

$$\mathrm{d}L_x[X, Y]f = [X, Y](f \circ L_x) = X(\mathrm{d}L_x Y)f - Y(\mathrm{d}L_x X)f = (XY - YX)f = [X, Y]f,$$

i.e., $[X, Y]$ is again a left invariant vector field if X, Y are. Now, if $X_e, Y_e \in T_e G$, we put $[X_e, Y_e] = [X, Y]_e$.

Definition A.2.10 (Lie algebra). Given a Lie group G, the *Lie algebra* $\mathfrak{g}$ is the vector space $T_e G$ with the bracket $[\cdot, \cdot]$.

Note. The elements in the Lie algebra $\mathfrak{g}$ will be thought of either as vectors in $T_e G$ or as left invariant vector fields on G.

To introduce a left invariant metric on g, take any arbitrary inner product $\langle \cdot, \cdot\rangle_e$ on $\mathfrak{g}$ and define

$$\langle u, v\rangle_x \coloneqq \langle (\mathrm{d}L_{x^{-1}})_x(u), (\mathrm{d}L_{x^{-1}})_x(v)\rangle_e \tag{A.1}$$

for $x \in G$, $u, v \in T_x G$. Since L_x depends differentiably on x, this is actually a Riemannian metric, which is clearly left invariant.

Remark. We can also construct a right invariant metric on G, and if G is compact, G possesses a bi-invariant metric.

One important characterization for G having a bi-invariant metric is that the inner product that the metric determines on $\mathfrak{g}$ satisfies the following relation.

Proposition A.2.1. If G has a bi-invariant metric, then for any $U, V, X \in \mathfrak{g}$, the inner product that the metric determines on $\mathfrak{g}$ satisfies

$$\langle [U, X], V\rangle = -\langle U, [V, X]\rangle .$$

Proof. See do Carmo [FC13, Page 40, 41]. ■

The important point about this relation is that it characterizes the bi-invariant metrics of G in the following sense.

Remark. If a positive bilinear form $\langle\cdot,\cdot\rangle_e$ defined on $\mathfrak{g}$ satisfies this relation, then the Riemannian metrics defined on G by Equation A.1 is bi-invariant.

A.2.3 Lie Subalgebra

Consider (h_t^X) be a local 1-parameter group for a vector field X, and let $\Gamma(T\mathcal{M})$ still denotes the set of all vector fields, but now view it as just an $\mathbb{R}$-vector space. Then, we revise Definition A.2.10 as follows.

Definition A.2.11 (Lie algebra*)**.** Let $\mathcal{M}$ be a smooth manifold, the $(\Gamma(T\mathcal{M}), [\cdot,\cdot])$ is the *Lie algebra.*

This induces the following.

Definition A.2.12 (Lie subalgebra)**.** Let $X_1, \ldots, X_n$ be n vector fields on $\mathcal{M}$ such that for all i, j,

$$[X_i, X_j] = C_{ij}^k X_k$$

for $C_{ij}^k \in \mathbb{R}$. Then, $L \coloneqq (\operatorname{span}_{\mathbb{R}}(\{X_1, \ldots, X_n\}), [\cdot,\cdot])$ is called a *Lie subalgebra.*

Notation (Structure constant)**.** C_{ij}^k in Definition A.2.12 are called *structure constants.*

Example. On S^2, given $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$, we have

$$(\operatorname{span}_{\mathbb{R}}(\{X_1, X_2, X_3\}), [\cdot,\cdot]) = \mathrm{so}(3).$$

Definition A.2.13 (Symmetry)**.** A finite-dimensional Lie subalgebra $(L, [\cdot,\cdot])$ is said to be a *symmetry* of a metric tensor field g if for every $X \in L$ and $t \in \mathbb{R}$,

$$g\big((h_t^X)_*(A), (h_t^X)_*(B)\big) = g(A, B).$$

This means that $(h_t^X)_*$ defines an isometry.

Note. Or equivalently, $(h_t^X)^* g = g$ where for $\varphi\colon \mathcal{M} \to \mathcal{M}$,

$$(\varphi^* g)(X, Y) \coloneqq g(\varphi_p(X), \varphi_p(Y)).$$

A.2.4 Lie Derivatives

Observe that for all $X \in L$ with the corresponding local 1-parameter group (h_t^X), if

$$\mathcal{L}_X \coloneqq \lim_{t\to\infty} \frac{(h_t^X)^* g - g}{t} = 0,$$

then L is a symmetry of g.

Definition A.2.14 (Lie derivative)**.** The *Lie derivative* $\mathcal{L}$ on a smooth manifold $\mathcal{M}$ sends a pair of a vector field X and a (p, q)-tensor field to a (p, q)-tensor field such that

(a) $\mathcal{L}_X f = Xf$;

(b) $\mathcal{L}_X Y = [X, Y]$;

(c) $\mathcal{L}_X(T + S) = \mathcal{L}_X T + \mathcal{L}_X S$;

(d) $\mathcal{L}_X(T(\omega, Y)) = (\mathcal{L}_X T)(\omega, Y) + T(\mathcal{L}_X \omega, Y) + T(\omega, \mathcal{L}_X Y)$, similarly for any other valence of T;

(e) $\mathcal{L}_{X+Y} T = \mathcal{L}_X T + \mathcal{L}_Y T$.

Remark. ∇_X is $C^\infty(\mathcal{M})$-linear in the lower slot, while $\mathcal{L}_X$ is not.

Intuition. Study neighboring fibers using a local 1-parameter group of diffeomorphisms $(\psi_t)_{t \in I}$.

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