

MATH597

Analysis II

Pingbang Hu

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\stackrel{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition.

Additionally, we'll use [FF99] as our main text, while using [Tao13] and [Axl19] as supplementary references.

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Lecture 1: σ -algebra

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1 Measure

Example. Before we start, we first see some examples.

1. Let $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) := \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which is the *power set* of X . We see that

$$\#X = n \implies \#\mathcal{P}(X) = 2^n$$

for $n < \infty$.

2. If $n = \infty$, say $X = \mathbb{N}$, then

$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while \mathbb{N} is a countable set. We can see this as follows. Consider

$$\phi: \mathcal{P}(\mathbb{N}) \rightarrow [0, 1], \quad A \mapsto 0.a_1a_2a_3 \dots \text{ (base 2),}$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example, A can be $A = \{2, 3, 6, \dots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have

$$\#\mathcal{P}(\mathbb{N}) \geq \# [0, 1].$$

But since $[0, 1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X . Hence, we are intrigued to define a map μ such that

$$\mu: \mathcal{P}(X) \rightarrow [0, \infty].$$

Example. We first see some examples.

1. Let $X = \{0, 1, 2\}$. Then we want to define $\mu: \mathcal{P}(X) \rightarrow [0, \infty]$, we can have

- $\mu(A) = \#A$. Then we have
 - $\mu(\{0, 1\}) = 2$
 - $\mu(\{0\}) = 1$
- $\mu(A) = \sum_{i \in A} 2^i$. Then we have
 - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$

2. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu: \mathcal{P}(\mathbb{N}) \rightarrow [0, \infty]$, we can have

- $\mu(A) = \#A$. Then we have
 - $\mu(\{2, 3, 4, 5, \dots\}) = \infty = \mu(\{\text{even numbers}\})$
- $\mu(A) = e^{-1} \sum_{i \in A} \frac{1}{i!}$. Then we have
 - $\mu(\{0, 2, 4, 6, \dots\}) = e^{-1} (1 + \frac{1}{2!} + \frac{1}{3!} + \dots)$
- $\mu(A) = \sum_{i \in A} a_i$

3. Let $X = \mathbb{R}$. Then we want to define $\mu: \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$, we can have

- $\mu(A) = \#A$
- $\mu((a, b)) = b - a$.

Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

Answer. No!

- $\mu((a, b)) = e^b - e^a$.

Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

Answer. No!

We immediately see the problems. To extend our native measure method into \mathbb{R} is hard and will cause something counter-intuitive!¹ Hence, rather than define measurement on *all* subsets in the power set of X , we only focus on *some* subsets. In other words, we want to define

$$\mu: \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \rightarrow [0, \infty].$$

1.1 σ -algebras

Definition 1.1 (σ -algebra). Let X be a set. A collection \mathcal{A} of subsets of X , i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a σ -algebra on X if

- $\emptyset \in \mathcal{A}$.
- \mathcal{A} is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- \mathcal{A} is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\emptyset}_{\in \mathcal{A}}$ and \mathcal{A} is closed under complement.
- $\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c$, namely \mathcal{A} is closed under countable intersections.
- $A_1 \cup A_2 \cup \dots \cup A_n = A_1 \cup A_2 \cup \dots \cup A_n \cup \emptyset \cup \emptyset \cup \dots$, hence \mathcal{A} is closed under finite unions and intersections.

Lecture 2: Measure

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Example. Again, we first see some examples.

1. Let $\mathcal{A} = \mathcal{P}(X)$, which is the power σ -algebra.
2. Let $\mathcal{A} = \{\emptyset, X\}$, which is a trivial σ -algebra.
3. Let $B \subset X$, $B \neq \emptyset$, $B \neq X$. Then we see that $\mathcal{A} = \{\emptyset, B, B^c, X\}$ is a σ -algebra.

Lemma 1.1. Let \mathcal{A}_α , $\alpha \in I$, be a family of σ -algebra on X . Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is a σ -algebra on X .

¹https://en.wikipedia.org/wiki/Banach-Tarski_paradox

Remark. Notice that I may be an uncountable intersection.

Proof. A simple proof can be made as follows. Firstly, $\emptyset \in \mathcal{A}_\alpha$ for every α clearly. Moreover, closure under complement and countable unions for every \mathcal{A}_α implies the same must be true for $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$. Hence, $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ -algebra. ■

The above allows us to give the following definition.

Definition 1.2 (Generation of σ -algebra). Given $\mathcal{E} \subset \mathcal{P}(X)$, where \mathcal{E} is not necessarily a σ -algebra. Let $\langle \mathcal{E} \rangle$ be the intersection of all σ -algebras on X containing \mathcal{E} , then we call $\langle \mathcal{E} \rangle$ the σ -algebra generated by \mathcal{E} .

Remark. Clearly, $\langle \mathcal{E} \rangle$ is the smallest σ -algebra containing \mathcal{E} , and it is unique. To check the uniqueness, we suppose there are two different $\langle \mathcal{E} \rangle_1$ and $\langle \mathcal{E} \rangle_2$ generated from \mathcal{E} . It's easy to show

$$\langle \mathcal{E} \rangle_1 \subseteq \langle \mathcal{E} \rangle_2,$$

and by symmetry, they are equal.

Example. We see that $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle$.

Lemma 1.2. We have

1. Given \mathcal{A} a σ -algebra, $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \mathcal{A}$
2. $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$

Proof. We'll see that after proving the first claim, the second follows smoothly.

1. The first claim is trivial, since we know that $\langle \mathcal{E} \rangle$ is the smallest σ -algebra containing \mathcal{E} , then if $\mathcal{E} \subset \mathcal{A}$, we clearly have $\langle \mathcal{E} \rangle \subset \mathcal{A}$ by the definition.
2. The second claim is also easy. From the first claim and the definition, we have

$$\mathcal{E} \subset \mathcal{F} \subset \langle \mathcal{F} \rangle \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle.$$

■

At this point, we haven't put any specific structure on X . Now we try to describe those spaces with good structure, which will give the space some nice properties.

Definition 1.3 (Borel σ -algebra). For a topological space X , the *Borel σ -algebra on X* , denoted as $\mathcal{B}(X)$, is the σ -algebra generated by the collection of all open sets in X .

Example. We see that $\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a, b) \mid a < b; a, b \in \mathbb{R}\}$.
- $\mathcal{E}_2 = \{[a, b] \mid a < b; a, b \in \mathbb{R}\}$ since $[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.
- $\mathcal{E}_3 = \{(a, b] \mid a < b; a, b \in \mathbb{R}\}$ since $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$.
- $\mathcal{E}_4 = \{[a, b) \mid a < b; a, b \in \mathbb{R}\}$ since $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b)$.
- $\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}$ since $(a, \infty) = \bigcup_{n=1}^{\infty} (a, a + n)$.
- $\mathcal{E}_6 = \{[a, \infty) \mid a \in \mathbb{R}\}$ since $[a, \infty) = \bigcup_{n=1}^{\infty} [a, a + n)$.
- $\mathcal{E}_7 = \{(-\infty, b) \mid b \in \mathbb{R}\}$ since $(-\infty, b) = \bigcup_{n=1}^{\infty} (b - n, b)$.
- $\mathcal{E}_8 = \{(-\infty, b] \mid b \in \mathbb{R}\}$ since $(-\infty, b] = \bigcup_{n=1}^{\infty} (b - n, b]$.

Proposition 1.1. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \dots, 8$.

Proof. Firstly, we see that $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \implies \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$ by Lemma 1.2. Secondly, by definition, $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$ where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R}\}.$$

It's enough to show $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$ since if so, $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$, and clearly $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$, then we will have $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$. Let $O \subset \mathbb{R}$ be an open set, i.e., $O \in \mathcal{E}$. We claim that every open set in \mathbb{R} is a countable union of disjoint open intervals.²

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

where I_j open interval with the form of $(a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$.

For example, \mathcal{E}_1 is trivially true, and

$$(a, b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{[a + \frac{1}{n}, b - \frac{1}{n}]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for \mathcal{E}_2 and

$$(a, \infty) = \bigcup_{k=1}^{\infty} (a, a + k)$$

²<https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals>

shows the case for \mathcal{E}_5 . It's now straightforward to check open intervals are in $\langle \mathcal{E}_i \rangle$ for every i . ■

Now, to put a structure on a space, we define the following.

Definition 1.4 (Measurable space). (X, \mathcal{A}) is called a *measurable space*, and $E \in \mathcal{A}$ is called an *\mathcal{A} -measurable set*.

1.2 Measures

With the definition of measurable space, we now can refine our measure function μ as follows.

Definition 1.5 (Measure). Given a measurable space on (X, \mathcal{A}) , a *measure* is a function μ such that

$$\mu: \mathcal{A} \rightarrow [0, \infty]$$

with

1. $\mu(\emptyset) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_1, A_2, \dots \in \mathcal{A}$ are **disjoint**. We call this *Countable additivity*.

We denote (X, \mathcal{A}, μ) a *measure space*.

Notation. We denote $[0, \infty] := [0, \infty) \cup \{\infty\}$.

Remark. The motivation of why we only want *countable additivity* but not uncountable additivity can be seen by the following example. We'll consider the most intuitive measure on $\mathbb{R}, \mathcal{B}(\mathbb{R})$.

Since we have

$$(0, 1] = \left(\frac{1}{2}, 1\right] \cup \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left(\frac{1}{8}, \frac{1}{4}\right] \cup \dots$$

and also

$$(0, 1] = \bigcup_{x \in (0, 1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2}, 1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4}, \frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8}, \frac{1}{4}])} + \dots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0, 1]} 0$$

since $\mu(x) = 0$ for $x \in \mathbb{R}$, which is clearly not what we want.

Example. We see some examples.

1. For any (X, \mathcal{A}) , we let $\mu(A) := \#A$. This is called *counting measure*.
2. Let $x_0 \in X$. For any (X, \mathcal{A}) , the *Dirac measure at x_0* is

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A. \end{cases}$$

3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$,

$$\mu(A) = \sum_{i \in A} a_i,$$

where $a_1, a_2, \dots \in [0, \infty)$.

Lecture 3: Construct a Measure

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Note. If $A, B \in \mathcal{A}$ and $A \subset B$, then

$$\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

Theorem 1.1. Given (X, \mathcal{A}, μ) be a measure space.

1. (monotonicity) $A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B)$.
2. (countable subadditivity) $A_1, A_2, \dots \in \mathcal{A} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$
3. (continuity from below/ monotone convergence theorem (MCT) for sets)

$$\begin{cases} A_1, A_2, \dots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \dots \end{cases} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

4. (continuity from above)

$$\begin{cases} A_1, A_2, \dots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \dots \\ \mu(A_1) < \infty \end{cases} \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Proof. We prove this theorem one by one.

1. Since $A \subset B$, hence we have

$$\mu(B) = \mu\left(\underbrace{(B \setminus A) \cup A}_{\text{disjoint}}\right) \stackrel{!}{=} \underbrace{\mu(B \setminus A) + \mu(A)}_{\geq 0} \geq \mu(A).$$

2. This should be trivial from [countable additivity](#) with the fact that $\mu(A) \geq 0$ for all A .

DIY!

3. Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$, then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

are a disjoint union and $B_i \in \mathcal{A}$, hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i).$$

With $\mu\left(\bigcup_{i=1}^n B_i\right) = \mu(A_n)$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n B_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

4. Let $E_i = A_1 \setminus A_i \implies E_i \in \mathcal{A}$, $E_1 \subset E_2 \subset \dots$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right),$$

which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i\right) \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$$

since $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu(A_1) < \infty$. Then from [continuity from below](#), we further have

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(E_n) = \mu(A_1) - \lim_{n \rightarrow \infty} (\mu(A_1) - \mu(A_n)).$$

From monotonicity, we see that $\mu(A_n) \leq \mu(A_1) < \infty$, hence we can split the limit and further get

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu(A_1) + \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \mu(A_n).$$

■

Example. Given $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting measure})$. Then we see

- $A_n = \{n, n+1, n+2, \dots\} \implies \mu(A_n) = \infty$
- $A_1 \supset A_2 \supset A_3 \supset \dots$
- $\bigcap_{i=1}^{\infty} A_i = \emptyset \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = 0$

Remark. We see that in this case, since $\mu(A_1) \not\leq \infty$, hence [continuity from above](#) doesn't hold.

We now try to characterize some properties of a measure space.

Definition 1.6 (μ -null, μ -subnull, Complete measure space). Given (X, \mathcal{A}, μ)

- $A \subset X$ is a μ -null set if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- $A \subset X$ is a μ -subnull set if $\exists \mu$ -null set B such that $A \subset B$. Note that A is not necessarily \mathcal{A} -measurable.
- (X, \mathcal{A}, μ) is a *complete* measure space if every μ -subnull set is \mathcal{A} -measurable.

There are some useful terminologies we'll use later relating to μ -null.

Definition 1.7 (Almost everywhere). Given (X, \mathcal{A}, μ) , a statement $P(x)$, $x \in X$ holds μ -almost everywhere (*a.e.*) if the set

$$\{x \in X : P(x) \text{ does not hold}\}$$

is μ -null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

Definition 1.8 (finite measure). Given (X, \mathcal{A}, μ)

- μ is a *finite measure* if $\mu(X) < \infty$.
- μ is a σ -finite measure if $X = \bigcup_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.

Exercise. Every measure space can be **completed**. Namely, we can always find a bigger σ -algebra to complete the space.

1.3 Outer Measures

We start by giving a definition.

Definition 1.9 (Outer measure). An *outer measure* on X is a map

$$\mu^*: \mathcal{P}(X) \rightarrow [0, \infty]$$

such that

- $\mu^*(\emptyset) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$
- (countable subadditivity) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ for every $A_i \subset X$.

Example. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

is an outer measure due to the [Proposition 1.2](#) we're going to show.

Remark. We see that an outer measure need not be a measure. Check the [Definition 1.5](#) for a measure function.

Proposition 1.2. Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\emptyset, X \in \mathcal{E}$. Let

$$\rho: \mathcal{E} \rightarrow [0, \infty]$$

such that $\rho(\emptyset) = 0$. Then

$$\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

is an outer measure on X .

Note. Recall the Tonelli's Theorem³ for series:

If $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Specifically, in [\[Tao13\]](#) Theorem 0.0.2.

Lecture 4: Carathéodory extension Theorem

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As previously seen. We now prove the [Proposition 1.2](#).

Proof. We need to prove

³https://en.wikipedia.org/wiki/Fubini%27s_theorem

- μ^* is well-defined. i.e., inf is taken over a non-empty set. This is trivial since $X \in \mathcal{E}$ and $X \supset A$ for any $A \in \mathcal{E}$.
- $\mu^*(\emptyset) = 0$. Since $\emptyset \in \mathcal{E}$ and

$$\mu^*(\emptyset) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \emptyset \right\} = 0$$

since $\rho(\emptyset) = 0$ for all i and further, by Squeeze Theorem⁴, we see that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(\emptyset) = 0.$$

- $A \subset B \implies \mu^*(A) \leq \mu^*(B)$. We simply show this by contradiction. Suppose $A \subset B$ and $\mu^*(A) > \mu^*(B)$, then by definition of μ^* , we have

$$\begin{aligned} \mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\} \\ &> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B \right\} = \mu^*(B). \end{aligned}$$

Now, let $B = (B \setminus A) \cup A$, then we have

$$\begin{aligned} \mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\} \\ &> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B \setminus A) \cup A \right\} = \mu^*(B). \end{aligned}$$

Now, since $B \setminus A \supseteq \emptyset$, then this inequality can't hold, hence a contradiction⁵.

- Countable subadditivity. Let $A_1, A_2, \dots \in X$. If one of $\mu^*(A_n) = \infty$, then result holds. So we may assume $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Now, fix any $\epsilon > 0$, we will show that

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon.$$

For each $n \in \mathbb{N}$, $\exists E_{n,1}, E_{n,2}, \dots \in \mathcal{E}$ such that

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n$$

and

$$\mu^*(A_n) + \frac{\epsilon}{2^n} \geq \sum_{k=1}^{\infty} \rho(E_{n,k}).$$

Then we see that

$$\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k) \in \mathbb{N}^2} E_{k,n},$$

⁴https://en.wikipedia.org/wiki/Squeeze_theorem

⁵This is an important trick!!

which implies

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{(n,k) \in \mathbb{N}^2} \rho(E_{k,n}) \stackrel{!}{=} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{k,n}) \leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right)$$

from the inequality just derived. Now, since the last term is just

$$\sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon,$$

hence we finally have

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon$$

for arbitrarily small fixed $\epsilon > 0$, hence the subadditivity is proved. ■

Definition 1.10 (Carathéodory measurable). Let μ^* be an outer measure on X . We say $A \subset X$ is *Carathéodory measurable* (*C-measurable*) with respect to μ^* if

$$\forall E \subset X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Lemma 1.3. Let μ^* be an outer measure on X . Suppose B_1, \dots, B_N are disjoint C-measurable sets. Then,

$$\forall E \subset X, \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \right) \right) = \sum_{i=1}^N \mu^*(E \cap B_i).$$

Proof. Since we have

$$\begin{aligned} \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \right) \right) &= \mu^*(E' \cap B_1) + \mu^*(E' \setminus B_1) \stackrel{6}{=} \\ &= \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \cap B_1 \right) \right) + \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \right) \cap B_1^c \right) \\ &= \mu^*(E \cap B_1) + \mu^* \left(E \cap \left(\bigcup_{i=2}^N B_i \right) \right) \end{aligned}$$

where the equality comes from the fact that B_1 is C-measurable and disjoint from $B_i, i \neq 1$. Then, we simply iterate this argument and have the result. ■

Remark. This implies that if we restrict an outer measure on C-measurable set, then it becomes finite additive.

⁶Here, $E' := E \cap \left(\bigcup_{i=1}^N B_i \right)$ for the simplicity of notation.

Theorem 1.2 (Carathéodory extension Theorem). Let μ^* be an outer measure on X . Let \mathcal{A} be the collection of C-measurable sets (with respect to μ^*). Then,

1. \mathcal{A} is a σ -algebra on X .
2. $\mu = \mu^*|_{\mathcal{A}}$ is a measure on (X, \mathcal{A}) .
3. (X, \mathcal{A}, μ) is a complete measure space.

Proof. We divide the proof in several steps.

1. We show \mathcal{A} is a σ -algebra by showing

- (a) $\emptyset \in \mathcal{A}$. To show this, we simply check that \emptyset is C-measurable. We see that

$$\forall_{E \subset X} \mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E \setminus \emptyset) = \mu^*(E),$$

which just shows $\emptyset \in \mathcal{A}$.

- (b) \mathcal{A} closed under complements. This is equivalent to say that if A is C-measurable, so is A^c . We see that if A is C-measurable, then for every $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Observing that $E \cap A = E \setminus A^c$ and $E \setminus A = E \cap A^c$, hence

$$\mu^*(E) = \mu^*(E \setminus A^c) + \mu^*(E \cap A^c).$$

We immediately see that above implies $A^c \in \mathcal{A}$.

- (c) \mathcal{A} closed under countable unions.

Note. To show \mathcal{A} closed under countable unions, we show that \mathcal{A} is closed under:

finite unions $\xRightarrow{\text{then}}$ countable disjoint unions $\xRightarrow{\text{then}}$ countable unions.

- We show \mathcal{A} is closed under finite unions.

Claim. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

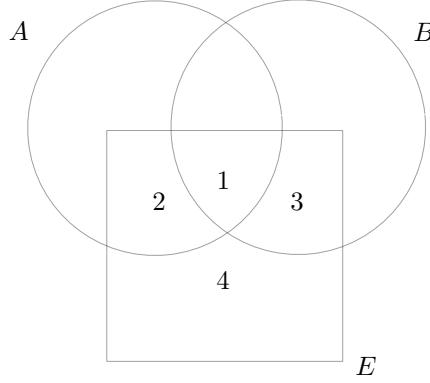
Fix $E \subset X$ arbitrary. We need to show that

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)),$$

i.e.,

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

given $A, B \in \mathcal{A}$.



- Since A is \mathcal{C} -measurable,
 - * $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$
 - * $\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$
- Since B is \mathcal{C} -measurable,
 - * $\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$

Hence, we have

$$\begin{aligned} \mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4). \end{aligned}$$

- We show \mathcal{A} is closed under countable disjoint unions.

Let $A_1, A_2, \dots \in \mathcal{A}$ and disjoint. Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^{\infty} A_i\right),$$

hence we only need to show another way around.

Fix $N \in \mathbb{N}$, we have $\bigcup_{n=1}^N A_n \in \mathcal{A}$ since N is finite, and

$$\begin{aligned} \mu^*(E) &= \mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right) + \mu^*\left(E \setminus \left(\bigcup_{n=1}^N A_n\right)\right) \\ &\geq \underbrace{\sum_{n=1}^N \mu^*(E \cap A_n)}_{\stackrel{!}{=} \mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right)} + \underbrace{\mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right)}_{\leq \mu^*\left(E \setminus \left(\bigcup_{n=1}^N A_n\right)\right)} \\ &\stackrel{!}{=} \mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right) + \mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right) \end{aligned}$$

Now, take $N \rightarrow \infty$ then we are done.

- We show \mathcal{A} is closed under countable unions.

DIY

The proof will be *continued*...

Lecture 5: Hahn-Kolmogorov Theorem

14 Jan. 11:00

Firstly, we see a stronger version of [Lemma 1.3](#) we have seen before.

Lemma 1.4. Let μ^* be an outer measure on X . Suppose B_1, B_2, \dots are disjoint C-measurable sets. Then,

$$\forall E \subset X, \mu^* \left(E \cap \left(\bigcup_{i=1}^{\infty} B_i \right) \right) = \sum_{i=1}^{\infty} \mu^* (E \cap B_i).$$

Proof.

$$\sum_{n=1}^{\infty} \mu^* (E \cap B_i) \geq \mu^* \left(E \cap \bigcup_{n=1}^{\infty} B_n \right) \geq \mu^* \left(E \cap \left(\bigcup_{n=1}^N B_n \right) \right) \stackrel{!}{=} \sum_{n=1}^N \mu^* (E \cap B_n).$$

Now, we just take $N \rightarrow \infty$ (or note that $N \in \mathbb{N}$ is arbitrary, we then get the result according to Squeeze Theorem⁷). ■

Let's continue the proof of [Theorem 1.2](#).

2. Since from [Definition 1.5](#), we need to show

- $\mu(\emptyset) = 0$. This means that we need to show $\mu^*|_{\mathcal{A}}(\emptyset) = 0$. Since $\emptyset \in \mathcal{A}$ and μ^* is an outer measure, hence from the [property](#) of outer measure, it clearly holds.
- [Countable additivity](#) of μ^* on \mathcal{A} follows from the [Lemma 1.4](#) with $E = X$

3. HW. ■

1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets \mathcal{E} and any ρ such that it assigns measure on \mathcal{E} , then induces an outer measure by [Proposition 1.2](#), finally complete the outer measure by [Theorem 1.2](#).

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.2}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to σ -algebra we are working on till now.

⁷https://en.wikipedia.org/wiki/Squeeze_theorem

Definition 1.11 (Algebra). Let X be a set. A collection \mathcal{A} of subsets of X , i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on X* if

- $\emptyset \in \mathcal{A}$.
- \mathcal{A} is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- \mathcal{A} is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^n A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an algebra and a σ -algebra is whether they are closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an [outer measure](#).

Definition 1.12 (Pre-measure). Let \mathcal{A}_0 be an [algebra](#) on X . We say

$$\mu_0: \mathcal{A}_0 \rightarrow [0, \infty]$$

is a *pre-measure* if

1. $\mu_0(\emptyset) = 0$
2. (finite additivity) $\mu_0\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu_0(A_i)$ if $A_1, \dots, A_n \in \mathcal{A}_0$ are disjoint.
3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, disjoint, then

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 1.5. (1) + (3) \implies (2) in [Definition 1.12](#).

Proof. It's easy to see that since μ_0 is monotone. ■

Theorem 1.3 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra \mathcal{A}_0 on X . Let μ^* be the outer measure induced by (\mathcal{A}_0, μ_0) in [Proposition 1.2](#). Let \mathcal{A} and μ be the [Carathéodory \$\sigma\$ -algebra](#) and measure for μ^* , then (\mathcal{A}, μ) extends (\mathcal{A}_0, μ_0) . i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts.

- We first show $\mathcal{A} \supset \mathcal{A}_0$. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is \mathcal{C} -measurable, i.e.,

$$\forall E \subset X \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We first fix an $E \subset X$. From countable subadditivity of μ^* , we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

Fix $\epsilon > 0$. By the [Proposition 1.2](#) of μ^* , $\exists B_1, B_2, \dots \in \mathcal{A}_0$, $\bigcup_{n=1}^{\infty} B_n \supset E$ such that

$$\mu^*(E) + \epsilon \stackrel{!}{\geq} \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left(\underbrace{\mu_0(B_n \cap A)}_{\in \mathcal{A}_0} + \underbrace{\mu_0(B_n \cap A^c)}_{\in \mathcal{A}_0} \right)$$

by the [finite additivity](#) of μ_0 . Note that

$$\left\{ \begin{array}{l} \bigcup_{n=1}^{\infty} (B_n \cap A) \supset E \cap A \\ \bigcup_{n=1}^{\infty} (B_n \cap A^c) \subset E \cap A^c \end{array} \right. \implies \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

since

$$\mu^*(E \cap A) \leq \mu^* \left(\bigcup_{n=1}^{\infty} (B_n \cap A) \right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A)$$

and

$$\mu^*(E \cap A^c) \leq \mu^* \left(\bigcup_{n=1}^{\infty} (B_n \cap A^c) \right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A^c).$$

We then see that for any $\epsilon > 0$, the inequality

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

holds, hence so does

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

which implies $\mathcal{A} \supset \mathcal{A}_0$.

The proof will be [continued](#)...

Lecture 6: Hahn-Kolmogorov Theorem and Extension.

18 Jan. 11:00

Let's continue the proof of [Theorem 1.3](#).

- Let $A \in \mathcal{A}_0$, we want to show that

$$\mu(A) = \mu_0(A).$$

– Firstly, let

$$B_i = \begin{cases} A, & \text{if } i = 1 \\ \emptyset, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,$$

hence $\bigcup_{i=1}^{\infty} B_i = A$, then we see that

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu_0(B_i) = \mu_0(A)$$

from the [definition](#) of μ^* and [countable additivity within the algebra](#) of μ_0 .

– Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A \cap B_1 \in \mathcal{A}_0$, $C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right) \in \mathcal{A}_0$ for $i \geq 2$ since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

are disjoint countable unions, by [countable additivity within the algebra](#), we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i) \implies \mu_0(A) \leq \mu^*(A)$$

by taking the infimum from the [definition](#) of μ^* .

Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every $A \in \mathcal{A}_0$, which implies

$$\mu(A) = \mu_0(A)$$

for every $A \in \mathcal{A}_0$ from [Theorem 1.2](#), where we extend μ^* to μ respect to \mathcal{A}_0 . ■

Definition 1.13 (HK extension). (\mathcal{A}, μ) obtained from [Theorem 1.3](#) is the *Hahn-Kolmogorov extensions* of (\mathcal{A}_0, μ_0) .

We can show the uniqueness of HK extension.

Theorem 1.4 (Uniqueness of HK extension). Let \mathcal{A}_0 be an algebra on X , μ_0 be a pre-measure on \mathcal{A}_0 . Let (\mathcal{A}, μ) be the HK extension of (\mathcal{A}_0, μ_0) . Let (\mathcal{A}', μ') be another extension of (\mathcal{A}_0, μ_0) . Then if μ_0 is [σ-finite](#), $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Note. Notice that $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$ since they both extend \mathcal{A}_0 .

Proof. Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing the arbitrary cover of A and using the [definition](#) of μ^* .

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) < \infty$, and fix $\epsilon > 0$. Then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup_{i=1}^{\infty} B_i \supset A$ such that

$$\mu(A) + \epsilon = \mu^*(A) + \epsilon \stackrel{!}{\geq} \sum_{i=1}^{\infty} \mu_0(B_i) \stackrel{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

This implies that

$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$

where the first equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

$$\mu(B) = \lim_{N \rightarrow \infty} \mu\left(\bigcup_{i=1}^N B_i\right) \stackrel{8}{=} \lim_{N \rightarrow \infty} \mu'\left(\bigcup_{i=1}^N B_i\right) = \mu'(B),$$

hence,

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \stackrel{9}{\leq} \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon$$

for arbitrary ϵ , so we conclude $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is σ -finite, so we know $X = \bigcup_{n=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ such that

$$\mu_0(X_n) < \infty.$$

Replacing X_n by $X_1 \cup \dots \cup X_n \in \mathcal{A}_0$, we may assume that

$$X_1 \subset X_2 \subset \dots$$

Then,

$$\forall_{n \in \mathbb{N}} \mu(A \cap X_n) < \infty \stackrel{!}{\implies} \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

From the continuity of measure, we then have

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A \cap X_n) \leq \lim_{n \rightarrow \infty} \mu'(A \cap X_n) = \mu'(A).$$

■

⁸ $\mu = \mu'$ on \mathcal{A}_0 .

⁹From the first part.

Corollary 1.1. Let μ_0 be a pre-measure on algebra \mathcal{A}_0 on X . Suppose μ_0 is σ -finite, then

$\exists!$ measure μ on $\langle \mathcal{A}_0 \rangle$ that extends \mathcal{A}_0 .

Furthermore,

- The completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of (\mathcal{A}_0, μ_0) .

-

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \forall_{i \in \mathbb{N}} \bigcup_{i=1}^{\infty} B_i \supset A \right\}$$

for all $A \in \langle \bar{\mathcal{A}}_0 \rangle$.

Lecture 7: Borel Measures

21 Jan. 11:00

1.5 Borel Measures on \mathbb{R}

We first introduce so-called *distribution function*.

Definition 1.14 (Distribution function). An increasing^a function

$$F: \mathbb{R} \rightarrow \mathbb{R}$$

and right-continuous. F is then a *distribution function*.

^aHere, increasing means $F(x) \leq F(y)$ for $x < y$.

Example. Here are some examples of right-continuous functions.

1. $F(x) = x$.

2. $F(x) = e^x$.

3. Define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

4. Let $\mathbb{Q} := \{r_1, r_2, \dots\}$. Define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$

and

$$F(x) := \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$

Then F is a distribution function (hence right-continuous).

Note. If F is increasing, and

$$F(\infty) := \lim_{x \nearrow \infty} F(x), \quad F(-\infty) := \lim_{x \searrow -\infty} F(x)$$

exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.¹⁰

Now, we can define a *Borel measure* on $(X, \mathcal{B}(\mathbb{R}))$.

Definition 1.15 (Borel measure). A *Borel measure* is any measure μ defined on the σ -algebra of Borel sets.

Definition 1.16 (Locally finite). Let X be a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Note. Some authors will require a [Borel measure](#) equipped with the [locally finite](#) property. But formally, this is not so common.

Lemma 1.6. Let μ be a [locally finite Borel measure](#) on \mathbb{R} , then

$$F_\mu(x) = \begin{cases} \mu((0, x]), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu((x, 0]), & \text{if } x < 0 \end{cases}$$

is a [distribution function](#).

Proof. To show F_μ is increasing, consider $x < y$ such that

$$F_\mu(x) \leq F_\mu(y)$$

by considering

- $x > 0$: Then $F_\mu(x) = \mu((0, x])$ and

$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$

- $x = 0$: Then $F_\mu(x) = 0$ and

$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$

since $y > 0$.

- $x < 0$: Follows the same argument with $x > 0$.

¹⁰There are [distributions](#) [FF99] Ch9., but these are different from distribution functions.

Now, we need to show F_μ is right-continuous. Firstly, assume that $x \geq 0$, then we see that

$$F_\mu(x) = \mu((0, x]) = \mu((0, x^+])$$

from the fact that a measure is right-continuous.¹¹ Now, if $x \leq 0$, the same argument follows since multiplying -1 will not change the fact that a measure is continuous. ■

Definition 1.17 (Half intervals). We call

$$\emptyset, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

half-intervals.

Lemma 1.7. Let \mathcal{H} be the collection of finite disjoint unions of [half-intervals](#). Then, \mathcal{H} is an [algebra](#) on \mathbb{R} .

Proof. We see that

- $\emptyset \in \mathcal{H}$. Clearly.
- To show \mathcal{H} is closed under complements, we have
 - $\emptyset^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$.
 - $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$.¹²
 - $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$.
 - $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$.
 - $(-\infty, \infty)^c = \emptyset \in \mathcal{H}$.
- \mathcal{H} is closed under finite unions, clearly.

■

¹¹Actually, a measure is always continuous.

¹²Since it's a two disjoint union of half intervals.

Proposition 1.3 (Distribution function defines a pre-measure). Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be a [distribution function](#). For a [half interval](#) I , define

$$\ell(I) := \ell_F(I) = \begin{cases} 0, & \text{if } I = \emptyset; \\ F(b) - F(a), & \text{if } I = (a, b]; \\ F(\infty) - F(a), & \text{if } I = (a, \infty]; \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b]; \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 := \mu_{0,F}$ as

$$\mu_{0,F}: \mathcal{H} \rightarrow [0, \infty]$$

by

$$\mu_0(A) = \sum_{k=1}^N \ell(I_k) \text{ if } A = \bigcup_{k=1}^N I_k,$$

where A is a finite disjoint union of [half intervals](#) I_1, \dots, I_N . Then, μ_0 is a [pre-measure](#) on \mathcal{H} .

Proof. We see that

1. μ_0 is well-defined.
2. $\mu_0(\emptyset) = 0$.
3. μ_0 is finite additive.
4. μ_0 is countable additive within \mathcal{H} .

Suppose $A \in \mathcal{H}$ where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.¹³

Focus on the case $I = (a, b]$. Let

$$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

which is a disjoint union. Then we only need to check

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

- Since $(a, b] \supset \bigcup_{n=1}^N (a_n, b_n]$ for any fixed $N \in \mathbb{N}$, hence

$$\forall_{N \in \mathbb{N}} F(b) - F(a) \geq \sum_{n=1}^N (F(b_n) - F(a_n)).$$

¹³Since \mathcal{H} is only a collection of *finite* disjoint [half intervals](#), hence after considering $A = I$, we can apply the same argument iteratively and stop in finite steps. Formally, we can consider $H \in \mathcal{H}$, $H = \bigcup_{i=1}^{\infty} A^i$, where A^i being a [half interval](#). Then by the above argument, we have $A^i = I^i$ and so on.

By letting $N \rightarrow \infty$, we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

- Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

$$F(a') - F(a) < \epsilon.$$

For each $n \in \mathbb{N}$, $\exists b'_n > b_n$ such that

$$F(b'_n) - F(b_n) < \frac{\epsilon}{2^n}.$$

Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b'_n),$$

hence

$$\exists_{N \in \mathbb{N}} [a', b] \subset \bigcup_{n=1}^N (a_n, b'_n),^{14}$$

which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^N F(b'_n) - F(a_n).$$

Finally, we see that

$$\begin{aligned} F(b) - F(a) &\leq F(b) - F(a') + \epsilon \\ &\leq \sum_{n=1}^{\infty} (F(b'_n) - F(a_n)) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\epsilon}{2^n} \right) + \epsilon \\ &= \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) + 2\epsilon \end{aligned}$$

for any fixed $\epsilon > 0$, hence

$$F(b) - F(a) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

Combine these two inequalities, we have

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

as we desired.

¹⁴This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.



Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8: Lebesgue-Stieltjes Measure on \mathbb{R}

24 Jan. 11:00

To classify all measures, we now see this last theorem to complete the task.

Theorem 1.5 (Locally finite Borel measures on \mathbb{R}). We have

1. $F: \mathbb{R} \rightarrow \mathbb{R}$ a **distribution function**, then there exists a **unique locally finite Borel measure** μ_F on \mathbb{R} satisfying

$$\mu_F((a, b]) = F(b) - F(a)$$

for every $a < b$.

2. Suppose $F, G: \mathbb{R} \rightarrow \mathbb{R}$ are **distribution functions**. Then,

$$\mu_F = \mu_G$$

on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof.



HW.

Remark. **Theorem 1.5** simply states that given a **distribution function**, if we restrict our attention on **locally finite** measures on \mathbb{R} following our usual convention, then it defines the measure on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

1.6 Lebesgue-Stieltjes Measure on \mathbb{R}

We see that

F distribution function $\xRightarrow{!} \mu_F$ on Carathéodory σ -algebra $\mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R})$.

Furthermore, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 1.18 (Lebesgue-Stieltjes measure). Given a **distribution function** F , we define

- μ_F on \mathcal{A}_{μ_F} is called the *Lebesgue-Stieltjes measure* corresponding to F .
- Special case: $F(x) = x \implies$ Lebesgue measure (\mathcal{L}, m) , where \mathcal{L} is called *Lebesgue σ -algebra*, and m is called *Lebesgue measure*.

Remark. Recall that \mathcal{L} is induced by [Theorem 1.2](#), namely given m , for all $A \subset \mathbb{R}$, we have

$$\mathcal{L} := \left\{ A \subset \mathbb{R} \mid \forall_{E \subset \mathbb{R}} m(A) = m(A \cap E) + m(A \setminus E) \right\}$$

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).^{15}$$

Example. We first see some examples.

1. $\mu_F((a, b]) = F(b) - F(a)$. Then

- $\mu_F(\{a\}) = F(a) - F(a^-)$
- $\mu_F([a, b]) = F(b) - F(a^-)$
- $\mu_F((a, b)) = F(b^-) - F(a)$

2. We define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Then

- $\mu_F(\{0\}) = 1$
- $\mu_F(\mathbb{R}) = 1$
- $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. This is easy to see since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, hence

$$\begin{aligned} \mu_F(\mathbb{R} \setminus \{0\}) &= \mu_F((-\infty, 0) \cup (0, \infty)) \\ &= \underbrace{\mu_F((-\infty, 0))}_{0-0^{16}} + \underbrace{\mu_F((0, \infty))}_{1-1^{17}} = 0. \end{aligned}$$

We call that μ_F is the *Dirac measure* at 0.

3. Denote $\mathbb{Q} = \{r_1, r_2, \dots\}$, and we define

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n. \end{cases}$$

Then

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$.

HW

¹⁵Some text will use x^- and x^+ instead of x^- and x^+ , respectively.

¹⁶It follows from $F(0^-) - F(-\infty) = 0 - 0 = 0$.

¹⁷It follows from $F(\infty) - F(0) = 1 - 1 = 0$.

4. If F is continuous at a , then $\mu_F(\{a\}) = 0$.
5. $F(x) = x$, then recall that we denote $\mu_F := m$, and we have
 - $m((a, b]) = m((a, b)) = m([a, b]) = b - a$.
6. $F(x) = e^x$
 - $\mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$.

Remark. We see that the first two examples are *discrete measures*.

Example (Middle thirds Cantor set). Let $C := \bigcap_{n=1}^{\infty} K_n$, where we have

$$\begin{aligned}
 K_0 &:= [0, 1] \\
 K_1 &:= K_0 \setminus \left(\frac{1}{3}, \frac{2}{3} \right) \\
 K_2 &:= K_1 \setminus \left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \\
 &\vdots \\
 K_n &:= K_{n-1} \setminus \bigcup_{k=1}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).
 \end{aligned}$$

We see that C is uncountable and with $m(C) = 0$. Firstly, since $x \in C$ if and only if $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for some $a_n \in \{0, 2\}$. Hence, we can instead formulate K_n by

$$K_n = \bigcup_{\substack{a_i \in \{0, 2\} \\ 1 \leq i \leq n}} \left[\sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n} \right].$$

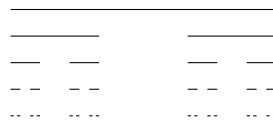


Figure 1: The top line corresponds to K_0 , and then K_1 , etc.

1.6.1 Cantor Function

Consider F as follows. We define a function F to be 0 to the left of 0, and 1 to the right of 1. Then, define F to be $\frac{1}{2}$ on $(\frac{1}{3}, \frac{2}{3})$, $\frac{1}{4}$ on $(\frac{1}{9}, \frac{2}{9})$, $\frac{3}{4}$ on $(\frac{7}{9}, \frac{8}{9})$ and so on. This is so-called *Cantor Function*. We can show F is continuous and increasing, which makes F a distribution function.

HW

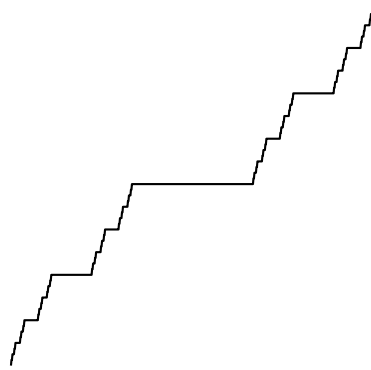


Figure 2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

Cantor Measure μ_F		Lebesgue Measure m
$\mu_F(\mathbb{R} \setminus C) = 0$		$m(\mathbb{R} \setminus C) = \infty > 0$
$\mu_F(C) = 1$	\iff	$m(C) = 0$
$\mu_F(\{a\}) = 0$		$m(\{a\}) = 0$

Remark. μ_F and m are said to be **singular** to each other.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 1.8. Let μ be [Lebesgue-Stieltjes measure](#) on \mathbb{R} . Then we have

$$\begin{aligned} \mu(A) &\stackrel{!}{=} \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

for every $A \in \mathcal{A}_\mu$

Proof. The second equality follows from the continuity of the measure. ■

Remark. This is similar to

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - 1/n], \quad (a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n].$$

Lecture 9: Properties of Lebesgue-Stieltjes measure

26 Jan. 11:00

As previously seen. Let $X \subset [0, \infty]$. Recall that

•

$$\alpha = \sup X < \infty \iff \begin{cases} \forall_{x \in X} \alpha \geq x \\ \forall_{\epsilon > 0} \exists_{x \in X} x + \epsilon \geq \alpha. \end{cases}$$

•

$$\alpha = \sup X = \infty \iff \forall_{L > 0} \exists_{x \in X} x \geq L.$$

This should be useful latter on.

Theorem 1.6. Let μ be [Lebesgue-Stieltjes measure](#). Then, for all $A \in \mathcal{A}_\mu$,

1. (outer regularity) $\mu(A) = \inf\{\mu(O) \mid O \supset A, O \text{ is open}\}$
2. (inner regularity) $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$

Proof. We check them separately.

1. DIY

2. Let $s := \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$, then by [monotonicity](#), we have $\mu(A) \geq s$. To show the other direction, we consider

- A is a bounded set.

Then $\bar{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, \bar{A} is also bounded $\implies \mu(\bar{A}) < \infty$. Fix $\epsilon > 0$, then by [outer regularity](#), there exists an open $O \supset \bar{A} \setminus A$, and $\mu(O) - \mu(\bar{A} \setminus A) = \mu(O \setminus (\bar{A} \setminus A)) \leq \epsilon$. Let $K := \underbrace{A \setminus O}_{K \subset A} = \underbrace{\bar{A} \setminus O}_{\text{compact}}$, we

show that

$$\mu(K) \geq \mu(A) - \epsilon.$$

DIY

- A is an unbounded set with $\mu(A) < \infty$.

Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \dots$, then

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) < \infty.$$

- A is an unbounded set with $\mu(A) = \infty$.

We can show that

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) = \infty.$$

Fix $L > 0$, then $\exists N$ such that $\mu(A_N) \geq L$.

■

Definition 1.19 (G_δ -set, F_σ -set). Let X be a topological space. Then

- A G_δ -set is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open.
- A F_σ -set is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.7. Let μ be a Lebesgue-Stieltjes measure. Then $TFAE^a$:

1. $A \in \mathcal{A}_\mu$
2. $A = G \setminus M$, G is a G_δ -set, M is a μ -null set.
3. $A = F \setminus N$, F is a F_σ -set, N is a μ -null set.

^aTFAE: The following are equivalent.

Proof. We see that (2.) \implies (1.) and (3.) \implies (1.) are clear.

- (1.) \implies (3.)

– Assume $\mu(A) < \infty$. From the inner regularity, we have

$$\forall n \in \mathbb{N} \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

Let $F = \bigcup_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ -null.

Check!

– Assume $\mu(A) = \infty$. Let $A = \bigcup_{k \in \mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

$$\forall k \in \mathbb{Z} \ A_k = F_k \cup N_k, \ A = \underbrace{\left(\bigcup_k F_k \right)}_{F_\sigma\text{-set}} \cup \underbrace{\left(\bigcup_k N_k \right)}_{\mu\text{-null}}.$$

- (1.) \implies (2.)

We see that

$$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

■

Proposition 1.4. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0 \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using [outer regularity](#) and the fact that every open set is $\bigcup_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. ■ [DIY](#)

We now see some properties of [Lebesgue measure](#).

Theorem 1.8. Let $A \in \mathcal{L}$, then we have $A + s \in \mathcal{L}$, $rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,

$$m(A + s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. ■ [DIY](#)

Example. We now see some examples.

1. Let $\mathbb{Q} =: \{r_i\}_{i=1}^{\infty}$ which is dense in \mathbb{R} . Let $\epsilon > 0$, and

$$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i} \right).$$

We see that O is open and dense in \mathbb{R} . But we see

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$

Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$

2. There exists uncountable set A with $m(A) = 0$.
3. There exists A with $m(A) > 0$ but A contains no non-empty open intervals.
4. There exists $A \notin \mathcal{L}$. e.g. Vitali set.¹⁸
5. There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

Lecture 10: Integration

26 Jan. 11:00

2 Integration

2.1 Measurable Function

We start with a definition.

Definition 2.1 (Measurable space). A *measurable space* or *Borel space* is a tuple of a set X and a σ -algebra \mathcal{A} on X , denoted by (X, \mathcal{A}) .

Definition 2.2 (Measurable function). Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$ are [measurable spaces](#). Then we say $f: X \rightarrow Y$ is $(\mathcal{A}, \mathcal{B})$ -*measurable* if

$$\forall B \in \mathcal{B} \quad f^{-1}(B) \in \mathcal{A}.$$

¹⁸https://en.wikipedia.org/wiki/Vitali_set

Lemma 2.1. Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then,

$$f: X \rightarrow Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall E \in \mathcal{E} \ f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part (\implies) is clear. On the other direction, we consider the following. Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $E \subset \mathcal{D}$ by assumption
- \mathcal{D} is a σ -algebra

Check!

hence, we see that $\langle \mathcal{E} \rangle \subset \mathcal{D}$. ■

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup_{\alpha} E_{\alpha}\right) = \bigcup_{\alpha} f^{-1}(E_{\alpha})$

Definition 2.3 (\mathcal{A} -measurable). Let (X, \mathcal{A}) be a measurable space. Then,

$$\left. \begin{array}{l} f: X \rightarrow \mathbb{R} \\ f: X \rightarrow \overline{\mathbb{R}} \\ f: X \rightarrow \mathbb{C} \end{array} \right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{ \begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f: X \rightarrow \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{array} \right.$$

Notation. Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$
- $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$.

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \implies$ every function is \mathcal{A} -measurable.
- $\mathcal{A} = \{\emptyset, X\} \implies$ only \mathcal{A} -measurable functions are constant functions.

Lemma 2.2. Given $f: X \rightarrow \mathbb{R}$, TFAE.

1. f is \mathcal{A} -measurable
2. $\forall a \in \mathbb{R}, f^{-1}((a, \infty)) \in \mathcal{A}$
3. $\forall a \in \mathbb{R}, f^{-1}([a, \infty)) \in \mathcal{A}$
4. $\forall a \in \mathbb{R}, f^{-1}((-\infty, a)) \in \mathcal{A}$
5. $\forall a \in \mathbb{R}, f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from the lemma we just saw. ■

Property. Given $f, g: X \rightarrow \mathbb{R}$ and f is \mathcal{A} -measurable, then

1. $\phi: \mathbb{R} \rightarrow \mathbb{R}$, \mathcal{A} -measurable¹⁹, then

$$\phi \circ f: X \rightarrow \mathbb{R}$$

is \mathcal{A} -measurable.

2. $-f$, $3f$, f^2 , $|f|$ are all \mathcal{A} -measurable, and $\frac{1}{f}$ is \mathcal{A} -measurable if $f(x) \neq 0, \forall x \in X$.

3. $f + g$ is \mathcal{A} -measurable. We see this from

$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} (f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty))).$$

4. $f \cdot g$ is \mathcal{A} -measurable. We see this from

$$f(x)g(x) = \frac{1}{2} ((f(x) + g(x))^2 - f(x)^2 - g(x)^2).$$

5. We see that

$$(f \vee g)(x) := \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) := \min\{f(x), g(x)\}$$

are \mathcal{A} -measurable.

6. Let $f_n: X \rightarrow \overline{\mathbb{R}}$ be \mathcal{A} -measurable. Then

$$\sup_{n \in \mathbb{N}} f_n, \inf_{n \in \mathbb{N}} f_n, \limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n$$

are \mathcal{A} -measurable.

Proof. Consider $\sup_{n \in \mathbb{N}} f_n =: g$, then

$$g^{-1}((a, \infty]) = \bigcup_{n \in \mathbb{N}} f_n^{-1}((a, \infty])$$

for $\sup_{n \in \mathbb{N}} f_n(x) = g(x) > a$. A similar argument can prove the case of $\inf_{n \in \mathbb{N}} f_n$. check

And notice that $\limsup_{n \rightarrow \infty} f_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n$, then the similar argument also proves this case. ■

7. If $\lim_{n \rightarrow \infty} f_n(x)$ converges for every $x \in X$, then f is \mathcal{A} -measurable.

Example. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous

$\implies f$ is Borel measurable

$\implies f$ is Lebesgue measurable

since the preimage of an open set of a continuous function is open, then we consider $f^{-1}((a, \infty))$.

¹⁹In this case, we also call it *Borel measurable*.

Definition 2.4 (f^+ , f^-). For $f: X \rightarrow \overline{\mathbb{R}}$, let $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$.^a

^ai.e., $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \min\{-f(x), 0\}$

Remark. If $\text{supp } f^+ \cap \text{supp } f^- = \emptyset$ and $f(x) = f^+(x) - f^-(x)$, then

f is \mathcal{A} -measurable $\iff f^+, f^-$ are \mathcal{A} -measurable.

Notation. $\text{supp } f$ means the support of f , which is the set of domain which makes f being non-zero.

Definition 2.5 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator) function* of E is

$$\chi_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is \mathcal{A} -measurable $\iff E \in \mathcal{A}$.

Definition 2.6 (Simple function). Let (X, \mathcal{A}) be a measurable space. Then a *simple function* $\phi: X \rightarrow \mathbb{C}$ that is \mathcal{A} -measurable and takes only finitely many values.

Remark. We see that if

$$\phi(X) = \{c_1, \dots, c_N\},$$

and

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \implies \phi = \sum_{i=1}^N \underbrace{c_i}_{\neq \pm\infty} \underbrace{\mathbb{1}_{E_i}}_{\in \mathcal{A}}.$$

Lecture 11: Integration of nonnegative functions

31 Jan. 11:00

As previously seen. For a \mathcal{A} -measurable ϕ , c_i can actually be in \mathbb{C} .

Theorem 2.1. Given (X, \mathcal{A}) be a measurable space, and let $f: X \rightarrow [0, \infty]$, the following is equivalent.

1. f is \mathcal{A} -measurable function.
2. There exists simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$ such that

$$\forall x \in X \quad \lim_{n \rightarrow \infty} \phi_n(x) = f(x)$$

i.e., f is a pointwise upward limit of simple functions.

Proof. It's clear that (2.) \implies (1.) from the fact that $f(x) = \sup_n \phi_n(x)$. Now, we want to show that (1.) \implies (2.). Assume f is \mathcal{A} -measurable, and fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. Then for $0 \leq k \leq 2^{2^n} - 1$, $E_{n,k} = f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}]) \in \mathcal{A}$.

Then, define ϕ_n be

$$\phi_n = \sum_{k=0}^{2^{2^n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n},$$

we have

- $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$ for every $x \in X$
- $\forall x \in X \setminus F_n$, we have $0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$

Furthermore, we see that

$$F_1 \supset F_2 \supset \dots, \quad \bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\}),$$

then

- $x \in f^{-1}([0, \infty]) = X \setminus \bigcap_{n=1}^{\infty} F_n \implies \lim_{n \rightarrow \infty} \phi_n(x) = f(x)$
- $x \in f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} F_n \implies f_n(x) \geq 2^n \implies \lim_{n \rightarrow \infty} \phi_n(x) = \infty = f(x)$

■

Corollary 2.1. If f is bounded on a set $A \subset \mathbb{R}$, i.e., $\exists b > 0$ such that

$$\forall_{x \in A} |f(x)| \leq L,$$

then $\phi_n \rightarrow f$ uniformly on A .

Proof.

■

DIY

Corollary 2.2. If $f: X \rightarrow \mathbb{C}$ is a measurable function if and only if there exists simple functions $\phi_n: X \rightarrow \mathbb{C}$ such that

$$0 \leq |\phi_1(x)| \leq |\phi_2(x)| \leq \dots \leq |f(x)|$$

with

$$\forall_{x \in X} \lim_{n \rightarrow \infty} \phi_n(x) = f(x).$$

Proof.

■

DIY

2.2 Integration of nonnegative functions

We start with a definition.

Definition 2.7 (Integration of nonnegative function). Let (X, \mathcal{A}, μ) be a [measure space](#), and $\phi: X \rightarrow [0, \infty]$ such that

$$\phi = \sum_{i=1}^N c_i \mathbb{1}_{E_i}$$

be a [simple function](#). Define

$$\int \phi \, dx = \int \phi \, d\mu = \int_X \phi \, d\mu = \sum_{i=1}^N c_i \mu(E_i).$$

Furthermore, for $A \in \mathcal{A}$,

$$\int_A \phi = \int_A \phi \, d\mu = \int \phi \mathbb{1}_A \, d\mu$$

Note. Notice that in the expression $\sum_{i=1}^N c_i \mu(E_i)$ we're using the convention such that $0 \cdot \infty = 0$, and in the function $\phi \mathbb{1}_A$ is also a [simple function](#) since both ϕ and $\mathbb{1}_A$ are [simple function](#).

Proposition 2.1. Suppose we have $\phi, \psi \geq 0$ be two [simple functions](#). Then,

- [Definition 2.7](#) is well-defined.
- $\int c\phi = c \int \phi$ for $c \in [0, \infty)$.
- $\int \phi + \psi = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x)$ for all $x \implies \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi \, d\mu$ is a [measure](#) on (X, \mathcal{A}) .

Proof.



DIY

Definition 2.8 (Generalization of Integration of nonnegative function). Given (X, \mathcal{A}, μ) with $f: X \rightarrow [0, \infty]$ be [A-measurable](#). Define

$$\int f = \int f \, d\mu = \sup \left\{ \int \phi \mid 0 \leq \phi \leq f \text{ such that } \phi \text{ is simple} \right\}.$$

- If f is a [simple function](#), the [Definition 2.7](#) and [Definition 2.8](#) of $\int f$ are the same.
- $\int cf = c \int f$ for $c \in [0, \infty)$.

- If $f \geq g \geq 0 \implies \int f \geq \int g$.
- But $\int f + g = \int f + \int g$ is not trivial.

Theorem 2.2 (Monotone Convergence Theorem (MCT)). Given (X, \mathcal{A}, μ) be a [measure space](#). Then if

- $f_n: X \rightarrow [0, \infty]$ be [A-measurable](#) for every $n \in \mathbb{N}$
- $0 \leq f_1(x) \leq f_2(x) \leq \dots$ for every $x \in X$
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for every $x \in X$,

we have

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

Proof. Note that $\lim_{n \rightarrow \infty} f_n(x)$ converges for every $x \in X$ and $\lim_{n \rightarrow \infty} \int f_n$ converges. Then

- $f_n \leq f \implies \int f_n \leq \int f \implies \lim_{n \rightarrow \infty} \int f_n \leq \int f$.
- Fix a simple function $0 \leq \phi \leq f$, then it's enough to show $\lim_{n \rightarrow \infty} \int f_n \geq \int \phi$.

We first fix $\alpha = (0, 1)$, then it's also enough to show

$$\lim_{n \rightarrow \infty} \int f_n \geq \alpha \int \phi.$$

Let $A_n := \{x \in X \mid f_n(x) \geq \alpha \phi(x)\}$, then since f_n is [measurable](#),

- $A_n \in \mathcal{A}$
- $A_1 \subset A_2 \subset A_3 \subset \dots$
- $\bigcup_{n=1}^{\infty} A_n = X$

Check!

We then have

$$\int f_n \geq \int f_n \mathbb{1}_{A_n} \geq \int \alpha \phi \mathbb{1}_{A_n} = \alpha \nu(A_n)$$

where $\nu(A) = \int_A \phi$ is a [measure](#). This implies

$$\lim_{n \rightarrow \infty} \int f_n \geq \alpha \lim_{n \rightarrow \infty} \nu(A_n) \stackrel{20}{=} \alpha \nu(X) = \alpha \int \phi.$$

■

Corollary 2.3. Let $f, g \geq 0$ be [measurable](#), then

$$\int f + g = \int f + \int g.$$

²⁰This follows from the continuity of [measure from below](#)

Proof. There exists simple functions ϕ_n and ψ_n such that

- $0 \leq \phi_1 \leq \phi_2 \leq \dots$ and $\phi_n \rightarrow f$ pointwise
- $0 \leq \psi_1 \leq \psi_2 \leq \dots$ and $\psi_n \rightarrow f$ pointwise

Then,

$$\int (f + g) \stackrel{!}{=} \lim_{n \rightarrow \infty} \int (\phi_n + \psi_n) = \lim_{n \rightarrow \infty} \int \phi_n + \int \psi_n \stackrel{!}{=} \int f + \int g.$$

■

Lecture 12

2 Feb. 11:00

We start with a useful corollary.

Corollary 2.4 (Tonelli's theorem for series and integrals). Given $g_n \geq 0$ for every $n \in \mathbb{N}$ and let g_n be measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Remark. Recall that we have seen [two series case](#) before. We'll later see two integrals case.

Proof. Let $f_N := \sum_{n=1}^N g_n$ such that $\lim_{N \rightarrow \infty} f_N = \sum_{n=1}^{\infty} g_n =: f$, then since $g_n \geq 0$, we have $0 \leq f_1 \leq f_2 \leq \dots$ with

$$\lim_{N \rightarrow \infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

By [Theorem 2.2](#), we have

$$\lim_{N \rightarrow \infty} \underbrace{\int \sum_{n=1}^N g_n}_{f_N} = \int \underbrace{\sum_{n=1}^{\infty} g_n}_f.$$

Now, since the terms in the limit on the left-hand side is just a finite sum, by [Corollary 2.3](#), we have

$$\underbrace{\lim_{N \rightarrow \infty} \sum_{n=1}^N \int g_n}_{\sum_{n=1}^{\infty} \int g_n} = \lim_{N \rightarrow \infty} \int \sum_{n=1}^N g_n = \int \sum_{n=1}^{\infty} g_n,$$

hence

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

■

Theorem 2.3 (Fatou's Lemma). Suppose $f_n \geq 0$ and measurable, then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Remark. Recall that

$$\liminf_{n \rightarrow \infty} f_n := \lim_{k \rightarrow \infty} \inf_{n \geq k} f_n = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f_n$$

and

$$\exists \lim_{n \rightarrow \infty} a_n \iff \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

Proof. Let $g_k = \inf_{n \geq k} f_n$, then g_k is measurable and $0 \leq g_1 \leq g_2 \leq \dots$. Now, from Theorem 2.2, we have

$$\int \lim_{k \rightarrow \infty} g_k = \lim_{k \rightarrow \infty} \int g_k.$$

Notice that the left-hand side is just $\int \liminf_{n \rightarrow \infty} f_n$, while the right-hand side is just $\lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n$, i.e.,

$$\int \liminf_{n \rightarrow \infty} f_n = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n.$$

We see that we want to take the inf outside the integral on the right-hand side. Observe that

$$\forall_{m \geq k} \inf_{n \geq k} f_n \leq f_m \implies \forall_{m \geq k} \int \inf_{n \geq k} f_n \leq \int f_m \implies \int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m.$$

Then, we have

$$\int \liminf_{n \rightarrow \infty} f_n = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n \leq \lim_{k \rightarrow \infty} \inf_{m \geq k} \int f_m = \liminf_{m \rightarrow \infty} \int f_m.$$

■

Example. Given $(\mathbb{R}, \mathcal{L}, m)$.

1. **Escape to horizontal infinity.** Let $f_n := \mathbb{1}_{(n, n+1)}$. We immediately see that
 - $f_n \rightarrow 0$ pointwise
 - $\int f_n = 1$ for every n
 - $\int f = 0$
2. **Escape to width infinity.** Let $f_n := \frac{1}{n} \mathbb{1}_{(0, n)}$.
3. **Escape to vertical infinity.** Let $f_n := n \mathbb{1}_{(0, \frac{1}{n})}$.

Lemma 2.3 (Markov's inequality). Let $f \geq 0$ be measurable. Then

$$\forall_{c \in (0, \infty)} \mu(\{x \mid f(x) \geq c\}) \leq \frac{1}{c} \int f.$$

Proof. Denote $\{x \mid f(x) \geq c\} =: E$, then

$$f(x) \geq c \mathbb{1}_E(x) \implies \int f \geq c \int \mathbb{1}_E = c \cdot \mu(E).$$

■

Remark. Notice that $E = f^{-1}([c, \infty))$, hence E is measurable.

Proposition 2.2. Let $f \geq 0$ be measurable. Then,

$$\int f = 0 \iff f = 0 \text{ a.e..}$$

i.e.,

$$\int f \, d\mu = 0 \iff \begin{cases} \mu(A) = 0 \\ A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty)). \end{cases}$$

Proof. Firstly, assume that $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^N c_i \mathbb{1}_{E_i}$$

where E_i are disjoint and $c_i \in (0, \infty)$. Then,

$$\begin{aligned} \int \phi &= \sum_{i=1}^N c_i \mu(E_i) = 0 \\ \iff \mu(E_1) &= \dots = \mu(E_N) = 0 \\ \iff \mu(A) &= 0, \quad A = \bigcup_{i=1}^N E_i. \end{aligned}$$

Now, assume that f is a general function where $f \geq 0$ is the only constraint.

1. Assume $\mu(A) = 0$ (i.e., $f = 0$ a.e.). Let $0 \leq \phi \leq f$, where ϕ is simple. Then

$$\forall_{x \in A^c} \phi(x) = 0$$

since $f(x) = 0, \forall x \in A^c$. This implies that $\phi = 0$ a.e. since $\mu(A) = 0$, so $\int \phi = 0$. We then have

$$\int f = 0$$

from Definition 2.8.

2. Assume $\int f = 0$. Let $A_n = f^{-1}([\frac{1}{n}, \infty])$. Then we see that

- $A_1 \subset A_2 \subset \dots$
- $\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} [\frac{1}{n}, \infty]\right) = f^{-1}((0, \infty)) = A$.

We then have

$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \stackrel{!}{\leq} n \int f = 0,$$

which further implies

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n) = 0$$

from the [continuity of measure from below](#). ■

Corollary 2.5. If $f, g \geq 0$ are both [measurable](#) and $f = g$ [a.e.](#), then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \neq g(x)\}$ ²¹. Then by assumption, $\mu(A) = 0$, hence

$$f \mathbb{1}_A = 0 \text{ [a.e.](#), } g \mathbb{1}_A = 0 \text{ [a.e.](#)..}$$

This further implies that

$$\begin{aligned} \int f &= \int f(\mathbb{1}_A + \mathbb{1}_{A^c}) \\ &\stackrel{!}{=} \int f \mathbb{1}_A + \int f \mathbb{1}_{A^c} \\ &= \int f \mathbb{1}_{A^c} = \int g \mathbb{1}_{A^c} \\ &= \int g \mathbb{1}_{A^c} + \int g \mathbb{1}_A = \int g. \end{aligned}$$
■

Corollary 2.6. Let $f_n \geq 0$ be [measurable](#). Then

1.
$$\left. \begin{array}{l} 0 \leq f_1 \leq f_2 \leq \dots \leq f \text{ [a.e.](#)} \\ \lim_{n \rightarrow \infty} f_n = f \text{ [a.e.](#)} \end{array} \right\} \implies \lim_{n \rightarrow \infty} \int f_n = \int f.$$
2.
$$\lim_{n \rightarrow \infty} f_n = f \text{ [a.e.](#)} \implies \int f \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Proof. ■

DIY

²¹ A is [measurable](#) indeed.

Appendix

References

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