

# MATH592

## Introduction to Algebraic Topology

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### Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course! In particular, I add some extra content which is not covered in lectures, things like [groupoid](#), [fibered coproduct](#), feel free to skip these content.

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## Lecture 1: Homotopies of Maps

05 Jan. 10:00

# 1 Foundation of Algebraic Topology

## 1.1 Homotopy

We start with the most important and fundamental concept, [homotopy](#).

**Definition 1.1 (Homotopy, homotopic, nullhomotopic).** Let  $X, Y$  be topological spaces. Let  $f, g: X \rightarrow Y$  continuous maps. Then a *homotopy* from  $f$  to  $g$  is a 1-parameter family of maps that continuously deforms  $f$  to  $g$ , i.e., it's a continuous function  $F: X \times I \rightarrow Y$ , where  $I = [0, 1]$ , such that

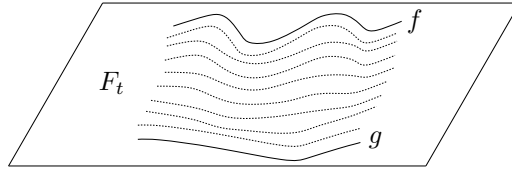
$$F(x, 0) = f(x), \quad F(x, 1) = g(x).$$

We often write  $F_t(x)$  for  $F(x, t)$ .

If a homotopy exists between  $f$  and  $g$ , we say they are *homotopic* and write

$$f \simeq g.$$

If  $f$  is homotopic to a constant map, we call it *nullhomotopic*.


 Figure 1: The continuous deforming from  $f$  to  $g$  described by  $F_t$ 

**Remark.** Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

**Example.** We first see some examples.

- Any two (continuous) maps with specification

$$f, g: X \rightarrow \mathbb{R}^n$$

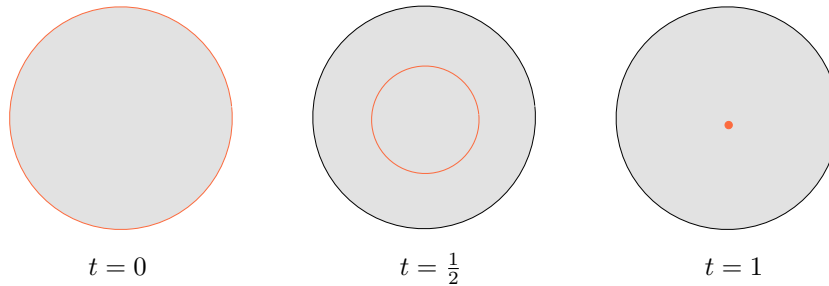
are **homotopic** by considering

$$F_t(x) = (1 - t)f(x) + tg(x).$$

We call it *the straight line homotopy*.

- Let  $S^1$  denotes the unit circle in  $\mathbb{R}^2$ , and  $D^2$  denotes the unit disk in  $\mathbb{R}^2$ . Then the inclusion  $f: S^1 \hookrightarrow D^2$  is **nullhomotopic** by considering

$$F_t(x) = (1 - t)f(x) + (t \cdot 0).$$


 Figure 2: The illustration of  $F_t(x)$ 

We see that there is a **homotopy** from  $f(x)$  to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a **nullhomotopy**.

- The maps

$$\begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not homotopy**.

**Remark.** It will essentially **flip** the orientation, hence we can't deform one to another continuously.

**Exercise.** We first see some exercises.

1. A subset  $S \subseteq \mathbb{R}^n$  is star-shaped if

$$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from  $x_0$  to  $x$  lies in  $S$ .

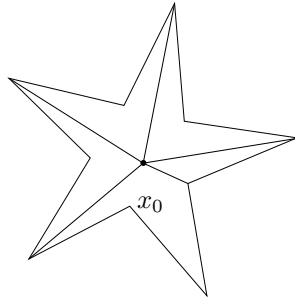


Figure 3: Star-shaped illustration

Show that  $\text{id}: S \rightarrow S$  is **nullhomotopic**.

**Answer.** Consider

$$F_t(x) := (1-t)x + tx_0,$$

which essentially just concentrates all points  $x$  to  $x_0$ . ■

2. Suppose

$$X \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_0} \end{array} Y \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_0} \end{array} Z$$

where

$$f_0 \simeq_{F_t} f_1, \quad g_0 \simeq_{G_t} g_1.$$

Show

$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

**Answer.** Consider  $I \times X \rightarrow Z$ , where

$$\begin{array}{ccccc} X \times I & \rightarrow & Y \times I & \rightarrow & Z \\ (x, t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

■

**Remark.** Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** **homotopic**?

**Answer.** We'll see! ■

## Lecture 2: Homotopy Equivalence

07 Jan. 10:00

**As previously seen.** Two maps  $f, g: X \rightarrow Y$  is **homotopy** if there exists a map

$$F_t(x): X \times I \rightarrow Y$$

with the properties

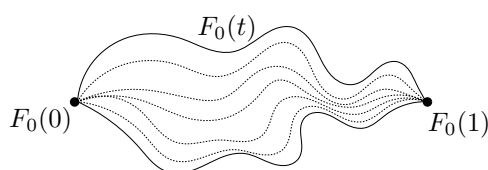
1. Continuous
2.  $F_0(x) = f(x)$
3.  $F_1(x) = g(x)$

**Remark.** The continuity of  $F_t$  is an even stronger condition for the continuity of  $F_t$  for a fixed  $t$ .

We now introduce another concept.

**Definition 1.2 (Homotopy relative).** Given two spaces  $X, Y$ , and let  $B \subseteq X$ . Then a **homotopy**  $F_t(x): X \rightarrow Y$  is called *homotopy relative  $B$*  (denotes  $\text{rel}B$ ) if  $F_t(b)$  is independent of  $t$  for all  $b \in B$ .

**Example.** Given  $X$  and  $B = \{0, 1\}$ . Then the **homotopy** of paths from  $[0, 1] \rightarrow X$  is  $\text{rel}\{0, 1\}$ .



### 1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

**Definition 1.3 (Homotopy equivalence, homotopy inverse).** A map  $f: X \rightarrow Y$  is a *homotopy equivalence* if  $\exists g: Y \rightarrow X$  such that

$$f \circ g \simeq \text{id}_Y, \quad g \circ f \simeq \text{id}_X.$$

We say that  $X, Y$  are *homotopy equivalent*, and  $g$  is called *homotopy inverse* of  $f$ .

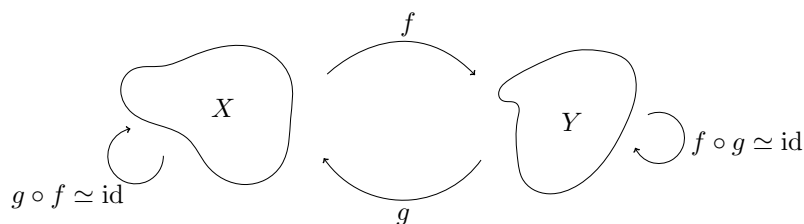
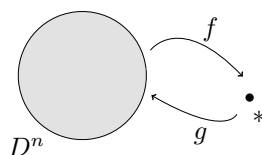


Figure 4: Homotopy Equivalence

If  $X, Y$  are [homotopy equivalent](#), then we say that they have the same *homotopy type*.

**Notation.** We denote a closed  $n$ -disk as  $D^n$ .

**Example.**  $D^n$  is [homotopy equivalent](#) to a point.



We see that  $f \circ g = \text{id}_*$  and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is [homotopic](#) to  $\text{id}_{D^n}$  by [straight line homotopy](#)  $F_t(x) = tx$ . Specifically, we see that this holds for any convex set.

**Definition 1.4 (Contractible).** We say that a space  $X$  is *contractible* if  $X$  is [homotopy equivalent](#) to a point.

The following proposition is added much after, which may use some concepts not yet covered.

**Proposition 1.1.** The followings are equivalent.

1.  $X$  is [contractible](#).
2.  $\forall x \in X, \text{id}_X \simeq c_x$ .
3.  $\exists x \in X, \text{id}_X \simeq c_x$ .

**Remark.** Note that the above notation  $c_x$  is introduced at [here](#).

*Proof.* We see that 2.  $\implies$  3. is obvious. We consider 3.  $\implies$  2. This follows the following general lemma.

**Lemma 1.1.** Given a topological space  $X$  such that  $\exists x \in X, \text{id}_X \simeq c_x$ , with  $f, g: Y \rightarrow X$ , then  $f \simeq g$ .

*Proof.* Let  $x \in X$  such that  $\text{id}_X \simeq c_x$ . Then

$$f = \text{id}_X \circ f \simeq c_x \circ f = c_x \circ g \simeq \text{id}_X \circ g = g.$$

■

Then, from this [Lemma 1.1](#), we see that assuming  $x_0 \in X$  such that  $\text{id}_X \simeq c_{x_0}$ , then consider  $c_x$  for all  $x \in X$ , then from [Lemma 1.1](#), we see that  $c_x \simeq \text{id}_X$ .

To show 3.  $\implies$  1., we let  $x_0 \in X$  such that  $\text{id}_X \simeq c_{x_0}$ .

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} \{*\}$$

Since  $g(*) = x_0$ , and

$$\begin{aligned} g \circ f: X &\rightarrow X \\ x &\mapsto x_0, \end{aligned}$$

which is just  $c_{x_0}$ , from the assumption we're done.

Now, we show 1.  $\implies$  3. Let

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} \{*\}$$

be a [homotopy equivalent](#), let  $g(*) = x_0$ . We see that  $c_{x_0} \simeq \text{id}_X$  since

$$g \circ f = c_{x_0} \simeq \text{id}_X.$$

■

Before doing exercises, we introduce two new concepts.

**Definition 1.5 (Retraction, retract).** Given  $B \subseteq X$ , a *retraction* from  $X$  to  $B$  is a map  $f: X \rightarrow B$  (or  $X \rightarrow B$ ) such that  $\forall b \in B$   $f(b) = b$ , namely  $r|_B = \text{id}_B$ . Or one can see this from

$$\begin{array}{ccc} B & \xrightarrow{i} & X \\ & \searrow r \circ i & \nearrow r \\ & & B \end{array}$$

where  $r$  is a retraction if and only if  $r \circ i = \text{id}_B$ , where  $i$  is an inclusion identity.

If  $r$  exists,  $B$  is a *retract* of  $X$ .

**Definition 1.6 (Deformation retraction).** Given  $X$  and  $B \subseteq X$ , a (strong) deformation retraction  $F_t: X \rightarrow X$  onto  $B$  is a homotopy  $\text{rel} B$  from the  $\text{id}_X$  to a retraction from  $X$  to  $B$ . i.e.,

$$\begin{aligned} F_0(x) &= x & \forall x \in X \\ F_1(x) &\in B & \forall x \in X \\ F_t(b) &= b & \forall t \forall b \in B. \end{aligned}$$

**Exercise.** We now see some problems.

1. Let  $X \simeq Y$ . Show  $X$  is path-connected if and only if  $Y$  is.

**Answer.** Suppose  $X$  is path-connected. Then we see that given two points  $x_1$  and  $x_2$  in  $X$ , there exists a path  $\gamma(t)$  with

$$\gamma: [0, 1] \rightarrow X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Since  $X \simeq Y$ , then there exists a pair of  $f$  and  $g$  such that  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  with

$$f \circ g \underset{F}{\simeq} \text{id}_Y, \quad g \circ f \underset{G}{\simeq} \text{id}_X.$$

(Notice the abuse of notation)

For any two  $y_1$  and  $y_2 \in Y$ , we want to construct a path  $\gamma'(t)$  such that

$$\gamma': [0, 1] \rightarrow Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

Firstly, we let  $g(y_1) =: x_1$  and  $g(y_2) =: x_2$ . From the argument above, we know there exists such a  $\gamma$  starting at  $x_1 = g(y_1)$  ending at  $x_2 = g(y_2)$ . Now, consider  $f(\gamma(t)) = (f \circ \gamma)(t)$  such that

$$f \circ \gamma: I \rightarrow Y, \quad f \circ \gamma(0) = y'_1, \quad f \circ \gamma(1) = y'_2,$$

we immediately see that  $y'_1$  and  $y'_2$  is path connected. Now, we claim that  $y_1$  and  $y'_1$  are path connected in  $Y$ , hence so are  $y_2$  and  $y'_2$ . To see this, note that

$$f \circ g \underset{F}{\simeq} \text{id}_Y,$$

which means that there exists  $F: Y \times I \rightarrow Y$  such that

$$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y'_1 \\ F(y_1, 1) = \text{id}_Y(y_1) = y_1. \end{cases}$$

Since  $F$  is continuous in  $I$ , we see that there must exist a path connects  $y_1$  and  $y'_1$ . The same argument applies to  $y_2$  and  $y'_2$ . Now, we see that the path

$$y_1 \rightarrow y'_1 \rightarrow y'_2 \rightarrow y_2$$

is a path in  $Y$  for any two  $y_1$  and  $y_2$ , which shows  $Y$  is path-connected. ■





Figure 5: Demonstration of the proof.

**Challenge:** One can further show that the connectedness is also preserved by any [homotopy equivalence](#).

**Corollary 1.1.** A [contractible](#) space is [path](#)-connected.

2. Show that if there exists [deformation retraction](#) from  $X$  to  $B \subseteq X$ , then  $X \simeq B$ .

## Lecture 3: Deformation Retraction

10 Jan. 10:00

**As previously seen.** A [deformation retraction](#) is a [homotopy](#) of maps  $\text{rel} B$   $X \rightarrow X$  from  $\text{id}_X$  to a [retraction](#) from  $X$  to  $B$ . Then  $B$  is a [deformation retract](#).

**Example.** We can also show

1.  $S^1$  is a [deformation retraction](#) of  $D^2 \setminus \{0\}$ . Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$



Figure 6: The [deformation retraction](#) of  $D^2 \setminus \{0\}$  is just to *enlarge* that hole and push all the interior of  $D^2$  to the boundary, which is  $S^1$ .

2.  $\mathbb{R}^n$  *deformation retracts* to 0. Indeed, since

$$F_t(x) = (1-t)x.$$

This implies that  $\mathbb{R}^n \simeq *$ , hence we see that

- dimension
- compactness
- etc.

are not *homotopy* invariants.

3.  $S^1$  is a *deformation retract* of a cylinder and a Möbius band.

For a cylinder, consider  $X \times I \rightarrow X$ . Define *homotopy* on a closed rectangle, then verify it induces map on quotient.

For a Möbius band, we define a *homotopy* on a closed rectangle, then verify that it respects the equivalence relation.

Finally, we use the universal property of quotient topology to argue that we get a *homotopy* on Möbius band.

**Upshot:** Möbius band  $\simeq S^1 \simeq$  cylinder, hence the orientability is not *homotopy* invariant.

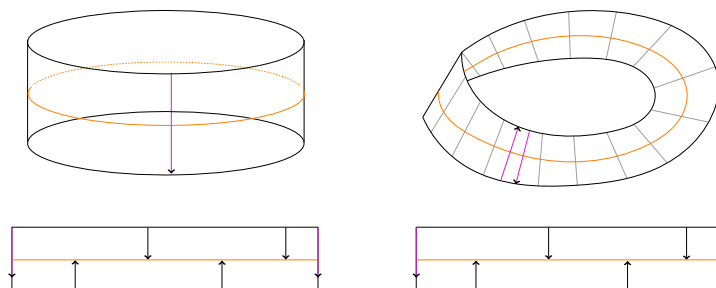


Figure 7: The *deformation retraction* for Cylinder and Möbius band

## Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

As previously seen. We saw that

- *homotopy equivalence*
- *homotopy* invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

### 1.3 CW Complexes

**Example.** Let's start with a few examples.

1. Constructing spheres:

- $S^1$  (up to homeomorphism<sup>1</sup>)



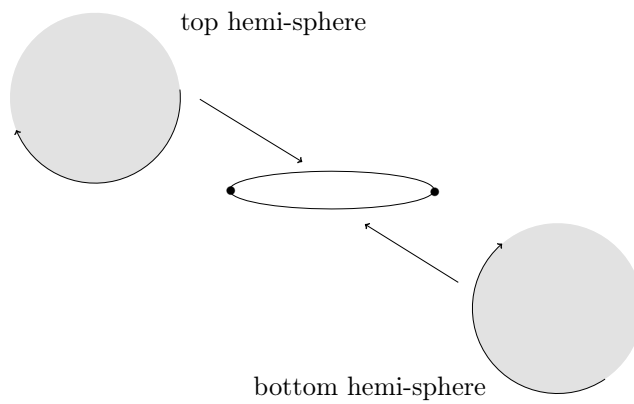
- $S^2$ 
  - glue boundary of 2-disk to a point
  - glue 2 disks onto a circle



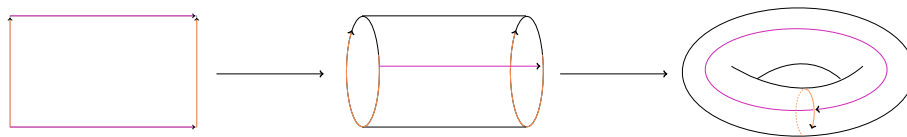
Figure 8: **Left:** Glue a 2-disk to a point along its boundary. **Right:** Glue 2 disks to  $S^1$ .

The gluing instruction to construct  $S^2$  in the right-hand side can be demonstrated as follows.

<sup>1</sup>This is just the term for isomorphism in topology.



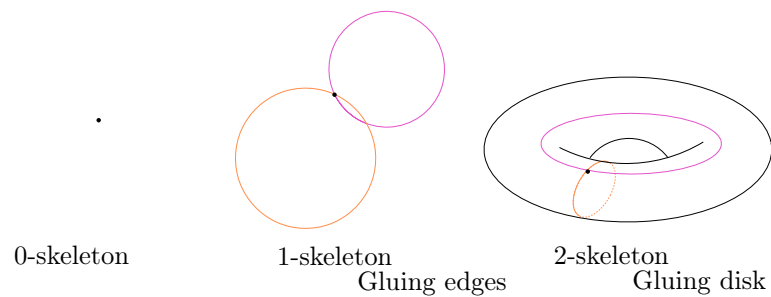
•  $T = S^1 \times S^1$



view as gluing instructions

vertex + 2 edges + 2-disks.

Specifically, we have




---

Formally, we have the following definition.

**Notation.** Let  $D^n$  denotes a closed n-disk (or n-ball)

$$D^n \simeq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

And let  $S^n$  denotes an  $n$ -sphere

$$S^n \simeq \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

Lastly, we call a point as a  $0$ -cell, and the interior of  $D^n$   $\text{int}(D^n)$  for  $n \geq 1$  as a  $n$ -cell.

**Definition 1.7 (CW Complex).** A *CW Complex* is a topological space constructed inductively as

1.  $X^0$  (the 0-skeleton) is a set of discrete points.
2. We inductively construct the  $n$ -skeleton  $X^n$  from  $X^{n-1}$  by attaching  $n$ -cells  $e_\alpha^n$ , where  $\alpha$  is the index.

The gluing instructions glued by an attaching map is that  $\forall \alpha, \exists$  continuous map  $\varphi_\alpha$

$$\varphi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1},$$

then

$$X^n = \left( X^{n-1} \amalg \coprod_\alpha D_\alpha^n \right) / x \sim \varphi_\alpha(x)$$

with identification  $x \sim \varphi_\alpha(x)$  for all  $x \in \partial D_\alpha^n$  with quotient topology.

3. We let  $X$  be defined as

$$X = \bigcup_{n=0} X^n,$$

and let  $\bar{w}$  denotes weak topology such that

$$u \subseteq X \text{ is open} \iff \forall n \ u \cap X^n \text{ is open}.$$

If all cells have dimension less than  $N$  and a  $\exists N$ -cell, then  $X = X^N$  and we call it  *$N$ -dimensional CW complex*.

**Remark.** We write  $X^{(n)}$  for  $n$ -skeleton if we need to distinguish from the Cartesian product.

**Example.** Let's look at some examples.

1. 0-dim **CW complex** is a discrete space.
2. 1-dim **CW complex** is a graph.
3. A **CW complex**  $X$  is finite if it has finitely many cells.

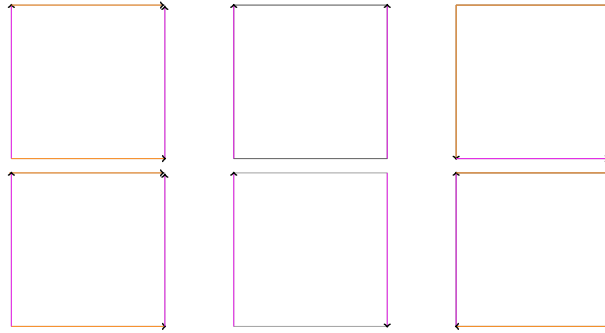
**Definition 1.8 (CW subcomplex).** A *CW subcomplex*  $A \subseteq X$  is a closed subset equal to a union of cells

$$e_\alpha^n = \text{int}(D_\alpha^n).$$

**Remark.** This inherits a **CW complex** structure.

Check the images of attaching maps.

**Exercise.** Given the following gluing instruction:



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere,  $\mathbb{R}P$ .

**Answer.** We see that

1. Torus
2. Cylinder
3. 2-sphere
4. Klein bottle
5. Möbius band
6.  $\mathbb{R}P$

**Notation.** We call the real projection space as  $\mathbb{R}P$ , and we also have so-called complex projection space, denote as  $\mathbb{C}P$ .

## Lecture 5: Operation on Spaces

14 Jan. 10:00

### 1.4 Operations on CW Complexes

#### 1.4.1 Products

We can consider the product of two **CW complex** given by a **CW complex** structure. Namely, given  $X$  and  $Y$  two **CW complexes**, we can take two cells  $e_\alpha^n$  from  $X$  and  $e_\beta^m$  from  $Y$  and form the product space  $e_\alpha^n \times e_\beta^m$ , which is homeomorphic to an  $(n+m)$ -cell. We then take these products as the cells for  $X \times Y$ .

Specifically, given  $X, Y$  are **CW complexes**, then  $X \times Y$  has a cell structure

$$\{e_\alpha^m \times e_\beta^n : e_\alpha^m \text{ is a } m\text{-cell on } X, e_\beta^n \text{ is an } n\text{-cell on } Y\}.$$

**Remark.** The product topology may not agree with the weak topology on the  $X \times Y$ . However, they do agree if  $X$  or  $Y$  is locally compact or if  $X$  and  $Y$  both have at most countably many cells.

#### 1.4.2 Wedge Sum

Given  $X, Y$  are **CW complexes**, and  $x_0 \in X^0, y_0 \in Y^0$  (only points). Then we define

$$X \vee Y = X \amalg Y$$

with quotient topology.

**Remark.**  $X \vee Y$  is a **CW complex**.

### 1.4.3 Quotients

Let  $X$  be a **CW complex**, and  $A \subseteq X$  **subcomplex** (closed union of cells), then

$$X / A$$

is a quotient space collapse  $A$  to one point and inherits a **CW complex** structure.

**Remark.**  $X / A$  is a **CW complex**.

0-skeleton

$$(X^0 - A^0) \coprod *$$

where  $*$  is a point for  $A$ . Each cell of  $X - A$  is attached to  $(X / A)^n$  by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n / A^n$$

**Example.** Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a **wedge** of two spheres.



2. We can take the torus and squish down a ring around the hole.



Figure 9: We see that  $X / A$  is [homotopy equivalent](#) to a 2-sphere [wedged](#) with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other points, forming a circle.

## Lecture 6: A Foray into Category Theory

19 Jan. 10:00

### 1.5 Category Theory

We start with a definition.

**Definition 1.9 (Category, object, morphism).** A *category*  $\mathcal{C}$  is 3 pieces of data

- A class of *objects*  $\text{Ob}(\mathcal{C})$
- $\forall X, Y \in \text{Ob}(\mathcal{C})$  a class of *morphisms* or arrows,  $\text{Hom}_{\mathcal{C}}(X, Y)$ .
- $\forall X, Y, Z \in \text{Ob}(\mathcal{C})$ , there exists a composition law

$$\text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z), \quad (f, g) \mapsto g \circ f$$

and 2 axioms

- Associativity.  $(f \circ g) \circ h = f \circ (g \circ h)$  for all [morphisms](#)  $f, g, h$  where composites are defined.
- Identity.  $\forall X \in \text{Ob}(\mathcal{C}) \exists \text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$  such that

$$f \circ \text{id}_X = f, \quad \text{id}_X \circ g = g$$

for all  $f, g$  where this makes sense.

Let's see some examples.

**Example.** We introduce some common [category](#).



$\mathcal{C}$	$\text{Ob}(\mathcal{C})$	$\text{Mor}(\mathcal{C})$
$\underline{\text{set}}$	Sets $X$	All maps of sets
$\underline{\text{fset}}$	Finite sets	All maps
$\underline{\text{Gp}}$	Groups	Group Homomorphisms
$\underline{\text{Ab}}$	Abelian groups	Group Homomorphisms
$\underline{k\text{-vect}}$	Vector spaces over $k$	$k$ -linear maps
$\underline{\text{Rng}}$	Rings	Ring Homomorphisms
$\underline{\text{Top}}$	Topological spaces	Continuous maps
$\underline{\text{Haus}}$	Hausdorff Spaces	Continuous maps
$\underline{\text{hTop}}$	Topological spaces	Homotopy classes of continuous maps
$\underline{\text{Top}^*}$	Based topological spaces <sup>2</sup>	Based maps <sup>3</sup>

**Remark.** Any **diagram** plus composition law.

$$\text{id}_A \hookrightarrow A \longrightarrow B \hookleftarrow \text{id}_B .$$

**Definition 1.10 (Monic, epic).** A **morphism**  $f: M \rightarrow N$  is *monic* if

$$\forall g_1, g_2 \quad f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$A \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} M \xrightarrow{f} N$$

Dually,  $f$  is *epic* if

$$\forall g_1, g_2 \quad g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$M \xrightarrow{f} N \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} B$$

**Lemma 1.2.** In  $\underline{\text{set}}, \underline{\text{Ab}}, \underline{\text{Top}}, \underline{\text{Gp}}$ , a map is **monic** if and only if  $f$  is injective, and **epic** if and only if  $f$  is surjective.

*Proof.* In  $\underline{\text{set}}$ , we prove that  $f$  is **monic** if and only if  $f$  is injective. Suppose  $f \circ g_1 = f \circ g_2$  and  $f$  is injective, then for any  $a$ ,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence  $g_1 = g_2$ .

<sup>2</sup>Topological spaces with a distinguished base point  $x_0 \in X$

<sup>3</sup>Continuous maps that presence base point  $f: (x, x_0) \rightarrow (y, y_0)$  such that

$$f: X \rightarrow Y, \quad f(x_0) = y_0$$

is continuous.

Now we prove another direction, with contrapositive. Namely, we assume that  $f$  is not injective and show that  $f$  is not **monic**. Suppose  $f(a) = f(b)$  and  $a \neq b$ , we want to show such  $g_i$  exists. This is easy by considering

$$g_1: * \mapsto a, \quad g_2: * \mapsto b.$$

■

### 1.5.1 Functor

After introducing the **category**, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

**Definition 1.11 (Functor).** Given  $\mathcal{C}, \mathcal{D}$  be two **categories**. A (covariant) *functor*  $F: \mathcal{C} \rightarrow \mathcal{D}$  is

1. a map on **objects**

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of **morphisms**

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(X) \rightarrow F(Y)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(f) \circ F(g)$

## Lecture 7: Functors

21 Jan. 10:00

**As previously seen.** Assume that we initially have a commutative diagram in  $\mathcal{C}$  as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & Z \end{array}$$

After applying  $F$ , we'll have

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ & \searrow F(g \circ f) = F(g) \circ F(f) & \downarrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in  $\mathcal{D}$ .

We can also have a so-called contravariant **functor**.

**Definition 1.12 (Contravariant functor).** Given  $\mathcal{C}, \mathcal{D}$  be two categories. A *contravariant functor*

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is

1. a map on objects

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(Y), F(X)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(Y) \rightarrow F(X)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a *contravariant functor*  $F$ , the diagram becomes

$$\begin{array}{ccc} F(X) & \xleftarrow{F(f)} & F(Y) \\ & \nwarrow F(g \circ f) = F(f) \circ F(g) & \uparrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in  $\mathcal{D}$ .

**Example.** Let see some examples.

1. Identity functor.

$$I: \mathcal{C} \rightarrow \mathcal{C}.$$

2. Forgetful functor.

•

$$F: \underline{\text{Gp}} \rightarrow \underline{\text{set}}, \quad G \mapsto G^4$$

such that

$$[f: G \rightarrow H] \mapsto [f: G \rightarrow H].$$

•

$$F: \underline{\text{Top}} \rightarrow \underline{\text{set}}, \quad X \mapsto X^5$$

such that

$$[f: X \rightarrow Y] \mapsto [f: X \rightarrow Y].$$

<sup>4</sup> $G$  is now just the underlying set of the group  $G$ .

<sup>5</sup> $X$  is now just the underlying set of the topological space  $X$ .

## 3. Free functor.

$$\begin{aligned} \underline{\text{set}} &\rightarrow \underline{k\text{-vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

i.e., vector space with basis  $s$  such that

$$[f: A \rightarrow B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

## 4.

$$\begin{aligned} \underline{k\text{-vect}} &\rightarrow \underline{k\text{-vect}} \\ V &\mapsto V^* = \text{Hom}_k(V, k) \end{aligned}$$

If we are working on a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functors.

## 1.

$$\begin{aligned} \underline{\text{Top}}^* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0) \end{aligned}$$

where  $\pi_1$  is so-called *fundamental group*.

## 2.

$$\begin{aligned} \underline{\text{Top}} &\rightarrow \underline{\text{Ab}} \\ X &\mapsto H_p(X) \end{aligned}$$

where  $H_p$  is so-called  $p^{\text{th}}$  *homology*.

Let's see the formal definition.

## 1.6 Free Groups

**Definition 1.13 (Free group).** Given a set  $S$ , the *free group* is a group  $F_S$  on  $S$  with a map  $S \rightarrow F_S$  satisfying the universal property.

If  $G$  is any group,  $f: S \rightarrow G$  is any map of sets,  $f$  extends uniquely to group homomorphism  $\bar{f}: F_S \rightarrow G$ .

$$\begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \bar{f}: \text{gp hom} \\ & & G \end{array}$$

**Note.** This defines a *natural bijection*

$$\mathrm{Hom}_{\mathrm{set}}(S, \mathcal{U}(G)) \cong \mathrm{Hom}_{\mathrm{Grp}}(F_S, G),$$

where  $\mathcal{U}(G)$  is the **forgetful functor** from the **category** of groups to the **category** of sets. This is the statement that the **free functor** and the forgetful functor are **adjoint**; specifically that the **free functor** is the left **adjoint** (appears on the left in the Hom above).

**Definition 1.14 (Adjoint functor).** A **free** and **forgetful functor** is *adjoints*.

**Remark.** Whenever we state a universal property for an **object** (plus a map), an **object** (plus a map) may or may not exist. If such **object** exists, then it defines the **object uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the **object** (plus a map).

**Lemma 1.3.** Universal property defines  $F_S$  (plus a map  $S \rightarrow F(S)$ ) uniquely up to unique isomorphism.

*Proof.* Fix  $S$ . Suppose

$$S \rightarrow F_S, \quad S \rightarrow \tilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{array}{ccc} S & \longrightarrow & \tilde{F}_S \\ & \searrow f & \downarrow \exists! \varphi \\ & & F_S \end{array} \quad \begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \psi \\ & & \tilde{F}_S \end{array}$$

We'll show  $\varphi$  and  $\psi$  are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \mathrm{id}_{F_S} & \searrow f \\ S & & \\ f \searrow & & \end{array} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \mathrm{id}_{\tilde{F}_S} & \searrow f \\ S & & \\ f \searrow & & \end{array}$$

Hence, we see that

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \psi & \searrow f \\ S & \longrightarrow & \tilde{F}_S \\ f \searrow & \downarrow \varphi & \nearrow f \\ & F_S & \end{array} \quad \varphi \circ \psi = \mathrm{id}_{F_S} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \varphi & \searrow f \\ S & \longrightarrow & F_S \\ f \searrow & \downarrow \psi & \nearrow f \\ & \tilde{F}_S & \end{array} \quad \psi \circ \varphi = \mathrm{id}_{\tilde{F}_S}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \text{id}_{F_S}, \quad \psi \circ \varphi = \text{id}_{\tilde{F}_S},$$

so  $\varphi$  and  $\psi$  are inverses (thus group isomorphism). ■

## Lecture 8: The Fundamental Group $\pi_1$

24 Jan. 10:00

**Example.** In [category](#) [Ab](#) [free](#) Abelian group on a set  $S$  is

$$\bigoplus_S \mathbb{Z}.$$

In [category](#) of fields, no such thing as [free field on  \$S\$](#) .

### 1.6.1 Constructing the Free Groups $F_S$

**Proposition 1.2.** The [free group](#) defined by the universal property exists.

*Proof.* We'll just give a construction below. First, we see the definition.

**Definition 1.15 (Word).** Fix a set  $S$ , and we define a *word* as a finite sequence (possibly  $\emptyset$ ) in the formal symbols

$$\{s, s^{-1} \mid s \in S\}.$$

Then we see that elements in  $F_S$  are equivalence classes of [words](#) with the equivalence relation being

- deleted  $ss^{-1}$  or  $s^{-1}s$ . i.e.,

$$vs^{-1}sw \sim vw$$

$$vss^{-1}w \sim vw$$

for every [word](#)  $v, w, s \in S$ ,

with the group operation being concatenation. ■

**Example.** Given [words](#)  $ab^{-1}, bba$ , their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

**Exercise.** There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of [words](#) has a unique *reduced form*, namely the representation.
3. Check that  $F_S$  satisfies the universal property with respect to the map

$$S \rightarrow F_S, \quad s \mapsto s.$$

## 2 The Fundamental Group

### 2.1 Path

We start with the definition.

**Definition 2.1 (Path).** A *path* in a space  $X$  is a continuous map

$$\gamma: I \rightarrow X$$

where  $I = [0, 1]$ .

**Definition 2.2 (Homotopy path).** A *homotopy of paths*  $\gamma_0, \gamma_1$  is a *homotopy* from  $\gamma_0$  to  $\gamma_1$  rel  $\{0, 1\}$ .



**Example.** Fix  $x_1, x_0 \in X$ , then  $\exists$  *homotopy of paths* is an equivalence relation on *paths* from  $x_0$  to  $x_1$  (i.e.,  $\gamma$  with  $\gamma(0) = x_0, \gamma(1) = x_1$ ).

**Definition 2.3 (Path composition).** For *paths*  $\alpha, \beta$  in  $X$  with  $\alpha(1) = \beta(0)$ , the *composition*<sup>a</sup>  $\alpha \cdot \beta$  is

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



<sup>a</sup>Also named *product*, *concatenation*.

**Remark.** By the pasting lemma, this is continuous, hence  $\alpha \cdot \beta$  is actually a *path* from  $\alpha(0)$  to  $\beta(1)$ .

**Definition 2.4 (Reparameterization).** Let  $\gamma: I \rightarrow X$  be a [path](#), then a *reparameterization* of  $\gamma$  is a [path](#)

$$\gamma': I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where  $\varphi$  is [continuous](#) and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

**Exercise.** A [path](#)  $\gamma$  is [homotopic rel \$\{0, 1\}\$](#)  to all of its [reparameterizations](#).

*Proof.* We show that  $\gamma$  and  $\gamma \circ \phi$  are [homotopic rel \$\{0, 1\}\$](#)  by showing that there exists a continuous  $F_t$  such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since  $\phi$  is continuous, so we define

$$F_t(x) = (1 - t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all  $x, t \in I$ .

Now, we check that  $F_t$  really gives us a [homotopic rel \$\{0, 1\}\$](#) . We have

$$\begin{aligned} F_t(0) &= (1 - t)\gamma(0) + t \cdot \gamma \circ \phi(0) = (1 - t)\gamma(0) + t \cdot \underbrace{\gamma(\phi(0))}_0 = \gamma(0), \\ F_t(1) &= (1 - t)\gamma(1) + t \cdot \gamma \circ \phi(1) = (1 - t)\gamma(1) + t \cdot \underbrace{\gamma(\phi(1))}_1 = \gamma(1), \end{aligned}$$

which shows that 0 and 1 are independent of  $t$ , hence  $\gamma$  and  $\gamma \circ \phi$  are [homotopic rel \$\{0, 1\}\$](#) . ■

**Exercise.** Fix  $x_0, x_1 \in X$ . Then [homotopy of paths](#) ([relative  \$\{0, 1\}\$](#) ) is an equivalence relation on [paths](#) from  $x_0$  to  $x_1$ .

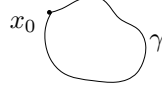
## 2.2 Fundamental Group and Groupoid

### 2.2.1 Fundamental Group



**Definition 2.5 (Fundamental Group).** Let  $X$  denotes the space and let  $x_0 \in X$  be the base point. The *fundamental group of  $X$  based at  $x_0$* , denoted by  $\pi_1(X, x_0)$ , is a group such that

- Elements: **Homotopy** classes  $\text{rel}\{0, 1\}$  of **paths**  $[\gamma]$  where  $\gamma$  is a **loop** with  $\gamma(0) = \gamma(1) = x_0$ <sup>a</sup>



- Operation: **Composition of paths**.
- Identity: Constant loop  $\gamma$  based at  $x_0$  such that

$$\gamma: I \rightarrow X, \quad t \mapsto x_0$$

- Inverses: The inverse  $[\gamma]^{-1}$  of  $[\gamma]$  is represented by the loop  $\bar{\gamma}$  such that

$$\bar{\gamma}(t) = \gamma(1 - t).$$



<sup>a</sup>We say  $\gamma$  is **based** at  $x_0$ .

*Proof.* We prove that

**Associativity.**  $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$ . We break this down into

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t), & t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t - 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t - 2), & t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t - 3), & t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t - 1), & t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, we define  $\phi: I \rightarrow I$  such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \frac{1}{2}\right], & t \in \left[0, \frac{1}{4}\right]; \\ t + \frac{1}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right], & t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{t+1}{2} \in \left[\frac{3}{4}, 1\right], & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and  $\phi(t)$  is continuous and satisfied  $\phi(0) = 0$  and  $\phi(1) = 1$ , which implies that the associativity holds.

**Identity.** We want to show that  $[\gamma \cdot c] = [\gamma]$ . Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & t \in \left[0, \frac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, consider  $\phi: I \rightarrow I$  such that

$$\phi(t) = \begin{cases} 2t, & t \in \left[0, \frac{1}{2}\right]; \\ 1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and  $\phi(t)$  is continuous and satisfied  $\phi(0) = 0$  and  $\phi(1) = 1$ .

**Inverses.** We want to show that  $\gamma \cdot \bar{\gamma} \simeq c$ , where  $\bar{\gamma}(t) = \gamma(1-t)$ . Firstly, we have

$$(\gamma \cdot \bar{\gamma})(t) = \begin{cases} \gamma(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2t), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider  $F_t$  given by

$$F_t(x) = \begin{cases} \gamma(2xt), & x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2xt), & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

If  $t = 0$ , we have

$$F_0(x) = \begin{cases} \gamma(0), & x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1), & x \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0$$

for all  $x \in I$ , namely  $F_0 = c$ , while when  $t = 1$ , we have

$$F_1(x) = \begin{cases} \gamma(2x), & x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2x), & x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \bar{\gamma})(x),$$

and we see that  $F_t$  is continuous since at  $x = \frac{1}{2}$ , we have

$$\gamma(2x) = \gamma(1) = \bar{\gamma}(0) = \bar{\gamma}(1-2x),$$

hence we see that  $F_t$  is the **homotopy** between  $\gamma \cdot \bar{\gamma}$  and  $c$ .

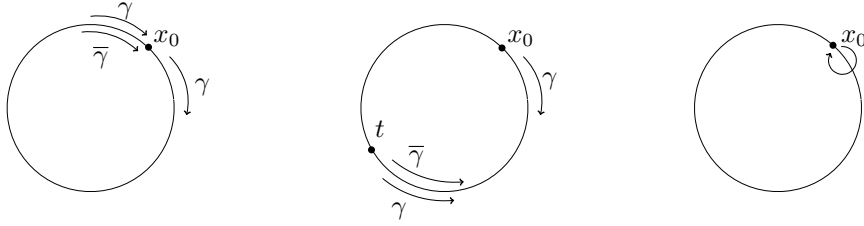


Figure 10: Illustration of  $F_t$ . Intuitively, the **path**  $\gamma \cdot \bar{\gamma}$  is  $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\bar{\gamma}} x_0$ . But now,  $F_t$  is  $x_0 \xrightarrow{\gamma} t \xrightarrow{\bar{\gamma}} x_0$ . We can think of this **homotopy** is *pulling back* the turning point along the original **path**.

■

**Theorem 2.1.** If  $X$  is **path**-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

**Remark.** We see that we can write  $\pi_1(X)$  up to isomorphism given this result.

*Proof.* To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if  $[h \cdot \gamma \cdot \bar{h}] = [h \cdot \gamma' \cdot \bar{h}]$ , then since we know that  $h^{-1} = \bar{h}$ , hence in the **fundamental group**  $\pi_1(X, x_0)$ , we see that

$$\bar{h} \cdot h \cdot \gamma \cdot \bar{h} \cdot h = \bar{h} \cdot h \cdot \gamma' \cdot \bar{h} \cdot h. \implies \gamma = \gamma'$$

as we desired.

- onto. We see that for every  $\alpha \in \pi_1(X, x_0)$ , there exists a  $\gamma \in \pi_1(X, x_0)$  such that

$$\gamma = \bar{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)^6$$

since  $h \cdot \gamma \cdot \bar{h} = \alpha$ .

We then see that the **fundamental group** of  $X$  does not depend on the choice of basepoint, only on the choice of the **path** component of the basepoint. If  $X$  is **path-connected**, it now makes sense to refer to *the fundamental group* of  $X$  and write  $\pi_1(X)$  for the abstract group (up to isomorphism). ■

**Exercise.** Composition of **paths** is well-defined on **homotopy** classes  $\text{rel}\{0, 1\}$ .

**Exercise.** If  $X$  is a contractible space, then  $X$  is **path-connected** and  $\pi_1(X)$  is trivial.

The followings are the properties about **homotopy path**. They are useful when we introduce **fundamental groupoid**.

**Lemma 2.1.** Given  $x_0, x_1, x_2 \in X$ ,  $\alpha, \alpha'$  are two **paths** from  $x_0$  to  $x_1$ , and  $\beta, \beta'$  are two **paths** from  $x_1$  to  $x_2$ . If  $\langle \alpha \rangle = \langle \alpha' \rangle$ ,  $\langle \beta \rangle = \langle \beta' \rangle$ , then  $\langle \alpha \cdot \beta \rangle = \langle \alpha' \cdot \beta' \rangle$ .

*Proof.* Given  $\alpha \simeq_F \alpha' \text{ rel}\{0, 1\}$ ,  $\beta \simeq_G \beta' \text{ rel}\{0, 1\}$ , then we want to prove

$$\alpha \cdot \beta \simeq \alpha' \cdot \beta' \text{ rel}\{0, 1\}.$$

This is done by using **homotopy**  $H: I \times I \rightarrow X$  such that it combines  $F(2s, t)$  and  $G(2s - 1, t)$ .



■

<sup>6</sup>Notice that this is indeed the case, one can verify this by the fact that  $h: x_0 \rightarrow x_1$  and  $\bar{h}: x_1 \rightarrow x_0$ .

**Lemma 2.2.** Let  $x_0, x_1, x_2, x_3 \in X$ ,  $\alpha$  is a path from  $x_0$  to  $x_1$ ,  $\beta$  is a path from  $x_1$  to  $x_2$ ,  $\gamma$  is a path from  $x_2$  to  $x_3$ . Then

$$\langle (\alpha \cdot \beta) \cdot \gamma \rangle = \langle \alpha \cdot (\beta \cdot \gamma) \rangle.$$

*Proof.* We can write out the homotopy by the following diagram.

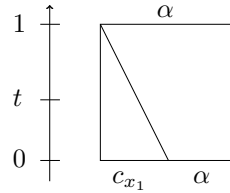


■

**Lemma 2.3.** Let  $X$  be a topological space, and  $x_0 \in X$ . Then for every path homotopy  $\langle \alpha \rangle$  from  $x_1$  to  $x_2$ , we have

$$\langle c_{x_1} \cdot \alpha \rangle = \langle \alpha \rangle = \langle \alpha \cdot c_{x_2} \rangle.$$

*Proof.* We only need to prove  $c_{x_1} \cdot \alpha \simeq \alpha \text{ rel } \{0, 1\}$ . The homotopy can be written out explicitly by the following diagram.



■

**Lemma 2.4.** For every path homotopy  $\langle \alpha \rangle$  from  $x_1$  to  $x_2$ , then

$$\langle \alpha \cdot \alpha^{-1} \rangle = \langle c_{x_1} \rangle, \quad \langle \alpha^{-1} \cdot \alpha \rangle = \langle c_{x_2} \rangle.$$

*Proof.* For the first case, we have the following diagram.



The second case follows similarly. ■

### 2.2.2 Fundamental Groupoid

This section is not covered in class, but it's a useful concept. The idea is that after giving [Definition 2.5](#), we see that we actually create a [fundamental group](#) at **every** point in  $X$ , furthermore, when we use [Theorem 2.1](#) if  $X$  is [path-connected](#), we actually **lose** some information about this space. Here is how we can store all the information.

**Notation (Constant loop).** We denote  $c_x$ , where  $x \in X$  such that

$$\begin{aligned} c_x : [0, 1] &\rightarrow X \\ t &\mapsto x \end{aligned}$$

as a *constant loop*.

**Definition 2.6 (Groupoid).** A [category](#)  $\mathcal{C}$  is a *groupoid* if any [morphisms](#) in  $\mathcal{C}$  is and isomorphism.

**Remark.** We'll soon see that for any topological space  $x$ , [Definition 2.5](#) defines a [groupoid](#), denoted by  $\Pi(X)$ .

**Definition 2.7 (Fundamental groupoid).** Let  $X$  denotes the space, then the [category](#)  $\Pi(X)$  is a *fundamental groupoid of  $X$*  such that

- $\text{Ob}(\Pi(X)) := X$
- $\text{Hom}(\Pi(X)) : \forall p, q \in \text{Ob}(\Pi(X)) = X,$

$$\text{Hom}_{\Pi(X)}(p, q) := \{\text{Paths from } p \text{ to } q\} / \sim.$$

- Composition: For every  $p, q, r \in \text{Ob}(\Pi(X)) = X,$

$$\begin{aligned} \circ : \text{Hom}_{\Pi(X)}(p, q) \times \text{Hom}_{\Pi(X)}(q, r) &\rightarrow \text{Hom}_{\Pi(X)}(p, r) \\ (\langle \alpha \rangle, \langle \beta \rangle) &\mapsto \langle \beta \rangle \circ \langle \alpha \rangle := \langle \alpha \cdot \beta \rangle. \end{aligned}$$

- Identity: For every  $p \in \text{Ob}(\Pi(X)) = X,$  we define  $1_p := \langle c_p \rangle \in \text{Hom}_{\Pi(X)}(p, p)$  be the constant loop based at  $p$  such that for every  $\langle \alpha \rangle \in \text{Hom}_{\Pi(X)}(p, q),$

$$\langle \alpha \rangle \circ \text{id}_p = \text{id}_q \circ \langle \alpha \rangle = \langle \alpha \rangle.$$

- Associativity: Given  $p, q, r, s \in \text{Ob}(\Pi(X)) = X,$  with the [paths](#)

$$p \xrightarrow{\langle \alpha \rangle} q \xrightarrow{\langle \beta \rangle} r \xrightarrow{\langle \gamma \rangle} s$$

Then

$$\langle \gamma \rangle \circ (\langle \beta \rangle \circ \langle \alpha \rangle) = (\langle \gamma \rangle \circ \langle \beta \rangle) \circ \langle \alpha \rangle.$$

*Proof.* Note that in [Definition 2.7](#), we need to show some of the definitions is indeed well-defined, and we also need to show that  $\Pi(X)$  is actually a [groupoid](#).

- Composition: Since if  $\alpha \simeq \alpha', \beta \simeq \beta',$  we have

$$\alpha \cdot \beta \simeq \alpha' \cdot \beta'$$

from [Lemma 2.1](#).

- Identity: It follows that

$$\langle \alpha \rangle \circ \text{id}_p = \langle c_p \cdot \alpha \rangle = \langle \alpha \rangle$$

from [Lemma 2.3](#). The left identity can be shown similarly.

- Associativity: It's trivial in the sense that all the [homotopy](#) can be easily derived from [Lemma 2.2](#).

Additionally, from [Lemma 2.4](#), we see that given  $\alpha$  is a [path](#) from  $p$  to  $q$ , then

$$\begin{cases} \langle \alpha^{-1} \cdot \alpha \rangle &= \langle c_q \rangle =: \text{id}_q \\ \langle \alpha \cdot \alpha^{-1} \rangle &= \langle c_p \rangle =: \text{id}_p. \end{cases}$$

Furthermore, since  $\langle \alpha^{-1} \cdot \alpha \rangle = \langle \alpha \rangle \circ \langle \alpha^{-1} \rangle$  and  $\langle \alpha \cdot \alpha^{-1} \rangle = \langle \alpha^{-1} \rangle \circ \langle \alpha \rangle,$  hence this means  $\Pi(X)$  is indeed a [groupoid](#). ■

**Remark.** Assume  $\mathcal{C}$  is a [groupoid](#), then for every  $x \in \text{Ob}(\mathcal{C})$ , we can define

$$\cdot : \text{Hom}_{\mathcal{C}}(x, x) \times \text{Hom}_{\mathcal{C}}(x, x) \rightarrow \text{Hom}_{\mathcal{C}}(x, x)$$

such that

$$(f, g) \mapsto f \cdot g := g \circ f.$$

We can prove that

$$(\text{Hom}_{\mathcal{C}}(x, x), \cdot)$$

defines a group  $\text{Aut}_{\mathcal{C}}(x)$  called the *isotropy group* of  $\mathcal{C}$  at  $x$ .

**Exercise.** For every  $x, y \in \text{Ob}(\mathcal{C})$ , if there exists  $f \in \text{Hom}_{\mathcal{C}}(x, y)$ , then  $f$  induces

$$f_* : \text{Aut}_{\mathcal{C}}(x) \xrightarrow{\sim} \text{Aut}_{\mathcal{C}}(y),$$

where  $f_*$  is a group homomorphism.

**Remark.** For every  $p \in X = \text{Ob}(\Pi(X))$ , we have

$$\text{Aut}_{\Pi(X)}(p) = \pi_1(X, p).$$

Firstly, since they're the same in the sense of **set**:

$$\text{Aut}_{\Pi(X)}(p) = \text{Hom}_{\Pi(X)}(p, p) = \{\text{Loops in } X \text{ based at } p\} / \sim = \pi_1(X, p).$$

Hence, we only need to verify their group composition agrees. But this is trivial, since for every two  $\langle \alpha \rangle, \langle \beta \rangle \in \text{Aut}_{\Pi(X)}(p)$ ,

$$\underbrace{\langle \alpha \rangle \cdot \langle \beta \rangle}_{\text{Composition from } \text{Aut}_{\Pi(X)}} = \langle \beta \rangle \circ \langle \alpha \rangle = \underbrace{\langle \alpha \cdot \beta \rangle}_{\text{Composition from } \pi_1}.$$

This implies that [Theorem 2.1](#) is just a particular example as a [groupoid](#).

## Lecture 9: Calculate Fundamental Group

26 Jan. 10:00



Figure 11: [Fundamental Group](#) is basically a *hole detector*!

### 2.3 Calculations with $\pi_1(S^n)$

Let's start with a basic but important theorem.



**Theorem 2.2 (The fundamental group of  $S^1$ ).** The fundamental group of  $S^1$  is

$$\pi_1(S^1) \cong \mathbb{Z},$$

and this identification is given by the [paths](#)

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))].$$

**Remark.** Intuitively, this winds around  $S^1$   $n$  times. The key to this proof was to understand  $S^1$  via the [covering space](#)  $\mathbb{R} \rightarrow S^1$ . We will talk about [covering spaces](#) much later.

*Proof.* With the help of [covering spaces](#) and the theorems build around which, we can define

$$\begin{aligned} p: \mathbb{R} &\rightarrow S^1, & x &\mapsto e^{2\pi i x}, \\ \varphi: \mathbb{Z} &\rightarrow \pi_1(S^1, 1), & n &\mapsto \langle p \circ \gamma_n \rangle, \end{aligned}$$

where  $p$  defined above is a [covering map](#). We need to show that this is well-defined.

From the definition of  $\varphi$ , we see that it's a homomorphism. But we also need to show

- $\varphi$  is a surjection. This is shown by [Corollary 3.1](#), specifically in the case of [path](#).
- $\varphi$  is an injection. This is shown by [Corollary 3.1](#), specifically in the case of [homotopy of paths](#).

■

**Theorem 2.3.** Given  $(X, x_0)$  and  $(Y, y_0)$ , then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\left[ \begin{array}{l} r: I \rightarrow X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{array} \right] \mapsto (r_X, r_Y).$$

*Proof.* Let  $Z \xrightarrow{f} X \times Y$  with  $z \mapsto (f_X(z), f_Y(z))$ . Then we have

$$f \text{ continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply above to

- [Paths](#)  $I \rightarrow X \times Y$ .
- [Homotopies of paths](#)  $I \times I \rightarrow X \times Y$ .

■

**Corollary 2.1 (The fundamental group of  $S^k$ ).** The torus  $T \cong S^1 \times S^1$  has fundamental group  $\pi_1(T) \cong \mathbb{Z}^2$ . Additionally, for a  $k$ -torus

$$\underbrace{S^1 \times S^1 \times \dots \times S^1}_{k \text{ times}} = (S^1)^k,$$

the fundamental group is then  $\mathbb{Z}^k$ , i.e.

$$\pi_1((S^1)^k) \cong \mathbb{Z}^k.$$

*Proof.* Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

**Remark.** One way to think of the  $k$ -torus is as a  $k$ -dimensional cube with opposite  $(k - 1)$ -dimensional faces identified by translation.

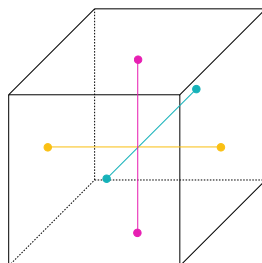


Figure 12: 3-torus with cube identified with parallel sides.

**Lemma 2.5.** Let  $f, g: X \rightarrow Y$  such that  $f \simeq_F g$ . Let  $x_0 \in X$ , then given

$$f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$$

$$g_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, g(x_0))$$

with  $\gamma: [0, 1] \rightarrow Y$ ,  $t \mapsto F(x_0, t)$ ,

$$\begin{aligned} \gamma_*: \pi_1(Y, f(x_0)) &\rightarrow \pi_1(Y, g(x_0)) \\ \langle \alpha \rangle &\mapsto \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle, \end{aligned}$$

the following diagram commutes.

$$\begin{array}{ccc} & \pi_1(Y, f(x_0)) & \\ f_* \nearrow & \downarrow \gamma_* & \searrow g_* \\ \pi_1(X, x_0) & & \pi_1(Y, g(x_0)) \end{array}$$

*Proof.* We want to prove that for any  $\langle \alpha \rangle \in \pi_1(X, x_0)$ , we have

$$\gamma_* \circ f_*(\langle \alpha \rangle) = g_*(\langle \alpha \rangle).$$

The left-hand side is just

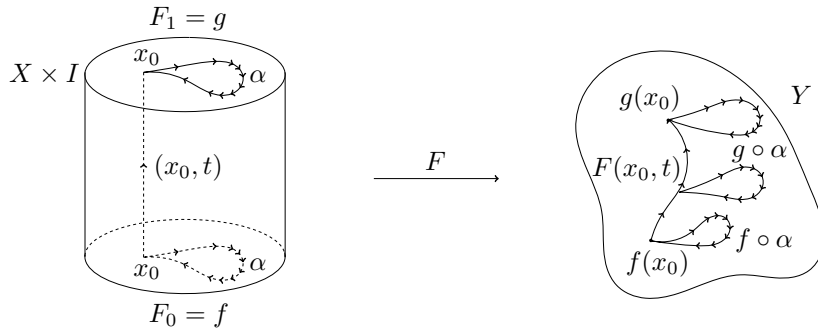
$$\gamma_* \circ f_*(\langle \alpha \rangle) = \gamma_*(\langle f \circ \alpha \rangle) = \langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \rangle,$$

while the right-hand side is just

$$g_*(\langle \alpha \rangle) = \langle g \circ \alpha \rangle.$$

That is, we now want to show

$$\langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \rangle = \langle g \circ \alpha \rangle.$$



We see that we can obtain a [homotopy](#)  $G: I \times I \rightarrow Y$  such that

$$G := F \circ (\alpha \times \text{id}),$$

where we define  $\alpha \times \text{id}$  by

$$\alpha \times \text{id}: I \times I \rightarrow X \times I, \quad (s, t) \mapsto (\alpha(s), t).$$



Figure 13:  $\alpha \times \text{id}$ 's image.

We see that by defining such  $G$ , we have the following.



To write out this [homotopy](#) explicitly, we see the following diagram.



■

**Theorem 2.4 (Fundamental group is a homotopy invariant).** If  $X, Y$  are homotopy equivalent, then their fundamental groups are isomorphic.

*Proof.*

HW.

**Remark.** This gives us a powerful tool to calculate  $\pi_1$ .

**Example.** We now see some examples.

1.  $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2.  $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong 0 \times \mathbb{Z} = \mathbb{Z}$  since

$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R},$$

which means that the generators are just loops around the hole intuitively.

## 2.4 Fundamental Group and Groupoid Define Functors

**Theorem 2.5 (Fundamental group defines a functor).**  $\pi_1$  is a functor such that

$$\begin{aligned} \pi_1: \underline{\text{Top}}_* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0). \end{aligned}$$

While on a map  $f: X \rightarrow Y$  taking base point  $x_0$  to  $y_0$ ,  $\pi_1$  induces a map

$$\begin{aligned} f_*: \pi_1(X, x_0) &\rightarrow \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

i.e.,

$$[f: X \rightarrow Y] \mapsto [f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))].$$

**Notation.** We usually write  $f_*$  if it's a covariant functor, while writing  $f^*$  if it's a contravariant functor.

*Proof.* We need to check

- well-defined on path homotopy classes.
- $f_*$  is a group homomorphism.

$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \frac{1}{2}\right] \\ f(\beta(1-2s)), & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

- $(\text{id}_{(X, x_0)})_* = \text{id}_{\pi_1(X, x_0)}$

- $(f_* \circ g_*) = (f \circ g)_*$

$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \implies f_*(g_*(\gamma)).$$

DIY

$$\begin{array}{ccc} (X, x_0) & \rightsquigarrow & \pi_1(X, x_0) \\ f \downarrow & & \downarrow f_* \\ (Y, y_0) & \rightsquigarrow & \pi_1(Y, y_0) \end{array}$$

■

**Remark.** We see that the construction of **fundamental group** is actually constructing a **functor**. Specifically,

$$\pi_1: \underline{\text{Top}}_* \rightarrow \underline{\text{Gp}}$$

such that

- on **objects**:

$$\forall (X, x_0) \in \text{Ob}(\underline{\text{Top}}_*), \quad \pi_1(X, x_0) = \text{fundamental group based at } x_0.$$

- on **morphisms**:

$$\forall f: (X, x_0) \rightarrow (Y, y_0), \quad \pi_1(f) = f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).$$

Our initial motivation is to construct a topological invariant, but we see that using  $\pi_1$ , we need an additional **base point**. But as you already imagined, the **fundamental groupoid** actually is a **functor** as well.

Before we proceed further, we need to see the **category of groupoid**, denoted by  $\underline{\text{Gpd}}$ .

**Definition 2.8 (Category of groupoid).** The *category of groupoid*, denoted as  $\underline{\text{Gpd}}$ , contains the following data.

- $\text{Ob}(\underline{\text{Gpd}})$ : **groupoids**.
- $\text{Hom}(\underline{\text{Gpd}})$ : **functors** between **groupoids**.
- Composition: For every  $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \in \text{Ob}(\underline{\text{Gpd}})$ ,

$$\mathfrak{X} \xrightarrow{F} \mathfrak{Y} \xrightarrow{G} \mathfrak{Z}$$

then  $G \circ F: \mathfrak{X} \rightarrow \mathfrak{Z}$  is a **functor** defined as

- on **objects**:  $\forall X \in \text{Ob}(\mathfrak{X})$ ,

$$G \circ F(X) := G(F(X)).$$

- on **morphisms**:  $\forall X, Y \in \text{Ob}(\mathfrak{X})$  and  $f: X \rightarrow Y$ ,

$$G \circ F(f) := G(F(f)).$$

- Identity. For every **groupoid**  $\mathfrak{X}$ , we define  $\text{id}_{\mathfrak{X}}: \mathfrak{X} \rightarrow \mathfrak{X}$ , where
  - $\forall X \in \text{Ob}(\mathfrak{X})$ ,  $\text{id}_{\mathfrak{X}}(X) = X$
  - $\forall f \in \text{Hom}(\mathfrak{X})$ ,  $\text{id}_{\mathfrak{X}}(f) = f$ .
- Associativity. Since the composition is defined based on two **functors** (given  $\mathfrak{X} \xrightarrow{F} \mathfrak{Y} \xrightarrow{G} \mathfrak{Z}$ ), this holds trivially.

*Proof.* We need to show that the composition is well-defined. Specifically, we need to check

- $G \circ F(\text{id}_X) = \text{id}_{G \circ F(X)}$ , since

$$G \circ F(\text{id}_X) = G(F(\text{id}_X)) = G(\text{id}_{F(X)}) = \text{id}_{G(F(X))} = \text{id}_{G \circ F(X)}.$$

- Given  $X_1, X_2, X_3 \in \text{Ob}(\mathfrak{X})$  and

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

we want to show  $G \circ F(g \circ f) = G \circ F(g) \circ G \circ F(f)$ . Firstly, since  $G$  is a **functor**, hence

$$G \circ F(g) \circ G \circ F(f) = G(F(g)) \circ G(F(f)) = G(F(g) \circ F(f)).$$

Again, since  $F$  is a functor, so we further have

$$G \circ F(g) \circ G \circ F(f) = G(F(g \circ f)) = G \circ F(g \circ f).$$

■

**Theorem 2.6 (Fundamental groupoid defines a functor).**  $\Pi$  is a functor such that

$$\Pi: \underline{\text{Top}} \rightarrow \underline{\text{Gpd}},$$

where

- on **objects**: For every  $X \in \text{Ob}(\underline{\text{Top}})$ ,

$$X \mapsto \Pi(X).$$

- on **morphisms**: for every  $X, Y \in \text{Ob}(\underline{\text{Top}})$ ,  $f: X \rightarrow Y$ , define a functor

$$\Pi(f): \Pi(X) \rightarrow \Pi(Y)$$

such that

- on **objects**: For every  $p \in \text{Ob}(\Pi(X)) = X$ ,  $\Pi(f)(p) = f(p)$ . i.e.,

$$\Pi(f): \underbrace{\text{Ob}(\Pi(X))}_X \rightarrow \underbrace{\text{Ob}(\Pi(Y))}_Y.$$

- on **morphisms**: For every  $\langle \alpha \rangle \in \text{Hom}_{\Pi(X)}(p, q)$ , define

$$\Pi(f)(\langle \alpha \rangle) := \langle f \circ \alpha \rangle \in \text{Hom}_{\Pi(Y)}(f(p), f(q)).$$

*Proof.* We need to check that the defined functor  $\Pi(f)$  satisfies

- $\Pi(f)(\text{id}_p) = \text{id}_{f(p)}$ . Indeed, since

$$\Pi(f)(\text{id}_p) = \Pi(f)(\langle c_p \rangle) = \langle f \circ d_p \rangle = \langle c_{f(p)} \rangle = \text{id}_{f(p)}.$$

- For every  $p, q, r \in X = \text{Ob}(\Pi(X))$ ,

$$p \xrightarrow{\langle \alpha \rangle} q \xrightarrow{\langle \beta \rangle} r$$

we want to show  $\Pi(f)(\langle \beta \rangle \circ \langle \alpha \rangle) = \Pi(f)(\langle \beta \rangle) \circ \Pi(f)(\langle \alpha \rangle)$ . Indeed, since

$$\Pi(f)(\langle \beta \rangle \circ \langle \alpha \rangle) = \Pi(f)(\langle \alpha \cdot \beta \rangle) = \langle f \circ (\alpha \cdot \beta) \rangle,$$

and

$$\Pi(f)(\langle \beta \rangle) \circ \Pi(f)(\langle \alpha \rangle) = \langle f \circ \beta \rangle \circ \langle f \circ \alpha \rangle = \langle (f \circ \alpha) \cdot (f \circ \beta) \rangle.$$

Since  $\langle f \circ (\alpha \cdot \beta) \rangle = \langle (f \circ \alpha) \cdot (f \circ \beta) \rangle$ , hence  $\Pi(f)$  is well-defined.

Now, we need to prove the same thing for  $\Pi$ , namely  $\Pi$  satisfies

- $\Pi(\text{id}_X) = \text{id}_{\Pi(X)}$  for all  $X \in \text{Ob}(\underline{\text{Top}})$ . This is trivial since

$$\Pi(\text{id}_X): \Pi(X) \rightarrow \Pi(X),$$

- on **objects**:  $p \mapsto \text{id}_X(p) = p$ .



– on **morphisms**:  $p \xrightarrow{\langle \alpha \rangle} q \mapsto \langle \text{id}_X \circ \alpha \rangle = \langle \alpha \rangle$ .

- For all  $X, Y, Z \in \text{Ob}(\underline{\text{Top}})$ ,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

then  $\Pi(g \circ f) = \Pi(g) \circ \Pi(f)$ . The diagrams are as follows.

$$\Pi(g \circ f): \Pi(X) \rightarrow \Pi(Z)$$

and

$$\Pi(X) \xrightarrow{\Pi(f)} \Pi(Y) \xrightarrow{\Pi(g)} \Pi(Z)$$

We see that this equality is in the sense of **functor**, hence we consider

– on **objects**: For every  $p \in \text{Ob}(\Pi(X)) = X$ ,  $\Pi(g \circ f)(p) = g \circ f(p)$  and

$$\Pi(g) \circ \Pi(f)(p) = \Pi(g)(\Pi(f)(p)) = \Pi(g)(f(p) = g(f(p))),$$

hence they're the same.

– on **morphisms**: For all  $\langle \alpha \rangle \in \text{Hom}_{\Pi(X)}(p, q)$ ,

$$* \Pi(g \circ f)(\langle \alpha \rangle) = \langle (g \circ f) \circ \alpha \rangle.$$

$$* \Pi(g) \circ \Pi(f)(\langle \alpha \rangle) = \Pi(g) \left( \underbrace{\Pi(f)(\langle \alpha \rangle)}_{\langle f \circ \alpha \rangle} \right) = \langle g \circ (f \circ \alpha) \rangle.$$

We see that they're the same.

■

## Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

The goal is to compute  $\pi_1(X)$  where  $X = A \cup B$  using the data

$$\pi_1(A), \pi_1(B), \pi_1(A \cap B).$$

### 2.5 Free Product

#### 2.5.1 Free Product

We first introduce a definition.

**Definition 2.9 (Free product).** Given some collections of groups  $\{G_\alpha\}_\alpha$ , the *free product*, denoted by  $*_\alpha G_\alpha$  is a group such that

- Elements: **Words** in  $\{g: g \in G_\alpha \text{ for any } \alpha\}$  modulo by the equivalence relation generated by

$$wg_i g_j v \sim w(g_i g_j)v$$

when both  $g_i, g_j \in G_\alpha$ . Also, for the identity element  $\text{id} = e_\alpha \in G_\alpha$  for any  $\alpha$  such that

$$we_\alpha v \sim wv.$$

Specifically,

$$*_\alpha G_\alpha := \{\text{words in } \{G_\alpha\}_\alpha\} / \sim.$$

- Operation: Concatenation of **words**.

**Remark.** In particular, we have the following universal property of  $*_\alpha G_\alpha$ . For every  $\alpha$ , there is a  $\iota_\alpha$  such that

$$\iota_\alpha: G_\alpha \rightarrow *_\alpha G_\alpha, \quad g \mapsto \bar{g},$$

where  $\iota_\alpha$  is a group homomorphism obviously. Further,  $(*_\alpha G_\alpha, \iota_\alpha)$  satisfies the following property: For every group  $H$  and a group homomorphism  $\varphi_\alpha: G_\alpha \rightarrow H$  for all  $\alpha$ , there exists an unique group homomorphism  $\varphi: *_\alpha G_\alpha \rightarrow H$  such that  $\varphi \circ \iota_\alpha = \varphi_\alpha$ , i.e., the following diagram commutes.



*Proof.* The proof is straightforward. Firstly, we define  $w = \overline{g_1 g_2 \dots g_n} \in *_\alpha G_\alpha$ ,  $g_i \in G_{\alpha_i}$ ,

$$\varphi(w) := \varphi_{\alpha_1}(g_1) \dots \varphi_{\alpha_n}(g_n).$$

Now, we just need to check

- It's well-defined, since  $\varphi_\alpha$  is a group homomorphism.
- $\varphi$  is a group homomorphism.
- $\varphi \circ \iota_\alpha = \varphi_\alpha$ .
- Such  $\varphi$  is unique. Suppose there exists another  $\psi: *_\alpha G_\alpha \rightarrow H$ , then

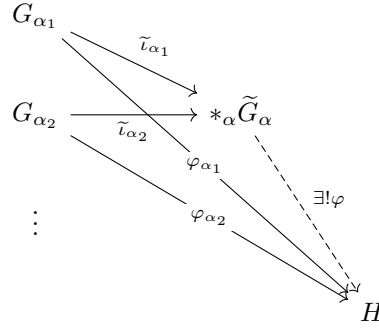
$$\varphi \circ \iota_\alpha = \varphi_\alpha \implies \forall_{g \in G_\alpha} \psi(\bar{g}) = \varphi_\alpha(g),$$

But then for every  $w = \overline{g_1 g_2 \dots g_n} \in *_\alpha G_\alpha$ ,  $g_i \in G_{\alpha_i}$ , we have

$$\psi(w) = \psi(\overline{g_1} \dots \overline{g_n}) = \psi(\overline{g_1}) \dots \psi(\overline{g_n}) = \psi_{\alpha_1}(\overline{g_1}) \dots \psi_{\alpha_n}(\overline{g_n}),$$

which is just  $\varphi$ . ■

**Remark.** We further claim that this universal property determines such [free product](#) uniquely. i.e., assume there are another group  $\tilde{G}$  and  $\tilde{\iota}_\alpha: G_\alpha \rightarrow \tilde{G}$ . Assume  $(\tilde{G}, \tilde{\iota}_\alpha)$  also satisfies the following property: For every group  $H$  and group homomorphism  $\varphi_\alpha: G_\alpha \rightarrow H$ , then there exists a unique group homomorphism  $\varphi: \tilde{G} \rightarrow H$  such that the following diagram commutes.



Then,  $\tilde{G} \cong *_\alpha G_\alpha$ .

*Proof.* Assume  $(\tilde{G}, \tilde{\iota}_\alpha)$  satisfies the universal property mentioned above. Then from the universal property and viewing  $\tilde{G}$  and  $*_\alpha G_\alpha$  as  $H$  separately, we obtain the following diagram.



We claim that

$$g \circ f = \text{id}, \quad f \circ g = \text{id}.$$

To see this, we simply apply the same observation, for example,



where  $g \circ f$  comes from the previous diagram. But notice that id let the diagram commutes also, and since it's unique, hence  $g \circ f = \text{id}$ . Similarly, we have  $f \circ g = \text{id}$ . ■

If you're careful enough, you may find out that all we're doing is just writing out a specific example of [Lemma 1.3](#)! Indeed, this is exactly the construction of a [free group](#).

**Definition 2.10 (Fibred coproduct).** Given a [category](#)  $\mathcal{C}$ , let  $f: Z \rightarrow X$ ,  $g: Z \rightarrow Y$ . The *fibred coproduct* between  $f$  and  $g$  is the data  $(W, p_1, p_2)$ , where  $W \in \text{Ob}(\mathcal{C})$ ,  $p_1: X \rightarrow W$ ,  $p_2: Y \rightarrow W$  satisfy the following.

- The diagram commutes.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \downarrow p_1 \\ Y & \xrightarrow{p_2} & W \end{array}$$

- For every  $u: X \rightarrow U$ ,  $v: Y \rightarrow U$  such that the following diagram commutes

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \downarrow p_1 \\ Y & \xrightarrow{p_2} & W \end{array} \quad \begin{array}{c} \searrow u \\ \downarrow \exists! h \\ \searrow v \end{array} \quad \begin{array}{c} \\ \\ U \end{array}$$

there exists a unique  $h: W \rightarrow U$  such that  $h \circ p_1 = u$ ,  $h \circ p_2 = v$ .

We say

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & W \end{array}$$

is a *Cocartesian* diagram.

**Exercise.** Prove that in a category  $\mathcal{C}$ , if the **fibred coproduct** of  $f$  and  $g$  exists

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \\ Y & & \end{array}$$

then such **fibred coproduct** is unique up to isomorphism.

**Remark.** If we reverse all the directions of **morphism**, then we have so-called *fibred product*.

**Example.** Let's see some example.

1. Let  $\mathcal{C} = \underline{\text{Top}}$ , and let  $X \in \text{Ob}(\underline{\text{Top}})$ . Given  $X_0, X_1 \in X$ , and  $\text{int}(X_0) \cup \text{int}(X_1) = X$ , if we have

$$\begin{aligned} i_0: X_0 &\hookrightarrow X, & i_1: X_1 &\hookrightarrow X \\ j_0: X_0 \cap X_1 &\hookrightarrow X_0, & j_1: X_0 \cap X_1 &\hookrightarrow X_1, \end{aligned}$$

then

$$\begin{array}{ccc} X_0 \cap X_1 & \xrightarrow{j_0} & X_0 \\ j_1 \downarrow & & \downarrow i_0 \\ X_1 & \xrightarrow{i_1} & X \end{array}$$

is a **cocartesian** diagram.

*Proof.* All we need to show is that given a topological space  $Y \in \underline{\text{Top}}$  and  $f: X_0 \rightarrow Y, g: X_1 \rightarrow Y$  in  $\underline{\text{Top}}$ , we have

$$f \circ j_0 = g \circ j_1.$$

$$\begin{array}{ccc} X_0 \cap X_1 & \xrightarrow{j_0} & X_0 \\ j_1 \downarrow & & \downarrow i_0 \\ X_1 & \xrightarrow{i_1} & X \end{array} \quad \begin{array}{c} \nearrow f \\ \searrow g \\ \exists! h \end{array}$$

We simply define  $h: X \rightarrow Y, x \mapsto h(x)$  such that

$$h(x) = \begin{cases} f(x), & \text{if } x \in X_0; \\ g(x), & \text{if } x \in X_1. \end{cases}$$

$h$  is clearly well-defined since the diagram commutes, so if  $x \in X_0 \cap X_1$ , then  $f(x) = g(x)$ . The only thing we need to show is that  $h$  is continuous. But this is obvious too since  $X = \text{int}(X_0) \cup \text{int}(X_1)$ , and

$$h|_{\text{int}(X_0)} = f|_{\text{int}(X_0)}, \quad h|_{\text{int}(X_1)} = g|_{\text{int}(X_1)}.$$

The uniqueness is trivial, hence this is indeed a **cocartesian** diagram.  $\blacksquare$

2. Let  $\mathcal{C} = \mathbf{Top}_*$ . Given  $p \in X_0 \cap X_1$ , where all other data are the same with the above example, we see that

$$\begin{array}{ccc} (X_0 \cap X_1, p) & \xrightarrow{j_0} & (X_0, p) \\ j_1 \downarrow & & \downarrow i_0 \\ (X_1, p) & \xrightarrow{i_1} & (X, p) \end{array}$$

is a **cocartesian** diagram.

3. Let  $\mathcal{C} = \mathbf{Gp}$ . Given  $P, G, H \in \mathbf{Ob}(\mathbf{Gp})$ , we claim that the **fibered coproduct** of  $i$  and  $j$  exists.

$$\begin{array}{ccc} P & \xrightarrow{i} & G \\ j \downarrow & & \\ H & & \end{array}$$

Consider  $G * H$  be the **free product** between  $G$  and  $H$ , with two inclusions

$$\iota_1: G \hookrightarrow G * H, \quad \iota_2: H \hookrightarrow G * H.$$

$$\begin{array}{ccc} P & \xrightarrow{i} & G \\ j \downarrow & & \downarrow \iota_1 \\ H & \xrightarrow{\iota_2} & G * H \end{array}$$

Let

$$N := \langle \{ \iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1} \mid x \in P \} \rangle,$$

we define

$$G *_p H = G * H / N.$$

$$\begin{array}{ccccc} P & \xrightarrow{i} & G & & \\ j \downarrow & & \downarrow \iota_1 & \searrow \tau & \\ H & \xrightarrow{\iota_2} & G * H & \xrightarrow{\pi} & G *_p H \\ & \searrow \nu & & & \end{array}$$

We claim that

$$\begin{array}{ccc} P & \xrightarrow{i} & G \\ j \downarrow & & \downarrow \tau \\ H & \xrightarrow{\nu} & G *_p H \end{array}$$

is a **cocartesian** diagram in  $\mathbf{Gp}$ .

*Proof.* Firstly, since it's just an outer diagram from above, hence it commutes. So we only need to prove this diagram satisfies the second diagram.

Given any group  $K$ , for every  $f: G \rightarrow K$ ,  $g: H \rightarrow K$  such that the following diagram commutes.

$$\begin{array}{ccc}
 P & \xrightarrow{i} & G \\
 j \downarrow & & \downarrow \tau \\
 H & \xrightarrow{\nu} & G *_p H
 \end{array}
 \begin{array}{c}
 \nearrow f \\
 \searrow g \\
 \text{---} h \text{---}
 \end{array}
 \rightarrow K$$

We want to prove that there exists a unique  $h: G *_p H \rightarrow K$  such that this diagram still commutes. The idea is simple, from the universal property of  $G * H$ , we see that there exists a unique  $\tilde{h}: G * H \rightarrow K$  such that

$$\tilde{h} \circ \iota_1 = f, \quad \tilde{h} \circ \iota_2 = g.$$

$$\begin{array}{ccc}
 P & \xrightarrow{i} & G \\
 j \downarrow & & \downarrow \tau \\
 H & \xrightarrow{\nu} & G *_p H
 \end{array}
 \begin{array}{c}
 \nearrow f \\
 \searrow g \\
 \text{---} h \text{---}
 \end{array}
 \rightarrow G *_p H$$

$\begin{array}{ccc} & G * H & \\ \iota_1 \swarrow & & \searrow \iota_2 \\ H & & G \end{array}$

We see that we can actually factor  $\tilde{h}$  through  $\pi$ , as long as  $\ker(\tilde{h}) \supset \ker(\pi)$ . Now, since

$$\ker(\pi) = \langle \{ \iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1} \mid x \in p \} \rangle,$$

we see that the kernel of  $\pi$  is indeed in the kernel of  $\tilde{h}$  since for every  $x \in P$ ,

$$\tilde{h}(\iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1}) = \underbrace{\tilde{h} \circ \iota_1}_{f} \circ i(x) \cdot \underbrace{\tilde{h} \circ \iota_2}_{g} \circ j(x)^{-1} = 1,$$

which implies  $\ker(\tilde{h}) \supset \ker(\pi)$ .

$$\begin{array}{ccc}
 G * H & \xrightarrow{\pi} & K \\
 \tilde{h} \downarrow & & \\
 G *_p H & &
 \end{array}$$

We then see that there exists a unique  $h: G *_p H \rightarrow K$  such that the above diagram commutes. ■

### 2.5.2 Free Product with Amalgamation

After seeing the above examples, the following definition should make sense.

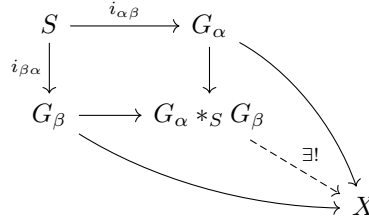
**Definition 2.11 (Free product with amalgamation).** If two groups  $G_\alpha$  and  $G_\beta$  have a common subgroup  $S_{\{\alpha,\beta\}}$ <sup>a</sup>, given two inclusion maps<sup>b</sup>  $i_{\alpha\beta}: S_{\{\alpha,\beta\}} \rightarrow G_\alpha$  and  $i_{\beta\alpha}: S_{\{\alpha,\beta\}} \rightarrow G_\beta$ , the *free product with amalgamation*  ${}_\alpha *_S G_\alpha$  is defined as  ${}^*_\alpha G_\alpha$  modulo the normal subgroup generated by

$$\{i_{\alpha\beta}(s_{\{\alpha,\beta\}})i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \mid s_{\{\alpha,\beta\}} \in S_{\{\alpha,\beta\}}\},$$

Namely<sup>c</sup>,

$${}_\alpha *_S G_\alpha = {}^*_\alpha G_\alpha / \langle i_{\alpha\beta}(s_{\{\alpha,\beta\}})i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \rangle$$

and satisfies the universal property



<sup>a</sup>In general, we don't need  $S_{\{\alpha,\beta\}}$  to be a subgroup.

<sup>b</sup>We don't actually need  $i_{\alpha\beta}, i_{\beta\alpha}$  to be inclusive as well.

<sup>c</sup>i.e.,  $i_{\alpha\beta}(s)$  and  $i_{\beta\alpha}(s)$  will be identified in the quotient.

**Remark.** We see that

- We can then write out words such as  $g_\alpha \cdot s \cdot g_\beta$  for  $s \in S$ , and view  $s$  as an element of  $G_\alpha$  or  $G_\beta$ . In fact, we can do this construction even when  $i_\alpha$  and  $i_\beta$  are not injective, though this means we are not working with a subgroup.
- Aside, in Top, the same universal property defines union



for  $A, B$  are open subsets and the inclusion of intersection.

## 2.6 Seifert-Van Kampen Theorem

With Definition 2.11, we can now see the important theorem.



**Theorem 2.7 (Seifert-Van Kampen Theorem).** Given  $(X, x_0)$  such that  $X = \bigcup_{\alpha} A_{\alpha}$  with

- $A_{\alpha}$  are open and **path**-connected and  $\forall \alpha \ x_0 \in A_{\alpha}$
- $A_{\alpha} \cap A_{\beta}$  is **path**-connected for all  $\alpha, \beta$ .

Then there exists a surjective group homomorphism

$$*_\alpha: \pi_1(A_{\alpha}, x_0) \rightarrow \pi_1(X, x_0).$$

If we additionally have  $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$  where they are all **path**-connected for every  $\alpha, \beta, \gamma$ , then

$$\pi_1(X, x_0) \cong *_\alpha \pi_1(A_{\alpha} \cap A_{\beta}, x_0) \pi_1(A_{\alpha}, x_0)$$

associated to all maps  $\pi_a(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha}), \pi_1(A_{\beta})$  induced by inclusions of spaces. i.e.,  $\pi_1(X, x_0)$  is a quotient of the **free product**  $*_{\alpha} \pi_1(A_{\alpha})$  where we have

$$(i_{\alpha\beta})_*: \pi_1(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha})$$

which is induced by the inclusion  $i_{\alpha\beta}: A_{\alpha} \cap A_{\beta} \rightarrow A_{\alpha}$ . We then take the quotient by the normal subgroup generated by

$$\{(i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_{\alpha} \cap A_{\beta})\}.$$

We'll defer the **proof** of **Theorem 2.7** until we get familiar with this theorem.

**Example.** We first see a great visualization of the **Theorem 2.7**.



Intuitively we see the **fundamental group** of  $X$ , which is built by gluing  $A$  and  $B$  along their intersection. As the **fundamental group** of  $A$  and  $B$  glued along the **fundamental group** of their intersection. In essence,  $\pi_1(X, x_0)$  is the quotient of  $\pi_1(A) * \pi_1(B)$  by relations to impose the condition that loops like  $\gamma$  lying in  $A \cap B$  can be viewed as elements of either  $\pi_1(A)$  or  $\pi_1(B)$ .

**Remark.** We can use a more abstract way to describe **Theorem 2.7**. Specifically, in the case that  $n = 2$ , i.e.,  $X = \bigcup_{i=1}^2 A_i$ , we let  $A_i =: X_i$ , then we have the

following. The functor  $\pi_1: \underline{\text{Top}}_* \rightarrow \underline{\text{Gp}}$  maps the [cocartesian](#) diagram in  $\underline{\text{Top}}_*$  to a [cocartesian](#) diagram in  $\underline{\text{Gp}}$  as follows.

$$\begin{array}{ccc} (X_0 \cap X_1, x_0) & \xrightarrow{j_0} & (X_0, x_0) \\ \downarrow j_1 & & \downarrow i_0 \\ (X_1, x_0) & \xrightarrow{i_1} & (X, x_0) \end{array} \quad \xrightarrow{\pi_1} \quad \begin{array}{ccc} \pi_1(X_0 \cap X_1, x_0) & \xrightarrow{(j_0)_*} & \pi_1(X_0, x_0) \\ \downarrow (j_1)_* & & \downarrow (i_0)_* \\ \pi_1(X_1, x_0) & \xrightarrow{(i_1)_*} & \pi_1(X, x_0) \end{array}$$

Then, simply from the property of [cocartesian](#) diagram, we see that

$$\pi_1(X, x_0) \cong \pi_1(X_0, x_0) *_{\pi_1(X_0 \cap X_1, x_0)} \pi_1(X_1, x_0).$$

Additionally, there is a more general version of [Theorem 2.7](#), which is defined on [groupoid](#). The theorem is stated in [Appendix A.1](#) with the proof.

With this more general version and the proof of which, we can apply it to [Theorem 2.7](#). But one question is that, the above proof works in  $\underline{\text{Gpd}}$  rather than in  $\underline{\text{Gp}}$ . We now see how to generalize a group to a [groupoid](#).

For any group  $G$ , we can define a [groupoid](#), denoted as  $G$  also, as follows.

- $\text{Ob}(G) = \{\text{pt}\}$ , a one point set.
- $\text{Hom}(G) = \{g \in G\}$ .
- Composition: We define

$$g \circ h := h \cdot g.$$

We see that the associativity of group elements implies the associativity of composition defined above, and since there is an identity element in  $G$ , hence we also have an identity [morphism](#), these two facts ensure that  $G$  is an [category](#).

Furthermore, since for every  $g \in G$ , there is a  $g^{-1} \in G$ , hence every [morphism](#) is an isomorphism, which implies  $G$  is a [groupoid](#).

With this, we see that we can view the following diagram in the [category](#) of [groupoid](#)  $\underline{\text{Gpd}}$ .

$$\begin{array}{ccc} \pi_1(X_0 \cap X_1, x_0) & \xrightarrow{(j_0)_*} & \pi_1(X_0, x_0) \\ \downarrow (j_1)_* & & \downarrow (i_0)_* \\ \pi_1(X_1, x_0) & \xrightarrow{(i_1)_*} & \pi_1(X, x_0) \end{array}$$

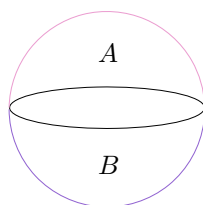
And to prove [Theorem 2.7](#), we only need to show this diagram is [cocartesian](#). This version of proof is given in [Appendix A.2](#).

## Lecture 11: Group Presentations

31 Jan. 10:00

**Example.** We now see some applications of [Theorem 2.7](#).

1. We can use [Seifert Van Kampen Theorem](#) to compute the [fundamental group](#) of  $S^2$ . We see that



We see that  $\pi_1(S^2)$  must be a quotient of  $\pi_1(A) * \pi_1(B)$ , but since  $A, B \simeq D^2$ , we know that  $\pi_1(A)$  and  $\pi_1(B)$  are both zero groups, thus  $\pi_1(A) * \pi_1(B)$  is the zero group, and  $\pi_1(S^2)$  is also the zero group.

**Remark.** Note that the inclusion of  $A \cap B \rightarrow A$  induces the zero map  $\pi_1(A \cap B) \rightarrow \pi_1(A)$ , which cannot be an injection. In fact, we know that  $\pi_1(A \cap B) \cong \mathbb{Z}$  since  $A \cap B \simeq S^1$ .

2. In the case of torus, consider the following.

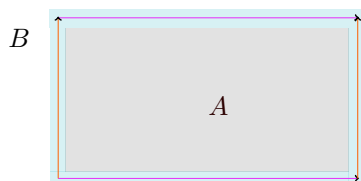


Figure 14:  $A$  is the interior, while  $B$  is the neighborhood of the boundary.

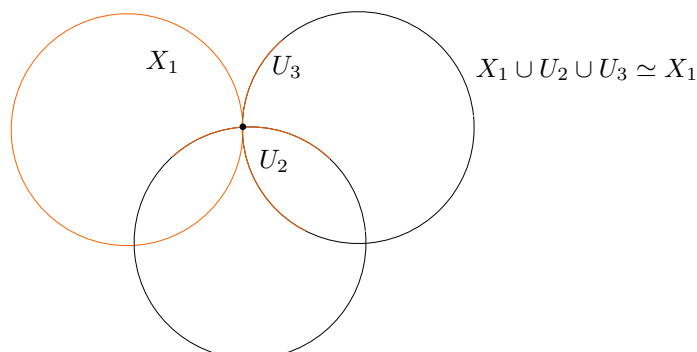
Now note that  $A \simeq D^2$  and  $B \simeq S^1 \vee S^1$ , and since it's a thickening of the two loops around the torus in both ways, this suggests the question of how do we find  $\pi_1(B)$ ? We grab a bit of knowledge from [Seifert Van Kampen Theorem](#) before we continue.

**Exercise.** Suppose we have [path](#)-connected spaces  $(X_\alpha, x_\alpha)$ , and we take their [wedge sum](#)  $\bigvee_\alpha X_\alpha$  by identifying the points  $x_\alpha$  to a single point  $x$ . We also suppose a mild condition for all  $\alpha$ , the point  $x_\alpha$  is a [deformation retract](#) of some neighborhood of  $x_\alpha$ .

For example, this doesn't work if we choose the *bad point* on the Hawaiian earring. Then we can use [Seifert Van Kampen Theorem](#) to show that

$$\pi_1 \left( \bigvee_\alpha X_\alpha, x \right) \cong *_{\alpha} \pi_1 (X_\alpha, x_\alpha).$$

*Proof.* If we denote



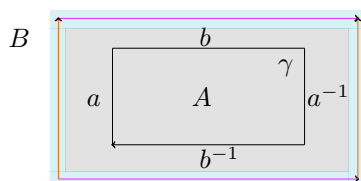
as  $C_n$ , then  $\pi_1(C_n) \cong F_n$ . Then we apply [Theorem 2.7](#) to  $A_\alpha = X_\alpha \cup_\beta U_\beta$ . Specifically, take  $A_\alpha = X_\alpha \cup_\beta U_\beta \simeq X_\alpha$ , where  $U_\beta$  is a neighborhood of  $x_\beta$  which [deformation retracts](#) to  $x_\beta$ . This makes  $A_\alpha$  open as desired. ■

**Corollary 2.2.** The [wedge sum](#) of circles  $\pi_1(\bigvee_{\alpha \in A} S^1) = *_\alpha \mathbb{Z}$  is a [free group](#) on  $A$ . In particular, when  $A$  is finite, the [fundamental group](#) of a bouquet of circles is the [free group](#) on  $|A|$ .

Returning to the [example of torus](#), we see that

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$

Further, we know that  $\pi_1(A \cap B) \rightarrow \pi_1(A)$  is the zero map. We need to understand  $\pi_1(A \cap B) \rightarrow \pi_1(B)$ . To do so we need to understand how we're able to identify  $\pi_1(S^1 \vee S^1)$  with  $F_2$  and how we identify  $\pi_1(S^1)$  with  $\mathbb{Z}$ . We update our [Figure 14](#) to talk about this.



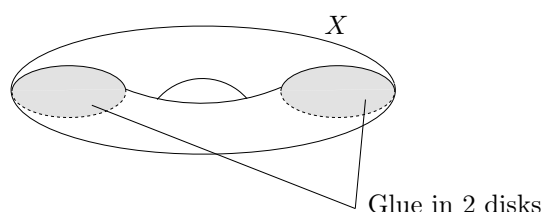
From this, we have

$$\pi_1(A \cap B) \rightarrow \pi_1(B) \cong F_{a,b}, \quad \gamma \mapsto aba^{-1}b^{-1}.$$

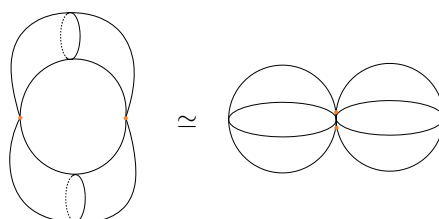
By [Seifert Van Kampen Theorem](#), we identify the image of  $\gamma$  in  $\pi_1(B)[aba^{-1}b^{-1}]$  with its image in  $\pi_1(A)$ , which is just trivial. Therefore, we have

$$\pi_1(T^2) = F_{a,b} / \langle aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2.$$

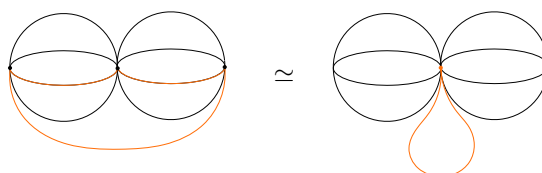
- Let's see the last example which illustrate the power of [Seifert Van Kampen Theorem](#). Start with a torus, and we glue in two disks into the hollow inside.



We'll call this space  $X$ , and our goal is to find  $\pi_1(X)$ . We can place a [CW complex](#) structure on this space so that each disk is a [subcomplex](#). Then, we take quotient of each disk to a point without changing the [homotopy type](#), hence  $X$  is [homotopy](#) to



By the same property, we can expand one of those points into an interval, and then contract the red [path](#) as follows.



This is exactly  $S^2 \vee S^2 \vee S^1$ . With [Seifert Van Kampen Theorem](#), we have

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}.$$

**Exercise.** Consider  $\mathbb{R}^2 \setminus \{x_1, \dots, x_n\}$ , that is the plane punctured at  $n$  points. Then  $X \simeq \bigvee_n S^1$ , so then

$$\pi_1(X) \simeq F_n.$$

One way to do this is to convince yourself that you can do a [deformation retract](#) the plane onto the following [wedge](#).



Figure 15: [Deformation retract](#)  $X$  onto [wedge](#).

## 2.7 Group Presentation

In order to go further, we introduce the concept of *group presentation*.

**Definition 2.12 (Group presentation).** A *presentation*  $\langle S \mid R \rangle$  of a group  $G$  is

- $S$ : set of *generators*
- $R$ : set of *relators* ([words](#) in a generator and inverses)

such that

$$G \cong F_S / \langle R \rangle,$$

where  $\langle R \rangle$  is a subgroup normally generated by the elements of  $R$ .

**Definition 2.13 (Finite presentation).** If  $S$  and  $R$  are both finite, then  $G = \langle S \mid R \rangle$  is a *finite presentation* if  $S, R$  are, and we say that  $G$  is *finitely presented*.

**Note.** One way to think about whether  $G$  is [finitely presented](#) is that if  $r$  is a [word](#) in  $R$  then  $r = 1$ , where 1 is the identity of  $G$ .

**Example.** We see that

1.  $F_2 = \langle a, b \mid \rangle$
2.  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid \overline{aba^{-1}b^{-1}} \rangle$
3.  $\mathbb{Z}/3\mathbb{Z} = \langle a \mid a^3 \rangle$
4.  $S_3 = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$

**Theorem 2.8.** Any group  $G$  has a [presentation](#).

*Proof.* We first choose a generating set  $S$  for  $G$ . Notice that we can even choose  $S = G$  directly. From the universal property of [free group](#), we see that there exists a surjective map  $\varphi: F_S \rightarrow G, s \mapsto s$ . Now, let  $R$  be the generating set for  $\ker(\varphi)$ , by the first isomorphism theorem<sup>7</sup>,  $G \cong F_S / \ker \varphi$ . In fact, we have  $G = \langle S \mid R \rangle$ .

Specifically,  $i: S \rightarrow G$  with  $\iota: S \rightarrow F_S$ , we have  $\varphi \circ \iota = i$ .

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F_S \\ & \searrow i & \downarrow \exists! \varphi \\ & & G \end{array}$$

■

**Remark.** The advantages of using [group presentation](#) are that given  $G = \langle S \mid R \rangle$ , it's now easy to define a homomorphism  $\psi: G \rightarrow H$  given a map  $\varphi: S \rightarrow H$ ,  $\psi$  extends to a group homomorphism  $G \rightarrow H$  if and only if  $\psi$  vanishes on  $R$ , i.e.,  $\psi(r) = 1$  for all  $r \in R$ . We see an example to illustrate this.

**Example.** If we have  $G = \langle a, b \mid aba \rangle$ , a map  $\varphi: \{a, b\} \rightarrow H$  gives a group homomorphism if and only if

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H.$$

This essentially uses the universal property of quotients.

---

**Remark.** It's sometimes easy to calculate  $G^{\text{Ab}}$

$$G^{\text{Ab}} = \langle S \mid R, \text{commutators in } S \rangle.$$

**Example.** Suppose all relations in  $R$  are commutators, so  $R \subseteq [G, G]$ . Then,

$$G^{\text{Ab}} = (F_S)^{\text{Ab}} = \bigoplus_S \mathbb{Z}.$$

**Remark.** The disadvantages are that this is computationally **very difficult**.

---

**Example.** Given  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ , let

$$\psi: \{a, b\} \rightarrow H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1_H \in H.$$

Namely, this is a [presentation](#) of the trivial group, but this is entirely unclear.

## Lecture 12: Presentations for $\pi_1$ of CW Complexes

2 Feb. 10:00

Let's first see an exercise.

**Exercise.** Consider  $G_1 = \langle S_1 \mid R_1 \rangle$  and  $G_2 = \langle S_2 \mid R_2 \rangle$ . Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\} \rangle$
- $G_1 *_H G_2$  where  $f_1: H \rightarrow G_1$  and  $f_2: H \rightarrow G_2$ . Then we have  

$$G_1 *_H G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{f_1(h)f_2(h)^{-1} \mid h \in H\} \rangle.$$

### 2.7.1 Presentations for $\pi_1$ of CW Complexes

For  $X$  a **CW complex**, we have

1. A 1-dimensional **CW complex** has free  $\pi_1$  (call its generators as  $a_1, \dots, a_n$ ).
2. Gluing a 2-disk by its boundary along a word  $w$  in the generators *kills*  $w$  in  $\pi_1$ . We then get a **presentation** for  $\pi_1(X^2)$  given by

$$\langle a_1, \dots, a_n \mid w \text{ for each 2-cell in } X_2 \rangle.$$

3. Gluing in any higher dimensional cells along their boundary will not change  $\pi_1$ . That is, in a **CW complex**, we have  $\pi_1(X) = \pi_1(X^2)$ .

**Remark.** We can write the above more precise.

1. Find free generators  $\{a_i\}_{i \in I}$  for  $\pi_1(X^1)$ .
2. For each 2-disk  $D_\alpha^2$ , write attaching map as word  $w_\alpha$  in  $a_i$ . i.e.,

$$\pi_1(X^2) = \langle a_i \mid w_\alpha \rangle.$$

3.  $\pi_1(X) = \pi_1(X^2)$ .

**Example.** Given  $G = \mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$ , then we take a loop and then wind a 2-disk around the loop  $a$  for  $n$  times.



Figure 16: For  $G = \mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$ , we wind the boundary around  $a$  for  $n$  times.

<sup>7</sup>[https://en.wikipedia.org/wiki/Isomorphism\\_theorems](https://en.wikipedia.org/wiki/Isomorphism_theorems)



We then see that given a group  $G$  with [presentation](#)  $\langle S \mid R \rangle$ , one can construct a 2-dimensional [CW complex](#) with  $\pi_1 = G$  by

- Set  $X^1 = \bigvee_{s \in S} S^1$
- For each relation  $r \in R$ , glue in a 2-disk along loops specified by the [word](#)  $r$ .

Every group is then  $\pi_1$  of some space.

**Theorem 2.9.** If  $X$  is a [CW complex](#) and  $\iota_1: X^1 \hookrightarrow X$  and  $\iota_2: X^2 \hookrightarrow X$ , then  $(\iota_1)_*$  surjects onto  $\pi_1$  and  $(\iota_2)_*$  is an isomorphism on  $\pi_1$ .

*Proof.*

HW

**Definition 2.14 (Graph, subgraph, tree, maximal tree).** We import some topological definitions of graph theoretic concepts.

- A *graph* is a 1-dimensional [CW complex](#).
- A *subgraph* is a [subcomplex](#).
- A *tree* is a contractible [graph](#).
- A [tree](#) in [graph](#)  $X$  (necessarily a [subgraph](#)) is *maximal* or *spanning* if it contains all the vertices.

**Theorem 2.10.** Every connected [graph](#) has a [maximal tree](#). Every [tree](#) is contained in a [maximal tree](#).

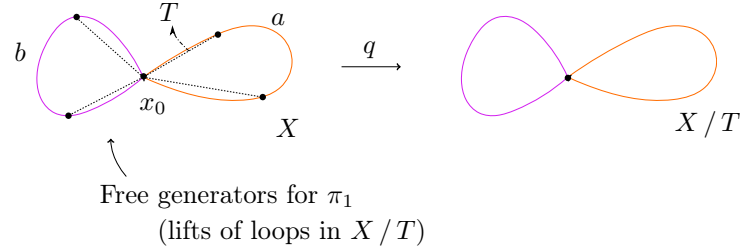
**Corollary 2.3.** Suppose  $X$  is a connected [graph](#) with basepoint  $x_0$ . Then  $\pi_1(X, x_0)$  is a [free group](#).

Furthermore, we can give a [presentation](#) for  $\pi_1(X, x_0)$  by finding a [spanning tree](#)  $T$  in  $X$ . The generators of  $\pi_1$  will be indexed by cells  $e_\alpha \in X - T$ , and  $e_\alpha$  will correspond to a loop that passes through  $T$ , traverses  $e_\alpha$  once, then returns to the basepoint  $x_0$  through  $T$ .

*Proof.* The idea is simple.  $X$  is [homotopy equivalent](#) to  $X/T$  via previous work on the homework,  $T$  contains all the vertices, so the quotient has a single vertex. Thus, it is a [wedge](#) of circles, and each  $e_\alpha$  projects to a loop in  $X/T$ .



The current plan is to calculate the **fundamental group** of **CW complexes**. For now, we need to see that the **fundamental group** of a 1-skeleton (a graph) can be found by taking a **maximal tree**, and then quotienting out the space by the **tree** to get a **wedge** of circles.



We now prove that the **maximal trees** exist. Recall that  $X$  is a quotient of

$$X^0 \coprod_{\alpha} I_{\alpha}.$$

Each subset  $U$  is open if and only if it intersects each edge  $\bar{e}_{\alpha}$  in an open subset. A map  $X \rightarrow Y$  if and only if its restriction to each edge  $\bar{e}_{\alpha}$  is continuous. Now, take  $X_0$  to be a **subgraph**. Our goal is to construct a **subgraph**  $Y$  with

- $X_0 \subset Y \subset X$
- $Y$  **deformation retracts** to  $X_0$
- $Y$  contains all vertices of  $X$ .

So if we take  $X_0$  to be a vertex, then  $Y$  is our **tree** and we're done!

Our strategy now is to build a sequence  $X_0 \subset X_1 \subset \dots$  and correspondingly,  $Y_0 \subset Y_1 \subset \dots$ . We start with  $X_0$  and inductively define

$$X_i := X_{i-1} \cup \text{all edges } \bar{e}_{\alpha} \text{ with one or both vertices in } X_{i-1}.$$

We then see that  $X = \bigcup_i X_i$ .<sup>8</sup> Now, let  $Y_0 = X_0$ . By induction, we'll assume that  $Y_i$  is a **subgraph** of  $X_i$  such that

Check.

- $Y_i$  contains all vertices of  $X_i$ .
- $Y_i$  **deformation retracts** to  $Y_{i-1}$ .

We can then construct  $Y_{i+1}$  by taking  $Y_i$  and adding to it one edge to adjoin every vertex of  $X_{i+1}$ , namely

$$Y_{i+1} := Y_i \cup \text{one edge to adjoin every vertex of } X_i^9$$

We then see that  $Y_{i+1}$  **deformation retracts** to  $Y_i$  by just smashing down each edge. Now, we can show that  $Y$  **deformation retracts** to  $Y_0 = X_0$  by performing the **deformation retraction** from  $Y_i$  to  $Y_{i-1}$  during the time interval  $[1/2^i, 1/2^{i-1}]$ . ■

<sup>8</sup>[HPM02] do this by arguing the union on the right is both open and closed.

<sup>9</sup>This is possible if we assume Axiom of Choice.

**Example.** Let

- $S^n$ : decompose into 2 open disks
- $A_1$ : neighborhood of top hemisphere
- $A_2$ : neighborhood of lower hemisphere

We see that  $A_1 \cap A_2 \simeq S^{n-1}$ , where we need  $n \geq 2$  to let  $S^{n-1}$  be connected. We then have

$$\pi_1(S^n) \cong 0 \underset{\pi_1(A_1 \cap A_2)}{*} 0 = 0.$$

On the other hand, if  $n \geq 3$ , then we see that

$$S^n = D^n \cup * / \sim.$$

Since 2-skeleton is a point, thus  $\pi_1(S^n) = 0$ .

## Lecture 13: Proof of Seifert-Van-Kampen Theorem

4 Feb. 10:00

### 2.8 Proof of Seifert-Van-Kampen Theorem

Let's start to prove Theorem 2.7.

*Proof.* The outline of the proof is the following. Let  $X = \bigcup_{\alpha} A_{\alpha}$  where  $A_{\alpha}$  are open, path-connected and contain the bluepoint  $x_0$ . We also must guarantee that  $A_{\alpha} \cap A_{\beta}$  is path-connected.

1. Since we have a map induced by the inclusions:

$$\Phi: \underset{\alpha}{*} \pi_1(A_{\alpha}, x_0) \rightarrow \pi_1(X, x_0).$$

We want to show that  $\phi$  is surjective. Take some  $\gamma: I \rightarrow X$ , then by the compactness of the interval  $I$ , we can show that there is a partition  $I$  with  $s_1 < \dots < s_n$  so that

$$\alpha|_{s_i, s_{i+1}} =: \alpha_i$$

has image in  $A_{\alpha_i}$  for some  $\alpha_i$ .<sup>10</sup> Specifically, since

- $A_{\alpha}$  is open for all  $\alpha$
- $I$  is compact,

then for all  $i$ , we choose a path  $h_i$  from  $x_0$  to  $\gamma(s_i)$  in  $A_{\sigma_{i-1}} \cap A_{\alpha_i}$ , using path-connectedness of the pairwise intersections. Now, take  $\gamma$  and write it as

$$\gamma = (\gamma_1 \cdot \bar{h}_1) \cdot (\bar{h}_1 \cdot \gamma_2) \cdot \dots \cdot (\gamma_{n-1} \cdot \bar{h}_{n-1}) \cdot (h_{n-1} \cdot \gamma_n).$$

Observe that each of these paths is fully contained in  $A_{\alpha_i}$ , so this implies that  $\gamma \in \text{Im}(\Phi)$ , therefore  $\Phi$  is surjective.

<sup>10</sup>This is a good exercise for point-set topology.

2. For the next step, we'll show that the second part of [Theorem 2.7](#). Assume that our triple intersections are [path-connected](#). We want to show that  $\ker(\Phi)$  is generated by

$$(i_{\alpha\beta})_*(\omega)(i_{\beta\alpha})_*(\omega)^{-1},$$

where

$$i_{\alpha\beta}: A_\alpha \cap A_\beta \hookrightarrow A_\alpha$$

for all loops  $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$ .

Before we go further, we'll need some definition.

**Definition 2.15 (Factorization).** A *factorization* of a [homotopy](#) class  $[f] \in \pi_1(X, x_0)$  is a formal product

$$[f_1][f_2] \dots [f_\ell]$$

with  $[f_i] \in \pi_1(A_\alpha, x_0)$  such that

$$f \simeq f_1 \cdot f_2 \cdot \dots \cdot f_\ell.$$

We showed that every  $[f]$  has a [factorization](#) in step 1 already. Now we want to show that two [factorizations](#)

$$[f_1] \cdot \dots \cdot [f_\ell] \text{ and } [f'_1] \cdot \dots \cdot [f'_{\ell'}]$$

of  $[f]$  must be related by two moves:

- (a)  $[f_i] \cdot [f_{i+1}] = [f_i \cdot f_{i+1}]$  if  $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$ . Namely, the relation defining the [free product](#) of groups.
- (b)  $[f_i]$  can be viewed as an element of  $\pi_1(A_\alpha, x_0)$  or  $\pi_1(A_\beta, x_0)$  whenever

$$[f_i] \in \pi_1(A_\alpha \cap A_\beta, x_0).$$

This is the relation defining the [amalgamated free product](#).

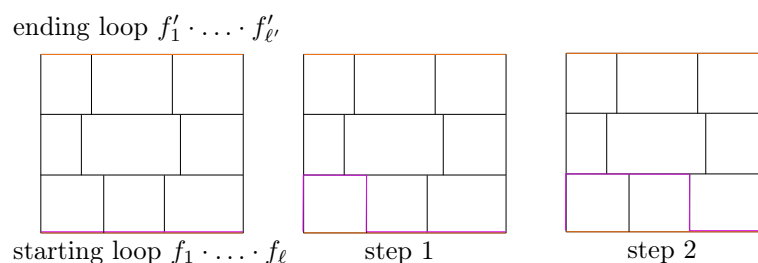
Now, let  $F_t: I \times I \rightarrow X$  be a [homotopy](#) from  $f_1 \dots f_\ell$  to  $f'_1 \dots f'_{\ell'}$ , since they both represent  $[f]$ . We subdivide  $I \times I$  into rectangles  $R_{ij}$  so that

$$F(R_{ij}) \subseteq A_{\alpha_{ij}} =: A_{ij}$$

for some  $\alpha_{ij}$  using compactness. We also argue that we can perturb the corners of the squares so that a corner lies only in three of the  $A_\alpha$ 's indexed by adjacent rectangles.

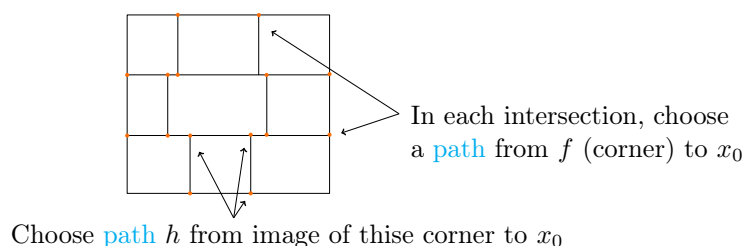
$A_{31}$	$A_{32}$	$A_{33}$
$A_{21}$	$A_{22}$	$A_{23}$
$A_{11}$	$A_{12}$	$A_{13}$

We also argue that we can set up our subdivision so that the partition of the top and bottom intervals must correspond with the two **factorizations** of  $[f]$ . We then perform our **homotopy** one rectangle at a time.



**Idea:** Argue that **homotoping** over a single rectangle has the effect of using allowable moves to modify the **factorization**.

At each triple intersection, choose a **path** from  $f$  (corner) to  $x_0$  which lies in the triple intersection, so we use the assumption that the triple intersections are **path-connected**.



Along the top and bottom, we make choices compatible with the two **factorizations**. It's now an exercise to check that these choices result in **homotoping** across a rectangle gives a new **factorization** related by an allowable move.

■

## Lecture 14: Covering Spaces

7 Feb. 10:00

### 3 Covering Spaces

#### 3.1 Lifting Properties

As always, we start with a definition.

**Definition 3.1 (Covering space).** A *covering space*  $\tilde{X}$  of  $X$  is a space  $\tilde{X}$  and a map  $p: \tilde{X} \rightarrow X$  such that  $\forall x \in X \exists$  neighborhood  $u_x$  with  $p^{-1}(u_x)$  the disjoint union of open sets

$$\coprod_{\alpha} u_{\alpha}$$

such that

$$p|_{u_{\alpha}} : u_{\alpha} \rightarrow u_x$$

is a homeomorphism for every  $\alpha$ .



We sometimes call  $p$  as *covering map*.

Although we already investigate into [covering spaces](#) quite a lot in homework, but a terminology is still worth mentioning.

**Definition 3.2 (Evenly covered).** Let  $p: \tilde{X} \rightarrow X$  be a continuous map of spaces. Then an open subset  $U \subseteq X$  is called *evenly covered by  $p$*  if

$$p|_{V_i} : V_i \rightarrow U$$

is a homeomorphism.

We call the parts  $V_i$  of the partition  $\coprod_i V_i$  of  $p^{-1}(U)$  *slices*.

**Remark.** We see that  $p$  is a [covering map](#) if and only if every point  $x \in X$  has a neighborhood which is [evenly covered](#).

We immediately have the following proposition.

**Proposition 3.1 (Homotopy lifting property).** The covering spaces satisfy the *homotopy lifting property* such that the following diagram commutes.

$$\begin{array}{ccc}
 X \times \{0\} & \xrightarrow{\tilde{F}_0} & \tilde{Y} \\
 \downarrow & \nearrow \exists! \tilde{F}_t & \downarrow p \\
 X \times I & \xrightarrow{F_t} & Y
 \end{array}$$

*Proof.* We already proved this in homework! ■

**Corollary 3.1 (Path lifting property).** For each path  $\gamma: I \rightarrow X$  in  $X$ ,  $\tilde{x}_0 \in p^{-1}(\gamma(0))$  such that there exists a unique lift  $\tilde{\gamma}$  starting at  $\tilde{x}_0$ .

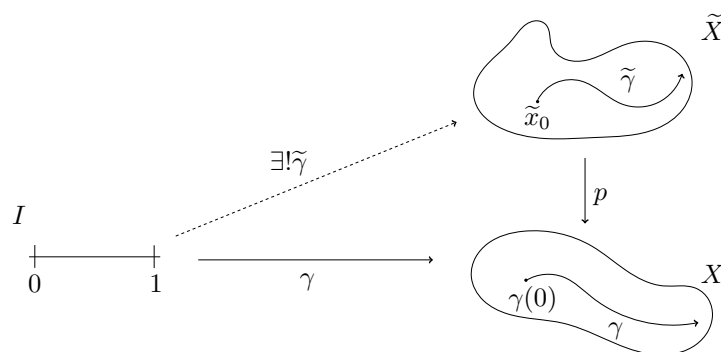
And for each path homotopy  $I \times I \rightarrow X$ , there exists a unique path homotopy  $\tilde{\gamma}: I \times I \rightarrow \tilde{X}$  starting at  $\tilde{x}_0$ .



Though we can directly use Proposition 3.1 to prove this, but we can see some insight by directly proving this.

*Proof.* We prove them separately.

**Lifting a path.** Assume that we have the following lift.



We first prove that a path will be lifted uniquely to a path  $\tilde{\gamma}$  from  $\tilde{x}_0$ . For every

$x \in X$ , there exists an open neighborhood  $U_x$  such that

$$p^{-1}(U_x) = \coprod_{\alpha} U_{x_\alpha},$$

where for every  $\alpha$ ,

$$p|_{U_{x_\alpha}} : U_{x_\alpha} \rightarrow U_x$$

is a homeomorphism. We see that  $\{U_x \mid x \in X\}$  is an open cover of  $X$ , hence

$$\{\gamma^{-1}(U_x) \mid x \in X\}$$

is an open cover of  $[0, 1]$ . Note that since  $[0, 1]$  is a compact metric space, from Lebesgue Lemma<sup>11</sup>, there exists a partition of  $[0, 1]$  such that

$$0 = t_0 < t_1 < \dots < t_k = 1$$

such that for every  $i$ ,  $[t_i, t_{i+1}] \subset \gamma^{-1}(U_x)$  for some  $x$ . Without loss of generality, we assume that  $[t_i, t_{i+1}] \subset \gamma^{-1}(U_{x_i})$ , i.e.,

$$\gamma([t_i, t_{i+1}]) \subset U_{x_i}.$$



Now, since  $p(\tilde{x}_0) = \gamma(0)$  for  $\gamma_0 \in U_{x_1}$  and  $\tilde{x}_0 \in p^{-1}(U_{x_1})$ , we may assume  $\tilde{x}_0 \in U_{x_1 \alpha_1}$ . Consider **lifting** the first segment, namely  $\gamma([0, t_1])$ .



<sup>11</sup>[https://en.wikipedia.org/wiki/Lebesgue%27s\\_number\\_lemma](https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma)



Specifically, let  $\tilde{\gamma}_1(t) = \left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ \gamma(t)$  for  $0 \leq t \leq t_1$ , we see that

$$\tilde{\gamma}_1: [0, t_1] \rightarrow \tilde{X}$$

is a **lift** of  $\gamma|_{[0, t_1]}$  from  $\tilde{x}_0$ . We claim that this **lift** is unique. Consider there exists another **lift** from  $\tilde{x}_0$   $\tilde{\tilde{\gamma}}_1: [0, t_1] \rightarrow \tilde{X}$ , then since

- $\tilde{\tilde{\gamma}}_1(0) = \tilde{x}_0$
- $\tilde{\tilde{\gamma}}_1$  is continuous
- $\tilde{x}_0 \in U_{x_1\alpha_1}$ ,

we see that  $\tilde{\tilde{\gamma}}_1([0, t_1]) \subset U_{x_1\alpha_1}$ , which implies

$$\begin{array}{ccc} [0, t_1] & \xrightarrow{\tilde{\tilde{\gamma}}_1} & U_{x_1\alpha_1} \\ & \searrow \gamma|_{[0, t_1]} & \downarrow p|_{U_{x_1\alpha_1}} \\ & & U_{x_1} \end{array} \implies \tilde{\tilde{\gamma}}_1 = \left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ \gamma|_{[0, t_1]} = \tilde{\gamma}_1,$$

hence this **lift** is unique. Now, we see that we can simply repeat this argument, namely replacing  $t_i$  by  $t_{i+1}$ ,  $\tilde{\gamma}_i(t_i)$  by  $\tilde{\gamma}_{i+1}(t_{i+1})$  and so on. Since this partition is finite, hence in finitely many steps, we obtain a unique **path homotopy**  $\tilde{\gamma}$  by concatenating all  $\tilde{\gamma}_i$  starting at  $\tilde{x}_0$ .

**Lifting a path homotopy.** We now consider lifting a **path homotopy**. Consider

$$\gamma_1 \underset{\tilde{F}}{\simeq} \gamma_2 \text{ rel } \{0, 1\}$$

we'll show that  $\tilde{\gamma}_1 \underset{\tilde{F}}{\simeq} \tilde{\gamma}_2 \text{ rel } \{0, 1\}$  where  $p \circ \tilde{F} = F$ . Firstly, we denote  $x_0 := \gamma_1(0) = \gamma_2(0)$ , such that



We claim that it's sufficient to show that there exists a continuous  $\tilde{F}: I \times I \rightarrow X$  such that  $p \circ \tilde{F} = F$ , and  $\tilde{F}(\{0\} \times I) = x_0$ . It's because

$$p \circ \tilde{F}_0 = F_0 = \gamma_1, \quad p \circ \tilde{F}_1 = F_1 = \gamma_2$$

where  $\tilde{F}_0, \tilde{F}_1$  is  $\gamma_1, \gamma_2$ 's **lifting** starting at  $\tilde{x}_0$ , respectively. And since  $p \circ \tilde{F} = F$ , we have

$$p(\tilde{F}(\{1\} \times I)) = x_1 \implies \tilde{F}(\{1\} \times I) \subset p^{-1}(\{x_1\}),$$

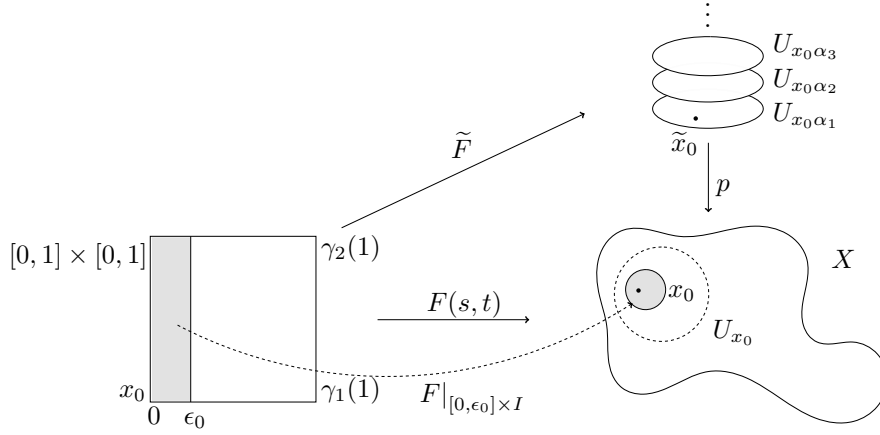
which implies  $\exists \tilde{x}_1 \in p^{-1}(\{x_1\})$  such that  $\tilde{F}(\{1\} \times I) = \tilde{x}_1$  since we know that  $p^{-1}(\{x_1\})$  is a discrete points set and  $\tilde{F}$  is assumed to be continuous, and  $\{1\} \times I$  is connected. We now show  $\tilde{F}$  exists.

We define

$$\begin{aligned} \tilde{F}: I \times I &\rightarrow X \\ (s, t) &\mapsto \tilde{F}_t(s), \end{aligned}$$

where  $\tilde{F}_t: [0, 1] \rightarrow \tilde{X}$  is a **lift** starting at  $\tilde{x}_0$  of  $F_t: [0, 1] \rightarrow X, s \mapsto F(s, t)$ . Obviously,  $p \circ \tilde{F} = F$  from the uniqueness of the **lift** of a path, and also,  $\tilde{F}(\{0\} \times I) = \tilde{x}_0$  holds trivially, hence we only need to show  $\tilde{F}$  is continuous.

1. We show that  $\exists \epsilon_0 > 0$  such that  $\tilde{F}|_{[0, \epsilon_0] \times I}$  is continuous.



Since  $F$  is continuous, we see that there exists an open neighborhood  $U_{x_0}$  of  $x_0$  such that  $p^{-1}(U_{x_0}) = \coprod_{\alpha} U_{x_0 \alpha}$ , where

$$p|_{U_{x_0 \alpha}}: U_{x_0 \alpha} \xrightarrow{\cong} U_{x_0}.$$

Since  $F^{-1}(U_{x_0})$  is an open set contain  $\{0\} \times I$ , there exists a  $\epsilon_0 > 0$  such that  $[0, \epsilon_0] \times I \subset F^{-1}(U_{x_0})$ ,<sup>12</sup> which implies

$$F([0, \epsilon_0] \times I) \subset U_{x_0}.$$

<sup>12</sup>Notice that we're working on product topology here.

Note that  $x_0 \in U_{x_0}$  and  $p(\tilde{x}_0) = x_0$ , we may assume  $\tilde{x}_0 \in U_{x_0\alpha_1}$ . Consider  $\left(p|_{U_{x_0\alpha_1}}\right)^{-1} \circ F|_{[0,\epsilon_0] \times I}$ , which is a **lift** of  $F|_{[0,\epsilon_0] \times I}$ . We claim that

$$\left(p|_{U_{x_0\alpha_1}}\right)^{-1} \circ F|_{[0,\epsilon_0] \times I} = \tilde{F}|_{[0,\epsilon_0] \times I}.$$

This is because for every  $t \in I$ ,

$$s \mapsto \left(p|_{U_{x_0\alpha_1}}\right)^{-1} \circ F|_{[0,\epsilon_0] \times I}(s, t)$$

is a **lift** starting at  $\tilde{x}_0$ ; also, for every  $t \in I$ ,

$$s \mapsto \tilde{F}|_{[0,\epsilon_0] \times I}(s, t)$$

is a **lift** of  $F_t$  starting at  $\tilde{x}_0$ . From the uniqueness of the **lift** of **paths**, we see that they're equal. Note that this implies  $\tilde{F}$  is now continuous at  $[0, \epsilon_0] \times I$ , since  $F$  is continuous and  $p|_{U_{x_0\alpha_1}}$  is a homeomorphism, hence continuous, then from

$$\tilde{F}|_{[0,\epsilon_0] \times I} = \underbrace{\left(p|_{U_{x_0\alpha_1}}\right)^{-1}}_{\text{continuous}} \circ \underbrace{F|_{[0,\epsilon_0] \times I}}_{\text{continuous}},$$

we see that  $\tilde{F}$  is indeed continuous at  $[0, \epsilon_0] \times I$ .

2. We now prove that  $\tilde{F}: I \times I \rightarrow \tilde{X}$  is continuous. Assume there exists  $(s_0, t_0) \in I \times I$  such that  $\tilde{F}$  is discontinuous at  $(s_0, t_0)$ . Then consider

$$0 < \epsilon_0 \leq \underbrace{\inf \left\{ s \mid \tilde{F} \text{ is discontinuous at } s, t_0 \right\}}_{\exists s_0 \Rightarrow \neq \emptyset} =: s_1,$$

where the first inequality is from the first step.

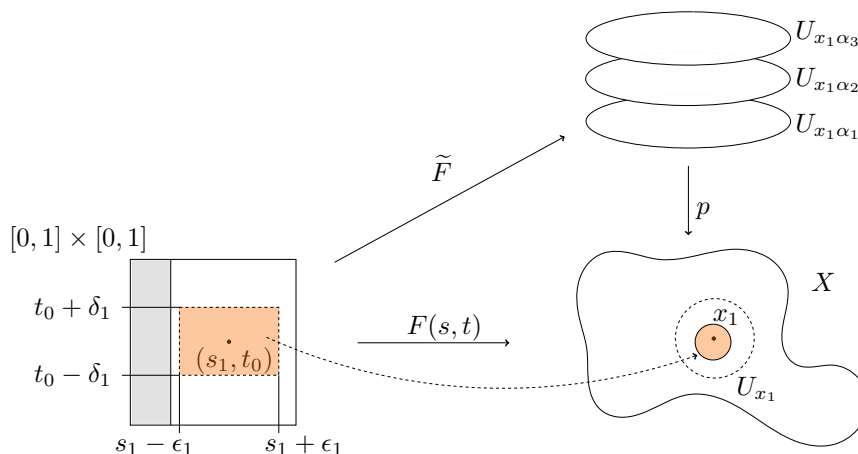


Let  $x_1 := F(s_1, t_0)$ ,  $\tilde{x}_1 := \tilde{F}(s_1, t_0)$ , then there exists an open neighborhood  $U_{x_1}$  in  $X$  such that  $x_1 \in U_{x_1} = \coprod_{\alpha} U_{x_1\alpha}$ , where

$$p|_{U_{x_1\alpha}} : U_{x_1\alpha} \xrightarrow{\cong} U_{x_1}.$$

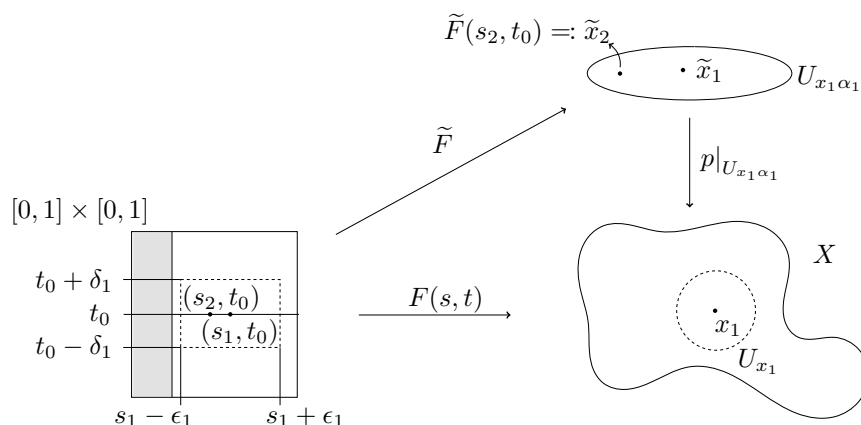
Since  $F$  is continuous, there exists an  $\epsilon_1 > 0$ ,  $\delta_1 > 0$  such that

$$F((s_1 - \epsilon_1, s_1 + \epsilon_1) \times (t_0 - \delta_1, t_0 + \delta_1))^{13} \subset U_{x_1}.$$



We may assume  $\tilde{x}_1 \in U_{x_1\alpha_1}$ . Then, we see that  $\tilde{F}_{t_0}$  is a **lift** of  $F_{t_0}$ , which means  $\tilde{F}_{t_0}$  is continuous, hence there exists an  $s_2$  such that  $s_1 - \epsilon_1 < s_2 < s_1$  such that

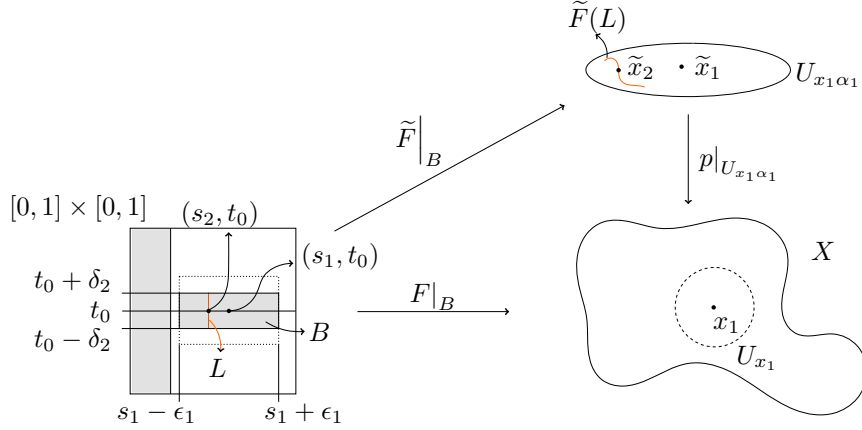
$$\tilde{F}(s_2, t_0) \in U_{x_1\alpha_1}.$$



<sup>13</sup>Notice that here we're considering **open** box.

We see that  $\tilde{F}$  is continuous at  $(s_2, t_0)$ , hence there exists a  $\delta_2 > 0$  such that

$$\tilde{F}(\{s_2\} \times (t_0 - \delta_2, t_0 + \delta_2)) \subset U_{x_1\alpha_1}.^{14}$$



Now, observe that  $\tilde{F}(B) \subset U_{x_1\alpha_1}$ . To see this, consider a fixed  $t \in (t_0 - \delta_2, t_0 + \delta_2)$ , then the map  $\tilde{F}$  is

$$[s_1 - \epsilon_1, s_1 + \epsilon_1] \rightarrow \tilde{X}, \quad s \mapsto \tilde{F}(s, t) = \tilde{F}_t(s).$$

Specifically,

$$\tilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1]) \subset p^{-1}(U_{x_1}) = \coprod_{\alpha} U_{x_1\alpha},$$

with the fact that  $\tilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1])$  is connected, and  $\tilde{F}_t(s_2) \in U_{x_1\alpha_1}$  with  $\tilde{F}_t$  is a [lift](#) of  $F_t$ , hence continuous, so

$$\tilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1]) \subset U_{x_1\alpha_1}.$$

This is true for every  $t \in [t_0 - \delta_2, t_0 + \delta_2]$ , hence  $\tilde{F}|_B \subset U_{x_1\alpha_1}$ . Now, since

$$p|_{U_{x_1\alpha_1}} \circ \tilde{F}|_B = F|_B,$$

and

$$\left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ F|_B : B \rightarrow U_{x_1\alpha_1},$$

so

$$p|_{U_{x_1\alpha_1}} \circ \left(\left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ F|_B\right) = F|_B$$

<sup>14</sup>Note that here we can also consider a closed interval, which matches what we're going to do. Namely, we're going to construct a **closed** box  $B$ . But this is just a technical detail.

obviously. Since  $p|_{U_{x_1\alpha_1}}$  is a homeomorphism, we have

$$\tilde{F}|_B = \underbrace{\left(p|_{U_{x_1\alpha_1}}\right)^{-1}}_{\text{continuous}} \circ \underbrace{F|_B}_{\text{continuous}},$$

hence we have  $\tilde{F}|_B$  is continuous, which leads to a contradiction since

$$s_1 = \inf \left\{ s \mid \tilde{F} \text{ is discontinuous at } s, t_0 \right\},$$

while  $\tilde{F}$  is continuous for all  $B$ , hence we see that  $\tilde{F} : I \times I \rightarrow \tilde{X}$  is continuous.<sup>15</sup>

■

**Example.** Let see some examples.

1. Covers of  $S^1 \vee S^1$ .



Note that in each cover (those three on the top), the black dot is the preimage of  $\{x_0\}$ , namely  $p_i^{-1}(\{x_0\})$ .

**Remark.** We see that for each  $p_i^{-1}(\{x_0\})$ , there are exactly

- one  $a$  edge goes out
- one  $b$  edge goes out
- one  $a$  edge goes in
- one  $b$  edge goes in

It turns out that there are much more covers of  $S^1 \vee S^1$ , as long as this main property is satisfied.

<sup>15</sup>There is a tricky situation, namely while  $s_1 = 1$ . But this can be considered also.

**Proposition 3.2.** Let

$$p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a **covering map**. Then

1.  $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$  is injective.
2.  $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0) = \{[\gamma] \mid \text{Lift } \tilde{\gamma} \text{ starting at } \tilde{x}_0 \text{ is a loop.}\}$ .

*Proof.* We prove this one by one.

1. Suppose  $\tilde{\gamma} \in \pi_1(\tilde{X}, \tilde{x}_0)$  is in  $\ker(p_*)$ . Then

$$[\gamma] = p_*([\tilde{\gamma}]) = [p \circ \tilde{\gamma}].$$

Let  $\gamma_t$  be a **nullhomotopy** from  $\gamma$  to the constant loop  $c_{x_0} \text{ rel}\{0, 1\}$ . We can then **lift**  $\gamma_t$  to  $\tilde{\gamma}_t$  where  $\tilde{\gamma}_0 = \tilde{\gamma}$ . Now, we claim that

- $\tilde{\gamma}$  is a **homotopy rel** $\{0, 1\}$ .
- $\tilde{\gamma}_1$  is the constant loop  $c_{\tilde{x}_0}$ .

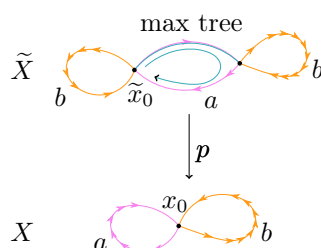
$$\begin{array}{ccc} & \tilde{X} & \\ \tilde{\gamma} \nearrow & \downarrow p & \\ I & \xrightarrow{\gamma} & X \end{array} \quad \begin{array}{ccc} & \tilde{X} & \\ \tilde{\gamma}_t \nearrow & \downarrow p & \\ I \times I & \xrightarrow{\gamma_t} & X \end{array}$$

We see that the above diagrams prove the first claim, since we know that the left and right edge of  $I \times I$  maps to  $x_0$  under  $\gamma_t$ , and  $c_{\tilde{x}_0}$  **lifts** this, so by uniqueness  $t \mapsto \tilde{\gamma}_t(0)$  and  $t \mapsto \tilde{\gamma}_t(1)$  must be constant **paths** at  $\tilde{x}_0$  as desired.

Then the **lift**  $\tilde{\gamma}_t$  is a **homotopy of paths** to the constant loop, so  $[\tilde{\gamma}] = 1$ .

2. Let see an example to show the idea of the proof.

**Example.** Given



Then

$$p_*\pi_1 = \langle b, a^2, ab\bar{a} \rangle \subseteq \pi_1(X) = \langle a, b \mid \rangle.$$

■

**Proposition 3.3 (Lifting criterion).** Let  $p: (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$  be [covering map](#). Given

- $f: (X, x_0) \rightarrow (Y, y_0)$ ;
- $X$  is [path-connected](#), locally [path-connected](#),

then a [lift](#)

$$\tilde{f}: (X, x_0) \rightarrow (\tilde{Y}, \tilde{y}_0)$$

exists if and only if

$$f_* (\pi_1(X, x_0)) \subseteq p_* (\pi_1(\tilde{Y}, \tilde{y}_0)).$$

$$\begin{array}{ccc} & (\tilde{Y}, \tilde{y}_0) & \\ \exists \tilde{f} \nearrow & \downarrow p & \\ (X, x_0) & \xrightarrow{f} & (Y, y_0) \end{array} \quad \begin{array}{ccc} & \pi_1(\tilde{Y}, \tilde{y}_0) & \\ \tilde{f}_* \nearrow & \downarrow p_* & \\ \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \end{array}$$

## Lecture 15: Lifting

9 Feb. 10:00

Before proving [Proposition 3.3](#), we first see an application.

**Example.** Prove that every continuous map  $f: \mathbb{R}P^2 \rightarrow S^1$  is [nullhomotopic](#).

*Proof.* If we can show that there is a [lift](#)  $\tilde{f}: \mathbb{R}P^2 \rightarrow \mathbb{R}$  of  $f$ , then we're done since we can apply the [straight line nullhomotopy](#) on  $\mathbb{R}$ , i.e.,

$$\begin{array}{ccc} & \mathbb{R} & \\ \tilde{f} \nearrow & \downarrow p & \\ \mathbb{R}P^2 & \xrightarrow{f} & S^1 \end{array}$$

and consider  $f = p \circ \tilde{f}$  compose [nullhomotopy](#) with  $p$ , so  $f \simeq$  constant map. Specifically, since  $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$  and  $\pi_1(S^1) = \mathbb{Z}$ , hence

$$f_* (\pi_1(\mathbb{R}P^2)) = 0$$

since  $\mathbb{Z}$  has no (nonzero) torsion. So it [lifts](#) by [Proposition 3.3](#). ■

Now we can proof [Proposition 3.3](#).

*Proof.* We prove two directions as follows.

**Necessary.** We see that we can [factorize](#)  $f_*$  as

$$f_* = p_* \circ \tilde{f}_*$$

follows from the [functoriality](#) of  $\pi_1$ .



**Sufficient.** Let  $x \in X$ . Choose a path  $\gamma$  from  $x_0$  to  $x$  by the assumption that  $X$  is path-connected. Then,  $f\gamma$  has a unique lift starting at  $\tilde{y}_0$ , denote by  $\tilde{f}\gamma$ . Now, define

$$\tilde{f}(x) = \tilde{f}\gamma(1).$$

Then, we need to check

1.  $\tilde{f}$  is well-defined. Suppose  $\gamma, \gamma'$  are paths in  $X$  from  $x_0$  to  $x$ . We want to show

$$\tilde{f}\gamma'(1) = \tilde{f}\gamma(1).$$

Since  $\gamma \cdot \overline{\gamma'}$  is a loop in  $X$  at  $x_0$ , we know that  $[(f\gamma) \cdot (f\overline{\gamma'})]$  is a class of loops in  $Y$  in  $\text{Im}(f_*)$ . By hypothesis, this class of loops is in  $\text{Im}(p_*)$ . It lifts to a loop which is based at  $\tilde{y}_0$ . By uniqueness of lifts, this loop lifting  $(f\gamma) \cdot (f\overline{\gamma'})$  to  $\tilde{Y}$  must be equal to the lifts  $\tilde{f}\gamma \cdot \overline{\tilde{f}\gamma'}$  with a common value at  $t = 1/2$ . Hence,  $\tilde{f}\gamma(1) = \tilde{f}\gamma'(1)$  as desired, namely the endpoints agree.



## Lecture 16: Proving Proposition 3.3

11 Feb. 10:00

2.  $\tilde{f}$  is continuous. Choose  $x \in X$  and a neighborhood  $\tilde{U}$  of  $\tilde{f}(x)$  in  $\tilde{Y}$ . Note that we can choose  $\tilde{U}$  small enough to  $p|_{\tilde{U}}$  is homeomorphism to  $U$  in  $Y$ . Now, there exists a neighborhood  $V$  of  $x$  in  $X$  with  $f(V) \subseteq U$ .



The goal is  $\tilde{f}(V) \subseteq \tilde{U}$ . Without loss of generality, we can assume that  $V$  is path-connected. Then,

$$\tilde{f}\gamma \cdot \tilde{f}\alpha = [\widetilde{f\gamma \cdot f\alpha}].$$

Hence,

$$\tilde{f}\alpha = (p|_{\tilde{U}})^{-1} \circ f \circ \alpha,$$

where  $(p|_{\tilde{U}})^{-1}$ 's image is in  $\tilde{U}$ , so

$$\tilde{f}(x') = f\gamma \cdot f\alpha(1) \in \tilde{U},$$

which implies

$$\tilde{f}(V) \subseteq \tilde{U}.$$

■

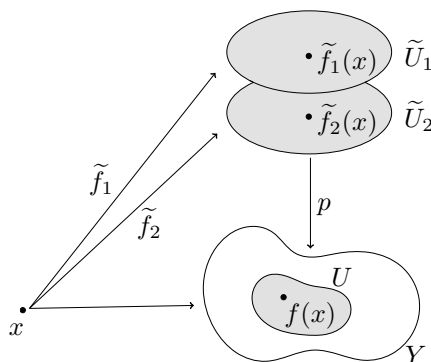
**Proposition 3.4 (Uniqueness of lifts).** Let  $p: \tilde{Y} \rightarrow Y$  be a covering map with  $X$  is a connected space. If two lifts  $\tilde{f}_1, \tilde{f}_2$  of the same map  $f$  agree at a single point, then they agree everywhere.



*Proof.* Let  $S$  being

$$S := \{x \in X \mid \tilde{f}_1(x) = \tilde{f}_2(x)\}.$$

We want to show that  $S$  is both closed and open, so if  $S$  is nonempty,  $S = X$ .



We see that  $\tilde{U}_1$  and  $\tilde{U}_2$  are slices of  $p^{-1}(U)$ , where  $U$  is evenly covered neighborhood of  $f(x)$ .

1. If  $\tilde{f}_1(x) \neq \tilde{f}_2(x)$ . Then  $\tilde{U}_1, \tilde{U}_2$  are disjoint. Since  $\tilde{f}_1, \tilde{f}_2$  are continuous, there exists a neighborhood  $N$  of  $x$  with

$$\tilde{f}_1(N) \subseteq \tilde{U}_1, \quad \tilde{f}_2(N) \subseteq \tilde{U}_2,$$

with the fact that they're disjoint, so  $x$  is an interior point of  $S^c$ .

2. If  $\tilde{f}_1(x) = \tilde{f}_2(x)$ . Then  $\tilde{U}_1 = \tilde{U}_2$ . Choose  $N$  as before, then we have

$$\tilde{f}_1(n) = (p|_{\tilde{U}_1})^{-1}(f(n)) = \tilde{f}_2(n),$$

hence  $x \in \text{int}(S)$ . ■

## 3.2 Deck Transformation

We now want to introduce a special kind of transformation.

**Definition 3.3 (Isomorphism of Covers).** Given covering maps

$$p_1: \tilde{X}_1 \rightarrow X, \quad p_2: \tilde{X}_2 \rightarrow X,$$

an *isomorphism of covers* is a homeomorphism

$$f: \tilde{X}_1 \rightarrow \tilde{X}_2$$

such that  $p_1 = p_2 \circ f$ .

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{f} & \tilde{X}_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

**Exercise.** This defines equivalent relation on [covers](#) of  $X$ .

**Definition 3.4 (Deck transformation).** Given a [covering map](#)  $p: \tilde{X} \rightarrow X$ , the [isomorphisms of covers](#)  $\tilde{X} \rightarrow \tilde{X}$  are called *Deck transformation*.

Furthermore, we'll let  $G(\tilde{X})$  denotes the *set of deck transformations*.

**Note.** Note that we've suppressed the data of  $p$  in the notation, but this data is essential to what a [deck transformation](#) is, when this is unclear we write  $G(\tilde{X}, p)$ .

## Lecture 17: Deck Transformation

14 Feb. 10:00

**Example.** Let's see some examples.

1. [Deck transformations](#)  $G(\tilde{X})$  are a subgroup of the group of homeomorphisms of  $\tilde{X}$ .

2. Given the **cover**  $p: \mathbb{R} \rightarrow S^1$ .
  - **Deck maps**: translation by  $n \in \mathbb{Z}$  units.
  - $G(\mathbb{R}) \cong \mathbb{Z}$
3. Given the **cover**  $p_n: S^1 \rightarrow S^1$  be an  $n$ -sheeted cover.
  - **Deck maps**: rotation by  $2\pi/n$ .
  - $G(S^1, p_n) \cong \mathbb{Z} / n\mathbb{Z}$



Figure 17:  $p_n: S^1 \rightarrow S^1$  be an  $n$ -sheeted **cover**, where  $n = 3$ .

**Exercise (Deck Transformation is determined by the image of one point).** Given  $X, \tilde{X}$  are **path**-connected, locally **path**-connected, **deck map** is determined by the image of any one point.

**Answer.**

$$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow f & \downarrow p \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

**Corollary 3.2.** If a **deck transformation** has a fixed point, it is the identity transformation.

**Exercise.** Let  $X$  be connected. Given a **deck transformation**  $\tau: \tilde{X} \rightarrow \tilde{X}$ ,  $\tau$  defines a permutation of  $p^{-1}(\{x_0\})$ . If this permutation has a fixed point, then it is the identity.

**Definition 3.5 (Regular, Normal).** A **covering space**  $p: \tilde{X} \rightarrow X$  is *regular* or *normal* if  $\forall x_0 \in X, \forall \tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$ , there exists a **deck transformation** such that

$$\tilde{x}_0 \mapsto \tilde{x}_1.$$



Figure 18: Covers of  $S^1 \vee S^1$ . The left one is **regular**, while the right one is not since there is no automorphism from  $\tilde{x}_0$  to  $\tilde{x}_1$  or  $\tilde{x}_2$ .

**Remark.** A **regular cover** is *as symmetric as possible*.

**Exercise.** **Regular** means that the group  $G(\tilde{X})$  acts transitively on  $p^{-1}(\{x_0\})$ . Explain why we cannot ask for more than this:

$G(\tilde{X})$  cannot induce the full symmetric group on  $p^{-1}(\{x_0\})$  provided that  $|p^{-1}(\{x_0\})| > 2$ .

**Answer.** The key is uniqueness.

**Definition 3.6 (Normalizer).** Given  $G$  as a group,  $H \subseteq G$  is a subgroup of  $G$ . Then the *normalizer* of  $H$ , denoted by  $N(H)$ , is defined as

$$N(H) := \{g \in G \mid gH = Hg\}.$$

**Exercise.** We can prove the followings.

1.  $N(H)$  is a subgroup.
2.  $H \leq N(H)$ .
3.  $H$  is normal in  $N(H)$ .
4. If  $H \leq G$  is normal,  $N(H) = G$ .
5.  $N(H)$  is the largest subgroup (under containment) of  $G$  containing  $H$  as normal subgroup.

**Proposition 3.5.** Given  $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  be a **cover**, and  $\tilde{X}, X$  are **path**-connected, locally **path**-connected. Let

$$H = p_* \left( \pi_1(\tilde{X}, \tilde{x}_0) \right) \subseteq \pi_1(X, x_0).$$

Then

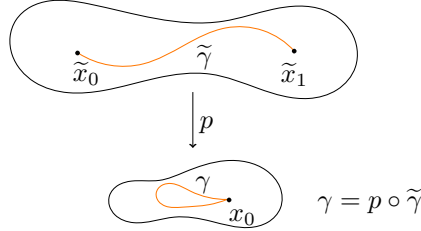
1.  $p$  is **normal** if and only if  $H \subset \pi_1(X, x_0)$  is normal.
2. We have

$$G(\tilde{X}) \cong N(H) / H,$$

where  $G(\tilde{X})$  are **Deck maps**, and  $N(H)$  is the **normalizer** of  $H$  in  $\pi_1(X, x_0)$ .

**Remark.** A fact is worth noting is the following. Let  $\tilde{\gamma}$  be a path  $\tilde{x}_1$  to  $\tilde{x}_0$ . Then

$$p_* \left( \pi_1(\tilde{X}, \tilde{x}_0) \right) = [\gamma] p_* \left( \pi_1(\tilde{X}, \tilde{x}_1) \right) [\gamma^{-1}].$$



## Lecture 18: Proving Proposition 3.5

16 Feb. 10:00

Now let's prove Proposition 3.5

*Proof.* Let  $X, x_0$  be the base space and  $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$  where  $p: \tilde{X} \rightarrow X$  is a covering map. Further, let  $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ .

In homework, given  $(X, x_0), \tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$  if we change the basepoint from  $\pi_1(\tilde{X}, \tilde{x}_0)$  to  $\pi_1(\tilde{X}, \tilde{x}_1)$ , then we have the induced subgroups of the base spaces fundamental group are conjugate by some loop  $[\gamma] \in \pi_1(X, x_0)$ , i.e.,

$$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = [\gamma] \cdot p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cdot [\gamma]^{-1}$$

where  $\gamma$  is lifted to a path from  $\tilde{x}_0$  to  $\tilde{x}_1$ .

Therefore,  $[\gamma] \in N(H)$  if and only if  $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ , and this holds if and only if there is a deck transformation taking  $\tilde{x}_0$  to  $\tilde{x}_1$  by the classification of based covering spaces in the homework.<sup>16</sup> This shows that  $p$  is a normal cover if and only if  $H$  is normal, which proves the first claim.

We then define a map  $\Phi$  such that

$$\Phi: N(H) \rightarrow G(\tilde{X})[\gamma], \quad \cdot \mapsto \tau$$

where  $\tau$  lifts to a path from  $\tilde{x}_0$  to  $\tilde{x}_1$  and  $\tau$  is a deck transformation mapping  $\tilde{x}_0$  to  $\tilde{x}_1$ , which will be uniquely defined by the uniqueness of lifts with specified base points. We now need to check

1.  $\Phi$  is surjective.
2.  $\ker(\Phi) = H$ .
3.  $\Phi$  is a group homomorphism.

If we can prove all the above, then, from the result follows directly from the first isomorphism theorem.<sup>17</sup>

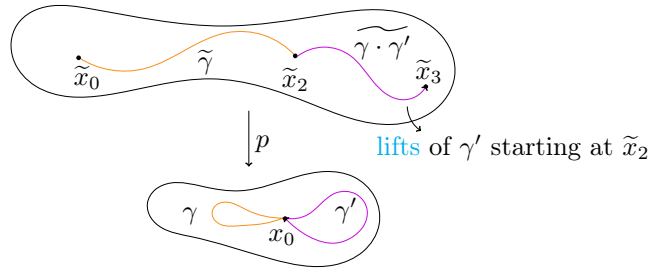
<sup>16</sup>Alternatively, we can use the lifting criterion.

<sup>17</sup>[https://en.wikipedia.org/wiki/Isomorphism\\_theorems](https://en.wikipedia.org/wiki/Isomorphism_theorems)

1. We've proved that  $\Phi$  is surjective before in our work above.
2.  $\Phi([\gamma])$  is the identity if and only if  $\tau$  sends  $\tilde{x}_0$  to  $\tilde{x}_0$ , meaning that  $[\gamma]$  **lifts** to a loop. Then by our characterization of the **fundamental group** downstairs:

$$\ker(\Phi) = \{[\gamma] \mid [\gamma] \text{ lifts to a loop}\} = H.$$

3. Suppose we have loops  $[\gamma_1] \xrightarrow{\Phi} \tau_1$  and  $[\gamma_2] \xrightarrow{\Phi} \tau_2$ . We claim that  $\gamma_1 \cdot \gamma_2$  **lifts** to  $\tilde{\gamma}_1 \cdot \tau(\tilde{\gamma}_2)$ .



It's an exercise to check that the **lift** of  $\gamma_2$  starting at  $\tilde{x}_1$  is exactly  $\phi_1(\tilde{\gamma}_2)$ , where  $\tilde{\gamma}_2$  is a **lift** starting at  $\tilde{x}_0$ .

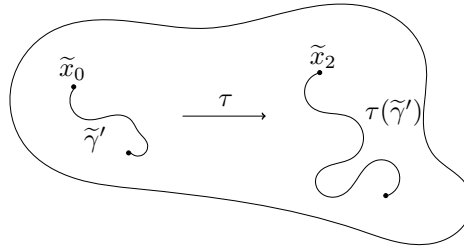


Figure 19: Must be **lift** of  $\gamma'$  starting at  $\tilde{x}_2$

The idea is that by uniqueness of **lifts** we'll have the desired claim. We then just observe that this **path**  $\tilde{\gamma}_1 \cdot \tau_1(\tilde{\gamma}_2)$  is a **path** from  $\tilde{x}_0$  to  $\tau_1(\tilde{\gamma}_2(1)) = \tau_1(\tau_2(\tilde{x}_0))$ , so the image must be a **deck transformation** sending  $\tilde{x}_0$  to  $\tau_1(\tau_2(\tilde{x}_0))$ . But then  $\tau_1 \circ \tau_2$  maps  $\tilde{x}_0$  to this same point, and from **this exercise**, we know that the **deck transformations** are determined by where they send a single point, hence we're done.

■

**Corollary 3.3.** If  $p$  is a **normal covering**, then  $G(\tilde{X}) \cong \pi_1(X, x_0) / H$ .

**Definition 3.7 (Universal covering).** A [cover](#)  $p: \tilde{X} \rightarrow X$  is called a *universal covering* if  $\tilde{X}$  is simply connected.

**Corollary 3.4.** If  $\tilde{X}$  is the [universal cover](#), then  $G(\tilde{X}) \cong \pi_1(X, x_0)$ .

**Exercise.** Whether  $\text{Im}(p_*)$  is normal is independent of the basepoint in  $\tilde{X}$  and  $X$ .

So,  $p$  is normal if and only if  $G(\tilde{X})$  is transitive on  $p^{-1}(x_0)$  for at least one  $x_0 \in X$ .

**Exercise.** Let  $\Sigma g$  be the genus  $g$  surface. Prove that  $\Sigma g$  has a normal  $n$ -sheeted [path-connected cover](#) for every  $n$ .

## Lecture 19: Simplex

18 Feb. 10:00

### 4 Homology

#### 4.1 Motivation for Homology

Informally, the higher [homotopy](#) groups is defined as

$$\pi_n(X, x_0): I^n \rightarrow (X, x_0), \quad \partial I^n \mapsto x_0.$$



We see that it's extremely hard to compute higher [fundamental group](#). Hence instead, we will study the higher dimensional structure of  $X$  via *homology*.

- **Cons.**
  - The definition is more opaque at first encounter.
- **Pros.**
  - Lots of computational tools
  - Functional
  - Abelian Groups



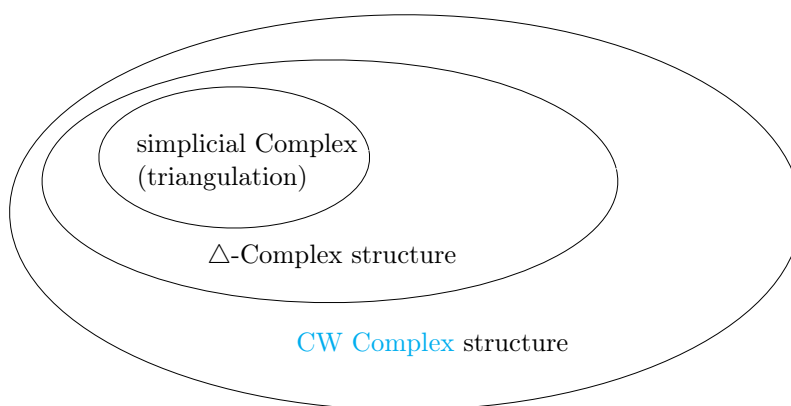
**Remark.** More like  $\pi_n$  for  $n > 1$ .

- No basepoints
- Can compute using **CW** structure.
- Good properties. For example,  $H_n = 0$  if  $n > \dim X$

## 4.2 simplicial Homology

### 4.2.1 $\Delta$ -Simplex

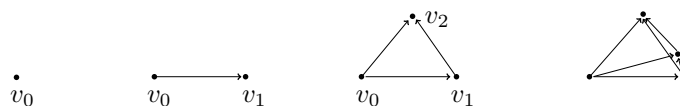
This is a stricter version of a **CW complex** which allows us to decompose our spaces into cells. In terms of how things fit together, we have the following diagram.



Now we try to give the definition.

**Definition 4.1 (Simplex).** We see that

- 0-simplex. A point.
- 1-simplex. Interval.
- 2-simplex. Triangle.
- 3-simplex. Tetrahedron.
- $n$ -simplex. The convex hull of  $(n + 1)$ -points position in  $\mathbb{R}^n$ .



**Remark.** We see that

- The top of which is the 2-disk and remember cell structure (edges and vertices) and remember orientation (ordering on vertices).
- The top of which is the 3-disk and cells and the orientation.

Further,

- We can view [simplices](#) as both *combinatorial* and *topological* objects.

An alternative definition can be done.

**Definition 4.2 (Standard simplex).** We say that an  $n$ -dimensional *standard simplex*, denoted by  $\Delta^n$  is

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1 \right\}.$$

We'll call such a simplex as *standard  $n$ -simplex*.



**Remark.** In our definition, the [simplices](#) will implicitly come with a choice of ordering of the vertices as

$$\Delta^n = [v_0, v_1, \dots, v_n]$$

such that the convex hull of these points is taken with this ordering.

## Lecture 20: Simplicial Complex

21 Feb. 10:00

**Definition 4.3 (Subsimplex).** A *subsimplex* of a [simplex](#)  $\sigma$  combinatorially, it's a subset of the vertices; while topologically, it's the convex hull of the subset of vertices.



**Definition 4.4 (Face).** A *face* of a [simplex](#)  $\Delta^n$  is a [subsimplex](#) of 1 dimensional lower than  $\Delta^n$  (codimension 1).

**Definition 4.5 (Boundary).** The *boundary*  $\partial\sigma$  of a **simplex**  $\sigma$  is the union of its **faces**.

**Definition 4.6 (Open simplex).** The *open simplex* of  $\Delta$  is defined as

$$\mathring{\Delta}^n := \Delta^n / \partial\Delta^n.$$

**Definition 4.7 ( $\Delta$ -Complex).** A  $\Delta$ -*complex* structure on  $X$  is a collection of maps

$$\sigma_\alpha: \Delta^n \rightarrow X$$

such that

1.  $\sigma_\alpha|_{\mathring{\Delta}^n}$  injective, each point of  $X$  is in the image of exactly one such map.
2. Each restriction of  $\sigma_\alpha$  to a **face** coincides with a map

$$\sigma_\beta: \Delta^{n-1} \rightarrow X.$$

3. A set  $A \subseteq X$  is open if and only if  $\sigma_\alpha^{-1}(A)$  is open in  $\mathring{\Delta}^n$  for all  $\sigma_\alpha$ , i.e.,  $X$  is a quotient

$$\coprod_{n,\alpha} \Delta_\alpha^n \xrightarrow{\coprod \sigma_\alpha} X.$$

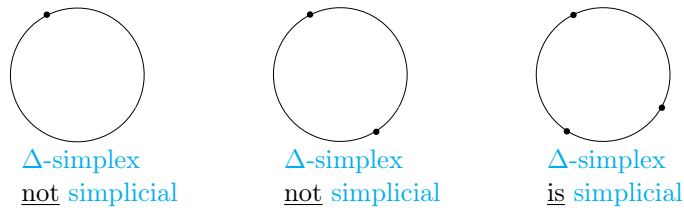
**Exercise.** A  $\Delta$ -**complex**  $X$  is a **CW complex**  $W$  characteristic maps  $\sigma_\alpha$  with extra constraints on the attaching maps.

**Note.** We see that the second condition of **Definition 4.7** implies that attaching maps injective on interior of **faces**.

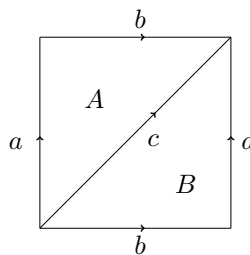
**Definition 4.8 (simplicial complex).** A *simplicial complex* is a  $\Delta$ -**complex** such that

- $\sigma_\alpha$  must map every **face** to a different  $(n-1)$ -**simplex**.
- Every **simplex** is uniquely determined by its vertex set.
- Any  $(n+1)$  vertices in  $X^0$  is the vertex set of at most 1 **simplex**.

**Remark.** With **Definition 4.8**, we see the followings.



**Example.** The torus with the following edges,  $a, b, c$  and the gluing in triangles  $A$  and  $B$  can be seen as follows.



For this  $\Delta$ -complex, notice that we've glued down a triangle whose vertices are all identified. This is not allowed in a simplicial complex / triangulation.

**Remark.** The minimum number of triangles in a simplicial complex structure is 14.

## Lecture 21: Homology

23 Feb. 10:00

### 4.3 Homology

To demonstrate how the definition of homology arise, we first see the idea behind it. Fix a space  $X$  which equips with the  $\Delta$ -complex structure. Then, we define  $C_n(X)$  to be the free Abelian group on the  $n$ -simplices of  $X$ . That is,

$$C_n(X) = \left\{ \text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha: \Delta^n \rightarrow X \right\}.$$

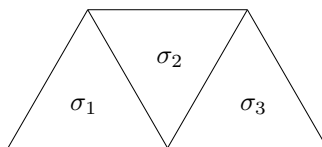
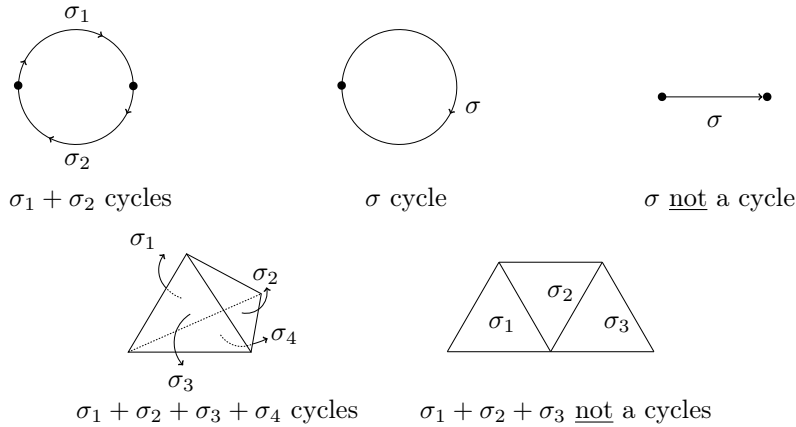


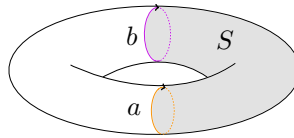
Figure 20:  $C_2(X) = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \oplus \mathbb{Z}\sigma_3$ .

Then, the  $n$ -th homology group will be a subquotient of  $C_n(X)$ , where the heuristic/imprecise idea is

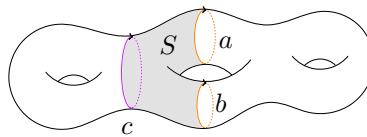
- Take subgroup of  $C_n$  of *cycles*. These are sums of *simplices* satisfying a combinatorial condition on the boundary gluing maps to ensure that they *close up*, i.e., they have no *boundary*.



- To take the quotient, we consider two cycles to be equivalent if their difference is a *boundary*. For example, in the case of torus,  $a$  is homologous to  $b$  since  $a - b$  is the *boundary* of the shaded subsurface  $S$  on of the torus below.



In fact,  $a$  and  $b$  are *homotopic* (which will imply they're homologous essentially), but two loops do not need to be *homotopic* to be homologous. For example, in the figure below,  $a + b$  is homologous to  $c$ , since  $a + b - c$  is the *boundary* of  $S$  ( $a + b$ <sup>18</sup> and  $c$  are not *homotopic*).



Let's now see the formal definition.

<sup>18</sup>Which isn't even a loop

**Definition 4.9 (Chain group).** We define the *chain group*  $C_n(X)$  of order  $n$  to be the *free Abelian group* on the  *$n$ -simplices* of  $X$  such that

$$C_n(X) := \left\{ \text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha: \Delta^n \rightarrow X \right\}.$$

**Definition 4.10 (Boundary homomorphism).** A map  $\partial_n: C_n(X) \rightarrow C_{n-1}(X)$  is called a *boundary homomorphism* such that

$$\begin{aligned} \partial_n: C_n(X) &\rightarrow C_{n-1}(X) \\ [\sigma_\alpha] &\mapsto \sum_{i=1}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}, \end{aligned}$$

which defines the map on the basis, and we extend it linearly.

**Example.** We give some lower dimensions examples of [Definition 4.10](#) to motivate the general definition.

- For  $n = 1$ ,  $\partial_1: C_1(X) \rightarrow C_0(X)$  such that

$$[\sigma_\alpha: [v_0, v_1] \rightarrow X] \mapsto \sigma_\alpha|_{[v_1]} - \sigma_\alpha|_{[v_0]}.$$

- For  $n = 2$ ,  $\partial_2: C_2(X) \rightarrow C_1(X)$  such that

$$[\sigma_\alpha: [v_0, v_1, v_2] \rightarrow X] \mapsto \sigma_\alpha|_{[v_1, v_2]} - \sigma_\alpha|_{[v_0, v_2]} + \sigma_\alpha|_{[v_0, v_1]}.$$

**Lemma 4.1.** For any  $n \geq 2$ , we have

$$\begin{array}{ccccc} C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \xrightarrow{\partial_{n-1}} & C_{n-2}(X) \\ & \searrow & \xrightarrow{\partial_{n-1} \circ \partial_n = 0} & & \end{array}$$

*Proof.* Since all  $C_i$  are *free Abelian group*, hence we only need to consider  $\partial_{n-1} \circ \partial_n(\sigma) = 0$  for a generator  $\sigma$ . Given a generator  $\sigma$ , the result follows from directly applying the definition and with some calculation. ■

**Definition 4.11 (Chain complex).** A *chain complex*  $(C_*, d_*)$  is a collection of maps such that

$$\dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

of *Abelian groups* and group homomorphism such that

$$d_{n-1} \circ d_n = 0.$$

We call  $C_n$  the  *$n$ -th chain group* and  $d_n$  the  *$n$ -th differential*.

**Remark.** We see that

- [Lemma 4.1](#) guarantees that our [simplicial chain groups](#) form a [chain complex](#).
- [Definition 4.11](#) means that  $\ker(d_n)$  contains  $\text{Im}(d_{n+1})$ , since  $d_n \circ d_{n+1} = 0$ .

**Definition 4.12 (Exact).** We say that the sequence is *exact at  $C_n$*  provided that  $\ker(d_n) = \text{Im}(d_{n+1})$ . A [chain complex](#) is *exact* if it is *exact at each point*.

**Definition 4.13 (Homology group).** The  $n^{\text{th}}$  *homology group of a chain complex*  $(C_*, d_*)$ , denoted as  $H_n$  or  $H_n(C_*)$ , is the quotient

$$H_n := \ker(d_n) / \text{Im}(d_{n+1}).$$

**Remark.** The [homology group](#) measures how far the [chain complex](#) is from being [exact](#) at  $C_n$ .

With what we have just defined, it's natural to define [homology groups](#) of spaces  $X$  with a  [\$\Delta\$ -complex](#) structure.

**Definition 4.14 (Homology class).** We say  $\ker(\partial_n)$  is the subgroup of **cycles** in  $C_n(X)$ , and  $\text{Im}(\partial_{n+1})$  is the subgroup of **boundaries** in  $C_n(X)$ . We then set

$$H_n(X) := \ker(\partial_n) / \text{Im}(\partial_{n+1}) = \{\text{cycles}\} / \{\text{boundaries}\}.$$

In other words, it's the [homology](#) of the [chain complex](#)

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

where we take it to be 0 in all negative indices, namely

$$\dots \xrightarrow{\partial_3} C_{n+1} \xrightarrow{\partial_2} C_n \xrightarrow{\partial_1} C_{n-1} \xrightarrow{\partial_0} 0$$

We then call the elements of  $H_n(X)$  as *homology classes*.

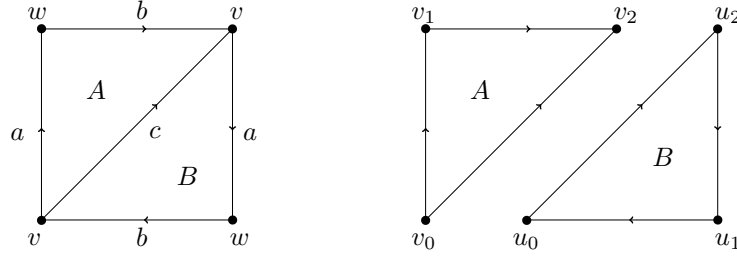
## Lecture 22: Calculation of Homology

25 Feb. 10:00

### 4.4 Calculation of Homology

We start from some calculation about [homology group](#) of some spaces.

**Example.** Let  $X = \mathbb{R}P^2$ .



We see that we have

- $C_0 = \mathbb{Z}\langle v, w \rangle$
- $C_1 = \mathbb{Z}\langle a, b, c \rangle$
- $C_2 = \mathbb{Z}\langle A, B \rangle = \mathbb{Z}A \oplus \mathbb{Z}B$

The chain complex is then

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Where we let  $A = [v_0, v_1, v_2]$  and  $B = [u_0, u_1, u_2]$ , then

$$\partial_2 : \begin{cases} A & \mapsto b - c + a \\ B & \mapsto -a - c - b \end{cases}, \quad \partial_1 : \begin{cases} a & \mapsto w - v \\ b & \mapsto v - w \\ c & \mapsto v - v = 0 \end{cases}$$

We can also calculate the image and the kernel at  $C_i$ , i.e.,

$$\begin{aligned} C_2 : \text{Im } \partial_3 &= 0, & \ker \partial_2 &= 0, \\ C_1 : \text{Im } \partial_2 &= \langle 2c, b - c + a \rangle, & \ker \partial_1 &= \langle b + a, c \rangle, \\ C_0 : \text{Im } \partial_1 &= \langle v, w \rangle, & \ker \partial_0 &= \langle v - w \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} H_0 &\cong \mathbb{Z}\langle v, w \rangle / \mathbb{Z}\langle v - w \rangle \cong \mathbb{Z} \\ H_1 &\cong \mathbb{Z}\langle b + a, c \rangle / \mathbb{Z}\langle 2c, b + a - c \rangle \cong \mathbb{Z}\langle b + a - c, c \rangle / \mathbb{Z}\langle 2c, b + a - c \rangle \cong \mathbb{Z} / 2\mathbb{Z} \\ H_2 &= 0 \end{aligned}$$

**Remark.** Given a basis for a free Abelian group  $\langle b_1, \dots, b_n \rangle$  we can replace  $b_i$  with

$$b_i \pm m_1 b_1 \pm \dots \pm \widehat{m_i b_i} \pm \dots \pm m_n b_n.$$

**Remark.** Warning! Care is needed when doing *change of bases* over  $\mathbb{Z}$ . For example, if  $b_1, b_2$  is a basis for  $A \subseteq \mathbb{Z}^n$ , then  $b_1 - b_2, b_1 + b_2$  is not a basis, it is an index-2 subgroup. The key to this is that  $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  has determinant  $-2$  (not unit in  $\mathbb{Z}$ ).



We can transform a basis for a [free group](#) into a different basis by applying a matrix of determinant  $\pm 1$ . If we apply a matrix of determinant  $D$  we will obtain generators for a subgroup of index  $|D|$ .

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \pm m_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \pm m_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \pm m_{i-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \pm m_{i+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \pm m_n & 0 & \cdots & 1 \end{bmatrix}$$

As a summary, we have the following procedures to compute  $H_n(X)$ .

1. Choose  [\$\Delta\$ -complex](#) structure on  $X$ . (Later: We will prove  $H_*(X)$  is independent of the choice of  [\$\Delta\$ -complex](#) structure)
2. Choose orientations on each [simplex](#) (Any choice is okay, but you must commit to a choice, or you will make a sign error!)
3. For each  [\$n\$ -simplex](#)  $\sigma$  compute  $\partial_n(\sigma)$  (careful with signs!)
4.  $\text{Im } \partial_n = \langle \partial_n(\sigma) \mid \sigma \text{ an } n\text{-simplex} \rangle$ . Use linear algebra to compute  $\ker(\partial_n)$ .
5. For each  $n$  compute  $H_n(X) = \ker \partial_n / \text{Im } \partial_{n+1}$ . Be careful that any change-of-variables map you apply is invertible over  $\mathbb{Z}$ .

## Lecture 23: Singular Homology

07 Mar. 10:00

### 4.5 Singular Homology

**Definition 4.15 (Singular simplex).** A *singular  $n$ -simplex* in a space  $X$  is a continuous map

$$\sigma: \Delta^n \rightarrow X.$$

**Definition 4.16 (Singular chain).** Let  $C_n(X)$  be the [free group](#) on [singular  \$n\$ -simplices](#) in  $X$ . The *singular  $n$ -chains* with [boundary maps](#):

$$\begin{aligned} \partial_n: C_n(X) &\rightarrow C_{n-1}(X) \\ \sigma &\mapsto \sum_{i=1}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v}_i, \dots, v_n]}. \end{aligned}$$

This gives us a *singular chain complex*

**Definition 4.17 (singular homology group).** The *singular homology groups* are the [homology groups](#) of this [singular chain complex](#) given as

$$H_n(X) = \ker \partial_n / \text{Im } \partial_{n+1}.$$

Since the generating sets for  $C_n(X)$  are almost always hugely uncountable, it's almost impossible to compute with these. However, it does give us a definition that does not depend on any other structure than the topology of  $X$ , making it useful for developing theory.

**Note.** The heuristic is that, we interpret a **chain**  $\sigma_1 \pm \sigma_2 \pm \dots \pm \sigma_k$  as a map from a  **$\Delta$ -complex** to  $X$ . For example with  $\sigma_1 + \sigma_2$ .

Where we've glued  $[v_1, v_2]$  of  $\sigma_1$  to  $[v_0, v_2]$  of  $\sigma_2$  if  $\sigma_1|_{[v_1, v_2]}$  and  $\sigma_2|_{[v_0, v_2]}$  are the same singular  **$n$ -chain** with opposite signs.

*Goals:*

- **Singular homology** is a **homotopy** invariant
- **Singular** and simplicial homology groups are isomorphic.

**Exercise.** Check that if  $X$  has **path** components  $\{X_\alpha\}$  then

$$H_n(X) \cong \bigoplus_{\alpha} H_n(X_\alpha).$$

**Exercise.** If  $X = \{*\}$ , then

$$H_n(X) = \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$

**Exercise.** If  $X$  is **path**-connected, then  $H_0(X) \cong \mathbb{Z}$

## 4.6 Functoriality and Homotopy Invariance

**Definition 4.18 (Induced map on chains).** For a given continuous map  $f: X \rightarrow Y$  we can consider the *map  $f_\#$  induced by chains* as

$$\begin{aligned} f_\# : C_n(X) &\rightarrow C_n(Y) \\ [\sigma : \Delta^n \rightarrow X] &\mapsto [f \circ \sigma : \Delta^n \rightarrow Y]. \end{aligned}$$

**Definition 4.19 (Chain map).** Given two **chain complexes**  $(C_*, \partial_*)$  and  $(D_*, \delta_*)$ , a *chain map* between them is a collection of group homomorphisms  $g_n: C_n \rightarrow D_n$  such that the following diagram commutes.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \dots \\ & & \downarrow g_{n+1} & & \downarrow g_n & & \downarrow g_{n-1} & & \\ \dots & \xrightarrow{\delta_{n+2}} & D_{n+1} & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} & \xrightarrow{\delta_{n-1}} & \dots \end{array}$$

i.e. we have that  $\delta_n \circ g_n = g_{n-1} \circ \partial_n$ .

**Exercise.** We have that  $f_\# \partial = \partial f_\#$ . In other words,  $f_\#$  is a **chain map**. Thus, by the homework  $f_\#$  induces a group homomorphism on the **homology groups**. We write this as  $f_*: H_n(X) \rightarrow H_n(Y)$  for all  $n$ .

**Exercise.** We have functoriality, i.e.  $(f \circ g)_* = f_* \circ g_*$ . Also, we have that  $(\text{id}_X)_* = \text{id}_{H_n(X)}$ .

**Theorem 4.1 (Homology group defines a functor).** The  $n$ -th homology group  $H_n: X \mapsto H_n(X)$  gives a functor from Top to Ab. This follows from the two exercises above.

**Theorem 4.2 (Functoriality is homotopy invariant).** If  $f, g: X \rightarrow Y$  are homotopic, then they will induce the same map on homology

$$f_* = g_*: H_n(X) \rightarrow H_n(Y).$$

**Exercise.** Theorem 4.1 and Theorem 4.2 imply that  $H_n$  is a homotopy invariant.

To prove the second theorem, we introduce some homological algebra.

**Definition 4.20 (Chain homotopy).** Given chain complexes  $(A_*, d_*^A)$  and  $(B_*, d_*^B)$  and chain maps  $f_*, g_*: A_* \rightarrow B_*$ . A *chain homotopy* from  $f$  to  $g$  is a sequence of group homomorphisms  $\psi_n: A_n \rightarrow B_{n+1}$  such that

$$f_n - g_n = d_{n+1}^B \circ \psi_n + \psi_{n-1} d_n^A$$

In a diagram, letting  $h_n = f_n - g_n$ , we have the following.

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{d_{n+2}^A} & A_{n+1} & \xrightarrow{d_{n+1}^A} & A_n & \xrightarrow{d_n^A} & A_{n-1} \xrightarrow{d_{n-1}^A} \dots \\
 & & \downarrow h_{n+1} & \swarrow \psi_n & \downarrow h_n & \swarrow \psi_{n-1} & \downarrow h_{n-1} \\
 \dots & \xrightarrow{d_{n+2}^B} & B_{n+1} & \xrightarrow{d_{n+1}^B} & B_n & \xrightarrow{d_n^B} & B_{n-1} \xrightarrow{d_{n-1}^B} \dots
 \end{array}$$

This diagram does not commute, but it shows everything that is going on. However, the **red** map is the sum of the **green** maps composed up.

# Appendix

## A Additional Proofs

### A.1 Seifert-Van Kampen Theorem on Groupoid

**Theorem A.1 (Seifert-Van Kampen Theorem on groupoid).** Given  $X_0, X_1, X$  as topological spaces with  $X_0 \cup X_1 = X$ . Then the functor  $\Pi: \underline{\text{Top}} \rightarrow \underline{\text{Gpd}}$  maps the [cocartesian](#) diagram in  $\underline{\text{Top}}_*$  to a [cocartesian](#) diagram in  $\underline{\text{Gp}}$  as follows.

$$\begin{array}{ccccc} (X_0 \cap X_1, x_0) & \xrightarrow{j_0} & (X_0, x_0) & & \Pi(X_0 \cap X_1) \xrightarrow{\Pi(j_0)} \Pi(X_0) \\ j_1 \downarrow & & \downarrow i_0 & \xrightarrow{\Pi} & \Pi(j_1) \downarrow \quad \downarrow \Pi(i_0) \\ (X_1, x_0) & \xrightarrow{i_1} & (X, x_0) & & \Pi(X_1) \xrightarrow{\Pi(i_1)} \Pi(X) \end{array}$$

**Note.** Notice that  $X_0, X_1, X$  don't need to be [path](#)-connected in particular.

Surprisingly, the proof of [Appendix A.1](#) is much more elegant with the elementary proof of [Theorem 2.7](#), hence we give the proof here.

*Proof.* Let  $\mathcal{G} \in \text{Ob}(\underline{\text{Gpd}})$  a [groupoid](#), and given [functors](#)

$$F: \Pi(X_0) \rightarrow \mathcal{G}, \quad G: \Pi(X_1) \rightarrow \mathcal{G}$$

such that

$$\begin{array}{ccc} \Pi(X_0 \cap X_1) & \xrightarrow{\Pi(j_0)} & \Pi_1(X_0) \\ \Pi(j_1) \downarrow & & \downarrow \Pi(i_0) \\ \Pi_1(X_1) & \xrightarrow{\Pi(i_1)} & \Pi_1(X) \end{array} \quad \begin{array}{c} \xrightarrow{F} \\ \searrow \exists! K \\ \xrightarrow{G} \end{array} \mathcal{G}$$

We now only need to prove that there exists a unique [functor](#)  $K: \Pi(X) \rightarrow \mathcal{G}$  such that the above diagram commutes.

We can define  $K$  as

- on [objects](#): For all  $x \in \text{Ob}(\Pi(X)) = X$ ,

$$K(x) = \begin{cases} F(x), & \text{if } x \in X_0; \\ G(x), & \text{if } x \in X_1. \end{cases}$$

This is well-defined since the diagram (without  $K$ ) commutes.

- on [morphisms](#): For every  $p, q \in X$ ,  $\langle \gamma \rangle : p \rightarrow q$  in  $\text{Hom}_{\Pi(X)}(p, q)$ , we need to define  $K(\langle \gamma \rangle) \in \text{Hom}_{\mathcal{G}}(K(p), K(q))$ . Our strategy is for every path  $\gamma$

from  $p$  to  $q$ , we define  $\tilde{K}(\gamma) \in \text{Hom}_{\mathcal{G}}(K(p), K(q))$ . Then if we also have  $\tilde{K}(\gamma) = \tilde{K}(\gamma')$  for  $\gamma \simeq \gamma' \text{ rel}\{0, 1\}$ , then we can just let

$$K(\langle \gamma \rangle) := \tilde{K}(\gamma).$$

Now we start to construct  $\tilde{K}$ .

Given a path  $\gamma: [0, 1] \rightarrow X$ ,  $\gamma(0) = p, \gamma(1) = q$ . Since  $\text{int}(X_0) \cup \text{int}(X_1) = X$ , we see that

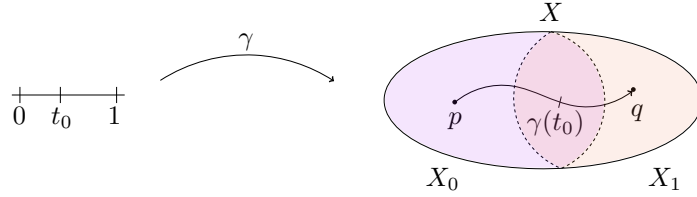
$$\gamma^{-1}(\text{int}(X_0)) \cup \gamma^{-1}(\text{int}(X_1)) = [0, 1].$$

From Lebesgue Lemma<sup>19</sup>, there exists a finite partition

$$0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1$$

such that for every  $i$ ,

$$\gamma([t_{i-1}, t_i]) \subset \text{int}(X_0) \text{ or } \text{int}(X_1).$$



Now, let  $\gamma_i: [0, 1] \rightarrow X, t \mapsto \gamma((1-t)t_{i-1} + t \cdot t_i)$ , we see that  $\gamma_i$  is either a [path](#) in  $X_0$  or  $X_1$ . We then define  $\tilde{K}(\gamma) := \tilde{K}(\gamma_m) \circ \tilde{K}(\gamma_{m-1}) \circ \dots \circ \tilde{K}(\gamma_1) \in \text{Hom}_{\mathcal{G}}(K(p), K(q))$  such that

$$\tilde{K}(\gamma_i) = \begin{cases} F(\langle \gamma_i \rangle), & \text{if } \gamma_i \subset X_0; \\ G(\langle \gamma_i \rangle), & \text{if } \gamma_i \subset X_1. \end{cases}$$

We need to prove that  $\tilde{K}(\gamma)$  does not depend on the partition. It's sufficient to prove that for any partition

$$0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1,$$

we consider any **finer** partition

$$0 = t_0 = t_{10} < t_{11} < \dots < t_{1K_1} = t_1 = t_{20} < t_{21} < \dots < t_{mK_m} = t_m = 1.$$

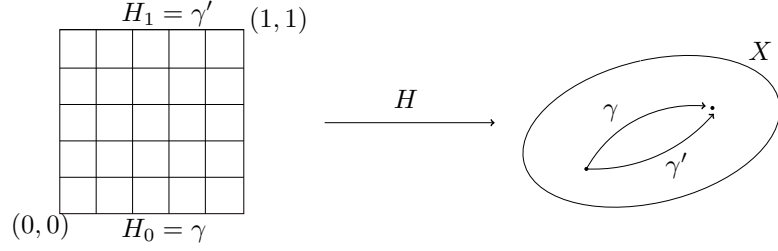
As before, we denote  $\gamma_{ij}: [0, 1] \rightarrow X, t \mapsto \gamma((1-t)t_{ij-1} + t \cdot t_{ij})$ . It's clear that as long as

$$\tilde{K}(\gamma_i) = \tilde{K}(\gamma_{iK_i}) \circ \tilde{K}(\gamma_{iK_i-1}) \circ \dots \circ \tilde{K}(\gamma_{i0}),$$

<sup>19</sup>[https://en.wikipedia.org/wiki/Lebesgue%27s\\_number\\_lemma](https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma)

then our claim is proved. But this is immediate since  $F$  and  $G$  are [functor](#) and for any  $i$ , we only use either  $F$  or  $G$  all the time.

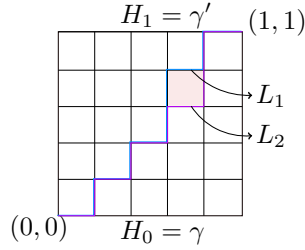
Now we prove  $\gamma \underset{H}{\simeq} \gamma' \text{ rel}\{0, 1\}$ , then  $\tilde{K}(\gamma) = \tilde{K}(\gamma')$ . This is best shown by some diagram.



The left-hand side represents a partition  $\mathcal{P}$  of  $[0, 1] \times [0, 1]$  such that every small square's image in  $X$  under  $H$  is either entirely in  $X_0$  or in  $x_1$ . Consider all paths from  $(0, 0)$  to  $(1, 1)$  such that it only goes right or up. We see that for any such path  $L$ , consider

$$\gamma_L: [0, 1] \rightarrow L, \quad t \mapsto \gamma_L(t).$$

We let  $\Gamma_L: H|_L \circ \gamma_L: [0, 1] \rightarrow X$ , we see that  $\Gamma_L$  is a [path](#) from  $p$  to  $q$ . Now, if for two paths  $L_1$  and  $L_2$  such that they only differ from a square.



We claim that  $\gamma_{L_1}, \gamma_{L_2}$  are two [paths](#) from  $p$  to  $q$ , and  $\tilde{K}(\Gamma_{L_1}) = \tilde{K}(\Gamma_{L_2})$ . Now, we denote  $\Gamma_0$  and  $\Gamma_1$  as follows.

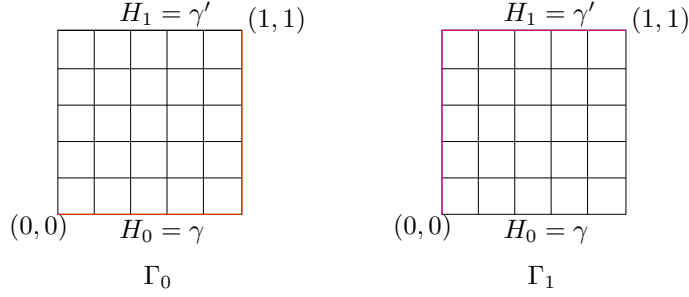


Figure 21: The definition of  $\Gamma_0$  and  $\Gamma_1$ .

It's clearly that by only finitely many steps, we can transform  $\Gamma_0$  to  $\Gamma_1$ , hence

$$\tilde{K}(\Gamma_0) = \tilde{K}(\Gamma_1).$$

Finally, we observe that

$$\tilde{K}(\gamma_0) = \tilde{K}(\Gamma_0) = \tilde{K}(\Gamma_1) = \tilde{K}(\gamma_1).$$

If we now define  $K(\langle \gamma \rangle) = \tilde{K}(\gamma)$ , then  $K: \text{Mor}(\Pi(X)) \rightarrow \text{Mor}(\mathcal{G})$ , then it's well-defined.

We now prove  $K: \Pi(X) \rightarrow \mathcal{G}$  is indeed a **functor**. But this is immediate from the definition of  $K$ , namely it'll send identity to identity and the composition associates.

Also, we need to prove that the following diagram commutes.

$$\begin{array}{ccc}
 \Pi(X_0 \cap X_1) & \xrightarrow{\Pi(j_0)} & \Pi_1(X_0) \\
 \Pi(j_1) \downarrow & & \downarrow \Pi(i_0) \\
 \Pi_1(X_1) & \xrightarrow{\Pi(i_1)} & \Pi_1(X) \\
 & \searrow G & \downarrow K \\
 & & \mathcal{G}
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow F \\
 \searrow K
 \end{array}$$

But this is again trivial.

Finally, we need to show that such  $K$  is unique. This is the same as the proof of [Lemma 1.3](#), hence the proof is done.  $\blacksquare$

## A.2 An alternative proof of **Seifert Van-Kampen Theorem**

**Theorem A.2.** We claim that the diagram

$$\begin{array}{ccc} \pi_1(X_0 \cap X_1, x_0) & \xrightarrow{(j_0)_*} & \pi_1(X_0, x_0) \\ (j_1)_* \downarrow & & \downarrow (i_0)_* \\ \pi_1(X_1, x_0) & \xrightarrow{(i_1)_*} & \pi_1(X, x_0) \end{array}$$

is **cocartesian**.

*Proof.* The basic idea is that, for this diagram,

$$\begin{array}{ccc} \Pi(X_0 \cap X_1) & \longrightarrow & \Pi(X_0) \\ \downarrow & & \downarrow \\ \Pi(X_1) & \longrightarrow & \Pi(X) \end{array}$$

we want to construct a **morphism**  $r: \Pi(Z) \rightarrow \pi_1(Z, p)$  in  $\mathbf{Gpd}$  such that  $Z = X_0 \cap X_1, X_0, X_1, X$ . For every  $x \in Z$ , we fix a **path**  $\gamma_x$  such that it connects  $p$  and  $x$  and satisfies

1. If  $x \in X_0 \cap X_1$ , then  $\text{Im}(\gamma_x) \subset X_0 \cap X_1$
2. If  $x \in X_0$ , then  $\text{Im}(\gamma_x) \subset X_0$
3. If  $x \in X_1$ , then  $\text{Im}(\gamma_x) \subset X_1$
4.  $\gamma_p = c_p$

The proof is given in [https://www.bilibili.com/video/BV1P7411N7fW?p=38&spm\\_id\\_from=pageDriver](https://www.bilibili.com/video/BV1P7411N7fW?p=38&spm_id_from=pageDriver). ■

If have  
time.

## B Abelian Group

This section aims to give some reference about **Abelian groups**, specifically for **free Abelian group**, which is used heavily when discussing homology.

### B.1 Abelian Group

**Definition B.1 (Abelian group).** A group  $(G, \cdot)$  is an *Abelian group* if for every  $a, b \in G$ , we have

$$a \cdot b = b \cdot a.$$

We often denote  $\cdot$  as  $+$  if  $(G, \cdot)$  is a **Abelian group**.



---

**Definition B.2 (Product of groups).** Given two groups  $(G, \cdot), (H, \cdot)$ , the *product of  $G$  and  $H$* , denoted by  $G \times H$  is defined as

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

and

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot g_2, h_1 \cdot h_2).$$

**Notation.** For simplicity, given an index set  $I$ , we'll denote the order pair  $(g_{\alpha_1}, g_{\alpha_2}, \dots)$  as  $(g_{\alpha})_{\alpha \in I}$ . Note that the latter notation can handle the case that  $I$  is either countable or uncountable, while the former can only handle the countable case.

**Definition B.3 (Direct product).** Given  $(G_{\alpha}, +)$ ,  $\alpha \in I$  as a collection of [Abelian group](#), we define their *direct product* as

$$\left( \prod_{\alpha \in I} G_{\alpha}, + \right),$$

where

$$\prod_{\alpha \in I} G_{\alpha} = \{(g_{\alpha})_{\alpha \in I} \mid g_{\alpha} \in G_{\alpha}\}$$

and  $\forall (g_{\alpha}), (h_{\alpha}) \in \prod_{\alpha \in I} G_{\alpha}$

$$(g_{\alpha}) + (h_{\alpha}) := g_{\alpha} + h_{\alpha}$$

for all  $\alpha \in I$ .

Specifically, if  $I$  is finite, namely there are only finitely many [Abelian groups](#)

$(G_1, +), \dots, (G_n, +)$ , and  $\left( \prod_{i=1}^n G_i, + \right)$  can be denoted as

$$(G_1 \times \dots \times G_n, +).$$

**Definition B.4 (External direct sum).** Given a collection of [Abelian groups](#)  $\{G_{\alpha}\}_{\alpha \in I}$ , the *external direct sum* of them, denoted as  $(\bigoplus_{\alpha \in I} G_{\alpha}, +)$  as

$$\bigoplus_{\alpha \in I} G_{\alpha} := \left\{ (g_{\alpha})_{\alpha \in I} \mid \forall_{\alpha \in I} g_{\alpha} \in G_{\alpha}, \# \text{ non-zero elements in } g_{\alpha} < \infty \right\}.$$

And for every  $(g_{\alpha}), (h_{\alpha}) \in \bigoplus_{\alpha \in I} G_{\alpha}$ ,

$$(g_{\alpha}) + (h_{\alpha}) := g_{\alpha} + h_{\alpha}$$

for all  $\alpha \in I$ .<sup>a</sup>

---

<sup>a</sup>This may not be the best notation: What we're really trying to say is  $(g_{\alpha})_{\alpha \in I} + (h_{\alpha})_{\alpha \in I} := g_i + h_i$  for all  $i \in I$ .

---

**Note.** We see that

$$\bigoplus_{\alpha \in I} G_\alpha \subset \prod_{\alpha \in I} G_\alpha.$$

Additionally, we also have

$$\left( \bigoplus_{\alpha \in I} G_\alpha, + \right) < \left( \prod_{\alpha \in I} G_\alpha, + \right).$$

**Remark.** We see that the operation  $+$  is indeed closed since the sum of  $g, g' \in \bigoplus_{\alpha \in I} G_\alpha$  will have only finitely non-zero elements if  $g, g'$  both have only finitely many non-zero elements.

We see that if  $I$  is a finite index set, given a collection of [Abelian group](#)  $\{G_\alpha\}_{\alpha \in I}$ , then

$$G_1 \times \dots \times G_n = G_1 \oplus \dots \oplus G_n.$$

**Definition B.5 (Internal direct sum).** Given an [Abelian group](#)  $G$ , and a collection of the subgroups  $\{G_\alpha\}_{\alpha \in I}$  of  $G$ , we say  $G$  is an *internal direct sum* of  $\{G_\alpha\}_{\alpha \in I}$  if for any  $g \in G$ , we can write

$$g = \sum_{\alpha \in I} g_\alpha$$

**uniquely**, where  $g_\alpha \in G_\alpha$  has only finitely many non-zero elements. In this case, we denote

$$G = \bigoplus_{\alpha \in I} G_\alpha.$$

Intuitively, the [external direct sum](#) is to build a new group based on the given collection of groups  $\{G_\alpha\}_{\alpha \in I}$ , while the internal direct sum is to express an **already known** group  $G$  with an **already known** collection of groups  $\{G_\alpha\}_{\alpha \in I}$ .

**Remark (Relation between Internal and External direct sum).** Given an [Abelian group](#)  $G$  and its [internal direct sum](#) decomposition  $\bigoplus_{\alpha \in I} G_\alpha$ ,  $G$  is isomorphic to the [external direct sum](#) of  $\{G_\alpha\}_{\alpha \in I}$ . We see this from the following group homomorphism:

$$\forall_{g \in G} \quad g = \sum_{\alpha \in I} g_\alpha \mapsto (g_\alpha)_{\alpha \in I}.$$

Conversely, given a collection of [Abelian group](#)  $\{G_\alpha\}_{\alpha \in I}$ , and let  $G = \bigoplus_{\alpha \in I} G_\alpha$  as the [external direct sum](#) of  $\{G_\alpha\}$ , denote  $i_{\alpha_0} : G_{\alpha_0} \rightarrow \bigoplus_{\alpha \in I} G_\alpha$  as a canonical embedding

$$g_{\alpha_0} \mapsto i_{\alpha_0}(g_{\alpha_0}) = (h_\alpha)_{\alpha \in I},$$

where

$$h_\alpha = \begin{cases} g_{\alpha_0}, & \text{if } \alpha_0 = \alpha; \\ 0, & \text{if } \alpha_0 \neq \alpha \end{cases}$$

given  $\alpha_0$ . Then

$$G'_{\alpha_0} := i_{\alpha_0}(G_{\alpha_0}) < \bigoplus_{\alpha \in I} G_{\alpha}$$

and  $G$  is the **internal direct sum** of  $G'_{\alpha_0}$ ,  $\alpha_0 \in I$ . This is because  $\forall g = (g_{\alpha})_{\alpha \in I} \in G (= \bigoplus_{\alpha \in I} G_{\alpha})$ , we have

$$g = \sum_{\alpha \in I} i_{\alpha}(g_{\alpha}).$$

Note that the above sum is well-defined since there are only finitely many non-zero elements for each  $g_{\alpha}$ . And additionally, we can see the uniqueness of this decomposition by defining  $\pi_{\alpha_0}$  such that

$$\pi_{\alpha_0}: \bigoplus_{\alpha \in I} G_{\alpha} \rightarrow G_{\alpha_0}, \quad (g_{\alpha})_{\alpha \in I} \mapsto g_{\alpha_0},$$

then  $\pi_{\alpha} \circ i_{\alpha} = \text{id}_{G_{\alpha}}$ ,  $\pi_{\alpha} \circ i_{\beta} = 0$  for all  $\beta \neq \alpha$  and

$$\pi_{\beta}(g) = \pi_{\beta} \left( \sum_{\alpha \in I} i_{\alpha}(g_{\alpha}) \right) = \sum_{\alpha \in I} \pi_{\beta} \circ i_{\alpha}(g_{\alpha}) = \pi_{\beta} \circ i_{\beta}(g_{\beta}) = g_{\beta}$$

for all  $\beta \in I$ , where the second equality is because this summation is finite. Hence, we have

$$g = \sum_{\alpha \in I} i_{\alpha}(\pi_{\alpha}(g)).$$

**Definition B.6.** Given two **Abelian groups**  $G, H$ , we define  $\text{Hom}(G, H)$  as

$$\text{Hom}(G, H) := \{f: G \rightarrow H \mid f \text{ is a group homomorphism}\},$$

then we can define

$$\begin{aligned} +: \text{Hom}(G, H) \times \text{Hom}(G, H) &\rightarrow \text{Hom}(G, H) \\ (\varphi, \psi) &\mapsto \varphi + \psi, \end{aligned}$$

where

$$(\varphi + \psi)(g) := \varphi(g) + \psi(g).$$

**Remark (Relation between direct sum and direct product).** Given a collection of **Abelian groups**  $\{G_{\alpha}\}_{\alpha \in I}$ , and another **Abelian group**  $H$ , there exists a  $\varphi$  such that

$$\begin{aligned} \varphi: \text{Hom} \left( \bigoplus_{\alpha \in I} G_{\alpha}, H \right) &\rightarrow \prod_{\alpha \in I} \text{Hom}(G_{\alpha}, H) \\ f &\mapsto \varphi(f) := (f_{\alpha})_{\alpha \in I} \end{aligned}$$

where  $f_{\alpha} = f \circ i_{\alpha}$ , where  $i_{\alpha}$  is the canonical embedding from  $G_{\alpha}$  to  $\bigoplus_{\alpha \in I} G_{\alpha}$ . We claim that  $\varphi$  is an isomorphism.

- $\varphi$  is injective. This is obvious since  $\ker(\varphi) = 0$  from the fact that if  $\varphi(f) = 0$ , then  $f_{\alpha} = 0$  for all  $\alpha$ , hence  $f$  is 0.

- $\varphi$  is surjective. For every  $(f_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} \text{Hom}(G_\alpha, H)$ , we define

$$f: \bigoplus_{\alpha \in I} G_\alpha \rightarrow H$$

$$\sum_{\alpha \in I} g_\alpha \mapsto \sum_{\alpha \in I} f_\alpha(g_\alpha).$$

We see that  $f \in \text{Hom}(\bigoplus_{\alpha \in I} G_\alpha, H)$  and  $\varphi(f) = (f_\alpha)_{\alpha \in I}$ .

This shows that

$$\text{Hom}\left(\bigoplus_{\alpha \in I} G_\alpha, H\right) \cong \prod_{\alpha \in I} \text{Hom}(G_\alpha, H).$$

**Exercise.** We can show that

$$\text{Hom}\left(H, \prod_{\alpha \in I} G_\alpha\right) \cong \prod_{\alpha \in I} \text{Hom}(H, G_\alpha).$$

Note the order in the Hom matters.

## B.2 Free Abelian Group

**Definition B.7 (Free Abelian group).** Given an Abelian group  $(G, +)$ , we say  $G$  is a *free Abelian group* if there exists a collection of elements  $\{g_\alpha\}_{\alpha \in J}$  in  $G$  such that  $\{g_\alpha\}_{\alpha \in J}$  forms a **basis** of  $G$ , i.e., for all  $g \in G$ ,  $\exists! n_\alpha \in \mathbb{Z}$  for all  $\alpha \in J$  such that

$$g = \sum_{\alpha \in J} n_\alpha g_\alpha$$

with finitely many non-zero  $n_\alpha$ .

**Remark.** If  $G$  is a free Abelian group, and  $\{g_\alpha\}_{\alpha \in J}$  is a basis, then for every  $\alpha \in J$ ,  $\langle g_\alpha \rangle$  is an infinite cyclic group since

$$n \cdot g_\alpha = 0 = 0 \cdot g_\alpha \implies n = 0.$$

And from Definition B.7, we have

$$G = \bigoplus_{\alpha \in J} \langle g_\alpha \rangle.$$

Conversely, assume there are a collection of infinite cyclic group  $\langle g_\alpha \rangle$  for  $\alpha \in I$  in  $G$  such that

$$G = \bigoplus_{\alpha \in I} \langle g_\alpha \rangle,$$

then  $\{g_\alpha\}_{\alpha \in I}$  is a basis of  $G$ , hence  $G$  is a free Abelian group.

---

**Proposition B.1.** If  $G$  is an [Abelian group](#), then the following are equivalent.

1.  $G$  is a [free Abelian group](#).
2.  $G$  is an [internal direct sum](#) of some infinite cyclic groups.
3.  $G$  is isomorphic to the [external direct sum](#) of some additive groups of integers  $\mathbb{Z}$ .

*Proof.* We see that 1.  $\iff$  2. is already proved. And for 2.  $\iff$  3., this follows directly from the [relation between internal and external direct sum](#). ■

Now, consider  $G$  as a [free Abelian group](#), then

$$u: G \xrightarrow{\cong} \bigoplus_{\alpha \in I} \mathbb{Z}$$

for some  $I$ . Denote  $e_\alpha := i_\alpha(1) \in \bigoplus_{\alpha \in I} \mathbb{Z}$ , where  $i_\alpha: \mathbb{Z} \rightarrow \bigoplus_{\alpha \in I} \mathbb{Z}$  is the canonical embedding, i.e.,  $e_\alpha = (g_\alpha)_{\alpha \in I} \in \bigoplus_{\alpha \in I} \mathbb{Z}$ , where

$$g_\beta = \begin{cases} 1, & \text{if } \beta = \alpha; \\ 0, & \text{if } \beta \neq \alpha. \end{cases}$$

Moreover, denote  $\epsilon_\alpha$  as the image of  $e_\alpha$  under the isomorphism  $u$ , namely  $\epsilon_\alpha = u^{-1}(e_\alpha)$ , then  $\{\epsilon_\alpha\}_{\alpha \in I}$  is a basis of  $G$ .

Now, for every [Abelian group](#)  $H$ , we have

$$\begin{array}{ccc} \text{Hom}(G, H) & \xleftarrow[\cong]{\circ u} & \text{Hom}\left(\bigoplus_{\alpha \in I} \mathbb{Z}, H\right) \\ & \searrow \cong & \downarrow \varphi \\ & & \prod_{\alpha \in I} \text{Hom}(\mathbb{Z}, H) \\ & & \downarrow \cong \\ & & \prod_{\alpha \in I} H \end{array} \quad \begin{array}{ccc} f & \xrightarrow{\quad} & f \circ u^{-1} \\ & \nwarrow & \downarrow \\ & & (f \circ u^{-1} \circ i_\alpha)_{\alpha \in I} \\ & & \downarrow \\ & & (f \circ u^{-1} \circ i_\alpha(1))_{\alpha \in I} \end{array}$$

where  $\varphi$  is the homeomorphism defined in [here](#), and the homeomorphism

$$\prod_{\alpha \in I} \text{Hom}(\mathbb{Z}, H) \xrightarrow{\cong} \prod_{\alpha \in I} H$$

is trivial since every  $f \in \prod_{\alpha \in I} \text{Hom}(\mathbb{Z}, H)$  corresponds to  $f(1) \in H$  uniquely. We see that

$$f \circ u^{-1} \circ i_\alpha(1) = f \circ u^{-1}(e_\alpha) = f(\epsilon_\alpha).$$

In other words, for all [Abelian group](#)  $H$ , a morphism from the set  $\{\epsilon_\alpha\}_{\alpha \in I}$  to  $H$  can be uniquely extended to the group a homomorphism from  $G$  to  $H$ .

---

**Remark.** This means, to determine  $\text{Hom}(G, H)$ , we only need to determine where each base element in  $G$  will map to in  $H$ , and this is why it's *free*.

---

We now want to generate **free Abelian group** by a set. Roughly speaking, given a set  $S$ , we can generate a **free Abelian group**  $Z$  by defining

$$Z := \left\{ \sum_{x \in S} n_x x \mid n_x \in \mathbb{Z}, \# \text{ non-zero elements in } n_x < \infty \right\}$$

with the naturally defined  $+$ . Formally, we have the following.

**Definition B.8 (Free Abelian group generated by a set).** Given a set  $S$ , the **free Abelian group** generated by  $S$   $(Z, +)$  is defined as

$$Z := \{f: S \rightarrow \mathbb{Z} \mid \text{only finitely many } x \in S \text{ such that } f(x) \neq 0\},$$

with

$$\begin{aligned} +: Z \times Z &\rightarrow Z \\ (f, g) &\mapsto f + g. \end{aligned}$$

**Remark.**  $\{\phi_x \mid x \in S\}$  forms a basis of  $Z$ , where  $\phi_x: S \rightarrow \mathbb{Z}$  such that

$$y \mapsto \phi_x(y) = \begin{cases} 1, & \text{if } y = x; \\ 0, & \text{if } y \neq x \end{cases}$$

is the characteristic function at  $x$ . We see this by for all  $f \in Z$ ,  $f = \sum_{x \in S} f(x) \phi_x$ , which is uniquely defined. Hence,  $(Z, +)$  is a **free Abelian group**.

**Note.** Note that

$$\begin{aligned} S &\xrightarrow{1:1} \{\phi_x \mid x \in S\} \\ x &\mapsto \phi_x. \end{aligned}$$

Hence, we often denote the element  $\sum_{x \in S} \underbrace{n_x}_{f(x)} \phi_x$  in  $Z$  as

$$\sum_{x \in S} n_x \cdot x.$$

**Theorem B.1 (The universal property of free Abelian group generated by a set).** Denote a canonical embedding  $i: S \rightarrow Z$ ,  $x \mapsto \phi_x$ . Then for all **Abelian group**  $H$  and  $f: S \rightarrow H$ , there exists a unique group homomorphism

$$\tilde{f}: Z \rightarrow H$$

such that  $\tilde{f} \circ i = f$ .

*Proof.* We define

$$\tilde{f}\left(\sum_{x \in S} n_x \cdot x\right) := \sum_{x \in S} n_x f(x),$$

and the uniqueness is obvious. ■

Note that we can use the above [universal property](#) to describe a [free Abelian group](#) since we have the following.

**Proposition B.2.** Given  $Z'$  as another [Abelian group](#) and  $i': S \rightarrow Z'$  as another canonical embedding such that for all [Abelian group](#)  $H$  and  $f: S \rightarrow H$ , there exists a unique group homomorphism  $\tilde{f}: Z' \rightarrow H$  such that  $\tilde{f} \circ i' = f$ , then

$$Z' \cong Z.$$

Namely, we can describe a [free Abelian group](#) by its [universal property](#) uniquely up to isomorphism.

**Theorem B.2.** Assume  $G$  is a [free Abelian group](#). Assume there exists a finite basis  $\{g_1, \dots, g_n\}$  of  $G$ , and also assume that there exists another basis  $\{h_\alpha\}_{\alpha \in I}$ . Then we have

$$\text{card}(I) < \infty,$$

specifically, we have

$$\text{card}(I) = n.$$

*Proof.* Firstly, we observe that if we can show

$$\text{card}(I) \leq n,$$

then by swapping  $\{h_\alpha\}_{\alpha \in I}$  and  $\{g_\alpha\}_{\alpha \in I}$ , we will have  $\text{card}(I) = n$ .

Suppose  $I$  is an infinite set, then we can find  $h_{\alpha_1}, \dots, h_{\alpha_m}$  such that  $m > n$  and  $h_{\alpha_i} \neq h_{\alpha_j}$  for  $i \neq j$ . Then since  $\{g_\alpha\}_{\alpha \in I}$  is a basis, we have

$$h_{\alpha_i} = \sum_{j=1}^n k_i^j g_j, \forall i = 1, \dots, m.$$

Specifically, we have

$$\begin{pmatrix} h_{\alpha_1} \\ \vdots \\ h_{\alpha_m} \end{pmatrix} = \underbrace{\begin{pmatrix} k_1^1 & k_1^2 & \dots & k_1^n \\ \vdots & & \ddots & \vdots \\ k_m^1 & k_m^2 & \dots & k_m^n \end{pmatrix}}_{K \in M_{m \times n}(\mathbb{Z}) \subset M_{m \times n}(\mathbb{Q})} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix},$$

where  $k_i^j \in \mathbb{Z}$ . From linear algebra, we know that there exists  $(r_1, \dots, r_m) \in \mathbb{Q}^m \setminus \{0\}$  such that

$$(r_1, \dots, r_m)K = (0, \dots, 0).$$

---

Multiplying both sides with the common multiple of the denominator of  $r_i$ , we see that there exists  $(\ell_1, \dots, \ell_m) \in \mathbb{Z}^m \setminus \{\vec{0}\}$  such that

$$\begin{aligned} (\ell_1, \dots, \ell_m)K &= (0, \dots, 0) \\ \implies (\ell_1, \dots, \ell_m) \begin{pmatrix} h_{\alpha_1} \\ \vdots \\ h_{\alpha_m} \end{pmatrix} &= (\ell_1, \dots, \ell_m)K \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = (0, \dots, 0) \\ \implies \sum_{i=1}^m \ell_i h_{\alpha_i} &= \vec{0} \text{ for } (\ell_1, \dots, \ell_m) \in \mathbb{Z}^m \setminus \{\vec{0}\} \not\Rightarrow \\ \implies \text{card}(I) &< \infty. \end{aligned}$$

From the same argument, we see that  $\text{card}(I) \leq n \implies \text{card}(I) = n$ . ■

**Remark.** Furthermore, one can prove that if  $G$  is a [free Abelian group](#), then we can prove that any two bases of  $G$  are equinumerous, which handle the case that the basis is an infinite set.

This induces the following definition.

**Definition B.9 (Rank).** Let  $G$  be a [free Abelian group](#), the *rank* of  $G$  is the cardinality of any basis of  $G$ .

### B.3 Finitely Generated Abelian Group

Since we're going to encounter some group as

$$\mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z},$$

so it's useful to look into those finitely generated [Abelian group](#).

Let's start with a definition.

**Definition B.10 (Torsion subgroup).** Given an [Abelian group](#)  $G$ , we say that  $g \in G$  has finite order if  $\exists n \in \mathbb{Z}$  such that  $n \cdot g = 0$ . Specifically, we say that

$$T := \{g \in G \mid g \text{ has finite order}\}$$

is a *torsion subgroup*.

If  $T = 0$  given  $G$ , we say that  $G$  is *torsion free*.

**Note.** Note that  $T$  is indeed a subgroup, since for any  $g_1, g_2 \in T$ ,  $g_1 + g_2 \in T$  from the fact that it still has finite order.

**Remark.** If  $G$  is a [free Abelian group](#), then  $G$  is [torsion free](#). Conversely, if  $G$  is [torsion free](#), we can't deduce  $G$  is a [free Abelian group](#). We see this from  $(\mathbb{Q}, +)$ . Firstly, we see that  $\mathbb{Q}$  is [torsion free](#). Now, suppose  $\mathbb{Q}$  is a [free Abelian group](#), then there exists a basis  $\{r_\alpha\}_{\alpha \in I}$  of  $\mathbb{Q}$  such that  $|I| > 1$ . Now, consider



---

$\alpha_1, \alpha_2 \in I$  such that  $\alpha_1, \alpha_2 \in I$ , for  $r_{\alpha_1}, r_{\alpha_2}$ , there exists  $n, m \in \mathbb{Z}$  and  $n, m \neq 0$  such that

$$nr_{\alpha_1} + mr_{\alpha_2} = 0 \implies n = m = 0 \not\downarrow$$

### B.3.1 Classification of Finitely generated Abelian Group

Given a finitely generated [Abelian group](#)  $G$ , we may assume its generators are  $g_1, \dots, g_n$ . Let  $F$  be

$$F := \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n \text{ times}},$$

then there are a natural surjective homomorphism

$$\varphi: F \rightarrow G, \quad e_i \mapsto g_i$$

where  $e_i = (0, \dots, 0, \underset{i^{th}}{1}, 0, \dots, 0)$ . Now, let  $K := \ker \varphi$ , we have

$$G \cong F / K.$$

Then we have the following lemma.

**Lemma B.1.**  $K$  is a finitely generated [Abelian group](#).

*Proof.*

$\mathbb{Z}$  is Noetherian,  $F$  is a finitely generated  $\mathbb{Z}$ -module  
 $\implies F$  is Noetherian module  
 $\implies K$  as a sub-module of  $F$  is a finitely generated  $\mathbb{Z}$ -module  
 $\implies K$  is a finitely generated [Abelian group](#).

Please refer all the concepts above from [\[AM94\]](#). ■

Hence, we may assume the generators of  $K$  as  $b_1, \dots, b_m$ . From the definition of  $K$ , we can further express  $b_i$  as

$$b_i = (b_{i1}, b_{i2}, \dots, b_{in}) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}_{n \times n}$$

for all  $i = 1, \dots, m$ . Denote all such row vectors  $b_i$  in a matrix  $B$ , namely

$$B := \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix} \in M_{m \times n}(\mathbb{Z}),$$

then we have

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = B \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

---

**Multiply a matrix on the right-hand side.** Now, consider a  $p \in \text{GL}(n; \mathbb{Z})$ , then

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = B \cdot \underbrace{P P^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}}_{\text{new basis}} = (BP) \cdot \begin{pmatrix} e'_1 \\ \vdots \\ e'_n \end{pmatrix},$$

where

$$P^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} =: \begin{pmatrix} e'_1 \\ \vdots \\ e'_n \end{pmatrix}.$$

We see that  $B \cdot P$  is the coefficient matrix of generators  $b_1, \dots, b_m$  under the new basis  $e'_1, \dots, e'_n$ .

**Multiply a matrix on the left-hand side.** For a  $A \in \text{GL}(m; \mathbb{Z})$ , then

$$\begin{pmatrix} b'_1 \\ \vdots \\ b'_m \end{pmatrix} = Q \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = QB \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix},$$

since  $Q$  is invertible, hence  $b'_1, \dots, b'_m$  are also generators of  $K$ . We see that  $QB$  is the coefficient matrix of new generators  $b'_1, \dots, b'_m$  under basis  $e_1, \dots, e_n$ .

**Generally** ,  $Q \cdot B \cdot P$  is the matrix representation of a particular set of  $F$ 's generators under a particular basis.

**Proposition B.3.** There exists  $P \in \text{GL}(n; \mathbb{Z})$  and  $Q \in \text{GL}(m; \mathbb{Z})$  such that

$$Q \cdot B \cdot P = \begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

where  $d_i \in \mathbb{Z}^+$  and  $d_1 \mid d_2 \mid \dots \mid d_k$ .

*Proof.* In fact,  $P, Q$  can be taken as the multiplication of the following three types of square matrices:

- $P_{ij}$ :

$$P_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1_{(ij)} \\ & & & \ddots & \\ & 1_{(ji)} & & 0 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

---

where the effect of multiplying  $P_{ij}$  from the right is *swapping column  $i, j$* .

- $P_i(c)$ , where  $c$  is the identity in  $\mathbb{Z}$ , i.e.,  $c = \pm 1$  :

$$P_i(c) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & c_{(ii)} & \\ & & & \ddots \\ & & & & 1 \end{pmatrix},$$

where the effect of multiplying  $P_i(c)$  from the right is *multiplying  $c$  to column  $i$* .

- $P_{ij}(a)$ ,  $a \in \mathbb{Z}$ :

$$P_{ij} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & a_{(ij)} & \\ & & & \ddots \\ & & & & 1 \end{pmatrix},$$

where the effect of multiplying  $P_{ij}(a)$  from the right is *adding  $a$  times column  $i$  to column  $j$* .

We see that these are *elementary column transformations* in linear algebra. In particular, if we multiply these matrices from the left, then it's called *elementary row transformations*.

That is to say, we're going to show

$$B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix}$$

can become

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 \\ & & & & \ddots \end{pmatrix},$$

$d_i \in \mathbb{Z}^+$ ,  $d_1 \mid d_2 \mid \dots \mid d_k$  from *elementary column/row transformations*.

We now show the steps to make this happens.

Step 1. Using elementary transformations, we make  $b_{11} > 0$ .

Step 2. Using elementary transformations, we make  $b_{11}$  become a divisor of all elements in the first column and row.

We see that if  $b_{11} \nmid b_{1i}$  for  $i \neq 1$ , we have  $b_{1i} = r \cdot b_{11} + s$  where  $0 < s < b_{11}$ . Then we add  $(-r)$  times the  $1^{th}$  column to the  $i^{th}$  column and swapping the  $1^{th}$  and the  $i^{th}$  column, which makes  $B$  becomes

$$\begin{pmatrix} s & \dots \\ \vdots & \ddots \end{pmatrix},$$

---

for  $0 < s < b_{11}$ . Since  $\text{card}(\{n \in \mathbb{Z} \mid 0 < n < b_{11}\}) < \infty$ , hence in finitely many steps we can make  $B$  becomes

$$\begin{pmatrix} d_1 & \cdots \\ \vdots & \ddots \end{pmatrix},$$

where  $d_1$  is a divisor of all other elements in the first column and row.

Step 3. Using elementary transformations, we can multiply the first row by a proper integer and add it to the other rows, do the same but for columns also, then we can make  $B$  becomes

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B_1 & \\ 0 & & & \end{pmatrix}.$$

Step 4. We iteratively apply Step 1. to step 3., we make  $B$  into

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

where  $d_i \in \mathbb{Z}^+$ .

Step 5. Using elementary transformations, by swapping columns and rows, we may assume  $d_1 \leq d_2 \leq \dots \leq d_k$ .

Step 6. Using elementary transformations, we can make  $B$  into

$$\begin{pmatrix} d'_1 & & & \\ & \ddots & & \\ & & d'_\ell & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

such that  $0 < d'_1 \leq \dots \leq d'_\ell$ ,  $d'_1 \mid d'_2 \mid \dots \mid d'_\ell$  since if  $d_1 \nmid d_i$  for some  $i \in \{2, \dots, k\}$ , then

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \rightarrow \begin{pmatrix} d_1 & d_i & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

then from Step 2., we have

$$\begin{pmatrix} s & \cdots \\ \vdots & \ddots \end{pmatrix}$$

where  $0 < s < d_1$  and  $s$  is a divisor of all other elements in the first row and column. Now, we repeat Step 3. to Step 5., we obtain

$$\begin{pmatrix} \tilde{d}_1 & & & \\ & \ddots & & \\ & & \tilde{d}_j & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

where  $\tilde{d}_1 \leq \dots \leq \tilde{d}_j$  such that  $\tilde{d}_1 < d_1$ . Since there are only finitely many integers which is smaller than  $d_1$ , we see that by repeating these steps, we can always make

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

into

$$\begin{pmatrix} \tilde{d}_1 & & & \\ & \ddots & & \\ & & \tilde{d}_p & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

such that  $d'_1 \mid d'_i$  for all  $i \neq 1$  and  $d'_1 \leq d'_2 \leq \dots \leq d'_p$ . By the same idea of Step 3., we have the desired matrix.

Since all operations are elementary and there are only finitely many of them, hence the result follows.  $\blacksquare$

From the definition of  $Q \cdot B \cdot P$  and [Proposition B.3](#), there exists a basis  $e'_1, \dots, e'_n$  of  $F$  such that  $K$  has finitely many generators  $d_1 e'_1, \dots, d_k e'_k$ , hence

$$G \cong \mathbb{Z} / d_1 \mathbb{Z} \oplus \mathbb{Z} / d_2 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / d_k \mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n-k \text{ times}}.$$

This leads to the following important theorem.

**Theorem B.3 (Fundamental theorem of finitely generated Abelian group).** Given a finitely generated [Abelian group](#), either  $G$  is a [free Abelian group](#), or there exists a unique set of  $\{m_i \in \mathbb{Z} \mid m_i > 1, i = 1, \dots, t\}$  such that  $m_1 \mid m_2 \mid \dots \mid m_t$  and a unique non-negative integer  $s$  such that

$$G \cong \mathbb{Z} / m_1 \mathbb{Z} \oplus \mathbb{Z} / m_2 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}}.$$

*Proof.* We need to show both uniqueness and existence.

---

**Existence.** From [Proposition B.3](#), we obtain a basis  $e'_1, \dots, e'_n$  of  $F$  and a basis  $d_1 e'_1, \dots, d_k e'_k$  in  $K$  such that  $d_1 \mid \dots \mid d_k$ . Let

$$(d_1, \dots, d_k) = (1, \dots, 1, m_1, \dots, m_t),$$

which implies

$$\begin{aligned} G &\cong F / K \\ &\cong \mathbb{Z} / d_1 \mathbb{Z} \oplus \mathbb{Z} / d_2 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / d_k \mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \\ &= \mathbb{Z} / 1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / 1 \mathbb{Z} \oplus \mathbb{Z} / m_1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \\ &= \mathbb{Z} / m_1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{\exists! s \text{ times}}. \end{aligned}$$

**Uniqueness.** Under the isomorphism  $\mathbb{Z} / m_1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}}$ ,

we see that

$$\mathbb{Z} / m_1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z}$$

corresponds to  $G$ 's [torsion subgroup](#)  $T$ , which implies

$$G / T \cong \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}},$$

which further implies  $G / T$  is a [free Abelian group](#) with

$$\text{rk} \left( G / T \right) = s,$$

which proves the uniqueness of  $s$ .

The proof of the uniqueness of  $m_i$  are long and tedious, we refer to [\[Arm13\]](#). ■

**Definition B.11 (Invariant factor).** We call  $m_1, \dots, m_t$  obtained from [Theorem B.3](#) the *invariant factor*.

**Lemma B.2.** Given a positive integer  $m$  such that

$$m = p_1^{n_1} \cdot \dots \cdot p_s^{n_s}$$

where  $p \in \mathcal{P}$  are all prime and  $p_i \neq p_j$  for  $i \neq j$ , with  $n_i \in \mathbb{Z}^+$  for all  $i$ . Then

$$\mathbb{Z} / m \mathbb{Z} \cong \mathbb{Z} / p_1^{n_1} \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / p_s^{n_s} \mathbb{Z}.$$

*Proof.* We define  $\phi$  as

$$\begin{aligned} \phi: \mathbb{Z} / m \mathbb{Z} &\rightarrow \mathbb{Z} / p_1^{n_1} \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / p_s^{n_s} \mathbb{Z} \\ \bar{n} &\mapsto (n + \langle p_1^{n_1} \rangle, \dots, n + \langle p_s^{n_s} \rangle). \end{aligned}$$

Then  $\bar{n} \in \ker \phi \iff \forall_i p_i^{n_i} \mid n \iff m \mid n \iff \bar{n} = \bar{0}$ . This means  $\ker \phi = 0$ , hence  $\phi$  is an injection.

We now prove  $\phi$  is a surjection. It's sufficient to prove that for all  $i$ ,

$$(0, \dots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \dots, 0) \in \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_s^{n_s}\mathbb{Z},$$

there exists an  $\bar{n}$  such that

$$\phi(\bar{n}) = (0, \dots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \dots, 0).$$

Notice that for all  $i \neq j$ ,  $\langle p_i^{n_i} \rangle + \langle p_j^{n_j} \rangle \in \mathbb{Z}$ , hence there exists  $u_j \in \langle p_i^{n_i} \rangle$  and  $v_j \in \langle p_j^{n_j} \rangle$  such that  $u_j + v_j = 1$ . Let  $n$  as

$$n = \prod_{i \neq j} (1 - u_j),$$

then

$$n + \langle p_i^{n_i} \rangle = 1 + \langle p_i^{n_i} \rangle, \quad n + \langle p_j^{n_j} \rangle = 0 + \langle p_j^{n_j} \rangle.$$

Above implies

$$\phi(\bar{n}) = (0, \dots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \dots, 0),$$

hence  $\phi$  surjects, so

$$\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_s^{n_s}\mathbb{Z}.$$

■

Combine [Theorem B.3](#) and [Lemma B.2](#), we see that we now only have

$$G \cong \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}},$$

we can further decompose  $G$  into

$$G \cong \mathbb{Z}/p_1^{s_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_k^{s_k}\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}},$$

where  $p_1, \dots, p_k$  are primes (which may includes repeated terms),  $s_i \in \mathbb{Z}^+$  for all  $i$ .

**Definition B.12 (Elementary divisors).** The set

$$\{p_1^{s_1}, \dots, p_k^{s_k}\}$$

are called *elementary divisors* of  $G$ .

**Theorem B.4 (Uniqueness of elementary divisors).** [Elementary divisors](#) of a group  $G$  is unique.

*Proof.* Please refer to [\[Arm13\]](#).

■

## C Homological Algebra

**As previously seen.** Given two [Abelian groups](#)  $A, B$  and the group homomorphism  $\varphi: A \rightarrow B$ , then we have

- $\ker \varphi = \{x \in A \mid \varphi(x) = 0\}$
- $\operatorname{Im} \varphi = \{\varphi(x) \mid x \in A\}$
- $\operatorname{coker} \varphi := B / \operatorname{Im} \varphi$
- $\operatorname{coIm} \varphi := A / \ker \varphi$

Consider a sequence of [Abelian](#) group homomorphism

$$\dots \longrightarrow A_{i-1} \xrightarrow{\phi_{i-1}} A_i \xrightarrow{\phi_i} A_{i+1} \longrightarrow \dots$$

We denote this sequence as  $S$ .

**Definition C.1 (Exact).** We say  $S$  is *exact* at  $A_i$  if

$$\operatorname{Im} \phi_{i-1} = \ker \phi_i.$$

**Definition C.2 (Exact sequence).** We call  $S$  is an *exact sequence* if it's exact at  $A_i$  for all  $i$ .

**Remark.** Specifically, consider the following two situations.

- We say

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots$$

is an [exact sequence](#) if it's exact at  $A_i$  for all  $i \geq 1$ .

- We say

$$\dots \longrightarrow A_{-2} \longrightarrow A_{-1} \longrightarrow A_0$$

is an [exact sequence](#) if it's exact at  $A_i$  for all  $i \leq -1$ .

**Remark.** Denote  $\circ$  as a trivial [Abelian group](#), then

$A \xrightarrow{\phi} B \longrightarrow \circ$  is an [exact sequence](#)  $\iff \phi$  is a surjective homomorphism;

conversely,

$\circ \longrightarrow B \xrightarrow{\phi} A$  is an [exact sequence](#)  $\iff \phi$  is an injective homomorphism.

**Definition C.3 (Short exact sequence).** A *short exact sequence* is an [exact sequence](#) such that it has the following form

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ.$$



---

**Remark.** Let  $B \xrightarrow{\psi} C$  as a surjective homomorphism and  $K = \ker \psi$ , and we denote  $K \xrightarrow{i} B$  as an injection. Then

$$\circ \longrightarrow K \xrightarrow{i} B \xrightarrow{\psi} C \longrightarrow \circ$$

is a [short exact sequence](#). Conversely, if

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

is a [short exact sequence](#), then  $\phi$  is an injective homomorphism since it is exact at  $A$ , and  $\psi$  is a surjective homomorphism since it is exact at  $C$ , and  $\phi(A) = \ker \psi$  since it is exact at  $B$ . This implies  $\phi: A \rightarrow \phi(A) = \ker \psi$  is a group homeomorphism.

**Example.** We see some examples.

1. Given  $A, B$  as [Abelian groups](#), then

$$\circ \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\text{Proj}_2} B \longrightarrow \circ$$

$$a \xrightarrow{i} (a, 0)$$

$$(a, b) \xrightarrow{\text{Proj}_2} b$$

is a [short exact sequence](#).

2. We see that

$$\circ \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\text{Proj}_2} \mathbb{Z}/n\mathbb{Z} \longrightarrow \circ$$

$$k \longmapsto k \cdot n$$

for  $n \in \mathbb{Z}_{\geq 1}$  is a [short exact sequence](#).

**Definition C.4 (Isomorphism between sequences).** Given  $A_\bullet$  and  $B_\bullet$  defined as two sequences of [Abelian group](#) homomorphisms

$$A_\bullet : \dots \longrightarrow A_i \xrightarrow{\phi_i} A_{i+1} \longrightarrow \dots$$

and

$$B_\bullet : \dots \longrightarrow B_i \xrightarrow{\psi_i} B_{i+1} \longrightarrow \dots$$

And we say a morphism  $\alpha$  from  $A_\bullet$  to  $B_\bullet$  is a series of group homomorphisms  $\alpha_i : A_i \rightarrow B_i$  for all  $i \in \mathbb{Z}$  such that the following diagram commutes.

$$\begin{array}{ccccccc} \dots & \longrightarrow & A_i & \xrightarrow{\phi_i} & A_{i+1} & \longrightarrow & \dots \\ & & \downarrow \alpha_i & & \downarrow \alpha_{i+1} & & \\ \dots & \longrightarrow & B_i & \xrightarrow{\psi_i} & B_{i+1} & \longrightarrow & \dots \end{array}$$

Additionally, if for all  $i$ ,  $\alpha_i$  is a group homeomorphism, then we say  $\alpha : A_\bullet \rightarrow B_\bullet$  is a homeomorphism.

**Definition C.5 (Split short exact sequence).** Given a [short exact sequence](#)

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

we say it is *split* if there exists a group homeomorphism  $\theta : B \rightarrow A \oplus C$  such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus C & \longrightarrow & C \longrightarrow 0 \end{array}$$

is the [isomorphism](#) between these two [short exact sequences](#).

**Remark.** Given [split short exact sequence](#)

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

and  $\theta$  defined in [Definition C.5](#), let  $i : A \rightarrow A \oplus C$ ,  $a \mapsto (a, 0)$  and  $j : C \rightarrow A \oplus C$ ,  $c \mapsto (0, c)$  are two canonical embeddings, then we have

$$A \oplus C = i(A) \oplus j(C).$$

Consider  $\theta^{-1} : A \oplus C \xrightarrow{\cong} B$ , then

$$B = \theta^{-1}(i(A)) \oplus \theta^{-1}(j(C)).$$

Since the diagram in [Definition C.5](#) commutes, hence

$$\theta^{-1}(i(A)) = \theta^{-1} \circ i(A) = \phi(A),$$

hence

$$B = \phi(A) \oplus \underbrace{\theta^{-1}(j(C))}_D,$$

which implies  $\psi|_D : D \rightarrow C$  is a group homeomorphism. We see that

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

[split](#) implies  $B = \phi(A) \oplus D$  and  $\psi|_D : D \xrightarrow{\cong} C$ .

Conversely, if  $B = \phi(A) \oplus D$  and  $\psi|_D : D \xrightarrow{\cong} C$ , then there exists a  $\theta$

$$\begin{aligned} \theta : B &\rightarrow A \oplus C \\ \phi(a) + d &\mapsto (a, \psi(d)) \end{aligned}$$

for  $a \in A, d \in D$  such that

$$\begin{array}{ccccccc} \circ & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow \circ \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ \circ & \longrightarrow & A & \longrightarrow & A \oplus C & \longrightarrow & C \longrightarrow \circ \end{array}$$

$$\begin{array}{ccc} \phi(a) + d & \longmapsto & \psi(d) \\ \downarrow & & \downarrow \\ (a, \psi(d)) & \longmapsto & \psi(d) \end{array}$$

commutes.

Hence, for a [short exact sequence](#)  $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$  is [split](#) if and only if  $B = \phi(A) \oplus D$  and  $\psi|_D : D \xrightarrow{\cong} C$ .

Remarkably, let  $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$  is a [split short exact sequence](#), then  $D$  constructed above is not unique. To see this, consider

$$\circ \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\text{Proj}_2} \mathbb{Z} \longrightarrow \circ$$

$$n \longmapsto (n, 0)$$

$$(n, m) \longmapsto m$$

We have  $\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus j(\mathbb{Z})$  where  $j : \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}, n \mapsto (0, n)$ . We see that we can let  $D := j(\mathbb{Z})$ . Meanwhile, we can also let

$$D := \{(n, n) \mid n \in \mathbb{Z}\} < \mathbb{Z} \oplus \mathbb{Z}$$

such that  $\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus D$ .

**Example (Non-split short exact sequence).** We see that

$$\begin{aligned} 0 &\longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\text{Proj}_2} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \\ k &\longmapsto k \cdot n \end{aligned}$$

is not a [split short exact sequence](#), since if it is, then

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z} / n\mathbb{Z} &\cong \mathbb{Z} \\ (0, 1) &\mapsto k, \end{aligned}$$

which is a contradiction since  $\mathbb{Z}$  is [torsion-free](#) while  $\mathbb{Z} \oplus \mathbb{Z} / n\mathbb{Z}$  is not.

**Lemma C.1 (Splitting lemma).** If  $0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$  is a [short exact sequence](#), then the following are equivalent.

1. This [short exact sequence](#) [splits](#).
2.  $\exists p: B \rightarrow A$  such that  $p \circ \phi = \text{id}_A$ .
3.  $\exists q: C \rightarrow B$  such that  $\psi \circ q = \text{id}_C$ .

*Proof.* • 1.  $\implies$  2. Let  $\theta: B \xrightarrow{\cong} A \oplus C$  such that it's the [isomorphism](#) which makes the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \xrightarrow{i} & A \oplus C & \longrightarrow & C \longrightarrow 0 \\ & & & \swarrow \text{Proj}_1 & & & \end{array}$$

Then we let  $p := \text{Proj}_1 \circ \theta$ , then

$$p \circ \phi = \text{Proj}_1 \circ \theta \circ \phi = \text{Proj}_1 \circ i = \text{id}_A.$$

- 1.  $\implies$  3. Let  $\theta: B \xrightarrow{\cong} A \oplus C$  such that it's the [isomorphism](#) which makes the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus C & \xrightarrow{\text{Proj}_2} & C \longrightarrow 0 \\ & & & & & \swarrow j & \end{array}$$

Then we let  $q := \theta^{-1} \circ j$ , then for all  $c \in C$ , we have

$$\psi \circ q(c) = \psi(\theta^{-1}(j(c))) = \text{Proj}_2 \circ \theta(\theta^{-1}(j(c))) = \text{Proj}_2(j(c)) = c,$$

hence  $\psi \circ q = \text{id}_C$ .

- 2.  $\implies$  1. We have

$$\circ \longrightarrow A \xrightleftharpoons[p]{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

where  $p \circ \phi = \text{id}_A$ . We claim that  $B = \phi(A) \oplus \ker(p)$  since for every  $b \in B$ ,  $\phi(p(b)) \in \phi(A)$ , and

$$b = \underbrace{\phi(p(b))}_{\in \phi(A)} + \underbrace{(b - \phi(p(b)))}_{\in \ker(p)}$$

from the fact that

$$p(b - \phi(p(b))) = p(b) - p \circ \phi(p(b)) = p(b) - p(b) = 0.$$

We need to show the uniqueness also. Suppose  $b = \phi(a_1) + d_1 = \phi(a_2) + d_2$ ,  $a_1, a_2 \in A$ ,  $d_1, d_2 \in \ker(p)$ . We see that

$$\phi(a_1 - a_2) = d_2 - d_1 \implies p(\phi(a_1 - a_2)) = 0 \implies a_1 = a_2 \implies d_1 = d_2.$$

Finally, we claim that

$$\psi|_{\ker(p)} : \ker(p) \rightarrow C$$

is a group homeomorphism. But it's obvious that  $\psi|_{\ker(p)}$  are both surjective and injective.

- 3.  $\implies$  1. We have

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightleftharpoons[q]{\psi} C \longrightarrow \circ$$

where  $\psi \circ q = \text{id}_C$ . We claim that  $B = \phi(A) \oplus q(C)$  since for every  $b \in B$ ,

$$b = \underbrace{(b - q(\psi(b)))}_{\in \ker(\psi) = \text{Im}(\phi)} + \underbrace{q(\psi(b))}_{\in q(C)},$$

which implies  $B = \phi(A) + q(C)$ . We can also prove that

$$B = \phi(A) \oplus q(C)$$

similarly. ■

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