# MATH592
# Introduction to Algebraic Topology

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February 3, 2022

**Abstract**

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

## Contents



### Lecture 1: Homotopies of Maps

05 Jan. 10:00

# 1 Foundation of Algebraic Topology

## 1.1 Homotopy

**Definition 1.1 (Homotopy).** Let $X$, $Y$ be topological spaces. Let $f$, $g\colon X \to Y$ continuous maps. Then a *homotopy* from $f$ to $g$ is a $1-$parameter family of maps that continuously deforms $f$ to $g$, i.e., it's a continuous function $F\colon X \times I \to Y$, where $I = [0, 1]$, such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x).$$

We often write $F_t(x)$ for $F(x, t)$.

If a homotopy exists between $f$ and $g$, we say they are *homotopic* and write

$$f \simeq g.$$

If $f$ is homotopic to a constant map, we call it *nullhomotopic*.

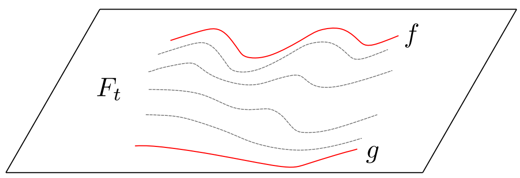



Figure 1: The continuous deforming from $f$ to $g$ described by $F_t$

**Remark.** Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

**Example.** We first see some examples.

1. Any two maps (continuous) with specification

$$f,\ g\colon X \to \mathbb{R}^n$$

   are homotopic by considering

$$F_t(x) = (1-t)f(x) + tg(x).$$

   We call it *the straight line homotopy.*

2. Let $S^1$ denotes the unit circle in $\mathbb{R}^2$, and $D^2$ denotes the unit disk in $\mathbb{R}^2$. Then the inclusion $f\colon S^1 \hookrightarrow D^2$ is nullhomotopic by considering

$$F_t(x) = (1-t)f(x)(+t \cdot 0).$$

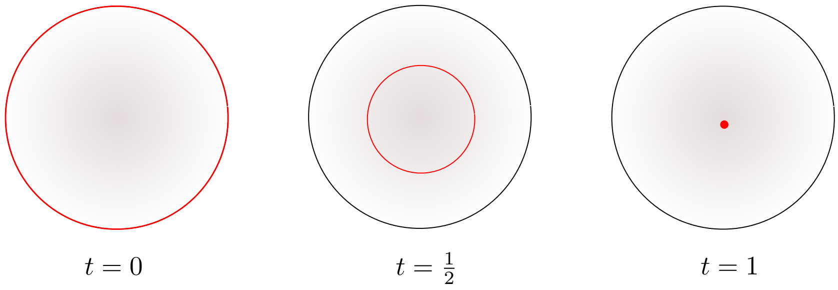



Figure 2: The illustration of $F_t(x)$

We see that there is a homotopy from $f(x)$ to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a nullhomotopy.

3. The maps

$$\begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not** homotopy.

**Remark.** It will essentially **flip** the orientation, hence we can't deform one to another continuously.

**Exercise.** We first see some exercises.

1. A subset $S \subseteq \mathbb{R}^n$ is star-shaped if

$$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from $x_0$ to $x$ lies in $S$.

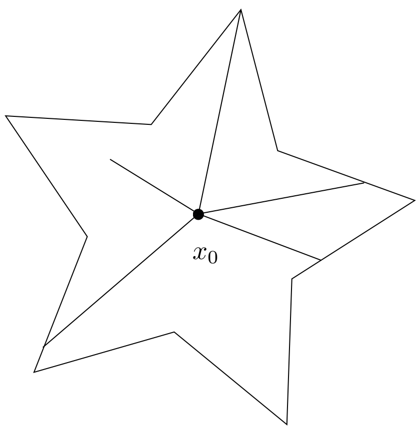



Figure 3: Star-shaped illustration

Show that $\mathrm{id}\colon S \to S$ is nullhomotopic.

**Answer.** Consider

$$F_t(x) \coloneqq (1-t)x + tx_0,$$

which essentially just concentrates all points $x$ to $x_0$.

2. Suppose

$$X \xrightarrow[f_0]{f_1} Y \xrightarrow[g_0]{g_1} Z \ .$$

where

$$f_0 \underset{F_t}{\simeq} f_1, \quad g_0 \underset{G_t}{\simeq} g_1.$$

Show

$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

**Answer.** Consider $I \times X \to Z$. Then

$$\begin{array}{ccccc} X \times I & \to & Y \times I & \to & Z \\ (x,t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

**Remark.** Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** homotopic?

**Answer.**

## Lecture 2: Homotopy Equivalence

07 Jan. 10:00

**As previously seen.** Two maps $f, g\colon X \to Y$ is homotopy if there exists a map

$$F_t(x)\colon X \times I \to Y$$

with the properties

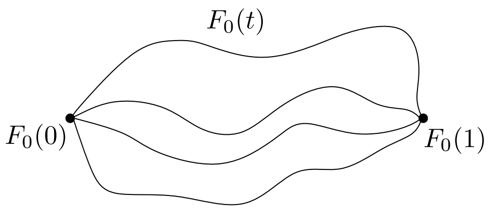
1. Continuous
2. $F_0(x) = f(x)$
3. $F_1(x) = g(x)$

**Remark.** The continuity of $F_t$ is an even stronger condition for the continuity of $F_t$ for a fixed $t$.

We now introduce another concept.

**Definition 1.2 (Homotopy relative).** Given two spaces $X, Y$, and let $B \subseteq X$. Then a homotopy $F_t(x)\colon X \to Y$ is called *homotopy relative* $B$ (denotes $\mathrm{rel}B$) if $F_t(b)$ is independent of $t$ for all $b$.

**Example.** Let $X = [0,1]$ and $B = \{0,1\}$. Then the homotopy of paths from $[0,1] \to X$ is $\text{rel}\{0,1\}$.




## 1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

**Definition 1.3 (Homotopy Equivalence).** A map $f\colon X \to Y$ is a *homotopy equivalence* if $\exists g\colon Y \to X$ such that

$$f \circ g \simeq \mathrm{id}_Y, \quad g \circ f \simeq \mathrm{id}_X.$$

We say that $X, Y$ are *homotopy equivalent*, and $g$ is called *homotopy inverse* of $f$.

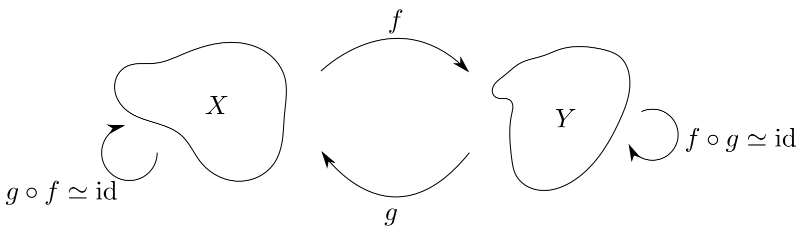
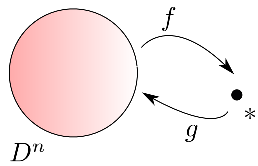



Figure 4: Homotopy Equivalence

If $X, Y$ are called *homotopy equivalent*, then we say that they have the same *homotopy type*.

**Notation.** We denote a closed $n$-disk as $D^n$.

**Example.** $D^n$ is homotopy equivalent to a point.




We see that $f \circ g = \mathrm{id}_*$ and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is homotopic to $\mathrm{id}_{D^n}$ by straight-line homotopy $F_t(x) = tx$.

**Note.** We say that a space is *contractible* if $H$ is homotopy equivalent to a point.

Before doing exercises, we introduce two new concepts.

**Definition 1.4 (Retraction).** Given $B \subseteq X$, a *retraction* from $X$ to $B$ is a map $f : X \to X$ (or $X \to B$) such that $\forall b \in B$ $f(b) = b$, namely $r|_B = \mathrm{id}_B$. Or one can see this from

$$\begin{tikzcd} B \arrow[r, hook, "i"] \arrow[rr, bend right, "r \circ i"'] & X \arrow[r, "r"] & B \end{tikzcd}$$

where $r$ is a retraction if and only if $r \circ i = \mathrm{id}_B$, where $i$ is an inclusion identity. If $r$ exists, $B$ is a retract of $X$.

**Definition 1.5 (Deformation retraction).** Given $X$ and $B \subseteq X$, a *(strong) deformation retraction* $F_t : X \to X$ onto $B$ is a homotopy rel$B$ from the $\mathrm{id}_X$ to a *retraction* from $X$ to $B$. i.e.,

$$\begin{aligned} F_0(x) &= x \quad && \forall x \in X \\ F_1(x) &\in B \quad && \forall x \in X \\ F_t(b) &= b \quad && \forall t \ \forall b \in B. \end{aligned}$$

**Exercise.** We now see some problems.

1. Let $X \simeq Y$. Show $X$ is path-connected if and only if $Y$ is.

   **Answer.** Suppose $X$ is path-connected. Then we see that given two points $x_1$ and $x_2$ in $X$, there exists a path $\gamma(t)$ with

   $$\gamma : [0,1] \to X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Since $X \simeq Y$, then there exists a pair of $f$ and $g$ such that $f\colon X \to Y$ and $g\colon Y \to X$ with

$$f \circ g \underset{F}{\simeq} \mathrm{id}_Y, \quad g \circ f \underset{G}{\simeq} \mathrm{id}_X.$$

(Notice the abuse of notation)

For any two $y_1$ and $y_2 \in Y$, we want to construct a path $\gamma'(t)$ such that

$$\gamma'\colon [0,1] \to Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

Firstly, we let $g(y_1) =: x_1$ and $g(y_2) =: x_2$. From the argument above, we know there exists such a $\gamma$ starting at $x_1 = g(y_1)$ ending at $x_2 = g(y_2)$. Now, consider $f(\gamma(t)) = (f \circ \gamma)(t)$ such that

$$f \circ \gamma\colon I \to Y, \quad f \circ \gamma(0) = y_1', \quad f \circ \gamma(1) = y_2',$$

we immediately see that $y_1'$ and $y_2'$ is path connected. Now, we claim that $y_1$ and $y_1'$ are path connected in $Y$, hence so are $y_2$ and $y_2'$. To see this, note that

$$f \circ g \underset{F}{\simeq} \mathrm{id}_Y,$$

which means that there exists $F\colon Y \times I \to Y$ such that

$$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y_1' \\ F(y_1, 1) = \mathrm{id}_Y(y_1) = y_1. \end{cases}$$

Since $F$ is continuous in $I$, we see that there must exist a path connects $y_1$ and $y_1'$. The same argument applies to $y_2$ and $y_2'$. Now, we see that the path

$$y_1 \to y_1' \to y_2' \to y_2$$

is a path in $Y$ for any two $y_1$ and $y_2$, which shows $Y$ is path-connected.

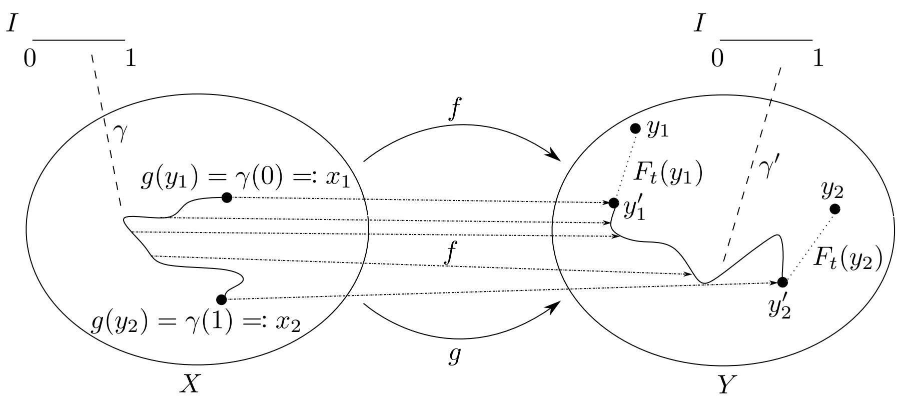



Figure 5: Demonstration of the proof

**Challenge**: One can further show that the connectedness is also preserved by any homotopy equivalence.

2. Show that if there exists deformation retraction from $X$ to $B \subseteq X$, then $X \simeq B$.

## Lecture 3: Deformation Retraction

10 Jan. 10:00

**As previously seen.** A *deformation retraction* is a homotopy of maps rel$B$ $X \to X$ from $\mathrm{id}_X$ to a retraction from $X$ to $B$. Then $B$ is a *deformation retract*.

**Example.** We can also show

1. $S^1$ is a deformation retraction of $D^2 \setminus \{0\}$. Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$

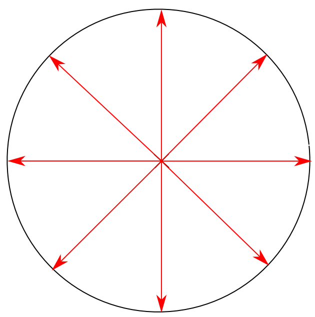


Figure 6: The deformation retraction of $D^{2\setminus\{0\}}$ is just to *enlarge* that hold and push all the interior of $D^2$ to the boundary, which is $S^1$

2. $\mathbb{R}^n$ deformation retracts to 0. Indeed, since

$$F_t(x). = (1-t)x.$$

This implies that $\mathbb{R}^n \simeq *$, hence we see that

- dimension
- compactness
- etc.

are <u>not</u> homotopy invariants.

3. $S^1$ is a deformation retract of a cylinder and a Möbius band.

For a cylinder, consider $X \times I \to X$. Define homotopy on a closed rectangle, then verify it induces map on quotient.

For a Möbius band, we define a homotopy on a closed rectangle, then verify that it respect the equivalence relation.

Finally, we use the universal property of quotient topology to argue that we get a homotopy on Möbius band.

**Upshot**: Möbius band $\simeq S^1 \simeq$ cylinder, hence the orientability is <u>not</u> homotopy invariant.

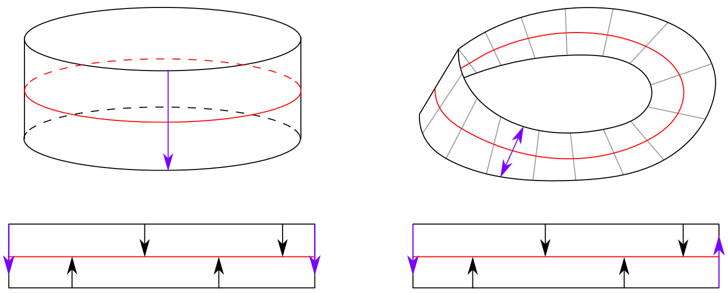


Figure 7: The deformation retraction for Cylinder and Möbius band

## Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

**As previously seen.** We saw that

- homotopy equivalence
- homotopy invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

### 1.3 CW Complexes

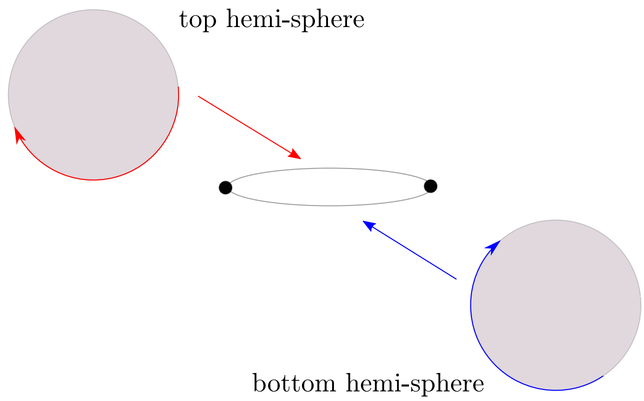
**Example.** Let's start with a few examples.

1. Constructing spheres:
   - $S^1$ (up to homeomorphism)

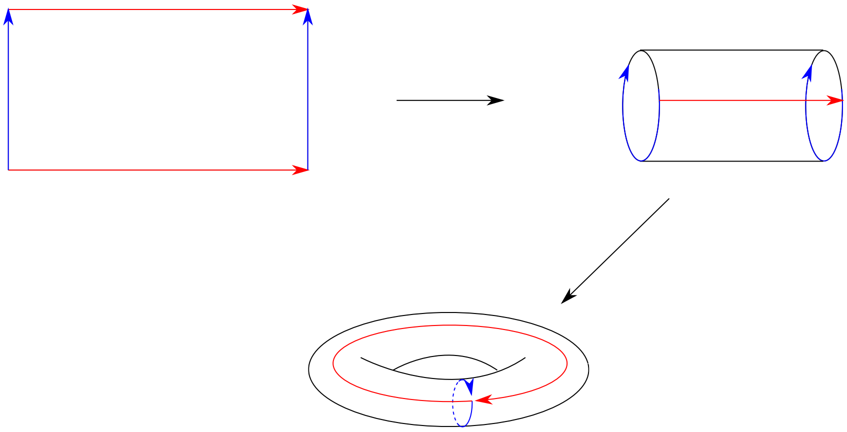
- $S^2$
  - glue boundary of 2-disk to a point
  - glue 2 disks onto a circle

Figure 8: **Left**: Glue a 2-disk to a point along its boundary. **Right**: Glue 2 disks to $S^1$.

The gluing instruction to construct $S^2$ in the right-hand side can be demonstrated as follows.



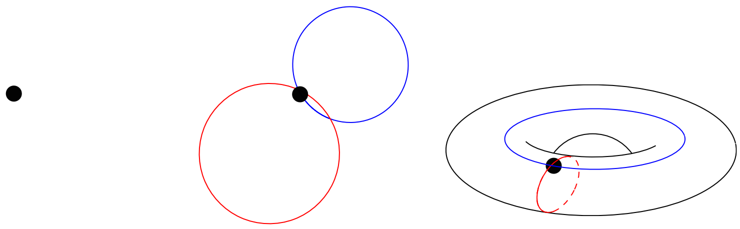

- $T = S^1 \times S^1$



view as gluing instructions

$$\text{vertex } + 2 \text{ edges } + 2\text{-disks.}$$

Specifically, we have



0-skeleton 1-skeleton 2-skeleton

Glue in edges Glue in disk

---

Formally, we have the following definition.

**Notation.** Let $D^n$ denotes a closed n-disk (or n-ball)

$$D^n \simeq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

And let $S^n$ denotes an $n$-sphere

$$S^n \simeq \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\}.$$

Lastly, we call a point as a *0-cell*, and the interior of $D^n$ $\mathrm{int}(D^n)$ for $n \geq 1$ as a *$n$-cell*.

**Definition 1.6 (CW Complex).** A *CW Complex* is a topological space constructed inductively as

1. $X^0$ (the 0-skeleton) is a set of discrete points.
2. We inductively construct the n-skeleton $X^n$ from $X^{n-1}$ by attaching $n$-cells $e^n_\alpha$, where $\alpha$ is the index.

   The gluing instructions glued by an attaching map is that $\forall\alpha$, $\exists$ continuous map $\varphi_\alpha$
   $$\varphi_\alpha\colon \partial D^n_\alpha \to X^{n-1},$$
   then
   $$X^n = \left(X^{n-1} \coprod_\alpha D^n_\alpha\right) \Big/ x \sim \varphi_\alpha(x)$$
   with identification $x \sim \varphi_\alpha(x)$ for all $x \in \partial D^n_\alpha$ with quotient topology.
3. $$X = \bigcup_{n=0} X^n,$$
   and let $\overline{w}$ denotes weak topology. Then
   $$u \subseteq X \text{ is open } \iff \forall n\ u \cap X^n \text{ is open }.$$
   If all cells have dimension less than $N$ and a $\exists N$-cell, then $X = X^N$ and we call it $N$-dim CW complex.

**Remark.** We write $X^{(n)}$ for $n$-skeleton if we need to distinguish from the Cartesian product.

**Example.** Let's look at some examples.

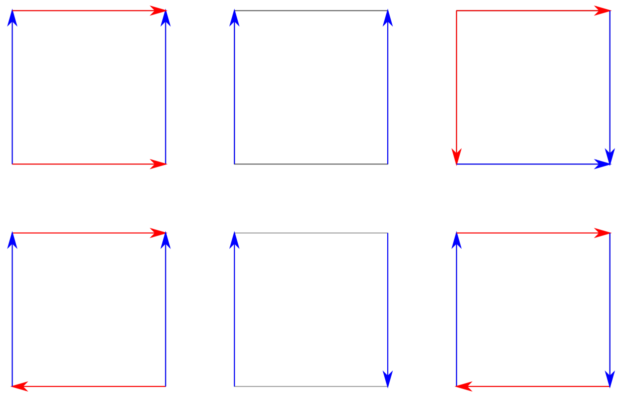
1. 0-dim CW complex is a discrete space.
2. 1-dim CW complex is a graph.
3. A CW complex $X$ is finite if it has finitely many cells.

**Definition 1.7 (CW subcomplex).** A *CW subcomplex* $A \subseteq X$ is a closed subset equal to a union of cells
$$e^n_\alpha = \operatorname{int}\left(D^n_\alpha\right).$$

**Remark.** This inherits a CW complex structure.

Check the images of attaching maps.

**Exercise.** Given the following gluing instruction:



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere, $\mathbb{R}P$.

**Answer.** We see that

| | | |
|---|---|---|
| 1. Torus | 2. Cylinder | 3. 2-sphere |
| 4. Klein bottle | 5. Möbius band | 6. $\mathbb{R}P$ |

**Notation.** We call the real projection space as $\mathbb{R}P$, and we also have so-called complex projection space, denote as $\mathbb{C}P$.

## Lecture 5: Operation on Spaces

14 Jan. 10:00

## 1.4 Operations on CW Complexes

### 1.4.1 Products

We can consider the product of two CW complexes given by a CW complex structure. Namely, given $X$ and $Y$ two CW complexes, we can take two cells $e^n_\alpha$ from $X$ and $e^m_\beta$ from $Y$ and form the product space $e^n_\alpha \times e^m_\beta$, which is homeomorphic to an $n+m$-cell. We then take these products as the cells for $X \times Y$.

Specifically, given $X$, $Y$ are CW complexes, then $X \times Y$ has a cell structure

$$\{e^m_\alpha \times e^n_\alpha \colon e^m_\alpha \text{ is a } m\text{-cell on } X, e^n_\alpha \text{ is an } n\text{-cell on } Y\}.$$

**Remark.** The product topology may not agree with the weak topology on the $X \times Y$. However, they do agree if $X$ or $Y$ is locally compact <u>or</u> if $X$ and $Y$ both have at most countably many cells.

**Note.** Notice that if the product is wild enough, then the product topology may not agree with the weak topology.

### 1.4.2 Wedge Sum

Given $X$, $Y$ are CW complexes, and $x_0 \in X^0$, $y_0 \in Y^0$ (only points). Then we define

$$X \vee Y = X \coprod Y$$

with quotient topology.

**Remark.** $X \vee Y$ is a CW complex.

### 1.4.3 Quotients

Let $X$ be a CW complex, and $A \subseteq X$ subcomplex (closed union of cells), then

$$X \big/ A$$

is a quotient space collapse $A$ to one point and inherits a CW complex structure.

**Remark.** $X \,/\, A$ is a CW complex.

0-skeleton

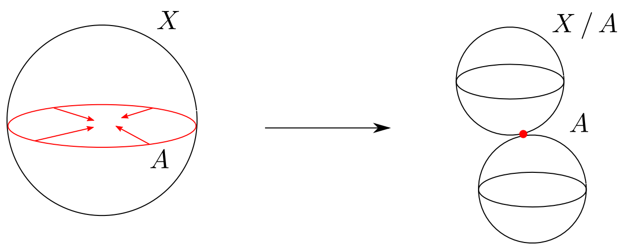
$$(X^0 - A^0) \coprod *$$

where $*$ is a point for $A$. Each cell of $X - A$ is attached to $(X \,/\, A)^n$ by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n \,/\, A^n$$

**Example.** Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a wedge of two spheres.




2. We can take the torus and squish down a ring around the hole.

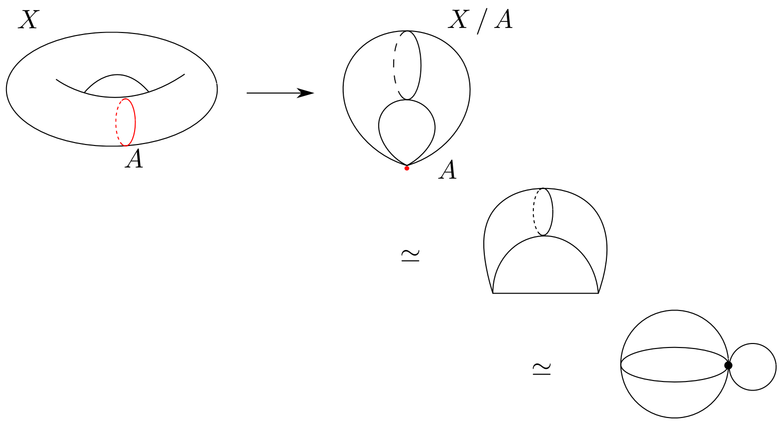



Figure 9: We see that $X \,/\, A$ is homotopy equivalent to a 2-sphere wedged with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other point, forming a circle.

## Lecture 6: A Foray into Category Theory

19 Jan. 10:00

### 1.5 Category Theory

We start with a definition.

**Definition 1.8 (Object, Morphism).** A category $\mathscr{C}$ is 3 pieces of data

- A class of objects $\mathrm{Ob}(\mathscr{C})$
- $\forall X, Y \in \mathrm{Ob}(\mathscr{C})$ a class of morphisms or arrows, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- $\forall X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, there exists a composition law

$$\begin{aligned}\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) &\to \mathrm{Hom}(X, Z)\\ (f, g) &\mapsto g \circ f\end{aligned}$$

and 2 axioms

- Associativity. $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms $f, g, h$ where composites are defined.
- Identity. $\forall X \in \mathrm{Ob}(\mathscr{C})$ $\exists \mathrm{id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g$$

for all $f, g$ where this makes sense.

Let's see some examples.

**Example.** We introduce some common category.

| $\mathcal{C}$ | $\mathrm{Ob}(\mathcal{C})$ | $\mathrm{Mor}(\mathcal{C})$ |
|---|---|---|
| set | Sets $X$ | All maps of sets |
| fset | Finite sets | All maps |
| Gp | Groups | Group Homomorphisms |
| Ab | Abelian groups | Group Homomorphisms |
| $k-$vect | Vector spaces over $k$ | $k$-linear maps |
| Rng | Rings | Ring Homomorphisms |
| Top | Topological spaces | Continuous maps |
| Haus | Hausdorff Spaces | Continuous maps |
| hTop | Topological spaces | Homotopy classes of continuous maps |
| $\underline{\mathrm{Top}^*}$ | Based topological spaces[1] | Based maps[2] |

**Remark.** Any **diagram** plus composition law.

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[r] & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd} .$$

**Definition 1.9 (monic, epic).** A morphism $f\colon M \to N$ is *monic* if

$$\forall g_1, g_2\ f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$\begin{tikzcd} A \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

Dually, $f$ is *epic* if

$$\forall g_1, g_2\ g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & B \end{tikzcd}$$

**Lemma 1.1.** In set, Ab, Top, Gp, a map is monic if and only if $f$ is injective, and epic if and only if $f$ is surjective.

*Proof.* In set, we prove that $f$ is monic if and only if $f$ is injective. Suppose $f \circ g_1 = f \circ g_2$ and $f$ is injective, then for any $a$,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence $g_1 = g_2$.

[1]Topological spaces with a distinguished base point $x_0 \in X$
[2]Continuous maps that presence base point $f\colon (x, x_0) \to (y, y_0)$ such that

$$f\colon X \to Y, \quad f(x_0) = y_0$$

is continuous.

Now we prove another direction, with contrapositive. Namely, we assume that $f$ is not injective and show that $f$ is not monic. Suppose $f(a) = f(b)$ and $a \neq b$, we want to show such $g_i$ exists. This is easy by considering

$$g_1 : * \mapsto a, \quad g_2 : * \mapsto b.$$

■

### 1.5.1 Functor

After introducing the category, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

**Definition 1.10 (Functor).** Given $\mathscr{C}, \mathscr{D}$ be two categories. A (covariant) *functor*

$$F : \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F \colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)) \\ [f \colon X \to Y] &\mapsto [F(f) \colon F(X) \to F(Y)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(f) \circ F(g)$

## Lecture 7: Functors

21 Jan. 10:00

**As previously seen.** Assume that we initially have a commutative diagram in $\mathscr{C}$ as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

After applying $F$, we'll have

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)=F(g)\circ F(f)"'] & F(Y) \arrow[d, "F(g)"] \\ & F(Z) \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

We can also have a so-called contravariant functor.

**Definition 1.11 (Contravariant functor).** Given $\mathscr{C}, \mathscr{D}$ be two categories. A contravariant *functor*

$$F\colon \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F\colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X)) \\ [f\colon X \to Y] &\mapsto [F(f)\colon F(Y) \to F(X)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a contravariant functor $F$, the diagram becomes

$$\begin{tikzcd} F(X) & F(Y) \arrow[l, "F(f)"'] \\ & F(Z) \arrow[u, "F(g)"'] \arrow[ul, "F(g\circ f)=F(f)\circ F(g)"] \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

**Example.** Let see some examples.

1. Identity functor.

$$I\colon \mathscr{C} \to \mathscr{C}.$$

2. Forgetful functors.

   -

$$\begin{aligned} F\colon \underline{\mathrm{Gp}} &\to \underline{\mathrm{set}} \\ G &\mapsto G\,\text{[3]} \\ [f\colon G \to H] &\mapsto [f\colon G \to H] \end{aligned}$$

   -

$$\begin{aligned} F\colon \underline{\mathrm{Top}} &\to \underline{\mathrm{set}} \\ X &\mapsto X\,\text{[4]} \\ [f\colon X \to Y] &\mapsto [f\colon X \to Y] \end{aligned}$$

[3] $G$ is now just the underlying set of the group $G$.

3. Free functors.

$$\begin{aligned} \underline{\text{set}} &\to \underline{k\text{-vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

i.e., vector space with basis $s$

$$[f\colon A \to B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

4. 

$$\begin{aligned} \underline{k\text{-vect}} &\to \underline{k\text{-vect}} \\ V &\mapsto V^* = \mathrm{Hom}_k(V, k) \end{aligned}$$

If we are working on a basis, then we have

$$A \mapsto A^T.$$

---

Specifically, we care about two functors.

1. 

$$\begin{aligned} \underline{\text{Top}^*} &\to \underline{\text{Gp}} \\ (X, x_0) &\mapsto \Pi_1(X, x_0) \end{aligned}$$

where $\Pi_1$ is so-called *fundamental group.*

2. 

$$\begin{aligned} \underline{\text{Top}} &\to \underline{\text{Ab}} \\ X &\mapsto \text{Hp}(X) \end{aligned}$$

where Hp is so-called $p^{th}$ *homology.*

Let see the formal definition.

## 1.6 Free Groups

**Definition 1.12 (Free group).** Given a set $S$, the *free group* is a group $F_S$ on $S$ with a map $S \to F_S$ satisfying the universal property.

If $G$ is any group, $f\colon S \to G$ is any map of sets, $f$ extends uniquely to group homomorphism $\overline{f}\colon F_S \to G$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists! \overline{f}\colon \text{gp hom}"] \\ & G \end{tikzcd}$$

[4] $X$ is now just the underlying set of the topological space $X$.

**Note.** This defines a *natural bijection*

$$\mathrm{Hom}_{\underline{\text{set}}}(S, \mathscr{U}(G)) \cong \mathrm{Hom}_{\underline{\text{Gp}}}(F_S, G),$$

where $\mathscr{U}(G)$ is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are adjoint; specifically that the free functor is the left adjoint (appears on the left in the Hom's above).

**Definition 1.13 (Adjoints functor).** A free and forgetful functors are *adjoints.*

**Remark.** Whenever we state a universal property for an object (plus a map), an object (plus a map) may or may not exist. If such object exists, then it defines the object **uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the object (plus a map).

**Lemma 1.2.** Universal property defines $F_S$ (plus a map $S \to F(S)$ ) uniquely up to unique isomorphism.

*Proof.* Fix $S$. Suppose

$$S \to F_S, \quad S \to \widetilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, dashed, "\exists!\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

We'll show $\varphi$ and $\psi$ are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{tikzcd} & F_S \arrow[dd, "\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[dd, "\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & \widetilde{F}_S \end{tikzcd}$$

Hence, we see that

$$\begin{tikzcd} & F_S \arrow[d, "\psi"] \arrow[dd, bend left, "\varphi\circ\psi=\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, "\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[d, "\varphi"] \arrow[dd, bend left, "\psi\circ\varphi=\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, "\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \mathrm{id}_{F_S}, \qquad \psi \circ \varphi = \mathrm{id}_{\widetilde{F}_S},$$

so $\varphi$ and $\psi$ are inverses (thus group isomorphism). ∎

## Lecture 8: The Fundamental Group $\pi_1$

24 Jan. 10:00

**Example.** In category <u>Ab</u> free Abelian group on a set $S$ is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on $S$** .

### 1.6.1 Constructing the Free Groups $F_S$

**Proposition 1.1.** The free group defined by the universal property exists.

*Proof.* We'll just give a construction below. First, we see the definition.

**Definition 1.14.** Fix a set $S$, and we define a <u>word</u> as a finite sequence (possibly $\varnothing$) in the formal symbols

$$\left\{s, s^{-1} \mid s \in S\right\}.$$

Then we see that elements in $F_S$ are equivalence classes of words with the equivalence relation being

- delete $ss^{-1}$ or $s^{-1}s$. i.e.,

  $$\begin{aligned} vs^{-1}sw &\sim vw \\ vss^{-1}w &\sim vw \end{aligned}$$

  for every word $v, w, s \in S$,

with the group operation being concatenation. ∎

**Example.** Given words $ab^{-1}, bba$, their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

**Exercise.** There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that $F_S$ satisfies the universal property with respect to the map

   $$S \to F_S, \quad s \mapsto s.$$

# 2 The Fundamental Group

## 2.1 Definition

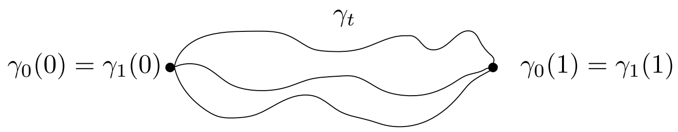
We start with the definition.

**Definition 2.1 (Path).** A *path* in a space $X$ is a continuous map

$$\gamma\colon I \to X$$

where $I = [0,1]$.

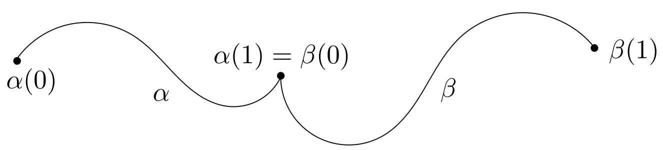
**Definition 2.2 (Homotopy path).** A *homotopy of paths* $\gamma_0$, $\gamma_1$ is a homotopy from $\gamma_0$ to $\gamma_1$ rel$\{0,1\}$.




**Example.** Fix $x_1, x_0 \in X$, then $\exists$ homotopy of paths is an equivalence relation on paths from $x_0$ to $x_1$ (i.e., $\gamma$ with $\gamma(0) = x_0, \gamma(1) = x_1$).

**Definition 2.3 (Path composition).** For paths $\alpha, \beta$ in $X$ with $\alpha(1) = \beta(0)$, the *composition*[a] $\alpha \cdot \beta$ is

$$(\alpha \cdot \beta)(t) \coloneqq \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \dfrac{1}{2}\right] \\ \beta(2t-1), & \text{if } t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$




[a] Also named *product*, *concatenation*.

**Remark.** By the pasting lemma, this is continuous, hence $\alpha \cdot \beta$ is actually a path from $\alpha(0)$ to $\beta(1)$.

**Definition 2.4 (Reparameterization).** Let $\gamma \colon I \to X$ be a path, then a *reparameterization* of $\gamma$ is a path

$$\gamma' \colon I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where $\varphi$ is continuous and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

**Exercise.** A path $\gamma$ is homotopic rel$\{0,1\}$ to all of its reparameterizations.

*Proof.* We show that $\gamma$ and $\gamma \circ \phi$ are homotopic rel$\{0,1\}$ by showing that there exists a continuous $F_t$ such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since $\phi$ is continuous, so we define

$$F_t(x) = (1-t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all $x, t \in I$.

Now, we check that $F_t$ really gives us a homotopic rel$\{0,1\}$. We have

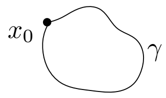
$$\begin{aligned} F_t(0) &= (1-t)\gamma(0) + t \cdot \gamma \circ \phi(0) = (1-t)\gamma(0) + t \cdot \gamma(\underbrace{\phi(0)}_{0}) = \gamma(0), \\ F_t(1) &= (1-t)\gamma(1) + t \cdot \gamma \circ \phi(1) = (1-t)\gamma(1) + t \cdot \gamma(\underbrace{\phi(1)}_{1}) = \gamma(1), \end{aligned}$$

which shows that 0 and 1 are independent of $t$, hence $\gamma$ and $\gamma \circ \phi$ are homotopic rel$\{0,1\}$. ∎

**Exercise.** Fix $x_1, x_1 \in X$. Then Homotopy of paths (relative $\{0,1\}$) is an equivalence relation on paths from $x_0$ to $x_1$.

**Definition 2.5 (Fundamental Group).** Let $X$ denotes the space and let $x_0 \in X$ be the base point. The *fundamental group of $X$ based at $x_0$*, denoted by $\pi_1(X, x_0)$, is a group such that

- Elements: Homotopy classes rel$\{0,1\}$ of paths $[\gamma]$ where $\gamma$ is a **loop** with $\gamma(0) = \gamma(1) = x_0$[a]

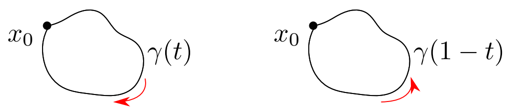



- Operation: Composition of paths.
- Identity: Constant loop $\gamma$ based at $x_0$ such that

$$\gamma \colon I \to X, \quad t \mapsto x_0$$

- Inverses: The inverse $[\gamma]^{-1}$ of $[\gamma]$ is represented by the loop $\overline{\gamma}$ such that

$$\overline{\gamma}(t) = \gamma(1-t).$$




[a] We say $\gamma$ is **based** at $x_0$.

*Proof.* We prove that

- Associativity: $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$. We break this down into

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t-2), & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t-3), & \text{if } t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t-1), & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, we define $\phi\colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \frac{1}{2}\right], & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ t + \frac{1}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right], & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{t+1}{2} \in \left[\frac{3}{4}, 1\right], & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$, which implies that the associativity holds.

- Identity: We want to show that $[\gamma \cdot c] = [\gamma]$. Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, consider $\phi\colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t, & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ 1, & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$.

- Inverses: We want to show that $\gamma \cdot \overline{\gamma} \simeq c$, where $\overline{\gamma}(t) = \gamma(1-t)$. Firstly, we have

$$(\gamma \cdot \overline{\gamma})(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2t), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider $F_t$ given by

$$F_t(x) = \begin{cases} \gamma(2xt), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2xt), & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

If $t = 0$, we have

$$F_0(x) = \begin{cases} \gamma(0), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0$$

for all $x \in I$, namely $F_0 = c$, while when $t = 1$, we have

$$F_1(x) = \begin{cases} \gamma(2x), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2x), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \overline{\gamma})(x),$$

and we see that $F_t$ is continuous since at $x = \frac{1}{2}$, we have

$$\gamma(2x) = \gamma(1) = \overline{\gamma}(0) = \overline{\gamma}(1-2x),$$

hence we see that $F_t$ is the homotopy between $\gamma \cdot \overline{\gamma}$ and $c$.

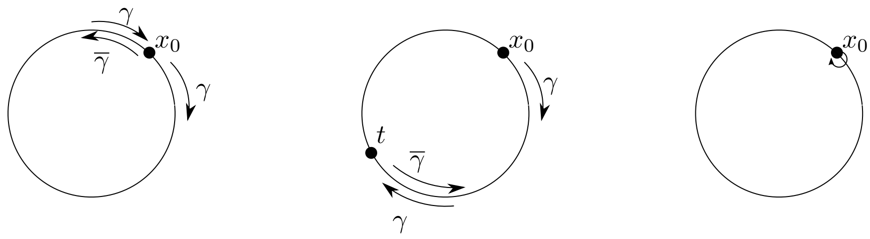



Figure 10: Illustration of $F_t$. Intuitively, the path $\gamma \cdot \overline{\gamma}$ is $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\overline{\gamma}} x_0$. But now, $F_t$ is $x_0 \xrightarrow{\gamma} t \xrightarrow{\overline{\gamma}} x_0$. We can think of this homotopy is *pulling back* the turning point along the original path.

■

**Theorem 2.1.** If $X$ is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

**Remark.** We often write $\pi_1(X)$ up to isomorphism.

*Proof.* To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if $[h \cdot \gamma \cdot \overline{h}] = [h \cdot \gamma' \cdot \overline{h}]$, then since we know that $h^{-1} = \overline{h}$, hence in the fundamental group $\pi_1(X, x_0)$, we see that

$$\overline{h} \cdot h \cdot \gamma \cdot \overline{h} \cdot h = \overline{h} \cdot h \cdot \gamma' \cdot \overline{h} \cdot h. \implies \gamma = \gamma'$$

  as we desired.

- onto. We see that for every $\alpha \in \pi_1(X, x_0)$, there exists a $\gamma \in \pi_1(X, x_0)$ such that

$$\gamma = \overline{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)^{5}$$

  since $h \cdot \gamma \cdot \overline{h} = \alpha$.

[5] Notice that this is indeed the case, one can verify this by the fact that $h\colon x_0 \to x_1$ and $\overline{h}\colon x_1 \to x_0$.

We then see that the fundamental group of $X$ does not depend on the choice of basepoint, only on the choice of the path component of the basepoint. If $X$ is path-connected, it now makes sense to refer to *the* fundamental group of $X$ and write $\pi_1(X)$ for the abstract group (up to isomorphism).

■

**Exercise.** Composition of paths is well-defined on homotopy classes rel$\{0, 1\}$.

**Exercise.** If $X$ is a contractible space, then $X$ is path connected and $\pi_1(X)$ is trivial.

## Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

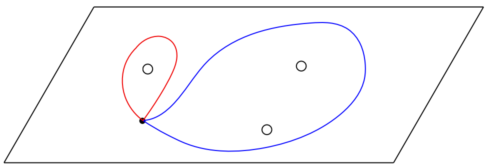


Figure 11: Fundamental Group is basically a *hole detector*!

### 2.2 Calculations with $\pi_1(S^n)$

Let's start with a simple theorem.

**Theorem 2.2.** $\pi_1(S^1) \cong \mathbb{Z}$, and this identification is given by the paths

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))].$$

**Remark.** Intuitively, this winds around $S^1$ $n$ times. The key to this proof was to understand $S^1$ via the covering space $\mathbb{R} \to S^1$. We will talk about covering spaces more later.

*Proof.* ■ HW

**Theorem 2.3.** Given $(X, x_0)$ and $(Y, y_0)$, then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\begin{bmatrix} r\colon I \to X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{bmatrix} \mapsto (r_X, r_Y),$$

where $\gamma$ is continuous $\iff$ $f_X, f_Y$ are continuous.

*Proof.* Let $Z \xrightarrow{f} X \times Y$ with $z \overset{f}{\mapsto} (f_X(z), f_Y(z))$. Then we have

$$\text{continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply above to

- Paths $I \to X \times Y$.
- Homotopies of paths $I \times I \to X \times Y$.

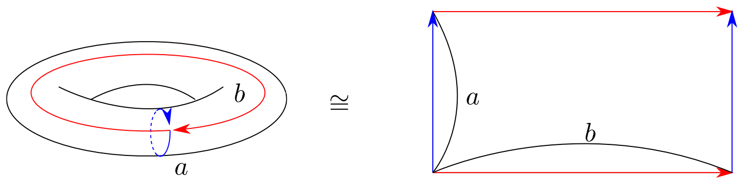
■

**Corollary 2.1.** The torus $T \cong S^1 \times S^1$ has fundamental group $\pi_1(T) \cong \mathbb{Z}^2$. Additionally, for a $k$-torus $\underbrace{S^1 \times S^1 \times \ldots \times S^1}_{k \text{ times}} = (S^1)^k$, the fundamental group is then $\mathbb{Z}^k$, i.e.

$$\pi_1\left((S^1)^k\right) \cong \mathbb{Z}^k.$$

*Proof.* Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$




■

**Remark.** One way to think of the $k$-torus is as a $k$-dimensional cube with opposite $(k-1)$-dimensional faces identified by translation.

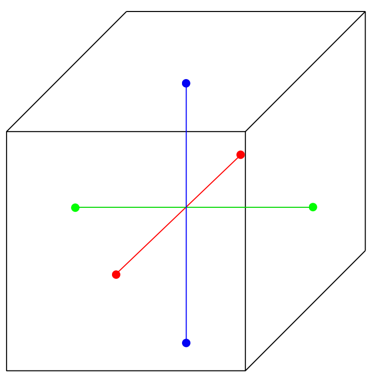


Figure 12: 3-torus with cube identified with parallel sides.

**Example.** We now see some examples.

1. $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2. $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong 0 \times \mathbb{Z} = \mathbb{Z}$ since
$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R},$$
which means that the generators are just loops around the hold intuitively.

**Theorem 2.4.** $\pi_1$ is a functor such that
$$\begin{aligned} \pi_1 \colon \underline{\mathrm{Top}_*} &\to \underline{\mathrm{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0). \end{aligned}$$
A map $f\colon X \to Y$ taking base point $x_0$ to $y_0$ induces a map
$$\begin{aligned} f_* \colon \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$
i.e.,
$$[f\colon X \to Y] \mapsto [f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))].$$

**Notation.** We usually write $f_*$ if it's a covarant functor, while writing $f^*$ if it's an contravariant functor.

*Proof.* We need to check

- well-defined on path homotopy classes.
- $f_*$ is a group homomorphism.
$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \frac{1}{2}\right] \\ f(\beta(1-2s)), & \text{if } s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$
- $\left(\mathrm{id}_{(X,x_0)}\right)_* = \mathrm{id}_{\pi_1(X,x_0)}$
- $(f_* \circ g_*) = (f \circ g)_*$
$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \implies f_*(\gamma_*(\gamma)).$$

DIY

■

## Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

The goal is to compute $\pi_1(X)$ where $X = A \cup B$ using the data
$$\pi_1(A), \pi_1(B), \pi_1(A \cap B).$$

We first introduce a definition.

**Definition 2.6 (Free product with amalgamation).** Given some collections of groups $\{G_\alpha\}_\alpha$, the *free product*, denoted by $\underset{\alpha}{*}G_\alpha$ is a group such that

- Elements: Words in $\{g\colon g \in G_\alpha \text{ for any } \alpha\}$ modulo by the equivalence relation generated by

$$wg_ig_jv \sim w(g_ig_j)v$$

when both $g_i, g_j \in G_\alpha$. Also, for the identity element $\mathrm{id} = e_\alpha \in G_\alpha$ for any $\alpha$ such that

$$we_\alpha v \sim wv.$$

- Operation: Concatenation of words.

Furthermore, if two groups $G_\alpha$ and $G_\beta$ have a common subgroup $S_{\{\alpha,\beta\}}$[a], given two inclusion maps[b] $i_{\alpha\beta}\colon S_{\{\alpha,\beta\}} \to G_\alpha$ and $i_{\beta\alpha}\colon S_{\{\alpha,\beta\}} \to G_\beta$, the *free product with amalgamation* ${}_\alpha{*}_S G_\alpha$ is defined as $\underset{\alpha}{*}G_\alpha$ modulo the normal subgroup generated by

$$\left\{ i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \mid s_{\{\alpha,\beta\}} \in S_{\{\alpha,\beta\}} \right\},$$

Namely[c],

$$ {}_\alpha{*}_S\, G_\alpha = \underset{\alpha}{*}G_\alpha \Big/ \left\langle i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \right\rangle$$

and satisfies the universal property

$$\begin{tikzcd}
S \arrow[r, "i_{\alpha\beta}"] \arrow[d, "i_{\beta\alpha}"'] & G_\alpha \arrow[d] \arrow[ddr, bend left] & \\
G_\beta \arrow[r] \arrow[drr, bend right] & G_\alpha *_S G_\beta \arrow[dr, dashed, "\exists!"] & \\
 & & X
\end{tikzcd}$$

[a] In general, we don't need $S_{\{\alpha,\beta\}}$ to be a subgroup.
[b] We don't actually need $i_{\alpha\beta}, i_{\beta\alpha}$ to be inclusive as well.
[c] i.e., $i_{\alpha\beta}(s)$ and $i_{\beta\alpha}(s)$ will be identified in the quotient.

**Remark.** We see that

- We can then write out words such as $g_1g_2sg_3$ for $s \in S$, and view $s$ as an element of $G_\alpha$ or $G_\beta$. In fact, we can do this construction even when $i_\alpha$ and $i_\beta$ are not injective, though this means we are not working with a subgroup.

- Aside, in <u>Top</u>, the same universal property defines union

$$\begin{tikzcd}
A \cap B \arrow[r, "i_\alpha"] \arrow[d, "i_\beta"'] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \cup B \arrow[dr, dashed, "\exists!"] & \\
 & & X
\end{tikzcd}$$

  for $A, B$ are open subsets and the inclusion of intersection.

**Theorem 2.5 (Seifert-Van Kampen Theorem).** Given $(X, x_0)$ such that $X = \bigcup\limits_{\alpha} A_\alpha$ with

- $A_\alpha$ are open and path-connected and $\forall \alpha\ x_0 \in A_\alpha$
- $A_\alpha \cap A_\beta$ is path-connected for all $\alpha, \beta$.

Then there exists a surjective group homomorphism

$$\underset{\alpha}{*}\colon \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0).$$

If we additionally have $A_\alpha \cap A_\beta \cap A_\gamma$ where they are all path-connected for every $\alpha, \beta, \gamma$, then

$$\pi_1(X, x_0) \cong_\alpha *_{\pi_1(A_\alpha \cap A_\beta, x_0)} \pi_1(A_\alpha, x_0)$$

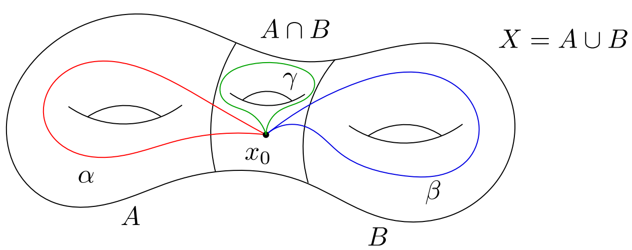
associated to all maps $\pi_a(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha), \pi_1(A_\beta)$ induced by inclusions of spaces. i.e., $\pi_1(X, x_0)$ is a quotient of the free product $*_\alpha \pi_1(A_\alpha)$ where we have

$$(i_{\alpha\beta})_*\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A + \alpha)$$

which is induced by the inclusion $i_{\alpha\beta}\colon A_\alpha \cap A_\beta \to A_\alpha$. We then take the quotient by the normal subgroup generated by

$$\{(i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_\alpha \cap A_\beta)\}.$$

**Example.** We first see a great visualization of the Theorem 2.5.



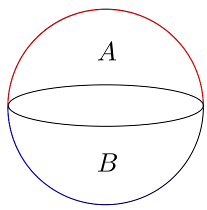
Intuitively we see the fundamental group of $X$, which is built by gluing $A$ and $B$ along their intersection. As the fundamental group of $A$ and $B$ glued along the fundamental group of their intersection. In essence, $\pi_1(X, x_0)$ is the quotient of $\pi_1(A) * \pi_1(B)$ by relations to impose the condition that loops like $\gamma$ lying in $A \cap B$ can be viewed as elements of either $\pi_1(A)$ or $\pi_1(B)$.

## Lecture 11: Group Presentations

31 Jan. 10:00

**Example.** We now see some applications of Theorem 2.5.

1. We can use Seifert Van Kampen Theorem to compute the fundamental group of $S^2$. We see that




We see that $\pi_1(S^2)$ must be a quotient of $\pi_1(A) * \pi_1(B)$, but since $A, B \simeq D^2$, we know that $\pi_1(A)$ and $\pi_1(B)$ are both zero groups, thus $\pi_1(A) * \pi_1(B)$ is the zero group, and $\pi_1(S^2)$ is also the zero group.

**Remark.** Note that the inclusion of $A \cap B \to A$ induces the zero map $\pi_1(A \cap B) \to \pi_1(A)$, which cannot be an injection. In fact, we know that $\pi_1(A \cap B) \cong \mathbb{Z}$ since $A \cap B \simeq S^1$.

2. In the case of torus, consider the following.

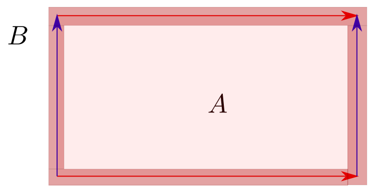



Figure 13: $A$ is the interior, while $B$ is the neighborhood of the boundary.

Now note that $A \simeq D^2$ and $B \simeq S^1 \vee S^1$, and since it's a thickening of the two loops around the torus in both ways, this suggests the question of how do we find $\pi_1(B)$? We grab a bit of knowledge from Seifert Van Kampen Theorem before we continue.

---

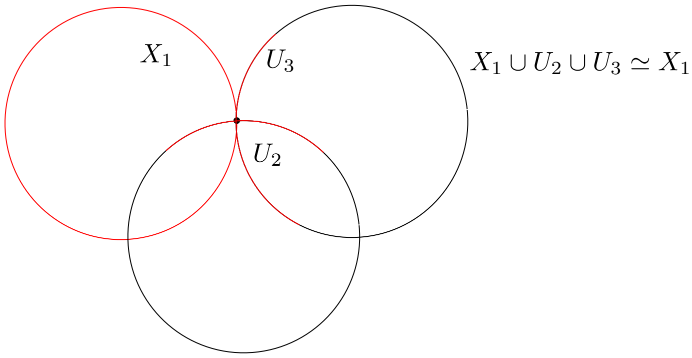
**Exercise.** Suppose we have path connected spaces $(X_\alpha, x_\alpha)$, and we take their wedge sum $\bigvee_\alpha X_\alpha$ by identifying the points $x_\alpha$ to a single point $x$.

We also suppose a mild condition for all $\alpha$, the point $x_\alpha$ is a deformation retract of some neighborhood of $x_\alpha$.

For example, this doesn't work if we choose the *bad point* on the Hawaiian earring. Then we can use Seifert Van Kampen Theorem to show that

$$\pi_1\left(\bigvee_\alpha X_\alpha, x\right) \cong \underset{\alpha}{*}\pi_1\left(X_\alpha, x_\alpha\right).$$

*Proof.* If we denote




as $C_n$, then $\pi_1(C_n) \cong F_n$. Then we apply Theorem 2.5 to $A_\alpha = X_\alpha \cup_\beta U_\beta$ Specifically, take $A_\alpha = X_\alpha \cup_\beta U_\beta \simeq X_\alpha$, where $U_\beta$ is a neighborhood of $x_\beta$ which deformation retracts to $x_\beta$. This makes $A_\alpha$ open as desired. ∎

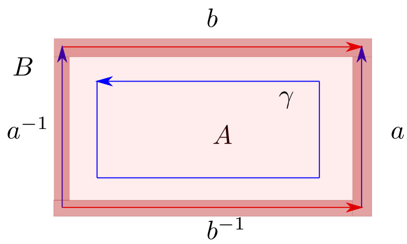
> **Corollary 2.2.** The wedge sum of circles $\pi_1(\bigvee_{\alpha\in A} S^1) = *_\alpha\mathbb{Z}$ is a free group on $A$. In particular, when $A$ is finite, the fundamental group of a bouquet of circles is the free group on $|A|$.

---

Returning to the example of torus, we see that

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$

Further, we know that $\pi_1(A\cap B) \to \pi_1(A)$ is the zero map. We need to understand $\pi_1(A\cap B) \to \pi_1(B)$. To do so we need to understand how we're able to identify $\pi_1(S^1 \vee S_1)$ with $F_2$ and how we identify $\pi_1(S^1)$ with $\mathbb{Z}$. We update our Figure 13 to talk about this.



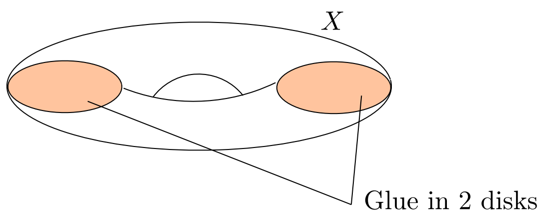

From this, we have

$$\pi_1(A \cap B) \to \pi_1(B) \cong F_{a,b}, \quad \gamma \mapsto aba^{-1}b^{-1}.$$

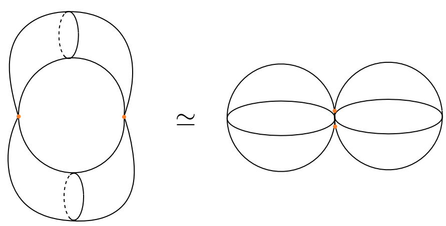
By Seifert Van Kampen Theorem, we identify the image of $\gamma$ in $\pi_1(B)[aba^{-1}b^{-1}]$ with its image in $\pi_1(A)$, which is just trivial. Therefore, we have

$$\pi_1(T^2) = {F_{a,b}}\big/{\langle aba^{-1}b^{-1} \rangle} \cong \mathbb{Z}^2.$$

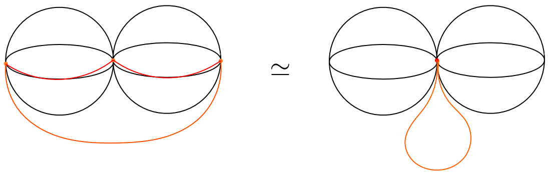
3. Let's see the last example which illustrate the power of Seifert Van Kampen Theorem. Start with a torus, and we glue in two disks into the hollow inside.




We'll call this space $X$, and out goal is to find $\pi_1(X)$. We can place a CW complex structure on this space so that each disk is a subcomplex. Then, we take quotient of each disk to a point without changing the homotopy type, hence $X$ is homotopy to




By the same property, we can expand one of those points into an interval, and then contract the red path as follows.



This is exactly $S^2 \vee S^2 \vee S^1$. With Seifert Van Kampen Theorem, we have

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}.$$

**Exercise.** Consider $\mathbb{R}^2 \setminus \{x_1, \dots, x_n\}$, that is the plane punctured at $n$ points. Then $X \simeq \bigvee_n S^1$, so then

$$\pi_1(X) \simeq F_n.$$

One way to do this is to convince yourself that you can do a deformation retract the plane onto the following wedge.

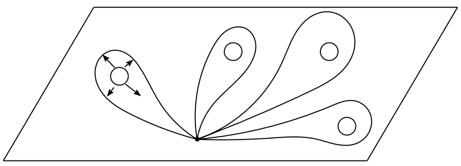


Figure 14: Deformation retract $X$ onto wedge.

## 2.3 Group Presentation

In order to go further, we introduce the concept of *group presentation*.

**Definition 2.7 (Group presentation).** A *presentation* $\langle S \mid R\rangle$ of a group $G$ is

- $S$: set of *generators*
- $R$: set of *relaters* (words in a generator and inverses)

such that

$$G \cong {}^{F_S}\big/_{\langle R\rangle},$$

where $\langle R\rangle$ is a subgroup normally generated by the elements of $R$.

Notice that $\langle S \mid R\rangle$ is finite if $S, R$ are, and $G$ is *finitely presented* if there exists a finite presentation.

**Note.** One way to think about whether $G$ is finitely presented is that if $r$ is a word in $R$ then $r = 1$, where 1 is the identity of $G$.

**Example.** We see that

1. $F_2 = \langle a, b \mid \, \rangle$
2. $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle$
3. $\mathbb{Z} \,/\, 3\mathbb{Z} = \left\langle a \mid a^3 \right\rangle$
4. $S_3 = \left\langle a, b \mid a^2, b^2, (ab)^3 \right\rangle$

**Theorem 2.6.** Any group $G$ has a presentation.

*Proof.* We first choose a generating set $S$ for $G$. Notice that we can even choose $S = G$ directly. From the universal property of free group, we see that there exists a surjective map $\varphi\colon F_S \to G, s \mapsto s$. Now, let $R$ be the generating set for $\ker(\varphi)$, by the first isomorphism theorem[6], $G \cong F_S \,/\, \ker\varphi$. In fact, we have $G = \langle S \mid R \rangle$. ∎

---

**Remark.** The advantages are that given $G = \langle S \mid R \rangle$, it's now easy to define a homomorphism $\psi\colon G \to H$ given a map $\varphi\colon S \to H$, $\psi$ extends to a group homomorphism $G \to H$ if and only if $\psi$ vanishes on $R$, i.e., $\phi(r) = 0$ for all $r \in R$. We see an example to illustrate this.

**Example.** If we have $G = \langle a, b \mid aba \rangle$, a map $\varphi\colon \{a, b\} \to H$ gives a group homomorphism if and only if

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H.$$

This essentially uses the universal property of quotients.

---

**Remark.** It's sometimes easy to calculate $G^{\mathrm{Ab}}$

$$G^{\mathrm{Ab}} = \langle S \mid R, \text{commutators in } S \rangle\,.$$

**Example.** Suppose all relations in $R$ are commutators, so $R \subseteq [G, G]$. Then,

$$G^{\mathrm{Ab}} = (F_S)^{\mathrm{Ab}} = \bigoplus_S \mathbb{Z}.$$

**Remark.** The disadvantages are that, the computationally **very difficult**.

---

**Example.** Given $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle$, let

$$\psi\colon \{a, b\} \to H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1_H \in H.$$

Namely, this is a presentation of the trivial group, but this is entirely unclear.

[6] https://en.wikipedia.org/wiki/Isomorphism_theorems

## Lecture 12: Presentations for $\pi_1$ of CW Complexes

2 Feb. 10:00

Let's first see an exercise.

**Exercise.** Consider $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\} \rangle$
- $G_1 *_H G_2$ where $f_1 \colon H \to G_1$ and $f_2 \colon H \to G_2$. Then we have

$$G_1 *_H G_2 = \left\langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \left\{ f_1(h) f_2(h)^{-1} \mid h \in H \right\} \right\rangle .$$

### 2.4 Presentations for $\pi_1$ of CW Complexes

For $X$ a CW complex, we have

1. A 1-dimensional CW complex has free $\pi_1$ (call its generators as $a_1, \ldots, a_n$).
2. Gluing a 2-disk by its boundary along a word $w$ in the generators *kills* $w$ in $\pi_1$. We then get a presentation for $\pi_1(X^2)$ given by

$$\langle a_1, \ldots, a_n \mid w \text{ for each 2-cell in } X_2 \rangle .$$

3. Gluing in any higher dimensional cells along their boundary will not change $\pi_1$. That is, in a CW complex, we have $\pi_1(X) = \pi_1(X^2)$.

**Remark.** We can write the above more precise.

1. Find free generators $\{a_i\}_{i \in I}$ for $\pi_1(X^1)$.
2. For each 2-disk $D^2_\alpha$, write attaching map as word $w_\alpha$ in $a_i$. i.e.,

$$\pi_1(X^2) = \langle a_i \mid w_\alpha \rangle .$$

3. $\pi_1(X) = \pi_1(X^2)$.

**Example.** Given $G = \mathbb{Z} / n\mathbb{Z} = \langle a, a^n \rangle$, then we take a loop and then wind a 2-disk around the loop $a$ for $n$ times.

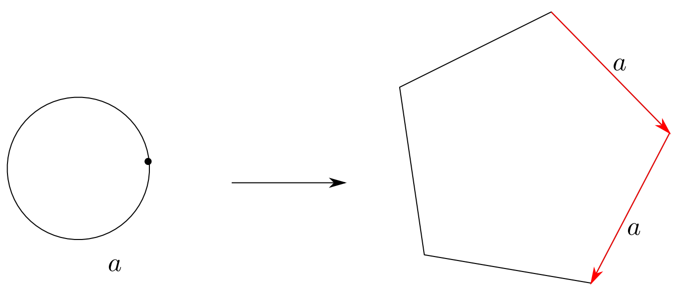



Figure 15: For $G = \mathbb{Z} / n\mathbb{Z} = \langle a \mid a^n \rangle$, we wind the boundary around $a$ for $n$ times.

We then see that given a group $G$ with presentation $\langle S \mid R \rangle$, one can construct a 2-dimensional CW complex with $\pi_1 = G$ by

- Set $X^1 = \bigvee_{s \in S} S^1$
- For each relation $r \in R$, glue in a 2-disk along loops specified by the word $r$.

Every group is then $\pi_1$ of some space.

**Theorem 2.7.** If $X$ is a CW complex and $\iota_1 \colon X^1 \hookrightarrow X$ and $\iota \colon X^2 \hookrightarrow X$, then $(\iota_1)_*$ surjects onto $\pi_1$ and $(\iota_2)_*$ is an isomorphism on $\pi_1$.

*Proof.* ■ HW

**Definition 2.8.** We import some topological definitions of graph theoretic concepts.

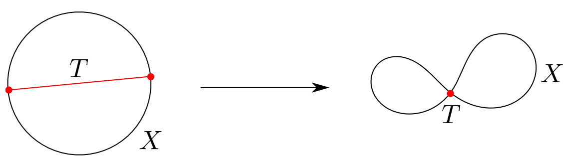
- A *graph* is a 1-dimensional CW complex.
- A *subgraph* is a subcomplex.
- A *tree* is a contractible graph.
- A tree in graph $X$ (necessarily a subgraph) is *maximal* or *spanning* if it contains all the vertices.

**Theorem 2.8.** Every connected graph has a maximal tree. Every tree is contained in a maximal tree.

**Corollary 2.3.** Suppose $X$ is a connected graph with basepoint $x_0$. Then $\pi_1(X, x_0)$ is a free group.

Furthermore, we can give a presentation for $\pi_1(X, x_0)$ by finding a spanning tree $T$ in $X$. The generators of $\pi_1$ will be indexed by cells $e_\alpha \in X - T$, and $e_\alpha$ will correspond to a loop that passes through $T$, traverses $e_\alpha$ once, then returns to the basepoint $x_0$ through $T$.

*Proof.* The idea is simple. $X$ is homotopy equivalent to $X \,/\, T$ via previous work on the homework, $T$ contains all the vertices, so the quotient has a single vertex. Thus, it is a wedge of circles, and each $e_\alpha$ projects to a loop in $X \,/\, T$.



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$\blacksquare$

**Example.** Let

- $S^n$: decompose into 2 open disks
- $A_1$: neighborhood of top hemisphere
- $A_2$: neighborhood of lower hemisphere

We see that $A_1 \cap A_2 \simeq S^{n-1}$, where we need $n \geq 2$ to let $S^{n-1}$ be connected. We then have

$$\pi_1(S^n) \cong 0 \underset{\pi_1(A_1 \cap A_2)}{*} 0 = 0.$$

On the other hand, if $n \geq 3$, then we see that

$$S^n = {}^{D^n \cup *}\big/_{\sim}.$$

Since 2-skeleton is a point, thus $\pi_1(S^n) = 0$.

## Lecture 13

4 Feb. 10:00

# Appendix

# References

[HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com/books?id=BjKs86kosqgC.