MATH597
Analysis II

Pingbang Hu

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\overset{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition we're using to deduce that particular equality.

Notice that there are some proofs is **intended** left as assignments, and for completeness, I put them in Appendix A, use it in your **own risks**! You'll lose the chance to practice and really understand the materials.

Additionally, we'll use [FF99] as our main text, while using [Tao13] and [Axl19] as supplementary references.

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Chapter 1

Measure

Lecture 1: σ-algebra

Before we start, we first see some examples. 05 Jan. 11:00

Example. Let $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) \coloneqq \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which is the *power set* of X. We see that

$$\#X = n \Rightarrow \#\mathcal{P}(X) = 2^n$$

for $n < \infty$.

Example. If $n = \infty$, say $X = \mathbb{N}$, then

$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while $\mathbb{N}$ is a countable set. We can see this as follows. Consider

$$\phi\colon \mathcal{P}(\mathbb{N}) \to [0, 1], \quad A \mapsto 0.a_1a_2a_3 \ldots (\text{base } 2),$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example, A can be $A = \{2, 3, 6, \ldots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have

$$\#\mathcal{P}(\mathbb{N}) \geq \#[0, 1].$$

But since $[0, 1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X. Hence, we are intriguing to define a map μ such that

$$\mu\colon \mathcal{P}(X) \to [0, \infty].$$

Example. Let $X = \{0, 1, 2\}$. Then we want to define $\mu\colon \mathcal{P}(X) \to [0, \infty]$, we can have

- $\mu(A) = \#A$. Then we have
  - $\mu(\{0, 1\}) = 2$
  - $\mu(\{0\}) = 1$
- $\mu(A) = \sum\limits_{i \in A} 2^i$. Then we have

- $\mu(\{0,1\}) = 2^0 + 2^1 = 3$

Example. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$, we can have

- $\mu(A) = \#A$. Then we have
  - $\mu(\{2,3,4,5,\dots\}) = \infty = \mu(\{\text{even numbers}\})$
- $\mu(A) = e^{-1} \sum\limits_{i \in A} \frac{1}{i!}$. Then we have
  - $\mu(\{0,2,4,6,\dots\}) = e^{-1}\left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right)$
- $\mu(A) = \sum\limits_{i \in A} a_i$

Example. Let $X = \mathbb{R}$. Then we want to define $\mu \colon \mathcal{P}(\mathbb{R}) \to [0, \infty]$, we can have

- $\mu(A) = \#A$
- $\mu((a,b)) = b - a$.

  Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

  Answer. No! ■
- $\mu((a,b)) = e^b - e^a$.

  Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

  Answer. No! ■

We immediately see the problems. To extend our native measure method into $\mathbb{R}$ is hard and will cause something counter-intuitive![1] Hence, rather than define measurement on *all* subsets in the power set of X, we only focus on *some* subsets. In other words, we want to define

$$\mu \colon \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \to [0, \infty] .$$

1.1 σ-algebras

We start from the definition of the most fundamental element in measure theory.

Definition 1.1.1 (σ-algebra)**.** Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra on X* if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\varnothing}_{\in \mathcal{A}}$ and $\mathcal{A}$ is closed under complement.
- $\bigcap\limits_{i=1}^{\infty} A_i = \left(\bigcup\limits_{i=1}^{\infty} A_i^c\right)^c$, namely $\mathcal{A}$ is <u>closed under countable intersections</u>.

[1] https://en.wikipedia.org/wiki/Banach-Tarski_paradox

- $A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \varnothing \cup \varnothing \cup \ldots$, hence $\mathcal{A}$ is closed under finite unions and intersections.

An immediate definition can be given. We now define so-called *Borel set*.

Definition 1.1.2 (Borel set). Given a topological space X, a *Borel set* is any set in X that can be formed from open sets through the operations of countable union, countable intersection and relative complement.

Lecture 2: Measure

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Example. Again, we first see some examples.

1. Let $\mathcal{A} = \mathcal{P}(X)$, which is the power σ-algebra.
2. Let $\mathcal{A} = \{\varnothing, X\}$, which is a trivial σ-algebra.
3. Let $B \subset X$, $B \neq \varnothing$, $B \neq X$. Then we see that $\mathcal{A} = \{\varnothing, B, B^c, X\}$ is a σ-algebra.

Lemma 1.1.1. Let $\mathcal{A}_\alpha$, $\alpha \in I$, be a family of σ-algebra on X. Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is a σ-algebra on X.

Proof. A simple proof can be made as follows. Firstly, $\varnothing \in \mathcal{A}_\alpha$ for every α clearly. Moreover, closure under complement and countable unions for every $\mathcal{A}_\alpha$ implies the same must be true for $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$. Hence, $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ-algebra.

Remark. Notice that I may be an uncountable intersection.

∎

The above allows us to give the following definition.

Definition 1.1.3 (Generation of σ-algebra). Given $\mathcal{E} \subset \mathcal{P}(X)$, where $\mathcal{E}$ is not necessarily a σ-algebra. Let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X containing $\mathcal{E}$, then we call $\langle \mathcal{E} \rangle$ the *σ-algebra generated by $\mathcal{E}$*.

Remark. Clearly, $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$, and it is unique. To check the uniqueness, we suppose there are two different $\langle \mathcal{E} \rangle_1$ and $\langle \mathcal{E} \rangle_2$ generated from $\mathcal{E}$. It's easy to show

$$\langle \mathcal{E} \rangle_1 \subseteq \langle \mathcal{E} \rangle_2 \,,$$

and by symmetry, they are equal.

Example. We see that $\{\varnothing, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle$.

Lemma 1.1.2. We have

1. Given $\mathcal{A}$ a σ-algebra, $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \Rightarrow \langle \mathcal{E} \rangle \subset \mathcal{A}$
2. $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \Rightarrow \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$

Proof. We'll see that after proving the first claim, the second follows smoothly.

1. The first claim is trivial, since we know that $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$, then if $\mathcal{E} \subset \mathcal{A}$, we clearly have $\langle \mathcal{E} \rangle \subset \mathcal{A}$ by the definition.

2. The second claim is also easy. From the first claim and the definition, we have

$$\mathcal{E} \subset \mathcal{F} \subset \langle \mathcal{F} \rangle \Rightarrow \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle .$$

■

At this point, we haven't put any specific structure on X. Now we try to describe those spaces with good structure, which will give the space some nice properties.

Definition 1.1.4 (Borel σ-algebra)**.** For a topological space X, the *Borel σ-algebra on X*, denotes as $\mathcal{B}(X)$, is the σ-algebra generated by the collection of all open sets in X.

Example. We see that $\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a, b) \mid a < b; a, b \in \mathbb{R}\}$.
- $\mathcal{E}_2 = \{[a, b] \mid a < b; a, b \in \mathbb{R}\}$ since $[a, b] = \bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.
- $\mathcal{E}_3 = ((a, b] \mid a < b; a, b \in \mathbb{R})$ since $(a, b] = \bigcap\limits_{n=1}^{\infty} (a, b + \frac{1}{n})$.
- $\mathcal{E}_4 = ([a, b) \mid a < b; a, b \in \mathbb{R})$ since $[a, b) = \bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b)$.
- $\mathcal{E}_5 = ((a, \infty) \mid a \in \mathbb{R})$ since $(a, \infty) = \bigcup\limits_{n=1}^{\infty} (a, a + n)$.
- $\mathcal{E}_6 = ([a, \infty) \mid a \in \mathbb{R})$ since $[a, \infty) = \bigcup\limits_{n=1}^{\infty} [a, a + n)$.
- $\mathcal{E}_7 = ((-\infty, b) \mid b \in \mathbb{R})$ since $(-\infty, b) = \bigcup\limits_{n=1}^{\infty} (b - n, b)$.
- $\mathcal{E}_8 = ((-\infty, b] \mid a \in \mathbb{R})$ since $(-\infty, b] = \bigcup\limits_{n=1}^{\infty} (b - n, b]$.

Proposition 1.1.1. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \ldots, 8$.

Proof. Firstly, we see that $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \Rightarrow \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$ by Lemma 1.1.2. Secondly, by definition, $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$ where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R}\ \} .$$

It's enough to show $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$ since if so, $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$, and clearly $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$, then we will have $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$. Let $O \subset \mathbb{R}$ be an open set, i.e., $O \in \mathcal{E}$. We claim that every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.[a]

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

where I_j open interval with the form of $(a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$.

For example, $\mathcal{E}_1$ is trivially true, and

$$(a,b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{[a+\frac{1}{n}, b-\frac{1}{n}]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for $\mathcal{E}_2$ and

$$(a,\infty) = \bigcup_{k=1}^{\infty} (a, a+k)$$

shows the case for $\mathcal{E}_5$. It's now straightforward to check open intervals are in $\langle \mathcal{E}_i \rangle$ for every i. ■

[a] https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals

Now, to put a structure on a space, we define the following.

Definition 1.1.5 (Measurable space, $\mathcal{A}$-measurable set). A *measurable space* or *Borel space* is a tuple of a set X and a σ-algebra $\mathcal{A}$ on X, denoted by $(X, \mathcal{A})$.

Definition 1.1.6 (Measurable set). Given a measurable space $(X, \mathcal{A})$, every $E \in \mathcal{A}$ is a so-called $\mathcal{A}$*−measurable set.*

1.2 Measures

With the definition of measurable space, we now can refine our measure function μ as follows.

Definition 1.2.1 (Measure, Measure space). Given a measurable space on $(X, \mathcal{A})$, a *measure* is a function μ such that

$$\mu\colon \mathcal{A} \to [0, \infty]$$

with

1. $\mu(\varnothing) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_1, A_2, \ldots \in \mathcal{A}$ are **disjoint**. We call this *Countable additivity.*

We denote $(X, \mathcal{A}, \mu)$ a *measure space*.

Notation. We denote $[0,\infty] \coloneqq [0,\infty) \cup \{\infty\}$.

Remark. The motivation of why we only want countable additivity but not uncountable additivity can be seen by the following example. We'll consider the most intuitive measure on $\mathbb{R}, \mathcal{B}(\mathbb{R})$.

Since we have

$$(0,1] = (1/2, 1] \cup (1/4, 1/2] \cup (1/8, 1/4] \cup \ldots$$

and also

$$(0,1] = \bigcup_{x \in (0,1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2},1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4},\frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8},\frac{1}{4}])} + \ldots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0,1]} 0$$

since $\mu(x) = 0$ for $x \in \mathbb{R}$, which is clearly not what we want.

Example. We see some examples.

1. For any $(X, \mathcal{A})$, we let $\mu(A) := \#A$. This is called *counting measure*.
2. Let $x_0 \in X$. For any $(X, \mathcal{A})$, the *Dirac-Delta measure at x_0* is
$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A. \end{cases}$$
3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$,
$$\mu(A) = \sum_{i \in A} a_i,$$
where $a_1, a_2, \ldots \in [0, \infty)$.

Lecture 3: Construct a Measure

Note. If $A, B \in \mathcal{A}$ and $A \subset B$, then 10 Jan. 11:00

$$\mu(B \setminus A) + \mu(A) = \mu(B) \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

Theorem 1.2.1. Given $(X, \mathcal{A}, \mu)$ be a measure space.

1. Monotonicity.
$$A, B \in \mathcal{A}, A \subset B \Rightarrow \mu(A) \leq \mu(B).$$
2. Countable subadditivity.
$$A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$
3. Continuity from below/ monotone convergence theorem (MCT) for sets.
$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \ldots \end{cases} \Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$
4. Continuity from above.
$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \ldots \\ \mu(A_1) < \infty \end{cases} \Rightarrow \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

Proof. We prove this theorem one by one.

1. Since $A \subset B$, hence we have
$$\mu(B) = \mu\Big(\underbrace{\underline{(B \setminus A)} \cup \underline{A}}_{\text{disjoint}}\Big) \overset{!}{=} \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A).$$

2. This should be trivial from countable additivity with the fact that $\mu(A) \geq 0$ for all A. DIY!
3. Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$, then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

are a disjoint union and $B_i \in \mathcal{A}$, hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i).$$

With $\mu\left(\bigcup_{i=1}^{n} B_i\right) = \mu(A_n)$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu(A_n).$$

4. Let $E_i = A_1 \setminus A_i \Rightarrow E_i \in \mathcal{A}$, $E_1 \subset E_2 \subset \ldots$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right),$$

which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i\right) \Rightarrow \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$$

since $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu(A_1) < \infty$. Then from continuity from below, we further have

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right).$$

From monotonicity, we see that $\mu(A_n) \leq \mu(A_1) < \infty$, hence we can split the limit and further get

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu(A_1) + \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(A_n).$$

■

Example. Given $(\mathbb{N}, \mathcal{P}(\mathbb{N})$, counting measure). Then we see

- $A_n = \{n, n+1, n+2, \ldots\} \Rightarrow \mu(A_n) = \infty$
- $A_1 \supset A_2 \supset A_3 \supset \ldots$
- $\bigcap_{i=1}^{\infty} A_i = \varnothing \Rightarrow \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = 0$

Remark. We see that in this case, since $\mu(A_1) \not< \infty$, hence continuity from above doesn't hold.

We now try to characterize some properties of a measure space.

Definition 1.2.2 (μ-null, μ-subnull, Complete measure space)**.** Given $(X, \mathcal{A}, \mu)$,

- $A \subset X$ is a *μ-null set* if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- $A \subset X$ is a *μ-subnull set* if $\exists$$\mu$-null set$B$ such that $A \subset B$.
- $(X, \mathcal{A}, \mu)$ is a *complete* measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Note. We see that for a μ-subnull set, it's not necessary $\mathcal{A}$-measurable.

There are some useful terminologies we'll use later relating to μ-null.

Definition 1.2.3 (Almost everywhere)**.** Given $(X, \mathcal{A}, \mu)$, a statement $P(x)$, $x \in X$ holds *μ-almost everywhere (a.e.)* if the set

$$\{x \in X : P(x) \text{ does not hold}\}$$

is μ-null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

Definition 1.2.4 (Finite measure)**.** , Given $(X, \mathcal{A}, \mu)$

- μ is a *finite measure* if $\mu(X) < \infty$.
- μ is a *σ-finite measure* if $X = \bigcup_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.

Exercise. Every measure space can be **completed**. Namely, we can always find a bigger σ-algebra to complete the space.

1.3 Outer Measures

We start by giving a definition.

Definition 1.3.1 (Outer measure)**.** An *outer measure* on X is a map

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that

- $\mu^*(\varnothing) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$
- (countable subadditivity) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ for every $A_i \subset X$.

Example. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : \bigcup_{i=1}^{\infty}(a_i, b_i) \supset A\right\}$$

is an outer measure due to the Proposition 1.3.1 we're going to show.

Remark. We see that an outer measure need not be a measure. Check Definition 1.2.1.

Proposition 1.3.1. Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\varnothing, X \in \mathcal{E}$. Let

$$\rho\colon \mathcal{E} \to [0,\infty]$$

such that $\rho(\varnothing) = 0$. Then

$$\mu^*(A) \coloneqq \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E},\ \bigcup_{i=1}^{\infty} E_i \supset A\right\}$$

is an outer measure on X.

Theorem 1.3.1 (Tonelli's Theorem for series)**.** Recall the Tonelli's Theorem[a] for series, i.e., if $a_{ij} \in [0,\infty]$, $\forall i,j \in \mathbb{N}$, then

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

[a] https://en.wikipedia.org/wiki/Fubini%27s_theorem

Proof. Read Tao[Tao13] Theorem 0.0.2. ■

Lecture 4: Carathéodory extension Theorem

We now prove Proposition 1.3.1. 12 Jan. 11:00

Proof of Proposition 1.3.1. We need to prove

- μ^* is well-defined. i.e., inf is taken over a non-empty set. This is trivial since $X \in \mathcal{E}$ and $X \supset A$ for any $A \in \mathcal{E}$.
- $\mu^*(\varnothing) = 0$. Since $\varnothing \in \mathcal{E}$ and

$$\mu^*(\varnothing) = \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \varnothing\right\} = 0$$

  since $\rho(\varnothing) = 0$ for all i and further, by Squeeze Theorem, we see that $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \rho(\varnothing) = 0$.
- $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$. We simply show this by contradiction. Suppose $A \subset B$ and $\mu^*(A) > \mu^*(B)$, then by definition of μ^*, we have

$$\begin{aligned}\mu^*(A) &= \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A\right\}\\ &> \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B\right\} = \mu^*(B).\end{aligned}$$

  Now, let $B \eqqcolon (B\setminus A)\cup A$, then we have

$$\begin{aligned}\mu^*(A) &= \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A\right\}\\ &> \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B\setminus A)\cup A\right\} = \mu^*(B).\end{aligned}$$

  Now, since $B \setminus A \supseteq \varnothing$, then this inequality can't hold, hence a contradiction $\lightning$.

- Countable subadditivity. Let $A_1, A_2, \ldots \in X$. If one of $\mu^*(A_n) = \infty$, then result holds. So we may assume $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Now, fix any $\epsilon > 0$, we will show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon.$$

For each $n \in \mathbb{N}$, $\exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ such that

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n$$

and

$$\mu^*(A_n) + \frac{\epsilon}{2^n} \geq \sum_{k=1}^{\infty} \rho(E_{n,k}).$$

> **Remark.** This is an important trick! We often set the error term as $\epsilon/2^n$ instead of ϵ as in above.

Then we see that

$$\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{k,n},$$

which implies

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{(n,k)\in\mathbb{N}^2} \rho\left(E_{k,n}\right) \overset{!}{=} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \rho(E_{k,n}) \leq \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right)$$

from the inequality just derived. Now, since the last term is just

$$\sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon,$$

hence we finally have

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon$$

for arbitrarily small fixed $\epsilon > 0$, hence the subadditivity is proved.

■

Definition 1.3.2 (Carathéodory measurable)**.** Let μ^* be an outer measure on X. We say $A \subset X$ is *Carathéodory measurable (C-measurable) with respect to* μ^* if

$$\forall E \subset X,\ \mu^*(E) = \mu^*\left(E \cap A\right) + \mu^*\left(E \setminus A\right).$$

Lemma 1.3.1. Let μ^* be an outer measure on X. Suppose $B_1, \ldots, B_N$ are disjoint C-measurable sets. Then,

$$\forall E \subset X,\ \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) = \sum_{i=1}^{N} \mu^*\left(E \cap B_i\right).$$

Proof. Since we have

$$\begin{aligned}\mu^*\left(E\cap\left(\bigcup_{i=1}^N B_i\right)\right) &= \mu^*\left(E'\cap B_1\right)+\mu^*\left(E'\setminus B_1\right)\\ &= \mu^*\left(E\cap\left(\bigcup_{i=1}^N B_i\cap B_1\right)\right)+\mu^*\left(E\cap\left(\bigcup_{i=1}^N B_i\right)\cap B_1^c\right)\\ &= \mu^*(E\cap B_1)+\mu^*\left(E\cap\left(\bigcup_{i=2}^N B_i\right)\right)\end{aligned}$$

where the equality comes from the fact that B_1 is C-measurable and disjoint from B_i, $i\neq 1$. Then, we simply iterate this argument and have the result. Note that in the first inequality, we define $E' := E\cap\left(\bigcup_{i=1}^N B_i\right)$ for the simplicity of notation. ■

Remark. This implies that if we restrict an outer measure on a C-measurable set, then it becomes finite additive.

Theorem 1.3.2 (Carathéodory extension Theorem). Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets (with respect to μ^*). Then,

1. $\mathcal{A}$ is a σ-algebra on X.
2. $\mu = \mu^*|_{\mathcal{A}}$ is a measure on $(X,\mathcal{A})$.
3. $(X,\mathcal{A},\mu)$ is a complete measure space.

Proof. We divide the proof in several steps.

1. We show $\mathcal{A}$ is a σ-algebra by showing
   (a) $\varnothing\in\mathcal{A}$. To show this, we simply check that $\varnothing$ is C-measurable. We see that

   $$\underset{E\subset X}{\forall}\ \mu^*(E)=\mu^*(E\cap\varnothing)+\mu^*(E\setminus\varnothing)=\mu^*(E),$$

   which just shows $\varnothing\in\mathcal{A}$.

   (b) $\mathcal{A}$ closed under complements. This is equivalent to say that if A is C-measurable, so is A^c. We see that if A is C-measurable, then for every $E\subset X$,

   $$\mu^*(E)=\mu^*(E\cap A)+\mu^*(E\setminus A).$$

   Observing that $E\cap A=E\setminus A^c$ and $E\setminus A=E\cap A^c$, hence

   $$\mu^*(E)=\mu^*(E\setminus A^c)+\mu^*(E\cap A^c).$$

   We immediately see that above implies $A^c\in\mathcal{A}$.

   (c) $\mathcal{A}$ closed under countable unions.

   Note. To show $\mathcal{A}$ closed under countable unions, we show that $\mathcal{A}$ is closed under:

   $$\text{finite unions}\overset{\text{then}}{\Rightarrow}\text{countable }\underline{\text{disjoint}}\text{ unions}\overset{\text{then}}{\Rightarrow}\text{countable unions.}$$

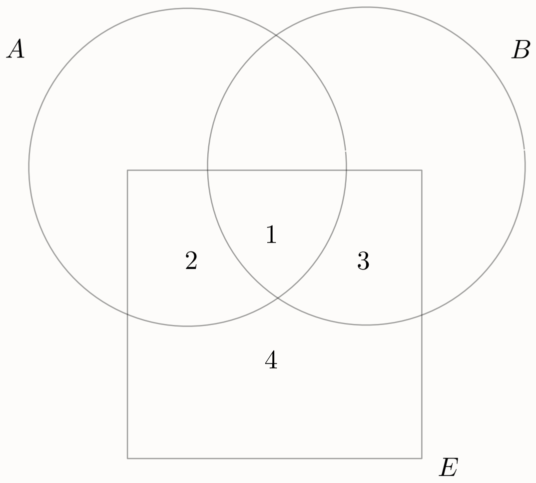
   - We show $\mathcal{A}$ is closed under finite unions.
     Claim. $A,B\in\mathcal{A}\Rightarrow A\cup B\in\mathcal{A}$.
     Fix $E\subset X$ arbitrary. We need to show that

     $$\mu^*(E)=\mu^*(E\cap(A\cup B))+\mu^*(E\setminus(A\cup B)),$$

i.e.,

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

given $A, B \in \mathcal{A}$.



- Since A is C-measurable,
  - $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$
  - $\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$
- Since B is C-measurable,
  - $\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$

Hence, we have

$$\begin{aligned}\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).\end{aligned}$$

- We show $\mathcal{A}$ is closed under countable disjoint unions.
  Let $A_1, A_2, \ldots \in \mathcal{A}$ and disjoint. Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^{\infty} A_i\right),$$

hence we only need to show another way around.

Fix $N \in \mathbb{N}$, we have $\bigcup\limits_{n=1}^{N} A_n \in \mathcal{A}$ since N is finite, and

$$\begin{aligned}\mu^*(E) &= \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} A_n\right)\right) + \mu^*\left(E \setminus \left(\bigcup_{n=1}^{N} A_n\right)\right) \\ &\geq \underbrace{\sum_{n=1}^{N} \mu^*(E \cap A_n)}_{\overset{!}{=}\mu^*\left(E \cap \left(\bigcup_{n=1}^{N} A_n\right)\right)} + \underbrace{\mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right)}_{\leq \mu^*\left(E \setminus \left(\bigcup_{n=1}^{N} A_n\right)\right)}.\end{aligned}$$

Now, take $N \to \infty$ then we are done.

- We show $\mathcal{A}$ is closed under countable unions. DIY

The proof will be continued...

Lecture 5: Hahn-Kolmogorov Theorem

Firstly, we see a stronger version of Lemma 1.3.1 we have seen before. 14 Jan. 11:00

Lemma 1.3.2. Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots$ are disjoint C-measurable sets. Then,

$$\forall E \subset X,\ \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \sum_{i=1}^{\infty} \mu^*\left(E \cap B_i\right).$$

Proof.

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_i) \geq \mu^*\left(E \cap \bigcup_{n=1}^{\infty} B_n\right) \geq \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} B_n\right)\right) \overset{!}{=} \sum_{n=1}^{N} \mu^*\left(E \cap B_n\right).$$

Now, we just take $N \to \infty$.[a] ■

[a] Note that $N \in \mathbb{N}$ is arbitrary, we then get the result according to Squeeze Theorem

Let's continue the proof of Theorem 1.3.2.

Proof of Theorem 1.3.2 (cont.) The 1. is proved, now we prove 2. and 3.

2. Since from Definition 1.2.1, we need to show
   - $\mu(\varnothing) = 0$. This means that we need to show $\mu^*|_{\mathcal{A}}(\varnothing) = 0$. Since $\varnothing \in \mathcal{A}$ and μ^* is an outer measure, hence from the property of outer measure, it clearly holds.
   - Countable additivity of μ^* on $\mathcal{A}$ follows from the Lemma 1.3.2 with $E = X$
3. The proof is given in Theorem A.1.1.

■

1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets $\mathcal{E}$ and any ρ such that it assigns measure on $\mathcal{E}$, then it induces an outer measure by Proposition 1.3.1, finally complete the outer measure by Theorem 1.3.2.

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.3.1}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.3.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to σ-algebra we are working on till now.

Definition 1.4.1 (Algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on* X if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an algebra and a σ-algebra is whether they closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an outer measure.

Definition 1.4.2 (Pre-measure). Let $\mathcal{A}_0$ be an algebra on X. We say

$$\mu_0 \colon \mathcal{A}_0 \to [0, \infty]$$

is a *pre-measure* if

1. $\mu_0(\varnothing) = 0$
2. (finite additivity) $\mu_0\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu_0(A_i)$ if $A_1, \dots, A_n \in \mathcal{A}_0$ are disjoint.
3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, disjoint, then

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 1.4.1. (1) + (3) $\Rightarrow$ (2) in Definition 1.4.2.

Proof. It's easy to see that since μ_0 is monotone. ■

Theorem 1.4.1 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in Proposition 1.3.1. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^*, then $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$. i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts.

- We first show $\mathcal{A} \supset \mathcal{A}_0$. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is C-measurable, i.e.,

$$\forall E \subset X \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We first fix an $E \subset X$. From countable subadditivity of μ^*, we have

$$\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

Fix $\epsilon > 0$. By the Proposition 1.3.1 of μ^*, $\exists B_1, B_2, \dots \in \mathcal{A}_0$, $\bigcup_{n=1}^{\infty} B_n \supset E$ such that

$$\mu^*(E) + \epsilon \overset{!}{\ge} \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \Bigg(\mu_0(\underbrace{B_n \cap A}_{\in \mathcal{A}_0}) + \mu_0(\underbrace{B_n \cap A^c}_{\in \mathcal{A}_0}) \Bigg)$$

by the finite additivity of μ_0. Note that

$$\begin{cases} \bigcup_{n=1}^{\infty} (B_n \cap A) \supset E \cap A \\ \bigcup_{n=1}^{\infty} (B_n \cap A^c) \subset E \cap A^c \end{cases} \Rightarrow \mu^*(E) + \epsilon \ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

since

$$\mu^*(E \cap A) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n \cap A)$$

and

$$\mu^*(E \cap A^c) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A^c)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n \cap A^c).$$

We then see that for any $\epsilon > 0$, the inequality

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

holds, hence so does

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

which implies $\mathcal{A} \supset \mathcal{A}_0$.

The proof will be continued...

Lecture 6: Hahn-Kolmogorov Theorem and Extension.

Let's continue the proof of Theorem 1.4.1. 18 Jan. 11:00

Proof of Theorem 1.4.1 (cont.) We proved the first part already, now we prove the part left.

- Let $A \in \mathcal{A}_0$, we want to show that

$$\mu(A) = \mu_0(A).$$

  - Firstly, let

$$B_i = \begin{cases} A, & \text{if } i = 1 \\ \varnothing, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,$$

    hence $\bigcup_{i=1}^{\infty} B_i = A$, then we see that

$$\mu^*(A) \leq \sum_{i=1}^{\infty}\mu_0(B_i) = \mu_0(A)$$

    from the definition of μ^* and countable additivity within the algebra of μ_0.

  - Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A \cap B_1 \in \mathcal{A}_0$, $C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j\right) \in \mathcal{A}_0$ for $i \geq 2$ since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

    are disjoint countable unions, by countable additivity within the algebra, we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty}\mu_0(C_i) \leq \sum_{i=1}^{\infty}\mu_0(B_i) \Rightarrow \mu_0(A) \leq \mu^*(A)$$

    by taking the infimum from the definition of μ^*.

  Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every $A \in \mathcal{A}_0$, which implies

$$\mu(A) = \mu_0(A)$$

for every $A \in \mathcal{A}_0$ from Theorem 1.3.2, where we extend μ^* to μ respect to $\mathcal{A}_0$.

■

Definition 1.4.3 (HK extension). $(\mathcal{A}, \mu)$ obtained from Theorem 1.4.1 is the *Hahn-Kolmogorov extensions* of $(\mathcal{A}_0, \mu_0)$.

We can show the uniqueness of HK extension.

Theorem 1.4.2 (Uniqueness of HK extension). Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$. Let $(\mathcal{A}, \mu)$ be the HK extension of $(\mathcal{A}_0, \mu_0)$. Let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$. Then if μ_0 is σ-finite, $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Proof. First, we note the following.

Note. Notice that $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$ since they both extend $\mathcal{A}_0$.

Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing the arbitrary cover of A and using the definition of μ^*.

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) < \infty$, and fix $\epsilon > 0$. Then there exists $B_i \in \mathcal{A}_0$ with $B \coloneqq \bigcup_{i=1}^{\infty} B_i \supset A$ such that

$$\mu(A) + \epsilon = \mu^*(A) + \epsilon \overset{!}{\geq} \sum_{i=1}^{\infty} \mu_0(B_i) \overset{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

  This implies that

$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$

  where the first equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

$$\mu(B) = \lim_{N \to \infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) = \lim_{N \to \infty} \mu'\left(\bigcup_{i=1}^{N} B_i\right) = \mu'(B)$$

  where the middle equality follows from $\mu = \mu'$ on $\mathcal{A}_0$, hence,

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon$$

  for arbitrary ϵ, so we conclude $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is σ-finite, so we know $X = \bigcup_{n=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ such that

$$\mu_0(X_n) < \infty.$$

  Replacing X_n by $X_1 \cup \ldots \cup X_n \in \mathcal{A}_0$, we may assume that

$$X_1 \subset X_2 \subset \ldots.$$

Then,

$$\underset{n\in\mathbb{N}}{\forall}\ \mu(A\cap X_n)<\infty \overset{!}{\Rightarrow} \mu(A\cap X_n)\leq \mu'(A\cap X_n).$$

From the continuity of measure, we then have

$$\mu(A)=\lim_{n\to\infty}\mu(A\cap X_n)\leq \lim_{n\to\infty}\mu'(A\cap X_n)=\mu'(A).$$

■

Corollary 1.4.1. Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then

$$\exists!\text{ measure } \mu \text{ on } \langle\mathcal{A}_0\rangle \text{ that extends } \mathcal{A}_0.$$

Furthermore,

- The completion of $(X,\langle\mathcal{A}_0\rangle,\mu)$ is the HK extension of $(\mathcal{A}_0,\mu_0)$.
- $$\mu(A)=\inf\left\{\sum_{i=1}^{\infty}\mu_0(B_i)\mid B_i\in\mathcal{A}_0, \underset{i\in\mathbb{N}}{\forall}\ \bigcup_{i=1}^{\infty}B_i\supset A\right\}$$

  for all $A\in\overline{\langle\mathcal{A}_0\rangle}$.

Lecture 7: Borel Measures

1.5 Borel Measures on $\mathbb{R}$

21 Jan. 11:00

We first introduce so-called *distribution function*.

Definition 1.5.1 (Distribution function). An increasing[a] function

$$F\colon\mathbb{R}\to\mathbb{R}$$

and right-continuous. F is then a *distribution function*.

[a] Here, increasing means $F(x)\leq F(y)$ for $x<y$.

Example. Here are some examples of right-continuous functions.

1. $F(x)=x$.
2. $F(x)=e^x$.
3. Define
$$F(x)=\begin{cases}1, & \text{if } x\geq 0\\ 0, & \text{if } x<0.\end{cases}$$
4. Let $\mathbb{Q}\coloneqq\{r_1,r_2,\ldots\}$. Define
$$F_n(x)=\begin{cases}1, & \text{if } x\geq r_n\\ 0, & \text{if } x<r_n,\end{cases}$$
and
$$F(x)\coloneqq\sum_{n=1}^{\infty}\frac{F_n(x)}{2^n}.$$
Then F is a distribution function (hence right-continuous). This is shown in Lemma A.1.1.

Note. If F is increasing, and

$$F(\infty) \coloneqq \lim_{x \nearrow \infty} F(x), \quad F(-\infty) \coloneqq \lim_{x \searrow \infty} F(x)$$

exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.[a]

[a]There are distributions [FF99] Ch9., but these are different from distribution functions.

Now, we can define a *Borel measure* on $(X, \mathcal{B}(\mathbb{R}))$.

Definition 1.5.2 (Borel mesaure)**.** A *Borel measure* is any measure μ defined on the σ-algebra of Borel sets.

Definition 1.5.3 (Locally finite)**.** Let X be a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Note. Some authors will require a Borel measure equipped with the locally finite property. But formally, this is not so common.

Lemma 1.5.1. Let μ be a locally finite Borel measure on $\mathbb{R}$, then

$$F_\mu(x) = \begin{cases} \mu\left((0, x]\right), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu\left((x, 0]\right), & \text{if } x < 0 \end{cases}$$

is a distribution function.

Proof. To show F_μ is increasing, consider $x < y$ such that

$$F_\mu(x) \leq F_\mu(y)$$

by considering

- $x > 0$: Then $F_\mu(x) = \mu((0, x])$ and

$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$

- $x = 0$: Then $F_\mu(x) = 0$ and

$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$

  since $y > 0$.

- $x < 0$: Follows the same argument with $x > 0$.

Now, we need to show F_μ is right-continuous. Firstly, assume that $x \geq 0$, then we see that

$$F_\mu(x) = \mu((0, x]) = \mu((0, x^+])$$

from the fact that a measure is right-continuous.[a] Now, if $x \leq 0$, the same argument follows since multiplying -1 will not change the fact that a measure is continuous. ∎

[a]Actually, a measure is always continuous.

Definition 1.5.4 (Half intervals). We call

$$\varnothing, (a,b], (a,\infty), (-\infty,b], (-\infty,\infty)$$

half-intervals.

Lemma 1.5.2. Let $\mathcal{H}$ be the collection of finite disjoint unions of half-intervals. Then, $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. We see that

- $\varnothing \in \mathcal{H}$. Clearly.
- To show $\mathcal{H}$ is closed under complements, we have
  - $\varnothing^c = \mathbb{R} = (-\infty,\infty) \in \mathcal{H}$.
  - $(a,b]^c = (-\infty,a] \cup (a,\infty) \in \mathcal{H}$.[a]
  - $(a,\infty)^c = (-\infty,a] \in \mathcal{H}$.
  - $(-\infty,b]^c = (b,\infty) \in \mathcal{H}$.
  - $(-\infty,\infty)^c = \varnothing \in \mathcal{H}$.
- $\mathcal{H}$ is closed under finite unions, clearly.

∎

[a] Since it's a two disjoint union of half intervals.

Proposition 1.5.1 (Distribution function defines a pre-measure). Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function. For a half interval I, define

$$\ell(I) \coloneqq \ell_F(I) = \begin{cases} 0, & \text{if } I = \varnothing; \\ F(b) - F(a), & \text{if } I = (a,b]; \\ F(\infty) - F(a), & \text{if } I = (a,\infty]; \\ F(b) - F(-\infty), & \text{if } I = (-\infty,b]; \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty,\infty). \end{cases}$$

Define $\mu_0 \coloneqq \mu_{0,F}$ as

$$\mu_{0,F}\colon \mathcal{H} \to [0,\infty]$$

by

$$\mu_0(A) = \sum_{k=1}^{N} \ell(I_k) \text{ if } A = \bigcup_{k=1}^{N} I_k,$$

where A is a finite disjoint union of half intervals $I_1, \ldots, I_N$. Then, μ_0 is a pre-measure on $\mathcal{H}$.

Proof. We see that

1. μ_0 is well-defined.
2. $\mu_0(\varnothing) = 0$.
3. μ_0 is finite additive.
4. μ_0 is countable additivity within $\mathcal{H}$.

   Suppose $A \in \mathcal{H}$ where $A = \bigcup\limits_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.

Remark. Since $\mathcal{H}$ is only a collection of *finite* disjoint half intervals, hence after considering $A = I$, we can apply the same argument iteratively and stop in finite steps. Formally, we can consider $H \in \mathcal{H}$, $H = \bigcup_{i=1}^{\infty} A^i$, where A^i being a half interval. Then by the above argument, we have $A^i = I^i$ and so on.

Focus on the case $I = (a, b]$. Let

$$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

which is a disjoint union. Then we only need to check

$$F(b) - F(a) = \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right).$$

- Since $(a, b] \supset \bigcup_{n=1}^{N} (a_n, b_n]$ for any fixed $N \in \mathbb{N}$, hence

  $$\forall_{N\in\mathbb{N}}\ F(b) - F(a) \geq \sum_{n=1}^{N} \left(F(b_n) - F(a_n)\right).$$

  By letting $N \to \infty$, we have

  $$F(b) - F(a) \geq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right).$$

- Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

  $$F(a') - F(a) < \epsilon.$$

  For each $n \in \mathbb{N}$, $\exists b'_n > b_n$ such that

  $$F(b'_n) - F(b_n) < \frac{\epsilon}{2^n}.$$

  Then, we have

  $$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b'_n),$$

  hence

  $$\exists_{N\in\mathbb{N}}\ [a', b] \subset \bigcup_{n=1}^{N} (a_n, b'_n),$$

  which is only finitely many unions now.

  Remark. This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.

  In this case, we have

  $$F(b) - F(a') \leq \sum_{n=1}^{N} F(b'_n) - F(a_n).$$

Finally, we see that

$$\begin{aligned} F(b)-F(a) &\leq F(b)-F(a')+\epsilon \\ &\leq \sum_{n=1}^{\infty}(F(b_n')-F(a_n))+\epsilon \\ &\leq \sum_{n=1}^{\infty}\left(F(b_n)-F(a_n)+\frac{\epsilon}{2^n}\right)+\epsilon \\ &= \sum_{n=1}^{\infty}(F(b_n)-F(a_n))+2\epsilon \end{aligned}$$

for any fixed $\epsilon > 0$, hence

$$F(b)-F(a) \leq \sum_{n=1}^{\infty}(F(b_n)-F(a_n)).$$

Combine these two inequalities, we have

$$F(b)-F(a) = \sum_{n=1}^{\infty}(F(b_n)-F(a_n))$$

as we desired.

■

Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8: Lebesgue-Stieltjes Measure on $\mathbb{R}$

To classify all measures, we now see this last theorem to complete the task. 24 Jan. 11:00

Theorem 1.5.1 (Locally finite Borel measures on $\mathbb{R}$)**.** We have

1. $F\colon \mathbb{R}\to\mathbb{R}$ a distribution function, then there exists a **unique** *locally finite Borel measure* μ_F on $\mathbb{R}$ satisfying
$$\mu_F((a,b]) = F(b)-F(a)$$
for every $a < b$.
2. Suppose $F, G\colon \mathbb{R}\to\mathbb{R}$ are distribution functions. Then,
$$\mu_F = \mu_G$$
on $\mathcal{B}(\mathbb{R})$ if and only if $F-G$ is a constant function.

Proof. ■ HW.

Remark. Theorem 1.5.1 simply states that given a distribution function, if we restrict our attention on locally finite measures on $\mathbb{R}$ following our usual convention, then it defines the measure on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

1.6 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that

$$F \text{ distribution function} \overset{!}{\Rightarrow} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 1.6.1 (Lebesgue-Stieltjes measure). Given a distribution function F, we say μ_F on $\mathcal{A}_{\mu_F}$ is called the *Lebesgue-Stieltjes measure* corresponding to F.

Definition 1.6.2 (Lebesgue measure, Lebesgue σ-algebra). From Definition 1.6.1, if $F(x) = x$, then the induced $(\mathcal{A}_{\mu_F}, \mu_F)$ is denoted as $(\mathcal{L}, m)$, where $\mathcal{L}$ is called *Lebesgue σ-algebra*, and m is called *Lebesgue measure*.

Remark. Recall that $\mathcal{L}$ is induced by Theorem 1.3.2, namely given m, for all $A \subset \mathbb{R}$, we have

$$\mathcal{L} := \left\{ A \subset \mathbb{R} \mid \underset{E \subset \mathbb{R}}{\forall}\ m(A) = m(A \cap E) + m(A \setminus E) \right\}.$$

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).$$

Some text will use $x-$ and $x+$ instead of x^- and x^+, respectively.

Example (Discrete measure). $\mu_F((a, b]) = F(b) - F(a)$. Then

- $\mu_F(\{a\}) = F(a) - F(a^-)$
- $\mu_F([a, b]) = F(b) - F(a^-)$
- $\mu_F((a, b)) = F(b^-) - F(a)$

This is so-called *discrete measure*.

Example (Dirac measure). We define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Then

- $\mu_F(\{0\}) = 1$
- $\mu_F(\mathbb{R}) = 1$
- $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. This is easy to see since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, hence

$$\mu_F(\mathbb{R} \setminus \{0\}) = \mu_F((-\infty, 0) \cup (0, \infty)) = \underbrace{\mu_F((-\infty, 0))}_{0-0^{2}} + \underbrace{\mu_F((0, \infty))}_{1-1^{3}} = 0,$$

  where $\mu_F((-\infty, 0)) = 0$ follows from $F(0^-) - F(-\infty) = 0 - 0 = 0$, while $\mu_F((0, \infty)) = 0$ follows from $F(\infty) - F(0) = 1 - 1 = 0$.

We call that μ_F is the *Dirac measure* at 0.

Example. Denote $\mathbb{Q} = \{r_1, r_2, \ldots\}$, and we define

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r. \end{cases}$$

Then

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$.

This is shown in Lemma A.1.2.

Example. If F is continuous at a, then $\mu_F(\{a\}) = 0$.

Example. $F(x) = x$, then recall that we denote $\mu_F \coloneqq m$, and we have

- $m((a, b]) = m((a, b)) = m([a, b]) = b - a$.

Example. $F(x) = e^x$

- $\mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$.

Example (Middle thirds Cantor set)**.** Let $C \coloneqq \bigcap_{n=1}^{\infty} K_n$, where we have

$$\begin{aligned} K_0 &\coloneqq [0, 1] \\ K_1 &\coloneqq K_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \\ K_2 &\coloneqq K_1 \setminus \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \\ &\vdots \\ K_n &\coloneqq K_{n-1} \setminus \bigcup_{k=1}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3^{k+2}}{3^{n+1}}\right). \end{aligned}$$

We see that C is uncountable and with $m(C) = 0$. And observe that $x \in C$ if and only if $x = \sum_{n=1}^{\infty} \frac{a_n}{3}$ for some $a_n \in \{0, 2\}$. Hence, we can instead formulate K_n by

$$K_n = \bigcup_{\substack{a_i \in \{0,2\} \\ 1 \leq i \leq n}} \left[\sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n}\right].$$

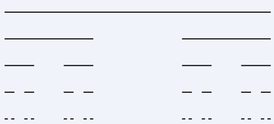


Figure 1.1: The top line corresponds to K_0, and then K_1, etc.

The proof of $m(C) = 0$ is given in Lemma A.1.3.

Cantor Function

Consider F as follows. We define a function F to be 0 to the left of 0, and 1 to the right of 1. Then, define F to be $\frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$, $\frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9}\right)$ and so on. This is so-called *Cantor Function*. We can show F is continuous and increasing, which makes F a distribution function. Also, we see that the measure this F induced is called *Cantor measure*.

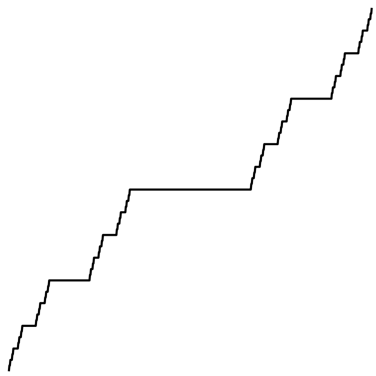


Figure 1.2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

Cantor Measure μ_F		Lebesgue Measure m
$\mu_F(\mathbb{R} \setminus C) = 0$		$m(\mathbb{R} \setminus C) = \infty > 0$
$\mu_F(C) = 1$	$\Leftrightarrow$	$m(C) = 0$
$\mu_F(\{a\}) = 0$		$m(\{a\}) = 0$

Remark. μ_F and m are said to be **singular** to each other.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 1.7.1. Let μ be Lebesgue-Stieltjes measure on $\mathbb{R}$. Then we have

$$\begin{aligned}\mu(A) &\overset{!}{=} \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}\end{aligned}$$

for every $A \in \mathcal{A}_\mu$

Proof. The second equality follows from the continuity of the measure. ■

Remark. This is similar to

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - 1/n], \quad (a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n].$$

Lecture 9: Properties of Lebesgue-Stieltjes measure

26 Jan. 11:00

As previously seen. Let $X \subset [0, \infty]$. Recall that

- Finite supremum.
$$\alpha = \sup X < \infty \Leftrightarrow \begin{cases} \underset{x \in X}{\forall} \ \alpha \geq x \\ \underset{\epsilon > 0}{\forall} \ \underset{x \in X}{\exists} \ x + \epsilon \geq \alpha. \end{cases}$$
- Infinite supremum.
$$\alpha = \sup X = \infty \Leftrightarrow \underset{L > 0}{\forall} \ \underset{x \in X}{\exists} \ x \geq L.$$

This should be useful latter on.

Theorem 1.7.1 (Regularity). Let μ be Lebesgue-Stieltjes measure. Then, for all $A \in \mathcal{A}_\mu$,

1. (outer regularity) $\mu(A) = \inf\{\mu(O) \mid O \supset A, O \text{ is open}\}$
2. (inner regularity) $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$

Proof. We check them separately.

1. DIY
2. Let $s := \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$, then by monotonicity, we have $\mu(A) \geq s$. To show the other direction, we consider
   - A is a bounded set.
     Then $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, $\overline{A}$ is also bounded $\Rightarrow \mu(\overline{A}) < \infty$. Fix $\epsilon > 0$, then by outer regularity, there exists an open $O \supset \overline{A} \setminus A$, and $\mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \epsilon$. Let $K := \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$, we show that
     $$\mu(K) \geq \mu(A) - \epsilon.$$
     DIY
   - A is an unbounded set with $\mu(A) < \infty$.
     Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \ldots$, then
     $$\lim_{n \to \infty} \mu(A_n) = \mu(A) < \infty.$$
   - A is an unbounded set with $\mu(A) = \infty$.
     We can show that
     $$\lim_{n \to \infty} \mu(A_n) = \mu(A) = \infty.$$
     Fix $L > 0$, then $\exists N$ such that $\mu(A_N) \geq L$.

$\blacksquare$

Definition 1.7.1 (G_δ-set, F_σ-set). Let X be a topological space. Then

- A *G_δ-set* is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open.
- A *F_σ-set* is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.7.2. Let μ be a Lebesgue-Stieltjes measure. Then *TFAE*[a]:

1. $A \in \mathcal{A}_\mu$
2. $A = G \setminus M$, G is a G_δ-set, M is a μ-null set.
3. $A = F \setminus N$, F is a F_σ-set, N is a μ-null set.

[a] *TFAE*: The following are equivalent.

Proof. We see that (2.) $\Rightarrow$ (1.) and (3.) $\Rightarrow$ (1.) are clear.

- (1.) $\Rightarrow$ (3.)
  - Assume $\mu(A) < \infty$. From the inner regularity, we have

    $$\forall n \in \mathbb{N}\ \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

    Let $F = \bigcup_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ-null. Check!
  - Assume $\mu(A) = \infty$. Let $A = \bigcup_{k\in\mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

    $$\forall k \in \mathbb{Z}\ A_k = F_k \cup N_k,\ A = \underbrace{\left(\bigcup_k F_k\right)}_{F_\sigma\text{-set}} \cup \underbrace{\left(\bigcup_k N_k\right)}_{\mu\text{-null}}.$$

- (1.) $\Rightarrow$ (2.)
  We see that

  $$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

$\blacksquare$

Proposition 1.7.1. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0\ \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using outer regularity and the fact that every open set is $\bigcup_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. $\blacksquare$ DIY

We now see some properties of Lebesgue measure.

Theorem 1.7.3. Let $A \in \mathcal{L}$, then we have $A + s \in \mathcal{L}$, $rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,

$$m(A + s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. $\blacksquare$ DIY

Example. We now see some examples.

1. Let $\mathbb{Q} =: \{r_i\}_{i=1}^{\infty}$ which is dense in $\mathbb{R}$. Let $\epsilon > 0$, and

$$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i}\right).$$

We see that O is open and dense[a] in $\mathbb{R}$. But we see

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$

Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$

2. There exists uncountable set A with $m(A) = 0$.
3. There exists A with $m(A) > 0$ but A contains no non-empty open intervals.
4. There exists $A \notin \mathcal{L}$. e.g. Vitali set.[b]
5. There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

[a] https://en.wikipedia.org/wiki/Dense_set
[b] https://en.wikipedia.org/wiki/Vitali_set

Chapter 2

Integration

Lecture 10: Integration

2.1 Measurable Function

26 Jan. 11:00

We start with a definition.

Definition 2.1.1 (Measurable function)**.** Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$ are measurable spaces. Then we say $f\colon X \to Y$ is $(\mathcal{A}, \mathcal{B})$-*measurable* if

$$\underset{B \in \mathcal{B}}{\forall} f^{-1}(B) \in \mathcal{A}.$$

Remark. If $\mathcal{A}$ and $\mathcal{B}$ are given, we'll sometimes say f is measurable if it'll not cause any confusions.

Lemma 2.1.1. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, and suppose $\mathcal{B} = \langle \mathcal{E} \rangle$ for some $\mathcal{E} \subset Y$. Then,

$$f\colon X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \Leftrightarrow \underset{E \in \mathcal{E}}{\forall} f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part ($\Rightarrow$) is clear. On the other direction, we consider the following. Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $\mathcal{E} \subset \mathcal{D}$ by assumption
- $\mathcal{D}$ is a σ-algebra

Check!

hence, we see that $\langle \mathcal{E} \rangle = \mathcal{B} \subset \mathcal{D}$ from Lemma 1.1.2. The result then follows from the definition of $(\mathcal{A}, \mathcal{B})$-measurable. ∎

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup_\alpha E_\alpha\right) = \bigcup_\alpha f^{-1}(E_\alpha)$

Definition 2.1.2 ($\mathcal{A}$-measurable)**.** Let $(X, \mathcal{A})$ be a measurable space. Then,

$$\left.\begin{array}{l} f\colon X \to \mathbb{R} \\ f\colon X \to \overline{\mathbb{R}} \\ f\colon X \to \mathbb{C} \end{array}\right\} \text{ is } \mathcal{A}\text{-}\textit{measurable} \text{ if } \begin{cases} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \operatorname{Re} f, \operatorname{Im} f\colon X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{cases}$$

Notation. Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$

- $\mathcal{B}(\overline{\mathbb{R}}) = \left\{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\right\}$.
- $\operatorname{Re} f$ is the real part of f, while $\operatorname{Im} f$ is the imaginary part of f.

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \Rightarrow$ Every function is $\mathcal{A}$-measurable.
- $\mathcal{A} = \{\varnothing, X\} \Rightarrow$ The only $\mathcal{A}$-measurable functions are constant functions.

Definition 2.1.3 (Lebesgue measurable)**.** A *Lebesgue measurable function* f is a measurable function

$$f\colon (\mathbb{R}, \mathcal{L}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})).$$

Lemma 2.1.2. Given $f\colon X \to \mathbb{R}$, *TFAE.*

1. f is $\mathcal{A}$-measurable
2. $\forall a \in \mathbb{R},\ f^{-1}((a, \infty)) \in \mathcal{A}$
3. $\forall a \in \mathbb{R},\ f^{-1}([a, \infty)) \in \mathcal{A}$
4. $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a)) \in \mathcal{A}$
5. $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from Lemma 2.1.1 we just saw. ∎

Remark (Operations preserve $\mathcal{A}$-measurability)**.** Given $f, g\colon X \to \mathbb{R}$ and is $\mathcal{A}$-measurable, then

1. $\phi\colon \mathbb{R} \to \mathbb{R}$, $\mathcal{A}$-measurable[a], then
$$\phi \circ f\colon X \to \mathbb{R}$$
is $\mathcal{A}$-measurable.
2. $-f,\ 3f,\ f^2,\ |f|$ are all $\mathcal{A}$-measurable, and $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) \neq 0, \forall x \in X$.
3. $f + g$ is $\mathcal{A}$-measurable. We see this from
$$(f+g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r, \infty)) \cap g^{-1}((a-r, \infty))\right)$$
with Lemma 2.1.2.
4. $f \cdot g$ is $\mathcal{A}$-measurable. We see this from
$$f(x)g(x) = \frac{1}{2}\left((f(x)+g(x))^2 - f(x)^2 - g(x)^2\right).$$
5. We see that
$$(f \vee g)(x) \coloneqq \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) \coloneqq \min\{f(x), g(x)\}$$
are $\mathcal{A}$-measurable.
6. Let $f_n\colon X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable. Then
$$\sup_{n \in \mathbb{N}} f_n,\ \inf_{n \in \mathbb{N}} f_n,\ \limsup_{n \to \infty} f_n,\ \liminf_{n \to \infty} f_n$$
are $\mathcal{A}$-measurable.

Proof. Consider $\sup_{n\in\mathbb{N}} f_n =: g$, then

$$g^{-1}((a,\infty]) = \bigcup_{n\in\mathbb{N}} f_n^{-1}((a,\infty])$$

for $\sup_n f_n(x) = g(x) > a$. A similar argument can prove the case of $\inf_{n\in\mathbb{N}} f_n$.

check

[a]In this case

And notice that $\limsup_{n\to\infty} f_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} f_n$, then the similar argument also proves this case. ∎

7. If $\lim_{n\to\infty} f_n(x)$ converges for every $x \in X$, then f is $\mathcal{A}$-measurable.
8. If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous

   $\Rightarrow$ f is Borel measurable

   $\Rightarrow$ f is Lebesgue measurable

   since the preimage of an open set of a continuous function is open, then we consider $f^{-1}((a,\infty))$.

Definition 2.1.4 (Support). The *support* of function $f\colon X \to \overline{\mathbb{R}}$ is

$$\operatorname{supp} f := \{x \in X \mid f(x) \neq 0\}.$$

Definition 2.1.5 (Positive and Negative part). For $f\colon X \to \overline{\mathbb{R}}$, let $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$,[a] where we call f^+ the *positive part* of f while f^- the *negative part* of f.

[a]i.e., $f^+(x) = \max\{f(x), 0\}, \quad f^-(x) = \max\{-f(x), 0\}$

Remark. If $\operatorname{supp} f^+ \cap \operatorname{supp} f^- = \varnothing$ and $f(x) = f^+(x) - f^-(x)$, then

$$f \text{ is } \mathcal{A}\text{-measurable} \Leftrightarrow f^+, f^- \text{ are } \mathcal{A}\text{-measurable}.$$

Definition 2.1.6 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator) function* of E is

$$\mathcal{X}_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is $\mathcal{A}$-measurable $\Leftrightarrow E \in \mathcal{A}$.

Definition 2.1.7 (Simple function). Let $(X, \mathcal{A})$ be a measurable space. Then a *simple function* $\phi\colon X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

Remark. We see that if

$$\phi(X) = \{c_1, \ldots, c_N\},$$

then

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \Rightarrow \phi = \sum_{i=1}^N \underbrace{c_i}_{\neq\pm\infty} \mathbb{1}_{\underbrace{E_i}_{\in\mathcal{A}}}.$$

Lecture 11: Integration of nonnegative functions

31 Jan. 11:00

As previously seen. For a simple function ϕ, c_i can actually be in $\mathbb{C}$.

Theorem 2.1.1. Given a measurable space $(X, \mathcal{A})$ and let $f\colon X \to [0, \infty]$, the followings are equivalent.

1. f is a $\mathcal{A}$-measurable function.
2. There exists simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ such that

$$\underset{x\in X}{\forall} \lim_{n\to\infty} \phi_n(x) = f(x)$$

i.e., f is a pointwise upward limit of simple functions.

Proof. We'll prove both directions.

- It's clear that (2.) $\Rightarrow$ (1.) from the fact that $f(x) = \sup_n \phi_n(x)$ and the remark.
- We want to show that (1.) $\Rightarrow$ (2.). Assume f is $\mathcal{A}$-measurable, and fix $n \in \mathbb{N}$.
  Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. Also, for $0 \leq k \leq 2^{2n} - 1$, $E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}$.
  Then, define ϕ_n be

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n},$$

  we have
  - $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ for every $x \in X$
  - $\forall x \in X \setminus F_n$, we have $0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$

  Furthermore, we see that

$$F_1 \supset F_2 \supset \ldots, \quad \bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\}),$$

  then
  - $x \in f^{-1}([0, \infty]) = X \setminus \bigcap_{n=1}^{\infty} F_n \Rightarrow \lim_{n\to\infty} \phi_n(x) = f(x)$
  - $x \in f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} F_n \Rightarrow f_n(x) \geq 2^n \Rightarrow \lim_{n\to\infty} \phi_n(x) = \infty = f(x)$

■

Corollary 2.1.1. If f is bounded on a set $A \subset \mathbb{R}$, i.e., $\exists L > 0$ such that

$$\underset{x\in A}{\forall} |f(x)| \leq L,$$

then there exists a sequence of simple functions $\{\phi_n\}$ such that $\phi_n \to f$ uniformly on A.

Proof. ■

DIY

Corollary 2.1.2. If $f\colon X \to \mathbb{C}$ is a measurable function if and only if there exists simple functions $\phi_n\colon X \to \mathbb{C}$ such that

$$0 \leq |\phi_1(x)| \leq |\phi_2(x)| \leq \ldots \leq |f(x)|$$

with

$$\underset{x\in X}{\forall} \lim_{n\to\infty} \phi_n(x) = f(x).$$

Proof. ■ DIY

2.2 Integration of Nonnegative Functions

We start with our first definition about integral.

Definition 2.2.1 (Integration of nonnegative funciton)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\phi\colon X \to [0, \infty]$ such that

$$\phi = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}$$

be a simple function. Define

$$\int \phi = \int \phi \,\mathrm{d}\mu = \int_X \phi \,\mathrm{d}\mu = \sum_{i=1}^{N} c_i \mu(E_i).$$

Furthermore, for $A \in \mathcal{A}$,

$$\int_A \phi = \int_A \phi \,\mathrm{d}\mu = \int \phi \mathbb{1}_A \,\mathrm{d}\mu.$$

Note. Note that

- In the expression $\sum_{i=1}^{N} c_i \mu(E_i)$, we're using the convention $0 \cdot \infty = 0$.
- The function $\phi \mathbb{1}_A$ is also a simple function since both ϕ and $\mathbb{1}_A$ are simple function.

Proposition 2.2.1. Suppose we have $\phi, \psi \geq 0$ be two simple functions. Then,

- Definition 2.2.1 is well-defined.
- $\int c\phi = c \int \phi$ for $c \in [0, \infty)$.
- $\int \phi + \psi = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x)$ for all $x \Rightarrow \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi \,\mathrm{d}\mu$ is a measure on $(X, \mathcal{A})$.

Proof. ■ DIY

Definition 2.2.2 (Generatlization of Integration of nonnegative function)**.** Given $(X, \mathcal{A}, \mu)$ with $f\colon X \to [0, \infty]$ be $\mathcal{A}$-measurable. Define

$$\int f = \int f \,\mathrm{d}\mu = \sup \left\{ \int \phi \colon 0 \leq \phi \leq f \text{ such that } \phi \text{ is simple} \right\}.$$

Note. Note that

- If f is a simple function, the Definition 2.2.1 and Definition 2.2.2 of $\int f$ are the same.
- $\int cf = c \int f$ for $c \in [0, \infty)$.
- If $f \geq g \geq 0 \Rightarrow \int f \geq \int g$.
- But $\int f + g = \int f + \int g$ is not trivial.

Theorem 2.2.1 (Monotone Convergence Theorem)**.** Given $(X, \mathcal{A}, \mu)$ be a measure space. Then if

- $f_n \colon X \to [0, \infty]$ be $\mathcal{A}$-measurable for every $n \in \mathbb{N}$;
- $0 \leq f_1(x) \leq f_2(x) \leq \ldots$ for every $x \in X$;
- $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$,

we have

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. Note that if $\lim_{n\to\infty} \int f_n$ exists, then it's equal to $\sup_n \int f_n$.

Then

- $f_n \leq f \Rightarrow \int f_n \leq \int f \Rightarrow \lim_{n\to\infty} \int f_n \leq \int f$.
- Fix a simple function $0 \leq \phi \leq f$, then it's enough to show $\lim_{n\to\infty} \int f_n \geq \int \phi$.

  We first fix $\alpha = (0, 1)$, then it's also enough to show

  $$\lim_{n\to\infty} \int f_n \geq \alpha \int \phi.$$

  Let $A_n \coloneqq \{x \in X \mid f_n(x) \geq \alpha\phi(x)\}$, then since f_n is measurable,

  - $A_n \in \mathcal{A}$
  - $A_1 \subset A_2 \subset A_3 \subset \ldots$
  - $\bigcup_{n=1}^{\infty} A_n = X$

  Check!

  We then have

  $$\int f_n \geq \int f_n \mathbb{1}_{A_n} \geq \int \alpha\phi\mathbb{1}_{A_n} = \alpha \int_{A_n} \phi = \alpha\nu(A_n)$$

  where $\nu(A) = \int_A \phi$ is a measure. This implies

  $$\lim_{n\to\infty} \int f_n \geq \alpha \lim_{n\to\infty} \nu(A_n) \overset{!}{=} \alpha\nu(X) = \alpha \int \phi.$$

∎

Corollary 2.2.1 (Linearity of nonnegative integral)**.** Let $f, g \geq 0$ be measurable, then

$$\int f + g = \int f + \int g.$$

Proof. There exists simple functions ϕ_n and ψ_n such that

- $0 \leq \phi_1 \leq \phi_2 \leq \ldots$ and $\phi_n \to f$ pointwise
- $0 \leq \psi_1 \leq \psi_2 \leq \ldots$ and $\psi_n \to g$ pointwise

Then,

$$\int (f+g) \overset{!}{=} \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \int \phi_n + \int \psi_n \overset{!}{=} \int f + \int g.$$

∎

Lecture 12: Fatou's Lemma

We start with a useful corollary. 2 Feb. 11:00

Corollary 2.2.2 (Tonelli's theorem for nonnegative series and integrals)**.** Given $g_n \geq 0$ for every $n \in \mathbb{N}$ and let g_n be measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Proof. Let $f_N := \sum_{n=1}^{N} g_n$ such that $\lim_{N\to\infty} f_N \sum_{n=1}^{\infty} g_n =: f$, then since $g_n \geq 0$, we have $0 \leq f_1 \leq f_2 \leq \ldots$ with

$$\lim_{N\to\infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

By Theorem 2.2.1, we have

$$\lim_{N\to\infty} \int \underbrace{\sum_{n=1}^{N} g_n}_{f_N} = \int \underbrace{\sum_{n=1}^{\infty} g_n}_{f}.$$

Now, since the terms in the limit on the left-hand side is just a finite sum, by Corollary 2.2.1, we have

$$\underbrace{\lim_{N\to\infty} \sum_{n=1}^{N}}_{\sum_{n=1}^{\infty}} \int g_n = \lim_{N\to\infty} \int \sum_{n=1}^{N} g_n = \int \sum_{n=1}^{\infty} g_n,$$

hence

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

∎

Remark. Recall that we have seen two series case before. We'll later see two integrals cases.

Theorem 2.2.2 (Fatou's Lemma)**.** Suppose $f_n \geq 0$ and measurable, then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Remark. Recall that

$$\liminf_{n\to\infty} f_n := \lim_{k\to\infty} \inf_{n\geq k} f_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} f_n$$

and

$$\exists \lim_{n\to\infty} a_n \Leftrightarrow \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Proof. Let $g_k = \inf_{n\geq k} f_n$, then g_k is measurable and $0 \leq g_1 \leq g_2 \leq \ldots$. Now, from Theorem 2.2.1, we have

$$\int \lim_{k\to\infty} g_k = \lim_{k\to\infty} \int g_k.$$

Notice that the left-hand side is just $\int \liminf_{n\to\infty} f_n$, while the right-hand side is just $\lim_{k\to\infty} \int \inf_{n\geq k} f_n$, i.e.,

$$\int \liminf_{n\to\infty} f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n.$$

We see that we want to take the inf outside the integral on the right-hand side. Observe that

$$\forall_{m\geq k} \inf_{n\geq k} f_n \leq f_m \Rightarrow \forall_{m\geq k} \int \inf_{n\geq k} f \leq \int f_m \Rightarrow \int \inf_{n\geq k} f_n \leq \inf_{m\geq k} \int f_m.$$

Then, we have

$$\int \liminf_{n\to\infty} f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n \leq \lim_{k\to\infty} \inf_{m\geq k} \int f_m = \liminf_{m\to\infty} \int f_m.$$

■

Example (Escape to horizontal infinity). Given $(\mathbb{R}, \mathcal{L}, m)$, let $f_n \coloneqq \mathbb{1}_{(n,n+1)}$. We immediately see that

- $f_n \to 0$ pointwise
- $\int f_n = 1$ for every n
- $\int f = 0$

From Theorem 2.2.2, we have a strict inequality

$$0 = \int \liminf_{n\to\infty} f_n, \liminf_{n\to\infty} \int f_n = 1.$$

Example (Escape to width infinity). Given $(\mathbb{R}, \mathcal{L}, m)$, let $f_n \coloneqq \frac{1}{n}\mathbb{1}_{(0,n)}$.

Example (Escape to vertical infinity). Given $(\mathbb{R}, \mathcal{L}, m)$, let $f_n \coloneqq n\mathbb{1}_{(0,\frac{1}{n})}$.

Lemma 2.2.1 (Markov's inequiality). Let $f \geq 0$ be measurable. Then

$$\underset{c\in(0,\infty)}{\forall} \mu(\{x \mid f(x) \geq c\}) \leq \frac{1}{c}\int f.$$

Proof. Denote $\{x \mid f(x) \geq c\} =: E$, then

$$f(x) \geq c\mathbb{1}_E(x) \Rightarrow \int f \geq c\int \mathbb{1}_E = c \cdot \mu(E).$$

■

Remark. Notice that $E = f^{-1}([c, \infty])$, hence E is measurable.

Proposition 2.2.2. Let $f \geq 0$ be measurable. Then,

$$\int f = 0 \Leftrightarrow f = 0 \text{ a.e.}.$$

i.e.,

$$\int f \,\mathrm{d}\mu = 0 \Leftrightarrow \mu(A) = 0$$

where $A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$.

Proof. Firstly, assume that $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}$$

where E_i are disjoint and $c_i \in (0, \infty)$. Then,

$$\begin{aligned}\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0 \\ \Leftrightarrow \mu(E_1) = \ldots = \mu(E_N) = 0 \\ \Leftrightarrow \mu(A) = 0,\ A = \bigcup_{i=1}^{N} E_i.\end{aligned}$$

Now, assume that f is a general function where $f \geq 0$ is the only constraint.

1. Assume $\mu(A) = 0$ (i.e., $f = 0$ a.e.). Let $0 \leq \phi \leq f$, where ϕ is simple. Then

$$\underset{x \in A^c}{\forall}\ \phi(x) = 0$$

since $f(x) = 0$, $\forall x \in A^c$. This implies that $\phi = 0$ a.e. since $\mu(A) = 0$, so $\int \phi = 0$. We then have

$$\int f = 0$$

from Definition 2.2.2.

2. Assume $\int f = 0$. Let $A_n = f^{-1}\left(\left[\frac{1}{n}, \infty\right]\right)$. Then we see that

- $A_1 \subset A_2 \subset \ldots$
- $\bigcup\limits_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup\limits_{n=1}^{\infty} \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0, \infty)) = A.$

We then have

$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \overset{!}{\leq} n \int f = 0,$$

which further implies

$$\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0$$

from the continuity of measure from below.

■

Corollary 2.2.3. If $f, g \geq 0$ are both measurable and $f = g$ a.e., then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \neq g(x)\}$[a]. Then by assumption, $\mu(A) = 0$, hence

$$f \mathbb{1}_A = 0 \text{ a.e.}, \quad g \mathbb{1}_A = 0 \text{ a.e.}.$$

This further implies that

$$\begin{aligned}\int f = \int f(\mathbb{1}_A + \mathbb{1}_{A^c}) &\overset{!}{=} \int f \mathbb{1}_A + \int f \mathbb{1}_{A^c} \\ &= \int f \mathbb{1}_{A^c} = \int g \mathbb{1}_{A^c} = \int g \mathbb{1}_{A^c} + \int g \mathbb{1}_A = \int g.\end{aligned}$$

■

[a] A is measurable indeed.

Corollary 2.2.4. Let $f_n \geq 0$ be measurable. Then

1. $$\left.\begin{array}{l} 0 \leq f_1 \leq f_2 \leq \ldots \leq f \text{ a.e.} \\ \lim\limits_{n\to\infty} f_n = f \text{ a.e.} \end{array}\right\} \Rightarrow \lim_{n\to\infty} \int f_n = \int f.$$

2. $\lim\limits_{n\to\infty} f_n = f$ a.e. $\Rightarrow \int f \leq \liminf\limits_{n\to\infty} \int f_n$.

Proof. ■ DIY

Remark. Almost all the theorems we've proved can be replaced by theorems dealing with almost everywhere condition.

Lecture 13: Integration of Complex Functions

2.3 Integration of Complex Functions

4 Feb. 11:00

As usual, we start with a definition.

Definition 2.3.1 (Integrable). Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f\colon X \to \overline{\mathbb{R}}$ and $g\colon X \to \mathbb{C}$ be measurable.[a]

Then f, g are called *integrable* if $\int |f| < \infty$ and $\int |g| < \infty$, and we define

$$\int f = \int f^+ - \int f^-, \quad \int g = \int \operatorname{Re} g + i \int \operatorname{Im} g.$$

Furthermore, for $f\colon X \to \overline{\mathbb{R}}$, we define

$$\int f = \begin{cases} \infty, & \text{if } \int f^+ = \infty, \int f^- < \infty; \\ -\infty, & \text{if } \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

[a]Recall that for a complex-valued function like g, this means that both $\operatorname{Re} g$ and $\operatorname{Im} g$ are measurable.

We now see a lemma.

Lemma 2.3.1. Let $f, g\colon X \to \overline{\mathbb{R}}$ or $\mathbb{C}$ integrable. Assume that $f(x) + g(x)$ is well-defined for all $x \in X$.[a]

Then we have

1. $f + g$, cf for all $c \in \mathbb{C}$ are integrable.
2. $\int f + g = \int f + \int g$. This is not trivial since $(f+g)^+ \neq f^+ + g^+$.
3. $\left|\int f\right| \leq \int |f|$.

[a]That is, we never see $\infty + (-\infty)$ or $(-\infty) + \infty$.

Proof. Check [FF99] page 53. ■

Lemma 2.3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be an integrable function on X. Then

1. f is finite a.e.. i.e., $\{x \in X \mid |f(x)| = \infty\}$ is a null set.
2. The set $\{x \in X \mid f(x) \neq 0\}$ is σ-finite.

Proof. ■ HW 5 Q8 by Lemma 2.2.1

Proposition 2.3.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, then

1. If h is integrable on X, then

$$\underset{E\in\mathcal{A}}{\forall} \int_E h = 0 \Leftrightarrow \int |h| = 0 \Leftrightarrow h = 0 \text{ a.e.}$$

2. If f, g are integrable on X, then

$$\underset{E\in\mathcal{A}}{\forall} \int_E f = \int_E g \Leftrightarrow f = g \text{ a.e.}$$

Proof. We prove this one by one.

1. We see that the second equivalence is done in Proposition 2.2.2, hence we prove the first equivalence only. Since we have

$$\int |h| = 0 \Rightarrow \left|\int_E h\right| \leq \int_E |h| \leq \int |h| = 0,$$

which shows one implication. Now assume that $\int_E h = 0$ for all $E \in \mathcal{A}$, then we can write h as

$$h = u + iv = (u^+ - u^-) + i(v^+ - v^-).$$

Let $B := \{x \in X \mid u^+(x) > 0\}$, then by assumption, we have

$$0 = \int_B h = \operatorname{Re}\int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+,$$

hence $u^+ = 0$ almost everywhere. Similarly, we have u^-, v^+, v^- are all zero almost everywhere. This gives us that h is zero almost everywhere as desired.

2. DIY

∎

Theorem 2.3.1 (Dominated Convergence Theorem). Let $(X, \mathcal{A}, \mu)$ be a measure space, and

- Let f_n be integrable on X.
- $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere.
- There is a $g\colon X \to [0, \infty]$ such that g is integrable and

$$\underset{n\in\mathbb{N}}{\forall} |f_n(x)| \leq g(x) \text{ a.e.}$$

Then we have

$$\lim_{n\to\infty} \int f_n = \int f = \int \lim_{n\to\infty} f_n.$$

Proof. Let F be the countable union of null set on which the three conditions may fail. Then we see that after modifying the definition of f_n, f and g on F, we may assume that all three conditions hold everywhere since modifying on a null set does not change the integral.

We now consider the $\overline{\mathbb{R}}$-valued case only. Note that the second and the third conditions imply that f is integrable since $|f| \leq g(x)$. We then see that $g + f_n \geq 0$ and $g - f_n \geq 0$ because $-g \leq f_n \leq g$. From Theorem 2.2.2, we have

Check $\mathbb{C}$-valued case

$$\int g + f \leq \liminf_{n\to\infty} \int g + f_n, \quad \int g - f \leq \liminf_{n\to\infty} \int g - f_n.$$

From the linearity of integral, we have

$$\int g + \int f \leq \int g + \liminf_{n\to\infty} \int f_n, \quad \int g - \int f \leq \int g - \liminf_{n\to\infty} \int f_n.$$

Now, since $\int g < \infty$, we can cancel it, which gives

$$\int f \leq \liminf_{n\to\infty} \int f_n, \quad -\int f \leq \liminf_{n\to\infty} \int -f_n = -\limsup_{n\to\infty} \int f_n,$$

which implies

$$\int f \leq \liminf_{n\to\infty} f_n \leq \limsup_{n\to\infty} \int f_n \leq \int f.$$

This shows that the limit exists, and the desired result indeed holds. ■

Corollary 2.3.1 (Tonelli's theorem for series and integrals)**.** Suppose f_n are integrable functions such that

$$\sum_{n=1}^{\infty} \int |f_n| < \infty,$$

then we have

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

Proof. Take $G(x)$ to be

$$G(x) \coloneqq \sum_{n=1}^{\infty} |f_n(x)|,$$

then we see

$$G(x) \geq |F_N(x)|$$

where

$$F_N(x) \coloneqq \sum_{n=1}^{N} f_n(x).$$

By Corollary 2.2.2, we have

$$\int G(x) = \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Lastly, from Theorem 2.3.1, the result follows. ■

Remark. Compare to Corollary 2.2.2, we see that we further generalize the result!

Lecture 14: L^1 Space

2.4 L^1 Space

7 Feb. 11:00

We now introduce another space called L^p *spaces*, which are function spaces defined using a natural generalization of the p-norm for finite-dimensional vector spaces. We sometimes call it Lebesgue spaces also.

Before we start, we need to define a *norm*.

Definition 2.4.1 (Seminorm)**.** Let V be a vector space over filed $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* on V is

$$\|\cdot\| : V \to [0, \infty)$$

such that

- $\|cv\| = |c|\,\|v\|$ for every $v \in V$ and every scalar c.
- $\|v + w\| \leq \|v\| + \|w\|$ for every $v, w \in V$.

Definition 2.4.2 (Norm). A *norm* is a seminorm with

- $\|v\| = 0 \Leftrightarrow v = 0$.

Lemma 2.4.1. A normed vector space is a metric space with metric

$$\rho(v, w) = \|v - w\|.$$

Proof. ■ DIY

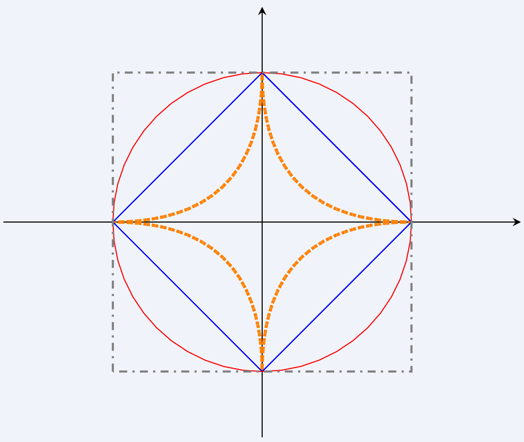
Example (p-norm). $V = \mathbb{R}^d$ with

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}, & \text{if } p \in [0, \infty); \\ \max_{1 \leq i \leq d} |x_i|, & \text{if } p = \infty \end{cases}$$

is a normed vector space. The unit ball

$$\{x \in \mathbb{R}^d \mid \|x\|_p \leq 1\}$$

for different p has the following figures.



Remark. All $\|\cdot\|_p$ norms induce the same topology. i.e., if U is open in p-norm, it is open in p'-norm as well.

Note. Recall that we say f is integrable means

$$\int |f| < \infty,$$

and if $f = g$ a.e., then

$$\int f = \int g$$

Definition 2.4.3 (L^1 Space). Given $(X, \mathcal{A}, \mu)$,

$$f \in L^1(X, \mathcal{A}, \mu) \left(= L^1(X, \mu) = L^1(X) = L^1(\mu)\right)$$

means that f is an integrable function on X.

Lemma 2.4.2. $L^1(X, \mathcal{A}, \mu)$ is a vector space with seminorm

$$\|f\|_1 = \int |f| \, .$$

Proof. ■

Check this is indeed a seminorm.

Definition 2.4.4 (L^1 Space with equivalence class). Define $f \sim g$ if $f = g$ a.e., then

$$L^1(X, \mathcal{A}, \mu) \Big/ \sim \; = L^1(X, \mathcal{A}, \mu),$$

i.e., we simply denote the collection of equivalence classes by itself.[a]

[a] By some abusing of notation of L^1.

Remark. We have

- With Definition 2.4.4, $L^1(X, \mathcal{A}, \mu)$ is a normed vector space.
- We say that the L^1-*metric* $\rho(f, g)$ is simply

$$\rho(f, g) = \int |f - g| \, .$$

Dense Subsets of L^1

Note. Recall the definition of a *dense set*[a].

[a] https://en.wikipedia.org/wiki/Dense_set

Definition 2.4.5 (Step function). A *step function* on $\mathbb{R}$ is

$$\psi = \sum_{i=1}^{N} c_i \mathbb{1}_{I_i},$$

where I_i is an interval.

Notation. We denote the collection of continuous functions with compact support by $C_c(\mathbb{R})$.

Theorem 2.4.1. We have the following.

1. {integrable simple functions} is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1-metric).
2. $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$, where μ is a Lebesgue-Stieltjes-measure. Then the set of integrable simple functions is dense in $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu)$.
3. $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

Proof. We prove this one by one.

1. Since there exists simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$, where $\phi_n \to f$ pointwise. Then

by Theorem 2.3.1, we have

$$\lim_{n\to\infty}\int \underbrace{|f_n - f|}_{\leq|\phi_n|+|f|\leq 2|f|} = 0$$

where $2|f|$ is in L^1.

2. Let $\mathbb{1}_E$ approximate by $\sum_{i=1}^{\infty} c_i \mathbb{1}_{I_i}$. From Theorem 1.7.1 for Lebesgue-Stieltjes-measure,

$$\forall \epsilon' > 0\ \exists I = \bigcup_{i=1}^{N} I_i \text{ such that } \mu(E \triangle I) \leq \epsilon'.$$

3. To approximate $\mathbb{1}_{(a,b)}$, we simply consider function $g \in C_c(\mathbb{R})$ such that

$$\int \left|\mathbb{1}_{(a,b)} - g\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

■

Lecture 15: Riemann Integral

9 Feb. 11:00

2.5 Riemann Integrability

We are now working in $(\mathbb{R}, \mathcal{L}, m)$. Let's first revisit the definition of Riemann Integral. Let P be a partition of $[a,b]$ as

$$P = \{a = t_0 < t_1 < \ldots < t_k = b\}.$$

Then the *lower Riemann sum* of f using P is equal to L_P, which is defined as

$$L_P = \sum_{i=1}^{K}\left(\inf_{[t_{i-1},t_i]} f\right)(t_i - t_{i-1}),$$

and the *upper Riemann sum* of f using P is equal to U_P, which is defined as

$$U_P = \sum_{i=1}^{K}\left(\sup_{[t_{i-1},t_i]} f\right)(t_i - t_{i-1}).$$

Then we call

- *Lower Riemann integral* of $f = \underline{I} = \sup_P L_P$
- *Upper Riemann integral* of $f = \overline{I} = \inf_P U_P$

Definition 2.5.1 (Riemann (Darboux) integrable). A bounded function $f\colon [a,b] \to \mathbb{R}$ is called *Riemann (Darboux) integrable* if

$$\underline{I} = \overline{I}$$

If so, then $\underline{I} = \overline{I} = \int_a^b f(x)\,\mathrm{d}x$.

Note. We see that

- If $P \subset P'$, then
$$L_P \leq L_{P'}, \quad U_{P'} \leq U_P.$$
- Recall that continuous functions on $[a,b]$ are Reimann integrable on $[a,b]$.

Theorem 2.5.1. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then

1. If f is Reimann integrable, then f is Lebesgue measurable, thus Lebesgue integrable. Further,

$$\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathrm{d}m.$$

2. If f is Reimann integrable $\Leftrightarrow$ f is continuous Lebesgue a.e.[a]

[a]Here, we mean that the set where f is discontinuous is a null set under Lebesgue measure.

Proof. There exists $P_1 \subset P_2 \subset \ldots$ such that $L_{P_n} \nearrow \underline{I}$ and $U_{P_n} \searrow \overline{I}$.

> **Note.** Here, we took refinements of P_n if needed.

Now, define simple (step) functions

- $\phi_n = \sum_{i=1}^{K} \left(\inf_{[t_{i-1},t_i]} f \right) \mathbb{1}_{(t_{i-1},t_i]}$
- $\psi_n = \sum_{i=1}^{K} \left(\sup_{[t_{i-1},t_i]} f \right) \mathbb{1}_{(t_{i-1},t_i]}$

if $P_n = \{a = t_0 < t_1 < \ldots < t_K\}$. Let $\phi := \sup_n \phi_n$ and $\psi := \inf_n \psi_n$. We then see that ϕ, ψ are Lebesgue (Borel) measurable function.

> **Note.** Note that
>
> - $\exists M > 0$ such that $\forall_{n\in\mathbb{N}}\ \lvert\phi_n\rvert, \lvert\psi_n\rvert \leq M\mathbb{1}_{[a,b]}$
> - $\int \phi_n\,\mathrm{d}m = L_{P_n}$, $\int \psi_n\,\mathrm{d}m = U_{P_n}$

By Theorem 2.3.1 and the fact that $M\mathbb{1}_{[a,b]} \in L^1(\mathbb{R}, \mathcal{L}, m)$, we have

$$\underline{I} = \lim_{n\to\infty} \int \phi_n\,\mathrm{d}m = \int \phi\,\mathrm{d}m, \quad \overline{I} = \int \psi\,\mathrm{d}m.$$

Thus,

$$f \text{ is Riemann integrable} \Leftrightarrow \int \phi = \int \psi \Leftrightarrow \int (\psi - \phi) = 0 \Leftrightarrow \psi = \phi \text{ Lebesgue a.e.}$$

■

2.6 Modes of Convergence

As we should already see, there are different *modes* of convergence. Let's formalize them.

Definition 2.6.1 (Pointwise, Uniformly convergence)**.** Let

$$f_n, f\colon X \to \mathbb{C},$$

and $S \subset X$. Then we say

- $f_n \to f$ *pointwise* on S if

$$\forall_{x\in S}\ \forall_{\epsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{n\geq N}\ \lvert f_n(x) - f(x)\rvert < \epsilon.$$

- $f_n \to f$ *uniformly* on S if

$$\forall_{\epsilon>0} \exists_{N\in\mathbb{N}} \forall_{x\in S} \forall_{n\geq N} |f_n(x) - f(x)| < \epsilon.$$

Remark. We see that we can replace $\forall\epsilon > 0$ by $\forall k \in \mathbb{N}$ with ϵ changing to $\frac{1}{k}$.

Lemma 2.6.1. Let $B_{n,k}$ be

$$B_{n,k} := \left\{ x \in X \mid |f_n(x) - f(x)| < \frac{1}{k} \right\}.$$

Then

1. $f_n \to f$ pointwise on S if and only if
$$S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$
2. $f_n \to f$ uniformly on S if and only if $\exists N_1, N_2, \ldots \in \mathbb{N}$ such that
$$S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

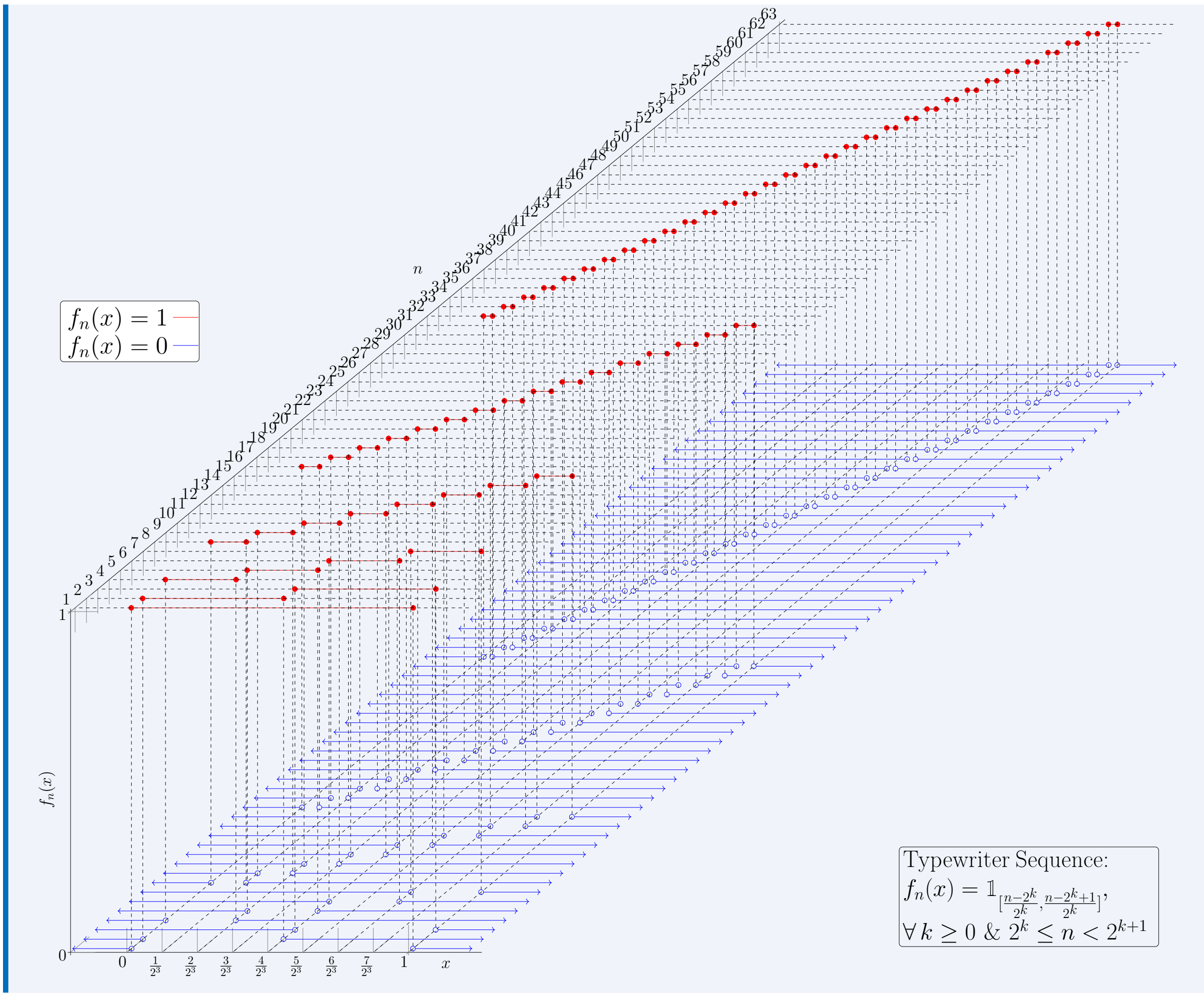
Proof. This essentially follows from Definition 2.6.1. ■

Definition 2.6.2 (Converges a.e., Converges in L^1). Let $(X, \mathcal{A}, \mu)$ be a measure space. Assuming that f_n, f are measurable function, then

1. $f_n \to f$ *almost everywhere* means
$$\exists \text{ null set } E \text{ such that } f_n \to f \text{ pointwise on } E^c.$$
2. $f_n \to f$ *in* L^1 means
$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

Example. Given $(\mathbb{R}, \mathcal{L}, m)$ and let $f = 0$. We see the followings.

1. $f_n = \mathbb{1}_{(n,n+1)}$
2. $f_n = \frac{1}{n}\mathbb{1}_{(0,n)}$
3. $f_n = n\mathbb{1}_{(0,\frac{1}{n})}$
4. **Typewriter functions**.



Lecture 16: Modes of Convergence

Let's start with a proposition. 11 Feb. 11:00

Proposition 2.6.1 (Fast L^1 convergence leads to a.e. convergence)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n, f are all measurable functions on X. Then

$$\sum_{n=1}^{\infty} \|f_n - f\|_1 < \infty \Rightarrow f_n \to f \quad \text{a.e.}$$

Proof. Let

$$E \coloneqq \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c = \{x \in X \mid f_n(x) \nrightarrow f(x)\}.$$

By Lemma 2.2.1, we see that

$$\forall_k \forall_N \, \mu\left(B_{n,k}^c\right) \leq k \int |f_n - f| \Rightarrow \forall_k \, \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) \leq \sum_{n=N}^{\infty} k \|f_n - f\|_1 \to 0$$

as $N \to \infty$. Now, by continuity of measure from above,

$$\forall_k \, \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = \lim_{N \to \infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \Rightarrow \mu(E) = 0$$

since $f_n \to f$ pointwise on E^c. ∎

Corollary 2.6.1. Given $\{f_n\}_n$ such that $f_n \to f$ in L^1, there exists a subsequence $\{f_{n_j}\}_{n_j}$ where $f_{n_j} \to f$ a.e.

Proof. Since

$$\forall_{j\in\mathbb{N}} \ \forall_{n_j\in\mathbb{N}} \ \left\|f_{n_j} - f\right\|_1 \leq \frac{1}{j^2}.$$

Then,

$$\sum_{j=1}^{\infty} \left\|f_{n_j} - f\right\|_1 < \infty.$$

From Proposition 2.6.1, we have the desired result. ∎

Definition 2.6.3 (Converge in measure)**.** Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. Then $f_n \to f$ *in measure* means

$$\forall_{\epsilon>0} \lim_{n\to\infty} \mu\left(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}\right) = 0.$$

Example. Let $f_n = n\mathbb{1}_{(0,\frac{1}{n})}$ and $f = 0$. We see that

$$\forall \epsilon > 0 \ \{x \in X \mid |f_n(x) - f(x)| > \epsilon\} = \left(0, \frac{1}{n}\right),$$

$f_n \to 0$ in measure. (Recall that $f_n \nrightarrow 0$ in L^1)

Remark. We see that

$$\begin{tikzcd}
 & f_n \to f \text{ in } L^1 \arrow[r, Rightarrow, shift left, "\text{Check}"] \arrow[dd, Leftrightarrow, "/" description] & f_n \to f \text{ in measure} \arrow[l, Rightarrow, shift left, "/" description] \arrow[dd, Rightarrow, "\text{Read}"] \arrow[ddl, Leftrightarrow, "/" description] \\
f_n \to f \text{ fast in } L^1 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & f_n \to f \text{ a.e.} & \exists f_{n_j} \to f \text{ a.e.}
\end{tikzcd}$$

Definition 2.6.4 (Uniformly a.e., Almost uniformly)**.** Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$.

1. $f_n \to f$ *uniformly almost everywhere* means $\exists$null set F such that $f_n \to f$ uniformly on F^c.
2. $f_n \to f$ *almost uniformly* means $\forall \epsilon > 0 \ \exists F \in \mathcal{A}$ such that $\mu(F) < \epsilon$, $f_n \to f$ uniformly on F^c.

Lemma 2.6.2. We have

$$f_n \to f \text{ uniformly on } S \Leftrightarrow \exists N_1, N_2, \ldots \in \mathbb{N} \ S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Theorem 2.6.1 (Egorov's Theorem)**.** Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. Suppose $\mu(X) < \infty$, then

$$f_n \to f \text{ a.e.} \Leftrightarrow f_n \to f \text{ almost uniformly}.$$

Proof. We prove two directions.

- $\Longleftarrow$ DIY

- $\Rightarrow$ Fix $\epsilon > 0$. We see that

$$\begin{aligned} f_n \to f \text{ a.e.} &\Rightarrow \mu\left(\bigcup_{k=1}^{\infty}\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \\ &\Rightarrow \underset{k}{\forall}\ \mu\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0. \end{aligned}$$

From continuity of measure from above and $\mu(X) < \infty$, we further have

$$\underset{k}{\forall}\ \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \Rightarrow \underset{k}{\forall}\ \underset{N_k\in\mathbb{N}}{\exists}\ \mu\left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c\right) < \frac{\epsilon}{2^k}.$$

Now, let

$$F \coloneqq \bigcup_{k=1}^{\infty}\bigcup_{n=N_k}^{\infty} B_{n,k}^c,$$

we see that $\mu(F) < \epsilon$, hence $f_n \to f$ uniformly.

■

Chapter 3

Product Measure

3.1 Product σ-algebra

Before we start, we see the setup.

- Product space.
$$X = \prod_{\alpha \in I} X_\alpha$$
where $x = (x_\alpha)_{\alpha \in I} \in X$.
- Coordinate map.
$$\pi_\alpha : X \to X_\alpha.$$

Now we see the formal definition.

Definition 3.1.1 (Product σ-algebra). Let $(X_\alpha, \mathcal{A}_\alpha)$ be a measurable space for all $\alpha \in I$. Then a *product σ-algebra* on $X = \prod_{\alpha \in I} X_\alpha$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha \in I} \pi_\alpha^{-1}(\mathcal{A}_\alpha) \right\rangle,$$

where

$$\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \left\{ \pi_\alpha^{-1}(E) \mid E \in \mathcal{A}_\alpha \right\}.$$

Notation. We denote $I = \{1, \ldots, d\} \Rightarrow X = \prod_{i=1}^{d} X_i, x = (x_1, \ldots, x_d)$. Also,

$$\bigotimes_{i=1}^{d} \mathcal{A}_i = \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_d.$$

Lemma 3.1.1. If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \forall_i \ E_i \in \mathcal{A}_i \right\} \right\rangle.$$

Proof. If $E_i \in \mathcal{A}_i$, then $\pi_i^{-1}(E_i) = \prod_{j=1}^{\infty} E_j$, where $E_j = X$ for $j \neq i$. On the other hand, since

$$\prod_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} \pi_i^{-1}(E_i),$$

from Lemma 1.1.2, the result follows. ■

Lecture 17: Product Measure

We now see a lemma. 14 Feb. 11:00

Lemma 3.1.2. Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle$ for every $\alpha \in I$. Then

1. $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$
2. $\bigotimes_\alpha \mathcal{A}_\alpha = \langle \bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$
3. If I is countable, then
$$\bigotimes_{i=1}^\infty \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^\infty E_i \mid \forall_i \ E_i \in \mathcal{E}_i \right\} \right\rangle$$

Proof. We prove this one by one.

1. Note that for $f\colon Y \to Z$, and $\mathcal{B}$ be a σ-algebra on Z, then $f^{-1}(\mathcal{B})$ is also a σ-algebra.[a] Hence, π_α^{-1} is a σ-algebra on X, i.e.,
$$\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \overset{!}{\Rightarrow} \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha).$$
To show the other direction, let $\mathcal{M}$ being
$$\mathcal{M} = \left\{ B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \right\}.$$
We now check
   - $\mathcal{M}$ is a σ-algebra. Check (Easy)!
   - $\mathcal{E}_\alpha \subset \mathcal{M}$. This is true by definition of $\mathcal{M}$.

   Thus, $\langle \mathcal{E}_\alpha \rangle = \mathcal{A}_\alpha \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha$, $E \in \mathcal{M}$, implying
$$\pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle,$$
   i.e., $\mathcal{A}_\alpha \subset \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.
2. DIY
3. DIY

■

[a] Since f^{-1}

Theorem 3.1.1. Suppose $X_1, \ldots, X_d$ are metric spaces. Let $X = \prod_{i=1}^d X_i$ with product metric defined as
$$\rho(x, y) = \sum_{i=1}^d \rho_i(x_i, y_i).$$
Then,

1. $\bigotimes_{i=1}^d \mathcal{B}(X_i) \subset \mathcal{B}(X)$
2. If in addition, each X_i has a countable dense subset,
$$\bigoplus_{i=1}^d \mathcal{B}(X_i) = \mathcal{B}(X).$$

Proof. ■ DIY

Remark. We see that

- $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$
- let $f = u + iv \colon X \to \mathbb{C}$, and $\mathcal{A}$ be a σ-algebra on X. Then

$$\underset{E\in\mathcal{B}(\mathbb{R})}{\forall} u^{-1}(E), v^{-1}(E) \in \mathcal{A} \Leftrightarrow f^{-1}(F) \in \mathcal{A}, \forall\ F \in \mathcal{B}(\mathbb{C})$$

  with $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Definition 3.1.2 (x-section, y-section)**.** Let X, Y be two sets. Then

1. For $E \subset X \times Y$,

$$E_x = \{y \in Y \mid (x, y) \in E\}, \quad E^y = \{x \in X \mid (x, y) \in E\}.$$

2. For $f \colon X \times Y \to \mathbb{C}$, define

$$f_x \colon Y \to \mathbb{C}, \quad f^y \colon X \to \mathbb{C}$$

   by

$$f_x(y) = f(x, y) = f^y(x).$$

Example. We see that

$$(\mathbb{1}_E)_x = \mathbb{1}_{E_x}$$

and

$$(\mathbb{1}_E)^y = \mathbb{1}_{E^y}.$$

Proposition 3.1.1. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, then

1. If $E \in \mathcal{A} \otimes \mathcal{B}$, then

$$\underset{x\in X}{\forall}\ \underset{y\in Y}{\forall}\ E_x \in \mathcal{B}, E^y \in \mathcal{A}.$$

2. If $f \colon X \times Y \to \mathbb{C}$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then

$$\underset{x\in X}{\forall}\ \underset{y\in Y}{\forall}\ f_x \text{ is } \mathcal{B}\text{-measurable}, f^y \text{ is } \mathcal{A}\text{-measurable}.$$

Proof. We prove this one by one.

1. Let $\mathcal{F} := \left\{E \subset X \times Y \mid \underset{x\in X}{\forall}\ \underset{y\in Y}{\forall}\ E_x \in \mathcal{B}, E^y \in \mathcal{A}\right\}$, then
   - $\mathcal{F}$ is a σ-algebra.
     - $\varnothing \in \mathcal{F}$.
     - $(E^c)_x = E_x^c$.
     - $\left(\bigcup_{j=1}^{\infty} E_j\right)_x = \bigcup_{j=1}^{\infty} (E_j)_x$.

     And the same is true for y.
   - Let $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$, which is again easy to show from definition.

   Hence, we see that $\langle \mathcal{R}_0 \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{F}$.

2. Since
$$(f_x)^{-1}(B) = (f^{-1}(B))_x$$
and
$$(f^y)^{-1}(B) = (f^{-1}(B))^y,$$
the result follows from 1.

■

3.2 Product Measures

We start with the definition.

Definition 3.2.1 (Rectangle). Given two measurable spaces, a *(measurable) rectangle* is $R = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, we let
$$\mathcal{R}_0 \coloneqq \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$
and
$$\mathcal{R} \coloneqq \left\{\bigcup_{i=1}^{N} R_i \mid N \in \mathbb{N}, R_1, \ldots, R_N \text{ disjoint rectangles}\right\}.$$

Note. Whenever we're talking about rectangle, they're always measurable.

Lemma 3.2.1. $\mathcal{R}$ is an algebra, and
$$\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}.$$

Proof. Simply observe that
$$(A \times B)^c = (A^c \times Y) \cup (A \times B)$$
■

DIY

Lecture 18: Monotone Class

Let's start with a theorem.

16 Feb. 11:00

Theorem 3.2.1. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces. Then

1. There is a measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying
$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$
for every $A \in \mathcal{A}, B \in \mathcal{B}$.
2. If μ, ν are σ-finite , then $\mu \times \nu$ is unique.

Proof. We prove this one by one.

1. Define $\mu \colon \mathcal{R} \to [0, \infty]$ by $\mu(A \times B) = \mu(A)\nu(B)$, and extending linearly, we have
$$\pi(A \times B) = \mu(A)\nu(B),$$
hence
$$\pi\left(\coprod_{i=1}^{N} A_i \times B_i\right) = \sum_{i=1}^{n} \pi(A_i \times B_i).$$

We claim that π is a pre-measure. To show this, it's enough to check that $\pi(A \times B) = \sum_{n=1}^{\infty} \pi(A_n \times B_n)$ if $A \times B = \coprod_n A_n \times B_n$. Since $A_n \times B_n$ are disjoint, so

$$\mathbb{1}_{A\times B}(x,y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n\times B_n}(x,y).$$

Thus,

$$\mathbb{1}_A(x)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y).$$

Integrating with respect to x, and applying Theorem 1.3.1, we have

$$\int_X \mathbb{1}_A(x)\mathbb{1}_B(y)\,\mathrm{d}\mu(x) = \sum_{n=1}^{\infty}\int_x \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y)\,\mathrm{d}\mu(x),$$

which implies

$$\mu(A)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mu(A_n)\mathbb{1}_{B_n}(y)$$

for every y. We can then integrate again with respect to y and apply Theorem 1.3.1, we have

$$\int_Y \mu(A)\mathbb{1}_B(y)\,\mathrm{d}\nu(y) = \sum_{n=1}^{\infty}\int_Y \mu(A_n)\mathbb{1}_{B_n}(y)\,\mathrm{d}\nu(y),$$

which gives us

$$\mu(A)\nu(B) = \sum_{n=1}^{\infty}\mu(A_n)\nu(B_n).$$

Hence, we see that μ is indeed a pre-measure, so Theorem 1.4.1 gives $\mu \times \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on $\mathcal{R}$.

2. If μ, ν are σ-finite, then π is σ-finite on $\mathcal{R}$, then Theorem 1.4.2 applies. Moreover, we have that

$$(\mu\times\nu)(E) = \inf\left\{\sum_{i=1}^{\infty}\mu(A_i)\nu(B_i) \mid E \subset \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B}\right\}.$$

■

3.3 Monotone Class Lemma

Let's start with a definition.

Definition 3.3.1 (Monotone Class). If X is a set, and $C \subset \mathcal{P}(X)$, we say that C is a *monotone class* on X if

- C is closed under countable increasing unions.
- C is closed under countable decreasing intersections.

Example. We see that

1. Every σ-algebra is a monotone class.
2. If C_α are (arbitrarily many) monotone classes on a set X, then $\bigcap_\alpha C_\alpha$ is a monotone class.

Furthermore, if $\mathcal{E} \subset \mathcal{P}(X)$, there is a unique smallest monotone class containing $\mathcal{E}$, denoted by $\langle \mathcal{E} \rangle$, which follows the same idea as in Definition 1.1.3.

Theorem 3.3.1 (Monotone Class Lemma). Suppose $\mathcal{A}_0$ is an algebra on X. Then $\langle\mathcal{A}_0\rangle$[a] is the monotone class generated by $\mathcal{A}_0$.

[a] $\langle\mathcal{A}_0\rangle$ is the σ-algebra generated by $\mathcal{A}_0$ by Definition 1.1.3.

Proof. Let $\mathcal{A} = \langle\mathcal{A}_0\rangle$ and let $\mathcal{C}$ be the monotone class generated by $\mathcal{A}_0$. Since $\mathcal{A}$ is a σ-algebra, it's a monotone class. Note that it contains $\mathcal{A}_0$, hence $\mathcal{A} \supset \mathcal{C}$.

To show $\mathcal{C} \supset \mathcal{A}$, it's enough to show that $\mathcal{C}$ is a σ-algebra. We check that

1. $\varnothing \in \mathcal{A}_0 \subseteq \mathcal{C}$.
2. Let $\mathcal{C}' = \{E \subset X \mid E^c \in \mathcal{C}\}$.
   - $\mathcal{C}'$ is a monotone class.
   - $\mathcal{A}_0 \subset \mathcal{C}'$ because if $E \in \mathcal{A}_0$, then $E^c \in \mathcal{A}_0$, so $E^c \in \mathcal{C}$, thus $E \in \mathcal{C}'$.

   We see that $\mathcal{C}' \subset \mathcal{C}'$, so $\mathcal{C}$ is closed under complements.
3. For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.
   - $\mathcal{D}(E) \subset \mathcal{C}$.
   - $\mathcal{D}(E)$ is a monotone class.
   - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. We see this by picking $F \in \mathcal{A}_0$, then $E \cup F \in \mathcal{A}_0 \supset \mathcal{C}$.

   Hence, $C = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.
4. Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\}$. That is $\mathcal{D} = \{E \in \mathcal{C} \mid E \cup F, \forall F \in \mathcal{C}\}$. Then we have
   - $A_0 \subset \mathcal{D}$ by 3.
   - $\mathcal{D}$ is a monotone class.
   - $\mathcal{D} \subset \mathcal{C}$ by definition.

   Thus, $\mathcal{D} = \mathcal{C}$, so if $E, F \in \mathcal{C}$, then $E \cup F \in \mathcal{C}$. This implies that $\mathcal{C}$ is closed under finite unions.
5. Now to show that $\mathcal{C}$ is closed under countable unions, let $E_1, E_2, \ldots \in \mathcal{C}$. We may then define

   $$F_N = \bigcup_{n=1}^{N} E_n \in \mathcal{C}.$$

   Then we see that $F_1 \subset F_2 \subset \ldots$, hence $\bigcup_N F_N \in \mathcal{C}$. But this simply implies

   $$\bigcup_N F_N = \bigcup_n E_n,$$

   so we're done.

■

Lecture 19: Fubini-Tonelli's Theorem

18 Feb. 11:00

As previously seen. If $E \in A \otimes B \Rightarrow E_x \in \mathcal{B}, E^y \in \mathcal{A}\ \forall x \in X, \forall y \in Y$. Note that the reverse is not true.

3.4 Fubini-Tonelli Theorem

We start with a theorem.

Theorem 3.4.1 (Tonelli's theorem for characteristic functions). Given $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure space. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$, then

1. $\alpha(x) \coloneqq \nu(E_x) \colon X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.
2. $\beta(x) \coloneqq \mu(E^y) \colon Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.
3. $(\mu \times \nu)(E) = \int_X \nu(E_x) \,\mathrm{d}\mu(x) = \int_Y \mu(E^y) \,\mathrm{d}\nu(y)$.

Proof. We prove this one by one.

1. Assume that μ, ν are finite measure. Let

$$C \coloneqq \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{ Conditions 1., 2., 3., hold}\}.$$

It's enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset C$. We further observe that from the Theorem 3.3.1 and the fact that $\mathcal{R}$ is an algebra, it's also enough to show that

- $\mathcal{R} \subset C$.
- C is a monotone class.

From condition 1.,

$$\alpha(x) = \nu\left((A \times B)_x\right) = \begin{cases} \nu(B), & \text{if } x \in A; \\ 0, & \text{if } x \notin A \end{cases} = \nu(B)\mathbb{1}_A.$$

And from condition 2.,

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

and

$$\int_X \nu((A \times B)_x) \,\mathrm{d}\mu(x) = \nu(B)\mu(A).$$

Let $E_n \in C$, $E_1 \subset E_2 \subset \ldots$. We need to show $E = \bigcup\limits_{n=1}^{\infty} E_n \in C$. We now see that

$$\begin{aligned} E_x = \bigcup_{n=1}^{\infty} (E_n)_x, (E_1)_x \subset (E_2)_x \subset \ldots \\ \Rightarrow \alpha(x) = \nu(E_n)_x \overset{!}{=} \lim_{n \to \infty} \nu((E_n)_x) \ \forall x \in X. \end{aligned}$$

This implies that 1. is proved.

For 3., we see that

$$\begin{aligned} (\mu \times \nu)(E) &\overset{!}{=} \lim_{n \to \infty} (\mu \times \nu)(E_n) \\ &= \lim_{n \to \infty} \int_X \nu((E_n)_x) \,\mathrm{d}\mu(x) \\ &\overset{!}{=} \int_X \nu(E_x) \,\mathrm{d}\mu(x). \end{aligned}$$

Now let $F_n \in C$, $F_1 \supset F_2 \supset \ldots$. We need to show that $F = \bigcap\limits_{n=1}^{\infty} F_n \in C$. Instead of using Theorem 2.2.1, we now want to use Theorem 2.3.1, which is applicable since $\mu(X), \nu(Y) < \infty$ by assumption. Then assume that μ, ν are σ-finite, then

$$X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n), \begin{cases} X_1 \subset X_2 \subset \ldots, & \mu(X_k) < \infty \\ Y_1 \subset Y_2 \subset \ldots, & \nu(Y_k) < \infty. \end{cases}$$

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$\blacksquare$

Theorem 3.4.2 (Fubini-Tonelli's Theorem)**.** Given two σ-finite measure space $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$, we have the following two versions.

(Tonelli) If $f \colon X \times Y \to [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then

1. $g(x) := \int_Y f(x, y) \,\mathrm{d}\nu(y)$, $X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.
2. $h(x) := \int_X f(x, y) \,\mathrm{d}\mu(x)$, $Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.
3. We have

$$\int_{X \times Y} f \,\mathrm{d}(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \,\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x, y) \,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y).$$

(Fubini) If $f \in L^1(X \times Y, \mu \times \nu)$, then

1. $f_x \in L^1(Y, \nu)$ for μ-a.e. x, and $g(x) \in L^1(X, \mu)$ defined μ-a.e.
2. $f^y \in L^1(X, \mu)$ for ν-a.e. y, and $h(x) \in L^1(Y, \nu)$ defined μ-a.e.
3. The iterated integral formulas hold. Namely, we have

$$\int_{X \times Y} f \,\mathrm{d}(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \,\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x, y) \,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y).$$

Proof. Read [FF99]. $\blacksquare$

Lecture 20: Lebesgue Measure on $\mathbb{R}^d$

3.5 Lebesgue Measure on $\mathbb{R}^d$

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Example. $(\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m)$ is not complete.

- Let $A \in \mathcal{L}$, $A \neq \varnothing$, $m(A) = 0$.
- Let $B \subset [0, 1]$, $B \notin \mathcal{L}$ (Vital set for example).
- Let $E = A \times B$, $F = A \times [0, 1]$.

We see that $E \subset F$, $F \in \mathcal{L} \otimes \mathcal{L}$, $(m \times m)(F) = m(A)m([0, 1]) = 0$, i.e., F is a null set. But E is **not** $\mathcal{L} \otimes \mathcal{L}$-measurable-function since otherwise, its sections are all measurable.

Definition 3.5.1. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the *completion* of

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \ldots \times m),$$

which is same as the *completion* of

$$(\mathbb{R}^d, \mathcal{L} \otimes \ldots \otimes \mathcal{L}, m \times \ldots m).$$

Remark. We see that

$$\mathcal{L}^d \supsetneq \mathcal{L} \otimes \ldots \otimes \mathcal{L} = \left\langle \left\{ \prod_{i=1}^d E_i \mid E_i \in \mathcal{L} \right\} \right\rangle.$$

Definition 3.5.2 (General rectangle)**.** A *rectangle* in $\mathbb{R}^d$ is $R = \prod_{i=1}^{d} E_i$ where $E_i \in \mathcal{B}(\mathbb{R})$.

Definition 3.5.3. We let

$$m^d(E) := \inf\left\{\sum_{k=1}^{\infty} m^d(R_k) \mid E \subset \bigcup_{k=1}^{\infty} R_k, R_k \text{ is rectangles}\right\}.$$

Theorem 3.5.1. Let $E \subset \mathcal{L}^d$. Then

1. $m^d(E) = \inf\left\{m^d(0) \mid \text{open } O \supset E\right\} = \sup\left\{m^d(K) \mid \text{compact } K \subset E\right\}$.
2. $E = A_1 \cup N_1 = A_2 \setminus N_2$, where A_1 is F_σ, A_2 is G_δ , and N_i are null.
3. If $m^d(E) < \infty$, $\forall \epsilon > 0$, $\exists R_1, \ldots, R_m$ rectangles whose sides are intervals such that

$$m^d\left(E \triangle \left(\bigcup_{i=1}^{m} R_i\right)\right) < \epsilon.$$

Proof. Similar to $d = 1$ case. ■

Theorem 3.5.2. Integrable step functions and $C_c(\mathbb{R}^d)$, the collection of continuous functions, are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$

Proof. See [FF99]. ■

Theorem 3.5.3. Lebesgue measure in $\mathbb{R}^d$ is translation-invariant.

Proof. See [FF99]. ■

Theorem 3.5.4 (Effect of linear transformation on Lebesgue measure)**.** If $T \in \mathrm{GL}(\mathbb{R}^d)$, $e \in \mathcal{L}^d$, then $T(E)$ is measurable and

$$m(T(E)) = |\det T| \cdot m(E).$$

Proof. See [FF99]. ■

Chapter 4

Differentiation on Euclidean Space

As previously seen. Given $f\colon [a,b] \to \mathbb{R}$, there are two versions of **fundamental theorem of calculus**:

1. $$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a).$$

2. $$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x),$$

   which follows from
   $$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t = f(x) = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} f(t)\,\mathrm{d}t.$$

> **Remark.** We see that
> $$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} (f(t)-f(x))\,\mathrm{d}t = 0 = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} (f(t)-f(x))\,\mathrm{d}t,$$
> where we have
> $$f(x) = \frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t.$$
> This generalized to $f\colon \mathbb{R}^d \to \mathbb{R}$, namely
> $$\lim_{r\to 0^+}\frac{1}{\mathrm{vol}\,(B(x,r))}\int_{B(x,r)} (f(t)-f(x))\,\underbrace{\mathrm{d}t}_{\mathbb{R}^d} \overset{?}{=} 0.$$

4.1 Hardy-Littlewood Maximal Function

We first see our notation.

Notation. Given a(n) (open) ball in $\mathbb{R}^d$, $B = B(a,r)$, denote $cB = B(a,cr)$ for $c > 0$.

> **Lemma 4.1.1 (Vitali-type covering lemma).** Let $B_1, \ldots, B_k$ be a finite collection of open balls in $\mathbb{R}^d$. Then there exists a sub-collection $B'_1, \ldots, B'_m$ of <u>disjoint</u> open balls such that
> $$\bigcup_{i=1}^{m} (3B'_j) \supset \bigcup_{i=1}^{k} B_i.$$

Proof. Greedy Algorithm. ∎

Lecture 21: Hardy-Littlewood Maximal Function and Inequality

25 Feb. 11:00

Notation. We let

$$\int_E f \, \mathrm{d}m = \int_E f(x) \, \mathrm{d}x.$$

The problem we're working on is

$$\frac{1}{m(B(w,r))} \int_{B(x,r)} f(y) \, \mathrm{d}y \overset{r\to 0}{\underset{?}{\longrightarrow}} f(x).$$

Definition 4.1.1 (Locally integrable). Given $f\colon \mathbb{R}^d \to \mathbb{C}$ be Lebesgue measurable function. Then we say f is *locally integrable* if for every compact $K \subset \mathbb{R}^d$,

$$\int_K |f| \, \mathrm{d}m < \infty.$$

We write $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Definition 4.1.2 (Hardy-Littlewood maximal function). Given $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the *Hardy-Littlewood maximal function* for f is defined as

$$\mathrm{H}f(x) \coloneqq \sup \left\{ \mathrm{A}_r(x) \mid r > 0 \right\},$$

where

$$\mathrm{A}_r(x) \coloneqq \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| \, \mathrm{d}y.$$

Note. We note that $\mathrm{A}_r(\cdot)$ means *averaging function*.

Lemma 4.1.2. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then

1. $\mathrm{A}_r(x)$ is jointly continuous for $(x, r) \in \mathbb{R}^d \times (0, \infty)$.
2. $\mathrm{H}f(x)$ is Borel measurable.

Proof. We outline the proof.

1. Let $(x, r) \to (x^*, r^*) \Rightarrow \mathrm{A}_r(x) \to \mathrm{A}_{r^*}(x^*)$. Let (x_n, r_n) be any sequence which converges to x^*, r^*, then we consider $\lim\limits_{n\to\infty} \mathrm{A}_{r_n}(x_n)$ and we can calculate

$$\int \underbrace{|f(y)| \, \mathbb{1}_{B(x_n, r_n)}(y)}_{\coloneqq h_n(y)},$$

then we apply Theorem 2.3.1 to h_n.

2. Observe that

$$(\mathrm{H}f)^{-1}(\underbrace{(a,\infty)}_{\text{open}}) = \bigcup_{r>0} \mathrm{A}_r^{-1}((a,\infty))$$

is open, since $\mathrm{A}_r^{-1}((a,\infty))$ is open from the 1. Note that the equality comes from the fact that $\mathrm{H}f = \sup_r \mathrm{A}_r$.

■

Theorem 4.1.1 (Hardy-Littlewood maximal inequality). There exists $C_d > 0$ such that for every $f \in L^1(\mathbb{R}^d)$,

$$\underset{\alpha>0}{\forall}\ m\left(\left\{x \in \mathbb{R}^d \mid \mathrm{H}f(x) > \alpha\right\}\right) \leq \frac{C_d}{\alpha} \int |f(x)| \,\mathrm{d}x.$$

Proof. We first fix $f \in L^1$ and $\alpha > 0$. We define

$$E \coloneqq \{x \mid \mathrm{H}f(x) > \alpha\},$$

which is a Borel measurable set by Lemma 4.1.2. Then

$$x \in E \Rightarrow \underset{r_x>0}{\exists}\ \mathrm{A}_{r_x}(x) > \alpha \Rightarrow m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x,r_x)} |f(y)| \,\mathrm{d}y.$$

From inner regularity, we have

$$m(E) = \sup\{m(K) \mid \text{compact } K \subset E\}.$$

Let $K \subset E$ be compact, then

$$K \subset \bigcup_{x\in K} B(x, r_x) \overset{K \text{ compact}}{\Rightarrow} K \subset \bigcup_{i=1}^{N} B_i \overset{!}{\Rightarrow} K \subset \bigcup_{i=1}^{m} \{3B'_j\}.$$

From here, we further have

$$m(K) \leq \sum_{i=1}^{m} m(3B'_j) = 3^d \sum_{j=1}^{m} m(B'_j) \leq \frac{3^d}{\alpha} \sum_{j=1}^{m} \int_{B'_j} |f(y)| \,\mathrm{d}y.$$

Now, since $B'_i, \ldots, B'_m$ are disjoint, hence we finally have

$$m(K) \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \,\mathrm{d}y.$$

■

Lecture 22: Lebesgue Differentiation Theorem

We should compare the Hardy-Littlewood maximal inequality to Markov's inequality. Namely, there exists $C_d > 0$ (can take 3^d) such that for all $f \in L^1(\mathbb{R}^d)$, $\alpha > 0$, we have 07 Mar. 11:00

$$\begin{cases} m(\{x \mid \mathrm{H}f(x) > \alpha\}) \leq \dfrac{C_d}{\alpha} \displaystyle\int |f|; \\ m(\{x \mid |f(x)| > \alpha\}) \leq \dfrac{1}{\alpha} \displaystyle\int |f|. \end{cases}$$

4.2 Lebesgue Differentiation Theorem

We start with a theorem!

Theorem 4.2.1 (Lebesgue Differentiation Theorem). Let $f \in L^1$, then

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \,\mathrm{d}y = 0$$

for a.e. x.

Proof. The result holds for $f \in C_c(\mathbb{R}^d)$, namely for those continuous functions with **compact** support. This is because for any $\epsilon > 0$, if r is small and $|f(y) - f(x)| < \epsilon$, then

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y < \epsilon.$$

Now, let $f \in L^1(\mathbb{R}^d)$ and fix $\epsilon > 0$. By density, there exists $g \in C_c(\mathbb{R}^d)$ with $\|f - g\|_1 < \epsilon$. We then have

$$\int_{B_r(x)} |f(y) - f(x)| \, \mathrm{d}y \leq \int_{B_r(x)} |f(y) - g(y)| \, \mathrm{d}y + \int_{B_r(x)} |g(y) - g(x)| \, \mathrm{d}y + \int_{B_r(x)} |g(x) - f(x)| \, \mathrm{d}y.$$

Note. We use $B_r(x)$ above to denote $B(x,r)$ for spacing reason only. Nothing tricky here.

Divide all of these by $m(B(x,r))$, and take $\limsup\limits_{r \to \infty}$, we need to understand the error terms, namely

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - g(x)| \, \mathrm{d}y = |g(x) - f(x)|$$

and

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g(y)| \, \mathrm{d}y \leq (\mathrm{H}(f - g))(x).$$

We define

$$Q(x) \coloneqq \limsup_{r \to \infty} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y.$$

We want to show $m(\{x \in X \mid Q(x) > 0\}) = 0$. Let $E_\alpha = \{x \in X \mid Q(x) > \alpha\}$. It is enough to show $m(E_\alpha) = 0$ for all $\alpha > 0$ because $\{x \in X \mid Q(x) > 0\} = \bigcup_n E_{\frac{1}{n}}$. We know by the above that

$$Q(x) \leq (\mathrm{H}(f - g))(x) + 0 + |g(x) - f(x)| \, .$$

Therefore,

$$E_\alpha \subset \{x \in X \mid (\mathrm{H}(f - g))(x) > \alpha/2\} \cup \{x \in X \mid |g(x) - f(x)| > \alpha/2\}.$$

By the Hardy-Littlewood maximal inequality and Markov's inequality, we have

$$\begin{cases} m(\{x \mid (\mathrm{H}(f - g))(x) > \alpha/2\}) \leq \dfrac{2C_d}{\alpha} \displaystyle\int |f - g| \, ; \\ m(\{x \mid |g(x) - f(x)| > \alpha/2\}) \leq \dfrac{2}{\alpha} \displaystyle\int |f - g| \, . \end{cases}$$

Thus,

$$0 \leq m(E_\alpha) \leq \frac{2C_d}{\alpha} \|f - g\|_1 + \frac{2}{\alpha} \|f - g\|_1 \leq \frac{2(C_d + 1)}{\alpha} \epsilon.$$

Taking $\epsilon \to 0$, $m(E_\alpha)$ does not depend on ϵ and g, hence $m(E_\alpha) = 0$. ■

Corollary 4.2.1. Theorem 4.2.1 also holds for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Proof. Using the fact that m^d is σ-finite, and apply Theorem 4.2.1. Specifically, partition $\mathbb{R}^d$ into countably many compact sets K_i and apply Theorem 4.2.1 to $f \mathbb{1}_{K_i}$ for all i. ■

Corollary 4.2.2. For $f \in L^1_{\mathrm{loc}}$, we have

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, \mathrm{d}y = f(x)$$

for a.e. x.

Proof. . Use that

$$f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x)\, \mathrm{d}y$$

and the triangle inequality. ■

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Definition 4.2.1 (Lebesgue point). Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the point $x \in \mathbb{R}^d$ is called a *Lebesgue point of f* if

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, \mathrm{d}y = 0.$$

Remark. Corollary 4.2.1 tells us that almost all points in $\mathbb{R}^d$ in $\mathbb{R}^d$ are Lebesgue points for f.

Definition 4.2.2 (Shrink nicely). We say that $\{E_r\}_{r>0}$ *shrinks nicely* to x as $r \to 0$ if $E_r \subset B(x,r)$ and

$$\underset{c>0}{\exists}\ c \cdot m(B(x,r)) \leq m(E_r).$$

Corollary 4.2.3. Suppose E_r shrink nicely to 0, and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, and x is a Lebesgue point. Then

$$\begin{cases} \lim\limits_{r\to 0} \dfrac{1}{m(E_r)} \displaystyle\int_{E_r+x} |f(y) - f(x)|\, \mathrm{d}y = 0; \\ \lim\limits_{r\to 0} \dfrac{1}{m(E_r)} \displaystyle\int_{E_r+x} |f(y)|\, \mathrm{d}y = f(x). \end{cases}$$

Corollary 4.2.4. If $f \in L^1_{\text{loc}}(\mathbb{R})$, then $F(x) = \int_0^x f(y)\, \mathrm{d}y$ is differentiable and $F'(x) = f(x)$ almost everywhere.

Chapter 5

Normed Vector Space

Lecture 23: Metric, normed and L^p Spaces

5.1 Metric Spaces and Normed Spaces

09 Mar. 11:00

We have seen the definition of a norm before, now we formally introduce the concept of *metric*.

Definition 5.1.1 (Metric). Let Y be a set, a function $\rho\colon Y \times Y \to [0, \infty)$ is a *metric* on Y if

- $\rho(x, y) = \rho(y, x)$ for all $x, y \in Y$.
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in Y$.
- $\rho(x, y) = 0$ if and only if $x = y$.

Note. The followings make sense in a metric space.

1. Open/closed balls.
2. Open/closed sets.
3. Convergence sequences ($x_n \to x$ with respect to ρ if and only if $\lim\limits_{n\to\infty} \rho(x_n, x) = 0$).
4. Continuous functions.

Example. We have the following metric spaces.

1. $\mathbb{Q}$ with $\rho(x, y) = |x - y|$.
2. $\mathbb{R}$ with $\rho(x, y) = |x - y|$.
3. $\mathbb{R}_+$ with $\rho(x, y) = |\ln(y/x)|$.
4. $\mathbb{R}^d$ with
$$\rho_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p}$$
and
$$\rho_\infty(x, y) = \max_{1\leq i\leq d} |x_i - y_i| .$$
These all give the same open sets, hence they are topologically equivalent.
5. $C([0, 1])$ with
$$\rho_p(f, g) = \left(\int_0^1 |f - g|^p\right)^{1/p}$$

and

$$\rho_\infty(f,g) = \max_{x\in[0,1]} |f(x)-g(x)|\,.$$

6. Let $(X,\mathcal{A},\mu)$ be a measure space with $\mu(X)<\infty$. Let Y be the set of measurable functions on X, then

$$\rho(f,g) = \int \min\{|f(x)-g(x)|\,,1\}\,\mathrm{d}\mu(x)$$

is a metric and $f_n\to f$ in ρ if and only if $f_n\to f$ in measure.

Let V be a vector space over scalar field $K=\mathbb{R}$ or $K=\mathbb{C}$.

As previously seen (Metric induced by a norm)**.** Recall the definition of seminorm and norm. We see that a norm induces a metric

$$\rho(v,w) := \|v-w\|\,,$$

and we have

$$v_n\to v \Leftrightarrow \lim_{n\to\infty}\|v_n-v\| = 0.$$

Example. We first see some common examples of normed vector space.

1. $L^1(X,\mathcal{A},\mu)$ with $\|f\|_1 := \int |f|\,\mathrm{d}\mu$.
2. $C([0,1])$ with $\|f\|_1 := \int_0^1 |f(x)|\,\mathrm{d}x$, $\|f\|_\infty := \max_{0\le x\le 1}|f(x)|$.
3. For $\mathbb{R}^d$ and $0<p<\infty$, we have

$$\|x\|_p := \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}, \qquad \|x\|_\infty := \max_{1\le i\le d}|x_i|\,.$$

5.2 L^p Space

It turns out that we can generalize L^1 into L^p.

Definition 5.2.1 (L^p space)**.** Given a measure space $(X,\mathcal{A},\mu)$ and a measurable function f and p such that $0<p<\infty$, we define a seminorm $\|\cdot\|_p$ such that

$$\|f\|_p := \left(\int_X |f|^p\,\mathrm{d}\mu\right)^{1/p},$$

which induces the so-called L^p *space* $L^p(X,\mathcal{A},\mu)$, where

$$L^p(X,\mathcal{A},\mu) := \left\{f \mid \|f\|_p < \infty\right\}.$$

Remark. Note that $\|\cdot\|_p$ is only a seminorm. But if we identity functions which are equal almost everywhere, then it's indeed a norm.

Example. $(\mathbb{R},\mathcal{L},m)$ has $f(x)=x^{-\alpha}\mathbb{1}_{(1,\infty)}(x)\in L^p$ if and only if $\alpha p>1$. In contrast, $g(X) = x^{-\beta}\mathbb{1}_{(0,1)}(x)\in L^p$ if and only if $\beta p<1$.

Similar to Definition 5.2.1, we have the following.

Definition 5.2.2 (ℓ^p space)**.** If $(X,\mathcal{P}(X),\nu)$ is equipped with the counting measure, then we say it's

an ℓ^p *space* such that
$$\ell^p(X) := L^p(X, \mathcal{P}(X), \nu).$$

Remark. We are interested in $\ell^p(\mathbb{N})$ in particular. We have
$$\ell^p := \ell^p(\mathbb{N}) = \left\{ a = (a_1, a_2, \ldots) \mid \|a\|_p = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} < \infty \right\}.$$

Lemma 5.2.1. $L^p(X, \mathcal{A}, \nu)$ is a vector space for all $p \in (0, \infty)$.

Proof. We verify the following.

- $c \cdot f \in L^p(X, \mathcal{A}, \mu)$ for $c \in \mathbb{R}$. Indeed, since
$$\|cf\|_p = \left(\int |cf|^p \,\mathrm{d}\mu \right)^{1/p} = |c| \, \|f\|_p < \infty \Leftrightarrow \|f\|_p < \infty,$$
which implies $c \cdot f \in L^p(X, \mathcal{A}, \mu)$.
- $f + g \in L^p(X, \mathcal{A}, \mu)$. Indeed, since for any real numbers α, β, we have
$$(\alpha + \beta)^p \leq (2 \cdot \max\{|\alpha|, |\beta|\})^p = 2^p \cdot \max\{|\alpha|^p, |\beta|^p\} \leq 2^p \left(|\alpha|^p + |\beta|^p \right),$$
which implies that for $f, g \in L^p(X, \mathcal{A}, \mu)$, we have
$$\|f + g\|_p < \infty \Leftrightarrow \|f + g\|_p^p = \int |f + g|^p \,\mathrm{d}\mu \leq 2^p \int (|f|^p + |g|^p) < \infty.$$
This further implies
$$\|f + g\|_p < \infty \Leftrightarrow \|f\|_p, \|g\|_p < \infty,$$
which is what we want.

■

We see that in the above derivation, it doesn't give us the triangle inequality, namely
$$\|f + g\|_p \leq \|f\|_p + \|g\|_p,$$
hence we need some new results.

Theorem 5.2.1 (Hölder's inequality). Let $1 < p < \infty$, and let $q := p/(p-1)$ so that $1/p + 1/p = 1$. Then we have
$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. We prove this in steps.

1. Note that
$$t \leq \frac{t^p}{p} + 1 - \frac{1}{p} = \frac{t^p}{p} + \frac{1}{q}$$
for all $t \geq 0$. Hence, by taking $F(t) := t - t^p/p$ and $t \geq 0$, we see that the maximum of F implies the above inequality.
2. Young's Inequality.[a] We have
$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$
for $\alpha, \beta > 0$. This follows by taking $t := \alpha/\beta^{q-1}$ in the first inequality we obtained.

3. Without loss of generality, we can assume that $0 < \|f\|_p, \|g\|_q < \infty$. Now, consider $F(x) = f(x)/\|f\|_p$, $G(x) = g(x)/\|g\|_q$. We know that $\|F\|_p = 1 = \|G\|_q$. Then by Young's Inequality, we have

$$\int |F(x)G(x)|\,\mathrm{d}\mu \leq \int \frac{|F(x)|^p}{p} + \frac{|G(x)|^q}{q} \Rightarrow \frac{\|fg\|_1}{\|f\|_p\|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

which implies our desired result.

■

[a] https://en.wikipedia.org/wiki/Young's_inequality_for_products

Example. For $p = q = 2$, $X = \{1, \ldots, d\}$ with μ being the counting measure, then for any $x, y \in \mathbb{R}^d$, we have

$$\sum_{i=1}^{d} |x_i y_i| \leq \sqrt{\sum_{i=1}^{d} x_i^2}\sqrt{\sum_{i=1}^{d} y_i^2}$$

We now see how we can obtain the desired triangle inequality.

Theorem 5.2.2 (Minkowski's Inequality)**.** Let $1 \leq p < \infty$, then for $f, g \in L^p$,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. For $p = 1$, it's easy since it's just triangle inequality. Now, we assume that $1 < p < \infty$, and we may assume also that $\|f+g\| \neq 0$ without loss of generality. Then

$$\begin{aligned}\int |f(x)+g(x)|^p &\leq \int |f(x)+g(x)|^{p-1}\left(|f(x)|+|g(x)|\right)\\ &\leq \left(\int |f+g|^{(p-1)q}\right)^{1/q}\left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right]\\ &\leq \left(\int |f+g|^p\right)^{1/q}\left(\|f\|_p + \|g\|_p\right).\end{aligned}$$

We then see that

$$\underbrace{\left(|f(x)+g(x)|^p\right)^{1-1/q}}_{(|f(x)+g(x)|^p)^{1/p}} \leq \|f\|_p + \|g\|_p,$$

which is just $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. ■

Lecture 24: Embedding L^p Space

11 Mar. 11:00

Definition 5.2.3 (Essential supremum)**.** For a measurable function f on $(X, \mathcal{A}, \mu)$, we define

$$\begin{aligned}S &:= \{\alpha \geq 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\}\\ &= \{\alpha \geq 0 \mid |f(x)| \leq \alpha \text{ a.e.}\}.\end{aligned}$$

Then, we say that the *essential supremum of* f, denoted as $\|f\|_\infty$, is defined as

$$\|f\|_\infty := \begin{cases}\inf S, & \text{if } S \neq \varnothing;\\ \infty, & \text{if } S = \varnothing.\end{cases}$$

Definition 5.2.4 (L^∞ space). Let $L^\infty(X, \mathcal{A}, \mu)$ be

$$L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\}.$$

Definition 5.2.5 (ℓ^∞ space). We let ℓ^∞ be defined as

$$\ell^\infty = L^\infty(\mathcal{N}, \mathcal{P}(\mathcal{N}), \nu),$$

where ν is the counting measure.

Example. Consider $(\mathbb{R}, \mathcal{L}, m)$. Then

$$\begin{aligned} f(x) &= \frac{1}{x}\mathbb{1}_{(0,\infty)}(x) \notin L^\infty; \\ g(x) &= x\mathbb{1}_{\mathbb{Q}}(x) + \frac{1}{1+x^2} \in L^\infty. \end{aligned}$$

If f is continuous on $(\mathbb{R}, \mathcal{L}, m)$, then $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. For $a \in \ell^\infty$, we have $\|a\|_\infty = \sup_{i\in\mathbb{N}} |a_i|$, and sequences in ℓ^∞ are exactly the bounded sequences.

Lemma 5.2.2. We have the following.

1. Suppose $f \in L^\infty(X, \mathcal{A}, \mu)$. Then,
$$\begin{cases} \mu(\{x \mid |f(x)| > \alpha\}) = 0, & \text{if } \alpha \geq \|f\|_\infty; \\ \mu(\{x \mid |f(x)| > \alpha\}) > 0, & \text{if } \alpha < \|f\|_\infty. \end{cases}$$
2. $|f(x)| \leq \|f\|_\infty$ almost everywhere.
3. $f \in L^\infty$ if and only if there exists a bounded measurable function g such that $f = g$ almost everywhere.

Proof. . ■ DIY

Theorem 5.2.3. We have the following.

1. $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.
2. $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
3. $f_n \to f$ in L^∞ if and only if $f_n \to f$ uniformly almost everywhere.

Proof. We'll do one implication in 3. Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$. Let $A = \bigcup_n A_n$, we see that $\mu(A) = 0$ as well. DIY 1. and 2.

For $x \in A^c$ and for every n, we have

$$|f_n(x) - f(x)| \leq \|f_n - f\|_\infty.$$

Given $\epsilon > 0$, there is an N so that

$$\|f_n - f\| < \epsilon$$

for all $n \geq N$. But then for all $x \in A^c$, $|f_n(x) - f(x)| < \epsilon$ as well. ■

Remark. The motivation for 1. is that

$$\frac{1}{1} + \frac{1}{\infty} = 1,$$

and we want to have the similar result as in Theorem 5.2.1.

Proposition 5.2.1. We have the following.

1. For $1 \leq p < \infty$, the collection of simple functions with finite measure support is dense in $L^p(X, \mathcal{A}, \mu)$.
2. For $1 \leq p < \infty$, the collection of step functions with finite measure support is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$, so is $C_c(\mathbb{R})$.
3. For $p = \infty$, the collection of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.

Proof. ■ DIY

Remark. Note that $C_c(\mathbb{R})$ is **not** dense in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

5.3 Embedding Properties of L^p Spaces

Definition 5.3.1 (Equivalent norm)**.** Two norms $\|\cdot\|, \|\cdot\|'$ on V are *equivalent* if there exists $c_1, c_2 > 0$, such that

$$c_1 \|v\| \leq \|v\|' \leq c_2 \|v\|$$

for all $v \in V$.

Note. We see that

1. These norms gives the same topological properties (open sets, closed sets, convergence, etc.).
2. Definition 5.3.1 is an equivalence relation on norms.

Example. For $\mathbb{R}^d$ we have the norms $\|\cdot\|_p$ for $1 \leq p \leq \infty$. All of these are equivalent. We see that for $1 \leq p < \infty$,

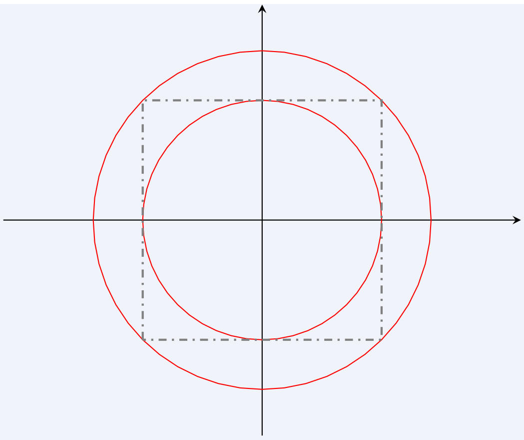
$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p} \leq (d \|x\|_\infty^p)^{1/p} = d^{1/p} \|x\|_\infty .$$

Also,

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p} \geq (\|x\|_\infty^p)^{1/p} = \|x\|_\infty .$$

Thus, $\|\cdot\|_p$ is equivalent to $\|\cdot\|_\infty$ for every $1 \leq p < \infty$, and transitivity gives that they are all equivalent.

Another way of thinking of this, by assuming $v \neq 0$, and scaling by some t, we may assume v lies on the unit circle in one of the norms. Then we are squeezing a unit circle in $\|\cdot\|'$ between two circles of radius c_1, c_2 in $\|\cdot\|$. In picture, we have to show that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent, we have



since the circles in $\|\cdot\|_\infty$ are squares.

Example. Fro $1 \le p, q \le \infty$, we have $L^p(\mathbb{R}, m)$-norm and $L^q(\mathbb{R}, m)$-norm are not equivalent, even worse, we have that

$$L^p(\mathbb{R}, m) \not\subseteq L^1(\mathbb{R}, m), \quad L^p(\mathbb{R}, m) \not\supseteq L^1(\mathbb{R}, m).$$

Lecture 25: Banach Spaces

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Proposition 5.3.1. Suppose $\mu(X) < \infty$, then for every $0 < p < q \le \infty$, $L^q \subseteq L^p$.

Proof. Suppose $q < \infty$, then

$$\int |f|^p \le \left(\int (|f|^p)^{q/p}\right)^{p/q} \left(\int 1^{q/(q-p)}\right)^{1-p/q} = \left(\int |f|^q\right)^{p/q} \mu(x)^{1-p/q} < \infty$$

where we split $\int |f|^p$ into $\int |f|^p \cdot 1$. From Hölder's inequality with $q/p > 1$, we have

$$\|f\|_p \le \|f\|_q \, \mu(X)^{1/p - 1/q} < \infty.$$

The case that $q = \infty$ is left as an exercise. ■

DIY

Proposition 5.3.2. If $0 < p < q \le \infty$, then $\ell^p \subseteq \ell^q$.

Proof. When $q = \infty$, we have

$$\|a\|_\infty^p = \left(\sup_i |a_i|\right)^p = \sup_i |a_i|^p \le \sum_{i=1}^\infty |a_i|^p.$$

Thus $\|a\|_\infty \le \|a\|_p$.

When $q < \infty$, we see that

$$\sum_{i=1}^\infty |a_i|^q = \sum_{i=1}^\infty |a_i|^p \cdot |a_i|^{q-p} \le \|a\|_\infty^{q-p} \sum_{i=1}^\infty |a_i|^p \le \|a\|_p^{q-p} \cdot \|a\|_p^p = \|a\|_p^q.$$

Therefore,

$$\|a\|_q \le \|a\|_p.$$

■

Proposition 5.3.3. For all $0 < p < q < r \le \infty$, $L^p \cap L^r \subseteq L^q$.

Proof. ■ DIY

5.4 Banach Spaces

Let's start with a definition.

Definition 5.4.1 (Cauchy sequence). Let Y, ρ be a metric space. We call x_n a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\rho(x_n, x_m) < \epsilon$.

Note. Convergent sequence are Cauchy.

Definition 5.4.2 (Complete). A metric space (Y, ρ) is called *complete* if every Cauchy sequence in Y converges.

Example. We first see some examples.

1. We see that $\mathbb{Q}$ with $\rho(x,y) = |x-y|$ is **not** complete, but $\mathbb{R}$ with the same metric is complete.
2. $C([0,1])$ with $\rho(f,g) = \|f-g\|_\infty$ is complete, but with $\rho(f,g) = \int |f-g|$ is not.

Definition 5.4.3 (Banach space). A *Banach space* is a complete normed vector space.

Remark. Namely, a vector space equipped with a norm whose metric induced by the norm is complete.

Theorem 5.4.1. Let $(V, \|\cdot\|)$ be a normed space. Then,

$$V \text{ is complete} \Leftrightarrow \text{every absolutely convergent series is convergent}.$$

i.e., if $\sum_{i=1}^{\infty} \|v_i\| < \infty$, then $\left\{\sum_{i=1}^{N} v_i\right\}_{N\in\mathbb{N}}$ converges to some $s \in V$.

Before we prove Theorem 5.4.1, we first see one of the result based on this theorem.[1]

Theorem 5.4.2 (Riesz-Fischer theorem). For every $1 \leq p \leq \infty$, we have $L^p(X, \mathcal{A}, \mu)$ is complete, hence a Banach space.

Proof. We prove this in steps.

1. We first prove this for $1 \leq p < \infty$. Suppose $f_n \in L^p$ and $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

   We need to show that there is an $F \in L^p$ such that $\left\|\sum_{n=1}^{N} f_n - F\right\|_p \to 0$ as $N \to \infty$. i.e., we need to show

   (a) $\sum_{n=1}^{\infty} f_n(x)$ is convergent a.e. In fact, we can show $\int \sum_{n=1}^{\infty} |f_n(x)| < \infty$.

   Let $G(x) = \sum_{n=1}^{\infty} |f_n(x)| = \sup_N \sum_{n=1}^{N} |f_n(x)|$, $G\colon X \to [0,\infty]$. Also, let $G_N(x) = \sum_{n=1}^{N} |f_n(x)|$. Then, we have

$$0 \leq G_1 \leq G_2 \leq \ldots \leq G,$$

[1] The proof can be found in here.

and $G_N \to G$. Furthermore,

$$0 \leq G_1^p \leq G_2^p \leq \ldots \leq G^p,$$

and $G_N^p \to G^p$. From monotone convergence theorem,

$$\int G^p = \lim_{N\to\infty} \int G_N^p.$$

From Minkowski inequality, we further have

$$\|G_N\|_p \leq \sum_{n=1}^{N} \|f_n\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p \coloneqq B < \infty.$$

Thus,

$$\int G(x)^p = \lim_{N\to\infty} \int G_N^p = \lim_{N\to\infty} \|G_N\|_p^p \leq B^p < \infty.$$

We see that G is finite a.e. as desired. This implies that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ a.e., so $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. Now, we simply let

$$F(x) = \begin{cases} \sum_{n=1}^{\infty} f_n(x), & \text{if it converges;} \\ 0, & \text{otherwise.} \end{cases}$$

(b) $F \in L^p$, where $F(x) \coloneqq \sum_{n=1}^{\infty} f_n(x)$ a.e. and say is zero elsewhere.

This is clear since

$$|F(x)| \leq G(x) \Rightarrow \int |F|^p \leq \int G^p < \infty,$$

hence $F \in L^p$.

(c) $\left\| \sum_{n=1}^{N} f_n - F \right\|_p \to 0$ as $N \to \infty$.

We now see that

$$\left| \sum_{n=1}^{N} f_n(x) - F(x) \right|^p \leq \left(\sum_{n=1}^{\infty} |f_n(x)| + |F(x)| \right)^p \leq (2G(x))^p.$$

Since $2G \in L^p$, so $2G^p \in L^1$. Thus, by dominated convergence theorem, we have

$$\lim_{N\to\infty} \int \left| \sum_{n=1}^{N} f_n(x) - F(x) \right|^p \,\mathrm{d}x = 0.$$

This implies

$$\left\| \sum_{n=1}^{N} f_n - F \right\|_p \to 0$$

as $N \to \infty$.

2. The case that $1 \leq p \leq \infty$ is left as an exercise. DIY

■

Lecture 26: Bounded Linear Transformations

We now prove Theorem 5.4.1, completing the proof of Theorem 5.4.2 since the latter relies on this result. 16 Mar. 11:00

Proof of Theorem 5.4.1. We prove it by proving two directions.

($\Rightarrow$) Suppose V is complete, and fix an absolutely convergent series $\sum_n v_n$. Define $s_N = \sum_{n=1}^N v_n$. It suffices to show the partial sums are a Cauchy Sequence.

Fix $\epsilon > 0$, then because $\sum_{n=1}^\infty \|v_n\| < \infty$, there is a $K \in \mathbb{N}$ so that

$$\sum_{n=K}^\infty \|v_n\| < \epsilon.$$

Now let $M > N > K$, we see that

$$\|s_M - s_N\| = \left\| \sum_{n=N+1}^M v_n \right\| \leq \sum_{n=N+1}^M \|v_n\| \leq \sum_{n=N}^\infty \|v_n\| < \epsilon,$$

so this is Cauchy.

($\Leftarrow$) Now suppose $v_n, n \in \mathbb{N}$ is a Cauchy sequence. For all $j \in \mathbb{N}$, there exists an $N_j \in \mathbb{N}$ such that

$$\|v_n - v_m\| < \frac{1}{2^j}$$

for all $n, m \geq N_j$. Without loss of generality, we may assume $N_1 < N_2 < \ldots$.

Let $w_1 = v_{N_1}$, $w_j = v_{N_j} - v_{N_j - 1}$ for $j \geq 2$. Therefore,

$$\sum_{j=1}^\infty \|w_j\| \leq \|v_{N_1}\| + \sum_{j=2}^\infty \frac{1}{2^{j-1}} < \infty.$$

Thus, $\sum_{j=1}^k w_j \to s \in V$ as $k \to \infty$. But by telescoping, we have

$$v_{N_k} = \sum_{j=1}^k w_j \to s.$$

Now we claim that since v_n is Cauchy, so $v_n \to s$.

Explicitly, take $\epsilon > 0$, and let k be large enough so that $\|v_{N_k} - s\| < \epsilon$ and $1/2^k < \epsilon$. Then if $n > N_k$ then

$$\|v_n - s\| \leq \|v_n - v_{N_k}\| + \|v_{N_k} - s\| < \epsilon + \epsilon = 2\epsilon.$$

Thus, $v_n \to s$. ∎

5.5 Bounded Linear Transformations

Definition 5.5.1 (Bounded linear transformation). Given two normed vector spaces $(V, \|\cdot\|)$, $(W, \|\cdot\|')$, a linear map $T \colon V \to W$ is called a *bounded map* if there exists $c \geq 0$ such that

$$\|Tv\|' \leq c\|v\|$$

for all $v \in V$.

Proposition 5.5.1. Suppose $T \colon (V, \|\cdot\|) \to (W, \|\cdot\|')$ is a linear map. Then the followings are equivalent.

1. T is continuous.
2. T is continuous at 0.
3. T is a bounded map.

Proof. 1. $\Rightarrow$ 2. is clear. For 2. $\Rightarrow$3., take $\epsilon = 1$, then there exists a $\delta > 0$ such that $\|Tu\|' < 1$ if $\|u\| < \delta$.

Now take an arbitrary $\|v\| \in V$, $v \neq 0$. Let $u = \frac{\delta}{2\|v\|}v$. Then $\|u\| < \delta$. Therefore,

$$\|Tu\|' < 1 \Rightarrow \frac{\delta}{2\|v\|}\|Tv\|' < 1 \Rightarrow \|Tv\|' < \frac{2}{\delta}\|v\|.$$

Then $2/\delta$ is our constant.

For 3. $\Rightarrow$ 1., fix $v_0 \in V$. Then for some constant c

$$\|Tv - Tv_0\|' = \|T(v - v_0)\|' \leq c\|v - v_0\|.$$

Thus, T is continuous, as when $v \to v_0$ the right-hand side goes to zero, and so $Tv \to Tv_0$. ■

Example. Let's see some examples.

1. We can look at

$$\begin{aligned} T\colon \ell^1 &\to \ell^1 \\ (a_1, a_2, \ldots) &\mapsto (a_2, a_3, \ldots). \end{aligned}$$

   Then clearly $\|Ta\|_1 \leq \|a\|_1$, so T is a bounded linear transformation.

2. We can also look at $S\colon (C([-1,1]), \|\cdot\|_1) \to \mathbb{C}$, where $Sf = f(0)$. S is not a bounded linear transformation, because we can make

$$\begin{cases} \|Sf\| &= |f(0)| = n \\ \|f\|_1 &= 1 \end{cases}$$

   for every $n \in \mathbb{N}$ (take f's graph to be a skinny triangle shooting up to n at 0).

3. But $U\colon (C([-1,1]), \|\cdot\|_\infty) \to \mathbb{C}$ defined by $Uf = f(0)$ is a bounded linear transformation, because $|f(0)| \leq \|f\|_\infty$.

4. Let A be an $n \times m$ matrix. Then $T\colon \mathbb{R}^m \to \mathbb{R}^n$ defined by $v \mapsto Av$ is a bounded linear transformation.

   Explicitly this is

$$(Tv)_i = (Av)_i = \sum_{j=1}^{m} A_{ij}v_j.$$

5. Let $K(x,y)$ be a continuous function on $[0,1] \times [0,1]$. We'll define

$$T\colon (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$$

   by

$$(Tf)(x) = \int_0^1 K(x,y)f(y)\,\mathrm{d}y.$$

   This is an analogue of matrix multiplication (K is like a continuous matrix). This is a bounded linear transformation.

6. Let us look at $T\colon L^1(\mathbb{R}) \to (C(\mathbb{R}), \|\cdot\|_\infty)$ defined by

$$(Tf)(t) = \int_{-\infty}^{\infty} e^{-itx}f(x)\,\mathrm{d}x$$

   that is the Fourier transform of f.

7. $T\colon (C^\infty[0,1], \|\cdot\|_\infty) \to (C^\infty[0,1], \|\cdot\|_\infty)$. Define

$$(Tf)(x) = f'(x).$$

This is not a bounded linear transformation. In contrast, S, defined on the same spaces

$$(Sf)(x) = \int_0^x f(t)\,\mathrm{d}t$$

is bounded.

Definition 5.5.2 (Operator norm). Let $L(V, W)$ be defined as a vector space such that

$$L(V,W) \coloneqq \{T\colon V \to W \mid T \text{ is a bounded linear transformation}\}.$$

Then for $T \in L(V, W)$, the *operator norm* of T is

$$\begin{aligned}\|T\| &\coloneqq \inf\{c \geq 0 \mid \|Tv\|'' \leq c\,\|v\|' \text{ for all } v \in V\}\\ &= \sup\left\{\frac{\|Tv\|''}{\|v\|'} \mid v \neq 0, v \in V\right\}\\ &= \sup\left\{\|Tv\|'' \mid \|v\|' = 1, v \in V\right\}.\end{aligned}$$

Lemma 5.5.1. We have that

1. The three definitions of $\|T\|$ above are all equal.
2. $(L(V,W), \|\cdot\|)$ is indeed a normed space.

Proof. ■ DIY

Lecture 27: Dual Space

18 Mar. 11:00

As previously seen. From Definition 5.5.2, we have that

$$\|Tv\|'' \leq \|T\|\,\|v\|'.$$

Remark. Notice that this Definition 5.5.2 is only for bounded linear transformation.

Theorem 5.5.1. If W is complete, then $L(V, W)$ is complete.

Proof. Suppose T_n is a Cauchy sequence in $L(V, W)$. Fix $v \in V$ and let $w_n \coloneqq T_n v \in W$, we then have

$$\|w_n - w_m\| = \|T_n v - T_m v\| = \|(T_n - T_m)v\| \leq \underbrace{\|T_n - T_m\|}_{\to 0}\,\underbrace{\|v\|}_{\substack{\text{fixed}\\ \text{value}}}.$$

Thus, w_n is Cauchy, so it converges since W is complete. We call its unique limit Tv. This makes $T\colon V \to W$ into a function. We must show it is a bounded linear transformation and that $\|T_n - T\| \to 0$. ■ DIY

5.6 Dual of L^p Spaces

Example. Let $w \in \mathbb{R}^d$, and denote the inner product between w and $v \in \mathbb{R}^d$ by

$$v \cdot w := \langle v, w \rangle .$$

Then we can consider

$$\max\{v \cdot w \mid \|v\|_2 = 1\} = \|w\|_2 .$$

If $w \in \mathbb{C}^d$, this is similar since

$$\max\{|v \cdot w| \mid \|v\|_2 = 1\} = \|w\|_2 .$$

These maximums are achieved by $v = \frac{\overline{w}}{\|w\|}$ if $w \neq 0$.

Proposition 5.6.1. Let $1/p + 1/q = 1$ with $1 \leq q < \infty$. For every $g \in L^q$,

$$\|g\|_q = \sup\left\{\left|\int fg\right| \mid \|f\|_p = 1\right\}.$$

Suppose μ is σ-finite. Then the result also holds for $q = \infty$, $p = 1$.

Proof. By Hölder's inequality, we know that

$$\left|\int fg\right| \leq \int |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q = \|g\|_q .$$

Thus, the supremum is less or equal to $\|g\|_q$.

1. Let

$$f(x) = \frac{|g(x)|^{q-1} \cdot \overline{\operatorname{sgn}(g(x))}}{\|g\|_q^{q-1}}$$

Then $\int |f|^p = 1$, and $\int fg = \|g\|_q$. Check

> **Note.** For $\alpha \in \mathbb{C}$, $\operatorname{sgn}(\alpha) := e^{i\theta}$ where $\alpha = |\alpha| e^{i\theta}$.

2. The case that μ is σ-finite and $q = \infty, p = 1$ can be shown. DIY

■

Remark. One could use the above to prove Minkowski's inequality (as it only uses Hölder's inequality).

Definition 5.6.1 (Dual space)**.** For a normed space $(V, \|\cdot\|)$, its *dual space* is $V^* = L(V, \mathbb{R})$ or $V^* = L(V, \mathbb{C})$.

Remark. Namely, the dual space of V contains bounded linear transformations with codomain being the scalar field.

Definition 5.6.2 (Linear functional)**.** Given a normed space $(V, \|\cdot\|)$, $\ell \in V^*$ is called a *linear functional* on V. i.e.,

- $\ell\colon V \to \mathbb{R}$ (or $\mathbb{C}$).
- ℓ is linear.
- There exists a $c \geq 0$ such that $|\ell(v)| = c\|v\|$.

Note. V^* is always a Banach space (even if V is not complete).

Corollary 5.6.1. We have the followings.

1. Let $1/p + 1/q = 1, 1 \leq q < \infty$. For $g \in L^q$ define $\ell_g \in L^p \to \mathbb{C}$ by
$$\ell_g(f) = \int fg.$$
Then $\ell_g \in (L^p)^*$. Furthermore, $\|\ell_g\| = \|g\|_q$.
2. If μ is σ-finite then this also holds for $q = \infty, p = 1$.

Proof. ℓ_g is clearly linear in f because the integral is linear. Then Proposition 5.6.1 gives in both 1. and 2. that
$$\|g\|_q = \sup\{|\ell_g(f)| \mid \|g\|_p = 1\} = \|\ell_g\|$$
and so ℓ_g is a bounded linear transformation with the desired properties. ∎

Theorem 5.6.1. We have the followings.

1. Let $1/p + 1/q = 1$, $1 \leq q < \infty$. The map $T\colon L^q \to (L^p)^*$ given by $Tg = \ell_g$ is an isometric[a] linear isomorphism.

   This means that

   - T is a bounded linear transformation.
   - T is bijective.
   - T is norm-preserving.
2. If μ is σ-finite then this also holds for $q = \infty, p = 1$.

[a] A map T is called isometric if for a given g, $\|Tg\| = \|g\|$.

Proof. We have already proved this is isometric in Corollary 5.6.1, it is clearly linear, and isometry implies injectivity.

We will prove that it is surjective later. ∎

Fix!!!

Note. Even if μ is σ-finite we might not have $L^1 \cong (L^\infty)^*$.

Also note that $L^2 \cong (L^2)^*$, and for all $1 < p < \infty$ we have $(L^p)^{**} \cong L^p$.

Chapter 6

Signed and Complex Measures

Lecture 28: Signed Measure

21 Mar. 11:00

As previously seen. Suppose $f\colon X \to [0, \infty]$ is a measurable function on $(X, \mathcal{A}, \mu)$.

We can define $\nu(E) = \int_E f \,\mathrm{d}\mu$ for $E \in \mathcal{A}$, and ν is a measure on $(X, \mathcal{A})$.

This gives a map from the set of non-negative measurable functions on X to measures on X. This is injective if we identify functions which are equal almost everywhere. But it is not necessarily surjective. We can then think of measures as a generalization of functions.

For an example, think of a Dirac-Delta measure on $\mathbb{R}$. This is not the Lebesgue integral of any non-negative measurable function.

What if instead we took $f\colon X \to \mathbb{R}$, $\overline{\mathbb{R}}$ or $\mathbb{C}$. We could take the same construction to get $\nu(E) = \int_E f \,\mathrm{d}\mu$, but this is no longer a measure as it can take $\mathbb{R}$, $\overline{\mathbb{R}}$ or $\mathbb{C}$ values.

6.1 Signed Measures

Definition 6.1.1. Let $(X, \mathcal{A})$ be a measurable space. A *signed measure* is $\nu\colon \mathcal{A} \to [-\infty, \infty)$ or $\nu\colon \mathcal{A} \to (-\infty, \infty]$ such that

- $\nu(\varnothing) = 0$.
- If $A_1, A_2, \ldots \in \mathcal{A}$ are disjoint then

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$$

  where the series on the right-hand side converges absolutely if

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \in (-\infty, \infty).$$

  This means the series does not depend on rearrangement.

Example. Consider

1. ν is a positive measure (i.e., measure), then ν is a signed measure.
2. If we have positive measures μ_1, μ_2 such that either $\mu_1(X) < \infty$ or $\mu_2(X) < \infty$, then $\nu = \mu_1 - \mu_2$ is a signed measure.
3. If $f\colon X \to \bar{\mathbb{R}}$ on a measure space $(X, \mathcal{A}, \mu)$ such that $\int_X f^+ \,\mathrm{d}\mu < \infty$ or $\int_X f^- \,\mathrm{d}\mu < \infty$, we can

define

$$\nu(E) = \int_E f \, \mathrm{d}\mu$$

and this will be a signed measure.

Note. The following weird things happen with signed measures.

1. $A \subseteq B$ does not imply $\nu(A) \leq \nu(B)$, as $\nu(B) = \nu(A) + \nu(B \setminus A)$, and $\nu(B \setminus A)$ may be negative.
2. If $A \subseteq B$ and $\nu(A) = \infty$, then $\nu(B) = \infty$, because $\nu(B \setminus A) \in (-\infty, \infty]$.
3. Similarly, if $A \subseteq B$ and $\nu(A) = -\infty$ then $\nu(B) = -\infty$.

Lemma 6.1.1. If ν is a signed measure on $(X, \mathcal{A})$, then we have

1. Continuity from below. If $E_n \in \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \cdots$ then
$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} \nu(E_N).$$
2. Continuity from above. If $E_n \in \mathcal{A}$, $E_1 \supseteq E_2 \supseteq \cdots$, and $-\infty < \nu(E_1) < \infty$ then
$$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} \nu(E_N).$$

Proof. Read [FF99]. ∎

Definition 6.1.2 (Positive, negative, null set for a signed measure)**.** Let ν be a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$, then we say that

1. E is *positive for* ν if for all $F \subseteq E$, $\nu(F) \geq 0$.
2. E is *negative for* ν if for all $F \subseteq E$, $\nu(F) \leq 0$.
3. E is *null for* ν if for all $F \subseteq E$, $\nu(F) = 0$.

Note. We see that

1. If E is a positive set, $F \subseteq E$, then $\nu(F) \leq \nu(E)$.
2. If E is a negative set, $F \subseteq E$, then $\nu(F) \geq \nu(E)$.

Lemma 6.1.2. Let ν be a signed measure on $(X, \mathcal{A})$, then

1. If E is positive, $G \subseteq E$ is measurable, then G is positive.
2. If E is negative, $G \subseteq E$ is measurable, then G is negative.
3. If E is null, $G \subseteq E$ is measurable, then G is positive.
4. $E_1, E_2, \ldots$ are positive sets, then $\bigcup_{i=1}^{\infty} E_i$ is positive.

Proof. ∎

DIY

Lemma 6.1.3. Suppose that ν is a signed measure with $\nu \colon \mathcal{A} \to [-\infty, \infty)$. Suppose $E \in \mathcal{A}$ and $0 < \nu(E) < \infty$, then there exists a measurable $A \subseteq E$ such A is a positive set and $\nu(A) > 0$.

Proof. If E is positive, we're done. Otherwise, there exist measurable subsets with negative measure. Let $n_1 \in \mathbb{N}$ be the least such n_1 such that there exists $E_1 \subseteq E$ with $\nu(E_1) < -1/n_1$.

If $E \setminus E_1$ is positive, we're done. Else we can inductively define $n_2, n_3, \ldots$ as well as $E_2, E_3, \ldots$.

Explicitly, if $E \setminus \bigcup_{i=1}^{k-1} E_i$ is not positive, let n_k be the least such that there exists $E_k \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ with $\nu(E_k) < -1/n_k$.

Note then that if $n_k \geq 2$, for all $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ we have that $\nu(B) \geq -\frac{1}{n_k - 1}$.

Now let $A = E \setminus \bigcup_{i=1}^{\infty} E_i$. Since $E = A \cup (\bigcup_i E_i)$ we have by countable additivity that

$$0 < \nu(E) = \nu(A) + \sum_{k=1}^{\infty} \nu(E_k) < \nu(A).$$

Furthermore, $\nu(E), \nu(A)$ are both in $(0, \infty)$, and we see that

$$0 < \nu(E) \leq \nu(A) - \sum_{k=1}^{\infty} \frac{1}{n_k}.$$

Therefore, the sum on the right-hand side must converge, meaning that $1/n_k \to 0$ as $k \to \infty$. That is $\lim_{k\to\infty} n_k = \infty$.

Now if $B \subseteq A$, then $B \subseteq E \setminus \bigcup_{i=1}^{\infty} E_i$. Therefore, $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$. By the note above, for large enough k such that $n_k \geq 2$ we have

$$\nu(B) \geq \frac{-1}{n_k - 1},$$

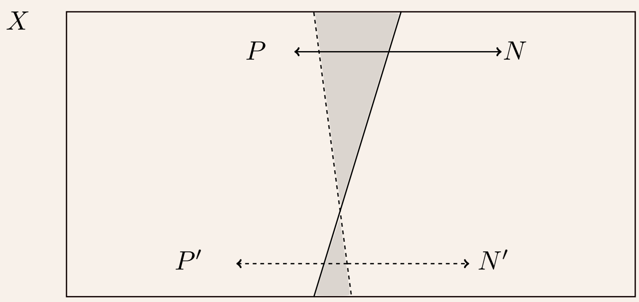
then taking $k \to \infty$ we have $\nu(B) \geq 0$, and so A is a positive set as desired. ∎

Theorem 6.1.1 (Hahn decomposition theorem)**.** If ν is a signed measure on $(X, \mathcal{A})$, then there exist $P, N \in \mathcal{A}$ such that

$$P \cap N = \varnothing, \quad P \cup N = X,$$

where P is positive for ν, N is negative for ν.

Furthermore, if P', N' are another such pair, then $P \triangle P' (= N \triangle N')$ is null for ν.



Lecture 29: Hahn and Jordan Decomposition Theorem

We now prove Theorem 6.1.1. 23 Mar. 11:00

Proof. We first show the uniqueness. We see that $P \setminus P' \subseteq P, P \setminus P' \subseteq N'$. Thus, $P \setminus P' \subseteq P \cap N'$ is both positive and negative, hence $P \setminus P'$ is null.

Similarly, for $P' \setminus P$, and then their union $P \triangle P'$ is null as well.

To show the existence, without loss of generality suppose $\nu \colon \mathcal{A} \to [-\infty, \infty)$. If not, consider $-\nu$. Let

$$s \coloneqq \sup\{\nu(E) \mid E \in \mathcal{A} \text{ is a positive set}\},$$

which is a nonempty supremum because $\varnothing$ is positive. Then there exist $P_1, P_2, \ldots$ positive sets such that $\lim\limits_{n\to\infty} \nu(P_n) = s$.

Then we have that $P = \bigcup_n P_n$ is positive by Lemma 6.1.2. We then have $\nu(P) \leq s$ and

$\nu(P) = \nu(P_n) + \nu(P \setminus P_n) \geq \nu(P_n)$. Thus,

$$\nu(P) \geq \lim_{n\to\infty} \nu(P_n) = s.$$

Hence, $\nu(P) = s$ and the supremum is in fact a max. We then know that $s = \nu(P) < \infty$ because ν does not attain the value infinity.

Now let $N = X \setminus P$. We claim that N is negative. If not then there exists a measurable $E \subseteq N$ with $\nu(E) > 0$. By assumption, $\nu(E) < \infty$. Then $0 < \nu(E) < \infty$, so by Lemma 6.1.3 there exists a measurable $A \subseteq E$ such that A is positive and $\nu(A) > 0$.

But we then know that

$$\nu(P \cup A) = \nu(P) + \nu(A) > \nu(P)$$

which is a contradiction since $P \cup A$ is a positive set, and $\nu(P)$ is maximal. Therefore, N is negative, and the theorem holds.

Finally, if P', N' is another pair of sets as in the statement of the theorem, we have

$$P \setminus P' \subset P, \quad P \setminus P' \subset N',$$

so that $P \setminus P'$ is both positive and negative, hence null; likewise for $P' \setminus P$. ∎

Definition 6.1.3 (Singular)**.** If μ, ν are signed measures on $(X, \mathcal{A})$, then we say μ and ν are *singular to each other*, denoted as $\mu \perp \nu$, if there exists $E, F \in \mathcal{A}$ such that $E \cap F = \varnothing, E \cup F = X$, F is null for μ, E is null for ν.

Example. Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

1. The Lebesgue measure m.
2. The Cantor measure μ_C defined by the Cantor function.
3. The discrete measure $\mu_D = \delta_1 + 2\delta_{-1}$.

We then see that

1. Take $E = \mathbb{R} \setminus \{-1, 1\}, F = \{1, -1\}$ to see that $m \perp \mu_D$.
2. Take $E = \mathbb{R} \setminus K$ and $F = K$ where K is the Cantor set to see that $m \perp \mu_C$.
3. We can also see that $\mu_C \perp \mu_D$.

Theorem 6.1.2 (Jordan decomposition theorem)**.** Let ν be a signed measure on $(X, \mathcal{A})$. Then there exists unique positive measures ν^+, ν^- on $(X, \mathcal{A})$ such that for all $E \in \mathcal{A}$ we have

$$\nu(E) = \nu^+(E) - \nu^-(E)$$

and

$$\nu^+ \perp \nu^-.$$

Proof. For existence, we take $\nu^+(E) := \nu(E \cap P), \nu^-(E) := -\nu(E \cap N)$ where P, N is the Hahn decomposition of X.

If there exists μ^+, μ^- such that $\nu = \mu^+ + \mu^-$ and $\mu^+ \perp \mu^-$, let $E, F \in \mathcal{A}$ be such that $E \cap F = \varnothing$, $E \cup F = X$, and $\mu^+(F) = \mu^-(E) = 0$. Then we have that $X = E \cup F$ is another Hahn decomposition for ν, so $P \triangle E$ is ν-null. Therefore, for any $A \in \mathcal{A}$, $\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A)$, hence $\mu^+ = \nu^+$. Likewise, we have $\nu^- = \mu^-$. ∎

Lecture 30: Absolutely Continuous Measures

25 Mar. 11:00

Example. For an example of Theorem 6.1.2, let $(X, \mathcal{A}, \mu)$ be a measure space, $f\colon X \to \overline{\mathbb{R}}$, and

$\nu(E) = \int_E f \,\mathrm{d}\mu$. Then

$$\nu^+(E) = \int_E f^+ \,\mathrm{d}\mu, \quad \nu^-(E) = \int_E f^- \,\mathrm{d}\mu.$$

Definition 6.1.4 (Positive, negative variation)**.** Given a signed measure ν on $(X, \mathcal{A})$ and its Jordan decomposition $\nu = \nu^+ - \nu^-$, we call ν^+ the *positive variation* of ν, and ν^- the *negative variation* of ν.

Definition 6.1.5 (Total variation)**.** Let ν be a signed measure on $(X, \mathcal{A})$. The *total variation measure of* ν is $|\nu| := \nu^+ + \nu^-$.

Remark. This is a positive measure on X.

Example. In the above example, $|\nu|(E) = \int_E |f| \,\mathrm{d}\mu$.

Lemma 6.1.4. We have the following

1. $|\nu(E)| \leq |\nu|(E)$.
2. E is ν-null if and only if E is $|\nu|$-null.
3. If κ is another signed measure, then $\kappa \perp \nu$ if and only if $\kappa \perp |\nu|$ if and only if $\kappa \perp \nu^+$ and $\kappa \perp \nu^-$.

Proof. ■ DIY

Definition 6.1.6 (Finite signed measure)**.** A signed measure ν is *finite* if $|\nu|$ is a finite measure, and similarly for σ-finite.

Remark. This holds if and only if ν^+, ν^- are both finite (resp. σ-finite) measures.

6.2 Absolutely Continuous Measures

Definition 6.2.1 (Absolutely continuous)**.** Let μ be a positive measure, ν be a signed measure, both on $(X, \mathcal{A})$. We say that ν is *absolutely continuous with respect to* μ, denoted as $\nu \ll \mu$, provided that for all $E \in \mathcal{A}$, $\mu(E) = 0$ implies $\nu(E) = 0$.

Remark. This is equivalent to every μ-null set being ν-null.

Example. If $(X, \mathcal{A}, \mu)$, $f \colon X \to \overline{\mathbb{R}}$, $\nu(E) = \int_E f \,\mathrm{d}\mu$, then $\nu \ll \mu$.

Notation. $\mathrm{d}\nu = f \,\mathrm{d}\mu$ means ν is a signed measure defined by

$$\nu(E) = \int_E f \,\mathrm{d}\mu.$$

Lemma 6.2.1. If μ is a positive measure, ν is a signed measure on $(X, \mathcal{A})$, then

1. $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll< \mu$.

2. $\nu \ll \mu$ and $\nu \perp \mu$ implies $\nu = 0$.

Proof. DIY 1.

For 2., write $X = A \cup B$, $A \cap B = \varnothing$, A μ-null, B ν-null. Then

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap A).$$

Then $E \cap A \subseteq A$, so $\nu(E \cap A) = 0$. By absolute continuity, $\nu(E \cap A) = 0$, thus $\nu(E) = 0$. ■

Theorem 6.2.1 (Radon-Nikodym theorem)**.** Suppose μ is a σ-finite positive measure, ν is a σ-finite signed measure, and suppose $\nu \ll \mu$. Then there exists $f\colon X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\nu = f\,\mathrm{d}\mu$, in other words $\nu(E) = \int_E f\,\mathrm{d}\mu$.

If g is another such function with $\mathrm{d}\nu = g\,\mathrm{d}\mu$ then $f = g$ μ-a.e..

Proof. We'll prove a more general form called Lebesgue Radon Nikodym theorem, which is a more general theorem compare to Theorem 6.2.1. ■

Definition 6.2.2 (Randon-Nikodym derivative)**.** Suppose $\nu \ll \mu$. The *Radon-Nikodym derivative of ν with respect to μ* is a function

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\colon X \to \overline{\mathbb{R}}$$

such that

$$\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu$$

for all $E \in \mathcal{A}$.

Remark. i.e. we have $\mathrm{d}\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu$.

Note. By Theorem 6.2.1, such a function exists and is unique up to equivalence μ-a.e. in the σ-finite case.

Example. Say $F(X) = e^{2x}\colon \mathbb{R} \to \mathbb{R}$. This is continuous and strictly increasing, so we may define a Lebesgue-Stieltjes measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

This is defined to be the unique locally finite measure satisfying $\mu_F([a,b]) = F(b) - F(a) = e^{2b} - e^{2a}$. Then one can check that

$$\mu_F(E) = \int_E 2e^{2x}\,\mathrm{d}x$$

by uniqueness and the classical fundamental theorem of calculus, since the right-hand side is a locally finite Borel measure, and $\kappa([a,b]) = e^{2b} - e^{2a}$, thus $\mu_F = \kappa$.

Therefore, $\mu_F \ll m$ and $\frac{\mathrm{d}\mu_F}{\mathrm{d}m} = 2e^{2x} = \frac{\mathrm{d}F}{\mathrm{d}x}$.

Example. Let $C(X)\colon \mathbb{R} \to \mathbb{R}$ be the Cantor function. Then $C'(x) = 0$ outside the Cantor set. But we don't always have

$$\mu_C(E) \neq \int_E 0\,\mathrm{d}x.$$

So the candidate derivative is 0, but this fails. In particular,

$$C(b) - C(a) \neq \int_a^b C'(x)\,\mathrm{d}x.$$

In fact, $\mu_C \not\ll m$ because $\mu_C \perp m$ and $\mu_C \neq 0$.

Thus, the existence of a derivative almost everywhere and continuity is not enough to guarantee a version of the fundamental theorem of calculus holds.

Lecture 31: Lebesgue-Radon-Nikodym Theorem

28 Mar. 11:00

Lemma 6.2.2. Let μ, ν be finite positive measures on $(X, \mathcal{A})$. Then either

1. $\nu \perp \mu$.
2. There exists an $\epsilon > 0$, an $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for the measure $\nu - \epsilon\mu$, i.e., for all $G \subseteq F$, $\nu(G) \geq \epsilon\mu(G)$.

Proof. Let $\kappa_n = \nu - (1/n)\mu$. By Theorem 6.1.1 we have $X = P_n \cup N_n$ for P_n positive, N_n negative for κ_n. Also, we let $P = \bigcup_n P_n$, $N = \bigcap_n N_n = X \setminus P$, then $X = P \cup N$.

We see that for any n we have $\kappa_n(N) \leq 0$ because $N \subseteq N_n$. Thus,

$$0 \leq \nu(N) \leq \frac{1}{n}\mu(N),$$

which implies $\nu(N) = 0$. Because ν is positive for any $N' \subseteq N$ we have $0 \leq \nu(N') \leq \nu(N)$, and thus $\nu(N') = 0$. This shows N is null for ν. Now, we see that

- If $\mu(P) = 0$, then $\nu \perp \mu$.
- If $\mu(P) \neq 0$, then we have $\mu(P) > 0$ hence $\mu(P_n) > 0$ for some n. With $F = P_n$ and $\epsilon = 1/n$, then F is a positive set for $\kappa_n = \nu - (1/n)\mu$ as desired.

■

Theorem 6.2.2 (Lebesgue-Radon-Nikodym theorem)**.** Let μ be a σ-finite positive measure, ν a σ-finite signed measure on $(X, \mathcal{A})$. Then there are unique σ-finite signed measures λ, ρ on $(X, \mathcal{A})$ such that

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \nu = \lambda + \rho.$$

Furthermore, there exists a measurable function $f \colon X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\rho = f \,\mathrm{d}\mu$.[a] And if there is another g such that $\mathrm{d}\rho = g \,\mathrm{d}\mu$, then $f = g$ μ-a.e.

[a] That is for all $E \in \mathcal{A}$, $\rho(E) = \int_E f \,\mathrm{d}\mu$.

Proof. We prove it step by step.

1. Assume μ, ν are finite positive measures. We first prove the existence of λ, f, and $\mathrm{d}\rho = f = \mathrm{d}\mu$.

   Let

$$\begin{aligned} \mathscr{F} &= \left\{ g \colon X \to [0, \infty] \mid \int_E g \,\mathrm{d}\mu \leq \nu(E), \forall E \in \mathcal{A} \right\} \\ &= \{ g \colon X \to [0, \infty] \mid \mathrm{d}\nu - g \,\mathrm{d}\mu \text{ is a positive measure} \}. \end{aligned}$$

   This set is nonempty since $g = 0 \in \mathscr{F}$. Let $s = \sup\{\int_X g \,\mathrm{d}\mu \mid g \in \mathscr{F}\}$.

   Claim. There is an $f \in \mathscr{F}$ such that $s = \int_X f \,\mathrm{d}\mu$.

Proof. If $g, h \in \mathscr{F}$, we can define $u(x) = \max\{g(x), h(x)\}$, then $u \in \mathscr{F}$. This can be seen by letting $A = \{x \mid g(x) \geq h(x)\}$, then

$$\begin{aligned}\int_E u \,\mathrm{d}\mu &= \int_{E\cap A} g \,\mathrm{d}\mu + \int_{E\cap A^c} h \,\mathrm{d}\mu \\ &\leq \nu(E\cap A) + \nu(E\cap A^c) = \nu(E).\end{aligned}$$

There exist measurable functions $g_1, g_2, \ldots \in \mathscr{F}$ such that

$$\lim_{n\to\infty} \int_X g_n \,\mathrm{d}\mu = s.$$

We can replace g_2 by $\max(g_1, g_2)$, g_3 by $\max(g_1, g_2, g_3)$. Generally,

$$g_n \leftarrow \max(g_1, g_2, \ldots, g_n),$$

so that we may assume $0 \leq g_1 \leq g_2 \leq \ldots$.
Then we still know that $\lim\limits_{n\to\infty} \int_X g_n \,\mathrm{d}\mu = s$, as all the relevant integrals are bounded above by s. Now let $f(x) = \sup_n g_n(x) = \lim\limits_{n\to\infty} g_n(x)$, by Monotone convergence theorem,

$$\int_E f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_E g_n \,\mathrm{d}\mu \leq \nu(E).$$

Thus, $f \in \mathscr{F}$, and when $E = X$ we get $\int_X f \,\mathrm{d}\mu = s$ as desired. ∎

Let $\rho(E) := \int_E f \,\mathrm{d}\mu$, then we of course have $\rho \ll \mu$, and also, we know

$$0 \leq \rho(X) = \int_X f \,\mathrm{d}\mu \leq \nu(X) < \infty.$$

Thus, ρ is a finite positive measure, so we can define $\lambda(E) := \nu(E) - \rho(E)$, then

$$\lambda(E) = \nu(E) - \int_E f \,\mathrm{d}\mu \geq 0$$

because $f \in \mathscr{F}$. Thus, λ is also a positive measure, and $\lambda(X) \leq \nu(X) < \infty$. It remains to show the following.

Claim. $\lambda \perp \mu$.

Proof. Suppose not, by Lemma 6.2.2, there exists $\epsilon > 0$, $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for $\lambda - \epsilon\mu$.

Then this says that $\mathrm{d}\lambda - \epsilon \mathbb{1}_F \,\mathrm{d}\mu$ is a positive measure, that is,

$$\mathrm{d}\nu - f\,\mathrm{d}\mu - \epsilon \mathbb{1}_F \,\mathrm{d}\mu$$

is a positive measure. But, this will break maximality of f, specifically, let $g(x) = f(x) + \epsilon \mathbb{1}_F(x)$. Then for all $E \in \mathcal{A}$ we have

$$\begin{aligned}\int_E g\,\mathrm{d}\mu &= \int_E f\,\mathrm{d}\mu + \epsilon\mu(E \cap F) \\ &= \nu(E) - \lambda(E) + \epsilon\mu(E \cap F) \\ &\leq \nu(E) - \lambda(E \cap F) + \epsilon\mu(E \cap F) \leq \nu(E)\end{aligned}$$

since $\lambda(E \cap F) - \epsilon\mu(E \cap F) \geq 0$. Thus, $g \in \mathscr{F}$. We then see that

$$s \geq \int_X g\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X \epsilon \mathbb{1}_F \,\mathrm{d}\mu = s + \epsilon\mu(F) > s,$$

which is a contradiction. ■

We see that the existence of λ, f, and $\mathrm{d}\rho = f\,\mathrm{d}\mu$ is proved. As for uniqueness, if there are λ' and f' such that $\mathrm{d}\nu = \mathrm{d}\lambda' + f'\,\mathrm{d}\mu$, we then have

$$\mathrm{d}\lambda - \mathrm{d}\lambda' = (f' - f)\,\mathrm{d}\mu.$$

But we see that $\lambda - \lambda' \perp \mu$ while $(f' - f)\,\mathrm{d}\mu \ll \mathrm{d}\mu$, hence Check!

$$\mathrm{d}\lambda - \mathrm{d}\lambda' = (f' - f)\,\mathrm{d}\mu = 0,$$

so $\lambda = \lambda'$ and $f = f'$ μ-a.e. by Proposition 2.3.1.

2. Suppose that μ and ν are σ-finite measures. Then X is a countable disjoint union of μ-finite sets and a countable disjoint union of ν-finite sets. By taking intersections of these we obtain a disjoint sequence $\{A_j\} \subset \mathcal{A}$ such that $\mu(A_j)$ and $\nu(A_j)$ are finite for all j and $X = \bigcup_j A_j$. Define $\mu_j(E) = \mu(E \cap A_j)$ and $\nu_j(E) = \nu(E \cap A_j)$, then by the reasoning above, for each j we have
$$\mathrm{d}\nu_j = \mathrm{d}\lambda_j + f_j\,\mathrm{d}\mu_j$$
where $\lambda_j \perp \mu_j$. Since $\mu_j(A_j^c) = \nu_j(A_j^c) = 0$, we have
$$\lambda_j(A_j^c) = \nu_j(A_j^c) - \int_{A_j^c} f\,\mathrm{d}\mu_j = 0,$$
and we may assume that $f_j = 0$ on A_j^c. Let $\lambda = \sum_j \lambda_j$ and $f = \sum_j f_j$, we then have
$$\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}\mu, \quad \lambda \perp \mu,$$
and $\mathrm{d}\lambda$ and $f\,\mathrm{d}\mu$ are σ-finite, as desired. As for uniqueness, it's the same as for the first case.

3. We now consider the general case. If ν is a signed measure, we apply the preceding argument to ν^+ and ν^- and subtract the results.

■

Remark. Notationally, we may write $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}\mu$, where $\mathrm{d}\lambda$ and $\mathrm{d}\mu$ are singular to each other.

Lecture 32: Lebesgue Differentiation Theorem for Regular Borel Measures

We now do an example to illustrate Theorem 6.2.2. 30 Mar. 11:00

Example. Let $\mu = m$, $\nu = \mu_F$ (Lebesgue-Stieltjes measure for F). We'll define $F(x)$ by

$$F(x) = \begin{cases} e^{3x}, & \text{if } x \leq 0; \\ 1, & \text{if } 0 < x < 1; \\ 5, & \text{if } x \geq 1. \end{cases}$$

Then we will have that

$$\mu_F(E) = \int_{E \cap \mathbb{R}_{<0}} 3e^{3x} \,\mathrm{d}x + 4\delta_1(E).$$

It is enough to check on $(-\infty, x]$ because these are locally finite Borel measures on $\mathbb{R}$.

Then we have $\mu_F = \mathrm{d}\rho + \mathrm{d}\lambda = f \,\mathrm{d}m + \mathrm{d}\lambda$ where $f = \mathbb{1}_{\mathbb{R}_{<0}} 3e^{3x}$ and $\lambda = 4\delta_1$, $\lambda \perp m$.

Specifically, we call such a decomposition *Lebesgue decomposition* of ν with respect to μ. Now, with the condition $\nu \ll \mu$, Theorem 6.2.2 implies that $\mathrm{d}\nu = f \,\mathrm{d}\mu$ for some f, which is exactly the statement of Theorem 6.2.1. And, it should be clear now that the definition of Radon Nikodym derivative of ν with respect to μ, denoted as $\mathrm{d}\nu / \mathrm{d}\mu$, is just f in this case.

As previously seen. If $\nu = \nu^+ - \nu^-$, we defined the total variation $|\nu| = \nu^+ + \nu^-$. Then we have $|\nu(E)| \leq |\nu|(E)$.

6.3 Lebesgue Differentiation Theorem for Regular Borel Measures

Definition 6.3.1 (Regular)**.** A Borel signed measure ν on $\mathbb{R}^d$ is called *regular* if

1. (compact finite) $|\nu|(K) < \infty$ for all compact K.
2. (outer regularity) We have outer regularity

$$|\nu|(E) = \inf\{|\nu|(U) \mid \text{ open } U \supseteq E\}$$

for every Borel set E.

Example. We see that

1. Any Lebesgue-Stieltjes measure on $\mathbb{R}$ has this property from Theorem 1.7.1, so is the difference between two of them (at least if one of them is finite).
2. The Lebesgue measure on $\mathbb{R}^d$ is regular.

Note. From compact finiteness, if ν is regular then it is σ-finite.

Lemma 6.3.1. $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ if and only if $\mathrm{d}\nu = f \,\mathrm{d}m$ is regular.

Proof. We prove this in two directions.

- Suppose $\mathrm{d}\nu = f \,\mathrm{d}m$ is regular. Then

$$|\nu|(K) = \int_K |f| \,\mathrm{d}m < \infty$$

for all compact K, thus $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

- Suppose $f \in L^1_{\text{loc}(\mathbb{R}^d)}$. This condition is clearly equivalent to compact finiteness. If this holds, then the outer regularity may be verified directly as follows. Suppose that E is a bounded Borel set. Given $\delta > 0$, by Theorem 3.5.1, there is a bounded open $U \supset E$ such that $m(U) < m(E) + \delta$ and hence $m(U \setminus E) < \delta$. But then, given $\epsilon > 0$, there is[a] an open $U \supset E$ such that

$$\int_{U \setminus E} f \, \mathrm{d}m < \epsilon$$

and hence

$$\int_U f \, \mathrm{d}m < \int_E f \, \mathrm{d}m + \epsilon.$$

The case of unbounded E follows easily by writing

$$E = \bigcup_{j=1}^{\infty} E_j$$

where E_j is bounded and finding an open $U_j \supset E_j$ such that

$$\int_{U_j \setminus E_j} f \, \mathrm{d}m < \epsilon 2^{-j}.$$

■

[a] This follows from [FF99] Corollary 3.6.

As previously seen. Recall the Lebesgue differentiation theorem, here we had that if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ implies that for Lebesgue almost every x,

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) \, \mathrm{d}y = f(x)$$

for any $\{E_r\}$ shrinks nicely to x.

Corollary 6.3.1. Let ρ be a regular signed Borel measure on $\mathbb{R}^d$. Suppose $\rho \ll m$. Then $\mathrm{d}\rho = f \, \mathrm{d}m$ for some $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, So then for Lebesgue almost every x we have

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) \, \mathrm{d}y = f(x).$$

Writing this nicely, using established notation, this is

$$\lim_{r \to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for every $\{E_r\}$ shrinks nicely to x.

Proposition 6.3.1. Let λ be a regular positive Borel measure on $\mathbb{R}^d$. Suppose $\lambda \perp m$. Then for Lebesgue almost every x, we have

$$\lim_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for every $\{E_r\}$ shrinking to x nicely (equivalently, shrinking to 0 nicely).

Proof. It is enough to consider $E_r = B(x, r)$. We wish to prove that

$$G := \left\{ x \mid \limsup_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} \neq 0 \right\} = \bigcup_{n=1}^{\infty} G_n$$

where

$$G_n := \left\{ x \mid \limsup_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} > \frac{1}{n} \right\}$$

such that $m(G) = 0$. We see that it suffices to show $m(G_n) = 0$ for all n. Since $\lambda \perp m$, so we know there exists A, B such that $\mathbb{R}^d = A \cup B$ disjoint with $\lambda(A) = 0$, $m(B) = 0$. Thus, it suffices to show $m(G_n \cap A) = 0$.[a]

Fix $\epsilon > 0$, since λ is regular, there exists an open set $U \supseteq A$ such that $\lambda(U) \leq \lambda(A) + \epsilon = \epsilon$. We see that for every $x \in G_n \cap A$, there is an $r_x > 0$ such that $\lambda(B(x, r_x))/m(B(x, r_x)) > 1/n$ where $B(x, r_x) \subseteq U$.

Let $K \subseteq G_n \cap A$, compact. Then $K \subseteq \bigcup_{x\in K} B(x, r_x)$. By compactness, we can take a finite sub-cover, and then use Lemma 4.1.1 to find disjoint $B_1, B_2, \ldots, B_N$ such that each of B_i is in the form of $B(x_i, r_{x_i})$ and $K \subseteq \bigcup_i 3B_i$. Therefore,

$$m(K) \leq 3^d \sum_{i=1}^{N} m(B_i) \leq 3^d n \sum_{i=1}^{N} \lambda(B_i) = 3^d n \lambda\left(\bigcup_{i=1}^{N} B_i \right) \leq 3^d n \lambda(U) = 3^d n \epsilon.$$

By inner regularity, $m(G_n \cap A) \leq 3^d n\epsilon$ for any $\epsilon > 0$. Taking $\epsilon \to 0$ yields $m(G_n \cap A) = 0$, so then $m(G_n) = 0$ as desired. ■

[a] Alternatively, we can simply define G_n over A instead of $\mathbb{R}^d$, as in Folland[FF99].

Lecture 33: Monotone Differentiation Theorem

1 Apr. 11:00

As previously seen. We have that if $\rho \ll m$ is regular then

$$\lim_{r\to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for Lebesgue almost every x, where $\{E_r\}$ shrinks nicely to x. Likewise, if $\lambda \perp m$ regular (positive measure) then

$$\lim_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for Lebesgue almost every x, where $\{E_r\}$ shrinks nicely to x.

From this, we can easily deduce the following important result.

Theorem 6.3.1 (Lebesgue differentiation theorem for regular measures)**.** Let ν be a regular Borel signed measure on $\mathbb{R}^d$. Then $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}m$, $\lambda \perp m$ by Theorem 6.2.2. Then for Lebesgue almost every x,

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every $\{E_r\}$ shrinks nicely to x.

Proof. It must be checked that ν regular implies $\lambda, f\,\mathrm{d}m$ are regular. Check!

In particular, since $f \in L^1_{\mathrm{loc}}$, so from Theorem 4.2.1 and its corollary (Corollary 4.2.1, Corollary 4.2.2), we see that it suffices to show that if λ is regular and $\lambda \perp m$, then for Lebesgue a.e. x,

$$\lim_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} \to 0$$

when $\{E_r\}$ shrinks nicely to x. It also suffices to take $E_r = B(r, x)$ and to assume that λ is positive, since for some $\alpha > 0$, we have

$$\left| \frac{\lambda(E_r)}{m(E_r)} \right| \leq \frac{|\lambda|(E_r)}{m(E_r)} \leq \frac{|\lambda|(B(r,x))}{m(E_r)} \leq \frac{|\lambda|(B(r,x))}{\alpha m(B(r,x))}.$$

Therefore, if $|\lambda|(E_r)/m(E_r) \to 0$, so does $|\lambda(E_r)/m(E_r)|$, hence $\lambda(E_r)/m(E_r)$. We see that the result then follows directly from Proposition 6.3.1. ■

6.4 Monotone Differentiation Theorem

We first formalize one ambiguous notation we used long time ago with discussing distribution function. Namely, $F(x^+), F(x^-)$.

Definition 6.4.1 ($F(x^+), F(x^-)$)**.** For $F\colon \mathbb{R} \to \mathbb{R}$ that is monotonically increasing, we denote

$$F(x^+) = \lim_{y \to x^+} F(y), \quad F(x^-) = \lim_{y \to x^-} F(y).$$

Remark. We see that if F is monotonically increasing, then $F(x^+), F(x^-)$ exist and are

$$\inf_{y > x} F(y), \quad \sup_{y < x} F(y)$$

respectively since it's bounded below/above respectively by $F(x)$.

Lemma 6.4.1. If $F\colon \mathbb{R} \to \mathbb{R}$ is monotonically increasing, then

$$D = \{x \in \mathbb{R} \mid F \text{ is discontinuous at } x\}$$

is a countable set.

Proof. $x \in D$ if and only if $F(x^+) > F(x^-)$. For each $x \in D$, let $I_x = (F(x^-), F(x^+))$, not empty. Also, if $x, y \in D$, $x \neq y$, then I_x, I_y are disjoint. Now, for $|x| < N$, I_x lie in the interval $(F(-N), F(N))$. Hence,

$$\sum_{|x| < N} \left[F(x^+) - F(x^-) \right] \leq F(N) - F(-N) < \infty,$$

which implies that

$$D \cap (-N, N) = \left\{ x \in (-N, N) \mid F(x^+) \neq F(x^-) \right\}$$

is countable. Since this is true for all N, the result follows. ■

Theorem 6.4.1 (Monotone Differentiation Theorem)**.** Let F be an increasing function, then

- F is differentiable Lebesgue almost everywhere.
- $G(x) := F(x^+)$[a] is differentiable almost everywhere.
- $G' = F'$ almost everywhere

[a]Observe that G is increasing and right-continuous.

Proof. Start with G, which is increasing and right-continuous on $\mathbb{R}$. There is then a Lebesgue-Stieltjes measure μ_G on $\mathbb{R}$, thus it is regular on $\mathbb{R}$. We see

$$\frac{G(x+h) - G(x)}{h} = \begin{cases} \dfrac{\mu_G((x, x+h])}{m((x, x+h])}, & \text{if } h > 0; \\[2ex] \dfrac{\mu_G((x+h, x])}{m((x+h, x])}, & \text{if } h < 0. \end{cases}$$

Note that both $\{(x, x+h]\}$ and $\{(x+h, x]\}$ shrink nicely to x as $|h| \to 0$. By Theorem 6.3.1 (since these shrink nicely), we then know that these both converge for Lebesgue almost every x to some common limit $f(x)$. Hence, G' exists Lebesgue almost everywhere. We now show that by defining

$H \coloneqq G - F$, H' exists and equals zero a.e.

Observe that $H(x) = G(x) - F(x) \geq 0$, and we see that

$$\{x \mid H(x) > 0\} \subseteq \{x \mid F \text{ is discontinuous at } x\}.$$

The latter set is then countable by Lemma 6.4.1, hence we can write $\{x \mid H(x) > 0\} = \{x_n\}$. Then let

$$\mu \coloneqq \sum_n H(x_n)\delta_{x_n}.$$

This is a Borel measure, so we check if it is locally finite. Indeed, since

$$\mu((-N, N)) = \sum_{-N<x_n<N} H(x_n) \leq G(N) - F(-N) < \infty,$$

where checking the inequality just consists of seeing that the intervals $(F(x_n), G(x_n))$ are disjoint and is a subset of $(F(-N), G(N))$, so

$$\sum_{-N<x_n<N} H(x_n) = \mu\left(\bigcup_n (F(x_n), G(x_n))\right) \leq \mu((F(-N), G(N))).$$

Thus, μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, so it is regular.

Remark. Special to $\mathbb{R}$, we have

$$\begin{aligned} \text{that locally finite Borel} &\Rightarrow \text{Lebesgue-Stieltjes} \\ &\Rightarrow \text{regular} \\ &\Rightarrow \text{outer regularity}. \end{aligned}$$

Also, we have $\mu \perp m$ since $m(E) = \mu(E^c) = 0$ where $E = \{x_n\}$. Then we have that

$$\left|\frac{H(x+h) - H(x)}{h}\right| \leq \frac{H(x+h) + H(x)}{|h|} \leq \frac{\mu((x-2h, x+2h))}{|h|},$$

which goes to 0 for Lebesgue almost every x by Theorem 6.3.1 and that $\mu \perp m$.

Thus, H is differentiable almost everywhere and $H' = 0$ almost everywhere, which implies F is differentiable almost everywhere and $F' = G'$ almost everywhere. ∎

Proposition 6.4.1. Suppose F is an increasing function, then F' exists almost everywhere and is measurable, then

$$\int_a^b F'(x) \,\mathrm{d}x \leq F(b) - F(a).$$

Example. Take $F(x)$ to be 0 on $x \leq 0$, 1 on $x > 0$. Then $F'(x) = 0$ almost everywhere. So

$$\int_{-1}^1 F'(x) \,\mathrm{d}x = 0 < 1 = F(1) - F(-1).$$

Even if F is continuous we might not have equality. Take $F(x)$ to be the Cantor function. Then $F'(x) = 0$ almost everywhere, but

$$\int_0^1 F'(x) \,\mathrm{d}x = 0 < 1 = F(1) - F(0).$$

Lecture 34: Functions of Bounded Variation

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Proof of Proposition 6.4.1. Let

$$G(x) := \begin{cases} F(a), & \text{if } x < a; \\ F(x), & \text{if } a \le x \le b; \\ F(b), & \text{if } x > b. \end{cases}$$

Then G is increasing. We define

$$g_n(x) = \frac{G(x + 1/n) - G(x)}{1/n} \to F'(x)$$

for almost every $x \in [a, b]$. We note that $g_n(x) \ge 0$.

Theorem 2.2.2 tells us that

$$\int_a^b F'(x) \, \mathrm{d}x = \int_a^b \liminf_{n\to\infty} g_n(x) \, \mathrm{d}x \le \liminf_{n\to\infty} \int_a^b g_n(x) \, \mathrm{d}x.$$

We then evaluate

$$\begin{aligned} \int_a^b g_n(x) &= n \left(\int_{a+1/n}^{b+1/n} G(x) \, \mathrm{d}x - \int_a^b G(x) \, \mathrm{d}x \right) \\ &= n \left(\int_b^{b+1/n} G(x) \, \mathrm{d}x - \int_a^{a+1/n} G(x) \, \mathrm{d}x \right) \\ &\le n \left(G\left(b + \frac{1}{n}\right) \cdot \frac{1}{n} - G(a) \cdot \frac{1}{n} \right) \\ &= F(b) - F(a). \end{aligned}$$

Therefore,

$$\int_a^b F'(x) \, \mathrm{d}x \le F(b) - F(a).$$

■

6.5 Functions of Bounded Variation

Definition 6.5.1 (Total variation function)**.** For $F\colon \mathbb{R} \to \mathbb{R}$, the *total variation function of F* is $T_F\colon \mathbb{R} \to [0, \infty]$ defined by

$$T_F(x) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| \;\middle|\; -\infty < x_0 < x_1 < \ldots < x_n = x \right\}$$

where $n \in \mathbb{N}$.

Lemma 6.5.1. $T_F(b)$ is equal to

$$T_F(a) + \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| \;\middle|\; a = x_0 < x_1 < \ldots < x_n = b \right\}$$

where $n \in \mathbb{N}$ if $a < b$.

Proof. The idea is that the sums in the Definition 6.1.5 of T_F are made bigger if the additional subdivision points x_j are added. Hence, if $a < b$, $T_F(b)$ is unaffected if we assume that a is always one of the subdivision points. ■

Remark. T_F is increasing.

Definition 6.5.2 (Bounded variation). We say that F is of *bounded variation*, denoted as $F \in BV$, provided that

$$T_F(\infty) = \lim_{x\to\infty} T_F(x) < \infty.$$

Similarly, $F \in BV([a,b])$ means that

$$\sup\left\{\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b\right\} < \infty.$$

Remark. We see the following.

1. If F is of bounded variation, then F is bounded.
2. $F(x) = \sin x$ is not of bounded variation, but it is of bounded variation over any $[a,b]$.
3. For $F(x)$ defined as
$$F(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0 \end{cases}$$
is not of bounded variation of $[a,b]$ if $a < 0 < b$ because the harmonic series does not converge.

Before we see more properties of bounded variation function, we introduce a useful characterization of a function.

Definition 6.5.3 (Lipschitz). A function $F\colon [a,b] \to \mathbb{C}$ is called *Lipschitz* provided that there exists an $M \geq 0$ such that $|F(x) - F(y)| \leq M\,|x - y|$.

Remark. We have the following.

1. If F, G are of bounded variation, $\alpha F + \beta G$ are of bounded variation.
2. If F is increasing and bounded, then F is a function of bounded variation.
3. If F is Lipschitz on $[a,b]$, then $F \in BV([a,b])$.
4. If F is differentiable, and F' is bounded on $[a,b]$, then F is Lipschitz (mean value theorem), so it is in $BV([a,b])$.
5. If $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ for $f \in L^1(\mathbb{R})$, then $F \in BV$.

   Namely,
$$\begin{aligned}\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^{n} \left|\int_{x_{i-1}}^{x_i} f(t)\,\mathrm{d}t\right| \\ &\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,\mathrm{d}t \\ &= \int_{x_0}^{x_n} |f(t)|\,\mathrm{d}t \\ &\leq \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t < \infty.\end{aligned}$$

Lemma 6.5.2. If $F \in BV$, then T_F is bounded, increasing, $T_F(-\infty) = 0$.

Proof. ■ DIY

Lemma 6.5.3. $F \in BV$, then $T_F \pm F$ are increasing and bounded and.

Proof. Let $x < y$ and fix $\epsilon > 0$, then there are points $x_0 < x_1 < \cdots < x_n = x$ such that

$$T_F(x) \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \epsilon.$$

Furthermore,

$$T_F(y) \geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \, .$$

Then, since $\pm(F(y) - F(x)) \leq |F(y) - F(x)|$, we have

$$T_F(y) \pm (F(y) - F(x)) \geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \geq T_F(x) - \epsilon,$$

hence

$$T_F(y) \pm F(y) \geq T_F(x) \pm F(x) - \epsilon.$$

Taking $\epsilon \to 0$ yields the result. ∎

Remark. Thus, any $F \in BV$ can be written as

$$F = \frac{T_F + F}{2} - \frac{T_F - F}{2}$$

which is a difference of increasing and bounded functions.

Theorem 6.5.1. F is of bounded variation if and only if $F = F_1 - F_2$ for F_1, F_2 increasing and bounded.

Proof. The forward implication is given by the Lemma 6.5.3. The other direction follows from the examples we gave. ∎ check!

Corollary 6.5.1 (Bounded Variation Differentiation)**.** $F \in BV$ implies that F is differentiable almost everywhere. Furthermore,

1. $F(x^+), F(x^-)$ exist for all x as do $F(-\infty), F(\infty)$.
2. The set of discontinuities of F is countable.
3. $G(x) = F(x^+)$ is differentiable and $G' = F'$ almost everywhere.
4. $F' \in L^1(\mathbb{R}, m)$ (i.e. $F \in L^1_{\text{loc}}(\mathbb{R})$) for every $a < b$.

Proof. ∎ DIY

Appendix

Appendix A

Additional Proofs

A.1 Measure

This section gives all additional proofs in chapter 1.

Theorem A.1.1 (Theorem 1.3.2 3.). Under the setup of Theorem 1.3.2, $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We see this in two parts.

Claim. If $A \subset X$ satisfies $\mu^*(A) = 0$, then A is Carathéodory measurable with respect to μ^*.

Proof. If $A \subset X$ and $\mu^*(A) = 0$, where μ^* is an outer measure on X, we'll show that A is Carathéodory measurable with respect to μ^*.

Equivalently, we want to show that for any $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Firstly, noting that $(E \cap A) \subset A$, and by monotonicity of μ^*, we see that

$$\mu^*(E \cap A) \leq \mu^*(A) = 0,$$

and since $\mu^* \geq 0$, hence $\mu^*(E \cap A) = 0$. Now, we only need to show that

$$\mu^*(E) = \mu^*(E \setminus A).$$

Since $E \setminus A = E \cap A^c$, and hence we have $E \cap A^c \subset E$, so

$$\mu^*(E) \geq \mu^*(E \setminus A).$$

To show another direction, we note that

$$\mu^*(E) \leq \mu^*(E \cup A) = \mu^*((E \setminus A) \cup A) \leq \mu^*(E \setminus A),$$

hence we conclude that A is Carathéodory measurable with respect to μ^* if $\mu^*(A) = 0$. ■

Claim. If A is μ-subnull, then $A \in \mathcal{A}$.

Proof. Let $\mathcal{A}$ denotes the Carathéodory σ-algebra, and $\mu := \mu^*|_{\mathcal{A}}$. We want to show if A is μ-subnull, then $A \in \mathcal{A}$.

Firstly, if A is μ-subnull, then there exists a $B \in \mathcal{A}$ such that $A \subset B$ and $\mu(B) = 0$. But since from the monotonicity of μ^*, we further have

$$0 = \mu(B) = \mu^*(B) \geq \mu^*(A),$$

hence $\mu^*(A) = 0$.

From the first claim, we immediately see that A is Carathéodory measurable with respect to μ^*, which implies A is in Carathéodory σ-algebra, hence $A \in \mathcal{A}$. ■

We see that the second claim directly proves that $(X, \mathcal{A}, \mu)$ is a complete measure space. ■

Lemma A.1.1. The function F defined in this example is a distribution function

Proof. We define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n \end{cases}$$

where $\{r_1, r_2, \ldots\} = \mathbb{Q}$, and

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} = \sum_{n; r_n \leq x} \frac{1}{2^n}$$

is both increasing and right-continuous.

- Increasing. Consider $x < y$. We see that

  $$F(x) = \sum_{n; r_n \leq x} \frac{1}{2^n} \leq \sum_{n; r_n \leq y} \frac{1}{2^n} = F(y)$$

  clearly.[a]

- Right-continuous. We want to show $F(x^+) = F(x)$. Let $x^+(\epsilon) := x + \epsilon$ with $\epsilon > 0$, we'll show that

  $$\lim_{\epsilon \to 0} F(x^+(\epsilon)) = \lim_{\epsilon \to 0} F(x + \epsilon) = F(x).$$

  Firstly, we have

  $$F(x^+(\epsilon)) - F(x) = \sum_{n; r_n \leq x + \epsilon} \frac{1}{2^n} - \sum_{n; r_n \leq x} \frac{1}{2^n} = \sum_{n; x < r_n \leq x + \epsilon} \frac{1}{2^n},$$

  and we want to show

  $$\lim_{\epsilon \to 0} F(x^+(\epsilon)) - F(x) = \lim_{\epsilon \to 0} \sum_{n; x < r_n \leq x + \epsilon} \frac{1}{2^n} = 0.$$

  Remark. The strict is crucial to show the result, since if $x = r_k$ for some fixed k, then we can't make the summation arbitrarily small.

  [a] This is trivial since we're always going to sum more strictly positive terms in $F(y)$ than in $F(x)$.

  Before we show how we choose ϵ,[b] we see that

  $$\sum_{n=k}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

Now, we observe that

$$\sum_{n; x < r_n \le x+\epsilon} \frac{1}{2^n} \le \sum_{n=\arg\min_k x < r_k \le x+\epsilon}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

With this observation, it should be fairly easy to see that we can choose ϵ based on how small we want to make 2^{1-k} be,[c] and we indeed see that

$$\lim_{k\to\infty} 2^{1-k} = 0,$$

which implies that F is right-continuous by squeeze theorem.

■

[b] To be precise, how ϵ depends on r_n.
[c] We're referring to $\epsilon - \delta$ proof approach.

Lemma A.1.2. The function F defined in this example satisfies

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$

given in this example.

Proof. We prove them one by one. And notice that F is indeed a distribution function as we proved in Lemma A.1.1.

1. To show $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$, we first note that

$$\{r\} = \bigcap_{a-1 \le x < r} (x, r].$$

Then, we see that

$$\mu_F(\{r\}) = \mu_F\left(\bigcap_{a-1 \le x < a} (x, r]\right),$$

where each $(x, r] \in \mathcal{A}$ and $(x, r] \supset (y, r]$ whenever $r - 1 \le x \le y < r$. Notice that we implicitly assign the order of the index by the order of x. Then, we see that $\mu_F(r-1, r] < \infty$.[a] Then, from continuity from above, we see that

$$\mu_F(\{r\}) = \lim_{i\to\infty} \mu_F((x_i, r]),$$

where we again implicitly assign an order to x as the usual order on $\mathbb{R}$ by given index i. It's then clear that as $i \to \infty$, $x_i \to r$. From the definition of F, we see that

$$F((x_i, r]) = F(r) - F(x_i) = \sum_{n; r_n \le r} \frac{1}{2^n} - \sum_{n; r_n \le x_i} \frac{1}{2^n} = \sum_{n; x_i < r_n \le r} \frac{1}{2^n}.$$

It's then clear that since $r \in \mathbb{Q}$, there exists an i' such that $r_{i'} = r$. Then, we immediately see that no matter how close $x_i \to r$, this sum is at least

$$\frac{1}{2^{i'}}$$

for a fixed i'. Hence, we conclude that $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$.

2. Now, we show $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$. Firstly, we claim that

$$\mu_F(\mathbb{Q}) = 1$$

and

$$\mu_F(\mathbb{R}) = 1$$

as well. Since $\mu_F(\mathbb{Q}) = 1$ is clear, we note that the second one essentially follows from the fact that we can write

$$\mathbb{R} = \lim_{N\to\infty} \bigcup_{i=1}^{N} (a-i, a+i]$$

for any $a \in \mathbb{R}$, say 0. From continuity from below, we have

$$\mu_F\left(\bigcup_{i=1}^{\infty}(-i,+i]\right) = \lim_{n\to\infty} \mu_F((-n,n]) = \sum_{n;r_n\in\mathbb{Q}} \frac{1}{2^n} = 1.$$

Given the above, from countable additivity of μ_F, we have

$$\mu_F(\mathbb{R}\setminus\mathbb{Q}) + \underbrace{\mu_F(\mathbb{Q})}_{1} = \underbrace{\mu_F(\mathbb{R})}_{1} \Rightarrow \mu_F(\mathbb{R}\setminus\mathbb{Q}) = 0$$

as we desired.

■

[a] This will be $\mu(A_1)$ in the condition of continuity from above. Furthermore, since $\mathbb{Q}$ is countable, hence $F(x) < \infty$ is promised.

Lemma A.1.3 (Cantor set has measure 0)**.** Let C denotes the middle thirds Cantor set, then the Lebesgue measure of C is 0. i.e.,

$$m(C) = 0.$$

Proof. Since we're removing $\frac{1}{3}$ of the whole interval at each n, we see that the measure of those removing parts, denoted by r, is

$$m(r) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = 1.$$

Then, by countable additivity of m, we see that

$$m(C) = m([0,1]) - m(r) = 1 - 1 = 0.$$

■

A.2 Integration

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