MATH597
Analysis II

Pingbang Hu

April 14, 2022

Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\overset{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition we're using to deduce that particular equality.

Notice that there are some proofs is **intended** left as assignments, and for completeness, I put them in Appendix A, use it in your **own risks**! You'll lose the chance to practice and really understand the materials.

Additionally, we'll use Folland[FF99] as our main text, while using Tao[Tao13] and Axler[Axl19] as supplementary references.

Contents

Chapter 1

Measure

Lecture 1: σ-algebra

Before we start, we first see some examples. 05 Jan. 11:00

Example (Finite power set). Let $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) \coloneqq \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\},$$

which is the *power set* of X. We see that

$$\#X = n \Rightarrow \#\mathcal{P}(X) = 2^n$$

for $n < \infty$.

Example (Infinite power set). If $n = \infty$, say $X = \mathbb{N}$, then

$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while $\mathbb{N}$ is a countable set. We can see this as follows. Consider

$$\phi\colon \mathcal{P}(\mathbb{N}) \to [0,1], \quad A \mapsto 0.a_1a_2a_3\ldots (\text{base } 2),$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example, A can be $A = \{2, 3, 6, \ldots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have

$$\#\mathcal{P}(\mathbb{N}) \geq \#[0,1].$$

But since $[0,1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X. Hence, we are intriguing to define a map μ such that

$$\mu\colon \mathcal{P}(X) \to [0,\infty].$$

Example. Let $X = \{0,1,2\}$. Then we want to define $\mu\colon \mathcal{P}(X) \to [0,\infty]$, we can have

- $\mu(A) = \#A$. Then we have
  - $\mu(\{0,1\}) = 2$
  - $\mu(\{0\}) = 1$

- $\mu(A) = \sum\limits_{i \in A} 2^i$. Then we have
  - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$

Example. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu\colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$, we can have

- $\mu(A) = \# A$. Then we have
  - $\mu(\{2, 3, 4, 5, \dots\}) = \infty = \mu(\{\text{even numbers}\})$
- $\mu(A) = e^{-1} \sum\limits_{i \in A} \frac{1}{i!}$. Then we have
  - $\mu(\{0, 2, 4, 6, \dots\}) = e^{-1} \left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right)$
- $\mu(A) = \sum\limits_{i \in A} a_i$

Example. Let $X = \mathbb{R}$. Then we want to define $\mu\colon \mathcal{P}(\mathbb{R}) \to [0, \infty]$, we can have

- $\mu(A) = \# A$
- $\mu((a, b)) = b - a$.

  Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

  Answer. No! ⊛

- $\mu((a, b)) = e^b - e^a$.

  Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

  Answer. No! ⊛

We immediately see the problems. To extend our native measure method into $\mathbb{R}$ is hard and will cause something counter-intuitive![1] Hence, rather than define measurement on *all* subsets in the power set of X, we only focus on *some* subsets. In other words, we want to define

$$\mu\colon \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \to [0, \infty]\,.$$

1.1 σ-algebras

We start from the definition of the most fundamental element in measure theory.

Definition 1.1.1 (σ-algebra)**.** Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra on X* if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\varnothing}_{\in \mathcal{A}}$ and $\mathcal{A}$ is closed under complement.

[1] https://en.wikipedia.org/wiki/Banach-Tarski_paradox

- $\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c$, namely $\mathcal{A}$ is closed under countable intersections.
- $A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \varnothing \cup \varnothing \cup \ldots$, hence $\mathcal{A}$ is closed under finite unions and intersections.

Note. The definition of σ-algebra should remind us the definition of topological basis, and this is indeed the case. We can consider a topological space and put some structure on the σ-algebra $\mathcal{A}$, which gives us the following.

Definition 1.1.2 (Borel set)**.** Given a topological space X, a *Borel set* is any set in X that can be formed from open sets through the operations of countable union, countable intersection and relative complement.

Lecture 2: Measure

07 Jan. 11:00

Example. We first see some examples.

(1) Let $\mathcal{A} = \mathcal{P}(X)$, which is the power σ-algebra.

(2) Let $\mathcal{A} = \{\varnothing, X\}$, which is a trivial σ-algebra.

(3) Let $B \subset X$, $B \neq \varnothing$, $B \neq X$. Then we see that $\mathcal{A} = \{\varnothing, B, B^c, X\}$ is a σ-algebra.

Lemma 1.1.1. Let $\mathcal{A}_\alpha$, $\alpha \in I$, be a family of σ-algebra on X. Then

$$\bigcap_{\alpha\in I} \mathcal{A}_\alpha$$

is a σ-algebra on X.

Proof. A simple proof can be made as follows. Firstly, $\varnothing \in \mathcal{A}_\alpha$ for every α clearly. Moreover, closure under complement and countable unions for every $\mathcal{A}_\alpha$ implies the same must be true for $\bigcap_{\alpha\in I} \mathcal{A}_\alpha$. Hence, $\bigcap_{\alpha\in I} \mathcal{A}_\alpha$ is a σ-algebra.

Remark. Notice that I may be an uncountable intersection.

■

The above allows us to give the following definition.

Definition 1.1.3 (Generation of σ-algebra)**.** Given $\mathcal{E} \subset \mathcal{P}(X)$, where $\mathcal{E}$ is not necessarily a σ-algebra. Let $\langle\mathcal{E}\rangle$ be the intersection of all σ-algebras on X containing $\mathcal{E}$, then we call $\langle\mathcal{E}\rangle$ the *σ-algebra generated by $\mathcal{E}$*.

Remark. Clearly, $\langle\mathcal{E}\rangle$ is the smallest σ-algebra containing $\mathcal{E}$, and it is unique. To check the uniqueness, we suppose there are two different $\langle\mathcal{E}\rangle_1$ and $\langle\mathcal{E}\rangle_2$ generated from $\mathcal{E}$. It's easy to show

$$\langle\mathcal{E}\rangle_1 \subseteq \langle\mathcal{E}\rangle_2\,,$$

and by symmetry, they are equal.

Example. We see that $\{\varnothing, B, B^c, X\} = \langle\{B\}\rangle = \langle\{B^c\}\rangle$.

Lemma 1.1.2. We have

(1) Given $\mathcal{A}$ a σ-algebra, $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \Rightarrow \langle \mathcal{E} \rangle \subset \mathcal{A}$

(2) $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \Rightarrow \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$

Proof. We'll see that after proving the first claim, the second follows smoothly.

(1) The first claim is trivial, since we know that $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$, then if $\mathcal{E} \subset \mathcal{A}$, we clearly have $\langle \mathcal{E} \rangle \subset \mathcal{A}$ by the definition.

(2) The second claim is also easy. From the first claim and the definition, we have

$$\mathcal{E} \subset \mathcal{F} \subset \langle \mathcal{F} \rangle \Rightarrow \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle .$$

■

At this point, we haven't put any specific structure on X. Now we try to describe those spaces with good structure, which will give the space some nice properties.

Definition 1.1.4 (Borel σ-algebra)**.** For a topological space X, the *Borel σ-algebra on* X, denotes as $\mathcal{B}(X)$, is the σ-algebra generated by the collection of all open sets in X.

Example. We see that $\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a, b) \mid a < b; a, b \in \mathbb{R}\}$.
- $\mathcal{E}_2 = \{[a, b] \mid a < b; a, b \in \mathbb{R}\}$ since $[a, b] = \bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.
- $\mathcal{E}_3 = ((a, b] \mid a < b; a, b \in \mathbb{R})$ since $(a, b] = \bigcap\limits_{n=1}^{\infty} (a, b + \frac{1}{n})$.
- $\mathcal{E}_4 = ([a, b) \mid a < b; a, b \in \mathbb{R})$ since $[a, b) = \bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b)$.
- $\mathcal{E}_5 = ((a, \infty) \mid a \in \mathbb{R})$ since $(a, \infty) = \bigcup\limits_{n=1}^{\infty} (a, a + n)$.
- $\mathcal{E}_6 = ([a, \infty) \mid a \in \mathbb{R})$ since $[a, \infty) = \bigcup\limits_{n=1}^{\infty} [a, a + n)$.
- $\mathcal{E}_7 = ((-\infty, b) \mid b \in \mathbb{R})$ since $(-\infty, b) = \bigcup\limits_{n=1}^{\infty} (b - n, b)$.
- $\mathcal{E}_8 = ((-\infty, b] \mid a \in \mathbb{R})$ since $(-\infty, b] = \bigcup\limits_{n=1}^{\infty} (b - n, b]$.

Proposition 1.1.1. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \ldots, 8$ in the above example.

Proof. Firstly, we see that $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \Rightarrow \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$ by Lemma 1.1.2. Secondly, by definition, $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$ where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R}\ \} .$$

It's enough to show $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$ since if so, $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$, and clearly $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$, then we will have $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$. Let $O \subset \mathbb{R}$ be an open set, i.e., $O \in \mathcal{E}$. We claim that every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.[a]

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

where I_j open interval with the form of $(a,b), (-\infty, b), (a, \infty), (-\infty, \infty)$.

For example, $\mathcal{E}_1$ is trivially true, and

$$(a,b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{[a + \frac{1}{n}, b - \frac{1}{n}]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for $\mathcal{E}_2$ and

$$(a, \infty) = \bigcup_{k=1}^{\infty} (a, a+k)$$

shows the case for $\mathcal{E}_5$. It's now straightforward to check open intervals are in $\langle \mathcal{E}_i \rangle$ for every i. ■

[a] https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals

Now, to put a structure on a space, we define the following.

Definition. Given a space X, we have the following.

Definition 1.1.5 (Measurable space). A *measurable space* is a tuple of X and a σ-algebra $\mathcal{A}$ on X, denoted by $(X, \mathcal{A})$.

In particular, if the σ-algebra is the Borel σ-algebra of X, then we give it a special name.

Definition 1.1.6 (Borel space). A *Borel space* is a tuple of X and $\mathcal{B}(X)$, denoted by $(X, \mathcal{B}(X))$.

Remark. This means that X implicitly has some topological structure.

Definition 1.1.7 ($\mathcal{A}$-measurable set). Given a measurable space $(X, \mathcal{A})$, every $E \in \mathcal{A}$ is a so-called *$\mathcal{A}-$measurable set.*

1.2 Measures

With the definition of measurable space, we now can refine our measure function μ as follows.

Definition 1.2.1 (Measure). Given a measurable space on $(X, \mathcal{A})$, a *measure* is a function

$$\mu \colon \mathcal{A} \to [0, \infty]$$

such that

- (null empty set) $\mu(\varnothing) = 0$.
- (countable additivity) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_1, A_2, \ldots \in \mathcal{A}$ are disjoint.

Definition 1.2.2 (Measure space). We denote $(X, \mathcal{A}, \mu)$ as so-called a *measure space* given μ is the measure on $(X, \mathcal{A})$.

Notation. We denote $[0, \infty] \coloneqq [0, \infty) \cup \{\infty\}$.

Remark. We only want countable additivity but not uncountable additivity.

Proof. Consider the most intuitive measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Since we have

$$(0,1] = (1/2,1] \cup (1/4,1/2] \cup (1/8,1/4] \cup \dots$$

and also

$$(0,1] = \bigcup_{x \in (0,1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2},1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4},\frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8},\frac{1}{4}])} + \dots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0,1]} 0$$

since $\mu(x) = 0$ for $x \in \mathbb{R}$, which is clearly not what we want. ⊛

Example (Counting measure). For any $(X, \mathcal{A})$, we let $\mu(A) := \#A$. This is the so-called *counting measure*.

Example (Dirac-Delta measure). Let $x_0 \in X$. For any $(X, \mathcal{A})$, the *Dirac-Delta measure at* x_0 is

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A \end{cases}$$

for every $A \in \mathcal{A}$.

Example. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, given $A \in \mathcal{P}(\mathbb{N})$,

$$\mu(A) = \sum_{i \in A} a_i$$

where $a_1, a_2, \dots \in [0, \infty)$.

Lecture 3: Construct a Measure

After seeing examples of measures, we now want to construct it from ground up. 10 Jan. 11:00

Note. If $A, B \in \mathcal{A}$ and $A \subset B$, then

$$\mu(B \setminus A) + \mu(A) = \mu(B) \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

Theorem 1.2.1. Given $(X, \mathcal{A}, \mu)$ be a measure space. Then the following hold.

(1) Monotonicity.

$$A, B \in \mathcal{A}, A \subset B \Rightarrow \mu(A) \leq \mu(B).$$

(2) Countable subadditivity.

$$A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

(3) Continuity from below.

$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \ldots \end{cases} \Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n).$$

(4) Continuity from above.

$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \ldots \\ \mu(A_1) < \infty \end{cases} \Rightarrow \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. We prove this theorem one by one.

(1) Since $A \subset B$, hence we have

$$\mu(B) = \mu\Big(\underbrace{\underline{(B \setminus A)} \cup \underline{A}}_{\text{disjoint}}\Big) \overset{!}{=} \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A).$$

(2) This should be trivial from countable additivity with the fact that $\mu(A) \geq 0$ for all A. DIY!

(3) Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$, then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

are a disjoint union and $B_i \in \mathcal{A}$, hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i).$$

With $\mu\left(\bigcup_{i=1}^{n} B_i\right) = \mu(A_n)$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu(A_n).$$

(4) Let $E_i = A_1 \setminus A_i \Rightarrow E_i \in \mathcal{A}$, $E_1 \subset E_2 \subset \ldots$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right),$$

which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i\right) \Rightarrow \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$$

since $\mu\left(\bigcup\limits_{i=1}^{\infty} E_i\right) \leq \mu(A_1) < \infty$. Then from continuity from below, we further have

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right).$$

From monotonicity, we see that $\mu(A_n) \leq \mu(A_1) < \infty$, hence we can split the limit and further get

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu(A_1) + \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(A_n).$$

■

Note. Sometimes we also call continuity from below property as monotone convergence theorem for sets. We'll later see the important monotone convergence theorem for integral, which is in different content.

Remark (Condition of continuity from above). The condition $\mu(A_1) < \infty$ in continuity from above is necessary.

Proof. Given $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure. Then we see

- $A_n = \{n, n+1, n+2, \ldots\} \Rightarrow \mu(A_n) = \infty$
- $A_1 \supset A_2 \supset A_3 \supset \ldots$
- $\bigcap\limits_{i=1}^{\infty} A_i = \varnothing \Rightarrow \mu\left(\bigcap\limits_{i=1}^{\infty} A_i\right) = 0$

We see that in this case, since $\mu(A_1) \not< \infty$, hence continuity from above doesn't hold. ⊛

We now try to characterize some properties of a measure space.

Definition. Given $(X, \mathcal{A}, \mu)$, we have the following.

Definition 1.2.3 (μ-null set). $A \subset X$ is a *μ-null set* if $A \in \mathcal{A}$ and $\mu(A) = 0$.

Definition 1.2.4 (μ-subnull set). $A \subset X$ is a *μ-subnull set* if there exists a μ-null set B such that $A \subset B$.

Definition 1.2.5 (Complete measure space). $(X, \mathcal{A}, \mu)$ is a *complete measure space* if every μ-subnull set is $\mathcal{A}$-measurable.

Note. We see that for a μ-subnull set, it's not necessary $\mathcal{A}$-measurable if the measure space is not complete.

Remark. From the property of measure, the condition for a measure space $(X, \mathcal{A}, \mu)$ being complete is equivalent to saying that every μ-subnull set is a μ-null set.

Proof. This follows from the monotonicity of a measure and the fact that a measure is always non-negative. Finally, a μ-subnull set is always in $\mathcal{A}$. ⊛

There are some useful terminologies we'll use later relating to μ-null.

Definition 1.2.6 (Almost everywhere). Given $(X, \mathcal{A}, \mu)$, a statement $P(x)$, $x \in X$ holds μ-*almost everywhere (a.e.)* if the set

$$\{x \in X : P(x) \text{ does not hold}\}$$

is μ-null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

Definition. Given $(X, \mathcal{A}, \mu)$, we have the following.

Definition 1.2.7 (Finite measure). μ is a *finite measure* if $\mu(X) < \infty$.

Definition 1.2.8 (σ-finite measure). μ is a σ-*finite measure* if $X = \bigcup_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.

Exercise. Every measure space can be completed. Namely, we can always find a bigger σ-algebra to complete the space.

1.3 Outer Measures

As we said before, we're now going to construct a measure. And a modern way to do this is to start with something called *outer measure*.

Definition 1.3.1 (Outer measure). An *outer measure* on X is a function

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that

- (null empty set) $\mu^*(\varnothing) = 0$.
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.
- (countable subadditivity) $\mu^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) \leq \sum\limits_{i=1}^{\infty} \mu^*(A_i)$ for every $A_i \subset X$.

Example. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : \bigcup_{i=1}^{\infty}(a_i, b_i) \supset A\right\}$$

is an outer measure.

Proof. This follows directly from the Proposition 1.3.1 we're going to show. ⊛

Remark. We see that an outer measure need not be a measure.

Proposition 1.3.1. Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\varnothing, X \in \mathcal{E}$. Let

$$\rho : \mathcal{E} \to [0, \infty]$$

such that $\rho(\varnothing) = 0$. Then

$$\mu^*(A) \coloneqq \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E},\ \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

is an outer measure on X.

Theorem 1.3.1 (Tonelli's Theorem for series)**.** Recall the Tonelli's Theorem[a] for series, i.e., if $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

[a] https://en.wikipedia.org/wiki/Fubini%27s_theorem

Proof. Read Tao[Tao13] Theorem 0.0.2. ∎

Lecture 4: Carathéodory Extension Theorem

12 Jan. 11:00

As previously seen. Last time we skip the proof of Proposition 1.3.1, which is a quite important theorem for building a measure. To see this, we note that from Proposition 1.3.1, given a positive function ρ defined on a subset of the power set of X with $\rho(\varnothing) = 0$, we can induce an outer measure from ρ.

Note. We'll see later that how can we further induce a natural measure from the induced outer measure.

We now prove Proposition 1.3.1.

Proof of Proposition 1.3.1. We need to prove the following.

Claim. μ^* is well-defined, i.e., inf is taken over a non-empty set.

Proof. This is trivial since $X \in \mathcal{E}$ and $X \supset A$ for any $A \in \mathcal{E}$. ⊛

Claim. Null empty set holds, i.e., $\mu^*(\varnothing) = 0$.

Proof. Since $\varnothing \in \mathcal{E}$ and

$$\mu^*(\varnothing) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \varnothing \right\} = 0$$

since $\rho(\varnothing) = 0$ for all i and further, by Squeeze Theorem, we see that $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \rho(\varnothing) = 0$. ⊛

Claim. Monotonicity holds, i.e., $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$.

Proof. We show this by contradiction. Suppose $A \subset B$ and $\mu^*(A) > \mu^*(B)$, then by definition of μ^*, we have

$$\begin{aligned}\mu^*(A) &= \inf\left\{\sum_{i=1}^{\infty}\rho(E_i)\colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A\right\}\\ &> \inf\left\{\sum_{i=1}^{\infty}\rho(E_i)\colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B\right\} = \mu^*(B).\end{aligned}$$

Now, let $B =: (B \setminus A) \cup A$, then we have

$$\begin{aligned}\mu^*(A) &= \inf\left\{\sum_{i=1}^{\infty}\rho(E_i)\colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A\right\}\\ &> \inf\left\{\sum_{i=1}^{\infty}\rho(E_i)\colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B\setminus A)\cup A\right\} = \mu^*(B).\end{aligned}$$

Now, since $B \setminus A \supseteq \varnothing$, then this inequality can't hold, hence a contradiction↯. ⊛

Claim. Countable subadditivity holds, i.e., $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty}\mu^*(A_i)$ for every $A_i \subset X$.

Proof. Let $A_1, A_2, \ldots \in X$. If one of $\mu^*(A_n) = \infty$, then result holds. So we may assume $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Now, fix any $\epsilon > 0$, we will show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\mu^*(A_n) + \epsilon.$$

For each $n \in \mathbb{N}$, $\exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ such that $\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n$ and $\mu^*(A_n) + \epsilon/2^n \geq \sum_{k=1}^{\infty}\rho(E_{n,k})$.

Remark. This is an important trick! We often set the error term as $\epsilon/2^n$ instead of ϵ as in above to accommodate the summation over a countable set.

Then we see that

$$\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{k,n},$$

which implies

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{(n,k)\in\mathbb{N}^2}\rho\left(E_{k,n}\right) \overset{!}{=} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\rho(E_{k,n}) \leq \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right)$$

from the inequality just derived. Now, since the last term is just

$$\sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) = \sum_{n=1}^{\infty}\mu^*(A_n) + \epsilon,$$

hence we finally have

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\mu^*(A_n) + \epsilon$$

for arbitrarily small fixed $\epsilon > 0$, hence the subadditivity is proved. ⊛

■

Definition 1.3.2 (Carathéodory measurable). Let μ^* be an outer measure on X. We say $A \subset X$ is *Carathéodory measurable with respect to* μ^* if

$$\forall E \subset X,\ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Note. We sometimes write *C-measurable* instead of Carathéodory measurable for convenience.

Lemma 1.3.1. Let μ^* be an outer measure on X. Suppose $B_1, \ldots, B_N$ are disjoint C-measurable sets. Then,

$$\forall E \subset X,\ \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) = \sum_{i=1}^{N} \mu^*(E \cap B_i).$$

Proof. Since we have

$$\begin{aligned}\mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) &= \mu^*(E' \cap B_1) + \mu^*(E' \setminus B_1)\\ &= \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i \cap B_1\right)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right) \cap B_1^c\right)\\ &= \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_{i=2}^{N} B_i\right)\right)\end{aligned}$$

where the equality comes from the fact that B_1 is C-measurable and disjoint from B_i, $i \neq 1$. Then, we simply iterate this argument and have the result. Note that in the first inequality, we define $E' := E \cap \left(\bigcup_{i=1}^{N} B_i\right)$ for the simplicity of notation. ■

Remark. This implies that if we restrict an outer measure on a C-measurable set, then it becomes finite additive.

Theorem 1.3.2 (Carathéodory extension Theorem). Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets (with respect to μ^*). Then,

(1) $\mathcal{A}$ is a σ-algebra on X.

(2) $\mu = \mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.

(3) $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We divide the proof in several steps.

(1) We show $\mathcal{A}$ is a σ-algebra by showing

- We first show $\varnothing \in \mathcal{A}$.

  Claim. $\varnothing \in \mathcal{A}$.

  Proof. To show this, we simply check that $\varnothing$ is C-measurable. We see that

  $$\underset{E \subset X}{\forall}\ \mu^*(E) = \mu^*(E \cap \varnothing) + \mu^*(E \setminus \varnothing) = \mu^*(E),$$

  which just shows $\varnothing \in \mathcal{A}$. ⊛

- Then we show $\mathcal{A}$ is closed under complements.

  Claim. $\mathcal{A}$ closed under complements.

Proof. This is equivalent to say that if A is C-measurable, so is A^c. We see that if A is C-measurable, then for every $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Observing that $E \cap A = E \setminus A^c$ and $E \setminus A = E \cap A^c$, hence

$$\mu^*(E) = \mu^*(E \setminus A^c) + \mu^*(E \cap A^c).$$

We immediately see that above implies $A^c \in \mathcal{A}$. ⊛

- We now show $\mathcal{A}$ is closed under countable unions.

Note. To show $\mathcal{A}$ closed under countable unions, we show that $\mathcal{A}$ is closed under:

$$\text{finite unions} \overset{\text{then}}{\Rightarrow} \text{countable } \underline{\text{disjoint}} \text{ unions} \overset{\text{then}}{\Rightarrow} \text{countable unions.}$$

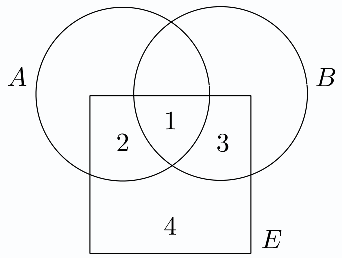
Hence, we first show $\mathcal{A}$ is closed under finite unions.

Claim. $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.

Proof. Fix $E \subset X$ arbitrary. We need to show that

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)),$$

i.e., $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$ given $A, B \in \mathcal{A}$ and the following figure.



- Since A is C-measurable,
  - $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$
  - $\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$
- Since B is C-measurable,
  - $\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$

Hence, we have

$$\begin{aligned}\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4).\end{aligned}$$

⊛

We now show $\mathcal{A}$ is closed under countable <u>disjoint</u> unions.

Claim. $\mathcal{A}$ is closed under countable disjoint unions.

Proof. Let $A_1, A_2, \ldots \in \mathcal{A}$ and disjoint. Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^{\infty} A_i\right),$$

hence we only need to show another way around.

Fix $N \in \mathbb{N}$, we have $\bigcup\limits_{n=1}^{N} A_n \in \mathcal{A}$ since N is finite, and

$$\begin{aligned}\mu^*(E) &= \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} A_n\right)\right) + \mu^*\left(E \setminus \left(\bigcup_{n=1}^{N} A_n\right)\right) \\ &\geq \underbrace{\sum_{n=1}^{N} \mu^*(E \cap A_n)}_{\overset{!}{=}\mu^*\left(E\cap\left(\bigcup_{n=1}^{N} A_n\right)\right)} + \underbrace{\mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right)}_{\leq \mu^*\left(E\setminus\left(\bigcup_{n=1}^{N} A_n\right)\right)}.\end{aligned}$$

Now, take $N \to \infty$ then we are done. ⊛

We can then extend this to the case of countable unions.

Exercise. Show $\mathcal{A}$ is closed under countable unions.

Answer. ⊛

DIY

Above shows that $\mathcal{A}$ is a σ-algebra.

The proof will be continued...

Lecture 5: Hahn-Kolmogorov Theorem

Firstly, we see a stronger version of Lemma 1.3.1 we have seen before.

14 Jan. 11:00

Lemma 1.3.2. Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots$ are disjoint C-measurable sets. Then,

$$\forall E \subset X,\ \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i).$$

Proof.

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_i) \geq \mu^*\left(E \cap \bigcup_{n=1}^{\infty} B_n\right) \geq \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} B_n\right)\right) \overset{!}{=} \sum_{n=1}^{N} \mu^*(E \cap B_n).$$

Now, we just take $N \to \infty$ and since $N \in \mathbb{N}$ is arbitrary, we then get the result according to Squeeze Theorem. ■

Let's continue the proof of Theorem 1.3.2.

Proof of Theorem 1.3.2 (cont.) The 1. is proved, now we prove 2. and 3.

2. Since from Definition 1.2.1, to show μ is a measure, we need to show the following.
   - Null empty set property.

Claim. $\mu(\varnothing) = 0$

Proof. This means that we need to show $\mu^*|_{\mathcal{A}}(\varnothing) = 0$. Since $\varnothing \in \mathcal{A}$ and μ^* is an outer measure, hence from the property of outer measure, it clearly holds. ⊛

- Countable additivity property.

Claim. μ^* on $\mathcal{A}$ has Countable additivity property.

Proof. It follows from Lemma 1.3.2 with $E = X$ ⊛

3. The proof is given in Theorem A.1.1.

■

1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets $\mathcal{E}$ and any ρ such that it assigns measure on $\mathcal{E}$, then it induces an outer measure by Proposition 1.3.1, finally complete the outer measure by Theorem 1.3.2.

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.3.1}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.3.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to σ-algebra we are working on till now.

Definition 1.4.1 (Algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on* X if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup\limits_{i=1}^{n} A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an algebra and a σ-algebra is whether they closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an outer measure.

Definition 1.4.2 (Pre-measure). Let $\mathcal{A}_0$ be an algebra on X. A *pre-measure* on X with respect to $\mathcal{A}_0$ is a function

$$\mu_0 \colon \mathcal{A}_0 \to [0, \infty]$$

such that

- (null empty set) $\mu_0(\varnothing) = 0$
- (finite additivity) $\mu_0\left(\bigcup\limits_{i=1}^{n} A_i\right) = \sum\limits_{i=1}^{n} \mu_0(A_i)$ if $A_1, \ldots, A_n \in \mathcal{A}_0$ are <u>disjoint</u>.
- (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and $A = \bigcup\limits_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, <u>disjoint</u>, then
$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 1.4.1. The null empty set property and countable additivity within the algebra implies finite additivity in Definition 1.4.2.

Proof. It's easy to see that since μ_0 is monotone. ■

Theorem 1.4.1 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in Proposition 1.3.1. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^*, then $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$, i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts. We first show that $\mathcal{A} \supset \mathcal{A}_0$.

Claim. $\mathcal{A} \supset \mathcal{A}_0$.

Proof. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is C-measurable, i.e.,

$$\forall E \subset X \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We first fix an $E \subset X$. From countable subadditivity of μ^*, we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

Fix $\epsilon > 0$. By the Proposition 1.3.1 of μ^*, $\exists B_1, B_2, \ldots \in \mathcal{A}_0$, $\bigcup\limits_{n=1}^{\infty} B_n \supset E$ such that

$$\mu^*(E) + \epsilon \overset{!}{\geq} \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left(\mu_0(\underbrace{B_n \cap A}_{\in \mathcal{A}_0}) + \mu_0(\underbrace{B_n \cap A^c}_{\in \mathcal{A}_0}) \right)$$

by the finite additivity of μ_0. Note that

$$\left\{ \begin{array}{l} \bigcup\limits_{n=1}^{\infty} (B_n \cap A) \supset E \cap A \\ \bigcup\limits_{n=1}^{\infty} (B_n \cap A^c) \subset E \cap A^c \end{array} \right. \Rightarrow \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

since

$$\mu^*(E \cap A) \leq \mu^* \left(\bigcup_{n=1}^{\infty} (B_n \cap A) \right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A)$$

and

$$\mu^*(E \cap A^c) \leq \mu^* \left(\bigcup_{n=1}^{\infty} (B_n \cap A^c) \right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A^c).$$

We then see that for any $\epsilon > 0$, the inequality

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

holds, hence so does

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

which implies $\mathcal{A} \supset \mathcal{A}_0$. ⊛

The proof will be continued...

Lecture 6: Hahn-Kolmogorov Theorem and Extension.

Let's continue the proof of Theorem 1.4.1. 18 Jan. 11:00

Proof of Theorem 1.4.1 (cont.) We proved the first part already, now we prove the part left.

Claim. $\mu|_{\mathcal{A}_0} = \mu_0$.

Proof. Let $A \in \mathcal{A}_0$, we want to show that

$$\mu(A) = \mu_0(A).$$

- Firstly, let

$$B_i = \begin{cases} A, & \text{if } i = 1 \\ \varnothing, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,$$

hence $\bigcup_{i=1}^{\infty} B_i = A$, then we see that

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu_0(B_i) = \mu_0(A)$$

from the definition of μ^* and countable additivity within the algebra of μ_0.

- Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A \cap B_1 \in \mathcal{A}_0$, $C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right) \in \mathcal{A}_0$ for $i \geq 2$ since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

are disjoint countable unions, by countable additivity within the algebra, we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i) \Rightarrow \mu_0(A) \leq \mu^*(A)$$

by taking the infimum from the definition of μ^*.

Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every $A \in \mathcal{A}_0$, which implies $\mu(A) = \mu_0(A)$ for every $A \in \mathcal{A}_0$ from Theorem 1.3.2, where we extend μ^* to μ respect to $\mathcal{A}_0$. ⊛

■

Definition 1.4.3 (Hahn-Kolmogorov extension). $(\mathcal{A}, \mu)$ obtained from Theorem 1.4.1 is the *Hahn-Kolmogorov extensions* of $(\mathcal{A}_0, \mu_0)$.

Note. We sometimes say *HK extension* instead of Hahn-Kolmogorov extensions for simplicity.

We can show the uniqueness of HK extension.

Theorem 1.4.2 (Uniqueness of HK extension). Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$. Let $(\mathcal{A}, \mu)$ be the HK extension of $(\mathcal{A}_0, \mu_0)$. Let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$. Then if μ_0 is σ-finite, $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Proof. First, we note the following.

Note. Notice that $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$ since they both extend $\mathcal{A}_0$.

Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

We'll show this by showing two inequalities. We first show that $\mu^*(A) \geq \mu'(A)$.

Claim. $\mu^*(A) \geq \mu'(A)$.

Proof. It's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing the arbitrary cover of A and using the definition of μ^*. ⊛

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

Claim. $\mu(A) \leq \mu'(A)$.

Proof. We split this into two cases.

- Assume $\mu(A) < \infty$, and fix $\epsilon > 0$. Then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup_{i=1}^{\infty} B_i \supset A$ such that

  $$\mu(A) + \epsilon = \mu^*(A) + \epsilon \overset{!}{\geq} \sum_{i=1}^{\infty} \mu_0(B_i) \overset{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

  This implies that

  $$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon,$$

  where the equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

  $$\mu(B) = \lim_{N\to\infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) = \lim_{N\to\infty} \mu'\left(\bigcup_{i=1}^{N} B_i\right) = \mu'(B)$$

  where the middle equality follows from $\mu = \mu'$ on $\mathcal{A}_0$, hence,

  $$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon$$

  for arbitrary ϵ, so we conclude $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is σ-finite, so we know $X = \bigcup_{n=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ such that $\mu_0(X_n) < \infty$. Replacing X_n by $X_1 \cup \ldots \cup X_n \in \mathcal{A}_0$, we may assume that $X_1 \subset X_2 \subset \ldots$. Then,

  $$\forall_{n\in\mathbb{N}}\ \mu(A \cap X_n) < \infty \overset{!}{\Rightarrow} \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

  By continuity from above, we have

  $$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) \leq \lim_{n\to\infty} \mu'(A \cap X_n) = \mu'(A).$$

Combine above two inequalities, the result follows. ⊛

■

Corollary 1.4.1. Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then there exists a unique measure μ on $\langle \mathcal{A}_0 \rangle$ that extends $\mathcal{A}_0$.

Furthermore,

(1) The completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.[a]

(2) $\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid \forall_{i \in \mathbb{N}} B_i \in \mathcal{A}_0, \bigcup_{i=1}^{\infty} B_i \supset A \right\}$ for all $A \in \overline{\langle \mathcal{A}_0 \rangle}$.

[a] This really means the pair of the σ-algebra and the measure in the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$, not the whole tuple of the measurable-space.

Lecture 7: Borel Measures

1.5 Borel Measures on $\mathbb{R}$

21 Jan. 11:00

As previously seen. Recall that when we say something is **Borel**, we assume some sort of topological structure on the underlying space implicitly. In our context, we're considering the usual topology on $\mathbb{R}^n$ specifically. We'll focus on one dimensional case for now.

Definition 1.5.1 (Distribution function)**.** An increasing[a] function

$$F : \mathbb{R} \to \mathbb{R}$$

and right-continuous. F is then a *distribution function.*

[a] Here, increasing means $F(x) \leq F(y)$ for $x < y$.

Example. Here are some examples of right-continuous functions.

(1) $F(x) = x$.

(2) $F(x) = e^x$.

(3) $F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$

(4) Let $\mathbb{Q} \coloneqq \{r_1, r_2, \ldots\}$. Define $F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n, \end{cases}$ and $F(x) \coloneqq \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}$.

Then F is a distribution function (hence right-continuous). This is shown in Lemma A.1.1.

Note. If F is increasing, and

$$F(\infty) \coloneqq \lim_{x \nearrow \infty} F(x), \quad F(-\infty) \coloneqq \lim_{x \searrow \infty} F(x)$$

exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.[a]

[a] There are distributions [FF99] Ch9., but these are different from distribution functions.

Now, we see a new definition which is essential to our discussion.

Definition 1.5.2 (Borel mesaure)**.** A *Borel measure* is any measure μ defined on the σ-algebra of Borel sets.

Since we're now considering a topological space, hence it's reasonable to define the following because we have the concept of compact set now.

Definition 1.5.3 (Locally finite)**.** Let X be a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Note. Some authors will require a Borel measure equipped with the locally finite property. But formally, this is not so common.

Lemma 1.5.1. Let μ be a locally finite Borel measure on $\mathbb{R}$, then

$$F_\mu(x) = \begin{cases} \mu\left((0,x]\right), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu\left((x,0]\right), & \text{if } x < 0 \end{cases}$$

is a distribution function.

Proof. We need to show two things.

Claim. F_μ is increasing.

Proof. To show F_μ is increasing, consider $x < y$ such that

$$F_\mu(x) \leq F_\mu(y)$$

by considering

- $x > 0$: Then $F_\mu(x) = \mu((0,x])$ and
$$F_\mu(y) = \mu((0,y]) = \mu((0,x] \cup (x,y]) \geq \mu((0,x]) = F_\mu(x).$$
- $x = 0$: Then $F_\mu(x) = 0$ and
$$F_\mu(y) = \mu((0,y]) \geq 0 = F_\mu(0)$$
since $y > 0$.
- $x < 0$: Follows the same argument with $x > 0$.

⊛

We now show F_μ is right-continuous.

Claim. F_μ is right-continuous.

Proof. Firstly, assume that $x \geq 0$, then we see that

$$F_\mu(x) = \mu((0,x]) = \mu((0,x^+])$$

from the fact that a measure is right-continuous.[a] Now, if $x \leq 0$, the same argument follows since multiplying -1 will not change the fact that a measure is continuous. ⊛

[a]Actually, a measure is always continuous.

■

Definition 1.5.4 (Half intervals)**.** We call $\varnothing, (a,b], (a,\infty), (-\infty,b]$, and $(-\infty,\infty)$ *half-intervals.*

Lemma 1.5.2. Let $\mathcal{H}$ be the collection of finite disjoint unions of half-intervals. Then, $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. We observe that $\varnothing \in \mathcal{H}$ and $\mathcal{H}$ is closed under finite unions is obvious, hence we only need to show that $\mathcal{H}$ is closed under complements.

Claim. $\mathcal{H}$ is closed under complements.

Proof. We have

- $\varnothing^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$.
- $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$ since it's a two disjoint union of half intervals.
- $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$.
- $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$.
- $(-\infty, \infty)^c = \varnothing \in \mathcal{H}$.

⊛

■

Proposition 1.5.1 (Distribution function defines a pre-measure)**.** Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function. For a half interval I, define

$$\ell(I) \coloneqq \ell_F(I) = \begin{cases} 0, & \text{if } I = \varnothing; \\ F(b) - F(a), & \text{if } I = (a, b]; \\ F(\infty) - F(a), & \text{if } I = (a, \infty]; \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b]; \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 \coloneqq \mu_{0,F}\colon \mathcal{H} \to [0, \infty]$ by

$$\mu_0(A) = \sum_{k=1}^{N} \ell(I_k) \text{ if } A = \bigcup_{k=1}^{N} I_k,$$

where A is a finite disjoint union of half intervals $I_1, \dots, I_N$. Then, μ_0 is a pre-measure on $\mathcal{H}$.

Proof. Firstly, we note that μ_0 is well-defined. And also, μ_0 satisfies null empty set and Finite additivity properties clearly. The only nontrivial part needs a proof is the Countable additivity within $\mathcal{H}$ properties. To show that Countable additivity within $\mathcal{H}$ holds, we proceed as follows.

Suppose $A \in \mathcal{H}$ where $A = \bigcup\limits_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.

Remark. Since $\mathcal{H}$ is only a collection of *finite* disjoint half intervals, hence after considering $A = I$, we can apply the same argument iteratively and stop in finite steps. Formally, we can consider $H \in \mathcal{H}$, $H = \bigcup_{i=1}^{\infty} A^i$, where A^i being a half interval. Then by the above argument, we have $A^i = I^i$ and so on.

Focus on the case $I = (a, b]$. Let $(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n]$, which is a disjoint union. Then we only need to check $F(b) - F(a) = \sum\limits_{n=1}^{\infty} (F(b_n) - F(a_n))$.

Claim. $F(b) - F(a) \geq \sum\limits_{n=1}^{\infty} (F(b_n) - F(a_n))$.

Proof. Since $(a,b] \supset \bigcup_{n=1}^{N}(a_n,b_n]$ for any fixed $N\in\mathbb{N}$, hence

$$\underset{N\in\mathbb{N}}{\forall}\ F(b)-F(a)\geq\sum_{n=1}^{N}\left(F(b_n)-F(a_n)\right).$$

By letting $N\to\infty$, we have $F(b)-F(a)\geq\sum_{n=1}^{\infty}(F(b_n)-F(a_n))$. ⊛

Claim. $F(b)-F(a)\leq\sum_{n=1}^{\infty}(F(b_n)-F(a_n))$.

Proof. Fix $\epsilon>0$. Since F is right-continuous, $\exists a'>a$ such that $F(a')-F(a)<\epsilon$. For each $n\in\mathbb{N}$, $\exists b_n'>b_n$ such that

$$F(b_n')-F(b_n)<\frac{\epsilon}{2^n}.$$

Then, we have $[a',b]\subset\bigcup_{n=1}^{\infty}(a_n,b_n')$, hence $\underset{N\in\mathbb{N}}{\exists}\ [a',b]\subset\bigcup_{n=1}^{N}(a_n,b_n')$, which is only finitely many unions now.

> **Remark.** This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.

In this case, we have

$$F(b)-F(a')\leq\sum_{n=1}^{N}F(b_n')-F(a_n).$$

Finally, we see that

$$\begin{aligned}F(b)-F(a)&\leq F(b)-F(a')+\epsilon\\&\leq\sum_{n=1}^{\infty}(F(b_n')-F(a_n))+\epsilon\\&\leq\sum_{n=1}^{\infty}\left(F(b_n)-F(a_n)+\frac{\epsilon}{2^n}\right)+\epsilon=\sum_{n=1}^{\infty}(F(b_n)-F(a_n))+2\epsilon\end{aligned}$$

for any fixed $\epsilon>0$, hence

$$F(b)-F(a)\leq\sum_{n=1}^{\infty}(F(b_n)-F(a_n)).$$

> **Remark.** It's again the $\epsilon/2^n$ trick we saw before!

⊛

Combine these two inequalities, we have

$$F(b)-F(a)=\sum_{n=1}^{\infty}(F(b_n)-F(a_n))$$

as we desired. ∎

Lecture 8: Lebesgue-Stieltjes Measure on $\mathbb{R}$

To classify all measures, we now see this last theorem to complete the task. 24 Jan. 11:00

Theorem 1.5.1 (Locally finite Borel measures on $\mathbb{R}$). We have

(1) $F\colon \mathbb{R} \to \mathbb{R}$ a distribution function, then there exists a **unique** *locally finite Borel measure* μ_F on $\mathbb{R}$ satisfying
$$\mu_F((a,b]) = F(b) - F(a)$$
for every $a < b$.

(2) Suppose $F, G\colon \mathbb{R} \to \mathbb{R}$ are distribution functions. Then,
$$\mu_F = \mu_G$$
on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. ■ HW.

Remark. Theorem 1.5.1 simply states that given a distribution function, if we restrict our attention on locally finite measures on $\mathbb{R}$ following our usual convention, then it defines the measure on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

1.6 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that
$$F \text{ distribution function} \overset{!}{\Rightarrow} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we have
$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 1.6.1 (Lebesgue-Stieltjes measure). Given a distribution function F, we say μ_F on $\mathcal{A}_{\mu_F}$ is called the *Lebesgue-Stieltjes measure* corresponding to F.

Definition. From Definition 1.6.1, if $F(x) = x$, then the induced $(\mathcal{A}_{\mu_F}, \mu_F)$ is denoted as $(\mathcal{L}, m)$.

Definition 1.6.2 (Lebesgue measure). m is called *Lebesgue measure*.

Definition 1.6.3 (Lebesgue σ-algebra). $\mathcal{L}$ is called *Lebesgue σ-algebra*.

Remark. Recall that $\mathcal{L}$ is induced by Theorem 1.3.2, namely given m, for all $A \subset \mathbb{R}$, we have
$$\mathcal{L} \coloneqq \left\{ A \subset \mathbb{R} \mid \underset{E \subset \mathbb{R}}{\forall} m(A) = m(A \cap E) + m(A \setminus E) \right\}.$$

Note. We see that since F is right-continuous and increasing, hence
$$F(x^-) \leq F(x) = F(x^+).$$
Some text will use $x-$ and $x+$ instead of x^- and x^+, respectively.

We now see some examples.

Example (Discrete measure). $\mu_F((a,b]) = F(b) - F(a)$. Then

- $\mu_F(\{a\}) = F(a) - F(a^-)$

- $\mu_F([a,b]) = F(b) - F(a^-)$
- $\mu_F((a,b)) = F(b^-) - F(a)$

This is so-called *discrete measure*.

Example (Dirac measure). We define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Then

- $\mu_F(\{0\}) = 1$
- $\mu_F(\mathbb{R}) = 1$
- $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. This is easy to see since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, hence

$$\mu_F(\mathbb{R} \setminus \{0\}) = \mu_F((-\infty, 0) \cup (0, \infty)) = \underbrace{\mu_F((-\infty, 0))}_{0-0} + \underbrace{\mu_F((0, \infty))}_{1-1} = 0,$$

  where $\mu_F((-\infty, 0)) = 0$ follows from $F(0^-) - F(-\infty) = 0 - 0 = 0$, while $\mu_F((0, \infty)) = 0$ follows from $F(\infty) - F(0) = 1 - 1 = 0$.

We call μ_F the *Dirac measure* at 0.

Example. Denote $\mathbb{Q} = \{r_1, r_2, \ldots\}$, and we define

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r. \end{cases}$$

Then

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$.

This is shown in Lemma A.1.2.

Example. If F is continuous at a, then $\mu_F(\{a\}) = 0$.

Example (Lebesgue measure). $F(x) = x$, then recall that we denote $\mu_F \coloneqq m$, and we have

- $m((a,b]) = m((a,b)) = m([a,b]) = b - a$.

Example. $F(x) = e^x$

- $\mu_F((a,b]) = \mu_F((a,b)) = e^b - e^a$.

Example (Middle thirds Cantor set). Let $C := \bigcap_{n=1}^{\infty} K_n$, where we have

$$
\begin{aligned}
K_0 &:= [0,1] \\
K_1 &:= K_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \\
K_2 &:= K_1 \setminus \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \\
&\vdots \\
K_n &:= K_{n-1} \setminus \bigcup_{k=1}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3^{k+2}}{3^{n+1}}\right).
\end{aligned}
$$

We see that C is uncountable and with $m(C) = 0$. And observe that $x \in C$ if and only if $x = \sum_{n=1}^{\infty} \frac{a_n}{3}$ for some $a_n \in \{0, 2\}$. Hence, we can instead formulate K_n by

$$
K_n = \bigcup_{\substack{a_i \in \{0,2\} \\ 1 \le i \le n}} \left[\sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n}\right].
$$

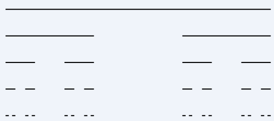


Figure 1.1: The top line corresponds to K_0, and then K_1, etc.

The proof of $m(C) = 0$ is given in Lemma A.1.3.

Cantor Function

Consider F as follows. We define a function F to be 0 to the left of 0, and 1 to the right of 1. Then, define F to be $\frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$, $\frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9}\right)$ and so on. This is so-called *Cantor Function*. We can show F is continuous and increasing, which makes F a distribution function. Also, we see that the measure this F induced is called *Cantor measure*.

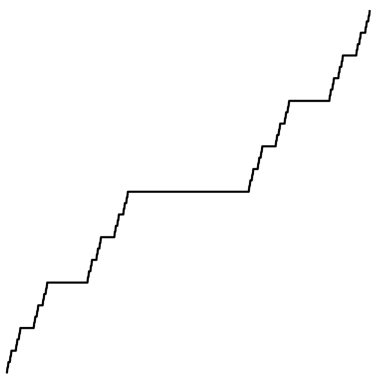


Figure 1.2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

Cantor Measure μ_F		Lebesgue Measure m
$\mu_F(\mathbb{R}\setminus C)=0$		$m(\mathbb{R}\setminus C)=\infty>0$
$\mu_F(C)=1$	$\Leftrightarrow$	$m(C)=0$
$\mu_F(\{a\})=0$		$m(\{a\})=0$

Remark. μ_F and m are said to be **singular** to each other.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 1.7.1. Let μ be Lebesgue-Stieltjes measure on $\mathbb{R}$. Then we have

$$\mu(A)\overset{!}{=}\inf\left\{\sum_{i=1}^{\infty}\mu((a_i,b_i])\mid\bigcup_{i=1}^{\infty}(a_i,b_i]\supset A\right\}=\inf\left\{\sum_{i=1}^{\infty}\mu((a_i,b_i))\mid\bigcup_{i=1}^{\infty}(a_i,b_i)\supset A\right\}$$

for every $A\in\mathcal{A}_\mu$

Proof. The second equality follows from the continuity of the measure. ∎

Remark. This is similar to

$$(a,b)=\bigcup_{n=1}^{\infty}(a,b-1/n],\quad (a,b]=\bigcap_{n=1}^{\infty}(a,b+1/n].$$

Lecture 9: Properties of Lebesgue-Stieltjes measure

26 Jan. 11:00

As previously seen. Let $X\subset[0,\infty]$. Recall that

- Finite supremum.

$$\alpha=\sup X<\infty\Leftrightarrow\begin{cases}\forall_{x\in X}\ \alpha\geq x\\ \forall_{\epsilon>0}\ \exists_{x\in X}\ x+\epsilon\geq\alpha.\end{cases}$$

- Infinite supremum.

$$\alpha=\sup X=\infty\Leftrightarrow\forall_{L>0}\ \exists_{x\in X}\ x\geq L.$$

This should be useful latter on.

Theorem 1.7.1 (Regularity). Let μ be Lebesgue-Stieltjes measure. Then, for all $A\in\mathcal{A}_\mu$,

(1) (outer regularity) $\mu(A)=\inf\{\mu(O)\mid O\supset A, O\text{ is open}\}$

(2) (inner regularity) $\mu(A)=\sup\{\mu(K)\mid K\subset A, K\text{ is compact}\}$

Proof. We check them separately.

(1) DIY

(2) Let $s:=\sup\{\mu(K)\mid K\subset A, K\text{ is compact}\}$, then by monotonicity, we have $\mu(A)\geq s$. To show the other direction, we consider

Claim. Inner regularity holds if A is a bounded set.

Proof. Then $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, $\overline{A}$ is also bounded $\Rightarrow \mu(\overline{A}) < \infty$. Fix $\epsilon > 0$, then by outer regularity, there exists an open $O \supset \overline{A} \setminus A$, and $\mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \epsilon$. Let $K := \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$, we show that

$$\mu(K) \geq \mu(A) - \epsilon.$$

⊛

DIY

Claim. Inner regularity holds if A is an unbounded set with $\mu(A) < \infty$.

Proof. Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \ldots$, then

$$\lim_{n \to \infty} \mu(A_n) = \mu(A) < \infty.$$

⊛

Claim. Inner regularity holds if A is an unbounded set with $\mu(A) = \infty$.

Proof. We can show that

$$\lim_{n \to \infty} \mu(A_n) = \mu(A) = \infty.$$

Fix $L > 0$, then $\exists N$ such that $\mu(A_N) \geq L$. ⊛

■

Definition. Let X be a topological space. Then

Definition 1.7.1 (G_δ-set)**.** A *G_δ-set* is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open.

Definition 1.7.2 (F_σ-set)**.** A *F_σ-set* is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.7.2. Let μ be a Lebesgue-Stieltjes measure. Then *TFAE*[a]:

(1) $A \in \mathcal{A}_\mu$

(2) $A = G \setminus M$, G is a G_δ-set, M is a μ-null set.

(3) $A = F \setminus N$, F is a F_σ-set, N is a μ-null set.

[a] *TFAE*: The following are equivalent.

Proof. We see that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are clear.

Claim. (1) $\Rightarrow$ (3).

Proof. We consider two cases.

- Assume $\mu(A) < \infty$. From the inner regularity, we have

$$\forall n \in \mathbb{N}\ \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

  Let $F = \bigcup_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ-null. Check!

- Assume $\mu(A) = \infty$. Let $A = \bigcup_{k \in \mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

$$\forall k \in \mathbb{Z}\ A_k = F_k \cup N_k,\ \ A = \underbrace{\left(\bigcup_k F_k\right)}_{F_\sigma\text{-set}} \cup \underbrace{\left(\bigcup_k N_k\right)}_{\mu\text{-null}}.$$

⊛

Claim. (1) ⇒ (2).

Proof. We see that

$$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

⊛

■

Proposition 1.7.1. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0\ \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using outer regularity and the fact that every open set is $\bigcup_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. ■ DIY

We now see some properties of Lebesgue measure.

Theorem 1.7.3. Let $A \in \mathcal{L}$, then we have $A + s \in \mathcal{L}$, $rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,

$$m(A+s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. ■ DIY

Example. We now see some examples.

(1) Let $\mathbb{Q} =: \{r_i\}_{i=1}^{\infty}$ which is dense in $\mathbb{R}$. Let $\epsilon > 0$, and

$$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i}\right).$$

We see that O is open and dense[a] in $\mathbb{R}$. But we see

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$

Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$

(2) There exists uncountable set A with $m(A) = 0$.

(3) There exists A with $m(A) > 0$ but A contains no non-empty open intervals.

(4) There exists $A \notin \mathcal{L}$. e.g. Vitali set.[b]

(5) There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

[a] https://en.wikipedia.org/wiki/Dense_set
[b] https://en.wikipedia.org/wiki/Vitali_set

Chapter 2

Integration

Lecture 10: Integration

2.1 Measurable Function

26 Jan. 11:00

We start with a definition.

Definition 2.1.1 (Measurable function). Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$ are measurable spaces. Then we say $f \colon X \to Y$ is $(\mathcal{A}, \mathcal{B})$-*measurable* if

$$\underset{B \in \mathcal{B}}{\forall} f^{-1}(B) \in \mathcal{A}.$$

Remark. If $\mathcal{A}$ and $\mathcal{B}$ are given, we'll sometimes say f is measurable if it'll not cause any confusions.

Lemma 2.1.1. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, and suppose $\mathcal{B} = \langle \mathcal{E} \rangle$ for some $\mathcal{E} \subset Y$. Then,

$$f \colon X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \Leftrightarrow \underset{E \in \mathcal{E}}{\forall} f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part ($\Rightarrow$) is clear. On the other direction, we consider the following. Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $\mathcal{E} \subset \mathcal{D}$ by assumption
- $\mathcal{D}$ is a σ-algebra

Check!

hence, we see that $\langle \mathcal{E} \rangle = \mathcal{B} \subset \mathcal{D}$ from Lemma 1.1.2. The result then follows from the definition of $(\mathcal{A}, \mathcal{B})$-measurable. ■

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup_\alpha E_\alpha\right) = \bigcup_\alpha f^{-1}(E_\alpha)$

Definition 2.1.2 ($\mathcal{A}$-measurable). Let $(X, \mathcal{A})$ be a measurable space. Then,

$$\left.\begin{aligned} f &\colon X \to \mathbb{R} \\ f &\colon X \to \overline{\mathbb{R}} \\ f &\colon X \to \mathbb{C} \end{aligned}\right\} \text{ is } \mathcal{A}\text{-}\textit{measurable} \text{ if } \begin{cases} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \operatorname{Re} f, \operatorname{Im} f \colon X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable}. \end{cases}$$

Notation. Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$

- $\mathcal{B}(\overline{\mathbb{R}}) = \left\{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\right\}$.
- $\operatorname{Re} f$ is the real part of f, while $\operatorname{Im} f$ is the imaginary part of f.

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \Rightarrow$ Every function is $\mathcal{A}$-measurable.
- $\mathcal{A} = \{\varnothing, X\} \Rightarrow$ The only $\mathcal{A}$-measurable functions are constant functions.

There are two very common kinds of measurable functions are worth mentioning.

Definition. Given a measurable function f, we have the following.

Definition 2.1.3 (Lebesgue measurable function)**.** f is a *Lebesgue measurable function* if $f\colon (\mathbb{R}, \mathcal{L}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$.

Definition 2.1.4 (Borel measurable function)**.** f is a *Borel measurable function* if $f\colon (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Lemma 2.1.2. Given $f\colon X \to \mathbb{R}$, *TFAE.*

(1) f is $\mathcal{A}$-measurable

(2) $\forall a \in \mathbb{R},\ f^{-1}((a, \infty)) \in \mathcal{A}$

(3) $\forall a \in \mathbb{R},\ f^{-1}([a, \infty)) \in \mathcal{A}$

(4) $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a)) \in \mathcal{A}$

(5) $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from Lemma 2.1.1 we just saw. ∎

Remark (Operations preserve $\mathcal{A}$-measurability)**.** Given $f, g\colon X \to \mathbb{R}$ and is $\mathcal{A}$-measurable, then

(1) $\phi\colon \mathbb{R} \to \mathbb{R}$, $\mathcal{A}$-measurable, then
$$\phi \circ f\colon X \to \mathbb{R}$$
is $\mathcal{A}$-measurable.

(2) $-f,\ 3f,\ f^2,\ |f|$ are all $\mathcal{A}$-measurable, and $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) \neq 0, \forall x \in X$.

(3) $f + g$ is $\mathcal{A}$-measurable. We see this from
$$(f+g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty))\right)$$
with Lemma 2.1.2.

(4) $f \cdot g$ is $\mathcal{A}$-measurable. We see this from
$$f(x)g(x) = \frac{1}{2}\left((f(x) + g(x))^2 - f(x)^2 - g(x)^2\right).$$

(5) We see that
$$(f \vee g)(x) \coloneqq \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) \coloneqq \min\{f(x), g(x)\}$$
are $\mathcal{A}$-measurable.

(6) Let $f_n\colon X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable. Then

$$\sup_{n\in\mathbb{N}} f_n,\ \inf_{n\in\mathbb{N}} f_n,\ \limsup_{n\to\infty} f_n,\ \liminf_{n\to\infty} f_n$$

are $\mathcal{A}$-measurable.

Proof. Consider $\sup_{n\in\mathbb{N}} f_n =: g$, then

$$g^{-1}((a,\infty]) = \bigcup_{n\in\mathbb{N}} f_n^{-1}((a,\infty])$$

for $\sup_n f_n(x) = g(x) > a$. A similar argument can prove the case of $\inf_{n\in\mathbb{N}} f_n$.

check

And notice that $\limsup_{n\to\infty} f_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} f_n$, then the similar argument also proves this case.

⊛

(7) If $\lim_{n\to\infty} f_n(x)$ converges for every $x \in X$, then f is $\mathcal{A}$-measurable.

(8) If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous $\Rightarrow$ f is Borel measurable $\Rightarrow$ f is Lebesgue measurable since the preimage of an open set of a continuous function is open, then we consider $f^{-1}((a,\infty))$.

Definition 2.1.5 (Support). The *support* of function $f\colon X \to \overline{\mathbb{R}}$ is

$$\operatorname{supp} f \coloneqq \{x \in X \mid f(x) \neq 0\}.$$

Definition. For $f\colon X \to \overline{\mathbb{R}}$, let $f^+ \coloneqq f \vee 0$ and $f^- \coloneqq (-f) \vee 0$, i.e., $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$. Then we have the following.

Definition 2.1.6 (Positive part). f^+ is the *positive part* of f.

Definition 2.1.7 (Negative part). f^- is the *negative part* of f.

Remark. If $\operatorname{supp} f^+ \cap \operatorname{supp} f^- = \varnothing$ and $f(x) = f^+(x) - f^-(x)$, then

$$f \text{ is } \mathcal{A}\text{-measurable} \Leftrightarrow f^+, f^- \text{ are } \mathcal{A}\text{-measurable}.$$

Definition 2.1.8 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator) function* of E is

$$\mathcal{X}_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is $\mathcal{A}$-measurable $\Leftrightarrow E \in \mathcal{A}$.

Definition 2.1.9 (Simple function). Let $(X, \mathcal{A})$ be a measurable space. Then a *simple function* $\phi\colon X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

Remark. We see that if $\phi(X) = \{c_1, \dots, c_N\}$, then

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \Rightarrow \phi = \sum_{i=1}^{N} \underbrace{c_i}_{\neq\pm\infty} \mathbb{1}_{\underbrace{E_i}_{\in\mathcal{A}}}.$$

Lecture 11: Integration of nonnegative functions

31 Jan. 11:00

As previously seen. For a simple function ϕ, c_i can actually be in $\mathbb{C}$.

Theorem 2.1.1. Given a measurable space $(X, \mathcal{A})$ and let $f\colon X \to [0, \infty]$, the following are equivalent.

(1) f is a $\mathcal{A}$-measurable function.

(2) There exists simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ such that

$$\underset{x\in X}{\forall} \lim_{n\to\infty} \phi_n(x) = f(x)$$

i.e., f is a pointwise upward limit of simple functions.

Proof. We'll prove both directions.

Claim. (2) $\Rightarrow$ (1).

Proof. It's clear from the fact that $f(x) = \sup_n \phi_n(x)$ and the remark. ⊛

Claim. (1) $\Rightarrow$ (2).

Proof. Assume f is $\mathcal{A}$-measurable, and fix $n \in \mathbb{N}$.
Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. Also, for $0 \leq k \leq 2^{2n} - 1$, $E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}$.
Then, define ϕ_n be

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n},$$

we have

- $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ for every $x \in X$
- $\forall x \in X \setminus F_n$, we have $0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$

Furthermore, we see that

$$F_1 \supset F_2 \supset \ldots, \quad \bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\}),$$

then

- $x \in f^{-1}([0, \infty]) = X \setminus \bigcap_{n=1}^{\infty} F_n \Rightarrow \lim_{n\to\infty} \phi_n(x) = f(x)$
- $x \in f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} F_n \Rightarrow f_n(x) \geq 2^n \Rightarrow \lim_{n\to\infty} \phi_n(x) = \infty = f(x)$

⊛

■

Corollary 2.1.1. If f is bounded on a set $A \subset \mathbb{R}$, i.e., $\exists L > 0$ such that

$$\underset{x\in A}{\forall} |f(x)| \leq L,$$

then there exists a sequence of simple functions $\{\phi_n\}$ such that $\phi_n \to f$ uniformly on A.

Proof. ■ DIY

Corollary 2.1.2. If $f\colon X\to\mathbb{C}$ is a measurable function if and only if there exists simple functions $\phi_n\colon X\to\mathbb{C}$ such that

$$0\leq|\phi_1(x)|\leq|\phi_2(x)|\leq\ldots\leq|f(x)|$$

with

$$\forall_{x\in X}\lim_{n\to\infty}\phi_n(x)=f(x).$$

Proof. ■ DIY

2.2 Integration of Nonnegative Functions

We start with our first definition about integral.

Definition 2.2.1 (Integration of nonnegative funciton)**.** Let $(X,\mathcal{A},\mu)$ be a measure space, and $\phi\colon X\to[0,\infty]$ such that

$$\phi=\sum_{i=1}^{N}c_i\mathbb{1}_{E_i}$$

be a simple function. Define

$$\int\phi=\int\phi\,\mathrm{d}\mu=\int_X\phi\,\mathrm{d}\mu=\sum_{i=1}^{N}c_i\mu(E_i).$$

Furthermore, for $A\in\mathcal{A}$,

$$\int_A\phi=\int_A\phi\,\mathrm{d}\mu=\int\phi\mathbb{1}_A\,\mathrm{d}\mu.$$

Note. Note that

- In the expression $\sum_{i=1}^{N}c_i\mu(E_i)$, we're using the convention $0\cdot\infty=0$.
- The function $\phi\mathbb{1}_A$ is also a simple function since both ϕ and $\mathbb{1}_A$ are simple function.

Proposition 2.2.1. Suppose we have $\phi,\psi\geq0$ be two simple functions. Then,

(1) Definition 2.2.1 is well-defined.

(2) $\int c\phi=c\int\phi$ for $c\in[0,\infty)$.

(3) $\int\phi+\psi=\int\phi+\int\psi$.

(4) $\phi(x)\geq\psi(x)$ for all $x\Rightarrow\int\phi\geq\int\psi$.

(5) $\nu(A)=\int_A\phi\,\mathrm{d}\mu$ is a measure on $(X,\mathcal{A})$.

Proof. ■ DIY

Definition 2.2.2 (Generatlization of Integration of nonnegative function)**.** Given $(X,\mathcal{A},\mu)$ with $f\colon X\to[0,\infty]$ be $\mathcal{A}$-measurable. Define

$$\int f=\int f\,\mathrm{d}\mu=\sup\left\{\int\phi\colon 0\leq\phi\leq f\text{ such that }\phi\text{ is simple}\right\}.$$

Note. Note that

- If f is a simple function, the Definition 2.2.1 and Definition 2.2.2 of $\int f$ are the same.
- $\int cf = c\int f$ for $c \in [0, \infty)$.
- If $f \geq g \geq 0 \Rightarrow \int f \geq \int g$.
- But $\int f + g = \int f + \int g$ is not trivial.

Theorem 2.2.1 (Monotone Convergence Theorem)**.** Given $(X, \mathcal{A}, \mu)$ be a measure space. Then if

- $f_n \colon X \to [0, \infty]$ be $\mathcal{A}$-measurable for every $n \in \mathbb{N}$;
- $0 \leq f_1(x) \leq f_2(x) \leq \ldots$ for every $x \in X$;
- $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$,

we have

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. Note that if $\lim\limits_{n\to\infty} \int f_n$ exists, then it's equal to $\sup_n \int f_n$.
Then

- $f_n \leq f \Rightarrow \int f_n \leq \int f \Rightarrow \lim\limits_{n\to\infty} \int f_n \leq \int f$.
- Fix a simple function $0 \leq \phi \leq f$, then it's enough to show $\lim\limits_{n\to\infty} \int f_n \geq \int \phi$.

  We first fix $\alpha = (0, 1)$, then it's also enough to show

  $$\lim_{n\to\infty} \int f_n \geq \alpha \int \phi.$$

  Let $A_n \coloneqq \{x \in X \mid f_n(x) \geq \alpha\phi(x)\}$, then since f_n is measurable,

  - $A_n \in \mathcal{A}$
  - $A_1 \subset A_2 \subset A_3 \subset \ldots$
  - $\bigcup\limits_{n=1}^{\infty} A_n = X$ Check!

  We then have

  $$\int f_n \geq \int f_n \mathbb{1}_{A_n} \geq \int \alpha\phi\mathbb{1}_{A_n} = \alpha \int_{A_n} \phi = \alpha\nu(A_n)$$

  where $\nu(A) = \int_A \phi$ is a measure. This implies

  $$\lim_{n\to\infty} \int f_n \geq \alpha \lim_{n\to\infty} \nu(A_n) \overset{!}{=} \alpha\nu(X) = \alpha \int \phi.$$

∎

Corollary 2.2.1 (Linearity of nonnegative integral)**.** Let $f, g \geq 0$ be measurable, then

$$\int f + g = \int f + \int g.$$

Proof. There exists simple functions ϕ_n and ψ_n such that

- $0 \leq \phi_1 \leq \phi_2 \leq \ldots$ and $\phi_n \to f$ pointwise

- $0 \le \psi_1 \le \psi_2 \le \ldots$ and $\psi_n \to g$ pointwise

Then,
$$\int (f+g) \overset{!}{=} \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \int \phi_n + \int \psi_n \overset{!}{=} \int f + \int g.$$
■

Lecture 12: Fatou's Lemma

We start with a useful corollary. 2 Feb. 11:00

Corollary 2.2.2 (Tonelli's theorem for nonnegative series and integrals)**.** Given $g_n \ge 0$ for every $n \in \mathbb{N}$ and let g_n be measurable, then
$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Proof. Let $f_N \coloneqq \sum\limits_{n=1}^{N} g_n$ such that $\lim\limits_{N\to\infty} f_N \sum\limits_{n=1}^{\infty} g_n \eqqcolon f$, then since $g_n \ge 0$, we have $0 \le f_1 \le f_2 \le \ldots$ with
$$\lim_{N\to\infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$
By Theorem 2.2.1, we have
$$\lim_{N\to\infty} \int \underbrace{\sum_{n=1}^{N} g_n}_{f_N} = \int \underbrace{\sum_{n=1}^{\infty} g_n}_{f}.$$
Now, since the terms in the limit on the left-hand side is just a finite sum, by Corollary 2.2.1, we have
$$\underbrace{\lim_{N\to\infty} \sum_{n=1}^{N}}_{\sum_{n=1}^{\infty}} \int g_n = \lim_{N\to\infty} \int \sum_{n=1}^{N} g_n = \int \sum_{n=1}^{\infty} g_n,$$
hence
$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$
■

Remark. Recall that we have seen two series case before. We'll later see two integrals cases.

Theorem 2.2.2 (Fatou's Lemma)**.** Suppose $f_n \ge 0$ and measurable, then
$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n.$$

Proof. Before we start we note the following.

Remark. Recall that
$$\liminf_{n\to\infty} f_n \coloneqq \lim_{k\to\infty} \inf_{n\ge k} f_n = \sup_{k\in\mathbb{N}} \inf_{n\ge k} f_n$$
and
$$\exists \lim_{n\to\infty} a_n \Leftrightarrow \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Let $g_k = \inf_{n\ge k} f_n$, then g_k is measurable and $0 \le g_1 \le g_2 \le \ldots$. Now, from Theorem 2.2.1, we have

$$\int \lim_{k\to\infty} g_k = \lim_{k\to\infty} \int g_k.$$

Notice that the left-hand side is just $\int \liminf_{n\to\infty} f_n$, while the right-hand side is just $\lim_{k\to\infty} \int \inf_{n\geq k} f_n$, i.e.,

$$\int \liminf_{n\to\infty} f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n.$$

We see that we want to take the inf outside the integral on the right-hand side. Observe that

$$\forall_{m\geq k} \inf_{n\geq k} f_n \leq f_m \Rightarrow \forall_{m\geq k} \int \inf_{n\geq k} f \leq \int f_m \Rightarrow \int \inf_{n\geq k} f_n \leq \inf_{m\geq k} \int f_m.$$

Then, we have

$$\int \liminf_{n\to\infty} f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n \leq \lim_{k\to\infty} \inf_{m\geq k} \int f_m = \liminf_{m\to\infty} \int f_m.$$

■

Example (Escape to horizontal infinity). Given $(\mathbb{R}, \mathcal{L}, m)$, let $f_n := \mathbb{1}_{(n,n+1)}$. We immediately see that

- $f_n \to 0$ pointwise
- $\int f_n = 1$ for every n
- $\int f = 0$

From Theorem 2.2.2, we have a strict inequality

$$0 = \int \liminf_{n\to\infty} f_n, \liminf_{n\to\infty} \int f_n = 1.$$

Example (Escape to width infinity). Given $(\mathbb{R}, \mathcal{L}, m)$, let $f_n := \frac{1}{n}\mathbb{1}_{(0,n)}$.

Example (Escape to vertical infinity). Given $(\mathbb{R}, \mathcal{L}, m)$, let $f_n := n\mathbb{1}_{(0,\frac{1}{n})}$.

Lemma 2.2.1 (Markov's inequiality). Let $f \geq 0$ be measurable. Then

$$\forall_{c\in(0,\infty)} \mu(\{x \mid f(x) \geq c\}) \leq \frac{1}{c}\int f.$$

Proof. Denote $\{x \mid f(x) \geq c\} =: E$, then

$$f(x) \geq c\mathbb{1}_E(x) \Rightarrow \int f \geq c\int \mathbb{1}_E = c \cdot \mu(E).$$

■

Remark. Notice that $E = f^{-1}([c,\infty])$, hence E is measurable.

Proposition 2.2.2. Let $f \geq 0$ be measurable. Then,

$$\int f = 0 \Leftrightarrow f = 0 \text{ a.e.}.$$

i.e.,

$$\int f \, \mathrm{d}\mu = 0 \Leftrightarrow \mu(A) = 0$$

where $A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$.

Proof. Firstly, assume that $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}$$

where E_i are disjoint and $c_i \in (0, \infty)$. Then,

$$\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0 \Leftrightarrow \mu(E_1) = \ldots = \mu(E_N) = 0 \Leftrightarrow \mu(A) = 0, \; A = \bigcup_{i=1}^{N} E_i.$$

Now, assume that f is a general function where $f \geq 0$ is the only constraint, and we consider two cases.

- Assume $\mu(A) = 0$ (i.e., $f = 0$ a.e.). Let $0 \leq \phi \leq f$, where ϕ is simple. Then

  $$\underset{x \in A^c}{\forall} \; \phi(x) = 0$$

  since $f(x) = 0$, $\forall x \in A^c$. This implies that $\phi = 0$ a.e. since $\mu(A) = 0$, so $\int \phi = 0$. We then have

  $$\int f = 0$$

  from Definition 2.2.2.

- Assume $\int f = 0$. Let $A_n = f^{-1}\left(\left[\frac{1}{n}, \infty\right]\right)$. Then we see that
  - $A_1 \subset A_2 \subset \ldots$
  - $\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty}\left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0, \infty)) = A$.

  We then have

  $$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \overset{!}{\leq} n \int f = 0,$$

  which further implies

  $$\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0$$

  from the continuity of measure from below.

■

Corollary 2.2.3. If $f, g \geq 0$ are both measurable and $f = g$ a.e., then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \neq g(x)\}$[a]. Then by assumption, $\mu(A) = 0$, hence

$$f \mathbb{1}_A = 0 \text{ a.e.}, \quad g \mathbb{1}_A = 0 \text{ a.e.}.$$

This further implies that

$$\begin{aligned}\int f = \int f(\mathbb{1}_A + \mathbb{1}_{A^c}) &\overset{!}{=} \int f\mathbb{1}_A + \int f\mathbb{1}_{A^c} \\ &= \int f\mathbb{1}_{A^c} = \int g\mathbb{1}_{A^c} = \int g\mathbb{1}_{A^c} + \int g\mathbb{1}_A = \int g.\end{aligned}$$

■

[a] A is measurable indeed.

Corollary 2.2.4. Let $f_n \geq 0$ be measurable. Then

(1) $\left.\begin{array}{l} 0 \leq f_1 \leq f_2 \leq \ldots \leq f \text{ a.e.} \\ \lim\limits_{n\to\infty} f_n = f \text{ a.e.} \end{array}\right\} \Rightarrow \lim\limits_{n\to\infty} \int f_n = \int f.$

(2) $\lim\limits_{n\to\infty} f_n = f$ a.e. $\Rightarrow \int f \leq \liminf\limits_{n\to\infty} \int f_n$.

Proof. ■

DIY

Remark. Almost all the theorems we've proved can be replaced by theorems dealing with almost everywhere condition.

Lecture 13: Integration of Complex Functions

2.3 Integration of Complex Functions

4 Feb. 11:00

As usual, we start with a definition.

Definition 2.3.1 (Integrable). Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f\colon X \to \overline{\mathbb{R}}$ and $g\colon X \to \mathbb{C}$ be measurable.[a]

Then f, g are called *integrable* if $\int |f| < \infty$ and $\int |g| < \infty$, and we define

$$\int f = \int f^+ - \int f^-, \quad \int g = \int \operatorname{Re} g + i \int \operatorname{Im} g.$$

Furthermore, for $f\colon X \to \overline{\mathbb{R}}$, we define

$$\int f = \begin{cases} \infty, & \text{if } \int f^+ = \infty, \int f^- < \infty; \\ -\infty, & \text{if } \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

[a] Recall that for a complex-valued function like g, this means that both $\operatorname{Re} g$ and $\operatorname{Im} g$ are measurable.

We now see a lemma.

Lemma 2.3.1. Let $f, g\colon X \to \overline{\mathbb{R}}$ or $\mathbb{C}$ integrable. Assume that $f(x) + g(x)$ is well-defined for all $x \in X$.[a] Then we have

(1) $f + g$, cf for all $c \in \mathbb{C}$ are integrable.

(2) $\int f + g = \int f + \int g$. This is not trivial since $(f+g)^+ \neq f^+ + g^+$.

(3) $\left|\int f\right| \leq \int |f|$.

[a] That is, we never see $\infty + (-\infty)$ or $(-\infty) + \infty$.

Proof. Check [FF99] page 53. ■

Lemma 2.3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be an integrable function on X. Then

(1) f is finite a.e.. i.e., $\{x \in X \mid |f(x)| = \infty\}$ is a null set.

(2) The set $\{x \in X \mid f(x) \neq 0\}$ is σ-finite.

Proof. ■

HW 5 Q8 by Lemma 2.2.1

Proposition 2.3.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, then

(1) If h is integrable on X, then

$$\underset{E\in\mathcal{A}}{\forall} \int_E h = 0 \Leftrightarrow \int |h| = 0 \Leftrightarrow h = 0 \text{ a.e.}$$

(2) If f, g are integrable on X, then

$$\underset{E\in\mathcal{A}}{\forall} \int_E f = \int_E g \Leftrightarrow f = g \text{ a.e.}$$

Proof. We prove this one by one.

(1) We see that the second equivalence is done in Proposition 2.2.2, hence we prove the first equivalence only. Since we have

$$\int |h| = 0 \Rightarrow \left|\int_E h\right| \leq \int_E |h| \leq \int |h| = 0,$$

which shows one implication. Now assume that $\int_E h = 0$ for all $E \in \mathcal{A}$, then we can write h as

$$h = u + iv = (u^+ - u^-) + i(v^+ - v^-).$$

Let $B \coloneqq \{x \in X \mid u^+(x) > 0\}$, then by assumption, we have

$$0 = \int_B h = \mathrm{Re} \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+,$$

hence $u^+ = 0$ almost everywhere. Similarly, we have u^-, v^+, v^- are all zero almost everywhere. This gives us that h is zero almost everywhere as desired.

(2)

DIY

■

Theorem 2.3.1 (Dominated Convergence Theorem)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, and

- Let f_n be integrable on X.
- $\lim\limits_{n\to\infty} f_n(x) = f(x)$ almost everywhere.
- There is a $g\colon X \to [0, \infty]$ such that g is integrable and

$$\underset{n\in\mathbb{N}}{\forall} \ |f_n(x)| \leq g(x) \text{ a.e.}$$

Then we have

$$\lim_{n\to\infty} \int f_n = \int f = \int \lim_{n\to\infty} f_n.$$

Proof. Let F be the countable union of null set on which the three conditions may fail. Then we see that after modifying the definition of f_n, f and g on F, we may assume that all three conditions hold everywhere since modifying on a null set does not change the integral.

We now consider the $\overline{\mathbb{R}}$-valued case only. Note that the second and the third conditions imply that f is integrable since $|f| \leq g(x)$. We then see that $g + f_n \geq 0$ and $g - f_n \geq 0$ because $-g \leq f_n \leq g$. From Theorem 2.2.2, we have

Check $\mathbb{C}$-valued case

$$\int g + f \leq \liminf_{n\to\infty} \int g + f_n, \quad \int g - f \leq \liminf_{n\to\infty} \int g - f_n.$$

From the linearity of integral, we have

$$\int g + \int f \leq \int g + \liminf_{n\to\infty} \int f_n, \quad \int g - \int f \leq \int g - \liminf_{n\to\infty} \int f_n.$$

Now, since $\int g < \infty$, we can cancel it, which gives

$$\int f \leq \liminf_{n\to\infty} \int f_n, \quad -\int f \leq \liminf_{n\to\infty} \int -f_n = -\limsup_{n\to\infty} \int f_n,$$

which implies

$$\int f \leq \liminf_{n\to\infty} f_n \leq \limsup_{n\to\infty} \int f_n \leq \int f.$$

This shows that the limit exists, and the desired result indeed holds. ∎

Corollary 2.3.1 (Tonelli's theorem for series and integrals)**.** Suppose f_n are integrable functions such that

$$\sum_{n=1}^{\infty} \int |f_n| < \infty,$$

then we have

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

Proof. Take $G(x)$ to be

$$G(x) \coloneqq \sum_{n=1}^{\infty} |f_n(x)|,$$

then we see $G(x) \geq |F_N(x)|$ where $F_N(x) \coloneqq \sum_{n=1}^{N} f_n(x)$. By Corollary 2.2.2, we have

$$\int G(x) = \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Lastly, from Theorem 2.3.1, the result follows. ∎

Remark. Compare to Corollary 2.2.2, we see that we further generalize the result!

Lecture 14: L^1 Space

7 Feb. 11:00

2.4 L^1 Space

We now introduce another space called L^p *spaces*, which are function spaces defined using a natural generalization of the p-norm for finite-dimensional vector spaces. We sometimes call it Lebesgue spaces also.

Before we start, we need to define a *norm*.

Definition 2.4.1 (Norm). Let V be a vector space over filed $\mathbb{R}$ or $\mathbb{C}$. A *norm* is a seminorm, defined as

Definition 2.4.2 (Seminorm). A *seminorm* on V is

$$\|\cdot\| : V \to [0, \infty)$$

such that

- $\|cv\| = |c| \|v\|$ for every $v \in V$ and every scalar c.
- $\|v + w\| \leq \|v\| + \|w\|$ for every $v, w \in V$.

with an additional condition

- $\|v\| = 0 \Leftrightarrow v = 0$.

Lemma 2.4.1. A normed vector space is a metric space with metric

$$\rho(v, w) = \|v - w\|.$$

Proof. ∎ DIY

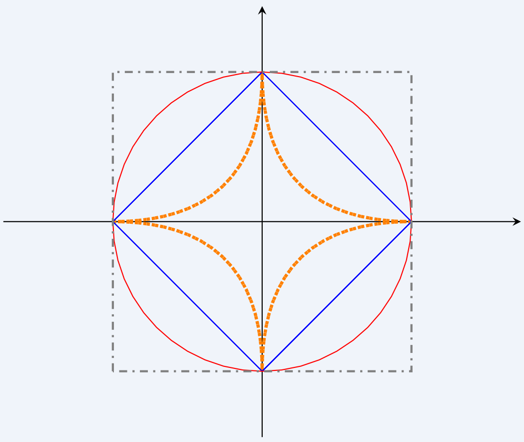
Example (p-norm). $V = \mathbb{R}^d$ with

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}, & \text{if } p \in [0, \infty); \\ \max_{1 \leq i \leq d} |x_i|, & \text{if } p = \infty \end{cases}$$

is a normed vector space. The unit ball

$$\{x \in \mathbb{R}^d \mid \|x\|_p \leq 1\}$$

for different p has the following figures.



Remark. All p-norms induce the same topology. i.e., if U is open in p-norm, it is open in p'-norm as well.

Note. Recall that we say f is integrable means

$$\int |f| < \infty,$$

and if $f = g$ a.e., then

$$\int f = \int g$$

Definition 2.4.3 (L^1 Space). Given $(X, \mathcal{A}, \mu)$,

$$f \in L^1(X, \mathcal{A}, \mu) \left(= L^1(X, \mu) = L^1(X) = L^1(\mu)\right)$$

means that f is an integrable function on X.

Lemma 2.4.2. $L^1(X, \mathcal{A}, \mu)$ is a vector space with seminorm

$$\|f\|_1 = \int |f| .$$

Proof. ■

Check this is indeed a seminorm.

Definition 2.4.4 (L^1 Space with equivalence class). Define $f \sim g$ if $f = g$ a.e., then

$$L^1(X, \mathcal{A}, \mu) \Big/ \sim \; = L^1(X, \mathcal{A}, \mu),$$

i.e., we simply denote the collection of equivalence classes by itself.[a]

[a] By some abusing of notation of L^1.

Remark. We have

- With Definition 2.4.4, $L^1(X, \mathcal{A}, \mu)$ is a normed vector space.
- We say that the L^1-*metric* $\rho(f, g)$ is simply

$$\rho(f, g) = \int |f - g| .$$

Dense Subsets of L^1

Note. Recall the definition of a *dense set*[a].

[a] https://en.wikipedia.org/wiki/Dense_set

Definition 2.4.5 (Step function). A *step function* on $\mathbb{R}$ is

$$\psi = \sum_{i=1}^{N} c_i \mathbb{1}_{I_i},$$

where I_i is an interval.

Notation. We denote the collection of continuous functions with compact support by $C_c(\mathbb{R})$.

Theorem 2.4.1. We have the following.

(1) {integrable simple functions} is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1-metric).

(2) $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$, where μ is a Lebesgue-Stieltjes-measure. Then the set of integrable simple functions is dense in $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu)$.

(3) $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

Proof. We prove this one by one.

(1) Since there exists simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$, where $\phi_n \to f$ pointwise. Then by Theorem 2.3.1, we have

$$\lim_{n\to\infty} \int \underbrace{|f_n - f|}_{\leq |\phi_n| + |f| \leq 2|f|} = 0$$

where $2\,|f|$ is in L^1.

(2) Let $\mathbb{1}_E$ <u>approximate</u> by $\sum_{i=1}^{\infty} c_i \mathbb{1}_{I_i}$. From Theorem 1.7.1 for Lebesgue-Stieltjes-measure,

$$\forall \epsilon' > 0\ \exists I = \bigcup_{i=1}^{N} I_i \text{ such that } \mu(E \triangle I) \leq \epsilon'.$$

(3) To approximate $\mathbb{1}_{(a,b)}$, we simply consider function $g \in C_c(\mathbb{R})$ such that

$$\int \left|\mathbb{1}_{(a,b)} - g\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

∎

Lecture 15: Riemann Integral

2.5 Riemann Integrability

9 Feb. 11:00

We are now working in $(\mathbb{R}, \mathcal{L}, m)$. Let's first revisit the definition of Riemann Integral. Let P be a partition of $[a, b]$ as

$$P = \{a = t_0 < t_1 < \ldots < t_k = b\}.$$

Then the *lower Riemann sum* of f using P is equal to L_P, which is defined as

$$L_P = \sum_{i=1}^{K} \left(\inf_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1}),$$

and the *upper Riemann sum* of f using P is equal to U_P, which is defined as

$$U_P = \sum_{i=1}^{K} \left(\sup_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1}).$$

Then we call

- *Lower Riemann integral* of $f = \underline{I} = \sup_P L_P$
- *Upper Riemann integral* of $f = \overline{I} = \inf_P U_P$

Definition 2.5.1 (Riemann (Darboux) integrable). A <u>bounded</u> function $f\colon [a, b] \to \mathbb{R}$ is called *Rie-*

mann (Darboux) integrable if $\underline{I} = \overline{I}$. If so, then we write

$$\underline{I} = \overline{I} = \int_a^b f(x)\,\mathrm{d}x.$$

Note. We see that

- If $P \subset P'$, then
$$L_P \leq L_{P'}, \quad U_{P'} \leq U_P.$$
- Recall that continuous functions on $[a,b]$ are Reimann integrable on $[a,b]$.

Theorem 2.5.1. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then

(1) If f is Reimann integrable, then f is Lebesgue measurable, thus Lebesgue integrable. Further,
$$\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathrm{d}m.$$

(2) If f is Reimann integrable $\Leftrightarrow$ f is continuous Lebesgue a.e.[a]

[a] Here, we mean that the set where f is discontinuous is a null set under Lebesgue measure.

Proof. There exists $P_1 \subset P_2 \subset \ldots$ such that $L_{P_n} \nearrow \underline{I}$ and $U_{P_n} \searrow \overline{I}$.

Note. Here, we took refinements of P_n if needed.

Now, define simple (step) functions

- $\phi_n = \sum_{i=1}^{K} \left(\inf_{[t_{i-1},t_i]} f \right) \mathbb{1}_{(t_{i-1},t_i]}$
- $\psi_n = \sum_{i=1}^{K} \left(\sup_{[t_{i-1},t_i]} f \right) \mathbb{1}_{(t_{i-1},t_i]}$

if $P_n = \{a = t_0 < t_1 < \ldots < t_K\}$. Let $\phi \coloneqq \sup_n \phi_n$ and $\psi \coloneqq \inf_n \psi_n$. We then see that ϕ, ψ are Lebesgue (Borel) measurable function.

Note. Note that

- $\exists M > 0$ such that $\forall_{n\in\mathbb{N}}\ |\phi_n|, |\psi_n| \leq M\mathbb{1}_{[a,b]}$
- $\int \phi_n\,\mathrm{d}m = L_{P_n}$, $\int \psi_n\,\mathrm{d}m = U_{P_n}$

By Theorem 2.3.1 and the fact that $M\mathbb{1}_{[a,b]} \in L^1(\mathbb{R}, \mathcal{L}, m)$, we have

$$\underline{I} = \lim_{n\to\infty} \int \phi_n\,\mathrm{d}m = \int \phi\,\mathrm{d}m, \quad \overline{I} = \lim_{n\to\infty} \int \psi_n\,\mathrm{d}m = \int \psi\,\mathrm{d}m.$$

Thus,

$$f \text{ is Riemann integrable} \Leftrightarrow \int \phi = \int \psi \Leftrightarrow \int (\psi - \phi) = 0 \Leftrightarrow \psi = \phi \text{ Lebesgue a.e.}$$

∎

2.6 Modes of Convergence

As we should already see, there are different *modes* of convergence. Let's formalize them.

Definition. Let $f_n, f\colon X \to \mathbb{C}$, and $S \subset X$. Then we have the following definitions.

Definition 2.6.1 (Pointwise convergence). $f_n \to f$ *pointwise* on S if

$$\underset{x\in S}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n\geq N}{\forall}\ |f_n(x) - f(x)| < \epsilon.$$

Definition 2.6.2 (Uniformly convergence). $f_n \to f$ *uniformly* on S if

$$\underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{x\in S}{\forall}\ \underset{n\geq N}{\forall}\ |f_n(x) - f(x)| < \epsilon.$$

Remark. We see that we can replace $\forall \epsilon > 0$ by $\forall k \in \mathbb{N}$ with ϵ changing to $\frac{1}{k}$.

Lemma 2.6.1. Let $B_{n,k}$ be

$$B_{n,k} \coloneqq \left\{x \in X \mid |f_n(x) - f(x)| < \frac{1}{k}\right\}.$$

Then

(1) $f_n \to f$ pointwise on S if and only if

$$S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$

(2) $f_n \to f$ uniformly on S if and only if $\exists N_1, N_2, \ldots \in \mathbb{N}$ such that

$$S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Proof. This essentially follows from Definition 2.6.1. ∎

Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space. Assuming that f_n, f are measurable functions, then we have the following.

Definition 2.6.3 (Converge almost everywhere). $f_n \to f$ *almost everywhere* means

$$\exists \text{ null set } E \text{ such that } f_n \to f \text{ pointwise on } E^c.$$

Definition 2.6.4 (Converge in L^1). $f_n \to f$ *in L^1* means

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

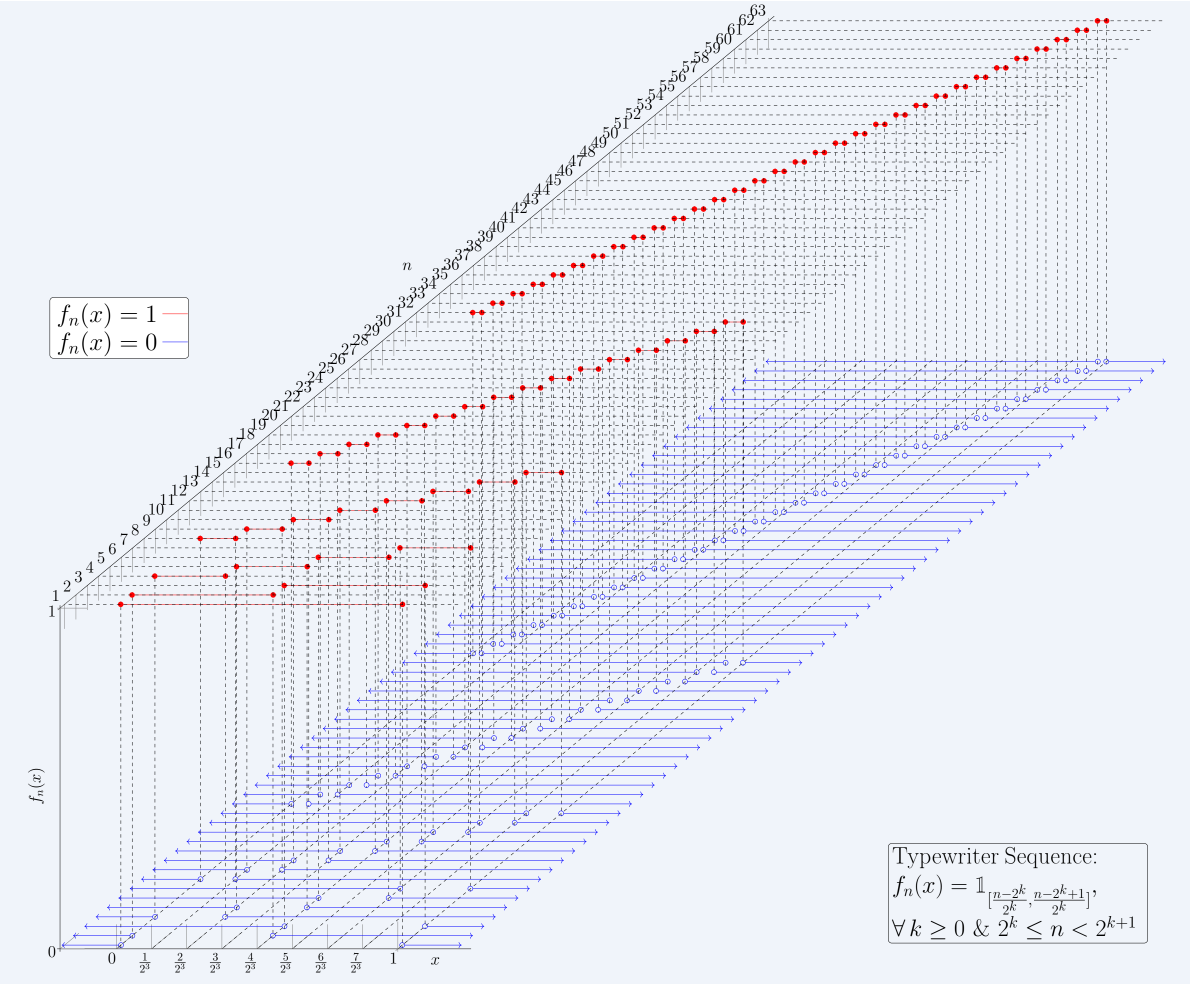
Example. Given $(\mathbb{R}, \mathcal{L}, m)$ and let $f = 0$. Consider the following functions.

(1) $f_n = \mathbb{1}_{(n,n+1)}$

(2) $f_n = \frac{1}{n}\mathbb{1}_{(0,n)}$

(3) $f_n = n\mathbb{1}_{(0,\frac{1}{n})}$

(4) **Typewriter functions**.



Proof. We see that different function sequences converge in different senses.

Exercise. Classify in what senses do (1), (2), (3) and the **type write** function converge.

⊛

Lecture 16: Modes of Convergence

Let's start with a proposition. 11 Feb. 11:00

Proposition 2.6.1 (Fast L^1 convergence leads to a.e. convergence)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n, f are all measurable functions on X. Then

$$\sum_{n=1}^{\infty} \|f_n - f\|_1 < \infty \Rightarrow f_n \to f \quad \text{a.e.}$$

Proof. Let

$$E := \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c = \{x \in X \mid f_n(x) \nrightarrow f(x)\}.$$

By Lemma 2.2.1, we see that

$$\underset{k}{\forall} \underset{N}{\forall} \mu\left(B_{n,k}^c\right) \leq k \int |f_n - f| \Rightarrow \underset{k}{\forall} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) \leq \sum_{n=N}^{\infty} k \|f_n - f\|_1 \to 0$$

as $N \to \infty$. Now, by continuity of measure from above,

$$\underset{k}{\forall}\ \mu\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = \lim_{N\to\infty}\mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \Rightarrow \mu(E) = 0$$

since $f_n \to f$ pointwise on E^c. ■

Corollary 2.6.1. Given $\{f_n\}_n$ such that $f_n \to f$ in L^1, there exists a subsequence $\{f_{n_j}\}_{n_j}$ where $f_{n_j} \to f$ a.e.

Proof. Since

$$\underset{j\in\mathbb{N}}{\forall}\ \underset{n_j\in\mathbb{N}}{\forall}\ \left\|f_{n_j} - f\right\|_1 \leq \frac{1}{j^2}.$$

Then,

$$\sum_{j=1}^{\infty}\left\|f_{n_j} - f\right\|_1 < \infty.$$

From Proposition 2.6.1, we have the desired result. ■

Definition 2.6.5 (Converge in measure)**.** Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. Then $f_n \to f$ *in measure* if

$$\underset{\epsilon>0}{\forall}\ \lim_{n\to\infty}\mu\left(\{x\in X \mid |f_n(x) - f(x)| \geq \epsilon\}\right) = 0.$$

Example. Let $f_n = n\mathbb{1}_{(0,\frac{1}{n})}$ and $f = 0$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, then $f_n \to f$ in measure.

Proof. We see that

$$\forall \epsilon > 0\ \ \{x \in X \mid |f_n(x) - f(x)| > \epsilon\} = \left(0, \frac{1}{n}\right),$$

$f_n \to 0$ in measure. (Recall that $f_n \nrightarrow 0$ in L^1) ⊛

Remark. We see that

$$\begin{tikzcd}
& f_n \to f \text{ in } L^1 \arrow[r, Rightarrow, shift left, "\text{Check}"] \arrow[dd, Leftrightarrow, "/" marking] & f_n \to f \text{ in measure} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[dd, Rightarrow, "\text{Read}"] \\
f_n \to f \text{ fast in } L^1 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
& f_n \to f \text{ a.e.} \arrow[uur, Leftrightarrow, "/" marking] & \exists f_{n_j} \to f \text{ a.e.}
\end{tikzcd}$$

Finally, we have the following.

Definition. Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$.

> **Definition 2.6.6** (Uniformly almost everywhere)**.** $f_n \to f$ *uniformly almost everywhere* if $\exists$null set F such that $f_n \to f$ uniformly on F^c.

> **Definition 2.6.7** (Almost uniformly)**.** $f_n \to f$ *almost uniformly* if $\forall \epsilon > 0\ \exists F \in \mathcal{A}$ such that $\mu(F) < \epsilon$, $f_n \to f$ uniformly on F^c.

Lemma 2.6.2. We have

$$f_n \to f \text{ uniformly on } S \Leftrightarrow \exists N_1, N_2, \ldots \in \mathbb{N} \; S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Theorem 2.6.1 (Egorov's Theorem)**.** Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. Suppose $\mu(X) < \infty$, then

$$f_n \to f \text{ a.e.} \Leftrightarrow f_n \to f \text{ almost uniformly}.$$

Proof. We prove two directions.

($\Leftarrow$)

DIY

($\Rightarrow$) Fix $\epsilon > 0$. We see that

$$f_n \to f \text{ a.e.} \Rightarrow \mu\left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \Rightarrow \forall_k \, \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0.$$

From continuity of measure from above and $\mu(X) < \infty$, we further have

$$\forall_k \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \Rightarrow \forall_k \, \exists_{N_k \in \mathbb{N}} \, \mu\left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c\right) < \frac{\epsilon}{2^k}.$$

Now, let

$$F \coloneqq \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c,$$

we see that $\mu(F) < \epsilon$, hence $f_n \to f$ uniformly. ■

Chapter 3

Product Measure

3.1 Product σ-algebra

Before we start, we see the setup.

- Product space.
$$X = \prod_{\alpha \in I} X_\alpha$$
where $x = (x_\alpha)_{\alpha \in I} \in X$.
- Coordinate map.
$$\pi_\alpha : X \to X_\alpha.$$

Now we see the formal definition.

Definition 3.1.1 (Product σ-algebra). Let $(X_\alpha, \mathcal{A}_\alpha)$ be a measurable space for all $\alpha \in I$. Then a *product σ-algebra* on $X = \prod_{\alpha \in I} X_\alpha$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha \in I} \pi_\alpha^{-1}(\mathcal{A}_\alpha) \right\rangle,$$

where $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \{\pi_\alpha^{-1}(E) \mid E \in \mathcal{A}_\alpha\}$.

Notation. We denote $I = \{1, \ldots, d\} \Rightarrow X = \prod_{i=1}^{d} X_i, x = (x_1, \ldots, x_d)$. Also,

$$\bigotimes_{i=1}^{d} \mathcal{A}_i = \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_d.$$

Lemma 3.1.1. If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \forall_i\, E_i \in \mathcal{A}_i \right\} \right\rangle.$$

Proof. If $E_i \in \mathcal{A}_i$, then $\pi_i^{-1}(E_i) = \prod_{j=1}^{\infty} E_j$, where $E_j = X$ for $j \neq i$. On the other hand, since

$$\prod_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} \pi_i^{-1}(E_i),$$

from Lemma 1.1.2, the result follows. ■

Lecture 17: Product Measure

We now see a lemma. 14 Feb. 11:00

Lemma 3.1.2. Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle$ for every $\alpha \in I$. Then

(1) $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$

(2) $\bigotimes_\alpha \mathcal{A}_\alpha = \left\langle \bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$

(3) If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \forall_i\, E_i \in \mathcal{E}_i \right\} \right\rangle$$

Proof. We prove this one by one.

(1) Note that for $f \colon Y \to Z$, and $\mathcal{B}$ be a σ-algebra on Z, then $f^{-1}(\mathcal{B})$ is also a σ-algebra.[a] Hence, π_α^{-1} is a σ-algebra on X, i.e.,

$$\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \overset{!}{\Rightarrow} \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha).$$

To show the other direction, let $\mathcal{M}$ being

$$\mathcal{M} = \left\{ B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle \right\}.$$

We now check

- $\mathcal{M}$ is a σ-algebra. Check (Easy)!
- $\mathcal{E}_\alpha \subset \mathcal{M}$. This is true by definition of $\mathcal{M}$.

[a]Since f^{-1}(

Thus, $\langle \mathcal{E}_\alpha \rangle = \mathcal{A}_\alpha \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha$, $E \in \mathcal{M}$, implying

$$\pi_\alpha^{-1}(E) \in \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle,$$

i.e., $\mathcal{A}_\alpha \subset \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$.

(2) DIY

(3) DIY

■

Theorem 3.1.1. Suppose $X_1, \ldots, X_d$ are metric spaces. Let $X = \prod_{i=1}^{d} X_i$ with product metric defined as

$$\rho(x, y) = \sum_{i=1}^{d} \rho_i(x_i, y_i).$$

Then,

(1) $\bigotimes_{i=1}^{d} \mathcal{B}(X_i) \subset \mathcal{B}(X)$

(2) If in addition, each X_i has a countable dense subset,

$$\bigoplus_{i=1}^{d} \mathcal{B}(X_i) = \mathcal{B}(X).$$

Proof. $\blacksquare$ DIY

Remark. We see that

- $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$
- let $f = u + iv \colon X \to \mathbb{C}$, and $\mathcal{A}$ be a σ-algebra on X. Then

$$\underset{E \in \mathcal{B}(\mathbb{R})}{\forall} u^{-1}(E), v^{-1}(E) \in \mathcal{A} \Leftrightarrow f^{-1}(F) \in \mathcal{A}, \forall\ F \in \mathcal{B}(\mathbb{C})$$

  with $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

We first focus on 2 dimensional case. Specifically, we think of our coordinate is x and y on $\mathbb{R}^2$.

Definition. Let X, Y be two sets, then we have the following.

Definition 3.1.2 (x-section, y-section for set)**.** For $E \subset X \times Y$,

$$E_x = \{y \in Y \mid (x, y) \in E\}, \quad E^y = \{x \in X \mid (x, y) \in E\},$$

where E_x is called the *x-section of E*, while E_y is called the *y-section of E*.

Definition 3.1.3 (x-section, y-section for function)**.** For $f \colon X \times Y \to \mathbb{C}$, define $f_x \colon Y \to \mathbb{C}$, $\quad f^y \colon X \to \mathbb{C}$ by

$$f_x(y) = f(x, y) = f^y(x),$$

where $f_x(y)$ is called the *x-section of f*, while $f_y(x)$ is called the *y-section of f*.

Example. We see that

$$(\mathbb{1}_E)_x = \mathbb{1}_{E_x}$$

and

$$(\mathbb{1}_E)^y = \mathbb{1}_{E^y}.$$

Proposition 3.1.1. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, then

(1) If $E \in \mathcal{A} \otimes \mathcal{B}$, then

$$\underset{x \in X}{\forall}\ \underset{y \in Y}{\forall}\ E_x \in \mathcal{B}, E^y \in \mathcal{A}.$$

(2) If $f \colon X \times Y \to \mathbb{C}$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then

$$\underset{x \in X}{\forall}\ \underset{y \in Y}{\forall}\ f_x \text{ is } \mathcal{B}\text{-measurable}, f^y \text{ is } \mathcal{A}\text{-measurable}.$$

Proof. We prove this one by one.

(1) Let $\mathcal{F} := \left\{ E \subset X \times Y \mid \underset{x \in X}{\forall}\ \underset{y \in Y}{\forall}\ E_x \in \mathcal{B}, E^y \in \mathcal{A} \right\}$, then

- $\mathcal{F}$ is a σ-algebra.
  - $\varnothing \in \mathcal{F}$.
  - $(E^c)_x = E_x^c$.
  - $\left(\bigcup_{j=1}^{\infty} E_j \right)_x = \bigcup_{j=1}^{\infty} (E_j)_x$.

  And the same is true for y.

- Let $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$, which is again easy to show from definition.

Hence, we see that $\langle R_0 \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{F}$.

(2) Since

$$(f_x)^{-1}(B) = (f^{-1}(B))_x$$

and

$$(f^y)^{-1}(B) = (f^{-1}(B))^y,$$

the result follows from 1.

■

3.2 Product Measures

We start with the definition.

Definition 3.2.1 (Rectangle). Given two measurable spaces, a *(measurable) rectangle* is $R = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, we let

$$\mathcal{R}_0 := \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$

and

$$\mathcal{R} := \left\{ \bigcup_{i=1}^{N} R_i \mid N \in \mathbb{N}, R_1, \ldots, R_N \text{ disjoint rectangles} \right\}.$$

Note. Whenever we're talking about rectangle, they're always measurable.

Lemma 3.2.1. $\mathcal{R}$ is an algebra, and

$$\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}.$$

Proof. Simply observe that

$$(A \times B)^c = (A^c \times Y) \cup (A \times B)$$

■ DIY

Lecture 18: Monotone Class

Let's start with a theorem. 16 Feb. 11:00

Theorem 3.2.1. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces. Then

(1) There is a measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

for every $A \in \mathcal{A}, B \in \mathcal{B}$.

(2) If μ, ν are σ-finite , then $\mu \times \nu$ is unique.

Proof. We prove this one by one.

(1) Define $\mu \colon \mathcal{R} \to [0, \infty]$ by $\mu(A \times B) = \mu(A)\nu(B)$, and extending linearly, we have

$$\pi(A \times B) = \mu(A)\nu(B),$$

hence

$$\pi\left(\coprod_{i=1}^{N} A_i \times B_i\right) = \sum_{i=1}^{n} \pi(A_i \times B_i).$$

We claim that π is a pre-measure. To show this, it's enough to check that $\pi(A \times B) = \sum_{n=1}^{\infty} \pi(A_n \times B_n)$ if $A \times B = \coprod_n A_n \times B_n$. Since $A_n \times B_n$ are disjoint, so

$$\mathbb{1}_{A\times B}(x, y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n\times B_n}(x, y).$$

Thus,

$$\mathbb{1}_A(x)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y).$$

Integrating with respect to x, and applying Theorem 1.3.1, we have

$$\int_X \mathbb{1}_A(x)\mathbb{1}_B(y)\,\mathrm{d}\mu(x) = \sum_{n=1}^{\infty} \int_x \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y)\,\mathrm{d}\mu(x),$$

which implies

$$\mu(A)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mu(A_n)\mathbb{1}_{B_n}(y)$$

for every y. We can then integrate again with respect to y and apply Theorem 1.3.1, we have

$$\int_Y \mu(A)\mathbb{1}_B(y)\,\mathrm{d}\nu(y) = \sum_{n=1}^{\infty} \int_Y \mu(A_n)\mathbb{1}_{B_n}(y)\,\mathrm{d}\nu(y),$$

which gives us

$$\mu(A)\nu(B) = \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n).$$

Hence, we see that μ is indeed a pre-measure, so Theorem 1.4.1 gives $\mu \times \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on $\mathcal{R}$.

(2) If μ, ν are σ-finite, then π is σ-finite on $\mathcal{R}$, then Theorem 1.4.2 applies. Moreover, we have that

$$(\mu \times \nu)(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \mid E \subset \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B}\right\}.$$

■

3.3 Monotone Class Lemma

Let's start with a definition.

Definition 3.3.1 (Monotone Class). If X is a set, and $C \subset \mathcal{P}(X)$, we say that C is a *monotone class* on X if

- C is closed under countable increasing unions.
- C is closed under countable decreasing intersections.

Example. Every σ-algebra is a monotone class.

Example. If C_α are (arbitrarily many) monotone classes on a set X, then $\bigcap_\alpha C_\alpha$ is a monotone class. Furthermore, if $\mathcal{E} \subset \mathcal{P}(X)$, there is a unique smallest monotone class containing $\mathcal{E}$, denoted by $\langle \mathcal{E} \rangle$, which follows the same idea as in Definition 1.1.3.

Theorem 3.3.1 (Monotone Class Lemma)**.** Suppose $\mathcal{A}_0$ is an algebra on X. Then $\langle \mathcal{A}_0 \rangle$[a] is the monotone class generated by $\mathcal{A}_0$.

[a] $\langle \mathcal{A}_0 \rangle$ is the σ-algebra generated by $\mathcal{A}_0$ by Definition 1.1.3.

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$ and let $\mathcal{C}$ be the monotone class generated by $\mathcal{A}_0$. Since $\mathcal{A}$ is a σ-algebra, it's a monotone class. Note that it contains $\mathcal{A}_0$, hence $\mathcal{A} \supset \mathcal{C}$.

To show $\mathcal{C} \supset \mathcal{A}$, it's enough to show that $\mathcal{C}$ is a σ-algebra. We check that

1. $\varnothing \in \mathcal{A}_0 \subseteq \mathcal{C}$.
2. Let $\mathcal{C}' = \{E \subset X \mid E^c \in \mathcal{C}\}$.
   - $\mathcal{C}'$ is a monotone class.
   - $\mathcal{A}_0 \subset \mathcal{C}'$ because if $E \in \mathcal{A}_0$, then $E^c \in \mathcal{A}_0$, so $E^c \in \mathcal{C}$, thus $E \in \mathcal{C}'$.

   We see that $\mathcal{C}' \subset \mathcal{C}'$, so $\mathcal{C}$ is closed under complements.
3. For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.
   - $\mathcal{D}(E) \subset \mathcal{C}$.
   - $\mathcal{D}(E)$ is a monotone class.
   - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. We see this by picking $F \in \mathcal{A}_0$, then $E \cup F \in \mathcal{A}_0 \supset \mathcal{C}$.

   Hence, $C = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.
4. Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\}$. That is $\mathcal{D} = \{E \in \mathcal{C} \mid E \cup F, \forall F \in \mathcal{C}\}$. Then we have
   - $A_0 \subset \mathcal{D}$ by 3.
   - $\mathcal{D}$ is a monotone class.
   - $\mathcal{D} \subset \mathcal{C}$ by definition.

   Thus, $\mathcal{D} = \mathcal{C}$, so if $E, F \in \mathcal{C}$, then $E \cup F \in \mathcal{C}$. This implies that $\mathcal{C}$ is closed under finite unions.
5. Now to show that $\mathcal{C}$ is closed under countable unions, let $E_1, E_2, \ldots \in \mathcal{C}$. We may then define

   $$F_N = \bigcup_{n=1}^{N} E_n \in \mathcal{C}.$$

   Then we see that $F_1 \subset F_2 \subset \ldots$, hence $\bigcup_N F_N \in \mathcal{C}$. But this simply implies

   $$\bigcup_N F_N = \bigcup_n E_n,$$

   so we're done.

■

Lecture 19: Fubini-Tonelli's Theorem

18 Feb. 11:00

As previously seen. If $E \in A \otimes B \Rightarrow E_x \in \mathcal{B}, E^y \in \mathcal{A}\ \forall x \in X, \forall y \in Y$. Note that the reverse is not true.

3.4 Fubini-Tonelli Theorem

We start with a theorem.

Theorem 3.4.1 (Tonelli's theorem for characteristic functions). Given $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure space. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$, then

(1) $\alpha(x) \coloneqq \nu(E_x) \colon X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.

(2) $\beta(x) \coloneqq \mu(E^y) \colon Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.

(3) $(\mu \times \nu)(E) = \int_X \nu(E_x) \,\mathrm{d}\mu(x) = \int_Y \mu(E^y) \,\mathrm{d}\nu(y)$.

Proof. We prove this in two cases.

1. Assume that μ, ν are finite measures. Let

$$C \coloneqq \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{ Conditions (1), (2), (3) hold}\}.$$

It's enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset C$. We further observe that from the Theorem 3.3.1 and the fact that $\mathcal{R}$ is an algebra, it's also enough to show that

- $\mathcal{R} \subset C$.
- C is a monotone class.

From condition (1),

$$\alpha(x) = \nu\left((A \times B)_x\right) = \begin{cases} \nu(B), & \text{if } x \in A; \\ 0, & \text{if } x \notin A \end{cases} = \nu(B) \mathbb{1}_A.$$

And from condition (2),

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

and

$$\int_X \nu((A \times B)_x) \,\mathrm{d}\mu(x) = \nu(B)\mu(A).$$

Let $E_n \in C$, $E_1 \subset E_2 \subset \ldots$. We need to show $E = \bigcup\limits_{n=1}^{\infty} E_n \in C$. We now see that

$$E_x = \bigcup_{n=1}^{\infty} (E_n)_x, (E_1)_x \subset (E_2)_x \subset \ldots \Rightarrow \alpha(x) = \nu(E_n)_x \overset{!}{=} \lim_{n \to \infty} \nu((E_n)_x) \ \forall x \in X.$$

This implies that (1) is proved.

For (3), we see that

$$(\mu \times \nu)(E) \overset{!}{=} \lim_{n \to \infty} (\mu \times \nu)(E_n) = \lim_{n \to \infty} \int_X \nu((E_n)_x) \,\mathrm{d}\mu(x) \overset{!}{=} \int_X \nu(E_x) \,\mathrm{d}\mu(x).$$

Now let $F_n \in C$, $F_1 \supset F_2 \supset \ldots$. We need to show that $F = \bigcap\limits_{n=1}^{\infty} F_n \in C$. Instead of using Theorem 2.2.1, we now want to use Theorem 2.3.1, which is applicable since $\mu(X), \nu(Y) < \infty$ by assumption.

2. Assume μ and ν are σ-finite measures. We then have a sequence $\{X_n \times Y_n\}$ of rectangles of with only finite measure. Now, just consider if $E \in \mathcal{A} \otimes \mathcal{B}$, 1. applies to $E \cap (X_n \times Y_n)$ for each n, with

$$X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n), \begin{cases} X_1 \subset X_2 \subset \ldots, & \mu(X_k) < \infty \\ Y_1 \subset Y_2 \subset \ldots, & \nu(Y_k) < \infty, \end{cases}$$

we have

$$\mu \times \nu(E \cap (X_n \times Y_n)) = \int \mathbb{1}_{X_n}(x).\nu(E_x \cap Y_n) \, \mathrm{d}\nu(x) = \int \mathbb{1}_{Y_n}(y)\mu(E^y \cap X_n) \, \mathrm{d}\nu(y).$$

By applying Theorem 2.2.1, the result follows.

■

Theorem 3.4.2 (Fubini-Tonelli's Theorem)**.** Given two σ-finite measure space $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$, we have the following.

Theorem (Tonelli's Theorem)**.** If $f \colon X \times Y \to [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then

(1) $g(x) \coloneqq \int_Y f(x, y) \, \mathrm{d}\nu(y)$, $X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.

(2) $h(x) \coloneqq \int_X f(x, y) \, \mathrm{d}\mu(x)$, $Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.

(3) We have

$$\int_{X \times Y} f \, \mathrm{d}(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \, \mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x, y) \, \mathrm{d}\mu(x) \right) \mathrm{d}\nu(y).$$

Theorem (Fubini's Theorem)**.** If $f \in L^1(X \times Y, \mu \times \nu)$, then

(1) $f_x \in L^1(Y, \nu)$ for μ-a.e. x, and $g(x) \in L^1(X, \mu)$ defined μ-a.e.

(2) $f^y \in L^1(X, \mu)$ for ν-a.e. y, and $h(x) \in L^1(Y, \nu)$ defined μ-a.e.

(3) The iterated integral formulas hold. Namely, we have

$$\int_{X \times Y} f \, \mathrm{d}(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \, \mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x, y) \, \mathrm{d}\mu(x) \right) \mathrm{d}\nu(y).$$

Proof. Read [FF99]. ■

Remark. The Fubini and Tonelli's theorem are frequently used in tandem. Say that if one want to reverse the order of integration in a double integral $\iint f \, \mathrm{d}\mu \, \mathrm{d}\nu$. We first verify that $\int |f| \, \mathrm{d}(\mu \times \nu) < \infty$ by using Tonelli's theorem to evaluate this integral as an iterated integral. Then, we apply Fubini theorem to conclude that

$$\iint f \, \mathrm{d}\mu \, \mathrm{d}\nu = \iint f \, \mathrm{d}\nu \, \mathrm{d}\mu.$$

Lecture 20: Lebesgue Measure on $\mathbb{R}^d$

3.5 Lebesgue Measure on $\mathbb{R}^d$

21 Feb. 11:00

Example. $(\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m)$ is not complete.

Proof.

- Let $A \in \mathcal{L}$, $A \neq \varnothing$, $m(A) = 0$.
- Let $B \subset [0, 1]$, $B \notin \mathcal{L}$ (Vital set for example).
- Let $E = A \times B$, $F = A \times [0, 1]$.

We see that $E \subset F$, $F \in \mathcal{L} \otimes \mathcal{L}$, $(m \times m)(F) = m(A)m([0, 1]) = 0$, i.e., F is a null set. But E is **not** $\mathcal{L} \otimes \mathcal{L}$-measurable-function since otherwise, its sections are all measurable. ⊛

Definition 3.5.1. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the *completion* of

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \ldots \times m),$$

which is same as the *completion* of

$$(\mathbb{R}^d, \mathcal{L} \otimes \ldots \otimes \mathcal{L}, m \times \ldots m).$$

Remark. We see that

$$\mathcal{L}^d \supsetneq \mathcal{L} \otimes \ldots \otimes \mathcal{L} = \left\langle \left\{ \prod_{i=1}^d E_i \mid E_i \in \mathcal{L} \right\} \right\rangle.$$

Definition 3.5.2 (Rectangle in $\mathbb{R}^d$)**.** A *rectangle in* $\mathbb{R}^d$ is $R = \prod_{i=1}^d E_i$ where $E_i \in \mathcal{B}(\mathbb{R})$.

Definition 3.5.3 (Lebesgue measure in $\mathbb{R}^d$)**.** We let the *Lebesgue measure in* $\mathbb{R}^d$, denoted as m^d, defined as

$$m^d(E) := \inf \left\{ \sum_{k=1}^\infty m^d(R_k) \mid E \subset \bigcup_{k=1}^\infty R_k, R_k \text{ is rectangles} \right\}.$$

Theorem 3.5.1. Let $E \subset \mathcal{L}^d$. Then

(1) $m^d(E) = \inf \left\{ m^d(0) \mid \text{open } O \supset E \right\} = \sup \left\{ m^d(K) \mid \text{compact } K \subset E \right\}$.

(2) $E = A_1 \cup N_1 = A_2 \setminus N_2$, where A_1 is F_σ, A_2 is G_δ , and N_i are null.

(3) If $m^d(E) < \infty$, $\forall \epsilon > 0$, $\exists R_1, \ldots, R_m$ rectangles whose sides are intervals such that

$$m^d \left(E \triangle \left(\bigcup_{i=1}^m R_i \right) \right) < \epsilon.$$

Proof. Similar to $d = 1$ case. ■

Theorem 3.5.2. Integrable step functions and $C_c(\mathbb{R}^d)$, the collection of continuous functions, are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$

Proof. See [FF99]. ■

Theorem 3.5.3. Lebesgue measure in $\mathbb{R}^d$ is translation-invariant.

Proof. See [FF99]. ■

Theorem 3.5.4 (Effect of linear transformation on Lebesgue measure)**.** If $T \in \mathrm{GL}(\mathbb{R}^d)$, $e \in \mathcal{L}^d$, then $T(E)$ is measurable and

$$m(T(E)) = |\det T| \cdot m(E).$$

Proof. See [FF99]. ■

Chapter 4

Differentiation on Euclidean Space

As previously seen. Given $f\colon [a,b] \to \mathbb{R}$, there are two versions of **fundamental theorem of calculus**:

(1)
$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a).$$

(2)
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x),$$

which follows from
$$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t = f(x) = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} f(t)\,\mathrm{d}t.$$

Remark. We see that
$$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} \left(f(t) - f(x)\right)\,\mathrm{d}t = 0 = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} \left(f(t) - f(x)\right)\,\mathrm{d}t,$$
where we have
$$f(x) = \frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t.$$
This generalized to $f\colon \mathbb{R}^d \to \mathbb{R}$, namely
$$\lim_{r\to 0^+}\frac{1}{\mathrm{vol}\left(B(x,r)\right)}\int_{B(x,r)} \left(f(t) - f(x)\right)\underbrace{\mathrm{d}t}_{\mathbb{R}^d} \overset{?}{=} 0.$$

4.1 Hardy-Littlewood Maximal Function

We first see our notation.

Notation. Given a(n) (open) ball in $\mathbb{R}^d$, $B = B(a,r)$, denote $cB = B(a,cr)$ for $c > 0$.

Lemma 4.1.1 (Vitali-type covering lemma)**.** Let $B_1, \ldots, B_k$ be a finite collection of open balls in $\mathbb{R}^d$. Then there exists a sub-collection $B_1', \ldots, B_m'$ of <u>disjoint</u> open balls such that
$$\bigcup_{i=1}^{m}(3B_j') \supset \bigcup_{i=1}^{k} B_i.$$

Proof. Greedy Algorithm. ■

Lecture 21: Hardy-Littlewood Maximal Function and Inequality

25 Feb. 11:00

Notation. We let
$$\int_E f \,\mathrm{d}m = \int_E f(x)\,\mathrm{d}x.$$

The problem we're working on is
$$\frac{1}{m(B(w,r))}\int_{B(x,r)} f(y)\,\mathrm{d}y \overset{r\to 0}{\underset{?}{\longrightarrow}} f(x).$$

Definition 4.1.1 (Locally integrable). Given $f\colon \mathbb{R}^d \to \mathbb{C}$ be Lebesgue measurable function. Then we say f is *locally integrable* if for every compact $K \subset \mathbb{R}^d$,
$$\int_K |f| \,\mathrm{d}m < \infty.$$
In this case, we write $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Definition 4.1.2 (Hardy-Littlewood maximal function). Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, the *Hardy-Littlewood maximal function* for f is defined as
$$\mathrm{H}f(x) := \sup\left\{\mathrm{A}_r(x) \mid r > 0\right\},$$
where
$$\mathrm{A}_r(x) := \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)| \,\mathrm{d}y.$$

Note. We note that $\mathrm{A}_r(\cdot)$ means *averaging function* over an open ball with radius being r.

Lemma 4.1.2. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then

(1) $\mathrm{A}_r(x)$ is jointly continuous for $(x,r) \in \mathbb{R}^d \times (0,\infty)$.

(2) $\mathrm{H}f(x)$ is Borel measurable.

Proof. We outline the proof.

(1) Let $(x,r) \to (x^*,r^*) \Rightarrow \mathrm{A}_r(x) \to \mathrm{A}_{r^*}(x^*)$. Let (x_n,r_n) be any sequence which converges to x^*, r^*, then we consider $\lim\limits_{n\to\infty} \mathrm{A}_{r_n}(x_n)$ and we can calculate
$$\int \underbrace{|f(y)|\, \mathbb{1}_{B(x_n,r_n)}(y)}_{:=h_n(y)},$$
then we apply Theorem 2.3.1 to h_n.

(2) Observe that
$$(\mathrm{H}f)^{-1}\underbrace{((a,\infty))}_{\text{open}} = \bigcup_{r>0} \mathrm{A}_r^{-1}\left((a,\infty)\right)$$
is open, since $\mathrm{A}_r^{-1}\left((a,\infty)\right)$ is open from the 1. Note that the equality comes from the fact that $\mathrm{H}f = \sup_r \mathrm{A}_r$.

■

Theorem 4.1.1 (Hardy-Littlewood maximal inequality)**.** There exists $C_d > 0$ such that for every $f \in L^1(\mathbb{R}^d)$,

$$\underset{\alpha>0}{\forall}\ m\left(\left\{x \in \mathbb{R}^d \mid \mathrm{H}f(x) > \alpha\right\}\right) \leq \frac{C_d}{\alpha} \int |f(x)| \,\mathrm{d}x.$$

Proof. We first fix $f \in L^1$ and $\alpha > 0$. We define

$$E \coloneqq \{x \mid \mathrm{H}f(x) > \alpha\},$$

which is a Borel measurable set by Lemma 4.1.2. Then

$$x \in E \Rightarrow \underset{r_x>0}{\exists}\ \mathrm{A}_{r_x}(x) > \alpha \Rightarrow m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x,r_x)} |f(y)| \,\mathrm{d}y.$$

From inner regularity, we have

$$m(E) = \sup\{m(K) \mid \text{compact } K \subset E\}.$$

Let $K \subset E$ be compact, then

$$K \subset \bigcup_{x\in K} B(x, r_x) \overset{K \text{ compact}}{\Rightarrow} K \subset \bigcup_{i=1}^{N} B_i \overset{!}{\Rightarrow} K \subset \bigcup_{i=1}^{m} \{3B_j'\}.$$

From here, we further have

$$m(K) \leq \sum_{i=1}^{m} m(3B_j') = 3^d \sum_{j=1}^{m} m(B_j') \leq \frac{3^d}{\alpha} \sum_{j=1}^{m} \int_{B_j'} |f(y)| \,\mathrm{d}y.$$

Now, since $B_i', \ldots, B_m'$ are disjoint, hence we finally have

$$m(K) \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \,\mathrm{d}y.$$

■

Lecture 22: Lebesgue Differentiation Theorem

We should compare the Hardy-Littlewood maximal inequality to Markov's inequality. Namely, there exists $C_d > 0$ (can take 3^d) such that for all $f \in L^1(\mathbb{R}^d)$, $\alpha > 0$, we have 07 Mar. 11:00

$$\begin{cases} m(\{x \mid \mathrm{H}f(x) > \alpha\}) \leq \dfrac{C_d}{\alpha} \displaystyle\int |f|; \\ m(\{x \mid |f(x)| > \alpha\}) \leq \dfrac{1}{\alpha} \displaystyle\int |f|. \end{cases}$$

4.2 Lebesgue Differentiation Theorem

We start with a theorem!

Theorem 4.2.1 (Lebesgue Differentiation Theorem)**.** Let $f \in L^1$, then

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \,\mathrm{d}y = 0$$

for a.e. x.

Proof. The result holds for $f \in C_c(\mathbb{R}^d)$, namely for those continuous functions with **compact** support. This is because for any $\epsilon > 0$, if r is small and $|f(y) - f(x)| < \epsilon$, then

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y < \epsilon.$$

Now, let $f \in L^1(\mathbb{R}^d)$ and fix $\epsilon > 0$. By density, there exists $g \in C_c(\mathbb{R}^d)$ with $\|f - g\|_1 < \epsilon$. We then have

$$\int_{B_r(x)} |f(y) - f(x)| \, \mathrm{d}y \leq \int_{B_r(x)} |f(y) - g(y)| \, \mathrm{d}y + \int_{B_r(x)} |g(y) - g(x)| \, \mathrm{d}y + \int_{B_r(x)} |g(x) - f(x)| \, \mathrm{d}y.$$

Note. We use $B_r(x)$ above to denote $B(x,r)$ for spacing reason only. Nothing tricky here.

Divide all of these by $m(B(x,r))$, and take $\limsup_{r \to \infty}$, we need to understand the error terms, namely

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - g(x)| \, \mathrm{d}y = |g(x) - f(x)|$$

and

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g(y)| \, \mathrm{d}y \leq (\mathrm{H}(f-g))(x).$$

We define

$$Q(x) \coloneqq \limsup_{r \to \infty} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y.$$

We want to show $m(\{x \in X \mid Q(x) > 0\}) = 0$. Let $E_\alpha = \{x \in X \mid Q(x) > \alpha\}$. It is enough to show $m(E_\alpha) = 0$ for all $\alpha > 0$ because $\{x \in X \mid Q(x) > 0\} = \bigcup_n E_{\frac{1}{n}}$. We know by the above that

$$Q(x) \leq (\mathrm{H}(f-g))(x) + 0 + |g(x) - f(x)| \, .$$

Therefore,

$$E_\alpha \subset \{x \in X \mid (\mathrm{H}(f-g))(x) > \alpha/2\} \cup \{x \in X \mid |g(x) - f(x)| > \alpha/2\}.$$

By the Hardy-Littlewood maximal inequality and Markov's inequality, we have

$$\begin{cases} m(\{x \mid (\mathrm{H}(f-g))(x) > \alpha/2\}) \leq \dfrac{2C_d}{\alpha} \displaystyle\int |f-g| \, ; \\ m(\{x \mid |g(x) - f(x)| > \alpha/2\}) \leq \dfrac{2}{\alpha} \displaystyle\int |f-g| \, . \end{cases}$$

Thus,

$$0 \leq m(E_\alpha) \leq \frac{2C_d}{\alpha} \|f-g\|_1 + \frac{2}{\alpha} \|f-g\|_1 \leq \frac{2(C_d+1)}{\alpha} \epsilon.$$

Taking $\epsilon \to 0$, $m(E_\alpha)$ does not depend on ϵ and g, hence $m(E_\alpha) = 0$. ∎

Corollary 4.2.1. Theorem 4.2.1 also holds for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Proof. Using the fact that m^d is σ-finite, and apply Theorem 4.2.1. Specifically, partition $\mathbb{R}^d$ into countably many compact sets K_i and apply Theorem 4.2.1 to $f \mathbb{1}_{K_i}$ for all i. ∎

Corollary 4.2.2. For $f \in L^1_{\text{loc}}$, we have

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, \mathrm{d}y = f(x)$$

for a.e. x.

Proof. . Use that

$$f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x) \, \mathrm{d}y$$

and the triangle inequality. ■

DIY

Definition 4.2.1 (Lebesgue point)**.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the point $x \in \mathbb{R}^d$ is called a *Lebesgue point of* f if

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y = 0.$$

Remark. Corollary 4.2.1 tells us that almost all points in $\mathbb{R}^d$ in $\mathbb{R}^d$ are Lebesgue points for f.

Definition 4.2.2 (Shrink nicely)**.** We say that $\{E_r\}_{r>0}$ *shrinks nicely* to x as $r \to 0$ if $E_r \subset B(x,r)$ and

$$\underset{c>0}{\exists} \ c \cdot m(B(x,r)) \leq m(E_r).$$

Corollary 4.2.3. Suppose E_r shrink nicely to 0, and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, and x is a Lebesgue point. Then

$$\begin{cases} \lim\limits_{r \to 0} \dfrac{1}{m(E_r)} \displaystyle\int_{E_r + x} |f(y) - f(x)| \, \mathrm{d}y = 0; \\ \lim\limits_{r \to 0} \dfrac{1}{m(E_r)} \displaystyle\int_{E_r + x} |f(y)| \, \mathrm{d}y = f(x). \end{cases}$$

Corollary 4.2.4. If $f \in L^1_{\text{loc}}(\mathbb{R})$, then $F(x) = \int_0^x f(y) \, \mathrm{d}y$ is differentiable and $F'(x) = f(x)$ almost everywhere.

Chapter 5

Normed Vector Space

Lecture 23: Metric, normed and L^p Spaces

5.1 Metric Spaces and Normed Spaces

09 Mar. 11:00

We have seen the definition of a norm before, now we formally introduce the concept of *metric*.

Definition 5.1.1 (Metric). Let Y be a set, a function $\rho\colon Y \times Y \to [0, \infty)$ is a *metric* on Y if

- $\rho(x, y) = \rho(y, x)$ for all $x, y \in Y$.
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in Y$.
- $\rho(x, y) = 0$ if and only if $x = y$.

Note. The following make sense in a metric space.

(1) Open/closed balls.

(2) Open/closed sets.

(3) Convergence sequences ($x_n \to x$ with respect to ρ if and only if $\lim\limits_{n\to\infty} \rho(x_n, x) = 0$).

(4) Continuous functions.

Example. We have the following metric spaces.

(1) $\mathbb{Q}$ with $\rho(x, y) = |x - y|$.

(2) $\mathbb{R}$ with $\rho(x, y) = |x - y|$.

(3) $\mathbb{R}_+$ with $\rho(x, y) = |\ln(y/x)|$.

(4) $\mathbb{R}^d$ with

$$\rho_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p}$$

and

$$\rho_\infty(x, y) = \max_{1\leq i\leq d} |x_i - y_i| .$$

These all give the same open sets, hence they are topologically equivalent.

(5) $C([0, 1])$ with

$$\rho_p(f, g) = \left(\int_0^1 |f - g|^p\right)^{1/p}$$

and

$$\rho_\infty(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|.$$

(6) Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Let Y be the set of measurable functions on X, then

$$\rho(f,g) = \int \min\{|f(x) - g(x)|, 1\} \,\mathrm{d}\mu(x)$$

is a metric and $f_n \to f$ in ρ if and only if $f_n \to f$ in measure.

Let V be a vector space over scalar field $K = \mathbb{R}$ or $K = \mathbb{C}$.

As previously seen (Metric induced by a norm)**.** Recall the definition of seminorm and norm. We see that a norm induces a metric

$$\rho(v,w) := \|v - w\|,$$

and we have

$$v_n \to v \Leftrightarrow \lim_{n\to\infty} \|v_n - v\| = 0.$$

Example. We first see some common examples of normed vector space.

(1) $L^1(X, \mathcal{A}, \mu)$ with $\|f\|_1 := \int |f| \,\mathrm{d}\mu$.

(2) $C([0,1])$ with $\|f\|_1 := \int_0^1 |f(x)| \,\mathrm{d}x$, $\|f\|_\infty := \max_{0\le x\le 1} |f(x)|$.

(3) For $\mathbb{R}^d$ and $0 < p < \infty$, we have

$$\|x\|_p := \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}, \qquad \|x\|_\infty := \max_{1\le i\le d} |x_i|.$$

5.2 L^p Space

It turns out that we can generalize L^1 into L^p.

Definition 5.2.1 (L^p space)**.** Given a measure space $(X, \mathcal{A}, \mu)$ and a measurable function f and p such that $0 < p < \infty$, we define a seminorm $\|\cdot\|_p$ such that

$$\|f\|_p := \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p},$$

which induces the so-called L^p *space* $L^p(X, \mathcal{A}, \mu)$, where

$$L^p(X, \mathcal{A}, \mu) := \left\{ f \mid \|f\|_p < \infty \right\}.$$

Remark. Note that $\|\cdot\|_p$ is only a seminorm. But if we identity functions which are equal almost everywhere, then it's indeed a norm.

Example. $(\mathbb{R}, \mathcal{L}, m)$ has $f(x) = x^{-\alpha}\mathbb{1}_{(1,\infty)}(x) \in L^p$ if and only if $\alpha p > 1$. In contrast, $g(X) = x^{-\beta}\mathbb{1}_{(0,1)}(x) \in L^p$ if and only if $\beta p < 1$.

Similar to Definition 5.2.1, we have the following.

Definition 5.2.2 (ℓ^p space)**.** If $(X, \mathcal{P}(X), \nu)$ is equipped with the counting measure, then we say it's an ℓ^p *space* such that

$$\ell^p(X) := L^p(X, \mathcal{P}(X), \nu).$$

Remark. We are interested in $\ell^p(\mathbb{N})$ in particular. We have

$$\ell^p := \ell^p(\mathbb{N}) = \left\{ a = (a_1, a_2, \dots) \mid \|a\|_p = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} < \infty \right\}.$$

Lemma 5.2.1. $L^p(X, \mathcal{A}, \nu)$ is a vector space for all $p \in (0, \infty)$.

Proof. We verify the following.

- $c \cdot f \in L^p(X, \mathcal{A}, \mu)$ for $c \in \mathbb{R}$. Indeed, since

$$\|cf\|_p = \left(\int |cf|^p \,\mathrm{d}\mu \right)^{1/p} = |c| \, \|f\|_p < \infty \Leftrightarrow \|f\|_p < \infty,$$

  which implies $c \cdot f \in L^p(X, \mathcal{A}, \mu)$.

- $f + g \in L^p(X, \mathcal{A}, \mu)$. Indeed, since for any real numbers α, β, we have

$$(\alpha + \beta)^p \leq (2 \cdot \max\{|\alpha|, |\beta|\})^p = 2^p \cdot \max\{|\alpha|^p, |\beta|^p\} \leq 2^p \left(|\alpha|^p + |\beta|^p \right),$$

  which implies that for $f, g \in L^p(X, \mathcal{A}, \mu)$, we have

$$\|f + g\|_p < \infty \Leftrightarrow \|f + g\|_p^p = \int |f + g|^p \,\mathrm{d}\mu \leq 2^p \int (|f|^p + |g|^p) < \infty.$$

  This further implies

$$\|f + g\|_p < \infty \Leftrightarrow \|f\|_p, \|g\|_p < \infty,$$

  which is what we want.

■

We see that in the above derivation, it doesn't give us the triangle inequality, namely

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

hence we need some new results.

Theorem 5.2.1 (Hölder's inequality)**.** Let $1 < p < \infty$, and let $q := p/(p-1)$ so that $1/p + 1/p = 1$. Then we have

$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. We prove this in steps.

Claim. We have

$$t \leq \frac{t^p}{p} + 1 - \frac{1}{p} = \frac{t^p}{p} + \frac{1}{q}$$

for all $t \geq 0$.

Proof. By taking $F(t) := t - t^p/p$ and $t \geq 0$, we see that the maximum of F implies the above inequality. ⊛

Claim (Young's Inequality). We have

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

for $\alpha, \beta > 0$.[a]

[a] https://en.wikipedia.org/wiki/Young's_inequality_for_products

Proof. This follows by taking $t := \alpha/\beta^{q-1}$ in the first inequality we obtained. ⊛

Then, without loss of generality, we can assume that $0 < \|f\|_p, \|g\|_q < \infty$. Now, consider $F(x) = f(x)/\|f\|_p$, $G(x) = g(x)/\|g\|_q$. We know that $\|F\|_p = 1 = \|G\|_q$. Then by Young's Inequality, we have

$$\int |F(x)G(x)| \,\mathrm{d}\mu \leq \int \frac{|F(x)|^p}{p} + \frac{|G(x)|^q}{q} \Rightarrow \frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

which implies our desired result. ∎

Example. For $p = q = 2$, $X = \{1, \dots, d\}$ with μ being the counting measure, then for any $x, y \in \mathbb{R}^d$, we have

$$\sum_{i=1}^d |x_i y_i| \leq \sqrt{\sum_{i=1}^d x_i^2}\sqrt{\sum_{i=1}^d y_i^2}$$

We now see how we can obtain the desired triangle inequality.

Theorem 5.2.2 (Minkowski's Inequality). Let $1 \leq p < \infty$, then for $f, g \in L^p$,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

Proof. For $p = 1$, it's easy since it's just triangle inequality. Now, we assume that $1 < p < \infty$, and we may assume also that $\|f+g\| \neq 0$ without loss of generality. Then

$$\begin{aligned}\int |f(x)+g(x)|^p &\leq \int |f(x)+g(x)|^{p-1}\left(|f(x)|+|g(x)|\right)\\ &\leq \left(\int |f+g|^{(p-1)q}\right)^{1/q}\left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right]\\ &\leq \left(\int |f+g|^p\right)^{1/q}\left(\|f\|_p + \|g\|_p\right).\end{aligned}$$

We then see that

$$\underbrace{\left(|f(x)+g(x)|^p\right)^{1-1/q}}_{(|f(x)+g(x)|^p)^{1/p}} \leq \|f\|_p + \|g\|_p ,$$

which is just $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. ∎

Lecture 24: Embedding L^p Space

11 Mar. 11:00

Definition 5.2.3 (Essential supremum). For a measurable function f on $(X, \mathcal{A}, \mu)$, we define

$$S := \{\alpha \geq 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\} = \{\alpha \geq 0 \mid |f(x)| \leq \alpha \text{ a.e.}\}.$$

Then, we say that the *essential supremum of* f, denoted as $\|f\|_\infty$, is defined as

$$\|f\|_\infty := \begin{cases} \inf S, & \text{if } S \neq \varnothing; \\ \infty, & \text{if } S = \varnothing. \end{cases}$$

Definition 5.2.4 (L^∞ space)**.** Let $L^\infty(X, \mathcal{A}, \mu)$ be

$$L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\}.$$

Definition 5.2.5 (ℓ^∞ space)**.** We let ℓ^∞ be defined as

$$\ell^\infty = L^\infty(\mathcal{N}, \mathcal{P}(\mathcal{N}), \nu),$$

where ν is the counting measure.

Example. Consider $(\mathbb{R}, \mathcal{L}, m)$. Then

$$f(x) = \frac{1}{x}\mathbb{1}_{(0,\infty)}(x) \notin L^\infty;$$
$$g(x) = x\mathbb{1}_{\mathbb{Q}}(x) + \frac{1}{1+x^2} \in L^\infty.$$

If f is continuous on $(\mathbb{R}, \mathcal{L}, m)$, then $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. For $a \in \ell^\infty$, we have $\|a\|_\infty = \sup_{i\in\mathbb{N}} |a_i|$, and sequences in ℓ^∞ are exactly the bounded sequences.

Lemma 5.2.2. We have the following.

(1) Suppose $f \in L^\infty(X, \mathcal{A}, \mu)$. Then,

$$\begin{cases} \mu(\{x \mid |f(x)| > \alpha\}) = 0, & \text{if } \alpha \geq \|f\|_\infty; \\ \mu(\{x \mid |f(x)| > \alpha\}) > 0, & \text{if } \alpha < \|f\|_\infty. \end{cases}$$

(2) $|f(x)| \leq \|f\|_\infty$ almost everywhere.

(3) $f \in L^\infty$ if and only if there exists a bounded measurable function g such that $f = g$ almost everywhere.

Proof. ■

DIY

Theorem 5.2.3. We have the following.

(1) $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.

(2) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(3) $f_n \to f$ in L^∞ if and only if $f_n \to f$ uniformly almost everywhere.

Proof. We'll do one implication in (3). Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$. Let $A = \bigcup_n A_n$, we see that $\mu(A) = 0$ as well.

DIY (1) and (2)

For $x \in A^c$ and for every n, we have

$$|f_n(x) - f(x)| \leq \|f_n - f\|_\infty.$$

Given $\epsilon > 0$, there is an N so that $\|f_n - f\| < \epsilon$ for all $n \geq N$. But then for all $x \in A^c$, $|f_n(x) - f(x)| < \epsilon$ as well. ■

Remark. The motivation for 1. is that
$$\frac{1}{1}+\frac{1}{\infty}=1,$$
and we want to have the similar result as in Theorem 5.2.1.

Proposition 5.2.1. We have the following.

(1) For $1 \leq p < \infty$, the collection of simple functions with finite measure support is dense in $L^p(X, \mathcal{A}, \mu)$.

(2) For $1 \leq p < \infty$, the collection of step functions with finite measure support is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$, so is $C_c(\mathbb{R})$.

(3) For $p = \infty$, the collection of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.

Proof. ■ DIY

Remark. Note that $C_c(\mathbb{R})$ is **not** dense in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

5.3 Embedding Properties of L^p Spaces

Definition 5.3.1 (Equivalent norm). Two norms $\|\cdot\|, \|\cdot\|'$ on V are *equivalent* if there exists $c_1, c_2 > 0$, such that
$$c_1 \|v\| \leq \|v\|' \leq c_2 \|v\|$$
for all $v \in V$.

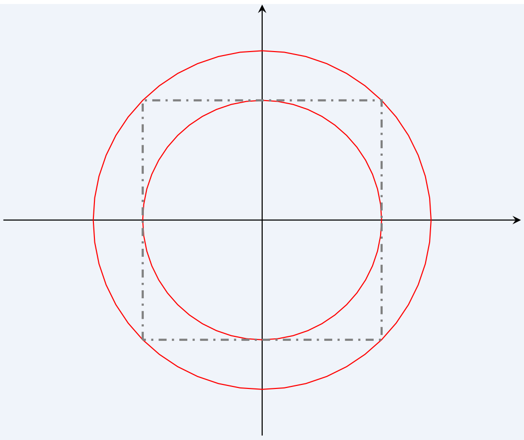
Note. We see that

(1) These norms gives the same topological properties (open sets, closed sets, convergence, etc.).

(2) Definition 5.3.1 is an equivalence relation on norms.

Example. For $\mathbb{R}^d$ we have the norms $\|\cdot\|_p$ for $1 \leq p \leq \infty$. All of these are equivalent. We see that for $1 \leq p < \infty$,
$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p} \leq (d \|x\|_\infty^p)^{1/p} = d^{1/p} \|x\|_\infty .$$
Also,
$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p} \geq (\|x\|_\infty^p)^{1/p} = \|x\|_\infty .$$
Thus, $\|\cdot\|_p$ is equivalent to $\|\cdot\|_\infty$ for every $1 \leq p < \infty$, and transitivity gives that they are all equivalent.

Another way of thinking of this, by assuming $v \neq 0$, and scaling by some t, we may assume v lies on the unit circle in one of the norms. Then we are squeezing a unit circle in $\|\cdot\|'$ between two circles of radius c_1, c_2 in $\|\cdot\|$. In picture, we have to show that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent, we have



since the circles in $\|\cdot\|_\infty$ are squares.

Example. For $1 \leq p, q \leq \infty$, we have $L^p(\mathbb{R}, m)$-norm and $L^q(\mathbb{R}, m)$-norm are not equivalent, even worse, we have that

$$L^p(\mathbb{R}, m) \nsubseteq L^1(\mathbb{R}, m), \quad L^p(\mathbb{R}, m) \nsupseteq L^1(\mathbb{R}, m).$$

Lecture 25: Banach Spaces

14 Mar. 11:00

Proposition 5.3.1. Suppose $\mu(X) < \infty$, then for every $0 < p < q \leq \infty$, $L^q \subseteq L^p$.

Proof. Suppose $q < \infty$, then

$$\int |f|^p \leq \left(\int (|f|^p)^{q/p} \right)^{p/q} \left(\int 1^{q/(q-p)} \right)^{1-p/q} = \left(\int |f|^q \right)^{p/q} \mu(x)^{1-p/q} < \infty$$

where we split $\int |f|^p$ into $\int |f|^p \cdot 1$. From Hölder's inequality with $q/p > 1$, we have

$$\|f\|_p \leq \|f\|_q \, \mu(X)^{1/p - 1/q} < \infty.$$

The case that $q = \infty$ is left as an exercise. ■

DIY

Proposition 5.3.2. If $0 < p < q \leq \infty$, then $\ell^p \subseteq \ell^q$.

Proof. We consider two cases.

- When $q = \infty$, we have

$$\|a\|_\infty^p = \left(\sup_i |a_i| \right)^p = \sup_i |a_i|^p \leq \sum_{i=1}^{\infty} |a_i|^p .$$

  Thus, $\|a\|_\infty \leq \|a\|_p$.

- When $q < \infty$, we see that

$$\sum_{i=1}^{\infty} |a_i|^q = \sum_{i=1}^{\infty} |a_i|^p \cdot |a_i|^{q-p} \leq \|a\|_\infty^{q-p} \sum_{i=1}^{\infty} |a_i|^p \leq \|a\|_p^{q-p} \cdot \|a\|_p^p = \|a\|_p^q .$$

  Therefore,

$$\|a\|_q \leq \|a\|_p .$$

■

Proposition 5.3.3. For all $0 < p < q < r \leq \infty$, $L^p \cap L^r \subseteq L^q$.

Proof. ■ DIY

5.4 Banach Spaces

Let's start with a definition.

Definition 5.4.1 (Cauchy sequence)**.** Let Y, ρ be a metric space. We call x_n a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\rho(x_n, x_m) < \epsilon$.

Note. Convergent sequence are Cauchy.

Definition 5.4.2 (Complete)**.** A metric space (Y, ρ) is called *complete* if every Cauchy sequence in Y converges.

Example. We first see some examples.

(1) We see that $\mathbb{Q}$ with $\rho(x, y) = |x - y|$ is **not** complete, but $\mathbb{R}$ with the same metric is complete.

(2) $C([0,1])$ with $\rho(f, g) = \|f - g\|_\infty$ is complete, but with $\rho(f, g) = \int |f - g|$ is not.

Definition 5.4.3 (Banach space)**.** A *Banach space* is a complete normed vector space.

Remark. Namely, a vector space equipped with a norm whose metric induced by the norm is complete.

Theorem 5.4.1. Let $(V, \|\cdot\|)$ be a normed space. Then,

$$V \text{ is complete} \Leftrightarrow \text{every absolutely convergent series is convergent.}$$

i.e., if $\sum_{i=1}^{\infty} \|v_i\| < \infty$, then $\left\{\sum_{i=1}^{N} v_i\right\}_{N\in\mathbb{N}}$ converges to some $s \in V$.

Before we prove Theorem 5.4.1, we first see one of the result based on this theorem.[1]

Theorem 5.4.2 (Riesz-Fischer theorem)**.** For every $1 \leq p \leq \infty$, we have $L^p(X, \mathcal{A}, \mu)$ is complete, hence a Banach space.

Proof. We prove this in two cases.

- We first prove this for $1 \leq p < \infty$. Suppose $f_n \in L^p$ and $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

  We need to show that there is an $F \in L^p$ such that $\left\|\sum_{n=1}^{N} f_n - F\right\|_p \to 0$ as $N \to \infty$. i.e., we need to show the following.

  1. $\sum_{n=1}^{\infty} f_n(x)$ is convergent a.e. In fact, we can show this by showing the following.

[1]The proof can be found in here.

Claim. We have
$$\int \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Proof. Let $G(x) = \sum\limits_{n=1}^{\infty} |f_n(x)| = \sup_N \sum\limits_{n=1}^{N} |f_n(x)|$, $G\colon X \to [0, \infty]$. Also, let $G_N(x) = \sum\limits_{n=1}^{N} |f_n(x)|$. Then, we have

$$0 \leq G_1 \leq G_2 \leq \ldots \leq G,$$

and $G_N \to G$. Furthermore,

$$0 \leq G_1^p \leq G_2^p \leq \ldots \leq G^p,$$

and $G_N^p \to G^p$. From monotone convergence theorem,

$$\int G^p = \lim_{N\to\infty} \int G_N^p.$$

From Minkowski inequality, we further have

$$\|G_N\|_p \leq \sum_{n=1}^{N} \|f_n\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p \coloneqq B < \infty.$$

Thus,
$$\int G(x)^p = \lim_{N\to\infty} \int G_N^p = \lim_{N\to\infty} \|G_N\|_p^p \leq B^p < \infty.$$

We see that G is finite a.e. as desired. This implies that $\sum\limits_{n=1}^{\infty} |f_n(x)| < \infty$ a.e., so $\sum\limits_{n=1}^{\infty} f_n(x)$ converges a.e. Now, we simply let

$$F(x) = \begin{cases} \sum\limits_{n=1}^{\infty} f_n(x), & \text{if it converges;} \\ 0, & \text{otherwise.} \end{cases}$$

⊛

2. $F \in L^p$, where $F(x) \coloneqq \sum\limits_{n=1}^{\infty} f_n(x)$ a.e. and say is zero elsewhere.

Claim. F defined in this way is indeed in L^p.

Proof. This is clear since

$$|F(x)| \leq G(x) \Rightarrow \int |F|^p \leq \int G^p < \infty,$$

hence $F \in L^p$. ⊛

3. We then show the last condition we need to check.

Claim. $\left\| \sum\limits_{n=1}^{N} f_n - F \right\|_p \to 0$ as $N \to \infty$.

Proof. We now see that

$$\left|\sum_{n=1}^{N} f_n(x) - F(x)\right|^p \leq \left(\sum_{n=1}^{\infty} |f_n(x)| + |F(x)|\right)^p \leq (2G(x))^p.$$

Since $2G \in L^p$, so $2G^p \in L^1$. Thus, by dominated convergence theorem, we have

$$\lim_{N\to\infty} \int \left|\sum_{n=1}^{N} f_n(x) - F(x)\right|^p \,\mathrm{d}x = 0.$$

This implies

$$\left\|\sum_{n=1}^{N} f_n - F\right\|_p \to 0$$

as $N \to \infty$. ⊛

- Now assume $1 \leq p \leq \infty$. DIY

■

Lecture 26: Bounded Linear Transformations

We now prove Theorem 5.4.1, completing the proof of Theorem 5.4.2 since the latter relies on this result. 16 Mar. 11:00

Proof of Theorem 5.4.1. We prove it by proving two directions.

($\Rightarrow$) Suppose V is complete, and fix an absolutely convergent series $\sum_n v_n$. Define $s_N = \sum_{n=1}^N v_n$. It suffices to show the partial sums are a Cauchy Sequence.

Fix $\epsilon > 0$, then because $\sum_{n=1}^\infty \|v_n\| < \infty$, there is a $K \in \mathbb{N}$ so that

$$\sum_{n=K}^{\infty} \|v_n\| < \epsilon.$$

Now let $M > N > K$, we see that

$$\|s_M - s_N\| = \left\|\sum_{n=N+1}^{M} v_n\right\| \leq \sum_{n=N+1}^{M} \|v_n\| \leq \sum_{n=N}^{\infty} \|v_n\| < \epsilon,$$

so this is Cauchy.

($\Leftarrow$) Now suppose $v_n, n \in \mathbb{N}$ is a Cauchy sequence. For all $j \in \mathbb{N}$, there exists an $N_j \in \mathbb{N}$ such that

$$\|v_n - v_m\| < \frac{1}{2^j}$$

for all $n, m \geq N_j$. Without loss of generality, we may assume $N_1 < N_2 < \ldots$.

Let $w_1 = v_{N_1}$, $w_j = v_{N_j} - v_{N_j - 1}$ for $j \geq 2$. Therefore,

$$\sum_{j=1}^{\infty} \|w_j\| \leq \|v_{N_1}\| + \sum_{j=2}^{\infty} \frac{1}{2^{j-1}} < \infty.$$

Thus, $\sum_{j=1}^k w_j \to s \in V$ as $k \to \infty$. But by telescoping, we have

$$v_{N_k} = \sum_{j=1}^{k} w_j \to s.$$

Now we claim that since v_n is Cauchy, so $v_n \to s$.

Explicitly, take $\epsilon > 0$, and let k be large enough so that $\|v_{N_k} - s\| < \epsilon$ and $1/2^k < \epsilon$. Then if $n > N_k$ then

$$\|v_n - s\| \leq \|v_n - v_{N_k}\| + \|v_{N_k} - s\| < \epsilon + \epsilon = 2\epsilon.$$

Thus, $v_n \to s$. ∎

5.5 Bounded Linear Transformations

Definition 5.5.1 (Bounded linear transformation)**.** Given two normed vector spaces $(V, \|\cdot\|)$, $(W, \|\cdot\|')$, a linear map $T \colon V \to W$ is called a *bounded map* if there exists $c \geq 0$ such that

$$\|Tv\|' \leq c \|v\|$$

for all $v \in V$.

Proposition 5.5.1. Suppose $T \colon (V, \|\cdot\|) \to (W, \|\cdot\|')$ is a linear map. Then the following are equivalent.

(1) T is continuous.

(2) T is continuous at 0.

(3) T is a bounded map.

Proof. (1) $\Rightarrow$ (2) is clear.

Claim. (2) $\Rightarrow$ (3).

Proof. Take $\epsilon = 1$, then there exists a $\delta > 0$ such that $\|Tu\|' < 1$ if $\|u\| < \delta$.

Now take an arbitrary $\|v\| \in V$, $v \neq 0$. Let $u = \frac{\delta}{2\|v\|} v$. Then $\|u\| < \delta$. Therefore,

$$\|Tu\|' < 1 \Rightarrow \frac{\delta}{2\|v\|} \|Tv\|' < 1 \Rightarrow \|Tv\|' < \frac{2}{\delta} \|v\|.$$

Then $2/\delta$ is our constant. ⊛

Claim. (3) $\Rightarrow$ (1).

Proof. Fix $v_0 \in V$. Then for some constant c

$$\|Tv - Tv_0\|' = \|T(v - v_0)\|' \leq c \|v - v_0\|.$$

Thus, T is continuous, as when $v \to v_0$ the right-hand side goes to zero, and so $Tv \to Tv_0$. ⊛

∎

Example. Let's see some examples.

(1) We can look at

$$\begin{aligned} T \colon \ell^1 &\to \ell^1 \\ (a_1, a_2, \ldots) &\mapsto (a_2, a_3, \ldots). \end{aligned}$$

Then clearly $\|Ta\|_1 \leq \|a\|_1$, so T is a bounded linear transformation.

(2) We can also look at $S \colon (C([-1, 1]), \|\cdot\|_1) \to \mathbb{C}$, where $Sf = f(0)$. S is not a bounded linear

transformation, because we can make

$$\begin{cases} \|Sf\| & = |f(0)| = n \\ \|f\|_1 & = 1 \end{cases}$$

for every $n \in \mathbb{N}$ (take f's graph to be a skinny triangle shooting up to n at 0).

(3) But $U\colon (C([-1,1]), \|\cdot\|_\infty) \to \mathbb{C}$ defined by $Uf = f(0)$ is a bounded linear transformation, because $|f(0)| \leq \|f\|_\infty$.

(4) Let A be an $n \times m$ matrix. Then $T\colon \mathbb{R}^m \to \mathbb{R}^n$ defined by $v \mapsto Av$ is a bounded linear transformation.

Explicitly this is

$$(Tv)_i = (Av)_i = \sum_{j=1}^{m} A_{ij} v_j.$$

(5) Let $K(x,y)$ be a continuous function on $[0,1] \times [0,1]$. We'll define

$$T\colon (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$$

by

$$(Tf)(x) = \int_0^1 K(x,y) f(y) \,\mathrm{d}y.$$

This is an analogue of matrix multiplication (K is like a continuous matrix). This is a bounded linear transformation.

(6) Let us look at $T\colon L^1(\mathbb{R}) \to (C(\mathbb{R}), \|\cdot\|_\infty)$ defined by

$$(Tf)(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \,\mathrm{d}x$$

that is the Fourier transform of f.

(7) $T\colon (C^\infty[0,1], \|\cdot\|_\infty) \to (C^\infty[0,1], \|\cdot\|_\infty)$. Define

$$(Tf)(x) = f'(x).$$

This is not a bounded linear transformation. In contrast, S, defined on the same spaces

$$(Sf)(x) = \int_0^x f(t) \,\mathrm{d}t$$

is bounded.

Definition 5.5.2 (Operator norm). Let $L(V,W)$ be defined as a vector space such that

$$L(V,W) := \{T\colon V \to W \mid T \text{ is a bounded linear transformation}\}.$$

Then for $T \in L(V,W)$, the *operator norm* of T is

$$\begin{aligned} \|T\| &:= \inf\{c \geq 0 \mid \|Tv\|'' \leq c\,\|v\|' \text{ for all } v \in V\} \\ &= \sup\left\{\frac{\|Tv\|''}{\|v\|'} \mid v \neq 0, v \in V\right\} \\ &= \sup\left\{\|Tv\|'' \mid \|v\|' = 1, v \in V\right\}. \end{aligned}$$

Lemma 5.5.1. We have that

(1) The three definitions of $\|T\|$ above are all equal.

(2) $(L(V, W), \|\cdot\|)$ is indeed a normed space.

Proof. $\blacksquare$ DIY

Lecture 27: Dual Space

18 Mar. 11:00

As previously seen. From Definition 5.5.2, we have that

$$\|Tv\|'' \leq \|T\| \, \|v\|' .$$

Remark. Notice that this Definition 5.5.2 is only for bounded linear transformation.

Theorem 5.5.1. If W is complete, then $L(V, W)$ is complete.

Proof. Suppose T_n is a Cauchy sequence in $L(V, W)$. Fix $v \in V$ and let $w_n := T_n v \in W$, we then have

$$\|w_n - w_m\| = \|T_n v - T_m v\| = \|(T_n - T_m)v\| \leq \underbrace{\|T_n - T_m\|}_{\to 0} \underbrace{\|v\|}_{\substack{\text{fixed} \\ \text{value}}} .$$

Thus, w_n is Cauchy, so it converges since W is complete. We call its unique limit Tv. This makes $T \colon V \to W$ into a function. We must show it is a bounded linear transformation and that $\|T_n - T\| \to 0$. $\blacksquare$ DIY

5.6 Dual of L^p Spaces

Example. Let $w \in \mathbb{R}^d$, and denote the inner product between w and $v \in \mathbb{R}^d$ by

$$v \cdot w := \langle v, w \rangle .$$

Then we can consider

$$\max\{v \cdot w \mid \|v\|_2 = 1\} = \|w\|_2 .$$

If $w \in \mathbb{C}^d$, this is similar since

$$\max\{|v \cdot w| \mid \|v\|_2 = 1\} = \|w\|_2 .$$

These maximums are achieved by $v = \frac{\overline{w}}{\|w\|}$ if $w \neq 0$.

Proposition 5.6.1. Let $1/p + 1/q = 1$ with $1 \leq q < \infty$. For every $g \in L^q$,

$$\|g\|_q = \sup\left\{ \left| \int fg \right| \mid \|f\|_p = 1 \right\} .$$

Suppose μ is σ-finite, then the result also holds for $q = \infty$, $p = 1$.

Proof. By Hölder's inequality, we know that

$$\left| \int fg \right| \leq \int |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q = \|g\|_q .$$

Thus, the supremum is less or equal to $\|g\|_q$.

(1) Let
$$f(x) = \frac{|g(x)|^{q-1} \cdot \overline{\operatorname{sgn}(g(x))}}{\|g\|_q^{q-1}}$$
Then $\int |f|^p = 1$, and $\int fg = \|g\|_q$. Check

> **Note.** For $\alpha \in \mathbb{C}$, $\operatorname{sgn}(\alpha) := e^{i\theta}$ where $\alpha = |\alpha| e^{i\theta}$.

(2) The case that μ is σ-finite and $q = \infty, p = 1$ can be shown. DIY

■

Remark. One could use the above to prove Minkowski's inequality (as it only uses Hölder's inequality).

Definition 5.6.1 (Dual space). For a normed space $(V, \|\cdot\|)$, its *dual space* is $V^* = L(V, \mathbb{R})$ or $V^* = L(V, \mathbb{C})$.

Remark. Namely, the dual space of V contains bounded linear transformations with codomain being the scalar field.

Definition 5.6.2 (Linear functional). Given a normed space $(V, \|\cdot\|)$, $\ell \in V^*$ is called a *linear functional* on V. i.e.,

- $\ell \colon V \to \mathbb{R}$ (or $\mathbb{C}$).
- ℓ is linear.
- There exists a $c \geq 0$ such that $|\ell(v)| = c\|v\|$.

Note. V^* is always a Banach space (even if V is not complete).

Corollary 5.6.1. We have the following.

(1) Let $1/p + 1/q = 1, 1 \leq q < \infty$. For $g \in L^q$ define $\ell_g \in L^p \to \mathbb{C}$ by
$$\ell_g(f) = \int fg.$$
Then $\ell_g \in (L^p)^*$. Furthermore, $\|\ell_g\| = \|g\|_q$.

(2) If μ is σ-finite, then this also holds for $q = \infty, p = 1$.

Proof. ℓ_g is clearly linear in f because the integral is linear. Then Proposition 5.6.1 gives in both (1) and (2) that
$$\|g\|_q = \sup\{|\ell_g(f)| \mid \|g\|_p = 1\} = \|\ell_g\|$$
and so ℓ_g is a bounded linear transformation with the desired properties. ■

Theorem 5.6.1. We have the following.

(1) Let $1/p + 1/q = 1$, $1 \leq q < \infty$. The map $T \colon L^q \to (L^p)^*$ given by $Tg = \ell_g$ is an isometric[a] linear isomorphism. This means that

- T is a bounded linear transformation.

- T is bijective.
- T is norm-preserving.

(2) If μ is σ-finite then this also holds for $q = \infty, p = 1$.

[a] A map T is called isometric if for a given g, $\|Tg\| = \|g\|$.

Proof. We have already proved this is isometric in Corollary 5.6.1, it is clearly linear, and isometry implies injectivity.

We will prove that it is surjective later. ■

Fix!!!

Note. Even if μ is σ-finite we might not have $L^1 \cong (L^\infty)^*$.

Also note that $L^2 \cong (L^2)^*$, and for all $1 < p < \infty$ we have $(L^p)^{**} \cong L^p$.

Chapter 6

Signed and Complex Measures

Lecture 28: Signed Measure

21 Mar. 11:00

As previously seen. Suppose $f\colon X \to [0,\infty]$ is a measurable function on $(X, \mathcal{A}, \mu)$.

We can define $\nu(E) = \int_E f \,\mathrm{d}\mu$ for $E \in \mathcal{A}$, and ν is a measure on $(X, \mathcal{A})$. This gives a map from the set of non-negative measurable functions on X to measures on X. This is injective if we identify functions which are equal almost everywhere. But it is not necessarily surjective. We can then think of measures as a generalization of functions.

For an example, think of a Dirac-Delta measure on $\mathbb{R}$. This is not the Lebesgue integral of any non-negative measurable function.

What if instead we took $f\colon X \to \mathbb{R}$, $\overline{\mathbb{R}}$ or $\mathbb{C}$? We could take the same construction to get $\nu(E) = \int_E f \,\mathrm{d}\mu$, but this is no longer a measure as it can take $\mathbb{R}$, $\overline{\mathbb{R}}$ or $\mathbb{C}$ values.

6.1 Signed Measures

Definition 6.1.1. Let $(X, \mathcal{A})$ be a measurable space. A *signed measure* is a function

$$\nu\colon \mathcal{A} \to [-\infty, \infty) \text{ or } \nu\colon \mathcal{A} \to (-\infty, \infty]$$

such that

- $\nu(\varnothing) = 0$.
- If $A_1, A_2, \ldots \in \mathcal{A}$ are disjoint then

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$$

where the series on the right-hand side converges absolutely if $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \in (-\infty, \infty)$.

> **Remark.** This means the series does not depend on rearrangement if our function ν takes finite value on the set $\bigcup_i A_i$.

Example. Consider

(1) ν is a positive measure (i.e., measure), then ν is a signed measure.

(2) If we have positive measures μ_1, μ_2 such that either $\mu_1(X) < \infty$ or $\mu_2(X) < \infty$, then $\nu = \mu_1 - \mu_2$ is a signed measure.

(3) If $f\colon X \to \overline{\mathbb{R}}$ on a measure space $(X, \mathcal{A}, \mu)$ such that $\int_X f^+ \,\mathrm{d}\mu < \infty$ or $\int_X f^- \,\mathrm{d}\mu < \infty$, we can

define

$$\nu(E) = \int_E f \, \mathrm{d}\mu$$

and this will be a signed measure.

Note. The following weird things happen with signed measures.

(1) $A \subseteq B$ does not imply $\nu(A) \leq \nu(B)$, as $\nu(B) = \nu(A) + \nu(B \setminus A)$, and $\nu(B \setminus A)$ may be negative.

(2) If $A \subseteq B$ and $\nu(A) = \infty$, then $\nu(B) = \infty$, because $\nu(B \setminus A) \in (-\infty, \infty]$.

(3) Similarly, if $A \subseteq B$ and $\nu(A) = -\infty$ then $\nu(B) = -\infty$.

Lemma 6.1.1. If ν is a signed measure on $(X, \mathcal{A})$, then we have the following.

(1) Continuity from below. If $E_n \in \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \cdots$ then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} \nu(E_N).$$

(2) Continuity from above. If $E_n \in \mathcal{A}$, $E_1 \supseteq E_2 \supseteq \cdots$, and $-\infty < \nu(E_1) < \infty$ then

$$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} \nu(E_N).$$

Proof. Read [FF99]. ∎

Definition. Let ν be a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$, then we say that

Definition 6.1.2 (Positive set for a signed measure)**.** E is *positive for* ν if for all $F \subseteq E$, $\nu(F) \geq 0$.

Definition 6.1.3 (Negative set for a signed measure)**.** E is *negative for* ν if for all $F \subseteq E$, $\nu(F) \leq 0$.

Definition 6.1.4 (Null set for a signed measure)**.** E is *null for* ν if for all $F \subseteq E$, $\nu(F) = 0$.

Note. We see that

(1) If E is a positive set, $F \subseteq E$, then $\nu(F) \leq \nu(E)$.

(2) If E is a negative set, $F \subseteq E$, then $\nu(F) \geq \nu(E)$.

Lemma 6.1.2. Let ν be a signed measure on $(X, \mathcal{A})$, then

(1) If E is positive, $G \subseteq E$ is measurable, then G is positive.

(2) If E is negative, $G \subseteq E$ is measurable, then G is negative.

(3) If E is null, $G \subseteq E$ is measurable, then G is null.

(4) $E_1, E_2, \ldots$ are positive sets, then $\bigcup_{i=1}^{\infty} E_i$ is positive.

Proof. The first three are trivial from their definition. For 4., if $E_1, \ldots$ are positive sets, let

$F_n := E_n \setminus \bigcup_{j=1}^{n-1} E_j$. Then $F_n \subset E_n$, so F_n is positive sets from 1., hence if $E \subset \bigcup_{j=1}^{\infty} E_j$, then

$$\nu(E) = \sum_{j=1}^{\infty} \nu(E \cap F_j) \geq 0$$

as desired. ∎

Lemma 6.1.3. Suppose that ν is a signed measure with $\nu\colon \mathcal{A} \to [-\infty, \infty)$. Suppose $E \in \mathcal{A}$ and $0 < \nu(E) < \infty$, then there exists a measurable $A \subseteq E$ such A is a positive set and $\nu(A) > 0$.

Proof. If E is positive, we're done. Otherwise, there exist measurable subsets with negative measure. Let $n_1 \in \mathbb{N}$ be the least such n_1 such that there exists $E_1 \subseteq E$ with $\nu(E_1) < -1/n_1$.

If $E \setminus E_1$ is positive, we're done. Else we can inductively define $n_2, n_3, \dots$ as well as $E_2, E_3, \dots$.

Explicitly, if $E \setminus \bigcup_{i=1}^{k-1} E_i$ is not positive, let n_k be the least such that there exists $E_k \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ with $\nu(E_k) < -1/n_k$.

Note then that if $n_k \geq 2$, for all $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ we have that $\nu(B) \geq -\frac{1}{n_k - 1}$.

Now let $A = E \setminus \bigcup_{i=1}^{\infty} E_i$. Since $E = A \cup (\bigcup_i E_i)$ we have by countable additivity that

$$0 < \nu(E) = \nu(A) + \sum_{k=1}^{\infty} \nu(E_k) < \nu(A).$$

Furthermore, $\nu(E), \nu(A)$ are both in $(0, \infty)$, and we see that

$$0 < \nu(E) \leq \nu(A) - \sum_{k=1}^{\infty} \frac{1}{n_k}.$$

Therefore, the sum on the right-hand side must converge, meaning that $1/n_k \to 0$ as $k \to \infty$. That is $\lim_{k \to \infty} n_k = \infty$.

Now if $B \subseteq A$, then $B \subseteq E \setminus \bigcup_{i=1}^{\infty} E_i$. Therefore, $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$. By the note above, for large enough k such that $n_k \geq 2$ we have

$$\nu(B) \geq \frac{-1}{n_k - 1},$$

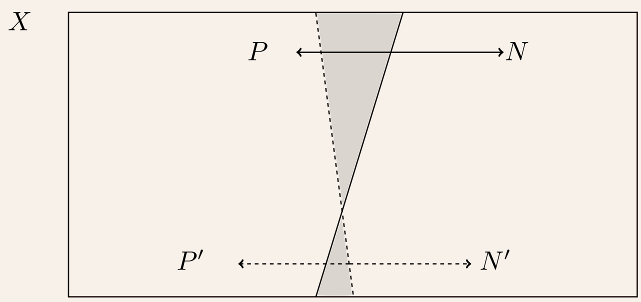
then taking $k \to \infty$ we have $\nu(B) \geq 0$, and so A is a positive set as desired. ∎

Theorem 6.1.1 (Hahn decomposition theorem). If ν is a signed measure on $(X, \mathcal{A})$, then there exist $P, N \in \mathcal{A}$ such that

$$P \cap N = \varnothing, \quad P \cup N = X,$$

where P is positive for ν, N is negative for ν.

Furthermore, if P', N' are another such pair, then $P \triangle P' (= N \triangle N')$ is null for ν.



Lecture 29: Hahn and Jordan Decomposition Theorem

We now prove Theorem 6.1.1. 23 Mar. 11:00

Proof of Theorem 6.1.1. We first show the uniqueness. We see that $P \setminus P' \subseteq P, P \setminus P' \subseteq N'$. Thus, $P \setminus P' \subseteq P \cap N'$ is both positive and negative, hence $P \setminus P'$ is null.

Similarly, for $P' \setminus P$, and then their union $P \triangle P'$ is null as well.

To show the existence, without loss of generality suppose $\nu \colon \mathcal{A} \to [-\infty, \infty)$. If not, consider $-\nu$. Let

$$s := \sup\{\nu(E) \mid E \in \mathcal{A} \text{ is a positive set}\},$$

which is a nonempty supremum because $\varnothing$ is positive. Then there exist $P_1, P_2, \ldots$ positive sets such that $\lim_{n\to\infty} \nu(P_n) = s$.

Then we have that $P = \bigcup_n P_n$ is positive by Lemma 6.1.2. We then have $\nu(P) \leq s$ and $\nu(P) = \nu(P_n) + \nu(P \setminus P_n) \geq \nu(P_n)$. Thus,

$$\nu(P) \geq \lim_{n\to\infty} \nu(P_n) = s.$$

Hence, $\nu(P) = s$ and the supremum is in fact a max. We then know that $s = \nu(P) < \infty$ because ν does not attain the value infinity.

Now let $N = X \setminus P$. We claim that N is negative. If not then there exists a measurable $E \subseteq N$ with $\nu(E) > 0$. By assumption, $\nu(E) < \infty$. Then $0 < \nu(E) < \infty$, so by Lemma 6.1.3 there exists a measurable $A \subseteq E$ such that A is positive and $\nu(A) > 0$.

But we then know that

$$\nu(P \cup A) = \nu(P) + \nu(A) > \nu(P)$$

which is a contradiction since $P \cup A$ is a positive set, and $\nu(P)$ is maximal. Therefore, N is negative, and the theorem holds.

Finally, if P', N' is another pair of sets as in the statement of the theorem, we have

$$P \setminus P' \subset P, \quad P \setminus P' \subset N',$$

so that $P \setminus P'$ is both positive and negative, hence null; likewise for $P' \setminus P$. ∎

Definition 6.1.5 (Singular). If μ, ν are signed measures on $(X, \mathcal{A})$, then we say μ and ν are *singular to each other*, denoted as $\mu \perp \nu$, if there exists $E, F \in \mathcal{A}$ such that $E \cap F = \varnothing, E \cup F = X$, F is null for μ, E is null for ν.

Example. Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

- The Lebesgue measure m.
- The Cantor measure μ_C induced by the Cantor function.
- The discrete measure $\mu_D = \delta_1 + 2\delta_{-1}$.

We then see that (1) $m \perp \mu_D$. (2) $m \perp \mu_c$. (3) $\mu_C \perp \mu_D$.

Proof. We see them as follows.

(1) Take $E = \mathbb{R} \setminus \{-1, 1\}, F = \{1, -1\}$ to see that $m \perp \mu_D$.

(2) Take $E = \mathbb{R} \setminus K$ and $F = K$ where K is the Cantor set to see that $m \perp \mu_C$.

(3) We can also see that $\mu_C \perp \mu_D$.

⊛

Theorem 6.1.2 (Jordan decomposition theorem). Let ν be a signed measure on $(X, \mathcal{A})$. Then there exists unique positive measures ν^+, ν^- on $(X, \mathcal{A})$ such that for all $E \in \mathcal{A}$ we have

$$\nu(E) = \nu^+(E) - \nu^-(E)$$

and $\nu^+ \perp \nu^-$.

Proof. For existence, we take $\nu^+(E) \coloneqq \nu(E \cap P), \nu^-(E) \coloneqq -\nu(E \cap N)$ where P, N is the Hahn decomposition of X.

If there exists μ^+, μ^- such that $\nu = \mu^+ + \mu^-$ and $\mu^+ \perp \mu^-$, let $E, F \in \mathcal{A}$ be such that $E \cap F = \varnothing$, $E \cup F = X$, and $\mu^+(F) = \mu^-(E) = 0$. Then we have that $X = E \cup F$ is another Hahn decomposition for ν, so $P \triangle E$ is ν-null. Therefore, for any $A \in \mathcal{A}$, $\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A)$, hence $\mu^+ = \nu^+$. Likewise, we have $\nu^- = \mu^-$. ■

Lecture 30: Absolutely Continuous Measures

25 Mar. 11:00

Example. For an example of Theorem 6.1.2, let $(X, \mathcal{A}, \mu)$ be a measure space, $f \colon X \to \overline{\mathbb{R}}$, and $\nu(E) = \int_E f \,\mathrm{d}\mu$. Then

$$\nu^+(E) = \int_E f^+ \,\mathrm{d}\mu, \quad \nu^-(E) = \int_E f^- \,\mathrm{d}\mu.$$

Definition. Given a signed measure ν on $(X, \mathcal{A})$ and its Jordan decomposition $\nu = \nu^+ - \nu^-$.

Definition 6.1.6 (Positive variation)**.** We call ν^+ the *positive variation* of ν.

Definition 6.1.7 (Negative variation)**.** We call ν^- the *negative variation* of ν.

Definition 6.1.8 (Total variation)**.** The *total variation measure of* ν, denoted as $|\nu|$, is defined as $|\nu| \coloneqq \nu^+ + \nu^-$.

Remark. There is always a positive measure on X.

Proof. Consider the total variation $|\nu|$ for an arbitrary signed measure ν. ⊛

Example. In the above example, $|\nu|(E) = \int_E |f| \,\mathrm{d}\mu$.

Lemma 6.1.4. We have the following

(1) $|\nu(E)| \leq |\nu|(E)$.

(2) E is ν-null if and only if E is $|\nu|$-null.

(3) If κ is another signed measure, then $\kappa \perp \nu$ if and only if $\kappa \perp |\nu|$ if and only if $\kappa \perp \nu^+$ and $\kappa \perp \nu^-$.

Proof. ■ DIY

Definition 6.1.9 (Finite signed measure)**.** A signed measure ν is *finite* if $|\nu|$ is a finite measure, and similarly for σ-finite.

Remark. This holds if and only if ν^+, ν^- are both finite (resp. σ-finite) measures.

6.2 Absolutely Continuous Measures

Definition 6.2.1 (Absolutely continuous measure)**.** Let μ be a positive measure, ν be a signed measure, both on $(X, \mathcal{A})$. We say that ν is *absolutely continuous with respect to* μ, denoted as $\nu \ll \mu$, provided that for all $E \in \mathcal{A}$, $\mu(E) = 0$ implies $\nu(E) = 0$.

Remark. This is equivalent to every μ-null set being ν-null.

Example. If $(X, \mathcal{A}, \mu)$, $f\colon X \to \overline{\mathbb{R}}$, $\nu(E) = \int_E f \,\mathrm{d}\mu$, then $\nu \ll \mu$.

Notation. $\mathrm{d}\nu = f \,\mathrm{d}\mu$ means ν is a signed measure defined by

$$\nu(E) = \int_E f \,\mathrm{d}\mu.$$

Lemma 6.2.1. If μ is a positive measure, ν is a signed measure on $(X, \mathcal{A})$, then

(1) $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll< \mu$.

(2) $\nu \ll \mu$ and $\nu \perp \mu$ implies $\nu = 0$.

Proof. DIY (1)

For (2), write $X = A \cup B$, $A \cap B = \varnothing$, A μ-null, B ν-null. Then

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap A).$$

Then $E \cap A \subseteq A$, so $\nu(E \cap A) = 0$. By absolute continuity, $\nu(E \cap A) = 0$, thus $\nu(E) = 0$. ∎

Theorem 6.2.1 (Radon-Nikodym theorem)**.** Suppose μ is a σ-finite positive measure, ν is a σ-finite signed measure, and suppose $\nu \ll \mu$. Then there exists $f\colon X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\nu = f \,\mathrm{d}\mu$, in other words $\nu(E) = \int_E f \,\mathrm{d}\mu$.

If g is another such function with $\mathrm{d}\nu = g \,\mathrm{d}\mu$ then $f = g$ μ-a.e..

Proof. We'll prove a more general form called Lebesgue Radon Nikodym theorem, which is a more general theorem compare to this theorem. ∎

Definition 6.2.2 (Randon-Nikodym derivative)**.** Suppose $\nu \ll \mu$. The *Radon-Nikodym derivative of* ν *with respect to* μ is a function

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\colon X \to \overline{\mathbb{R}}$$

such that

$$\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,\mathrm{d}\mu$$

for all $E \in \mathcal{A}$.

Remark. i.e. we have $\mathrm{d}\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,\mathrm{d}\mu$ in our notation.

Note. By Theorem 6.2.1, such a function exists and is unique up to equivalence μ-a.e. in the σ-finite case.

Example. Let $F(x) = e^{2x} \colon \mathbb{R} \to \mathbb{R}$, then

$$\frac{\mathrm{d}F}{\mathrm{d}m} = 2e^{2x}.$$

Proof. Since F is continuous and strictly increasing, so we may define a Lebesgue-Stieltjes measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

This is defined to be the unique locally finite measure satisfying

$$\mu_F([a,b]) = F(b) - F(a) = e^{2b} - e^{2a}.$$

Then one can check that

$$\mu_F(E) = \int_E 2e^{2x}\,\mathrm{d}x.$$

By uniqueness and the classical fundamental theorem of calculus, since the right-hand side is a locally finite Borel measure, and $\kappa([a,b]) = e^{2b} - e^{2a}$, thus $\mu_F = \kappa$. Therefore, $\mu_F \ll m$ and we have

$$\frac{\mathrm{d}\mu_F}{\mathrm{d}m} = 2e^{2x} = \frac{\mathrm{d}F}{\mathrm{d}x}.$$

⊛

Example. Let $C(X) \colon \mathbb{R} \to \mathbb{R}$ be the Cantor function. Then $C'(x) = 0$ outside the Cantor set, but we don't always have

$$\mu_C(E) \neq \int_E 0\,\mathrm{d}x.$$

So the candidate derivative is 0, but this fails.

Proof. In particular,

$$C(b) - C(a) \neq \int_a^b C'(x)\,\mathrm{d}x.$$

In fact, $\mu_C \not\ll m$ because $\mu_C \perp m$ and $\mu_C \neq 0$.

Thus, the existence of a derivative almost everywhere and continuity is not enough to guarantee a version of the fundamental theorem of calculus to hold. ⊛

Lecture 31: Lebesgue-Radon-Nikodym Theorem

28 Mar. 11:00

Lemma 6.2.2. Let μ, ν be finite positive measures on $(X, \mathcal{A})$. Then either

- $\nu \perp \mu$.
- There exists an $\epsilon > 0$, an $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for the measure $\nu - \epsilon\mu$, i.e., for all $G \subseteq F$, $\nu(G) \geq \epsilon\mu(G)$.

Proof. Let $\kappa_n = \nu - (1/n)\mu$. By Theorem 6.1.1 we have $X = P_n \cup N_n$ for P_n positive, N_n negative for κ_n. Also, we let $P = \bigcup_n P_n$, $N = \bigcap_n N_n = X \setminus P$, then $X = P \cup N$.

We see that for any n we have $\kappa_n(N) \leq 0$ because $N \subseteq N_n$. Thus,

$$0 \leq \nu(N) \leq \frac{1}{n}\mu(N),$$

which implies $\nu(N) = 0$. Because ν is positive for any $N' \subseteq N$ we have $0 \leq \nu(N') \leq \nu(N)$, and thus $\nu(N') = 0$. This shows N is null for ν. Now, we see that

- If $\mu(P) = 0$, then $\nu \perp \mu$.
- If $\mu(P) \neq 0$, then we have $\mu(P) > 0$ hence $\mu(P_n) > 0$ for some n. With $F = P_n$ and $\epsilon = 1/n$, then F is a positive set for $\kappa_n = \nu - (1/n)\mu$ as desired.

■

Theorem 6.2.2 (Lebesgue-Radon-Nikodym theorem). Let μ be a σ-finite positive measure, ν a σ-finite signed measure on $(X, \mathcal{A})$. Then there are unique σ-finite signed measures λ, ρ on $(X, \mathcal{A})$ such that

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \nu = \lambda + \rho.$$

Furthermore, there exists a measurable function $f \colon X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\rho = f \, \mathrm{d}\mu$.[a] And if there is another g such that $\mathrm{d}\rho = g \, \mathrm{d}\mu$, then $f = g$ μ-a.e.

[a] That is for all $E \in \mathcal{A}$, $\rho(E) = \int_E f \, \mathrm{d}\mu$.

Proof. We prove it step by step.

(1) Assume μ, ν are finite positive measures. We first prove the existence of λ, f, and $\mathrm{d}\rho = f = \mathrm{d}\mu$. Let

$$\begin{aligned}\mathscr{F} &= \left\{ g \colon X \to [0, \infty] \mid \int_E g \, \mathrm{d}\mu \leq \nu(E), \forall E \in \mathcal{A} \right\} \\ &= \{ g \colon X \to [0, \infty] \mid \mathrm{d}\nu - g \, \mathrm{d}\mu \text{ is a positive measure} \}.\end{aligned}$$

This set is nonempty since $g = 0 \in \mathscr{F}$. Let $s = \sup\{ \int_X g \, \mathrm{d}\mu \mid g \in \mathscr{F} \}$.

Claim. There is an $f \in \mathscr{F}$ such that $s = \int_X f \, \mathrm{d}\mu$.

Proof. If $g, h \in \mathscr{F}$, we can define $u(x) = \max\{g(x), h(x)\}$, then $u \in \mathscr{F}$. This can be seen by letting $A = \{x \mid g(x) \geq h(x)\}$, then

$$\int_E u \, \mathrm{d}\mu = \int_{E \cap A} g \, \mathrm{d}\mu + \int_{E \cap A^c} h \, \mathrm{d}\mu \leq \nu(E \cap A) + \nu(E \cap A^c) = \nu(E).$$

There exist measurable functions $g_1, g_2, \ldots \in \mathscr{F}$ such that

$$\lim_{n \to \infty} \int_X g_n \, \mathrm{d}\mu = s.$$

We can replace g_2 by $\max(g_1, g_2)$, g_3 by $\max(g_1, g_2, g_3)$. Generally,

$$g_n \leftarrow \max(g_1, g_2, \ldots, g_n),$$

so that we may assume $0 \leq g_1 \leq g_2 \leq \ldots$.

Then we still know that $\lim\limits_{n \to \infty} \int_X g_n \, \mathrm{d}\mu = s$, as all the relevant integrals are bounded above by s. Now let $f(x) = \sup_n g_n(x) = \lim\limits_{n \to \infty} g_n(x)$, by monotone convergence theorem,

$$\int_E f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_E g_n \, \mathrm{d}\mu \leq \nu(E).$$

Thus, $f \in \mathscr{F}$, and when $E = X$ we get $\int_X f \, \mathrm{d}\mu = s$ as desired. ⊛

Let $\rho(E) \coloneqq \int_E f \, \mathrm{d}\mu$, then we of course have $\rho \ll \mu$, and also, we know

$$0 \leq \rho(X) = \int_X f \, \mathrm{d}\mu \leq \nu(X) < \infty.$$

Thus, ρ is a finite positive measure, so we can define $\lambda(E) \coloneqq \nu(E) - \rho(E)$, then

$$\lambda(E) = \nu(E) - \int_E f \, \mathrm{d}\mu \geq 0$$

because $f \in \mathscr{F}$. Thus, λ is also a positive measure, and $\lambda(X) \leq \nu(X) < \infty$. It remains to show the following.

Claim. $\lambda \perp \mu$.

Proof. Suppose not, by Lemma 6.2.2, there exists $\epsilon > 0$, $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for $\lambda - \epsilon\mu$.

Then this says that $\mathrm{d}\lambda - \epsilon \mathbb{1}_F \,\mathrm{d}\mu$ is a positive measure, that is,

$$\mathrm{d}\nu - f\,\mathrm{d}\mu - \epsilon \mathbb{1}_F \,\mathrm{d}\mu$$

is a positive measure. But, this will break maximality of f, specifically, let $g(x) = f(x) + \epsilon \mathbb{1}_F(x)$. Then for all $E \in \mathcal{A}$ we have

$$\begin{aligned}
\int_E g\,\mathrm{d}\mu &= \int_E f\,\mathrm{d}\mu + \epsilon\mu(E \cap F) \\
&= \nu(E) - \lambda(E) + \epsilon\mu(E \cap F) \\
&\leq \nu(E) - \lambda(E \cap F) + \epsilon\mu(E \cap F) \leq \nu(E)
\end{aligned}$$

since $\lambda(E \cap F) - \epsilon\mu(E \cap F) \geq 0$. Thus, $g \in \mathscr{F}$. We then see that

$$s \geq \int_X g\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X \epsilon \mathbb{1}_F \,\mathrm{d}\mu = s + \epsilon\mu(F) > s,$$

which is a contraction. ⊛

We see that the existence of λ, f, and $\mathrm{d}\rho = f\,\mathrm{d}\mu$ is proved. As for uniqueness, if there are λ' and f' such that $\mathrm{d}\nu = \mathrm{d}\lambda' + f'\,\mathrm{d}\mu$, we then have

$$\mathrm{d}\lambda - \mathrm{d}\lambda' = (f' - f)\,\mathrm{d}\mu.$$

But we see that $\lambda - \lambda' \perp \mu$ while $(f' - f)\,\mathrm{d}\mu \ll \mathrm{d}\mu$, hence Check!

$$\mathrm{d}\lambda - \mathrm{d}\lambda' = (f' - f)\,\mathrm{d}\mu = 0,$$

so $\lambda = \lambda'$ and $f = f'$ μ-a.e. by Proposition 2.3.1.

(2) Suppose that μ and ν are σ-finite measures. Then X is a countable disjoint union of μ-finite sets and a countable disjoint union of ν-finite sets. By taking intersections of these we obtain a disjoint sequence $\{A_j\} \subset \mathcal{A}$ such that $\mu(A_j)$ and $\nu(A_j)$ are finite for all j and $X = \bigcup_j A_j$. Define $\mu_j(E) = \mu(E \cap A_j)$ and $\nu_j(E) = \nu(E \cap A_j)$, then by the reasoning above, for each j we have

$$\mathrm{d}\nu_j = \mathrm{d}\lambda_j + f_j\,\mathrm{d}\mu_j$$

where $\lambda_j \perp \mu_j$. Since $\mu_j(A_j^c) = \nu_j(A_j^c) = 0$, we have

$$\lambda_j(A_j^c) = \nu_j(A_j^c) - \int_{A_j^c} f\,\mathrm{d}\mu_j = 0,$$

and we may assume that $f_j = 0$ on A_j^c. Let $\lambda = \sum_j \lambda_j$ and $f = \sum_j f_j$, we then have

$$\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}\mu, \quad \lambda \perp \mu,$$

and $\mathrm{d}\lambda$ and $f\,\mathrm{d}\mu$ are σ-finite, as desired. As for uniqueness, it's the same as for the first case.

(3) We now consider the general case. If ν is a signed measure, we apply the preceding argument to ν^+ and ν^- and subtract the results.

■

Remark. Notationally, we may write $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}\mu$, where $\mathrm{d}\lambda$ and $\mathrm{d}\mu$ are singular to each other.

Lecture 32: Lebesgue Differentiation Theorem for Regular Borel Measures

We now do an example to illustrate Theorem 6.2.2. 30 Mar. 11:00

Example. Let $\mu = m$, $\nu = \mu_F$ (Lebesgue-Stieltjes measure for F). We'll define $F(x)$ by

$$F(x) = \begin{cases} e^{3x}, & \text{if } x \leq 0; \\ 1, & \text{if } 0 < x < 1; \\ 5, & \text{if } x \geq 1. \end{cases}$$

Then we have

$$\mu_F(E) = \int_{E\cap\mathbb{R}_{<0}} 3e^{3x}\,\mathrm{d}x + 4\delta_1(E).$$

Proof. It is enough to check on $(-\infty, x]$ because these are locally finite Borel measures on $\mathbb{R}$. Then we have

$$\mu_F = \mathrm{d}\rho + \mathrm{d}\lambda = f\,\mathrm{d}m + \mathrm{d}\lambda$$

where $f = \mathbb{1}_{\mathbb{R}_{<0}} 3e^{3x}$ and $\lambda = 4\delta_1$, $\lambda \perp m$. ⊛

Specifically, we call such a decomposition *Lebesgue decomposition* of ν with respect to μ. Now, with the condition $\nu \ll \mu$, Theorem 6.2.2 implies that $\mathrm{d}\nu = f\,\mathrm{d}\mu$ for some f, which is exactly the statement of Theorem 6.2.1. And, it should be clear now that the definition of Radon Nikodym derivative of ν with respect to μ, denoted as $\mathrm{d}\nu/\,\mathrm{d}\mu$, is just f in this case.

As previously seen. If $\nu = \nu^+ - \nu^-$, we defined the total variation $|\nu| = \nu^+ + \nu^-$. Then we have $|\nu(E)| \leq |\nu|(E)$.

6.3 Lebesgue Differentiation Theorem for Regular Borel Measures

Definition 6.3.1 (Regular)**.** A Borel signed measure ν on $\mathbb{R}^d$ is called *regular* if

- (compact finite) $|\nu|(K) < \infty$ for all compact K.
- (outer regularity) We have outer regularity

$$|\nu|(E) = \inf\{|\nu|(U) \mid \text{ open } U \supseteq E\}$$

for every Borel set E.

Example. We see that

(1) Any Lebesgue-Stieltjes measure on $\mathbb{R}$ has this property from Theorem 1.7.1, so is the difference between two of them (at least if one of them is finite).

(2) The Lebesgue measure on $\mathbb{R}^d$ is regular.

Note. From compact finiteness, if ν is regular then it is σ-finite.

Lemma 6.3.1. $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ if and only if $\mathrm{d}\nu = f\,\mathrm{d}m$ is regular.

Proof. We prove this in two directions.

($\Leftarrow$) Suppose $\mathrm{d}\nu = f\,\mathrm{d}m$ is regular. Then

$$|\nu|(K) = \int_K |f|\,\mathrm{d}m < \infty$$

for all compact K, thus $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

($\Rightarrow$) Suppose $f \in L^1_{\text{loc}(\mathbb{R}^d)}$. This condition is clearly equivalent to compact finiteness. If this holds, then the outer regularity may be verified directly as follows. Suppose that E is a bounded Borel set. Given $\delta > 0$, by Theorem 3.5.1, there is a bounded open $U \supset E$ such that $m(U) < m(E) + \delta$ and hence $m(U \setminus E) < \delta$. But then, given $\epsilon > 0$, there is[a] an open $U \supset E$ such that

$$\int_{U\setminus E} f\,\mathrm{d}m < \epsilon$$

and hence

$$\int_U f\,\mathrm{d}m < \int_E f\,\mathrm{d}m + \epsilon.$$

The case of unbounded E follows easily by writing $E = \bigcup_{j=1}^\infty E_j$ where E_j is bounded and finding an open $U_j \supset E_j$ such that

$$\int_{U_j\setminus E_j} f\,\mathrm{d}m < \epsilon 2^{-j}.$$

■

[a] This follows from [FF99] Corollary 3.6.

As previously seen. Recall the Lebesgue differentiation theorem, here we had that if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ implies that for Lebesgue almost every x,

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)\,\mathrm{d}y = f(x)$$

for any $\{E_r\}$ shrinks nicely to x.

Corollary 6.3.1. Let ρ be a regular signed Borel measure on $\mathbb{R}^d$. Suppose $\rho \ll m$. Then $\mathrm{d}\rho = f\,\mathrm{d}m$ for some $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, So then for Lebesgue almost every x we have

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)\,\mathrm{d}y = f(x).$$

Writing this nicely, using established notation, this is

$$\lim_{r\to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for every $\{E_r\}$ shrinks nicely to x.

Proposition 6.3.1. Let λ be a regular positive Borel measure on $\mathbb{R}^d$. Suppose $\lambda \perp m$. Then for Lebesgue almost every x, we have

$$\lim_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for every $\{E_r\}$ shrinking to x nicely (equivalently, shrinking to 0 nicely).

Proof. It is enough to consider $E_r = B(x, r)$. We wish to prove that

$$G := \left\{ x \mid \limsup_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} \neq 0 \right\} = \bigcup_{n=1}^\infty G_n$$

where

$$G_n := \left\{ x \mid \limsup_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} > \frac{1}{n} \right\}$$

such that $m(G) = 0$. We see that it suffices to show $m(G_n) = 0$ for all n. Since $\lambda \perp m$, so we know there exists A, B such that $\mathbb{R}^d = A \cup B$ disjoint with $\lambda(A) = 0$, $m(B) = 0$. Thus, it suffices to show $m(G_n \cap A) = 0$.

Note. Alternatively, we can simply define G_n over A instead of $\mathbb{R}^d$, as in Folland[FF99].

Claim. Given a A and B defined above induced from Theorem 6.2.2, $m(G_n \cap A) = 0$ for all n

Proof. Fix $\epsilon > 0$, since λ is regular, there exists an open set $U \supseteq A$ such that $\lambda(U) \leq \lambda(A) + \epsilon = \epsilon$. We see that for every $x \in G_n \cap A$, there is an $r_x > 0$ such that $\lambda(B(x, r_x))/m(B(x, r_x)) > 1/n$ where $B(x, r_x) \subseteq U$.

Let $K \subseteq G_n \cap A$, compact. Then $K \subseteq \bigcup_{x\in K} B(x, r_x)$. By compactness, we can take a finite sub-cover, and then use Lemma 4.1.1 to find disjoint $B_1, B_2, \ldots, B_N$ such that each of B_i is in the form of $B(x_i, r_{x_i})$ and $K \subseteq \bigcup_i 3B_i$. Therefore,

$$m(K) \leq 3^d \sum_{i=1}^{N} m(B_i) \leq 3^d n \sum_{i=1}^{N} \lambda(B_i) = 3^d n \lambda\left(\bigcup_{i=1}^{N} B_i\right) \leq 3^d n \lambda(U) = 3^d n \epsilon.$$

By inner regularity, $m(G_n \cap A) \leq 3^d n\epsilon$ for any $\epsilon > 0$. Taking $\epsilon \to 0$ yields $m(G_n \cap A) = 0$, so then $m(G_n) = 0$ as desired. ⊛

■

Lecture 33: Monotone Differentiation Theorem

1 Apr. 11:00

As previously seen. We have that if $\rho \ll m$ is regular then

$$\lim_{r\to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for Lebesgue almost every x, where $\{E_r\}$ shrinks nicely to x. Likewise, if $\lambda \perp m$ regular (positive measure) then

$$\lim_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for Lebesgue almost every x, where $\{E_r\}$ shrinks nicely to x.

From this, we can easily deduce the following important result.

Theorem 6.3.1 (Lebesgue differentiation theorem for regular measures)**.** Let ν be a regular Borel signed measure on $\mathbb{R}^d$. Then $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}m$, $\lambda \perp m$ by Theorem 6.2.2. Then for Lebesgue almost every x,

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every $\{E_r\}$ shrinks nicely to x.

Proof. It must be checked that ν regular implies $\lambda, f\,\mathrm{d}m$ are regular. In particular, since $f \in L^1_{\mathrm{loc}}$, so from Theorem 4.2.1 and its corollary (Corollary 4.2.1, Corollary 4.2.2), we see that it suffices to show that if λ is regular and $\lambda \perp m$, then for Lebesgue a.e. x, Check!

$$\lim_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} \to 0$$

when $\{E_r\}$ shrinks nicely to x. It also suffices to take $E_r = B(r, x)$ and to assume that λ is positive, since for some $\alpha > 0$, we have

$$\left|\frac{\lambda(E_r)}{m(E_r)}\right| \leq \frac{|\lambda|(E_r)}{m(E_r)} \leq \frac{|\lambda|(B(r,x))}{m(E_r)} \leq \frac{|\lambda|(B(r,x))}{\alpha m(B(r,x))}.$$

Therefore, if $|\lambda|(E_r)/m(E_r) \to 0$, so does $|\lambda(E_r)/m(E_r)|$, hence $\lambda(E_r)/m(E_r)$. We see that the result then follows directly from Proposition 6.3.1. ■

6.4 Monotone Differentiation Theorem

We first formalize one ambiguous notation we used long time ago with discussing distribution function. Namely, $F(x^+), F(x^-)$.

Definition. For a $F\colon \mathbb{R} \to \mathbb{R}$ that is monotonically increasing, we have the following.

Definition 6.4.1 ($F(x^+)$). We define $F(x^+) = \lim_{y \to x^+} F(y)$.

Definition 6.4.2 ($F(x^-)$). We define $F(x^-) = \lim_{y \to x^-} F(y)$.

Remark. We see that if F is monotonically increasing, then $F(x^+), F(x^-)$ exist and are

$$\inf_{y > x} F(y), \quad \sup_{y < x} F(y)$$

respectively since it's bounded below/above respectively by $F(x)$.

Lemma 6.4.1. If $F\colon \mathbb{R} \to \mathbb{R}$ is monotonically increasing, then

$$D = \{x \in \mathbb{R} \mid F \text{ is discontinuous at } x\}$$

is a countable set.

Proof. $x \in D$ if and only if $F(x^+) > F(x^-)$. For each $x \in D$, let $I_x = (F(x^-), F(x^+))$, not empty. Also, if $x, y \in D$, $x \neq y$, then I_x, I_y are disjoint. Now, for $|x| < N$, I_x lie in the interval $(F(-N), F(N))$. Hence,

$$\sum_{|x|<N} \left[F(x^+) - F(x^-)\right] \leq F(N) - F(-N) < \infty,$$

which implies that

$$D \cap (-N, N) = \left\{x \in (-N, N) \mid F(x^+) \neq F(x^-)\right\}$$

is countable. Since this is true for all N, the result follows. ■

Theorem 6.4.1 (Monotone Differentiation Theorem). Let F be an increasing function, then

- F is differentiable Lebesgue almost everywhere.
- $G(x) := F(x^+)$[a] is differentiable almost everywhere.
- $G' = F'$ almost everywhere

[a]Observe that G is increasing and right-continuous.

Proof. Start with $G(x) := F(x^+)$, which is increasing and right-continuous on $\mathbb{R}$. There is then a

Lebesgue-Stieltjes measure μ_G on $\mathbb{R}$, thus it is regular on $\mathbb{R}$. We see

$$\frac{G(x+h)-G(x)}{h} = \begin{cases} \dfrac{\mu_G((x,x+h])}{m((x,x+h])}, & \text{if } h>0; \\ \dfrac{\mu_G((x+h,x])}{m((x+h,x])}, & \text{if } h<0. \end{cases}$$

Note that both $\{(x,x+h]\}$ and $\{(x+h,x]\}$ shrink nicely to x as $|h|\to 0$. By Theorem 6.3.1 (since these shrink nicely), we then know that these both converge for Lebesgue almost every x to some common limit $f(x)$. Hence, G' exists Lebesgue almost everywhere. We now show that by defining $H \coloneqq G - F$, H' exists and equals zero a.e.

Observe that $H(x) = G(x) - F(x) \geq 0$, and we see that

$$\{x \mid H(x) > 0\} \subseteq \{x \mid F \text{ is discontinuous at } x\}.$$

The latter set is then countable by Lemma 6.4.1, hence we can write $\{x \mid H(x)>0\} = \{x_n\}$. Then let

$$\mu \coloneqq \sum_n H(x_n)\delta_{x_n}.$$

This is a Borel measure, so we check if it is locally finite. Indeed, since

$$\mu((-N,N)) = \sum_{-N<x_n<N} H(x_n) \leq G(N) - F(-N) < \infty,$$

where checking the inequality just consists of seeing that the intervals $(F(x_n), G(x_n))$ are disjoint and is a subset of $(F(-N), G(N))$, so

$$\sum_{-N<x_n<N} H(x_n) = \mu\left(\bigcup_n (F(x_n), G(x_n))\right) \leq \mu((F(-N), G(N))).$$

Thus, μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, so it is regular.

Remark. Special to $\mathbb{R}$, we have that

$$\text{locally finite Borel} \Rightarrow \text{Lebesgue-Stieltjes} \Rightarrow \text{regular} \Rightarrow \text{outer regularity}.$$

Also, we have $\mu \perp m$ since $m(E) = \mu(E^c) = 0$ where $E = \{x_n\}$. Then we have that

$$\left|\frac{H(x+h)-H(x)}{h}\right| \leq \frac{H(x+h)+H(x)}{|h|} \leq \frac{\mu((x-2h, x+2h))}{|h|},$$

which goes to 0 for Lebesgue almost every x by Theorem 6.3.1 and that $\mu \perp m$.

Thus, H is differentiable almost everywhere and $H' = 0$ almost everywhere, which implies F is differentiable almost everywhere and $F' = G'$ almost everywhere. ■

Proposition 6.4.1. Suppose F is an increasing function, then F' exists almost everywhere and is measurable, then

$$\int_a^b F'(x)\,\mathrm{d}x \leq F(b) - F(a).$$

Example. The inequality can't be made into equality in Proposition 6.4.1 by the given condition, or even if F is continuous in addition.

Proof. Take $F(x)$ to be 0 on $x \leq 0$, 1 on $x > 0$. Then $F'(x) = 0$ almost everywhere. So

$$\int_{-1}^1 F'(x)\,\mathrm{d}x = 0 < 1 = F(1) - F(-1).$$

Even if F is continuous we might not have equality. Take $F(x)$ to be the Cantor function. Then $F'(x) = 0$ almost everywhere, but

$$\int_0^1 F'(x)\,\mathrm{d}x = 0 < 1 = F(1) - F(0).$$

⊛

Lecture 34: Functions of Bounded Variation

4 Apr. 11:00

Proof of Proposition 6.4.1. Let

$$G(x) \coloneqq \begin{cases} F(a), & \text{if } x < a; \\ F(x), & \text{if } a \leq x \leq b; \\ F(b), & \text{if } x > b. \end{cases}$$

Then G is increasing. We define

$$g_n(x) = \frac{G(x + 1/n) - G(x)}{1/n} \to F'(x)$$

for almost every $x \in [a, b]$. We note that $g_n(x) \geq 0$. Theorem 2.2.2 tells us that

$$\int_a^b F'(x)\,\mathrm{d}x = \int_a^b \liminf_{n\to\infty} g_n(x)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_a^b g_n(x)\,\mathrm{d}x.$$

We then evaluate

$$\begin{aligned} \int_a^b g_n(x) &= n\left(\int_{a+1/n}^{b+1/n} G(x)\,\mathrm{d}x - \int_a^b G(x)\,\mathrm{d}x\right) \\ &= n\left(\int_b^{b+1/n} G(x)\,\mathrm{d}x - \int_a^{a+1/n} G(x)\,\mathrm{d}x\right) \\ &\leq n\left(G\left(b + \frac{1}{n}\right)\cdot\frac{1}{n} - G(a)\cdot\frac{1}{n}\right) = F(b) - F(a). \end{aligned}$$

Therefore,

$$\int_a^b F'(x)\,\mathrm{d}x \leq F(b) - F(a).$$

■

6.5 Functions of Bounded Variation

Definition 6.5.1 (Total variation function)**.** For $F\colon \mathbb{R} \to \mathbb{R}$, the *total variation function of F* is $T_F\colon \mathbb{R} \to [0, \infty]$ defined by

$$T_F(x) = \sup\left\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, -\infty < x_0 < x_1 < \ldots < x_n = x\right\}.$$

Lemma 6.5.1. $T_F(b)$ is equal to

$$T_F(a) + \sup\left\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, a = x_0 < x_1 < \ldots < x_n = b\right\}$$

if $a < b$.

Proof. The idea is that the sums in the Definition 6.1.8 of T_F are made bigger if the additional subdivision points x_j are added. Hence, if $a < b$, $T_F(b)$ is unaffected if we assume that a is always one of the subdivision points. ■

Remark. T_F is increasing.

Definition 6.5.2 (Bounded variation)**.** We say that F is of *bounded variation*, denoted as $F \in BV$, provided that

$$T_F(\infty) = \lim_{x\to\infty} T_F(x) < \infty.$$

Similarly, $F \in BV([a, b])$ means that

$$\sup\left\{\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b\right\} < \infty.$$

Remark. We see the following.

(1) If F is of bounded variation, then F is bounded.

(2) $F(x) = \sin x$ is not of bounded variation, but it is of bounded variation over any $[a, b]$.

(3) For $F(x)$ defined as

$$F(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0 \end{cases}$$

is not of bounded variation of $[a, b]$ if $a < 0 < b$ because the harmonic series does not converge.

Before we see more properties of bounded variation function, we introduce a useful characterization of a function.

Definition 6.5.3 (Lipschitz)**.** A function $F\colon [a, b] \to \mathbb{C}$ is called *Lipschitz* if there exists an $M \geq 0$ such that

$$|F(x) - F(y)| \leq M\,|x - y|\,.$$

Remark. We have the following.

(1) If F, G are of bounded variation, $\alpha F + \beta G$ are of bounded variation.

(2) If F is increasing and bounded, then F is a function of bounded variation.

(3) If F is Lipschitz on $[a, b]$, then $F \in BV([a, b])$.

(4) If F is differentiable, and F' is bounded on $[a, b]$, then F is Lipschitz (mean value theorem), so it is in $BV([a, b])$.

In particular, we have the following.

Remark. If $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ for $f \in L^1(\mathbb{R})$, then $F \in BV$.

Proof. We see this by

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} \left|\int_{x_{i-1}}^{x_i} f(t)\,\mathrm{d}t\right| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,\mathrm{d}t = \int_{x_0}^{x_n} |f(t)|\,\mathrm{d}t \leq \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t,$$

which is finite since $f \in L^1(\mathbb{R})$. ⊛

Lemma 6.5.2. If $F \in BV$, then T_F is bounded, increasing, $T_F(-\infty) = 0$.

Proof. ■ DIY

Lemma 6.5.3. $F \in BV$, then $T_F \pm F$ are increasing and bounded and.

Proof. Let $x < y$ and fix $\epsilon > 0$, then there are points $x_0 < x_1 < \cdots < x_n = x$ such that

$$T_F(x) \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \epsilon.$$

Furthermore,

$$T_F(y) \geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)|.$$

Then, since $\pm(F(y) - F(x)) \leq |F(y) - F(x)|$, we have

$$T_F(y) \pm (F(y) - F(x)) \geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \geq T_F(x) - \epsilon,$$

hence

$$T_F(y) \pm F(y) \geq T_F(x) \pm F(x) - \epsilon.$$

Taking $\epsilon \to 0$ yields the result. ■

Remark. Thus, any $F \in BV$ can be written as

$$F = \frac{T_F + F}{2} - \frac{T_F - F}{2}$$

which is a difference of increasing and bounded functions.

Theorem 6.5.1. F is of bounded variation if and only if $F = F_1 - F_2$ for F_1, F_2 increasing and bounded.

Proof. The forward implication is given by the Lemma 6.5.3. The other direction follows from the examples we gave. ■ check!

Corollary 6.5.1 (Bounded Variation Differentiation). $F \in BV$ implies that F is differentiable almost everywhere. Furthermore,

(1) $F(x^+), F(x^-)$ exist for all x as do $F(-\infty), F(\infty)$.

(2) The set of discontinuities of F is countable.

(3) $G(x) = F(x^+)$ is differentiable and $G' = F'$ almost everywhere.

(4) $F' \in L^1(\mathbb{R}, m)$ (i.e. $F \in L^1_{\text{loc}}(\mathbb{R})$) for every $a < b$.

Proof. ■ DIY

Lecture 35: Continue on Functions of Bounded Variation

6 Apr. 11:00

Definition 6.5.4 (Normalized bounded variation). A function $G \in BV$ is said to have *normalized bounded variation*, denoted as $G \in NBV$ provided that G is right continuous and $G(-\infty) = 0$.

Example. If F is increasing and bounded, F right continuous, $F(-\infty) = 0$.
$F(x) = \int_{-\infty}^{x} f(t)\,dt$, $f \in L^1(\mathbb{R})$. Midterm gave F is uniformly continuous.

Lemma 6.5.4. If $F \in BV$ is right continuous, then $T_F \in NBV$.

Proof. T_F is bounded, increasing, and satisfies $T_F(-\infty) = 0$ by Lemma 6.5.2. Thus, $T_F \in BV$.

Hence, we just need to check that T_F is right continuous. Suppose not, then there is a point $a \in \mathbb{R}$ such that $c := T_F(a^+) - T_F(a) > 0$.

Fix $\epsilon > 0$, since $F(x)$ and $g(x) := T_F(x^+)$ are right-continuous, there exists a $\delta > 0$ such that for $y \in (a, a+\delta]$ we have

$$|F(y) - F(a)| - \epsilon, \quad |g(y) - g(a)| < \epsilon.$$

We then have that

$$T_F(y) - T_F(a^+) \leq T_F(y^+) - T_F(a^+) < \epsilon.$$

There exist $a = x_0 < x_1 < \cdots < x_n = a + \delta$ such that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \geq T_F(a+\delta) - T_F(a) - \frac{c}{4} \geq T_F(a^+) - T_F(a) - \frac{c}{4} = \frac{3c}{4}.$$

Then $|F(x_1) - F(a)| < \epsilon$ so we have

$$\sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4} - \epsilon.$$

There exist $a = t_0 < \cdots < t_k = x_1$ such that

$$\sum_{i=1}^{k} |F(t_i) - F(t_{i-1})| \geq T_F(x_1) - T_F(a) - \frac{c}{4} \geq \frac{3}{4}c.$$

Then as $[a, a+\delta] = [a, x_1] \cup [x_1, a+\delta]$ we see that

$$T_F(a+\delta) - T_F(a) \geq \sum_{j=1}^{k} |F(t_j) - F(t_{j-1})| + \sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}c - \epsilon + \frac{3}{4}c = \frac{3}{2}c - \epsilon.$$

Thus

$$\epsilon + c \geq T_F(a+\delta) - T_F(a^+) + T_F(a^+) - T_F(a) = T_F(a+\delta) - T_F(a) \geq \frac{3}{2}c - \epsilon$$

and

$$c \leq 4\epsilon.$$

Thus taking $\epsilon \to 0$ yields $c = 0$, which is a contradiction. ■

Corollary 6.5.2. $F \in NBV$ if and only if $F = F_1 - F_2$, $F_1, F_2 \in NBV$ and increasing.

Proof. $F = (T_F + F)/2 - (T_F - F)/2$. ■

Theorem 6.5.2. We have that

(1) Suppose that μ is a finite signed Borel measure on $\mathbb{R}$, then $F(x) = \mu((-\infty, x]) \in NBV$.

(2) $F \in NBV$ implies that there exists a unique finite signed Borel measure on $\mathbb{R}$ satisfying $\mu_F((-\infty, x]) = F(x)$.

Proof. We have

(1) Let $\mu = \mu^+ - \mu^-$, then $F = F^+ - F^-$, where $F^{\pm}(x) = \mu^{\pm}((-\infty, x])$, which are bounded, right continuous, $F^{\pm}(-\infty) = 0$, so $F^{\pm} \in NBV$.

(2) Let $F \in NBV$, then $F = F_1 - F_2$, $F_1, F_2 \in NBV$ and increasing. Then define μ_{F_1}, μ_{F_2} by Lebesgue-Stieltjes measure, and set $\mu_F := \mu_{F_1} - \mu_{F_2}$.

Show Uniqueness in HW

■

Proposition 6.5.1. We have the following.

(1) If $F \in NBV$, then F is differentiable almost everywhere, $F' \in L^1(\mathbb{R}, m)$.

(2) $\mu_F + \lambda + F'\,m$ for some measure λ satisfying $\lambda \perp m$.

(3) $\mu_F \perp m$ if and only if $F' = 0$ Lebesgue almost everywhere.

(4) $\mu_F \ll m$ if and only if $\int_{-\infty}^{x} F'(t)\,t = F(x) - F(-\infty) = F(x)$.

Proof.

Check (1), (2), (3)

For (4), we have

$$\begin{aligned}
\mu_F \ll m &\Leftrightarrow \lambda = 0 \\
&\Leftrightarrow \mu_F = F'\,m \\
&\Leftrightarrow \mu_F(E) = \int_E F'\,m \ \ \forall \text{ Borel } E \\
&\Leftrightarrow F(x) = \mu_F((-\infty, x]) = \int_{-\infty}^{x} F'(t)\,t, \quad \forall x \in \mathbb{R}.
\end{aligned}$$

The last converse comes from the uniqueness of Theorem 6.5.2 above. ■

Lecture 36: Absolutely Continuous Functions

6.5.1 Absolutely Continuous Functions

8 Apr. 11:00

We start with a definition.

Definition 6.5.5 (Absolutely continuous function)**.** We say that $F\colon \mathbb{R} \to \mathbb{R}$ is *absolutely continuous*, denoted as $F \in AC$, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $(a_1, b_1), \ldots, (a_N, b_N)$ are finitely many disjoint open intervals satisfying $\sum_{n=1}^{N}(b_n - a_n) < \delta$, then

$$\sum_{n=1}^{N} \lvert F(b_n) - F(a_n) \rvert < \epsilon.$$

Note. Do not be confused between Definition 6.5.5 and Definition 6.2.1.

Lemma 6.5.5. We have that

(1) If F is absolutely continuous, then it is uniformly continuous.

(2) If F is Lipschitz, then F is absolutely continuous.

(3) $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$, $f \in L^1$, is absolutely continuous.

Proof. We prove this one by one.

(1) We simply take $N = 1$.

(2) This is trivial.

(3) We write this out as

$$\sum_{n=1}^{N} |F(b_n) - F(a_n)| = \sum_{n=1}^{N} \left| \int_{a_n}^{b_n} f(t)\,\mathrm{d}t \right| \le \sum_{n=1}^{N} \int_{a_n}^{b_n} |f(t)|\,\mathrm{d}t = \int_E |f(t)|\,\mathrm{d}t$$

where $E = \bigcup_{n=1}^{N}(a_n, b_n)$, so $m(E) = \sum_{n=1}^{N}(b_n - a_n)$. From Midterm Q1, if $f \in L^1(X, \mu)$, for all $\epsilon > 0$, there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |f| < \epsilon$. This directly implies that this function is absolutely continuous.

■

Example. The Cantor function F is uniformly continuous. However, we will see that it is not absolutely continuous.

Proposition 6.5.2. Suppose $F \in NBV$, then F is absolutely continuous if and only if $\mu_F \ll m$.

Proof. We prove two directions.

($\Leftarrow$) Suppose $\mu_F \ll m$. Then $F(x) = \int_{-\infty}^{x} F'(t)\,\mathrm{d}t$, and $F' \in L^1(\mathbb{R}, m)$, by Proposition 6.5.1. Therefore, $F \in AC$.

($\Rightarrow$) Now suppose $F \in AC$. Note that since F is continuous,

$$\mu_F((a,b)) = \lim_{n\to\infty} \mu_F((a, b - 1/n]) = \lim_{n\to\infty} F(b - 1/n) - F(a) = F(b) - F(a).$$

We let E be a Borel set with $m(E) = 0$. Fix $\epsilon > 0$, we will show $|\mu_F(E)| \le \epsilon$. Let $\delta > 0$ be the constant from $F \in AC$.

We know that there exists open $U_1 \supseteq U_2 \supseteq \cdots \supseteq E$ such that $\lim\limits_{n\to\infty} m(U_n) = m(E) = 0$, and open $V_1 \supseteq V_2 \supseteq \cdots \supseteq E$ such that $\lim\limits_{n\to\infty} \mu_F(V_n) = \mu_F(E)$ by regularity.

Let $O_n = U_n \cap V_n$, then $O_1 \supseteq O_2 \supseteq \cdots \supseteq E$, and by monotonicity (for μ_F decomposing into pos/neg first)

$$\lim_{n\to\infty} m(O_n) = m(E) = 0, \quad \lim_{n\to\infty} \mu_F(O_n) = \mu_F(E).$$

Thus without loss of generality, we may assume $m(O_1) < \delta$. Each O_n is a countable union of disjoint intervals

$$O_n = \bigcup_{k=1}^{\infty} (a_k^n, b_k^n)\,.$$

For any N we also have

$$\sum_{k=1}^{N} (b_k^n - a_k^n) \le m(O_n) \le m(O_1) < \delta.$$

Therefore,

$$\left| \mu_F\left(\bigcup_{k=1}^{N} (a_k^n, b_k^n) \right) \right| = \left| \sum_{k=1}^{N} \mu_F((a_k^n, b_k^n)) \right| \le \sum_{k=1}^{N} |\mu_F((a_k^n, b_k^n))| \le \sum_{k=1}^{N} |F(b_k^n) - F(a_k^n)| < \epsilon,$$

which implies

$$|\mu_F(O_n)| = \lim_{N\to\infty} \left| \mu_F\left(\bigcup_{k=1}^{N} (a_k^n, b_k^n) \right) \right| \le \epsilon$$

and

$$|\mu_F(E)| = \lim_{n\to\infty} |\mu_F(O_n)| \leq \epsilon.$$

By taking $\epsilon \to 0$ we have $\mu_F(E) = 0$, hence $\mu_F \ll m$. ■

Corollary 6.5.3. $F \in NBV \cap AC$ if and only if $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ for some $f \in L^1(\mathbb{R}, m)$. If this holds, we have $f = F'$ Lebesgue almost everywhere.

Lemma 6.5.6. If $F \in AC([a, b])$, then $F \in NBV([a, b])$.

Proof. ■

DIY

Theorem 6.5.3 (Fundamental Theorem of Calculus). For $F \in [a, b] \to \mathbb{R}$, the following are equivalent.

(1) $F \in AC([a, b])$.

(2) $F(x) - F(a) = \int_a^x f(t)\,\mathrm{d}t$ for some $f \in L^1([a, b], m)$.

(3) F is differentiable almost everywhere on $[a, b]$ and $F(x) - F(a) = \int_a^b F'(t)\,\mathrm{d}t$.

Proof. This follows directly from **??**. ■

Definition. Let μ be a finite signed Borel measure on $\mathbb{R}$.

Definition 6.5.6 (Discrete measure). μ is called a *discrete measure* if there is a countable set $\{x_n\}$ and $c_n \neq 0$ such that $\sum_{n=1}^{\infty} |c_n| < \infty$ and $\mu = \sum_n c_n \delta_{x_n}$.

Note. δ_{x_n} is the Dirac delta measure at x_n.

Definition 6.5.7 (Continuous measure). μ is called a *continuous measure* if $\mu(\{a\}) = 0$ for all $a \in \mathbb{R}$.

Lemma 6.5.7. Given a finite signed Borel measure μ,

(1) Any $\mu = \mu_d + \mu_c$, where μ_d is discrete, μ_c is continuous are uniquely determined.

(2) μ discrete implies $\mu \perp m$.

(3) $\mu \ll m$ implies μ is continuous.

Corollary 6.5.4. For μ a finite signed Borel measure on $\mathbb{R}$, we have that

$$\mu = \mu_d + \mu_{ac} + \mu_{sc}$$

where μ_d is discrete, μ_{ac} is absolutely continuous, and μ_{sc} is continuous and singularly to m.

Chapter 7

Hilbert Spaces

Lecture 37: Hilbert Spaces

7.1 Inner Product Spaces

11 Apr. 11:00

Definition 7.1.1 (Inner product). Let V be a (complex) vector space. An *inner product* is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

such that

- $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in V$, and $\alpha, \beta \in \mathbb{C}$.
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for every $x, y \in V$.
- $\langle x, x \rangle \in [0, \infty)$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.

Note. Note that we have conjugate linearity in the second argument, i.e.,

$$\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$$

for any $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.

Proof. This follows from

$$\langle x, \alpha y + \beta z \rangle = \overline{\langle \alpha y + \beta z, x \rangle} = \overline{\alpha \langle y, x \rangle + \beta \langle z, x \rangle} = \overline{\alpha}\overline{\langle y, x \rangle} + \overline{\beta}\overline{\langle z, x \rangle}. = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle .$$

⊛

Example. We have the following examples.

- $\mathbb{R}^d$ with $\langle x, y \rangle = x \cdot y = \sum_{i=1}^d x_i y_i$.
- $\mathbb{C}^d$ with $\langle x, y \rangle = \sum_{i=1}^d x_i \overline{y_i}$.
- $L^2(X, \mu)$ with $\langle f, g \rangle = \int_X f \overline{g} \,\mathrm{d}\mu$. Note by Theorem 5.2.1,

$$\left\lvert \int_X f \overline{g} \right\rvert \leq \lVert f \overline{g} \rVert_1 \leq \lVert f \rVert_2 \lVert g \rVert_2 < \infty$$

because $1/2 + 1/2 = 1$.

- A special case is ℓ^2, where we have

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

Note. Note that

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$$

Theorem 7.1.1 (Cauchy-Schwarz Inequality)**.** Given an inner product space, $|\langle x, y\rangle| \leq \|x\| \|y\|$.

Proof. This is clear if $\langle x, y\rangle = 0$. Assume $\langle x, y\rangle \neq 0$, then for every $\alpha \in \mathbb{C}$, we know that

$$0 \leq \|\alpha x - y\|^2 = |\alpha|^2 \|x\|^2 - 2\operatorname{Re}\alpha\langle x, y\rangle + \|y\|^2.$$

Write $\langle x, y\rangle = |\langle x, y\rangle| e^{i\theta}$, and take $\alpha = e^{-i\theta}t$ for arbitrary $t \in \mathbb{R}$. Then, the right-hand side gives

$$0 \leq \|x\|^2 t^2 - 2|\langle x, y\rangle| t + \|y\|^2.$$

Note this is a real quadratic function of t, with at most one real root. Thus, the discriminant $\Delta \leq 0$. Specifically, we have

$$\Delta = 4|\langle x, y\rangle|^2 - 4\|x\|^2\|y\|^2 \leq 0 \Leftrightarrow |\langle x, y\rangle|^2 \leq \|x\|^2\|y\|^2 \Leftrightarrow |\langle x, y\rangle| \leq \|x\| \|y\|.$$

■

Definition 7.1.2 (Induced norm from inner product)**.** Given an inner product space V, let

$$\|x\| \coloneqq \sqrt{\langle x, x\rangle},$$

which is so-call the *norm induced from the inner product*.

Proof. We need to check that this actually defines a norm. We check the following.

Claim. $\|x\| = 0 \Leftrightarrow x = 0$ for all $x \in V$.

Proof. This follows from the definition of an inner product. ⊛

Claim. $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in V$.

Proof. This follows from

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\overline{\alpha}\langle x, x\rangle} = |\alpha| \|x\|.$$

⊛

Claim (Triangle inequality)**.** $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Proof. The triangle inequality is less obvious, and comes from Theorem 7.1.1. Namely,

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\left|\langle x, y\rangle\right| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2\end{aligned}$$

Taking square root on both sides, we have

$$\|x+y\| \leq \|x\| + \|y\|.$$

⊛

■

Theorem 7.1.2 (Parallelogram law)**.** Let V be a normed vector space. Then, $\|\cdot\|$ is induced by an inner product if and only if

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in V$.

Proof. We show two directions.

($\Rightarrow$) This follows from

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$$

and

$$\|x \pm iy\|^2 = \|x\|^2 \pm 2\operatorname{Im}\langle x, y\rangle + \|y\|^2.$$

($\Leftarrow$) Firstly, we define

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \|x+iy\|^2 - i\|x-iy\|^2\right)$$

as motivated by the above relationship.

Exercise. Check this inner product is indeed inducing the desired norm.

■

Example. Consider $L^p(\mathbb{R}, m)$, $f = \mathbb{1}_{(0,1)}$, $g = \mathbb{1}_{(1,2)}$. We see the parallelogram law is satisfied only when $p = 2$.

Remark. Hence, $L^p(\mathbb{R}, m)$ is only an inner product space when $p = 2$.

Since we're doing real analysis, we want to deal with limits. It turns out that with an inner product space, we can say something more compare to the case of a normed vector space. We now illustrate this.

Definition. Given a vector space V with either a norm or an inner product, we have the followings.

Definition 7.1.3 (Strong convergence). We say that $x_n \in V$ *converges to* $x \in V$ *strongly* if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Definition 7.1.4 (Weak convergnece). We say that $x_n \in V$ *converges to* $x \in V$ *weakly* if for any fixed $y \in V$,

$$\lim_{n\to\infty} \langle x_n - x, y\rangle = 0.$$

Lemma 7.1.1 (Strong convergence implies weak convergence). Suppose V is an inner product space. If $x_n \to x$ strongly, then $x_n \to x$ *weakly*.

Proof. By Cauchy-Schwarz inequality,

$$0 \leq |\langle x_n - x, y\rangle| \leq \|x_n - x\| \cdot \|y\|.$$

Since $\|x_n - x\| \to 0$ and $\|y\|$ is constant in n, from the Squeeze theorem, we have

$$\langle x_n - x, y\rangle \to 0$$

as $n \to \infty$. ■

Example. Consider ℓ^2, $x_n = (0, \ldots, 0, 1, 0, \ldots)$ and $x = 0$. Then x_n does not converge strongly to any vector, but it does converge to 0 weakly.

Proof. If we fix $y \in \ell^2$, then

$$\langle x_n - x, y\rangle = \overline{y}_n$$

which goes to 0 as $n \to \infty$ because $\sum_n |y_n|^2 < \infty$. Therefore, $x_n \to 0$ weakly, but we see that

$$\|x_n - 0\| = \|x_n\| = 1.$$

Thus, $x_n \not\to 0$ strongly. ⊛

7.1.1 Orthonormal Bases

Definition 7.1.5 (Orthogonal). Two vectors x, y are *orthogonal* if $\langle x, y\rangle = 0$, denoted as $x \perp y$.

Remark. Do not confuse between this notation and Definition 6.1.5.

Lemma 7.1.2 (Pythagorean Theorem). If $x_1, \ldots, x_n \in V$, $\langle x_i, x_j\rangle = 0$ for all $i \neq j$, then

$$\left\|\sum_{i=1}^n x_i\right\|^2 = \sum_{i=1}^n \|x_i\|^2.$$

Proof. Use that $\|x + y\| = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$ and induct. ■

Definition 7.1.6 (Orthonormal set). We call $\{e_i\}_{i\in I}$ an *orthonormal set* if

$$\langle e_i, e_j\rangle = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j. \end{cases}$$

Lecture 38: Orthonormal Basis

Let's start with a lemma. 13 Apr. 11:00

Lemma 7.1.3 (Best approximation). Let $e_1, \ldots, e_N$ be orthonormal, then for $x \in V$, let $\alpha_i = \langle x, e_i \rangle$, then

$$\left\| x - \sum_{i=1}^{N} \alpha_i e_i \right\| \leq \left\| x - \sum_{i=1}^{N} \beta_i e_i \right\|$$

for all $\beta_1, \ldots, \beta_N \in \mathbb{C}$. This is the *best approximation* to x within the span of $e_1, \ldots, e_N$.

Note. We can also think of it as an orthogonal projection.

Proof. Let $z = x - \sum_{i=1}^{N} \alpha_i e_i$, $w = \sum_{i=1}^{N} (\alpha_i - \beta_i) e_i$. Note that for all $n = 1, \ldots, N$, we have

$$\langle z, e_n \rangle = \langle x, e_n \rangle - \alpha_n = 0.$$

Thus, $\langle z, w \rangle = 0$. So by the Pythagorean theorem,

$$\|z + w\|^2 = \|z\|^2 + \|w\|^2 \geq \|z\|,$$

which proves the result. ■

Lemma 7.1.4 (Bessel's inequality). Let $\{e_i\}_1^\infty$ be an orthonormal set. For $x \in V$, let $\alpha_i = \langle x, e_i \rangle$. Then,

(1) We have that

$$\|x\|^2 = \left\| x - \sum_{i=1}^{N} \alpha_i e_i \right\| x^2 + \sum_{i=1}^{N} |\alpha_i|^2$$

for all $N \in \mathbb{N}$.

(2) $\sum_{i=1}^{\infty} |\alpha_i|^2 \leq \|x\|^2$, referred to as *Bessel's inequality*.

Remark. These actually hold even for an uncountable collection.

Proof. (2) follows from (1), for (1), we see that

$$\begin{aligned}
\left\| x - \sum_{i=1}^{N} \alpha_i e_i \right\| x^2 &= \|x\|^2 - 2 \operatorname{Re} \left\langle x, \sum_{i=1}^{N} \alpha_i e_i \right\rangle + \left\| x \sum_{i=1}^{N} \alpha_i e_i \right\| x^2 \\
&= \|x\|^2 - 2 \sum_{i=1}^{N} \operatorname{Re} \overline{\alpha_i} \langle x, e_i \rangle + \sum_{i=1}^{N} |\alpha_i|^2 \\
&= \|x\|^2 - 2 \sum_{i=1}^{N} |\alpha_i|^2 + \sum_{i=1}^{N} |\alpha_i|^2 \\
&= \|x\|^2 - \sum_{i=1}^{N} |\alpha_i|^2.
\end{aligned}$$

■

Definition 7.1.7 (Orthonormal basis). An orthonormal set $\{e_i\}$ is said to be an *orthonormal basis* of V if $\overline{W} = V$, where

$$W = \left\{ \sum_{i=1}^{N} \beta_i e_i \mid N \in \mathbb{N}, \beta_1, \ldots, \beta_N \in \mathbb{C} \right\}$$

is the subspace of finite linear combinations. In other words, for all $x \in V$ and for every $\epsilon > 0$, there exists $w \in W$ such that $\|x - w\| < \epsilon$.

Example. For $\mathbb{C}^d$, the orthonormal basis is $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ for $i = 1, \ldots, d$.

Example. For ℓ^2, the orthonormal basis is the countably many $e_i = (0, \ldots, 0, 1, 0, \ldots)$ for $i \in \mathbb{N}$.

Definition 7.1.8 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Note. Namely, a Banach space with an inner product.

Example. $\mathbb{R}^d, \mathbb{C}^d, L^2(X, \mathcal{A}, \mu), \ell^2$ are Hilbert spaces.

Proof. Then f_n is Cauchy, but its natural limit is discontinuous. ⊛

Example. $C([0,1]) \subseteq L^2(X, \mathcal{A}, \mu)$ is **not** a Hilbert space.

Proof. We see this by the fact that it is not complete. Take a function f_n so that f_n is zero from 0 to $1/2$ and 1 from $1/2 + 1/n$ to 1, connected continuously line. ⊛

Theorem 7.1.3. Let $\mathcal{H}$ be a Hilbert space. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal set. The following are equivalent.

(1) $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis.

(2) If $x \in \mathcal{H}$ and $\langle x, e_i \rangle = 0$ for all i, then $x = 0$.

(3) If $x \in \mathcal{H}$, then $s_N \coloneqq \sum_{i=1}^{N} \alpha_i e_i \to x$ strongly where $\alpha_i = \langle x, e_i \rangle$.

(4) If $x \in \mathcal{H}$, then $\|x\|^2 = \sum_{i=1}^{\infty} |\alpha_i|^2$.

Proof. We prove this as follows.

Claim. (3) $\Rightarrow$ (4).

Proof. From Lemma 7.1.4, we have

$$\|x\|^2 = \|x - s_N\|^2 + \sum_{i=1}^{N} |\alpha_i|^2 .$$

Taking $N \to \infty$ and noting $s_N \to x$ strongly gives

$$\|x\|^2 = \lim_{N \to \infty} \sum_{i=1}^{N} |\alpha_i|^2 = \sum_{i=1}^{\infty} |\alpha_i|^2 .$$

⊛

Claim. (4) $\Rightarrow$ (1).

Proof. Using the same equality

$$\|x\|^2 = \|x - s_N\|^2 + \sum_{i=1}^{N} |\alpha_i|^2$$

and taking $N \to \infty$ yields $\|x - s_N\|^2 \to 0$ so $\|x - s_N\| \to 0$. Therefore, $s_N \to x$ strongly, yielding that x can be approximated by finite linear combinations as desired. ⊛

Claim. (1) ⇒ (2).

Proof. Fix $x \in \mathcal{H}$, and fix $\epsilon > 0$. Then by (1), there exists a $y = \sum_{i=1}^{k} \beta_i e_i$ such that $\|x - y\| < \epsilon$.

By the best approximation lemma, $\|x - s_k\| \leq \|x - y\| < \epsilon$. If $\langle x, e_i \rangle = 0$ for all i, then $s_k = 0$, so $\|x\| < \epsilon$.

Taking $\epsilon \to 0$ would yield $\|x\| = 0$, implying $x = 0$. ⊛

Claim. (2) ⇒ (3).

Proof. Bessel's inequality gives

$$\sum_{i=1}^{\infty} |\alpha_i|^2 \leq \|x\|^2 < \infty.$$

We now see that for $N > M$,

$$\|s_N - s_M\|^2 = \left\| x \sum_{i=M+1}^{N} \alpha_i e_i \right\| x^2 = \sum_{i=M+1}^{N} |\alpha_i|^2 \to 0$$

as $N > M \to \infty$, by convergence of the series. This implies that $\{s_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}$.

Since $\mathcal{H}$ is complete, there is a vector y such that $s_N \to y$ strongly.

Problem. Is $y = x$?

Answer. Fix $i \in \mathbb{N}$, consider $\langle y - x, e_i \rangle$. We see that

$$\langle y - x, e_i \rangle = \langle y - s_N, e_i \rangle + \langle s_N - x, e_i \rangle .$$

We can compute that for $N > i$, that

$$\langle s_N - x, e_i \rangle = \alpha_i - \langle x, e_i \rangle = 0.$$

Hence, $\langle y - x, e_i \rangle = \langle y - s_N, e_i \rangle$. Because strong convergence implies weak convergence, taking $N \to \infty$ yields that $\langle y - x, e_i \rangle = 0$ for all $i \in \mathbb{N}$. Therefore, by the assumption of (2), $y - x = 0$, so $x = y$ and we're done. ⊛

⊛

■

Remark. Note that for everything except (2) ⇒ (3) we did not use the Hilbert space property. When $\mathcal{H}$ is replaced by any inner product space V we only have

$$(3) \Rightarrow (4) \Rightarrow (1) \Rightarrow (2).$$

Note. The (4) in Theorem 7.1.3 is called the *Planchenel identity*.

Definition 7.1.9 (Separable)**.** A metric space is called *separable* if there exists a countable dense subset.

Example. $\mathbb{R}^d \supseteq \mathbb{Q}^d$ and ℓ^p with $1 \leq p < \infty$, but not $p = \infty$.

Proof. To do this, consider sequences of rational numbers. ⊛

Example. $L^p(\mathbb{R}, m)$ is separable for $1 \leq p < \infty$.

Proof. Take step functions with rational heights and rational endpoints to intervals. ⊛

Theorem 7.1.4. Every separable Hilbert space has a countable orthonormal basis.

Proof. Gram-Schmidt algorithm[a] will construct such an orthonormal basis explicitly. ■

[a] https://en.wikipedia.org/wiki/Gram-Schmidt_process

Note. The cardinality of an orthonormal basis is determined by the space, and we can call this the dimension of the Hilbert space.

Appendix

Appendix A

Additional Proofs

A.1 Measure

This section gives all additional proofs in chapter 1.

Theorem A.1.1 (Theorem 1.3.2 3.). Under the setup of Theorem 1.3.2, $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We see this in two parts.

Claim. If $A \subset X$ satisfies $\mu^*(A) = 0$, then A is Carathéodory measurable with respect to μ^*.

Proof. If $A \subset X$ and $\mu^*(A) = 0$, where μ^* is an outer measure on X, we'll show that A is Carathéodory measurable with respect to μ^*.

Equivalently, we want to show that for any $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Firstly, noting that $(E \cap A) \subset A$, and by monotonicity of μ^*, we see that

$$\mu^*(E \cap A) \leq \mu^*(A) = 0,$$

and since $\mu^* \geq 0$, hence $\mu^*(E \cap A) = 0$. Now, we only need to show that

$$\mu^*(E) = \mu^*(E \setminus A).$$

Since $E \setminus A = E \cap A^c$, and hence we have $E \cap A^c \subset E$, so

$$\mu^*(E) \geq \mu^*(E \setminus A).$$

To show another direction, we note that

$$\mu^*(E) \leq \mu^*(E \cup A) = \mu^*((E \setminus A) \cup A) \leq \mu^*(E \setminus A),$$

hence we conclude that A is Carathéodory measurable with respect to μ^* if $\mu^*(A) = 0$. ⊛

Claim. If A is μ-subnull, then $A \in \mathcal{A}$.

Proof. Let $\mathcal{A}$ denotes the Carathéodory σ-algebra, and $\mu := \mu^*|_{\mathcal{A}}$. We want to show if A is μ-subnull, then $A \in \mathcal{A}$.

Firstly, if A is μ-subnull, then there exists a $B \in \mathcal{A}$ such that $A \subset B$ and $\mu(B) = 0$. But since from the monotonicity of μ^*, we further have

$$0 = \mu(B) = \mu^*(B) \geq \mu^*(A),$$

hence $\mu^*(A) = 0$.

From the first claim, we immediately see that A is Carathéodory measurable with respect to μ^*, which implies A is in Carathéodory σ-algebra, hence $A \in \mathcal{A}$. ⊛

We see that the second claim directly proves that $(X, \mathcal{A}, \mu)$ is a complete measure space. ■

Lemma A.1.1. The function F defined in this example is a distribution function

Proof. We define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n \end{cases}$$

where $\{r_1, r_2, \ldots\} = \mathbb{Q}$, and

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} = \sum_{n; r_n \leq x} \frac{1}{2^n}$$

is both increasing and right-continuous.

- Increasing. Consider $x < y$. We see that

$$F(x) = \sum_{n; r_n \leq x} \frac{1}{2^n} \leq \sum_{n; r_n \leq y} \frac{1}{2^n} = F(y)$$

clearly.[a]

- Right-continuous. We want to show $F(x^+) = F(x)$. Let $x^+(\epsilon) := x + \epsilon$ with $\epsilon > 0$, we'll show that

$$\lim_{\epsilon \to 0} F(x^+(\epsilon)) = \lim_{\epsilon \to 0} F(x + \epsilon) = F(x).$$

Firstly, we have

$$F(x^+(\epsilon)) - F(x) = \sum_{n; r_n \leq x+\epsilon} \frac{1}{2^n} - \sum_{n; r_n \leq x} \frac{1}{2^n} = \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n},$$

and we want to show

$$\lim_{\epsilon \to 0} F(x^+(\epsilon)) - F(x) = \lim_{\epsilon \to 0} \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} = 0.$$

Remark. The strict is crucial to show the result, since if $x = r_k$ for some fixed k, then we can't make the summation arbitrarily small.

[a] This is trivial since we're always going to sum more strictly positive terms in $F(y)$ than in $F(x)$.

Before we show how we choose ϵ,[b] we see that

$$\sum_{n=k}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

Now, we observe that

$$\sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} \leq \sum_{n = \arg\min_k \, x < r_k \leq x+\epsilon}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

With this observation, it should be fairly easy to see that we can choose ϵ based on how small we want to make 2^{1-k} be,[c] and we indeed see that

$$\lim_{k\to\infty} 2^{1-k} = 0,$$

which implies that F is right-continuous by squeeze theorem.

■

[b] To be precise, how ϵ depends on r_n.
[c] We're referring to $\epsilon - \delta$ proof approach.

Lemma A.1.2. The function F defined in this example satisfies

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$

given in this example.

Proof. We prove them one by one. And notice that F is indeed a distribution function as we proved in Lemma A.1.1.

(1) To show $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$, we first note that

$$\{r\} = \bigcap_{a-1\leq x<r} (x, r].$$

Then, we see that

$$\mu_F(\{r\}) = \mu_F\left(\bigcap_{a-1\leq x<a} (x, r]\right),$$

where each $(x, r] \in \mathcal{A}$ and $(x, r] \supset (y, r]$ whenever $r - 1 \leq x \leq y < r$. Notice that we implicitly assign the order of the index by the order of x. Then, we see that $\mu_F(r-1, r] < \infty$.[a] Then, from continuity from above, we see that

$$\mu_F(\{r\}) = \lim_{i\to\infty} \mu_F((x_i, r]),$$

where we again implicitly assign an order to x as the usual order on $\mathbb{R}$ by given index i. It's then clear that as $i \to \infty$, $x_i \to r$. From the definition of F, we see that

$$F((x_i, r]) = F(r) - F(x_i) = \sum_{n; r_n \leq r} \frac{1}{2^n} - \sum_{n; r_n \leq x_i} \frac{1}{2^n} = \sum_{n; x_i < r_n \leq r} \frac{1}{2^n}.$$

It's then clear that since $r \in \mathbb{Q}$, there exists an i' such that $r_{i'} = r$. Then, we immediately see that no matter how close $x_i \to r$, this sum is at least

$$\frac{1}{2^{i'}}$$

for a fixed i'. Hence, we conclude that $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$.

(2) Now, we show $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$. Firstly, we claim that

$$\mu_F(\mathbb{Q}) = 1$$

and

$$\mu_F(\mathbb{R}) = 1$$

as well. Since $\mu_F(\mathbb{Q}) = 1$ is clear, we note that the second one essentially follows from the fact that we can write

$$\mathbb{R} = \lim_{N\to\infty} \bigcup_{i=1}^{N} (a-i, a+i]$$

for any $a \in \mathbb{R}$, say 0. From continuity from below, we have

$$\mu_F\left(\bigcup_{i=1}^{\infty}(-i, +i]\right) = \lim_{n\to\infty} \mu_F((-n, n]) = \sum_{n; r_n \in \mathbb{Q}} \frac{1}{2^n} = 1.$$

Given the above, from countable additivity of μ_F, we have

$$\mu_F(\mathbb{R} \setminus \mathbb{Q}) + \underbrace{\mu_F(\mathbb{Q})}_{1} = \underbrace{\mu_F(\mathbb{R})}_{1} \Rightarrow \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$$

as we desired.

■

[a] This will be $\mu(A_1)$ in the condition of continuity from above. Furthermore, since $\mathbb{Q}$ is countable, hence $F(x) < \infty$ is promised.

Lemma A.1.3 (Cantor set has measure 0)**.** Let C denotes the middle thirds Cantor set, then the Lebesgue measure of C is 0. i.e.,

$$m(C) = 0.$$

Proof. Since we're removing $\frac{1}{3}$ of the whole interval at each n, we see that the measure of those removing parts, denoted by r, is

$$m(r) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = 1.$$

Then, by countable additivity of m, we see that

$$m(C) = m([0,1]) - m(r) = 1 - 1 = 0.$$

■

A.2 Integration

Bibliography

[Axl19] S. Axler. *Measure, Integration & Real Analysis*. Graduate Texts in Mathematics. Springer International Publishing, 2019. ISBN: 9783030331429. URL: https://books.google.com/books?id=8hCDyQEACAAJ.

[FF99] G.B. Folland and G.B.A. FOLLAND. *Real Analysis: Modern Techniques and Their Applications*. A Wiley-Interscience publication. Wiley, 1999. ISBN: 9780471317166. URL: https://books.google.com/books?id=uPkYAQAAIAAJ.

[Tao13] T. Tao. *An Introduction to Measure Theory*. Graduate studies in mathematics. American Mathematical Society, 2013. ISBN: 9781470409227. URL: https://books.google.com/books?id=SPGJjwEACAAJ.