MATH597
Analysis II

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\overset{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition.

Additionally, we'll use [FF99] as our main text, while using [Tao13] and [Axl19] as supplementary references.

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Lecture 8: Lebesgue-Stieltjes Measure on $\mathbb{R}$

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To classify all measures, we now see this last theorem to complete the task.

Theorem 0.1 (Locally finite Borel measures on $\mathbb{R}$). We have

1. $F\colon \mathbb{R} \to \mathbb{R}$ a distribution function, then there exists a **unique** *locally finite Borel measure* μ_F on $\mathbb{R}$ satisfying
$$\mu_F((a,b]) = F(b) - F(a)$$
for every $a < b$.
2. Suppose $F, G\colon \mathbb{R} \to \mathbb{R}$ are distribution functions. Then,
$$\mu_F = \mu_G$$
on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. ■ HW.

Remark. Theorem 0.1 simply states that given a distribution function, if we restrict our attention on locally finite measures on $\mathbb{R}$ following our usual convention, then it defines the measure on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

0.1 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that

$$F \text{ distribution function } \overset{!}{\Longrightarrow} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 0.1 (Lebesgue-Stieltjes measure). Given a distribution function F, we define

- μ_F on $\mathcal{A}_{\mu_F}$ is called the *Lebesgue-Stieltjes measure* corresponding to F.
- Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{L}, m)$, where $\mathcal{L}$ is called *Lebesgue σ-algebra*, and m is called *Lebesgue measure*.

Remark. Recall that $\mathcal{L}$ is induced by **??**, namely given m, for all $A \subset \mathbb{R}$, we have

$$\mathcal{L} := \left\{ A \subset \mathbb{R} \mid \underset{E\subset\mathbb{R}}{\forall}\ m(A) = m(A \cap E) + m(A \setminus E) \right\}$$

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).$$[1]

Example. We first see some examples.

1. $\mu_F((a, b]) = F(b) - F(a)$. Then
   - $\mu_F(\{a\}) = F(a) - F(a^-)$
   - $\mu_F([a, b]) = F(b) - F(a^-)$
   - $\mu_F((a, b)) = F(b^-) - F(a)$
2. We define
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$
   Then

[1] Some text will use $x-$ and $x+$ instead of x^- and x^+, respectively.

- $\mu_F(\{0\}) = 1$
- $\mu_F(\mathbb{R}) = 1$
- $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. This is easy to see since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, hence

$$\begin{aligned}\mu_F(\mathbb{R} \setminus \{0\}) &= \mu_F((-\infty, 0) \cup (0, \infty)) \\ &= \underbrace{\mu_F((-\infty, 0))}_{0-0^{2}} + \underbrace{\mu_F((0, \infty))}_{1-1^{3}} = 0.\end{aligned}$$

We call that μ_F is the *Dirac measure* at 0.

3. Denote $\mathbb{Q} = \{r_1, r_2, \ldots\}$, and we define

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r. \end{cases}$$

Then HW

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$.

4. If F is continuous at a, then $\mu_F(\{a\}) = 0$.
5. $F(x) = x$, then recall that we denote $\mu_F := m$, and we have
   - $m((a, b]) = m((a, b)) = m([a, b]) = b - a$.
6. $F(x) = e^x$
   - $\mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$.

Remark. We see that the first two examples are *discrete measures*.

Example (Middle thirds Cantor set). Let $C := \bigcap\limits_{n=1}^{\infty} K_n$, where we have

$$\begin{aligned}K_0 &:= [0, 1] \\ K_1 &:= K_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \\ K_2 &:= K_1 \setminus \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \\ &\vdots \\ K_n &:= K_{n-1} \setminus \bigcup_{k=1}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3^{k+2}}{3^{n+1}}\right).\end{aligned}$$

We see that C is uncountable and with $m(C) = 0$. Firstly, since $x \in C$ if and only if $x = \sum\limits_{n=1}^{\infty} \frac{a_n}{3}$ for some $a_n \in \{0, 2\}$.

[2] It follows from $F(0^-) - F(-\infty) = 0 - 0 = 0$.
[3] It follows from $F(\infty) - F(0) = 1 - 1 = 0$.

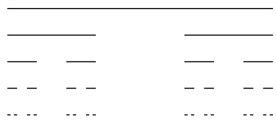

Figure 1: The top line corresponds to K_0, and then K_1, etc.

0.1.1 Cantor Function

Consider F as follows. We define a function F to be 0 to the left of 0, and 1 to the right of 1. Then, define F to be $\frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$, $\frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9}\right)$ and so on. This is so-called *Cantor Function*. We can show F is continuous and increasing, which makes F a distribution function. HW

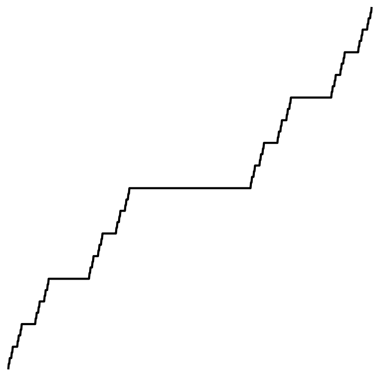

Figure 2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

Cantor Measure μ_F		Lebesgue Measure m
$\mu_F(\mathbb{R} \setminus C) = 0$		$m(\mathbb{R} \setminus C) = \infty > 0$
$\mu_F(C) = 1$	$\iff$	$m(C) = 0$
$\mu_F(\{a\}) = 0$		$m(\{a\}) = 0$

Remark. μ_F and m are said to be **singular** to each other.

0.2 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 0.1. Let μ be Lebesgue-Stieltjes measure on $\mathbb{R}$. Then we have

$$\mu(A) \overset{!}{=} \inf\left\{\sum_{i=1}^{\infty}\mu((a_i,b_i]) \mid \bigcup_{i=1}^{\infty}(a_i,b_i]\supset A\right\}$$
$$= \inf\left\{\sum_{i=1}^{\infty}\mu((a_i,b_i)) \mid \bigcup_{i=1}^{\infty}(a_i,b_i)\supset A\right\}$$

for every $A\in\mathcal{A}_\mu$

Proof. The second equality follows from the continuity of the measure. ∎

Remark. This is similar to

$$(a,b)=\bigcup_{n=1}^{\infty}(a,b-1/n],\quad (a,b]=\bigcap_{n=1}^{\infty}(a,b+1/n].$$

Lecture 9: Properties of Lebesgue-Stieltjes measure

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As previously seen. Let $X\subset[0,\infty]$. Recall that

- $$\alpha=\sup X<\infty \iff \begin{cases}\underset{x\in X}{\forall}\ \alpha\geq x\\ \underset{\epsilon>0}{\forall}\ \underset{x\in X}{\exists}\ x+\epsilon\geq\alpha.\end{cases}$$

- $$\alpha=\sup X=\infty \iff \underset{L>0}{\forall}\ \underset{x\in X}{\exists}\ x\geq L.$$

This should be useful latter on.

Theorem 0.2. Let μ be Lebesgue-Stieltjes measure. Then, for all $A\in\mathcal{A}_\mu$,

1. (outer regularity) $\mu(A)=\inf\{\mu(O)\mid O\supset A, O\text{ is open}\}$
2. (inner regularity) $\mu(A)=\sup\{\mu(K)\mid K\subset A, K\text{ is compact}\}$

Proof. We check them separately.

1. DIY
2. Let $s:=\sup\{\mu(K)\mid K\subset A, K\text{ is compact}\}$, then by monotonicity, we have $\mu(A)\geq s$. To show the other direction, we consider
   - A is a bounded set.

     Then $\overline{A}\in\mathcal{B}(\mathbb{R})\subset\mathcal{A}_\mu$, $\overline{A}$ is also bounded $\implies$ $\mu(\overline{A})<\infty$. Fix $\epsilon>0$, then by outer regularity, there exists an open $O\supset\overline{A}\setminus A$, and $\mu(O)-\mu(\overline{A}\setminus A)=\mu(O\setminus(\overline{A}\setminus A))\leq\epsilon$. Let $K:=\underbrace{A\setminus O}_{K\subset A}=\underbrace{\overline{A}\setminus O}_{\text{compact}}$, we show that

     $$\mu(K)\geq\mu(A)-\epsilon.$$

DIY

- A is an unbounded set with $\mu(A) < \infty$.

  Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \ldots$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) < \infty.$$

- A is an unbounded set with $\mu(A) = \infty$.

  We can show that

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) = \infty.$$

  Fix $L > 0$, then $\exists N$ such that $\mu(A_N) \geq L$.

■

Definition 0.2 (G_δ**-set,** F_σ**-set).** Let X be a topological space. Then

- A G_δ*-set* is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open.
- A F_σ*-set* is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 0.3. Let μ be a Lebesgue-Stieltjes measure. Then *TFAE*[a]:

1. $A \in \mathcal{A}_\mu$
2. $A = G \setminus M$, G is a G_δ-set, M is a μ-null set.
3. $A = F \setminus N$, F is a F_σ-set, N is a μ-null set.

[a] *TFAE*: The following are equivalent.

Proof. We see that (2.) $\implies$ (1.) and (3.) $\implies$ (1.) are clear.

- (1.) $\implies$ (3.)
  - Assume $\mu(A) < \infty$. From the inner regularity, we have

$$\forall n \in \mathbb{N}\ \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

    Let $F = \bigcup_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ-null. Check!

  - Assume $\mu(A) = \infty$. Let $A = \bigcup_{k\in\mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

$$\forall k \in \mathbb{Z}\ A_k = F_k \cup N_k,\ \ A = \underbrace{\left(\bigcup_k F_k\right)}_{F_\sigma\text{-set}} \cup \underbrace{\left(\bigcup_k N_k\right)}_{\mu\text{-null}}.$$

- (1.) $\implies$ (2.)

  We see that
  $$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

■

Proposition 0.1. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0 \; \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using outer regularity and the fact that every open set is $\bigcup_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. ■ DIY

We now see some properties of Lebesgue measure.

Theorem 0.4. Let $A \in \mathcal{L}$, then we have $A + s \in \mathcal{L}$, $rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,
$$m(A + s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. ■ DIY

Example. We now see some examples.

1. Let $\mathbb{Q} =: \{r_i\}_{i=1}^{\infty}$ which is dense in $\mathbb{R}$. Let $\epsilon > 0$, and
   $$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i} \right).$$
   We see that O is open and dense in $\mathbb{R}$. But we see
   $$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$
   Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$
2. There exists uncountable set A with $m(A) = 0$.
3. There exists A with $m(A) > 0$ but A contains no non-empty open intervals.
4. There exists $A \notin \mathcal{L}$. e.g. Vitali set.[4]
5. There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

Lecture 10: Integration

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1 Integration

1.1 Measurable Function

We start with a definition.

Definition 1.1 (Measurable space). A *measurable space* or *Borel space* is a tuple of a set X and a σ-algebra $\mathcal{A}$ on X, denoted by $(X, \mathcal{A})$.

Definition 1.2 (Measurable function). Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$ are measurable spaces. Then we say $f\colon X \to Y$ is $(\mathcal{A}, \mathcal{B})$*-measurable* if

$$\forall B \in \mathcal{B}\ f^{-1}(B) \in \mathcal{A}.$$

Lemma 1.1. Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then,

$$f\colon X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall E \in \mathcal{E}\ f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part ($\implies$) is clear. On the other direction, we consider the following. Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $E \subset \mathcal{D}$ by assumption
- $\mathcal{D}$ is a σ-algebra

Check!

hence, we see that $\langle \mathcal{E} \rangle \subset \mathcal{D}$. ∎

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup_\alpha E_\alpha\right) = \bigcup_\alpha f^{-1}(E_\alpha)$

Definition 1.3 ($\mathcal{A}$-measurable). Let $(X, \mathcal{A})$ be a measurable space. Then,

$$\left.\begin{array}{l} f\colon X \to \mathbb{R} \\ f\colon X \to \overline{\mathbb{R}} \\ f\colon X \to \mathbb{C} \end{array}\right\} \text{ is } \mathcal{A}\text{-}\textit{measurable} \text{ if } \begin{cases} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f\colon X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{cases}$$

Notation. Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$
- $\mathcal{B}(\overline{\mathbb{R}}) = \left\{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\right\}$.

[4] https://en.wikipedia.org/wiki/Vitali_set

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \implies$ every function is $\mathcal{A}$-measurable.
- $\mathcal{A} = \{\varnothing, X\} \implies$ only $\mathcal{A}$-measurable functions are constant functions.

Lemma 1.2. Given $f \colon X \to \mathbb{R}$, *TFAE.*

1. f is $\mathcal{A}$-measurable
2. $\forall a \in \mathbb{R}$, $f^{-1}((a, \infty)) \in \mathcal{A}$
3. $\forall a \in \mathbb{R}$, $f^{-1}([a, \infty)) \in \mathcal{A}$
4. $\forall a \in \mathbb{R}$, $f^{-1}((-\infty, a)) \in \mathcal{A}$
5. $\forall a \in \mathbb{R}$, $f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from the lemma we just saw. ∎

Property. Given $f, g \colon X \to \mathbb{R}$ and is $\mathcal{A}$-measurable, then

1. $\phi \colon \mathbb{R} \to \mathbb{R}$, $\mathcal{A}$-measurable[5], then
$$\phi \circ f \colon X \to \mathbb{R}$$
is $\mathcal{A}$-measurable.
2. $-f,\ 3f,\ f^2,\ |f|$ are all $\mathcal{A}$-measurable, and $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) \neq 0, \forall x \in X$.
3. $f + g$ is $\mathcal{A}$-measurable. We see this from
$$(f+g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty)) \right).$$
4. $f \cdot g$ is $\mathcal{A}$-measurable. We see this from
$$f(x)g(x) = \frac{1}{2} \left((f(x) + g(x))^2 - f(x)^2 - g(x)^2 \right).$$
5. We see that
$$(f \vee g)(x) \coloneqq \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) \coloneqq \min\{f(x), g(x)\}$$
are $\mathcal{A}$-measurable.
6. Let $f_n \colon X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable. Then
$$\sup_{n \in \mathbb{N}} f_n,\ \inf_{n \in \mathbb{N}} f_n,\ \limsup_{n \to \infty} f_n,\ \liminf_{n \to \infty} f_n$$
are $\mathcal{A}$-measurable.

[5] In this case, we also call it *Borel measurable*.

Proof. Consider $\sup_{n\in\mathbb{N}} f_n =: g$, then

$$g^{-1}((a,\infty]) = \bigcup_{n\in\mathbb{N}} f_n^{-1}((a,\infty])$$

for $\sup_n f_n(x) = g(x) > a$. A similar argument can prove the case of $\inf_{n\in\mathbb{N}} f_n$. check

And notice that $\limsup_{n\to\infty} f_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} f_n$, then the similar argument also proves this case. ■

7. If $\lim_{n\to\infty} f_n(x)$ converges for every $x \in X$, then f is $\mathcal{A}$-measurable.

   Example. If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous

   $\implies$ f is Borel measurable

   $\implies$ f is Lebesgue measurable

   since the preimage of an open set of a continuous function is open, then we consider $f^{-1}((a,\infty))$.

Definition 1.4 (f^+, f^-)**.** For $f\colon X \to \overline{\mathbb{R}}$, let $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$.[a]

[a] i.e., $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \min\{-f(x), 0\}$

Remark. If $\text{supp} f^+ \cap \text{supp} f^- = \varnothing$ and $f(x) = f^+(x) - f^-(x)$, then

$$f \text{ is } \mathcal{A}\text{-measurable} \iff f^+, f^- \text{ are } \mathcal{A}\text{-measurable}.$$

Notation. $\text{supp} f$ means the support of f, which is the set of domain which makes f being non-zero.

Definition 1.5 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator) function* of E is

$$\mathcal{X}_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is $\mathcal{A}$-measurable $\iff$ $E \in \mathcal{A}$.

Definition 1.6 (Simple function). Let $(X, \mathcal{A})$ be a measurable space. Then a *simple function* $\phi\colon X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

Remark. We see that if

$$\phi(X) = \{c_1, \ldots, c_N\},$$

and

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \implies \phi = \sum_{i=1}^{N} \underbrace{c_i}_{\neq \pm\infty} \mathbb{1}_{\underbrace{E_i}_{\in \mathcal{A}}}.$$

Lecture 11

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Appendix

References

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