MATH635
Riemannian Geometry

Pingbang Hu

January 7, 2023

Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by Lydia Bieri. Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [FC13] as our reference; while not required, but highly recommended have on.

This course is taken in Winter 2023, and the date on the covering page is the last updated time.

Contents

Chapter 1

Manifolds

Lecture 1: A Foray to Smooth Manifolds

1.1 Differentiable Manifolds

5 Jan. 14:30

1.1.1 Topological Manifolds

Let's start with a common definition.

Definition 1.1.1 (Topological manifold). A *topological manifold* $\mathcal{M}$ of dimension n is a (topological) Hausdorff space such that each point $p \in \mathcal{M}$ has a neighborhood U homeomorphic via $\varphi \colon U \to U'$ to an open subset $U' \subseteq \mathbb{R}^n$.

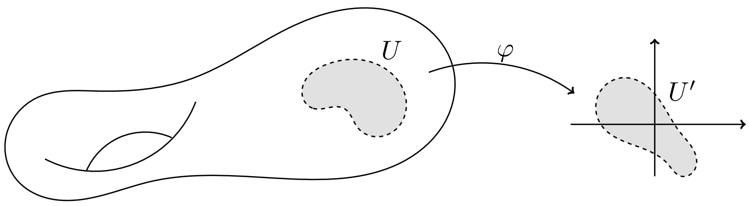
Definition 1.1.2 (Local coordinate map). For every $p \in \mathcal{M}$, the corresponding homeomorphism φ is called the *local coordinate map*.

Definition 1.1.3 (Local coordinate). The pull-back $(x^1, \dots, x^n)$ of the local coordinate map φ from $\mathbb{R}^n$ is called the *local coordinates* on U, given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

Definition 1.1.4 (Coordinate chart). The pair (U, φ) is called a *(coordinate) chart* on M.

In other words, a topological manifold can be thought of as a space such that it looks like $\mathbb{R}^n$ locally.



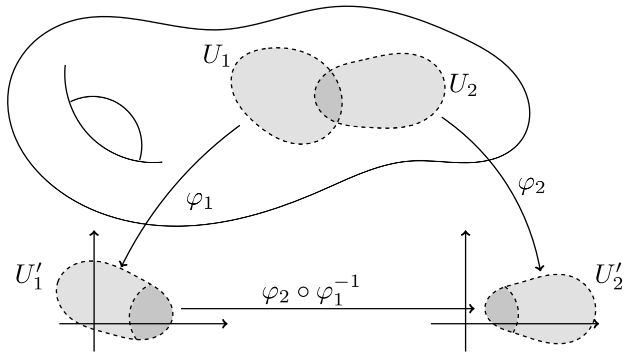
Definition 1.1.5 (Atlas). An *atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ for a manifold $\mathcal{M}$ is a collection of charts such that $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$ are an open covering of $\mathcal{M}$, i.e., $\mathcal{M} = \bigcup_\alpha U_\alpha$.

In other words, for all $p \in \mathcal{M}$, there exists a neighborhood $U \subseteq \mathcal{M}$ and homeomorphism $h \colon U \to U' \subseteq \mathbb{R}^n$ open.

Definition 1.1.6 (Locally finite). An atlas is said to be *locally finite* if each point $p \in \mathcal{M}$ is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as $\mathbb{R}^n$, there's no clear way to make sense of differentiability of a manifold. But thankfully, we now have an explicit relation to the ambient space $\mathbb{R}^n$ via φ_α. To formalize, let $\mathcal{A}$ be an atlas for a manifold $\mathcal{M}$, and assume that $(U_1, \varphi_1), (U_2, \varphi_2)$ are 2 elements of $\mathcal{A}$. Then clearly, the map $\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a homeomorphism between 2 open sets of Euclidean spaces since both φ_1 and φ_2 are homeomorphism. Due to this map's importance, it has its own name.

Definition 1.1.7 (Coordinate transition)**.** The map $\varphi_2 \circ \varphi_1^{-1}$ is called the *coordinate transition* of $\mathcal{A}$ for the pair of charts $(U_1, \varphi_1), (U_2, \varphi_2)$.



1.1.2 Differentiable Structures

Notice that the coordinate transitions are from $\mathbb{R}^n$ to $\mathbb{R}^n$; hence differentiability makes sense now, which induces the following.

Definition 1.1.8 (Differentiable atlas)**.** The atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is *differentiable* if all transitions are differentiable.

Remark. Here, the differentiability depends on the content. Sometimes, we may want it to be C^∞, and sometimes may be C^k for some finite k. On the other hand, smooth always refers to C^∞. We'll use them interchangeably if it's clear which case we're referring to.

Definition 1.1.9 (Equivalence atlas)**.** Two atlases $\mathcal{U}, \mathcal{V}$ of a manifold are equivalent if for every $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$,

$$\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

Notation (Smoothly compatible)**.** Two charts (U, φ) and (V, ψ) are *smoothly compatible* if either $U \cap V = \varnothing$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

This suggests the following.

Definition 1.1.10 (Smooth structure)**.** A *smooth structure* on $\mathcal{M}$ is an equivalence class $\mathcal{U}$ of coordinate atlas with the property that all transition functions are diffeomorphisms.

Remark. We can also use the *maximal* differentiable atlas to be our differentiable structure.

Definition 1.1.11 (Smooth manifold)**.** A *smooth manifold* is a manifold $\mathcal{M}$ with a smooth structure.

In this way, we can do calculus on smooth manifolds! Furthermore, it now makes sense to say that a function $f\colon \mathcal{M} \to \mathbb{R}$ is differentiable (or C^∞) by considering differentiability of $f \circ \varphi^{-1}$ around p.

Notation. The collection of smooth functions on smooth manifold $\mathcal{M}$ is denoted by $C^\infty(\mathcal{M}, \mathbb{R})$, or $C^k(\mathcal{M}, \mathbb{R})$.

Remark. The class $C^\infty(\mathcal{M}, \mathbb{R})$ consists of functions with property is well-defined.

Proof. Let $\mathcal{A}$ be any given atlas from equivalence class that defines the smooth structure, and as we have shown, if $(U, \varphi) \in \mathcal{A}$, then $f \circ \varphi^{-1}$ is a smooth function on $\mathbb{R}^n$. This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.1.9 requirement that defines the equivalent manifolds. ⊛

1.1.3 Orientation

Another essential property of a manifold is its orientability.

Definition. Consider an atlas $\mathcal{A}$ for a differentiable manifold $\mathcal{M}$.

Definition 1.1.12 (Oriented)**.** $\mathcal{A}$ is *oriented* if all transitions have positive functional determinant.

Definition 1.1.13 (Orientable)**.** $\mathcal{M}$ is *orientable* if $\mathcal{A}$ is an oriented atlas.

Motivated by the above definitions, we see that we can actually use an atlas to define an orientation.

Definition 1.1.14 (Orientation)**.** Let $\mathcal{M}$ be an orientable manifold. Then a oriented differentiable structure is called an *orientation* of $\mathcal{M}$.

If $\mathcal{M}$ possesses an orientation, we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with Definition 1.1.12.

Remark. Two differentiable structures obeying Definition 1.1.12 determine the same orientation if the union again satisfying Definition 1.1.12.

Remark. If $\mathcal{M}$ is orientable and connected, then there exists exactly 2 distinct orientations on $\mathcal{M}$.

Now, we can see some examples of smooth manifolds.

Example (Sphere)**.** The sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by

$$S^n = \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1 \right\}.$$

Consider $U_i^+ = \{x \in S^n \mid x_i > 0\}$, $U_i^- = \{x \in S^n \mid x_i < 0\}$ for $i = 1, \dots, n+1$, and $h_i^\pm \colon U_i^\pm \to \mathbb{R}^n$ such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, .., x_{n+1}).$$

Note that the minimum charts needed to cover S^n is 2.

Example. Let $\mathcal{M} = U \subseteq \mathbb{R}^n$, then $\{(U, \varphi)\}$ is a smooth structure with $\varphi = \mathbb{1}$.

Example. Open sets of C^∞-manifolds are C^∞-manifolds.

Example (General linear group). $\mathrm{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, open.

Example (Real projective space). $\mathbb{R}P^n = S^n \,/\, \sim$ where $x \sim -x$ with $\pi \colon S^n \to \mathbb{R}P^n$, $x \mapsto [x]$.

Proof. π is a homeomorphism on each U_i^+ for $i = 1, \ldots, n+1$, with

$$\left\{ \left(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1} \right), i = 1, \ldots, n+1 \right\}$$

is a C^∞-atlas for $\mathbb{R}P^n$. ⊛

Note. Observe that $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} \,/\, \sim$.

Example (Grassmannian manifold). Given m, n, $G(n, m)$ is the set of all n-dimensional subspaces of $\mathbb{R}^{n+m}$.

Appendix

Bibliography

[FC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013. ISBN: 9780817634902. URL: https://books.google.com/books?id=ct91XCWkWEUC.