# MATH635
# Riemannian Geometry

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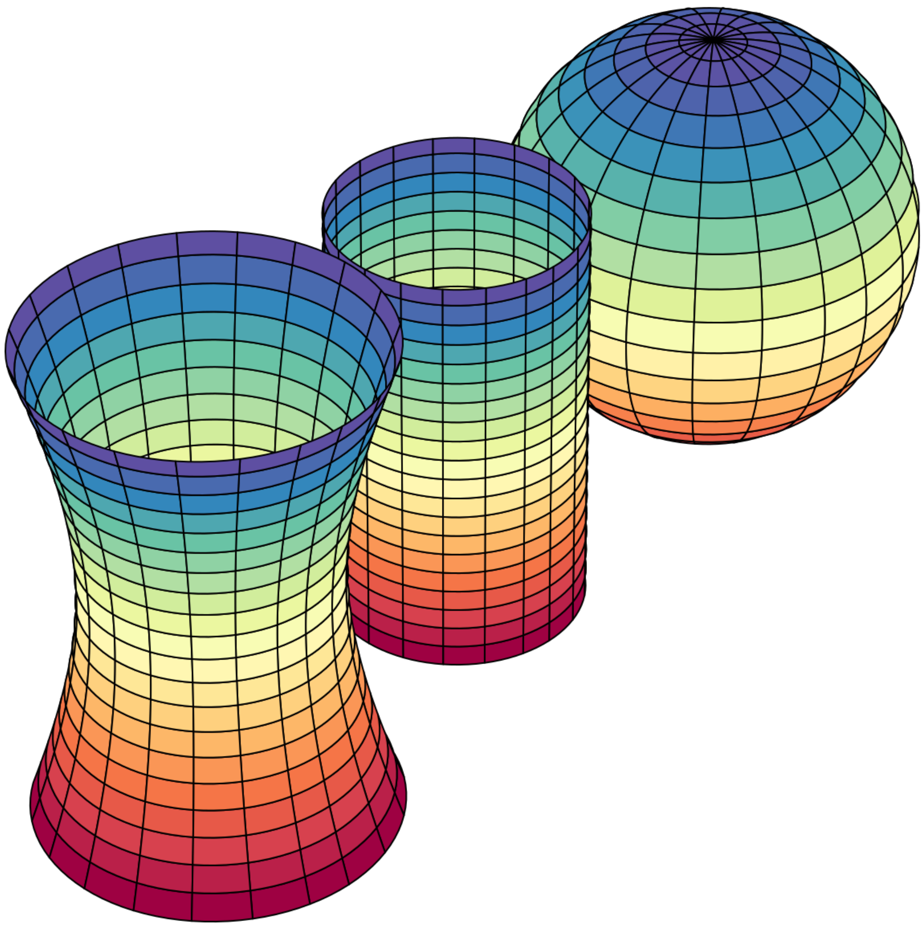
**Abstract**

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by Lydia Bieri. Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [FC13] as our reference; while not required, but highly recommended have on. Apart from this, I also found [Sch15] very useful.



This course is taken in Winter 2023, and the date on the covering page is the last updated time.

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# Chapter 1

# Smooth Manifolds

## Lecture 1: A Foray to Smooth Manifolds

### 1.1 Topological Manifolds

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Let's start with a common definition.

**Definition 1.1.1** (Topological manifold). A *topological manifold* $\mathcal{M}$ of dimension $n$ is a (topological) Hausdorff space such that each point $p \in \mathcal{M}$ has a neighborhood $U$ homeomorphic via $\varphi \colon U \to U'$ to an open subset $U' \subseteq \mathbb{R}^n$.

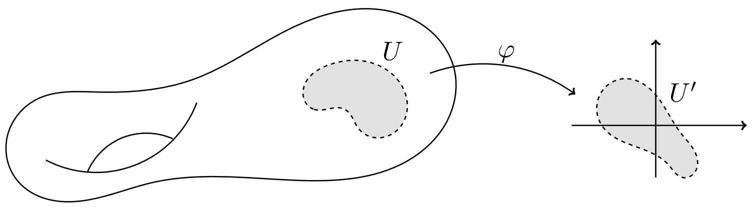
**Definition 1.1.2** (Local coordinate map). For every $p \in \mathcal{M}$, the corresponding homeomorphism $\varphi$ is called the *local coordinate map*.

**Definition 1.1.3** (Local coordinate). The pull-back $(x^1, \dots, x^n)$ of the local coordinate map $\varphi$ from $\mathbb{R}^n$ is called the *local coordinates* on $U$, given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

**Definition 1.1.4** (Coordinate chart). The pair $(U, \varphi)$ is called a *(coordinate) chart* on $M$.

In other words, a topological manifold can be thought of as a space such that it looks like $\mathbb{R}^n$ locally.



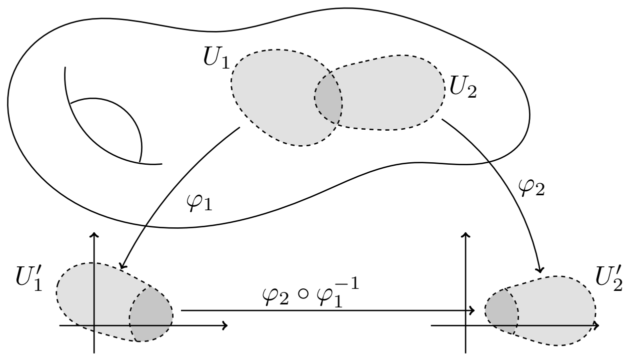

**Definition 1.1.5** (Atlas). An *atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ for a manifold $\mathcal{M}$ is a collection of charts such that $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$ are an open covering of $\mathcal{M}$, i.e., $\mathcal{M} = \bigcup_\alpha U_\alpha$.

In other words, for all $p \in \mathcal{M}$, there exists a neighborhood $U \subseteq \mathcal{M}$ and homeomorphism $h \colon U \to U' \subseteq \mathbb{R}^n$ open.

**Definition 1.1.6** (Locally finite). An atlas is said to be *locally finite* if each point $p \in \mathcal{M}$ is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as $\mathbb{R}^n$, there's no clear way to make sense of differentiability of a manifold. But thankfully, we now have an explicit relation to the ambient space $\mathbb{R}^n$ via $\varphi_\alpha$. To formalize, let $\mathcal{A}$ be an atlas for a manifold $\mathcal{M}$, and assume that $(U_1, \varphi_1), (U_2, \varphi_2)$ are 2 elements of $\mathcal{A}$. Then clearly, the map $\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a homeomorphism between 2 open sets of Euclidean spaces since both $\varphi_1$ and $\varphi_2$ are homeomorphism. Due to this map's importance, it has its own name.

**Definition 1.1.7** (Coordinate transition). The map $\varphi_2 \circ \varphi_1^{-1}$ is called the *coordinate transition* of $\mathcal{A}$ for the pair of charts $(U_1, \varphi_1), (U_2, \varphi_2)$.




## 1.2 Differentiable Manifolds

Notice that the coordinate transitions are from $\mathbb{R}^n$ to $\mathbb{R}^n$; hence differentiability makes sense now, which induces the following.

**Definition 1.2.1** (Differentiable atlas). The atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is *differentiable* if all transitions are differentiable.

**Remark.** Here, the differentiability depends on the content. Sometimes, we may want it to be $C^\infty$, and sometimes may be $C^k$ for some finite $k$. On the other hand, smooth always refers to $C^\infty$. We'll use them interchangeably if it's clear which case we're referring to.

**Definition 1.2.2** (Equivalence atlas). Two atlases $\mathcal{U}, \mathcal{V}$ of a manifold are equivalent if for every $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$,

$$\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

**Notation** (Smoothly compatible). Two charts $(U, \varphi)$ and $(V, \psi)$ are *smoothly compatible* if either $U \cap V = \varnothing$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

This suggests the following.

**Definition 1.2.3** (Smooth structure). A *smooth structure* on $\mathcal{M}$ is an equivalence class $\mathcal{U}$ of coordinate atlas with the property that all transition functions are diffeomorphisms.

**Remark.** We can also use the *maximal* differentiable atlas to be our differentiable structure.

**Definition 1.2.4** (Smooth manifold). A *smooth manifold* is a manifold $\mathcal{M}$ with a smooth structure.

In this way, we can do calculus on smooth manifolds! Furthermore, it now makes sense to say that a function $f \colon \mathcal{M} \to \mathbb{R}$ is differentiable (or $C^\infty$) by considering differentiability of $f \circ \varphi^{-1}$ around $p$.

**Notation.** The collection of smooth functions on smooth manifold $\mathcal{M}$ is denoted by $C^\infty(\mathcal{M}, \mathbb{R})$, or $C^k(\mathcal{M}, \mathbb{R})$.

**Remark.** The class $C^\infty(\mathcal{M}, \mathbb{R})$ consists of functions with property is well-defined.

**Proof.** Let $\mathcal{A}$ be any given atlas from equivalence class that defines the smooth structure, and as we have shown, if $(U, \varphi) \in \mathcal{A}$, then $f \circ \varphi^{-1}$ is smooth on $\mathbb{R}^n$. This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.2.2 requirement that defines the equivalent manifolds. ⊛

### 1.2.1 Orientation

Another essential property of a manifold is its orientability.

**Definition.** Consider an atlas $\mathcal{A}$ for a differentiable manifold $\mathcal{M}$.

**Definition 1.2.5** (Oriented)**.** $\mathcal{A}$ is *oriented* if all transitions have positive functional determinant.

**Definition 1.2.6** (Orientable)**.** $\mathcal{M}$ is *orientable* if $\mathcal{A}$ is an oriented atlas.

Motivated by the above definitions, we see that we can actually use an atlas to define an orientation.

**Definition 1.2.7** (Orientation)**.** Let $\mathcal{M}$ be an orientable manifold. Then a oriented differentiable structure is called an *orientation* of $\mathcal{M}$.

If $\mathcal{M}$ possesses an orientation, we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with Definition 1.2.5.

**Remark.** Two differentiable structures obeying Definition 1.2.5 determine the same orientation if the union again satisfying Definition 1.2.5.

**Remark.** If $\mathcal{M}$ is orientable and connected, then there exists exactly 2 distinct orientations on $\mathcal{M}$.

Now, we can see some examples of smooth manifolds.

**Example** (Sphere)**.** The sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by

$$S^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\right\}.$$

Consider $U_i^+ = \{x \in S^n \mid x_i > 0\}$, $U_i^- = \{x \in S^n \mid x_i < 0\}$ for $i = 1, \ldots, n+1$, and $h_i^\pm : U_i^\pm \to \mathbb{R}^n$ such that

$$h_i^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, .., x_{n+1}).$$

Note that the minimum charts needed to cover $S^n$ is 2.

**Example.** Let $\mathcal{M} = U \subseteq \mathbb{R}^n$, then $\{(U, \varphi)\}$ is a smooth structure with $\varphi = \mathbb{1}$.

**Example.** Open sets of $C^\infty$-manifolds are $C^\infty$-manifolds.

**Example** (General linear group)**.** $\mathrm{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, open.

**Example** (Real projective space)**.** $\mathbb{R}P^n = S^n / \sim$ where $x \sim -x$ with $\pi \colon S^n \to \mathbb{R}P^n$, $x \mapsto [x]$.

**Proof.** $\pi$ is a homeomorphism on each $U_i^+$ for $i = 1, \ldots, n+1$, with

$$\left\{ \left( \pi(U_i^+), \varphi_i^+ \circ \pi^{-1} \right), i = 1, \ldots, n+1 \right\}$$

is a $C^\infty$-atlas for $\mathbb{R}P^n$. ⊛

**Note.** Observe that $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} \ / \sim$.

# Lecture 2: Maps Between Smooth Manifolds

## 1.2.2 Smooth Maps

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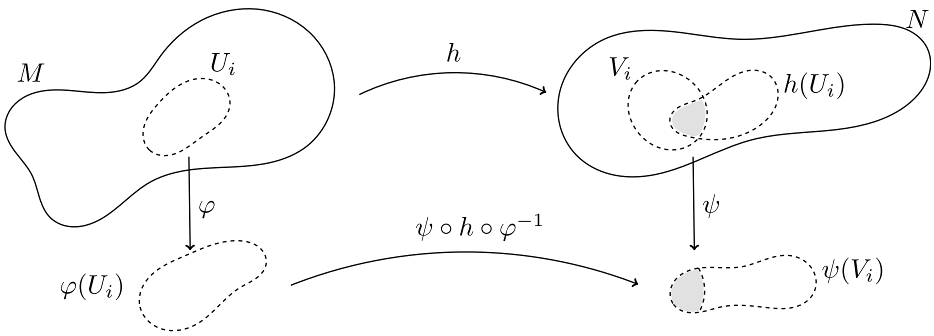
We can now consider the maps between manifolds, specifically, the smooth manifolds.

**Definition 1.2.8** (Smooth function). Let $M, N$ be two smooth manifolds, and let $\mathcal{U}$ be locally finite atlas from the equivalence class that gives the smooth structure on $M$, and let $\mathcal{V}$ be the corresponding for $N$. A map $h\colon M \to N$ is said to be *smooth* if each map in the collection

$$\left\{ \psi \circ h \circ \varphi^{-1} \colon h(U) \cap V \neq \varnothing \right\},$$

where $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$ is $C^\infty$-differentiable as a map from one Euclidean space to another.




**Remark.** Equivalence relation guarantees that Definition 1.2.8 depends only on the smooth structure of $M, N$, but not on the chosen representative coordinate atlas.

**Definition.** Consider two smooth manifolds $M, N$ and a smooth homeomorphism $h\colon M \to N$ with smooth inverse.

**Definition 1.2.9** (Diffeomorphic). The two manifolds $M, N$ are said to be *diffeomorphic*.

**Definition 1.2.10** (Diffeomorphism). The map $h$ is said to be a *diffeomorphism*.

Let $M_1, M_2$ be two smooth manifolds, and let $\varphi\colon M_1 \to M_2$ be a diffeomorphism. Then

(a) $M_1$ is orientable if and only if $M_2$ is orientable.

(b) If in addition, $M_1$ and $M_2$ are both connected and oriented, then $\varphi$ induces an orientation on $M_2$ that may or may not coincide with the initial orientation of $M_2$.

If the induced orientation coincides, then we say $\varphi$ preserves the orientation, otherwise $\varphi$ reverses the orientation.

### 1.2.3 Grassmannian Manifold

Before proceeding, let's consider an interesting smooth manifold.

**Definition 1.2.11** (Grassmannian manifold). Given $m, n \in \mathbb{N}$, the so-called *Grassmannian manifold* $G(n, m)$ is the set of all $n$-dimensional subspaces of $\mathbb{R}^{n+m}$.

**Note.** $G(1, m)$ is just $\mathbb{R}P^m$, and $G(0, m)$, $G(n, 0)$ are one-point sets.

As we will soon see, $G(n, m)$ has the smooth structure of an $mn$-dimensional manifold.

**Intuition.** We obtain the structure by exhibiting an atlas whose transitions are diffeomorphisms.

Firstly, we give $G(n, m)$ a suitable topology, i.e., the metric topology. Let $\Pi \in G(n, m)$, and let $\mathcal{L}(\Pi, \Pi^\perp)$ denote the $mn$-dimensional space of linear maps from $\Pi$ to $\Pi^\perp$. Define the map

$$\varphi_\Pi \colon \mathcal{L}(\Pi, \Pi^\perp) \to G(n, m), \qquad \varphi_\Pi(\alpha) = (\mathbb{1}_\Pi \oplus \alpha)\,(\Pi)$$

where $\mathbb{1}_\Pi \oplus \alpha$ is regarded as a map $\Pi \to \Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.[1] Clearly, $\varphi_\Pi$ is injective, and thus, $\big(\mathcal{L}(\Pi, \Pi^\perp), \varphi_\Pi\big)$ is an $mn$-dimensional chart of $G(n, m)$.

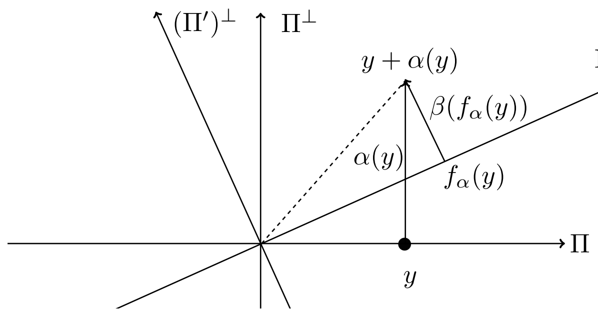
**Remark.** The images $\varphi_\Pi\big(\mathcal{L}(\Pi, \Pi^\perp)\big)$ cover $G(n, m)$.

**Example.** $\Pi = \varphi_\Pi(0) \in \varphi_\Pi\big(\mathcal{L}(\Pi, \Pi^\perp)\big)$.

We can now prove that these charts are mutually compatible. Let $\Pi, \Pi' \in G(n, m)$, and let $P, P'$ be orthogonal projections from $\mathbb{R}^{n+m}$ onto $\Pi, \Pi'$ respectively. Firstly,

$$F = \varphi_{\Pi'}^{-1}\varphi_\Pi \colon \varphi_\Pi^{-1}\big(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))\big) \to \varphi_{\Pi'}^{-1}\big(\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))\big)$$

is smooth.




Consider $\alpha \in \mathcal{L}(\Pi, \Pi^\perp)$, and $\beta \in \mathcal{L}(\Pi', (\Pi')^\perp)$, then for $\alpha, \beta$, the equality $F(\alpha) = \beta$ means that $\varphi_\Pi(\alpha) = \varphi_{\Pi'}(\beta)$. Let $f_\alpha \colon \Pi \to \Pi'$ be defined by

$$f_\alpha = P' \circ (\mathbb{1}_\Pi \oplus \alpha).$$

We need to check

(a) $f_\alpha$ is invertible, and

(b) $\forall y \in \Pi$, $y + \alpha(y) = f_\alpha(y) + \beta(f_\alpha(y))$.

**Note.** The condition that $\det f_\alpha \neq 0$ gives an exact description of the subset $\varphi_{\Pi^{-1}}\big(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))\big)$ of $\mathcal{L}(\Pi, \Pi^\perp)$, which is therefore open.

For $\beta$, it is $\big(\mathbb{1}_{\Pi'} \oplus \beta\big) \circ f_\alpha = \mathbb{1}_\Pi \oplus \alpha$, and hence

$$\beta = F(\alpha) = (\mathbb{1}_\Pi \oplus \alpha) \circ f_\alpha^{-1} - \mathbb{1}_{\Pi'}.$$

It follows by the construction that the image of $\beta$ is contained in $(\Pi')^\perp$.

[1] In other words, $\varphi_\Pi(\alpha)$ is the graph of $\alpha$ in $\Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.

**Remark.** We obtain an infinite atlas for $G(n,m)$ with charts labeled by $\Pi \in G(n,m)$. But it's suffices to consider only $\binom{n+m}{n}$ charts corresponding to subspaces $\Pi$ spanned with $n$ coordinate axes.

We now introduce two notions.

**Definition 1.2.12** (Closed manifold). A manifold is *closed* if it is compact and without boundary.

**Definition 1.2.13** (Open manifold). A manifold is *open* if it has only non-compact components without boundary.

**Lemma 1.2.1.** If $M$ can be covered by two coordinate neighborhoods $V_1, V_2$ such that $V_1 \cap V_2$ is connected, then $M$ is orientable.

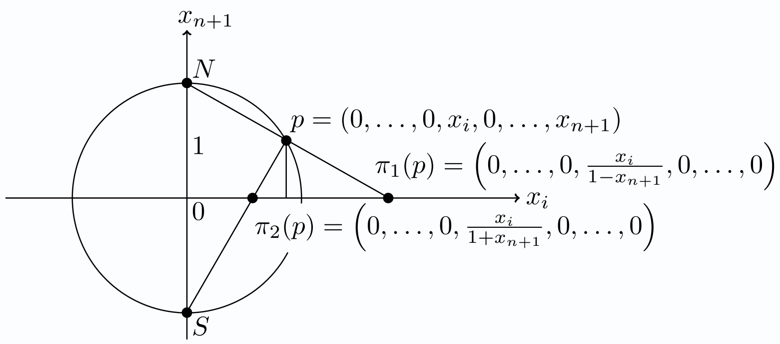
**Proof.** The determinant of the differential of the coordinate change $\neq 0$, so it does not change sign in $V_1 \cap V_2$. If it's negative at a single point, it's enough to change the sign of one of the coordinates to make it positive at that point, hence on $V_1 \cap V_2$. ∎

**Example.** Let $S^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\right\} \subseteq \mathbb{R}^{n+1}$ is orientable.

**Proof.** Let $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$, consider given $p = (0, \dots, 0, x_i, 0, \dots, x_{n+1})$, then $\pi_1 \colon S^n \setminus \{N\} \to \mathbb{R}^n$ given by

$$\pi_1(p) = \left(0, \dots, 0, \frac{x_i}{1 - x_{n+1}}, 0, \dots, 0\right)$$

to be the stereographic projection from the north pole $N$.




More generally, it takes $p(x_1, \dots, x_{n+1}) \in S^n - \{N\}$ into the intersection at the hyperplane $x_{n+1} = 0$ with the line passing through $p$ ad $N$. In this way, we have

$$\pi_1(x_1, \dots, x_n) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right),$$

hence $\pi_1 \colon S^n \setminus \{N\} \to \mathbb{R}^n$ is differentiable, and is injective. Similarly, $\pi_2 \colon S^n \setminus \{S\} \to \mathbb{R}^n$ for $S$ can also be defined and everything holds similarly. We see that these two parametrizations $(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})$ cover $S^n$. The change of coordinate is given by

$$y_j = \frac{x_j}{1 - x_{n+1}} \leftrightarrow y_j' = \frac{x_j}{1 + x_{n+1}}, \ (y_1, \dots, y_n) \in \mathbb{R}^n, \ j = 1, \dots, n,$$

where

$$y_j' = \frac{y_j}{\sum_{i=1}^n y_i^2}.$$

This implies that $\{(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})\}$ is a differentiable structure for $S^n$. Now, consider $\pi_1^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^n) = S^n \setminus \{N \cup S\}$, which is connected, and hence $S^n$ is orientable, and the above structure gives an orientation of $S^n$. ⊛

# Lecture 3: Complex Manifolds, Tangent Spaces and Bundles

Let's look at two more examples about orientation. 12 Jan. 14:30

**Example.** Let $A\colon S^n \to S^n$ be the antipodal map given by $A(p) = -p$ for $p \in \mathbb{R}^{n+1}$. It's easy to see that $A$ is differentiable with $A^2 = \mathbb{1}$. Furthermore, $A$ is diffeomorphism of $S^n \subseteq \mathbb{R}^{n+1}$. We see that

- if $n$ is even, $A$ reverses the orientation;
- if $n$ is odd, $A$ preserves the orientation.

**Example.** $G(k, n)$ is orientable if and only if $n$ is even or $n = 1$.

Finally, we introduce the notion of complex manifolds.

**Definition 1.2.14** (Complex manifold)**.** A *complex manifold* $\mathcal{M}$ of complex dimension $d$ ($\dim_{\mathbb{C}} \mathcal{M} = d$) is a differentiable manifold of (real) dimension $2d$ ($\dim_{\mathbb{R}} \mathcal{M} = 2d$) whose charts take values in open subsets of $\mathbb{C}^d$ with holomorphic chart transitions.

**As previously seen.** The chart transitions $z_\beta \circ z_\alpha^{-1}\colon z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ is holomorphic if $\partial z_\beta^j / \partial \overline{z_\alpha^k} = 0$ for all $j, k$ where

$$\frac{\partial}{\partial \overline{z^k}} = \frac{1}{2}\left(\frac{\partial}{\partial \overline{x^k}} + i \frac{\partial}{\partial \overline{y^k}}\right).$$

**Remark.** Complex Grassmannians $G_{\mathbb{C}}(k, n)$ are all orientable. More generally, complex manifolds are always orientable because holomorphic maps always have positive functional determinant.

## 1.3 Partition of Unity

We state, without proof, of an important lemma about the partition of unity.

**Definition 1.3.1** (Partition of unity)**.** Let $\mathcal{M}$ be a differentiable manifold, and let $(U_\alpha)_{\alpha \in \mathcal{A}}$ be an open covering of $\mathcal{M}$. Then a *partition of unity* is a locally finite refinement $(V_\beta)_{\beta \in \mathcal{B}}$ of $(U_\alpha)$ and $C^\infty$-functions $\varphi_\beta\colon \mathcal{M} \to \mathbb{R}$ with

(a) $\operatorname{supp}(\varphi_\beta) \subseteq V_\beta$ for all $\beta \in \mathcal{B}$;

(b) $0 \leq \varphi_\beta(x) \leq 1$ for all $x \in \mathcal{M}$, $\beta \in \mathcal{B}$;

(c) $\sum_{\beta \in \mathcal{B}} \varphi_\beta = 1$ for all $x \in \mathcal{M}$.[a]

[a] There are only finitely many non-vanishing summands of each point, since only finitely many $\varphi_\beta$ are non-zero of any given point as the covering $(V_\beta)$ is locally finite.

**Lemma 1.3.1** (Partition of unity)**.** Let $\mathcal{M}$ be a differentiable manifold, and let $(U_\alpha)_{\alpha \in \mathcal{A}}$ be an open covering of $\mathcal{M}$. Then there exists a partition of unity subordinate to $(U_\alpha)$,

## 1.4 Tangent and Cotangent Spaces

### 1.4.1 Tangent Spaces in Euclidean Spaces

To discuss the concept of calculus between manifolds formally, we start with our discussion in Euclidean spaces, where we naturally have the coordinates for every point.

**Definition.** Let $\mathcal{M}$ be a Euclidean manifold of dimension $d$, $x = (x^1, \ldots, x^d)$ be Euclidean coordinates of $\mathbb{R}^d$, and $x_0 \in \Omega \subseteq \mathbb{R}^d$ where $\Omega$ is open.

**Definition 1.4.1** (Tangent space of Euclidean space)**.** The *tangent space* $T_{x_0}\Omega$ *of* $\Omega$ *at* $x_0$ is the vector space $\{x_0\} \times E$[a] spanned by the basis $(\partial/\partial x^1, \ldots, \partial/\partial x^d)$.

[a] $E$ is a $d$-dimensional Euclidean space.

**Definition 1.4.2** (Tangent vector of Euclidean space)**.** The elements in the tangent space of Euclidean spaces is called *tangent vectors*.

Before proceeding, we introduce a shorthand notation.

**Notation** (Einstein notation)**.** The *Einstein notation* abbreviates the summation $\sum_i v^i x_i$ as $v^i x_i$, where we implicitly sum over the upper and lower index.

**Definition 1.4.3** (Differential of Euclidean space)**.** If $\Omega \subseteq \mathbb{R}^d$, $\Omega' \subseteq \mathbb{R}^d$ are open, and $f\colon \Omega \to \Omega'$ is differentiable, then the *differential* $\mathrm{d}f(x_0)$ for $x_0 \in \Omega$ is the induced linear map between tangent spaces

$$\mathrm{d}f(x_0)\colon T_{x_0}\Omega \to T_{f(x_0)}\Omega', \quad v = v^i \frac{\partial}{\partial x^i} \mapsto v^i \frac{\partial f^j}{\partial x^i}(x_0)\frac{\partial}{\partial f^j}.$$

**Definition 1.4.4** (Tangent bundle of Euclidean space)**.** The *tangent bundle* is defined as $T\Omega := \bigsqcup_{x\in\Omega} T_x\Omega \cong \Omega \times E \cong \Omega \times \mathbb{R}^d$, which is an open subset of $\mathbb{R}^d \times \mathbb{R}^d$.

**Note** (Total space)**.** $T\Omega$ is also called the *total space*.

**Remark.** Given a tangent bundle $T\Omega$, we define $\pi$ to be the projection $\pi\colon T\Omega \to \Omega$ given by $\pi(x, v) = x$. This makes $T\Omega$ naturally a differentiable manifold.

With the notion of tangent bundle, given $f\colon \Omega \to \Omega'$, we can also define $\mathrm{d}f\colon T\Omega \to T\Omega'$ as

$$\left(x, v^i \frac{\partial}{\partial x^i}\right) \mapsto \left(f(x), v^i \frac{\partial f^j}{\partial x^i}(x_0)\frac{\partial}{\partial f^j}\right).$$

**Notation.** We often write $\mathrm{d}f(x)(v)$ instead of $\mathrm{d}f(x, v)$ to coincide with the notation of differential.

In particular, for $v = v^i \partial/\partial x^i$, we have

$$\mathrm{d}f(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R},$$

and we write $v(f)(x)$ for $\mathrm{d}f(x)(v)$.

### 1.4.2 Tangent Spaces in Manifolds

We now try to formally define the tangent space on a smooth manifold. A natural idea is the following.

**Intuition.** Let $\mathcal{M}^d$ be a differentiable manifold with a chart $x\colon U \to \Omega \subseteq \mathbb{R}^d$ and $p \in U \subseteq \mathcal{M}$ where $U$ is open. The *tangent space* $T_p\mathcal{M}$ of $\mathcal{M}$ at $p$ *should be* represented in the chart $x$ by $T_{x(p)}x(U)$.

To see that the above are well-defined, i.e., $T_p\mathcal{M}$ are independent of the choice of charts, let $x'\colon U' \to \mathbb{R}^d$ to be another chart with $p \in U' \subseteq \mathcal{M}$ where $U'$ is also open. Denote $\Omega \coloneqq x(U)$, and $\Omega' \coloneqq x'(U')$, then the transition map

$$x' \circ x^{-1}\colon x(U \cap U') \to x'(U \cap U')$$

induces a vector space isomorphism

$$L \coloneqq \mathrm{d}(x' \circ x^{-1})(x(p))\colon T_{x(p)}\Omega \to T_{x'(p)}\Omega',$$

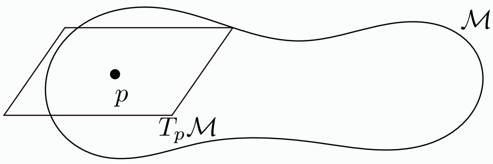
such that $v \in T_{x(p)}\Omega$ and $L(v) \in T_{x'(p)}\Omega'$ represent the same tangent vector in $T_p\mathcal{M}$.

**Remark.** A tangent vector in $T_p\mathcal{M}$ is given by the family of the coordinate representations.

Now, we want to define the similar notion of differential of Euclidean spaces. Let consider a simple case first, where we let $f\colon \mathcal{M} \to \mathbb{R}$ to be a differentiable function, and assume that the tangent vector $w \in T_p\mathcal{M}$ is represented by $v \in T_{x(p)}x(U)$.

**Intuition.** We want to define $\mathrm{d}f(p)$ as a linear map from $T_p\mathcal{M} \to \mathbb{R}$. In chart $x$, let $w \in T_p\mathcal{M}$ be given as $v = v^i\partial/\partial x^i \in T_{x(p)}x(U)$. Say that $\mathrm{d}f(p)(w)$ in this chart represented by

$$\mathrm{d}(f \circ x^{-1})(x(p))(v).$$




**Remark.** $T_p\mathcal{M}$ is a vector space of dimension $d$ isomorphic to $\mathbb{R}^d$, where the isomorphism depends on choice of chart.

**Intuition.** Pull functions on $\mathcal{M}$ back by a chart to an open subset of $\mathbb{R}^d$, differentiate there.

In order to obtain a tangent space which does not depend on charts, we need to have transformation behavior under change of charts. Let $F\colon \mathcal{M}^d \to \mathcal{N}^c$ be a differentiable map where $\mathcal{M}, \mathcal{N}$ are smooth manifolds. Then we want to represent $\mathrm{d}F$ in local charts $x\colon U \subseteq \mathcal{M} \to \mathbb{R}^d, y\colon V \subseteq \mathcal{N} \to \mathbb{R}^c$ by $\mathrm{d}(y \circ F \circ x^{-1})$. The local coordinates on $U$ is given by $(x^1, \ldots, x^d)$, and on $V$ is $(F^1, \ldots, F^c)$ such that

$$F(x) = \left(F^1(x^1, \ldots, x^d), \ldots, F^c(x^1, \ldots, x^d)\right).$$

Then, $\mathrm{d}F$ induces a linear map $\mathrm{d}F\colon T_p\mathcal{M} \to T_{F(x)}\mathcal{N}$ which in our coordinate representation is given by the matrix

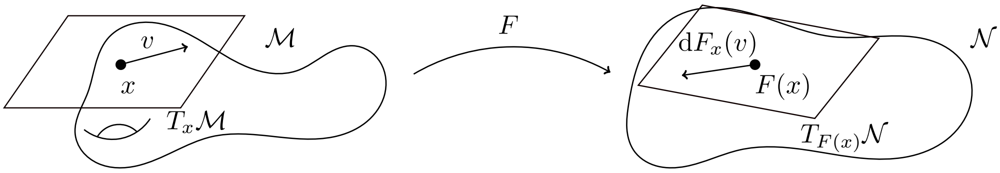
$$\left(\frac{\partial F^\alpha}{\partial x^i}\right)_{\substack{\alpha=1,\ldots,c\\ i=1,\ldots,d}},$$

and a change of charts is then just the base change at tangent spaces: if

$$\begin{aligned}(x^1, \ldots, x^d) &\mapsto (\xi^1, \ldots, \xi^d)\\ (F^1, \ldots, F^c) &\mapsto (\phi^1, \ldots, \phi^c)\end{aligned}$$

are coordinate changes, then $\mathrm{d}F$ represented in the new coordinates is given by

$$\left(\frac{\partial \phi^\beta}{\partial \xi^j}\right) = \left(\frac{\partial \phi^\beta}{\partial F^\alpha}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial x^i}{\partial \xi^j}\right).$$




# Lecture 4: Tangent Bundles, Vector Fields, and Submanifolds

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**Definition.** Let $\mathcal{M}^d$ be a differentiable manifold with a chart $x\colon U \to \Omega \subseteq \mathbb{R}^d$ and $p \in U \subseteq \mathcal{M}$ where $U$ is open. On $\{(x, v) \mid v \in T_{x(p)}\Omega\}$, we define an equivalence relation by $(x, v) \sim (y, w)$ if and only if $w = \mathrm{d}(y \circ x^{-1})v$.

**Definition 1.4.5** (Tangent space)**.** The space of equivalence classes is called the *tangent space* $T_p\mathcal{M}$ at point $p$ to $\mathcal{M}$.

**Definition 1.4.6** (Tangent vector)**.** The elements in the tangent space is called *tangent vectors*.

**Remark.** $T_p\mathcal{M}$ naturally caries the structure of a vector space.

Now, $T\mathcal{M}$ is defined as

$$T\mathcal{M} \coloneqq \coprod_{p\in\mathcal{M}} T_p\mathcal{M}.$$

Recall the projection $\pi\colon T\mathcal{M} \to \mathcal{M}$ with $\pi(V) = p$ for $V \in T_p\mathcal{M}$. Then we can define the following.

**Definition 1.4.7** (Derivation)**.** If $x\colon U \to \mathbb{R}^d$ be a chart for $\mathcal{M}$, and let $TU = \coprod_{p\in U} T_pU$. Then we define the *derivation* $\mathrm{d}x\colon TU \to Tx(U) \coloneqq \coprod_{p\in x(U)} T_p\mathcal{M}$ by $w \mapsto \mathrm{d}x(\pi(w))(w) \in T_{x(\pi(w))}x(U)$.

The transition maps

$$\mathrm{d}x' \circ (\mathrm{d}x)^{-1} = \mathrm{d}(x' \circ x^{-1})$$

are differentiable. $\pi$ is local represented by $x \circ \pi \circ \mathrm{d}x^{-1}$ maps $(x_0, v) \in Tx(U)$ to $x_0$.

**Definition 1.4.8** (Tangent bundle)**.** The triple $(T\mathcal{M}, \pi, \mathcal{M})$ is called the *tangent bundle* of $\mathcal{M}$.

**Definition 1.4.9** (Total space)**.** $T\mathcal{M}$ is called the *total space* of the tangent bundle.

We can choose the courses (the initial) topology for total space $T\mathcal{M}$ such that $\pi$ is continuous. Furthermore, we can construct a $C^\infty$-atlas $\mathcal{A}_{T\mathcal{M}}$ on $T\mathcal{M}$ from the $C^\infty$-atlas $\mathcal{A}$ of $\mathcal{M}$. Specifically, consider $\mathcal{A}_{T\mathcal{M}} \coloneqq \{(TU, \xi_x) \mid (U, x) \in \mathcal{A}\}$ where $\xi_x\colon TU \to \mathbb{R}^{2\cdot d}$ such that

$$x \mapsto \left((x^1 \circ \pi)(x), \dots, (x^d \circ \pi)(x), (\mathrm{d}x^1)_{\pi(x)}(X), \dots, (\mathrm{d}x^d)_{\pi(x)}(X)\right).$$

**Intuition.** We know that $X = X_x^i(\partial/\partial x^i)_{\pi(x)}$, and we might tempt to write $X^i$ as the last $d$ components. But we write it in the above way is because

$$(\mathrm{d}x^j)_{\pi(x)}(X) = (\mathrm{d}x^j)_{\pi(x)}\left(X_x^i\left(\frac{\partial}{\partial x^i}\right)_{\pi(x)}\right) = X_x^i\delta_i^j = X_x^j.$$

**Note.** We can check that $\xi_x^{-1}$ exists, and it's also smooth, hence $T\mathcal{M}$ has a natural topology and a $C^\infty$-atlas making it a $2\dim\mathcal{M}$-dimensional smooth manifold.

### 1.4.3 Cotangent Spaces

Another important objects is the cotangent spaces.

**Definition.** Let $\mathcal{M}^d$ be a differentiable manifold, and $T_p\mathcal{M}$ be the tangent space at $p$ to $\mathcal{M}$.

**Definition 1.4.10** (Cotangent space). The *cotangent space* $T_p^*\mathcal{M}$ to $\mathcal{M}$ is the dual of $T_p\mathcal{M}$, i.e., $T_p^*\mathcal{M} = (T_p\mathcal{M})^*$.

**Definition 1.4.11** (Cotangent vector). The elements in the cotangent space is called *cotangent vectors*.

**Remark.** $T_p^*\mathcal{M}$ is the space of 1-forms on $T_p\mathcal{M}$.

**Notation** (Covariant vector). The cotangent vectors are also called *covariant vectors*.

**Notation** (Contravariant vector). The tangent vectors are also called *contravariant vectors*.

Similarly, we can define the projection $\pi\colon T^*\mathcal{M} \to \mathcal{M}$ with $\pi(\omega) = p$ for $\omega \in T_p^*\mathcal{M}$, and we have the following.

**Definition 1.4.12** (Cotangent bundle). The triple $(T^*\mathcal{M}, \pi, \mathcal{M})$ is called the *cotangent bundle* of $\mathcal{M}$.

## 1.5 Vector Fields and Brackets

### 1.5.1 Vector Fields

We now introduce the notion of vector field.

**Definition 1.5.1** (Vector field). A *(tangent) vector field* $X$ on a differentiable manifold $\mathcal{M}$ is a correspondence associating to each point $p \in \mathcal{M}$ a vector $X(p) \in T_p\mathcal{M}$, i.e., $X\colon \mathcal{M} \to T\mathcal{M}$.

**Remark.** Naturally, we say that the field $X$ is differentiable if the map $X$ is differentiable.

Considering a local chart $x\colon U \subseteq \mathbb{R}^n \to \mathcal{M}$, we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i},$$

where $a_i\colon U \to \mathbb{R}$ are functions on $U$ for $i = 1, \ldots, n$, and $\{\partial/\partial x_i\}_i$ is the basis associated to $x$.

**Remark.** $X$ is differentiable if and only if $a_i$ are differentiable for some (and, therefore, for any) $x$.

It's convenient to think of a vector field as a mapping $X\colon \mathcal{D} \to \mathcal{F}$ from the set $\mathcal{D}$ of differentiable functions on $\mathcal{M}$ to the set $\mathcal{F}$ of the functions on $\mathcal{M}$, defined by

$$(Xf)(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p),$$

where $f$ is implicitly denoting the expression of $f$ in the chart $x$.

**Intuition.** This idea of a vector as a directional derivative is precisely what was used to define the notion of tangent vector.

**Remark.** $Xf$ does not depend on the choice of $x$.

**Remark.** $X$ is differentiable if and only if $X: \mathcal{D} \to \mathcal{D}$, i.e., $Xf \in \mathcal{D}$ for all $f \in \mathcal{D}$.

Observe that if $\varphi: \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, $v \in T_p\mathcal{M}$ and $f$ differentiable function in a neighborhood of $\varphi(p)$, we have

$$(\mathrm{d}\varphi(v)f)\varphi(p) = v(f \circ \varphi)(p)$$

since by letting $\alpha: (-\epsilon, \epsilon) \to \mathcal{M}$ be a differentiable curve with $\alpha'(0) = v$, $\alpha(0) = p$,[2] then

$$(\mathrm{d}\varphi(v)f)\varphi(p) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi \circ \alpha)\bigg|_{t=0} = v(f \circ \varphi)(p).$$

### 1.5.2 Brackets

By viewing $X$ as an operator on $\mathcal{D}$, we can consider the iterates of $X$, i.e, given differentiable fields $X$ and $Y$ and $f: M \to \mathbb{R}$ being a differentiable function, consider $X(Yf)$ and $Y(Xf)$.

**Note.** In general, $X(Yf)$ (and hence $Y(Xf)$) is not a field.

**Proof.** It involves derivatives of order higher than one. ⊛

But we have the following.

**Lemma 1.5.1.** Let $X, Y$ be differentiable vector fields on a smooth manifold $\mathcal{M}$. Then there exists a unique vector field $Z$ such that for all $f \in \mathcal{D}$, $Zf = (XY - YX)f$.

**Proof.** See do Carmo [FC13, Chapter 0, Lemma 5.2]. ■

This $Z$ is called the bracket.

**Definition 1.5.2** (Bracket)**.** Given two differentiable vector fields $X, Y$ on a smooth manifold $\mathcal{M}$, the *bracket* of $X$ and $Y$ is defined by

$$[X, Y] := XY - YX.$$

Clearly, $[X, Y]$ is differentiable.

**Proposition 1.5.1.** If $X, Y$ and $Z$ are differentiable vector fields on $\mathcal{M}$, $a, b \in \mathbb{R}$, $f, g$ are differentiable functions, then we have the following.

(a) $[X, Y] = -[Y, X]$ (*anti-commutativity*),

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (*linearity*),

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (*Jacobi identity*),

(d) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

**Proof.** See do Cargo [FC13, Chapter 0, Proposition 5.3]. ■

**Example.** $[\partial/\partial x^i, \partial/\partial x^j] = 0$ for $i = j$.

## 1.6 Submanifolds, Immersions, and Embeddings

We now study the relation between manifolds.

[2] This is the way do Carmo [FC13] used to define tangent vectors.

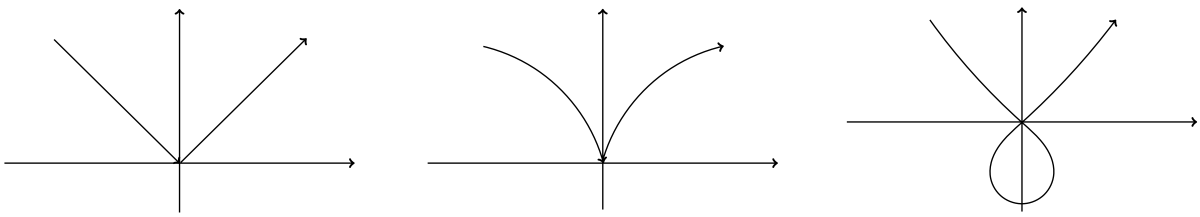
**Definition 1.6.1** (Immersion). Let $\mathcal{M}^m, \mathcal{N}^n$ be smooth manifolds. A differentiable mapping $\varphi\colon \mathcal{M} \to \mathcal{N}$ is an *immersion* if

$$\mathrm{d}\varphi_p\colon T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$$

is injective for every $p \in \mathcal{M}$.

**Definition 1.6.2** (Embedding). An immersion $\varphi\colon \mathcal{M} \to \mathcal{N}$ is an *embedding* if it is also a homeomorphism onto $\varphi(\mathcal{M}) \subseteq \mathcal{N}$, with $\varphi(\mathcal{M})$ having the subspace topology induced from $\mathcal{N}$.

**Definition 1.6.3** (Submanifold). If the inclusion $\iota\colon \mathcal{M} \hookrightarrow \mathcal{N}$ between two manifolds is an embedding, then $\mathcal{M}$ is a *submanifold* of $\mathcal{N}$.



(a) Non-differentiable curve. (b) Non-immersion curve. (c) Non-embedding curve.

Figure 1.1: Three simple examples

**Lemma 1.6.1.** Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ to be an immersion and $x \in \mathcal{M}$.[a] Then there exists a neighborhood $U$ of $x$ and a chart $(V, y)$ on $\mathcal{N}$ with $f(x) \in V$ such that $f|_U$ is a differentiable embedding and $y^{m+1}(p) = \ldots = y^n(p) = 0$ for all $p \in f(U \cap V)$.

[a] Hence, $n \geq m$.

**Proof.** In the local coordinates $(z^1, \ldots, z^n)$ on $\mathcal{N}$, and $(x^1, \ldots, x^m)$ on $\mathcal{M}$, without loss of generality,[a] let

$$\left(\frac{\partial z^\alpha(f(x))}{\partial x^i}\right)_{i,\alpha=1,\ldots,m}$$

be non-singular. Consider

$$F(z, x) := \left(z^1 - f^1(x), \ldots, z^n - f^n(x)\right),$$

which has maximal rank in $x^1, \ldots, x^m, z^{m+1}, \ldots, z^n$. By the implicit function theorem, locally, there exists a map $\varphi\colon U \to \mathbb{R}^n$ such that

$$(z^1, \ldots, z^m) \mapsto (\varphi^1(z^1, \ldots, z^m), \ldots, \varphi^n(z^1, \ldots, z^m)) = x$$

such that $F(z, x) = 0$, i.e.,

$$\varphi^i(z^1, \ldots, z^m) = \begin{cases} x^i, & \text{if } i = 1, \ldots, m; \\ z^i, & \text{if } i = m+1, \ldots, n, \end{cases}$$

for which

$$\left(\frac{\partial \varphi^i}{\partial z^\alpha}\right)_{\alpha,i=1,\ldots,m}$$

has maximal rank. Now, we choose a new coordinate

$$\begin{aligned}(y^1, \ldots, y^n) = \big(&\varphi^1(z^1, \ldots, z^m), \ldots, \varphi^m(z^1, \ldots, z^m), \\ &z^{m+1} - \varphi^{m+1}(z^1, \ldots, z^m), \ldots, z^n - \varphi^n(z^1, \ldots, z^m)\big).\end{aligned}$$

Then, we have $z = f(x) \Leftrightarrow F(z, x) = 0$, i.e., $(y^1, \ldots, y^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$, proving the result. ■

[a] Since $\mathrm{d}f(x)$ is injective.

**Lemma 1.6.2.** Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ be a differentiable map such that $m \geq n$ with $p \in \mathcal{N}$. Let $\mathrm{d}f(x)$ has rank $n$ for all $x \in \mathcal{M}$ with $f(x) = p$. Then $f^{-1}(p)$ is the union of differentiable submanifolds of $\mathcal{M}$ of dimension $m - n$.

**Remark.** Let $\mathcal{N}^n$ be a smooth manifold, and let $1 \leq m \leq n$. Then an arbitrary subset $\mathcal{M} \subseteq \mathcal{N}$ has the structure of differentiable submanifold of $\mathcal{N}$ of dimension $m$ if and only if for all $p \in \mathcal{M}$, there exists a smooth chart $(U, \varphi)$ of $\mathcal{N}$ such that $p \in U$, $\varphi(p) = 0$, $\varphi(U)$ is open, and

$$\varphi(U \cap \mathcal{M}) = (-\epsilon, +\epsilon)^n \times \{0\}^{n-m},$$

where $(-\epsilon, +\epsilon)^n$ is the cube. Noticeably, the $C^\infty$-manifold structure of $\mathcal{M}$ is uniquely determined.

**Remark.** Let $\mathcal{M} \subseteq \mathcal{N}$ be a differentiable submanifold of $\mathcal{N}$, and let $\iota\colon \mathcal{M} \hookrightarrow \mathcal{N}$ be the inclusion. Then, for $p \in \mathcal{M}$, $T_p\mathcal{M}$ can be considered as subspace of $T_p\mathcal{N}$, namely as the image of $\mathrm{d}\iota(T_p\mathcal{M})$.

**Lemma 1.6.3.** Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ be a differentiable map such that $m \geq n$ with $p \in \mathcal{N}$. Let $\mathrm{d}f(x)$ has rank $n$ for all $x \in \mathcal{M}$ with $f(x) = p$. For the submanifold $X = f^{-1}(p)$ and for $q \in X$, it is true that

$$T_qX = \ker \mathrm{d}f(q) \subseteq T_q\mathcal{M}.$$

# Chapter 2

# Riemannian Manifolds

## Lecture 5: Riemannian Manifolds

In this chapter, we start our discussion on Riemannian manifolds. 19 Jan. 14:30

## 2.1 Riemannian Metrics

We start by defining the Riemannian metric.

**Definition 2.1.1** (Riemannian metric)**.** A *Riemannian metric* $g$ on a differentiable manifold $\mathcal{M}$ is given by a scalar product $I$ on each $T_p\mathcal{M}$ which depends smoothly on the base point $p$.

**Definition 2.1.2** (Riemannian manifold)**.** A *Riemannian manifold* $(\mathcal{M}, g)$ is a smooth manifold $\mathcal{M}$ equipped with a Riemannian metric $g$.

Let $x = (x^1, \ldots, x^d)$ be the local coordinates. In these, a metric is represented by a positive definite symmetric matrix $\left(g_{ij}(x)\right)_{i,j=1,\ldots,d}$, i.e., $g_{ij} = g_{ji}$, and $g_{ij}\xi^i\xi^j > 0$ for all $\xi = (\xi^1, \ldots, \xi^d) \neq 0$ with coefficients smoothly depending on $x$.

### 2.1.1 Transformation Behavior

We now see that the smoothness does not depend on coordinates, i.e., the smooth dependence on the base point (as required in Definition 2.1.1) can be represented in the local coordinates. Given 2 tangent vectors $v, w \in T_p\mathcal{M}$ with coordinate representations $(v^1, \ldots, v^d), (w^1, \ldots, w^d)$ given by $x$ such that $v = v^i \frac{\partial}{\partial x^i}$ and $w = w^i \frac{\partial}{\partial x^i}$, their product is

$$\langle v, w\rangle \coloneqq g_{ij}(x(p))v^i w^j.$$

In particular,

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = g_{ij}.$$

**Remark.** The length of $v$ is given as $\|v\| \coloneqq \langle v, v\rangle^{1/2}$.

Let $y = f(x)$ define different local coordinates. In these, $v, w$ are given as

$$(\widetilde{v}^1, \ldots, \widetilde{v}^d), (\widetilde{w}^1, \ldots, \widetilde{w}^d)$$

with $\widetilde{v}^j = v^i \frac{\partial f^j}{\partial x^i}$ and $\widetilde{w}^j = w^i \frac{\partial f^j}{\partial x^i}$. Denote the metric in new coordinates $y$ by $h_{k\ell}(y)$, then we have

$$h_{k\ell}(f(x))\widetilde{v}^k\widetilde{w}^\ell = \langle v, w\rangle = g_{ij}(x)v^i w^j.$$

Plug everything in, we have

$$h_{k\ell}(f(x))\frac{\partial f^k}{\partial x^i}\frac{\partial f^\ell}{\partial x^j}v^i w^j = g_{ij}(x)v^i w^j.$$

We see that this holds for any tangent vectors $v, w$, therefore,

$$h_{k\ell}(f(x))\frac{\partial f^k}{\partial x^i}\frac{\partial f^\ell}{\partial x^j} = g_{ij}(x),$$

which is the transformation behavior under coordinates changes.

**Remark.** This shows that the smoothness does not depend on the choice of coordinates!

**Example.** Consider the Euclidean space $\Omega$, then given $v, w \in T_p\Omega$, we have

$$\langle v, w\rangle = \delta_{ij}v^i w^j = v^i w_i.$$

**Theorem 2.1.1.** Every differentiable manifold can be equipped with a Riemannian metric.

**Proof.** From Lemma 1.3.1, there exists a differentiable partition of unity $\{f_\alpha\}$ of $\mathcal{M}$ subordinate to a covering $\{V_\alpha\}$ of $\mathcal{M}$. Consider the induced metric $\langle\cdot,\cdot\rangle^\alpha$ of the system of local coordinates on each $V_\alpha$. Then, for every $p \in M$, a Riemannian metric $\langle\cdot,\cdot\rangle_p$ can be defined naturally as

$$\langle u, v\rangle_p = \sum_\alpha f_\alpha(p)\,\langle u, v\rangle_p^\alpha$$

for all $u, v \in T_pM$. Given the fact that $\{f_\alpha\}$ is the partition of unity, we know that

(a) $f_\alpha \geq 0$, and $f_\alpha = 0$ on $\overline{V}_\alpha^c$,

(b) $\sum_\alpha f_\alpha(p) = 1$ for all $p$ on $M$,

it's then immediate that the defined is indeed a Riemannian metric. ■

### 2.1.2 Isometry

After introducing any type of mathematical structure, we must introduce a notion of when two objects are the same, hence we now characterize $g$.

**Definition 2.1.3** (Isometry). A diffeomorphism $h\colon \mathcal{M} \to \mathcal{N}$ is an *isometry* between two Riemannian manifolds if it preserves the Riemannian metric, i.e., for $p \in \mathcal{M}$, $v, w \in T_p\mathcal{M}$,

$$\langle v, w\rangle_{\mathcal{M}} = \langle \mathrm{d}h(v), \mathrm{d}h(w)\rangle_{\mathcal{N}}.$$

**Definition 2.1.4** (Local isometry). A diffeomorphism $h\colon \mathcal{M} \to \mathcal{N}$ is a *local isometry* between two Riemannian manifolds if for every $p \in \mathcal{M}$, there exists a neighborhood $U$ such that $h|_U : U \to h(U)\colon \mathcal{M} \to \mathcal{N}$ is an isometry and $h(U) \subseteq \mathcal{N}$ is open.

If's common to say that a Riemannian manifold $\mathcal{M}$ is locally isometric to a Riemannian manifold $\mathcal{N}$ if for every $p \in \mathcal{M}$, there exists a neighborhood $U$ of $p$ in $\mathcal{M}$ and a local isometry $f\colon U \to f(U) \subseteq \mathcal{N}$.

**Example** (Euclidean space). The *Euclidean space of dimension $n$* $\mathcal{M} = \mathbb{R}^n$ with $\partial/\partial x_i$ identified with $e_i = (0, \ldots, 1, \ldots, 0)$ is with the metric

$$\langle e_i, e_j\rangle = \delta_{ij}.$$

The Riemannian geometry of this space is metric Euclidean geometry.

**Example** (Lie group). See Appendix A.4 for reference.

## 2.2 Geodesics

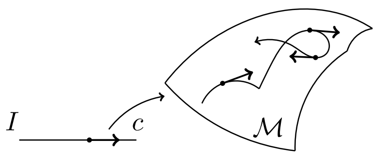
This is the first focus on the study of Riemannian geometry, i.e., the geodesics. The up-shot is that a geodesic minimizes the arc length for points *sufficiently close* (in a sense to be made precise); in addition, if a curve minimizes arc length between any two of its points, it is a geodesic.

### 2.2.1 Vector Fields along Curves

We are now going to show how a Riemannian metric can be used to calculate the length of a curve.

**Definition 2.2.1** (Curve). A (parametrized) *curve* is a differentiable mapping $c\colon I \subseteq \mathbb{R} \to \mathcal{M}$ to a smooth manifold $\mathcal{M}$.

**Note.** A parametrized curve can admit self-intersections as well as corners.




**Definition 2.2.2** (Vector field along a curve). A *(smooth) vector field $X$ along a curve $c\colon I \subseteq \mathbb{R} \to \mathcal{M}$* on a smooth manifold $\mathcal{M}$ is defined as $X\colon I \to T\mathcal{M}$ such that $X(t) \in T_{c(t)}\mathcal{M}$ for all $t \in I$.

**Notation.** The set of smooth vector fields along $c$ is denoted as $\chi_c(\mathcal{M})$.

**Note.** To say $V$ is differentiable means that for any differentiable function $f$ on $\mathcal{M}$, the function $t \mapsto V(t)f$ is a differentiable function on $I$.

**Example** (Velocity field). The vector field along $c$ $\mathrm{d}c/\mathrm{d}t \coloneqq \mathrm{d}c(\mathrm{d}/\mathrm{d}t)$ is called the *velocity field* or *tangent vector field*.

**Remark.** A vector field along $c$ can't necessarily be extended to a vector field on an open set of $\mathcal{M}$.

**Notation** (Segment). The restriction of a curve $c$ to a closed interval $[a, b] \subseteq I$ is called a *segment*.

### 2.2.2 Lengths and Energies

We're interested in the following two quantities.

**Definition.** Let $\gamma\colon [a, b] \to \mathcal{M}$ be a curve on a Riemannian manifold $(\mathcal{M}, g)$.

**Definition 2.2.3** (Length). The *length* of $\gamma$ is defined as

$$L(\gamma) \coloneqq \int_a^b \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \right\| \,\mathrm{d}t.$$

**Definition 2.2.4** (Energy). The *energy* of $\gamma$ is defined as

$$E(\gamma) \coloneqq \frac{1}{2} \int_a^b \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \right\|^2 \mathrm{d}t.$$

We now want to compute $L(\gamma)$, $E(\gamma)$ in local coordinates. Let the local coordinates be

$$(x^1(\gamma(t)), \ldots, x^d(\gamma(t))),$$

we write

$$\dot{x}^i(t) = \frac{\mathrm{d}}{\mathrm{d}t}(x^i(\gamma(t))).$$

Then, we have

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)}\,\mathrm{d}t, \quad E(\gamma) = \frac{1}{2}\int_a^b g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)\,\mathrm{d}t.$$

**Definition 2.2.5** (Distance)**.** Given a Riemannian manifold $(\mathcal{M}, g)$, the *distance* between 2 points $p, q \in \mathcal{M}$ is defined as

$$d(p,q) \coloneqq \inf\left\{L(\gamma) \mid \gamma\colon [a,b] \to \mathcal{M} \text{ piecewise curve with } \gamma(a) = p, \gamma(b) = q\right\}.$$

**Note.** Any 2 points $p, q \in \mathcal{M}$ can be connected by a piecewise curve, hence $d(p,q)$ always exists.

**Corollary 2.2.1.** The topology of $\mathcal{M}$ induced by the distance function $d$ coincides with the original manifold topology of $\mathcal{M}$.

**Lemma 2.2.1.** If $\gamma\colon [a,b] \to \mathcal{M}$ is a curve, and $\psi\colon [\alpha,\beta] \to [a,b]$ is a reparametrization, then $L(\gamma \circ \psi) = L(\gamma)$.

**Proof.** This can be proved by computation, and the take-away is that the length functional is invariant under parameter changes. ■

### 2.2.3 Euler-Lagrange Equations

We want to find a curve which minimizes the length between sufficiently close two points. It turns out that instead of working with length directly, we should work with energy instead.

**Notation.** Let's first fix some common notations.

(a) $\left(g^{ij}\right)_{i,j=1,\ldots,d} = (g_{ij})^{-1}_{i,j=1,\ldots,d}$.[a]

(b) $g_{j\ell,k} \coloneqq \frac{\partial}{\partial x^k} g_{j\ell}$.

[a] Technically, $g^{-1}$ is not an inverse: $g$ is a $(0,2)$-tensor field, while $g^{-1}$ is a $(2,0)$-tensor field.

**Note.** In the above notations, we have $g^{i\ell}g_{\ell j} = \delta^i_j$.

And the following is particularly important.

**Notation** (Christoffel symbol)**.** The *Christoffel symbol* is defined for all $i$ as

$$\Gamma^i_{jk} \coloneqq \frac{1}{2}g^{i\ell}\left(g_{j\ell,k} + g_{k\ell,j} - g_{jk,\ell}\right).$$

**Remark.** The notion of $\Gamma$ is a bit cryptic at first, and we will come back to this after. Now, just treat it as a calculation tool.

The up-shot is that the Euler-Lagrange equations for the energy $E$ has a nice form, and the solution of which has exactly the properties we want, hence we define it as geodesics.

**Proposition 2.2.1.** The Euler-Lagrange equations for the energy $E$ are

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0 \text{ for } i = 1, \dots, d. \tag{2.1}$$

**Proof.** The Euler-Lagrange equations of a functional[a]

$$I(x) = \int_a^b f(t, x(t), \dot{x}(t)) \,\mathrm{d}t$$

are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0$$

for $i = 1, \dots, d$. Just by plugging in, we obtain for $E$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ik}(x(t))\dot{x}^k(t) + g_{ji}(x(t))\dot{x}^j(t)\right) - g_{jk,i}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$$

for $i = 1, \dots, d$. Hence,

$$g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j + g_{ik,\ell}\dot{x}^\ell\dot{x}^k + g_{ji,\ell}\dot{x}^\ell\dot{x}^j - g_{jk,i}\dot{x}^\ell\dot{x}^j = 0$$

Rename some indices and use $g_{ij} = g_{ji}$, we have that

$$2g_{\ell m}\ddot{x}^m + (g_{k\ell,j} + g_{j\ell,k} - g_{jk,\ell})\,\dot{x}^j\dot{x}^k = 0$$

for $\ell = 1, \dots, d$. Hence, we have

$$g^{i\ell}g_{\ell m}\ddot{x}^m + \frac{1}{2}g^{i\ell}\,(g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\,\dot{x}^j\dot{x}^k = 0$$

for $i = 1, \dots, d$. Finally, observe that $g^{i\ell}g_{\ell m} = \delta_{im}$, i.e., $g^{i\ell}g_{\ell m}\ddot{x}^m = \ddot{x}^i$, hence the claim follows. ■

[a] The Lagrangian is $\mathcal{L} = \frac{1}{2}g_{jk}\dot{x}^j\dot{x}^k$.

Finally, we define the geodesics as the solution of Equation 2.1.

**Definition 2.2.6** (Geodesic). A curve $\gamma\colon [a, b] \to \mathcal{M}$ that obeys Equation 2.1 is called a *geodesic*.

**Intuition.** From Proposition 2.2.1, we naturally define geodesic by the solution of Equation 2.1, which is the critical points of energy.[a]

[a] In fact, we can also start from length and get the same thing, which might be more natural.

### 2.2.4 Solving The Euler-Lagrangian Equations

To solve this via the variational principal, we first define the action functional.

**Definition 2.2.7** (Action functional). Let $\mathcal{L}$ be the Lagrangian, then the *action functional*

$$I[w(\cdot)] \coloneqq \int_0^t \mathcal{L}(\dot{w}(s), w(s)) \,\mathrm{d}s$$

is defined for functions $w(\cdot) = (w^1(\cdot), \dots w^n(\cdot))$ of the admissible class

$$\mathcal{A} = \left\{w(\cdot) \in C^2([0, t]; \mathbb{R}^n) \mid w(0) = y, w(t) = x\right\}.$$

**Example.** Clearly, both length and energy are action functionals.

From the calculus of variation, we can find a curve $x(\cdot) \in \mathcal{A}$ such that $I[x(\cdot)] = \min_{w(\cdot)\in\mathcal{A}} I[w(\cdot)]$.

**Theorem 2.2.1** (Euler-Lagrangian equations)**.** The solution $x(\cdot)$ from $I[x(\cdot)] = \min_{w(\cdot)\in\mathcal{A}} I[w(\cdot)]$ solves the system of Euler-Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(D_{\dot{x}}\mathcal{L}(\dot{x}(s), x(s)) + D_x\mathcal{L}(\dot{x}(s), x(s))\right) = 0$$

for $0 \leq s \leq t$.

## Lecture 6: Geodesics and the Exponential Map

Now, we draw some relations between length and energy and see why starting from energy makes sense. 24 Jan. 14:30

**Proposition 2.2.2.** For all curves $\gamma\colon [a, b] \to \mathcal{M}$,

$$\mathcal{L}(\gamma)^2 \leq 2(b-a)E(\gamma)$$

with equality if and only if $\|\mathrm{d}\gamma/\mathrm{d}t\|$ is a constant.

**Proof.** From Hölder's inequality,

$$\int_a^b \left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\| \mathrm{d}t \leq (b-a)^{1/2}\left(\int_a^b \left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\|^2 \mathrm{d}t\right)^{1/2}$$

with equality if and only if $\|\mathrm{d}\gamma/\mathrm{d}t\|$ is a constant. ■

**Example.** Let

$$\mathcal{L}(q, x) = \frac{1}{2}m|q|^2 - V(x)$$

with $m > 0, q = \dot{x}$, the Euler-Lagrangian equations is given by $m\ddot{x}(s) = F(x(s))$ for $F := -\mathrm{D}V$.

**As previously seen.** Regular curves can be parametrized by arc length with unit speed $\|\mathrm{d}\gamma/\mathrm{d}t\| = \|\dot{\gamma}\| \equiv 1$.

**Lemma 2.2.2.** Each geodesic is parametrized proportionally to the arc length.[a]

[a] This means that we have constant speed, i.e., $\|\dot{\gamma}\|$ is a constant.

**Proof.** For a solution $x(t)$ of $\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$ (i.e., the geodesic), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\dot{x}, \dot{x}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)\right) = 0.$$

■

Our goal now is to minimize the length within class of regular smooth curves. Notice that the length and the energy functionals are invariants under parameter changes, which means that it's enough to look at curves parametrized by arc length.

**Theorem 2.2.2.** Let $\mathcal{M}$ be a Riemannian manifold, $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$. Then there exists an $\epsilon > 0$ and a unique geodesic such that $c\colon [0, \epsilon] \to \mathcal{M}$ with $c(0) = p$ and $\dot{c}(0) = v$. In addition, $c$ smoothly depend on $p, v$.

**Proof.** Since Equation 2.1 is a system of second order ODE, by Picard-Lindelöf theorem, we have local existence and uniqueness of the solution with prescribed initial values and derivative such that the solution depends smoothly on $p, v$. ■

If $x(t)$ is the solution of Equation 2.1, then $x(\lambda t)$ is also a solution for any constant $\lambda \in \mathbb{R}$. Denote geodesic from Theorem 2.2.2 by $c_v$, then

$$c_v(t) = c_{\lambda v}(t/\lambda)$$

for $\lambda > 0$, $t \in [0, \epsilon]$, and hence $c_{\lambda v}$ defined on $[0, \epsilon/\lambda]$.

**Remark.** Since $c_v$ depends smoothly on $v$, the set $\{v \in T_p\mathcal{M} \mid \|v\| = 1\}$ is compact, hence there exists $\epsilon_0 > 0$ such that for $\|v\| = 1$, $c_v$ defined at least on $[0, \epsilon_0]$, implying that for all $w \in T_p\mathcal{M}$ with $\|w\| \leq \epsilon_0$, $c_w$ is defined at least on $[0, 1]$.

## 2.3 Exponential Maps

The above discussion permits us to introduce the concept of the exponential map in the following manner.

**Definition 2.3.1** (Exponential map). Let $(\mathcal{M}, g)$ be a Riemannian manifold, $p \in \mathcal{M}$, and $V_p := \{v \in T_p\mathcal{M} \mid c_v \text{ defined on } [0, 1]\}$. The *exponential map of* $\mathcal{M}$ *at* $p$, $\exp_p\colon V_p \to \mathcal{M}$, is defined as $v \mapsto c_v(1)$.

Clearly, exp is differentiable, and we shall utilize the restriction of exp to an open subset of the tangent space $T_q\mathcal{M}$, i.e., we define

$$\exp_p\colon B(0, \epsilon) \subseteq T_p\mathcal{M} \to \mathcal{M},$$

where $B(0, \epsilon)$ is an open ball with center at the origin 0 of $T_p\mathcal{M}$ of radius $\epsilon$. It's easy to see that $\exp_p$ is differentiable and that $\exp_p(0) = q$.

**Intuition.** Geometrically, $\exp_p(v)$ is a point of $\mathcal{M}$ obtained by going out the length equal to $|v|$, starting from $p$, along a geodesic which passes through $p$ with velocity equal to $v/|v|$.

**Proposition 2.3.1.** The exponential map $\exp_p$ maps a neighborhood of $0 \in T_p\mathcal{M}$ diffeomorphically onto a neighborhood of $p \in \mathcal{M}$.

**Proof.** We see that

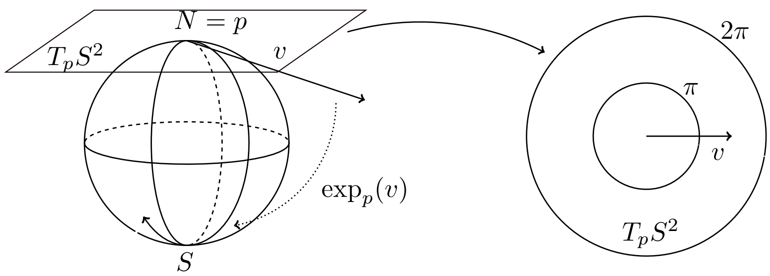
$$\mathrm{d}(\exp_p)_0(v) = \frac{\mathrm{d}}{\mathrm{d}t} \exp_p(tv)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} c_{tv}(1)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} c_v(t)\bigg|_{t=0} = v,$$

i.e., $\mathrm{d}(\exp_p)_0$ is the identity of $T_q\mathcal{M}$. By the inverse function theorem, $\exp_p$ is a local diffeomorphism on a neighborhood of 0. ■

**Example.** Let $\mathcal{M} = \mathbb{R}^n$, then the exponential map is the identity.[a]

[a] With the usual identification of $T_p\mathbb{R}^n$ at $p$ with $\mathbb{R}^n$.

For $\mathcal{M} = S^2$, we see that




Now we know that $\exp_p\colon B(0, \epsilon) \subseteq T_p\mathcal{M} \to \mathcal{M}$ maps diffeomorphically onto its image, we then define the following.

**Definition 2.3.2** (Normal coordinate). Given an exponential map $\exp_p\colon B(0,\epsilon) \to \mathcal{M}$, let $(e_1, \dots, e_n)$ be the orthonormal basis of $T_p\mathcal{M}$. Then the associated local coordinates are the *normal coordinates*.

Given $p \in \mathcal{M}^n$, $0 \in \mathbb{R}^n$, we have

$$g_{ij}(p) = \delta_{ij}, \quad \Gamma^k_{ij}(p) = 0, \quad g_{ij,k} = 0$$

for all $i, j, k$.[1]

**Note.** The first derivative vanishes, so locally, the manifold looks Euclidean.

**Theorem 2.3.1.** For all $p \in \mathcal{M}$, there exists $\rho > 0$ such that the Riemannian polar coordinates may be introduced on $B(p,\rho) = \{q \in \mathcal{M} \mid d(p,q) \leq \rho\}$. For any such $\rho$ and $q \in \partial B(p,\rho)$, there exists a unique geodesic of shortest length ($= \rho$) from $p$ to $q$. And in the polar coordinates, this geodesic is given by the straight line $x(t) = (t, \varphi_0)$, $0 \leq t \leq \rho$, with $q$ represented by coordinates $(\rho, \varphi_0)$, $\varphi_0 \in S^{d-1}$.

**Proof.** Take an arbitrary curve from $p$ to $q$, namely $c(t) = \big(r(t), \varphi(t)\big)$, $0 \leq t \leq T$, which does not have to be entirely be contained in $B(p,\rho)$. Let $t_0$ be defined as

$$t_0 \coloneqq \inf\left\{t \leq T \mid d(x(t), p) \geq \rho\right\}.$$

Then $t_0 \leq T$ such that $c|_{[0,t_0]}$ lies entirely in $B(p,\rho)$. We want to show that

(a) $L\left(c|_{[0,t_0]}\right) \geq \rho$, and

(b) $L\left(c|_{[0,t_0]}\right) = \rho$ only for a straight line in the polar coordinates,

where

$$L\left(c|_{[0,t_0]}\right) \coloneqq \int_0^{t_0} \sqrt{g_{ij}(c(t))\dot{c}^i\dot{c}^j}\,\mathrm{d}t.$$

Observe that $g_{r\varphi} = 0$, with $g_{\varphi\varphi}$ being positive definite, hence

$$L\left(c|_{[0,t_0]}\right) \geq \int_0^{t_0} \sqrt{g_{rr}(c(t))\dot{r}\dot{r}}\,\mathrm{d}t = \int_0^{t_0} \lvert\dot{r}\rvert\,\mathrm{d}t \geq \int_0^{t_0} \dot{r}\,\mathrm{d}t = r(t_0) = \rho,$$

where we know that $g_{rr} \equiv 1$. ∎

**Remark** (Compact manifold). For compact manifold, from Theorem 2.3.1, we can prove that Riemannian polar coordinates can be introduced. Also, there exists $\rho_0 > 0$ such that for any 2 points $p, q \in \mathcal{M}$ with $d(p,q) \leq \rho_0$ can be connected by minimizing geodesic.

# Lecture 7: Hopf-Rinow Theorem

## 2.4 Hopf-Rinow Theorem

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We have shown the following in the homework.

**Theorem 2.4.1.** Let $(\mathcal{M}, g)$ be a compact Riemannian manifold.

(a) Any 2 points $p, q \in \mathcal{M}$ can be connected by a minimizing geodesic.

(b) For all $p \in \mathcal{M}$, the exponential map $\exp_p$ is defined on all of $T_p\mathcal{M}$ and any geodesic may be extended indefinitely in each direction.

[1] Note that this only holds at $p$. We will come back to this when we formally introduce the linear connection.

We now want to generalize it. However, this is not true in the most general setting, and we need one more requirement.

**Definition 2.4.1** (Geodesically complete). A Riemannian manifold $(\mathcal{M}, g)$ is *geodesically complete* if for all $p \in \mathcal{M}$, $\exp_p$ is defined on all of $T_p\mathcal{M}$.

In other words, a Riemannian manifold $\mathcal{M}$ is geodesically complete if any geodesic $c(t)$ with $c(0) = p$ can be extended for all $t \in \mathbb{R}$. Then, we have the following.

**Theorem 2.4.2** (Hopf-Rinow theorem). Let $(\mathcal{M}, g)$ be a compact Riemannian manifold, then the following statements are equivalent.

(a) $\mathcal{M}$ is complete as a metric space.[a]

(b) The closed and bounded subsets of $\mathcal{M}$ are compact.

(c) There exists $p \in \mathcal{M}$ such that $\exp_p$ is defined on all $T_p\mathcal{M}$.

(d) $\mathcal{M}$ is geodesically complete.

Furthermore, (d) (and hence (a), (b), and (c)) implies

(e) for two points $p, q \in \mathcal{M}$ can be joined by a minimizing geodesic, i.e., geodesic of the shortest distance $d(p, q)$.

[a] Hence, equivalently, complete as a topological space w.r.t. the underlying topology.

**Proof.** We start by proving (d) implies (e). Let $\mathcal{M}$ be geodesically complete, and let $r := d(p, q)$, and let $\rho$ be as in the corollary from handout for HW1. Let $p_0 \in \partial B(p, \rho)$ be a point where the continuous functional $d(q, \cdot)$ attains its minimum on the compact set $\partial B(p, \rho)$. Then, for some $V \in T_p\mathcal{M}$,

$$p_0 = \exp_p \rho V.$$

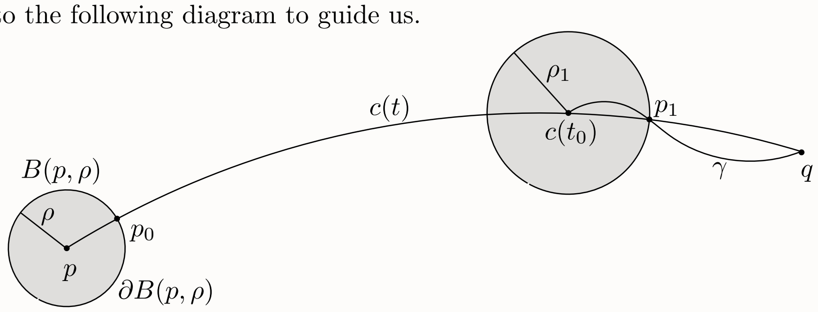
Consider the geodesic $c(t) = \exp_p tV$, by showing

$$c(r) = q,$$

$c|_{[0,r]}$ will be the shortest geodesic from $p$ to $q$. We start by defining

$$I := \{t \in [0, r] \mid d(c(t), q) = r - t\},$$

and referring to the following diagram to guide us.




Now, we want to show that $I = [0, r]$, which will follow from showing that $I$ is open.

**Note.** $I$ is not empty since by definition it contains 0 and $r$. Further, $I$ is closed by continuity.

Let $t_0 \in I$, and let $\rho_1 > 0$ be the radius as in the corollary, without loss of generality, $\rho_1 < r - t_0$. Let $p_1 \in \partial B(c(t_0), \rho_1)$ be the point where the continuous functional $d(q, \cdot)$ attains its minimum on the compact set $\partial B(c(t_0), \rho_1)$. By the triangle inequality,

$$d(p, q) \leq d(p, p_1) + d(p_1, q).$$

For every curve $\gamma$ from $c(t_0)$ to $q$, there exists $\gamma(t) \in \partial B(c(t_0), \rho_1)$, hence

$$L(\gamma) \geq \underbrace{d(c(t_0), \gamma(t))}_{\rho_1} + d(\gamma(t), q) = \rho_1 + d(p_1, q),$$

implying $d(q, c(t_0)) \geq \rho_1 + d(p_1, q)$. But from the triangle inequality, we actually have

$$d(q, c(t_0)) = \rho_1 + d(p_1, q) \Leftrightarrow d(p_1, q) = \underbrace{d(q, c(t_0))}_{r-t_0} - \rho_1,$$

hence $d(p_1, p) \geq r - (r - t_0 - \rho_1) = t_0 + \rho_1$, i.e., this is a minimizing curve!

On the other hand, there exists a curve from $p$ to $p_1$ of length $t_1 + \rho_1$ since it's composed by the portion from $p$ to $c(t_0)$ along $c(t)$ and the portion being the geodesic from $c(t_0)$ to $p_1$ of length $\rho_1$. Then, by the theorem we have proved in the HW1#5, this curve is a geodesic curve. Finally, from the uniqueness of geodesic with the given extra data, this geodesic coincides with $c$. Hence,

$$p_1 = c(t_0 + \rho_1),$$

with $d(p_1, q) = r - t_0 - \rho_1$,

$$d(c(t_0 + \rho_1), q) = d(p_1, q) = r - t_0 - \rho = r - (t_0 + \rho_1),$$

thus $t_0 + \rho_1 \in I$, hence $I$ is open, i.e., $I = [0, r]$, so $c(r) = q$ follows.

# Lecture 8: Injectivity Radius and Vector Bundles

In the proof we did last time, the last step can be shown via [FC13, Corollary 3.9]. 31 Jan. 14:30

**Proof of Hopf-Rinow theorem (Continued).** We see that (d) implies (e), hence we only need to show that (a), (b), (c), and (d) are equivalent.

- (d) $\Rightarrow$ (c) is trivial.
- (c) $\Rightarrow$ (b): Let $K \subseteq \mathcal{M}$ be closed and bounded. As $K$ bounded, $K \subseteq B(p, r)$ for some $r > 0$. Then any point in $B(p, r)$ can be joined with $p$ by geodesic of length $\leq r$, and $B(p, r)$ is the image of the compact ball in $T_p\mathcal{M}$ of radius $r$ under continuous map $\exp_p$, hence $B(p, r)$ is compact. As $K$ closed and $K \subseteq B(p, r)$, $K$ is compact.
- (b) $\Rightarrow$ (a): Let $(p_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be a Cauchy sequence, so it's bounded, and by (b), its closure is compact. It contains a convergent subsequence, so it converges, i.e., $\mathcal{M}$ is complete.
- (a) $\Rightarrow$ (d): Let $c$ be a geodesic in $\mathcal{M}$, parametrized by arc length defined on a maximal interval $I$. Since $I$ s non-empty, and we can show that $I$ is both open and closed. Exercise

■

It's worth mentioning that we do have uniqueness after choosing $p_0$, in other words, after choosing $p_0$, everything is fixed, so the non-uniqueness really comes from the initial choose of $p_0$.

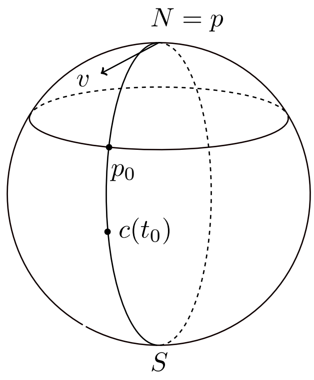



Figure 2.1: Consider $S^2$, after fixing $p_0$, $c(t_0)$ is extended uniquely.

## 2.5 Injectivity Radius

Consider the following.

**Definition 2.5.1** (Injectivity radius)**.** Let $\mathcal{M}$ be a Riemannian manifold, and $p \in \mathcal{M}$. The *injectivity radius* $i(p)$ of $p$ is

$$i(p) := \sup \left\{ \rho > 0 \mid \exp_p \text{ defined on } B(0, \rho) \subseteq T_p\mathcal{M} \text{ and injective} \right\}.$$

Similarly, the *injectivity radius* $i(\mathcal{M})$ of $\mathcal{M}$ is defined as $i(\mathcal{M}) := \inf_{p \in \mathcal{M}} i(p)$.

**Example** (Sphere)**.** $i(S^n) = \pi$.

**Example** (Torus)**.** $i(T^n) = 1/2$.

Any manifold carries a complete Riemannian metric.
If $(\mathcal{M}, g_1)$ is not complete, we can find $g_2$ such that $(\mathcal{M}, g_2)$ is complete.

**Example** (Hyperbolic half-plane)**.** The half-plane $P = \left\{ (x, y) \in \mathbb{R}^2 \mid y > 0 \right\}$ equipped with metric induced by the Euclidean metric on $\mathbb{R}^2$, which is not complete.

However, it becomes complete when equipped with the following metric

$$\frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2).$$

In fact, $P$ with the above metric is called the *hyperbolic half-plane* $H^2$, and we can extend it to $H^n$.

Another question we may ask is the following.

**Problem.** Is the converse of Hopf-Rinow theorem true? I.e., can we show that (e) implies (d)?

**Answer.** No! Any 2 points in the open half-sphere can be joint by a unique minimal geodesic, but this manifold is not geodesically complete. ⊛

**Example.** The injectivity radius of $H^n$ is $\infty$.

**Remark.** Given a compact $\mathcal{M}$, the injectivity radius is always $> 0$ by continuity argument.

Now, given a complete but not compact $\mathcal{M}$, the injectivity radius can be 0.

**Example.** Take the quotient of the Poincaré half-plane by the translations

$$(x, y) \mapsto (x + n, y), \quad n \in \mathbb{Z}.$$

We then obtain a complete Riemannian manifold $\mathcal{M}$ with $i(\mathcal{M}) = 0$.

**Note.** Finding lower bounds for $i(\mathcal{M})$ introduces curvature estimates.

## 2.6 Bundles and Fields

Let's first introduce the theory of bundles, which allows us to introduce the notion of vector fields, which is a more general notion of tensor fields. And noticeably, nearly every structure we can put on a Riemannian manifold will be in the form of tensor fields.

**Example.** Given a tangent vector field $X$ of a smooth manifold $\mathcal{M}$ is where we simply associate $X(p)$ to a tangent vector:

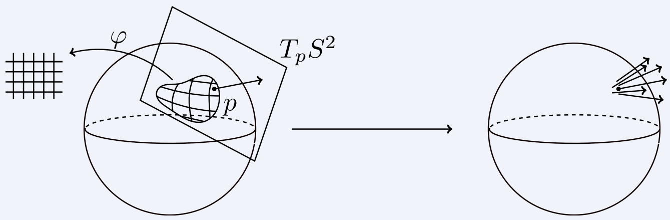



Figure 2.2: Given $\mathcal{M} = S^2$, a vector field assigns every point a "point" in the associated "space." In this case, a tangent vector field associates every $p$ a vector in the corresponding tangent space.

Recall the tangent bundle $(T\mathcal{M}, \pi, \mathcal{M})$, where we only take the name "bundle" for granted and don't know why it is: however, we should see that it helps us construct the vector field, since it captures the idea of "for every point $p$, we have an associated space $T_p\mathcal{M}$," which is exactly what we need here. This idea generalizes quite easily.

### 2.6.1 Bundles

We start by introducing the notion of bundles.

**Definition 2.6.1** (Bundle). A *bundle* is a tuple $(E, \pi, \mathcal{M})$ consists of the total space $E$, the base space $\mathcal{M}$, and the bundle projection $\pi\colon E \to \mathcal{M}$.

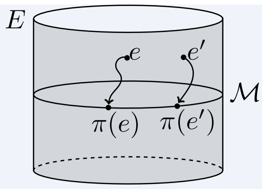
**Definition 2.6.2** (Total space). The differentiable manifold $E$ is called the *total space*.

**Definition 2.6.3** (Base space). The differentiable manifold $\mathcal{M}$ is called the *base space*.

**Definition 2.6.4** (Bundle projection). The (differentiable) continuous surjection $\pi\colon E \to \mathcal{M}$ is called the *bundle projection*.

**Note.** We see that a tangent bundle $(T\mathcal{M}, \pi, \mathcal{M})$ is actually a bundle.

**Example.** Let $E$ be a cylinder, $\mathcal{M}$ be a circle.




As we can see, the number of possible $\pi$ is enormous, as long as it's surjective and smooth.

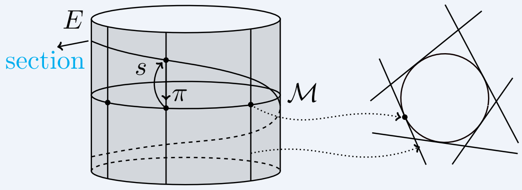
**Notation.** Sometimes, we will just denote a bundle as $E \xrightarrow{\pi} \mathcal{M}$, or even more compactly, just $\pi$ since it captures all the data.

**Definition 2.6.5** (Fiber)**.** Given a bundle $(E, \pi, \mathcal{M})$, the *fiber* over $p \in \mathcal{M}$ under $\pi$ is the preimage of a $\{p\}$, i.e., $\pi^{-1}(\{p\})$.

**Definition 2.6.6** (Section)**.** A *section* of a bundle $(E, \pi, \mathcal{M})$ is a differentiable map $s \colon \mathcal{M} \to E$ such that $\pi \circ s = \mathrm{id}_{\mathcal{M}}$.

**Remark.** We see that a section $s$ encodes lots of information of a bundle, since $s$ includes $E, \mathcal{M}$, and the condition deal with $\pi$.

**Example.** Again let $E$ be a cylinder, $\mathcal{M}$ be a circle. This time, we choose $\pi$ to be the trivial one.




We see that in this way, this bundle really captures all the tangent spaces structure of a circle!

### 2.6.2 Vector Bundles

Then, we're interested in the so-called vector bundle.

**Definition 2.6.7** (Vector bundle)**.** A (differentiable) *vector bundle* of rank $n$ is a bundle $(E, \pi, \mathcal{M})$ such that each fiber $E_x := \pi^{-1}(x)$ of $x \in \mathcal{M}$ carries a structure of an $n$-dimensional (real) vector space, and local triviality condition holds.

> **Definition 2.6.8** (Local trivialization)**.** For all $x \in \mathcal{M}$, the *local trivialization* $(U, \varphi)$ consists a neighborhood $U$ and diffeomorphism $\varphi \colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that for all $y \in U$,
>
> $$\varphi_y := \varphi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^n$$
>
> is a vector space isomorphism.

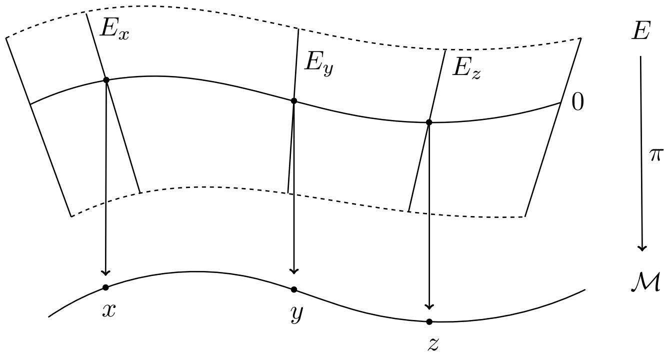



Figure 2.3: An illustration of vector bundle $(E, \pi, \mathcal{M})$.

**Definition 2.6.9** (Trivial). A vector bundle is *trivial* if it's isomorphic to $\mathcal{M} \times \mathbb{R}^n$.[a]

[a] $n$ is the rank of the vector bundle.

**Intuition.** Local trivialization shows that *locally* $\pi$ looks like the projection of $U \times \mathbb{R}^n$ on $U$.

**Definition 2.6.10** (Bundle chart). The pair $(\varphi, U)$ is the *bundle chart* in local trivialization.

**Remark.** From Definition 2.6.7, vector bundle is locally, but not necessarily globally a product of base space and the fiber.

**Intuition.** We may look at a vector bundle as a family of vector spaces, all isomorphic to a fixed $\mathbb{R}^n$, "parametrized" (locally trivially) by a manifold.

### 2.6.3 Vector Fields

We can now introduce the notion of vector fields in terms of section.

**Definition 2.6.11** (Vector field). A (smooth) *vector field* $X$ is a smooth section of a bundle.

**Note.** We see that a smooth tangent vector field is indeed a smooth vector field with the bundle being the tangent bundle.

**Notation.** Since we will nearly always be talking about tangent vector fields, we will abuse the notation a bit and just simply call it vector fields. But always keep in mind that more broadly, a vector field should be a section of a bundle, not always $T\mathcal{M}$.

## Lecture 9: Tensors and Connections

### 2.6.4 Tensor Fields 2 Feb. 14:30

We can introduce the notion of "tensor fields" in a brute-fore way.[2] To do this, given a vector space $V$, we first introduce tensors.

[2] See Appendix A.3.2 for another view point.

**Definition 2.6.12** (Tensor). Let $V$ be a vector space of dimension $m < \infty$, and the dual space $V^*$.[a] Then the vector space of the *$r$-times contravariant and $s$-times covariant tensors over $V$*, denoted as $T^r_s(V)$, is the vector field defined as

$$T^r_s(V) = \{T \colon \underbrace{V^* \times \ldots \times V^*}_{r} \times \underbrace{V \times \ldots \times V}_{s} \to \mathbb{R}\} = \underbrace{V \otimes \ldots \otimes V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s}.$$

[a] I.e., $V^* := \{\lambda \colon V \to \mathbb{R} \mid \lambda \text{ linear}\}$.

**Notation.** Let $\mathcal{M}^n$ be a smooth manifold and $\pi \colon E \to \mathcal{M}$ a smooth vector bundle, then the set of sections is denoted as

$$\Gamma(E) := \{s \in C^\infty(\mathcal{M}, E) \mid \pi \circ s = \mathrm{id}_{\mathcal{M}}\}.$$

**Example.** Consider the vector bundle $(T\mathcal{M}, \pi, \mathcal{M})$, then $\Gamma(T\mathcal{M}) := \{\text{vector fields on } \mathcal{M}\}$.

**Example.** $\Gamma(\Lambda_s\mathcal{M}) := \{s\text{-forms on } \mathcal{M}\}$ with $\Lambda_s\mathcal{M} = \Lambda^s\left(\bigcup_{p\in\mathcal{M}} T_p^*\mathcal{M}\right)$.[a]

[a] Here, $\Lambda^s(V^*) := \left\{A \in T^0_s(V) \mid A \text{ skew-symmetric}\right\}$, where $s \in \mathbb{N}$.

Then, we have the following.

**Definition 2.6.13** (Tensor field). The *$(r, s)$-tensor fields* on $\mathcal{M}$ is defined as elements in $\Gamma(T^r_s\mathcal{M})$ with $T^r_s\mathcal{M} = \bigcup_{p\in\mathcal{M}} T^r_s(T_p\mathcal{M})$.

**Example.** A Riemannian metric $g$ on $\mathcal{M}$ is a $(0, 2)$-tensor field, i.e., $g \in \Gamma(T^0_2(\mathcal{M}))$ for all $p \in \mathcal{M}$.

**Proof.** Since $g_p \colon T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$, so by regarding $p$ as the argument of the map $g$, $g \colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to C\infty(\mathcal{M})$. ⊛

**Note.** It's in fact unnecessary to have such a general Definition 2.6.13 on a Riemannian manifold.

**Proof.** Since given a Riemannian metric $g$, it associates to each $X \in \Gamma(T\mathcal{M})$ a unique $\omega \in \Gamma(T^*\mathcal{M})$ given by

$$\omega(Y) = g(X, Y)$$

for all $X, Y \in \Gamma(T\mathcal{M})$. ⊛

## 2.7 Other Metrics

Finally, we discuss some other metrics we may let a manifold equipped with.

**Definition 2.7.1** (Pseudo-Riemannian metric). A *pseudo-Riemannian metric* on a differentiable manifold $\mathcal{M}$ is a $(0, 2)$-tensor field $g \in \Gamma(T^0_2(\mathcal{M}))$ for all $p \in \mathcal{M}$ with

(a) $g(X, Y) = g(Y, X)$ for all $X, Y \in T\mathcal{M}$;

(b) for all $p \in \mathcal{M}$, $g_p$ is non-degenerate bilinear form on $T_p\mathcal{M}$, i.e., $g_p(X, Y) = 0$ for all $X, Y \in T_p\mathcal{M}$ if and only if $Y = 0$.

**Note.** A pseudo Riemannian metric is actually a Riemannian metric if it's positive definite at every $p \in \mathcal{M}$.

**Definition 2.7.2** (Lorentzian metric). A *Lorentzian metric* $g$ is a continuous assignment of a non-degenerate[a] quadratic form $g_p$ of index 1[b] in $T_p\mathcal{M}$ for all $p \in \mathcal{M}$.

[a] $g_p(X, Y) = 0$ for all $Y \in T_p\mathcal{M}$ implies $X = 0$.
[b] It means that the maximal dimension of a subspace of $T_p\mathcal{M}$ on which $g_p$ is negative definite is 1.

An equivalent definition is the following.

**Definition 2.7.3** (Lorentzian). A quadratic form $g_p$ in $T_p\mathcal{M}$ is *Lorentzian* if there exists a vector $V \in T_p\mathcal{M}$ such that $g_p(V, V) < 0$ while setting $\Sigma_V = \{X \mid g_p(X, V) = 0\}$ such that $g_p|_{\Sigma_V}$[a] is positive definite.

[a] The $g_p$-orthogonal complement of $V$.

**Example** (Minkowski space). The Minkowski space on $\mathbb{R}^4$ is the prototypical example from physics (flat spacetime). Namely, the metric is given by the quadratic form

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with the coordinates being $(t, x, y, z)$.

# Chapter 3

# Connections and Curvatures

So far, we saw that a vector field $X$ can be used to provide a directional derivative since it gives us a tangent vector at each point smoothly. Now, we will introduce a new symbol $\nabla$ where we let

$$\nabla_X f \coloneqq Xf$$

for $f \in C^\infty(\mathcal{M})$.

**Problem.** Does this notation overkill? We already know that $Xf = (\mathrm{d}f)(X)$!

**Answer.** No! While $\nabla, X \colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, while $\mathrm{d}f \colon \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$, we can generalize $\nabla_X$ to act from vector fields to vector fields! The insight is that if $X$ can be extended naturally (without providing any extra structures), then we certainly won't bother introducing a new symbol. However, as you might guess, to let $\nabla$ doing this, we do need to provide extra structures, and $\nabla$ stands exactly for these extra structures! ⊛

In some sense, this new notions $\nabla$ allows us to "connect" tangent spaces, which allows us to make sense of "curvatures" and other geometric property of a Riemannian manifold.

## 3.1 Levi-Civita Connections

We start by talking about linear connections, and then realize that after specifying a Riemannian metric $g$, with an additional (technical) assumption, a unique linear connection, defined as Levi-Civita connections, exists for any Riemannian manifold. In other words, specifying $g$ is the same as specifying the "shape of the space." We'll make sense of all these on the way.

### 3.1.1 Affine Connections

We first formulate a *wish list* of properties which the $\nabla_X$ should have. Any remaining freedom in choosing $\nabla$ will need to be provided as additional structures beyond the structures on $\mathcal{M}$ we already have.

**Definition 3.1.1 (Linear connection).** A *linear connection* (*affine connection*) on a smooth manifold $\mathcal{M}$ is a bilinear map

$$\nabla \colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M}),$$

which is denoted by $\nabla(X, Y) = \nabla_X Y$ and which satisfies

(a) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$;

(b) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;

(c) $\nabla_X fY = f\nabla_X Y + X(f)Y$;

for all vector fields $X, Y, Z \in \Gamma(T\mathcal{M})$ and $f, g \in C^\infty(\mathcal{M})$.

**Remark.** Definition 3.1.1 (c) shows that this is actually a local notion as we will see.

**Note.** There's a similar notation called covariant derivative, denoted by D, satisfies similar properties as a linear connection. Hence, we often write D and $\nabla$ interchangeably.[a]

[a] $\nabla$ is more general than D; however, we treat them as the same as suggested by Proposition 3.4.1.

Now, one might be wondering that, after fixing these rules we want, how much freedom is left? To see this, let's first do some calculations...

### 3.1.2 Connection Coefficients

Choose a system of coordinates $(x_1, \ldots, x_n)$ at $p \in \mathcal{M}$, we can write $X = X^i \frac{\partial}{\partial x_i}, Y = Y^j \frac{\partial}{\partial x_j}$, then

$$\nabla_X Y = \nabla_{X^i \frac{\partial}{\partial x_i}} \left( Y^j \frac{\partial}{\partial x_j} \right) = X^i Y^j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + X^i \frac{\partial}{\partial x_i}(Y^j) \frac{\partial}{\partial x_j}.$$

Now, we see that $\nabla_{\partial / \partial x_i} \frac{\partial}{\partial x_j}$ is another vector field, hence can again write

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} =: \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

in terms of the basis with a new set of coefficients $\Gamma$.

**Notation** (Connection coefficient)**.** The coefficients $\Gamma_{ij}^k$ is called the *connection coefficients*.[a]

[a] It's tempting to say that the connection coefficients are the same as Christoffel symbols since we're using the same symbols. Indeed, they are! For a deeper understanding, see Appendix A.1.

**Note.** It's clear that $\Gamma_{ij}^k$ are differentiable and charts-dependent and hence $\nabla$ is local.

Finally, we have

$$\nabla_X Y = \left( X^i Y^j \Gamma_{ij}^k + X(Y^k) \right) \frac{\partial}{\partial x_k} \Rightarrow (\nabla_X Y)^k = X(Y^k) + \Gamma_{ij}^k X^i Y^j,$$

meaning that we have $(\dim \mathcal{M})^3$ many $\Gamma$'s (freedom) when choosing $\Gamma_{ij}^k$ with Definition 3.1.1.[1]

**Remark.** One might ask what about other tensor fields? Fortunately, the same set of $\Gamma$'s fix the action of $\nabla$ on any tensor fields.

**Proof.** The key observation is that if we define $\nabla_{\frac{\partial}{\partial x^j}}(\mathrm{d}x^i) =: \Sigma_{jk}^i \mathrm{d}x^k$, then

$$\nabla_{\frac{\partial}{\partial x^j}} \left( \mathrm{d}x^i \left( \frac{\partial}{\partial x^k} \right) \right) = \begin{cases} \frac{\partial}{\partial x^j}(\delta_k^i) = 0; \\ \left( \nabla_{\frac{\partial}{\partial x^j}} \mathrm{d}x^i \right) \frac{\partial}{\partial x^k} + \mathrm{d}x^i \underbrace{\left( \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \right)}_{\Gamma_{jk}^\ell \frac{\partial}{\partial x^\ell}}, \end{cases}$$

leading to

$$\left( \nabla_{\frac{\partial}{\partial x^j}} \mathrm{d}x^i \right) \frac{\partial}{\partial x^k} = -\mathrm{d}x^i \left( \Gamma_{jk}^\ell \frac{\partial}{\partial x^\ell} \right) \Rightarrow \left( \nabla_{\frac{\partial}{\partial x^j}} \mathrm{d}x^i \right)_k = -\Gamma_{jk}^i$$

since $\mathrm{d}x^i \frac{\partial}{\partial x^\ell} = \delta_\ell^i$. ⊛

In summary, we have

$$\begin{cases} (\nabla_X Y)^k = X(Y^k) + \Gamma_{ij}^k X^i Y^j, & \text{if } Y \text{ is a vector field;} \\ (\nabla_X \omega)_k = X(\omega_k) - \Gamma_{ik}^j X^i \omega_j, & \text{if } \omega \text{ is a co-vector field.} \end{cases}$$

[1] This is for a particular domain $U$.

### 3.1.3 Levi-Civita Connections

The basic insight is that, after choosing a particular connection (remember that we have freedom to choose $\Gamma$'s), the space is basically fixed: i.e., the shape (curvature) of the space is determined by the choice of $\nabla$! We now formalize this idea. A particularly natural notion related to "curvature" is the torsion, defined as follows.

**Definition 3.1.2** (Torsion)**.** The *torsion* $T$ of a linear connection $\nabla$ is the $(1,2)$-tensor field

$$T(\omega, X, Y) \coloneqq \omega\left(\nabla_X Y - \nabla_Y X - [X, Y]\right).$$

**Notation.** We usually write this as $T(X, Y)$ by neglecting $\omega$.

**Remark.** $T$ is actually $C^\infty$-linear in each entry,[a] hence a tensor field.

[a]See Appendix A.3.2.

**Proof.** Since $T(f\cdot\omega, X, Y) = f\cdot\omega(\ldots) = fT(\omega, X, Y)$ and $T(\omega+\psi, X, Y) = \ldots = T(\omega, X, Y) + T(\psi, X, Y)$, and also

$$\begin{aligned} T(\omega, fX, Y) &= \omega\left(\nabla_{fX} Y - \nabla_Y (fX) - [fX, Y]\right) \\ &= \omega(f\nabla_X Y - (Yf)X - f\nabla_Y X - f[X,Y] + (Yf)X) = f\cdot T(\omega, X, Y) \end{aligned}$$

since

$$([fX, Y])g = f\cdot X(Yg) - Y(f\cdot Xg) = f\cdot X(Yg) - (Yf)(Xg) - f\cdot Y(Xg) = (f\cdot[X,Y] - (Yf)X)\, g.$$

Finally, we claim that the additivity at $X$ holds, with $T(\omega, X, Y) = -T(\omega, Y, X)$, we're done. ⊛

**Intuition.** Definition 3.1.2 makes sense (in such a form) since this will make $T$ actually a tensor field. For example, without the Lie bracket term, we don't have the linearity at $X$ (hence $Y$).

**Definition 3.1.3** (Torsion-free)**.** A linear connection $\nabla$ is *torsion-free* if $T = 0$.

**Notation** (symmetric)**.** A torsion-free $\nabla$ is sometimes said to be *symmetric*.

In a chart,

$$T^i_{jk} \coloneqq T\left(\mathrm{d}x^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \Gamma^i_{jk} - \Gamma^i_{kj} = 2\Gamma^i_{[jk]},$$

hence if $T = 0$, we can interchange the lower two indexes of $\Gamma^k_{ij}$, i.e., $\Gamma^k_{ij} = \Gamma^k_{ji}$.

**Definition 3.1.4** (Riemannian)**.** Let $\nabla$ be a linear connection and $g$ be a Riemannian metric on $\mathcal{M}$. Then $\nabla$ is *Riemannian* (or *metric*) if for all $X, Y, Z \in \Gamma(T\mathcal{M})$,[a]

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

[a]We view $g(X, Y) \in C^\infty(\mathcal{M})$ as suggested by Appendix A.3.2.

**Notation** (Compatible)**.** A Riemannian $\nabla$ is sometimes said to be *compatible*.

**Remark.** Equivalently, Definition 3.1.4 can be formulated as $\nabla g = 0$.

We are now able to state the fundamental theorem of this section.

**Theorem 3.1.1** (Levi-Civita). On each Riemannian manifold $(\mathcal{M}, g)$, there exists a unique Riemannian, torsion-free connection $\nabla$ on $T\mathcal{M}$ determined by the *Koszul formula*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X\langle Y, Z\rangle - Z\langle X, Y\rangle + Y\langle Z, X\rangle - \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle\right). \quad (3.1)$$

**Proof sketch.** Firstly, we can show that every Riemannian and torsion-free connection satisfies Equation 3.1, which implies uniqueness. For existence, we verify that the unique map $\nabla\colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ given by Equation 3.1 is Riemannian and torsion-free. ■

Finally, we define the following.

**Definition 3.1.5** (Levi-Civita connection). The *Levi-Civita connection* is the unique linear connection $\nabla$ defined by the Koszul formula.

**Remark.** This means, given a Riemannian metric $g$, with the condition of torsion-free, the shape of the space is also fixed since there's a unique linear connection $\nabla$ such that $T = \nabla g = 0$.

# Lecture 10: Curvatures and Flow of Vector Fields

## 3.2 Riemannian Curvatures

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Given all these definitions, we can now introduce the notion of "curvatures." Consider the following.

**Definition 3.2.1** (Riemannian curvature). The *Riemannian curvature* $R$ of a Levi-Civita connection $\nabla$ is the $(1, 3)$-tensor field[a]

$$R(\omega, Z, X, Y) \coloneqq \omega\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right).$$

[a] $R$ is indeed $C^\infty$-linear in each entry,[b] although we omit the proof here.

**Notation.** We usually write this as $R(X, Y)Z$ by emphasizing $Z$ and neglecting $\omega$.

**Note.** In do Carmo [FC13], the corresponding definition of $R$ differs from Definition 3.2.1 by a sign.

**Example** (Euclidean space). If $\mathcal{M} = \mathbb{R}^n$ (with the "flat" $\nabla$), $R(X, Y)Z = 0$, $\forall\ X, Y, Z \in \Gamma(T\mathbb{R}^n)$.

**Proof.** Since given $Z = (z_1, \ldots, z_n)$ with the components from natural coordinates of $\mathbb{R}^n$, $\nabla_X Z = (Xz_1, \ldots, Xz_n)$, then $\nabla_Y \nabla_X Z = (YXz_1, \ldots, YXz_n)$, hence $R(X, Y)Z = 0$. ⊛

**Intuition.** $R(X, Y)Z$ is trying to measure how much $\mathcal{M}$ deviates from being Euclidean.

Another way to look at this is that, consider a system of coordinates $\{x_i\}$ around $p \in \mathcal{M}$. Since $[\partial/\partial x_i, \partial/\partial x_j] = 0$,[2] we have

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = (\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}})\frac{\partial}{\partial x_k}.$$

**Intuition.** $R(X, Y)Z$ is trying to measure the non-commutativity of the covariant derivative.

Consider expressing things in a chart $(U, x)$ at $p \in \mathcal{M}$. Let $\partial/\partial x_i = X_i$, then

$$R(X_i, X_j)X_k \eqqcolon R_{ijk}^{\ell} X_\ell$$

[2] For second derivative, we can exchange the order due to smoothness.

as how we define connection coefficients, i.e., $R^{\ell}_{ijk}$ are components of $R$ in $(U, x)$.[3] If $X = u^i X_i, Y = v^j X_j, Z = w^k X_k$, from the linearity of $R$,

$$R(X,Y)Z = R^{\ell}_{ijk} u^i v^j w^k X_\ell.$$

Then the above computation formally can be written as follows.

**Remark (Algebraic significant of Riemannian curvature).** Since

$$(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) = R(\cdot, Z, X, Y) + \nabla_{[X,Y]} Z,$$

by letting $\nabla_i \coloneqq \nabla_{\frac{\partial}{\partial x^i}}, \nabla_j \coloneqq \nabla_{\frac{\partial}{\partial x^j}}$, in a chart $(U, x)$, we have

$$(\nabla_i \nabla_j Z)^k - (\nabla_j \nabla_i Z)^k = R^k_{\ell ij} Z^\ell + \underbrace{\nabla_{\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]}}_{=0} Z = R^k_{\ell ij},$$

i.e., the components of $R$ contains all the information of how $\nabla_i$ and $\nabla_j$ fail to commute.

Finally, it's worth-noting that

$$\langle R(X_i, X_j) X_k, X_\ell \rangle = R^s_{ijk} g_{\ell s} = R_{ijk\ell}.$$

### 3.2.1 Identities

There are many important identities related to $R$, and we should see some of them.

**Proposition 3.2.1 (First Bianchi identity).** Given the Riemannian curvature tensor $R$, for all vector fields $X, Y, Z$,

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0;$$

or equivalently, $R_{k\ell ij} + R_{kij\ell} + R_{kj\ell i} = 0$.

**Proof.** See do Carmo [FC13, Proposition 2.4] (and also homework 2). ■

**Proposition 3.2.2 (Second Bianchi identity).** Given the Riemannian curvature tensor $R$,

$$\frac{\partial}{\partial x^h} R_{k\ell ij} + \frac{\partial}{\partial x^k} R_{\ell hij} + \frac{\partial}{\partial x^\ell} R_{hkij} = 0;$$

or equivalently, $\nabla_{[\alpha} R_{\beta\gamma]\delta\epsilon} \coloneqq \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0$.[a]

[a] This notation is a bit cryptic: see Ricci calculus.

**Proof.** See homework 2. ■

**Proposition 3.2.3.** Given the Riemannian curvature tensor $R$,

(a) $R(X,Y)Z - -R(Y,X)Z$, i.e., $R_{k\ell ij} = -R_{k\ell ji}$;

(b) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$, i.e., $R_{k\ell ij} = -R_{\ell kij}$;

(c) $\langle R(X,Y)Z, W\rangle = -\langle R(Y,X)Z, W\rangle$, i.e., $R_{k\ell ij} = -R_{\ell kji}$;

(d) $\langle R(X,Y)Z, W\rangle = -\langle R(Z,W)X, Y\rangle$, i.e., $R_{k\ell ij} = R_{ij\ell k}$.

**Proof.** See do Carmo [FC13, Proposition 2.5] (and also homework 2). ■

[3] do Carmo [FC13, Page 93] shows that $R^{\ell}_{ijk} = \Gamma^p_{ik}\Gamma^\ell_{jp} - \Gamma^p_{jk}\Gamma^\ell_{ip} + \Gamma^\ell_{ik,j} - \Gamma^\ell_{jk,i}$ (note the sign difference).

### 3.2.2 Other Curvatures

There are other notions of curvature, but they all depend on the Riemannian curvature, and appearing to be some sorts of "average" of $R$.

**Definition 3.2.2** (Riemannian-Christoffel curvature)**.** The *Riemannian-Christoffel curvature* is defined by

$$R_{k\ell ij} \coloneqq g_{km} R^m_{\ell ij} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k} \right\rangle.$$

**Definition 3.2.3** (Ricci curvature)**.** The *Ricci curvature* is defined by $R_{ab} = g^{cm} R_{camb} = R^m_{amb}$.

**Definition 3.2.4** (Ricci scalar curvature)**.** The *(Ricci) scalar curvature* is defined by $R = g^{ab} R_{ab}$.

**Note.** For a more formal treatment, see do Carmo [FC13, §4.4].[a]

[a]Notice that the order in do Carmo [FC13] is a bit different: it introduces sectional curvature first.

## 3.3 Flows of Vector Fields

Let $\mathcal{M}$ be a smooth manifold, and $X$ a vector field on $\mathcal{M}$. Then $X$ defines $1^{st}$ order differential equations[4]

$$\dot{c} = X(c).$$

And this ODE has a solution, as guaranteed by Proposition 3.3.1.

**Proposition 3.3.1.** For all $p \in \mathcal{M}^d$, there exists an open interval $I = I_p \subseteq \mathbb{R}$ with $0 \in I_p$ such that a smooth curve $c\colon I_p \to \mathcal{M}$ solves

$$\begin{cases} \dfrac{\mathrm{d}c(t)}{\mathrm{d}t} = X(c(t)), \quad t \in I; \\ c(0) = p. \end{cases}$$

Further, the solution depends smoothly on the initial data (i.e., $p$).[a]

[a]This directly follows from ODE theory.

*Proof.* For all $p \in \mathcal{M}$, we want to find an open interval $I = I_p$ around $0 \in \mathbb{R}$ and a solution of the following ODE for $c\colon I \to \mathcal{M}$:

$$\begin{cases} \dfrac{\mathrm{d}c(t)}{\mathrm{d}t} = X(c(t)), \quad t \in I; \\ c(0) = p. \end{cases}$$

We can check in local coordinates that this is a system of ODE. In such coordinates, let $c(t)$ be given by $c(t) = \left(c^1(t), c^2(t), \dots, c^d(t)\right)$. Let $X =: X^i \partial/\partial x^i$, then the above system becomes

$$\frac{\mathrm{d}c^i(t)}{\mathrm{d}t} = X^i\big(c(t)\big), \quad i = 1, \dots, d.$$

From the Picard-Lindelöf theorem, with the initial data $c(0) = p$, there is a unique solution. ■

**Proposition 3.3.2.** For all $p \in \mathcal{M}$, there exists an open neighborhood $U$ of $p$ and an open interval $I_p$ with $0 \in I_p$ such that for all $q \in U$, the curve $c_q$ with

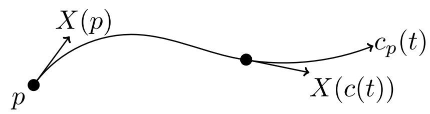
$$\dot{c}_q(t) = X(c_q(t)), \quad c_q(0) = q$$

[4]If $\dim \mathcal{M} > 1$, it is a system of $1^{st}$-order differential equations.

is defined on $I$ and the map $c\colon I \times U \to \mathcal{M}$, $(t, q) \mapsto c_q(t)$ is smooth.

Proposition 3.3.2 suggests the following definition.

**Definition 3.3.1** (Local flow). The map $c_q(t)\colon I \times U \to \mathcal{M}$, $(t, q) \mapsto c_q(t)$ from Proposition 3.3.2 is called the *local flow* of the vector field $X$.




**Definition 3.3.2** (Integral curve). The local flow $c_q(t)$ is called the *integral curve* of $X$ through $q$.

### 3.3.1 Local 1-Parameter Groups

Now, given a local flow $c_q(t)$ of a vector field $X$, by fixing $t$, we can vary $q$ and see the following.

**Theorem 3.3.1.** Let $\varphi_t(q) \coloneqq c_q(t)$ such that $\varphi_t \circ \varphi_s(q) = \varphi_{t+s}(q)$ for $s, t, (t+s) \in I_q$. If $\varphi_t$ is defined on $U \subseteq \mathcal{M}$, it maps $U$ diffeomorphically onto its image.

We see that $\varphi_t$ defines a family of diffeomorphism around $p$, which gives the following.

**Definition 3.3.3** (Local 1-parameter group). A family $(\varphi_t)_{t\in I}$ of diffeomorphism from $\mathcal{M}$ to $\mathcal{M}$ satisfying Theorem 3.3.1 is called a *local* 1-*parameter group* of diffeomorphisms.

In general, a local 1-parameter group needs not be extendible to a group because the maximum interval $I = I_q$ in Definition 3.3.3 need not be all of $\mathbb{R}$.

**Example.** Let $\mathcal{M} = \mathbb{R}$, $X(t) = \tau^2 \mathrm{d}/\mathrm{d}\tau$. Then the solution of $\dot{c}(t) = c^2(t)$ is not defined over all $\mathbb{R}$.

To get the whole group structure, consider the following.

**Theorem 3.3.2.** Let $X$ be a vector field on a smooth manifold $\mathcal{M}$ with a compact support. Then the corresponding local flow is defined for every $q \in \mathcal{M}$ and $t \in \mathbb{R}$, and the local 1-parameter group becomes a group of diffeomorphisms.

**Proof.** By using $\operatorname{supp}(X) \subseteq K$, $K$ compact, we can cover $K$ by a finite covering, then using Proposition 3.3.2, we're done. ∎

This leads to the following.

**Corollary 3.3.1.** On a compact differentiable manifold $\mathcal{M}$, any vector field generates a local 1-parameter group.

# Lecture 11: Geodesic & Cogeodesic Flows and Parallel Transport

### 3.3.2 Geodesic and Cogeodesic Flows

9 Feb. 14:30

A particularly interesting flow is the cogeodesic-flow, which can be constructed as follows. Let's first transform Equation 2.1 (which is a $2^{nd}$-ODE) into a $1^{st}$ order system on the cotangent bundle $T^*\mathcal{M}$, and locally trivialize $T^*\mathcal{M}$ by chart $T^*\mathcal{M}|_U \cong U \times \mathbb{R}^d$ with coordinates $(x^1, \ldots, x^d, p_1, \ldots, p_d)$. Now, set

$$H(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j, \tag{3.2}$$

**Theorem 3.3.3.** Equation 2.1 is equivalent to the system on $T^*\mathcal{M}$:

$$\begin{cases} \dot{x}^i = \dfrac{\partial H}{\partial p_i} g^{ij}(x) p_j; \\ \dot{p}_i = -\dfrac{\partial H}{\partial x^i} = -\dfrac{1}{2} g^{jk}_{,i}(x) p_j p_k. \end{cases} \tag{3.3}$$

**Proof.** This is just computation (recall that$g^{ik} g_{kj} = \delta^i_j$). ■

**Definition 3.3.4** (Cogeodesic flow)**.** The *cogeodesic flow* is the local flow determined by Equation 3.3.

**Definition 3.3.5** (Geodesic flow)**.** The geodesic flow on $T\mathcal{M}$ is obtained from the cogeodesic flow by the first equation in Equation 3.3.

Thus, the geodesic is the projection of the integral curve of the geodesic flow onto $\mathcal{M}$.

**Remark** (Hamiltonian flow)**.** The cogeodesic flow is a *Hamiltonian flow* for the Hamiltonian $H$.

**Proof.** By Equation 3.3, along the integral curves,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = H_{x^i} \dot{x}^i + H_{p_i} \dot{p}^i = -\dot{p}_i x \dot{x}^i + \dot{x}^i \dot{p}_i = 0.$$

Observe that the cogeodesic flow maps the set $E_\lambda \coloneqq \{(x, p) \in T^*\mathcal{M} \mid H(x, p) = \lambda\}$ onto itself for all $\lambda \geq 0$. ⊛

**Remark.** If $\mathcal{M}$ is compact, then all $E_\lambda$ are compact, then all geodesic flows are defined on all $E_\lambda$ for all $\lambda$.

**Remark.** $\mathcal{M} = \bigcup_{\lambda \geq 0} P E_\lambda$ for $P$ being the projection.

## 3.4 Parallelism

An important concept related to curvatures is "parallelism," which needs a formal introduction of covariant derivatives.[5] As a motivating example, the following is an equivalent definition of geodesic.

**Example** (Autoparallel)**.** The geodesic $c$ satisfies $\nabla_{\dot{c}} \dot{c} = 0$. This is called autoparallel.

**Proof.** In the local coordinates, we have $\dot{c} = \dot{c}^i \partial / \partial x^i$, and note that

$$\nabla_{\dot{c}} \dot{c} = \dot{c}^i \nabla_{\frac{\partial}{\partial x^i}} \dot{c}^j \frac{\partial}{\partial x^j} = \dot{c}^i \dot{c}^j \Gamma^k_{ij} \frac{\partial}{\partial x^k} + \ddot{c}^k \frac{\partial}{\partial x^k} = \left( \ddot{c}^k + \Gamma^k_{ij} \dot{c}^i \dot{c}^j \right) \frac{\partial}{\partial x^k} = 0 \tag{3.4}$$

since a geodesic is the solution of Equation 2.1. ⊛

To understand what $\nabla_{\dot{c}} \dot{c}$ is doing beyond just calculation, we need to understand parallel transports.

### 3.4.1 Covariant Derivatives

**As previously seen.** The set of smooth vector fields along $c$ is denoted as $\mathcal{X}_c(\mathcal{M})$.

We can now finally define covariant derivative formally.

[5] Although we say we're going to treat them the same as $\nabla$.

**Definition 3.4.1** (Covariant derivative)**.** The *covariant derivative* of $V$ along $c$ is the vector field $\mathrm{D}V/\mathrm{d}t$ in Proposition 3.4.1.

**As previously seen.** Let $X = X^i \frac{\partial}{\partial x_i}$, $V = V^k \frac{\partial}{\partial x_k}$, and let $\mathrm{D}$ be the Levi-Civita connection. Then

$$\mathrm{D}_V X = \mathrm{D}_V(X^i \frac{\partial}{\partial x_i}) = V(X^i)\frac{\partial}{\partial x_i} + X^i \underbrace{\mathrm{D}_V \frac{\partial}{\partial x_i}}_{V^k \mathrm{D}_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}} = V(X^i)\frac{\partial}{\partial x_i} + V^k X^i \Gamma^j_{ki}\frac{\partial}{\partial x_j}.$$

**Proposition 3.4.1** (Covariant derivative)**.** Let $(\mathcal{M}, g)$ be a Riemannian manifold, $\mathrm{D}$ the canonical (Levi-Civita) connection, and $c$ a smooth curve in $\mathcal{M}$. Then there exists a unique operator $\mathrm{D}/\mathrm{d}t$ defined as the vector space of vector fields along $c$ satisfying

(i) (a) $\frac{\mathrm{D}}{\mathrm{d}t}(fY)(t) = f'(t)Y(t) + f(t)\frac{\mathrm{D}}{\mathrm{d}t}Y(t)$ for all $f \in C^\infty(I)$ and $Y \in \mathcal{X}_c(\mathcal{M})$;

(b) $\frac{\mathrm{D}}{\mathrm{d}t}(V + W) = \frac{\mathrm{D}V}{\mathrm{d}t} + \frac{\mathrm{D}W}{\mathrm{d}t}$ for all $V, W \in \mathcal{X}_c(\mathcal{M})$;

(ii) if there exists a neighborhood of in $I$ such that $Y$ is the restriction to $c$ of a vector field $X$ defined on a neighborhood of $c(t_0)$ in $\mathcal{M}$, then $\frac{\mathrm{D}}{\mathrm{d}t}Y(t_0) = (\mathrm{D}_{c(t_0)}X)_{c(t_0)}$.

**Proof.** Consider defining such an operator $\mathrm{D}/\mathrm{d}t$ as

$$\frac{\mathrm{D}}{\mathrm{d}t}\left(Y^i(t)\frac{\partial}{\partial x_i}\right) = \frac{\mathrm{d}V^i}{\mathrm{d}t}\frac{\partial}{\partial x_i} + \dot{c}Y^i\Gamma^k_{ji}(c(t))\frac{\partial}{\partial x_k},$$

where $\dot{c} = \dot{c}^k \frac{\partial}{\partial x_k}$. This shows (i) (a) and (b) hold. Next, to show (ii), let $x$ be a smooth vector field in $\mathcal{M}$. Then the induced vector field along $c$ is given by $Y(t) = X_{c(t)}$, i.e., in terms of the coordinate basis, we have

$$Y(t) = Y^i(t)\frac{\partial}{\partial x_i}, \quad X_x = X^i(x)\frac{\partial}{\partial x_i}, \quad Y^i(t) = X^i(c(t)).$$

Then,

$$\begin{aligned}\mathrm{D}_{\dot{c}}X = \mathrm{D}_{\dot{c}}\left(X^i\frac{\partial}{\partial x_i}\right) &= \dot{c}(X^i)\frac{\partial}{\partial x_i} + X^i\mathrm{D}_{\dot{c}}\frac{\partial}{\partial x_i} = X^i\dot{c}^k \underbrace{\mathrm{D}_{\frac{\partial}{\partial x_k}}\frac{\partial}{\partial x_i}}_{\Gamma^\ell_{ki}\frac{\partial}{\partial x_\ell}} \\ &= \partial_t(X^i\circ c)\frac{\partial}{\partial x_i} + \dot{c}^k X^i\Gamma^\ell_{ki}\frac{\partial}{\partial x_\ell} = \partial_t(X^i\circ c)\frac{\partial}{\partial x_i} + \dot{c}^k Y^i\Gamma^\ell_{ki}\frac{\partial}{\partial x_\ell} = \frac{\mathrm{D}}{\mathrm{d}t}Y.\end{aligned}$$

■

**Problem 3.4.1.** Why not just define $\mathrm{D}Y/\mathrm{d}t$ by (ii)?

**Answer.** A vector field $Y$ along a curve may not always be extended to a neighborhood of $c$ in $\mathcal{M}$. But, in local coordinates,

$$Y(t) = \sum_{i=1}^n Y^i(t)\left(\frac{\partial}{\partial x^i}\right)_{c(t)},$$

which shows that a vector field along $c$ is always a linear combination of vector fields along $c$ that can be extended. ⊛

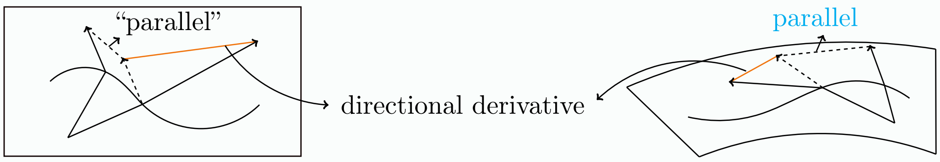
**Remark.** Proposition 3.4.1 shows that the choice of an linear connection on $\mathcal{M}$ leads to a bona fide (satisfying (a) and (b)) derivative of vector fields along curves. The notion of "connection" furnishes, therefore, a manner of differentiating vectors along curves.

### 3.4.2 Parallel Transports

Finally, we introduce the notion of parallel.

**Definition 3.4.2** (Parallel). A vector field $X$ on $\mathcal{M}$ along a curve $c$ is *parallel* (or *parallelly transported*) along $c$ if $\mathrm{D}X/\mathrm{d}t = 0$ for all $t \in I$.

**Intuition.** In the (flat) Euclidean space, we know what is "parallel," and hence we can define the directional derivative. But now the logic is reversed: we first define what is parallel in a curved space, and then we can make sense of directional derivative in a curved space!

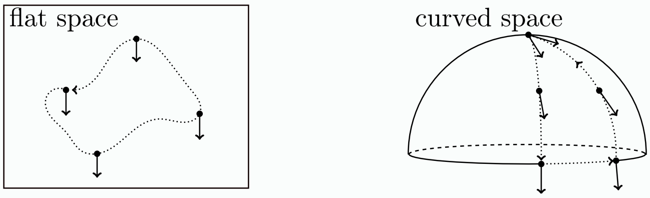



Given the definition of a parallel vector fields along curves, we can talk about parallel transport.

**Definition 3.4.3** (Parallel transport). The *parallel transport* from $c(0)$ to $c(t)$ along the curve $c$ in a Riemannian manifold $(\mathcal{M}, g)$ is the linear map $P_i \colon T_{c(0)}\mathcal{M} \to T_{c(t)}\mathcal{M}$ associating $v \in T_{c(0)}\mathcal{M}$ with $X_v(i) \in T_{c(i)}\mathcal{M}$ with $X_v$ being the parallel vector field along $c$ such that $X_v(0) = v$.

It's clear that how we can extend Definition 3.4.3 for a piece-wise smooth curve.

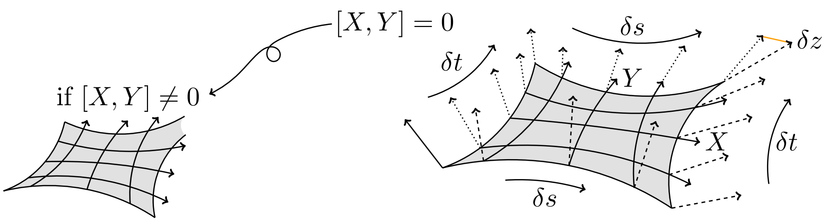
**Intuition.** When the space is flat, keeping the "arrow" (which defines a vector field) in one direction and moving around won't produce any changes, while when the space is curved, it will.




We make a surprising remark on the relation between Riemannian curvature and parallel transport.

**Remark** (Geometric significant of Riemannian curvature). The idea is that for a manifold with torsion free $\nabla$, if we parallel transporting along two paths on an infinitesimal patch (which induces $X, Y$) such that $[X, Y] = 0$, we can detect curvature in terms of $\delta z$, where[a]

$$(\delta z)^i = \ldots = R^i_{jk\ell} X^k Y^\ell Z^j \cdot \delta s \delta t + O(\delta s^2 \delta t, \delta s \delta t^2).$$




We will come back to this later.

[a] This is a deep theorem! In the ..., we use $T \equiv 0$.

**Proposition 3.4.2.** The parallel transport exists, uniquely.

**Proof.** do Carmo [FC13, Proposition 2.6] ∎

**Proposition 3.4.3.** Let $(\mathcal{M}, g)$ be a Riemannian manifold. The parallel transport defines for all $t$ an isometry from $T_{c(0)}\mathcal{M}$ onto $T_{c(t)}\mathcal{M}$; more generally, if $X, Y$ are vector fields along $c$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} g(x(t), y(t)) = g\left(\frac{\mathrm{D}X(t)}{\mathrm{d}t}, Y(t)\right) + g\left(X(t), \frac{\mathrm{D}Y(t)}{\mathrm{d}t}\right).$$

**Proof.** See do Carmo [FC13, Proposition 3.2] ■

### 3.4.3 Autoparallel Curves

Now we can formally introduce the notion of autoparallel.

**Definition 3.4.4** (Autoparallel)**.** Let $\nabla$ be a connection on $T\mathcal{M}$ of a differentiable manifold $\mathcal{M}$. A curve $c\colon I \to \mathcal{M}$ is called *autoparallel* (or *geodesic*) w.r.t. $\nabla$ if

$$\nabla_{\dot{c}}\dot{c} = 0.$$

**Intuition.** An autoparallel curve is the *straightest line* (hence geodesic) in the space w.r.t. $\nabla$!

**Remark** (Physical interpretation)**.** One can start from introducing $\nabla$, considering $\nabla_{\dot{c}}\dot{c} := 0$ (which is just Equation 2.1), and realize that we don't need to consider gravity as a force, rather a "curvature of spacetime," in order to make sense of Newton's first law, i.e., mass without forces will undergo a autoparallel curve.

**Example** (Euclidean plane)**.** Let $U = \mathbb{R}^2$, $x = \mathrm{id}_{\mathbb{R}^2}$, $\Gamma^i_{jk} = 0$, then $\ddot{c}^k = 0$ in Equation 3.4. Hence,

$$c^k(t) = a^k t + b^k \text{ for } a^k, b^k \in \mathbb{R}^d.$$

**Example** (Round sphere)**.** The geodesics on a "round sphere" are the great circles.

**Proof.** Consider a "unit round sphere" $\mathcal{M} = S^2$ with spherical coordinates $x(p) = (r, \theta, \varphi)$ such that $r = 1$, $\theta \in (0, \pi)$, and $\varphi \in 0, 2\pi$. The "roundness" is given by $\nabla_{\text{round}}$ where we specify (at one point)

$$\Gamma^1_{22} := -\sin\theta\cos\theta, \quad \Gamma^2_{21} = \Gamma^2_{12} := \cot\theta,$$

where we let $x^1(p) = \theta(p), x^2(p) = \varphi(p)$. The autoparallel equation tells us

$$\begin{cases} \ddot{\theta} + \Gamma^1_{22}\dot{\varphi}\dot{\varphi} = 0; \\ \ddot{\varphi} + 2\Gamma^2_{12}\dot{\theta}\dot{\varphi} = 0; \end{cases} \Leftrightarrow \begin{cases} \ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\varphi}\dot{\varphi} = 0; \\ \ddot{\varphi} + 2\cot(\theta)\dot{\theta}\dot{\varphi} = 0. \end{cases}$$

Then, we see that $\theta(t) = \pi/2$, $\varphi(t) = \omega t + \varphi_0$ is a solution.[a] Hence, we conclude that if we run at a constant speed around the great circle of $S^2$, it'll be autoparallel, hence a geodesic. ⊛

[a] Note that $\theta(t) = \pi/2$, $\varphi(t) = \omega t^2 + \varphi_0$ is not a solution.

Similarly, given any $\nabla$ on a space, we can find the straightest curve on which.

# Lecture 12: Tangent and Cotangent Bundles

## 3.5 More on Tangent and Cotangent Bundles

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Let $f\colon \mathcal{M} \to \mathcal{N}$ be a differentiable map between two differentiable manifolds, until now, we have only talked about how to transform tangent vectors or 1-form via $f$. Implicitly, these are just pullback $(f^*)$ and pushforward $(f_*)$, as we now define formally.

**Definition.** Let $f\colon \mathcal{M} \to \mathcal{N}$ be a smooth map between two smooth manifolds and $p \in \mathcal{M}$.

**Definition 3.5.1** (Pushforward)**.** The *pushforward* is the linear map $f_* \coloneqq \mathrm{d}f_p \colon T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$.

**Definition 3.5.2** (Pullback)**.** The *pullback* is the linear map $f^*\colon T^*_{f(p)}\mathcal{N} \to T^*_p\mathcal{M}$ where

$$(f^*\omega)(X) = \omega(f_* X)$$

for $\omega \in T^*_{f(p)}\mathcal{N}$ and $X \in T_p\mathcal{M}$.

In all, the following diagram commutes:

$$\begin{tikzcd} T_p^*\mathcal{M} \arrow[d, "\pi"'] & T_p^*\mathcal{N} \arrow[l, "f^*"'] \arrow[d, "\pi"] \\ \mathcal{M} \arrow[r, "f"'] & \mathcal{N} \end{tikzcd} \qquad \begin{tikzcd} T_p\mathcal{M} \arrow[r, "f_*"] \arrow[d, "\pi"'] & T_p\mathcal{N} \arrow[d, "\pi"] \\ \mathcal{M} \arrow[r, "f"'] & \mathcal{N} \end{tikzcd}$$

### 3.5.1 Pullbacks and Pushforwards on Bundles

Now, consider a vector bundle $(E, \pi, \mathcal{N})$ over $\mathcal{N}$, we want to use $f$ to "pull back" the vector bundle, i.e., construct a vector bundle, denote as $f^*E$, for which the fiber over $x \in \mathcal{M}$ is $E_{f(x)}$.

**Definition 3.5.3** (Pullback bnudle)**.** The *pullback bundle* $f^*E$ is the vector bundle over $\mathcal{M}$ with the bundle charts $(\varphi \circ f, f^{-1}(U))$ if $(\varphi, U)$ is the bundle charts of $E$.

Similarly, we can "push forward" a vector bundle $(E, \pi, \mathcal{M})$ over $\mathcal{M}$ via $f$ in the same fashion.

**Definition 3.5.4** (Pushforward bnudle)**.** The *pushforward bundle* $f_*E$ is the vector bundle over $\mathcal{N}$ with the bundle charts $(\varphi \circ f^{-1}, f(U))$ if $(\varphi, U)$ is the bundle charts of $E$.

**Note.** In Definition 3.5.4, it only makes sense if $\mathcal{M} \hookrightarrow \mathcal{N}$.

**Definition 3.5.5** (Bundle homomorphism)**.** Consider 2 vector bundles $(E_1, \pi_1, \mathcal{M}), (E_2, \pi_2, \mathcal{M})$ over $\mathcal{M}$, and let the differentiable map $f\colon E_1 \to E_2$ be fiber preserving, i.e., $\pi_2 \circ f = \pi_1$. If the fiber maps $f_x\colon E_{1,x} \to E_{2,x}$ is linear,[a] then $f$ is called a *bundle homomorphism*.

[a] I.e., vector homomorphisms.

**Definition 3.5.6** (Subbundle)**.** Let $(E, \pi, \mathcal{M})$ of rank $n$ be a vector bundle. Let $E^1 \subseteq E$, and assume that for all $x \in \mathcal{M}$, there exists a bundle chart $(\varphi, U)$ for $x \in U$ and

$$\varphi(\pi^{-1}(U) \cap E^1) = U \times \mathbb{R}^m \subseteq U \times \mathbb{R}^n$$

for $m \leq n$. Then the *subbundle* of $E$ of rank $m$ is the vector bundle $(E^1, \pi|_{E^1}, \mathcal{M})$.

**Example.** Consider $f\colon \mathcal{M} \hookrightarrow \mathcal{N}$ where $g_{\mathcal{N}}$ is a metric on $\mathcal{N}$. Then, $g_{\mathcal{N}}$ induces a metric $g_{\mathcal{M}}$ on $\mathcal{M}$ by $f$ since we can define

$$g_{\mathcal{M}}(X, Y) \coloneqq g_{\mathcal{N}}(f_*(X), f_*(Y)).$$

### 3.5.2 Pullbacks and Pushforwards of Vector Fields

Now, we consider to "pull back" or "push forward" a vector field, i.e., a section of a bundle.

**Definition 3.5.7** (Pushforward)**.** Let $\psi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism between smooth manifolds, and let $X$ be a vector field on $\mathcal{M}$. Then the *pushforward* vector field $Y = \psi_* X = \mathrm{d}\psi X$ on $\mathcal{N}$ is

$$Y(p) = \mathrm{d}\psi(X(\psi^{-1}(p))).$$

**Definition 3.5.8** (Pullback)**.** Let $\psi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism between smooth manifolds, and let $Y$ be a vector field on $\mathcal{N}$. Then the *pullback* vector field $X = \psi^* Y$ on $\mathcal{M}$ is just $X(p) = Y_{\psi(p)}$.

**Note.** We let $\psi$ be a diffeomorphism just for convenient: we can also consider a vector fields along curve when $\psi$ injects/surjects.

**Lemma 3.5.1.** For every differentiable function $f\colon \mathcal{N} \to \mathbb{R}$, $(\psi_* X)(f)(p) = X(f \circ \psi)(\psi^{-1} p)$.

**Lemma 3.5.2.** Let $X$ be a vector field on $\mathcal{M}$ and $\psi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. If the local 1-parameter group $(\varphi_t)_{t\in I}$ generated by $X$, then the local 1-parameter group generated by $\psi_* X$ is $\psi \circ \varphi_t \circ \psi^{-1}$.

### 3.5.3 Induced Bundle Metrics

Let $(\mathcal{M}, g)$ be a Riemannian manifold, then $g$ induces the *bundle metrics* on all vector bundles over $\mathcal{M}$: for $T^*\mathcal{M}$, it is given by

$$g(\omega, \eta) \coloneqq g^{ij}\omega_i \eta_i$$

for $\omega = \omega_i \mathrm{d}x^i, \eta = \eta_i \mathrm{d}x^i$. Hence, we can talk about the identification between $T\mathcal{M}$ and $T^*\mathcal{M}$ through $g$:

$$\begin{gathered} V = V^i \tfrac{\partial}{\partial x^i} \in T\mathcal{M} \\ \Updownarrow \\ \omega = \omega_j \mathrm{d}x^j \in T^*\mathcal{M} \end{gathered}$$

with $\omega_j = g_{ij} V^i$ (or $V^i = g^{ij}\omega_j$) such that

(a) $g(X, Y) = g_{ij} X^i Y^j$ for $X, Y \in T\mathcal{M}$;

(b) $g(\omega, \eta) = g^{ij}\omega_i \eta_j$ for $\omega, \eta \in T^*\mathcal{M}$.

Thus, for $V \in T_x\mathcal{M}$, there corresponds a 1-form $\omega \in T_x^*\mathcal{M}$ via the metric $\omega(Y) \coloneqq g(V, Y)$ for all $Y$, and we further have $\|\omega\| = \|V\|$.

We can also consider the coordinate transformation behavior. Let $(e_i)_{i=1,\dots,d}$ be a basis of $T_x\mathcal{M}$ and $(\omega^j)_{j=1,\dots,d}$ the dual basis of $T_x^*\mathcal{M}$, i.e., $w^j(e_i) = \delta_i^j$. Given $V = V^i e_i \in T_x\mathcal{M}$, $\eta = \eta_j \omega^j \in T_x^*\mathcal{M}$, we then have $\eta(V) = \eta_i V^i$. Now, consider bases $(e_i), (\omega^j)$ in the local coordinates, i.e., $e_i = \partial/\partial x^i$ and $\omega^j = \mathrm{d}x^j$. Let $f$ be a local coordinates change, then $V$ and $\eta$ transformed as

$$f_*(V) \coloneqq V^i \frac{\partial f^\alpha}{\partial x^i}\frac{\partial}{\partial f^\alpha}, \qquad f^*(\eta) \coloneqq \eta_j \frac{\partial x^j}{\partial f^\beta} \mathrm{d}f^\beta$$

correspondingly, and we see that

$$f^*(\eta)(f_*(V)) = \eta_j \frac{\partial x^j}{\partial f^\alpha} V^i \frac{\partial f^\alpha}{\partial x^i} = \eta_i V^i = \eta(V).$$

**Intuition.** The above means that

- the tangent vectors transform with the functional matrix of coordinates change;
- the cotangent vectors transform with the transposed inverse of the above matrix.

To compute the coordinates change $y \mapsto x(y)$ for $\omega = \omega_i \mathrm{d}x^i$, $\eta = \eta_i \mathrm{d}x^i$ with $\langle \omega, \eta \rangle = g^{ij}\omega_i \eta_j$, we have

$$\omega_i \mathrm{d}x^i = \omega_i \frac{\partial x^i}{\partial y^\alpha} \mathrm{d}y^\alpha =: \widetilde{\omega}_\alpha \mathrm{d}y^\alpha.$$

**As previously seen.** $g^{ij}$ is transformed as

$$h^{\alpha\beta} = g^{ij} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}.$$

Then, we see that $h^{\alpha\beta}\widetilde{\omega}_\alpha \widetilde{\eta}_\beta = g^{ij}\omega_i \eta_j$ and $\|\omega(x)\| = \sup\{\omega(x)(V) \mid V \in T_x\mathcal{M}, \|v\| = 1\}$.

**Remark.** If we consider $T\mathcal{M} \otimes T\mathcal{M}$, then metric is

$$\langle V \otimes Y, \xi \otimes \eta \rangle = g_{ij} V^i Y^i g_{k\ell} \omega^k \eta^\ell.$$

**As previously seen** (Lie derivative). Consider a vector field $X$ with a local 1-parameter group $(\psi_t)_{t \in I}$ and a tensor field $S$ on $\mathcal{M}$. The Lie derivative of $S$ in the direction of $X$ is defined as

$$\mathcal{L}_X S := \left.\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^* S)\right|_{t=0}.$$

# Lecture 13: Sectional Curvatures and Space Forms

Let $X = X^i \partial/\partial x^i$ be a vector field. Then consider $(\psi_t)_* X(\psi_t(X))$ to get a curve $X_t$ in $T_x\mathcal{M}$ for $t \in I$. 16 Feb. 14:30
By differentiate that curve, i.e.,

$$(\psi_t)_* \frac{\partial}{\partial x^i}(\psi_t(x)) = \frac{\partial \psi_t^k}{\partial x^i} \frac{\partial}{\partial x^k}.$$

**Note.** For $\varphi \colon \mathcal{M} \to \mathcal{N} := \mathcal{M}$ and $X$ and $\varphi(x)$ are in the same coordinate neighborhood,

$$\varphi_* \frac{\partial}{\partial x^i} = \frac{\partial \varphi^k}{\partial x^i} \frac{\partial}{\partial \varphi^k}$$

since $\frac{\partial}{\partial \varphi^k} = \frac{\partial}{\partial x^k}$.

On the other hand, let $\omega = \omega_i \mathrm{d}x^i$ be a 1-form, then we have

$$(\psi_t^*)(\omega)(x) = \omega_i(\psi_t(x)) \frac{\partial \psi_t^i}{\partial x^k} \mathrm{d}x^k,$$

which is a curve in $T_x^*\mathcal{M}$.

**Note.** For $\varphi \colon \mathcal{M} \to \mathcal{N}$,[a] with for the 1-form $\omega = \omega_i \mathrm{d}x^i$ on $\mathcal{N}$,

$$\varphi^* \omega = \omega_i(\varphi(x)) \frac{\partial z^i}{\partial x^k} \mathrm{d}x^k.$$

[a] $\varphi$ need not be a diffeomorphism.

Let $\varphi \colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism, $Y$ be a vector field on $\mathcal{N}$. Then, set

$$\varphi^* Y := (\varphi^{-1})_* Y,$$

and for other contravariant tensors, $\varphi^*$ can be defined in an analogous way.

**Example.** For a vector field $X$ and a local 1-parameter group $(\psi_t)_{t\in I}$, it is $(\psi_t^* X) = (\psi_t)_* X$.

## 3.6 Sectional Curvatures

Consider the following.

**Definition 3.6.1** (Sectional curvature). The *sectional curvature* of the plane spanned by the (linearly independent) tangent vectors $X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i} \in T_x\mathcal{M}$ of a Riemannian manifold $(\mathcal{M}, g)$ is

$$K(X \wedge Y) \coloneqq \frac{g(R(X,Y)Y,X)}{|X \wedge Y|^2}$$

where $|X \wedge Y|^2 = g(X,X)g(Y,Y) - g(X,Y)^2$.

**Intuition.** Given a vector space $V$ and $x, y \in V$, $|x \wedge y| \coloneqq \sqrt{|x|^2 + |y|^2 - \langle x, y\rangle^2}$ represents the area of the two-dimensional parallelogram spanned by $x, y$.

**Remark.** Sectional curvature determines the whole Riemannian curvature.

**Proof.** Given $g(R(X,Y)Z,W)$, we can express this entirely by $K$. See do Carmo [FC13, Lemma 3.3]. ⊛

**Remark** (Gauss curvature). For $\dim \mathcal{M} = 2$, $R_{ijk\ell} = K(g_{ik}g_{j\ell} - g_{ij}g_{k\ell})$ since $T_x\mathcal{M}$ contains only one plane, i.e., $T_x\mathcal{M}$ itself. In this case, $K$ is called the *Gauss curvature*.

In particular, the space form considers the space with constant sectional curvature.

**Definition 3.6.2** (Space form). A Riemannian manifold $(\mathcal{M}, g)$ is a *space form* if $K(X \wedge Y)$ is a constant for all linearly independent tangent vectors $X, Y \in T_p\mathcal{M}$ for all $p \in \mathcal{M}$.

> **Definition 3.6.3** (Spherical). A space form is called *spherical* if $K > 0$.
>
> **Definition 3.6.4** (Flat). A space form is called *flat* if $K = 0$.
>
> **Definition 3.6.5** (Hyperbolic). A space form is called *hyperbolic* if $K > 0$.

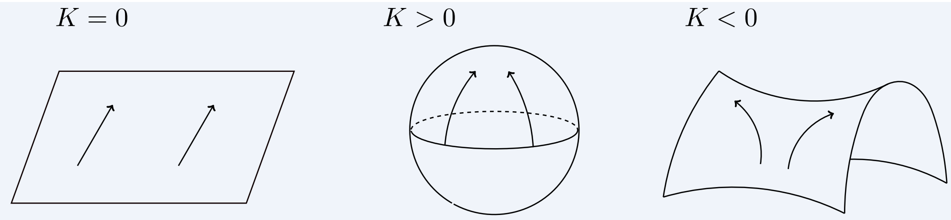
Generalize Definition 3.6.2 a bit, we have the so-called Einstein manifolds.

**Definition 3.6.6** (Einstein manifold). A Riemannian manifold $(\mathcal{M}, g)$ is called an *Einstein manifold* if $R_{ik} = cg_{ik}$ for a constant $c$.[a]

[a] Which does not depend on the choice of local coordinates.

**Remark.** Every space form is an Einstein manifold.

**Example.** $\mathbb{R}^n$ is flat, $S^n$ is spherical, and $\mathbb{H}^n$[a] is hyperbolic. And all are Einstein manifolds.




[a] The hyperbolic space.

**Definition 3.6.7** (Flat). A connection $\nabla$ on $T\mathcal{M}$ is *flat* if each point in $\mathcal{M}$ has a neighborhood $U$ with local coordinates for which all the coordinate vector fields $\partial/\partial x^i$ are parallel, i.e., $\nabla\partial/\partial x^i = 0$.

**Theorem 3.6.1.** A connection $\nabla$ on $T\mathcal{M}$ is flat if and only if its curvature and torsion vanish identically.

**Proof.** Flat connection implies $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$, hence all $\Gamma_{ij}^k = 0$, so $T, R$ vanish. Conversely, find the local coordinates such that $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ for all $i, j$ and use Frobenius theorem. ∎

**Example.** The following are flat manifolds with their usual shape, i.e., connections.

- $\mathbb{R}^n$.
- Torus $T^2$.
- Every 1-dimensional Riemannian manifold.
- Products of flat manifolds.
- Tori.

**Theorem 3.6.2** (Schur theorem). Let $(\mathcal{M}, g)$ be a Riemannian manifold with $\dim \mathcal{M} \geq 3$.

(a) If the sectional curvature of $\mathcal{M}$ is constant at each point, i.e., $K(X \wedge Y) = f(x)$ for $X, Y \in T_x\mathcal{M}$, then $f(x)$ is a constant on $\mathcal{M}$, hence $\mathcal{M}$ is a space form.

(b) If the Ricci curvature is a constant at each point, i.e., $R_{ik} = c(x)g_{ik}$, then $c(x)$ is a constant, hence $\mathcal{M}$ is an Einstein manifold.

**Remark.** Schur theorem says that the isotropy[a] of a Riemannian manifold implies the homogeneity.[b] Hence, a point-wise property implies a global one!

[a] I.e., the property that at each point, all directions are geometrically indistinguishable.
[b] I.e., all points are geometrically indistinguishable.

# Chapter 4

# Isometric Immersions

## 4.1 Covering Maps

**Definition 4.1.1** (Covering map)**.** Let $\mathcal{M}, \widetilde{\mathcal{M}}$ be 2 manifolds.a map $p\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ is a *covering map* if

(a) $p$ is smooth and surjective;

(b) for all $m \in \mathcal{M}$, there exists a neighborhood $U$ at $m$ in $\mathcal{M}$ with $p^{-1}(U) = \coprod_{i\in I} U_i$ with $p\colon U_i \to U$ being a diffeomorphism and $U_i$ are disjoint open subsets of $\widetilde{\mathcal{M}}$.

**Notation** (Covering space)**.** $\widetilde{\mathcal{M}}$ in Definition 4.1.1 is called the *covering space*.

**Notation** (Universal covering space)**.** A covering space is *universal* if it's simply connected.

**Definition 4.1.2** (Riemannian covering map)**.** Let $(\mathcal{M}, g), (\mathcal{N}, h)$ be Riemannian manifolds. A map $p\colon \mathcal{N} \to \mathcal{M}$ is a *Riemannian covering map* if $p$ is a smooth covering map and is a local isometry.

**Proposition 4.1.1.** Let $p\colon \mathcal{N} \to \mathcal{M}$ be a smooth covering map. For every Riemannian metric $g$ on $\mathcal{M}$, there exists a unique Riemannian metric $h$ on $\mathcal{N}$ such that $p$ is a Riemannian covering map.

**Note.** The converse of Proposition 4.1.1 is generally not true.

**Example.** Every space covers itself trivially.

**Example.** $\mathbb{R}$ is the universal covering space of $S^1$.

**Example.** $U(n)$ has universal covers $U(n) \times \mathbb{R}$.

**Example.** $S^n$ is a double cover for $\mathbb{R}P^n$ and is universal for $n > 1$.

## Lecture 14: Second Fundamental Forms

**Proposition 4.1.2.** Let $(\mathcal{N}, h)$ be a Riemannian manifold and $G$ be a free and proper group of isometries of $(\mathcal{N}, h)$, then there exists a unique Riemannian metric $g$ on the quotient manifold $\mathcal{M} = \mathcal{N} / G$ such that the connected projection $p\colon \mathcal{N} \to \mathcal{M}$ is a Riemannian covering map. 21 Feb. 14:30

**Proof.** Let $n, n' \in \mathcal{N}$ such that $n, n' \in p^{-1}(m)$ for $m \in \mathcal{M}$. Hence, there exists an isometry $f \in G$ such that $f(n) = n'$. Also, $p \circ f = p$, and $p$ is a local diffeomorphism, so we can define a scalar product $g_m$ on $T_m\mathcal{M}$: for all $u, v \in Tvm\mathcal{M}$,

$$g_m(u, v) = h_n\big((T_n p)^{-1}u, (T_n p)^{-1}v\big)$$

for $n \in p^{-1}(m)$. This does not depend on the choice of $n \in p^{-1}(m)$ since $(T_n p)^{-1} = T_n f \circ (T_n p)^{-1}$ and $T_n f$ is an isometry of the Euclidean vector spaces $T_n\mathcal{N}$ and $T_{n'}\mathcal{N}$. It can be shown that $g$ is smooth. Thus, we have constructed a metric $g$ on $\mathcal{M}$ such that $p$ is a Riemannian covering map, which is unique. ∎

**Definition 4.1.3** (Totally geodesic)**.** A submanifold $\mathcal{M}$ of $(\widetilde{m}, \widetilde{g})$ is called *totally geodesic* if for all $m \in \mathcal{M}$ and $v \in T_m\mathcal{M}$, the geodesic $c$ of $(\widetilde{M}, g)$ with $c(0) = m$ and $c'(0) = v$ is contained fully in $\mathcal{M}$.

**Proposition 4.1.3.** Let $p\colon (\mathcal{N}, h) \to (\mathcal{M}, g)$ be a Riemannian covering map. The geodesic of $(\mathcal{M}, g)$ are the projections of the geodesic in $(\mathcal{N}, h)$; and the geodesic of $(\mathcal{N}, h)$ are the liftings of those in $(\mathcal{M}, g)$.

**Proof.** Since $p$ is a local isometry, if $\gamma$ is a geodesic of $\mathcal{N}$, then $c = p \circ \gamma$ is also a geodesic of $\mathcal{M}$. From the uniqueness theorem for geodesics shows that these are indeed the only geodesics on $\mathcal{M}$. Conversely, if $p \circ \gamma$ is a geodesic in $\mathcal{M}$, then $\gamma$ is a geodesic in $\mathcal{N}$. ∎

**Example.** In Euclidean spaces, the totally geodesic submanifold are affine linear subspaces and their open subsets.

**Example.** Each closed geodesic in Riemannian manifolds defines a 1-dimensional compact totally geodesic submanifold.

**Example.** The totally geodesic submanifolds of $S^n \subseteq \mathbb{R}^{n+1}$ are the intersections of $S^n$ with linear subspaces of $\mathbb{R}^{n+1}$.

**Example.** In general, Riemannian manifolds do not have any totally geodesic submanifolds of dimensional $> 1$.

**Note.** We will see that $\mathcal{M}$ is totally geodesic in $\widetilde{M}$ if and only if all the $2^{nd}$-fundamental forms vanish identically.

## 4.2 The Second Fundamental Form

Let $\mathcal{M}^m \subseteq \mathcal{N}^n$ be two Riemannian manifolds, and we know that a metric on $N$ induces a metric on $\mathcal{M}$ naturally. Now, we want to see that given the Levi-Civita connection $\nabla^{\mathcal{N}}$ of $\mathcal{N}$, how to get $\nabla^{\mathcal{M}}$ of $\mathcal{M}$.

This is given by the central object $(\nabla^{\mathcal{N}}_X Y)^\top$ we will study in this chapter, where $\top\colon T_x\mathcal{N} \to T_x\mathcal{M}$ for $x \in \mathcal{M}$ is the orthogonal projection. We see the following.

**Theorem 4.2.1.** For $X, Y \in \Gamma(T\mathcal{M})$, $\nabla^{\mathcal{M}}_X Y = \big(\nabla^{\mathcal{N}}_X Y\big)^T$.

**Proof.** Firstly, we have to make sure that the right-hand side is defined. This can be done by extending vector fields $X, Y$ locally to a neighborhood of $\mathcal{M}$ in $\mathcal{N}$. Do this in the local coordinates around $x \in \mathcal{M}$ locally mapping $\mathcal{M}$ to $\mathbb{R}^m \subseteq \mathbb{R}^n$.

Specifically, the extension of $X = \xi^i(x)\partial/\partial x^i$ is

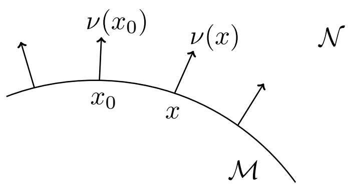
$$\widetilde{X}(x^1, \ldots, x^n) = \sum_{i=1}^{m} \xi^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}.$$

Then $\langle \widetilde{X}, \widetilde{Y}\rangle(x) = \langle X, Y\rangle(x)$ and $[\widetilde{X}, \widetilde{Y}](x) = [X, Y](x)$. From Levi-Civita theorem, the Koszul formula holds for both $\mathcal{N}$ and $\mathcal{M}$. Finally, we see that

- $(\nabla^{\mathcal{N}}_X Y)^\top$ does not depend on the chosen extensions: follows from the fact that the representation of $\nabla^{\mathcal{N}}$ is done by $\Gamma$;
- $(\nabla^{\mathcal{N}}_X Y)^\top$ defines a torsion-free connection on $\mathcal{M}$: as $\nabla^{\mathcal{N}}_X Y - \nabla^c al_Y X - [X, Y]$ vanishes, also the tangential part to $\mathcal{M}$ has to vanish.

■

Let $\nu(x)$ be a vector field in a neighborhood of $x_0 \in \mathcal{M} \subseteq \mathcal{N}$ that is orthogonal to $\mathcal{M}$, i.e., $\langle \nu(x), X\rangle = 0$ for all $X \in T_x\mathcal{M}$. Also, let $T_x\mathcal{M}^\perp$ be the orthogonal complement of $T_x\mathcal{M}$ in $T_x\mathcal{N}$, and $T\mathcal{M}^\perp$ with fiber $T_x\mathcal{M}^\perp$ of $x \in \mathcal{M}$.




**Notation** (Normal bundle). $T\mathcal{M}^\perp$ is the *normal bundle* of $\mathcal{M}$ in $\mathcal{N}$.

We see that $\langle \nu(x), X\rangle = 0$ for all $X \in T_x\mathcal{M}$ means $\nu(x) \in T_x\mathcal{M}^\perp$.

**Lemma 4.2.1.** $(\nabla^{\mathcal{N}}_X \nu)^\top(x)$ only depends on $\nu(x)$.

**Proof.** This follows directly from

$$(\nabla^{\mathcal{N}}_X f\nu)^\top(x) = (X(f)(x)\nu(x))^\top + f(x)(\nabla^{\mathcal{N}}_X \nu)^\top(x) = f(x)(\nabla^{\mathcal{N}}_X \nu)^\top(x)$$

for $f$ smooth, since $(X(f)(x)\nu(x))^\top = 0$. ■

**Definition 4.2.1** (Second fundamental tensor). The *second fundamental tensor* $S\colon T_x\mathcal{M} \times T_x\mathcal{M}^\perp \to T_x\mathcal{M}$ of $\mathcal{M}$ at point $x \in \mathcal{M}$ is defined by

$$S(X, \nu) = (\nabla^{\mathcal{N}}_X \nu)^\top.$$

**Lemma 4.2.2.** For $X, Y \in T_x\mathcal{M}$, $\ell_\nu(X, Y) \coloneqq \langle S(X, \nu), Y\rangle$ is symmetric in $X, Y$.

**Proof.** Since

$$\ell_\nu(X, Y) = \langle (\nabla^{\mathcal{N}}_X \nu)^\top, Y\rangle = \langle \nabla^{\mathcal{N}}_X \nu, Y\rangle = -\langle \nu, \nabla^{\mathcal{N}}_X Y\rangle$$

as $\nabla^{\mathcal{N}}$ is metric and $\langle \nu, Y\rangle = 0$. Now, since $\nabla^{\mathcal{N}}$ is torsion-free, we further have

$$\ell_\nu(X, Y) = -\langle \nu, \nabla^{\mathcal{N}}_Y X + [X, Y]\rangle = -\langle \nu, \nabla^{\mathcal{N}}_Y X\rangle - \langle \nu, [X, Y]\rangle = -\langle \nu, \nabla^{\mathcal{N}}_Y X\rangle$$

as $\nu \in T_x\mathcal{M}^\perp, [X, Y] \in T_x\mathcal{M}$, so $\langle \nu, [X, Y]\rangle = 0$. Finally, since again, $\nabla^{\mathcal{N}}$ is metric,

$$\ell_\nu(X, Y) = \langle \nabla^{\mathcal{N}}_Y \nu, X\rangle = \langle (\nabla^{\mathcal{N}}_Y \nu)^\top, X\rangle = \ell_\nu(Y, X).$$

■

**Definition 4.2.2** (Second fundamental form)**.** The *second fundamental form* $\ell_\nu(\cdot,\cdot)$ of $\mathcal{M}$ in $\mathcal{N}$ is defined as $\ell_\nu(X,Y) \coloneqq \langle S(X,\nu), Y\rangle$.

Now, fix a normal field $\nu$, and let $S_\nu(X) \coloneqq S(X,\nu)$, then

$$S_\nu \colon T_x\mathcal{M} \to T_x\mathcal{M}$$

is self-adjoint w.r.t. the metric $\langle\cdot,\cdot\rangle$ by Lemma 4.2.2.

**Definition.** Assume that $\langle \nu,\nu\rangle \equiv 1$, i.e., $\nu$ is the unit normal field, then $S_\nu$ has $m$ real eigenvalues.

**Definition 4.2.3** (Principal curvature)**.** The eigenvalues are called *principal curvatures* of $\mathcal{M}$ in direction $\nu$.

**Definition 4.2.4** (Principal curvature vector)**.** The corresponding eigenvectors are called *principal curvature vectors* of $\mathcal{M}$ in direction $\nu$.

**Definition 4.2.5** (Mean curvature)**.** The *mean curvature* of $\mathcal{M}$ in direction $\nu$ is defined by

$$H_\nu \coloneqq \frac{1}{m}\operatorname{Tr} S_\nu.$$

**Definition 4.2.6** (Gauss-Kronecker curvature)**.** The *Gauss-Kronecker curvature* of $\mathcal{M}$ in direction $\nu$ is defined by

$$K_\nu \coloneqq \det S_\nu.$$

## Lecture 15

Given a 1-form $\omega$, and vector fields $X, Y$, we have 23 Feb. 14:30

$$X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y).$$

For arbitrary tensors $S, T$, we similarly have

$$\nabla_X(S\otimes T) = \nabla_X S\otimes T + S\otimes \nabla_X T.$$

If $S$ is a $p$-times covariant tensor, and $Y_1,\ldots,Y_p$ vector fields,

$$(\nabla_X S)(Y_1,\ldots,Y_p) = X(S(Y_1,\ldots,Y_p)) - \sum_{i=1}^{p} S(Y_1,\ldots,Y_{i-1},\nabla_X Y_i, Y_{i+1},\ldots,Y_p).$$

For $T$ a $(p,q)$-tensor field,

$$\begin{aligned}(\nabla_Y T)(\alpha_1,\ldots,\alpha_q,X_1,\ldots,X_p) &= Y(T(\alpha_1,\ldots,\alpha_q,X_1,\ldots,X_p))\\ &\quad - \sum_{i=1}^{q} T(\alpha_1,\ldots,\nabla_Y\alpha_i,\ldots,\alpha_q,X_1,\ldots,X_p)\\ &\quad - \sum_{i=1}^{q} T(\alpha_1,\ldots,\alpha_q,X_1,\ldots,\nabla_Y X_i,\ldots,X_p).\end{aligned}$$

If $S = g_{ij}\mathrm{d}x^i\otimes \mathrm{d}x^j$, then $\nabla_X g = 0$ for all vector fields $X$.
Also,

$$\begin{aligned}(\mathcal{L}_X S)(Y_1,\ldots,Y_p) &= X(S(Y_1,\ldots,Y_p)) - \sum_{i=1}^{p} S(Y_1,\ldots,[X,Y_i],\ldots,Y_p)\\ &= (\nabla_X S)(Y_1,\ldots,Y_p) + \sum_{i=1}^{p} S(Y_i,\ldots,\nabla_{Y_i}X,\ldots,Y_p)\end{aligned}$$

since $\nabla$ is torsion-free, we have $\nabla_X Y_i - \nabla_{Y_i}X = [X,Y_i]$.

**Definition 4.2.7** (Killing field). Consider a Riemannian manifold $(\mathcal{M}, g)$, and $g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$. Then a vector field $X$ such that

$$\mathcal{L}_X g = 0$$

is called a *killing field* (or *infinitesimal isometry*).

**Lemma 4.2.3.** A vector field $X$ on $(\mathcal{M}, g)$ is a killing field if and only if the local 1-parameter group generated by $X$ consisted of local isometries.

**Lemma 4.2.4.** The killing fields of a Riemannian manifold constitute a Lie algebra.

Let $\dim \mathcal{N} = m + 1$, $\dim \mathcal{M} = m$, then for all $x \in \mathcal{M}$, there are exactly 2 normal vectors $\nu \in T_x\mathcal{M}^\perp$ with $\langle \nu, \nu \rangle \equiv 1$, i.e., $\nabla^{\mathcal{N}}_X \nu$ always tangential to $\mathcal{M}$.

**Remark.** $\nabla^{\mathcal{N}}_X \nu$ measures the "tilting velocity" with which $\nu$ is tilted relative to a fixed parallel vector field in $\mathcal{N}$, when on $\mathcal{M}$ in direction $X$.

**Theorem 4.2.2.** Given $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$ such that $\mathcal{M}$ is totally geodesic in $\widetilde{\mathcal{M}}$ if and only if all $2^{nd}$ fundamental form of $\mathcal{M}$ vanish identically.

**Proof.** Let $c\colon I \to \mathcal{M}$ be a geodesic in $\mathcal{M}$, i.e., $\nabla^{\mathcal{M}}_{\dot{c}} \dot{c} = 0$. By Theorem 4.2.1, we have that

$$\nabla^{\mathcal{M}}_{\dot{c}} \dot{c} = (\nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c})^\top = 0,$$

i.e., $c$ is a geodesic in $\widetilde{\mathcal{M}}$ if and only if $(\nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c})^\top = 0$, i.e.,

$$\left\langle \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c}, \nu \right\rangle = 0$$

for all $\nu \in T\mathcal{M}^\perp$. Notice that $\langle \dot{c}, \nu \rangle = 0$ and $\dot{c}\langle \dot{c}, \nu \rangle = \left\langle \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c}, \nu \right\rangle + \left\langle \dot{c}, \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \nu \right\rangle = 0$, we have

$$0 = \left\langle \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \dot{c}, \nu \right\rangle = \left\langle \dot{c}, \nabla^{\widetilde{\mathcal{M}}}_{\dot{c}} \nu \right\rangle = -\ell_\nu(\dot{c}, \dot{c}).$$

∎

**Note.** Theorem 4.2.2 holds for Lorentzian manifolds $(\widetilde{\mathcal{M}}, \widetilde{g})$.

**Example.** The initial value problem for Einstein equations. Given a $(\widetilde{\mathcal{M}}^4, \widetilde{g})$ a Lorentzian manifolds satisfying Einstein equations. $(\mathcal{M}^3, g)$ non-degenerate Riemannian manifold. If the $2^{nd}$ fundamental form of $\mathcal{M}^3$ in $\widetilde{\mathcal{M}}^4$ vanishes identically, then $\mathcal{M}^3$ is totally geodesic. This is a special case and not in general.

**Notation.** Greek indices $(\alpha, \beta, \ldots)$ occurring twice are summed over from 1 to $k$ for $X, Y, Z, W \in T_x\mathcal{M}$.

**Theorem 4.2.3** (Gauss equations). Let $\mathcal{N}$ be a Riemannian manifold with $\dim \mathcal{N} = n$, and let $\mathcal{M} \subseteq \mathcal{N}$ be a submanifold with $\dim \mathcal{M} = m$. Let $k = n - m$, and $x \in \mathcal{M}$, $\nu_1, \ldots, \nu_k$ be an orthonormal basis of $(T_x\mathcal{M})^\perp$, $S_\alpha :=^2_{\nu_\alpha}$, $\ell_\alpha := \ell_{\nu_\alpha}$, $\alpha = 1, \ldots, k$. Then,

$$R^{\mathcal{M}}(X, Y)Z - \left(R^{\mathcal{N}}(X, Y)Z\right)^\top = \ell_\alpha(Y, Z)S_\alpha(X) - \ell_\alpha(X, Z)S_\alpha(Y).$$

Thus, we also have

$$\left\langle R^{\mathcal{M}}(X,Y)Z,W\right\rangle - \left\langle R^{\mathcal{N}}(X,Y)Z,W\right\rangle = \ell_\alpha(Y,Z)\ell_\alpha(X,W) - \ell_\alpha(X,Z)\ell_\alpha(Y,W).$$

**Proof.** We can extend $X, Y, Z, W$, ad $\nu, \ldots, \nu_k$ to vector fields inn $T_{\mathcal{M}}$ and $T\mathcal{M}^\perp$, respectively. Let $\nu_\alpha$ be orthonormal, then

$$\nabla^{\mathcal{N}}_Y Z = (\nabla^{\mathcal{N}}_Y Z)^\top = (\nabla^{\mathcal{N}}_X Z)^\perp = \nabla^{\mathcal{M}}_Y Z + \left\langle \nu_\alpha, \nabla^{\mathcal{N}}_Y Z\right\rangle \nu_\alpha$$

as $\nu_\alpha$ form orthonormal basis of $T\mathcal{M}^\perp$. Hence,

$$\nabla^{\mathcal{N}}_X \nabla^{\mathcal{N}}_Y Z = \nabla^{\mathcal{N}}_X \nabla^{\mathcal{M}}_Y Z + X(\left\langle \nu_\alpha, \nabla^{\mathcal{N}}_Y Z\right\rangle)\nu_\alpha + \left\langle \nu_\alpha, \nabla^{\mathcal{N}}_Y Z\right\rangle \nabla^{\mathcal{N}}_X \nu_\alpha.$$

Then,

$$(\nabla^{\mathcal{N}}_X \nabla^{\mathcal{N}}_Y Z)^\top = \nabla^{\mathcal{M}}_X \nabla^{\mathcal{M}}_Y Z + \underbrace{\left\langle \nu_\alpha, \nabla^{\mathcal{N}}_Y Z\right\rangle}_{-\ell_\alpha(Y,Z)} \underbrace{(\nabla^{\mathcal{N}}_X \nu_\alpha)^\top}_{S_\alpha(X)} = \nabla^{\mathcal{M}}_X \nabla^{\mathcal{M}}_Y Z - \ell_\alpha(Y,Z)S_\alpha(X).$$

Analogously, we have

$$(\nabla^{\mathcal{N}}_Y \nabla^{\mathcal{N}}_X Z)^\top = \nabla^{\mathcal{M}}_Y \nabla^{\mathcal{M}}_X Z - \ell_\alpha(X,Z)S_\alpha(Y),$$

and also, we have

$$(\nabla^{\mathcal{N}}_{[X,Y]} Z)^\top = \nabla^{\mathcal{M}}_{[X,Y]} Z.$$

By collecting terms, we have

$$\begin{aligned}
&(\nabla^{\mathcal{N}}_X \nabla^{\mathcal{N}}_Y Z)^\top - (\nabla^{\mathcal{N}}_Y \nabla^{\mathcal{N}}_X Z)^\top - (\nabla^{\mathcal{N}}_{[X,Y]} Z)^\top \\
&\qquad = \nabla^{\mathcal{M}}_X \nabla^{\mathcal{M}}_Y Z - \nabla^{\mathcal{M}}_Y \nabla^{\mathcal{M}}_X Z - \nabla^{\mathcal{M}}_{[X,Y]} Z - \ell_\alpha(Y,Z)S_\alpha(X) + \ell_\alpha(X,Z)S_\alpha(Y),
\end{aligned}$$

equivalently,

$$R^{\mathcal{M}}(X,Y)Z - (R^{\mathcal{N}}(X,Y)Z)^\top = \ell_\alpha(Y,Z)S_\alpha(X) - \ell_\alpha(X,Z)S_\alpha(Y).$$

■

Theorem 4.2.3 tells us that for a surface $\mathcal{M}$ in $\mathbb{R}^3$, the Gauss-Kronecker curvature coincides with the Riemannian curvature of $\mathcal{M}$, which is independent of the embedding. Therefore, Gauss-Kronecker curvature does not depend on embeddings of $\mathcal{M}$ into $\mathbb{R}^3$.

**Remark** (Codazzi equations)**.** Let $\mathcal{M}^m \subseteq \mathcal{N}^{m+1}$ where $N$ is unit normal on $\mathcal{M}$

$$\langle R(X,Y)e_j, N\rangle = (\nabla^{\mathcal{M}}_X \ell)(Y, e_j) - (\nabla^{\mathcal{M}}_Y \ell)(X, e_j) = XrkY^i \nabla^{\mathcal{M}}_k \ell_{ij} - Y^k X^i \nabla^{\mathcal{M}}_k \ell_{ij},$$

i.e., $\langle R(X,Y)Z, N\rangle = (\nabla^{\mathcal{M}}_X \ell)(Y,Z) - (\nabla^{\mathcal{M}}_Y \ell)(X,Z)$.

# Appendix

# Appendix A

# Additional Notes

## A.1 Christoffel Symbols

See this

In this section, we dive deep into the notion of the Christoffel symbols $\Gamma$ in various ways.

In particular, we will see that $\Gamma$ are really just the corrections to an ordinary derivative on a "curved" manifold w.r.t. the Levi-Civita connection, i.e., in the context of torsion free and Riemannian connection $\nabla$, we have also defined the so-called connection coefficients, and we use the same notation $\Gamma$, and indeed they're the same.

### A.1.1 Geometric Interpretation

### A.1.2 Metric Interpretation

### A.1.3 A Visual Guide

## A.2 Tensor Calculus

## A.3 Algebra

This chapter will collect some notion about algebras which you might not be familiar with.

### A.3.1 Modules

**Definition A.3.1** (Left module)**.** Suppose $R$ is a ring with 1. A *left $R$-module $M$* consists of an Abelian group $(M, +)$ and n operation $\cdot\colon R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$,

(a) $r \cdot (x + y) = r \cdot x + r \cdot y$;

(b) $(r + s) \cdot x = r \cdot x + s \cdot x$;

(c) $(rs) \cdot x = r \cdot (s \cdot x)$;

(d) $1 \cdot x = x$.

**Note.** A *right $R$-module $M$* can also be defined similarly by consider $\cdot\colon M \times R \to M$.

**Definition A.3.2** (Module)**.** If $R$ is commutative, then the left and right $R$-module $M$ are the same, and we call $M$ a *module*.

**Intuition.** We're basically relaxing the notion of $\mathbb{F}$-vector field, but this time, the field $\mathbb{F}$ is replaced by a ring $R$.

**Remark.** The most noticeable difference between a module and a vector field is that a module usually don't have a basis.

### A.3.2 The $C^\infty(\mathcal{M})$-Module Viewpoint of Tensor Fields

The reason why we introduce the notion of module is because of the following: we can understand tensor-field better in the following way. Firstly, let's introduce the so-called tensor bundles.

**Definition A.3.3** (Tensor bundle)**.** A *tensor bundle* is a fiber bundle where the fiber is the product of any number of tangent spaces and/or cotangent spaces.

So in a tensor bundle, the fiber is a vector space and the tensor bundle is a special kind of vector bundle.[1] Then, recall how we introduce Definition 1.5.1:

**As previously seen.** A $(r, s)$-tensor field $T$ is just a section of a tensor bundle.

But there's actually a deeper explanation: observe that $\Gamma(T\mathcal{M}) = \{X : \text{vector fields on } \mathcal{M}\}$ is actually a $C^\infty(\mathcal{M})$-module:

**Claim.** $\Gamma(T\mathcal{M})$ carries a natural $C^\infty(\mathcal{M})$-module structure.

**Proof.** Firstly, observe that $C^\infty(\mathcal{M}) = \big((C^\infty(\mathcal{M}), +, \cdot)\big)$ is not a field but a ring.[a] Then, naturally, the $C^\infty(\mathcal{M})$-module $(\Gamma(T\mathcal{M}), \oplus, \odot)$ where

- $\oplus$: $(X \oplus \widetilde{X})(f) \coloneqq (Xf) + \widetilde{X}(f)$;
- $\odot$: $(g \odot X)(f) \coloneqq g \cdot X(f)$,

for $X, \widetilde{X} \in \Gamma(T\mathcal{M})$, $g, f \in C^\infty(\mathcal{M})$. ⊛

[a] Since given $f \in C^\infty(\mathcal{M})$, we might not have $f^{-1}$.

**Notation.** Notice that given a vector field $X \colon \mathcal{M} \to T\mathcal{M}$ with $p \mapsto X(p)$, we let

$$Xf \colon \mathcal{M} \to \mathbb{R}, \quad p \mapsto X(p)f.$$

This makes sense since we can't always do things globally, e.g., Hairy ball theorem. Specifically, we can't choose a basis $X_1, \dots, X_d \in \Gamma(T\mathcal{M})$ for our vector field globally as we already know. Similarly, we can define $\Gamma(T^*\mathcal{M})$, i.e., the set of "convector field"[2] is again a $C^\infty(\mathcal{M})$-module.

**Example.** Given $\omega \in \Gamma(T^*\mathcal{M})$ and $X \in \Gamma(T\mathcal{M})$, $\omega$ acts on $X$ to yield smooth functions by point-wise evaluation, i.e., we define

$$(\omega(X))\,(p) \coloneqq \omega(p)(X(p)).$$

Then, the action of $\omega$ on $X$ is a $C^\infty(\mathcal{M})$-linear map since

$$(\omega(fX))(p) = f(p)\omega(p)(X(p)) = (f\omega)(p)(X(p)) = (f\omega(X))(p)$$

for $f \in C^\infty(\mathcal{M})$. This suggests that we should not regard $\omega$ just as a section of $T^*\mathcal{M}$, but also a linear mapping of $X \in \Gamma(T\mathcal{M})$ into $C^\infty(\mathcal{M})$.

Then, in this view point, we have the following.

**Definition A.3.4** (Tensor field*)**.** A $(r, s)$-*tensor field* $T$ on a smooth manifold $\mathcal{M}$ is a $C^\infty(\mathcal{M})$

[1] There are vector bundles which are not tensor bundles.
[2] We won't define it formally, but it's defined similarly.

multilinear map

$$T\colon \underbrace{\Gamma(T^*\mathcal{M})\times\cdots\times\Gamma(T^*\mathcal{M})}_{r}\times\underbrace{\Gamma(T\mathcal{M})\times\ldots\Gamma(T\mathcal{M})}_{s}\to C^\infty(\mathcal{M}).$$

Comparing to Definition 2.6.13, this definition is more general!

**Example.** The linear connection $\nabla$ $(X,Y)\mapsto\nabla_X Y$ does not define a tensor field.

**Proof.** Since $\nabla$ is only $\mathbb{R}$-linear in $Y$. ⊛

# A.4 Lie Groups and Lie Algebra

## A.4.1 Lie Groups

Lie groups are an important topic to study for Riemannian geometry, hence we now introduce it.

**Definition A.4.1** (Lie group)**.** A *Lie group* is a group $G$ with a differentiable structure such that the mapping $G\times G\to G$ given by $(x,y)\to xy^{-1}$, $x,y\in G$, is differentiable.

**Definition** (Transformation)**.** Let $G$ be a Lie group.

**Definition A.4.2** (Left transformation)**.** The *translations from the left* $L_x\colon G\to G$ is defined as $L_x(y)=xy$.

**Definition A.4.3** (Right transformation)**.** The *translations from the right* $R_x\colon G\to G$ is defined as $R_x(y)=yx$.

**Remark.** Both $L_x$ and $R_x$ are diffeomorphisms.

In the following discussion, let $G$ be a Lie group. Turns out that $G$ admits some nice properties on left invariant vector fields.

**Definition** (Invariant of Riemannian metric)**.** Let $g$ be a Riemannian metric on $G$.

**Definition A.4.4** (Left invariant)**.** $g$ is *left invariant* if

$$\langle u,v\rangle_y=\langle \mathrm{d}(L_x)_y u,\mathrm{d}(L_x)_y v\rangle_{L_x(y)}$$

for all $x,y\in G$, $u,v\in T_yG$, i.e., $L_x$ is an isometry.

**Definition A.4.5** (Right invariant)**.** $g$ is *right invariant* if

$$\langle u,v\rangle_y=\langle \mathrm{d}(R_x)_y u,\mathrm{d}(R_x)_y v\rangle_{R_x(y)}$$

for all $x,y\in G$, $u,v\in T_yG$, i.e., $R_x$ is an isometry.

**Definition A.4.6** (Bi-invariant)**.** $g$ is *bi-invariant* if it's both right and left invariant.

**Definition** (Invariant of vector field)**.** Let $X$ be a vector field on $G$.

**Definition A.4.7** (Left invariant). $X$ is *left invariant* if $\mathrm{d}L_x X = X$ for all $x \in G$.

**Definition A.4.8** (Right invariant). $X$ is *right invariant* if $\mathrm{d}R_x X = X$ for all $x \in G$.

**Definition A.4.9** (Bi-invariant). $X$ is *bi-invariant* if it's both right and left invariant.

As we mentioned, the left invariant vector fields are completely determined by their values at a single point of $G$, which allows us to introduce an additional structure on the tangent space to the neutral element $e \in G$ in the following manner.

To each vector $X_e \in T_eG$, we associate the left invariant $X$ defined by

$$X_a \coloneqq \mathrm{d}L_a X_e, \quad a \in G.$$

### A.4.2 Lie Algebras

Let $X, Y$ be left invariant vector fields on $G$. Since for each $x \in G$ and for any differentiable function $f$ on $G$,

$$\mathrm{d}L_x[X,Y]f = [X,Y](f \circ L_x) = X(\mathrm{d}L_x Y)f - Y(\mathrm{d}L_x X)f = (XY - YX)f = [X,Y]f,$$

i.e., $[X,Y]$ is again a left invariant vector field if $X, Y$ are. Now, if $X_e, Y_e \in T_eG$, we put $[X_e, Y_e] = [X,Y]_e$.

**Definition A.4.10** (Lie algebra). Given a Lie group $G$, the *Lie algebra* $\mathfrak{g}$ is the vector space $T_eG$ with the bracket $[\cdot,\cdot]$.

**Note.** The elements in the Lie algebra $\mathfrak{g}$ will be thought of either as vectors in $T_eG$ or as left invariant vector fields on $G$.

To introduce a left invariant metric on $g$, take any arbitrary inner product $\langle \cdot,\cdot \rangle_e$ on $\mathfrak{g}$ and define

$$\langle u, v \rangle_x \coloneqq \langle (\mathrm{d}L_{x^{-1}})_x(u), (\mathrm{d}L_{x^{-1}})_x(v) \rangle_e \tag{A.1}$$

for $x \in G$, $u, v \in T_xG$. Since $L_x$ depends differentiably on $x$, this is actually a Riemannian metric, which is clearly left invariant.

**Remark.** We can also construct a right invariant metric on $G$, and if $G$ is compact, $G$ possesses a bi-invariant metric.

One important characterization for $G$ having a bi-invariant metric is that the inner product that the metric determines on $\mathfrak{g}$ satisfies the following relation.

**Proposition A.4.1.** If $G$ has a bi-invariant metric, then for any $U, V, X \in \mathfrak{g}$, the inner product that the metric determines on $\mathfrak{g}$ satisfies

$$\langle [U,X], V \rangle = - \langle U, [V,X] \rangle .$$

**Proof.** See do Carmo [FC13, Page 40, 41]. ∎

The important point about this relation is that it characterizes the bi-invariant metrics of $G$ in the following sense.

**Remark.** If a positive bilinear form $\langle \cdot,\cdot \rangle_e$ defined on $\mathfrak{g}$ satisfies this relation, then the Riemannian metrics defined on $G$ by Equation A.1 is bi-invariant.

### A.4.3 Lie Subalgebra

Consider $(h_t^X)$ be a local 1-parameter group for a vector field $X$, and let $\Gamma(T\mathcal{M})$ still denotes the set of all vector fields, but now view it as just an $\mathbb{R}$-vector space. Then, we revise Definition A.4.10 as follows.

**Definition A.4.11** (Lie algebra*). Let $\mathcal{M}$ be a smooth manifold, the $(\Gamma(T\mathcal{M}), [\cdot,\cdot])$ is the *Lie algebra*.

This induces the following.

**Definition A.4.12** (Lie subalgebra). Let $X_1, \dots, X_n$ be $n$ vector fields on $\mathcal{M}$ such that for all $i, j$,

$$[X_i, X_j] = C_{ij}^k X_k$$

for $C_{ij}^k \in \mathbb{R}$. Then, $L \coloneqq (\operatorname{span}_{\mathbb{R}}(\{X_1, \dots, X_n\}), [\cdot,\cdot])$ is called a *Lie subalgebra*.

**Notation** (Structure constant). $C_{ij}^k$ in Definition A.4.12 are called *structure constants*.

**Example.** On $S^2$, given $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$, we have

$$(\operatorname{span}_{\mathbb{R}}(\{X_1, X_2, X_3\}), [\cdot,\cdot]) = \mathrm{so}(3).$$

**Definition A.4.13** (Symmetry). A finite-dimensional Lie subalgebra $(L, [\cdot,\cdot])$ is said to be a *symmetry* of a metric tensor field $g$ if for every $X \in L$ and $t \in \mathbb{R}$,

$$g\big((h_t^X)_*(A), (h_t^X)_*(B)\big) = g(A, B).$$

This means that $(h_t^X)_*$ defines an isometry.

**Note.** Or equivalently, $(h_t^X)^* g = g$ where for $\varphi\colon \mathcal{M} \to \mathcal{M}$,

$$(\varphi^* g)(X, Y) \coloneqq g(\varphi_p(X), \varphi_p(Y)).$$

### A.4.4 Lie Derivatives

Observe that for all $X \in L$ with the corresponding local 1-parameter group $(h_t^X)$, if

$$\mathcal{L}_X \coloneqq \lim_{t \to \infty} \frac{(h_t^X)^* g - g}{t} = 0,$$

then $L$ is a symmetry of $g$.

**Definition A.4.14** (Lie derivative). The *Lie derivative* $\mathcal{L}$ on a smooth manifold $\mathcal{M}$ sends a pair of a vector field $X$ and a $(p, q)$-tensor field to a $(p, q)$-tensor field such that

(a) $\mathcal{L}_X f = Xf$;

(b) $\mathcal{L}_X Y = [X, Y]$;

(c) $\mathcal{L}_X (T + S) = \mathcal{L}_X T + \mathcal{L}_X S$;

(d) $\mathcal{L}_X (T(\omega, Y)) = (\mathcal{L}_X T)(\omega, Y) + T(\mathcal{L}_X \omega, Y) + T(\omega, \mathcal{L}_X Y)$, similarly for any other valence of $T$;

(e) $\mathcal{L}_{X+Y} T = \mathcal{L}_X T + \mathcal{L}_Y T$.

**Remark.** $\nabla_X$ is $C^\infty(\mathcal{M})$-linear in the lower slot, while $\mathcal{L}_X$ is not.

**Intuition.** Study neighboring fibers using a local 1-parameter group of diffeomorphisms $(\psi_t)_{t \in I}$.

# Bibliography


[FC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013. ISBN: 9780817634902. URL: https://books.google.com/books?id=ct91XCWkWEUC.

[Sch15] Frederic P Schuller. *International Winter School on Gravity and Light 2015*. Youtube. 2015. URL: https://www.youtube.com/playlist?list=PLFeEvEPtX_0S6vxxiiNPrJbLu9aK1UVC_.