MATH592
Introduction to Algebraic Topology

Pingbang Hu

February 2, 2022

Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

Contents

Lecture 1: Homotopies of Maps

05 Jan. 10:00

1 Foundation of Algebraic Topology

1.1 Homotopy

Definition 1.1 (Homotopy). Let X, Y be topological spaces. Let f, $g\colon X \to Y$ continuous maps. Then a *homotopy* from f to g is a $1-$parameter family of maps that continuously deforms f to g, i.e., it's a continuous function $F\colon X \times I \to Y$, where $I = [0,1]$, such that

$$F(x,0) = f(x), \quad F(x,1) = g(x).$$

We often write $F_t(x)$ for $F(x,t)$.

If a homotopy exists between f and g, we say they are *homotopic* and write

$$f \simeq g.$$

If f is homotopic to a constant map, we call it *nullhomotopic*.

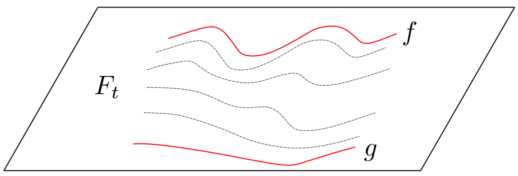


Figure 1: The continuous deforming from f to g described by F_t

Remark. Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

Example. We first see some examples.

1. Any two maps (continuous) with specification

   $$f,\ g\colon X \to \mathbb{R}^n$$

   are homotopic by considering

   $$F_t(x) = (1-t)f(x) + tg(x).$$

   We call it *the straight line homotopy.*

2. Let S^1 denotes the unit circle in $\mathbb{R}^2$, and D^2 denotes the unit disk in $\mathbb{R}^2$. Then the inclusion $f\colon S^1 \hookrightarrow D^2$ is nullhomotopic by considering

   $$F_t(x) = (1-t)f(x)(+t\cdot 0).$$

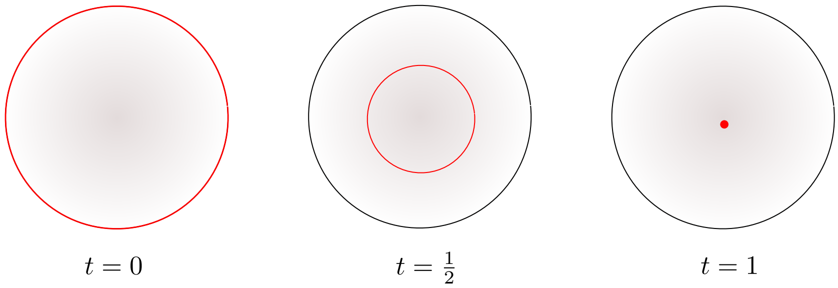


Figure 2: The illustration of $F_t(x)$

We see that there is a homotopy from $f(x)$ to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a nullhomotopy.

3. The maps

$$\begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not** homotopy.

Remark. It will essentially **flip** the orientation, hence we can't deform one to another continuously.

Exercise. We first see some exercises.

1. A subset $S \subseteq \mathbb{R}^n$ is star-shaped if

$$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from x_0 to x lies in S.

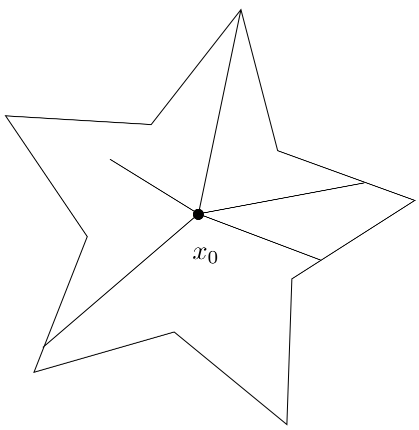


Figure 3: Star-shaped illustration

Show that $\mathrm{id}\colon S \to S$ is nullhomotopic.

Answer. Consider

$$F_t(x) \coloneqq (1-t)x + tx_0,$$

which essentially just concentrates all points x to x_0.

2. Suppose

$$X \underset{f_0}{\overset{f_1}{\longrightarrow}} Y \underset{g_0}{\overset{g_1}{\longrightarrow}} Z \ .$$

where

$$f_0 \underset{F_t}{\simeq} f_1, \quad g_0 \underset{G_t}{\simeq} g_1.$$

Show

$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

Answer. Consider $I \times X \to Z$. Then

$$\begin{array}{ccccc} X \times I & \to & Y \times I & \to & Z \\ (x, t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

Remark. Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** homotopic?

Answer.

Lecture 2: Homotopy Equivalence

07 Jan. 10:00

As previously seen. Two maps $f, g\colon X \to Y$ is homotopy if there exists a map

$$F_t(x)\colon X \times I \to Y$$

with the properties

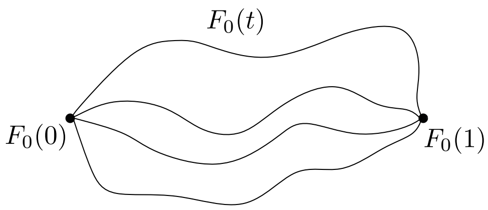
1. Continuous
2. $F_0(x) = f(x)$
3. $F_1(x) = g(x)$

Remark. The continuity of F_t is an even stronger condition for the continuity of F_t for a fixed t.

We now introduce another concept.

Definition 1.2 (Homotopy relative). Given two spaces X, Y, and let $B \subseteq X$. Then a homotopy $F_t(x)\colon X \to Y$ is called *homotopy relative* B (denotes $\mathrm{rel} B$) if $F_t(b)$ is independent of t for all b.

Example. Let $X = [0,1]$ and $B = \{0,1\}$. Then the homotopy of paths from $[0,1] \to X$ is $\text{rel}\{0,1\}$.



1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

Definition 1.3 (Homotopy Equivalence). A map $f\colon X \to Y$ is a *homotopy equivalence* if $\exists g\colon Y \to X$ such that

$$f \circ g \simeq \text{id}_Y, \quad g \circ f \simeq \text{id}_X.$$

We say that X, Y are *homotopy equivalent*, and g is called *homotopy inverse* of f.

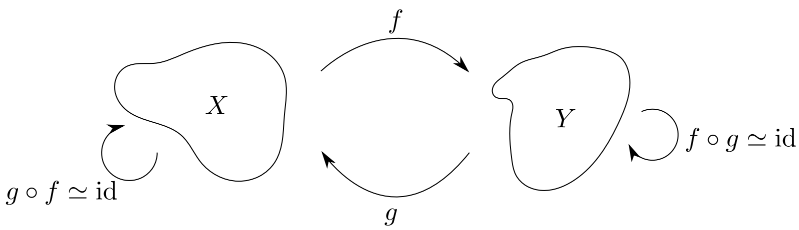
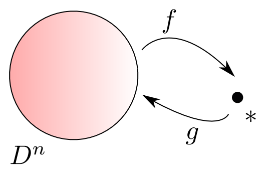


Figure 4: Homotopy Equivalence

If X, Y are called *homotopy equivalent*, then we say that they have the same *homotopy type*.

Notation. We denote a closed n-disk as D^n.

Example. D^n is homotopy equivalent to a point.



We see that $f \circ g = \mathrm{id}_*$ and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is homotopic to id_{D^n} by straight-line homotopy $F_t(x) = tx$.

Note. We say that a space is *contractible* if H is homotopy equivalent to a point.

Before doing exercises, we introduce two new concepts.

Definition 1.4 (Retraction). Given $B \subseteq X$, a *retraction* from X to B is a map $f\colon X \to X$ (or $X \to B$) such that $\forall b \in B$ $f(b) = b$, namely $r|_B = \mathrm{id}_B$. Or one can see this from

$$\begin{tikzcd} B \arrow[r, hook, "i"] \arrow[rr, bend right, "r\circ i"'] & X \arrow[r, "r"] & B \end{tikzcd}$$

where r is a retraction if and only if $r \circ i = \mathrm{id}_B$, where i is an inclusion identity. If r exists, B is a retract of X.

Definition 1.5 (Deformation retraction). Given X and $B \subseteq X$, a *(strong) deformation retraction* $F_t\colon X \to X$ onto B is a homotopy rel$B$ from the id_X to a *retraction* from X to B. i.e.,

$$\begin{aligned} F_0(x) &= x && \forall x \in X \\ F_1(x) &\in B && \forall x \in X \\ F_t(b) &= b && \forall t\ \forall b \in B. \end{aligned}$$

Exercise. We now see some problems.

1. Let $X \simeq Y$. Show X is path-connected if and only if Y is.

   Answer. Suppose X is path-connected. Then we see that given two points x_1 and x_2 in X, there exists a path $\gamma(t)$ with

   $$\gamma\colon [0,1] \to X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Since $X \simeq Y$, then there exists a pair of f and g such that $f\colon X \to Y$ and $g\colon Y \to X$ with
$$f \circ g \underset{F}{\simeq} \mathrm{id}_Y, \quad g \circ f \underset{G}{\simeq} \mathrm{id}_X.$$
(Notice the abuse of notation)

For any two y_1 and $y_2 \in Y$, we want to construct a path $\gamma'(t)$ such that
$$\gamma'\colon [0,1] \to Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

Firstly, we let $g(y_1) =: x_1$ and $g(y_2) =: x_2$. From the argument above, we know there exists such a γ starting at $x_1 = g(y_1)$ ending at $x_2 = g(y_2)$. Now, consider $f(\gamma(t)) = (f \circ \gamma)(t)$ such that
$$f \circ \gamma\colon I \to Y, \quad f \circ \gamma(0) = y_1', \quad f \circ \gamma(1) = y_2',$$
we immediately see that y_1' and y_2' is path connected. Now, we claim that y_1 and y_1' are path connected in Y, hence so are y_2 and y_2'. To see this, note that
$$f \circ g \underset{F}{\simeq} \mathrm{id}_Y,$$
which means that there exists $F\colon Y \times I \to Y$ such that
$$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y_1' \\ F(y_1, 1) = \mathrm{id}_Y(y_1) = y_1. \end{cases}$$

Since F is continuous in I, we see that there must exist a path connects y_1 and y_1'. The same argument applies to y_2 and y_2'. Now, we see that the path
$$y_1 \to y_1' \to y_2' \to y_2$$
is a path in Y for any two y_1 and y_2, which shows Y is path-connected.

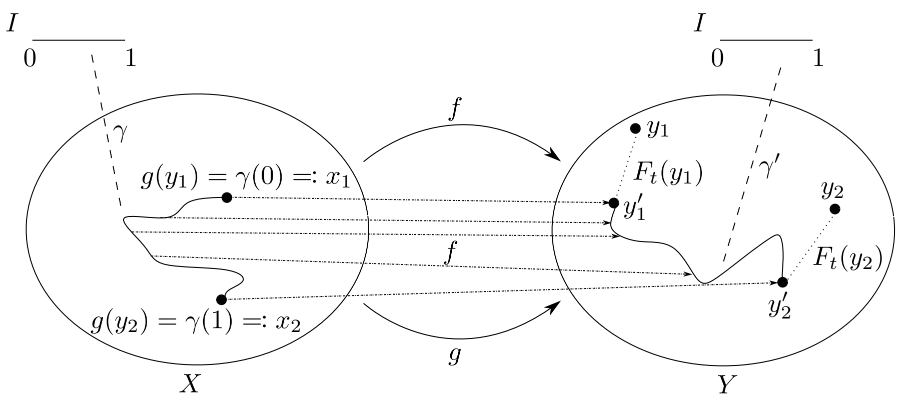


Figure 5: Demonstration of the proof

Challenge: One can further show that the connectedness is also preserved by any homotopy equivalence.

2. Show that if there exists deformation retraction from X to $B \subseteq X$, then $X \simeq B$.

Lecture 3: Deformation Retraction

10 Jan. 10:00

As previously seen. A *deformation retraction* is a homotopy of maps rel$B$ $X \to X$ from id_X to a retraction from X to B. Then B is a *deformation retract*.

Example. We can also show

1. S^1 is a deformation retraction of $D^2 \setminus \{0\}$. Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$

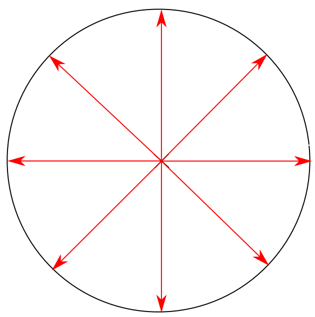


Figure 6: The deformation retraction of $D^{2 \setminus \{0\}}$ is just to *enlarge* that hold and push all the interior of D^2 to the boundary, which is S^1

2. $\mathbb{R}^n$ deformation retracts to 0. Indeed, since

$$F_t(x). = (1-t)x.$$

This implies that $\mathbb{R}^n \simeq *$, hence we see that

- dimension
- compactness
- etc.

are <u>not</u> homotopy invariants.

3. S^1 is a deformation retract of a cylinder and a Möbius band.

For a cylinder, consider $X \times I \to X$. Define homotopy on a closed rectangle, then verify it induces map on quotient.

For a Möbius band, we define a homotopy on a closed rectangle, then verify that it respect the equivalence relation.

Finally, we use the universal property of quotient topology to argue that we get a homotopy on Möbius band.

Upshot: Möbius band $\simeq S^1 \simeq$ cylinder, hence the orientability is <u>not</u> homotopy invariant.

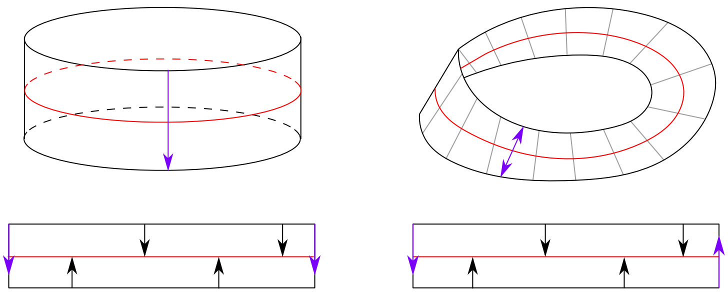


Figure 7: The deformation retraction for Cylinder and Möbius band

Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

As previously seen. We saw that

- homotopy equivalence
- homotopy invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

1.3 CW Complexes

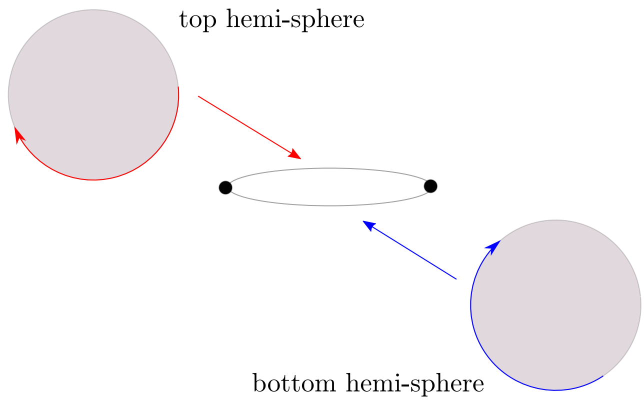
Example. Let's start with a few examples.

1. Constructing spheres:
   - S^1 (up to homeomorphism)

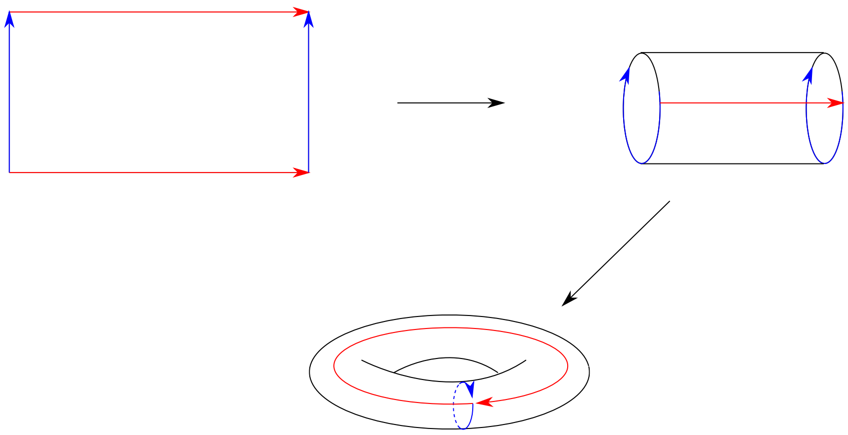
- S^2
  - glue boundary of 2-disk to a point
  - glue 2 disks onto a circle

Figure 8: **Left**: Glue a 2-disk to a point along its boundary. **Right**: Glue 2 disks to S^1.

The gluing instruction to construct S^2 in the right-hand side can be demonstrated as follows.



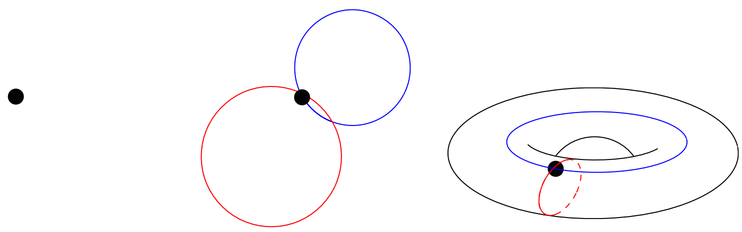
- $T = S^1 \times S^1$



view as gluing instructions

$$\text{vertex } + 2 \text{ edges } + 2\text{-disks}.$$

Specifically, we have



0-skeleton 1-skeleton 2-skeleton

Glue in edges Glue in disk

Formally, we have the following definition.

Notation. Let D^n denotes a closed n-disk (or n-ball)

$$D^n \simeq \{x \in \mathbb{R}^n :\ \|x\| \leq 1\}.$$

And let S^n denotes an n-sphere

$$S^n \simeq \left\{x \in \mathbb{R}^{n+1} :\ \|x\| = 1\right\}.$$

Lastly, we call a point as a *0-cell*, and the interior of D^n $\operatorname{int}(D^n)$ for $n \geq 1$ as a *n-cell*.

Definition 1.6 (CW Complex). A *CW Complex* is a topological space constructed inductively as

1. X^0 (the 0-skeleton) is a set of discrete points.
2. We inductively construct the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α, where α is the index.

   The gluing instructions glued by an attaching map is that $\forall\alpha$, $\exists$ continuous map φ_α

   $$\varphi_\alpha \colon \partial D^n_\alpha \to X^{n-1},$$

   then

   $$X^n = \left(X^{n-1} \coprod_\alpha D^n_\alpha \right) \Big/ x \sim \varphi_\alpha(x)$$

   with identification $x \sim \varphi_\alpha(x)$ for all $x \in \partial D^n_\alpha$ with quotient topology.
3. $$X = \bigcup_{n=0} X^n,$$

   and let $\overline{w}$ denotes weak topology. Then

   $$u \subseteq X \text{ is open } \iff \forall n\ u \cap X^n \text{ is open }.$$

   If all cells have dimension less than N and a $\exists N$-cell, then $X = X^N$ and we call it N-dim CW complex.

Remark. We write $X^{(n)}$ for n-skeleton if we need to distinguish from the Cartesian product.

Example. Let's look at some examples.

1. 0-dim CW complex is a discrete space.
2. 1-dim CW complex is a graph.
3. A CW complex X is finite if it has finitely many cells.

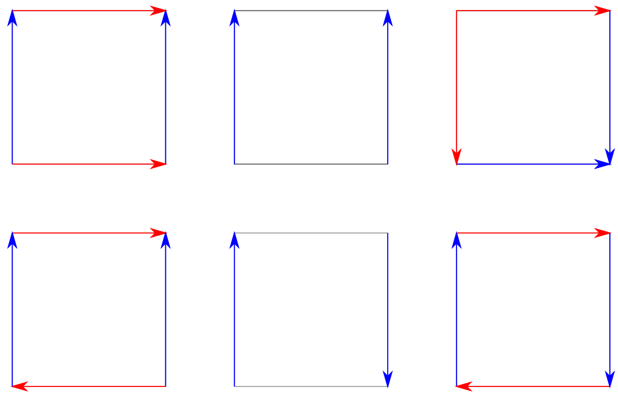
Definition 1.7 (CW subcomplex). A *CW subcomplex* $A \subseteq X$ is a closed subset equal to a union of cells

$$e^n_\alpha = \operatorname{int}(D^n_\alpha).$$

Remark. This inherits a CW complex structure.

Check the images of attaching maps.

Exercise. Given the following gluing instruction:



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere, $\mathbb{R}P$.

Answer. We see that

1. Torus	2. Cylinder	3. 2-sphere
4. Klein bottle	5. Möbius band	6. $\mathbb{R}P$

Notation. We call the real projection space as $\mathbb{R}P$, and we also have so-called complex projection space, denote as $\mathbb{C}P$.

Lecture 5: Operation on Spaces

14 Jan. 10:00

1.4 Operations on CW Complexes

1.4.1 Products

We can consider the product of two CW complexes given by a CW complex structure. Namely, given X and Y two CW complexes, we can take two cells e_α^n from X and e_β^m from Y and form the product space $e_\alpha^n \times e_\beta^m$, which is homeomorphic to an $n+m$-cell. We then take these products as the cells for $X \times Y$.

Specifically, given X, Y are CW complexes, then $X \times Y$ has a cell structure

$$\{e_\alpha^m \times e_\alpha^n \colon e_\alpha^m \text{ is a } m\text{-cell on } X, e_\alpha^n \text{ is an } n\text{-cell on } Y\}.$$

Remark. The product topology may not agree with the weak topology on the $X \times Y$. However, they do agree if X or Y is locally compact <u>or</u> if X and Y both have at most countably many cells.

Note. Notice that if the product is wild enough, then the product topology may not agree with the weak topology.

1.4.2 Wedge Sum

Given X, Y are CW complexes, and $x_0 \in X^0$, $y_0 \in Y^0$ (only points). Then we define

$$X \vee Y = X \coprod Y$$

with quotient topology.

Remark. $X \vee Y$ is a CW complex.

1.4.3 Quotients

Let X be a CW complex, and $A \subseteq X$ subcomplex (closed union of cells), then

$$X \big/ A$$

is a quotient space collapse A to one point and inherits a CW complex structure.

Remark. $X \,/\, A$ is a CW complex.

0-skeleton

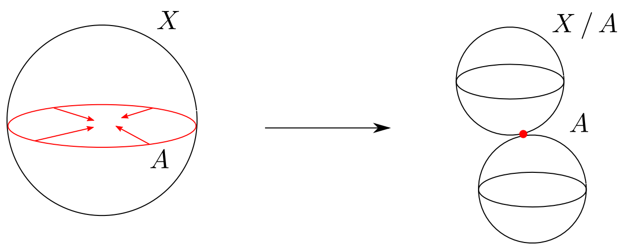
$$(X^0 - A^0) \coprod *$$

where $*$ is a point for A. Each cell of $X - A$ is attached to $(X \,/\, A)^n$ by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n \,/\, A^n$$

Example. Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a wedge of two spheres.



2. We can take the torus and squish down a ring around the hole.

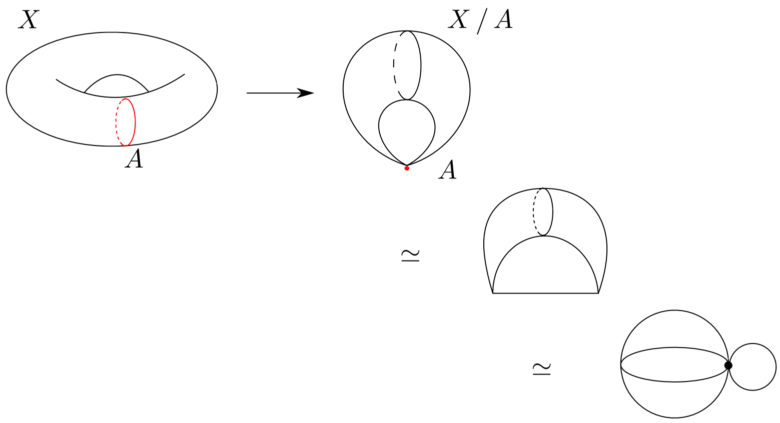


Figure 9: We see that $X\,/\,A$ is homotopy equivalent to a 2-sphere wedged with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other point, forming a circle.

Lecture 6: A Foray into Category Theory

19 Jan. 10:00

1.5 Category Theory

We start with a definition.

Definition 1.8 (Object, Morphism). A category $\mathscr{C}$ is 3 pieces of data

- A class of objects $\mathrm{Ob}(\mathscr{C})$
- $\forall X, Y \in \mathrm{Ob}(\mathscr{C})$ a class of morphisms or arrows, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- $\forall X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, there exists a composition law

$$\begin{aligned}\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) &\to \mathrm{Hom}(X, Z)\\ (f, g) &\mapsto g \circ f\end{aligned}$$

and 2 axioms

- Associativity. $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms f, g, h where composites are defined.
- Identity. $\forall X \in \mathrm{Ob}(\mathscr{C})\ \exists \mathrm{id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g$$

for all f, g where this makes sense.

Let's see some examples.

Example. We introduce some common category.

$\mathcal{C}$	Ob($\mathcal{C}$)	Mor($\mathcal{C}$)
$\underline{\text{set}}$	Sets X	All maps of sets
$\underline{\text{fset}}$	Finite sets	All maps
$\underline{\text{Gp}}$	Groups	Group Homomorphisms
$\underline{\text{Ab}}$	Abelian groups	Group Homomorphisms
$\underline{k-\text{vect}}$	Vector spaces over k	k-linear maps
$\underline{\text{Rng}}$	Rings	Ring Homomorphisms
$\underline{\text{Top}}$	Topological spaces	Continuous maps
$\underline{\text{Haus}}$	Hausdorff Spaces	Continuous maps
$\underline{\text{hTop}}$	Topological spaces	Homotopy classes of continuous maps
$\underline{\text{Top}^*}$	Based topological spaces[1]	Based maps[2]

Remark. Any **diagram** plus composition law.

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[r] & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd}.$$

Definition 1.9 (monic, epic). A morphism $f\colon M \to N$ is *monic* if

$$\forall g_1, g_2 \; f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$\begin{tikzcd} A \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

Dually, f is *epic* if

$$\forall g_1, g_2 \; g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & B \end{tikzcd}$$

Lemma 1.1. In $\underline{\text{set}}$, $\underline{\text{Ab}}$, $\underline{\text{Top}}$, $\underline{\text{Gp}}$, a map is monic if and only if f is injective, and epic if and only if f is surjective.

Proof. In $\underline{\text{set}}$, we prove that f is monic if and only if f is injective. Suppose $f \circ g_1 = f \circ g_2$ and f is injective, then for any a,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence $g_1 = g_2$.

[1] Topological spaces with a distinguished base point $x_0 \in X$

[2] Continuous maps that presence base point $f\colon (x, x_0) \to (y, y_0)$ such that

$$f\colon X \to Y, \quad f(x_0) = y_0$$

is continuous.

Now we prove another direction, with contrapositive. Namely, we assume that f is not injective and show that f is not monic. Suppose $f(a) = f(b)$ and $a \neq b$, we want to show such g_i exists. This is easy by considering

$$g_1 \colon * \mapsto a, \quad g_2 \colon * \mapsto b.$$

■

1.5.1 Functor

After introducing the category, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

Definition 1.10 (Functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A (covariant) *functor*

$$F \colon \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F \colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)) \\ [f \colon X \to Y] &\mapsto [F(f) \colon F(X) \to F(Y)] \end{aligned}$$

such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
- $F(f \circ g) = F(f) \circ F(g)$

Lecture 7: Functors

21 Jan. 10:00

As previously seen. Assume that we initially have a commutative diagram in $\mathscr{C}$ as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

After applying F, we'll have

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "F(g \circ f) = F(g) \circ F(f)"'] & F(Y) \arrow[d, "F(g)"] \\ & F(Z) \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

We can also have a so-called contravariant functor.

Definition 1.11 (Contravariant functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A contravariant *functor*

$$F : \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F \colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X)) \\ [f \colon X \to Y] &\mapsto [F(f) \colon F(Y) \to F(X)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a contravariant functor F, the diagram becomes

$$\begin{tikzcd} F(X) & F(Y) \arrow[l, "F(f)"'] \\ & F(Z) \arrow[u, "F(g)"'] \arrow[ul, "F(g\circ f)=F(f)\circ F(g)"] \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

Example. Let see some examples.

1. Identity functor.

$$I : \mathscr{C} \to \mathscr{C}.$$

2. Forgetful functors.

   -

$$\begin{aligned} F \colon \underline{\mathrm{Gp}} &\to \underline{\mathrm{set}} \\ G &\mapsto G^{3} \\ [f \colon G \to H] &\mapsto [f \colon G \to H] \end{aligned}$$

   -

$$\begin{aligned} F \colon \underline{\mathrm{Top}} &\to \underline{\mathrm{set}} \\ X &\mapsto X^{4} \\ [f \colon X \to Y] &\mapsto [f \colon X \to Y] \end{aligned}$$

[3] G is now just the underlying set of the group G.

3. Free functors.

$$\begin{aligned} \underline{\text{set}} &\to \underline{k\text{−vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

i.e., vector space with basis s

$$[f\colon A \to B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

4.

$$\begin{aligned} \underline{k\text{−vect}} &\to \underline{k\text{−vect}} \\ V &\mapsto V^* = \mathrm{Hom}_k(V, k) \end{aligned}$$

If we are working on a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functors.

1.

$$\begin{aligned} \underline{\text{Top}^*} &\to \underline{\text{Gp}} \\ (X, x_0) &\mapsto \Pi_1(X, x_0) \end{aligned}$$

where Π_1 is so-called *fundamental group.*

2.

$$\begin{aligned} \underline{\text{Top}} &\to \underline{\text{Ab}} \\ X &\mapsto \text{Hp}(X) \end{aligned}$$

where Hp is so-called p^{th} *homology.*

Let see the formal definition.

1.6 Free Groups

Definition 1.12 (Free group). Given a set S, the *free group* is a group F_S on S with a map $S \to F_S$ satisfying the universal property.

If G is any group, $f\colon \underline{S} \to G$ is any map of sets, f extends uniquely to group homomorphism $\overline{f}\colon F_S \to G$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\overline{f}\colon \text{gp hom}"] \\ & G \end{tikzcd}$$

[4] X is now just the underlying set of the topological space X.

Note. This defines a *natural bijection*

$$\mathrm{Hom}_{\underline{\mathrm{set}}}(S, \mathscr{U}(G)) \cong \mathrm{Hom}_{\underline{\mathrm{Gp}}}(F_S, G),$$

where $\mathscr{U}(G)$ is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are adjoint; specifically that the free functor is the left adjoint (appears on the left in the Hom's above).

> **Definition 1.13 (Adjoints functor).** A free and forgetful functors are *adjoints.*

Remark. Whenever we state a universal property for an object (plus a map), an object (plus a map) may or may not exist. If such object exists, then it defines the object **uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the object (plus a map).

> **Lemma 1.2.** Universal property defines F_S (plus a map $S \to F(S)$) uniquely up to unique isomorphism.

Proof. Fix S. Suppose

$$S \to F_S, \quad S \to \widetilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, dashed, "\exists!\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

We'll show φ and ψ are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{tikzcd} & F_S \arrow[dd, "\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[dd, "\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & \widetilde{F}_S \end{tikzcd}$$

Hence, we see that

$$\begin{tikzcd} & F_S \arrow[d, "\psi"] \arrow[dd, bend left, "\varphi\circ\psi=\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, "\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[d, "\varphi"] \arrow[dd, bend left, "\psi\circ\varphi=\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, "\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \mathrm{id}_{F_S}, \qquad \psi \circ \varphi = \mathrm{id}_{\widetilde{F}_S},$$

so φ and ψ are inverses (thus group isomorphism). ■

Lecture 8: The Fundamental Group π_1

24 Jan. 10:00

Example. In category <u>Ab</u> free Abelian group on a set S is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on S** .

1.6.1 Constructing the Free Groups F_S

Proposition 1.1. The free group defined by the universal property exists.

Proof. We'll just give a construction below. First, we see the definition.

Definition 1.14. Fix a set S, and we define a <u>word</u> as a finite sequence (possibly $\varnothing$) in the formal symbols

$$\left\{s, s^{-1} \mid s \in S\right\}.$$

Then we see that elements in F_S are equivalence classes of words with the equivalence relation being

- delete ss^{-1} or $s^{-1}s$. i.e.,

$$\begin{aligned} vs^{-1}sw &\sim vw \\ vss^{-1}w &\sim vw \end{aligned}$$

  for every word $v, w, s \in S$,

with the group operation being concatenation. ■

Example. Given words ab^{-1}, bba, their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

Exercise. There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that F_S satisfies the universal property with respect to the map

$$S \to F_S, \quad s \mapsto s.$$

2 The Fundamental Group

2.1 Definition

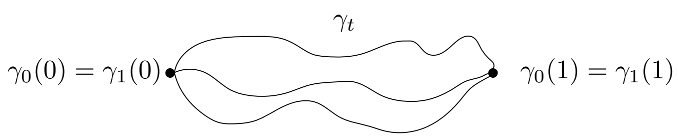
We start with the definition.

Definition 2.1 (Path). A *path* in a space X is a continuous map

$$\gamma\colon I \to X$$

where $I = [0,1]$.

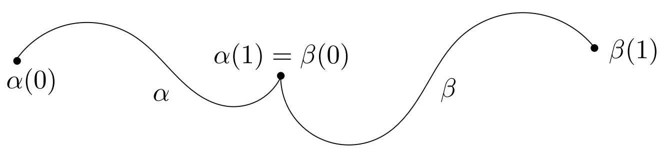
Definition 2.2 (Homotopy path). A *homotopy of paths* γ_0, γ_1 is a homotopy from γ_0 to γ_1 rel$\{0,1\}$.



Example. Fix $x_1, x_0 \in X$, then $\underline{\exists \text{ homotopy of paths}}$ is an equivalence relation on paths from x_0 to x_1 (i.e., γ with $\gamma(0) = x_0, \gamma(1) = x_1$).

Definition 2.3 (Path composition). For paths α, β in X with $\alpha(1) = \beta(0)$, the *composition*[a] $\alpha \cdot \beta$ is

$$(\alpha \cdot \beta)(t) \coloneqq \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



[a] Also named *product*, *concatenation*.

Remark. By the pasting lemma, this is continuous, hence $\alpha \cdot \beta$ is actually a path from $\alpha(0)$ to $\beta(1)$.

Definition 2.4 (Reparameterization). Let $\gamma\colon I \to X$ be a path, then a *reparameterization* of γ is a path

$$\gamma'\colon I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where φ is continuous and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

Exercise. A path γ is homotopic rel$\{0,1\}$ to all of its reparameterizations.

Proof. We show that γ and $\gamma \circ \phi$ are homotopic rel$\{0,1\}$ by showing that there exists a continuous F_t such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since ϕ is continuous, so we define

$$F_t(x) = (1-t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all $x, t \in I$.

Now, we check that F_t really gives us a homotopic rel$\{0,1\}$. We have

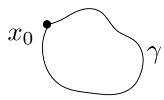
$$\begin{aligned} F_t(0) &= (1-t)\gamma(0) + t \cdot \gamma \circ \phi(0) = (1-t)\gamma(0) + t \cdot \gamma(\underbrace{\phi(0)}_{0}) = \gamma(0), \\ F_t(1) &= (1-t)\gamma(1) + t \cdot \gamma \circ \phi(1) = (1-t)\gamma(1) + t \cdot \gamma(\underbrace{\phi(1)}_{1}) = \gamma(1), \end{aligned}$$

which shows that 0 and 1 are independent of t, hence γ and $\gamma \circ \phi$ are homotopic rel$\{0,1\}$. ∎

Exercise. Fix $x_1, x_1 \in X$. Then Homotopy of paths (relative $\{0,1\}$) is an equivalence relation on paths from x_0 to x_1.

Definition 2.5 (Fundamental Group). Let X denotes the space and let $x_0 \in X$ be the base point. The *fundamental group of X based at x_0*, denoted by $\pi_1(X, x_0)$, is a group such that

- Elements: Homotopy classes rel$\{0,1\}$ of paths $[\gamma]$ where γ is a **loop** with $\gamma(0) = \gamma(1) = x_0$[a]

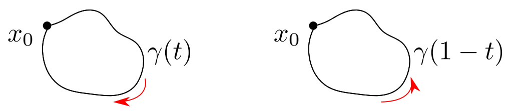


- Operation: Composition of paths.
- Identity: Constant loop γ based at x_0 such that

$$\gamma \colon I \to X, \quad t \mapsto x_0$$

- Inverses: The inverse $[\gamma]^{-1}$ of $[\gamma]$ is represented by the loop $\overline{\gamma}$ such that

$$\overline{\gamma}(t) = \gamma(1-t).$$



[a]We say γ is **based** at x_0.

Proof. We prove that

- Associativity: $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$. We break this down into

$$\gamma_1{\cdot}(\gamma_2{\cdot}\gamma_3)(t) = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t-2), & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t-3), & \text{if } t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1{\cdot}\gamma_2){\cdot}\gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t-1), & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, we define $\phi\colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \frac{1}{2}\right], & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ t + \frac{1}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right], & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{t+1}{2} \in \left[\frac{3}{4}, 1\right], & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$, which implies that the associativity holds.

- Identity: We want to show that $[\gamma \cdot c] = [\gamma]$. Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, consider $\phi\colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t, & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ 1, & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$.

- Inverses: We want to show that $\gamma \cdot \overline{\gamma} \simeq c$, where $\overline{\gamma}(t) = \gamma(1-t)$. Firstly, we have

$$(\gamma \cdot \overline{\gamma})(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2t), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider F_t given by

$$F_t(x) = \begin{cases} \gamma(2xt), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2xt), & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

If $t = 0$, we have

$$F_0(x) = \begin{cases} \gamma(0), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0$$

for all $x \in I$, namely $F_0 = c$, while when $t = 1$, we have

$$F_1(x) = \begin{cases} \gamma(2x), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2x), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \overline{\gamma})(x),$$

and we see that F_t is continuous since at $x = \frac{1}{2}$, we have

$$\gamma(2x) = \gamma(1) = \overline{\gamma}(0) = \overline{\gamma}(1-2x),$$

hence we see that F_t is the homotopy between $\gamma \cdot \overline{\gamma}$ and c.

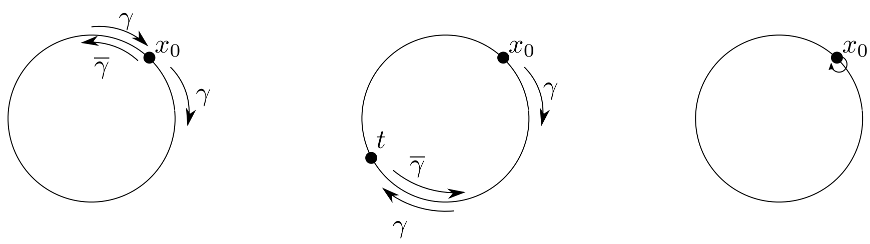


Figure 10: Illustration of F_t. Intuitively, the path $\gamma \cdot \overline{\gamma}$ is $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\overline{\gamma}} x_0$. But now, F_t is $x_0 \xrightarrow{\gamma} t \xrightarrow{\overline{\gamma}} x_0$. We can think of this homotopy is *pulling back* the turning point along the original path.

■

Theorem 2.1. If X is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Remark. We often write $\pi_1(X)$ up to isomorphism.

Proof. To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if $[h \cdot \gamma \cdot \overline{h}] = [h \cdot \gamma' \cdot \overline{h}]$, then since we know that $h^{-1} = \overline{h}$, hence in the fundamental group $\pi_1(X, x_0)$, we see that

$$\overline{h} \cdot h \cdot \gamma \cdot \overline{h} \cdot h = \overline{h} \cdot h \cdot \gamma' \cdot \overline{h} \cdot h. \implies \gamma = \gamma'$$

  as we desired.

- onto. We see that for every $\alpha \in \pi_1(X, x_0)$, there exists a $\gamma \in \pi_1(X, x_0)$ such that

$$\gamma = \overline{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)^5$$

  since $h \cdot \gamma \cdot \overline{h} = \alpha$.

[5] Notice that this is indeed the case, one can verify this by the fact that $h\colon x_0 \to x_1$ and $\overline{h}\colon x_1 \to x_0$.

We then see that the fundamental group of X does not depend on the choice of basepoint, only on the choice of the path component of the basepoint. If X is path-connected, it now makes sense to refer to *the* fundamental group of X and write $\pi_1(X)$ for the abstract group (up to isomorphism).

■

Exercise. Composition of paths is well-defined on homotopy classes rel$\{0, 1\}$.

Exercise. If X is a contractible space, then X is path connected and $\pi_1(X)$ is trivial.

Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

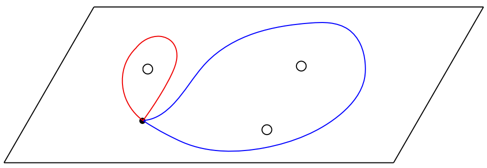


Figure 11: Fundamental Group is basically a *hole detector*!

2.2 Calculations with $\pi_1(S^n)$

Let's start with a simple theorem.

Theorem 2.2. $\pi_1(S^1) \cong \mathbb{Z}$, and this identification is given by the paths

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))].$$

Remark. Intuitively, this winds around S^1 n times. The key to this proof was to understand S^1 via the covering space $\mathbb{R} \to S^1$. We will talk about covering spaces more later.

Proof. ■ HW

Theorem 2.3. Given (X, x_0) and (Y, y_0), then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\begin{bmatrix} r \colon I \to X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{bmatrix} \mapsto (r_X, r_Y),$$

where γ is continuous $\iff$ f_X, f_Y are continuous.

Proof. Let $Z \xrightarrow{f} X \times Y$ with $z \xmapsto{f} (f_X(z), f_Y(z))$. Then we have

$$\text{continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply above to

- Paths $I \to X \times Y$.
- Homotopies of paths $I \times I \to X \times Y$.

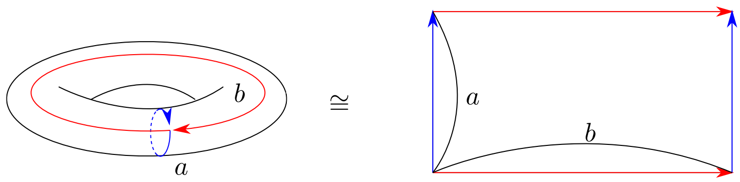
■

Corollary 2.1. The torus $T \cong S^1 \times S^1$ has fundamental group $\pi_1(T) \cong \mathbb{Z}^2$. Additionally, for a k-torus $\underbrace{S^1 \times S^1 \times \ldots \times S^1}_{k \text{ times}} = (S^1)^k$, the fundamental group is then $\mathbb{Z}^k$, i.e.

$$\pi_1\left((S^1)^k\right) \cong \mathbb{Z}^k.$$

Proof. Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

Remark. One way to think of the k-torus is as a k-dimensional cube with opposite $(k-1)$-dimensional faces identified by translation.

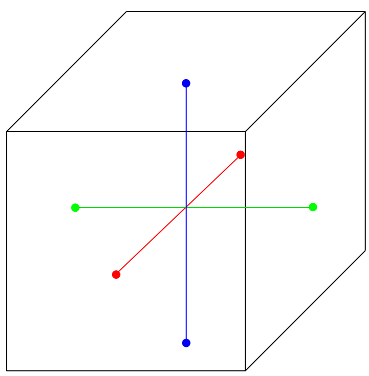


Figure 12: 3-torus with cube identified with parallel sides.

Example. We now see some examples.

1. $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2. $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong 0 \times \mathbb{Z} = \mathbb{Z}$ since
$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R},$$
which means that the generators are just loops around the hold intuitively.

Theorem 2.4. π_1 is a functor such that
$$\begin{aligned}\pi_1 \colon \underline{\mathrm{Top}_*} &\to \underline{\mathrm{Gp}}\\ (X, x_0) &\mapsto \pi_1(X, x_0).\end{aligned}$$
A map $f \colon X \to Y$ taking base point x_0 to y_0 induces a map
$$\begin{aligned}f_* \colon \pi_1(X, x_0) &\to \pi_1(Y, y_0)\\ [\gamma] &\mapsto [f \circ \gamma]\end{aligned}$$
i.e.,
$$[f \colon X \to Y] \mapsto [f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))]\,.$$

Notation. We usually write f_* if it's a covarant functor, while writing f^* if it's an contravariant functor.

Proof. We need to check

- well-defined on path homotopy classes.
- f_* is a group homomorphism.
$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \frac{1}{2}\right]\\ f(\beta(1-2s)), & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$
- $\left(\mathrm{id}_{(X,x_0)}\right)_* = \mathrm{id}_{\pi_1(X,x_0)}$
- $(f_* \circ g_*) = (f \circ g)_*$
$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \implies f_*(\gamma_*(\gamma)).$$

DIY

■

Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

The goal is to compute $\pi_1(X)$ where $X = A \cup B$ using the data
$$\pi_1(A), \pi_1(B), \pi_1(A \cap B).$$

We first introduce a definition.

Definition 2.6 (Free product with amalgamation). Given some collections of groups $\{G_\alpha\}_\alpha$, the *free product*, denoted by $\underset{\alpha}{*} G_\alpha$ is a group such that

- Elements: Words in $\{g \colon g \in G_\alpha \text{ for any } \alpha\}$ modulo by the equivalence relation generated by

$$wg_ig_jv \sim w(g_ig_j)v$$

  when both $g_i, g_j \in G_\alpha$. Also, for the identity element $\mathrm{id} = e_\alpha \in G_\alpha$ for any α such that

$$we_\alpha v \sim wv.$$

- Operation: Concatenation of words.

Furthermore, if two groups G_α and G_β have a common subgroup $S_{\{\alpha,\beta\}}$[a], given two inclusion maps[b] $i_{\alpha\beta} \colon S_{\{\alpha,\beta\}} \to G_\alpha$ and $i_{\beta\alpha} \colon S_{\{\alpha,\beta\}} \to G_\beta$, the *free product with amalgamation* ${}_{\alpha}{*}_S G_\alpha$ is defined as $\underset{\alpha}{*} G_\alpha$ modulo the normal subgroup generated by

$$\left\{ i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \mid s_{\alpha\beta} \in S_{\{\alpha,\beta\}} \right\},$$

Namely[c],

$${}_{\alpha}{*}_S\, G_\alpha = \underset{\alpha}{*} G_\alpha \Big/ \left\langle i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \right\rangle$$

and satisfies the universal property

$$\begin{tikzcd}
S \arrow[r, "i_{\alpha\beta}"] \arrow[d, "i_{\beta\alpha}"'] & G_\alpha \arrow[d] \arrow[ddr, bend left] & \\
G_\beta \arrow[r] \arrow[drr, bend right] & G_\alpha *_S G_\beta \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}$$

[a] In general, we don't need $S_{\{\alpha,\beta\}}$ to be a subgroup.
[b] We don't actually need $i_{\alpha\beta}, i_{\beta\alpha}$ to be inclusive as well.
[c] Namely, $i_\alpha(s)$ and $i_\beta(s)$ will be identified in the quotient.

Remark. We see that

- We can then write out words such as $g_1g_2sg_3$ for $s \in S$, and view s as an element of G_α or G_β. In fact, we can do this construction even when i_α and i_β are not injective, though this means we are not working with a subgroup.

- Aside, in Top, the same universal property defines union

$$
\begin{tikzcd}
A \cap B \arrow[r, "i_\alpha"] \arrow[d, "i_\beta"'] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \cup B \arrow[dr, dashed, "\exists!"] & \\
 & & X
\end{tikzcd}
$$

for A, B are open subsets and the inclusion of intersection.

Theorem 2.5 (Seifert-Van Kampen Theorem). Given (X, x_0) such that $X = \bigcup_\alpha A_\alpha$ with

- A_α are open and path-connected and $\forall \alpha\ x_0 \in A_\alpha$
- $A_\alpha \cap A_\beta$ is path-connected for all α, β.

Then there exists a surjective group homomorphism

$$\underset{\alpha}{*}\colon \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0).$$

If we additionally have $A_\alpha \cap A_\beta \cap A_\gamma$ where they are all path-connected for every α, β, γ, then

$$\pi_1(X, x_0) \cong_\alpha *_{\pi_1(A_\alpha \cap A_\beta, x_0)} \pi_1(A_\alpha, x_0)$$

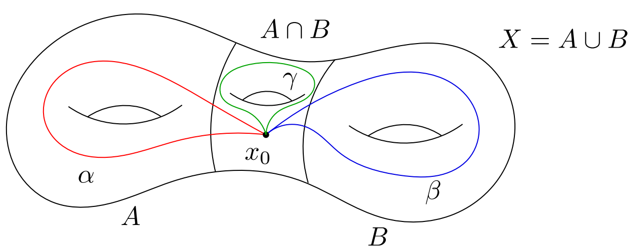
associated to all maps $\pi_a(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha), \pi_1(A_\beta)$ induced by inclusions of spaces. i.e., $\pi_1(X, x_0)$ is a quotient of the free product $*_\alpha \pi_1(A_\alpha)$ where we have

$$(i_{\alpha\beta})_*\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A + \alpha)$$

which is induced by the inclusion $i_{\alpha\beta}\colon A_\alpha \cap A_\beta \to A_\alpha$. We then take the quotient by the normal subgroup generated by

$$\left\{ (i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_\alpha \cap A_\beta) \right\}.$$

Example. We first see a great visualization of the Theorem 2.5.



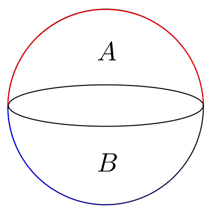
Intuitively we see the fundamental group of X, which is built by gluing A and B along their intersection. As the fundamental group of A and B glued along the fundamental group of their intersection. In essence, $\pi_1(X, x_0)$ is the quotient of $\pi_1(A) * \pi_1(B)$ by relations to impose the condition that loops like γ lying in $A \cap B$ can be viewed as elements of either $\pi_1(A)$ or $\pi_1(B)$.

Lecture 11: Group Presentations

31 Jan. 10:00

Example. We now see some applications of Theorem 2.5.

1. We can use Seifert Van Kampen Theorem to compute the fundamental group of S^2. We see that



   We see that $\pi_1(S^2)$ must be a quotient of $\pi_1(A) * \pi_1(B)$, but since $A, B \simeq D^2$, we know that $\pi_1(A)$ and $\pi_1(B)$ are both zero groups, thus $\pi_1(A) * \pi_1(B)$ is the zero group, and $\pi_1(S^2)$ is also the zero group.

   Remark. Note that the inclusion of $A \cap B \to A$ induces the zero map $\pi_1(A \cap B) \to \pi_1(A)$, which cannot be an injection. In fact, we know that $\pi_1(A \cap B) \cong \mathbb{Z}$ since $A \cap B \simeq S^1$.

2. In the case of torus, consider the following.

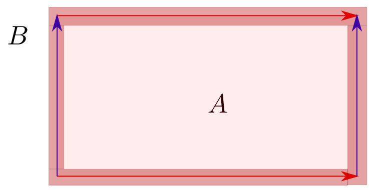


Figure 13: A is the interior, while B is the neighborhood of the boundary.

   Now note that $A \simeq D^2$ and $B \simeq S^1 \vee S^1$, and since it's a thickening of the two loops around the torus in both ways, this suggests the question of how do we find $\pi_1(B)$? We grab a bit of knowledge from Seifert Van Kampen Theorem before we continue.

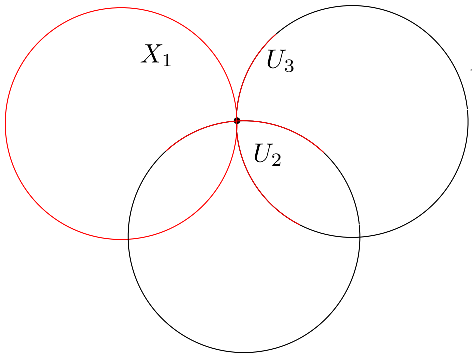
Exercise. Suppose we have path connected spaces (X_α, x_α), and we take their wedge sum $\bigvee_\alpha X_\alpha$ by identifying the points x_α to a single point x.

We also suppose a mild condition for all α, the point x_α is a deformation retract of some neighborhood of x_α.

For example, this doesn't work if we choose the *bad point* on the Hawaiian earring. Then we can use Seifert Van Kampen Theorem to show that

$$\pi_1\left(\bigvee_\alpha X_\alpha, x\right) \cong \mathop{*}_\alpha \pi_1\left(X_\alpha, x_\alpha\right).$$

Proof. If we denote



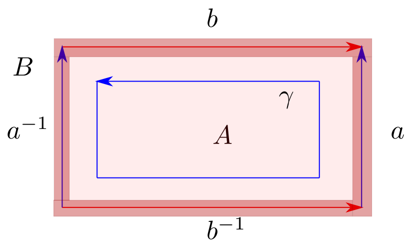
as C_n, then $\pi_1(C_n) \cong F_n$. Then we apply Theorem 2.5 to $A_\alpha = X_\alpha \cup_\beta U_\beta$ Specifically, take $A_\alpha = X_\alpha \cup_\beta U_\beta \simeq X_\alpha$, where U_β is a neighborhood of x_β which deformation retracts to x_β. This makes A_α open as desired. ∎

Corollary 2.2. The wedge sum of circles $\pi_1(\bigvee_{\alpha\in A} S^1) = *_\alpha\mathbb{Z}$ is a free group on A. In particular, when A is finite, the fundamental group of a bouquet of circles is the free group on $|A|$.

Returning to the example of torus, we see that

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$

Further, we know that $\pi_1(A \cap B) \to \pi_1(A)$ is the zero map. We need to understand $\pi_1(A \cap B) \to \pi_1(B)$. To do so we need to understand how we're able to identify $\pi_1(S^1 \vee S_1)$ with F_2 and how we identify $\pi_1(S^1)$ with $\mathbb{Z}$. We update our Figure 13 to talk about this.



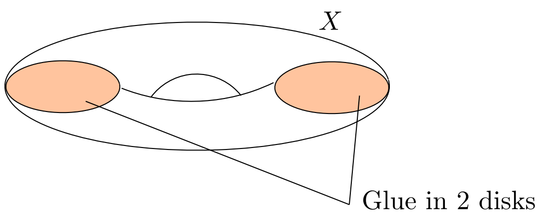
From this, we have

$$\pi_1(A \cap B) \to \pi_1(B) \cong F_{a,b}, \quad \gamma \mapsto aba^{-1}b^{-1}.$$

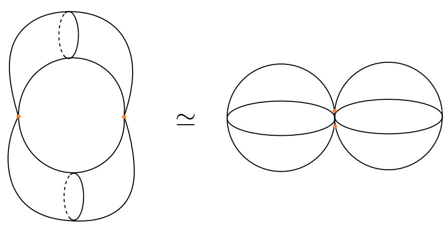
By Seifert Van Kampen Theorem, we identify the image of γ in $\pi_1(B)[aba^{-1}b^{-1}]$ with its image in $\pi_1(A)$, which is just trivial. Therefore, we have

$$\pi_1(T^2) = {F_{a,b}}\Big/\langle aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2.$$

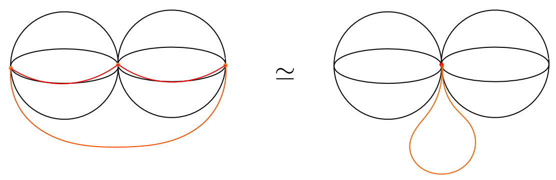
3. Let's see the last example which illustrate the power of Seifert Van Kampen Theorem. Start with a torus, and we glue in two disks into the hollow inside.



We'll call this space X, and out goal is to find $\pi_1(X)$. We can place a CW complex structure on this space so that each disk is a subcomplex. Then, we take quotient of each disk to a point without changing the homotopy type, hence X is homotopy to



By the same property, we can expand one of those points into an interval, and then contract the red path as follows.



This is exactly $S^2 \vee S^2 \vee S^1$. With Seifert Van Kampen Theorem, we have

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}.$$

Exercise. Consider $\mathbb{R}^2 \setminus \{x_1, \ldots, x_n\}$, that is the plane punctured at n points. Then $X \simeq \bigvee_n S^1$, so then

$$\pi_1(X) \simeq F_n.$$

One way to do this is to convince yourself that you can do a deformation retract the plane onto the following wedge.

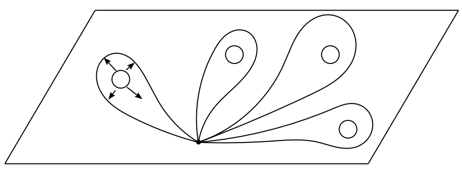


Figure 14: Deformation retract X onto wedge.

2.3 Group Presentation

In order to go further, we introduce the concept of *group presentation.*

Definition 2.7 (Group presentation). A *presentation* $\langle S \mid R\rangle$ of a group G is

- S: set of *generators*
- R: set of *relaters* (words in a generator and inverses)

such that

$$G \cong {}^{F_S}\big/_{\langle R\rangle},$$

where $\langle R\rangle$ is a subgroup normally generated by the elements of R.

Notice that $\langle S \mid R\rangle$ is finite if S, R are, and G is *finitely presented* if there exists a finite presentation.

Note. One way to think about whether G is finitely presented is that if r is a word in R then $r = 1$, where 1 is the identity of G.

Example. We see that

1. $F_2 = \langle a, b \mid \rangle$
2. $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle$
3. $\mathbb{Z} / 3\mathbb{Z} = \left\langle a \mid a^3 \right\rangle$
4. $S_3 = \left\langle a, b \mid a^2, b^2, (ab)^3 \right\rangle$

Theorem 2.6. Any group G has a presentation.

Proof. We first choose a generating set S for G. Notice that we can even choose $S = G$ directly. From the universal property of free group, we see that there exists a surjective map $\varphi\colon F_S \to G, s \mapsto s$. Now, let R be the generating set for $\ker(\varphi)$, by the first isomorphism theorem[6], $G \cong F_S / \ker\varphi$. In fact, we have $G = \langle S \mid R \rangle$. ■

Remark. The advantages are that given $G = \langle S \mid R \rangle$, it's now easy to define a homomorphism $\psi\colon G \to H$ given a map $\varphi\colon S \to H$, ψ extends to a group homomorphism $G \to H$ if and only if ψ vanishes on R, i.e., $\phi(r) = 0$ for all $r \in R$. We see an example to illustrate this.

Example. If we have $G = \langle a, b \mid aba \rangle$, a map $\varphi\colon \{a, b\} \to H$ gives a group homomorphism if and only if

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H.$$

This essentially uses the universal property of quotients.

Remark. It's sometimes easy to calculate G^{Ab}

$$G^{\mathrm{Ab}} = \langle S \mid R, \text{commutators in } S \rangle .$$

Example. Suppose all relations in R are commutators, so $R \subseteq [G, G]$. Then,

$$G^{\mathrm{Ab}} = (F_S)^{\mathrm{Ab}} = \bigoplus_S \mathbb{Z}.$$

Remark. The disadvantages are that, the computationally **very difficult**.

Example. Given $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle$, let

$$\psi\colon \{a, b\} \to H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1_H \in H.$$

Namely, this is a presentation of the trivial group, but this is entirely unclear.

[6] https://en.wikipedia.org/wiki/Isomorphism_theorems

Lecture 12: Presentations for π_1 of CW Complexes

2 Feb. 10:00

Let's first see an exercise.

Exercise. Consider $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\} \rangle$
- $G_1 *_H G_2$ where $f_1 \colon H \to G_1$ and $f_2 \colon H \to G_2$. Then we have

$$G_1 *_H G_2 = \left\langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \left\{ f_1(h) f_2(h)^{-1} \mid h \in H \right\} \right\rangle .$$

2.4 Presentations for π_1 of CW Complexes

For X a CW complex, we have

1. A 1-dimensional CW complex has free π_1 (call its generators as $a_1, \ldots, a_n$).
2. Gluing a 2-disk by its boundary along a word w in the generators *kills* w in π_1. We then get a presentation for $\pi_1(X^2)$ given by

$$\langle a_1, \ldots, a_n \mid w \text{ for each 2-cell in } X_2 \rangle .$$

3. Gluing in any higher dimensional cells along their boundary will not change π_1. That is, in a CW complex, we have $\pi_1(X) = \pi_1(X^2)$.

Remark. We can write the above more precise.

1. Find free generators $\{a_i\}_{i \in I}$ for $\pi_1(X^1)$.
2. For each 2-disk D^2_α, write attaching map as word w_α in a_i. i.e.,

$$\pi_1(X^2) = \langle a_i \mid w_\alpha \rangle .$$

3. $\pi_1(X) = \pi_1(X^2)$.

Example. Given $G = \mathbb{Z} / n\mathbb{Z} = \langle a, a^n \rangle$, then we take a loop and then wind a 2-disk around the loop a for n times.

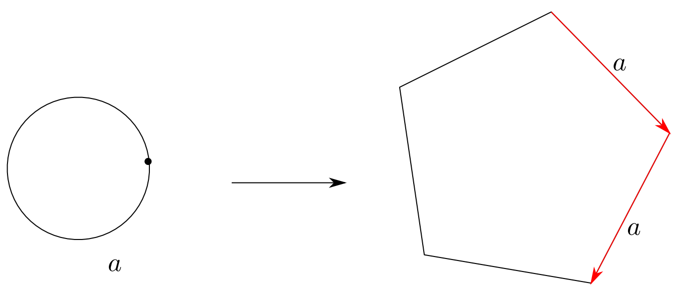


Figure 15: For $G = \mathbb{Z} / n\mathbb{Z} = \langle a \mid a^n \rangle$, we wind the boundary around a for n times.

We then see that given a group G with presentation $\langle S \mid R \rangle$, one can construct a 2-dimensional CW complex with $\pi_1 = G$ by

- Set $X^1 = \bigvee_{s \in S} S^1$
- For each relation $r \in R$, glue in a 2-disk along loops specified by the word r.

Every group is then π_1 of some space.

Theorem 2.7. If X is a Cw complex and $\iota_1 \colon X^1 \hookrightarrow X$ and $\iota \colon X^2 \hookrightarrow X$, then $(\iota_1)_*$ surjects onto π_1 and $(\iota_2)_*$ is an isomorphism on π_1.

Proof. ■ HW

Definition 2.8. We import some topological definitions of graph theoretic concepts.

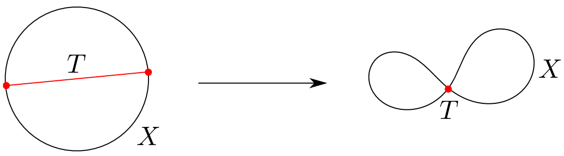
- A *graph* is a 1-dimensional CW complex.
- A *subgraph* is a subcomplex.
- A *tree* is a contractible graph.
- A tree in graph X (necessarily a subgraph) is *maximal* or *spanning* if it contains all the vertices.

Theorem 2.8. Every connected graph has a maximal tree. Every tree is contained in a maximal tree.

Corollary 2.3. Suppose X is a connected graph with basepoint x_0. Then $\pi_1(X, x_0)$ is a free group.

Furthermore, we can give a presentation for $\pi_1(X, x_0)$ by finding a spanning tree T in X. The generators of π_1 will be indexed by cells $e_\alpha \in X - T$, and e_α will correspond to a loop that passes through T, traverses e_α once, then returns to the basepoint x_0 through T.

Proof. The idea is simple. X is homotopy equivalent to $X \,/\, T$ via previous work on the homework, T contains all the vertices, so the quotient has a single vertex. Thus, it is a wedge of circles, and each e_α projects to a loop in $X \,/\, T$.



$\blacksquare$

Example. Let

- S^n: decompose into 2 open disks
- A_1: neighborhood of top hemisphere
- A_2: neighborhood of lower hemisphere

We see that $A_1 \cap A_2 \simeq S^{n-1}$, where we need $n \geq 2$ to let S^{n-1} be connected. We then have

$$\pi_1(S^n) \cong 0 \underset{\pi_1(A_1 \cap A_2)}{*} 0 = 0.$$

On the other hand, if $n \geq 3$, then we see that

$$S^n = {D^n \cup *}\big/{\sim}.$$

Since 2-skeleton is a point, thus $\pi_1(S^n) = 0$.

Appendix

References

[HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com/books?id=BjKs86kosqgC.