MATH602
Real Analysis II

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Abstract

This is a graduate level functional analysis taught by Joseph Conlon. The prerequisites include linear algebra, complex analysis and also real analysis. We'll use Peter Lax[Lax02] and Reed-Simon[RS80] as textbooks.

This course is taken in Fall 2022, and the date on the covering page is the last updated time.

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Chapter 1

Banach and Hilbert Spaces

Lecture 1: Introduction

We first briefly review different kinds of vector spaces. 30 Aug. 14:30

1.1 Linear Spaces

Let's first see the simplest (i.e., without structures) vector space called linear vector space.

Definition 1.1.1 (Linear vector space). A *linear vector space* E over a field $\mathbb{F}$ is a set with operations of addition and multiplication (by a scalar) such that it's closed under operations, and also the addition and scalar multiplication obey

(a) $u + v = v + u$ for $u, v \in E$

(b) $u + (v + w) = (u + v) + w$ for $u, v, w \in E$

(c) $\exists 0 \in E$ such that $0 + u = u + 0 = u$ for $u \in E$

(d) $\forall u \in E$, $\exists -u \in E$ such that $u + (-u) = 0$

(e) $\lambda(u + v) = \lambda u + \lambda v$ for $u, v \in E$, $\lambda \in \mathbb{F}$

(f) $(\lambda + \mu)u = \lambda u + \mu u$ for $u \in E$, $\lambda, \mu \in \mathbb{F}$

(g) $\lambda(\mu u) = (\lambda\mu)u$ for $u \in E$, $\lambda, \mu \in \mathbb{F}$

Remark. If $v, w \in E$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), then $\lambda v + \mu w \in E$.

Notation (Real and complex vector space). If E is over $\mathbb{F} = \mathbb{C}$, we usually call E a *complex vector space*; if $\mathbb{F} = \mathbb{R}$, we say E is a *real vector space*.

Example. $\mathbb{R}^n$ an n dimensional real linear vector space, $\mathbb{C}^n$ an n dimensional complex linear vector space.

We concentrate on ∞ dimensional linear vector space.

Example. Let K is a compact Hausdorff space, then

$$E = \{f \colon K \to \mathbb{R} \mid f(\cdot) \text{ is continuous}\}$$

is a ∞ dimensional **real** linear vector space.

Notation (Subspace). If E is a linear vector space, then we say $E_1 \subseteq E$ is a *subspace* if $E_1 \subseteq E$ and E_1 is itself a linear vector space. Moreover, if $E_1 \subsetneq E$, we say E_1 is a *proper subspace*.

Observe that a linear vector space can have many subspaces.

1.2 Quotient Spaces

Sometimes we don't care about vectors in some directions, hence we introduce the notion of quotient space.

Definition 1.2.1 (Quotient Space). The *quotient space* $E \,/\, E_1$ of two linear vector spaces E, E_1 such that $E_1 \subseteq E$ is the set of equivalence classes of vectors in E where equivalence is given by $x \sim y$ if $x - y \in E_1$. Additionally, denote $[x]$ as the equivalence class of $x \in E$, i.e., $[x] = x + E_1$.

One can see that quotient space $E \,/\, E_1$ is a linear vector space since if $x_1 + x_2 \in E$, $[x_1] + [x_2] = [x_1 + x_2]$, and also, $\lambda[x] = [\lambda x]$ for $\lambda \in \mathbb{R}$ or $\mathbb{C}$, i.e., $v, w \in E \,/\, E_1$, $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$ implies $\lambda v + \mu w \in E$.

The dimension of a quotient space is defined as follows.

Definition 1.2.2 (Codimension). If $E \,/\, E_1$ has finite dimension, then the dimension of $E \,/\, E_1$ is called the *codimension* of E_1 in E, denoted as $\operatorname{codim}(E_1)$.

Definition 1.2.2 is introduced since the way of defining dimensions for finite dimensional vector spaces doesn't work here. Recall Theorem 1.2.1 in the finite dimension case.

Theorem 1.2.1. If E is finite dimensional, then $\operatorname{codim}(E_1) + \dim(E_1) = \dim(E)$

We see that we may encounter something like $\infty - \infty$ if we define $\operatorname{codim}(E_1) \coloneqq \dim(E) - \dim(E_1)$, and indeed, Definition 1.2.2 is well-defined in this sense.

Example. There exists the case that $\dim(E) = \infty$, $\dim(E_1) < \infty$ where $\dim(E \,/\, E_1) < \infty$.

Proof. Let $E = \{f \colon K \to \mathbb{R} \mid f(\cdot) \text{ continuous}\}$ and $E_1 = \{f \in E \colon f(k_1) = 0\}$ for a fixed $k_1 \in K$. We see that the dimension of $E \,/\, E_1$ is exactly 1 since $E \,/\, E_1$ is the set of constant functions. ⊛

Definition 1.2.3 (Linear operator). A map $T \colon E \to F$ between linear spaces E and F is a *linear operator* if it preserves the properties of addition and multiplication by a scalar, i.e., for $v, w \in E$ and $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$,

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w).$$

Definition. Given a linear operator $T \colon E \to F$ we have the following.

Definition 1.2.4 (Kernel). The *kernel* of T is the subspace $\ker(T) = \{x \in E \mid Tx = 0\}$.

Definition 1.2.5 (Image). The *image* of T is the subspace $\operatorname{Im}(T) = \{Tx \in F \mid x \in E\}$.

1.3 Normed Spaces

Given a vector, we want to measure the length of which. This suggests the following definitions.

Definition 1.3.1 (Norm). Let E be a linear vector space. A *norm* $\|\cdot\| : E \to \mathbb{R}$ on E is a function from E to $\mathbb{R}$ with the properties:

(a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) $\|x+y\| \leq \|x\| + \|y\|$.

Notation (Dilation). We say that the second condition is the *dilation* property.

Definition 1.3.2 (Normed vector space). A linear vector space E equipped with a norm $\|\cdot\|$ is called a *normed vector space*, denoted by $(E, \|\cdot\|)$.

A similar notion called metric is also widely used, though the structure is slightly coarser.

As previously seen (Metric). Given a vector space E, the *metric* $d(\cdot,\cdot)\colon E \times E \to \mathbb{R}$ on E is a function form $E \times E$ to $\mathbb{R}$ with the properties:

(a) $d(x,y) \geq 0$. Also, $d(x,x) = 0$ and $d(x,y)$ implies $x = y$.

(b) $d(x,y) = d(y,x)$.

(c) $d(x,z) \leq d(x,y) + d(y,z)$.

As one can imagine, if we can measure the length of a vector (by a norm), we can also measure the distance between vectors (by a metric).

Remark (Induced metric space). A normed vector space $(E, \|\cdot\|)$ induces a *metric space* (E, d) with the induced metric $d(x,y) = \|x-y\|$.

Now we give some well-known examples of normed spaces.

Example (Bounded sequences ℓ^∞). Let ℓ_∞ be the space of bounded sequences $x = (x_1, x_2, \ldots)$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots$. Then we define $\|x\| = \|x\|_\infty = \sup_{i\geq 1} |x_i|$.

Example (Absolutely summable sequences ℓ_1). Let ℓ_1 be the space of absolutely summable sequences $x = (x_1, x_2, \ldots)$ and $\sum_{i=1}^\infty |x_i| < \infty$. Then we define $\|x\| = \|x\|_1 = \sum_{i=1}^\infty |x_i| < \infty$.

Example (Continuous functions $C(k)$). The space $C(k)$ of continuous functions $f\colon K \to \mathbb{R}$ where K is compact Hausdorff. Then we define $\|f\| = \|f\|_\infty = \sup_{x\in K} |f(x)|$.

1.3.1 Geometry of Normed Spaces

Now we can look into the structure of a normed space we're referring to without actually explaining what this really means previously. Intuitively, it's about the geometric properties of the spaces like how do balls, spheres and other shapes look like in that space when defining these shapes with norms.

Definition 1.3.3 (Ball). A (closed) *ball* centered at a point $x_0 \in E$ with radius $r > 0$ is the set $B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\}$.

Definition 1.3.4 (Sphere). The *sphere* centered at x_0 with radius $r > 0$ is the set $S(x_0, r) = \{x \in E \mid \|x - x_0\| = r\}$.

Note. We see that $S(x_0, r)$ is the **boundary** of $B(x_0, r)$, i.e., $S(x_0, r) = \partial B(x_0, r)$.

Let's first look at balls. In finite dimensional, all norms are equivalent, which is not true for infinite dimensional vector spaces. This has something to do with the geometry of balls.

Explicitly, balls can have different geometries depending on the properties of the norms. We see that a $\|\cdot\|_\infty$ can have multiple supporting hyperplane at the corner, while for a $\|\cdot\|_2$ can have only one at each point.

Remark. The unit balls for $\|\cdot\|_1$ looks like **squares**, where we have

$$B(0,1) = \{x = (x_1, x_2, \ldots) \mid -1 < y_\epsilon < 1 \text{ for all } \epsilon\}$$

such that $y_\epsilon = \sum_{i=1}^{\infty} \epsilon_i x_i$, $\epsilon_i = \pm 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots)$.

We see that different norms give different geometry, but they have important common features, most notably, convexity properties.

Definition 1.3.5 (Convex set)**.** Given E a linear vector space, a set $K \subset E$ is *convex* if for $x, y \in K$ and $0 \leq \lambda \leq 1$,

$$\lambda x + (1 - \lambda) y \in K.$$

Definition 1.3.6 (Convex function)**.** Given E a linear vector space, a function $f\colon E \to \mathbb{R}$ is called *convex* if for $x, y \in E$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Remark (Sublevel set)**.** If $f\colon E \to \mathbb{R}$ is a convex function, then for any $M \in \mathbb{R}$ the *sublevel set* $\{x \in E \mid f(x) \leq M\}$ is convex.

The upshot is that norms are convex, and the unit balls are convex as well.

Lecture 2: Banach Spaces and Completion

Let's first see a proposition. 01 Sep. 14:30

Proposition 1.3.1. Let $(E, \|\cdot\|)$ be a normed linear space, then the norm is convex and continuous.

Proof. Let $f\colon E \to \mathbb{R}$ be $f(x) = \|x\|$. Then $f(x) - f(y) = \|x\| - \|y\| \leq \|x - y\|$, which implies $|f(x) - f(y)| \leq \|x - y\|$ for $x, y \in E$, i.e., f is Lipschitz continuous hence continuous. For convexity, let $0 \leq \lambda \leq 1$, we have

$$f(\lambda x + (1-\lambda) y) = \|\lambda x + (1-\lambda) y\| \leq \|\lambda x\| + \|(1-\lambda) y\| = \lambda \|x\| + (1-\lambda) \|y\| = \lambda f(x) + (1-\lambda) f(y).$$

■

Note. Note that $f(\cdot) = \|\cdot\|$ is continuous implies the closed ball

$$B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\} = \{x \in E \mid f(x - x_0) \leq r\}$$

is closed in topology of E. Also, $f(\cdot)$ is convex implies $B(x_0, r)$ is convex.

Remark. If $f\colon E \to \mathbb{R}$ is convex, then the sets $\{x \in E \mid f(x) \leq M\}$ is also convex. However, it's possible to have non-convex functions f such that all sets $\{x \in E \mid f(x) \leq M\}$ are convex.

Proof. Take $f(x) = |x|^p$ for $x \in \mathbb{R}$ and $p > 0$. We see that f is convex if $p > 1$, and non-convex if $p < 1$. However, the sets $\{x \in \mathbb{R} \mid f(x) \leq M\}$ all convex since it's independent of p. ⊛

Lemma 1.3.1. Suppose $x \mapsto \|x\|$ satisfies

(a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) The unit ball $B(0,1)$ is convex.

Then $f(x) = \|x\|$ satisfies the triangle inequality $\|x+y\| \leq \|x\| + \|y\|$.

Proof. We see that if the third condition is true, the for $u, v \in B(0,1)$ and $0 < \lambda < 1$, we have $\lambda u + (1-\lambda)v \in B(0,1)$. Let $x, y \in E$, and

$$\lambda = \frac{\|x\|}{\|x\| + \|y\|} \Rightarrow 1 - \lambda = \frac{\|y\|}{\|x\| + \|y\|}.$$

By letting $u = x/\|x\|$, $v = y/\|y\|$ we see that

$$\lambda u + (1-\lambda)v = \frac{\|x\|}{\|x\| + \|y\|}\frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|}\frac{y}{\|y\|} \in B(0,1) \Rightarrow \left\| \frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|} \right\| \leq 1.$$

From the second condition, it follows that $\|x+y\| \leq \|x\| + \|y\|$, which is the triangle inequality. ■

Remark. If $x \mapsto \|x\|$ satisfies the first two condition and is convex, then it satisfies the triangle inequality.

Proof. Since $\frac{1}{2}\|x+y\| = \left\|\frac{x}{2} + \frac{y}{2}\right\| \leq \frac{1}{2}\|x\| + \frac{1}{2}\|y\|$. ⊛

Now, given a quotient space $E\,/\,E_1$, the question is can we try to define a norm?

Problem 1.3.1. On $E\,/\,E_1$, is $\|[x]\| \coloneqq \inf_{y \in E_1} \|x+y\|$ a norm?

Answer. We see that if $x \in \overline{E}_1 \setminus E_1$, then $\|[x]\| = 0$ but $0 \neq [x] \in E\,/\,E_1$. ⊛

We now see the difference from finite dimensional situation. All finite dimensional spaces E_1 are closed but not in general if E_1 has ∞ dimensions.

Example. Let $\ell_1(\mathbb{R})$ be the sequence of x_n for $n \geq 1$ in $\mathbb{R}$ such that $\sum_{i=1}^{\infty} |x_i| \leq \infty$. Define

$$\|x\|_1 \coloneqq \sum_{i=1}^{\infty} |x_i|,$$

and let E_1 be all sequences with finite number of the x_n are nonzero. We see that $\overline{E}_1 = \ell_1(\mathbb{R})$ is infinite dimensional.

Proposition 1.3.2. Let $(E, \|\cdot\|)$ be a normed space and $E_1 \subseteq E$, E_1 is closed. Then

$$\|\cdot\| : {}^{E}\!\big/_{E_1} \to \mathbb{R}, \quad \|[x]\| = \inf_{y \in E_1} \|x+y\|$$

is a norm on $E\,/\,E_1$.

Proof. If $\|[x]\| = 0$, then $\inf_{y \in E_1} \|x-y\| = 0$, which implies $x \in E_1$ since E_1 is closed, so $[x] = 0$. Also, since

$$\|\lambda[x]\| = \inf_{y \in E_1} \|\lambda x + y\| = \inf_{z \in E_1} \|\lambda x + \lambda z\| = |\lambda| \inf_{z \in E_1} \|x+z\| = |\lambda| \|[x]\|,$$

the dilation property is satisfied. Finally, for triangle inequality, we have

$$\|[x] + [y]\| = \inf_{x_1, y_1 \in E} \|x + y + x_1 + y_1\| \leq \inf_{x_1 \in E_1} \|x + x_1\| + \inf_{y_1 \in E_1} \|y + y_1\| = \|[x]\| + \|[y]\|.$$

■

Remark. This shows that the only obstacle for this kind of norm being an actual norm is whether E_1 is closed.

1.4 Banach Spaces

Turns out that a normed vector space is not enough in general, hence we introduce the following.

Definition 1.4.1 (Banach space)**.** A linear normed space is a *Banach space* if it's complete, i.e., every Cauchy sequence converges.

This implies that given a Banach space $(E, \|\cdot\|)$, if $\{x_n\}_{n\geq 1}$ is a sequence in E with the property such that $\lim\limits_{m\to\infty} \sup\limits_{n\geq m} \|x_n - x_m\| = 0$, then $\exists x_\infty \in E$ such that $\lim\limits_{n\to\infty} \|x_n - x_m\| = 0$ as well.

Example. The spaces ℓ_1, ℓ_∞ and $C(K)$ are Banach spaces.

1.4.1 Completion of Normed Space

We now show an important theorem which characterizes completeness in terms of convergence of series rather than sequences. We first see the definition.

Definition 1.4.2 (Absolutely summable)**.** Let E be a linear normed space and a sequence $\{x_i\}_{i\geq 1}$ in E. Then $\{x_i\}_{i\geq 1}$ is *absolutely summable* if $\sum_{i=1}^{\infty} \|x_i\| < \infty$.

Then, we have the following.

Theorem 1.4.1 (Criterion for completeness)**.** A normed space $(E, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in E converges.

Proof. We need to prove two directions.

$(\Rightarrow)$ Suppose E is a Banach space and $\{x_k\}_{k\geq 1}$ an absolutely summable series. Set $s_n = \sum_{k=1}^{n} x_k$ for $n \geq 1$, we want to show s_n is Cauchy, and if this is the case, completeness of E implies $\exists s_\infty$ and $\lim\limits_{n\to\infty} \|s_n - s_\infty\| = 0$. Let $n > m$, we see that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} \|x_k\| \leq \sum_{k=m+1}^{\infty} \|x_k\| .$$

Observe that $\lim\limits_{m\to\infty} \sum\limits_{k=m+1}^{\infty} \|x_k\| = 0$, we see that the sequence $\{s_n\}$ is Cauchy, hence it converges.

$(\Leftarrow)$ Conversely, suppose E is **not** complete. Then there exists a Cauchy sequence $\{x_n\}_{n\geq 1}$ which does not converge, implying no subsequence of $\{x_n\}_{n\geq 1}$ converges.[a] We now construct an absolutely summable series which does not converge.

Define $n(1) \geq 1$ such that $\left\|x_n - x_{n(1)}\right\| \leq \frac{1}{2}$ if $n \geq n(1)$, similarly, let $n(2) > n(1)$ be such that $\left\|x_n - x_{n(2)}\right\| \leq \frac{1}{2^2}$ if $n > n(2)$. In all, we have $n(1) < n(2) < n(3) < \ldots$ such that $\left\|x_n - x_{n(k)}\right\| \leq \frac{1}{2^k}$ if $n > n(k)$. Define $w_j \coloneqq x_{n(j+1)} - x_{n(j)}$ for $j = 1, 2, \ldots$. We see that

$$x_{n(m)} = x_{n(1)} + \sum_{j=1}^{m} w_j$$

for $m = 1, 2, \ldots$, and $\left\{x_{n(m)}\right\}$ does not converge, hence so does the series $\sum_{j=1}^{\infty} w_j$. However,

$\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$, which implies $\{w_j\}$ is absolutely summable. ■

[a] Otherwise, the whole sequence converges by the fact that it's Cauchy.

Theorem 1.4.2 (Completion). Suppose E is a normed space. Then there exists a Banach space $\hat{E}$ called *the completion* of E with the following properties:

(a) There exists a linear map $\iota\colon E \to \hat{E}$ such that $\|\iota x\| = \|x\|$.[a]

(b) $\mathrm{Im}(\iota)$ is dense in $\hat{E}$, and $\hat{E}$ is the smallest Banach space containing image of E.

[a] This is called an *isometric embedding* of E into $\hat{E}$.

Lecture 3: Banach, Inner Product Spaces

Notice that ℓ_1 and ℓ_∞ are Banach, and we want to generalize to ℓ_p with $1 < p < \infty$. For $x = \{x_n\}_{n\geq 1}$ in ℓ_p and if $\sum_{n=1}^{\infty} |x_n|^p < \infty$, for $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$, we want to show that $x \to \|x\|_p$ satisfies properties of a norm. The first two properties of a norm is easy check. As for triangle inequality, we have the following. 06 Sep. 14:30

Lemma 1.4.1 (Minkowski inequality). Let $1 \leq p < \infty$, for $x, y \in \ell_p$,

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p .$$

Proof. Recall that from Lemma 1.3.1, we only need to show that $B(0,1)$ is convex, where

$$B(0,1) = \left\{ x = \{x_n : n \geq 1\} \mid f(x) = \sum_{n=1}^{\infty} |x_n|^p \leq 1 \right\}.$$

But $f(x)$ is convex since $x \mapsto |x|^p$, $x \in \mathbb{R}$ is convex if $p \geq 1$, we're done. ■

Lemma 1.4.2 (Hölder's inequality). Let $1 < p < \infty$, for $x \in \ell_p$, $y \in \ell_q$, we have

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q$$

where $1/p + 1/q = 1$.

Proof. Note first that we can assume without loss of generality, $\sum_{j=1}^{\infty} |x_j|^p = \sum_{j=1}^{\infty} |y_j|^q = 1$. Then, result follows from the Young's inequality,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

for $x, y > 0$, $x, y \in \mathbb{R}$. ■

Remark (Legendre transform and the inequality). Young's inequality is a special case of the inequality

$$xy \leq f(x) + \mathcal{L}f(y)$$

where $\mathcal{L}f(\cdot)$ is the *Legendre transform* of $f(\cdot)$, i.e., $\mathcal{L}f(y) = \sup_x [xy - f(x)]$.

If f is convex, then the function $xy \mapsto xy - f(x)$ is concave so has unique maximum. And $\mathcal{L}f(\cdot)$ always convex even if $f(\cdot)$ is not. In particular, if $f(x) = x^p/p$, then $\mathcal{L}f(y) = y^q/q$.

Note. Minkowski inequality is usually proved via the Hölder's inequality.

Proof. To show this, since

$$\sum_{j=1}^{\infty} |x_j + y_j|^p \leq \sum_{j=1}^{\infty} |x_j|\, |x_j + y_j|^{p-1} + \sum_{j=1}^{\infty} |y_j|\, |x_j + y_j|^{p-1}.$$

Then Hölder's inequality implies

$$\sum_{j=1}^{\infty} |x_j|\, |x_j + y_j|^{p-1} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |x_j + y_j|^{(p-1)q} \right)^{1/q},$$

and similarly,

$$\sum_{j=1}^{\infty} |y_j|\, |x_j + y_j|^{p-1} \leq \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |x_j + y_j|^{(p-1)q} \right)^{1/q}.$$

Note that $(p-1)q = p$, hence by combining both, we have

$$\sum_{j=1}^{\infty} |x_j + y_j|^p \leq \left[\left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p} \right] \left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1/q},$$

i.e.,

$$\left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1-1/q} = \left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p},$$

proving the result. ⊛

Notice that Minkowski inequality and Hölder's inequality also hold for $1 \leq p \leq \infty$, or more generally, both hold for L^p spaces also. Let (Ω, Σ, μ) be a measure space and $L^p(\Omega, \Sigma, \mu)$ where all Σ measure functions $f\colon \Omega \to \mathbb{R}$ (or $\mathbb{C}$) such that $\int_\Omega |f|^p \,\mathrm{d}\mu < \infty$. Then, $L^p(\Omega, \Sigma, \mu)$ is a normed space with norm

$$\|f\|_p \coloneqq \left(\int_\Omega |f|^p \,\mathrm{d}\mu \right)^{1/p}.$$

It's more tricky to show that L^p is a Banach space, but it's indeed still the case.

Theorem 1.4.3 (Riesz-Fisher)**.** The space $L^p(\Omega, \Sigma, \mu)$ is a Banach space for $1 \leq p < \infty$.

Proof. Toward using Theorem 1.4.1, let $\{f_n\}_{n\geq 1}$ be an absolutely summable sequence in L^p. Then the norm satisfies

$$\left\| \sum_{k=1}^{N} f_k \right\|_p \overset{!}{\leq} \sum_{k=1}^{N} \|f_k\|_p \leq C < \infty \Rightarrow \int_\Omega \left| \sum_{k=1}^{N} f_k \right|^p \,\mathrm{d}\mu \leq C^p.$$

- Assume all f_k are non-negative. From monotone convergence theorem, we have

$$\lim_{N\to\infty} \int_\Omega \left(\sum_{k=1}^{N} f_k \right)^p \,\mathrm{d}\mu = \int_\Omega \left(\sum_{k=1}^{\infty} f_k \right)^p \,\mathrm{d}\mu \leq C^p.$$

Hence, $g = \sum_{k=1}^{\infty} f_k \in L^p$. We now want to show that $\sum_{k=1}^{N} f_k \to g$ in L^p. Set $r_n = \sum_{k=n+1}^{\infty} f_k$ where r_n is a decreasing sequence where $r_n \to 0$ a.e. and also

$$\int_\Omega r_1^p \,\mathrm{d}\mu < \infty.$$

This means that $\lim_{n\to\infty} \|r_n\|_p = 0$ by dominate convergence theorem.

- For arbitrary $f_k \colon \Omega \to \mathbb{R}$, write $f_k = f_k^+ + f_k^-$ where $f_k^+ = \sup(f_k, 0)$ and $f_k^- = \inf(f_k, 0)$. The sequence $\{f_k^+\}_{k\geq 1}$ are absolutely summable, and we just proceed as before. Similarly, if $f_k \colon \Omega \to \mathbb{C}$, we get the same result.

■

1.5 Inner Product Spaces

Indeed, a slightly stronger structure than a normed space equipped is the so-called inner product, since it actually induces a norm.

Definition 1.5.1 (Inner product)**.** Let E be a linear space over $\mathbb{C}$. An *inner product* $\langle\cdot,\cdot\rangle : E\times E \to \mathbb{C}$ is a function which has the following properties:

(a) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

(b) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for $a, b \in \mathbb{C}$.

(c) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

Notation (Real inner product)**.** We can also define inner products of spaces over $\mathbb{R}$ with no extra conjugation in the last property.

Definition 1.5.2 (Inner product space)**.** An *inner product space* is a linear space E with an inner product $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{C}$.

Definition 1.5.3 (Orthogonal)**.** Given a linear space E, $x, y \in E$ are *orthogonal* if $\langle x, y\rangle = 0$, denote as $x \perp y$.

Theorem 1.5.1 (Cauchy-Schwarz inequality)**.** Let $x, y \in E$ and an inner product $\langle\cdot,\cdot\rangle$, then

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}}.$$

Proof. Define $Q(t)$ by $Q(t) = \langle x + ty, x + ty\rangle = \langle y, y\rangle t^2 + 2t \operatorname{Re}\langle x, y\rangle + \langle x, x\rangle$ if $t \in \mathbb{R}$. Then we see that $Q(t) \geq 0$ with $t \in \mathbb{R}$, by looking at the discriminant, we have $(\operatorname{Re}\langle x, y\rangle)^2 \leq \langle x, x\rangle\langle y, y\rangle$. Finally, the result follows by choosing $\theta \in \mathbb{R}$ such that $\langle x, y\rangle = \operatorname{Re}\left\langle xe^{i\theta}, y\right\rangle$, we then see that

$$|\langle x, y\rangle| = \left|\operatorname{Re}\left\langle xe^{i\theta}, y\right\rangle\right| = |\operatorname{Re}\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle\langle y, y\rangle},$$

proving the result. ■

Corollary 1.5.1. The function $x \mapsto \|x\| := \langle x, x\rangle^{\frac{1}{2}}$ is a norm on E.

Proof. The first two properties of a norm is easy to verify, and the triangle inequality is a consequence of Cauchy-Schwarz inequality such that

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \overset{!}{\leq} \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

■

Remark (Pythagorean theorem)**.** The calculation in Corollary 1.5.1 clearly implies *Pythagorean the-*

orem, which states that if $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Example. The space ℓ_2 of square summable sequences $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$,

$$\langle x, y\rangle \coloneqq \sum_{j=1}^{\infty} x_j \overline{y}_j$$

defines an inner product.

Example (Canonical inner product on L^2)**.** The space $L^2(\Omega, \Sigma, \mu)$ of square integrable functions f, g,

$$\langle f, g\rangle = \int_\Omega f(x)\overline{g}(x) \,\mathrm{d}\mu(x)$$

defines an inner product. Furthermore, $\|f\|_2 = \langle f, f\rangle^{1/2}$.

Proof. The only non-trivial fact to prove is that $\langle f, g\rangle$ is finite, i.e., $f\overline{g}$ is integrable. Firstly, f^2, $\overline{f}^2$ and $(f+g)^2$ are all integrable since f, $\overline{g}$ and $f+\overline{g}$ are all in L^2, hence $f\overline{g}$ is also integrable. ⊛

Example. The space of $m \times n$ matrices $A = (a_{ij})$, $1 \leq i \leq m, 1 \leq j \leq n$. Then

$$\langle A, B\rangle = \operatorname{tr}(AB^*)$$

defines an inner product, where B^* is the Hermitian adjoint of B, i.e., for $B = (b_{ij})$, then $B^* = (b^*_{ij})$ for $b^*_{ij} = \overline{b}_{ji}$.

> **Remark** (Hilbert-Schmidt (Frobenius) norm)**.** Specifically, the norm corresponding to this inner product is
>
> $$\|A\|_{\mathrm{HS}} \coloneqq \left(\sum_{i,j}^{\infty} |a_{ij}|^2\right)^{1/2},$$
>
> which is known as the *Hilbert-Schmidt* or *Frobenius* norm.

1.5.1 Geometry of Inner Product Spaces

Indeed, the structure of an inner product space is much more interesting, since we can now consider the notion of angle between vectors.

> **As previously seen.** Recall that in Euclidean space $\mathbb{R}^n$, the inner product can be computed by the formula
>
> $$\langle x, y\rangle = \|x\| \, \|y\| \cos\theta(x, y)$$
>
> where $\theta(x, y)$ denotes the angle between x and y.

Similarly, we can define the angle between x, y in an inner product space by

$$\cos\theta(x, y) \coloneqq \frac{\langle x, y\rangle}{\|x\| \, \|y\|} \in [-1, 1]$$

where the range is ensured by Cauchy-Schwarz inequality, so it's well-defined. Though this concept this rarely used anyway. Indeed, the only useful case is when $\cos\theta = 0$, namely when x and y are perpendicular, or orthogonal.

But beyond orthogonality, there are other geometric properties in an inner product space captures by norms. Specifically, both parallelogram law and polarization identity hold, and the result is stated in terms of norm while they actually rely on the property of inner product.

Lemma 1.5.1 (Parallelogram law). Given E an inner product space, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Proof. Recall that $\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\operatorname{Re}\langle x,y\rangle + \|y\|^2$ and similarly, $\|x-y\|^2 = \|x\|^2 - 2\operatorname{Re}\langle x,y\rangle + \|y\|^2$, hence the result follows. ■

Lemma 1.5.2 (Polarization identity). Given E an inner product space, we have

$$\langle x,y\rangle = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right\}$$

Proof. The proof is just to expand the right-hand side in terms of inner product. ■

Remark. Polarization identity shows that the function $x \mapsto \|x\|^2$ determines the inner product.

Lecture 4: Orthogonality and Projection

1.6 Hilbert Spaces

08 Sep. 14:30

Just like the case of normed spaces, the inner product spaces are incomplete in general, hence we define the completed spaces of which, called Hilbert spaces.

Definition 1.6.1 (Hilbert space). A complete inner product space is called a *Hilbert space*.

Example. Both ℓ_2 and $L^2(\Omega,\Sigma,\mu)$ are normed spaces and complete, hence are Hilbert space.

1.6.1 Orthogonality

We'll soon see that the key notion in Hilbert space theory is orthogonality.

Definition 1.6.2 (Orthogonal complement). Let $A \subseteq \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space, then the *orthogonal complement* $A^\perp$ of A is

$$A^\perp := \{x \in \mathcal{H} \colon \langle x,y\rangle = 0 \text{ for } y \in A\}.$$

Remark. $A^\perp$ is also a Hilbert space, in particular, closed and $A^\perp \cap A \subseteq \{0\}$.

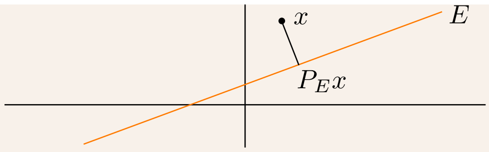
Proof. $A^\perp$ is closed linear subspace of $\mathcal{H}$ where the closure follows from the continuity of the function $x \mapsto \langle x,y\rangle$ for $x \in \mathcal{H}$ by looking at the inverse image of $\{0\}$. Also, for $x \in A^\perp \cap A$, $\langle x,x\rangle = 0$ implies $x = 0$. The reverse inclusion is false since A can be empty. ⊛

The fundamental theory of Hilbert spaces is Theorem 1.6.1.

Theorem 1.6.1 (Orthogonality principle). Assume $E \subseteq \mathcal{H}$ is a closed linear subspace of the Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$. Then we have the following.

(a) Then there exists a unique closest point $y = P_E x \in E$ to x, i.e., $\|x - P_E x\| = \inf_{y' \in E}\|x - y'\|$.

(b) The point $y = P_E x \in E$ is the unique vector such that $x - y \in E^\perp$.



Proof. Note that the function $y' \mapsto \|x - y'\|$ for $y' \in E$ is convex. We expect a minimizer y'.

(a) Let $y_n \in E$ for $n = 1, 2, \ldots$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} \|x - y_n\| = \inf_{y'\in E} \|x - y'\| =: d.$$

From parallelogram law, we have

$$\|y_n - y_m\|^2 + 4\|x - (y_n + y_m)/2\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2.$$

As $n, m \to \infty$, the right-hand side goes to $4d^2$. But since $\frac{1}{2}(y_n + y_m) \in E$, we have $\left\|x - \frac{1}{2}(y_n - y_m)\right\| \geq d$, so

$$\lim_{m\to\infty} \sup_{m\geq n} \|y_n - y_m\|^2 = 0,$$

which implies $\{y_n\}$ is a Cauchy sequence. As $\mathcal{H}$ is complete, we see that $y_n \to y_\infty \in E$, with $\|x - y_\infty\| = d$.

Now, with the fact that E is closed, we set $y_\infty = P_E x$ where y_∞ is unique since if $\|x - y_\infty\| = \|x - y'_\infty\| = d$, again by the parallelogram law where we now plug in y_∞ and y'_∞ instead of y_n and y_m as above, we see that $\|y_\infty - y'_\infty\| = 0$, hence $y_\infty = P_E x \in E$ is well-defined.

(b) We now show $P_E x$ is the unique vector $y \in E$ such that $x - y \perp E$, i.e., $x - y \in E^\perp$. Let $y' \in E$ and let $Q(t)$ be the quadratic

$$Q(t) := \langle x - P_E x + ty', x - P_E x + ty' \rangle = \|x - P_E x + ty'\|^2.$$

Since $t \mapsto Q(t)$ has a **strict** minimum at $t = 0$, which implies $Q'(0) = 0$, i.e., $\operatorname{Re}(x - P_E x, y') = 0$ for all $y' \in E$, which further implies $\langle x - P_E x, y' \rangle = 0$ for all $y' \in E$. This shows that $x - P_E x \in E^\perp$.

Finally, we need to show $P_E x \in E$ is the unique vector such $x - P_E x \in E^\perp$. This can be seen from $Q(t) = \|x - P_E x\|^2 + t^2\|y'\|^2$ for any $y' \in E$.

■

We see that orthogonality principle is actually quite surprising, since to show existence of such a closest point, we typically need

1. Compactness properties
2. Non-degeneracy properties for uniqueness

But here by using parallelogram law and the completeness of $\mathcal{H}$, we don't need these.

Remark. Orthogonality principle shows that the minimizer for the function $y' \mapsto \|x - y'\|$ for $y' \in E$ is characterized by the orthogonality condition, i.e., $x - y \perp E$ for some $y \in E$.

This suggests the following definition.

Definition 1.6.3 (Orthogonal projection)**.** Let $\mathcal{H}$ be a Hilbert space ad let $E \subseteq \mathcal{H}$ be a closed subspace. The *orthogonal projection operator* $P_E \colon \mathcal{H} \to E$ is given by $x \mapsto P_E x$ where $P_E x$ is defined uniquely via $x - P_E x \in E^\perp$.

The orthogonal projection is actually a so-called bounded linear map which defined below.

Definition 1.6.4 (Bounded linear map)**.** Given a mapping $A\colon \mathcal{B} \to \mathcal{B}$ on a Banach space $\mathcal{B}$, we say it's a *bounded linear map* if it's bounded and linear.

Definition 1.6.5 (Linear map)**.** The operator A is *linear* if for $x, y \in \mathcal{B}$, $a, b \in \mathbb{C}$,

$$A(ax + by) = aA(x) + bB(y).$$

Definition 1.6.6 (Bounded map)**.** The operator A is *bounded* if

$$\|A\| := \sup_{\|x\|=1} \|Ax\| < \infty.$$

Remark. Note that $\|Ax\| \leq \|A\| \|x\|$ for $x \in \mathcal{B}$.

We see that $P_E x$ is a bounded linear map $P_E\colon \mathcal{H} \to E \subseteq \mathcal{H}$ with the properties $P_E^2 = P_E$ and $\|x\|^2 = \|P_E x\|^2 + \|(I - P_E)x\|^2$ since $(I - P_E)x \perp P_E x$. The latter property shows that

$$\|P_E\| \leq 1, \quad \|(I - P_E)\| \leq 1,$$

and fact, $\|P_E\| = \|I - P_E\| = 1$. Also, $I - P_E$ is also an orthogonal projection onto $E^\perp$.

1.7 Fourier Series

Hilbert space gives a geometric framework for studying Fourier series. The classical Fourier analysis studies situations where a function $f\colon [-\pi, \pi] \to \mathbb{C}$ can be expanded as Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{1}{\sqrt{2\pi}} e^{ikt}$$

with the Fourier coefficients

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \,\mathrm{d}t.$$

In order to make Fourier analysis rigorous, we have to understand what functions f can be written as Fourier series, and in what sense the Fourier series converges. To do so, it's of great advantage to depart from this specific situation and carry out Fourier analysis in an abstract Hilbert space. Let $f(t)$ be a vector in the function space $L^2[-\pi, \pi]$, and the exponential functions e^{-ikt} will form a set of orthogonal vectors in this space. Then, Fourier series will become an orthogonal decomposition of a vector f w.r.t. an orthogonal system of coordinates.

1.7.1 Orthogonal Systems

We first give the definition.

Definition 1.7.1 (Orthogonal system)**.** A sequence $\{x_k\}_{k\geq 1}$ of non-zero vectors in a Hilbert space $\mathcal{H}$ is *orthogonal* if $\langle x_k, x_\ell \rangle = 0$ for all $\ell \neq k$.

Definition 1.7.2 (Orthonormal system)**.** An orthogonal system $\{x_k\}_{k\geq 1}$ is an *orthonormal system* if in addition, we have $\|x_k\| = 1$ for all k.

Write it in a more compact way, $\{x_k\}_{k\geq 1}$ is orthonormal if $\langle x_k, x_\ell \rangle = \delta_{k,\ell}$ where δ is the Kronecker delta. Here is an immediate generation given the remark.

Theorem 1.7.1 (Pythagorean theorem)**.** Let $\{x_k\}_{k\geq 1}$ be an orthogonal system in a Hilbert space $\mathcal{H}$.

Then for every $n \in \mathbb{N}$,

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n} \|x_k\|^2$$

Proof. From orthogonality,

$$\left\langle \sum_{k=1}^{n} x_k, \sum_{k=1}^{n} x_k \right\rangle = \sum_{k,j=1}^{n} \langle x_k, x_j \rangle = \sum_{k=1}^{n} \langle x_k, x_k \rangle,$$

proving the result ∎

We now see some examples.

Example (Canonical basis of ℓ_2). In the space ℓ_2, $x_k = (0, 0, \ldots, 1, 0, \ldots, 0) \in \ell_2$ for $k = 1, 2, \ldots$ is orthonormal system in ℓ_2.

Example (Fourier basis in L^2). In the space $L^2[-\pi, \pi]$, consider the exponential

$$e_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$$

for $t \in [-\pi, \pi]$. The set $\{e_k\}_{k=-\infty}^{\infty}$ is an orthonormal-system in $L^2[-\pi, \pi]$.

1.7.2 Fourier Series

We can further generalize Fourier series to any Hilbert space by letting $\{x_k\}_{k\geq 1}$ be an orthonormal set in $\mathcal{H}$ as follows.

Definition. Consider an orthonormal-system $\{x_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ and a vector $x \in \mathcal{H}$.

Definition 1.7.3 (Fourier series). The *Fourier series* of x w.r.t. $\{x_k\}_{k\geq 1}$ is the formal series

$$\sum_{k=1}^{\infty} \langle x, x_k \rangle x_k.$$

Definition 1.7.4 (Fourier coefficient). The coefficient $\langle x, x_k \rangle$ in the Fourier series are called *Fourier coefficients* of x.

To understand the convergence of Fourier series, we first focus on the finite case and study the partial sums of Fourier series. For $n = 1, 2, \ldots$, we define $S_n \colon \mathcal{H} \to E_n$ such that

$$S_n(x) = \sum_{k=1}^{n} \langle x, x_k \rangle x_k$$

for $x \in \mathcal{H}$ where $E_n = \operatorname{span}(\{x_1, \ldots, x_n\})$. We see that S_n is a linear operator and $S_n = P_{E_n}$ is the orthogonal projection onto E_n since $\langle x - S_n(x), x_k \rangle = 0$ for $k = 1, \ldots, n$, hence $S_n(x) \in E_n$ and $x - S_n(x) \perp E_n$.

Theorem 1.7.2 (Bessel's inequality). Let $\{x_k\}_k$ be an orthogonal system in a Hilbert space $\mathcal{H}$. Then for every $x \in \mathcal{H}$,

$$\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \leq \|x\|^2.$$

Proof. To estimate the size of $S_n(x)$, consider $x - S_n(x)$ and from Pythagorean theorem,

$$\|S_n(x)\|^2 + \|x - S_n(x)\|^2 = \|x\|^2 \Rightarrow \|S_n(x)\|^2 \leq \|x\|^2 .$$

On the other hand, again by Pythagorean theorem and orthogonality,

$$\|S_n(x)\|^2 = \sum_{k=1}^{n} \|\langle x, x_k\rangle x_k\|^2 = \sum_{k=1}^{n} |\langle x, x_k\rangle|^2 .$$

We see that by combining these two inequalities and let $n \to \infty$, we have the result. ■

Remark. In particular, we see that $\|S_n(x)\|^2 = \sum_{k=1}^{n} |\langle x, x_k\rangle|^2$, with $S_n = P_{E_n}$ we have $\|P_{E_n} x\|^2 \leq \|x\|^2$ for all $x \in \mathcal{H}$.

This implies the following.

Corollary 1.7.1. Let $\{x_k\}_{k\geq 1}$ be an orthonormal system in a Hilbert space $\mathcal{H}$. Then the Fourier series $\sum_k \langle x, x_k\rangle x_k$ for every $x \in \mathcal{H}$ converges in $\mathcal{H}$.

Proof. This follows directly from Bessel's inequality with the fact that the tail sum is Cauchy, i.e., we have

$$\left\|\sum_{k=n}^{m} x_k\right\|^2 = \sum_{k=n}^{m} \|x_k\|^2 \to 0$$

as $n, m \to \infty$ from Pythagorean theorem. ■

Corollary 1.7.1 tells us that Fourier series of x converge, but in fact, it needs not converge to x. But we still can compute the point where it converges to by considering Bessel's inequality, and the optimality is guaranteed by orthogonality principle.

Theorem 1.7.3 (Optimality of Fourier series). Let $\{x_k\}_k$ be an orthonormal system in a Hilbert space $\mathcal{H}$. Then the corresponding Fourier series $S_n(x) = \sum_{k=1}^{n} \langle x, x_k\rangle x_k$ converges, i.e., $\lim_{n\to\infty} S_n(x) = S_\infty(x)$ exists for $x \in \mathcal{H}$. Furthermore, $S_n = P_{E_n}$ for every n where E_n is the space spanned by $\{x_i\}_{i=1}^{n}$.[a]

[a] This includes $n = \infty$, where E_∞ is the **closure** of the space spanned by $\{x_k\}_{k\geq 1}$.

Proof. We show that the sequence $S_n(x)$ for $n = 1, 2, \ldots$ is Cauchy. This is because

$$\|S_n(x) - S_m(x)\|^2 = \sum_{k=m+1}^{n} |\langle x, x_k\rangle|^2 ,$$

and Bessel's inequality implies $\sum_{k=1}^{\infty} |\langle x, x_k\rangle|^2 \leq \|x\|^2$. Hence, for any $\epsilon > 0$, there exists $m(\epsilon)$ such that

$$\sum_{k=m(\epsilon)+1}^{\infty} |\langle x, x_k\rangle|^2 < \epsilon,$$

which implies $\|S_n(x) - S_m(x)\|^2 < \epsilon$ if $n > m(\epsilon)$, hence $\{S_n(x)\}_{n\geq 1}$ is Cauchy, implying $\lim_{n\to\infty} S_n(x) = S_\infty(x) \in \mathcal{H}$. Also, $S_\infty = P_{E_\infty}$ where E_∞ is the closure of the linear space generated by the sequence $\{x_k\}_{k\geq 1}$. ■

Remark. From orthogonality principle, we see that among all convergent series of the form $S = \sum_k a_k x_{vk}$, the approximation error $\|x - S\|$ is minimized by the Fourier series of x since it's the projection.

We finally note that the closeness of E_∞ makes sense since the self-dual of a set's orthogonal complement is itself if it's closed in the first place.

Lecture 5: Abstract Fourier Series

1.7.3 Orthonormal Bases

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It should be easy to identify an extra condition which makes the Fourier series of every vector x converges to x.

Definition 1.7.5 (Complete system). A system of vector $\{x_k\}_k$ in Hilbert space $\mathcal{H}$ is *complete* if the space spanned by $\{x_k\}_k$ is dense in $\mathcal{H}$, i.e., $\overline{\operatorname{span}(\{x_k\}_k)} = \mathcal{H}$.

Definition 1.7.6 (Orthonormal basis). A complete orthonormal system in a Hilbert space $\mathcal{H}$ is called an *orthonormal basis* of $\mathcal{H}$.

Theorem 1.7.4 (Fourier expansions). Let $\{x_k\}_k$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Then every vector $x \in \mathcal{H}$ can be expanded in its Fourier series

$$x = \sum_k \langle x, x_k \rangle x_k.$$

This is sometimes called Fourier inversion formula.

Proof. If an orthogonal set $\{x_k\}_k$ is complete, then $E_\infty = \mathcal{H}$, $P_{E_\infty} = I$. This implies $x = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k$ for $x \in \mathcal{H}$. ∎

Corollary 1.7.2 (Parseval's identity). Let $\{x_k\}_k$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Then

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 .$$

Proof. From Fourier expansion, we have $\|x\|^2 = \|P_{E_n} x\|^2 + \|(I - P_{E_n})\|^2$. By letting $n \to \infty$, we have

$$\|x\|^2 = \lim_{n\to\infty} \|P_{E_n} x\|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 = \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 .$$

∎

1.7.4 Gram-Schmidt Orthogonalization

Suppose $x_1, x_2, \ldots \in \mathcal{H}$ is a set of vectors and $E_n = \operatorname{span}(\{x_1, \ldots, x_n\})$. Then we can find an orthonormal set $\{y_k\}_{k\geq 1}$ in $\mathcal{H}$ such that $E_n = \operatorname{span}(\{y_1, y_2, \ldots, y_{m(n)}\})$ where $m(n) \leq n$.

Firstly, set $y_1 = x_1 / \|x_1\|$, and

$$y_n = \frac{(I - P_{E_{n-1}}) x_n}{\|(I - P_{E_{n-1}}) x_n\|}$$

if $x_n \notin E_{n-1}$, i.e., E_{n-1} is properly contained in E_n.

Remark. Proving completeness of a set of vectors $\{x_k\}_{k\geq 1}$ in $\mathcal{H}$ can be **non**-trivial.

We note that we can effectively compute the vectors $P_{E_n}(x_{n+1})$ since we know that $S_n(x)$ is the orthogonal projection of x onto $\operatorname{span}(\{y_k\})$, which is the partial sum of Fourier series

$$S_n(x) = \sum_{k=1}^{n} \langle x, y_k \rangle y_k.$$

As for $P_n(x)$, we see that it's the orthogonal projection onto the orthogonal complement, i.e.,

$$P_{E_n}(x) = x - S_n(x) = x - \sum_{k=1}^{n} \langle x, y_k \rangle y_k \Rightarrow P_{E_n}(x_{n+1}) = x_{n+1} - \sum_{k=1}^{n} \langle x_{n+1, y_k} \rangle y_k.$$

Let's now see some examples.

Example (Haar basis). We consider the *Haar basis* for $L^2([0,1])$. Let $h\colon (0,1) \to \mathbb{R}$ where

$$h(t) = \begin{cases} 1, & \text{if } 0 < t < 1/2; \\ -1, & \text{if } 1/2 < t < 1. \end{cases}$$

Extend $h(\cdot)$ by zero outside $(0,1)$, we get $h\colon \mathbb{R} \to \mathbb{R}$, $h(t) = 0$ if $t \notin (0,1)$, otherwise it's the same as above. The function $t \mapsto h(2^k t)$ has support in interval $0 < t < 2^{-k}$. Move the support to interval $\ell 2^{-k} < t < (\ell+1)2^{-k}$ by translation. Set

$$h_{k,\ell}(t) = h(2^k t - \ell), \quad \ell = 0, 1, \ldots, 2^k - 1.$$

The constant function plus functions $h_{k,\ell}$, $k = 0, 1, 2, \ldots$, $0 \leq \ell \leq 2^k - 1$ are a complete orthogonal set for $\mathcal{H} = L^2([0,1])$.

Proof. The span of the Haar functions includes characteristics functions χ_F for all dyadic intervals $[2^{-k}\ell, 2^{-k}(\ell+1)]$ for $\ell = 0, 1, \ldots, 2^{k-1}$, $k = 0, 1, \ldots$. If the set is **not** complete, then there exists $f \in L^2([0,1])$ such that

$$\int_F f \,\mathrm{d}t = 0$$

for all dyadic intervals F. Since we can approximate any measurable set $E \subseteq (0,1)$ by a union of dyadic intervals.

Intuition. An easy way to see this is to consider

$$\left\{ F \in \mathcal{B} \colon \int_F f \,\mathrm{d}t = 0 \right\},$$

which is the Borel subalgebra of $\mathcal{B}$, which indeed is a Borel algebra on $(0,1)$. Then observe that dyadic intervals generate all open intervals.

Hence, we see that $\int_F f \,\mathrm{d}t = 0$ for all measurable $F \subseteq (0,1)$. Let $F = \{t \in (0,1) \colon f(t) > 0\}$, if $m(F) > 0$, then

$$\int_F f \,\mathrm{d}t > 0.$$

Hence, a contradiction, so $m(F) = 0$. ⊛

Example (Fourier basis). Consider the Fourier basis $e_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$ for $k \in \mathbb{Z}$, $-\pi < t < \pi$. This is complete in $L^2([-\pi,\pi])$.

Proof. We use Stone-Weierstrass theorem and apply it to Fourier basis. All $e_k(\cdot)$ are in $C[-\pi,\pi]$, i.e., continuous functions $f\colon [-\pi,\pi] \to \mathbb{C}$. We know that $C([-\pi,\pi])$ is a Banach space with supremum norm $\|f\| \coloneqq \sup_{t \in [-\pi,\pi]} |f(t)|$. Stone-Weierstrass theorem implies density of the space spanned by $e_k(\cdot)$, $k \in \mathbb{Z}$ in $C([-\pi,\pi])$, hence the completeness in $L^2([-\pi,\pi])$ follows from the density of continuous functions in $L^2([-\pi,\pi])$. ⊛

Proposition 1.7.1. Let $\{x_k\}_k$ be a linear independent system in a Hilbert space $\mathcal{H}$. Then the system $\{y_k\}_k$ obtained by Gram-Schmidt orthogonalization of $\{x_k\}_k$ is an orthonormal system in $\mathcal{H}$, and

$$\operatorname{span}(\{y_k\}_{k=1}^n) = \operatorname{span}(\{x_k\}_{k=1}^n)$$

for all $n \in \mathbb{N}$.

Proof. The system $\{y_k\}_k$ is orthonormal by construction, and we obviously have the inclusion $\operatorname{span}(\{y_k\}_k) \subseteq \operatorname{span}(\{x_k\}_k)$. Furthermore, since the dimensions of these subspaces both equal n by construction, so they're indeed equal. ■

1.7.5 Existence of Orthogonal Bases

We see that from Proposition 1.7.1, we'll obtain that every Hilbert space that is not *too large* has an orthonormal basis. We call this Hilbert space separable.

Definition 1.7.7 (Separable)**.** A metric space is *separable* if it contains a countable dense subset.

Remark (Banach space)**.** For Banach space, separability follows from finding a countable set of vectors $\{x_k\}_k$ such that the span of $\{x_k\}_k$ is dense in E.

Chapter 2

Bounded Linear Operators

In this chapter we study certain transformations of Banach spaces. Because these spaces are linear, the appropriate transformations to study will be linear operators. Furthermore, since Banach spaces carry topology, it is most appropriate to study continuous transformations, i.e. continuous linear operators. They are also called bounded linear operators for the reasons that will become clear shortly.

2.1 Bounded Linear Functionals

Turns out that the case when the operators' range is $\mathbb{R}$ is interesting enough already, hence we study this case first.

2.1.1 Continuity and Boundedness

Definition. Let E be a linear space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1 (Linear functional)**.** A *linear functional* on E is a linear operator $f\colon E \to \mathbb{R}$ or $\mathbb{C}$ such that

$$f(ax + by) = af(x) + bf(y)$$

for $x, y \in E$, $a, b \in \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.2 (Bounded linear functional)**.** A linear functional $f(\cdot)$ is *bounded* if

$$\|f\| := \sup_{\|x\|=1} |f(x)| < \infty.$$

Clearly, the boundedness of $f(\cdot)$ implies $|f(x-y)| \leq \|f\| \|x-y\|$ for $x, y \in E$. Hence, $f(\cdot)$ is continuous and in fact Lipschitz continuous if it's boounded.

Remark. Conversely, if a linear functional is continuous then it is bounded.

Proof. Suppose $f(\cdot)$ is not bounded, then there exists a sequence $x_n \in E$ such that $|f(x_n)| \geq n \|x_n\|$ for $n = 1, 2, \ldots$. By linearity,

$$\left| f\left(\frac{x_n}{n \|x_n\|}\right) \right| \geq 1, \quad n = 1, 2, \ldots.$$

But we know $\lim\limits_{n\to\infty} \frac{x_n}{n\|x_n\|} = 0$ and $f(0) = 0$, hence $f(\cdot)$ is not continuous at 0. ⊛

2.1.2 Dual Spaces and Hyperplanes

Indeed, we have a special name for the space of all bounded linear functionals called dual spaces due to its importance.

Definition 2.1.3 (Dual space)**.** Let E be a normed space, then the space of all bounded linear functionals $f(\cdot)$ on E is called the *dual space* E^* of E.

The dual space is also a normed space with norm $\|f\| := \sup_{\|x\|=1} |f(x)|$, which is in fact a Banach space. And it is a Banach space even if the original E is not. This definition implies $|f(x)| \leq \|f\| \|x\|$ for $x \in E$, $f \in E^*$. Also, $\|f\|$ is the smallest number in this inequality that makes it valid for all $x \in X$.

Definition 2.1.4 (Hyperplane)**.** Let E be a linear space and $H \subseteq E$ is a subspace. Say H is a *hyperplane* if $\operatorname{codim}(H) = 1$, i.e., $\dim(E \,/\, H) = 1$.

The goal is to make an equivalence between bounded linear functionals on E and *closed* hyperplanes in E.

Problem 2.1.1. Does there exist a **non**-closed hyperplane?

Answer. We know that this is not the case in finite dimension. And this question is analogous to *does there exist a subset $F \subseteq \mathbb{R}$ which is **not** Lebesgue measurable?* The answer to this is yes in both cases. However, construction uses axiom of choice. ⊛

Turns out that there is a canonical correspondence between the linear functionals and the hyperplanes in E. This is clarified in the following.

Proposition 2.1.1 (Linear functionals and hyperplanes)**.** Let E be a linear space.

(a) For every linear functional f on E, $\ker(f)$ is a hyperplane in E. If E is a Banach space, and $f(\cdot)$ is bounded, then $\ker(f) = H$ is closed.

(b) If $f, g \neq 0$ are linear functionals on E such that $\ker(f) = \ker(g)$, then $f = ag$ for some $a \neq 0$.

(c) For every hyperplane $H \subseteq E$, there exists a linear functional $f \neq 0$ on E such that $\ker(f) = H$. If E is a Banach space and $\ker(f) = H$ is closed, then $f(\cdot)$ is bounded.

Lecture 6: Riesz Representation Theorem

Let's first see the proof of Proposition 2.1.1. 15 Sep. 14:30

Proof of Proposition 2.1.1. We prove them in order.

(a) Let $x, y \notin \ker(f)$, then $f(x), f(y) \neq 0$, meaning that there exists a scalar $\lambda \neq 0$ such that $f(x) = \lambda f(y)$, i.e., $x - \lambda y \in \ker(f)$. Hence, if $[x], [y] \in E \,/\, \ker(f)$, $[x] = \lambda[y]$, implying $\dim(E \,/\, \ker(f)) = 1$. Now, if f is bounded, then f is continuous, so $\ker(f) = f^{-1}(\{0\})$ is closed.

(b) Consider the induced functionals $\widetilde{f}, \widetilde{g} \colon E \,/\, H \to \mathbb{R}$ or $\mathbb{C}$ where $H = \ker(f) = \ker(g)$. This implies
$$\dim \left({}^{E}\Big/_{H} \right) = 1 \Rightarrow \widetilde{f} = a\widetilde{g} \text{ for some } a \neq 0 \Rightarrow f = ag.$$

(c) Assume $\dim(E \,/\, H) = 1$, so $E \,/\, H = \{a[x_0] \colon a \in \mathbb{C} \text{ (or } \mathbb{R})\}$ for some $x_0 \in E$. Then, for any $x \in E$, $[x] = a(x)[x_0]$ for some $a(x) \in \mathbb{C}$ or $\mathbb{R}$. Define $f(x) := a(x)$, we see that f is linear and $\ker(f) = H$. Now, if E is a Banach space and H is closed with $\dim(E \,/\, H) = 1$. Recall that $E \,/\, H$ is also a Banach space with norm $\|[x]\| = \inf_{y \in H} \|x + y\|$ for $x \in E$.[a] Let $\widetilde{f}$ be a linear functional on $E \,/\, H$. Since $\dim(E \,/\, H)$ is finite, $\widetilde{f}$ is continuous, implying $|\widetilde{f}([x])| \leq A \|[x]\|$ for all $x \in E$ for some scalar A. Finally, we define $f(x) = \widetilde{f}([x])$ for $x \in E$, then $\ker(f) = H$ and $|f(x)| \leq A \|[x]\| \leq A \|x\|$.

■

[a] We see now why we need the closure: otherwise we'll get a non-zero function with norm 0.

2.2 Representation Theorems

In concrete Banach spaces, the bounded linear functionals usually have a specific and useful form. Generally speaking, all linear functionals on function spaces (such as L^p and $C(K)$) act by integration of the function (with respect to some weight or measure). Similarly, all linear functionals on sequence spaces (such as ℓ_p) act by summation with weights.

We now start by characterizing bounded linear functionals on a Hilbert space $\mathcal{H}$.

Theorem 2.2.1 (Riesz representation theorem)**.** Let $\mathcal{H}$ be a Hilbert space. Then we have the following.

(a) For every $y \in \mathcal{H}$, then function $f(x) = \langle x, y\rangle$ for $x \in \mathcal{H}$ is a bounded linear functional on $\mathcal{H}$.

(b) If $f\colon \mathcal{H} \to \mathbb{C}$ or $\mathbb{R}$ is a bounded linear functional on $\mathcal{H}$, then there exists $y \in \mathcal{H}$ such that $f(x) = \langle x, y\rangle$ for all $x \in \mathcal{H}$. Hence, the dual $\mathcal{H}^*$ of $\mathcal{H}$ is isometric to $\mathcal{H}$.

Proof. We prove this in order.

(a) $f(x) = \langle x, y\rangle$ is clearly a linear functional. Boundedness follows form Cauchy-Schwarz inequality such that

$$|f(x)| = |\langle x, y\rangle| \leq \|x\| \, \|y\| \, ,$$

and we can achieve $\|f\| = \|y\|$ by setting $x = y / \|y\|$.

> **Note.** Note that there exists x_f such that $\|x_f\| = 1$ since $\|f\| = \sup_{\|x\|=1} |f(x)| = f(x_f)$, i.e., the supremum is achieved, although we're working on an infinite dimensional space. This property does not always hold for bounded linear functionals on Banach space since the unit ball can be not compact. But this holds for Hilbert space.

(b) Let $f\colon \mathcal{H} \to \mathbb{C}$ or $\mathbb{R}$ be a bounded linear functional on $\mathcal{H}$. Let $H = \ker(f)$, which is closed from Proposition 1.7.1. Let $H^\perp$ be the orthogonal complement of H, i.e., $\mathcal{H} = H \oplus H^\perp$. Then $\dim(\mathcal{H} \,/\, H) = 1 \Rightarrow \dim(H^\perp) = 1$. Choose $y' \in H^\perp$ such that $g(x) = \langle x, y'\rangle$, which is in $\mathcal{H}^*$ from (i). Furthermore, we see that $\ker(g) = \ker(f)$, so from Proposition 1.7.1, f and g are equal up to a constant $\lambda \in \mathbb{C}$ or $\mathbb{R}$, i.e., $f = \lambda g$. It follows that

$$f(x) = \lambda g(x) = \lambda \langle x, y'\rangle = \langle x, \lambda y'\rangle =: \langle x, y\rangle$$

for $y := \lambda y'$, hence we're done.[a]

■

[a]We can even show that y here is unique.

In a concise form, Riesz representation theorem can be realized as $\mathcal{H}^* = \mathcal{H}$. Given a Hilbert space $\mathcal{H}$, Riesz representation theorem identifies the dual space $\mathcal{H}^*$, which can be used to show Radon-Nikodym theorem.

Theorem 2.2.2 (Radon-Nikodym theorem)**.** Let μ, ν be two finite measures such that $v \ll \mu$, i.e., ν is absolutely continuous w.r.t, μ.[a] Then there exists $g \geq 0$ such that g is μ-integrable and

$$\nu(A) = \int_A g \, \mathrm{d}\mu$$

for A measurable.

[a]This means $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

Proof. Consider the linear functional $F\colon L^2(\mu) \to \mathbb{R}$ or $\mathbb{C}$ such that

$$F(f) = \int_\Omega f \, \mathrm{d}\mu.$$

Then we have $\|F(f)\| \leq \|f\|_2 \sqrt{\mu(\Omega)}$, i.e., F is also a bounded linear functional on $L^2(\mu+\nu)$, hence by Riesz representation theorem, there exists $h \in L^2(\mu+\nu)$ such that

$$F(f) = \int_\Omega fh \,\mathrm{d}(\mu+\nu)$$

for $f \in L^2(\mu+\nu)$, i.e.,

$$\int_\Omega f \,\mathrm{d}\mu = \int_\Omega fh \,\mathrm{d}\mu + \int_\Omega fh \,\mathrm{d}\nu \tag{2.1}$$

if $f \in L^2(\mu+\nu)$. This further implies

$$\int_\Omega fh \,\mathrm{d}\nu = \int_\Omega f[1-h] \,\mathrm{d}\mu \tag{2.2}$$

for $f \in L^2(\mu+\nu)$.

Claim. Such h satisfies $0 < h \leq 1$ μ-a.e., moreover, $(\mu+\nu)$-a.e.

Proof. We first note that $\mu(A) = 0 \Leftrightarrow \mu(A) + \nu(A) = 0$. Let $A = \{h \leq 0\}$, $f = \mathbb{1}_A$ be the characteristic function on A. Then Equation 2.1 implies

$$\int_A h(\,\mathrm{d}\mu + \,\mathrm{d}\nu) \leq 0 \Rightarrow \mu(A) = 0 \Rightarrow h > 0 \ \mu \text{ a.e.}$$

But since g is a positive function, so we also need $h \leq 1$. Again, set $B = \{h > 1\}$, $f = \mathbb{1}_B$. Then Equation 2.1 implies

$$\mu(B) = \int_B h\,(\,\mathrm{d}\mu + \,\mathrm{d}\nu) > \mu(B)$$

unless $\mu(B) = 0$. ⊛

Now, by using monotone convergence theorem, we conclude[a] that Equation 2.2 holds for all $f \geq 0$, $f \in L^2(\mu+\nu)$.[b] Finally, let $A \subseteq \Omega$ measurable and $hf = \chi_A$, from Equation 2.2,

$$\nu(A) = \int_A \frac{1-h}{h} \,\mathrm{d}\mu.$$

By letting $g \coloneqq 1 - h/h \Rightarrow g = \mathrm{d}\nu/\mathrm{d}\mu$, we're done. ■

[a] Consider $f_n(t) \coloneqq \min(f(t), n)$ and let $n \to \infty$.
[b] Both sides could be ∞.

Notation (Radon-Nikodym derivative)**.** g in Radon-Nikodym theorem is referred to as the *Radon-Nikodym derivative* where $g \coloneqq \mathrm{d}\nu/\mathrm{d}\mu$.

Note (Uniqueness)**.** The uniqueness of Radon-Nikodym derivatives can be shown via

$$\int_A g \,\mathrm{d}\mu = 0$$

for all μ-measurable A, i.e., $g = 0$ μ-a.e.

Another useful application of Riesz representation theorem is to characterize L^p and ℓ_p spaces and their dual L_p^* and ℓ_p^*. We first see the following.

Remark. Consider spaces $L^p(\Omega, \mu)$ for $1 \leq p \leq \infty$, then we have

$$L^q(\Omega, \Sigma, \mu) \subseteq (L^p(\Omega, \Sigma, \mu))^*$$

where $1/p + 1/q = 1$.

Proof. The easy part is that $g \in L^q$ induces a bounded linear functional on L^p by setting

$$F(f) = \int_\Omega fg \,\mathrm{d}\mu.$$

By Hölder's inequality, $|F(f)| \leq \|f\|_p \|g\|_q$, hence $\|F\| \leq \|g\|_q$. To show the equality and $\sup_{\|f\|_p} |F(f)|$ is attained for $1 < p < \infty$, we choose $f = g^{q-1} \operatorname{sgn}(g)$ since

$$F(f) = \int_\Omega |g|^q \,\mathrm{d}\mu = \|g\|_q^q,$$

and from $1/p + 1/q = 1 \Rightarrow q - 1 = q/p$, we have

$$\|f\|_p^p = \int |f|^p \,\mathrm{d}\mu = \int_\Omega |g|^q \,\mathrm{d}\mu = \|g\|_q^q \Rightarrow \|f\|_p = \|g\|_q^{q/p} = \|g\|_q^{q-1}.$$

This implies

$$F(f) = \int_\Omega |g|^q \,\mathrm{d}\mu \Rightarrow \|g\|_q^q = \|g\|_q \|f\|_p.$$

Note. We see that $\sup_{\|f\|_p = 1} |F(f)|$ is attained by taking $f = \operatorname{sgn}(g)$.

⊛

In particular, we have the following.

Theorem 2.2.3 ($L^{p*} = L^q$)**.** Consider the space $L^p = L^p(\Omega, \Sigma, \mu)$ with finite measure of σ-finite measure μ. Then for $1 \leq p < \infty$ and the conjugate exponent q of q.

(a) For every weight function $g \in L^q$, integration with weight

$$\int_\Omega fg \,\mathrm{d}\mu$$

for $f \in L^p$ is a bounded linear functional on L^p, and its norm is $\|G\| = \|g\|_q$.

(b) Conversely, every bounded linear functional $G \in L^{p*}$ can be represented as integration with weight for some unique weight function $g \in L^q$. Moreover, $\|G\| = \|g\|_q$.

Lecture 7: Hahn-Banach Theorem

Remark. When $p = 1$, the supremum is not attained necessarily. Consider $g \in L^\infty$, $F(f) := \int fg \,\mathrm{d}\mu$ is dual of L^1. If $g(\cdot)$ is continuous on $\mathbb{R}$ with unique maximum, then the supremum $\sup_{\|f\|_1} |F(f)|$ is not attained. In all, for $1 \leq p \leq \infty$, L^q contained in the dual of L^p. If $1 < p \leq \infty$, then $\sup_{\|f\|_p = 1} |F(f)|$ is attained. For $p = 1$, the supremum is not necessarily attained. 20 Sep. 14:30

Now, we're ready to prove Theorem 2.2.3.

Proof of Theorem 2.2.3. To show that the dual of L^p is L^q if $1 \leq p < \infty$ where $1/p + 1/q = 1$, we use Radon-Nikodym theorem. Suppose $E = L^p(\Omega, \Sigma, \mu)$ with $1 \leq p < \infty$ and $f \in E^*$. Just consider finite measure space, i.e., $\mu(\Omega) < \infty$. We define a measure ν on Σ by $\nu(A) := F(\chi_A)$ for $A \in \Sigma$, where χ_A is the characteristic function of A. We see that

$$\mu(A) = 0 \Rightarrow \nu(A) = 0 \Rightarrow \nu \ll \mu,$$

and Radon-Nikodym theorem implies

$$\nu(A) = \int_A g \, d\mu$$

for some $g =: \frac{d\nu}{d\mu} \in L^1(\Omega, \Sigma, \mu)$. Note that g may not be in L^q since $q > 1$. Hence, $F(f) = \int_\Omega f g \, d\mu$ for all simple function f assuming $g \geq 0$. Set $f = g^{q-1}$ with the fact that $|F(f)| \leq \|F\|_p \|f\|_p$. Recall that $q - 1 = q/p$, hence

$$\int g^q \, d\mu \leq \|F\|_p \left(\int g^q \, d\mu \right)^{1/p} \Rightarrow \|g\|_q^q \leq \|F\|_p \|g\|_q^{q/p} = \|F\|_p \|g\|_q^{q-1},$$

hence $\|g\|_q \leq \|F\|_p$.

> **Note.** We assume $g \geq 0$ is because ν is a sign measure, then if we have a bounded variation function, we can just break it into $\nu^+ + \nu^-$.

■

Remark. L^1 is a subset of $L^{\infty *}$ but not equal to it. If $F \colon L^\infty(\mu) \to \mathbb{C}$ is a bounded linear functional, then if $\Omega = K$ is a compact Hausdorff space, F induces a bounded linear functional on $C(K)$, i.e., the space of continuous functions on K. We see that $C(K) \subseteq L^\infty(K, \Sigma, \mu)$ where Σ is the Borel algebra on K.

Theorem 2.2.4. Let $E = C(K)$ be the space of continuous functions on compact Hausdorff space K. Then we have the following.

(a) For every Borel regular signed measure on K, the functional $F(f) = \int_K f \, d\mu$ is a bounded linear functional on K.

(b) Every bounded linear functional on $C(K)$ can be expressed as $F(f) = \int_K f \, d\mu$ for some measure μ, and $\|F\| = |\mu|(K)$, i.e., $TV(K)$.

In this case, the proof is much more difficult, and we omit the proof here.

2.3 Hahn-Banach Theorem

Hahn-Banach theorem allows one to extend continuous linear functionals f from a subspace to the whole normed space, while preserving the continuity of f. Hahn-Banach theorem is a major tool in functional analysis. Together with its variants and consequences, this result has applications in various areas of mathematics, computer science, economics and engineering.

Theorem 2.3.1 (Hahn-Banach theorem). Let E_0 be a subspace of a Banach space E. Then every $f_0 \colon E_0 \to \mathbb{R}$ or $\mathbb{C}$ has a continuous extension $f \colon E \to \mathbb{R}$ or $\mathbb{C}$ such that $\|f\| = \|f_0\|$.

Before proving this, let's first see some implications.

Theorem 2.3.2 (Supporting functional). Let E be a Banach space. For every $x \in E$, there exists $f \in E^*$ such that $\|f\| = 1$, $f(x) = \|x\|$. i.e., $\sup_{\|y\|=1} |f(y)|$ attained at $y = x$.

Proof. Consider dimension 1 space $E_0 = \operatorname{span}(x) = \{tx, t \in \mathbb{R} \text{ or } \mathbb{C}\}$. Define $f_0 \colon E_0 \to \mathbb{R}$ or $\mathbb{C}$ such that $f_0(tx) = t\|x\|$. We see that $\|f_0\| = 1$, and Hahn-Banach theorem implies there exists $f \in E^*$ with $\|f\| = 1$. We see that $f(x) = \|x\|$ explicitly attain the norm and $\|\cdot\|$ is clearly a continuous extension of $\|\cdot\|_{E_0} = f_0$ as required. ■

Remark (Geometric interpretation). Let B be a unit ball $\{x \in E \colon \|x\| \leq 1\}$ in a real Banach space E. Choose $x_0 \in \partial B$ such that $\|x_0\| = 1$. Then there exists $f \in E^*$, $\|f\| = 1$, $f(x) = \|x\|$. Let $H = \ker(f) + x_0$ where H intersects B at x_0, we see that H divides E into 2 disjoint subsets, while B lies in one of which.

Proof. Since $x \in B$ and $\|x\| < 1$ implies $|f(x)| \leq \|x\| < 1$, we have $f(x) < 1$, i.e., $B \subseteq \{x \colon f(x) < 1\}$ and $E = \{x \colon f(x) < 1\} \cup H \cup \{x \colon f(x) > 1\}$. ⊛

Note. Notice that we don't have uniqueness (as we don't have it in Hahn-Banach theorem) since a unit ball in L^∞ has corner, which will give multiple hyperplanes.

Lecture 8: Proof of Hahn-Banach Theorem and Duality

We now see the proof of Hahn-banach theorem. 20 Sep. 14:30

Proof of Theorem 2.3.1. We assume E is separable, otherwise we need transfinite induction. Let $\{x_n\}_{n \geq 1}$ have the property that its span is dense in E.

Intuition. Separability allows us to extend f_0 one dimension at a time. Now, if we can extend f_0 such that $E_0 \to E_0 + \{x_1\} \to E_0 + \{x_1, x_2\} \to \ldots \to E_0 + \operatorname{span}(\{x_n \colon n \geq 1\})$, then we can have $\|f\| = \|f_0\|$, with the final space is dense in E, we can extend f to E by continuity.

To extend f by 1 dimension, i.e., $E \to E + \{x_1\}$, first note that extension is determined by a single number $\gamma = f(x_1)$ since f is a linear functional. We want that $\|f\| = \|f_0\|$ such that the linear functional $f_0 \colon E_0 \to \mathbb{R}$ extends to $f \colon D_0 + \{x_1\} \to \mathbb{R}$, i.e., we want

$$|f_0(x_0) + \lambda\gamma| \leq \|x_0 + \lambda x_1\|$$

for $x_0 \in E$, $\lambda \in \mathbb{R}$. By dividing the inequality by $\lambda \neq 0$, it's sufficient to find γ such that $|f_0(x_0) + \gamma| \leq \|x_0 + x_1\|$, $x_0 \in E_0$.

Suppose f_0 is a real-valued function, we need

$$-\|x_0 + x_1\| \leq f_0(x_0) + \gamma \leq \|x_0 + x_1\|$$

for all $x_0 \in E_0$. Such a γ exists, provides $\|x_0 + x_1\| - f_0(x_0) \geq -\|x_0' + x_1\| - f_0(x_0')$ for all $x_0, x_0' \in E_0$. Furthermore, this is equivalent to write

$$f_0(x_0 - x_0') \leq \|x_0 + x_1\| + \|x_0' + x_1\|$$

for all $x_0, x_0' \in E_0$, i.e., $f_0(x_0 - x_0') \leq \|x_0 + x_1\| + \|-x_1 - x_0'\|$ for $x_0, x_0' \in E_0$. Recall that $\|f_0\| = 1$, we have

$$f_0(x_0 - x_0') \leq \|x_0 - x_0'\| \leq \|x_0 + x_1\| + \|-x_1 - x_0'\|.$$

For complex valued f, consider $f \colon E \to \mathbb{C}$ be a linear functional over $\mathbb{C}$ and let $g(x) = \operatorname{Re} f(x)$. Then $g \colon E \to \mathbb{R}$ is a real-valued linear functional. We see that $f(x) = g(x) - ig(ix)$ for all $x \in E$.[a] Conversely, if $g \colon E \to \mathbb{R}$ is a real linear functional on Banach space E over $\mathbb{C}$, then $f \colon E \to \mathbb{C}$ defined by $f(x) = g(x) - ig(ix)$, $x \in E$ is a complex linear functional on E.

But we need to be a bit careful since when we extend $f_0 \colon E_0 \to \mathbb{C}$, we're extending 2 real dimensions since for $g_0 = \operatorname{Re} f_0$, we need to do $E_0 \to E_0 + \{x_1\} \to E_0 \to \{x_1, x_2\}$. Again, define $f(\cdot) = g(\cdot) - ig(i\cdot)$, we want to show $|f| = \|f_0\|$. We use the fact that for $x \in E_0 + \{\lambda x_0 \colon \lambda \in \mathbb{C}\}$,

$$e^{i\theta} f(x) = f(xe^{i\theta})$$

for $\theta \in \mathbb{R}$. Choose θ such that $f(xe^{i\theta}) = g(xe^{i\theta})$, and since we already have $\left|g(xe^{i\theta})\right| \leq \|f_0\| \left\|xe^{i\theta}\right\|$, we see that $|f(x)| \leq \|f_0\| \|x\|$ for $x \in E_0 + \{\lambda x_1 \colon \lambda \in \mathbb{C}\}$. ■

[a] Since $f(ix) = if(x)$, hence $g(ix) = -\operatorname{Im} f(x)$.

Before we end this section, we see some corollaries of Hahn-Banach theorem. From supporting-functional theorem, we see that for every vector x, we indeed attain its norm on some functional $f \in E^*$, i.e., their supporting functional. But recall that the norm of a functional $f \in E^*$ is defined as

$$\|f\| := \sup_{x \neq 0} \frac{|f(x)|}{\|x\|},$$

and in general, f will not attain its norm on some vector x. This surprising observation leads to the following.

Corollary 2.3.1. For every vector x in a normed space E,

$$\|x\| = \max_{f \neq 0} \frac{|f(x)|}{\|f\|}$$

where the maximum is taken over all non-zero linear functionals $f \in E^*$.

Hahn-Banach theorem implies that there are enough bounded linear functionals $f \in E^*$ on every space E. One manifestation of this is the following.

Corollary 2.3.2 (Separation of points)**.** For every two vectors $x_1 \neq x_2$ in a normed space E, there exists a functional $f \in E^*$ such that $f(x_1) \neq f(x_2)$.

Proof. The supporting functional $f \in E^*$ of the vector $x = x_1 - x_2$ must satisfy

$$f(x_1 - x_2) = \|x_1 - x_2\| \neq 0,$$

as required. ■

2.3.1 Second Dual Space

Let E be a normed space, then the functionals f^* are designed to act on vectors $x \in E$ via

$$f : x \mapsto f(x).$$

But indeed, we can instead say that *vectors $x \in E$ act on functionals $f \in E^*$* via

$$x : f \mapsto f(x).$$

Thus, a vector $x \in E$ can itself be considered as a function from E^* to $\mathbb{R}$. Furthermore, this function x is clearly linear, so we may consider x as a linear functional on E^*. Also, the inequality

$$|f(x)| \leq \|x\| \, \|f\|$$

shows that this functional is bounded, so $x \in E^{**}$. We may instead write x as x^{**} for clarity. Note that the norm of x^{**} as a functional is $\|x^{**}\|_{E^{**}} \leq \|x\|$ since

$$\|x^{**}\| = \sup_{\substack{\|f\|=1 \\ f \in E^*}} |x^{**}(f)| = \sup_{\substack{\|f\|=1 \\ f \in E^*}} |f(x)| \leq \|x\|,$$

implying that $\|x^{**}\| \leq \|x\|$ for all $x \in E$. But from supporting functional $f \in E^*$ of x, we actually have

$$\|x^{**}\| = \|x\|,$$

i.e., we have a *canonical embedding* of E into E^{**}. The above discussion leads to the second dual space theorem.

Theorem 2.3.3 (Second dual space)**.** Let E be a normed space. Then E can be considered as a linear subspace of E^{**}. For this, a vector $x \in E$ is considered as a bounded linear functional on E^* via the action

$$x : f \mapsto f(x), \quad f \in E^*.$$

To characterize the canonical embedding, we have the following definition.

Definition 2.3.1 (Reflexive space)**.** A normed space E is called *reflexive space* if $E = E^{**}$ under the canonical embedding.

Example. L^p spaces for $1 < p < \infty$ are reflexive spaces.

Proof. We know that $L^{p*} = L^q$ where $1 \leq p < \infty$ for q being the conjugate index of p. ⊛

Example. L^p spaces for $p = 1$ or ∞ are not reflexive spaces

Proposition 2.3.1. Let E be a reflexive space, then every linear functional $f \in E^*$ attains its norm on E.

Proof. By reflexivity, the supporting functional of f is a vector $x \in E^{**} = E$, thus $\|x\| = 1$ and $f(x) = \|f\|$, as required. ■

Remark (James' theorem)**.** The converse of Proposition 2.3.1 is also true, i.e., if every functional $f \in E^*$ on a Banach space E attains its norm, then E is are reflexive.

Lecture 9: Hahn-Banach Theorem for Sublinear Functions

From Proposition 2.3.1, we see that to show a Banach space E is not reflexive, it's sufficient to find $f \in E^*$ such that $\sup_{\|x\|=1} |f(x)|$ is not attained. 27 Sep. 14:30

Example. Let $C([0,1])$ be the space of continuous functions $g\colon [0,1] \to \mathbb{C}$ with $\|g\| := \sup_{0\leq t\leq 1} |g(t)|$. Then for $f \in E^*$,

$$f(g) = \int_0^1 h(x)g(x)\,\mathrm{d}x$$

for

$$h(x) = \begin{cases} -1, & \text{if } 0 < x < \dfrac{1}{2}; \\ 1, & \text{if } \dfrac{1}{2} < x < 1. \end{cases}$$

Then we have $\|f\| = 1 = \sup_{\|g\|=1} |f(g)|$, but the supremum is not attained since g needs to be continuous.

2.4 Separation of Convex Sets

In this section, we can extend supporting functional theorem such that we now have it for arbitrary convex sets other than the unit ball. Since supporting functional theorem depends on Hahn-Banach theorem, so we should first generalize Hahn-Banach theorem.

2.4.1 Sublinear Functions

By looking into the proof of Hahn-Banach theorem, we see that we only used positive homogeneity and triangle inequality of the axiom of norm, which suggests we define the following.

Definition 2.4.1 (Sublinear)**.** Let E be a linear vector space. a function $\|\cdot\| : E \to [0, \infty)$ is *sublinear* if it satisfies

(a) $\|\lambda x\| = \lambda \|x\|$ for $\lambda \in \mathbb{R}^+$, $x \in E$.

(b) $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in E$.

Remark (Differences from norm). Note that for a sublinear function to be a norm, we need

(a) $\|-x\| = \|x\|$, $x \in E$

(b) $\|x\| = 0 \Rightarrow x = 0$.

Theorem 2.4.1 (Hahn-Banach theorem for sublinear functions). Let E_0 be a subspace of a linear vector space over $\mathbb{R}$. Let $\|\cdot\|$ be a sublinear functional on E, and $f_0 \colon E_0 \to \mathbb{R}$ be a linear functional on E_0 satisfying $f_0(x) \leq \|x\|$ for $x \in E_0$. Then f_0 admits an extension f to E such that $f(x) \leq \|x\|$ for $x \in E$.

Proof. The idea is the same from Hahn-Banach theorem. ∎

2.4.2 Geometric Properties of Sublinear Functions

We see that by considering sublinear functionals instead of norms offers us more flexibility in geometric applications. In particular, sublinear functionals arise as Minkowski functionals of convex sets.

Definition 2.4.2 (Absorbing). A subset K of a linear vector space is *absorbing* if

$$E = \bigcup_{t \geq 0} tK$$

where $tK \coloneqq \{tk \colon k \in K\}$.

Definition 2.4.3 (Minkowski functional). Let K be an absorbing convex subset of a linear vector space E such that $0 \in K$. Then the *Minkowski functional* $\|\cdot\|_K$ is defined as

$$\|x\|_K \coloneqq \inf \{t > 0 \colon x/t \in K\}.$$

Proposition 2.4.1. Let K be an absorbing convex subset of a linear vector space E such that $0 \in K$. Then Minkowski functional $\|x\|_K$ is a sublinear functional on E. Conversely, let $\|\cdot\|$ be a sublinear functional on a linear vector space E, then the sub-level set

$$K = \{x \in E \colon \|x\| \leq 1\}$$

is an absorbing convex set, and $0 \in K$.

Proof. To prove the forward direction, the main observation is that since $0 \in K$ and K is convex, then $x \in K \Rightarrow tx \in K$ if $0 \leq t < 1$. To show dilation, for $\lambda > 0$,

$$\|\lambda x\| = \inf \left\{ t > 0 \colon x \in \frac{t}{\lambda} K \right\} = \lambda \inf \{s > 0 \colon x \in sK\} = \lambda \|x\|.$$

To show triangle inequality, suppose $x \in tK$, $y \in sK$, then $x = tk_1$, $y = sk_2$ for some $k_1, k_2 \in K$. We then have

$$x + y = (t+s)\left(\frac{t}{t+s}k_1 + \frac{s}{t+s}k_2\right) = (t+s)k$$

for some $k \in K$ since K is convex, hence $x + y \in (t+s)K$, we then have $\|x+y\| \leq \|x\| + \|y\|$.

Now, if $\|\cdot\|$ is sublinear, then $K = \{x \in E \colon \|x\| \leq 1\}$ is absorbing, convex and $0 \in K$.[a] ∎

[a] $0 \in K$ since $\|0\| = 0$, while the convexity comes from the triangle inequality.

Remark. If $K \neq -K$, then $\exists x \in E$ with $\|x\| \neq \|-x\|$. If $K = E$, then $\|\cdot\| \equiv 0$.

2.4.3 Separation of Convex Sets

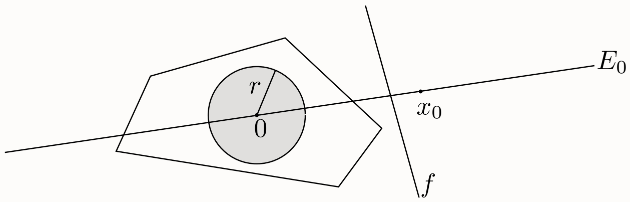
Hahn-Banach theroem has some remarkable geometric implications, which are grouped together under the name of *separation theorems*. Under mild topological requirements, these results guarantee that two convex sets A, B can always be separated by a hyperplane.

Theorem 2.4.2 (Separation of a point from a convex set). Let K be an open convex subset of a normed space E and $x_0 \notin K$. Then there exists a continuous linear functional $f\colon E \to \mathbb{R}$ with $f \neq 0$ and $f(x) < f(x_0)$ for $x \in K$.

Proof. By translation, we can assume without loss of generality that $0 \in K$. Since K is open, it is absorbing. Now, let $\|\cdot\|_K$ be the Minkowski functional, then

$$\|x\|_K \leq \frac{1}{r}\|x\|$$

for $x \in E$ if $B(0,r) \subseteq K$.



Proceed as in supporting functional theorem for unit ball, we define f_0 on $\operatorname{span}(\{x_0\})$ by

$$f_0(tx_0) = t\|x_0\|_K$$

for $t \in \mathbb{R}$. Then if $E_0 = \{\lambda x_0 \colon \lambda \in \mathbb{R}\}$, $f_0(x) \leq \|x\|_K$ for $x \in E_0$ (i.e., $\|\cdot\|_K$ dominates f_0) since for $t \geq 0$,

$$f_0(tx_0) = t\|x_0\|_K = \|tx_0\|_K;$$

while for $t \leq 0$,

$$f_0(tx_0) = t\|x_0\|_K \leq 0 \leq \|tx_0\|_K.$$

Then from Hahn-Banach theorem, we can extend f_0 to $f\colon E \to \mathbb{R}$ such that

$$f(x) \leq \|x\|_K \leq \frac{1}{r}\|x\|$$

for $x \in E$, hence $f \in E^*$. For separation, we see that if $x \in K$ (hence in E),

$$f(x) \leq \|x\|_K \leq 1 \leq \|x_0\|_K = f_0(x_0) = f(x_0),$$

hence $f(x) \leq f(x_0)$. To get a strict separation, since K is open, so $x + tv \in K$ for $x \in K$ and some $t > 0$ and all v with $\|v\| = 1$. Hence, for all $t = t_x > 0$, we have

$$f(x+tv) \leq f(x_0) \Rightarrow f(x) + t \sup_{\|v\|=1} f(v) \leq f(x_0).$$

With the fact that $f \neq 0$, so $\|f\| = \sup_{\|v\|=1} f(v) \neq 0$, we conclude that

$$f(x) < f(x_0).$$

■

A more general version holds.

Theorem 2.4.3 (Separation of convex sets). Let A, B be disjoint convex subsets of a Banach space E.

(a) If A is open, then there $\exists f\colon E \to \mathbb{R}$ such that $f(a) < f(b)$ for $a \in A$, $b \in B$.

(b) If A, B are closed and B is compact, then there $\exists f\colon E \to \mathbb{R}$ such that $\sup_{a\in A} f(a) < \inf_{b\in B} f(b)$.

Proof. We have the following.

(a) Let $K = A - B = \{a - b\colon a \in A, b \in B\}$, we then see that K is open, convex and $0 \notin K$. Since we can separate a point from a convex set, there exists $f \in E^*$ such that

$$f(a-b) < f(0) = 0$$

for $a \in A$, $b \in B$, hence $f(a) < f(b)$ for $a \in A$, $b \in B$.

(b) Let A be closed, B be compact. Then we have

$$d(A, B) = \inf\{\|x - y\| : x \in A, y \in B\} = r > 0.$$

Define $A_\delta \coloneqq \{x \in E\colon d(x, A) < \delta\}$ where A_δ is open. By setting $\delta \coloneqq r/2$, we have $A_\delta \cap B = 0$. From (a), we see that there exists $f \in E^*$ such that $f(x) < f(y)$ for $x \in A_\delta$, $y \in B$. Then $a \in A$ implies $a + \delta/2v \in A_\delta$ if $\|v\| = 1$, hence

$$f(a + \delta/2v) < f(b)$$

for $b \in B$. So

$$f(a) + \frac{\delta}{2} f(v) < f(b)$$

for $b \in B$, $\|v\| = 1$. Take the supremum over $\|v\| = 1$, we have $\sup_{\|v\|=1} |f(v)| = \delta > 0$, implying $f(a) < f(b) - \delta$, $a \in A$, $b \in B$. Finally, we have

$$\sup_{a\in A} f(a) < \inf_{b\in B} f(b).$$

■

Lecture 10: Adjoint Operators and Ergodic Theorem

Before ending this section, we have this final characterization of convex sets: they're intersections of half-spaces! 29 Sep. 14:30

Definition 2.4.4 (Half-space)**.** A *half-space* $H \subseteq E$ has the form of

$$H = \{x \in E\colon f(x) \leq \lambda\}$$

for $f \in E^*$, i.e., it is what lies on one side of a hyperplane.

Corollary 2.4.1. Let $K \subseteq E$ be closed convex set. Then K is the intersection of all half-spaces containing K.

Proof. Firstly, K is trivially contained in the intersection of the half-spaces that contain K. Denote such an intersection as S, then we have $K \subseteq S$. On the other hand, to show $K \supseteq S$, if $x_0 \notin K$, we show that there's a half-space contains K but not x_0, hence $x_0 \notin S$ too, i.e., $S \subseteq K$.

From separation of convex sets theorem with $A = K$ and $B = \{x_0\}$, there exists $f \in E^*$ such that $\lambda \coloneqq \sup_{k\in K} f(k) < f(x_0)$. We then see that the half-space $\{x \in E\colon f(x) \leq \lambda\}$ contains K but not x_0. ■

2.5 Bounded Linear Operators

Turns out that we can generalize the notion of linear functionals $f\colon E \to \mathbb{R}$ or $\mathbb{C}$ by further abstracting out the domain by another Banach space.

As one can imagine, several results for linear operators will be generalizations of those we have already seen for linear functionals, but there'll be important differences though. For example, a natural extension of Hahn-Banach theorem fails for linear operators.

Firstly, same as before, the operator norm is defined as follows, which is a norm on bounded linear operators.

Definition 2.5.1 (Operator norm)**.** Given an operator T, its *operator norm* is defined as

$$\|T\| := \sup_{\|x\|=1} \|Tx\| .$$

2.5.1 Continuity and Boundedness

As for Definition 2.1.2, we have the following.

Definition (Bounded linear operator)**.** Let X, Y be two Banach spaces and let T be a linear operator between X and Y. Then we say T is *bounded* if $\|T\| < \infty$.

Remark (Bounded operator)**.** We can also talk about boundedness of a(n) (nonlinear) operator T just the same as requiring $\|T\| < \infty$.

As before, given Definition 2.5.1, we always have

$$\|Tx\| \leq \|T\| \|x\|$$

for a linear operator $T\colon X \to Y$, $x \in X$.

Definition 2.5.2 (Lipschitz)**.** The operator T is called *Lipschitz* if

$$\|Tx - Ty\| \leq \|T\| \|x - y\|$$

for $x, y \in E$.

Remark (Continuity and Boundness)**.** Same as linear functionals, the continuity and boundedness of linear operators are equivalent.

2.5.2 Space of Operators

Let X and Y be normed space, and let $\mathcal{L}(X, Y)$ be the space of bounded linear operators $T\colon X \to Y$, then $\mathcal{L}(X, Y)$ is a Banach space under the norm $T \to \|T\|$.

Example. The dual space of E is just $E^* = \mathcal{L}(E, \mathbb{R})$.

Remark. In particular, we have

(a) $\|T\| = 0 \Leftrightarrow T = 0$.

(b) $\|\lambda T\| = |\lambda| \|T\|$ for $\lambda \in \mathbb{R}$ or $\mathbb{C}$, $T \in \mathcal{L}(X, Y)$.

(c) $\|T + S\| \leq \|T\| + \|S\|$, $T, S \in \mathcal{L}(X, Y)$.

(d) $\|TS\| \leq \|T\| \|S\|$, $T, S \in \mathcal{L}(X, Y)$.

2.5.3 Adjoint Operators

The concept of adjoint operators is a generalization of matrix transpose in linear algebra. Recall that if $A = (a_{ij})$ is an $n \times n$ matrix with complex entries, then the Hermitian transpose of A is an $n \times n$ matrix

$A^* = (\overline{a_{ij}})$. The transpose thus satisfies the identity

$$\langle A^* x, y\rangle = \langle x, Ay\rangle$$

for $x, y \in \mathbb{C}^n$. We now extend this to linear operators.

Definition 2.5.3 (Adjoint operator)**.** Let $T \in \mathcal{L}(X, Y)$, the *adjoint* $T^* \in \mathcal{L}(Y^*, X^*)$ of T is defined as

$$T^* f \colon X \to \mathbb{R} \text{ or } \mathbb{C}$$

for $f \in Y^*$, and $T^* f(x) = f(Tx)$ for $x \in X$.

We should note that T^* is indeed a bounded linear operator since

$$|T^* f(x)| = |f(Tx)| \leq \|f\| \, \|Tx\| \leq \|f\| \, \|T\| \, \|x\|$$

for $x \in X$, hence $T^* f$ is a linear functional where

$$\|T^* f\| = \sup_{\|x\|=1} |T^* f(x)| \leq \sup_{\|x\|=1} \|f\| \, \|T\| \, \|x\| = \|f\| \, \|T\| \, ,$$

hence, $T^* f \in X^*$ and $\|T^* f\| \leq \|T\| \, \|f\|$. So, we have $T^* \colon Y^* \to X^*$ with T^* being a linear operator and T^* is bounded with

$$\|T^*\| \leq \|T\| \, .$$

In fact, we can achieve equality, which is shown in Proposition 2.5.1.

Proposition 2.5.1. For every $T \in \mathcal{L}(X, Y)$, the adjoint T^* is in $\mathcal{L}(Y^*, X^*)$ with $\|T^*\| = \|T\|$.

Proof. Since

$$\begin{aligned}
\|T^*\| = \sup_{\|f\|_{Y^*}=1} \|T^* f\|_{X^*} &= \sup_{\|f\|_{Y^*}=1} \sup_{\|x\|_X=1} |T^* f(x)| \\
&= \sup_{\|f\|_{Y^*}=1} \sup_{\|x\|_X=1} |f(Tx)| = \sup_{\|x\|_X=1} \sup_{\|f\|_{Y^*}=1} |f(Tx)| \, .
\end{aligned}$$

By choosing f to be a supporting functional of Tx, $\sup_{\|f\|_{Y^*}=1} |f(Tx)| = \|Tx\|_{Y^*}$, hence

$$\|T^*\| = \sup_{\|x\|_X=1} \|Tx\|_{Y^*} = \|T\| \, .$$

■

Let's look at some properties of adjoint operators. Let $T, S \in \mathcal{L}(X, Y) \Rightarrow T^*, S^* \in \mathcal{L}(Y^*, X^*)$, then

(a) $(aT + bS)^* = aT^* + bS^*$, $a, b \in \mathbb{R}$ or $\mathbb{C}$. Also, $(aT)^* f(x) = f(aTx) = af(Tx) = aT^* f(x)$.

(b) $(ST)^* = T^* S^*$. This implies that if $T \in \mathcal{L}(X, X)$ is invertible, then $T^* \in \mathcal{L}(X^*, X^*)$ is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Remark (Adjoint operators on Hilbert spaces)**.** Specialize to Hilbert space $\mathcal{H}$, then by Riesz representation theorem, $\mathcal{H}^* \equiv \mathcal{H}$, i.e., $f \in \mathcal{H}^* \Leftrightarrow \exists y \in \mathcal{H}$ such that $f(x) = \langle x, y\rangle$ for $x \in \mathcal{H}$. Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$, and $T^* \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*)$ with

$$T^* f(x) = f(Tx) = \langle Tx, y\rangle$$

for $x, y \in \mathcal{H}$, $f \in \mathcal{H}^*$. By writing $T^* f(x) = \langle x, T^* y\rangle$, which defined $T^* y \colon \mathcal{H} \to \mathcal{H}$, hence $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for $x, y \in \mathcal{H}$. Clearly, T^* is a bounded linear operator on $\mathcal{H}$, i.e., $T^* \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*)$ since

$$\|T^*\| = \sup_{\|y\|=1} \|T^* y\| = \sup_{\|y\|=\|x\|=1} \langle x, T^* y\rangle = \sup_{\|y\|=\|x\|=1} \langle Tx, y\rangle = \|T\|$$

just like Proposition 2.5.1. We see that $T^* \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*) \Rightarrow T^* \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ via Riesz representation. Note that if $T^* \in \mathcal{L}(\mathcal{H}, \mathcal{H})$,

$$(aT)^* = \overline{a} T^*$$

for $a \in \mathbb{C}$.

Just as with Hilbert space, we have a generalized notion of orthogonality, which we call annihilator.

Definition 2.5.4 (Annihilator)**.** Let $A \subseteq X$ where X is a Banach space, then the *annihilator* $A^{\perp}$ of A is a subset of X^* defined as

$$A^{\perp} := \{f \in X^* \colon f(x) = 0, x \in A\}.$$

Note. $A^{\perp}$ is a closed linear subspace of X^*.

Proposition 2.5.2. Given two Banach spaces X and Y, let $T \in \mathcal{L}(X, Y)$ and $T^* \in \mathcal{L}(Y^*, X^*)$. Then $(\operatorname{Im} T)^{\perp}, \ker(T^*) \subseteq Y^*$ satisfy

$$(\operatorname{Im} T)^{\perp} = \ker(T^*).$$

Proof. Since $f \in (\operatorname{Im} T)^{\perp} \Leftrightarrow f(Tx) = 0$ for all $x \in X$, i.e., $T^* f(x) = 0 \Leftrightarrow T^* f = 0 \Leftrightarrow f \in \ker(T^*)$, proving the result. ∎

Corollary 2.5.1. Let $\mathcal{H}$ be a Hilbert space, and $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$. Then the orthogonal decomposition holds, i.e.,

$$\mathcal{H} = \overline{\operatorname{Im} T} \oplus \ker(T^*).$$

Proof. By Proposition 2.5.2, $\ker(T^*) = (\operatorname{Im} T)^{\perp}$. And since $\mathcal{H}$ is Hilbert space, $\overline{\operatorname{Im} T} = \operatorname{Im} T$ from the fact that if $E \subseteq \mathcal{H}$, $(E^{\perp})^{\perp} = \overline{E}$, hence $(\overline{\operatorname{Im} T})^{\perp} = \ker T^*$. Then by using orthogonality principle, the proof is complete. ∎

2.5.4 Ergodic Theory

We now see an application on ergodic theorems. Ergodic theorems allow one to compute space averages as time averages. Given a probability space $(\Omega, \mathcal{F}, P)$ with $P(\Omega) = 1$, let $T \colon \Omega \to \Omega$ be a measurable map, i.e., $T^{-1}A \in \mathcal{F}$ if $A \in \mathcal{F}$. Then, we define the following.

Definition 2.5.5 (Measure-preserving)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A transformation $T \colon \Omega \to \Omega$ is called *measure-preserving* if

$$P(T^{-1}A) = P(A)$$

for $A \in \mathcal{F}$, where $T^{-1}A = \{\omega \in \Omega \colon T\omega \in A\}$.

Let's first see some examples which illustrate the so-called *time and space averages*. We start with simple dynamical systems corresponding to rotation.

Example (Rotation)**.** Let $\Omega = [0, 1]$, P be the Lebesgue measure and $\mathcal{F}$ be Borel sets. Given $\lambda \in \mathbb{R}$, define

$$T\omega = \omega + \lambda \bmod 1.$$

This is equivalent to rotation on the unit circle through an angle $2\pi\lambda$, and we see that T is measure-preserving and one-to-one, and T^{-1} exists.

Example (Shift Operator)**.** Let $\Omega = [0, 1]$, P be the Lebesgue measure and $\mathcal{F}$ be Borel sets. Now, let

$$T\omega = 2\omega \bmod 1.$$

Then we see that T is just the shift operator on the binary representation, i.e., given $\omega = \sum_{j=1}^{\infty} \frac{a_j}{2^j}$

for $a_j = 0$ or 1, then

$$T\omega = \sum_{j=1}^{\infty} \frac{a_{j+1}}{2^j}.$$

Now, let the *dyadic interval* $I_{n,k}$ be defined as

$$I_{n,k} \coloneqq \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$$

for $1 \leq k \leq 2^n$, we have $T^{-1}I_{n,k} = I_{n+1,k} \cup I_{n+1,k+2^n}$, hence $P(T^{-1}I_{n,k}) = P(I_{n,k})$ for all dyadic intervals $I_{n,k}$. This implies

$$P(T^{-1}O) = P(O)$$

for all $O \in \mathcal{F}$, hence T is measure-preserving, but not one-to-one. In fact, T is a two-to-one mapping. The action of T is $[0, 1/2] \xrightarrow{T} [0, 1]$, $[1/2, 1] \xrightarrow{T} [0, 1]$. We see that T doubles the length of a dyadic interval. To summarize,

- T is measure-preserving since it is two-to-one.
- T is an expanding map, which is called hyperbolic.

Lecture 11: Ergodic Theorem and Open Mapping

Now, we're ready to discuss ergodic theorem formally. Suppose $T\colon \Omega \to \Omega$ is measure-preserving, we can associate operator U on $L^2(\Omega)$ by defining $Uf(\omega) = f(T\omega)$ for $f \in L^2(\Omega)$ and $\omega \in \Omega$. Notice that 4 Oct. 14:30

$$\int_\Omega f(T\omega) \,\mathrm{d}\mu(\omega) = \int_\Omega f(\omega) \,\mathrm{d}\mu(\omega)$$

for all $f \in L^1(\Omega)$,[1] so for $\varphi \in L^2(\Omega)$, $U\varphi(\omega) = \varphi(T\omega)$ and since

$$\langle U\varphi, U\psi\rangle = \int_\Omega \varphi(T\omega)\psi(T\omega) \,\mathrm{d}\mu(\omega) = \int_\Omega \varphi(\omega)\psi(\omega) \,\mathrm{d}\mu(\omega) = \langle \varphi, \psi\rangle$$

for $\varphi, \psi \in L^2(\Omega)$, we see that U is a bounded linear operator on $\mathcal{H} = L^2(\Omega)$ with $\|U\| = 1$, $\|U\varphi\| = \|\varphi\|$, $\varphi \in \mathcal{H}$. In addition, for $\varphi, \psi \in \mathcal{H}$, $\langle U\varphi, U\psi\rangle = \langle \varphi, \psi\rangle$ implies $\langle U^*U\varphi, \psi\rangle = \langle \varphi, \psi\rangle$, which further implies $U^*U = I$, so U is one-to-one. Let's first see one more definition before we proceed.

Definition 2.5.6 (Unitary operator). A *unitary operator* is a bounded linear operator $U\colon \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ such that U is surjective and for all $x, y \in \mathcal{H}$,

$$\langle Ux, Uy\rangle_{\mathcal{H}} = \langle x, y\rangle_{\mathcal{H}}.$$

Notice that U is not necessarily onto. However, if U is indeed onto, then $UU^* = U^*U = I$, implying that U is a unitary operator on $\mathcal{H}$ and invertible.

Note. U is invertible if and only if T is one-to-one.

Proof. Since U just need to be onto for U being invertible, with $U^*\varphi(\omega) = \varphi(T^{-1}\omega)$ for $\omega \in \Omega$, if T is one-to-one then T^{-1} is onto, implying U^* is onto, so is U. ⊛

Remark. $T\colon \Omega \to \Omega$ is one-to-one implies T is almost onto.

Proof. Let A be a set such that $T(\Omega) \subset A$, and hence $T^{-1}A = \Omega$ so $P(T^{-1}A) = P(\Omega) = 1$, implying that $P(A) = 1$, hence $P(\Omega \setminus A) = 0$. ⊛

In the case T is not invertible (e.g. a 2-1 mapping), one might expect a similar formula for U^*. In the shift operator example, $T_1\colon [0, 1/2] \to [0, 1]$, $T_2\colon [1/2, 1] \to [0, 1]$, and T_1, T_2 are invertible, we have

$$U^*\varphi(\omega) = \frac{1}{2}\left(\varphi(T_1^{-1}\omega) + \varphi(T_2^{-1}\omega)\right).$$

[1] This is true by letting $f = \mathbb{1}_A$ and then extend to $L^1(\Omega)$.

Definition 2.5.7 (Ergodic transformation)**.** A one-to-one, measure-preserving transformation T is *ergodic* if the only functions $f \in L^2(\Omega, \mathcal{F}, P)$ which satisfy $f(T\omega) = f(\omega)$ for almost all $\omega \in \Omega$ are the constant functions.

Remark (Eigenfunction)**.** Phrasing differently, a measure-preserving mapping $T \colon \Omega \to \Omega$ is ergodic if and only if the only eigenfunction $\varphi \in L^2(\Omega)$ of the corresponding operator U is the constant function, i.e. $U\varphi = \varphi$ implying φ is a constant.

Lemma 2.5.1. A measure-preserving mapping $T \colon \Omega \to \Omega$ is ergodic if and only if invariant sets of T have probability 0 or 1, i.e. if $A \in \mathcal{F}$ satisfies

$$P((A - T^{-1}A) \cup (T^{-1}A - A)) = 0,$$

then $P(A) = 0$ or $P(A) = 1$.

Proof. Assume T is not ergodic, then there exists $\varphi \in L^2(\Omega)$ such that $U\varphi = \varphi$. Hence, we can find $a, b \in \mathbb{R}$, $a < b$ such that $A = \{\omega \in \Omega \colon a < \varphi(\omega) < b\}$ has $0 < P(A) < 1$. However,

$$T^{-1}A = \{\omega : \ T\omega \in A\} = \{\omega : \ a < \varphi(T\omega) < b\} = \{\omega : \ a < \varphi(\omega) < b\} = A,$$

and thus A is invariant.

Conversely, suppose $A \in \mathcal{F}$, we have $A = T^{-1}A$ up to measure-zero sets and $0 < P(A) < 1$, then $\varphi = \mathbb{1}_A$ satisfies $U\varphi = \varphi \in L^2(\Omega)$ with the fact that φ is not constant., proving the result. ■

Proposition 2.5.3. Suppose $T \colon \Omega \to \Omega$ is measure-preserving and $\varphi \in L^2(\Omega)$, $\mathbb{E}[\varphi] = 0$, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \varphi(T^n \cdot) \to 0$$

in $L^2(\Omega)$.

Proof. Note it suffices to assume $\mathbb{E}[\varphi] = 0$. We want to show

$$\lim_{N \to \infty} \frac{1}{N} [I + U + U^2 + \ldots + U^{N-1}] \varphi(\cdot) = 0$$

in $L^2(\Omega)$. If φ is orthogonal to the constant function. Since $\mathbb{E}[\varphi] = 0$, then $\langle \varphi, 1 \rangle = 0$. Define a *derivative* operator on $L^2(\Omega)$ such that

$$D\varphi = (U - I)\varphi = \varphi(T\cdot) - \varphi(\cdot).$$

Using the fundamental theorem of calculus argument,

$$[I + U + U^2 + \ldots + U^{N-1}] D\varphi = (U^N - I)\varphi.$$

Hence,

$$\left\| \frac{I + U + U^2 + \ldots + U^{N-1}}{N} \varphi \right\| \leq \frac{2\left\|\psi\right\|}{N}$$

if $\varphi = D\psi$. In that case that as $N \to \infty$ is zero, i.e. if $\varphi \in \operatorname{Im}(D) \subset \mathcal{H} = L^2(\Omega)$, then we're done. Note also that

$$\left\| \frac{I + U + U^2 + \ldots + U^{N-1}}{N} \right\| \leq 1$$

since $\|U\| = 1$. Hence, converge to zero if $\varphi \in \overline{\mathrm{Im}(D)}$, which implies that there exists $\varphi_\epsilon \in \mathrm{Im}(D)$ such that $\|\varphi_\epsilon - \varphi\| < \epsilon$, i.e.,

$$\left\| \frac{I + U + \ldots + U^{N-1}}{N} (\varphi_\epsilon - \varphi) \right\| < \epsilon.$$

Recall $\overline{\mathrm{Im}(D)} \oplus \ker(D^*) = \mathcal{H} = L^2(\Omega)$. It suffices to show $\ker(D^*)$ is spanned by constant functions. Note T is ergodic implies $\ker(D)$ is spanned by constants, we have $D\varphi = 0 \Leftrightarrow U\varphi = \varphi$, and

$$(D^*\varphi = 0 \Leftrightarrow U^*\varphi = 0) \Rightarrow (\langle \varphi, U^*\varphi, \varphi \rangle = \langle \varphi, \varphi \rangle).$$

Therefore, we have $\langle U\varphi, \varphi \rangle = \langle \varphi, \varphi \rangle$, and also,

$$\int \varphi(T\omega)\varphi(\omega) \,\mathrm{d}P(\omega) = \int \varphi(\omega)^2 \,\mathrm{d}\omega = \int \varphi(T\omega)^2 \,\mathrm{d}\omega,$$

which implies

$$\frac{1}{2} \int [\varphi(T\omega)^2 + \varphi(\omega)^2] \,\mathrm{d}P(\omega) - \int \varphi(T\omega)\varphi(\omega) \,\mathrm{d}P(\omega) = 0.$$

i.e. $\frac{1}{2} \int [\varphi(T\omega) - \varphi(\omega)]^2 \,\mathrm{d}P(\omega) = 0$, which means

$$\varphi(T\omega) = \varphi(\omega), \ \omega \in \Omega.$$

i.e. $\varphi \equiv$ constant by ergodicity. ■

Theorem 2.5.1 (von Newmann ergodic theorem). Suppose $T \colon \Omega \to \Omega$ is measure-preserving, then for any $\varphi \in L^2(\Omega)$, one has

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \varphi(T^n \cdot) = \int_\Omega \varphi(\omega) \,\mathrm{d}P(\omega).$$

Remark. Convergence is in the $L^2(\Omega)$ sense, i.e. mean square.

Chapter 3

Main Principles of Functional Analysis

In this chapter, we'll study three of the fundamental theorems in functional analysis, which together with Hahn-Banach theorem, form the main principles of functional analysis. Those are the open mapping theorem, closed graph theorem and the uniform boundedness principle.

3.1 Open Mapping Theorem

Suppose $T\colon X \to Y$ is a bounded linear operator on Banach spaces, and T is injective and surjective, i.e. $T^{-1}\colon Y \to X$ exists. We'll soon see that the open mapping theorem implies T^{-1} is a bounded operator, where the main argument relies on Baire category theorem.

Definition 3.1.1 (Nowhere dense). A set S in a metric space M is *nowhere dense* if its closure $\overline{S}$ has empty interior.

Example (Cantor set). The Cantor set is a nowhere dense set.

Lecture 12: Open Mapping Theorem

Let's start with Baire category theorem. 6 Oct. 14:30

Proposition 3.1.1 (Baire category theorem). A complete metric space M is never the union of a countable number of nowhere dense sets.

Proof. We prove this by contradiction. Assume $M = \bigcup_{n=1}^{\infty} A_n$ with each A_n nowhere dense. Since A_1 is nowhere dense, so we can find $x_1 \in M - \overline{A}_1$. Furthermore, since $\overline{A}_1$ is closed, so we can find open ball B_1 centered at x_1 with radius less or equal to 1 such that $B_1 \cap A_1 = \varnothing$.

Similarly, A_2 is nowhere dense, so there exists $x_1 \in B_1 - \overline{A}_2$, with $\overline{A}_2$ closed, we can still find ball B_2 centered at x_2 with radius less or equal to $1/2$ such that

$$x_2 \in B_2 \subseteq \overline{B}_2 \subseteq B_1$$

and $B_2 \cap A_2 = \varnothing$. By induction, we can find a sequence $\{x_n\}_{n=1}^{\infty}$ and open balls B_n such that

$$x_{n+1} \in B_{n+1} \subseteq \overline{B}_{n+1} \subseteq B_n$$

where B_n has radius smaller than $1/2^{n-1}$ and $B_n \cap A_n = \varnothing$.

Now, since the sequence $\{x_n\}$ is Cauchy and M is complete, we know that $x_n \to x_\infty \in M$, so $x_\infty \in B_n$ for all n and hence $x_\infty \notin A_n$ for all n. This implies

$$M \neq \bigcup_{n=1}^{\infty} A_n,$$

which is a contradiction ↯ ■

We can now prove the central theorem in functional analysis, the open mapping theorem.

Theorem 3.1.1 (Open mapping theorem). Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Assume T is surjective, i.e., $T(X) = Y$, then T maps open sets in X to open sets in Y.

Proof. Let $B_X \coloneqq \{x \in X \mid \|x\| \leq 1\}$ be a unit ball in X, similarly B_Y be a unit ball in Y.

Claim. It's sufficient to show $T(B_X) \supseteq \epsilon B_Y$ for some $\epsilon > 0$.

Proof. To see this, let $U \subseteq X$ be an open set and $y \in TU$. We need to show TU contains a neighborhood of y. Let $x \in U$ such that $Tx = y$. Since U is open, so there exists $\delta > 0$ such that $U \supseteq x + \delta B_X$, so

$$TU \supseteq T(x + \delta B_X) = y + \delta T(B_X) \supseteq y + \delta \epsilon B_Y,$$

i.e., TU contains a neighborhood of y. ⊛

We now show $TB_X \supseteq \epsilon B_Y$ for some $\epsilon > 0$. Observe that $X = \bigcup_{n=1}^{\infty} nB_X$, hence

$$Y = TX = \bigcup_{n=1}^{\infty} nT(B_X).$$

From Baire category theorem, we know that there exists $n \geq 1$ such that $\overline{nT(B_X)}$ has non-empty interior, i.e., $\overline{TB_X}$ has non-empty interior too. Hence, there exists $y \in Y$, $\delta > 0$ such that $y + \delta B_Y \subseteq \overline{TB_X}$. With $TX = Y$, there exists $x \in X$ such that $Tx = y$, hence $\delta B_Y \subseteq \overline{T(B_X - \{x\})}$. Since $B_X - \{x\} \subseteq nB_X$ for some $n \geq 1$, meaning that $\delta B_Y \subseteq n\overline{TB_X}$, implying $\overline{TB_X} \supseteq \epsilon B_Y$ for some $\epsilon > 0$. Finally, we show that $\overline{TB_X} \subseteq T(2B_X)$, which will imply

$$TB_X \supseteq \frac{1}{2}\overline{TB_X} \supseteq \frac{\epsilon}{2} B_Y,$$

completes the proof. To see this, we use a scaling argument. Let $y \in \overline{TB_X}$, then there exists $x_1 \in B_X$ such that

$$y - Tx_1 \in \frac{\epsilon}{2} B_y \subseteq \overline{T\frac{1}{2}B_X}.$$

We can then choose $x_2 \in \frac{1}{2} B_X$ such that

$$y - Tx_1 - Tx_2 \in \frac{\epsilon}{4} B_Y \subseteq \overline{T\frac{1}{2^2}B_X}.$$

By induction, we can construct a sequence $\{x_n\}_{n \geq 1}$ such that

$$x_n \in \frac{1}{2^{n-1}} B_X, \quad y - \sum_{j=1}^{n} Tx_j \in \frac{\epsilon}{2^n} B_Y.$$

Then, $x = \sum_{j=1}^{\infty} x_n \in 2B_X$ where $Tx = y$. ■

3.1.1 Inverse Mapping Theorem

As an immediate consequence of the open mapping theorem, we have the inverse mapping theorem.

Theorem 3.1.2 (Inverse mapping theorem). Let $T \colon X \to Y$ be a bounded linear operator between Banach spaces X and Y which is both injective and surjective. Then T has a bounded inverse $T^{-1} \in \mathcal{L}(Y, X)$.

Proof. Since open mapping theorem states that the preimages of open sets under T^{-1} are open, hence T^{-1} is continuous. ■

Inverse mapping theorem is used to establish stability of solutions of linear equations. Consider a linear equation in x in a Banach space

$$Tx = b$$

for $T \in \mathcal{L}(X, Y)$ and $b \in Y$. Assume that a solution x exists and is unique for every b, then, from inverse mapping theorem, we see that the solution $x = x(b)$ is continuous w.r.t. b. In other words, the solution is stable under perturbations of b. In case T is not injective but is surjective, we can still apply inverse mapping theorem to the injectivization of T as follows.

Corollary 3.1.1 (Surjective operators are essentially quotient maps)**.** Let X, Y be Banach spaces. Then every surjective bounded linear operator $T \in \mathcal{L}(X, Y)$ is a composition of a quotient map and an isomorphism. Specifically,

$$T = \widetilde{T}q,$$

where $q\colon X \to X / \ker(T)$ is the quotient map, $\widetilde{T}\colon X / \ker(T) \to Y$ is an isomorphism.

Proof. Let $\widetilde{T}$ be the injectivization of T then by construction, $T = \widetilde{T}q$ and $\widetilde{T}$ is injective. Since T is surjective, $\widetilde{T}$ is also surjective. Hence, by inverse mapping theorem, $\widetilde{T}$ is an isomorphism. ■

3.1.2 Isomorphic Embeddings

Finally, as we know, the kernel of every bounded linear operators $T \in \mathcal{L}(X, Y)$ is always a closed subspace, while the image of T may or may not be closed. We can also characterize this.

Proposition 3.1.2. Given two Banach spaces X, Y and $T \in \mathcal{L}(X, Y)$, the following are equivalent.

(a) T is injective and $\operatorname{Im}(T)$ is closed.

(b) T is bounded below, i.e., $\exists c > 0$, $\lVert Tx\rVert \geq c\lVert x\rVert$ for all $x \in X$.

Proof. To show that (a) implies (b), we see that $T^{-1}\colon \operatorname{Im}(T) \to X$ is bounded since $\operatorname{Im}(T)$ is Banach space, from open mapping theorem,

$$\lVert T^{-1}y\rVert \leq c^{-1}\lVert y\rVert$$

for $y \in \operatorname{Im}(T)$, $c > 0$ being some constant. Set $y \coloneqq Tx$, then

$$\lVert Tx\rVert \geq c\lVert x\rVert$$

for $x \in X$, we're done. To show another direction, suppose T is bounded below, then T is injective since $Tx = 0$ implies $x = 0$. To see $\operatorname{Im}(T)$ is closed, let $x_n \in X$ for $n \geq 1$ be a sequence such that $\{Tx_n\}_{n\geq 1}$ is Cauchy such that $\lVert Tx_n - Tx_m\rVert \geq c\lVert x_n - x_m\rVert$ for all n, m, implying $\{x_n\}_{n\geq 1}$ is Cauchy, hence $x_n \to x_\infty \in X$, i.e., $Tx_n \to Tx_\infty \in \operatorname{Im}(T)$, proving the result. ■

3.2 Closed Graph Theorem

We now study the second main theorem in functional analysis, which characterizes the property of the graph of a bounded linear operator.

Definition 3.2.1 (Graph)**.** Let $T \in \mathcal{L}(X, Y)$ for X, Y being Banach spaces. Then the *graph* $\Gamma(T)$ of T is defined as

$$\Gamma(T) \coloneqq \{(x, Tx) \in X \times Y \mid x \in X\}.$$

Clearly, $\Gamma(T)$ is a linear subspace of the normed space $X \oplus Y$.

Definition 3.2.2 (Closed graph)**.** The graph $\Gamma(T)$ of T is *closed* if it is a closed subspace of $X \times Y$.

Hence, if $\{x_n\}_{n\geq 1}$ is a sequence in X such that both $\{x_n\}_{n\geq 1}$ and $\{Tx_n\}_{n\geq 1}$ are Cauchy, then there exists $x_\infty \in X$ such that $x_n \to x_\infty$ and $Tx_\infty \to y_\infty$ for $y_\infty = Tx_\infty$.

Theorem 3.2.1 (Closed graph theorem)**.** Let $T\colon X \to Y$ be a linear operator between Banach spaces X and Y. Then T is bounded (continuous) if and only if $\Gamma(T)$ is closed.

Proof. The forward direction is easy since if T is bounded, then $\Gamma(T)$ is closed.

Now assume $\Gamma(T)$ is closed, then we see that $\Gamma(T)$ is a Banach space, so we can now use open mapping theorem. Define a norm on $X \times Y$ by

$$\|(x, y)\| = \|x\| + \|y\|\,,$$

then $\Gamma(T)$ is a Banach space with this norm. Define $u\colon \Gamma(T) \to X$ by $u(x, Tx) = x$ for $x \in X$, then u is bounded wince $\|u\| \leq 1$. From open mapping theorem, we know that u is surjective and injective implies $u^{-1}\colon X \to \Gamma(T)$ is bounded, hence

$$\|u(x, Tx)\| \geq c\,\|(x, Tx)\|$$

for all $x \in X$ and some $c > 0$, i.e.,

$$\|x\| \geq c\,(\|x\| + \|Tx\|) \Rightarrow \|Tx\| \leq \left(\frac{1}{c} - 1\right)\|x\|$$

for all $x \in X$, so T is bounded. ∎

3.2.1 Symmetric Operators on Hilbert Spaces

One application to self-adjoint (symmetric) operator, i.e., $T^* = T$, on Hilbert space is the following.

Theorem 3.2.2 (Hellinger-Toeplitz theorem)**.** Let $T\colon \mathcal{H} \to \mathcal{H}$ be a linear operator. If T is self-adjoint, i.e., $\langle Tx, y\rangle = \langle x, Ty\rangle$ for $x, y \in \mathcal{H}$, then T is bounded.

Proof. From closed graph theorem, it suffices to show that for a self-adjoint operator T, $\Gamma(T)$ is closed. Let $\{x_n\}_{n\geq 1}$ in $\mathcal{H}$ such that $x_n \to x_\infty \in \mathcal{H}$ and $Tx_n \to y_\infty \in \mathcal{H}$, then we need to show $Tx_\infty \to y_\infty$. From the self-adjointness of T and the continuity of an inner product, for all $z \in \mathcal{H}$,

$$\langle z, y_\infty\rangle = \lim_{n\to\infty} \langle z, Tx_n\rangle = \lim_{n\to\infty} \langle Tz, x_n\rangle = \langle Tz, x_\infty\rangle = \langle z, Tx_\infty\rangle\,.$$

Since this holds for all $z \in \mathcal{H}$, we know that $Tx_\infty = y_\infty$, hence $\Gamma(T)$ is closed, so T is bounded. ∎

Hellinger-Toeplitz theorem identifies the source of considerable difficulties in mathematical physics since many natural operators such as differential, though satisfy the symmetry condition, but are unbounded, and hence Hellinger-Toeplitz theorem declares that such operators *can not be defined everywhere* on the Hilbert space.

Example. There are no useful notions of differentiation that would make all $f \in L^2$ differentiable.

Lecture 13: Principle of Uniform Boundedness

3.3 Principle of Uniform Boundedness

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The final consequence of open mapping theorem is the following, which completes the whole picture of functional analysis. We first see some definitions.

Definition. Let X, Y be Banach spaces and let $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ be a family of bounded linear operator from X to Y.

Definition 3.3.1 (Point-wise bounded)**.** $\mathcal{T}$ is *point-wise bounded*, if $\sup_{T\in\mathcal{T}} \|Tx\| < \infty$ for all $x \in X$.

Definition 3.3.2 (Uniformly bounded). $\mathcal{T}$ is *uniformly bounded* if $\sup_{T\in\mathcal{T}} \|T\| < \infty$.

Theorem 3.3.1 (Uniform boundedness theorem). Let X, Y be Banach spaces and let $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ be a family of bounded linear operator from X to Y such that it's point-wise bounded, then it's uniformly bounded.

Proof. Define $M\colon X \to \mathbb{R}$ by $M(x) = \sup_{T\in\mathcal{T}} \|Tx\|$ for $x \in X$, also, let $X_n := \{x \in X\colon M(x) \leq n\}$, we can then write $X = \bigcup_{n=1}^{\infty} X_n$. From Baire-category theorem, there exists $n \geq 1$ such that $\overline{X}_n$ has non-empty interior. Note that the function $x \mapsto M(x)$ for $x \in X$ is lower semi-continuous, i.e.,

$$M(x) \leq \liminf_{x_n \to x} M(x_n)$$

since

$$\|Tx\| \leq \lim_{n\to\infty} \|Tx_n\| \leq \liminf_{n\to\infty} M(x_n),$$

and by taking supremum over x, we have $M(x) \leq \liminf_{n\to\infty} M(x_n)$. Hence, we see that X_n is closed, i.e., $\overline{X}_n = X_n$, and we conclude X_n has non-empty interior. This implies $X_n \supseteq x_0 + \epsilon B_X$ for some $\epsilon > 0$ and $B_X := \{x \in X\colon \|x\| \leq 1\}$.

Since $M(\cdot)$ is symmetric and convex, i.e., $M(-x) = M(x)$ for $x \in X$ and $M(\lambda x + (1-\lambda)y) \leq \lambda M(x) + (1-\lambda)M(y)$ for $x, y \in X$, $0 < \lambda < 1$, we see that $X_n \supseteq x_0 + \epsilon B_X$. From symmetric, we also have $X_n \supseteq -x_0 + \epsilon B_X$. Then by convexity, we together have $X_n \supseteq \epsilon B_X$, hence

$$\|x\| \leq \epsilon \Rightarrow \sup_{T\in\mathcal{T}} \|Tx\| \leq n \Rightarrow \sup_{T\in\mathcal{T}} \|T\| \leq \frac{n}{\epsilon}.$$

∎

We note that in the above proof, we only use the completeness of X, not Y. So the uniform boundedness theorem still holds if X is a Banach space while Y is only a normed space.

Remark (Principle of condensation of singularities). The uniform Boundedness theorem is called *principle of condensation of singularities* by Banach and Steinhaus initially.

Proof. Suppose a family $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is not uniformly bounded, then the set of vectors

$$\{Tx\colon x \in B_X, T \in \mathcal{T}\}$$

is unbounded. We see that from the uniform boundedness theorem is not even point-wise bounded, so there exists *one* vector $x \in X$ with unbounded trajectory $\{Tx\colon T \in \mathcal{T}\}$. One can say that the unboundedness of the family $\mathcal{T}$ is condensated in a single *singularity* vector x. ⊛

3.3.1 Weak and Strong Boundedness

Principle of uniform boundedness can be used to check whether a given set in a Banach space is bounded in the following way. Firstly, let's see some definitions.

Definition. Let $A \subseteq X$ where X is a Banach space.

Definition 3.3.3 (Weakly bounded). A is *weakly bounded* if $\sup_{f\in X^*} |f(x)| < \infty$ for all $x \in A$.

Definition 3.3.4 (Strongly bounded). A is *strongly bounded* if $\sup_{x\in A} \|x\| < \infty$.

Corollary 3.3.1 (Weak boundedness implies strong boundedness). Let $A \subseteq X$ for X being a Banach space, and suppose A is weakly bounded. Then A is strongly bounded.

Proof. Firstly, we embed A into $A^{**} \subseteq X^{**}$ by considering the conical embedding $X \to X^{**}$, and we see that

$$\sup_{x^{**} \in A^{**}} |x^{**}(f)| < \infty$$

for all $f \in X^*$. From the uniform boundedness theorem, we have $\sup_{x^{**} \in A^{**}} \|x^{**}\| < \infty$, and with Hahn-Banach theorem, we have $\|x^{**}\| = \|x\|$ for all $x \in X$, proving the result. ■

Lecture 14: Midterm

Good luck! 13 Oct. 14:30

Lecture 15: Compactness in Banach Spaces

3.4 Compact Sets in Banach Spaces 20 Oct. 14:30

3.4.1 Schauder Basis

Definition 3.4.1 (Schauder basis). Let X be a separable Banach space. A sequence $\{x_k\}_{k\geq 1}$ is a *Schauder basis* for X for every $x \in X$ can be uniquely represented as a convergent series

$$x = \sum_{k=1}^{\infty} a_k x_k$$

for $a_k \in \mathbb{R}$ or $\mathbb{C}$.

Theorem 3.4.1. Let $\{x_k\}_{k\geq 1}$ be a Schauder basis for a Banach space X. Then there exists an $M \geq 0$ such that for all $n \geq 1$,

$$\left\| \sum_{k=1}^{n} a_k x_k \right\| \leq M \|x\| = M \left\| \sum_{k=1}^{\infty} a_k x_k \right\|$$

for $x \in X$.

Proof. Define a sequence space

$$E := \left\{ a = \{a_k\}_{k\geq 1} : \sum_{k=1}^{\infty} a_k x_k \text{ converges in } X \right\}$$

and for $a \in E$, define

$$\|a\| = \sup_{n\geq 1} \left\| \sum_{k=1}^{n} a_k x_k \right\| < \infty.$$

We see that $\|\cdot\|$ is a norm on E since $\|a\| = 0 \Rightarrow a = 0$ follows from the uniqueness property for Schauder basis and the fact that E is a Banach space, so E is complete.

Now, define an linear operator $T\colon E \to X$ by

$$Ta = \sum_{k=1}^{\infty} a_k x_k,$$

we have $\|Ta\| \leq \|a\|$, so T is also bounded, injective and surjective. From open mapping theorem,

$$T^{-1}\colon X \to E$$

is bounded such that $\|T^{-1}\| \leq M < \infty$, i.e.,

$$\|Ta\| \geq \frac{1}{M}\|a\|$$

for all $a \in E$. This is equivalent to say

$$\sup_{n\geq 1}\left\|\sum_{k=1}^{n} a_k x_k\right\| \leq M\left\|\sum_{k=1}^{\infty} a_k x_k\right\|.$$

■

Notation (Basis constant). The $M \geq 0$ in Theorem 3.4.1 is called the *basis constant*.

We see that we can define a partial sum operators for $n = 1, 2, \ldots$ such that

$$S_n \colon X \to X, \quad S_n(x) = \sum_{k=1}^{n} a_k x_k$$

for $x = \sum_{i=1}^{\infty} a_k x_k$. And we have shown that S_n is a bounded linear operator and $\sup_{n\geq 1}\|S_n\| < \infty$. Observe that $a_k = a_k(x)$ is a linear functional on X, which are called *biorthogonal functionals* of the basis $\{x_k\}_{k\geq 1}$ and denoted by x_k^* for $k \geq 1$. We now show $x_k^* \in X^*$, i.e., x_k^* is a bounded linear functional. to do this, we write

$$x_n^*(x)x_n = S_n(x) - S_{n-1}(x)$$

for $n \geq 1$. From Theorem 3.4.1, we have

$$\|x_n^*(x)x_n\| \leq \|S_n(x)\| + \|S_{n-1}(x)\| \leq 2M\|x\|,$$

hence we conclude that $x_n^* \in X^*$ and $\sup_{n\geq 1}\|x_n^*\|\|x_n\| < \infty$.

3.4.2 Compactness

We first review some properties of compactness.

Definition 3.4.2 (Compact). A subset A of a topological space is *compact* if every open cover of A has a finite subcover.

This means, given a cover $A \subseteq \bigcup_\alpha U_\alpha$ for some collection of open sets U_α, then $A \subseteq \bigcup_{k=1}^{n} U_{\alpha_k}$ for some finite subcollection.

Remark. Properties of compact sets:

(a) Compact sets of a Hausdorff space are closed.

(b) Closed subsets of compact sets are compact.

(c) The image of a compact set under a continuous function is compact.

(d) Continuous functions on compact sets are uniformly continuous and attain their maximum and minimum.

Definition 3.4.3 (Precompact). A set A is *precompact* if its closure $\overline{A}$ is compact.

Definition 3.4.4 (ϵ-net). Let A be a subset of a metric space X. Then a subset $\Omega_\epsilon \subseteq X$ is an *ϵ-net* for A if A can be covered by balls of radius ϵ centered at points of Ω_ϵ, i.e.,

$$A \subseteq \{y \colon d(y, x) < \epsilon \text{ for some } x \in \Omega_\epsilon\}.$$

Theorem 3.4.2. Let A be a subset of a complete metric space X, the following are equivalent.

(a) A is precompact.

(b) Every sequence $\{x_n\}$ in A has a Cauchy subsequence which converges in X.

(c) For every $\epsilon > 0$, there exists a finite ϵ-net for A.

Theorem 3.4.3 (Heine-Borel theorem). A subset A of a finite dimensional normed space X is precompact if and only if A is bounded.

We can extend Heine-Borel theorem to infinite dimensional spaces.

Lemma 3.4.1 (Approximation by finite dimensional subspaces). A subspace A of a normed space X is precompact if and only if A is bounded. And for every $\epsilon > 0$, there exists a finite dimensional subspace Y_ϵ of X containing an ϵ-net for A.

Proof. We first prove the necessity. Let A be precompact and $\epsilon > 0$. Then there exists a finite ϵ-net Ω_ϵ for A. Now, take $Y_\epsilon = \operatorname{span}(\Omega_\epsilon)$.

As for sufficiency, assume A is bounded, so $A \subseteq RB_X$ for some $R > 0$ where B_X is the unit ball $\{x \in X \colon \|x\| \le 1\}$. Hence, we can restrict to points of Ω_ϵ contained in $(R+\epsilon)B_{Y_\epsilon}$. Hence,

$$A \subseteq \{x \in X \colon d(x, (R+\epsilon)B_{Y_\epsilon}) < \epsilon\},$$

i.e., $(R+\epsilon)B_{Y_\epsilon}$ is compact.

In all, $(R+\epsilon)B_{Y_\epsilon}$ is covered by a finite collection of balls of radius ϵ. ∎

Theorem 3.4.4 (Riesz's theorem). The unit ball B_X of an infinite dimensional normed space is never compact.

Proof. Suppose $B_X = \{x \in X \colon \|x\| \le 1\}$ is compact. Then from Lemma 3.4.1, we can find a finite dimensional subspace Y containing an ϵ-net with $\epsilon = 1/2$ for B_X, i.e., $d(x, Y) \le 1/2$ for all $x \in B_X$ where X is infinite dimensional, Y is finite dimensional. This means the quotient space X / Y is nontrivial. Note that Y is a closed subspace of X. Hence norm on X induces a norm on X / Y by letting $x \in X$ and $[x] \in X / Y$ such that $\|[x]\| = 0.9$ where $\|[x]\| = \inf_{y \in Y} \|x - y\|$. Choose $\overline{y} \in Y$ such that $\|x - \overline{y}\| \le 1$, meaning that $x - \overline{y} \in B_X$ and $d(x - \overline{y}, Y) = 0.9 > 1/2$. ∎

Lecture 16: Strong Convergence

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Definition 3.4.5 (Strongly convergence). Let X, Y be Banach spaces, T_n, $n \ge 1$ be a sequence in $\mathcal{L}(X, Y)$ and $T \in \mathcal{L}(X, Y)$. We say the sequence $\{T_n\}_{n \ge 1}$ *converges strongly* to T if

$$\lim_{n \to \infty} \|T_n x - Tx\| = 0$$

for all $x \in X$.

Lemma 3.4.2. Let X, Y be Banach spaces, and $\{T_n\}_{n \ge 1}, T \in \mathcal{L}(X, Y)$. Suppose the sequence $\{T_n\}_{n \ge 1}$ converges strongly to T, then $\{T_n\}_{n \ge 1}$ converges to T on all precompact subsets $A \subseteq X$, i.e.,

$$\lim_{n \to \infty} \sup_{x \in A} \|T_n x - Tx\| = 0.$$

Proof. From uniform boundedness theorem, we know that $\sup_{n \ge 1} \|T_n\| < \infty$, i.e., $\exists M$ such that $\|T_n\| \le M$ for all $n \ge 1$. Let $\epsilon > 0$ and choose a finite ϵ-net Ω_ϵ for A. Since Ω_ϵ is finite, there exists N_ϵ such that

$$\|T_n y - Ty\| \le \epsilon$$

for $n \geq N_\epsilon$, $y \in \Omega_\epsilon$. Now, for an arbitrarily $x \in A$, there exists $y \in \Omega_\epsilon$ such that $\|x - y\| < \epsilon$. Then

$$\|T_n x - Tx\| \leq \|T_n y - Ty\| + \|(T_n - T)(x - y)\| \leq \epsilon + (\|T_n\| + \|T\|)\|x - y\| \leq \epsilon + 2M\epsilon$$

if $n \geq N_\epsilon$. This implies $\|T_n x - Tx\| \leq (2M+1)\epsilon$ for $n \geq N_\epsilon$ for all $x \in A$, hence we have uniform convergence on A. ■

Corollary 3.4.1. Let X be a Banach space with Shauder basis $\{x_k\}_{k\geq 1}$. A subset $A \subseteq X$ is precompact if and only if A is bounded and the basis expansion of vectors $x \in A$ converges uniformly, i.e.,

$$\lim_{n\to\infty} \sup_{x\in A} \|x - S_n x\| = 0.$$

Proof. From Lemma 3.4.2, if $S_n \to I$ strongly, implying uniform convergence if A is precompact. Conversely, for any $\epsilon > 0$, there exists n such that $\|x - S_n x\| < \epsilon$ for all $x \in A$, and $\operatorname{Im}(S_n)$ is finite dimensional and $\operatorname{Im}(S_n A)$ is bounded. Hence, there exists an ϵ-net Ω_ϵ for $\operatorname{Im}(S_n A)$, so A is covered by a finite 2ϵ-net, i.e., A is precompact. ■

3.5 Weak Topology

3.5.1 Weak Convergence

Definition 3.5.1 (Weakly convergence)**.** A sequence $\{x_n\}_{n\geq 1}$ in a Banach space X *converges weakly* to $x \in X$ if

$$\lim_{n\to\infty} f(x_n) = f(x)$$

for all $f \in X^*$.

Notation. If $\{x_n\}_{n\geq 1}$ converges to x weakly, we write $x_n \overset{w}{\to} x$.

Remark (Strong and weak)**.** As we have seen before (Definition 3.3.4, Definition 3.3.3 and Definition 3.4.5, Definition 3.5.1), the convention is that *strong* is for norm, while *weak* is for functional.

Proposition 3.5.1. Suppose the sequence $\{x_n\}_{n\geq 1}$ converges weakly to $x \in X$, then

(a) $\sup_{n\geq 1} \|x_n\| < \infty$.

(b) $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

(c) x is contained in the closure of the convex hull of the sequence $\{x_n\}_{n\geq 1}$, i.e., the smallest closed convex set containing the sequence.

Proof. Let's prove this one by one.

(a) For $y \in X$, let $y^{**} \in X^{**}$ be from the embedding $X \to X^{**}$, $y \mapsto y^{**}$ such that $\|y^{**}\| = \|y\|$. Then for $n \geq 1$, $x_n \in X$, so $x_n^{**} \in X^{**}$, we have

$$\sup_{n\geq 1} |x_n^{**}(f)| < \infty$$

since $f(x_n) \to f(x)$. Then, uniform boundedness theorem implies $\sup_{n\geq 1} \|x_n^{**}\| < \infty$. Now, since $\|x_n\| = \|x_n^{**}\|$, we conclude $\sup_{n\geq 1} \|x_n\| < \infty$.

(b) If $x_n \overset{w}{\to} x$ in X, by Hahn Banach theorem, there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$. Since $\|f\| = 1$,

$$f(x_n) \leq \|x_n\|$$

for $n \geq 1$. And since $x_n \xrightarrow{w} x$, we have $f(x_n) \to f(x) = \|x\|$, i.e.,

$$\liminf_{n\to\infty} \|x_n\| \geq \|x\|.$$

(c) To show x lies in the closure of the convex hull of $\{x_n\}_{n\geq 1}$, denoted it by K, we first note that K is a closed convex set. If $x \notin K$, by separating hyperplane theorem, there exists $f \in X^*$ such that $\sup_{y\in K} f(y) < f(x)$, and hence $\sup_{n\geq 1} f(x_n) < f(x)$. Since $\lim_{n\to\infty} f(x_n) = f(x)$, we have a contradiction.

■

Lemma 3.5.1. To show $x_n \xrightarrow{w} x$, it's sufficient to show $\lim_{n\to\infty} f(x_n) = f(x)$ for a dense subset of $f \in X^*$.

Proof. Let X be a Banach space and A a dense subset of X^*. Then $x_n \xrightarrow{w} X$ if and only if the sequence $\{x_n\}_{n\geq 1}$ is bounded in X and $\lim_{n\to\infty} f(x_n) = f(x)$ for $f \in A$.then the necessity follows from Proposition 3.5.1. To show the sufficiency, let $g \in X^*$, we need to show that

$$\lim_{n\to\infty} g(x_n) = g(x).$$

Let $\epsilon > 0$, and A is dense in X^*, so there exists $f \in A$ such that $\|g - f\| < \epsilon$. Then

$$\begin{aligned}
\limsup_{n\to\infty} |g(x_n - x)| &\leq \limsup_{n\to\infty} |f(x_n - x)| + \limsup_{n\to\infty} |(g-f)(x_{n-x})| \\
&\leq \|g - f\| \left(\sup_{n\geq 1} |x_n| + \|x\| \right) \\
&\leq 2M \|g - f\| \\
&\leq 2M\epsilon
\end{aligned}$$

where $M \coloneqq \sup_{n\geq 1} \|x_n\|$. We see that since $\epsilon > 0$ is arbitrary, so

$$\lim_{n\to\infty} |g(x_n - x)| = 0,$$

hence $x_n \xrightarrow{w} x$. ■

3.5.2 Weak Topology

Definition 3.5.2 (Weak topology)**.** The *weak topology* on a Banach space X is the weakest topology such that all maps $f \in X^*$ are continuous.

Recall that weak convergence means $\lim_{n\to\infty} f(x_n) = f(x)$. And if f is continuous at x_0, we know that the preimage of a ϵ-ball

$$\{x \in X \colon\ |f(x) - f(x_0)| < \epsilon\}$$

is open. Intuitively, the base of the weak topology are *cylinders* of the form

$$\{x \in X \colon\ |f_k(x - x_0)| < \epsilon, k = 1, 2, \dots, N\}$$

where $x_0 \in X$, $f_k \in X^*$, $k = 1, \dots, N$ for $\epsilon > 0$, $N \geq 1$.

Remark. This is equivalent to embedding X into an infinite product of $\mathbb{R}$ or $\mathbb{C}$, i.e.,

$$X \to \mathbb{R}^\infty, \quad x \mapsto [f(x) \colon f \in X^*]$$

where the coordinates of x in $\mathbb{R}^\infty$ are $f(x)$ given $f \in X^*$. Then, weak topology induced on products of reals. This is the same as in algebraic topology!

Proposition 3.5.2 (Weak closedness)**.** Let K be a convex subset of a Banach space. Then K is weakly closed if and only if K is strongly closed.

Proof. Weak closure implies strong closure does not involve convexity. Suppose $\{x_n\}_{n\geq 1} \subseteq K$, $x_n \stackrel{s}{\to} x$, then $x_n \stackrel{w}{\to} x$. If K is weak closed, then $x \in K$. ■

Appendix

Appendix A

Review

A.1 Midterm Review

A.1.1 Normed Spaces

Recall the normed spaces, and the properties of which. In particular, focus on convexity and note that $x \mapsto \|x\|$ is a convex function.

Example (Normed spaces). The spaces ℓ_p for $1 \leq p \leq \infty$ of sequences and $L^p(\Omega, \mathcal{F}, \mu)$ of integrable functions. Also, the space of continuous functions on compact Hausdorff space with supremum norm $C(K)$. Notice that

$$C(K) \subseteq L^\infty(K, \mathcal{F}).$$

Remark (Legendre transform). Recall the Legendre transform of convex functions. The most general form is that let X be a Banach space and X^* its dual space with a convex function $f\colon X \to \mathbb{R}$ and $f^*\colon X^* \to \mathbb{R}$. We have

$$f^*(y^*) = \sup_{x \in X} \left[y^*(x) - f^*(x)\right].$$

Notice that f^* is convex and lower semi-continuous where $f^*\colon X^* \to \mathbb{R} \cup \{+\infty\}$.

A.1.2 Quotient Spaces

Let X be a normed space and E be a subspace of X. Then $X \,/\, E = \{[x] = x + E \colon x \in X\}$ if E is closed, then $X \,/\, E$ is also a normed space with the norm $\|[x]\| \coloneqq \inf_{y \in E} \|x - y\|$..

Remark. E need to be closed since we need $\|[x]\| = 0 \Rightarrow [x] = 0$.

A.1.3 Banach Spaces

Any normed space E can be completed to a Banach space $\hat{E}$ by Theorem 1.4.2.

Example. ℓ_p and L^p are Banach spaces. For $x \in \ell_p$, $x = \{x_n, n \geq 1\}$ with

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

Notice that Minkowski inequality is the triangle inequality for ℓ_p and L^p, and we can prove this using Hölder's inequality where we have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

for $1/p + 1/q = 1$.

Remark (Proof of completeness of the ℓ_p spacees)**.** This is easy for ℓ_p, but for L^p, we need to use dominated convergence theorem.

A.1.4 Inner Product Spaces and Hilbert Spaces

Notice that the Hilbert spaces are the completion of inner product spaces. Recall the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

and the Cauchy-Schwartz inequality

$$|\langle x, y\rangle| \leq \|x\| \|y\| .$$

Orthogonality

Recall the orthogonal projection P_E onto a closed subspace $E \subseteq \mathcal{H}$ is $P_E x = x(y)$ where $x(y)$ is the minimizer of $\min_{y\in E} \|x-y\|$.

Remark. P_E is the projection, i.e., $P_E^2 g P_E$, and $I - P_E$ is proaction onto the orthogonal complement $E^\perp$ of E in $\mathcal{H}$ such that $\mathcal{H} = E \oplus E^\perp$. We see that

$$\|x\|^2 = \|P_E x\|^2 + \|(I-P_E)x\|^2$$

for $x \in \mathcal{H}$.

Consider the orthogonal or orthonormal sets of vectors x_k, $k = 1, 2, \ldots$ in $\mathcal{H}$ with the corresponding Fourier series being

$$S_n(x) := \sum_{k=1}^{n} \langle x, x_k\rangle x_k$$

such that

$$\|S_n(x)\|^2 = \sum_{k=1}^{n} |\langle x, x_k\rangle|^2 .$$

If the set $\{x_k\}_{k=1}^{\infty}$ is orthonormal, then $S_n = P_{E_n}$ where E_n is the span of $\{x_1, \ldots, x_n\}$, and

$$\|S_n x\|^2 = \|P_{E_n} x\|^2 \leq \|x\|^2 ,$$

which is the Bessel's inequality.

Remark. $S_n x \to S_\infty x$ in $\mathcal{H}$ where $S_\infty = P_{E_\infty}$ and E_∞ is the closure of spaces E_n, $n \geq 1$.

The orthonormal system $\{x_k\}_{k\geq 1}$ is complete if $E_\infty = \mathcal{H}$. In that case, $\|x\|^2 = \|P_{E_\infty} x\|^2 = \sum_{k=1}^{\infty} |\langle x, x_k\rangle|^2$.

Remark. Proving completeness can be difficult.

Example (Haar basis)**.** The Haar basis for $L^2([0,1])$ is the Fourier basis $e^{2\pi n i x}$, $n \in \mathbb{Z}$ for $L^2([0,1])$.

Proof. Let x_k, $k \geq 1$ be any arbitrary sequence of vectors in $\mathcal{H}$. We can then construct an orthonormal sequence y_k, $k \geq 1$ by applying Gram-Schmidt procedure. ⊛

A.1.5 Bounded Linear Functionals

Consider bounded linear functionals on a Banach space E, $f \in E^*$, $\|f\| = \sup_{\|x\|=1} |f(x)|$ and E^* is a Banach space. Recall that $f(\cdot)$ is essentially defined by $H = \ker(f)$ where H is a closed subspace of E with $\operatorname{codim}(H) = 1$, i.e., $\dim E / H = 1$ and we have

$$\widetilde{f} \colon {}^{E}\big/_{H} \to \mathbb{R}$$

is defined via $\widetilde{f}([x]) = f(x)$ for $x \in E$, and $\widetilde{f}(a[x]) = ca$ for some constant c.

A.1.6 Representation Theorem

The important representation theorem for bounded linear functionals is the Riesz representation theorem. The easiest case is $E = \mathcal{H}$ being a Hilbert space and $E^* \equiv \mathcal{H}$. This implies Radon-Nikodym theorem, where if we have $\nu \ll \mu$, then

$$\nu(E) = \int_E f \, \mathrm{d}\mu, \quad f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^1(\mu)$$

for ν, μ being finite measures. Furthermore, the Radon-Nikodym theorem implies the Riesz representation theorem for ℓ_p and L^p with $1 \leq p < \infty$.

> **Remark.** We have $E^* = \ell_q$ or L^q with $1/p + 1/q = 1$ for $1 \leq p < \infty$, and remarkably, $\ell_1^* = \ell_\infty$ but $\ell_\infty^* \neq \ell_1$.

> **Remark.** The Riesz representation theorem for $C(K)$ is space of bounded Borel measures where for $g \in C(K)^*$,
>
> $$g(f) = \int_K f \, \mathrm{d}\mu$$
>
> for $f \in C(K)$.

A.1.7 Hahn-Banach Theorem

Let E be a Banach space and E_0 be a subspace such that $f_0 \colon E_0 \to \mathbb{R}$ a bounded linear functional on E_0 such that $\|f_0\| < \infty$. Then there exists an extension f of f_0 to E with $\|f\| = \|f_0\|$.

> **Remark.** f is not necessary unique. Nevertheless, it is unique for Hilbert spaces, or ℓ_p, L^p with $1 < p < \infty$.

Reflexivity

Consider the embedding $E \to E^{**}$ such that $x \mapsto x^{**}$, then E is reflexive if the embedding is surjective. Also, E is reflexive implies that

$$\|f\| = \sup_{\|x\|=1} |f(x)| = f(x_f)$$

for some $x_f \in E$ with $\|x_f\| = 1$ for every $f \in E^*$.

> **Remark.** This is one way of showing some spaces is not reflexive.

Separation Theorem

Recall the separation theorem for convex sets from a point. Given a convex set K and a point $x_0 \notin K$, there is a hyperplane such that $f(x_0) > f(k)$ for all $k \in K$. The Minkowski functional for convex set essentially makes convex sets unit ball for some semi-norm.

Bibliography

[Lax02] P.D. Lax. *Functional Analysis*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2002. ISBN: 9780471556046. URL: https://books.google.com/books?id=18VqDwAAQBAJ.

[RS80] M. Reed and B. Simon. *I: Functional Analysis*. Methods of Modern Mathematical Physics. Elsevier Science, 1980. ISBN: 9780125850506. URL: https://books.google.com/books?id=hInvAAAAMAAJ.