MATH635
Riemannian Geometry

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Abstract

This is a graduate level differential geometry course focuses on Riemannian geometry.

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Contents

Chapter 1

Banach and Hilbert Spaces

Lecture 1: Introduction

We first briefly review different kinds of vector spaces. 30 Aug. 14:30

1.1 Linear Spaces

Let's first see the simplest (i.e., without structures) vector space called linear vector space.

Definition 1.1.1 (Linear vector space). A *linear vector space* E over a field $\mathbb{F}$ is a set with operations of addition and multiplication (by a scalar) such that it's closed under operations, and also the addition and scalar multiplication obey

(a) $u + v = v + u$ for $u, v \in E$

(b) $u + (v + w) = (u + v) + w$ for $u, v, w \in E$

(c) $\exists 0 \in E$ such that $0 + u = u + 0 = u$ for $u \in E$

(d) $\forall u \in E$, $\exists -u \in E$ such that $u + (-u) = 0$

(e) $\lambda(u + v) = \lambda u + \lambda v$ for $u, v \in E$, $\lambda \in \mathbb{F}$

(f) $(\lambda + \mu)u = \lambda u + \mu u$ for $u \in E$, $\lambda, \mu \in \mathbb{F}$

(g) $\lambda(\mu u) = (\lambda\mu)u$ for $u \in E$, $\lambda, \mu \in \mathbb{F}$

Remark. If $v, w \in E$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), then $\lambda v + \mu w \in E$.

Notation (Real and complex vector space). If E is over $\mathbb{F} = \mathbb{C}$, we usually call E a *complex vector space*; if $\mathbb{F} = \mathbb{R}$, we say E is a *real vector space*.

Example. Given $n \in \mathbb{N}$, $\mathbb{R}^n$ is an n dimensional real linear vector space.

Example. Given $n \in \mathbb{N}$, $\mathbb{C}^n$ is an n dimensional complex linear vector space.

We concentrate on ∞ dimensional linear vector space.

Example. Let K is a compact Hausdorff space, then

$$E = \{f \colon K \to \mathbb{R} \mid f(\cdot) \text{ is continuous}\}$$

is a ∞ dimensional real linear vector space.

Notation (Subspace)**.** If E is a linear vector space, then we say $E_1 \subseteq E$ is a *subspace* if $E_1 \subseteq E$ and E_1 is itself a linear vector space. Moreover, if $E_1 \subsetneq E$, we say E_1 is a *proper subspace*.

Observe that a linear vector space can have many subspaces.

1.2 Quotient Spaces

Sometimes we don't care about vectors in some directions, suggesting the notion of quotient space.

Definition 1.2.1 (Quotient Space)**.** The *quotient space* E / E_1 of two linear vector spaces E, E_1 such that $E_1 \subseteq E$ is the set of equivalence classes of vectors in E where equivalence is given by $x \sim y$ if $x - y \in E_1$. Additionally, denote $[x]$ as the equivalence class of $x \in E$, i.e., $[x] = x + E_1$.

One can see that quotient space E / E_1 is a linear vector space since if $x_1 + x_2 \in E$, $[x_1] + [x_2] = [x_1 + x_2]$, and also, $\lambda[x] = [\lambda x]$ for $\lambda \in \mathbb{R}$ or $\mathbb{C}$, i.e., $v, w \in E / E_1$, $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$ implies $\lambda v + \mu w \in E$.

The dimension of a quotient space is defined as follows.

Definition 1.2.2 (Codimension)**.** If E / E_1 has finite dimension, then the dimension of E / E_1 is called the *codimension* of E_1 in E, denoted as $\operatorname{codim}(E_1)$.

Definition 1.2.2 is introduced since the way of defining dimensions for finite dimensional vector spaces doesn't work here. Recall Theorem 1.2.1 in the finite dimension case.

Theorem 1.2.1. If E is finite dimensional, then $\operatorname{codim}(E_1) + \dim(E_1) = \dim(E)$

We see that we may encounter something like $\infty - \infty$ if we define $\operatorname{codim}(E_1) := \dim(E) - \dim(E_1)$, and indeed, Definition 1.2.2 is well-defined in this sense.

Example. There exists the case that $\dim(E) = \infty$, $\dim(E_1) < \infty$ where $\dim(E / E_1) < \infty$.

Proof. Let $E = \{f : K \to \mathbb{R} \mid f(\cdot) \text{ continuous}\}$ and $E_1 = \{f \in E : f(k_1) = 0\}$ for a fixed $k_1 \in K$. We see that the dimension of E / E_1 is exactly 1 since E / E_1 is the set of constant functions. ⊛

Definition 1.2.3 (Linear operator)**.** A map $T : E \to F$ between linear spaces E and F is a *linear operator* if it preserves the properties of addition and multiplication by a scalar, i.e., for $v, w \in E$ and $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$,

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w).$$

Definition. Given a linear operator $T : E \to F$ we have the following.

Definition 1.2.4 (Kernel)**.** The *kernel* of T is the subspace $\ker(T) = \{x \in E \mid Tx = 0\}$.

Definition 1.2.5 (Image)**.** The *image* of T is the subspace $\operatorname{Im}(T) = \{Tx \in F \mid x \in E\}$.

1.3 Normed Spaces

Given a vector, we want to measure the length of which. This suggests the following definitions.

Definition 1.3.1 (Norm)**.** Let E be a linear vector space. A *norm* $\|\cdot\| : E \to \mathbb{R}$ on E is a function from E to $\mathbb{R}$ with the properties:

(a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Notation (Dilation). We say that the second condition is the *dilation* property.

Definition 1.3.2 (Normed vector space). A linear vector space E equipped with a norm $\|\cdot\|$ is called a *normed vector space*, denoted by $(E, \|\cdot\|)$.

A similar notion called metric is also widely used, though the structure is slightly coarser.

As previously seen (Metric). Given a vector space E, the *metric* $d(\cdot,\cdot)\colon E \times E \to \mathbb{R}$ on E is a function form $E \times E$ to $\mathbb{R}$ with the properties:

(a) $d(x, y) \geq 0$. Also, $d(x, x) = 0$ and $d(x, y)$ implies $x = y$.

(b) $d(x, y) = d(y, x)$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

As one can imagine, if we can measure the length of a vector (by a norm), we can also measure the distance between vectors (by a metric).

Remark (Induced metric space). A normed vector space $(E, \|\cdot\|)$ induces a *metric space* (E, d) with the induced metric $d(x, y) = \|x - y\|$.

Now we give some well-known examples of normed spaces.

Example (Bounded sequences ℓ^∞). Let ℓ_∞ be the space of bounded sequences $x = (x_1, x_2, \ldots)$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots$. Then we define $\|x\| = \|x\|_\infty = \sup_{i \geq 1} |x_i|$.

Example (Absolutely summable sequences ℓ_1). Let ℓ_1 be the space of absolutely summable sequences $x = (x_1, x_2, \ldots)$ and $\sum_{i=1}^{\infty} |x_i| < \infty$. Then we define $\|x\| = \|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$.

Example (Continuous functions $C(k)$). The space $C(k)$ of continuous functions $f\colon K \to \mathbb{R}$ where K is compact Hausdorff. Then we define $\|f\| = \|f\|_\infty = \sup_{x \in K} |f(x)|$.

1.3.1 Geometry of Normed Spaces

Now we can look into the structure of a normed space we're referring to without actually explaining what this really means previously. Intuitively, it's about the geometric properties of the spaces like how do balls, spheres and other shapes look like in that space when defining these shapes with norms.

Definition 1.3.3 (Ball). A (closed) *ball* centered at a point $x_0 \in E$ with radius $r > 0$ is the set

$$B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\}.$$

Definition 1.3.4 (Sphere). The *sphere* centered at x_0 with radius $r > 0$ is the set

$$S(x_0, r) = \{x \in E \mid \|x - x_0\| = r\}.$$

Note. We see that $S(x_0, r)$ is the **boundary** of $B(x_0, r)$, i.e., $S(x_0, r) = \partial B(x_0, r)$.

Let's first look at balls. In finite dimensional, all norms are equivalent, which is not true for infinite dimensional vector spaces. This has something to do with the geometry of balls.

Explicitly, balls can have different geometries depending on the properties of the norms. We see that a $\|\cdot\|_\infty$ can have multiple supporting hyperplane at the corner, while for a $\|\cdot\|_2$ can have only one at each point.

Remark. The unit balls for $\|\cdot\|_1$ looks like **squares**, where we have

$$B(0,1) = \{x = (x_1, x_2, \dots) \mid -1 < y_\epsilon < 1 \text{ for all } \epsilon\}$$

such that $y_\epsilon = \sum_{i=1}^{\infty} \epsilon_i x_i$, $\epsilon_i = \pm 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \dots)$.

We see that different norms give different geometry, but they have important common features, most notably, convexity properties.

Definition 1.3.5 (Convex set)**.** Given E a linear vector space, a set $K \subset E$ is *convex* if for $x, y \in K$ and $0 \leq \lambda \leq 1$,

$$\lambda x + (1-\lambda) y \in K.$$

Definition 1.3.6 (Convex function)**.** Given E a linear vector space, a function $f \colon E \to \mathbb{R}$ is called *convex* if for $x, y \in E$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Remark (Sublevel set)**.** If $f \colon E \to \mathbb{R}$ is a convex function, then for any $M \in \mathbb{R}$ the *sublevel set* $\{x \in E \mid f(x) \leq M\}$ is convex.

The upshot is that norms are convex, and the unit balls are convex as well.

Appendix

Appendix A

Review

A.1 Midterm Review

A.1.1 Normed Spaces

Recall the normed spaces, and the properties of which. In particular, focus on convexity and note that $x \mapsto \|x\|$ is a convex function.

> **Example** (Normed spaces). The spaces ℓ_p for $1 \leq p \leq \infty$ of sequences and $L^p(\Omega, \mathcal{F}, \mu)$ of integrable functions. Also, the space of continuous functions on compact Hausdorff space with supremum norm $C(K)$. Notice that
> $$C(K) \subseteq L^\infty(K, \mathcal{F}).$$

> **Remark** (Legendre transform). Recall the Legendre transform of convex functions. The most general form is that let X be a Banach space and X^* its dual space with a convex function $f\colon X \to \mathbb{R}$ and $f^*\colon X^* \to \mathbb{R}$. We have
> $$f^*(y^*) = \sup_{x \in X} \left[y^*(x) - f^*(x) \right].$$
> Notice that f^* is convex and lower semi-continuous where $f^*\colon X^* \to \mathbb{R} \cup \{+\infty\}$.

A.1.2 Quotient Spaces

Let X be a normed space and E be a subspace of X. Then $X \,/\, E = \{[x] = x + E \colon x \in X\}$ if E is closed, then $X \,/\, E$ is also a normed space with the norm $\|[x]\| \coloneqq \inf_{y \in E} \|x - y\|$.

> **Remark.** E need to be closed since we need $\|[x]\| = 0 \Rightarrow [x] = 0$.

A.1.3 Banach Spaces

Any normed space E can be completed to a Banach space $\hat{E}$ by **??**.

> **Example.** ℓ_p and L^p are Banach spaces. For $x \in \ell_p$, $x = \{x_n, n \geq 1\}$ with
> $$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Notice that Minkowski inequality is the triangle inequality for ℓ_p and L^p, and we can prove this using Hölder's inequality where we have
$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$
for $1/p + 1/q = 1$.

Remark (Proof of completeness of the ℓ_p spacees). This is easy for ℓ_p, but for L^p, we need to use dominated convergence theorem.

A.1.4 Inner Product Spaces and Hilbert Spaces

Notice that the Hilbert spaces are the completion of inner product spaces. Recall the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

and the Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| .$$

Orthogonality

Recall the orthogonal projection P_E onto a closed subspace $E \subseteq \mathcal{H}$ is $P_E x = x(y)$ where $x(y)$ is the minimizer of $\min_{y\in E} \|x-y\|$.

Remark. P_E is the projection, i.e., $P_E^2 g P_E$, and $I - P_E$ is proaction onto the orthogonal complement $E^\perp$ of E in $\mathcal{H}$ such that $\mathcal{H} = E \oplus E^\perp$. We see that

$$\|x\|^2 = \|P_E x\|^2 + \|(I-P_E)x\|^2$$

for $x \in \mathcal{H}$.

Consider the orthogonal or orthonormal sets of vectors x_k, $k = 1, 2, \ldots$ in $\mathcal{H}$ with the corresponding Fourier series being

$$S_n(x) \coloneqq \sum_{k=1}^n \langle x, x_k\rangle \, x_k$$

such that

$$\|S_n(x)\|^2 = \sum_{k=1}^n |\langle x, x_k\rangle|^2 .$$

If the set $\{x_k\}_{k=1}^\infty$ is orthonormal, then $S_n = P_{E_n}$ where E_n is the span of $\{x_1, \ldots, x_n\}$, and

$$\|S_n x\|^2 = \|P_{E_n} x\|^2 \leq \|x\|^2 ,$$

which is the Bessel's inequality.

Remark. $S_n x \to S_\infty x$ in $\mathcal{H}$ where $S_\infty = P_{E_\infty}$ and E_∞ is the closure of spaces E_n, $n \geq 1$.

The orthonormal system $\{x_k\}_{k\geq 1}$ is complete if $E_\infty = \mathcal{H}$. In that case, $\|x\|^2 = \|P_{E_\infty} x\|^2 = \sum_{k=1}^\infty |\langle x, x_k\rangle|^2$.

Remark. Proving completeness can be difficult.

Example (Haar basis). The Haar basis for $L^2([0,1])$ is the Fourier basis $e^{2\pi n i x}$, $n \in \mathbb{Z}$ for $L^2([0,1])$.

Proof. Let x_k, $k \geq 1$ be any arbitrary sequence of vectors in $\mathcal{H}$. We can then construct an orthonormal sequence y_k, $k \geq 1$ by applying Gram-Schmidt procedure. ⊛

A.1.5 Bounded Linear Functionals

Consider bounded linear functionals on a Banach space E, $f \in E^*$, $\|f\| = \sup_{\|x\|=1} |f(x)|$ and E^* is a Banach space. Recall that $f(\cdot)$ is essentially defined by $H = \ker(f)$ where H is a closed subspace of E with $\operatorname{codim}(H) = 1$, i.e., $\dim E \,/\, H = 1$ and we have

$$\widetilde{f} \colon {}^{E}\!\big/_{H} \to \mathbb{R}$$

is defined via $\widetilde{f}([x]) = f(x)$ for $x \in E$, and $\widetilde{f}(a[x]) = ca$ for some constant c.

A.1.6 Representation Theorem

The important representation theorem for bounded linear functionals is the Riesz representation theorem. The easiest case is $E = \mathcal{H}$ being a Hilbert space and $E^* \equiv \mathcal{H}$. This implies Radon-Nikodym theorem, where if we have $\nu \ll \mu$, then

$$\nu(E) = \int_E f \,\mathrm{d}\mu, \quad f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^1(\mu)$$

for ν, μ being finite measures. Furthermore, the Radon-Nikodym theorem implies the Riesz representation theorem for ℓ_p and L^p with $1 \leq p < \infty$.

> **Remark.** We have $E^* = \ell_q$ or L^q with $1/p + 1/q = 1$ for $1 \leq p < \infty$, and remarkably, $\ell_1^* = \ell_\infty$ but $\ell_\infty^* \neq \ell_1$.

> **Remark.** The Riesz representation theorem for $C(K)$ is space of bounded Borel measures where for $g \in C(K)^*$,
>
> $$g(f) = \int_K f \,\mathrm{d}\mu$$
>
> for $f \in C(K)$.

A.1.7 Hahn-Banach Theorem

Let E be a Banach space and E_0 be a subspace such that $f_0 \colon E_0 \to \mathbb{R}$ a bounded linear functional on E_0 such that $\|f_0\| < \infty$. Then there exists an extension f of f_0 to E with $\|f\| = \|f_0\|$.

> **Remark.** f is not necessary unique. Nevertheless, it is unique for Hilbert spaces, or ℓ_p, L^p with $1 < p < \infty$.

Reflexivity

Consider the embedding $E \to E^{**}$ such that $x \mapsto x^{**}$, then E is reflexive if the embedding is surjective. Also, E is reflexive implies that

$$\|f\| = \sup_{\|x\|=1} |f(x)| = f(x_f)$$

for some $x_f \in E$ with $\|x_f\| = 1$ for every $f \in E^*$.

> **Remark.** This is one way of showing some spaces is not reflexive.

Separation Theorem

Recall the separation theorem for convex sets from a point. Given a convex set K and a point $x_0 \notin K$, there is a hyperplane such that $f(x_0) > f(k)$ for all $k \in K$. The Minkowski functional for convex set essentially makes convex sets unit ball for some semi-norm.