

MATH635
Riemannian Geometry

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Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by [Lydia Bieri](#). Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [\[FC13\]](#) as our reference; while not required, but highly recommended have on.

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Chapter 1

Manifolds

Lecture 1: A Foray to Smooth Manifolds

1.1 Differentiable Manifolds

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1.1.1 Topological Manifolds

Let's start with a common definition.

Definition 1.1.1 (Topological manifold). A *topological manifold* \mathcal{M} of dimension n is a (topological) Hausdorff space such that each point $p \in \mathcal{M}$ has a neighborhood U homeomorphic via $\varphi: U \rightarrow U'$ to an open subset $U' \subseteq \mathbb{R}^n$.

Definition 1.1.2 (Local coordinate map). For every $p \in \mathcal{M}$, the corresponding homeomorphism φ is called the *local coordinate map*.

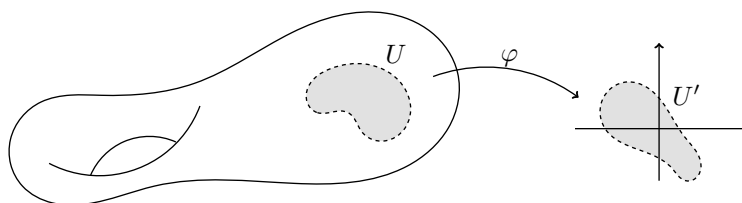
Definition 1.1.3 (Local coordinate). The pull-back (x^1, \dots, x^n) of the *local coordinate map* φ from \mathbb{R}^n is called the *local coordinates* on U , given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

Definition 1.1.4 (Coordinate chart). The pair (U, φ) is called a *coordinate chart*^a on M .

^aOr just a *chart*

In other words, a *topological manifold* can be thought of as a space such that it looks like \mathbb{R}^n locally.



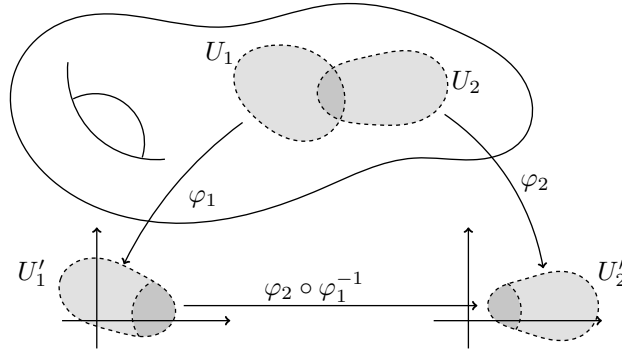
Definition 1.1.5 (Atlas). An *atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ for a *manifold* \mathcal{M} is a collection of *charts* such that $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$ are an open covering of \mathcal{M} , i.e., $\mathcal{M} = \bigcup_\alpha U_\alpha$.

In other words, for all $p \in \mathcal{M}$, there exists a neighborhood $U \subseteq \mathcal{M}$ and homeomorphism $h: U \rightarrow U' \subseteq \mathbb{R}^n$ open.

Definition 1.1.6 (Locally finite). An **atlas** is said to be *locally finite* if each point $p \in \mathcal{M}$ is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as \mathbb{R}^n , there's no clear way to make sense of differentiability of a **manifold**. But thankfully, we now have an explicit relation to the ambient space \mathbb{R}^n via φ_α . To formalize, let \mathcal{A} be an **atlas** for a **manifold** \mathcal{M} , and assume that $(U_1, \varphi_1), (U_2, \varphi_2)$ are 2 elements of \mathcal{A} . Then clearly, the map $\varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$ is a homeomorphism between 2 open sets of Euclidean spaces since both φ_1 and φ_2 are homeomorphism. Due to this map's importance, it has its own name.

Definition 1.1.7 (Coordinate transition). The map $\varphi_2 \circ \varphi_1^{-1}$ is called the *coordinate transition* of \mathcal{A} for the pair of **charts** $(U_1, \varphi_1), (U_2, \varphi_2)$.



1.1.2 Differentiable Structures

Notice that the **coordinate transitions** are from \mathbb{R}^n to \mathbb{R}^n ; hence differentiability makes sense now, which induces the following.

Definition 1.1.8 (Differentiable atlas). The **atlas** $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is *differentiable* if all **transitions** are differentiable.

Remark. Here, the differentiability depends on the content. Sometimes, we may want it to be C^∞ , and sometimes may be C^k for some finite k . On the other hand, smooth always refers to C^∞ . We'll use them interchangeably if it's clear which case we're referring to.

Definition 1.1.9 (Equivalence atlas). Two **atlases** \mathcal{U}, \mathcal{V} of a **manifold** are equivalent if for every $(U, \varphi) \in \mathcal{U}, (V, \psi) \in \mathcal{V}$,

$$\varphi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1}: \varphi(U \cap V) \rightarrow \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

Notation (Smoothly compatible). Two **charts** (U, φ) and (V, ψ) are *smoothly compatible* if either $U \cap V = \emptyset$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

This suggests the following.

Definition 1.1.10 (Smooth structure). A *smooth structure* on \mathcal{M} is an equivalence class \mathcal{U} of **coordinate atlas** with the property that all **transition functions** are diffeomorphisms.

Remark. In the literature, we can also use the *maximal differentiable atlas* to be our differentiable structure. Either way is fine.

Definition 1.1.11 (Smooth manifold). A *smooth manifold* is a manifold \mathcal{M} with a *smooth structure*.

In this way, we can do calculus on smooth manifolds! Furthermore, it now makes sense to say that a function $f: \mathcal{M} \rightarrow \mathbb{R}$ is differentiable (or C^∞) by considering differentiability of $f \circ \varphi^{-1}$ around p .

Notation. The collection of smooth functions on *smooth manifold* \mathcal{M} is denoted by $C^\infty(\mathcal{M}, \mathbb{R})$, or $C^k(\mathcal{M}, \mathbb{R})$.

Remark. The class $C^\infty(\mathcal{M}, \mathbb{R})$ consists of functions with property is well-defined.

Proof. Let \mathcal{A} be any given *atlas* from *equivalence class* that defines the *smooth structure*, and as we have shown, if $(U, \varphi) \in \mathcal{A}$, then $f \circ \varphi^{-1}$ is a smooth function on \mathbb{R}^n . This requirement defines the same set of smooth functions no matter the choice of representative *atlas* by the nature of *Definition 1.1.9* requirement that defines the equivalent *manifolds*. \otimes

1.1.3 Orientation

Another essential property of a *manifold* is its orientability.

Definition. Consider an *atlas* \mathcal{A} for a *differentiable manifold* \mathcal{M} .

Definition 1.1.12 (Oriented). \mathcal{A} is *oriented* if all *transitions* have positive functional determinant.

Definition 1.1.13 (Orientable). \mathcal{M} is *orientable* if \mathcal{A} is an *oriented atlas*.

Motivated by the above definitions, we see that we can actually use an *atlas* to define an *orientation*.

Definition 1.1.14 (Orientation). Let \mathcal{M} be an *orientable manifold*. Then a *oriented differentiable structure* is called an *orientation* of \mathcal{M} .

If \mathcal{M} possesses an *orientation*, we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with *Definition 1.1.12*.

Remark. Two *differentiable structures* obeying *Definition 1.1.12* determine the same *orientation* if the union again satisfying *Definition 1.1.12*.

Remark. If \mathcal{M} is *orientable* and connected, then there exists exactly 2 distinct *orientations* on \mathcal{M} .

Now, we can see some examples of *smooth manifolds*.

Example (Sphere). The sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Consider $U_i^+ = \{x \in S^n \mid x_i > 0\}$, $U_i^- = \{x \in S^n \mid x_i < 0\}$ for $i = 1, \dots, n+1$, and $h_i^\pm: U_i^\pm \rightarrow \mathbb{R}^n$ such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1}).$$

Note that the minimum charts needed to cover S^n is 2.

Example. Let $\mathcal{M} = U \subseteq \mathbb{R}^n$, then $\{(U, \varphi)\}$ is a **smooth structure** with $\varphi = \mathbb{1}$.

Example. Open sets of C^∞ -manifolds are C^∞ -manifolds.

Example. $\text{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, open.

Example. $\mathbb{R}P^n = S^n / \sim$ where $x \sim -x$ with $\pi: S^n \rightarrow \mathbb{R}P^n$, $x \mapsto [x]$.

Proof. π is a homeomorphism on each U_i^+ for $i = 1, \dots, n+1$, with

$$\{(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1}), i = 1, \dots, n+1\}$$

is a C^∞ -atlas for $\mathbb{R}P^n$.

⊛

Note. $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$.

Example (Grassmannian manifolds). Given m, n , $G(n, m)$ is the set of all n -dimensional subspaces of \mathbb{R}^{n+m} .

Appendix

Bibliography

- [FC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013. ISBN: 9780817634902. URL: <https://books.google.com/books?id=ct91XCWkWEUC>.