MATH635
Riemannian Geometry

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Abstract

This is a graduate level differential geometry course focuses on Riemannian geometry.

This course is taken in Winter 2023, and the date on the covering page is the last updated time.

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Chapter 1

Manifolds

Lecture 1: Introduction

1.1 Introduction

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Let's start with a common definition.

Definition 1.1.1 (Topological manifold). A *topological manifold* $\mathcal{M}$ of dimension n is a (topological) Hausdorff space such that each point $p \in \mathcal{M}$ has a neighborhood U homeomorphic to $U' \subseteq \mathbb{R}^n$ open.

> **Definition 1.1.2** (Coordinate chart). U' is called the *coordinate chart*.

> **Definition 1.1.3** (Local coordinate). The pull-back of the coordinate functions from $\mathbb{R}^n$ is called the *local coordinates*.

> **Definition 1.1.4** (Atlas). An *atlas* $\mathcal{A}$ is a collection such that $\mathcal{A} = \{U_\alpha, f_\alpha\}$ of charts for which the U_α are an open covering of $\mathcal{M}$, i.e., $\mathcal{M} = \bigcup_\alpha U_\alpha$, $U_\alpha \subseteq \mathcal{M}$ open.

In other words, for all $p \in \mathcal{M}$, there exists a neighborhood $U \subseteq \mathcal{M}$ and homeomorphism $h\colon U \to U' \subseteq \mathbb{R}^n$ open.

Definition 1.1.5 (Locally finite). An atlas (coordinate atlas) is said to be *locally finite* if each point $p \in \mathcal{M}$ contained in only finite collection of its open sets.

Definition 1.1.6 (Smooth manifold). Let $\mathcal{A}$ be a coordinate atlas for a manifold $\mathcal{M}$. Assume that $(U_1, \varphi_1), (U_2, \varphi_2)$ are 2 elements of $\mathcal{A}$. The map $\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a homeomorphism between 2 open sets of Euclidean spaces.

> **Definition 1.1.7** (Coodinate transition). The map $\varphi_2 \circ \varphi_1^{-1}$ is called the *coordinate transition* of $\mathcal{A}$ for the pair of charts $(U_1, \varphi_1), (U_2, \varphi_2)$.

The atlas $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}$ is called *differentiable* if all transitions are differentiable.
We can also talk about the equivalence between two atlases.

Definition 1.1.8 (Equivalence)**.** Two atlases $\mathcal{U}, \mathcal{V}$ are equivalent if the following holds: Assume $(U_1, \varphi_1) \in \mathcal{U}$, $(V_1, \varphi_2) \in \mathcal{V}$, then

$$\varphi_1 \circ \varphi_2^{-1} \colon \varphi_2(U_1 \cap V_2) \to \varphi_1(U_1 \cap V_2)$$

and

$$\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap V_2) \to \varphi_2(U_1 \cap V_2)$$

are diffeomorphisms between subsets of Euclidean spaces.

Definition 1.1.9 (Smooth structure)**.** A *smooth structure* on $\mathcal{M}$[a] is defined by an equivalence class $\mathcal{U}$ of coordinate atlas with property that all transition functions are diffeomorphisms. Then, the maximal differentiable atlas is our differentiable structure.

[a] Also called a *differentiable structure.*

A manifold $\mathcal{M}$ with a smooth structure is called a *smooth manifold.*[b]

[b] Also called a *differentiable manifold.*

In this way, we can do calculus on smooth manifolds! Furthermore, we can say that a function $f \colon \mathcal{M} \to \mathbb{R}$ is differentiable (or C^∞), and the collection of smooth functions of smooth manifold $\mathcal{M}$ is $C^\infty(\mathcal{M}, \mathbb{R})$, or $C^k(\mathcal{M}, \mathbb{R})$ in general.

Remark. The class $C^\infty(\mathcal{M}, \mathbb{R})$ consists of functions with property: Let $\mathcal{A}$ be any given atlas from equivalence class that defines the smooth structure. If $(U_1, \varphi_1) \in \mathcal{A}$, then $f \circ \varphi_1^{-1}$ is a smooth function on $\mathbb{R}^n$. This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.1.8 requirement that defines the equivalence manifolds.

Definition 1.1.10 (Orientation)**.** Consider an atlas for a differentiable manifold $\mathcal{M}$.

Definition 1.1.11 (Orientated)**.** The atlas is called *orientated* if all transitions have positive functional determinant.

Definition 1.1.12 (Orientable)**.** $\mathcal{M}$ is *orientable* if it possesses an orientated atlas.

Definition 1.1.13. Let $\mathcal{M}$ be an orientable manifold. Then a choice of a differentiable structure satisfying Definition 1.1.11 is called an *orientation* of $\mathcal{M}$, and then $\mathcal{M}$ is said to be *orientated.*

Remark. Two differentiable structures obeying Definition 1.1.11 determining the same orientation if the union again satisfying Definition 1.1.11.

Remark. If $\mathcal{M}$ is orientable and connected, then there exists exactly two distinct orientations on $\mathcal{M}$.

Example (Sphere)**.** The sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by

$$S^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 g1 \right\}.$$

Consider $U_i^+ = \{x \in S^n \mid x_i > 0\}$, $U_i^- = \{x \in S^n \mid x_i < 0\}$ for $i = 1, \ldots, n+1$, and $h_i^\pm \colon U_i^\pm \to \mathbb{R}^n$ such that

$$h_i^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, .., x_{n+1}).$$

Note that the minimum charts needed to cover S^n is 2.

Example. $\mathcal{M} = \mathbb{R}^n$.

Example. $U \subseteq \mathbb{R}^n$ with $\varphi = \mathbb{1}$.

Example. Open sets of C^∞-manifolds are C^∞-manifolds.

Example. $\mathrm{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, open.

Example. $\mathbb{R}P^n = S^n \,/\, \sim$ where $x \sim -x$ with $\pi\colon S^n \to \mathbb{R}P^n$, $x \mapsto [x]$.

Proof. π is a homeomorphism on each U_i^+ for $i = 1, \ldots, n+1$, with

$$\left\{\left(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1}\right), i = 1, \ldots, n+1\right\}$$

is a C^∞-atlas for $\mathbb{R}P^n$. ⊛

Note. $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} \,/\, \sim$.

Example (Grassmannian manifolds)**.** Given m, n, $G(n, m)$ is the set of all n-dimensional subspaces of $\mathbb{R}^{n+m}$.

Appendix

Appendix A

Review

A.1 Midterm Review

A.1.1 Normed Spaces

Recall the normed spaces, and the properties of which. In particular, focus on convexity and note that $x \mapsto \|x\|$ is a convex function.

Example (Normed spaces). The spaces ℓ_p for $1 \leq p \leq \infty$ of sequences and $L^p(\Omega, \mathcal{F}, \mu)$ of integrable functions. Also, the space of continuous functions on compact Hausdorff space with supremum norm $C(K)$. Notice that

$$C(K) \subseteq L^\infty(K, \mathcal{F}).$$

Remark (Legendre transform). Recall the Legendre transform of convex functions. The most general form is that let X be a Banach space and X^* its dual space with a convex function $f\colon X \to \mathbb{R}$ and $f^*\colon X^* \to \mathbb{R}$. We have

$$f^*(y^*) = \sup_{x \in X} \left[y^*(x) - f^*(x)\right].$$

Notice that f^* is convex and lower semi-continuous where $f^*\colon X^* \to \mathbb{R} \cup \{+\infty\}$.

A.1.2 Quotient Spaces

Let X be a normed space and E be a subspace of X. Then $X / E = \{[x] = x + E\colon x \in X\}$ if E is closed, then X / E is also a normed space with the norm $\|[x]\| \coloneqq \inf_{y \in E} \|x - y\|$.

Remark. E need to be closed since we need $\|[x]\| = 0 \Rightarrow [x] = 0$.

A.1.3 Banach Spaces

Any normed space E can be completed to a Banach space $\hat{E}$ by **??**.

Example. ℓ_p and L^p are Banach spaces. For $x \in \ell_p$, $x = \{x_n, n \geq 1\}$ with

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

Notice that Minkowski inequality is the triangle inequality for ℓ_p and L^p, and we can prove this using Hölder's inequality where we have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

for $1/p + 1/q = 1$.

Remark (Proof of completeness of the ℓ_p spacees)**.** This is easy for ℓ_p, but for L^p, we need to use dominated convergence theorem.

A.1.4 Inner Product Spaces and Hilbert Spaces

Notice that the Hilbert spaces are the completion of inner product spaces. Recall the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

and the Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \|y\| .$$

Orthogonality

Recall the orthogonal projection P_E onto a closed subspace $E \subseteq \mathcal{H}$ is $P_E x = x(y)$ where $x(y)$ is the minimizer of $\min_{y\in E} \|x-y\|$.

Remark. P_E is the projection, i.e., $P_E^2 g P_E$, and $I - P_E$ is proaction onto the orthogonal complement $E^\perp$ of E in $\mathcal{H}$ such that $\mathcal{H} = E \oplus E^\perp$. We see that

$$\|x\|^2 = \|P_E x\|^2 + \|(I - P_E)x\|^2$$

for $x \in \mathcal{H}$.

Consider the orthogonal or orthonormal sets of vectors x_k, $k = 1, 2, \ldots$ in $\mathcal{H}$ with the corresponding Fourier series being

$$S_n(x) \coloneqq \sum_{k=1}^{n} \langle x, x_k\rangle x_k$$

such that

$$\|S_n(x)\|^2 = \sum_{k=1}^{n} |\langle x, x_k\rangle|^2 .$$

If the set $\{x_k\}_{k=1}^\infty$ is orthonormal, then $S_n = P_{E_n}$ where E_n is the span of $\{x_1, \ldots, x_n\}$, and

$$\|S_n x\|^2 = \|P_{E_n} x\|^2 \leq \|x\|^2 ,$$

which is the Bessel's inequality.

Remark. $S_n x \to S_\infty x$ in $\mathcal{H}$ where $S_\infty = P_{E_\infty}$ and E_∞ is the closure of spaces E_n, $n \geq 1$.

The orthonormal system $\{x_k\}_{k\geq 1}$ is complete if $E_\infty = \mathcal{H}$. In that case, $\|x\|^2 = \|P_{E_\infty} x\|^2 = \sum_{k=1}^\infty |\langle x, x_k\rangle|^2$.

Remark. Proving completeness can be difficult.

Example (Haar basis)**.** The Haar basis for $L^2([0,1])$ is the Fourier basis $e^{2\pi n i x}$, $n \in \mathbb{Z}$ for $L^2([0,1])$.

Proof. Let x_k, $k \geq 1$ be any arbitrary sequence of vectors in $\mathcal{H}$. We can then construct an orthonormal sequence y_k, $k \geq 1$ by applying Gram-Schmidt procedure. ⊛

A.1.5 Bounded Linear Functionals

Consider bounded linear functionals on a Banach space E, $f \in E^*$, $\|f\| = \sup_{\|x\|=1} |f(x)|$ and E^* is a Banach space. Recall that $f(\cdot)$ is essentially defined by $H = \ker(f)$ where H is a closed subspace of E with $\operatorname{codim}(H) = 1$, i.e., $\dim E \,/\, H = 1$ and we have

$$\widetilde{f} \colon {}^{E}\big/_{H} \to \mathbb{R}$$

is defined via $\widetilde{f}([x]) = f(x)$ for $x \in E$, and $\widetilde{f}(a[x]) = ca$ for some constant c.

A.1.6 Representation Theorem

The important representation theorem for bounded linear functionals is the Riesz representation theorem. The easiest case is $E = \mathcal{H}$ being a Hilbert space and $E^* \equiv \mathcal{H}$. This implies Radon-Nikodym theorem, where if we have $\nu \ll \mu$, then

$$\nu(E) = \int_E f \, \mathrm{d}\mu, \quad f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^1(\mu)$$

for ν, μ being finite measures. Furthermore, the Radon-Nikodym theorem implies the Riesz representation theorem for ℓ_p and L^p with $1 \leq p < \infty$.

> **Remark.** We have $E^* = \ell_q$ or L^q with $1/p + 1/q = 1$ for $1 \leq p < \infty$, and remarkably, $\ell_1^* = \ell_\infty$ but $\ell_\infty^* \neq \ell_1$.

> **Remark.** The Riesz representation theorem for $C(K)$ is space of bounded Borel measures where for $g \in C(K)^*$,
>
> $$g(f) = \int_K f \, \mathrm{d}\mu$$
>
> for $f \in C(K)$.

A.1.7 Hahn-Banach Theorem

Let E be a Banach space and E_0 be a subspace such that $f_0 \colon E_0 \to \mathbb{R}$ a bounded linear functional on E_0 such that $\|f_0\| < \infty$. Then there exists an extension f of f_0 to E with $\|f\| = \|f_0\|$.

> **Remark.** f is not necessary unique. Nevertheless, it is unique for Hilbert spaces, or ℓ_p, L^p with $1 < p < \infty$.

Reflexivity

Consider the embedding $E \to E^{**}$ such that $x \mapsto x^{**}$, then E is reflexive if the embedding is surjective. Also, E is reflexive implies that

$$\|f\| = \sup_{\|x\|=1} |f(x)| = f(x_f)$$

for some $x_f \in E$ with $\|x_f\| = 1$ for every $f \in E^*$.

> **Remark.** This is one way of showing some spaces is not reflexive.

Separation Theorem

Recall the separation theorem for convex sets from a point. Given a convex set K and a point $x_0 \notin K$, there is a hyperplane such that $f(x_0) > f(k)$ for all $k \in K$. The Minkowski functional for convex set essentially makes convex sets unit ball for some semi-norm.