MATH597
Analysis II

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Contents

Lecture 1: Measures

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1 Measures

Example. Before we start, we first see some examples.

1. Let $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) \coloneqq \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which is the *power set* of X. We see that

$$\#X = n \implies \#\mathcal{P}(X) = 2^n$$

for $n < \infty$.

2. If $n = \infty$, say $X = \mathbb{N}$, then

$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while $\mathbb{N}$ is a countable set. We can see this as follows. Consider

$$\phi\colon \mathcal{P}(\mathbb{N}) \to [0, 1], \quad A \mapsto 0.a_1 a_2 a_3 \ldots (\text{base } 2),$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example, A can be $A = \{2, 3, 6, \ldots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have

$$\#\mathcal{P}(\mathbb{N}) \geq \# [0, 1].$$

But since $[0, 1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X. Hence, we are intriguing to define a map μ such that

$$\mu\colon \mathcal{P}(X) \to [0, \infty].$$

Example. We first see some examples.

1. Let $X = \{0, 1, 2\}$. Then we want to define $\mu\colon \mathcal{P}(X) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$. Then we have
     - $\mu(\{0, 1\}) = 2$
     - $\mu(\{0\}) = 1$
   - $\mu(A) = \sum\limits_{i \in A} 2^i$. Then we have
     - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$
2. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu\colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$. Then we have
     - $\mu(\{2, 3, 4, 5, \ldots\}) = \infty = \mu(\{\text{even numbers}\})$
   - $\mu(A) = e^{-1} \sum\limits_{i \in A} \frac{1}{i!}$. Then we have
     - $\mu(\{0, 2, 4, 6, \ldots\}) = e^{-1} \left(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right)$
   - $\mu(A) = \sum\limits_{i \in A} a_i$
3. Let $X = \mathbb{R}$. Then we want to define $\mu\colon \mathcal{P}(\mathbb{R}) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$
   - $\mu((a, b)) = b - a$.

     Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

     Answer. No!
   - $\mu((a, b)) = e^b - e^a$.

     Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

     Answer. No!

We immediately see the problems. To extend our native measure method into $\mathbb{R}$ is hard and will cause something counter-intuitive![1] Hence, rather than define measurement on *all* subsets in the power set of X, we only focus on *some* subsets. In other words, we want to define

$$\mu\colon \mathcal{P}(\mathbb{R}) \supset \mathcal{U} \to [0, \infty]\,.$$

1.1 $\sigma-$algebras

[1] https://en.wikipedia.org/wiki/Banach-Tarski_paradox

Definition 1. Let X be a set. A collection $\mathcal{U}$ of subsets of X, i.e., $\mathcal{U} \subset \mathcal{P}(X)$ is called a $\sigma-$*algebra on* X if

- $\phi \subset \mathcal{U}$
- $\mathcal{U}$ is closed under complements. i.e., if $A \in \mathcal{U}$, $A^c = X \setminus A \in \mathcal{U}$.
- $\mathcal{U}$ is closed under countable union. i.e., if $A_i \in \mathcal{U}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{U}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{U}$
- $\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i \right)^c$, namely $\mathcal{U}$ is closed under countable intersections.
- $A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \varnothing \cup \varnothing \cup \ldots$, hence $\mathcal{U}$ is closed under finite unions and intersections.