

MATH635  
Riemannian Geometry

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## Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by [Lydia Bieri](#). Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [[FC13](#)] as our reference; while not required, but highly recommended have on. Apart from this, I also found [[Sch15](#)] very useful.



This course is taken in Winter 2023, and the date on the covering page is the last updated time.

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# Chapter 1

## Smooth Manifolds

### Lecture 1: A Foray to Smooth Manifolds

#### 1.1 Topological Manifolds

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Let's start with a common definition.

**Definition 1.1.1 (Topological manifold).** A *topological manifold*  $\mathcal{M}$  of dimension  $n$  is a (topological) Hausdorff space such that each point  $p \in \mathcal{M}$  has a neighborhood  $U$  homeomorphic via  $\varphi: U \rightarrow U'$  to an open subset  $U' \subseteq \mathbb{R}^n$ .

**Definition 1.1.2 (Local coordinate map).** For every  $p \in \mathcal{M}$ , the corresponding homeomorphism  $\varphi$  is called the *local coordinate map*.

**Definition 1.1.3 (Local coordinate).** The pull-back  $(x^1, \dots, x^n)$  of the *local coordinate map*  $\varphi$  from  $\mathbb{R}^n$  is called the *local coordinates* on  $U$ , given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

**Definition 1.1.4 (Coordinate chart).** The pair  $(U, \varphi)$  is called a (*coordinate*) *chart* on  $\mathcal{M}$ .

In other words, a *topological manifold* can be thought of as a space such that it looks like  $\mathbb{R}^n$  locally.



**Definition 1.1.5 (Atlas).** An *atlas*  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$  for a *manifold*  $\mathcal{M}$  is a collection of *charts* such that  $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$  are an open covering of  $\mathcal{M}$ , i.e.,  $\mathcal{M} = \bigcup_\alpha U_\alpha$ .

In other words, for all  $p \in \mathcal{M}$ , there exists a neighborhood  $U \subseteq \mathcal{M}$  and homeomorphism  $h: U \rightarrow U' \subseteq \mathbb{R}^n$  open.

**Definition 1.1.6 (Locally finite).** An *atlas* is said to be *locally finite* if each point  $p \in \mathcal{M}$  is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as  $\mathbb{R}^n$ , there's no clear way to make sense of differentiability of a *manifold*. But thankfully, we now have an explicit relation to the ambient space  $\mathbb{R}^n$  via  $\varphi_\alpha$ . To formalize, let  $\mathcal{A}$  be an *atlas* for a *manifold*  $\mathcal{M}$ , and assume that  $(U_1, \varphi_1), (U_2, \varphi_2)$  are 2 elements

of  $\mathcal{A}$ . Then clearly, the map  $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$  is a homeomorphism between 2 open sets of Euclidean spaces since both  $\varphi_1$  and  $\varphi_2$  are homeomorphism. Due to this map's importance, it has its own name.

**Definition 1.1.7 (Coordinate transition).** The map  $\varphi_2 \circ \varphi_1^{-1}$  is called the *coordinate transition* of  $\mathcal{A}$  for the pair of charts  $(U_1, \varphi_1), (U_2, \varphi_2)$ .



## 1.2 Differentiable Manifolds

Notice that the *coordinate transitions* are from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ ; hence differentiability makes sense now, which induces the following.

**Definition 1.2.1 (Differentiable atlas).** The atlas  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$  is *differentiable* if all *transitions* are differentiable.

**Remark.** Here, the differentiability depends on the content. Sometimes, we may want it to be  $C^\infty$ , and sometimes may be  $C^k$  for some finite  $k$ . On the other hand, smooth always refers to  $C^\infty$ . We'll use them interchangeably if it's clear which case we're referring to.

**Definition 1.2.2 (Equivalence atlas).** Two atlases  $\mathcal{U}, \mathcal{V}$  of a manifold are equivalent if for every  $(U, \varphi) \in \mathcal{U}, (V, \psi) \in \mathcal{V}$ ,

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

**Notation (Smoothly compatible).** Two charts  $(U, \varphi)$  and  $(V, \psi)$  are *smoothly compatible* if either  $U \cap V = \emptyset$  or  $\psi \circ \varphi^{-1}$  is a diffeomorphism.

This suggests the following.

**Definition 1.2.3 (Smooth structure).** A *smooth structure* on  $\mathcal{M}$  is an equivalence class  $\mathcal{U}$  of *coordinate atlas* with the property that all *transition functions* are diffeomorphisms.

**Remark.** We can also use the *maximal differentiable atlas* to be our differentiable structure.

**Definition 1.2.4 (Smooth manifold).** A *smooth manifold* is a manifold  $\mathcal{M}$  with a *smooth structure*.

In this way, we can do calculus on *smooth manifolds*! Furthermore, it now makes sense to say that a function  $f : \mathcal{M} \rightarrow \mathbb{R}$  is differentiable (or  $C^\infty$ ) by considering differentiability of  $f \circ \varphi^{-1}$  around  $p$ .

**Notation.** The collection of smooth functions on [smooth manifold](#)  $\mathcal{M}$  is denoted by  $C^\infty(\mathcal{M}, \mathbb{R})$ , or  $C^k(\mathcal{M}, \mathbb{R})$ .

**Remark.** The class  $C^\infty(\mathcal{M}, \mathbb{R})$  consists of functions with property is well-defined.

**Proof.** Let  $\mathcal{A}$  be any given [atlas](#) from [equivalence class](#) that defines the [smooth structure](#), and as we have shown, if  $(U, \varphi) \in \mathcal{A}$ , then  $f \circ \varphi^{-1}$  is smooth on  $\mathbb{R}^n$ . This requirement defines the same set of smooth functions no matter the choice of representative [atlas](#) by the nature of [Definition 1.2.2](#) requirement that defines the equivalent [manifolds](#).  $\circledast$

### 1.2.1 Orientation

Another essential property of a [manifold](#) is its orientability.

**Definition.** Consider an [atlas](#)  $\mathcal{A}$  for a [differentiable manifold](#)  $\mathcal{M}$ .

**Definition 1.2.5 (Oriented).**  $\mathcal{A}$  is *oriented* if all [transitions](#) have positive functional determinant.

**Definition 1.2.6 (Orientable).**  $\mathcal{M}$  is *orientable* if  $\mathcal{A}$  is an [oriented atlas](#).

Motivated by the above definitions, we see that we can actually use an [atlas](#) to define an [orientation](#).

**Definition 1.2.7 (Orientation).** Let  $\mathcal{M}$  be an [orientable manifold](#). Then a [oriented differentiable structure](#) is called an *orientation* of  $\mathcal{M}$ .

If  $\mathcal{M}$  possesses an [orientation](#), we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with [Definition 1.2.5](#).

**Remark.** Two [differentiable structures](#) obeying [Definition 1.2.5](#) determine the same [orientation](#) if the union again satisfying [Definition 1.2.5](#).

**Remark.** If  $\mathcal{M}$  is [orientable](#) and connected, then there exists exactly 2 distinct [orientations](#) on  $\mathcal{M}$ .

Now, we can see some examples of [smooth manifolds](#).

**Example (Sphere).** The sphere  $S^n \subseteq \mathbb{R}^{n+1}$  given by

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Consider  $U_i^+ = \{x \in S^n \mid x_i > 0\}$ ,  $U_i^- = \{x \in S^n \mid x_i < 0\}$  for  $i = 1, \dots, n+1$ , and  $h_i^\pm: U_i^\pm \rightarrow \mathbb{R}^n$  such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1}).$$

Note that the minimum [charts](#) needed to cover  $S^n$  is 2.

**Example.** Let  $\mathcal{M} = U \subseteq \mathbb{R}^n$ , then  $\{(U, \varphi)\}$  is a [smooth structure](#) with  $\varphi = \text{id}$ .

**Example.** Open sets of  $C^\infty$ -[manifolds](#) are  $C^\infty$ -[manifolds](#).

**Example (General linear group).**  $\text{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ , open.

**Example (Real projective space).**  $\mathbb{R}P^n = S^n / \sim$  where  $x \sim -x$  with  $\pi: S^n \rightarrow \mathbb{R}P^n$ ,  $x \mapsto [x]$ .

**Proof.**  $\pi$  is a homeomorphism on each  $U_i^+$  for  $i = 1, \dots, n+1$ , with

$$\{(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1}), i = 1, \dots, n+1\}$$

is a  $C^\infty$ -atlas for  $\mathbb{R}P^n$ . \*

**Note.** Observe that  $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ .

## Lecture 2: Maps Between Smooth Manifolds

### 1.2.2 Smooth Maps

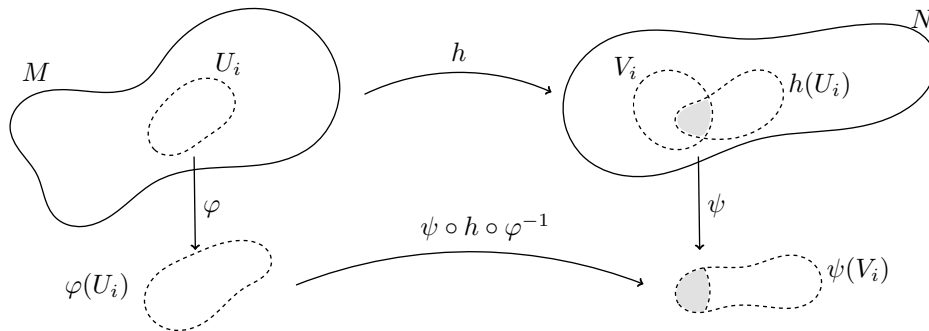
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We can now consider the maps between manifolds, specifically, the smooth manifolds.

**Definition 1.2.8** (Smooth function). Let  $M, N$  be two smooth manifolds, and let  $\mathcal{U}$  be locally finite atlas from the equivalence class that gives the smooth structure on  $M$ , and let  $\mathcal{V}$  be the corresponding for  $N$ . A map  $h: M \rightarrow N$  is said to be *smooth* if each map in the collection

$$\{\psi \circ h \circ \varphi^{-1} : h(U) \cap V \neq \emptyset\},$$

where  $(U, \varphi) \in \mathcal{U}$ ,  $(V, \psi) \in \mathcal{V}$  is  $C^\infty$ -differentiable as a map from one Euclidean space to another.



**Remark.** Equivalence relation guarantees that Definition 1.2.8 depends only on the smooth structure of  $M, N$ , but not on the chosen representative coordinate atlas.

**Definition.** Consider two smooth manifolds  $M, N$  and a smooth homeomorphism  $h: M \rightarrow N$  with smooth inverse.

**Definition 1.2.9** (Diffeomorphic). The two manifolds  $M, N$  are said to be *diffeomorphic*.

**Definition 1.2.10** (Diffeomorphism). The map  $h$  is said to be a *diffeomorphism*.

Let  $M_1, M_2$  be two smooth manifolds, and let  $\varphi: M_1 \rightarrow M_2$  be a diffeomorphism. Then

- (a)  $M_1$  is orientable if and only if  $M_2$  is orientable.
- (b) If in addition,  $M_1$  and  $M_2$  are both connected and oriented, then  $\varphi$  induces an orientation on  $M_2$  that may or may not coincide with the initial orientation of  $M_2$ .

If the induced orientation coincides, then we say  $\varphi$  preserves the orientation, otherwise  $\varphi$  reverses the orientation.

### 1.2.3 Grassmannian Manifold

Before proceeding, let's consider an interesting [smooth manifold](#).

**Definition 1.2.11 (Grassmannian manifold).** Given  $m, n \in \mathbb{N}$ , the so-called *Grassmannian manifold*  $G(n, m)$  is the set of all  $n$ -dimensional subspaces of  $\mathbb{R}^{n+m}$ .

**Note.**  $G(1, m)$  is just  $\mathbb{R}P^m$ , and  $G(0, m)$ ,  $G(n, 0)$  are one-point sets.

As we will soon see,  $G(n, m)$  has the [smooth structure](#) of an  $mn$ -dimensional [manifold](#).

**Intuition.** We obtain the [structure](#) by exhibiting an [atlas](#) whose [transitions](#) are [diffeomorphisms](#).

Firstly, we give  $G(n, m)$  a suitable topology, i.e., the metric topology. Let  $\Pi \in G(n, m)$ , and let  $\mathcal{L}(\Pi, \Pi^\perp)$  denote the  $mn$ -dimensional space of linear maps from  $\Pi$  to  $\Pi^\perp$ . Define the map

$$\varphi_\Pi: \mathcal{L}(\Pi, \Pi^\perp) \rightarrow G(n, m), \quad \varphi_\Pi(\alpha) = (\mathbb{1}_\Pi \oplus \alpha)(\Pi)$$

where  $\mathbb{1}_\Pi \oplus \alpha$  is regarded as a map  $\Pi \rightarrow \Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$ .<sup>1</sup> Clearly,  $\varphi_\Pi$  is injective, and thus,  $(\mathcal{L}(\Pi, \Pi^\perp), \varphi_\Pi)$  is an  $mn$ -dimensional [chart](#) of  $G(n, m)$ .

**Remark.** The images  $\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$  cover  $G(n, m)$ .

**Example.**  $\Pi = \varphi_\Pi(0) \in \varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$ .

We can now prove that these [charts](#) are mutually [compatible](#). Let  $\Pi, \Pi' \in G(n, m)$ , and let  $P, P'$  be orthogonal projections from  $\mathbb{R}^{n+m}$  onto  $\Pi, \Pi'$  respectively. Firstly,

$$F = \varphi_{\Pi'}^{-1} \varphi_\Pi: \varphi_\Pi^{-1}(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))) \rightarrow \varphi_{\Pi'}^{-1}(\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp)))$$

is smooth.



Consider  $\alpha \in \mathcal{L}(\Pi, \Pi^\perp)$ , and  $\beta \in \mathcal{L}(\Pi', (\Pi')^\perp)$ , then for  $\alpha, \beta$ , the equality  $F(\alpha) = \beta$  means that  $\varphi_\Pi(\alpha) = \varphi_{\Pi'}(\beta)$ . Let  $f_\alpha: \Pi \rightarrow \Pi'$  be defined by

$$f_\alpha = P' \circ (\mathbb{1}_\Pi \oplus \alpha).$$

We need to check

- (a)  $f_\alpha$  is invertible, and
- (b)  $\forall y \in \Pi, y + \alpha(y) = f_\alpha(y) + \beta(f_\alpha(y))$ .

**Note.** The condition that  $\det f_\alpha \neq 0$  gives an exact description of the subset  $\varphi_{\Pi'}^{-1}(\varphi_\Pi(\mathcal{L}(\Pi', (\Pi')^\perp)))$  of  $\mathcal{L}(\Pi, \Pi^\perp)$ , which is therefore open.

For  $\beta$ , it is  $(\mathbb{1}_{\Pi'} \oplus \beta) \circ f_\alpha = \mathbb{1}_\Pi \oplus \alpha$ , and hence

$$\beta = F(\alpha) = (\mathbb{1}_\Pi \oplus \alpha) \circ f_\alpha^{-1} - \mathbb{1}_{\Pi'}.$$

It follows by the construction that the image of  $\beta$  is contained in  $(\Pi')^\perp$ .

<sup>1</sup>In other words,  $\varphi_\Pi(\alpha)$  is the graph of  $\alpha$  in  $\Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$ .



**Remark.** We obtain an infinite atlas for  $G(n, m)$  with charts labeled by  $\Pi \in G(n, m)$ . But it's suffices to consider only  $\binom{n+m}{n}$  charts corresponding to subspaces  $\Pi$  spanned with  $n$  coordinate axes.

We now introduce two notions.

**Definition 1.2.12** (Closed manifold). A manifold is *closed* if it is compact and without boundary.

**Definition 1.2.13** (Open manifold). A manifold is *open* if it has only non-compact components without boundary.

**Lemma 1.2.1.** If  $M$  can be covered by two coordinate neighborhoods  $V_1, V_2$  such that  $V_1 \cap V_2$  is connected, then  $M$  is orientable.

**Proof.** The determinant of the differential of the coordinate change  $\neq 0$ , so it does not change sign in  $V_1 \cap V_2$ . If it's negative at a single point, it's enough to change the sign of one of the coordinates to make it positive at that point, hence on  $V_1 \cap V_2$ . ■

**Example.** Let  $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$  is orientable.

**Proof.** Let  $N = (0, \dots, 0, 1)$  and  $S = (0, \dots, 0, -1)$ , consider given  $p = (0, \dots, 0, x_i, 0, \dots, x_{n+1})$ , then  $\pi_1: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  given by

$$\pi_1(p) = \left(0, \dots, 0, \frac{x_i}{1 - x_{n+1}}, 0, \dots, 0\right)$$

to be the stereographic projection from the north pole  $N$ .



More generally, it takes  $p(x_1, \dots, x_{n+1}) \in S^n - \{N\}$  into the intersection at the hyperplane  $x_{n+1} = 0$  with the line passing through  $p$  and  $N$ . In this way, we have

$$\pi_1(x_1, \dots, x_n) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right),$$

hence  $\pi_1: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  is differentiable, and is injective. Similarly,  $\pi_2: S^n \setminus \{S\} \rightarrow \mathbb{R}^n$  for  $S$  can also be defined and everything holds similarly. We see that these two parametrizations  $(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})$  cover  $S^n$ . The change of coordinate is given by

$$y_j = \frac{x_j}{1 - x_{n+1}} \leftrightarrow y'_j = \frac{x_j}{1 + x_{n+1}}, \quad (y_1, \dots, y_n) \in \mathbb{R}^n, \quad j = 1, \dots, n,$$

where

$$y'_j = \frac{y_j}{\sum_{i=1}^n y_i^2}.$$

This implies that  $\{(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})\}$  is a **differentiable structure** for  $S^n$ . Now, consider  $\pi_1^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^n) = S^n \setminus \{N \cup S\}$ , which is connected, and hence  $S^n$  is **orientable**, and the above **structure** gives an **orientation** of  $S^n$ .  $\circledast$

## Lecture 3: Complex Manifolds, Tangent Spaces and Bundles

Let's look at two more examples about **orientation**.

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**Example.** Let  $A: S^n \rightarrow S^n$  be the antipodal map given by  $A(p) = -p$  for  $p \in \mathbb{R}^{n+1}$ . It's easy to see that  $A$  is differentiable with  $A^2 = 1$ . Furthermore,  $A$  is **diffeomorphism** of  $S^n \subseteq \mathbb{R}^{n+1}$ . We see that

- if  $n$  is even,  $A$  reverses the **orientation**;
- if  $n$  is odd,  $A$  preserves the **orientation**.

**Example.**  $G(k, n)$  is **orientable** if and only if  $n$  is even or  $n = 1$ .

Finally, we introduce the notion of **complex manifolds**.

**Definition 1.2.14 (Complex manifold).** A *complex manifold*  $\mathcal{M}$  of complex dimension  $d$  ( $\dim_{\mathbb{C}} \mathcal{M} = d$ ) is a **differentiable manifold** of (real) dimension  $2d$  ( $\dim_{\mathbb{R}} \mathcal{M} = 2d$ ) whose **charts** take values in open subsets of  $\mathbb{C}^d$  with holomorphic **chart transitions**.

**As previously seen.** The **chart transitions**  $z_{\beta} \circ z_{\alpha}^{-1}: z_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow z_{\beta}(U_{\alpha} \cap U_{\beta})$  is holomorphic if  $\partial z_{\beta}^j / \partial z_{\alpha}^k = 0$  for all  $j, k$  where

$$\frac{\partial}{\partial z^k} = \frac{1}{2} \left( \frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k} \right).$$

**Remark.** **Complex Grassmannians**  $G_{\mathbb{C}}(k, n)$  are all **orientable**. More generally, **complex manifolds** are always **orientable** because holomorphic maps always have positive functional determinant.

### 1.3 Partition of Unity

We state, without proof, of an important lemma about the **partition of unity**.

**Definition 1.3.1 (Partition of unity).** Let  $\mathcal{M}$  be a **differentiable manifold**, and let  $(U_{\alpha})_{\alpha \in \mathcal{A}}$  be an open covering of  $\mathcal{M}$ . Then a *partition of unity* is a **locally finite** refinement  $(V_{\beta})_{\beta \in \mathcal{B}}$  of  $(U_{\alpha})$  and  $C^{\infty}$ -functions  $\varphi_{\beta}: \mathcal{M} \rightarrow \mathbb{R}$  with

- $\text{supp}(\varphi_{\beta}) \subseteq V_{\beta}$  for all  $\beta \in \mathcal{B}$ ;
- $0 \leq \varphi_{\beta}(x) \leq 1$  for all  $x \in \mathcal{M}$ ,  $\beta \in \mathcal{B}$ ;
- $\sum_{\beta \in \mathcal{B}} \varphi_{\beta} = 1$  for all  $x \in \mathcal{M}$ .<sup>a</sup>

<sup>a</sup>There are only finitely many non-vanishing summands of each point, since only finitely many  $\varphi_{\beta}$  are non-zero of any given point as the covering  $(V_{\beta})$  is **locally finite**.

**Lemma 1.3.1 (Partition of unity).** Let  $\mathcal{M}$  be a **differentiable manifold**, and let  $(U_{\alpha})_{\alpha \in \mathcal{A}}$  be an open covering of  $\mathcal{M}$ . Then there exists a **partition of unity** subordinate to  $(U_{\alpha})$ ,

## 1.4 Tangent and Cotangent Spaces

### 1.4.1 Tangent Spaces in Euclidean Spaces

To discuss the concept of calculus between [manifolds](#) formally, we start with our discussion in Euclidean spaces, where we naturally have the coordinates for every point.

**Definition.** Let  $\mathcal{M}$  be a Euclidean [manifold](#) of dimension  $d$ ,  $x = (x^1, \dots, x^d)$  be Euclidean coordinates of  $\mathbb{R}^d$ , and  $x_0 \in \Omega \subseteq \mathbb{R}^d$  where  $\Omega$  is open.

**Definition 1.4.1** (Tangent space of Euclidean space). The *tangent space*  $T_{x_0}\Omega$  of  $\Omega$  at  $x_0$  is the vector space  $\{x_0\} \times E^a$  spanned by the basis  $(\partial/\partial x^1, \dots, \partial/\partial x^d)$ .

<sup>a</sup> $E$  is a  $d$ -dimensional Euclidean space.

**Definition 1.4.2** (Tangent vector of Euclidean space). The elements in the [tangent space of Euclidean space](#) is called *tangent vectors*.

Before proceeding, we introduce a shorthand notation.

**Notation** ([Einstein notation](#)). The *Einstein notation* abbreviates the summation  $\sum_i v^i x_i$  as  $v^i x_i$ , where we implicitly sum over the upper and lower index.

**Definition 1.4.3** (Differential of Euclidean space). If  $\Omega \subseteq \mathbb{R}^d$ ,  $\Omega' \subseteq \mathbb{R}^d$  are open, and  $f: \Omega \rightarrow \Omega'$  is differentiable, then the *differential*  $df(x_0)$  for  $x_0 \in \Omega$  is the induced linear map between [tangent spaces](#)

$$df(x_0): T_{x_0}\Omega \rightarrow T_{f(x_0)}\Omega', \quad v = v^i \frac{\partial}{\partial x^i} \mapsto v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}.$$

**Definition 1.4.4** (Tangent bundle of Euclidean space). The *tangent bundle* is defined as  $T\Omega := \bigsqcup_{x \in \Omega} T_x\Omega \cong \Omega \times E \cong \Omega \times \mathbb{R}^d$ , which is an open subset of  $\mathbb{R}^d \times \mathbb{R}^d$ .

**Note** ([Total space](#)).  $T\Omega$  is also called the *total space*.

**Remark.** Given a [tangent bundle](#)  $T\Omega$ , we define  $\pi$  to be the projection  $\pi: T\Omega \rightarrow \Omega$  given by  $\pi(x, v) = x$ . This makes  $T\Omega$  naturally a [differentiable manifold](#).

With the notion of [tangent bundle](#), given  $f: \Omega \rightarrow \Omega'$ , we can also define  $df: T\Omega \rightarrow T\Omega'$  as

$$\left(x, v^i \frac{\partial}{\partial x^i}\right) \mapsto \left(f(x), v^i \frac{\partial f^j}{\partial x^i}(x) \frac{\partial}{\partial f^j}\right).$$

**Notation.** We often write  $df(x)(v)$  instead of  $df(x, v)$  to coincide with the notation of [differential](#).

In particular, for  $v = v^i \partial/\partial x^i$ , we have

$$df(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R},$$

and we write  $v(f)(x)$  for  $df(x)(v)$ .

### 1.4.2 Tangent Spaces in Manifolds

We now try to formally define the [tangent space](#) on a [smooth manifold](#). A natural idea is the following.

**Intuition.** Let  $\mathcal{M}^d$  be a differentiable manifold with a chart  $x: U \rightarrow \Omega \subseteq \mathbb{R}^d$  and  $p \in U \subseteq \mathcal{M}$  where  $U$  is open. The tangent space  $T_p\mathcal{M}$  of  $\mathcal{M}$  at  $p$  should be represented in the chart  $x$  by  $T_{x(p)}x(U)$ .

To see that the above are well-defined, i.e.,  $T_p\mathcal{M}$  are independent of the choice of charts, let  $x': U' \rightarrow \mathbb{R}^d$  to be another chart with  $p \in U' \subseteq \mathcal{M}$  where  $U'$  is also open. Denote  $\Omega := x(U)$ , and  $\Omega' := x'(U')$ , then the transition map

$$x' \circ x^{-1}: x(U \cap U') \rightarrow x'(U \cap U')$$

induces a vector space isomorphism

$$L := d(x' \circ x^{-1})(x(p)): T_{x(p)}\Omega \rightarrow T_{x'(p)}\Omega',$$

such that  $v \in T_{x(p)}\Omega$  and  $L(v) \in T_{x'(p)}\Omega'$  represent the same tangent vector in  $T_p\mathcal{M}$ .

**Remark.** A tangent vector in  $T_p\mathcal{M}$  is given by the family of the coordinate representations.

Now, we want to define the similar notion of differential of Euclidean spaces. Let consider a simple case first, where we let  $f: \mathcal{M} \rightarrow \mathbb{R}$  to be a differentiable function, and assume that the tangent vector  $w \in T_p\mathcal{M}$  is represented by  $v \in T_{x(p)}x(U)$ .

**Intuition.** We want to define  $df(p)$  as a linear map from  $T_p\mathcal{M} \rightarrow \mathbb{R}$ . In chart  $x$ , let  $w \in T_p\mathcal{M}$  be given as  $v = v^i \partial/\partial x^i \in T_{x(p)}x(U)$ . Say that  $df(p)(w)$  in this chart represented by

$$d(f \circ x^{-1})(x(p))(v).$$



**Remark.**  $T_p\mathcal{M}$  is a vector space of dimension  $d$  isomorphic to  $\mathbb{R}^d$ , where the isomorphism depends on choice of chart.

**Intuition.** Pull functions on  $\mathcal{M}$  back by a chart to an open subset of  $\mathbb{R}^d$ , differentiate there.

In order to obtain a tangent space which does not depend on charts, we need to have transformation behavior under change of charts. Let  $F: \mathcal{M}^d \rightarrow \mathcal{N}^c$  be a differentiable map where  $\mathcal{M}, \mathcal{N}$  are smooth manifolds. Then we want to represent  $dF$  in local charts  $x: U \subseteq \mathcal{M} \rightarrow \mathbb{R}^d, y: V \subseteq \mathcal{N} \rightarrow \mathbb{R}^c$  by  $d(y \circ F \circ x^{-1})$ . The local coordinates on  $U$  is given by  $(x^1, \dots, x^d)$ , and on  $V$  is  $(F^1, \dots, F^c)$  such that

$$F(x) = (F^1(x^1, \dots, x^d), \dots, F^c(x^1, \dots, x^d)).$$

Then,  $dF$  induces a linear map  $dF: T_p\mathcal{M} \rightarrow T_{F(x)}\mathcal{N}$  which in our coordinate representation is given by the matrix

$$\left( \frac{\partial F^\alpha}{\partial x^i} \right)_{\substack{\alpha=1, \dots, c \\ i=1, \dots, d}},$$

and a change of charts is then just the base change at tangent spaces: if

$$\begin{aligned} (x^1, \dots, x^d) &\mapsto (\xi^1, \dots, \xi^d) \\ (F^1, \dots, F^c) &\mapsto (\phi^1, \dots, \phi^c) \end{aligned}$$

are coordinate changes, then  $dF$  represented in the new coordinates is given by

$$\left( \frac{\partial \phi^\beta}{\partial \xi^j} \right) = \left( \frac{\partial \phi^\beta}{\partial F^\alpha} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial x^i}{\partial \xi^j} \right).$$



## Lecture 4: Tangent Bundles, Vector Fields, and Submanifolds

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**Definition.** Let  $\mathcal{M}^d$  be a **differentiable manifold** with a **chart**  $x: U \rightarrow \Omega \subseteq \mathbb{R}^d$  and  $p \in U \subseteq \mathcal{M}$  where  $U$  is open. On  $\{(x, v) \mid v \in T_{x(p)}\Omega\}$ , we define an equivalence relation by  $(x, v) \sim (y, w)$  if and only if  $w = d(y \circ x^{-1})v$ .

**Definition 1.4.5 (Tangent space).** The space of equivalence classes is called the *tangent space*  $T_p\mathcal{M}$  at point  $p$  to  $\mathcal{M}$ .

**Definition 1.4.6 (Tangent vector).** The elements in the **tangent space** is called *tangent vectors*.

**Remark.**  $T_p\mathcal{M}$  naturally carries the structure of a vector space.

Now,  $T\mathcal{M}$  is defined as

$$T\mathcal{M} := \coprod_{p \in \mathcal{M}} T_p\mathcal{M}.$$

Recall the projection  $\pi: T\mathcal{M} \rightarrow \mathcal{M}$  with  $\pi(w) = p$  for  $w \in T_p\mathcal{M}$ . Then we can define the following.

**Definition 1.4.7 (Derivation).** If  $x: U \rightarrow \mathbb{R}^d$  be a **chart** for  $\mathcal{M}$ , and let  $TU = \coprod_{p \in U} T_pU$ . Then we define the *derivation*  $dx: TU \rightarrow Tx(U) := \coprod_{p \in x(U)} T_p\mathcal{M}$  by  $w \mapsto dx(\pi(w))(w) \in T_{x(\pi(w))}x(U)$ .

The transition maps

$$dx' \circ (dx)^{-1} = d(x' \circ x^{-1})$$

are differentiable.  $\pi$  is local represented by  $x \circ \pi \circ dx^{-1}$  maps  $(x_0, v) \in Tx(U)$  to  $x_0$ .

**Definition 1.4.8 (Tangent bundle).** The triple  $(T\mathcal{M}, \pi, \mathcal{M})$  is called the *tangent bundle* of  $\mathcal{M}$ .

**Definition 1.4.9 (Total space).**  $T\mathcal{M}$  is called the *total space* of the **tangent bundle**.

We can choose the courses (the initial) topology for **total space**  $T\mathcal{M}$  such that  $\pi$  is continuous. Furthermore, we can construct a  **$C^\infty$ -atlas**  $\mathcal{A}_{T\mathcal{M}}$  on  $T\mathcal{M}$  from the  **$C^\infty$ -atlas**  $\mathcal{A}$  of  $\mathcal{M}$ . Specifically, consider  $\mathcal{A}_{T\mathcal{M}} := \{(TU, \xi_x) \mid (U, x) \in \mathcal{A}\}$  where  $\xi_x: TU \rightarrow \mathbb{R}^{2 \cdot d}$  such that

$$x \mapsto ((x^1 \circ \pi)(x), \dots, (x^d \circ \pi)(x), (dx^1)_{\pi(x)}(X), \dots, (dx^d)_{\pi(x)}(X)).$$

**Intuition.** We know that  $X = X_x^i (\partial/\partial x^i)_{\pi(x)}$ , and we might tempt to write  $X^i$  as the last  $d$  components. But we write it in the above way is because

$$(dx^j)_{\pi(x)}(X) = (dx^j)_{\pi(x)} \left( X_x^i \left( \frac{\partial}{\partial x^i} \right)_{\pi(x)} \right) = X_x^i \delta_i^j = X_x^j.$$

**Note.** We can check that  $\xi_x^{-1}$  exists, and it's also smooth, hence  $T\mathcal{M}$  has a natural topology and a  **$C^\infty$ -atlas** making it a  $2 \dim \mathcal{M}$ -dimensional **smooth manifold**.

### 1.4.3 Cotangent Spaces

Another important objects is the [cotangent spaces](#).

**Definition.** Let  $\mathcal{M}^d$  be a [differentiable manifold](#), and  $T_p\mathcal{M}$  be the [tangent space](#) at  $p$  to  $\mathcal{M}$ .

**Definition 1.4.10** (Cotangent space). The *cotangent space*  $T_p^*\mathcal{M}$  to  $\mathcal{M}$  is the dual of  $T_p\mathcal{M}$ , i.e.,  $T_p^*\mathcal{M} = (T_p\mathcal{M})^*$ .

**Definition 1.4.11** (Cotangent vector). The elements in the [cotangent space](#) is called *cotangent vectors*.

**Remark.**  $T_p^*\mathcal{M}$  is the space of 1-forms on  $T_p\mathcal{M}$ .

**Notation** (Covariant vector). The [cotangent vectors](#) are also called *covariant vectors*.

**Notation** (Contravariant vector). The [tangent vectors](#) are also called *contravariant vectors*.

## 1.5 Vector Fields and Brackets

### 1.5.1 Vector Fields

We now introduce the notion of [vector field](#).

**Definition 1.5.1** (Vector field). A (*tangent*) *vector field*  $X$  on a [differentiable manifold](#)  $\mathcal{M}$  is a correspondence associating to each point  $p \in \mathcal{M}$  a vector  $X(p) \in T_p\mathcal{M}$ , i.e.,  $X: \mathcal{M} \rightarrow T\mathcal{M}$ .

**Remark.** Naturally, we say that the [field](#)  $X$  is differentiable if the map  $X$  is differentiable.

Considering a [local chart](#)  $x: U \subseteq \mathbb{R}^n \rightarrow \mathcal{M}$ , we can write

$$X(p) = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i},$$

where  $a_i: U \rightarrow \mathbb{R}$  are functions on  $U$  for  $i = 1, \dots, n$ , and  $\{\partial/\partial x_i\}_i$  is the basis associated to  $x$ .

**Remark.**  $X$  is differentiable if and only if  $a_i$  are differentiable for some (and, therefore, for any)  $x$ .

It's convenient to think of a [vector field](#) as a mapping  $X: \mathcal{D} \rightarrow \mathcal{F}$  from the set  $\mathcal{D}$  of differentiable functions on  $\mathcal{M}$  to the set  $\mathcal{F}$  of the functions on  $\mathcal{M}$ , defined by

$$(Xf)(p) = \sum_{i=1}^n a_i(p) \frac{\partial f}{\partial x_i}(p),$$

where  $f$  is implicitly denoting the expression of  $f$  in the [chart](#)  $x$ .

**Intuition.** This idea of a vector as a directional derivative is precisely what was used to define the notion of [tangent vector](#).

**Remark.**  $Xf$  does not depend on the choice of  $x$ .

**Remark.**  $X$  is differentiable if and only if  $X: \mathcal{D} \rightarrow \mathcal{D}$ , i.e.,  $Xf \in \mathcal{D}$  for all  $f \in \mathcal{D}$ .

Observe that if  $\varphi: \mathcal{M} \rightarrow \mathcal{M}$  is a **diffeomorphism**,  $v \in T_p\mathcal{M}$  and  $f$  differentiable function in a neighborhood of  $\varphi(p)$ , we have

$$(d\varphi(v)f)\varphi(p) = v(f \circ \varphi)(p)$$

since by letting  $\alpha: (-\epsilon, \epsilon) \rightarrow \mathcal{M}$  be a differentiable **curve** with  $\alpha'(0) = v$ ,  $\alpha(0) = p$ ,<sup>2</sup> then

$$(d\varphi(v)f)\varphi(p) = \left. \frac{d}{dt}(f \circ \varphi \circ \alpha) \right|_{t=0} = v(f \circ \varphi)(p).$$

### 1.5.2 Brackets

By viewing  $X$  as an operator on  $\mathcal{D}$ , we can consider the iterates of  $X$ , i.e, given differentiable **fields**  $X$  and  $Y$  and  $f: \mathcal{M} \rightarrow \mathbb{R}$  being a differentiable function, consider  $X(Yf)$  and  $Y(Xf)$ .

**Note.** In general,  $X(Yf)$  (and hence  $Y(Xf)$ ) is not a **field**.

**Proof.** It involves derivatives of order higher than one. ⊗

But we have the following.

**Lemma 1.5.1.** Let  $X, Y$  be differentiable **vector fields** on a **smooth manifold**  $\mathcal{M}$ . Then there exists a unique **vector field**  $Z$  such that for all  $f \in \mathcal{D}$ ,  $Zf = (XY - YX)f$ .

**Proof.** See do Carmo [FC13, Chapter 0, Lemma 5.2]. ■

This  $Z$  is called the **bracket**.

**Definition 1.5.2 (Bracket).** Given two differentiable **vector fields**  $X, Y$  on a **smooth manifold**  $\mathcal{M}$ , the **bracket** of  $X$  and  $Y$  is defined by

$$[X, Y] := XY - YX.$$

Clearly,  $[X, Y]$  is differentiable.

**Proposition 1.5.1.** If  $X, Y$  and  $Z$  are differentiable **vector fields** on  $\mathcal{M}$ ,  $a, b \in \mathbb{R}$ ,  $f, g$  are differentiable functions, then we have the following.

- (a)  $[X, Y] = -[Y, X]$  (*anti-commutativity*),
- (b)  $[aX + bY, Z] = a[X, Z] + b[Y, Z]$  (*linearity*),
- (c)  $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$  (*Jacobi identity*),
- (d)  $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$ .

**Proof.** See do Carmo [FC13, Chapter 0, Proposition 5.3]. ■

**Example.**  $[\partial/\partial x^i, \partial/\partial x^j] = 0$  for  $i = j$ .

## 1.6 Submanifolds, Immersions, and Embeddings

We now study the relation between **manifolds**.

**Definition 1.6.1 (Immersion).** Let  $\mathcal{M}^m, \mathcal{N}^n$  be **smooth manifolds**. A differentiable mapping  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$  is an *immersion* if

$$d\varphi_p: T_p\mathcal{M} \rightarrow T_{\varphi(p)}\mathcal{N}$$

<sup>2</sup>This is the way do Carmo [FC13] used to define **tangent vectors**.

is injective for every  $p \in \mathcal{M}$ .

**Definition 1.6.2 (Embedding).** An **immersion**  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$  is an *embedding* if it is also a homeomorphism onto  $\varphi(\mathcal{M}) \subseteq \mathcal{N}$ , with  $\varphi(\mathcal{M})$  having the subspace topology induced from  $\mathcal{N}$ .

**Definition 1.6.3 (Submanifold).** If the inclusion  $\iota: \mathcal{M} \hookrightarrow \mathcal{N}$  between two **manifolds** is an **embedding**, then  $\mathcal{M}$  is a *submanifold* of  $\mathcal{N}$ .

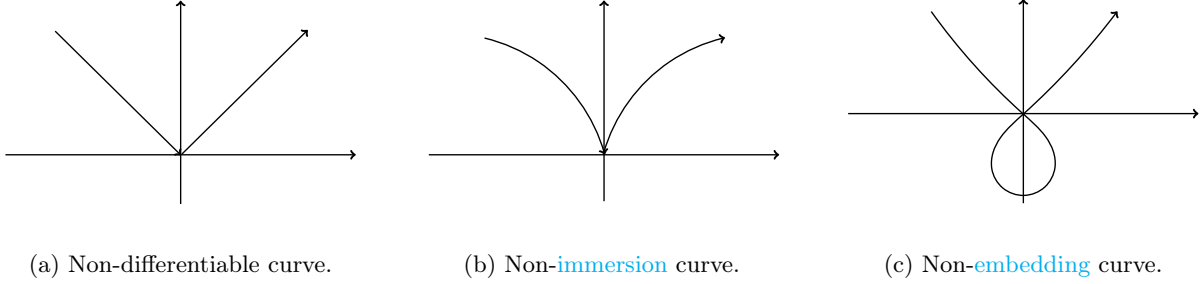


Figure 1.1: Three simple examples

**Lemma 1.6.1.** Let  $f: \mathcal{M}^m \rightarrow \mathcal{N}^n$  to be an **immersion** and  $x \in \mathcal{M}$ .<sup>a</sup> Then there exists a neighborhood  $U$  of  $x$  and a **chart**  $(V, y)$  on  $\mathcal{N}$  with  $f(x) \in V$  such that  $f|_U$  is a differentiable **embedding** and  $y^{m+1}(p) = \dots = y^n(p) = 0$  for all  $p \in f(U \cap V)$ .

<sup>a</sup>Hence,  $n \geq m$ .

**Proof.** In the local coordinates  $(z^1, \dots, z^n)$  on  $\mathcal{N}$ , and  $(x^1, \dots, x^m)$  on  $\mathcal{M}$ , without loss of generality,<sup>a</sup> let

$$\left( \frac{\partial z^\alpha(f(x))}{\partial x^i} \right)_{i, \alpha=1, \dots, m}$$

be non-singular. Consider

$$F(z, x) := (z^1 - f^1(x), \dots, z^m - f^m(x)),$$

which has maximal rank in  $x^1, \dots, x^m, z^{m+1}, \dots, z^n$ . By the **implicit function theorem**, locally, there exists a map  $\varphi: U \rightarrow \mathbb{R}^n$  such that

$$(z^1, \dots, z^m) \mapsto (\varphi^1(z^1, \dots, z^m), \dots, \varphi^n(z^1, \dots, z^m)) = x$$

such that  $F(z, x) = 0$ , i.e.,

$$\varphi^i(z^1, \dots, z^m) = \begin{cases} x^i, & \text{if } i = 1, \dots, m; \\ z^i, & \text{if } i = m+1, \dots, n, \end{cases}$$

for which

$$\left( \frac{\partial \varphi^i}{\partial z^\alpha} \right)_{\alpha, i=1, \dots, m}$$

has maximal rank. Now, we choose a new coordinate

$$(y^1, \dots, y^n) = (\varphi^1(z^1, \dots, z^m), \dots, \varphi^m(z^1, \dots, z^m), \\ z^{m+1} - \varphi^{m+1}(z^1, \dots, z^m), \dots, z^n - \varphi^n(z^1, \dots, z^m)).$$

Then, we have  $z = f(x) \Leftrightarrow F(z, x) = 0$ , i.e.,  $(y^1, \dots, y^n) = (x^1, \dots, x^n, 0, \dots, 0)$ , proving the result. ■

<sup>a</sup>Since  $df(x)$  is injective.



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**Lemma 1.6.2.** Let  $f: \mathcal{M}^m \rightarrow \mathcal{N}^n$  be a differentiable map such that  $m \geq n$  with  $p \in \mathcal{N}$ . Let  $df(x)$  has rank  $n$  for all  $x \in \mathcal{M}$  with  $f(x) = p$ . Then  $f^{-1}(p)$  is the union of differentiable submanifolds of  $\mathcal{M}$  of dimension  $m - n$ .

**Remark.** Let  $\mathcal{N}^n$  be a smooth manifold, and let  $1 \leq m \leq n$ . Then an arbitrary subset  $\mathcal{M} \subseteq \mathcal{N}$  has the structure of differentiable submanifold of  $\mathcal{N}$  of dimension  $m$  if and only if for all  $p \in \mathcal{M}$ , there exists a smooth chart  $(U, \varphi)$  of  $\mathcal{N}$  such that  $p \in U$ ,  $\varphi(p) = 0$ ,  $\varphi(U)$  is open, and

$$\varphi(U \cap \mathcal{M}) = (-\epsilon, +\epsilon)^n \times \{0\}^{n-m},$$

where  $(-\epsilon, +\epsilon)^n$  is the cube. Noticeably, the  $C^\infty$ -manifold structure of  $\mathcal{M}$  is uniquely determined.

**Remark.** Let  $\mathcal{M} \subseteq \mathcal{N}$  be a differentiable submanifold of  $\mathcal{N}$ , and let  $\iota: \mathcal{M} \hookrightarrow \mathcal{N}$  be the inclusion. Then, for  $p \in \mathcal{M}$ ,  $T_p\mathcal{M}$  can be considered as subspace of  $T_p\mathcal{N}$ , namely as the image of  $d\iota(T_p\mathcal{M})$ .

**Lemma 1.6.3.** Let  $f: \mathcal{M}^m \rightarrow \mathcal{N}^n$  be a differentiable map such that  $m \geq n$  with  $p \in \mathcal{N}$ . Let  $df(x)$  has rank  $n$  for all  $x \in \mathcal{M}$  with  $f(x) = p$ . For the submanifold  $X = f^{-1}(p)$  and for  $q \in X$ , it is true that

$$T_q X = \ker df(q) \subseteq T_q \mathcal{M}.$$

# Chapter 2

## Riemannian Manifolds

### Lecture 5: Riemannian Manifolds

In this chapter, we start our discussion on [Riemannian manifolds](#).

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#### 2.1 Riemannian Metrics

We start by defining the [Riemannian metric](#).

**Definition 2.1.1** (Riemannian metric). A *Riemannian metric*  $g$  on a [differentiable manifold](#)  $\mathcal{M}$  is given by a scalar product  $I$  on each  $T_p\mathcal{M}$  which depends smoothly on the base point  $p$ .

**Definition 2.1.2** (Riemannian manifold). A *Riemannian manifold*  $(\mathcal{M}, g)$  is a [smooth manifold](#)  $\mathcal{M}$  equipped with a [Riemannian metric](#)  $g$ .

Let  $x = (x^1, \dots, x^d)$  be the [local coordinates](#). In these, a [metric](#) is represented by a positive definite symmetric matrix

$$(g_{ij}(x))_{i,j=1,\dots,d},$$

i.e.,  $g_{ij} = g_{ji}$ , and  $g_{ij}\xi^i\xi^j > 0$  for all  $\xi = (\xi^1, \dots, \xi^d) \neq 0$  with coefficients smoothly depending on  $x$ .

##### 2.1.1 Transformation Behavior

We now see that the smoothness does not depend on [coordinates](#), i.e., the smooth dependence on the base point (as required in [Definition 2.1.1](#)) can be represented in the [local coordinates](#). Given 2 [tangent vectors](#)  $v, w \in T_p\mathcal{M}$  with [coordinate representations](#)  $(v^1, \dots, v^d), (w^1, \dots, w^d)$  given by  $x$  such that  $v = v^i \frac{\partial}{\partial x^i}$  and  $w = w^i \frac{\partial}{\partial x^i}$ , their product is

$$\langle v, w \rangle := g_{ij}(x(p))v^i w^j.$$

In particular,

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = g_{ij}.$$

**Remark.** The length of  $v$  is given as  $\|v\| := \langle v, v \rangle^{1/2}$ .

Let  $y = f(x)$  define different [local coordinates](#). In these,  $v, w$  are given as

$$(\tilde{v}^1, \dots, \tilde{v}^d), (\tilde{w}^1, \dots, \tilde{w}^d)$$

with  $\tilde{v}^j = v^i \frac{\partial f^j}{\partial x^i}$  and  $\tilde{w}^j = w^i \frac{\partial f^j}{\partial x^i}$ . Denote the [metric](#) in new [coordinates](#)  $y$  by  $h_{k\ell}(y)$ , then we have

$$h_{k\ell}(f(x))\tilde{v}^k \tilde{w}^\ell = \langle v, w \rangle = g_{ij}(x)v^i w^j.$$

Plug everything in, we have

$$h_{k\ell}(f(x)) \frac{\partial f^k}{\partial x^i} \frac{\partial f^\ell}{\partial x^j} v^i w^j = g_{ij}(x) v^i w^j.$$

We see that this holds for any **tangent vectors**  $v, w$ , therefore,

$$h_{k\ell}(f(x)) \frac{\partial f^k}{\partial x^i} \frac{\partial f^\ell}{\partial x^j} = g_{ij}(x),$$

which is the transformation behavior under **coordinates changes**.

**Remark.** This shows that the smoothness does not depend on the choice of coordinates!

**Example.** Consider the Euclidean space  $\Omega$ , then given  $v, w \in T_p\Omega$ , we have

$$\langle v, w \rangle = \delta_{ij} v^i w^j = v^i w_i.$$

**Theorem 2.1.1.** Every **differentiable manifold** can be equipped with a **Riemannian metric**.

**Proof.** From **Lemma 1.3.1**, there exists a differentiable **partition of unity**  $\{f_\alpha\}$  of  $\mathcal{M}$  subordinate to a covering  $\{V_\alpha\}$  of  $\mathcal{M}$ . Consider the induced **metric**  $\langle \cdot, \cdot \rangle^\alpha$  of the system of **local coordinates** on each  $V_\alpha$ . Then, for every  $p \in M$ , a **Riemannian metric**  $\langle \cdot, \cdot \rangle_p$  can be defined naturally as

$$\langle u, v \rangle_p = \sum_{\alpha} f_{\alpha}(p) \langle u, v \rangle_p^{\alpha}$$

for all  $u, v \in T_p M$ . Given the fact that  $\{f_\alpha\}$  is the **partition of unity**, we know that

- (a)  $f_\alpha \geq 0$ , and  $f_\alpha = 0$  on  $\overline{V_\alpha}^c$ ,
- (b)  $\sum_{\alpha} f_{\alpha}(p) = 1$  for all  $p$  on  $M$ ,

it's then immediate that the defined is indeed a **Riemannian metric**. ■

## 2.1.2 Isometry

After introducing any type of mathematical structure, we must introduce a notion of when two objects are the same.

**Definition 2.1.3 (Isometry).** A **diffeomorphism**  $h: \mathcal{M} \rightarrow \mathcal{N}$  is an **isometry** between two **Riemannian manifolds** if it preserves the **Riemannian metric**, i.e., for  $p \in \mathcal{M}$ ,  $v, w \in T_p \mathcal{M}$ ,

$$\langle v, w \rangle_{\mathcal{M}} = \langle dh(v), dh(w) \rangle_{\mathcal{N}}.$$

**Definition 2.1.4 (Local isometry).** A **diffeomorphism**  $h: \mathcal{M} \rightarrow \mathcal{N}$  is a **local isometry** between two **Riemannian manifolds** if for every  $p \in \mathcal{M}$ , there exists a neighborhood  $U$  such that  $h|_U: U \rightarrow h(U): \mathcal{M} \rightarrow \mathcal{N}$  is an **isometry** and  $h(U) \subseteq \mathcal{N}$  is open.

It's common to say that a **Riemannian manifold**  $\mathcal{M}$  is **locally isometric** to a **Riemannian manifold**  $\mathcal{N}$  if for every  $p \in \mathcal{M}$ , there exists a neighborhood  $U$  of  $p$  in  $\mathcal{M}$  and a **local isometry**  $f: U \rightarrow f(U) \subseteq \mathcal{N}$ .

Let's first look at an almost trivial example.

**Example (Euclidean space).** Let  $\mathcal{M} = \mathbb{R}^n$  with  $\partial/\partial x_i$  identified with  $e_i = (0, \dots, 1, \dots, 0)$ . The metric is given by

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

$\mathbb{R}^n$  is called *Euclidean space of dimension  $n$*  and the Riemannian geometry of this space is metric Euclidean geometry.

**Example** (Lie group). See [Appendix A.2](#) for reference.

## 2.2 Geodesics

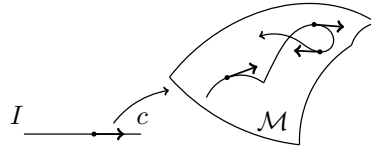
This is the first focus on the study of Riemannian geometry, i.e., the [geodesics](#). The up-shot is that a [geodesic](#) minimizes the [arc length](#) for points *sufficiently close* (in a sense to be made precise); in addition, if a [curve](#) minimizes [arc length](#) between any two of its points, it is a [geodesic](#).

### 2.2.1 Curves

We are now going to show how a [Riemannian metric](#) can be used to calculate the [length](#) of a [curve](#).

**Definition 2.2.1** (Curve). A differentiable mapping  $c: I \rightarrow \mathcal{M}$  of an open interval  $I \subseteq \mathbb{R}$  into a [differentiable manifold](#)  $\mathcal{M}$  is called a (parametrized) *curve*.

**Note.** A parametrized curve can admit self-intersections as well as corners.



**Definition 2.2.2** (Vector field along a curve). We say that a *vector field along a curve*  $c: I \rightarrow \mathcal{M}$  is a differentiable mapping that associates to every  $t \in I$  a [tangent vector](#)  $V(t) \in T_{c(t)}\mathcal{M}$ .

To say  $V$  is differentiable means that for any differentiable function  $f$  on  $\mathcal{M}$ , the function  $t \mapsto V(t)f$  is a differentiable function on  $I$ .

**Example** (Velocity field). The [vector field](#)  $dc(d/dt)$ , denoted by  $dc/dt$ , is called the *velocity field* or *tangent vector field*, of course.

**Remark.** A [vector field along  \$c\$](#)  cannot necessarily be extended to a [vector field](#) on an open set of  $\mathcal{M}$ .

**Notation** (Segment). The restriction of a [curve](#)  $c$  to a closed interval  $[a, b] \subseteq I$  is called a *segment*.

### 2.2.2 Lengths and Energies

We're interested in the following two quantities.

**Definition.** Let  $\gamma: [a, b] \rightarrow \mathcal{M}$  be a [curve](#) on a [Riemannian manifold](#)  $(\mathcal{M}, g)$ .

**Definition 2.2.3** (Length). The *length* of  $\gamma$  is defined as

$$L(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt.$$

**Definition 2.2.4** (Energy). The *energy* of  $\gamma$  is defined as

$$E(\gamma) := \frac{1}{2} \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt.$$

We now want to compute  $L(\gamma)$ ,  $E(\gamma)$  in **local coordinates**. Let the **local coordinates** be

$$(x^1(\gamma(t)), \dots, x^d(\gamma(t))),$$

we write

$$\dot{x}^i(t) = \frac{d}{dt}(x^i(\gamma(t))).$$

Then, we have

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t))) \dot{x}^i(t) \dot{x}^j(t)} dt, \quad E(\gamma) = \frac{1}{2} \int_a^b g_{ij}(x(\gamma(t))) \dot{x}^i(t) \dot{x}^j(t) dt.$$

**Definition 2.2.5 (Distance).** Given a **Riemannian manifold**  $(\mathcal{M}, g)$ , the *distance* between 2 points  $p, q \in \mathcal{M}$  is defined as

$$d(p, q) := \inf \{L(\gamma) \mid \gamma: [a, b] \rightarrow \mathcal{M} \text{ piecewise curve with } \gamma(a) = p, \gamma(b) = q\}.$$

**Note.** Any 2 points  $p, q \in \mathcal{M}$  can be connected by a piecewise **curve**, hence  $d(p, q)$  always exists.

**Corollary 2.2.1.** The topology of  $\mathcal{M}$  induced by the **distance function**  $d$  coincides with the original manifold topology of  $\mathcal{M}$ .

**Lemma 2.2.1.** If  $\gamma: [a, b] \rightarrow \mathcal{M}$  is a **curve**, and  $\psi: [\alpha, \beta] \rightarrow [a, b]$  is a change of parameter, then  $L(\gamma \circ \psi) = L(\gamma)$ .

**Proof.** This can be proved by computation, and the take-away is that the **length functional** is invariant under parameter changes. ■

### 2.2.3 Euler-Lagrange Equations

We want to find a **curve** which minimizes the **length** between sufficiently close two points. It turns out that instead of working with **length** directly, we should work with **energy** instead.

**Notation.** Let's first fix some common notations.

- (a)  $(g^{ij})_{i,j=1,\dots,d} = (g_{ij})_{i,j=1,\dots,d}^{-1}$ .
- (b)  $g_{j\ell,k} := \frac{\partial}{\partial x^k} g_{j\ell}$ .

**Note.** In the above notations, we have  $g^{i\ell} g_{\ell j} = \delta_j^i$ .

And the following is particularly important.

**Notation (Christoffel symbol).** The *Christoffel symbol* is defined for all  $i$  as

$$\Gamma_{jk}^i := \frac{1}{2} g^{i\ell} (g_{j\ell,k} + g_{k\ell,j} - g_{j\ell,k}).$$

**Remark.** The notion of  $\Gamma$  is a bit cryptic at first, and we will come back to this after. Now, just treat it as a calculation tool.

The up-shot is that the **Euler-Lagrange equations** for the **energy**  $E$  has a nice form, and the solution of which has exactly the properties we want, hence we define it as **geodesics**.

**Proposition 2.2.1.** The Euler-Lagrange equations for the energy  $E$  are

$$\ddot{x}^i(t) + \Gamma_{jk}^i(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0 \text{ for } i = 1, \dots, d. \quad (2.1)$$

**Proof.** The Euler-Lagrange equations of a functional<sup>a</sup>

$$I(x) = \int_a^b f(t, x(t), \dot{x}(t)) dt$$

are

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0$$

for  $i = 1, \dots, d$ . Just by plugging in, we obtain for  $E$ , we have

$$\frac{d}{dt} (g_{ik}(x(t))\dot{x}^k(t) + g_{ji}(x(t))\dot{x}^j(t)) - g_{jk,i}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$$

for  $i = 1, \dots, d$ . Hence,

$$g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j + g_{ik,\ell}\dot{x}^\ell\dot{x}^k + g_{ji,\ell}\dot{x}^\ell\dot{x}^j - g_{jk,i}\dot{x}^\ell\dot{x}^j = 0$$

Rename some indices and use  $g_{ij} = g_{ji}$ , we have that

$$2g_{\ell m}\ddot{x}^m + (g_{k\ell,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0$$

for  $\ell = 1, \dots, d$ . Hence, we have

$$g^{i\ell}g_{\ell m}\ddot{x}^m + \frac{1}{2}g^{i\ell}(g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0$$

for  $i = 1, \dots, d$ . Finally, observe that  $g^{i\ell}g_{\ell m} = \delta_{im}$ , i.e.,  $g^{i\ell}g_{\ell m}\ddot{x}^m = \ddot{x}^i$ , hence the claim follows. ■

<sup>a</sup>The Lagrangian is  $\mathcal{L} = \frac{1}{2}g_{jk}\dot{x}^j\dot{x}^k$ .

Finally, we define the geodesics as the solution of Equation 2.1.

**Definition 2.2.6 (Geodesic).** A curve  $\gamma: [a, b] \rightarrow \mathcal{M}$  that obeys Equation 2.1 is called a *geodesic*.

In other words, from Proposition 2.2.1, we naturally define geodesic by the solution of Equation 2.1, which is the critical points of energy. To do this via the variational principal, we first define the action functional.

**Definition 2.2.7 (Action functional).** Let  $\mathcal{L}$  be the Lagrangian, then the *action functional*

$$I[w(\cdot)] := \int_0^t \mathcal{L}(\dot{w}(s), w(s)) ds$$

is defined for functions  $w(\cdot) = (w^1(\cdot), \dots, w^n(\cdot))$  of the admissible class

$$\mathcal{A} = \{w(\cdot) \in C^2([0, t]; \mathbb{R}^n) \mid w(0) = y, w(t) = x\}.$$

From the calculus of variation, we can find a curve  $x(\cdot) \in \mathcal{A}$  such that  $I[x(\cdot)] = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)]$ .

**Theorem 2.2.1 (Euler-Lagrangian equations).** The solution  $x(\cdot)$  from  $I[x(\cdot)] = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)]$  solves the system of Euler-Lagrangian equations

$$\frac{d}{ds} (D_{\dot{x}}\mathcal{L}(\dot{x}(s), x(s)) + D_x\mathcal{L}(\dot{x}(s), x(s))) = 0$$

for  $0 \leq s \leq t$ .

## Lecture 6: Geodesic and the Exponential Map

Now, we draw some relations between [length](#) and [energy](#) and see why starting from [energy](#) makes sense. 24 Jan. 14:30

**Proposition 2.2.2.** For all [curves](#)  $\gamma: [a, b] \rightarrow \mathcal{M}$ ,

$$\mathcal{L}(\gamma)^2 \leq 2(b-a)E(\gamma)$$

with equality if and only if  $\|d\gamma/dt\|$  is a constant.

**Proof.** From [Hölder's inequality](#),

$$\int_a^b \left\| \frac{d\gamma}{dt} \right\| dt \leq (b-a)^{1/2} \left( \int_a^b \left\| \frac{d\gamma}{dt} \right\|^2 dt \right)^{1/2}$$

with equality if and only if  $\|d\gamma/dt\|$  is a constant. ■

**Example.** Let

$$\mathcal{L}(q, x) = \frac{1}{2}m|q|^2 - V(x)$$

with  $m > 0$ ,  $q = \dot{x}$ , the Euler-Lagrangian equations is given by  $m\ddot{x}(s) = F(x(s))$  for  $F := -DV$ .

**As previously seen.** Regular curves can be parametrized by [arc length](#) with unit speed  $\|d\gamma/dt\| = \|\dot{\gamma}\| \equiv 1$ .

**Lemma 2.2.2.** Each [geodesic](#) is parametrized proportionally to the [arc length](#).<sup>a</sup>

<sup>a</sup>This means that we have constant speed, i.e.,  $\|\dot{\gamma}\|$  is a constant.

**Proof.** For a solution of  $\ddot{x}^i(t) + \Gamma_{jk}^i(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$ ,

$$\frac{d}{dt} \langle \dot{x}, \dot{x} \rangle = \frac{d}{dt} (g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)) = 0.$$

■

Our goal now is to minimize the [length](#) within class of regular [smooth curves](#). Notice that the [length](#) and the [energy](#) functionals are invariants under parameter changes, which means that it's enough to look at [curves](#) parametrized by [arc length](#).

**Theorem 2.2.2.** Let  $\mathcal{M}$  be a [Riemannian manifold](#),  $p \in \mathcal{M}$  and  $v \in T_p\mathcal{M}$ . Then there exists an  $\epsilon > 0$  and a unique [geodesic](#) such that  $c: [0, \epsilon] \rightarrow \mathcal{M}$  with  $c(0) = p$  and  $\dot{c}(0) = v$ . In addition,  $c$  smoothly depend on  $p, v$ .

**Proof.** Since [Equation 2.1](#) is a system of second order ODE, by [Picard-Lindelöf theorem](#), we have local existence and uniqueness of the solution with prescribed initial values and derivative such that the solution depends smoothly on  $p, v$ . ■

If  $x(t)$  is the solution of [Equation 2.1](#), then  $x(\lambda t)$  is also a solution for any constant  $\lambda \in \mathbb{R}$ . Denote [geodesic](#) from [Theorem 2.2.2](#) by  $c_v$ , then

$$c_v(t) = c_{\lambda v}(t/\lambda)$$

for  $\lambda > 0$ ,  $t \in [0, \epsilon]$ , and hence  $c_{\lambda v}$  defined on  $[0, \epsilon/\lambda]$ .

**Remark.** Since  $c_v$  depends smoothly on  $v$ , the set  $\{v \in T_p\mathcal{M} \mid \|v\| = 1\}$  is compact, hence there exists  $\epsilon_0 > 0$  such that for  $\|v\| = 1$ ,  $c_v$  defined at least on  $[0, \epsilon_0]$ , implying that for all  $w \in T_p\mathcal{M}$  with  $\|w\| \leq \epsilon_0$ ,  $c_w$  is defined at least on  $[0, 1]$ .

## 2.3 Exponential Maps

The above discussion permits us to introduce the concept of the [exponential map](#) in the following manner.

**Definition 2.3.1 (Exponential map).** Let  $(\mathcal{M}, g)$  be a [Riemannian manifold](#),  $p \in \mathcal{M}$ , and  $V_p := \{v \in T_p\mathcal{M} \mid c_v \text{ defined on } [0, 1]\}$ . The *exponential map* of  $\mathcal{M}$  at  $p$ ,  $\exp_p: V_p \rightarrow \mathcal{M}$ , is defined as  $v \mapsto c_v(1)$ .

Clearly,  $\exp$  is differentiable, and we shall utilize the restriction of  $\exp$  to an open subset of the [tangent space](#)  $T_q\mathcal{M}$ , i.e., we define

$$\exp_p: B(0, \epsilon) \subseteq T_p\mathcal{M} \rightarrow \mathcal{M},$$

where  $B(0, \epsilon)$  is an open ball with center at the origin 0 of  $T_p\mathcal{M}$  of radius  $\epsilon$ . It's easy to see that  $\exp_p$  is differentiable and that  $\exp_p(0) = p$ .

**Intuition.** Geometrically,  $\exp_p(v)$  is a point of  $\mathcal{M}$  obtained by going out the [length](#) equal to  $|v|$ , starting from  $p$ , along a [geodesic](#) which passes through  $p$  with velocity equal to  $v/|v|$ .

**Proposition 2.3.1.** The [exponential map](#)  $\exp_p$  maps a neighborhood of  $0 \in T_p\mathcal{M}$  [diffeomorphically](#) onto a neighborhood of  $p \in \mathcal{M}$ .

**Proof.** We see that

$$d(\exp_p)_0(v) = \left. \frac{d}{dt} \exp_p(tv) \right|_{t=0} = \left. \frac{d}{dt} c_{tv}(1) \right|_{t=0} = \left. \frac{d}{dt} c_v(t) \right|_{t=0} = v,$$

i.e.,  $d(\exp_p)_0$  is the identity of  $T_p\mathcal{M}$ . By the inverse function theorem,  $\exp_p$  is a local [diffeomorphism](#) on a neighborhood of 0. ■

Now we know that  $\exp_p: B(0, \epsilon) \subseteq T_p\mathcal{M} \rightarrow \mathcal{M}$  maps [diffeomorphically](#) onto its image, we then define the following.

**Definition 2.3.2 (Normal coordinate).** Given an [exponential map](#)  $\exp_p: B(0, \epsilon) \rightarrow \mathcal{M}$ , let  $(e_1, \dots, e_n)$  be the orthonormal basis of  $T_p\mathcal{M}$ . Then the associated [local coordinates](#) are the *normal coordinates*.

Given  $p \in \mathcal{M}^n$ ,  $0 \in \mathbb{R}^n$ , we have

$$g_{ij}(p) = \delta_{ij}, \quad \Gamma_{ij}^k(p) = 0, \quad g_{ij,k} = 0$$

for all  $i, j, k$ .<sup>1</sup>

**Note.** The first derivative vanishes, so locally, the [manifold](#) looks Euclidean.

**Theorem 2.3.1.** For all  $p \in \mathcal{M}$ , there exists  $\rho > 0$  such that the Riemannian polar coordinates may be introduced on  $B(p, \rho) = \{q \in \mathcal{M} \mid d(p, q) \leq \rho\}$ . For any such  $\rho$  and  $q \in \partial B(p, \rho)$ , there exists a unique [geodesic](#) of shortest length ( $= \rho$ ) from  $p$  to  $q$ . And in the polar coordinates, this [geodesic](#) is given by the straight line  $x(t) = (t, \varphi_0)$ ,  $0 \leq t \leq \rho$ , with  $q$  represented by coordinates  $(\rho, \varphi_0)$ ,  $\varphi_0 \in S^{d-1}$ .

**Proof.** Take an arbitrary [curve](#) from  $p$  to  $q$ , namely  $c(t) = (r(t), \varphi(t))$ ,  $0 \leq t \leq T$ , which does not have to be entirely contained in  $B(p, \rho)$ . Let  $t_0$  be defined as

$$t_0 := \inf \{t \leq T \mid d(x(t), p) \geq \rho\}.$$

Then  $t_0 \leq T$  such that  $c|_{[0, t_0]}$  lies entirely in  $B(p, \rho)$ . We want to show that

$$(a) \quad L(c|_{[0, t_0]}) \geq \rho, \text{ and}$$

<sup>1</sup>Note that this only holds at  $p$ . We will come back to this when we formally introduce the [linear connection](#).



(b)  $L(c|_{[0,t_0]}) = \rho$  only for a straight line in the polar coordinates,

where

$$L(c|_{[0,t_0]}) := \int_0^{t_0} \sqrt{g_{ij}(c(t))\dot{c}^i\dot{c}^j} dt.$$

Observe that  $g_{r\varphi} = 0$ , with  $g_{\varphi\varphi}$  being positive definite, hence

$$L(c|_{[0,t_0]}) \geq \int_0^{t_0} \sqrt{g_{rr}(c(t))\dot{r}^2} dt = \int_0^{t_0} |\dot{r}| dt \geq \int_0^{t_0} \dot{r} dt = r(t_0) = \rho,$$

where we know that  $g_{rr} \equiv 1$ . ■

**Remark (Compact manifold).** For compact manifold, from [Theorem 2.3.1](#), we can prove that Riemannian polar coordinates can be introduced. Also, there exists  $\rho_0 > 0$  such that for any 2 points  $p, q \in \mathcal{M}$  with  $d(p, q) \leq \rho_0$  can be connected by minimizing [geodesic](#).

## Lecture 7: Hopf-Rinow Theorem

### 2.4 Hopf-Rinow Theorem

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We have shown the following in the homework.

**Theorem 2.4.1.** Let  $(\mathcal{M}, g)$  be a compact [Riemannian manifold](#).

- (a) Any 2 points  $p, q \in \mathcal{M}$  can be connected by a minimizing [geodesic](#).
- (b) For all  $p \in \mathcal{M}$ , the [exponential map](#)  $\exp_p$  is defined on all of  $T_p\mathcal{M}$  and any [geodesic](#) may be extended indefinitely in each direction.

We now want to generalize it. However, this is not true in the most general setting, and we need one more requirement.

**Definition 2.4.1 (Geodesically complete).** A [Riemannian manifold](#)  $(\mathcal{M}, g)$  is *geodesically complete* if for all  $p \in \mathcal{M}$ ,  $\exp_p$  is defined on all of  $T_p\mathcal{M}$ .

In other words, a [Riemannian manifold](#)  $\mathcal{M}$  is [geodesically complete](#) if any [geodesic](#)  $c(t)$  with  $c(0) = p$  can be extended for all  $t \in \mathbb{R}$ . Then, we have the following.

**Theorem 2.4.2 (Hopf-Rinow theorem).** Let  $(\mathcal{M}, g)$  be a compact [Riemannian manifold](#), then the following statements are equivalent.

- (a)  $\mathcal{M}$  is complete as a metric space.<sup>a</sup>
- (b) The closed and bounded subsets of  $\mathcal{M}$  are compact.
- (c) There exists  $p \in \mathcal{M}$  such that  $\exp_p$  is defined on all  $T_p\mathcal{M}$ .
- (d)  $\mathcal{M}$  is [geodesically complete](#).

Furthermore, (d) (and hence (a), (b), and (c)) implies

- (e) for two points  $p, q \in \mathcal{M}$  can be joined by a minimizing [geodesic](#), i.e., [geodesic](#) of the shortest [distance](#)  $d(p, q)$ .

<sup>a</sup>Hence, equivalently, complete as a topological space w.r.t. the underlying topology.

**Proof.** We start by proving (d) implies (e). Let  $\mathcal{M}$  be [geodesically complete](#), and let  $r := d(p, q)$ , and let  $\rho$  be as in the corollary from handout for HW1. Let  $p_0 \in \partial B(p, \rho)$  be a point where the

continuous functional  $d(q, \cdot)$  attains its minimum on the compact set  $\partial B(p, \rho)$ . Then, for some  $V \in T_p \mathcal{M}$ ,

$$p_0 = \exp_p \rho V.$$

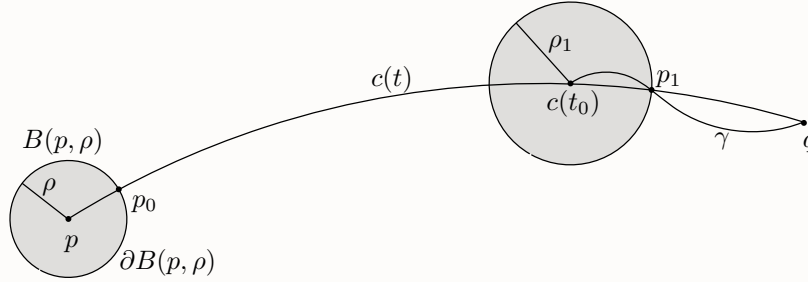
Consider the [geodesic](#)  $c(t) = \exp_p tV$ , by showing

$$c(r) = q,$$

$c|_{[0,r]}$  will be the shortest [geodesic](#) from  $p$  to  $q$ . We start by defining

$$I := \{t \in [0, r] \mid d(c(t), q) = r - t\},$$

and referring to the following diagram to guide us.



Now, we want to show that  $I = [0, r]$ , which will follow from showing that  $I$  is open.

**Note.**  $I$  is not empty since by definition it contains 0 and  $r$ . Further,  $I$  is closed by continuity.

Let  $t_0 \in I$ , and let  $\rho_1 > 0$  be the radius as in the corollary, without loss of generality,  $\rho_1 < r - t_0$ . Let  $p_1 \in \partial B(c(t_0), \rho_1)$  be the point where the continuous functional  $d(q, \cdot)$  attains its minimum on the compact set  $\partial B(c(t_0), \rho_1)$ . By the triangle inequality,

$$d(p, q) \leq d(p, p_1) + d(p_1, q).$$

For every curve  $\gamma$  from  $c(t_0)$  to  $q$ , there exists  $\gamma(t) \in \partial B(c(t_0), \rho_1)$ , hence

$$L(\gamma) \geq \underbrace{d(c(t_0), \gamma(t))}_{\rho_1} + d(\gamma(t), q) = \rho_1 + d(p_1, q),$$

implying  $d(q, c(t_0)) \geq \rho_1 + d(p_1, q)$ . But from the triangle inequality, we actually have

$$d(q, c(t_0)) = \rho_1 + d(p_1, q) \Leftrightarrow d(p_1, q) = \underbrace{d(q, c(t_0))}_{r-t_0} - \rho_1,$$

hence  $d(p_1, p) \geq r - (r - t_0 - \rho_1) = t_0 + \rho_1$ , i.e., this is a minimizing curve!

On the other hand, there exists a curve from  $p$  to  $p_1$  of length  $t_1 + \rho_1$  since it's composed by the portion from  $p$  to  $c(t_0)$  along  $c(t)$  and the portion being the [geodesic](#) from  $c(t_0)$  to  $p_1$  of length  $\rho_1$ . Then, by the theorem we have proved in the HW1#5, this curve is a [geodesic](#) curve. Finally, from the uniqueness of [geodesic](#) with the given extra data, this [geodesic](#) coincides with  $c$ . Hence,

$$p_1 = c(t_0 + \rho_1),$$

with  $d(p_1, q) = r - t_0 - \rho_1$ ,

$$d(c(t_0 + \rho_1), q) = d(p_1, q) = r - t_0 - \rho_1 = r - (t_0 + \rho_1),$$

thus  $t_0 + \rho_1 \in I$ , hence  $I$  is open, i.e.,  $I = [0, r]$ , so  $c(r) = q$  follows.

## Lecture 8: Injectivity Radius and Vector Bundles

In the proof we did last time, the last step can be shown via [FC13, Corollary 3.9].

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**Proof of Hopf-Rinow theorem (Continued).** We see that (d) implies (e), hence we only need to show that (a), (b), (c), and (d) are equivalent.

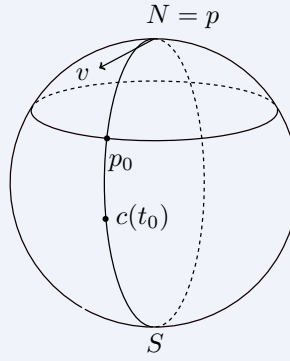
- (d)  $\Rightarrow$  (c) is trivial.
- (c)  $\Rightarrow$  (b): Let  $K \subseteq \mathcal{M}$  be closed and bounded. As  $K$  bounded,  $K \subseteq B(p, r)$  for some  $r > 0$ . Then any point in  $B(p, r)$  can be joined with  $p$  by **geodesic** of length  $\leq r$ , and  $B(p, r)$  is the image of the compact ball in  $T_p\mathcal{M}$  of radius  $r$  under continuous map  $\exp_p$ , hence  $B(p, r)$  is compact. As  $K$  closed and  $K \subseteq B(p, r)$ ,  $K$  is compact.
- (b)  $\Rightarrow$  (a): Let  $(p_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$  be a Cauchy sequence, so it's bounded, and by (b), its closure is compact. It contains a convergent subsequence, so it converges, i.e.,  $\mathcal{M}$  is **complete**.
- (a)  $\Rightarrow$  (d): Let  $c$  be a **geodesic** in  $\mathcal{M}$ , parametrized by arc length defined on a maximal interval  $I$ . Since  $I$  is non-empty, and we can show that  $I$  is both open and closed.

Exercise

■

It's worth mentioning that we do have uniqueness after choosing  $p_0$ , in other words, after choosing  $p_0$ , everything is fixed, so the non-uniqueness really comes from the initial choice of  $p_0$ .

**Example.** Consider  $S^2$ , after fixing  $p_0$ ,  $c(t_0)$  is extended uniquely.



## 2.5 Injectivity Radius

Consider the following.

**Definition 2.5.1** (Injectivity radius). Let  $\mathcal{M}$  be a **Riemannian manifold**, and  $p \in \mathcal{M}$ . The *injectivity radius*  $i(p)$  of  $p$  is

$$i(p) := \sup \{ \rho > 0 \mid \exp_p \text{ defined on } B(0, \rho) \subseteq T_p\mathcal{M} \text{ and injective} \}.$$

Similarly, the *injectivity radius*  $i(\mathcal{M})$  of  $\mathcal{M}$  is defined as  $i(\mathcal{M}) := \inf_{p \in \mathcal{M}} i(p)$ .

**Example (Sphere).**  $i(S^n) = \pi$ .

**Example (Torus).**  $i(T^n) = 1/2$ .

Any manifold carries a **complete Riemannian metric**.

If  $(\mathcal{M}, g_1)$  is not **complete**, we can find  $g_2$  such that  $(\mathcal{M}, g_2)$  is **complete**.

**Example (Hyperbolic half-plane).** The half-plane  $P = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$  equipped with metric induced by the Euclidean metric on  $\mathbb{R}^2$ , which is not **complete**.

However, it becomes **complete** when equipped with the following metric

$$\frac{1}{y^2}(dx^2 + dy^2).$$

In fact,  $P$  with the above metric is called the *hyperbolic half-plane*  $H^2$ , and we can extend it to  $H^n$ . Another question we may ask is the following.

**Problem.** Is the converse of **Hopf-Rinow theorem** true? I.e., can we show that (e) implies (d)?

**Answer.** No! Any 2 points in the open half-sphere can be joined by a unique minimal **geodesic**, but this manifold is not **geodesically complete**.  $\otimes$

**Example.** The **injectivity radius** of  $H^n$  is  $\infty$ .

**Remark.** Given a compact  $\mathcal{M}$ , the **injectivity radius** is always  $> 0$  by continuity argument.

Now, given a **complete** but not compact  $\mathcal{M}$ , the **injectivity radius** can be 0.

**Example.** Take the quotient of the Poincaré half-plane by the translations

$$(x, y) \mapsto (x + n, y), \quad n \in \mathbb{Z}.$$

We then obtain a **complete Riemannian manifold**  $\mathcal{M}$  with  $i(\mathcal{M}) = 0$ .

**Note.** Finding lower bounds for  $i(\mathcal{M})$  introduces curvature estimates.

## 2.6 Bundles and Fields

Let's first introduce the theory of **bundles**, which allows us to introduce the notion of **vector fields**, which is a more general notion of **tensor fields**. And noticeably, nearly every structure we can put on a **Riemannian manifold** will be in the form of **tensor fields**.

**Example.** Given a **tangent vector field**  $X$  of a **smooth manifold**  $\mathcal{M}$  is where we simply associate  $X(p)$  to a **tangent vector**:

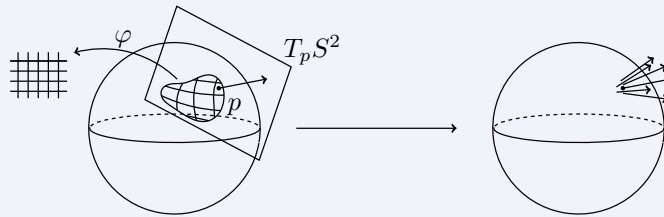


Figure 2.1: Given  $\mathcal{M} = S^2$ , a **vector field** assigns every point a “point” in the associated “space.” In this case, a **tangent vector field** associates every  $p$  a vector in the corresponding **tangent space**.

Recall the **tangent bundle**  $(T\mathcal{M}, \pi, \mathcal{M})$ , where we only take the name “**bundle**” for granted and don’t know why it is: however, we should see that it helps us construct the **vector field**, since it captures the idea of “for every point  $p$ , we have an associated space  $T_p\mathcal{M}$ ,” which is exactly what we need here. This idea generalizes quite easily.

### 2.6.1 Bundles

We start by introducing the notion of **bundles**.

**Definition 2.6.1 (Bundle).** A *bundle* is a tuple  $(E, \pi, \mathcal{M})$  consists of the **total space**  $E$ , the **base space**  $\mathcal{M}$ , and the **bundle projection**  $\pi: E \rightarrow \mathcal{M}$ .

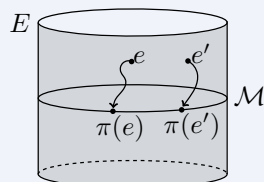
**Definition 2.6.2 (Total space).** The **differentiable manifold**  $E$  is called the *total space*.

**Definition 2.6.3 (Base space).** The **differentiable manifold**  $\mathcal{M}$  is called the *base space*.

**Definition 2.6.4 (Bundle projection).** The (differentiable) continuous surjection  $\pi: E \rightarrow \mathcal{M}$  is called the *bundle projection*.

**Note.** We see that a **tangent bundle**  $(T\mathcal{M}, \pi, \mathcal{M})$  is actually a **bundle**.

**Example.** Let  $E$  be a cylinder,  $\mathcal{M}$  be a circle.



As we can see, the number of possible  $\pi$  is enormous, as long as it's surjective and smooth.

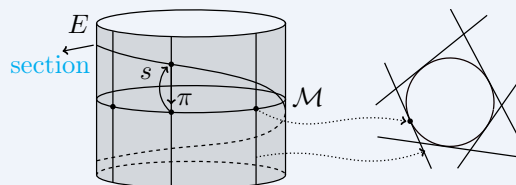
**Notation.** Sometimes, we will just denote a **bundle** as  $E \xrightarrow{\pi} \mathcal{M}$ , or even more compactly, just  $\pi$  since it captures all the data.

**Definition 2.6.5 (Fiber).** Given a **bundle**  $(E, \pi, \mathcal{M})$ , the *fiber* over  $p \in \mathcal{M}$  under  $\pi$  is the preimage of a  $\{p\}$ , i.e.,  $\pi^{-1}(\{p\})$ .

**Definition 2.6.6 (Section).** A *section* of a **bundle**  $(E, \pi, \mathcal{M})$  is a differentiable map  $s: \mathcal{M} \rightarrow E$  such that  $\pi \circ s = \text{id}_{\mathcal{M}}$ .

**Remark.** We see that a **section**  $s$  encodes lots of information of a **bundle**, since  $s$  includes  $E, \mathcal{M}$ , and the condition deal with  $\pi$ .

**Example.** Again let  $E$  be a cylinder,  $\mathcal{M}$  be a circle. This time, we choose  $\pi$  to be the trivial one.



We see that in this way, this **bundle** really captures all the **tangent spaces** structure of a circle!

## 2.6.2 Vector Bundles

Then, we're interested in the so-called **vector bundle**.

**Definition 2.6.7** (Vector bundle). A (differentiable) *vector bundle* of rank  $n$  is a *bundle*  $(E, \pi, \mathcal{M})$  such that each *fiber*  $E_x := \pi^{-1}(x)$  of  $x \in \mathcal{M}$  carries a structure of an  $n$ -dimensional (real) vector space, and *local triviality* condition holds.

**Definition 2.6.8** (Local trivialization). For all  $x \in \mathcal{M}$ , the *local trivialization*  $(U, \varphi)$  consists a neighborhood  $U$  and *diffeomorphism*  $\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  such that for all  $y \in U$ ,

$$\varphi_y := \varphi|_{E_y} : E_y \rightarrow \{y\} \times \mathbb{R}^n$$

is a vector space isomorphism.

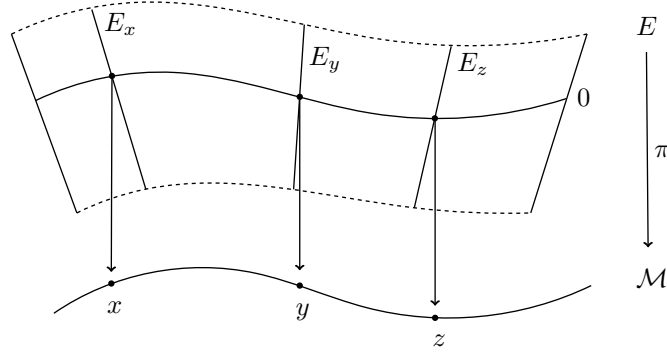


Figure 2.2: An illustration of *vector bundle*  $(E, \pi, \mathcal{M})$ .

**Definition 2.6.9** (Trivial). A *vector bundle* is *trivial* if it's isomorphic to  $\mathcal{M} \times \mathbb{R}^n$ .<sup>a</sup>

<sup>a</sup> $n$  is the rank of the *vector bundle*.

**Intuition.** The *local trivialization* shows that *locally* the map  $\pi$  looks like the *projection* of  $U \times \mathbb{R}^n$  on  $U$ .

**Definition 2.6.10** (Bundle chart). The pair  $(\varphi, U)$  is also called the *bundle chart* in *local trivialization*.

**Remark.** From *Definition 2.6.7*, *vector bundle* is locally, but not necessarily globally a product of *base space* and the *fiber*.

**Intuition.** We may look at a *vector bundle* as a family of vector spaces, all isomorphic to a fixed  $\mathbb{R}^n$ , “parametrized” (*locally trivially*) by a *manifold*.

### 2.6.3 Vector Fields

We can now introduce the notion of *vector fields* in terms of *section*.

**Definition 2.6.11** (Vector field). A (smooth) *vector field*  $X$  is a smooth *section* of a *bundle*.

**Note.** We see that a smooth *tangent vector field* is indeed a smooth *vector field* with the *bundle* being the *tangent bundle*.

**Notation.** Since we will nearly always be talking about [tangent vector fields](#), we will abuse the notation a bit and just simply call it [vector fields](#). But always keep in mind that more broadly, a [vector field](#) should be a [section](#) of a [bundle](#), not always  $T\mathcal{M}$ .

## Lecture 9: Tensors and Connections

### 2.6.4 Tensor Fields

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We can introduce the notion of “[tensor fields](#)” in a brute-force way.<sup>2</sup> To do this, given a vector space  $V$ , we first introduce [tensors](#).

**Definition 2.6.12 (Tensor).** Let  $V$  be a vector space of dimension  $m < \infty$ , and the dual space  $V^*$ .<sup>a</sup> Then the vector space of the  $r$ -times contravariant and  $s$ -times covariant tensors over  $V$ , denoted as  $T_s^r(V)$ , is the [vector field](#) defined as

$$T_s^r(V) = \{T: \underbrace{V^* \times \dots \times V^*}_r \times \underbrace{V \times \dots \times V}_s \rightarrow \mathbb{R}\} = \underbrace{V \otimes \dots \otimes V}_r \otimes \underbrace{V^* \otimes \dots \otimes V^*}_s.$$

<sup>a</sup>I.e.,  $V^* := \{\lambda: V \rightarrow \mathbb{R} \mid \lambda \text{ linear}\}$ .

**Notation.** Let  $\mathcal{M}^n$  be a [smooth manifold](#) and  $\pi: E \rightarrow \mathcal{M}$  a [smooth vector bundle](#), then the set of [sections](#) is denoted as

$$\Gamma(E) := \{s \in C^\infty(\mathcal{M}, E) \mid \pi \circ s = \text{id}_{\mathcal{M}}\}.$$

**Example.** Consider the [vector bundle](#)  $(T\mathcal{M}, \pi, \mathcal{M})$ , then  $\Gamma(T\mathcal{M}) := \{\text{vector fields on } \mathcal{M}\}$ .

**Example.**  $\Gamma(\Lambda_s \mathcal{M}) := \{s\text{-forms on } \mathcal{M}\}$  with  $\Lambda_s \mathcal{M} = \Lambda^s \left( \bigcup_{p \in \mathcal{M}} T_p^* \mathcal{M} \right)$ .<sup>a</sup>

<sup>a</sup>Here,  $\Lambda^s(V^*) := \{A \in T_s^0(V) \mid A \text{ skew-symmetric}\}$ , where  $s \in \mathbb{N}$ .

Then, we have the following.

**Definition 2.6.13 (Tensor field).** The  $(r, s)$ -tensor fields on  $\mathcal{M}$  is defined as elements in  $\Gamma(T_s^r \mathcal{M})$  with  $T_s^r \mathcal{M} = \bigcup_{p \in \mathcal{M}} T_s^r(T_p \mathcal{M})$ .

**Example.** A [Riemannian metric](#)  $g$  on  $\mathcal{M}$  is a  $(0, 2)$ -tensor field, i.e.,  $g \in \Gamma(T_2^0(\mathcal{M}))$  for all  $p \in \mathcal{M}$ .

**Proof.** Since  $g_p: T_p \mathcal{M} \times T_p \mathcal{M} \rightarrow \mathbb{R}$ . ⊗

## 2.7 Other Metrics

Finally, we discuss some other metrics we may let a [manifold](#) equipped with.

**Definition 2.7.1 (Pseudo-Riemannian metric).** A *pseudo-Riemannian metric* on a [differentiable manifold](#)  $\mathcal{M}$  is a  $(0, 2)$ -tensor field  $g \in \Gamma(T_2^0(\mathcal{M}))$  for all  $p \in \mathcal{M}$  with

- (a)  $g(X, Y) = g(Y, X)$  for all  $X, Y \in T\mathcal{M}$ ;
- (b) for all  $p \in \mathcal{M}$ ,  $g_p$  is non-degenerate bilinear form on  $T_p \mathcal{M}$ , i.e.,  $g_p(X, Y) = 0$  for all  $X, Y \in T_p \mathcal{M}$  if and only if  $Y = 0$ .

<sup>2</sup>See [Appendix A.3.2](#) for another view point.

**Note.** A **pseudo Riemannian metric** is actually a **Riemannian metric** if it's positive definite at every  $p \in \mathcal{M}$ .

**Definition 2.7.2 (Lorentzian metric).** A *Lorentzian metric*  $g$  is a continuous assignment of a non-degenerate<sup>a</sup> quadratic form  $g_p$  of index 1<sup>b</sup> in  $T_p\mathcal{M}$  for all  $p \in \mathcal{M}$ .

<sup>a</sup> $g_p(X, Y) = 0$  for all  $Y \in T_p\mathcal{M}$  implies  $X = 0$ .

<sup>b</sup>It means that the maximal dimension of a subspace of  $T_p\mathcal{M}$  on which  $g_p$  is negative definite is 1.

An equivalent definition is the following.

**Definition 2.7.3 (Lorentzian).** A quadratic form  $g_p$  in  $T_p\mathcal{M}$  is *Lorentzian* if there exists a vector  $V \in T_p\mathcal{M}$  such that  $g_p(V, V) < 0$  while setting  $\Sigma_V = \{X \mid g_p(X, V) = 0\}$  such that  $g_p|_{\Sigma_V}$ <sup>a</sup> is positive definite.

<sup>a</sup>The  $g_p$ -orthogonal complement of  $V$ .

**Example (Minkowski space).** The Minkowski space on  $\mathbb{R}^4$  is the prototypical example from physics (flat spacetime). Namely, the metric is given by the quadratic form

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with the coordinates being  $(t, x, y, z)$ .



## Chapter 3

# Connections and Curvatures

So far, we saw that a [vector field](#)  $X$  can be used to provide a directional derivative since it gives us a [tangent vector](#) at each point smoothly. Now, we will introduce a new symbol  $\nabla$  where we let

$$\nabla_X f := Xf$$

for  $f \in C^\infty(\mathcal{M})$ .

**Problem.** Does this notation overkill? We already know that  $Xf = (df)(X)$ !

**Answer.** No! While  $\nabla, X: C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ , while  $df: \Gamma(T\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ , we can generalize  $\nabla_X$  to act from [vector fields](#) to [vector fields](#)! The insight is that if  $X$  can be extended naturally (without providing any extra structures), then we certainly won't bother introducing a new symbol. However, as you might guess, to let  $\nabla$  doing this, we do need to provide extra structures, and  $\nabla$  stands exactly for these extra structures!  $\otimes$

In some sense, this new notions  $\nabla$  allows us to “connect” [tangent spaces](#), which allows us to make sense of “curvatures” and other geometric property of a [Riemannian manifold](#).

### 3.1 Affine Connections

We first formulate a *wish list* of properties which the  $\nabla_X$  should have. Any remaining freedom in choosing  $\nabla$  will need to be provided as additional structures beyond the structures on  $\mathcal{M}$  we already have.

**Definition 3.1.1 (Linear connection).** A *linear connection* (*affine connection*) on a [smooth manifold](#)  $\mathcal{M}$  is a bilinear map

$$\nabla: \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M}),$$

which is denoted by  $\nabla(X, Y) = \nabla_X Y$  and which satisfies

- (a)  $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$ ;
- (b)  $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$ ;
- (c)  $\nabla_X fY = f\nabla_X Y + X(f)Y$ ;

for all [vector fields](#)  $X, Y, Z \in \Gamma(T\mathcal{M})$  and  $f, g \in C^\infty(\mathcal{M})$ .

**Note.** There's a similar notation called [covariant derivative](#), denoted by  $D$ , satisfies similar properties as a [linear connection](#). Hence, we often write  $D$  and  $\nabla$  interchangeably.<sup>a</sup>

<sup>a</sup> $\nabla$  is more general than  $D$ ; however, we treat them as the same as suggested by [Proposition 3.4.3](#).

**Remark.** [Definition 3.1.1 \(c\)](#) shows that this is actually a local notion as we will see.

Now, one might be wondering that, after fixing these rules we want, how much freedom is left? To see this, let's first do some calculations...

### 3.1.1 Connection Coefficients

Choose a [system of coordinates](#)  $(x_1, \dots, x_n)$  at  $p \in \mathcal{M}$ , we can write  $X = X^i \frac{\partial}{\partial x_i}$ ,  $Y = Y^j \frac{\partial}{\partial x_j}$ , then

$$\nabla_X Y = \nabla_{X^i \frac{\partial}{\partial x_i}} \left( Y^j \frac{\partial}{\partial x_j} \right) = X^i Y^j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + X^i \frac{\partial}{\partial x_i} (Y^j) \frac{\partial}{\partial x_j}.$$

Now, we see that  $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$  is another [vector field](#), hence can again write

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} =: \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

in terms of the basis with a new set of coefficients  $\Gamma$ .

**Notation** (Connection coefficient). The coefficients  $\Gamma_{ij}^k$  is called the *connection coefficients*.<sup>a</sup>

<sup>a</sup>It's tempting to say that the [connection coefficients](#) are the same as [Christoffel symbols](#) since we're using the same symbols. Indeed, they are! For a deeper understanding, see [Appendix A.1](#).

**Note.** It's clear that  $\Gamma_{ij}^k$  are differentiable and [charts](#)-dependent and hence  $\nabla$  is local.

Finally, we have

$$\nabla_X Y = \left( X^i Y^j \Gamma_{ij}^k + X(Y^k) \right) \frac{\partial}{\partial x_k} \Rightarrow (\nabla_X Y)^k = X(Y^k) + \Gamma_{ij}^k X^i Y^j,$$

meaning that we have  $(\dim \mathcal{M})^3$  many  $\Gamma$ 's (freedom) when choosing  $\Gamma_{ij}^k$  with [Definition 3.1.1](#).<sup>1</sup>

**Remark.** One might ask what about other [tensor fields](#)? Fortunately, the same set of  $\Gamma$ 's fix the action of  $\nabla$  on any [tensor fields](#).

**Proof.** The key observation is that if we define  $\nabla_{\frac{\partial}{\partial x^j}} (dx^i) =: \Sigma_{jk}^i dx^k$ , then

$$\nabla_{\frac{\partial}{\partial x^j}} \left( dx^i \left( \frac{\partial}{\partial x^k} \right) \right) = \begin{cases} \frac{\partial}{\partial x^j} (\delta_k^i) = 0; \\ \left( \nabla_{\frac{\partial}{\partial x^j}} dx^i \right) \frac{\partial}{\partial x^k} + dx^i \left( \underbrace{\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}}_{\Gamma_{jk}^\ell \frac{\partial}{\partial x^\ell}} \right), \end{cases}$$

leading to

$$\left( \nabla_{\frac{\partial}{\partial x^j}} dx^i \right) \frac{\partial}{\partial x^k} = -dx^i \left( \Gamma_{jk}^\ell \frac{\partial}{\partial x^\ell} \right) \Rightarrow \left( \nabla_{\frac{\partial}{\partial x^j}} dx^i \right)_k = -\Gamma_{jk}^i$$

since  $dx^i \frac{\partial}{\partial x^\ell} = \delta_\ell^i$ . ⊗

In summary, we have

$$\begin{cases} (\nabla_X Y)^k = X(Y^k) + \Gamma_{ij}^k X^i Y^j, & \text{if } Y \text{ is a vector field;} \\ (\nabla_X \omega)_k = X(\omega_k) - \Gamma_{ik}^j X^i \omega_j, & \text{if } \omega \text{ is a co-vector field.} \end{cases}$$

## 3.2 Levi-Civita Connections

The basic insight is that, after choosing a particular [connection](#) (remember that we have freedom to choose  $\Gamma$ 's), the space is basically fixed: i.e., the shape (curvature) of the space is determined by the choice of  $\nabla$ ! We now formalize this idea. A particularly natural notion related to “curvature” is the [torsion](#), defined as follows.

<sup>1</sup>This is for a particular domain  $U$ .

**Definition 3.2.1 (Torsion).** The *torsion*  $T$  of a linear connection  $\nabla$  is the  $(1, 2)$ -tensor field

$$T(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X, Y]).$$

**Notation.** We usually write this as  $T(X, Y)$  by neglecting  $\omega$ .

**Remark.**  $T$  is actually  $C^\infty$ -linear in each entry,<sup>a</sup> hence a tensor field.

<sup>a</sup>See Appendix A.3.2.

**Proof.** Since  $T(f \cdot \omega, X, Y) = f \cdot \omega(\dots) = fT(\omega, X, Y)$  and  $T(\omega + \psi, X, Y) = \dots = T(\omega, X, Y) + T(\psi, X, Y)$ , and also

$$\begin{aligned} T(\omega, fX, Y) &= \omega(\nabla_{fX} Y - \nabla_Y(fX) - [fX, Y]) \\ &= \omega(f\nabla_X Y - (Yf)X - f\nabla_Y X - f[X, Y] + (Yf)X) = f \cdot T(\omega, X, Y) \end{aligned}$$

since

$$([fX, Y])g = f \cdot X(Yg) - Y(f \cdot Xg) = f \cdot X(Yg) - (Yf)(Xg) - f \cdot Y(Xg) = (f \cdot [X, Y] - (Yf)X)g.$$

Finally, we claim that the additivity at  $X$  holds, with  $T(\omega, X, Y) = -T(\omega, Y, X)$ , we're done.  $\otimes$

**Intuition.** Definition 3.2.1 makes sense (in such a form) since this will make  $T$  actually a tensor field. For example, without the Lie bracket term, we don't have the linearity at  $X$  (hence  $Y$ ).

**Definition 3.2.2 (Torsion-free).** A linear connection  $\nabla$  is *torsion-free* if  $T = 0$ .

In a chart,

$$T_{jk}^i := T\left(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \Gamma_{jk}^i - \Gamma_{kj}^i = 2\Gamma_{[jk]}^i.$$

**Definition 3.2.3 (Riemannian).** Let  $\nabla$  be a linear connection and  $g$  be a Riemannian metric on  $\mathcal{M}$ . Then  $\nabla$  is *Riemannian* (or *metric*) if

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

for all  $X, Y, Z \in \Gamma(T\mathcal{M})$ .<sup>a</sup>

<sup>a</sup>We view  $g(X, Y) \in C^\infty(\mathcal{M})$  as suggested by Appendix A.3.2.

**Proposition 3.2.1 (Koszul formula).** On each Riemannian manifold  $(\mathcal{M}, g)$ , there exists a unique Riemannian, torsion-free connection  $\nabla$  on  $T\mathcal{M}$  determined by the Koszul formula

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} (X \langle Y, Z \rangle - Z \langle X, Y \rangle + Y \langle Z, X \rangle - \langle X, [Y, Z] \rangle + \langle Z, [X, Y] \rangle + \langle Y, [Z, X] \rangle). \quad (3.1)$$

**Proof sketch.** Firstly, we can show that every Riemannian and torsion-free connection satisfies Equation 3.1, which implies uniqueness. For existence, we verify that the unique map  $\nabla: \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M})$  given by Equation 3.1 has the desired properties, i.e., satisfying Definition 3.1.1, Definition 3.2.2, and Definition 2.1.1.  $\blacksquare$

Finally, we define the following.

**Definition 3.2.4 (Levi-Civita connection).** The *Levi-Civita connection* is the unique linear connection  $\nabla$  defined by the Koszul formula.

## Lecture 10: Curvatures and Flow of Vector Fields

### 3.3 Curvatures

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Given all these definitions, we can now introduce the notion of “curvatures.”

#### 3.3.1 Riemannian Curvatures

Consider the following.

**Definition 3.3.1** (Riemannian curvature). The *Riemannian curvature*  $R$  of a [Levi-Civita connection](#)  $\nabla$  is the [\(1,3\)-tensor field](#)

$$R(\omega, Z, X, Y) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z).$$

**Remark.**  $R$  is actually  $C^\infty$ -linear in each entry,<sup>a</sup> although we omit the proof here.

<sup>a</sup>See [Appendix A.3.2](#).

**Notation.** We usually write this as  $R(X, Y)Z$  by emphasizing  $Z$  and neglecting  $\omega$ .

**Remark** (Algebraic significant of Riemannian curvature). Since  $(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) = R(\cdot, Z, X, Y) + \nabla_{[X,Y]} Z$ , by letting  $\nabla_i := \nabla_{\frac{\partial}{\partial x^i}}$ ,  $\nabla_j := \nabla_{\frac{\partial}{\partial x^j}}$ , in a [chart](#)  $(U, x)$ , we have

$$(\nabla_i \nabla_j Z)^k - (\nabla_j \nabla_i Z)^k = R_{\ell ij}^k Z^\ell + \underbrace{\nabla_{[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]} Z}_{=0}^k = R_{\ell ij}^k,$$

i.e., the components of  $R$  contains all the information of how  $\nabla_i$  and  $\nabla_j$  fail to commute.

Consider plugging in basis vectors  $\partial/\partial x^i$  into [Definition 3.3.1](#), with  $[\partial/\partial x^i, \partial/\partial x^k] = 0$ , we have

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^\ell} = R_{\ell ij}^k \frac{\partial}{\partial x^k}$$

where

$$R_{k\ell ij} := g_{km} R_{\ell ij}^m = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k} \right\rangle.$$

**Remark.** We have  $\langle R(X, Y)Z, W \rangle = -\langle R(X, Y)W, Z \rangle$  and  $R_{k\ell ij} = -R_{\ell k ij}$ , etc.

**Remark.**  $R_{ijk}^\ell = \Gamma_{ik}^p \Gamma_{jp}^\ell - \Gamma_{jk}^p \Gamma_{ip}^\ell + \Gamma_{ik,j}^\ell - \Gamma_{jk,i}^\ell$ .

**Definition 3.3.2** (Ricci curvature). The *Ricci curvature* is defined by  $R_{ab} = g^{cd} R_{cadb} = R_{adb}^d$ .

**Definition 3.3.3** (Scalar curvature). The *scalar curvature* is defined by  $R = g^{ab} R_{ab}$ .

**Proposition 3.3.1** (Bianchi identity). Given the [Riemannian curvature tensor](#)  $R$ ,

$$\nabla_{[\alpha} R_{\beta\gamma]\delta\epsilon} := \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0$$

We see that the notion of “curvatures” can vary.

### 3.4 Covariant Derivatives

An important concept related to curvatures is “parallelism,” which needs a formal introduction of **covariant derivatives** (although we say we’re going to treat them the same). As a motivating example, the following is an equivalent definition of **geodesic**.

**Example** (Autoparallel). The **geodesic**  $c$  satisfies  $\nabla_{\dot{c}}\dot{c} = 0$ . This is called **autoparallel**.

**Proof.** In the **local coordinates**, we have  $\dot{c} = \dot{c}^i \partial/\partial x^i$ , and note that

$$\nabla_{\dot{c}}\dot{c} = \dot{c}^i \nabla_{\frac{\partial}{\partial x^i}} \dot{c}^j \frac{\partial}{\partial x^j} = \dot{c}^i \dot{c}^j \Gamma_{ij}^k \frac{\partial}{\partial x^k} + \dot{c}^k \frac{\partial}{\partial x^k} = (\dot{c}^k + \Gamma_{ij}^k \dot{c}^i \dot{c}^j) \frac{\partial}{\partial x^k} = 0 \quad (3.2)$$

since a **geodesic** is the solution of **Equation 2.1**. \*

As the word suggests, it has something to do with “parallelism.” Indeed, to understand what  $\nabla_{\dot{c}}\dot{c}$  is doing beyond just calculation, we need to make sense of “**vector-fields along curves**.” This is done by introducing the **local flows** first.

#### 3.4.1 Flows of Vector Fields

Let  $\mathcal{M}$  be a **smooth manifold**, and  $X$  a **vector field** on  $\mathcal{M}$ . Then  $X$  defines  $1^{st}$  order differential equations<sup>2</sup>

$$\dot{c} = X(c).$$

And this ODE has a solution, as guaranteed by **Proposition 3.4.1**.

**Proposition 3.4.1.** For all  $p \in \mathcal{M}^d$ , there exists an open interval  $I = I_p \subseteq \mathbb{R}$  with  $0 \in I_p$  such that a **smooth curve**  $c: I_p \rightarrow \mathcal{M}$  solves

$$\begin{cases} \frac{dc(t)}{dt} = X(c(t)), & t \in I; \\ c(0) = p. \end{cases}$$

Further, the solution depends smoothly on the initial data (i.e.,  $p$ ).<sup>a</sup>

<sup>a</sup>This directly follows from ODE theory.

**Proof.** For all  $p \in \mathcal{M}$ , we want to find an open interval  $I = I_p$  around  $0 \in \mathbb{R}$  and a solution of the following ODE for  $c: I \rightarrow \mathcal{M}$ :

$$\begin{cases} \frac{dc(t)}{dt} = X(c(t)), & t \in I; \\ c(0) = p. \end{cases}$$

We can check in **local coordinates** that this is a system of ODE. In such **coordinates**, let  $c(t)$  be given by  $c(t) = (c^1(t), c^2(t), \dots, c^d(t))$ . Let  $X =: X^i \partial/\partial x^i$ , then the above system becomes

$$\frac{dc^i(t)}{dt} = X^i(c(t)), \quad i = 1, \dots, d.$$

From the **Picard-Lindelöf theorem**, with the initial data  $c(0) = p$ , there is a unique solution. ■

**Proposition 3.4.2.** For all  $p \in \mathcal{M}$ , there exists an open neighborhood  $U$  of  $p$  and an open interval  $I_p$  with  $0 \in I_p$  such that for all  $q \in U$ , the **curve**  $c_q$  with

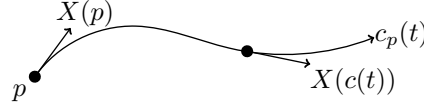
$$\dot{c}_q(t) = X(c_q(t)), \quad c_q(0) = q$$

is defined on  $I$  and the map  $c: I \times U \rightarrow \mathcal{M}$ ,  $(t, q) \mapsto c_q(t)$  is smooth.

**Proposition 3.4.2** suggests the following definition.

<sup>2</sup>If  $\dim \mathcal{M} > 1$ , it is a system of  $1^{st}$ -order differential equations.

**Definition 3.4.1** (Local flow). The map  $c_q(t): I \times U \rightarrow \mathcal{M}$ ,  $(t, q) \mapsto c_q(t)$  from Proposition 3.4.2 is called the *local flow* of the vector field  $X$ .



**Definition 3.4.2** (Integral curve). The local flow  $c_q(t)$  is called the *integral curve* of  $X$  through  $q$ .

### 3.4.2 Local 1-Parameter Groups

Now, fixing  $t$ , we can vary  $q$  and see the following.

**Theorem 3.4.1.** Let  $\varphi_t(q) := c_q(t)$  such that  $\varphi_t \circ \varphi_s(q) = \varphi_{t+s}(q)$  for  $s, t, (t+s) \in I_q$ . If  $\varphi_t$  is defined on  $U \subseteq \mathcal{M}$ , it maps  $U$  diffeomorphically onto its image.

We see that  $\varphi_t$  defines a family of diffeomorphism around  $p$ , which gives the following.

**Definition 3.4.3** (Local 1-parameter group). A family  $(\varphi_t)_{t \in I}$  of diffeomorphism from  $\mathcal{M}$  to  $\mathcal{M}$  satisfying Theorem 3.4.1 is called a *local 1-parameter group* of diffeomorphisms.

In general, a local 1-parameter group needs not be extendible to a group because the maximum interval  $I = I_q$  in Definition 3.4.3 need not be all of  $\mathbb{R}$ .

**Example.** Let  $\mathcal{M} = \mathbb{R}$ ,  $X(t) = \tau^2 d/d\tau$ . Then the solution of  $\dot{c}(t) = c^2(t)$  is not defined over all  $\mathbb{R}$ .

To get the whole group structure, consider the following.

**Theorem 3.4.2.** Let  $X$  be a vector field on a smooth manifold  $\mathcal{M}$  with a compact support. Then the corresponding local flow is defined for every  $q \in \mathcal{M}$  and  $t \in \mathbb{R}$ , and the local 1-parameter group becomes a group of diffeomorphisms.

**Proof.** By using  $\text{supp}(X) \subseteq K$ ,  $K$  compact, we can cover  $K$  by a finite covering, then using Proposition 3.4.2, we're done. ■

**Corollary 3.4.1.** On a compact differentiable manifold  $\mathcal{M}$ , any vector field generates a local 1-parameter group.

## Lecture 11: Geodesic & Cogeodesic Flows and Parallel Transport

### 3.4.3 Geodesic and Cogeodesic Flows

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A particular interesting flow is the *cogeodesic-flow*, which can be constructed as follows. Let's first transform Equation 2.1 (which is a  $2^{nd}$ -ODE) into a  $1^{st}$  order system on the cotangent bundle  $T^*\mathcal{M}$ , and locally trivialize  $T^*\mathcal{M}$  by chart  $T^*\mathcal{M}|_U \cong U \times \mathbb{R}^d$  with coordinates  $(x^1, \dots, x^d, p_1, \dots, p_d)$ . Set

$$H(x, p) = \frac{1}{2} g^{ij}(x) p_i p_j, \quad (3.3)$$

and recall  $g^{ik} g_{kj} = \delta_j^i$ .

**Theorem 3.4.3.** Equation 2.1 is equivalent to the system on  $T^*\mathcal{M}$ :

$$\begin{cases} \dot{x}^i = \frac{\partial H}{\partial p_i} g^{ij}(x) p_j; \\ \dot{p}_i = -\frac{\partial H}{\partial x^i} = -\frac{1}{2} g_{,i}^{jk}(x) p_j p_k. \end{cases} \quad (3.4)$$

**Proof.** This is just computation. ■

**Definition 3.4.4** (Cogeodesic flow). The *cogeodesic flow* is the [local flow](#) determined by Equation 3.4.

**Definition 3.4.5** (Geodesic flow). The [geodesic flow](#) on  $T\mathcal{M}$  is obtained from the [cogeodesic flow](#) by the first equation in Equation 3.4.

Thus, the [geodesic](#) is the projection of the [integral curve](#) of the [geodesic flow](#) onto  $\mathcal{M}$ .

**Remark** (Hamiltonian flow). The [cogeodesic flow](#) is a *Hamiltonian flow* for the Hamiltonian  $H$ .

**Proof.** By Equation 3.4, along the [integral curves](#),

$$\frac{dH}{dt} = H_{x^i} \dot{x}^i + H_{p_i} \dot{p}^i = -\dot{p}_i x \dot{x}^i + \dot{x}^i p_i = 0.$$

Observe that the [cogeodesic flow](#) maps the set

$$E_\lambda := \{(x, p) \in T^*\mathcal{M} \mid H(x, p) = \lambda\}$$

onto itself for all  $\lambda \geq 0$ . ⊗

**Remark.** If  $\mathcal{M}$  is compact, then all  $E_\lambda$  are compact, then all [geodesic flow](#) is defined on all of  $E_\lambda$  for all  $\lambda$ .

**Remark.**  $\mathcal{M} = \bigcup_{\lambda \geq 0} PE_\lambda$  for  $P$  being the projection.

### 3.4.4 Vector Fields along Curves

With these tools, we now introduce the [vector field along a curve](#).

**Definition 3.4.6** (Vector field along a curve). A (smooth) *vector field  $X$  along a curve  $c: I \subseteq \mathbb{R} \rightarrow \mathcal{M}$*  on a [smooth manifold](#)  $\mathcal{M}$  is defined as  $X: I \rightarrow T\mathcal{M}$  such that  $X(t) \in T_{c(t)}\mathcal{M}$  for all  $t \in I$ .

**Notation.** The set of smooth [vector fields along  \$c\$](#)  is denoted as  $\chi_c(\mathcal{M})$ .

### 3.4.5 Covariant Derivatives

We can now finally define [covariant derivative](#) formally, and see why we can use them interchangeably with [linear connection](#).

**Definition 3.4.7** (Covariant derivative). The *covariant derivative* of  $v$  along  $c$  is the [vector field](#)  $DV/dt$  in Proposition 3.4.3.

As previously seen. Let  $X = X^i \frac{\partial}{\partial x_i}$ ,  $V = V^k \frac{\partial}{\partial x_k}$ , and let  $D$  be the **Levi-Civita connection**. Then

$$D_V X = D_V \left( X^i \frac{\partial}{\partial x_i} \right) = V(X^i) \frac{\partial}{\partial x_i} + \underbrace{X^i D_V \frac{\partial}{\partial x_i}}_{V^k D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}} = V(X^i) \frac{\partial}{\partial x_i} + V^k X^i \Gamma_{ki}^j \frac{\partial}{\partial x_j}.$$

**Proposition 3.4.3 (Covariant derivative).** Let  $(\mathcal{M}, g)$  be a **Riemannian manifold**,  $D$  the canonical **(Levi-Civita) connection**, and  $c$  a **smooth curve** in  $\mathcal{M}$ . Then there exists a unique operator  $D/dt$  defined as the vector space of **vector fields along  $c$**  satisfying

- (i) (a)  $\frac{D}{dt}(fY)(t) = f'(t)Y(t) + f(t)\frac{D}{dt}Y(t)$  for all  $f \in C^\infty(I)$  and  $Y \in \mathcal{X}_c(\mathcal{M})$ ;
- (b)  $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$  for all  $V, W \in \mathcal{X}_c(\mathcal{M})$ ;
- (ii) if there exists a neighborhood of in  $I$  such that  $Y$  is the restriction to  $c$  of a **vector field  $X$**  defined on a neighborhood of  $c(t_0)$  in  $\mathcal{M}$ , then  $\frac{D}{dt}Y(t_0) = (D_{c(t_0)}X)_{c(t_0)}$ .

**Proof.** The **covariant derivative** along  $c$  is defined by

$$\frac{D}{dt} \left( Y^i(t) \frac{\partial}{\partial x_i} \right) = \frac{dY^i}{dt} \frac{\partial}{\partial x_i} + \dot{c} Y^i \Gamma_{ji}^k(c(t)) \frac{\partial}{\partial x_k},$$

where  $\dot{c} = \dot{c}^k \frac{\partial}{\partial x_k}$ . This shows (i) (a) and (b) hold. Next, to show (ii), let  $x$  be a smooth **vector field** in  $\mathcal{M}$ . Then the induced **vector field along  $c$**  is given by  $Y(t) = X_{c(t)}$ , i.e., in terms of the coordinate basis, we have

$$Y(t) = Y^i(t) \frac{\partial}{\partial x_i}, \quad X_x = X^i(x) \frac{\partial}{\partial x_i}, \quad Y^i(t) = X^i(c(t)).$$

Then,

$$\begin{aligned} D_i X &= D_i \left( X^i \frac{\partial}{\partial x_i} \right) = \dot{c}(X^i) \frac{\partial}{\partial x_i} + X^i D_i \frac{\partial}{\partial x_i} = X^i \dot{c}^k \underbrace{D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}}_{\Gamma_{ki}^\ell \frac{\partial}{\partial x_\ell}} \\ &= \partial_t(X^i \circ c) \frac{\partial}{\partial x_i} + \dot{c}^k X^i \Gamma_{ki}^\ell \frac{\partial}{\partial x_\ell} = \partial_t(X^i \circ c) \frac{\partial}{\partial x_i} + \dot{c}^k Y^i \Gamma_{ki}^\ell \frac{\partial}{\partial x_\ell} = \frac{D}{dt} Y. \end{aligned}$$

■

**Problem 3.4.1.** Why not just define  $DY/dt$  by (ii)?

**Answer.** A **vector field  $Y$  along a curve** may not always be extended to a neighborhood of  $c$  in  $\mathcal{M}$ . But, in **local coordinates**,

$$Y(t) = \sum_{i=1}^n Y^i(t) \left( \frac{\partial}{\partial x^i} \right)_{c(t)},$$

which shows that a **vector field along  $c$**  is always a linear combination of **vector fields along  $c$**  that can be extended. ⊛

**Remark.** Proposition 3.4.3 shows that the choice of an **linear connection** on  $\mathcal{M}$  leads to a bona fide (satisfying (a) and (b)) derivative of **vector fields along curves**. The notion of “connection” furnishes, therefore, a manner of differentiating vectors along **curves**.



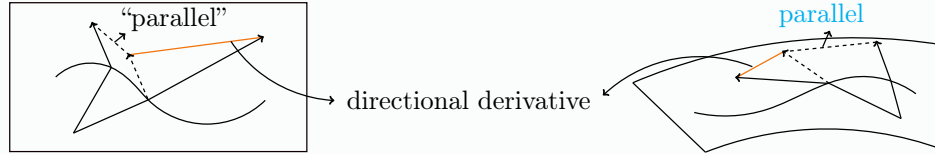
## 3.5 Parallel Transports

### 3.5.1 Parallel

Finally, we introduce the notion of [parallel](#).

**Definition 3.5.1 (Parallel).** A [vector field](#)  $X$  on  $\mathcal{M}$  along a [curve](#)  $c$  is *parallel* (or *parallelly transported*) along  $c$  if  $DX/dt = 0$  for all  $t \in I$ .

**Intuition.** In the ordinary Euclidean space, i.e., when the space is flat, we know what is “parallel,” and hence we can define the directional derivative.



But now, the logic is reversed: we first define what is [parallel](#) in a curved space, and then we can make sense of directional derivative in a curved space!

### 3.5.2 Parallel Transports

Given the definition of a [parallel vector fields along curves](#), we can talk about [parallel transport](#).

**Definition 3.5.2 (Parallel transport).** The *parallel transport* from  $c(0)$  to  $c(t)$  along the [curve](#)  $c$  in a [Riemannian manifold](#)  $(\mathcal{M}, g)$  is the linear map  $P_t: T_{c(0)}\mathcal{M} \rightarrow T_{c(t)}\mathcal{M}$  associating  $v \in T_{c(0)}\mathcal{M}$  with  $X_v(t) \in T_{c(t)}\mathcal{M}$  with  $X_v$  being the [parallel vector field along](#)  $c$  such that  $X_v(0) = v$ .

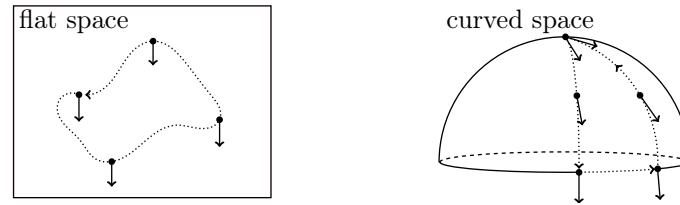
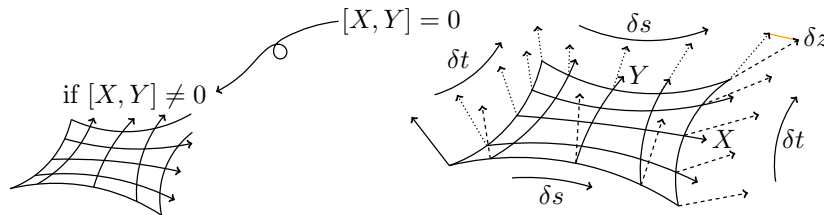


Figure 3.1: When the space is flat, keeping the “arrow” (which defines a [vector field](#)) in one direction and moving around won’t produce any changes, while when the space is curved, it will.

We make a surprising remark on the relation between [Riemannian curvature](#) and [parallel transport](#).

**Remark (Geometric significant of Riemannian curvature).** The idea is that for a [manifold](#) with [torsion free](#)  $\nabla$ , if we [parallel transporting](#) along two paths on an infinitesimal patch (which induces  $X, Y$ ) such that  $[X, Y] = 0$ , we can detect [curvature](#) in terms of  $\delta z$ , where  $(\delta z)^m = \dots = R_{nab}^m X^a Y^b Z^n \cdot \delta s \delta t + O(\delta s^2 \delta t, \delta s \delta t^2)$ .<sup>a</sup>



<sup>a</sup>This is a deep theorem! In the ..., we use  $T \equiv 0$ .

**Proposition 3.5.1.** The parallel transport defines for all  $t$  an isometry from  $T_{c(0)}\mathcal{M}$  onto  $T_{c(t)}\mathcal{M}$ . More generally, if  $X, Y$  are vector fields along  $c$ , then

$$\frac{d}{dt}g(x(t), y(t)) = g\left(\frac{DX(t)}{dt}, Y(t)\right) + g\left(X(t), \frac{DY(t)}{dt}\right).$$

### 3.5.3 Autoparallel

Now we can formally introduce the notion of autoparallel.

**Definition 3.5.3 (Autoparallel).** Let  $\nabla$  be a connection on  $T\mathcal{M}$  of a differentiable manifold  $\mathcal{M}$ . A curve  $c: I \rightarrow \mathcal{M}$  is called *autoparallel* (or *geodesic*) w.r.t.  $\nabla$  if

$$\nabla_{\dot{c}}\dot{c} = 0.$$

**Intuition.** An autoparallel curve is the *straightest line* in the space w.r.t.  $\nabla$ , hence it gives us a geodesic!

**Remark (Physical interpretation).** One can start from introducing  $\nabla$ , considering  $\nabla_{\dot{c}}\dot{c} := 0$  (which is just Equation 2.1), and realize that we don't need to consider gravity as a force, rather a "curvature of spacetime," in order to make sense of Newton's first law, i.e., mass without forces will undergo a autoparallel curve.

**Example (Euclidean plane).** Let  $U = \mathbb{R}^2$ ,  $x = \text{id}_{\mathbb{R}^2}$ ,  $\Gamma_{jk}^i = 0$ , then  $\ddot{c}^k = 0$  in Equation 3.2. Hence,

$$c^k(t) = a^k t + b^k \text{ for } a^k, b^k \in \mathbb{R}^d.$$

**Example (Round sphere).** The geodesics on a "round sphere" are the great circles.

**Proof.** Consider a "unit round sphere"  $\mathcal{M} = S^2$  with spherical coordinates  $x(p) = (r, \theta, \varphi)$  such that  $r = 1$ ,  $\theta \in (0, \pi)$ , and  $\varphi \in [0, 2\pi)$ . The "roundness" is given by  $\nabla_{\text{round}}$  where we specify (at one point  $x$ )

$$\Gamma_{22}^1 := -\sin\theta \cos\theta, \quad \Gamma_{21}^2 = \Gamma_{12}^2 := \cot\theta.$$

Furthermore, let  $x^1(p) = \theta(p)$ ,  $x^2(p) = \varphi(p)$ . We see that the autoparallel equation tells us

$$\begin{cases} \ddot{\theta} + \Gamma_{22}^1 \dot{\varphi}^2 = 0; \\ \ddot{\varphi} + 2\Gamma_{12}^2 \dot{\theta} \dot{\varphi} = 0; \end{cases} \Leftrightarrow \begin{cases} \ddot{\theta} - \sin(\theta) \cos(\theta) \dot{\varphi}^2 = 0; \\ \ddot{\varphi} + 2 \cot(\theta) \dot{\theta} \dot{\varphi} = 0. \end{cases}$$

Then, we see that  $\theta(t) = \pi/2$ ,  $\varphi(t) = \omega t + \varphi_0$  is a solution.<sup>a</sup> Hence, we conclude that if we run at a constant speed around the great circle of  $S^2$ , it'll be autoparallel, hence a geodesic.  $\circledast$

<sup>a</sup>Note that  $\theta(t) = \pi/2$ ,  $\varphi(t) = \omega t^2 + \varphi_0$  is not a solution.

Similarly, given any  $\nabla$  on a space, we can find the straightest curve on which.

## Lecture 12: Tangent and Cotangent Bundles

### 3.6 More on Tangent and Cotangent Bundles

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#### 3.6.1 Preliminaries

Let  $f: \mathcal{M} \rightarrow \mathcal{N}$  be a differentiable map between two differentiable manifolds, and let  $(E, \pi, \mathcal{N})$  be a vector bundle over  $\mathcal{N}$ . We want to pull back the bundle via  $f$ , i.e., construct a bundle  $f^*E$  for which the fiber over  $x \in \mathcal{M}$  is  $E_{f(x)}$ :

**Definition 3.6.1 (Pullback).** The *pullback bundle*  $f^*E$  is the bundle over  $\mathcal{M}$  with the bundle charts  $\varphi \circ f, f^{-1}(U)$  with  $(\varphi, U)$  being the bundle charts of  $E$ .

**Definition 3.6.2 (Bundle homomorphism).** Consider 2 vector bundles  $(E_1, \pi_1, \mathcal{M}), (E_2, \pi_2, \mathcal{M})$  over  $\mathcal{M}$ , and let the differentiable map  $f: E_1 \rightarrow E_2$  be fiber preserving, i.e.,  $\pi_2 \circ f = \pi_1$ . Let the fiber maps  $f_x: E_{1,x} \rightarrow E_{2,x}$  be linear.<sup>a</sup> Then  $f$  is called a *bundle homomorphism*.

<sup>a</sup>I.e., vector homomorphisms.

**Definition 3.6.3 (Subbundle).** Let  $(E, \pi, \mathcal{M})$  of rank  $n$  be a vector bundle. Let  $E^1 \subseteq E$ , and assume that for all  $x \in \mathcal{M}$ , there exists a bundle chart  $(\varphi, U)$  for  $x \in U$  and  $\varphi(\pi^{-1}(U) \cap E^1) = U \times \mathbb{R}^m \subseteq U \times \mathbb{R}^n$  for  $m \leq n$ . The so constructed vector bundle  $(E^1, \pi|_{E^1}, \mathcal{M})$  is called the *subbundle* of  $E$  of rank  $m$ .

### 3.6.2 Identifications

Let  $\mathcal{M}$  be a differentiable manifold.

**As previously seen.** The elements of  $T\mathcal{M}$  is  $(x, V)$  with  $x \in \mathcal{M}$ ,  $V$  the tangent vector to  $\mathcal{M}$  at  $x$ . Also, the tangent bundle is the section of  $T\mathcal{M}$ .

Also, the elements of  $T^*\mathcal{M}$  are called *cotangent vector*, i.e., linear functionals  $\alpha \in T_x^*\mathcal{M}$  such that  $\alpha: T_x\mathcal{M} \rightarrow \mathbb{R}$ . Then, the section of  $T^*\mathcal{M}$  are 1-forms, i.e.,  $\alpha: T\mathcal{M} \rightarrow \mathbb{R}$ ,  $\alpha_x: T_x\mathcal{M} \rightarrow \mathbb{R}$ .

A Riemannian metric induces bundle metrics on all tensor bundles over  $\mathcal{M}$ . Metric of  $T^*\mathcal{M}$  given by

$$\langle \omega, \eta \rangle = g(\omega, \eta) = g^{ij} \omega_i \eta_j$$

where  $\omega = \omega_i dx^i$ ,  $\eta = \eta_i dx^i$ . The identification between  $T\mathcal{M}$  such that  $T^*\mathcal{M}$  through Riemannian metric is given by

$$V = V^i \frac{\partial}{\partial x^i} \in T\mathcal{M} \text{ corresponds to } \omega = \omega_j dx^j \in T^*\mathcal{M}$$

with  $\omega_j = g_{ij} V^i$  or  $V^i = g^{ij} \omega_j$  with

$$(a) \quad g(X, Y) = g_{ij} X^i Y^j \text{ for } X, Y \in T\mathcal{M}.$$

$$(b) \quad g(\omega, \eta) = g^{ij} \omega_i \eta_j \text{ for } \omega, \eta \in T^*\mathcal{M}.$$

Then, for  $V \in T_x\mathcal{M}$ , there corresponds a 1-form  $\omega \in T_x^*\mathcal{M}$  via the metric  $\omega(Y) := g(V, Y)$  for all  $Y$ , and  $\|\omega\| = \|V\|$ . Let  $(e_i)_{i=1, \dots, d}$  be a basis of  $T_x\mathcal{M}$  and  $(\omega^j)_{j=1, \dots, d}$  the dual basis of  $T_x^*\mathcal{M}$ , i.e.,  $\omega^j(e_i) = \delta_i^j$ . Let  $V = V^i e_i \in T_x\mathcal{M}$ ,  $\eta = \eta_j \omega^j \in T_x^*\mathcal{M}$ , then  $\eta(V) = \eta_i V^i$ .

Consider basis  $(e_i), (\omega^j)$  in the local coordinates:  $e_i = \partial/\partial x^i$  and  $\omega^j = dx^j$ . Let  $f$  be a local coordinates change. Then  $V$  transform as

$$f_*(V) := V^i \frac{\partial f^\alpha}{\partial x^i} \frac{\partial}{\partial f^\alpha};$$

and  $\eta$  transforms as

$$f^*(\eta) := \eta_j \frac{\partial x^j}{\partial f^\beta} df^\beta.$$

Hence,

$$f^*(\eta)(f_*(V)) = \eta_j \frac{\partial x^j}{\partial f^\alpha} V^i \frac{\partial f^\alpha}{\partial x^i} = \eta_i V^i = \eta(V).$$

**Intuition.** The above means that

- the tangent vectors transform with the functional matrix of coordinates change;

- the **cotangent vectors** transform with the transposed inverse of the above matrix.

To compute the **coordinates** change  $y \mapsto x(y)$  for  $\omega = \omega_i dx^i$ ,  $\eta = \eta_i dx^i$  with  $\langle \omega, \eta \rangle = g^{ij} \omega_i \eta_j$ , we have

$$\omega_i dx^i = \omega_i \frac{\partial x^i}{\partial y^\alpha} dy^\alpha =: \tilde{w}_\alpha dy^\alpha,$$

and  $g^{ij}$  is transformed as

$$h^{\alpha\beta} = g^{ij} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}$$

and  $h^{\alpha\beta} \tilde{w}_\alpha \tilde{w}_\beta = g^{ij} \omega_i \eta_j$  and

$$\|\omega(x)\| = \sup \{ \omega(x)(V) \mid V \in T_x \mathcal{M}, \|v\| = 1 \}.$$

Consider  $T\mathcal{M} \otimes T\mathcal{M}$ , the metric is

$$\langle V \otimes Y, \xi \otimes \eta \rangle = g_{ij} V^i Y^j g_{kl} \xi^k \eta^l.$$

**Definition 3.6.4** (Lie derivative). Consider a **vector field**  $X$  with a **local 1-parameter group**  $(\psi_t)_{t \in I}$  of local **diffeomorphism**, and a tensor field  $S$  on  $\mathcal{M}$ . The *Lie derivative* of  $S$  in the direction of  $X$  is defined as

$$L_X S := \left. \frac{d}{dt} (\psi_t^* S) \right|_{t=0}.$$

**Definition 3.6.5** (Pushforward). Let  $\psi: \mathcal{M} \rightarrow \mathcal{N}$  be a **diffeomorphism** between two **differentiable manifolds**. Let  $X$  be a **vector field** on  $\mathcal{M}$ . Then define a **vector field**  $Y = \psi_* X$  on  $\mathcal{N}$  be by  $Y(p) = d\psi(X(\psi^{-1}(p)))$ .

**Lemma 3.6.1.** The following holds.

1. For every differentiable function  $f: \mathcal{N} \rightarrow \mathbb{R}$ , then  $(\psi_* X)(f)(p) = X(f \circ \psi)(\psi^{-1}p)$ .
2. Let  $X$  be a **vector field** on  $\mathcal{M}$ ,  $\psi: \mathcal{M} \rightarrow \mathcal{N}$  be a **diffeomorphism**. If the **local 1-parameter group** generated by  $X$  given by  $\varphi_t$ , then the local group generated by  $\psi_* X$  is  $\psi \circ \varphi_t \circ \psi^{-1}$ .

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Let  $X = X^i \partial / \partial x^i$  be a **vector field**. Then

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$$(\psi_t)_* \frac{\partial}{\partial x^i} (\psi_t(x)) = \frac{\partial \psi_t^k}{\partial x^i} \frac{\partial}{\partial x^k}.$$

For  $\mathcal{M} = \mathcal{N}$ ,  $X$  and  $\varphi(x)$  in the same coordinate neighborhood, we have

$$\frac{\partial}{\partial \varphi^k} = \frac{\partial}{\partial x^k}.$$

On the other hand, let  $\omega g \omega_i dx^i$  be a 1-form, then we have

$$(\psi_t^*)(\omega)(x) = \omega(\psi_t(x)) \frac{\partial \psi_t^i}{\partial x^k} dx^k,$$

which is the curve in  $T_x^* \mathcal{M}$ . For  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ , with the 1-form  $\omega = \omega_i dx^i$  on  $\mathcal{N}$ ,

$$\varphi^* \omega = \omega_i(\varphi(x)) \frac{\partial z^i}{\partial x^k} dx^k.$$

Let  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$  be a **diffeomorphism**,  $Y$  be a **vector field** on  $\mathcal{N}$ . Set

$$\varphi^* Y := (\varphi^{-1})_* Y.$$

**Remark.**  $\varphi^*$  can be defined in an analogous way in terms of contravariant tensors.

In particular, let  $X$  be a **vector field**, and a local 1-parameter group  $(\psi_t)_{t \in I}$ ,

$$(\psi_t^* X) = (\psi_t)_* X.$$

Let  $x \in \mathcal{M}$ ,  $X \in T_x \mathcal{M}$ , and  $df_x: T_x \mathcal{M} \rightarrow T_{f(x)} \mathcal{N}$ .  $df_x X$  or  $(f_*)_x X$  is called the pushforward of  $x$  by  $f$ .

Let  $\omega \in T^* \mathcal{N}$ , a 1-form on  $\mathcal{N}$ . Then

$$(f^* \omega)_x X = \omega_{f(x)}(df_x X)$$

is called pullback of  $\omega$  by  $f$ .

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & \mathcal{N} \\ \pi_{\mathcal{M}} \uparrow & & \uparrow \pi_{\mathcal{N}} \\ T\mathcal{M} & \xrightarrow{df} & T\mathcal{N} \end{array}$$

**Definition 3.6.6** (Sectional curvature). The *sectional curvature* of the plane spanned by the (linearly independent) **tangent vectors**  $X = X^i \partial / \partial x^i, Y = Y^i \partial / \partial x^i \in T_x \mathcal{M}$  of the **Riemannian manifold**  $\mathcal{M}$  is

$$K(X \wedge Y) = \frac{g(R(X, Y)Y, X)}{|X \wedge Y|^2}$$

where  $|X \wedge Y|^2 = g(X, X)g(Y, Y) - g(X, Y)^2$ .

**Remark.** **Sectional curvature** determines the whole **curvature tensor**.

**Proof.** Given  $g(R(X, Y)Z, W)$ , we can express this entirely by  $K$ . ⊛

**Remark** (Gauss curvature). For  $\dim \mathcal{M} = 2$ ,

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{ij}g_{kl})$$

where here,  $K$  is called the *Gauss curvature*. Since  $T_x \mathcal{M}$  contains only one plane, i.e.,  $T_x \mathcal{M}$  itself.

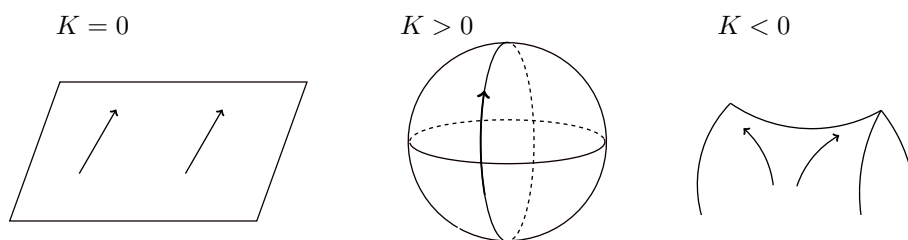
**Definition 3.6.7** (Space form). Let  $(\mathcal{M}, g)$  be a **Riemannian manifold**.  $\mathcal{M}$  is called a *space of constant sectional curvature* or a *space form* if  $K(X \wedge Y)$  is a constant for all linearly independent **tangent vectors**  $X, Y \in T_x \mathcal{M}$  and for all  $x \in \mathcal{M}$ .

**Definition 3.6.8** (Spherical). A **space form** is called *spherical* if  $K > 0$ .

**Definition 3.6.9** (Flat). A **space form** is called *flat* if  $K = 0$ .

**Definition 3.6.10** (Hyperbolic). A **space form** is called *hyperbolic* if  $K < 0$ .

**Definition 3.6.11** (Einstein manifold).  $\mathcal{M}$  is called an *Einstein manifold* if  $R_{ik} = cg_{ik}$  for  $c$  being a constant which does not depend on the choice of **local coordinates**.



**Remark.** Every manifold with constant sectional curvature is an Einstein manifold.

**Definition 3.6.12 (Flat).** A connection  $\nabla$  on  $T\mathcal{M}$  is called *flat* if each point in  $\mathcal{M}$  has a neighborhood  $U$  with local coordinates for which all the coordinate vector fields  $\partial/\partial x^i$  are parallel, i.e.,  $\nabla \partial/\partial x^i = 0$ .

**Theorem 3.6.1.** A connection  $\nabla$  on  $T\mathcal{M}$  is flat if and only if its curvature and torsion vanish identically.

**Proof.** Flat connection implies  $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ , hence all  $\Gamma_{ij}^k = 0$ , so  $T, R$  vanish. Conversely, find the local coordinates such that  $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$  for all  $i, j$ . Then, by computation with Frobenius theorem, we're done. ■

**Theorem 3.6.2 (Schur theorem).** Let  $(\mathcal{M}, g)$  be a Riemannian manifold with  $\dim \mathcal{M} \geq 3$ .

- (a) If the sectional curvature of  $\mathcal{M}$  is constant at each point, i.e.,  $K(X \wedge Y) = f(x)$  for  $X, Y \in T_x \mathcal{M}$ . Then  $f(x)$  is a constant on  $\mathcal{M}$ , so  $\mathcal{M}$  is a space form.
- (b) If the Ricci curvature is a constant at each point, i.e.,  $R_{ik} = c(x)g_{ik}$ , then  $c(x)$  is a constant and  $\mathcal{M}$  is an Einstein manifold.

**Remark.** Schur theorem says that the isotropy of a Riemannian manifold, i.e., the property that at each point, all directions are geometrically indistinguishable, implies the homogeneity, i.e., all points are geometrically indistinguishable.

**Note.** A point-wise property implies a global one!

### 3.6.3 Covering Maps

**Definition 3.6.13 (Covering map).** Let  $\mathcal{M}, \widetilde{\mathcal{M}}$  be 2 manifolds. a map  $p: \widetilde{\mathcal{M}} \rightarrow \mathcal{M}$  is a *covering map* if

- (a)  $p$  is smooth and surjective, and
- (b) for all  $m \in \mathcal{M}$ , there exists a neighborhood  $U$  at  $m$  in  $\mathcal{M}$  with  $p^{-1}(U) = \coprod_{i \in I} U_i$  with  $p: U_i \rightarrow U$  being a diffeomorphism and  $U_i$  are disjoint open subsets of  $\widetilde{\mathcal{M}}$ .

**Notation** (Covering space).  $\widetilde{\mathcal{M}}$  in Definition 3.6.13 is called the *covering space*.

**Notation** (Universal covering space). A covering space is *universal* if it's simply connected.

**Definition 3.6.14 (Riemannian covering map).** Let  $(\mathcal{M}, g), (\mathcal{N}, h)$  be 2 Riemannian manifolds. A map  $p: \mathcal{N} \rightarrow \mathcal{M}$  is a *Riemannian covering map* if  $p$  is a smooth covering map and is a local

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isometry.

**Proposition 3.6.1.** Let  $p: \mathcal{N} \rightarrow \mathcal{M}$  be a smooth covering map. For every Riemannian metric  $g$  on  $\mathcal{M}$ , there exists a unique Riemannian metric  $h$  on  $\mathcal{N}$  such that  $p$  is a Riemannian covering map.

**Note.** The converse of the above is generally not true.

**Example.** Every space covers itself trivially.

**Example.**  $\mathbb{R}$  is the universal covering space of  $S^1$ .

**Example.**  $U(n)$  has universal covers  $U(n) \times \mathbb{R}$ .

**Example.**  $S^n$  is a double cover for  $P_n(\mathbb{R})$  and is universal for  $n > 1$ .

**Proposition 3.6.2.** Let  $(\mathcal{N}, h)$  be a Riemannian manifold. Let  $G$  be a free and proper group of isometries of  $(\mathcal{N}, h)$ . Then there exists a unique Riemannian manifold  $(\mathcal{M}, g)$  on the quotient manifold  $\mathcal{M} = \mathcal{N}/G$  such that the connected projection  $p: \mathcal{N} \rightarrow \mathcal{M}$  is a Riemannian covering map.

# Appendix



# Appendix A

## Additional Notes

### A.1 Christoffel Symbols

In this section, we dive deep into the notion of the [Christoffel symbols](#)  $\Gamma$  in various ways.

In particular, we will see that  $\Gamma$  are really just the corrections to an ordinary derivative on a “curved” manifold w.r.t. the [Levi-Civita connection](#), i.e., in the context of [torsion free](#) and [Riemannian connection](#)  $\nabla$ , we have also defined the so-called [connection coefficients](#), and we use the same notation  $\Gamma$ , and indeed they’re the same.

See [this](#)

#### A.1.1 Geometric Interpretation

#### A.1.2 Metric Interpretation

#### A.1.3 A Visual Guide

### A.2 Lie Groups and Lie Algebra

#### A.2.1 Lie Groups

[Lie groups](#) are an important topic to study for Riemannian geometry, hence we now introduce it now.

**Definition A.2.1 (Lie group).** A *Lie group* is a group  $G$  with a [differentiable structure](#) such that the mapping  $G \times G \rightarrow G$  given by  $(x, y) \rightarrow xy^{-1}$ ,  $x, y \in G$ , is differentiable.

**Definition (Transformation).** Let  $G$  be a [Lie group](#).

**Definition A.2.2 (Left transformation).** The *translations from the left*  $L_x: G \rightarrow G$  is defined as  $L_x(y) = xy$ .

**Definition A.2.3 (Right transformation).** The *translations from the right*  $R_x: G \rightarrow G$  is defined as  $R_x(y) = yx$ .

**Remark.** Both  $L_x$  and  $R_x$  are [diffeomorphisms](#).

In the following discussion, let  $G$  be a [Lie group](#). Turns out that  $G$  admits some nice properties on [left invariant vector fields](#).

**Definition (Invariant of Riemannian metric).** Let  $g$  be a [Riemannian metric](#) on  $G$ .

**Definition A.2.4** (Left invariant).  $g$  is *left invariant* if

$$\langle u, v \rangle_y = \langle d(L_x)_y u, d(L_x)_y v \rangle_{L_x(y)}$$

for all  $x, y \in G$ ,  $u, v \in T_y G$ , i.e.,  $L_x$  is an **isometry**.

**Definition A.2.5** (Right invariant).  $g$  is *right invariant* if

$$\langle u, v \rangle_y = \langle d(R_x)_y u, d(R_x)_y v \rangle_{R_x(y)}$$

for all  $x, y \in G$ ,  $u, v \in T_y G$ , i.e.,  $R_x$  is an **isometry**.

**Definition A.2.6** (Bi-invariant).  $g$  is *bi-invariant* if it's both **right** and **left invariant**.

**Definition** (Invariant of vector field). Let  $X$  be a **vector field** on  $G$ .

**Definition A.2.7** (Left invariant).  $X$  is *left invariant* if  $dL_x X = X$  for all  $x \in G$ .

**Definition A.2.8** (Right invariant).  $X$  is *right invariant* if  $dR_x X = X$  for all  $x \in G$ .

**Definition A.2.9** (Bi-invariant).  $X$  is *bi-invariant* if it's both **right** and **left invariant**.

As we mentioned, the **left invariant vector fields** are completely determined by their values at a single point of  $G$ , which allows us to introduce an additional structure on the **tangent space** to the neutral element  $e \in G$  in the following manner.

To each **vector**  $X_e \in T_e G$ , we associate the **left invariant**  $X$  defined by

$$X_a := dL_a X_e, \quad a \in G.$$

## A.2.2 Lie Algebras

Let  $X, Y$  be **left invariant vector fields** on  $G$ . Since for each  $x \in G$  and for any differentiable function  $f$  on  $G$ ,

$$dL_x[X, Y]f = [X, Y](f \circ L_x) = X(dL_x Y)f - Y(dL_x X)f = (XY - YX)f = [X, Y]f,$$

i.e.,  $[X, Y]$  is again a **left invariant vector field** if  $X, Y$  are. Now, if  $X_e, Y_e \in T_e G$ , we put  $[X_e, Y_e] = [X, Y]_e$ .

**Definition A.2.10** (Lie algebra). The *Lie algebra* of  $G$ , denoted by  $\mathfrak{g}$ , is the vector space  $T_e G$  with the **bracket**  $[\cdot, \cdot]$ .

**Note.** The elements in the **Lie algebra**  $\mathfrak{g}$  will be thought of either as **vectors** in  $T_e G$  or as **left invariant vector fields** on  $G$ .

To introduce a **left invariant metric** on  $g$ , take any arbitrary inner product  $\langle \cdot, \cdot \rangle_e$  on  $\mathfrak{g}$  and define

$$\langle u, v \rangle_x := \langle (dL_{x^{-1}})_x(u), (dL_{x^{-1}})_x(v) \rangle_e \quad (\text{A.1})$$

for  $x \in G$ ,  $u, v \in T_x G$ . Since  $L_x$  depends differentiably on  $x$ , this is actually a **Riemannian metric**, which is clearly **left invariant**.

**Remark.** We can also construct a **right invariant metric** on  $G$ , and if  $G$  is compact,  $G$  possesses a **bi-invariant metric**.

One important characterization for  $G$  having a **bi-invariant metric** is that the inner product that the **metric** determines on  $\mathfrak{g}$  satisfies the following relation.

**Proposition A.2.1.** If  $G$  has a [bi-invariant metric](#), then for any  $U, V, X \in \mathfrak{g}$ , the inner product that the [metric](#) determines on  $\mathfrak{g}$  satisfies

$$\langle [U, X], V \rangle = -\langle U, [V, X] \rangle.$$

**Proof.** See do Carmo [FC13, Page 40, 41]. ■

The important point about this relation is that it characterizes the [bi-invariant metrics](#) of  $G$  in the following sense.

**Remark.** If a positive bilinear form  $\langle \cdot, \cdot \rangle_e$  defined on  $\mathfrak{g}$  satisfies this relation, then the [Riemannian metrics](#) defined on  $G$  by [Equation A.1](#) is [bi-invariant](#).

## A.3 Algebra

This chapter will collect some notion about algebras which you might not be familiar with.

### A.3.1 Modules

**Definition A.3.1 (Left module).** Suppose  $R$  is a ring with 1. A *left  $R$ -module*  $M$  consists of an Abelian group  $(M, +)$  and an operation  $\cdot : R \times M \rightarrow M$  such that for all  $r, s \in R$  and  $x, y \in M$ ,

- (a)  $r \cdot (x + y) = r \cdot x + r \cdot y$ ;
- (b)  $(r + s) \cdot x = r \cdot x + s \cdot x$ ;
- (c)  $(rs) \cdot x = r \cdot (s \cdot x)$ ;
- (d)  $1 \cdot x = x$ .

**Note.** A *right  $R$ -module*  $M$  can also be defined similarly by consider  $\cdot : M \times R \rightarrow M$ .

**Definition A.3.2 (Module).** If  $R$  is commutative, then the [left and right  \$R\$ -module](#)  $M$  are the same, and we call  $M$  a *module*.

**Intuition.** We're basically relaxing the notion of  $\mathbb{F}$ -vector field, but this time, the field  $\mathbb{F}$  is replaced by a ring  $R$ .

**Remark.** The most noticeable difference between a [module](#) and a vector field is that a [module](#) usually don't have a basis.

### A.3.2 The $C^\infty(\mathcal{M})$ -Module Viewpoint of Tensor Fields

The reason why we introduce the notion of [module](#) is because of the following: we can understand [tensor-field](#) better in the following way. Firstly, let's introduce the so-called [tensor bundles](#).

**Definition A.3.3 (Tensor bundle).** A *tensor bundle* is a [fiber bundle](#) where the [fiber](#) is the product of any number of [tangent spaces](#) and/or [cotangent spaces](#).

So in a [tensor bundle](#), the [fiber](#) is a vector space and the [tensor bundle](#) is a special kind of [vector bundle](#).<sup>1</sup> Then, recall how we introduce [Definition 1.5.1](#):

**As previously seen.** A  $(r, s)$ -[tensor field](#)  $T$  is just a [section](#) of a [tensor bundle](#).

<sup>1</sup>There are [vector bundles](#) which are not [tensor bundles](#).

But there's actually a deeper explanation: observe that  $\Gamma(TM) = \{X : \text{vector fields on } \mathcal{M}\}$  is actually a  $C^\infty(\mathcal{M})$ -module:

**Claim.**  $\Gamma(TM)$  carries a natural  $C^\infty(\mathcal{M})$ -module structure.

**Proof.** Firstly, observe that  $C^\infty(\mathcal{M}) = ((C^\infty(\mathcal{M}), +, \cdot))$  is not a field but a ring.<sup>a</sup> Then, naturally, the  $C^\infty(\mathcal{M})$ -module  $(\Gamma(TM), \oplus, \odot)$  where

- $\oplus: (X \oplus \tilde{X})(f) := (Xf) + \tilde{X}(f);$
- $\odot: (g \odot X)(f) := g \cdot X(f),$

for  $X, \tilde{X} \in \Gamma(TM)$ ,  $g, f \in C^\infty(\mathcal{M})$ . ⊛

<sup>a</sup>Since given  $f \in C^\infty(\mathcal{M})$ , we might not have  $f^{-1}$ .

**Notation.** Notice that given a vector field  $X: \mathcal{M} \rightarrow TM$  with  $p \mapsto X(p)$ , we let

$$Xf: \mathcal{M} \rightarrow \mathbb{R}, \quad p \mapsto X(p)f.$$

This makes sense since we can't always do things globally, e.g., **Hairy ball theorem**. Specifically, we can't choose a basis  $X_1, \dots, X_d \in \Gamma(TM)$  for our vector field globally as we already know. Similarly, we can define  $\Gamma(T^*\mathcal{M})$ , i.e., the set of "convector field"<sup>2</sup> is again a  $C^\infty(\mathcal{M})$ -module.

**Example.** Given  $\omega \in \Gamma(T^*\mathcal{M})$  and  $X \in \Gamma(TM)$ ,  $\omega$  acts on  $X$  to yield smooth functions by point-wise evaluation, i.e., we define

$$(\omega(X))(p) := \omega(p)(X(p)).$$

Then, the action of  $\omega$  on  $X$  is a  $C^\infty(\mathcal{M})$ -linear map since

$$(\omega(fX))(p) = f(p)\omega(p)(X(p)) = (f\omega)(p)(X(p)) = (f\omega(X))(p)$$

for  $f \in C^\infty(\mathcal{M})$ . This suggests that we should not regard  $\omega$  just as a section of  $T^*\mathcal{M}$ , but also a linear mapping of  $X \in \Gamma(TM)$  into  $C^\infty(\mathcal{M})$ .

Then, in this view point, we have the following.

**Definition A.3.4 (Tensor field\*).** A  $(r, s)$ -tensor field  $T$  on a smooth manifold  $\mathcal{M}$  is a  $C^\infty(\mathcal{M})$  multilinear map

$$T: \underbrace{\Gamma(T^*\mathcal{M}) \times \dots \times \Gamma(T^*\mathcal{M})}_r \times \underbrace{\Gamma(TM) \times \dots \times \Gamma(TM)}_s \rightarrow C^\infty(\mathcal{M}).$$

Comparing to Definition 2.6.13, this definition is more general!

**Example.** The linear connection  $\nabla(X, Y) \mapsto \nabla_X Y$  does not define a tensor field.

**Proof.** Since  $\nabla$  is only  $\mathbb{R}$ -linear in  $Y$ . ⊛

<sup>2</sup>We won't define it formally, but it's defined similarly.

# Bibliography

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