

MATH635
Riemannian Geometry

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Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by [Lydia Bieri](#). Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [\[FC13\]](#) as our reference; while not required, but highly recommended have on.

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Chapter 1

Manifolds

Lecture 1: A Foray to Smooth Manifolds

1.1 Differentiable Manifolds

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1.1.1 Topological Manifolds

Let's start with a common definition.

Definition 1.1.1 (Topological manifold). A *topological manifold* \mathcal{M} of dimension n is a (topological) Hausdorff space such that each point $p \in \mathcal{M}$ has a neighborhood U homeomorphic via $\varphi: U \rightarrow U'$ to an open subset $U' \subseteq \mathbb{R}^n$.

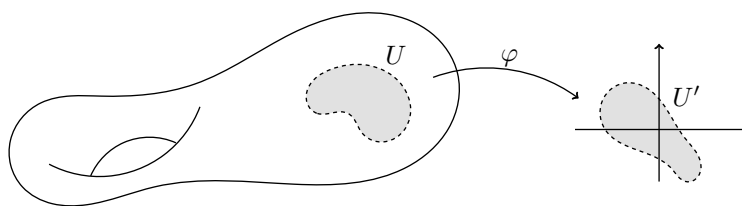
Definition 1.1.2 (Local coordinate map). For every $p \in \mathcal{M}$, the corresponding homeomorphism φ is called the *local coordinate map*.

Definition 1.1.3 (Local coordinate). The pull-back (x^1, \dots, x^n) of the *local coordinate map* φ from \mathbb{R}^n is called the *local coordinates* on U , given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

Definition 1.1.4 (Coordinate chart). The pair (U, φ) is called a *(coordinate) chart* on M .

In other words, a *topological manifold* can be thought of as a space such that it looks like \mathbb{R}^n locally.



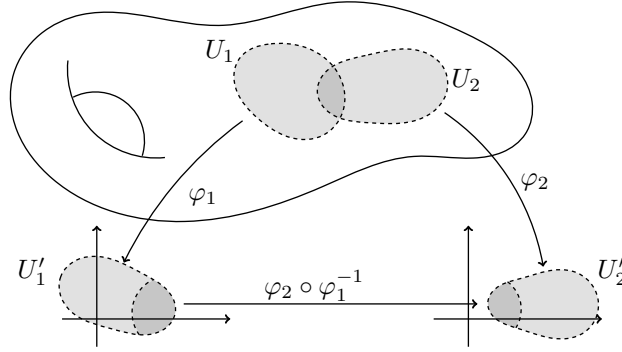
Definition 1.1.5 (Atlas). An *atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ for a *manifold* \mathcal{M} is a collection of *charts* such that $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$ are an open covering of \mathcal{M} , i.e., $\mathcal{M} = \bigcup_\alpha U_\alpha$.

In other words, for all $p \in \mathcal{M}$, there exists a neighborhood $U \subseteq \mathcal{M}$ and homeomorphism $h: U \rightarrow U' \subseteq \mathbb{R}^n$ open.

Definition 1.1.6 (Locally finite). An *atlas* is said to be *locally finite* if each point $p \in \mathcal{M}$ is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as \mathbb{R}^n , there's no clear way to make sense of differentiability of a **manifold**. But thankfully, we now have an explicit relation to the ambient space \mathbb{R}^n via φ_α . To formalize, let \mathcal{A} be an **atlas** for a **manifold** \mathcal{M} , and assume that $(U_1, \varphi_1), (U_2, \varphi_2)$ are 2 elements of \mathcal{A} . Then clearly, the map $\varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$ is a homeomorphism between 2 open sets of Euclidean spaces since both φ_1 and φ_2 are homeomorphism. Due to this map's importance, it has its own name.

Definition 1.1.7 (Coordinate transition). The map $\varphi_2 \circ \varphi_1^{-1}$ is called the *coordinate transition* of \mathcal{A} for the pair of **charts** $(U_1, \varphi_1), (U_2, \varphi_2)$.



1.1.2 Differentiable Structures

Notice that the **coordinate transitions** are from \mathbb{R}^n to \mathbb{R}^n ; hence differentiability makes sense now, which induces the following.

Definition 1.1.8 (Differentiable atlas). The **atlas** $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is *differentiable* if all **transitions** are differentiable.

Remark. Here, the differentiability depends on the content. Sometimes, we may want it to be C^∞ , and sometimes may be C^k for some finite k . On the other hand, smooth always refers to C^∞ . We'll use them interchangeably if it's clear which case we're referring to.

Definition 1.1.9 (Equivalence atlas). Two **atlases** \mathcal{U}, \mathcal{V} of a **manifold** are equivalent if for every $(U, \varphi) \in \mathcal{U}, (V, \psi) \in \mathcal{V}$,

$$\varphi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1}: \varphi(U \cap V) \rightarrow \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

Notation (Smoothly compatible). Two **charts** (U, φ) and (V, ψ) are *smoothly compatible* if either $U \cap V = \emptyset$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

This suggests the following.

Definition 1.1.10 (Smooth structure). A *smooth structure* on \mathcal{M} is an equivalence class \mathcal{U} of **coordinate atlas** with the property that all **transition functions** are diffeomorphisms.

Remark. We can also use the *maximal differentiable atlas* to be our differentiable structure.

Definition 1.1.11 (Smooth manifold). A *smooth manifold* is a manifold \mathcal{M} with a smooth structure.

In this way, we can do calculus on smooth manifolds! Furthermore, it now makes sense to say that a function $f: \mathcal{M} \rightarrow \mathbb{R}$ is differentiable (or C^∞) by considering differentiability of $f \circ \varphi^{-1}$ around p .

Notation. The collection of smooth functions on smooth manifold \mathcal{M} is denoted by $C^\infty(\mathcal{M}, \mathbb{R})$, or $C^k(\mathcal{M}, \mathbb{R})$.

Remark. The class $C^\infty(\mathcal{M}, \mathbb{R})$ consists of functions with property is well-defined.

Proof. Let \mathcal{A} be any given atlas from equivalence class that defines the smooth structure, and as we have shown, if $(U, \varphi) \in \mathcal{A}$, then $f \circ \varphi^{-1}$ is a smooth function on \mathbb{R}^n . This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.1.9 requirement that defines the equivalent manifolds. \circledast

1.1.3 Orientation

Another essential property of a manifold is its orientability.

Definition. Consider an atlas \mathcal{A} for a differentiable manifold \mathcal{M} .

Definition 1.1.12 (Oriented). \mathcal{A} is *oriented* if all transitions have positive functional determinant.

Definition 1.1.13 (Orientable). \mathcal{M} is *orientable* if \mathcal{A} is an oriented atlas.

Motivated by the above definitions, we see that we can actually use an atlas to define an orientation.

Definition 1.1.14 (Orientation). Let \mathcal{M} be an orientable manifold. Then a oriented differentiable structure is called an *orientation* of \mathcal{M} .

If \mathcal{M} possesses an orientation, we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with Definition 1.1.12.

Remark. Two differentiable structures obeying Definition 1.1.12 determine the same orientation if the union again satisfying Definition 1.1.12.

Remark. If \mathcal{M} is orientable and connected, then there exists exactly 2 distinct orientations on \mathcal{M} .

Now, we can see some examples of smooth manifolds.

Example (Sphere). The sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Consider $U_i^+ = \{x \in S^n \mid x_i > 0\}$, $U_i^- = \{x \in S^n \mid x_i < 0\}$ for $i = 1, \dots, n+1$, and $h_i^\pm: U_i^\pm \rightarrow \mathbb{R}^n$ such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1}).$$

Note that the minimum charts needed to cover S^n is 2.

Example. Let $\mathcal{M} = U \subseteq \mathbb{R}^n$, then $\{(U, \varphi)\}$ is a smooth structure with $\varphi = \text{id}$.

Example. Open sets of C^∞ -manifolds are C^∞ -manifolds.

Example (General linear group). $\mathrm{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, open.

Example (Real projective space). $\mathbb{R}P^n = S^n / \sim$ where $x \sim -x$ with $\pi: S^n \rightarrow \mathbb{R}P^n$, $x \mapsto [x]$.

Proof. π is a homeomorphism on each U_i^+ for $i = 1, \dots, n+1$, with

$$\{(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1}), i = 1, \dots, n+1\}$$

is a C^∞ -atlas for $\mathbb{R}P^n$. *

Note. Observe that $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$.

Lecture 2: Maps Between Smooth Manifolds

1.1.4 Smooth Maps

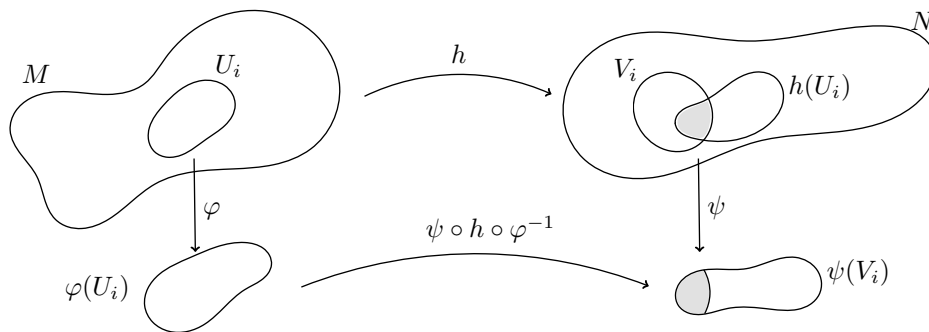
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We can now consider the maps between manifolds, specifically, the smooth manifolds.

Definition 1.1.15 (Smooth function). Let M, N be two smooth manifolds, and let \mathcal{U} be locally finite atlas from the equivalence class that gives the smooth structure on M , and let \mathcal{V} be the corresponding for N . A map $h: M \rightarrow N$ is said to be smooth if each map in the collection

$$\{\psi \circ h \circ \varphi^{-1}: h(U) \cap V \neq \emptyset\},$$

where $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$ is C^∞ -differentiable as a map from one Euclidean space to another.



Remark. Equivalence relation guarantees that Definition 1.1.15 depends only on the smooth structure of M, N , but not on the chosen representative coordinate atlas.

Definition. Consider two smooth manifolds M, N and a smooth homeomorphism $h: M \rightarrow N$ with smooth inverse.

Definition 1.1.16 (Diffeomorphic). The two manifolds M, N are said to be diffeomorphic.

Definition 1.1.17 (Diffeomorphism). The map h is said to be a diffeomorphism.

Let M_1, M_2 be two smooth manifolds, and let $\varphi: M_1 \rightarrow M_2$ be a diffeomorphism. Then the following hold.

- M_1 is orientable if and only if M_2 is orientable.
- If in addition, M_1 and M_2 are both connected and oriented, then φ induces an orientation on M_2 that may or may not coincide with the initial orientation of M_2 .

Check

If the induced **orientation** coincides, then we say φ preserves the **orientation**, otherwise φ reverses the **orientation**.

1.1.5 Grassmannian Manifold

Before proceeding, let's consider an interesting **smooth manifold**.

Definition 1.1.18 (Grassmannian manifold). Given $m, n \in \mathbb{N}$, the so-called *Grassmannian manifold* $G(n, m)$ is the set of all n -dimensional subspaces of \mathbb{R}^{n+m} .

Note. $G(1, m)$ is just $\mathbb{R}P^m$, and $G(0, m)$, $G(n, 0)$ are one-point sets.

As we will soon see, $G(n, m)$ has the **smooth structure** of an mn -dimensional **manifold**.

Intuition. We obtain the **structure** by exhibiting an **atlas** whose **transitions** are **diffeomorphisms**.

Firstly, we give $G(n, m)$ a suitable topology, i.e., the metric topology. Let $\Pi \in G(n, m)$, and let $\mathcal{L}(\Pi, \Pi^\perp)$ denote the mn -dimensional space of linear maps from Π to Π^\perp . Define the map

$$\varphi_\Pi: \mathcal{L}(\Pi, \Pi^\perp) \rightarrow G(n, m), \quad \varphi_\Pi(\alpha) = (\mathbb{1}_\Pi \oplus \alpha)(\Pi)$$

where $\mathbb{1}_\Pi \oplus \alpha$ is regarded as a map $\Pi \rightarrow \Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.¹ Clearly, φ_Π is injective, and thus, $(\mathcal{L}(\Pi, \Pi^\perp), \varphi_\Pi)$ is an mn -dimensional **chart** of $G(n, m)$.

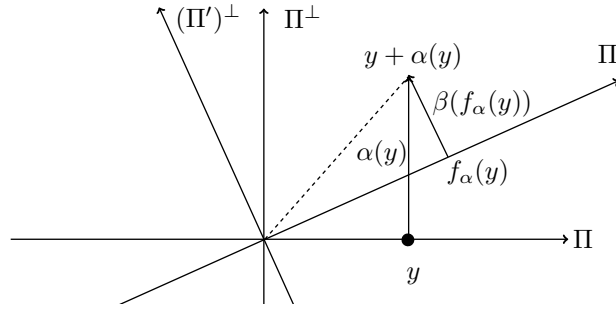
Remark. The images $\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$ cover $G(n, m)$.

Example. $\Pi = \varphi_\Pi(0) \in \varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$.

We can now prove that these **chart** are mutually **compatible**. Let $\Pi, \Pi' \in G(n, m)$, and let P, P' be orthogonal projections from \mathbb{R}^{n+m} onto Π, Π' respectively. Firstly,

$$F = \varphi_{\Pi'}^{-1} \varphi_\Pi: \varphi_\Pi^{-1}(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))) \rightarrow \varphi_{\Pi'}^{-1}(\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp)))$$

is smooth.



Consider $\alpha \in \mathcal{L}(\Pi, \Pi^\perp)$, and $\beta \in \mathcal{L}(\Pi', (\Pi')^\perp)$, then for α, β , the equality $F(\alpha) = \beta$ means that $\varphi_\Pi(\alpha) = \varphi_{\Pi'}(\beta)$. Let $f_\alpha: \Pi \rightarrow \Pi'$ be defined by

$$f_\alpha = P' \circ (\mathbb{1}_\Pi \oplus \alpha).$$

We need to check

- (a) f_α is invertible, and
- (b) $\forall y \in \Pi, y + \alpha(y) = f_\alpha(y) + \beta(f_\alpha(y))$.

¹In other words, $\varphi_\Pi(\alpha)$ is the graph of α in $\Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.

Note. The condition that $\det f_\alpha \neq 0$ gives an exact description of the subset

$$\varphi_{\Pi^{-1}}(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp)))$$

of $\mathcal{L}(\Pi, \Pi^\perp)$, which is therefore open.

For β , it is $(\mathbb{1}_{\Pi'} \oplus \beta) \circ f_\alpha = \mathbb{1}_\Pi \oplus \alpha$, and hence

$$\beta = F(\alpha) = (\mathbb{1}_\Pi \oplus \alpha) \circ f_\alpha^{-1} - \mathbb{1}_{\Pi'}.$$

It follows by the construction that the image of β is contained in $(\Pi')^\perp$.

Remark. We obtain an infinite atlas for $G(n, m)$ with charts labeled by $\Pi \in G(n, m)$. But it suffices to consider only $\binom{n+m}{n}$ charts corresponding to subspaces Π spanned with n coordinate axes.

1.2 Manifolds with Boundary

We first introduce two notions.

Definition 1.2.1 (Closed manifold). A manifold is *closed* if it is compact and without boundary.

Definition 1.2.2 (Open manifold). A manifold is *open* if it has only non-compact components without boundary.

Lemma 1.2.1. If M can be covered by two coordinate neighborhoods V_1, V_2 such that $V_1 \cap V_2$ is connected, then M is *orientable*.

Proof. The determinant of the differential of the coordinate change $\neq 0$, so it does not change sign in $V_1 \cap V_2$. If it's negative at a single point, it's enough to change the sign of one of the coordinates to make it positive at that point, hence on $V_1 \cap V_2$. ■

Example (S^n). Let $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$ is *orientable*.

Proof. Let $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$, consider given $p = (0, \dots, 0, x_i, 0, \dots, x_{n+1})$, then $\Pi_1: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$ given by

$$\Pi_1(p) = \left(0, \dots, 0, \frac{x_i}{1 - x_{n+1}}, 0, \dots, 0\right)$$

to be the stereographic projection from the north-pole N .

More generally, it takes $p(x_1, \dots, x_{n+1}) \in S^n - \{N\}$ into the intersection at the hyperplane $x_{n+1} = 0$ with the line passing through p and N . In this way, we have

$$\Pi_1(x_1, \dots, x_n) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right),$$

hence Π_1 is differentiable, and is injective. Similarly, Π_2 for S can also be defined and everything holds similarly. ⊛

Appendix

Bibliography

- [FC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013. ISBN: 9780817634902. URL: <https://books.google.com/books?id=ct91XCWkWEUC>.