MATH592
Introduction to Algebraic Topology

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Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

Contents

Lecture 6: A Foray into Category Theory

19 Jan. 10:00

0.1 Category Theory

We start with a definition.

Definition 0.1 (Object, Morphism). A category $\mathscr{C}$ is 3 pieces of data

- A class of objects $\mathrm{Ob}(\mathscr{C})$
- $\forall X, Y \in \mathrm{Ob}(\mathscr{C})$ a class of morphisms or arrows, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- $\forall X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, there exists a composition law

$$\begin{aligned} \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) &\to \mathrm{Hom}(X,Z) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

and 2 axioms

- Associativity. $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms f, g, h where composites are defined.
- Identity. $\forall X \in \mathrm{Ob}(\mathscr{C})$ $\exists \mathrm{id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g$$

for all f, g where this makes sense.

Let's see some examples.

Example. We introduce some common category.

$\mathcal{C}$	$\mathrm{Ob}(\mathcal{C})$	$\mathrm{Mor}(\mathcal{C})$
set	Sets X	All maps of sets
fset	Finite sets	All maps
Gp	Groups	Group Homomorphisms
Ab	Abelian groups	Group Homomorphisms
$k-$vect	Vector spaces over k	k-linear maps
Rng	Rings	Ring Homomorphisms
Top	Topological spaces	Continuous maps
Haus	Hausdorff Spaces	Continuous maps
hTop	Topological spaces	Homotopy classes of continuous maps
Top*	Based topological spaces[1]	Based maps[2]

Remark. Any **diagram** plus composition law.

$$\begin{tikzcd} A \arrow[r] \arrow[loop left, "\mathrm{id}_A"] & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd}.$$

[1] Topological spaces with a distinguished base point $x_0 \in X$

[2] Continuous maps that presence base point $f\colon (x, x_0) \to (y, y_0)$ such that

$$f\colon X \to Y, \quad f(x_0) = y_0$$

is continuous.

Definition 0.2 (monic, epic). A morphism $f\colon M \to N$ is *monic* if

$$\forall g_1, g_2 \; f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$\begin{tikzcd} A \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

Dually, f is *epic* if

$$\forall g_1, g_2 \; g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & B \end{tikzcd}$$

Lemma 0.1. In set, Ab, Top, Gp, a map is monic if and only if f is injective, and epic if and only if f is surjective.

Proof. In set, we prove that f is monic if and only if f is injective. Suppose $f \circ g_1 = f \circ g_2$ and f is injective, then for any a,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence $g_1 = g_2$.

Now we prove another direction, with contrapositive. Namely, we assume that f is not injective and show that f is not monic. Suppose $f(a) = f(b)$ and $a \neq b$, we want to show such g_i exists. This is easy by considering

$$g_1 : * \mapsto a, \quad g_2 : * \mapsto b.$$

■

0.1.1 Functor

After introducing the category, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

Definition 0.3 (Functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A (covariant) *functor*

$$F : \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F\colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)) \\ [f\colon X \to Y] &\mapsto [F(f)\colon F(X) \to F(Y)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(f) \circ F(g)$

Lecture 7: Functors

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As previously seen. Assume that we initially have a commutative diagram in $\mathscr{C}$ as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

After applying F, we'll have

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)=F(g)\circ F(f)"'] & F(Y) \arrow[d, "F(g)"] \\ & F(Z) \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

We can also have a so-called contravariant functor.

Definition 0.4 (Contravariant functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A contravariant *functor*

$$F\colon \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F\colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X)) \\ [f\colon X \to Y] &\mapsto [F(f)\colon F(Y) \to F(X)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a contravariant functor F, the diagram becomes

$$\begin{tikzcd} F(X) & F(Y) \arrow[l, "F(f)"'] \\ & F(Z) \arrow[u, "F(g)"'] \arrow[ul, "F(g\circ f)=F(f)\circ F(g)"] \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

Example. Let see some examples.

1. Identity functor.

$$I\colon \mathscr{C} \to \mathscr{C}.$$

2. Forgetful functors.

   - $$\begin{aligned} F\colon \underline{\mathrm{Gp}} &\to \underline{\mathrm{set}} \\ G &\mapsto G^{3} \\ [f\colon G \to H] &\mapsto [f\colon G \to H] \end{aligned}$$

   - $$\begin{aligned} F\colon \underline{\mathrm{Top}} &\to \underline{\mathrm{set}} \\ X &\mapsto X^{4} \\ [f\colon X \to Y] &\mapsto [f\colon X \to Y] \end{aligned}$$

[3] G is now just the underlying set of the group G.
[4] X is now just the underlying set of the topological space X.

3. Free functors.

$$\underline{\text{set}} \to \underline{k\text{−vect}}$$
$$s \mapsto \text{"free" } k\text{-vector space on } s$$

i.e., vector space with basis s

$$[f\colon A \to B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

4.

$$\underline{k\text{−vect}} \to \underline{k\text{−vect}}$$
$$V \mapsto V^* = \mathrm{Hom}_k(V, k)$$

If we are working in a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functors.

1.

$$\underline{\text{Top}^*} \to \underline{\text{Gp}}$$
$$(X, x_0) \mapsto \Pi_1(X, x_0)$$

where Π_1 is so-called *fundamental group.*

2.

$$\underline{\text{Top}} \to \underline{\text{Ab}}$$
$$X \mapsto \text{Hp}(X)$$

where Hp is so-called p^{th} *homology.*

Let see the formal definition.

0.2 Free Groups

Definition 0.5 (Free group). Given a set S, the *free group* is a group F_S on S with a map $S \to F_S$ satisfying the universal property.

If G is any group, $f\colon S \to G$ is any map of sets, f extends uniquely to group homomorphism $\overline{f}\colon F_S \to G$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\overline{f}\colon \text{gp hom}"] \\ & G \end{tikzcd}$$

Note. This defines a *natural bijection*

$$\mathrm{Hom}_{\underline{\mathrm{set}}}(S, \mathscr{U}(G)) \cong \mathrm{Hom}_{\underline{\mathrm{Gp}}}(F_S, G),$$

where $\mathscr{U}(G)$ is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are adjoint; specifically that the free functor is the left adjoint (appears on the left in the Hom's above).

> **Definition 0.6 (Adjoints functor).** A free and forgetful functors are *adjoints.*

Remark. Whenever we state a universal property for an object (plus a map), an object (plus a map) may or may not exist. If such object exists, then it defines the object **uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the object (plus a map).

> **Lemma 0.2.** Universal property defines F_S (plus a map $S \to F(S)$) uniquely up to unique isomorphism.

Proof. Fix S. Suppose

$$S \to F_S, \quad S \to \widetilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{tikzcd}
S \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, dashed, "\exists!\varphi"] \\
& F_S
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\psi"] \\
& \widetilde{F}_S
\end{tikzcd}$$

We'll show φ and ψ are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{tikzcd}
& F_S \arrow[dd, "\mathrm{id}_{F_S}"] \\
S \arrow[ur, "f"] \arrow[dr, "f"'] & \\
& F_S
\end{tikzcd}
\qquad
\begin{tikzcd}
& \widetilde{F}_S \arrow[dd, "\mathrm{id}_{\widetilde{F}_S}"] \\
S \arrow[ur, "f"] \arrow[dr, "f"'] & \\
& \widetilde{F}_S
\end{tikzcd}$$

Hence, we see that

$$\begin{tikzcd}
& F_S \arrow[d, "\psi"] \arrow[dd, bend left, "\varphi\circ\psi=\mathrm{id}_{F_S}"] \\
S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, "\varphi"] \\
& F_S
\end{tikzcd}
\qquad
\begin{tikzcd}
& \widetilde{F}_S \arrow[d, "\varphi"] \arrow[dd, bend left, "\psi\circ\varphi=\mathrm{id}_{\widetilde{F}_S}"] \\
S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, "\psi"] \\
& \widetilde{F}_S
\end{tikzcd}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \mathrm{id}_{F_S}, \qquad \psi \circ \varphi = \mathrm{id}_{\widetilde{F}_S},$$

so φ and ψ are inverses (thus group isomorphism). ■

Lecture 8: The Fundamental Group π_1

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Example. In category Ab free Abelian group on a set S is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on S** .

0.2.1 Constructing the Free Groups F_S

Proposition 0.1. The free group defined by the universal property exists.

Proof. We'll just give a construction below. First, we see the definition.

Definition 0.7. Fix a set S, and we define a word as a finite sequence (possibly $\varnothing$) in the formal symbols

$$\left\{ s, s^{-1} \mid s \in S \right\}.$$

Then we see that elements in F_S are equivalence classes of words with the equivalence relation being

- delete ss^{-1} or $s^{-1}s$. i.e.,

  $$vs^{-1}sw \sim vw$$
  $$vss^{-1}w \sim vw$$

  for every word $v, w, s \in S$,

with the group operation being concatenation. ■

Example. Given words ab^{-1}, bba, their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

Exercise. There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that F_S satisfies the universal property with respect to the map

$$S \to F_S, \quad s \mapsto s.$$

1 The Fundamental Group π_1

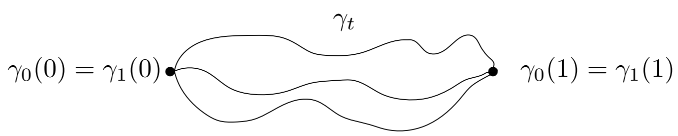
We start with the definition.

Definition 1.1 (Path). A *path* in a space X is a continuous map

$$\gamma \colon I \to X$$

where $I = [0, 1]$.

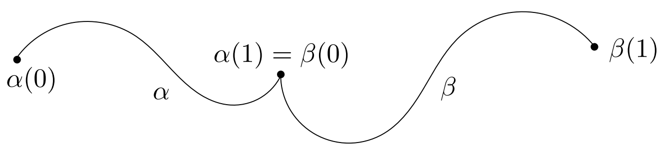
Definition 1.2 (Homotopy path). A *homotopy of paths* γ_0, γ_1 is a homotopy from γ_0 to γ_1 rel$\{0, 1\}$.



Example. Fix $x_1, x_0 \in X$, then $\exists$ homotopy of paths is an equivalence relation on paths from x_0 to x_1 (i.e., γ with $\gamma(0) = x_0, \gamma(1) = x_1$).

Definition 1.3 (Path composition). For paths α, β in X with $\alpha(1) = \beta(0)$, the *composition*[a] $\alpha \cdot \beta$ is

$$(\alpha \cdot \beta)(t) \coloneqq \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \dfrac{1}{2}\right] \\ \beta(2t-1), & \text{if } t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$



[a] Also named *product*, *concatenation*.

Remark. By the pasting lemma, this is continuous, hence $\alpha \cdot \beta$ is actually a path from $\alpha(0)$ to $\beta(1)$.

Definition 1.4 (Reparameterization). Let $\gamma\colon I \to X$ be a path, then a *reparameterization* of γ is a path

$$\gamma'\colon I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where φ is continuous and

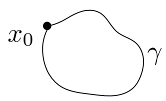
$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

Exercise. A path γ is homotopic rel$\{0,1\}$ to all of its reparameterizations. HW

Exercise. Fix $x_1, x_1 \in X$. Then Homotopy of paths (relative $\{0,1\}$) is an equivalence relation on paths from x_0 to x_1.

Definition 1.5 (Fundamental Group). Let X denotes the space and let $x_0 \in X$ be the base point. The *fundamental group of X based at x_0*, denoted by $\pi_1(X, x_0)$, is a group such that

- Elements: Homotopy classes rel$\{0,1\}$ of paths $[\gamma]$ where γ is a **loop** with $\gamma(0) = \gamma(1) = x_0$[a]

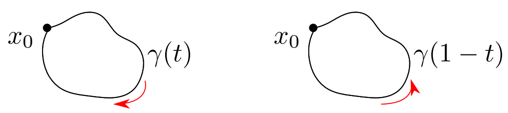


- Operation: Composition of paths.
- Identity: Constant loop γ based at x_0 such that

$$\gamma\colon I \to X, \quad t \mapsto x_0$$

- Inverses: The inverse $[\gamma]^{-1}$ of $[\gamma]$ is represented by the loop $\overline{\gamma}$ such that

$$\overline{\gamma}(t) = \gamma(1-t).$$



[a] We say γ is **based** at x_0.

Proof. We need to prove that the above define a group. ■ HW.

Theorem 1.1. If X is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Remark. We often write $\pi_1(X)$ up to isomorphism.

Proof. ■ HW.

Exercise. Composition of paths is well-defined on homotopy classes rel$\{0, 1\}$.

Exercise. If X is a contractible space, then X is path connected and $\pi_1(X)$ is trivial.

Lecture 9

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Appendix

References

[HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com/books?id=BjKs86kosqgC.