MATH635
Riemannian Geometry

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Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by Lydia Bieri. Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [FC13] as our reference; while not required, but highly recommended have on.

This course is taken in Winter 2023, and the date on the covering page is the last updated time.

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1.1 Differentiable Manifolds

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1.1.1 Topological Manifolds

Let's start with a common definition.

Definition 1.1.1 (Topological manifold). A *topological manifold* $\mathcal{M}$ of dimension n is a (topological) Hausdorff space such that each point $p \in \mathcal{M}$ has a neighborhood U homeomorphic via $\varphi \colon U \to U'$ to an open subset $U' \subseteq \mathbb{R}^n$.

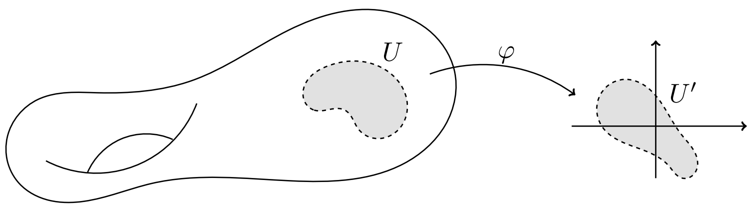
Definition 1.1.2 (Local coordinate map). For every $p \in \mathcal{M}$, the corresponding homeomorphism φ is called the *local coordinate map*.

Definition 1.1.3 (Local coordinate). The pull-back $(x^1, \dots, x^n)$ of the local coordinate map φ from $\mathbb{R}^n$ is called the *local coordinates* on U, given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

Definition 1.1.4 (Coordinate chart). The pair (U, φ) is called a *(coordinate) chart* on M.

In other words, a topological manifold can be thought of as a space such that it looks like $\mathbb{R}^n$ locally.



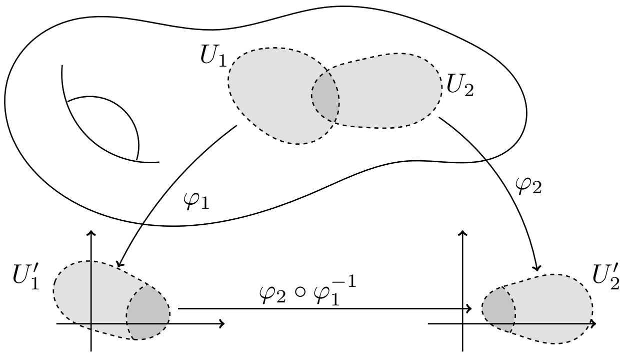
Definition 1.1.5 (Atlas). An *atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ for a manifold $\mathcal{M}$ is a collection of charts such that $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$ are an open covering of $\mathcal{M}$, i.e., $\mathcal{M} = \bigcup_\alpha U_\alpha$.

In other words, for all $p \in \mathcal{M}$, there exists a neighborhood $U \subseteq \mathcal{M}$ and homeomorphism $h \colon U \to U' \subseteq \mathbb{R}^n$ open.

Definition 1.1.6 (Locally finite). An atlas is said to be *locally finite* if each point $p \in \mathcal{M}$ is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as $\mathbb{R}^n$, there's no clear way to make sense of differentiability of a manifold. But thankfully, we now have an explicit relation to the ambient space $\mathbb{R}^n$ via φ_α. To formalize, let $\mathcal{A}$ be an atlas for a manifold $\mathcal{M}$, and assume that $(U_1, \varphi_1), (U_2, \varphi_2)$ are 2 elements of $\mathcal{A}$. Then clearly, the map $\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a homeomorphism between 2 open sets of Euclidean spaces since both φ_1 and φ_2 are homeomorphism. Due to this map's importance, it has its own name.

Definition 1.1.7 (Coordinate transition)**.** The map $\varphi_2 \circ \varphi_1^{-1}$ is called the *coordinate transition* of $\mathcal{A}$ for the pair of charts $(U_1, \varphi_1), (U_2, \varphi_2)$.



1.1.2 Differentiable Structures

Notice that the coordinate transitions are from $\mathbb{R}^n$ to $\mathbb{R}^n$; hence differentiability makes sense now, which induces the following.

Definition 1.1.8 (Differentiable atlas)**.** The atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is *differentiable* if all transitions are differentiable.

Remark. Here, the differentiability depends on the content. Sometimes, we may want it to be C^∞, and sometimes may be C^k for some finite k. On the other hand, smooth always refers to C^∞. We'll use them interchangeably if it's clear which case we're referring to.

Definition 1.1.9 (Equivalence atlas)**.** Two atlases $\mathcal{U}, \mathcal{V}$ of a manifold are equivalent if for every $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$,

$$\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

Notation (Smoothly compatible)**.** Two charts (U, φ) and (V, ψ) are *smoothly compatible* if either $U \cap V = \varnothing$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

This suggests the following.

Definition 1.1.10 (Smooth structure)**.** A *smooth structure* on $\mathcal{M}$ is an equivalence class $\mathcal{U}$ of coordinate atlas with the property that all transition functions are diffeomorphisms.

Remark. We can also use the *maximal* differentiable atlas to be our differentiable structure.

Definition 1.1.11 (Smooth manifold)**.** A *smooth manifold* is a manifold $\mathcal{M}$ with a smooth structure.

In this way, we can do calculus on smooth manifolds! Furthermore, it now makes sense to say that a function $f\colon \mathcal{M} \to \mathbb{R}$ is differentiable (or C^∞) by considering differentiability of $f \circ \varphi^{-1}$ around p.

Notation. The collection of smooth functions on smooth manifold $\mathcal{M}$ is denoted by $C^\infty(\mathcal{M}, \mathbb{R})$, or $C^k(\mathcal{M}, \mathbb{R})$.

Remark. The class $C^\infty(\mathcal{M}, \mathbb{R})$ consists of functions with property is well-defined.

Proof. Let $\mathcal{A}$ be any given atlas from equivalence class that defines the smooth structure, and as we have shown, if $(U, \varphi) \in \mathcal{A}$, then $f \circ \varphi^{-1}$ is a smooth function on $\mathbb{R}^n$. This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.1.9 requirement that defines the equivalent manifolds. ⊛

1.1.3 Orientation

Another essential property of a manifold is its orientability.

Definition. Consider an atlas $\mathcal{A}$ for a differentiable manifold $\mathcal{M}$.

Definition 1.1.12 (Oriented)**.** $\mathcal{A}$ is *oriented* if all transitions have positive functional determinant.

Definition 1.1.13 (Orientable)**.** $\mathcal{M}$ is *orientable* if $\mathcal{A}$ is an oriented atlas.

Motivated by the above definitions, we see that we can actually use an atlas to define an orientation.

Definition 1.1.14 (Orientation)**.** Let $\mathcal{M}$ be an orientable manifold. Then a oriented differentiable structure is called an *orientation* of $\mathcal{M}$.

If $\mathcal{M}$ possesses an orientation, we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with Definition 1.1.12.

Remark. Two differentiable structures obeying Definition 1.1.12 determine the same orientation if the union again satisfying Definition 1.1.12.

Remark. If $\mathcal{M}$ is orientable and connected, then there exists exactly 2 distinct orientations on $\mathcal{M}$.

Now, we can see some examples of smooth manifolds.

Example (Sphere)**.** The sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by

$$S^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1 \right\}.$$

Consider $U_i^+ = \{x \in S^n \mid x_i > 0\}$, $U_i^- = \{x \in S^n \mid x_i < 0\}$ for $i = 1, \ldots, n+1$, and $h_i^\pm \colon U_i^\pm \to \mathbb{R}^n$ such that

$$h_i^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, .., x_{n+1}).$$

Note that the minimum charts needed to cover S^n is 2.

Example. Let $\mathcal{M} = U \subseteq \mathbb{R}^n$, then $\{(U, \varphi)\}$ is a smooth structure with $\varphi = \mathbb{1}$.

Example. Open sets of C^∞-manifolds are C^∞-manifolds.

Example (General linear group). $\mathrm{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, open.

Example (Real projective space). $\mathbb{R}P^n = S^n / \sim$ where $x \sim -x$ with $\pi\colon S^n \to \mathbb{R}P^n$, $x \mapsto [x]$.

Proof. π is a homeomorphism on each U_i^+ for $i = 1, \ldots, n+1$, with

$$\left\{ \left(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1} \right), i = 1, \ldots, n+1 \right\}$$

is a C^∞-atlas for $\mathbb{R}P^n$. ⊛

Note. Observe that $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$.

Lecture 2: Maps Between Smooth Manifolds

1.1.4 Smooth Maps

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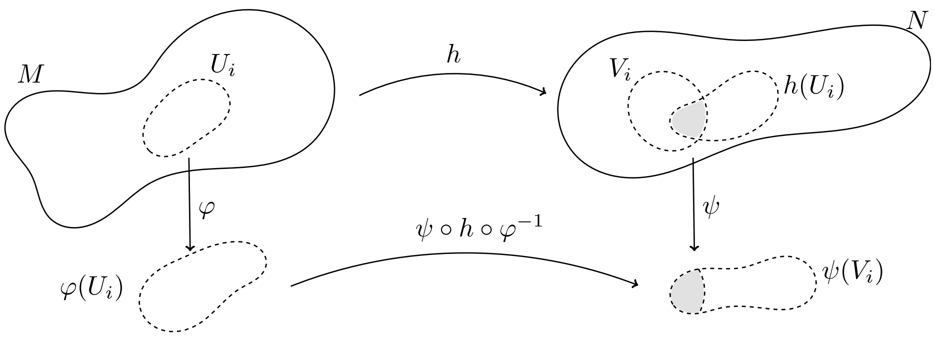
We can now consider the maps between manifolds, specifically, the smooth manifolds.

Definition 1.1.15 (Smooth function). Let M, N be two smooth manifolds, and let $\mathcal{U}$ be locally finite atlas from the equivalence class that gives the smooth structure on M, and let $\mathcal{V}$ be the corresponding for N. A map $h\colon M \to N$ is said to be *smooth* if each map in the collection

$$\left\{ \psi \circ h \circ \varphi^{-1} \colon h(U) \cap V \neq \varnothing \right\},$$

where $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$ is C^∞-differentiable as a map from one Euclidean space to another.



Remark. Equivalence relation guarantees that Definition 1.1.15 depends only on the smooth structure of M, N, but not on the chosen representative coordinate atlas.

Definition. Consider two smooth manifolds M, N and a smooth homeomorphism $h\colon M \to N$ with smooth inverse.

Definition 1.1.16 (Diffeomorphic). The two manifolds M, N are said to be *diffeomorphic*.

Definition 1.1.17 (Diffeomorphism). The map h is said to be a *diffeomorphism*.

Let M_1, M_2 be two smooth manifolds, and let $\varphi\colon M_1 \to M_2$ be a diffeomorphism. Then the following hold.

Check

(a) M_1 is orientable if and only if M_2 is orientable.

(b) If in addition, M_1 and M_2 are both connected and oriented, then φ induces an orientation on M_2 that may or may not coincide with the initial orientation of M_2.

If the induced orientation coincides, then we say φ preserves the orientation, otherwise φ reverses the orientation.

1.1.5 Grassmannian Manifold

Before proceeding, let's consider an interesting smooth manifold.

Definition 1.1.18 (Grassmannian manifold). Given $m, n \in \mathbb{N}$, the so-called *Grassmannian manifold* $G(n, m)$ is the set of all n-dimensional subspaces of $\mathbb{R}^{n+m}$.

Note. $G(1, m)$ is just $\mathbb{R}P^m$, and $G(0, m)$, $G(n, 0)$ are one-point sets.

As we will soon see, $G(n, m)$ has the smooth structure of an mn-dimensional manifold.

Intuition. We obtain the structure by exhibiting an atlas whose transitions are diffeomorphisms.

Firstly, we give $G(n, m)$ a suitable topology, i.e., the metric topology. Let $\Pi \in G(n, m)$, and let $\mathcal{L}(\Pi, \Pi^\perp)$ denote the mn-dimensional space of linear maps from Π to $\Pi^\perp$. Define the map

$$\varphi_\Pi \colon \mathcal{L}(\Pi, \Pi^\perp) \to G(n, m), \qquad \varphi_\Pi(\alpha) = (\mathbb{1}_\Pi \oplus \alpha)(\Pi)$$

where $\mathbb{1}_\Pi \oplus \alpha$ is regarded as a map $\Pi \to \Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.[1] Clearly, φ_Π is injective, and thus, $\left(\mathcal{L}(\Pi, \Pi^\perp), \varphi_\Pi\right)$ is an mn-dimensional chart of $G(n, m)$.

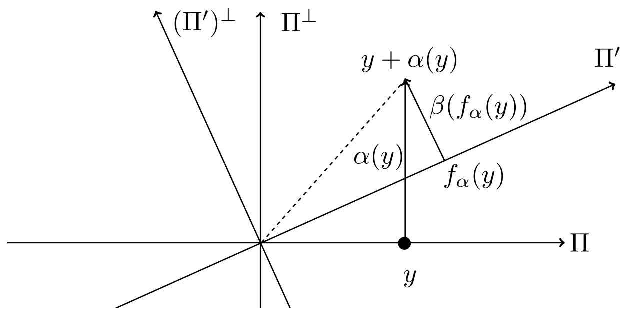
Remark. The images $\varphi_\Pi\left(\mathcal{L}(\Pi, \Pi^\perp)\right)$ cover $G(n, m)$.

Example. $\Pi = \varphi_\Pi(0) \in \varphi_\Pi\left(\mathcal{L}(\Pi, \Pi^\perp)\right)$.

We can now prove that these charts are mutually compatible. Let $\Pi, \Pi' \in G(n, m)$, and let P, P' be orthogonal projections from $\mathbb{R}^{n+m}$ onto Π, Π' respectively. Firstly,

$$F = \varphi_{\Pi'}^{-1}\varphi_\Pi \colon \varphi_\Pi^{-1}\left(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))\right) \to \varphi_{\Pi'}^{-1}\left(\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))\right)$$

is smooth.



Consider $\alpha \in \mathcal{L}(\Pi, \Pi^\perp)$, and $\beta \in \mathcal{L}(\Pi', (\Pi')^\perp)$, then for α, β, the equality $F(\alpha) = \beta$ means that $\varphi_\Pi(\alpha) = \varphi_{\Pi'}(\beta)$. Let $f_\alpha \colon \Pi \to \Pi'$ be defined by

$$f_\alpha = P' \circ (\mathbb{1}_\Pi \oplus \alpha).$$

We need to check

(a) f_α is invertible, and

(b) $\forall y \in \Pi$, $y + \alpha(y) = f_\alpha(y) + \beta(f_\alpha(y))$.

[1] In other words, $\varphi_\Pi(\alpha)$ is the graph of α in $\Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.

Note. The condition that $\det f_\alpha \neq 0$ gives an exact description of the subset

$$\varphi_{\Pi^{-1}}\left(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))\right)$$

of $\mathcal{L}(\Pi, \Pi^\perp)$, which is therefore open.

For β, it is $\left(\mathbb{1}_{\Pi'} \oplus \beta\right) \circ f_\alpha = \mathbb{1}_\Pi \oplus \alpha$, and hence

$$\beta = F(\alpha) = (\mathbb{1}_\Pi \oplus \alpha) \circ f_\alpha^{-1} - \mathbb{1}_{\Pi'}.$$

It follows by the construction that the image of β is contained in $(\Pi')^\perp$.

Remark. We obtain an infinite atlas for $G(n,m)$ with charts labeled by $\Pi \in G(n,m)$. But it's suffices to consider only $\binom{n+m}{n}$ charts corresponding to subspaces Π spanned with n coordinate axes.

1.1.6 Manifolds with Boundary

We first introduce two notions.

Definition 1.1.19 (Closed manifold)**.** A manifold is *closed* if it is compact and without boundary.

Definition 1.1.20 (Open manifold)**.** A manifold is *open* if it has only non-compact components without boundary.

Lemma 1.1.1. If M can be covered by two coordinate neighborhoods V_1, V_2 such that $V_1 \cap V_2$ is connected, then M is orientable.

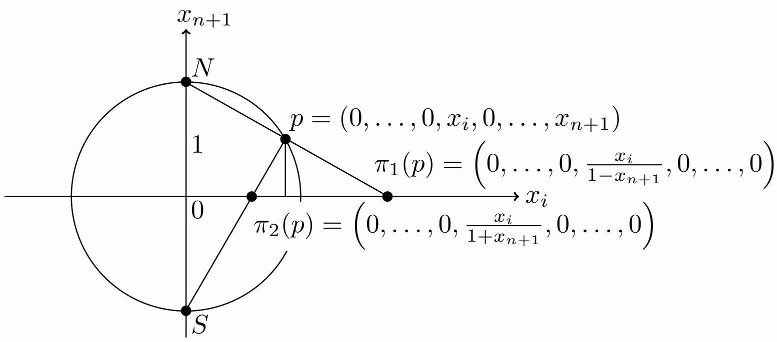
Proof. The determinant of the differential of the coordinate change $\neq 0$, so it does not change sign in $V_1 \cap V_2$. If it's negative at a single point, it's enough to change the sign of one of the coordinates to make it positive at that point, hence on $V_1 \cap V_2$. ■

Example. Let $S^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\right\} \subseteq \mathbb{R}^{n+1}$ is orientable.

Proof. Let $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$, consider given $p = (0, \dots, 0, x_i, 0, \dots, x_{n+1})$, then $\pi_1 \colon S^n \setminus \{N\} \to \mathbb{R}^n$ given by

$$\pi_1(p) = \left(0, \dots, 0, \frac{x_i}{1 - x_{n+1}}, 0, \dots, 0\right)$$

to be the stereographic projection from the north pole N.



More generally, it takes $p(x_1, \dots, x_{n+1}) \in S^n - \{N\}$ into the intersection at the hyperplane

$x_{n+1} = 0$ with the line passing through p ad N. In this way, we have

$$\pi_1(x_1, \ldots, x_n) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right),$$

hence $\pi_1 \colon S^n \setminus \{N\} \to \mathbb{R}^n$ is differentiable, and is injective. Similarly, $\pi_2 \colon S^n \setminus \{S\} \to \mathbb{R}^n$ for S can also be defined and everything holds similarly. We see that these two parametrizations $(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})$ cover S^n. The change of coordinate is given by

$$y_j = \frac{x_j}{1 - x_{n+1}} \leftrightarrow y_j' = \frac{x_j}{1 + x_{n+1}}, \ (y_1, \ldots, y_n) \in \mathbb{R}^n, \ j = 1, \ldots, n,$$

where

$$y_j' = \frac{y_j}{\sum_{i=1}^n y_i^2}.$$

This implies that $\left\{ (\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1}) \right\}$ is a differentiable structure for S^n. Now, consider $\pi_1^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^n) = S^n \setminus \{N \cup S\}$, which is connected, and hence S^n is orientable, and the above structure gives an orientation of S^n. ⊛

Lecture 3: Complex Manifolds, Tangent Spaces and Bundles

Let's look at two more examples about orientation. 12 Jan. 14:30

Example. Let $A \colon S^n \to S^n$ be the antipodal map given by $A(p) = -p$ for $p \in \mathbb{R}^{n+1}$. It's easy to see that A is differentiable with $A^2 = \mathbb{1}$. Furthermore, A is diffeomorphism of $S^n \subseteq \mathbb{R}^{n+1}$. We see that

- if n is even, A reverses the orientation;
- if n is odd, A preserves the orientation.

Example. $G(k, n)$ is orientable if and only if n is even or $n = 1$.

1.1.7 Complex Manifolds

Here we introduce the notion of complex manifold.

Definition 1.1.21 (Complex manifold). A *complex manifold* $\mathcal{M}$ of complex dimension d ($\dim_{\mathbb{C}} \mathcal{M} = d$) is a differentiable manifold of (real) dimension $2d$ ($\dim_{\mathbb{R}} \mathcal{M} = 2d$) whose charts take values in open subsets of $\mathbb{C}^d$ with holomorphic chart transitions.

As previously seen. The chart transitions $z_\beta \circ z_\alpha^{-1} \colon z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ is holomorphic if $\partial z_\beta^j / \partial \overline{z_\alpha^k} = 0$ for all j, k where

$$\frac{\partial}{\partial \overline{z^k}} = \frac{1}{2} \left(\frac{\partial}{\partial \overline{x^k}} + i \frac{\partial}{\partial \overline{y^k}} \right).$$

Remark. Complex Grassmannians $G_{\mathbb{C}}(k, n)$ are all orientable. More generally, complex manifolds are always orientable because holomorphic maps always have positive functional determinant.

1.1.8 Partition of Unity

We state, without proof, of an important lemma about the partition of unity.

Definition 1.1.22 (Partition of unity). Let $\mathcal{M}$ be a differentiable manifold, and let $(U_\alpha)_{\alpha \in \mathcal{A}}$ be an open covering of $\mathcal{M}$. Then a *partition of unity* is a locally finite refinement $(V_\beta)_{\beta \in \mathcal{B}}$ of (U_α) and

C^∞-functions $\varphi_\beta \colon \mathcal{M} \to \mathbb{R}$ with

(a) $\operatorname{supp}(\varphi_\beta) \subseteq V_\beta$ for all $\beta \in \mathcal{B}$;

(b) $0 \leq \varphi_\beta(x) \leq 1$ for all $x \in \mathcal{M}$, $\beta \in \mathcal{B}$;

(c) $\sum_{\beta\in\mathcal{B}} \varphi_\beta = 1$ for all $x \in \mathcal{M}$.[a]

[a] There are only finitely many non-vanishing summands of each point, since only finitely many φ_β are non-zero of any given point as the covering (V_β) is locally finite.

Lemma 1.1.2 (Partition of unity). Let $\mathcal{M}$ be a differentiable manifold, and let $(U_\alpha)_{\alpha\in\mathcal{A}}$ be an open covering of $\mathcal{M}$. Then there exists a partition of unity subordinate to (U_α),

1.2 Tangent Vectors

To discuss the concept of calculus between manifolds formally, we start with our discussion in Euclidean spaces, where we naturally have the coordinates for every point.

Definition 1.2.1 (Tangent space of Euclidean space). Given a d dimensional manifold $\mathcal{M}$, let $x = (x^1, \dots, x^d)$ be Euclidean coordinates of $\mathbb{R}^d$, and $x_0 \in \Omega \subseteq \mathbb{R}^d$ where Ω is open. The *tangent space* $T_{x_0}\Omega$ of Ω at the point x_0 is the vector space $\{x_0\} \times E$[a] spanned by the basis $(\partial/\partial x^1, \dots, \partial/\partial x^d)$.

[a] E is a d-dimensional Euclidean space.

Definition 1.2.2 (Derivative of Euclidean space). If $\Omega \subseteq \mathbb{R}^d$, $\Omega' \subseteq \mathbb{R}^d$ open, and $f \colon \Omega \to \Omega'$ differentiable, then we define the *derivative* $\mathrm{d}f(x_0)$ for $x_0 \in \Omega$ to be the induced linear map between tangent spaces

$$\mathrm{d}f(x_0) \colon T_{x_0}\Omega \to T_{f(x_0)}\Omega', \quad v = v^i \frac{\partial}{\partial x^i} \mapsto v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}.$$

Notation (Einstein notation). The *Einstein notation* abbreviates the summation $\sum_i v^i x_i$ as $v^i x_i$, where we implicitly sum over the upper and lower index.

Definition 1.2.3 (Tangent bundle of Euclidean space). The *tangent bundle* is defined as $T\Omega := \coprod_{x\in\Omega} T_x\Omega \cong \Omega \times E \cong \Omega \times \mathbb{R}^d$, which is an open subset of $\mathbb{R}^d \times \mathbb{R}^d$.

Note (Total space). $T\Omega$ is also called the *total space*.

Remark. Given a tangent bundle $T\Omega$, we define π to be the projection $\pi \colon T\Omega \to \Omega$ given by $\pi(x, v) = x$. This makes $T\Omega$ naturally a differentiable manifold.

We also have $\mathrm{d}f \colon T\Omega \to T\Omega'$ defined by

$$\left(x, v^i \frac{\partial}{\partial x^i}\right) \mapsto \left(f(x), v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}\right).$$

Notation. We often write $\mathrm{d}f(x)(v)$ instead of $\mathrm{d}f(x, v)$.

In particular, $f \colon \Omega \to \mathbb{R}$, the differentiable function for $v = v^i \partial/\partial x^i$, we have

$$\mathrm{d}f(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R},$$

and we write $v(f)(x)$ for $\mathrm{d}f(x)(v)$.

Let $\mathcal{M}$ be a differentiable manifold of dimension d, $p \in \mathcal{M}$. The tangent space of $\mathcal{M}$ at point p. Let $x\colon U \to \mathbb{R}^d$ be a chart with $p \in U \subseteq \mathcal{M}$, open. The tangent space $T_p\mathcal{M}$ is represented in the chart x by $T_{x(p)}x(U)$. Let $x'\colon U' \to \mathbb{R}^d$ to be another chart with $p \in U' \subseteq \mathcal{M}$, open. Denote $\Omega \coloneqq x(U)$, and $\Omega' \coloneqq x'(U')$, then the transition map

$$x' \circ x^{-1}\colon x(U \cap U') \to x'(U \cap U')$$

induces a vector space isomorphism

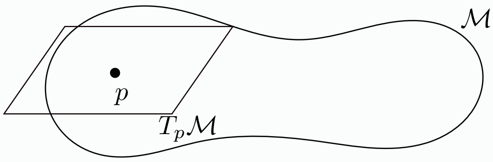
$$L \coloneqq \mathrm{d}(x' \circ x^{-1})(x(p))\colon T_{x(p)}\Omega \to T_{x'(p)}\Omega',$$

such that $v \in T_{x(p)}\Omega$ and $L(v) \in T_{x'(p)}\Omega'$ represent the same tangent vector in $T_p\mathcal{M}$.

Remark. A tangent vector in $T_p\mathcal{M}$ is given by the family of the coordinate representations.

Intuition. Let $f\colon \mathcal{M} \to \mathbb{R}$ be a differentiable function. Assume that the tangent vector $w \in T_p\mathcal{M}$ is represented by $v \in T_{x(p)}x(U)$. We want to define $\mathrm{d}f(p)$ as a linear map from $T_p\mathcal{M} \to \mathbb{R}$. In chart x, let $w \in T_p\mathcal{M}$ given as $v = v^i\partial/\partial x^i \in T_{x(p)}x(U)$. Say that $\mathrm{d}f(p)(w)$ in this chart represented by

$$\mathrm{d}(f \circ x^{-1})(x(p))(v).$$



Remark. $T_p\mathcal{M}$ is a vectors peace of dimension d isomorphic to $\mathbb{R}^d$, where the isomorphism depends on choice of chart.

Remark. Functions on $\mathcal{M}$: pull it back by a chart to an open subset of $\mathbb{R}^d$, differentiate there.

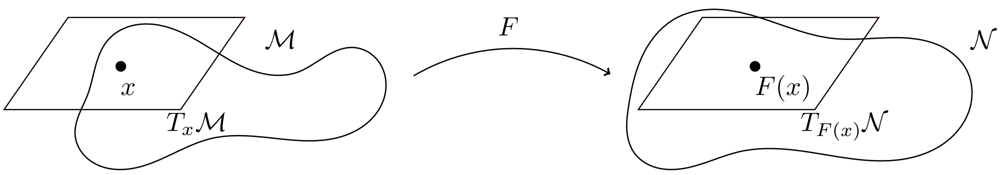
Remark. In order to obtain object not depending on chart, we need to have transformation behavior under chart changes.

Let $F\colon \mathcal{M} \to \mathcal{N}$ be a differentiable map where $\mathcal{M}, \mathcal{N}$ are smooth manifolds. Then we want to represent $\mathrm{d}F$ in local charts $x\colon U \subseteq \mathcal{M} \to \mathbb{R}^d, y\colon V \subseteq \mathcal{N} \to \mathbb{R}^c$ by $\mathrm{d}(y \circ F \circ x^{-1})$. The local coordinates on U is given by $(x^1, \dots, x^d)$, and on V is $(F^1, \dots, F^c)$ such that

$$F(x) = \big(F^1(x^1, \dots, x^d), \dots, F^c(x^1, \dots, x^d)\big).$$

Then, $\mathrm{d}F$ induces linear map $\mathrm{d}F\colon T_p\mathcal{M} \to T_{F(x)}\mathcal{N}$ which in our coordinate representation is given by matrix

$$\left(\frac{\partial F^\alpha}{\partial x^i}\right)_{\alpha=1,\dots,c\mathbb{l}g1,\dots,d}$$



a change of charts is then just the base change at tangent spaces. The transformation behavior: if

$$(x^1, \dots, x^d) \mapsto (\xi^1, \dots, \xi^d)$$
$$(F^1, \dots, F^c) \mapsto (\phi^1, \dots, \phi^c)$$

are coordinate changes, then $\mathrm{d}F$ represented in the new coordinates is given by

$$\left(\frac{\partial \phi^\beta}{\partial \xi^j}\right) = \left(\frac{\partial \phi^\beta}{\partial F^\alpha}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial x^i}{\partial \xi^j}\right).$$

Appendix

Bibliography

[FC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013. ISBN: 9780817634902. URL: https://books.google.com/books?id=ct91XCWkWEUC.