STAT575
Lrage Sample Theory

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Abstract

This is a graduate-level theoretical statistics course taught by Georgios Fellouris at University of Illinois Urbana-Champaign, aiming to provide an introduction to asymptotic analysis of various statistical methods, including weak convergence, Lindeberg-Feller CLT, asymptotic relative efficiency, etc.

We list some references of this course, although we will not follow any particular book page by page: *Asymnptotic Statistics* [Vaa98], *Asymptotic Theory of Statistics and Probability* [Das08], *A course in Large Sample Theory* [Fer17], *Approximation Theorems of Mathematical Statistics* [Ser09], and *Elements of Large-Sample Theory* [Leh04].

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Contents

1 Introduction **2**
1.1 Parametrized Approach . . . 2
1.2 Hypothesis Testing . . . 3

2 Modes of Convergence **4**
2.1 Different Modes of Convergence . . . 4
2.2 Weak Convergence . . . 7
2.3 Stochatsic Boundedness . . . 16

Chapter 1

Introduction

Lecture 1: Introduction to Large Sample Theory

Say we first collect n data points $x_1, \ldots, x_n \in \mathbb{R}^d$, large sample theory concerns with the limiting theory as $n \to \infty$. We may treat x_i as a realization of a random vector X_i on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. In this course, we will primarily consider the case that X_i's are i.i.d., i.e., independent and identically distributed from a distribution function, or the *cumulative density function* (CDF) F such that 16 Jan. 9:30

$$X = (X^1, \ldots, X^d) \sim F(x_1, \ldots, x_d) \equiv \mathbb{P}(X^1 \leq x_1, \ldots, X^d \leq x_d)$$

for all $x_i \in \mathbb{R}$. If we have access to F, we can compute the corresponding *probability density function* (PDF) $\mathbb{P}$, and then have access to $\mathbb{P}(X \in A)$ for all (measurable) $A \subseteq \mathbb{R}^d$ of interest.

Remark. If we know any of the above, we know every thing about the population.

Hence, the goal is to compute this by collecting data x_i's, which is a statistical inference problem.

1.1 Parametrized Approach

There are various ways of doing this task, one way is the so-called parametrized approach. By postulating a family of CDFs $\{F_\theta, \theta \in \Theta\}$ where Θ is often a subset of $\mathbb{R}^m$ for some m (generally $\neq n$), the goal is to select a member of this family that is the "closet", or the "best fit" to the truth, i.e., F, based on the data.

Note. To emphasize that this depends on the data, we sometimes write the function we found as $\hat{\theta}_n(x_1, \ldots, x_n)$ so that $F_{\hat{\theta}_n(x_1, \ldots, x_n)}$ is our proxy for F.

Now, assume that the family is initially given, the problem is then how to select $\hat{\theta}_n$.

Example. Fisher suggested that we should look at the maximum likelihood estimator (MLE).

The justification for MLE is not about finite n, but about its asymptotic behavior when $n \to \infty$. Specifically, we have the following theorem due to Fisher (informally stated).

Theorem 1.1.1 (Fisher). If $F \in \{F_\theta \colon \theta \in \Theta\}$, i.e., if $F = F_{\theta^*}$ for some $\theta^* \in \Theta$, then under certain conditions, $\hat{\theta}_n$ will be "close" to θ^* as $n \to \infty$. Under some other conditions, $\sqrt{n}(\hat{\theta}_n - \theta)$ is approximately Gaussian with variance being the "best possible" in some sense.

On the other hand, in the misspecified case, i.e., $F \notin \{F_\theta, \theta \in \Theta\}$, we can still compute the MLE, which leads to another justification for MLE since even in this case, $\hat{\theta}_n$ will still be "close" to θ^* such that F_{θ^*} is, in some sense, the "closest" to F among all possible F_θ (minimizing divergence, to be precise).

1.2 Hypothesis Testing

We will also develop theory for hypothesis testing for some hypothesis we're interested in, e.g., whether the data we collect is really i.i.d., or whether our proposed family is reasonable enough. Say now X_i's are scalar random variable with $\mathbb{E}[X] = \mu$, and we want to test the null hypothesis $H_0\colon \mu = 0$.

Example. Consider a controlled group Z and a treatment group Y, and we observe $Z_1, \dots, Z_n$, and $Y_1, \dots, Y_n$, respectively, and compute $X_i = Z_i - Y_i$ for all i. Testing H_0 on the distribution of X will show the effect of the treatment.

To do this, a well-known method is the so-called t-test. Let s_n to be the sample standard derivation, then we can compute

$$T_n = \frac{\overline{X}_n}{s_n/\sqrt{n}} \sim t_{n-1}$$

as long as X is Gaussian, i.e., the t-statistics for H_0. What if X is not an Gaussian? We will show that even if X is not Gaussian, this result is "approximately valid" when n is "large enough" as long as $\operatorname{Var}[X] < \infty$.

Remark (Sample Size)**.** When we say n is "large enough", what we mean really depends on how fast the underlying distribution will approach Gaussian as n grows. Hence, if we can say more about the underlying population, we can say more about when does n is "large enough"; otherwise such a limiting theory might be completely useless in practice.

What if now $\operatorname{Var}[X]$ doesn't exit? When the population has a heavy tail distribution, then second moment may not exit.

Example (Cauchy distribution)**.** The Cauchy distribution doesn't have finite moment of order greater than 1.

In this case, some other test is needed. A simple test would be looking at the sign of X_i, i.e., the sign test.

Example (Sign test)**.** We might reject H_0 if $\sum_{i=1}^n \mathbb{1}_{X_i>0}$ is large. Note that under H_0, $\sum_{i=1}^n \mathbb{1}_{X_i>0} \sim \operatorname{Bin}(n, 1/2)$, and this test is valid even if expectation doesn't exist.

We see that without saying anything about F, the sign test is valid even for $n = 3$ or 5 as the sum is exactly binomial distribution under H_0. Although simple and have good property, only looking at the sign of X_i might be too weak. A natural idea is to look at the absolute value of X_i.

Example (Wilcoxon's rank-sum test)**.** Let $R_{i,n}$ to be the rank of $|X_i|$, then consider the so-called *Wilcoxon's rank-sum test*

$$\sum_{i=1}^n \mathbb{1}_{X_i>0} R_{i,n}.$$

As one can imagine, the closed form of the above sum will be complicated; however, asymptotically, the above statics will follow Gaussian again, such that the rate of convergence doesn't depend on the underlying population.

Finally, we also ask how can we compare these different tests? This will also be addressed in this course.

Chapter 2

Modes of Convergence

Lecture 2: Modes of Convergence

18 Jan. 9:30

2.1 Different Modes of Convergence

Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, consider a sequence of d-dimensional random vectors (X_n) and a random vector X, i.e., $X_n, X \colon \Omega \to \mathbb{R}^d$. We now discuss different modes of convergence for (X_n).

Definition 2.1.1 (Point-wise converge). (X_n) *point-wise converges* to X, denoted as $X_n \to X$, if $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega$.[a]

[a] I.e., for every $\epsilon > 0$, there exists $n_0(\omega) \in \mathbb{N}$ such that for every $n \geq n_0$, $\|X_n(\omega) - X(\omega)\|_2 < \epsilon$.

Since we don't care about measure zero sets, we may instead consider the following.

Definition 2.1.2 (Almost-surely converge). (X_n) *almost-surely converges* to X, denoted as $X_n \overset{\text{a.s.}}{\to} X$, if $\mathbb{P}(X_n \to X) = 1$.[a]

[a] I.e., $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega \setminus N$ where $\mathbb{P}(N) = 0$.

However, this might still be too strong.

Definition 2.1.3 (Converge in probability). (X_n) *converges in probability* to X, denoted as $X_n \overset{p}{\to} X$, if for every $\epsilon > 0$, $\mathbb{P}(\|X_n - X\| > \epsilon) \to 0$ as $n \to \infty$.

Remark. $X_n \to X$ if and only if $\|X_n - X\| \to 0$. The same also holds for $\overset{p}{\to}$ and $\overset{\text{a.s.}}{\to}$.

A related notion is the following, where we now sum over n.

Definition 2.1.4 (Converge completely). (X_n) *converges completely* to X, denoted as $X_n \overset{\text{comp}}{\to} X$, if for every $\epsilon > 0$, $\sum_{n=1}^{\infty} \mathbb{P}(\|X_n - X\| > \epsilon) < \infty$.

Finally, we have the following.

Definition 2.1.5 (Converge in L^p). (X_n) *converges in L^p* to X for some $p > 0$, denoted as $X_n \overset{L^p}{\to} X$, if $\mathbb{E}\left[\|X_n - X\|^p\right] \to 0$ as $n \to \infty$.

2.1.1 Connection Between Modes of Convergence

We have the following connections between different modes of convergence.

$$\text{completely} \implies \text{almost-surely} \implies \text{in probability} \impliedby \text{in } L^p$$

To show the above, the following characterization for almost-surely convergence is useful.

Proposition 2.1.1. For a sequence of random vectors (X_n) and a random vector X, we have

$$\begin{aligned} X_n \overset{\text{a.s.}}{\to} X &\Leftrightarrow \mathbb{P}(\|X_k - X\| > \epsilon \text{ for some } k \geq n) \overset{n\to\infty}{\to} 0 \\ &\Leftrightarrow \mathbb{P}(\|X_n - X\| > \epsilon \text{ for infinitely many } n\text{'s}) = 0 \\ &\Leftrightarrow \mathbb{P}(\limsup_{n\to\infty} \|X_n - X\| > \epsilon) = 0, \end{aligned}$$

where the above holds for every $\epsilon > 0$.

From Proposition 2.1.1, it's clear that $\overset{\text{a.s.}}{\to}$ implies $\overset{p}{\to}$ since

$$\mathbb{P}(\|X_k - X\| > \epsilon \text{ for some } k \geq n\ ) \geq \mathbb{P}(\|X_n - X\| > \epsilon),$$

hence if the former goes to 0, so does the latter. On the other hand, $\overset{\text{comp}}{\to}$ implies $\overset{\text{a.s.}}{\to}$ follows from the third equivalence. Lastly, the convergence in L^p implies the convergence in probability since

$$\mathbb{P}(\|X_n - X\| > \epsilon) \leq \frac{1}{\epsilon^p} \mathbb{E}\left[\|X_n - X\|^p\right]$$

from Markov's inequality. However, the converse is not always true.

Theorem 2.1.1 (Dominated convergence theorem). If $X_n \overset{p}{\to} X$ and $\|X_n - X\| \leq Z$ for all $n \geq 1$ where $\mathbb{E}\left[\|Z\|^p\right] < \infty$, then $X_n \overset{L^p}{\to} X$.

Theorem 2.1.2 (Scheffé's theorem). If $X_n \overset{p}{\to} X$ and $\limsup_{n\to\infty} \mathbb{E}\left[\|X_n\|^p\right] \leq \mathbb{E}\left[\|X\|^p\right] < \infty$, then $X_n \overset{L^p}{\to} X$.

2.1.2 Applications to Statistics

Let $(X_n) \overset{\text{i.i.d.}}{\sim} F$ where F is a distribution function. Say we're interested in some aspect of F, for example, some parameter $\theta = T(F) \in \mathbb{R}^m$. By collecting data $X_1, \ldots, X_n$, we estimate θ by computing an estimator $\hat{\theta}_n$ of θ.[1] There are some properties we might want for $\hat{\theta}_n$.

Definition 2.1.6 (Consistent). $\hat{\theta}_n$ is *consistent* of θ if $\hat{\theta}_n \overset{p}{\to} \theta$ as $n \to \infty$.

Definition 2.1.7 (Strongly consistent). $\hat{\theta}_n$ is *strongly consistent* of θ if $\hat{\theta}_n \overset{\text{a.s.}}{\to} \theta$ as $n \to \infty$.

Definition 2.1.8 (Converge in mean squared error). $\hat{\theta}_n$ converges to θ *in mean squared error* if $\hat{\theta}_n \overset{L^2}{\to} \theta$.

Remark. When $d = 1$, $\mathbb{E}[(\hat{\theta}_n - \theta)^2] = \operatorname{Var}[\hat{\theta}_n] + (\mathbb{E}[\hat{\theta}_n - \theta])^2$. Therefore, $\hat{\theta}_n$ converges in mean squared error to θ if and only if $\mathbb{E}[\hat{\theta}_n] \to \theta$ and $\operatorname{Var}[\hat{\theta}_n] \to 0$.

Let's first see the most well-known estimation problem, the mean estimation.

Example (Mean esimation). Suppose $d = 1$, and let X be non-negative. Say we're interested in $\theta = \mathbb{E}\left[X\right]$. It's standard that in this case, we can compute $\mathbb{E}\left[X\right]$ by

$$\theta = \mathbb{E}\left[X\right] = \int_0^\infty \mathbb{P}(X > t)\,\mathrm{d}t = \int_0^\infty (1 - F(t))\,\mathrm{d}t.$$

If X has a PMF f, then $\mathbb{E}\left[X\right] = \sum_x x f(x) = \sum_x x \Delta F(x)$ where $f(x) = \Delta F(x) \equiv F(x) - F(x^-)$;

[1] $\hat{\theta}_n$ is a function of X_i's.

if X has a PDF f, then

$$\mathbb{E}[X] = \int_0^\infty x f(x)\,\mathrm{d}x = \int_0^\infty x F(\mathrm{d}x)$$

where $F(\mathrm{d}x) := f(x)\mathrm{d}x$ in a measure-theoretical sense.

Now, let $\hat{\theta}_n$ to be the sample mean, i.e., $\hat{\theta}_n = \overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. From the strong law of large number, $\overline{X}_n \overset{\text{a.s.}}{\to} \mathbb{E}[X]$, which implies that $\hat{\theta}_n$ is a strongly consistent estimator of θ.

On the other hand, if $\mathrm{Var}[X] < \infty$, then $\overline{X}_n \overset{L^2}{\to} \mathbb{E}[X]$, which further implies $\overline{X}_n \overset{p}{\to} \mathbb{E}[X]$, hence $\hat{\theta}_n$ is consistent.[a]

[a] The latter is true even without $\mathrm{Var}[X] = \infty$ as we expect.

Proof. We show the last statement. Since $\mathrm{Var}[X] < \infty$, then

$$\frac{\mathrm{Var}[X]}{n} = \mathrm{Var}\left[\overline{X}_n\right] = \mathbb{E}\left[\left(\overline{X} - \mathbb{E}[X]\right)^2\right] \to 0$$

as $n \to \infty$, which implies $\overline{X}_n \overset{p}{\to} \mathbb{E}[X]$. ⊛

Another interesting problem is the supremum estimation.

Example (Supremum estimation). Suppose there is a $\theta \in \mathbb{R}$ where distribution function F such that $F(\theta - \epsilon) < 1 = F(\theta)$ for all $\epsilon > 0$, i.e., $\theta = \sup_\omega X(\omega)$ since $\mathbb{P}(X \le \theta - \epsilon) = F(\theta - \epsilon)$ and $F(\theta) = \mathbb{P}(X \le \theta)$.[a] Then $\hat{\theta}_n = \max_{1 \le i \le n} X_i$ is indeed a strongly consistent estimator of θ.

[a] Such a distribution exists, for example, $\mathcal{U}(0, \theta)$.

Proof. We see that for any $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}(|\hat{\theta}_n - \theta| > \epsilon) &= \mathbb{P}(\hat{\theta}_n > \theta + \epsilon) + \mathbb{P}(\hat{\theta}_n < \theta - \epsilon) \\
&= \mathbb{P}\left(\bigcup_{i=1}^n \{X_i > \theta + \epsilon\}\right) + \mathbb{P}\left(\bigcap_{i=1}^n \{X_i < \theta - \epsilon\}\right) \\
&\le \sum_{i=1}^n \underbrace{\mathbb{P}(X > \theta + \epsilon)}_{0} + \prod_{i=1}^n \mathbb{P}(X_i < \theta - \epsilon) = (\mathbb{P}(X_1 < \theta - \epsilon))^n \le (F(\theta - \epsilon))^n \to 0
\end{aligned}$$

as $n \to \infty$ since $F(\theta - \epsilon) < 1$. This shows that $\hat{\theta}_n$ is indeed consistent. Moreover, since $\mathbb{P}(|\hat{\theta}_n - \theta| > \epsilon)$ decays exponentially, so this is absolutely summable, hence it's also strongly consistency. ⊛

Proving convergence of $\hat{\theta}_n$ is useful, but this might not be enough.

Example. Consider any deterministic sequence (a_n) in $\mathbb{R}$ which converges to 0. Adding a_n to $\hat{\theta}_n$ will not change the convergence of $\hat{\theta}_n$.

The above suggests that we should look at the *distribution* of $\hat{\theta}_n - \theta$ in order to say how does $\hat{\theta}_n \to \theta$.

Example (Mean estimation for Gaussian). Suppose $X \sim \mathcal{N}(\theta, 1)$. Then $\hat{\theta}_n = \overline{X}_n \sim \mathcal{N}(\theta, 1/n)$, i.e., $\sqrt{n}(\hat{\theta}_n - \theta) \sim \mathcal{N}(0, 1)$. This implies that we can write down a confidence interval (CI) such that $\hat{\theta}_n \pm 1.96/\sqrt{n}$ with 95% CI for $\hat{\theta}_n$.

Doing this for other kind of estimators and F is not that straightforward and will be challenging.

Remark. Let (X_n) and X be d-dimensional random vectors, $h\colon \mathbb{R}^d \to \mathbb{R}^m$, and $c \in \mathbb{R}^d$ constant.

(a) If $X_n \to c$, then $h(X_n) \to h(c)$ if h is continuous at c.[a] This also holds for $\overset{\text{a.s.}}{\to}$ and $\overset{p}{\to}$.

(b) If $X_n \to X$, then $h(X_n) \to h(X)$ if h is continuous. This also holds for $\overset{\text{a.s.}}{\to}$ and $\overset{p}{\to}$.

[a] This is an if and only if condition if this holds for any h.

Let's see some examples.

Example. If $d = 1$, and $X_n \to \theta \neq 0$. Then $1/X_n \to 1/\theta$ where

$$h(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x \neq 0; \\ c, & \text{if } x = 0 \end{cases}$$

for any $c \in \mathbb{R}$. The same holds for $\overset{\text{a.s.}}{\to}$ and $\overset{p}{\to}$.

Example. If $X_n \to X$ and $Y_n \to Y$, then $(X_n Y_n) \to (X, Y)$.[a] The same holds for $\overset{\text{a.s.}}{\to}$ and $\overset{p}{\to}$.

[a] The converse is also true since projections are continuous.

Proof. $\|(X_n, Y_n) - (X, Y)\| \to 0$ since $\|(X_n, Y_n) - (X, Y)\| \leq \|X_n - X\| + \|Y_n - Y\|$ for all $n \geq 1$.[a] The latter two terms goes to 0 (in whatever sense) by assumption. ⊛

[a] This can be seen from $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.

Lecture 3: Weak Convergence Portmanteau theorem

2.2 Weak Convergence

25 Jan. 9:30

All convergences we have discussed are in some senses "point-wise" but not "distribution-wise", and the latter is more powerful. Consider working with a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and the following.

Definition 2.2.1 (Total variation). The *total variation* distance between X and Y in Ω is defined as

$$\operatorname{TV}(X, Y) = \sup_{B \in \mathscr{F}} |\mathbb{P}(X \in B) - \mathbb{P}(Y \in B)|$$

Returning to our situation, consider a sequence or random variables (X_n) and a random variable X.

Remark. If X_n has density f_n and X has density f, then $\operatorname{TV}(X_n, X) = \frac{1}{2}\int |f_n - f|$.

Definition 2.2.2 (Convergence in total variation). (X_n) *converges in total variation* to X, denoted as $X_n \overset{\text{TV}}{\to} X$, if $\operatorname{TV}(X_n, X) \to 0$ as $n \to \infty$.

Remark. If X_n and X have densities f_n and f, $f_n \to f$ implies $X_n \overset{\text{TV}}{\to} X$ from Scheffé's theorem.

Note. The above could make sense even if X_n is defined on different $(\Omega_n, \mathscr{F}_n, \mathbb{P}_n)$ for every n.

Let's see some examples.

Example. Consider $X_n \sim \operatorname{Bin}(n, p_n)$ such that $np_n \to \lambda \in \mathbb{R}$. As this happens,

$$X_n \sim \operatorname{Bin}(n, p_n) \overset{\text{TV}}{\to} X \sim \operatorname{Pois}(\lambda).$$

Example. Let $X_n \sim f_{\theta_n}$ where $f_\theta(x) = f(x)e^{\theta x - \psi(\theta)}$ for some $\theta \in \Theta$. If $\theta_n \to \theta$, then $X_n \overset{\text{TV}}{\to} X \sim f_\theta$. For example, if $X_n \sim \operatorname{Pois}(\theta_n)$ and $\theta_n \to \theta$, then $X_n \overset{\text{TV}}{\to} X \sim \operatorname{Pois}(\theta)$.

However, convergence in total variation might be too strong to work with.

Example. Let $X_n \sim \mathcal{U}\{0, 1/n, \dots, (n-1)/n\}$, which should be converging to $X \sim \mathcal{U}(0,1)$. However, this doesn't happen in total variation distance as we can take B to be $\mathbb{Q}$.

This suggests that we should look at something weaker.

Definition 2.2.3 (Weak convergence)**.** (X_n) *converges weakly* to X, denoted as $X_n \overset{w}{\to} X$, if for all bounded continuous $g \colon \mathbb{R}^d \to \mathbb{R}$, $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$.

To see how is weak convergence compared to convergence in total variation, we revisit the above.

Example. Let $X_n \sim \mathcal{U}\{0, 1/n, \dots, (n-1)/n\}$, which should be converging to $X \sim \mathcal{U}(0,1)$. We have

$$\mathbb{E}[g(X_n)] = \sum_{k=0}^{n-1} g(k/n) \left(\frac{k+1}{n} - \frac{k}{n} \right) \to \int_0^1 g(x) \, \mathrm{d}x = \mathbb{E}[g(X)]$$

as g is bounded and continuous on $[0,1]$, hence Riemann integrable.

2.2.1 Portmanteau Theorem

The following is our main tool of proving weak convergence.

Theorem 2.2.1 (Portmanteau theorem)**.** The following are equivalent.

(a) $X_n \overset{w}{\to} X$.

(b) $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$ for all bounded Lipschitz $g \colon \mathbb{R}^d \to \mathbb{R}$.

(c) $\mathbb{P}(X \in A) \leq \liminf_{n\to\infty} \mathbb{P}(X_n \in A)$ for all $A \subseteq \mathbb{R}^d$ open.

(d) $\mathbb{P}(X \in A) \geq \limsup_{n\to\infty} \mathbb{P}(X_n \in A)$ for all $A \subseteq \mathbb{R}^d$ closed.

(e) $\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A)$ for all A such that $\mathbb{P}(X \in \partial A) = 0$.

Before we prove Portmanteau theorem, we should note that all our discussion can be extended to metric spaces from Euclidean spaces. Let's first recall some basic results for metric spaces.

Claim. Given a metric space (S, ρ), $\rho(\cdot, A)$ is Lipschitz for any $A \subseteq S$, i.e., for any $x, y \in S$,

$$|\rho(x, A) - \rho(y, A)| \leq \rho(x, y).$$

Proof. Since for any $z \in S$, $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$, hence $\rho(x,A) - \rho(y,A) \leq \rho(x,y)$ by taking the infimum over $z \in A$. Interchanging x and y gives another inequality. ⊛

Claim. Given a metric space (S, ρ), for any $A \subseteq S$, $x \in \overline{A} \Leftrightarrow \rho(x, A) = 0$.

Proof. If $x \in \overline{A}$, there exists (x_n) in A such that $\rho(x_n, x) \to 0$. Then for any $z \in A$, $\rho(x,z) \leq \rho(x, x_n) + \rho(x_n, z)$, implying

$$\rho(x, A) \leq \rho(x, x_n) + \rho(x_n, A) \to 0,$$

hence $\rho(x, A) = 0$. On the other hand, suppose $\rho(x, A) = 0$. As $\rho(x,A) = \inf_{y\in A} \rho(x,y)$, there exists (y_n) in A such that $\rho(x, y_n) \to \rho(x, A) = 0$, i.e., $x \in \overline{A}$. ⊛

The crucial lemma we're going to use to prove Portmanteau theorem is the following.

Lemma 2.2.1. Given a metric space (S, ρ) and let $A \subseteq S$ be a closed subset. Then there exists bounded Lipschitz $g_k \colon S \to \mathbb{R}$, decreasing in k such that $g_k(x) \searrow \mathbb{1}_A(x)$.

Proof. Since A is closed, $A = \overline{A}$ and

$$\mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x \in A \Leftrightarrow \rho(x, A) = 0; \\ 0, & \text{if } x \notin A \Leftrightarrow \rho(x, A) > 0. \end{cases}$$

Now, we let

$$g_k(x) = \begin{cases} 0, & \text{if } \rho(x, A) > \dfrac{1}{k}; \\ 1 - k\rho(x, A), & \text{otherwise;} \end{cases} = 1 - (k\rho(x, A) \wedge 1).$$

We see that

- if $x \in A$: $\mathbb{1}_A(x) = 1$, and $g_k(x) = 1$ since $\rho(x, A) = 0$;
- if $x \notin A$: $\mathbb{1}_A(x) = 0$, and $\rho(x, A) > 0$ since A closed, and $g_k(x) = 0$ for all large enough k.

Finally, it's clear that $g_k(x)$ takes values in $[0, 1]$, and we now show it's Lipschitz. We have

$$|g_k(x) - g_k(y)| = |(k\rho(x, A) \wedge 1) - (k\rho(y, A) \wedge 1)| \leq k\rho(x, y)$$

for all $x, y \in S$. ■

Then we can prove the Portmanteau theorem.

Proof of Theorem 2.2.1. (a) $\Rightarrow$ (b) is clear. And we start by proving (c) $\Leftrightarrow$ (d).

Claim. (c) $\Leftrightarrow$ (d).

Proof. We first prove that (d) $\Rightarrow$ (c). Since when A is open,

$$\begin{aligned} \mathbb{P}(X \in A) = 1 - \mathbb{P}(X \in A^c) &\leq 1 - \limsup_{n\to\infty} \mathbb{P}(X_n \in A^c) \qquad \text{(d)} \\ &= 1 - \limsup_{n\to\infty} (1 - \mathbb{P}(X_n \in A)) = \liminf_{n\to\infty} \mathbb{P}(X_n \in A). \end{aligned}$$

(c) $\Rightarrow$ (d) is exactly the same, hence (c) $\Leftrightarrow$ (d). ⊛

Next, we prove (b) $\Rightarrow$ (d), which gives us (a) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Leftrightarrow$ (c).

Claim. (b) $\Rightarrow$ (d).

Proof. From Lemma 2.2.1, there exists bounded Lipschitz $g_k \searrow \mathbb{1}_A$ such that for all closed A,

$$\mathbb{P}(X_n \in A) = \mathbb{E}\left[\mathbb{1}_A(X_n)\right] \leq \mathbb{E}\left[g_k(X_n)\right].$$

This is true for every k and n since $g_k \geq \mathbb{1}_A$, and by taking the limit as $n \to \infty$,

$$\limsup_{n\to\infty} \mathbb{P}(X_n \in A) \leq \limsup_{n\to\infty} \mathbb{E}\left[g_k(X_n)\right] = \mathbb{E}\left[g_k(X)\right]$$

from our assumption (b). Finally, as $k \to \infty$, it goes to $\mathbb{E}\left[\mathbb{1}_A(X)\right] = \mathbb{P}(X \in A)$ as desired. ⊛

The proof will be continued...

Lecture 4: Continuous Mapping Theorem

Before finishing the proof of Portmanteau theorem, we need one additional tool. 30 Jan. 9:30

Lemma 2.2.2. If $\{A_i\}_{i\in I}$ are pairwise disjoint events, then $\{i \in I : \mathbb{P}(A_i) > 0\}$ is countable.[a]

[a] Note that I can be uncountable.

Proof. Since we can write

$$\{i \in I \colon \mathbb{P}(A_i) > 0\} = \bigcup_{k=1}^{\infty} \left\{ i \in I \colon \mathbb{P}(A_i) \geq \frac{1}{k} \right\} =: \bigcup_{k=1}^{\infty} I_k,$$

hence it suffices to show $|I_k| < \infty$ for any $k \geq 1$. Indeed, for any k, $|I_k| \leq k$. Suppose not. Then there exists a countable $J_k \subseteq I_k$ such that $|J_k| > k$, implying

$$\mathbb{P}\left(\bigcup_{i \in J_k} A_i \right) = \sum_{i \in J_k} \mathbb{P}(A_i) \geq \frac{|J_k|}{k} > 1,$$

which is a contradiction. ∎

We now finish the proof of Portmanteau theorem.

Proof of Theorem 2.2.1 (cont.) We already proved (a) ⇒ (b) ⇒ (d) ⇔ (c).

Claim. (c) + (d) ⇒ (e).

Proof. We see that for any A, $A^o \subseteq A \subseteq \overline{A}$, and from (c),

$$\begin{aligned}\mathbb{P}(X \in A^o) &\leq \liminf_{n \to \infty} \mathbb{P}(X_n \in A^o) \leq \liminf_{n \to \infty} \mathbb{P}(X_n \in A) \\ &\leq \limsup_{n \to \infty} \mathbb{P}(X_n \in A) \leq \limsup_{n \to \infty} \mathbb{P}(X_n \in \overline{A}) \leq \mathbb{P}(X \in \overline{A})\end{aligned}$$

where the last step follows from (d). Finally, since

$$\mathbb{P}(X \in \overline{A}) - \mathbb{P}(X \in A^o) = \mathbb{P}(\{X \in \overline{A}\} \setminus \{X \in A^o\}) = \mathbb{P}(X \in (\overline{A} \setminus A^o)) = \mathbb{P}(X \in \partial A),$$

which is 0 by our assumption, i.e., inequalities above are all equalities. In particular, since

$$\liminf_{n \to \infty} \mathbb{P}(X_n \in A) \leq \lim_{n \to \infty} \mathbb{P}(X_n \in A) \leq \limsup_{n \to \infty} \mathbb{P}(X_n \in A)$$

and $\mathbb{P}(X \in A^o) \leq \mathbb{P}(X \in A) \leq \mathbb{P}(X \in \overline{A})$, $\mathbb{P}(X \in A) = \lim_{n \to \infty} \mathbb{P}(X_n \in A)$. ⊛

Finally, we prove the following.

Claim. (e) ⇒ (a).

Proof. For every $g \colon \mathbb{R}^d \to \mathbb{R}$ bounded and continuous, we want to show $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$. Suppose $g \geq 0$,[a] and let $K \geq g(x)$ for every $x \in \mathbb{R}^d$ (which exists since g is bounded), then

$$\mathbb{E}[g(X_n)] = \int_0^K \mathbb{P}(g(X_n) > t) \,\mathrm{d}t, \quad \mathbb{E}[g(X)] = \int_0^K \mathbb{P}(g(X) > t) \,\mathrm{d}t,$$

so we just need to prove the convergence of the above two integrals. From bounded convergence theorem, it suffices to show that for almost every $t \in [0, K]$,

$$\mathbb{P}(g(X_n) > t) \to \mathbb{P}(g(X) > t).$$

Observe that $\mathbb{P}(g(X_n) > t) = \mathbb{P}(X_n \in \{g > t\})$ and $\mathbb{P}(g(X) > t) = \mathbb{P}(X \in \{g > t\})$, so from (e) with $A := \{g > t\}$, it suffices to show $\mathbb{P}(X \in \partial\{g > t\}) = 0$ for almost all t. Firstly,

$$\mathbb{P}(X \in \partial\{g > t\}) = \mathbb{P}(X \in \overline{\{g > t\}} \setminus \{g > t\}^o) = \mathbb{P}(X \in \overline{\{g \geq t\}} \setminus \{g > t\}) = \mathbb{P}(g(X) = t).$$

Moreover, consider the events $\{g(X) = t\}_{t \in [0,K]}$, which are pairwise disjoint, hence Lemma 2.2.2 implies $\mathbb{P}(g(X) = t) = 0$ for all but countably many t's, exactly what we want to show. ⊛

[a] Otherwise, we consider $g = g^+ - g^-$ where $g^+ = \max(g, 0)$ and $g^- = \max(-g, 0)$, and everything follows.

This finishes the proof. ∎

2.2.2 Continuous Mapping Theorem

A common scenario is that given a nice function h (in terms of continuity), if $X_n \xrightarrow{w} X$, we want to know when will $h(X_n) \xrightarrow{w} h(X)$. To develop the theorem of this, we need some more facts about metric spaces.

As previously seen. Given two metric spaces (S, ρ), (S', ρ'), $g\colon S \to S'$ is continuous if $x_n \xrightarrow{\rho} x$ implies $g(x_n) \xrightarrow{\rho'} g(x)$, or for open $A \subseteq S'$, $g^{-1}(A) \subseteq S$ is open.

Notation. We sometimes write $g^{-1}(A) =: \{g \in A\}$.

It's clear that the following holds.

Note. If $g\colon S \to S'$ is continuous and $A \subseteq S'$ is closed, then $\overline{\{g \in A\}} = \{g \in \overline{A}\}$.

However, when g is not continuous and A is not closed, the situation is a bit more complicated. But at least we can first look at the set where g is continuous.

Notation (Continuous set)**.** For any $g\colon S \to S'$, we denote the *continuous set* as $C_g := \{x \in S \colon g \text{ is continuous at } x\}$.

Then we have the following.

Proposition 2.2.1. Given $g\colon S \to S'$ between metric spaces and $A \subseteq S'$,

$$C_g \cap \overline{\{g \in A\}} \subseteq \{g \in \overline{A}\}.$$

Proof. Let $x \in C_g \cap \overline{\{g \in A\}}$. Since $x \in \overline{\{g \in A\}}$, there exists $(x_n) \in \{g \in A\}$ such that $x_n \xrightarrow{\rho} x$. Moreover, $x \in C_g$ implies g is continuous at x, hence $g(x_n) \xrightarrow{\rho'} g(x)$, i.e., $g(x) \in \overline{A}$. ■

This allows us to prove the following theorem, which answers our main question in this section.

Theorem 2.2.2 (Continuous mapping theorem)**.** Consider $X_n \xrightarrow{w} X$ and $h\colon \mathbb{R}^d \to \mathbb{R}^m$. If $\mathbb{P}(X \in C_h) = 1$, then $h(X_n) \xrightarrow{w} h(X)$.

Proof. Let $A \subseteq \mathbb{R}^m$ be a closed set. Then from Portmanteau theorem (d), we need to show

$$\limsup_{n\to\infty} \mathbb{P}(h(X_n) \in A) \leq \mathbb{P}(h(X) \in A).$$

Since $\limsup_{n\to\infty} \mathbb{P}(h(X_n) \in A) = \limsup_{n\to\infty} \mathbb{P}(X_n \in \{h \in A\})$, implying

$$\limsup_{n\to\infty} \mathbb{P}(h(X_n) \in A) \leq \limsup_{n\to\infty} \mathbb{P}(X_n \in \overline{\{h \in A\}}) \leq \mathbb{P}(X \in \overline{\{h \in A\}}),$$

where the last inequality follows again from Portmanteau theorem (d) since $\overline{\{h \in A\}}$ is clearly closed and $X_n \xrightarrow{w} X$. Finally, as $\mathbb{P}(X \in C_h) = 1$,

$$\mathbb{P}(X \in \overline{\{h \in A\}}) = \mathbb{P}(X \in \overline{\{h \in A\}} \cap C_h) \leq \mathbb{P}(X \in \{h \in \overline{A}\})$$

from Proposition 2.2.1, i.e.,

$$\limsup_{n\to\infty} \mathbb{P}(h(X_n) \in A) \leq \mathbb{P}(X \in \{h \in \overline{A}\}) = \mathbb{P}(X \in \{h \in A\}) = \mathbb{P}(h(X) \in A)$$

since A is closed, hence we're done. ■

Example. Let $d = 1$ and $X_n \xrightarrow{w} X$ where X is continuous. Then $1/X_n \xrightarrow{w} 1/X$ and $X_n^2 \xrightarrow{w} X^2$.

Proof. For the case of $X^2 \xrightarrow{w} X^2$, continuous mapping theorem clearly applies with $h(x) = x^2$. For the first case, consider

$$h(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

This means $C_h = \mathbb{R} \setminus \{0\}$. Then, we just need to show $\mathbb{P}(X \in C_h) = 1$ and apply continuous mapping theorem. Observe that this is the same as asking $\mathbb{P}(X = 0) = 0$, which is true when X is continuous.[a] ⊛

[a] Even if X is not continuous, as long as this is true we can conclude the same thing.

Another useful theorem for proving weak convergence is the following.

Theorem 2.2.3 (Converging together). Let $X_n \xrightarrow{w} X$, and if Y_n on the same probability space as X_n such that $\|X_n - Y_n\| \xrightarrow{p} 0$, i.e., for all $\epsilon > 0$, $\mathbb{P}(\|X_n - Y_n\| > \epsilon) \to 0$ as $n \to \infty$. Then, $Y_n \xrightarrow{w} X$.

We first see some applications.

Corollary 2.2.1. If $Y_n \xrightarrow{p} X$, then $Y_n \xrightarrow{w} X$. The converse holds as long as $\mathbb{P}(X = c) = 1$ for some constant c.

Proof. By considering $X_n = X$ for all n, Theorem 2.2.3 implies that if $Y_n \xrightarrow{p} X$, $Y_n \xrightarrow{w} X$. Conversely, if $Y_n \xrightarrow{w} c$, from Portmanteau theorem (c), for any fixed $\epsilon > 0$,[a]

$$\underbrace{\mathbb{P}(c \in B(c, \epsilon))}_{1} \leq \liminf_{n\to\infty} \mathbb{P}(Y_n \in B(c, \epsilon)),$$

implying $\mathbb{P}(Y_n \in B(c, \epsilon)) \to 1$, i.e., $\mathbb{P}(\|Y_n - c\| < \epsilon) \to 1$. ■

[a] Recall that $B(c, \epsilon)$ is the open ball centered at c with radius ϵ.

Lecture 5: Convergence in Distribution and Weak Convergence

Now we prove Theorem 2.2.3. 1 Feb. 9:30

Proof. From Portmanteau theorem (b), we want to prove that $\mathbb{E}[g(Y_n)] \to \mathbb{E}[g(X)]$ for all bounded and Lipschitz $g\colon \mathbb{R}^d \to \mathbb{R}$. Specifically, let $|g(x)| \leq C$ for all $x \in \mathbb{R}^d$ and $|g(x) - g(y)| \leq K\|x - y\|$ for all $x, y \in \mathbb{R}^d$. From triangle inequality,

$$|\mathbb{E}[g(Y_n)] - \mathbb{E}[g(X)]| \leq |\mathbb{E}[g(Y_n)] - \mathbb{E}[g(X_n)]| + |\mathbb{E}[g(X_n)] - \mathbb{E}[g(X)]|.$$

Since $X_n \xrightarrow{w} X$, the second term goes to 0. As for the first term, since Y_n and X_n are in the same probability space, we see that

$$\begin{aligned}
|\mathbb{E}[g(Y_n)] - \mathbb{E}[g(X_n)]| &= |\mathbb{E}[g(Y_n) - g(X_n)]| \\
&\leq \mathbb{E}[|g(Y_n) - g(X_n)|] \\
&= \mathbb{E}\left[|g(Y_n) - g(X_n)| \cdot \mathbb{1}_{\|X_n - Y_n\| > \epsilon}\right] + \mathbb{E}\left[|g(Y_n) - g(X_n)| \cdot \mathbb{1}_{\|X_n - Y_n\| \leq \epsilon}\right] \\
&\leq 2C\mathbb{P}(\|X_n - Y_n\| > \epsilon) + K\epsilon\mathbb{P}(\|X_n - Y_n\| \leq \epsilon) \\
&\leq 2C\mathbb{P}(\|X_n - Y_n\| > \epsilon) + K\epsilon.
\end{aligned}$$

As $n \to \infty$, we finally have

$$\limsup_{n\to\infty} |\mathbb{E}[g(Y_n)] - \mathbb{E}[g(X)]| \leq K\epsilon$$

for all $\epsilon > 0$, by letting $\epsilon \to 0$, we're done. ■

We can now apply Theorem 2.2.3 to prove something similar as we have seen before in the case of convergence in probability.

As previously seen. $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ if and only if $(X_n, Y_n) \xrightarrow{p} (X, Y)$.

Now, in the case of weak convergence, from continuous mapping theorem, we see that if $(X_n, Y_n) \xrightarrow{w} (X, Y)$, then $X_n \xrightarrow{w} X$ and $Y_n \xrightarrow{w} Y$. However, the converse needs not be true.

Example. Consider a random variable X on $(\Omega, \mathscr{F}, \mathbb{P})$, and let $X_n = X$, $Y_n = -X$ for all $n \geq 1$. If $X \sim \mathcal{N}(0, 1)$, we see that $\mathbb{P}(X \in A) = \mathbb{P}(-X \in A)$ for all $A \subseteq \mathbb{R}^d$, implying $X_n \xrightarrow{w} X$ and $Y_n \xrightarrow{w} X$.

However, this does not imply $(X_n, Y_n) \xrightarrow{w} (X, X)$ since otherwise, by continuous mapping theorem, $X_n + Y_n \xrightarrow{w} X + X = 2X$, which is not true since $X_n + Y_n = 0$.

But in the case of Y is a constant, the converse is actually true, and the result is quite useful.

Theorem 2.2.4 (Slutsky's theorem)**.** If $X_n \xrightarrow{w} X$ in $\mathbb{R}^d$ and $Y_n \xrightarrow{p} c$ in $\mathbb{R}^m$,[a] then $(X_n, Y_n) \xrightarrow{w} (X, c)$.

[a] Recall that from Corollary 2.2.1, for a constant c, weak convergence is equivalent to convergence in probability.

Proof. Firstly, we show that $(X_n, c) \xrightarrow{w} (X, c)$. Indeed, since for every continuous and bounded $g\colon \mathbb{R}^{d+m} \to \mathbb{R}$, $\mathbb{E}\left[g(X_n, c)\right] \to \mathbb{E}\left[g(X, c)\right]$ follows directly from $X_n \xrightarrow{w} X$ with $g(\cdot, c)$ being continuous and bounded.

Secondly, we show that $\|(X_n, Y_n) - (X_n, c)\| \xrightarrow{p} 0$. This is easy since

$$\|(X_n, Y_n) - (X_n, c)\| \leq \|X_n - X_n\| + \|Y_n - c\| = \|Y_n - c\|,$$

which goes to 0 in probability as we wish. Combining both with Theorem 2.2.3 gives the result. ∎

Revisiting the counter-example, we see that now it's not the case when Y is a constant.

Corollary 2.2.2. If $X_n \xrightarrow{w} X$ and $Y_n \xrightarrow{p} c$ in $\mathbb{R}^d$, $X_n \pm Y_n \xrightarrow{w} X \pm c$, $X_n \cdot Y_n \xrightarrow{w} X \cdot c$. If $d = 1$ and $c \neq 0$, then $X_n / Y_n \xrightarrow{w} X / c$.

Proof. This follows directly from Slutsky's theorem and continuous mapping theorem. ∎

2.2.3 Convergence in Distribution

So far, the notions of convergence we have talked about applies to general probability space, which needs not to be in $\mathbb{R}^d$ in general. However, traditionally, the case in $\mathbb{R}^d$ is considered first.

Intuition. There's a conical ordering available in $\mathbb{R}^d$ to define F_X and F_{X_n}.

This allows us to define the following.

Definition 2.2.4 (Converge in distribution)**.** Let (X_n) and X be random variables in $\mathbb{R}^d$. Then (X_n) *converges in distribution* to X, denoted as $X_n \xrightarrow{D} X$, if for all $(t_1, \ldots, t_d) \in C_{F_X}$,

$$F_{X_n}(t_1, \ldots, t_d) \to F_X(t_1, \ldots, t_d).$$

Note. X_n and X (in $\mathbb{R}^d$) do not have to be on the same probability space.

Specifically, to see how this relates to what we have seen, recall that

$$F_{X_n}(t_1, \ldots, t_d) = \mathbb{P}(X_n^i \leq t_i, \forall 1 \leq i \leq d) = \mathbb{P}(X_n \in (-\infty, t_1] \times \cdots \times (-\infty, t_d]),$$

same for F_X. So this reduces to the form we're familiar with, i.e., $\mathbb{P}(X_n \in A)$ for some A.

Remark. $X_n \xrightarrow{\text{TV}} X$ implies $X_n \xrightarrow{D} X$.

Proof. Since $X_n \overset{\text{TV}}{\to} X$ means $\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A)$ uniformly in A, but $X_n \overset{D}{\to} X$ only requires the above holds for A in the form of $(-\infty, t_1] \times \cdots \times (-\infty, t_d]$, which is weaker. ⊛

There are more classical results that are worth mentioning.

Remark (De Moivre central limit theorem). Let $X_n \sim \operatorname{Bin}(n, p)$, then for every $t \in \mathbb{R}$, as $n \to \infty$,

$$\mathbb{P}\left(\frac{X_n - np}{\sqrt{np(1-p)}} \leq t\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} \,\mathrm{d}u = \Phi(t).$$

Proposition 2.2.2. Let X_n and X be in $\mathbb{Z}$ such that f_n and f are their corresponding PMF's, then

$$f_n \to f \Leftrightarrow X_n \overset{\text{TV}}{\to} X \Leftrightarrow X_n \overset{D}{\to} X.$$

Proof. The forward implications are clear, so we just need to show $X_n \overset{D}{\to} X$ implies $f_n \to f$. Since for every $t \in \mathbb{Z}$, since X_n and X are discrete in $\mathbb{Z}$,

$$f_n(t) = \mathbb{P}(X_n = t) = \mathbb{P}(X_n \leq t + \epsilon) - \mathbb{P}(X_n \leq t - \epsilon)$$

for some $\epsilon > 0$ small enough. Now, as $t \pm \epsilon$ are in C_X clearly, $X_n \overset{D}{\to} X$ implies

$$\mathbb{P}(X_n \leq t + \epsilon) \to \mathbb{P}(X \leq t + \epsilon),$$

and the same holds for $t - \epsilon$, hence

$$f_n(t) = \mathbb{P}(X_n = t) = \mathbb{P}(X_n \leq t+\epsilon) - \mathbb{P}(X_n \leq t-\epsilon) \to \mathbb{P}(X \leq t+\epsilon) - \mathbb{P}(X \leq t-\epsilon) = \mathbb{P}(X = t) = f(t).$$

As this holds for every $t \in \mathbb{Z}$, we're done. ■

Now, the problem one might have is the following.

Problem. Why not defined for all $t \in \mathbb{R}^d$, rather than $t \in C_{F_X}$?

Answer. Consider for $d = 1$ with $X = c \in \mathbb{R}$, i.e., F_X is the step function at c. To show $X_n \overset{D}{\to} c$, we don't have to show $\mathbb{P}(X_n \leq c) \to \mathbb{P}(X \leq c) = 1$. Otherwise, if we need to show this for all t, in particular, c, $X_n = c + 1/n$ would not satisfy this. ⊛

If $X_n \overset{d}{\to} X$ and X is continuous, then F_{X_n} converges to F_X not only point-wise, but uniformly.

Remark (Polya's theorem). If F_X is continuous, $X_n \overset{D}{\to} X$ is equivalent as

$$\sup_{t \in \mathbb{R}^d} \lvert F_{X_n}(t) - F_X(t) \rvert \to 0.$$

Now we have seen various remarks and clarifications about convergence in distribution, the upshot is that, it is actually just a renaming of weak convergence in $\mathbb{R}^d$!

Theorem 2.2.5. Given X_n and X in $\mathbb{R}^d$, then $X_n \overset{\text{w}}{\to} X$ if and only if $X_n \overset{D}{\to} X$.

Proof. We prove for the case of $d = 1$, then it's easy to see the same holds for $d \geq 1$. For the forward direction, we want to show that for all $t \in C_{F_X}$, $\mathbb{P}(X_n \leq t) \to \mathbb{P}(X \leq t)$. Note that $\mathbb{P}(X \leq t) = \mathbb{P}(X \in (-\infty, t])$ and $\mathbb{P}(X_n \leq t) = \mathbb{P}(X_n \in (-\infty, t])$, hence, from Portmanteau theorem (e) with $A = (-\infty, t]$, $X_n \overset{\text{w}}{\to} X$ is equivalently as saying $\mathbb{P}(X_n \leq t) \to \mathbb{P}(X \leq t)$ if

$$\mathbb{P}(X \in \partial(-\infty, t]) = \mathbb{P}(X \in \{t\}) = \mathbb{P}(X = t)$$

is 0. This is indeed the case since $t \in C_{F_X}$, hence we're done.

To show the backward direction, we need the following lemma.

Lemma 2.2.3. $X_n \xrightarrow{D} X$ if and only if for all $x \in \mathbb{R}^d$,

$$F_X(x^-) \leq \liminf_{n\to\infty} F_{X_n}(x^-) \leq \liminf_{n\to\infty} F_{X_n}(x) \leq \limsup_{n\to\infty} F_{X_n}(x) \leq F_X(x).$$

Proof. The backward direction is clear, so we prove the forward direction. When $x \in C_{F_X}$, we're clearly done, so consider $x \notin C_{F_X}$. Firstly, note that $|C_{F_X}^c|$ is countable, so there exists $(x_k) \nearrow x$ and $(y_k) \searrow x$, both in C_{F_X}. Hence, for all $n \geq 1$ and $k \geq 1$,

$$F_{X_n}(x_k) \leq F_{X_n}(x) \leq F_{X_n}(y_k)$$

as F_{X_n} is increasing. We now have for every $k \geq 1$,

$$\begin{aligned}
F_X(x_k) &= \lim_{n\to\infty} F_{X_n}(x_k) && x_k \in C_{F_X} \\
&\leq \liminf_{n\to\infty} F_{X_n}(x^-) \\
&\leq \liminf_{n\to\infty} F_{X_n}(x) && F_{X_n} \text{ is increasing} \\
&\leq \limsup_{n\to\infty} F_{X_n}(x) \\
&\leq \limsup_{n\to\infty} F_{X_n}(y_k) = F_X(y_k). && y_k \in C_{F_X}
\end{aligned}$$

By taking $k \to \infty$, $F_X(x_k) \to F_X(x^-)$, while $F_X(y_k) \to F_X(x)$,[a] and we're done.

[a]Recall that the distribution function is always right-continuous.

The proof will be continued...

Lecture 6: Stochastic Boundedness and Delta Theorem

Before we finish the proof of Theorem 2.2.5, we need recall one important characterization of lim inf. 2 Feb. 17:30

As previously seen. Given two real sequence x_n and y_n,

$$\liminf_{n\to\infty}(x_n + y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n,$$

where the equality holds when either x_n or y_n converges (not if and only if).

We can then finish the proof of Theorem 2.2.5.

Proof of Theorem 2.2.5 (cont.) Now we can prove the backward direction. Form Portmanteau theorem (c), it suffices to show that for every open $A \subseteq \mathbb{R}$, we have

$$\mathbb{P}(X \in A) \leq \liminf_{n\to\infty} \mathbb{P}(X_n \in A).$$

From the elementary analysis, we see that it suffices to show when $A = (a, b)$ since when $A \subseteq \mathbb{R}$ is open, one can write $A = \bigcup_{k=1}^{\infty}(a_k, b_k)$ where (a_k, b_k)'s disjoint, and have

$$\begin{aligned}
\mathbb{P}(X \in A) &= \sum_{k=1}^{\infty} \mathbb{P}(X \in (a_k, b_k)) \\
&\leq \sum_{k=1}^{\infty} \liminf_{n\to\infty} \mathbb{P}(X_n \in (a_k, b_k)) && \text{assume true for intervals} \\
&\leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} \mathbb{P}(X_n \in (a_k, b_k)) = \liminf_{n\to\infty} \mathbb{P}(X_n \in A),
\end{aligned}$$

where the last inequality follows from induction on $\liminf_{n\to\infty}(x_n+y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$. Now, we show that $\mathbb{P}(X \in A) \leq \liminf_{n\to\infty} \mathbb{P}(X_n \in A)$ when $A = (a, b)$.

Claim. $\mathbb{P}(X \in (a, b)) \leq \liminf_{n\to\infty} \mathbb{P}(X_n \in (a, b))$.

Proof. Observe that $\mathbb{P}(X \in (a, b)) = F_X(b^-) - F_X(a)$, with Lemma 2.2.3, we further have

$$\begin{aligned}
\mathbb{P}(X \in (a, b)) &= F_X(b^-) - F_X(a) \\
&\leq \liminf_{n\to\infty} F_{X_n}(b^-) - \left(\limsup_{n\to\infty} F_{X_n}(a)\right) \\
&\leq \liminf_{n\to\infty} F_{X_n}(b^-) + \liminf_{n\to\infty}(-F_{X_n}(a)) \\
&\leq \liminf_{n\to\infty} \left(F_{X_n}(b^-) - F_{X_n}(a)\right) = \liminf_{n\to\infty} \mathbb{P}(X_n \in (a, b)),
\end{aligned}$$

which proves the claim. ⊛

This proves the case of $d = 1$. ∎

Theorem 2.2.5 means that when talking about random vectors, we can use every result we have proved for the case of weak convergence. Let's see one application, which uses weak convergence's result but now prove something about the distribution.

Proposition 2.2.3. If $X_n \xrightarrow{D} X$ and $t_n \to t \in C_{F_X}$, then $\mathbb{P}(X_n \leq t_n) \to \mathbb{P}(X \leq t)$.

Proof. We see that from Corollary 2.2.2, $X_n - t_n \xrightarrow{w} X - t$, i.e., $X_n - t_n \xrightarrow{D} X - t$. Hence,

$$\mathbb{P}(X_n \leq t_n) = \mathbb{P}(X_n - t_n \leq 0) = F_{X_n - t_n}(0) \to F_{X-t}(0) = \mathbb{P}(X - t \leq 0)$$

as long as $0 \in C_{F_{X-t}}$, i.e., $\mathbb{P}(X - t = 0) = \mathbb{P}(X = t) = 0$, which is just $t \in C_{F_X}$ as we assumed. ∎

2.3 Stochatsic Boundedness

So far we have been talking about the notion of convergence, now we switch the gear a bit and consider boundedness. In this section, let $(X_i)_{i\in I}$ be a family of d-dimensional random vectors with the index set I, which can be either finite or infinite.

Definition 2.3.1 (Bounded in probability). $(X_i)_{i\in I}$ is said to be *bounded in probability* if for every $\epsilon > 0$, there exists an $M > 0$ such that for every $i \in I$,

$$\mathbb{P}(\lVert X_i \rVert \geq M) < \epsilon.$$

In other words, for every $\epsilon > 0$, there exists an $M > 0$ such that $\mathbb{P}(\lVert X_i \rVert < M) \geq 1 - \epsilon$ for every $i \in I$.

Intuition. For any arbitrary large probability close to 1 we want, one can find an upper-bound M on $\lVert X_i \rVert$ uniformly for all $i \in I$.

Note. When $X_i = X$ for every $i \in I$, $(X_i)_{i\in I}$ is trivially bounded in probability.

Proof. Since if not, there exists $\epsilon > 0$, for every $M > 0$, $\mathbb{P}(\lVert X \rVert \geq M) \geq \epsilon$. Then as $M \to \infty$, $\mathbb{P}(\lVert X \rVert = \infty) \geq \epsilon$, which is a contradiction since $\lVert X \rVert = \infty$. ⊛

Remark. When I is finite, $(X_i)_{i\in I}$ is also trivially bounded in probability. On the other hand, when I is infinite, by considering $X_n = n$ (deterministic), which is not bounded in probability anymore.

Remark. If $(X_i)_{i\in I}$ is bounded in L^p for some $p > 0$, i.e., $\sup_{i\in I}\mathbb{E}\left[\lVert X_i\rVert^p\right] < \infty$, then $(X_i)_{i\in I}$ is bounded in probability.

Proof. Since for any $\epsilon > 0$, from Markov's inequality,

$$\mathbb{P}(\lVert X_i\rVert > M) \leq \frac{\mathbb{E}\left[\lVert X_i\rVert^p\right]}{M^p},$$

which can be made less than ϵ since $\sup_{i\in I}\mathbb{E}\left[\lVert X_i\rVert^p\right] < \infty$, for M large enough it'll be satisfied. ⊛

2.3.1 Convergence and Boundedness

Recall the following fact in elementary analysis.

As previously seen. If a deterministic sequence in $\mathbb{R}$ converges, then it's bounded.

In our context, we might expect something like "if $X_n \xrightarrow{p} X$, then (X_n) is bounded in probability." In fact, we have the following "stronger" result where we only require convergence in distribution.

Proposition 2.3.1. If $X_n \xrightarrow{D} X$, then (X_n) is bounded in probability.

Proof. Fix an $\epsilon > 0$. There is an $M > 0$ such that $\mathbb{P}(\lVert X\rVert \geq M) < \epsilon$ since this is a single random vector. To relate this back to X_n, from Portmanteau theorem (d),

$$\epsilon > \mathbb{P}(\lVert X\rVert \geq M) = \mathbb{P}(X \in B^c(0, M)) \geq \limsup_{n\to\infty}\mathbb{P}(X_n \in B^c(0, M)) = \limsup_{n\to\infty}\mathbb{P}(\lVert X_n\rVert > M).$$

In other words,

$$\liminf_{n\to\infty}\mathbb{P}(\lVert X_n\rVert \leq M) > 1-\epsilon,$$

i.e., there exists an n_0 such that for every $n \geq n_0$, $\mathbb{P}(\lVert X_n\rVert \leq M) \geq 1-\epsilon$. As for those $n < n_0$, we can also find $M' > 0$ such that $\mathbb{P}(\lVert X_n\rVert \leq M') > 1-\epsilon$ for every $n < n_0$. Finally, by considering $M'' := \max(M, M')$, we have $\mathbb{P}(\lVert X_n\rVert \leq M'') > 1-\epsilon$, i.e., $\mathbb{P}(\lVert X_n\rVert > M) < \epsilon$ as desired. ∎

There is a kind of converse theorem holds called Prokhorov's theorem, but we won't prove it here. Another useful characterization that generalizes our intuition in $\mathbb{R}$ is the following.

Proposition 2.3.2. When $d = 1$, if $X_n \xrightarrow{p} 0$ and Y_n is bounded in probability, then $X_nY_n \xrightarrow{p} 0$.

Proof. Fix an $\epsilon > 0$. We want to show that $\mathbb{P}(\lvert X_nY_n\rvert > \epsilon) \to 0$. This is because

$$\begin{aligned}\mathbb{P}(\lvert X_nY_n\rvert > \epsilon) &= \mathbb{P}(\lvert X_nY_n\rvert > \epsilon, \lvert Y_n\rvert > M) + \mathbb{P}(\lvert X_nY_n\rvert > \epsilon, \lvert Y_n\rvert \leq M)\\ &\leq \mathbb{P}(\lvert Y_n\rvert > M) + \mathbb{P}(\lvert X_nY_n\rvert > \epsilon, \lvert Y_n\rvert \leq M) \leq \mathbb{P}(\lvert Y_n\rvert > M) + \mathbb{P}(\lvert X_n\rvert > \epsilon/M)\end{aligned}$$

for any M. Now, we see that

- since Y_n is bounded in probability, there's an $M > 0$ such that $\mathbb{P}(\lvert Y_n\rvert > M) < \epsilon$ for all n;
- since $X_n \xrightarrow{p} 0$, for the M (depends on the fixed ϵ) above, $\mathbb{P}(\lvert X_n\rvert > \epsilon/M) \to 0$ as $n \to \infty$.

We see that the second term always goes to 0, while the first term can always be upper-bounded by ϵ. Hence, by letting $\epsilon \to 0$, we're done. ∎

The analogy to the case in $\mathbb{R}$ is the following.

Intuition. In $\mathbb{R}$, if $a_n \to 0$ and b_n is bounded, $a_nb_n \to 0$.

We often write the following.

Notation. We write $X_n = o_p(1)$ for $X_n \xrightarrow{p} 0$, and $X_n = O_p(1)$ when (X_n) is bounded in probability.

Let's see one important application. Consider an estimator T_n of θ, and a deterministic sequence b_n which goes to ∞. In this case, we often have

$$b_n(T_n - \theta) \xrightarrow{D} Y.$$

Example. When $X_n \sim \mathrm{Bin}(n, p)$, then

$$\frac{X_n - np}{\sqrt{np(1-p)}} = \sqrt{\frac{n}{p(1-p)}}\left(\frac{X_n}{n} - p\right) \to Y \sim \mathcal{N}(0, 1)$$

This allows us to compute the rate of convergence and the limiting distribution. But what can we say when we care about $g(T_n)$ for a function g?

Theorem 2.3.1 (Delta method)**.** Let (T_n) be random vectors in $\mathbb{R}^d$ and a deterministic sequence $b_n \to \infty$ such that $b_n(T_n - \theta) \xrightarrow{D} Y$, then $T_n \xrightarrow{p} \theta$. Moreover, if $g\colon \mathbb{R}^d \to \mathbb{R}^m$ is differentiable at θ,

$$b_n(g(T_n) - g(\theta)) \xrightarrow{D} \nabla g(\theta) Y.$$

Proof. We first observe that $\|b_n(T_n - \theta)\| \in O_p(1)$ since $b_n(T_n - \theta) \xrightarrow{D} Y$, with continuous mapping theorem and the fact that $\|\cdot\|$ is continuous, $\|b_n(T_n - \theta)\| \xrightarrow{p} \|Y\|$, so $\|b_n(T_n - \theta)\| \in O_p(1)$ by Proposition 2.3.1. With this, as $b_n \to \infty$,

$$\|T_n - \theta\| = \frac{1}{b_n}\|b_n(T_n - \theta)\| = o(1)O_p(1) \xrightarrow{p} 0,$$

i.e., $T_n \xrightarrow{p} \theta$. For the second claim, since g is differentiable at θ, as $x \to \theta$,

$$\frac{g(x) - g(\theta) - \nabla g(\theta)(x - \theta)}{\|x - \theta\|} \to 0.$$

Let $r(x) \coloneqq g(x) - g(\theta) - \nabla g(\theta)(x - \theta)$ to be the remainder, and consider

$$h(x) = \begin{cases} 0, & \text{if } x = \theta; \\ \dfrac{r(x)}{\|x - \theta\|}, & \text{if } x \neq \theta, \end{cases}$$

which is continuous at θ. Rewriting everything, we have

$$r(x) = g(x) - g(\theta) - \nabla g(\theta)(x - \theta) = h(x)\|x - \theta\|$$

for every $x \in \mathbb{R}^d$. Now, let $x = T_n$, multiply both sides by b_n, and take the norm, we see that

$$\|b_n\left(g(T_n) - g(\theta)\right) - \nabla g(\theta) b_n(T_n - \theta)\| = \|h(T_n)\|\|b_n(T_n - \theta)\|.$$

We observe the following.

Claim. It suffices to show that the right-hand sides goes to 0 in probability.

Proof. Indeed, since it will imply that $b_n(g(T_n) - g(\theta))$ has the same weak limit as $\nabla g(\theta) b_n(T_n - \theta)$ from Theorem 2.2.3, i.e., $\nabla g(\theta) Y$ from our assumption with continuous mapping theorem. ⊛

It's enough to show $\|h(T_n)\| = o_p(1)$ since we know that $\|b_n(T_n - \theta)\| \in O_p(1)$. Since $T_n \xrightarrow{p} \theta$, $h(T_n) \xrightarrow{p} h(\theta)$ again by continuous mapping theorem with h being continuous at θ. This further implies $\|h(T_n)\| \xrightarrow{p} 0$ as we desired.[a] Combining the above, we prove the result. ■

[a] This involves continuous mapping theorem and Corollary 2.2.1 since $h(\theta) = 0$, a constant (so does its norm).

Hence, we see that the answer to our original question is rather simple: as $b_n(T_n - \theta) \xrightarrow{D} Y$,

$$b_n(g(T_n) - g(\theta)) \xrightarrow{D} \nabla g(\theta) \cdot Y$$

for any differentiable g at θ.

Appendix

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