

# MATH592

## Introduction to Algebraic Topology

Pingbang Hu

February 14, 2022

### Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

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## Lecture 1: Homotopies of Maps

05 Jan. 10:00

## 1 Foundation of Algebraic Topology

## 1.1 Homotopy

**Definition 1.1 (Homotopy, homotopic, nullhomotopic).** Let  $X, Y$  be topological spaces. Let  $f, g: X \rightarrow Y$  continuous maps. Then a *homotopy* from  $f$  to  $g$  is a 1-parameter family of maps that continuously deforms  $f$  to  $g$ , i.e., it's a continuous function  $F: X \times I \rightarrow Y$ , where  $I = [0, 1]$ , such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x).$$

We often write  $F_t(x)$  for  $F(x, t)$ .

If a homotopy exists between  $f$  and  $g$ , we say they are *homotopic* and write

$$f \simeq g.$$

If  $f$  is homotopic to a constant map, we call it *nullhomotopic*.

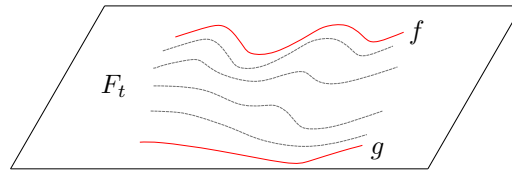


Figure 1: The continuous deforming from  $f$  to  $g$  described by  $F_t$

**Remark.** Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

**Example.** We first see some examples.

1. Any two maps (continuous) with specification

$$f, g: X \rightarrow \mathbb{R}^n$$

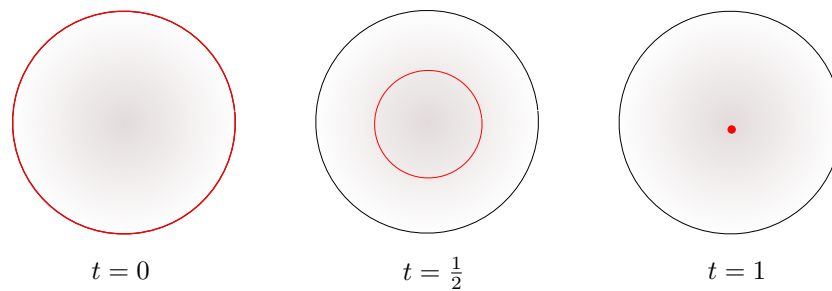
are *homotopic* by considering

$$F_t(x) = (1 - t)f(x) + tg(x).$$

We call it *the straight line homotopy*.

2. Let  $S^1$  denotes the unit circle in  $\mathbb{R}^2$ , and  $D^2$  denotes the unit disk in  $\mathbb{R}^2$ . Then the inclusion  $f: S^1 \hookrightarrow D^2$  is *nullhomotopic* by considering

$$F_t(x) = (1 - t)f(x) + t \cdot 0.$$


 Figure 2: The illustration of  $F_t(x)$ 

We see that there is a **homotopy** from  $f(x)$  to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a **nullhomotopy**.

3. The maps

$$\begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not** **homotopy**.

**Remark.** It will essentially **flip** the orientation, hence we can't deform one to another continuously.

**Exercise.** We first see some exercises.

1. A subset  $S \subseteq \mathbb{R}^n$  is star-shaped if

$$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from  $x_0$  to  $x$  lies in  $S$ .

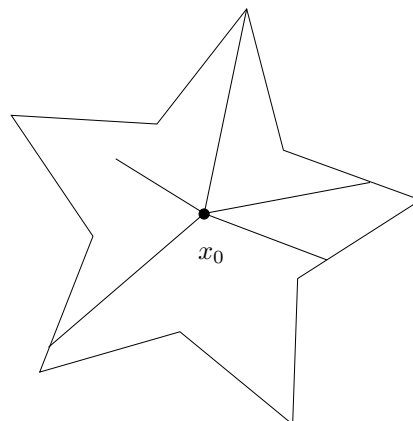


Figure 3: Star-shaped illustration

Show that  $\text{id}: S \rightarrow S$  is **nullhomotopic**.

**Answer.** Consider

$$F_t(x) := (1-t)x + tx_0,$$

which essentially just concentrates all points  $x$  to  $x_0$ .

2. Suppose

$$X \xrightarrow[f_0]{f_1} Y \xrightarrow[g_0]{g_1} Z$$

where

$$f_0 \simeq_{F_t} f_1, \quad g_0 \simeq_{G_t} g_1.$$

Show

$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

**Answer.** Consider  $I \times X \rightarrow Z$ . Then

$$\begin{array}{ccccc} X \times I & \rightarrow & Y \times I & \rightarrow & Z \\ (x, t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

**Remark.** Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** **homotopic**?

**Answer.**

## Lecture 2: Homotopy Equivalence

07 Jan. 10:00

**As previously seen.** Two maps  $f, g: X \rightarrow Y$  is **homotopy** if there exists a map

$$F_t(x): X \times I \rightarrow Y$$

with the properties

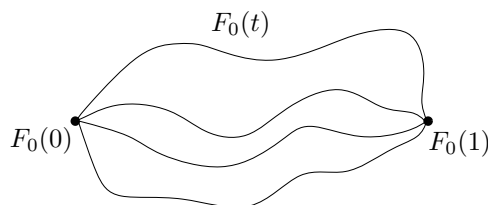
1. Continuous
2.  $F_0(x) = f(x)$
3.  $F_1(x) = g(x)$

**Remark.** The continuity of  $F_t$  is an even stronger condition for the continuity of  $F_t$  for a fixed  $t$ .

We now introduce another concept.

**Definition 1.2 (Homotopy relative).** Given two spaces  $X, Y$ , and let  $B \subseteq X$ . Then a **homotopy**  $F_t(x): X \rightarrow Y$  is called *homotopy relative  $B$*  (denotes  $\text{rel}B$ ) if  $F_t(b)$  is independent of  $t$  for all  $b$ .

**Example.** Let  $X = [0, 1]$  and  $B = \{0, 1\}$ . Then the [homotopy](#) of paths from  $[0, 1] \rightarrow X$  is  $\text{rel}\{0, 1\}$ .



## 1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

**Definition 1.3 (Homotopy equivalence, homotopy inverse).** A map  $f: X \rightarrow Y$  is a *homotopy equivalence* if  $\exists g: Y \rightarrow X$  such that

$$f \circ g \simeq \text{id}_Y, \quad g \circ f \simeq \text{id}_X.$$

We say that  $X, Y$  are *homotopy equivalent*, and  $g$  is called *homotopy inverse* of  $f$ .

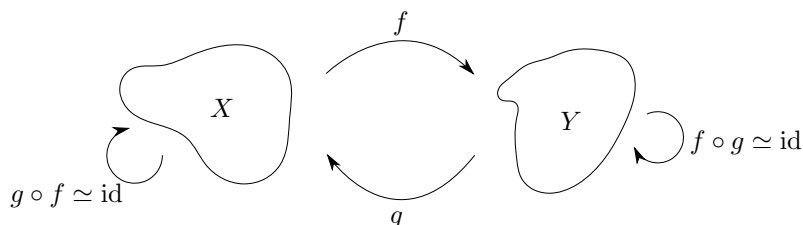


Figure 4: [Homotopy Equivalence](#)

If  $X, Y$  are [homotopy equivalent](#), then we say that they have the same *homotopy type*.

**Notation.** We denote a closed  $n$ -disk as  $D^n$ .

**Example.**  $D^n$  is [homotopy equivalent](#) to a point.



We see that  $f \circ g = \text{id}_*$  and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is [homotopic](#) to  $\text{id}_{D^n}$  by [straight line homotopy](#)  $F_t(x) = tx$ .

**Note.** We say that a space is *contractible* if  $H$  is [homotopy equivalent](#) to a point.

Before doing exercises, we introduce two new concepts.

**Definition 1.4 (Retraction, retract).** Given  $B \subseteq X$ , a *retraction* from  $X$  to  $B$  is a map  $f: X \rightarrow X$  (or  $X \rightarrow B$ ) such that  $\forall b \in B$   $f(b) = b$ , namely  $r|_B = \text{id}_B$ . Or one can see this from

$$\begin{array}{ccc} B & \xhookrightarrow{i} & X \xrightarrow{r} B \\ & \searrow r \circ i & \nearrow \end{array}$$

where  $r$  is a retraction if and only if  $r \circ i = \text{id}_B$ , where  $i$  is an inclusion identity. If  $r$  exists,  $B$  is a *retract* of  $X$ .

**Definition 1.5 (Deformation retraction).** Given  $X$  and  $B \subseteq X$ , a *(strong) deformation retraction*  $F_t: X \rightarrow X$  onto  $B$  is a [homotopy](#)  $\text{rel} B$  from the  $\text{id}_X$  to a [retraction](#) from  $X$  to  $B$ . i.e.,

$$\begin{aligned} F_0(x) &= x & \forall x \in X \\ F_1(x) &\in B & \forall x \in X \\ F_t(b) &= b & \forall t \forall b \in B. \end{aligned}$$

**Exercise.** We now see some problems.

1. Let  $X \simeq Y$ . Show  $X$  is path-connected if and only if  $Y$  is.

**Answer.** Suppose  $X$  is path-connected. Then we see that given two points  $x_1$  and  $x_2$  in  $X$ , there exists a path  $\gamma(t)$  with

$$\gamma: [0, 1] \rightarrow X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Since  $X \simeq Y$ , then there exists a pair of  $f$  and  $g$  such that  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  with

$$f \circ g \underset{F}{\simeq} \text{id}_Y, \quad g \circ f \underset{G}{\simeq} \text{id}_X.$$

(Notice the abuse of notation)

For any two  $y_1$  and  $y_2 \in Y$ , we want to construct a path  $\gamma'(t)$  such that

$$\gamma': [0, 1] \rightarrow Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

Firstly, we let  $g(y_1) =: x_1$  and  $g(y_2) =: x_2$ . From the argument above, we know there exists such a  $\gamma$  starting at  $x_1 = g(y_1)$  ending at  $x_2 = g(y_2)$ . Now, consider  $f(\gamma(t)) = (f \circ \gamma)(t)$  such that

$$f \circ \gamma: I \rightarrow Y, \quad f \circ \gamma(0) = y'_1, \quad f \circ \gamma(1) = y'_2,$$

we immediately see that  $y'_1$  and  $y'_2$  is path connected. Now, we claim that  $y_1$  and  $y'_1$  are path connected in  $Y$ , hence so are  $y_2$  and  $y'_2$ . To see this, note that

$$f \circ g \underset{F}{\simeq} \text{id}_Y,$$

which means that there exists  $F: Y \times I \rightarrow Y$  such that

$$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y'_1 \\ F(y_1, 1) = \text{id}_Y(y_1) = y_1. \end{cases}$$

Since  $F$  is continuous in  $I$ , we see that there must exist a path connects  $y_1$  and  $y'_1$ . The same argument applies to  $y_2$  and  $y'_2$ . Now, we see that the path

$$y_1 \rightarrow y'_1 \rightarrow y'_2 \rightarrow y_2$$

is a path in  $Y$  for any two  $y_1$  and  $y_2$ , which shows  $Y$  is path-connected.



Figure 5: Demonstration of the proof

**Challenge:** One can further show that the connectedness is also preserved by any [homotopy equivalence](#).

2. Show that if there exists [deformation retraction](#) from  $X$  to  $B \subseteq X$ , then  $X \simeq B$ .

### Lecture 3: Deformation Retraction

10 Jan. 10:00

**As previously seen.** A [deformation retraction](#) is a [homotopy](#) of maps  $\text{rel} B$   $X \rightarrow X$  from  $\text{id}_X$  to a [retraction](#) from  $X$  to  $B$ . Then  $B$  is a [deformation retract](#).

**Example.** We can also show

1.  $S^1$  is a [deformation retraction](#) of  $D^2 \setminus \{0\}$ . Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$

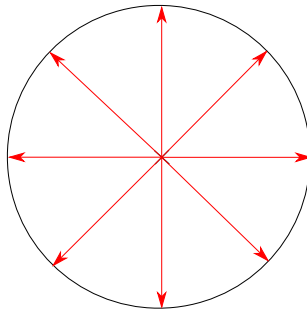


Figure 6: The [deformation retraction](#) of  $D^2 \setminus \{0\}$  is just to *enlarge* that hold and push all the interior of  $D^2$  to the boundary, which is  $S^1$

2.  $\mathbb{R}^n$  [deformation retracts](#) to 0. Indeed, since

$$F_t(x) = (1-t)x.$$

This implies that  $\mathbb{R}^n \simeq *$ , hence we see that

- dimension
- compactness
- etc.

are not [homotopy](#) invariants.

3.  $S^1$  is a [deformation retract](#) of a cylinder and a Möbius band.

For a cylinder, consider  $X \times I \rightarrow X$ . Define [homotopy](#) on a closed rectangle, then verify it induces map on quotient.

For a Möbius band, we define a [homotopy](#) on a closed rectangle, then verify that it respect the equivalence relation.



Finally, we use the universal property of quotient topology to argue that we get a [homotopy](#) on Möbius band.

**Upshot:** Möbius band  $\simeq S^1 \simeq$  cylinder, hence the orientability is not [homotopy](#) invariant.

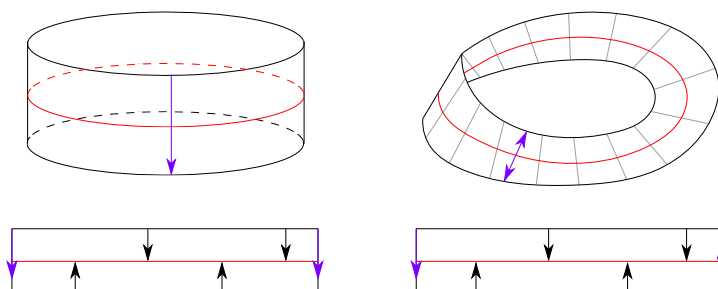


Figure 7: The [deformation retraction](#) for Cylinder and Möbius band

## Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

As previously seen. We saw that

- [homotopy equivalence](#)
- [homotopy](#) invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

### 1.3 CW Complexes

**Example.** Let's start with a few examples.

1. Constructing spheres:

- $S^1$  (up to homeomorphism<sup>1</sup>)

<sup>1</sup>This is just the term for isomorphism in topology.

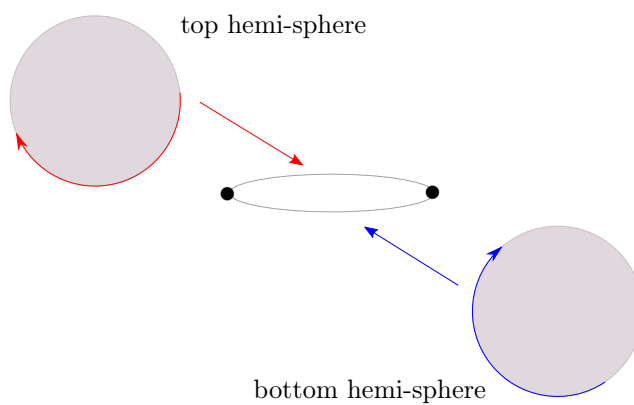


- $S^2$ 
  - glue boundary of 2-disk to a point
  - glue 2 disks onto a circle



Figure 8: **Left:** Glue a 2-disk to a point along its boundary. **Right:** Glue 2 disks to  $S^1$ .

The gluing instruction to construct  $S^2$  in the right-hand side can be demonstrated as follows.



- $T = S^1 \times S^1$



view as gluing instructions

vertex + 2 edges + 2-disks.

Specifically, we have




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Formally, we have the following definition.

**Notation.** Let  $D^n$  denotes a closed  $n$ -disk (or  $n$ -ball)

$$D^n \simeq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

And let  $S^n$  denotes an  $n$ -sphere

$$S^n \simeq \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

Lastly, we call a point as a  $0$ -cell, and the interior of  $D^n$   $\text{int}(D^n)$  for  $n \geq 1$  as a  $n$ -cell.

**Definition 1.6 (CW Complex).** A *CW Complex* is a topological space constructed inductively as

1.  $X^0$  (the 0-skeleton) is a set of discrete points.
2. We inductively construct the n-skeleton  $X^n$  from  $X^{n-1}$  by attaching  $n$ -cells  $e_\alpha^n$ , where  $\alpha$  is the index.

The gluing instructions glued by an attaching map is that  $\forall \alpha, \exists$  continuous map  $\varphi_\alpha$

$$\varphi_\alpha: \partial D_\alpha^n \rightarrow X^{n-1},$$

then

$$X^n = \left( X^{n-1} \coprod_{\alpha} D_\alpha^n \right) / x \sim \varphi_\alpha(x)$$

with identification  $x \sim \varphi_\alpha(x)$  for all  $x \in \partial D_\alpha^n$  with quotient topology.

- 3.

$$X = \bigcup_{n=0} X^n,$$

and let  $\bar{w}$  denotes weak topology. Then

$$u \subseteq X \text{ is open} \iff \forall n \ u \cap X^n \text{ is open}.$$

If all cells have dimension less than  $N$  and a  $\exists N$ -cell, then  $X = X^N$  and we call it  $N$ -dim CW complex.

**Remark.** We write  $X^{(n)}$  for  $n$ -skeleton if we need to distinguish from the Cartesian product.

**Example.** Let's look at some examples.

1. 0-dim **CW complex** is a discrete space.
2. 1-dim **CW complex** is a graph.
3. A CW complex  $X$  is finite if it has finitely many cells.

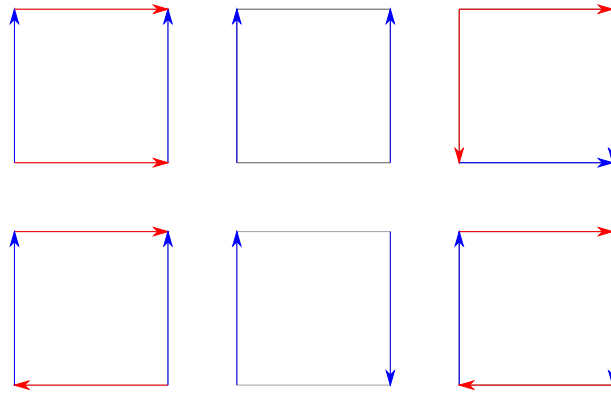
**Definition 1.7 (CW subcomplex).** A *CW subcomplex*  $A \subseteq X$  is a closed subset equal to a union of cells

$$e_\alpha^n = \text{int}(D_\alpha^n).$$

**Remark.** This inherits a **CW complex** structure.

**Exercise.** Given the following gluing instruction:

Check the images of attaching maps.



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere,  $\mathbb{R}P$ .

**Answer.** We see that

1. Torus
2. Cylinder
3. 2-sphere
4. Klein bottle
5. Möbius band
6.  $\mathbb{R}P$

**Notation.** We call the real projection space as  $\mathbb{R}P$ , and we also have so-called complex projection space, denote as  $\mathbb{C}P$ .

## Lecture 5: Operation on Spaces

14 Jan. 10:00

### 1.4 Operations on CW Complexes

#### 1.4.1 Products

We can consider the product of two CW complexes given by a [CW complex](#) structure. Namely, given  $X$  and  $Y$  two [CW complexes](#), we can take two cells  $e_\alpha^n$  from  $X$  and  $e_\beta^m$  from  $Y$  and form the product space  $e_\alpha^n \times e_\beta^m$ , which is homeomorphic to an  $n + m$ -cell. We then take these products as the cells for  $X \times Y$ .

Specifically, given  $X, Y$  are [CW complexes](#), then  $X \times Y$  has a cell structure

$$\{e_\alpha^m \times e_\alpha^n : e_\alpha^m \text{ is a } m\text{-cell on } X, e_\alpha^n \text{ is an } n\text{-cell on } Y\}.$$

**Remark.** The product topology may not agree with the weak topology on the  $X \times Y$ . However, they do agree if  $X$  or  $Y$  is locally compact or if  $X$  and  $Y$  both have at most countably many cells.

**Note.** Notice that if the product is wild enough, then the product topology may not agree with the weak topology.

### 1.4.2 Wedge Sum

Given  $X, Y$  are **CW complexes**, and  $x_0 \in X^0, y_0 \in Y^0$  (only points). Then we define

$$X \vee Y = X \amalg Y$$

with quotient topology.

**Remark.**  $X \vee Y$  is a **CW complex**.

### 1.4.3 Quotients

Let  $X$  be a **CW complex**, and  $A \subseteq X$  **subcomplex** (closed union of cells), then

$$X / A$$

is a quotient space collapse  $A$  to one point and inherits a **CW complex** structure.

**Remark.**  $X / A$  is a **CW complex**.

0-skeleton

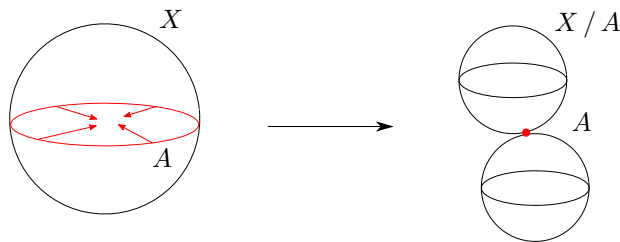
$$(X^0 - A^0) \amalg *$$

where  $*$  is a point for  $A$ . Each cell of  $X - A$  is attached to  $(X / A)^n$  by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n / A^n$$

**Example.** Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a **wedge** of two spheres.



2. We can take the torus and squish down a ring around the hole.



Figure 9: We see that  $X / A$  is [homotopy equivalent](#) to a 2-sphere wedged with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other points, forming a circle.

## Lecture 6: A Foray into Category Theory

19 Jan. 10:00

### 1.5 Category Theory

We start with a definition.

**Definition 1.8 (Category, object, morphism).** A *category*  $\mathcal{C}$  is 3 pieces of data

- A class of *objects*  $\text{Ob}(\mathcal{C})$
- $\forall X, Y \in \text{Ob}(\mathcal{C})$  a class of *morphisms* or arrows,  $\text{Hom}_{\mathcal{C}}(X, Y)$ .
- $\forall X, Y, Z \in \text{Ob}(\mathcal{C})$ , there exists a composition law

$$\begin{aligned} \text{Hom}(X, Y) \times \text{Hom}(Y, Z) &\rightarrow \text{Hom}(X, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

and 2 axioms

- Associativity.  $(f \circ g) \circ h = f \circ (g \circ h)$  for all [morphisms](#)  $f, g, h$  where composites are defined.
- Identity.  $\forall X \in \text{Ob}(\mathcal{C}) \exists \text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$  such that

$$f \circ \text{id}_X = f, \quad \text{id}_X \circ g = g$$

for all  $f, g$  where this makes sense.

Let's see some examples.

**Example.** We introduce some common [category](#).

$\mathcal{C}$	$\text{Ob}(\mathcal{C})$	$\text{Mor}(\mathcal{C})$
$\underline{\text{set}}$	Sets $X$	All maps of sets
$\underline{\text{fset}}$	Finite sets	All maps
$\underline{\text{Gp}}$	Groups	Group Homomorphisms
$\underline{\text{Ab}}$	Abelian groups	Group Homomorphisms
$\underline{k\text{-vect}}$	Vector spaces over $k$	$k$ -linear maps
$\underline{\text{Rng}}$	Rings	Ring Homomorphisms
$\underline{\text{Top}}$	Topological spaces	Continuous maps
$\underline{\text{Haus}}$	Hausdorff Spaces	Continuous maps
$\underline{\text{hTop}}$	Topological spaces	Homotopy classes of continuous maps
$\underline{\text{Top}^*}$	Based topological spaces <sup>2</sup>	Based maps <sup>3</sup>

**Remark.** Any **diagram** plus composition law.

$$\text{id}_A \hookrightarrow A \longrightarrow B \hookleftarrow \text{id}_B .$$

**Definition 1.9 (monic, epic).** A **morphism**  $f: M \rightarrow N$  is *monic* if

$$\forall g_1, g_2 \quad f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$A \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} M \xrightarrow{f} N$$

Dually,  $f$  is *epic* if

$$\forall g_1, g_2 \quad g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$M \xrightarrow{f} N \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} B$$

**Lemma 1.1.** In  $\underline{\text{set}}, \underline{\text{Ab}}, \underline{\text{Top}}, \underline{\text{Gp}}$ , a map is **monic** if and only if  $f$  is injective, and **epic** if and only if  $f$  is surjective.

*Proof.* In  $\underline{\text{set}}$ , we prove that  $f$  is **monic** if and only if  $f$  is injective. Suppose  $f \circ g_1 = f \circ g_2$  and  $f$  is injective, then for any  $a$ ,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence  $g_1 = g_2$ .

<sup>2</sup>Topological spaces with a distinguished base point  $x_0 \in X$

<sup>3</sup>Continuous maps that presence base point  $f: (x, x_0) \rightarrow (y, y_0)$  such that

$$f: X \rightarrow Y, \quad f(x_0) = y_0$$

is continuous.



Now we prove another direction, with contrapositive. Namely, we assume that  $f$  is not injective and show that  $f$  is not **monic**. Suppose  $f(a) = f(b)$  and  $a \neq b$ , we want to show such  $g_i$  exists. This is easy by considering

$$g_1: * \mapsto a, \quad g_2: * \mapsto b.$$

■

### 1.5.1 Functor

After introducing the **category**, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

**Definition 1.10 (Functor).** Given  $\mathcal{C}, \mathcal{D}$  be two **categories**. A (covariant) *functor*

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is

1. a map on **objects**

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of **morphisms**

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(X) \rightarrow F(Y)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(f) \circ F(g)$

## Lecture 7: Functors

21 Jan. 10:00

**As previously seen.** Assume that we initially have a commutative diagram in  $\mathcal{C}$  as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & Z \end{array}$$

After applying  $F$ , we'll have

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ & \searrow F(g \circ f) = F(g) \circ F(f) & \downarrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in  $\mathcal{D}$ .

We can also have a so-called contravariant **functor**.

**Definition 1.11 (Contravariant functor).** Given  $\mathcal{C}, \mathcal{D}$  be two categories. A *contravariant functor*

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is

1. a map on **objects**

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of **morphisms**

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(Y), F(X)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(Y) \rightarrow F(X)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a **contravariant functor**  $F$ , the diagram becomes

$$\begin{array}{ccc} F(X) & \xleftarrow{F(f)} & F(Y) \\ & \nwarrow F(g \circ f) = F(f) \circ F(g) & \uparrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in  $\mathcal{D}$ .

**Example.** Let see some examples.

1. Identity **functor**.

$$I: \mathcal{C} \rightarrow \mathcal{C}.$$

2. Forgetful **functor**.

•

$$\begin{aligned} F: \underline{\text{Gp}} &\rightarrow \underline{\text{set}} \\ G &\mapsto G^4 \\ [f: G \rightarrow H] &\mapsto [f: G \rightarrow H] \end{aligned}$$

•

$$\begin{aligned} F: \underline{\text{Top}} &\rightarrow \underline{\text{set}} \\ X &\mapsto X^5 \\ [f: X \rightarrow Y] &\mapsto [f: X \rightarrow Y] \end{aligned}$$

<sup>4</sup> $G$  is now just the underlying set of the group  $G$ .

## 3. Free functor.

$$\begin{aligned} \underline{\text{set}} &\rightarrow \underline{k\text{-vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

i.e., vector space with basis  $s$

$$[f: A \rightarrow B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

## 4.

$$\begin{aligned} \underline{k\text{-vect}} &\rightarrow \underline{k\text{-vect}} \\ V &\mapsto V^* = \text{Hom}_k(V, k) \end{aligned}$$

If we are working on a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functor.

## 1.

$$\begin{aligned} \underline{\text{Top}}^* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \Pi_1(X, x_0) \end{aligned}$$

where  $\Pi_1$  is so-called *fundamental group*.

## 2.

$$\begin{aligned} \underline{\text{Top}} &\rightarrow \underline{\text{Ab}} \\ X &\mapsto \text{Hp}(X) \end{aligned}$$

where Hp is so-called  $p^{\text{th}}$  *homology*.

Let see the formal definition.

## 1.6 Free Groups

We start with a definition.

**Definition 1.12 (Free group).** Given a set  $S$ , the *free group* is a group  $F_S$  on  $S$  with a map  $S \rightarrow F_S$  satisfying the universal property.

If  $G$  is any group,  $f: S \rightarrow G$  is any map of sets,  $f$  extends uniquely to group homomorphism  $\bar{f}: F_S \rightarrow G$ .

$$\begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \bar{f}: \text{gp hom} \\ & & G \end{array}$$

<sup>5</sup> $X$  is now just the underlying set of the topological space  $X$ .

**Note.** This defines a *natural bijection*

$$\mathrm{Hom}_{\mathrm{set}}(S, \mathcal{U}(G)) \cong \mathrm{Hom}_{\mathrm{Grp}}(F_S, G),$$

where  $\mathcal{U}(G)$  is the **forgetful functor** from the **category** of groups to the **category** of sets. This is the statement that the **free functor** and the forgetful functor are **adjoint**; specifically that the **free functor** is the left **adjoint** (appears on the left in the Hom above).

**Definition 1.13 (Adjoins functor).** A **free** and **forgetful** functors are *adjoints*.

**Remark.** Whenever we state a universal property for an **object** (plus a map), an **object** (plus a map) may or may not exist. If such **object** exists, then it defines the **object uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the **object** (plus a map).

**Lemma 1.2.** Universal property defines  $F_S$  (plus a map  $S \rightarrow F(S)$ ) uniquely up to unique isomorphism.

*Proof.* Fix  $S$ . Suppose

$$S \rightarrow F_S, \quad S \rightarrow \tilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{array}{ccc} S & \longrightarrow & \tilde{F}_S \\ & \searrow f & \downarrow \exists! \varphi \\ & & F_S \end{array} \quad \begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \psi \\ & & \tilde{F}_S \end{array}$$

We'll show  $\varphi$  and  $\psi$  are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \mathrm{id}_{F_S} & \searrow f \\ S & & \\ f \searrow & \downarrow & \nearrow \\ & F_S & \end{array} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \mathrm{id}_{\tilde{F}_S} & \searrow f \\ S & & \\ f \searrow & \downarrow & \nearrow \\ & \tilde{F}_S & \end{array}$$

Hence, we see that

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \psi & \searrow f \\ S & \longrightarrow & \tilde{F}_S \\ f \searrow & \downarrow \varphi & \nearrow \\ & F_S & \end{array} \quad \varphi \circ \psi = \mathrm{id}_{F_S} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \varphi & \searrow f \\ S & \longrightarrow & F_S \\ f \searrow & \downarrow \psi & \nearrow \\ & \tilde{F}_S & \end{array} \quad \psi \circ \varphi = \mathrm{id}_{\tilde{F}_S}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \text{id}_{F_S}, \quad \psi \circ \varphi = \text{id}_{\tilde{F}_S},$$

so  $\varphi$  and  $\psi$  are inverses (thus group isomorphism). ■

## Lecture 8: The Fundamental Group $\pi_1$

24 Jan. 10:00

**Example.** In category Ab **free** Abelian group on a set  $S$  is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on  $S$** .

### 1.6.1 Constructing the Free Groups $F_S$

**Proposition 1.1.** The **free group** defined by the universal property exists.

*Proof.* We'll just give a construction below. First, we see the definition.

**Definition 1.14.** Fix a set  $S$ , and we define a *word* as a finite sequence (possibly  $\emptyset$ ) in the formal symbols

$$\{s, s^{-1} \mid s \in S\}.$$

Then we see that elements in  $F_S$  are equivalence classes of words with the equivalence relation being

- delete  $ss^{-1}$  or  $s^{-1}s$ . i.e.,

$$vs^{-1}sw \sim vw$$

$$vss^{-1}w \sim vw$$

for every **word**  $v, w, s \in S$ ,

with the group operation being concatenation. ■

**Example.** Given words  $ab^{-1}, bba$ , their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

**Exercise.** There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that  $F_S$  satisfies the universal property with respect to the map

$$S \rightarrow F_S, \quad s \mapsto s.$$

## 2 The Fundamental Group

### 2.1 Definition

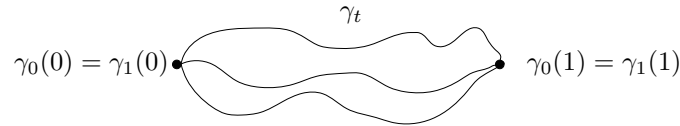
We start with the definition.

**Definition 2.1 (Path).** A *path* in a space  $X$  is a continuous map

$$\gamma: I \rightarrow X$$

where  $I = [0, 1]$ .

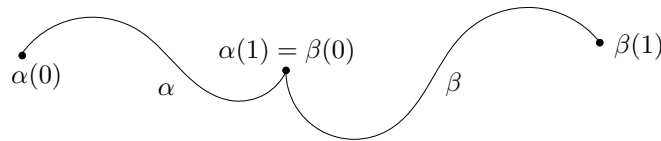
**Definition 2.2 (Homotopy path).** A *homotopy of paths*  $\gamma_0, \gamma_1$  is a *homotopy* from  $\gamma_0$  to  $\gamma_1$  rel  $\{0, 1\}$ .



**Example.** Fix  $x_1, x_0 \in X$ , then  $\exists$  *homotopy of paths* is an equivalence relation on *paths* from  $x_0$  to  $x_1$  (i.e.,  $\gamma$  with  $\gamma(0) = x_0, \gamma(1) = x_1$ ).

**Definition 2.3 (Path composition).** For *paths*  $\alpha, \beta$  in  $X$  with  $\alpha(1) = \beta(0)$ , the *composition*<sup>a</sup>  $\alpha \cdot \beta$  is

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



<sup>a</sup>Also named *product*, *concatenation*.

**Remark.** By the pasting lemma, this is continuous, hence  $\alpha \cdot \beta$  is actually a path from  $\alpha(0)$  to  $\beta(1)$ .

**Definition 2.4 (Reparameterization).** Let  $\gamma: I \rightarrow X$  be a path, then a *reparameterization* of  $\gamma$  is a path

$$\gamma': I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where  $\varphi$  is continuous and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

**Exercise.** A path  $\gamma$  is **homotopic**  $\text{rel}\{0, 1\}$  to all of its **reparameterizations**.

*Proof.* We show that  $\gamma$  and  $\gamma \circ \phi$  are **homotopic**  $\text{rel}\{0, 1\}$  by showing that there exists a continuous  $F_t$  such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since  $\phi$  is continuous, so we define

$$F_t(x) = (1-t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all  $x, t \in I$ .

Now, we check that  $F_t$  really gives us a **homotopic**  $\text{rel}\{0, 1\}$ . We have

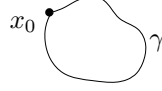
$$\begin{aligned} F_t(0) &= (1-t)\gamma(0) + t \cdot \gamma \circ \phi(0) = (1-t)\gamma(0) + t \cdot \underbrace{\gamma(\phi(0))}_0 = \gamma(0), \\ F_t(1) &= (1-t)\gamma(1) + t \cdot \gamma \circ \phi(1) = (1-t)\gamma(1) + t \cdot \underbrace{\gamma(\phi(1))}_1 = \gamma(1), \end{aligned}$$

which shows that 0 and 1 are independent of  $t$ , hence  $\gamma$  and  $\gamma \circ \phi$  are **homotopic**  $\text{rel}\{0, 1\}$ . ■

**Exercise.** Fix  $x_0, x_1 \in X$ . Then Homotopy of paths (**relative**  $\{0, 1\}$ ) is an equivalence relation on **paths** from  $x_0$  to  $x_1$ .

**Definition 2.5 (Fundamental Group).** Let  $X$  denotes the space and let  $x_0 \in X$  be the base point. The *fundamental group of  $X$  based at  $x_0$* , denoted by  $\pi_1(X, x_0)$ , is a group such that

- Elements: **Homotopy** classes  $\text{rel}\{0, 1\}$  of **paths**  $[\gamma]$  where  $\gamma$  is a **loop** with  $\gamma(0) = \gamma(1) = x_0$ <sup>a</sup>

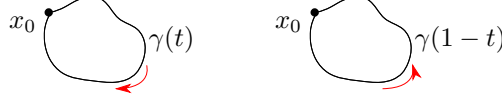


- Operation: **Composition of paths**.
- Identity: Constant loop  $\gamma$  based at  $x_0$  such that

$$\gamma: I \rightarrow X, \quad t \mapsto x_0$$

- Inverses: The inverse  $[\gamma]^{-1}$  of  $[\gamma]$  is represented by the loop  $\bar{\gamma}$  such that

$$\bar{\gamma}(t) = \gamma(1 - t).$$



<sup>a</sup>We say  $\gamma$  is **based** at  $x_0$ .

*Proof.* We prove that

- Associativity:  $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$ . We break this down into

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t - 2), & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t - 3), & \text{if } t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t - 1), & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



Then, we define  $\phi: I \rightarrow I$  such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \frac{1}{2}\right], & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ t + \frac{1}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right], & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{t+1}{2} \in \left[\frac{3}{4}, 1\right], & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and  $\phi(t)$  is continuous and satisfied  $\phi(0) = 0$  and  $\phi(1) = 1$ , which implies that the associativity holds.

- Identity: We want to show that  $[\gamma \cdot c] = [\gamma]$ . Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, consider  $\phi: I \rightarrow I$  such that

$$\phi(t) = \begin{cases} 2t, & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ 1, & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and  $\phi(t)$  is continuous and satisfied  $\phi(0) = 0$  and  $\phi(1) = 1$ .

- Inverses: We want to show that  $\gamma \cdot \bar{\gamma} \simeq c$ , where  $\bar{\gamma}(t) = \gamma(1-t)$ . Firstly, we have

$$(\gamma \cdot \bar{\gamma})(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2t), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider  $F_t$  given by

$$F_t(x) = \begin{cases} \gamma(2xt), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2xt), & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

If  $t = 0$ , we have

$$F_0(x) = \begin{cases} \gamma(0), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0$$

for all  $x \in I$ , namely  $F_0 = c$ , while when  $t = 1$ , we have

$$F_1(x) = \begin{cases} \gamma(2x), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2x), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \bar{\gamma})(x),$$

and we see that  $F_t$  is continuous since at  $x = \frac{1}{2}$ , we have

$$\gamma(2x) = \gamma(1) = \bar{\gamma}(0) = \bar{\gamma}(1-2x),$$

hence we see that  $F_t$  is the **homotopy** between  $\gamma \cdot \bar{\gamma}$  and  $c$ .

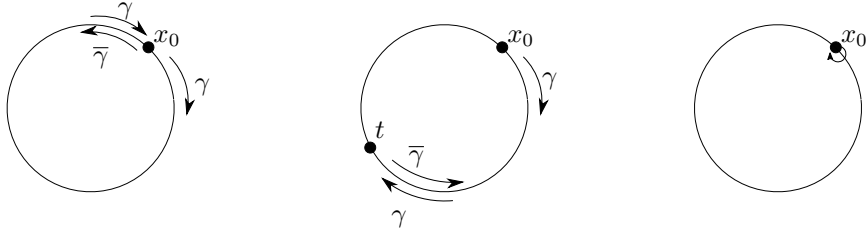


Figure 10: Illustration of  $F_t$ . Intuitively, the path  $\gamma \cdot \bar{\gamma}$  is  $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\bar{\gamma}} x_0$ . But now,  $F_t$  is  $x_0 \xrightarrow{\gamma} t \xrightarrow{\bar{\gamma}} x_0$ . We can think of this **homotopy** is *pulling back* the turning point along the original path.

■

**Theorem 2.1.** If  $X$  is **path**-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

**Remark.** We often write  $\pi_1(X)$  up to isomorphism.

*Proof.* To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if  $[h \cdot \gamma \cdot \bar{h}] = [h \cdot \gamma' \cdot \bar{h}]$ , then since we know that  $h^{-1} = \bar{h}$ , hence in the **fundamental group**  $\pi_1(X, x_0)$ , we see that

$$\bar{h} \cdot h \cdot \gamma \cdot \bar{h} \cdot h = \bar{h} \cdot h \cdot \gamma' \cdot \bar{h} \cdot h. \implies \gamma = \gamma'$$

as we desired.

- onto. We see that for every  $\alpha \in \pi_1(X, x_0)$ , there exists a  $\gamma \in \pi_1(X, x_0)$  such that

$$\gamma = \bar{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)^6$$

since  $h \cdot \gamma \cdot \bar{h} = \alpha$ .

<sup>6</sup>Notice that this is indeed the case, one can verify this by the fact that  $h: x_0 \rightarrow x_1$  and  $\bar{h}: x_1 \rightarrow x_0$ .

We then see that the **fundamental group** of  $X$  does not depend on the choice of basepoint, only on the choice of the **path** component of the basepoint. If  $X$  is **path**-connected, it now makes sense to refer to *the fundamental group* of  $X$  and write  $\pi_1(X)$  for the abstract group (up to isomorphism). ■

**Exercise.** Composition of paths is well-defined on **homotopy** classes  $\text{rel}\{0, 1\}$ .

**Exercise.** If  $X$  is a contractible space, then  $X$  is **path**-connected and  $\pi_1(X)$  is trivial.

## Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

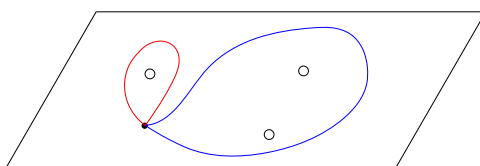


Figure 11: **Fundamental Group** is basically a *hole detector*!

### 2.1.1 Calculations with $\pi_1(S^n)$

Let's start with a simple theorem.

**Theorem 2.2.**  $\pi_1(S^1) \cong \mathbb{Z}$ , and this identification is given by the paths

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))].$$

**Remark.** Intuitively, this winds around  $S^1$   $n$  times. The key to this proof was to understand  $S^1$  via the covering space  $\mathbb{R} \rightarrow S^1$ . We will talk about covering spaces more later.

*Proof.*

HW

**Theorem 2.3.** Given  $(X, x_0)$  and  $(Y, y_0)$ , then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\left[ \begin{array}{l} r: I \rightarrow X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{array} \right] \mapsto (r_X, r_Y),$$

where  $\gamma$  is continuous  $\iff f_X, f_Y$  are continuous.

*Proof.* Let  $Z \xrightarrow{f} X \times Y$  with  $z \mapsto (f_X(z), f_Y(z))$ . Then we have

$$\text{continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply above to

- **Paths**  $I \rightarrow X \times Y$ .
- **Homotopies of paths**  $I \times I \rightarrow X \times Y$ .

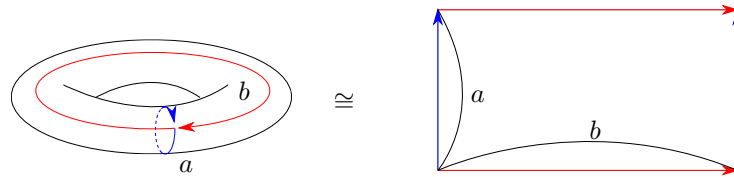
■

**Corollary 2.1.** The torus  $T \cong S^1 \times S^1$  has **fundamental group**  $\pi_1(T) \cong \mathbb{Z}^2$ . Additionally, for a  $k$ -torus  $\underbrace{S^1 \times S^1 \times \dots \times S^1}_{k \text{ times}} = (S^1)^k$ , the **fundamental group** is then  $\mathbb{Z}^k$ , i.e.

$$\pi_1((S^1)^k) \cong \mathbb{Z}^k.$$

*Proof.* Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

**Remark.** One way to think of the  $k$ -torus is as a  $k$ -dimensional cube with opposite  $(k-1)$ -dimensional faces identified by translation.

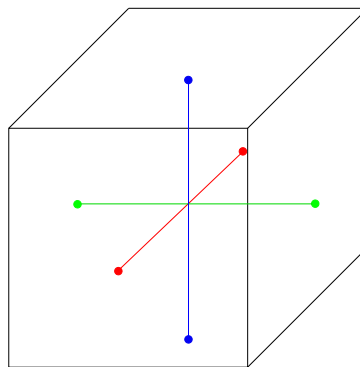


Figure 12: 3-torus with cube identified with parallel sides.

**Example.** We now see some examples.

1.  $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2.  $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong 0 \times \mathbb{Z} = \mathbb{Z}$  since

$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R},$$

which means that the generators are just loops around the hole intuitively.

**Theorem 2.4.**  $\pi_1$  is a **functor** such that

$$\begin{aligned} \pi_1: \underline{\text{Top}}_* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0). \end{aligned}$$

A map  $f: X \rightarrow Y$  taking base point  $x_0$  to  $y_0$  induces a map

$$\begin{aligned} f_*: \pi_1(X, x_0) &\rightarrow \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

i.e.,

$$[f: X \rightarrow Y] \mapsto [f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))].$$

**Notation.** We usually write  $f_*$  if it's a **covariant functor**, while writing  $f^*$  if it's a **contravariant functor**.

*Proof.* We need to check

- well-defined on **path homotopy** classes.
- $f_*$  is a group homomorphism.

$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \frac{1}{2}\right] \\ f(\beta(1-2s)), & \text{if } s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

- $(\text{id}_{(X, x_0)})_* = \text{id}_{\pi_1(X, x_0)}$
- $(f_* \circ g_*) = (f \circ g)_*$

$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \implies f_*(\gamma_*(\gamma)).$$

DIY



## Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

The goal is to compute  $\pi_1(X)$  where  $X = A \cup B$  using the data

$$\pi_1(A), \pi_1(B), \pi_1(A \cap B).$$

## 2.2 Seifert-Van Kampen Theorem

### 2.2.1 Free Product with Amalgamation

We first introduce a definition.

**Definition 2.6 (Free product with amalgamation).** Given some collections of groups  $\{G_\alpha\}_\alpha$ , the *free product*, denoted by  $*_\alpha G_\alpha$  is a group such that

- Elements: **Words** in  $\{g : g \in G_\alpha \text{ for any } \alpha\}$  modulo by the equivalence relation generated by

$$wg_i g_j v \sim w(g_i g_j) v$$

when both  $g_i, g_j \in G_\alpha$ . Also, for the identity element  $\text{id} = e_\alpha \in G_\alpha$  for any  $\alpha$  such that

$$we_\alpha v \sim wv.$$

- Operation: Concatenation of **words**.

Furthermore, if two groups  $G_\alpha$  and  $G_\beta$  have a common subgroup  $S_{\{\alpha, \beta\}}$ <sup>a</sup>, given two inclusion maps<sup>b</sup>  $i_{\alpha\beta} : S_{\{\alpha, \beta\}} \rightarrow G_\alpha$  and  $i_{\beta\alpha} : S_{\{\alpha, \beta\}} \rightarrow G_\beta$ , the *free product with amalgamation*  ${}_\alpha *_S G_\alpha$  is defined as  $*_\alpha G_\alpha$  modulo the normal subgroup generated by

$$\{i_{\alpha\beta}(s_{\{\alpha, \beta\}})i_{\beta\alpha}(s_{\{\alpha, \beta\}})^{-1} \mid s_{\{\alpha, \beta\}} \in S_{\{\alpha, \beta\}}\},$$

Namely<sup>c</sup>,

$${}_\alpha *_S G_\alpha = {}_\alpha * G_\alpha / \langle i_{\alpha\beta}(s_{\{\alpha, \beta\}})i_{\beta\alpha}(s_{\{\alpha, \beta\}})^{-1} \rangle$$

and satisfies the universal property

$$\begin{array}{ccc} S & \xrightarrow{i_{\alpha\beta}} & G_\alpha \\ i_{\beta\alpha} \downarrow & & \downarrow \\ G_\beta & \longrightarrow & G_\alpha *_S G_\beta \end{array} \quad \begin{array}{c} \searrow \\ \exists! \\ \rightarrow X \end{array}$$

<sup>a</sup>In general, we don't need  $S_{\{\alpha, \beta\}}$  to be a subgroup.

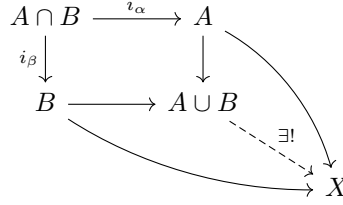
<sup>b</sup>We don't actually need  $i_{\alpha\beta}, i_{\beta\alpha}$  to be inclusive as well.

<sup>c</sup>i.e.,  $i_{\alpha\beta}(s)$  and  $i_{\beta\alpha}(s)$  will be identified in the quotient.

**Remark.** We see that

- We can then write out **words** such as  $g_\alpha \cdot s \cdot g_\beta$  for  $s \in S$ , and view  $s$  as an element of  $G_\alpha$  or  $G_\beta$ . In fact, we can do this construction even when  $i_\alpha$  and  $i_\beta$  are not injective, though this means we are not working with a subgroup.

- Aside, in Top, the same universal property defines union



for  $A, B$  are open subsets and the inclusion of intersection.

**Theorem 2.5 (Seifert-Van Kampen Theorem).** Given  $(X, x_0)$  such that  $X = \bigcup_{\alpha} A_{\alpha}$  with

- $A_{\alpha}$  are open and **path**-connected and  $\forall \alpha \ x_0 \in A_{\alpha}$
- $A_{\alpha} \cap A_{\beta}$  is **path**-connected for all  $\alpha, \beta$ .

Then there exists a surjective group homomorphism

$$*_\alpha: \pi_1(A_{\alpha}, x_0) \rightarrow \pi_1(X, x_0).$$

If we additionally have  $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$  where they are all **path**-connected for every  $\alpha, \beta, \gamma$ , then

$$\pi_1(X, x_0) \cong *_\alpha \pi_1(A_{\alpha} \cap A_{\beta}, x_0) \pi_1(A_{\alpha}, x_0)$$

associated to all maps  $\pi_a(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha}), \pi_1(A_{\beta})$  induced by inclusions of spaces. i.e.,  $\pi_1(X, x_0)$  is a quotient of the **free product**  $*_{\alpha} \pi_1(A_{\alpha})$  where we have

$$(i_{\alpha\beta})_*: \pi_1(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha})$$

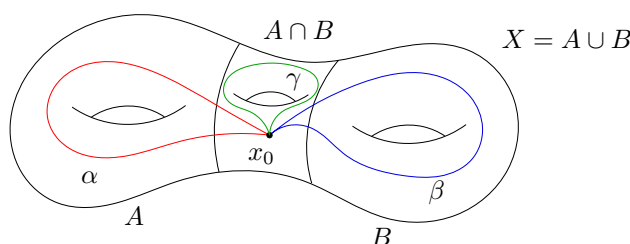
which is induced by the inclusion  $i_{\alpha\beta}: A_{\alpha} \cap A_{\beta} \rightarrow A_{\alpha}$ . We then take the quotient by the normal subgroup generated by

$$\{(i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_{\alpha} \cap A_{\beta})\}.$$

We'll defer the proof of [Theorem 2.5](#) until we get familiar with this theorem.<sup>7</sup>

**Example.** We first see a great visualization of the [Theorem 2.5](#).

<sup>7</sup>The proof can be found in [Section 2.4](#).



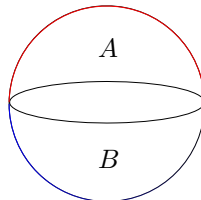
Intuitively we see the [fundamental group](#) of  $X$ , which is built by gluing  $A$  and  $B$  along their intersection. As the [fundamental group](#) of  $A$  and  $B$  glued along the [fundamental group](#) of their intersection. In essence,  $\pi_1(X, x_0)$  is the quotient of  $\pi_1(A) * \pi_1(B)$  by relations to impose the condition that loops like  $\gamma$  lying in  $A \cap B$  can be viewed as elements of either  $\pi_1(A)$  or  $\pi_1(B)$ .

## Lecture 11: Group Presentations

31 Jan. 10:00

**Example.** We now see some applications of [Theorem 2.5](#).

1. We can use [Seifert Van Kampen Theorem](#) to compute the [fundamental group](#) of  $S^2$ . We see that



We see that  $\pi_1(S^2)$  must be a quotient of  $\pi_1(A) * \pi_1(B)$ , but since  $A, B \simeq D^2$ , we know that  $\pi_1(A)$  and  $\pi_1(B)$  are both zero groups, thus  $\pi_1(A) * \pi_1(B)$  is the zero group, and  $\pi_1(S^2)$  is also the zero group.

**Remark.** Note that the inclusion of  $A \cap B \rightarrow A$  induces the zero map  $\pi_1(A \cap B) \rightarrow \pi_1(A)$ , which cannot be an injection. In fact, we know that  $\pi_1(A \cap B) \cong \mathbb{Z}$  since  $A \cap B \simeq S^1$ .

2. In the case of torus, consider the following.



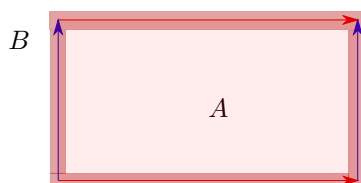


Figure 13:  $A$  is the interior, while  $B$  is the neighborhood of the boundary.

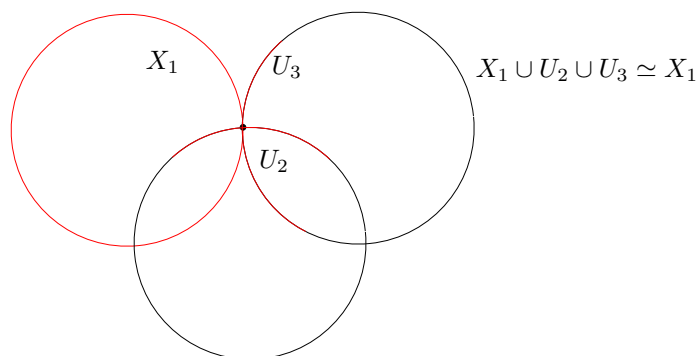
Now note that  $A \simeq D^2$  and  $B \simeq S^1 \vee S^1$ , and since it's a thickening of the two loops around the torus in both ways, this suggests the question of how do we find  $\pi_1(B)$ ? We grab a bit of knowledge from [Seifert Van Kampen Theorem](#) before we continue.

**Exercise.** Suppose we have [path](#)-connected spaces  $(X_\alpha, x_\alpha)$ , and we take their wedge sum  $\bigvee_\alpha X_\alpha$  by identifying the points  $x_\alpha$  to a single point  $x$ . We also suppose a mild condition for all  $\alpha$ , the point  $x_\alpha$  is a [deformation retract](#) of some neighborhood of  $x_\alpha$ .

For example, this doesn't work if we choose the *bad point* on the Hawaiian earring. Then we can use [Seifert Van Kampen Theorem](#) to show that

$$\pi_1 \left( \bigvee_\alpha X_\alpha, x \right) \cong *_\alpha \pi_1 (X_\alpha, x_\alpha).$$

*Proof.* If we denote



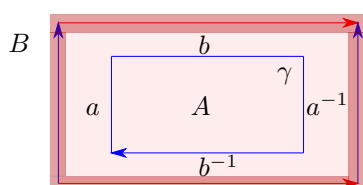
as  $C_n$ , then  $\pi_1(C_n) \cong F_n$ . Then we apply [Theorem 2.5](#) to  $A_\alpha = X_\alpha \cup_\beta U_\beta$ . Specifically, take  $A_\alpha = X_\alpha \cup_\beta U_\beta \simeq X_\alpha$ , where  $U_\beta$  is a neighborhood of  $x_\beta$  which [deformation retracts](#) to  $x_\beta$ . This makes  $A_\alpha$  open as desired. ■

**Corollary 2.2.** The wedge sum of circles  $\pi_1(\bigvee_{\alpha \in A} S^1) = *_\alpha \mathbb{Z}$  is a [free group](#) on  $A$ . In particular, when  $A$  is finite, the [fundamental group](#) of a bouquet of circles is the [free group](#) on  $|A|$ .

Returning to the [example of torus](#), we see that

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$

Further, we know that  $\pi_1(A \cap B) \rightarrow \pi_1(A)$  is the zero map. We need to understand  $\pi_1(A \cap B) \rightarrow \pi_1(B)$ . To do so we need to understand how we're able to identify  $\pi_1(S^1 \vee S^1)$  with  $F_2$  and how we identify  $\pi_1(S^1)$  with  $\mathbb{Z}$ . We update our [Figure 13](#) to talk about this.



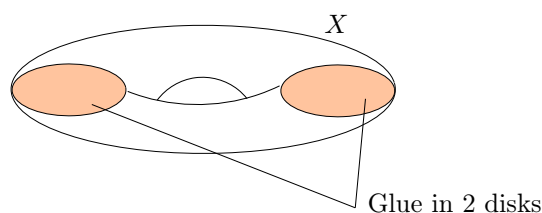
From this, we have

$$\pi_1(A \cap B) \rightarrow \pi_1(B) \cong F_{a,b}, \quad \gamma \mapsto aba^{-1}b^{-1}.$$

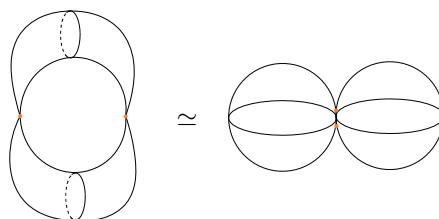
By [Seifert Van Kampen Theorem](#), we identify the image of  $\gamma$  in  $\pi_1(B)[aba^{-1}b^{-1}]$  with its image in  $\pi_1(A)$ , which is just trivial. Therefore, we have

$$\pi_1(T^2) = F_{a,b} / \langle aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2.$$

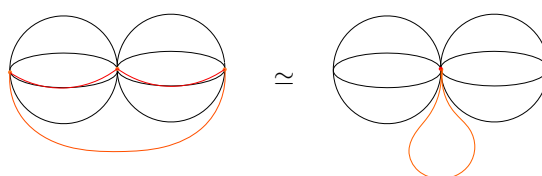
- Let's see the last example which illustrate the power of [Seifert Van Kampen Theorem](#). Start with a torus, and we glue in two disks into the hollow inside.



We'll call this space  $X$ , and our goal is to find  $\pi_1(X)$ . We can place a [CW complex](#) structure on this space so that each disk is a [subcomplex](#). Then, we take quotient of each disk to a point without changing the [homotopy type](#), hence  $X$  is [homotopy](#) to



By the same property, we can expand one of those points into an interval, and then contract the red path as follows.



This is exactly  $S^2 \vee S^2 \vee S^1$ . With [Seifert Van Kampen Theorem](#), we have

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}.$$

**Exercise.** Consider  $\mathbb{R}^2 \setminus \{x_1, \dots, x_n\}$ , that is the plane punctured at  $n$  points. Then  $X \simeq \bigvee_n S^1$ , so then

$$\pi_1(X) \simeq F_n.$$

One way to do this is to convince yourself that you can do a [deformation retract](#) the plane onto the following [wedge](#).

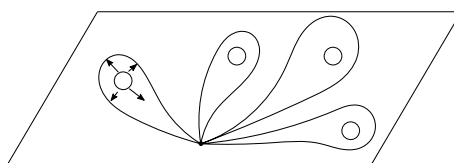


Figure 14: [Deformation retract](#)  $X$  onto wedge.

## 2.3 Group Presentation

In order to go further, we introduce the concept of *group presentation*.

**Definition 2.7 (Group presentation).** A presentation  $\langle S \mid R \rangle$  of a group  $G$  is

- $S$ : set of *generators*
- $R$ : set of *relators* (words in a generator and inverses)

such that

$$G \cong F_S / \langle R \rangle,$$

where  $\langle R \rangle$  is a subgroup normally generated by the elements of  $R$ .

**Definition 2.8 (Finite presentation).** If  $S$  and  $R$  are both finite, then  $G = \langle S \mid R \rangle$  is a *finite presentation* if  $S, R$  are, and we say that  $G$  is *finitely presented*.

**Note.** One way to think about whether  $G$  is finitely presented is that if  $r$  is a word in  $R$  then  $r = 1$ , where  $1$  is the identity of  $G$ .

**Example.** We see that

1.  $F_2 = \langle a, b \mid \rangle$
2.  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$
3.  $\mathbb{Z}/3\mathbb{Z} = \langle a \mid a^3 \rangle$
4.  $S_3 = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$

**Theorem 2.6.** Any group  $G$  has a presentation.

*Proof.* We first choose a generating set  $S$  for  $G$ . Notice that we can even choose  $S = G$  directly. From the universal property of free group, we see that there exists a surjective map  $\varphi: F_S \rightarrow G, s \mapsto s$ . Now, let  $R$  be the generating set for  $\ker(\varphi)$ , by the first isomorphism theorem<sup>8</sup>,  $G \cong F_S / \ker\varphi$ . In fact, we have  $G = \langle S \mid R \rangle$ . ■

**Remark.** The advantages of using group presentation are that given  $G = \langle S \mid R \rangle$ , it's now easy to define a homomorphism  $\psi: G \rightarrow H$  given a map  $\varphi: S \rightarrow H$ ,  $\psi$  extends to a group homomorphism  $G \rightarrow H$  if and only if  $\psi$  vanishes on  $R$ , i.e.,  $\phi(r) = 0$  for all  $r \in R$ . We see an example to illustrate this.

**Example.** If we have  $G = \langle a, b \mid aba \rangle$ , a map  $\varphi: \{a, b\} \rightarrow H$  gives a group homomorphism if and only if

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H.$$

This essentially uses the universal property of quotients.

<sup>8</sup>[https://en.wikipedia.org/wiki/Isomorphism\\_theorems](https://en.wikipedia.org/wiki/Isomorphism_theorems)

**Remark.** It's sometimes easy to calculate  $G^{\text{Ab}}$

$$G^{\text{Ab}} = \langle S \mid R, \text{commutators in } S \rangle.$$

**Example.** Suppose all relations in  $R$  are commutators, so  $R \subseteq [G, G]$ . Then,

$$G^{\text{Ab}} = (F_S)^{\text{Ab}} = \bigoplus_S \mathbb{Z}.$$

**Remark.** The disadvantages are that, this is computationally **very difficult**.

---

**Example.** Given  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ , let

$$\psi: \{a, b\} \rightarrow H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1_H \in H.$$

Namely, this is a [presentation](#) of the trivial group, but this is entirely unclear.

## Lecture 12: Presentations for $\pi_1$ of CW Complexes

2 Feb. 10:00

Let's first see an exercise.

**Exercise.** Consider  $G_1 = \langle S_1 \mid R_1 \rangle$  and  $G_2 = \langle S_2 \mid R_2 \rangle$ . Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\} \rangle$
- $G_1 *_H G_2$  where  $f_1: H \rightarrow G_1$  and  $f_2: H \rightarrow G_2$ . Then we have

$$G_1 *_H G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{f_1(h)f_2(h)^{-1} \mid h \in H\} \rangle.$$

### 2.3.1 Presentations for $\pi_1$ of CW Complexes

For  $X$  a [CW complex](#), we have

1. A 1-dimensional [CW complex](#) has free  $\pi_1$  (call its generators as  $a_1, \dots, a_n$ ).
2. Gluing a 2-disk by its boundary along a word  $w$  in the generators *kills*  $w$  in  $\pi_1$ . We then get a [presentation](#) for  $\pi_1(X^2)$  given by

$$\langle a_1, \dots, a_n \mid w \text{ for each 2-cell in } X_2 \rangle.$$

3. Gluing in any higher dimensional cells along their boundary will not change  $\pi_1$ . That is, in a [CW complex](#), we have  $\pi_1(X) = \pi_1(X^2)$ .

**Remark.** We can write the above more precise.

1. Find free generators  $\{a_i\}_{i \in I}$  for  $\pi_1(X^1)$ .
2. For each 2-disk  $D_\alpha^2$ , write attaching map as word  $w_\alpha$  in  $a_i$ . i.e.,

$$\pi_1(X^2) = \langle a_i \mid w_\alpha \rangle.$$

3.  $\pi_1(X) = \pi_1(X^2)$ .

**Example.** Given  $G = \mathbb{Z}/n\mathbb{Z} = \langle a, a^n \rangle$ , then we take a loop and then wind a 2-disk around the loop  $a$  for  $n$  times.

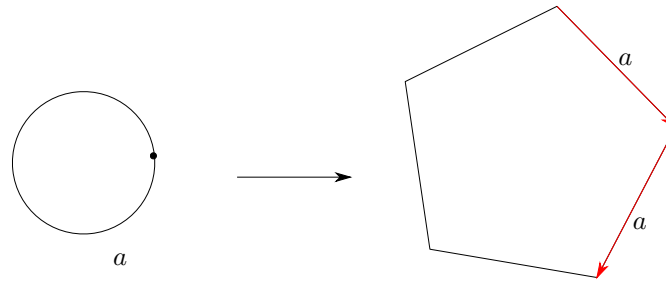


Figure 15: For  $G = \mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$ , we wind the boundary around  $a$  for  $n$  times.

We then see that given a group  $G$  with [presentation](#)  $\langle S \mid R \rangle$ , one can construct a 2-dimensional [CW complex](#) with  $\pi_1 = G$  by

- Set  $X^1 = \bigvee_{s \in S} S^1$
- For each relation  $r \in R$ , glue in a 2-disk along loops specified by the [word](#)  $r$ .

Every group is then  $\pi_1$  of some space.

**Theorem 2.7.** If  $X$  is a [CW complex](#) and  $\iota_1: X^1 \hookrightarrow X$  and  $\iota: X^2 \hookrightarrow X$ , then  $(\iota_1)_*$  surjects onto  $\pi_1$  and  $(\iota_2)_*$  is an isomorphism on  $\pi_1$ .

*Proof.*



HW

**Definition 2.9 (Graph, subgraph, tree, maximal tree).** We import some topological definitions of graph theoretic concepts.

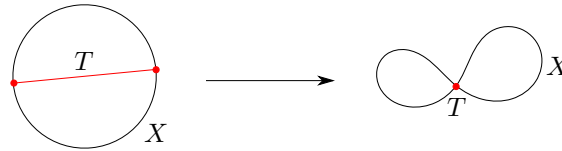
- A *graph* is a 1-dimensional [CW complex](#).
- A *subgraph* is a [subcomplex](#).
- A *tree* is a contractible [graph](#).
- A [tree](#) in [graph](#)  $X$  (necessarily a [subgraph](#)) is *maximal* or *spanning* if it contains all the vertices.

**Theorem 2.8.** Every connected graph has a maximal tree. Every tree is contained in a maximal tree.

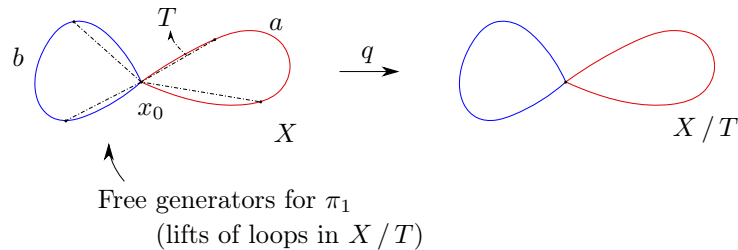
**Corollary 2.3.** Suppose  $X$  is a connected graph with basepoint  $x_0$ . Then  $\pi_1(X, x_0)$  is a free group.

Furthermore, we can give a presentation for  $\pi_1(X, x_0)$  by finding a spanning tree  $T$  in  $X$ . The generators of  $\pi_1$  will be indexed by cells  $e_\alpha \in X - T$ , and  $e_\alpha$  will correspond to a loop that passes through  $T$ , traverses  $e_\alpha$  once, then returns to the basepoint  $x_0$  through  $T$ .

*Proof.* The idea is simple.  $X$  is homotopy equivalent to  $X/T$  via previous work on the homework,  $T$  contains all the vertices, so the quotient has a single vertex. Thus, it is a wedge of circles, and each  $e_\alpha$  projects to a loop in  $X/T$ .



The current plan is to calculate the fundamental group of CW complexes. For now, we need to see that the fundamental group of a 1-skeleton (a graph) can be found by taking a maximal tree, and then quotienting out the space by the tree to get a wedge of circles.



We now prove that the maximal trees exist. Recall that  $X$  is a quotient of

$$X^0 \coprod_{\alpha} I_{\alpha}.$$

Each subset  $U$  is open if and only if it intersects each edge  $\bar{e}_\alpha$  in an open subset. A map  $X \rightarrow Y$  if and only if its restriction to each edge  $\bar{e}_\alpha$  is continuous. Now, take  $X_0$  to be a subgraph. Our goal is to construct a subgraph  $Y$  with

- $X_0 \subset Y \subset X$

- $Y$  deformation retracts to  $X_0$
- $Y$  contains all vertices of  $X$ .

So if we take  $X_0$  to be a vertex, then  $Y$  is out [tree](#) and we're done!

Our strategy now is to build a sequence  $X_0 \subset X_1 \subset \dots$  and correspondingly,  $Y_0 \subset Y_1 \subset \dots$ . We start with  $X_0$  and inductively define

$$X_i := X_{i-1} \bigcup \text{all edges } \bar{e}_\alpha \text{ with one or both vertices in } X_{i-1}.$$

We then see that  $X = \bigcup_i X_i$ .<sup>9</sup> Now, let  $Y_0 = X_0$ . By induction, we'll assume that  $Y_i$  is a [subgraph](#) of  $X_i$  such that

- $Y_i$  contains all vertices of  $X_i$ .
- $Y_i$  deformation retracts to  $Y_{i-1}$ .

We can then construct  $Y_{i+1}$  by taking  $Y_i$  and adding to it one edge to adjoin every vertex of  $X_{i+1}$ , namely

$$Y_{i+1} := Y_i \bigcup \text{one edge to adjoin every vertex of } X_i^{10}$$

We then see that  $Y_{i+1}$  deformation retracts to  $Y_i$  by just smashing down each edge. Now, we can show that  $Y$  deformation retracts to  $Y_0 = X_0$  by performing the deformation retraction from  $Y_i$  to  $Y_{i-1}$  during the time interval  $[1/2^i, 1/2^{i-1}]$ . ■

**Example.** Let

- $S^n$ : decompose into 2 open disks
- $A_1$ : neighborhood of top hemisphere
- $A_2$ : neighborhood of lower hemisphere

We see that  $A_1 \cap A_2 \simeq S^{n-1}$ , where we need  $n \geq 2$  to let  $S^{n-1}$  be connected. We then have

$$\pi_1(S^n) \cong 0 \underset{\pi_1(A_1 \cap A_2)}{*} 0 = 0.$$

On the other hand, if  $n \geq 3$ , then we see that

$$S^n = D^n \cup * / \sim.$$

Since 2-skeleton is a point, thus  $\pi_1(S^n) = 0$ .

## Lecture 13: Proof of Seifert-Van-Kampen Theorem

4 Feb. 10:00

### 2.4 Proof of Seifert-Van-Kampen Theorem

Let's start to prove [Theorem 2.5](#).

<sup>9</sup>[HPM02] do this by arguing the union on the right is both open and closed.

<sup>10</sup>This is possible if we assume Axiom of Choice.



*Proof.* The outline of the proof is the following. Let  $X = \bigcup_{\alpha} A_{\alpha}$  where  $A_{\alpha}$  are open, [path](#)-connected and contain the bluepoint  $x_0$ . We also must guarantee that  $A_{\alpha} \cap A_{\beta}$  is [path](#)-connected.

1. Since we have a map induced by the inclusions:

$$\Phi: *_{\alpha} \pi_1(A_{\alpha}, x_0) \rightarrow \pi_1(X, x_0).$$

We want to show that  $\phi$  is surjective. Take some  $\gamma: I \rightarrow X$ , then by the compactness of the interval  $I$ , we can show that there is a partition  $I$  with  $s_1 < \dots < s_n$  so that

$$\alpha|_{s_i, s_{i+1}} =: \alpha_i$$

has image in  $A_{\alpha_i}$  for some  $\alpha_i$ .<sup>11</sup> Specifically, since

- $A_{\alpha}$  is open for all  $\alpha$
- $I$  is compact,

then for all  $i$ , we choose a path  $h_i$  from  $x_0$  to  $\gamma(s_i)$  in  $A_{\sigma_{i-1}} \cap A_{\alpha_i}$ , using [path](#)-connectedness of the pairwise intersections. Now, take  $\gamma$  and write it as

$$\gamma = (\gamma_1 \cdot \bar{h}_1) \cdot (\bar{h}_1 \cdot \gamma_2) \cdot \dots \cdot (\gamma_{n-1} \cdot \bar{h}_{n-1}) \cdot (h_{n-1} \cdot \gamma_n).$$

Observe that each of these paths is fully contained in  $A_{\alpha_i}$ , so this implies that  $\gamma \in \text{Im}(\Phi)$ , therefore  $\Phi$  is surjective.

2. For the next step, we'll show that the second part of [Theorem 2.5](#). Assume that our triple intersections are [path](#)-connected. We want to show that  $\ker(\Phi)$  is generated by

$$(i_{\alpha\beta})_*(\omega)(i_{\beta\alpha})_*(\omega)^{-1},$$

where

$$i_{\alpha\beta}: A_{\alpha} \cap A_{\beta} \hookrightarrow A_{\alpha}$$

for all loops  $\omega \in \pi_1(A_{\alpha} \cap A_{\beta}, x_0)$ .

Before we go further, we'll need some definition.

**Definition 2.10 (Factorization).** A *factorization* of a [homotopy](#) class  $[f] \in \pi_1(X, x_0)$  is a formal product

$$[f_1][f_2] \dots [f_{\ell}]$$

with  $[f_i] \in \pi_1(A_{\alpha_i}, x_0)$  such that

$$f \simeq f_1 \cdot f_2 \cdot \dots \cdot f_{\ell}.$$

We showed that every  $[f]$  has a [factorization](#) in step 1 already. Now we want to show that two [factorizations](#)

$$[f_1] \cdot \dots \cdot [f_{\ell}] \text{ and } [f'_1] \cdot \dots \cdot [f'_{\ell'}]$$

of  $[f]$  must be related by two moves:

<sup>11</sup>This is a good exercise for point-set topology.

(a)  $[f_i] \cdot [f_{i+1}] = [f_i \cdot f_{i+1}]$  if  $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$ . Namely, the reaction defining the **free product** of groups.

(b)  $[f_i]$  can be viewed as an element of  $\pi_1(A_\alpha, x_0)$  or  $\pi_1(A_\beta, x_0)$  whenever

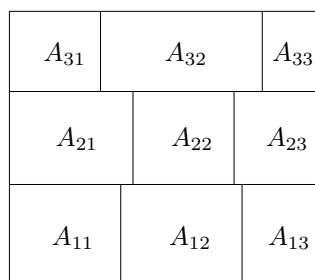
$$[f_i] \in \pi_1(A_\alpha \cap A_\beta, x_0).$$

This is the relation defining the **amalgamated free product**.

Now, let  $F_t: I \times I \rightarrow X$  be a **homotopy** from  $f_1 \dots f_\ell$  to  $f'_1 \dots f'_{\ell'}$ , since they both represent  $[f]$ . We subdivide  $I \times I$  into rectangles  $R_{ij}$  so that

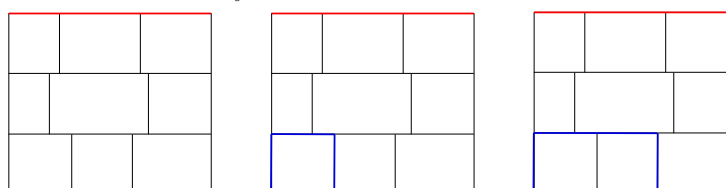
$$F(R_{ij}) \subseteq A_{\alpha_{ij}} =: A_{ij}$$

for some  $\alpha_{ij}$  using compactness. We also argue that we can perturb the corners of the squares so that a corner lies only in three of the  $A_\alpha$ 's indexed by adjacent rectangles.



We also argue that we can set up our subdivision so that the partition of the top and bottom intervals must correspond with the two **factorizations** of  $[f]$ . We then perform our **homotopy** one rectangle at a time.

ending loop  $f'_1 \dots f'_{\ell'}$



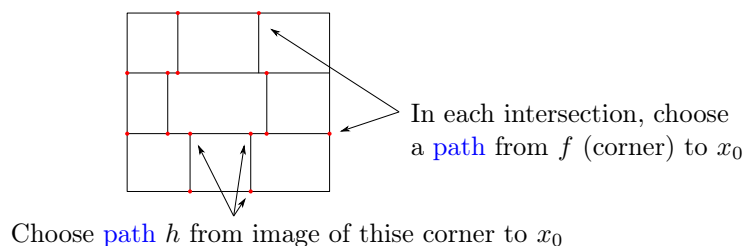
starting loop  $f_1 \dots f_\ell$

step 1

step 2

**Idea:** Argue that **homotoping** over a single rectangle has the effect of using allowable moves to modify the **factorization**.

At each triple intersection, choose a **path** from  $f$  (corner) to  $x_0$  which lies in the triple intersection, so we use the assumption that the triple intersections are **path-connected**.



Along the top and bottom, we make choices compatible with the two **factorizations**. It's now an exercise to check that these choices result in **homotoping** across a rectangle gives a new **factorization** related by an allowable move.

■

## Lecture 14: Covering Spaces

7 Feb. 10:00

### 3 Covering Spaces

#### 3.1 Lifting Properties

As always, we start with a definition.

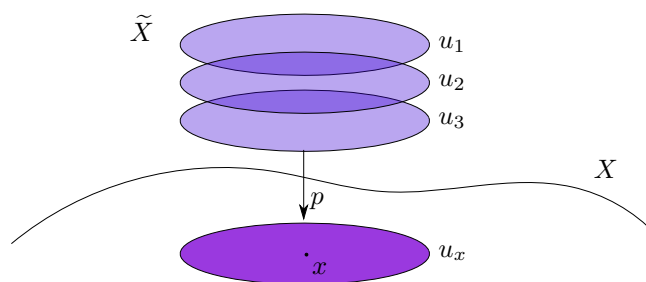
**Definition 3.1 (Covering space).** A *covering space*  $\tilde{X}$  of  $X$  is a space  $\tilde{X}$  and a map  $p: \tilde{X} \rightarrow X$  such that  $\forall x \in X \exists$  neighborhood  $u_x$  with  $p^{-1}(u_x)$  the disjoint union of open sets

$$\coprod_{\alpha} u_{\alpha}$$

such that

$$p|_{u_{\alpha}} : u_{\alpha} \rightarrow u_x$$

is a homeomorphism for every  $\alpha$ .



We sometimes call  $p$  as *covering map*.

Although we already investigate into [covering spaces](#) quite a lot in homework, but a terminology is still worth mentioning.

**Definition 3.2 (Evenly covered).** Let  $p: \tilde{X} \rightarrow X$  be a continuous map of spaces. Then an open subset  $U \subseteq X$  is called *evenly covered by  $p$*  if

$$p|_{V_i}: V_i \rightarrow U$$

is a homeomorphism.

We call the parts  $V_i$  of the partition  $\coprod_i V_i$  of  $p^{-1}(U)$  *slices*.

**Remark.** We see that  $p$  is a [covering map](#) if and only if every point  $x \in X$  has a neighborhood which is [evenly covered](#).

We immediately have the following proposition.

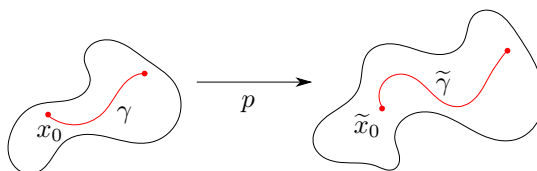
**Proposition 3.1 (Homotopy lifting property).** The [covering spaces](#) satisfy the [homotopy lifting property](#) such that the following diagram commutes.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{F}_0} & \tilde{Y} \\ \downarrow & \nearrow \exists \tilde{F}_t & \downarrow p \\ X \times I & \xrightarrow{F_t} & Y \end{array}$$

*Proof.* We already proved this in homework! ■

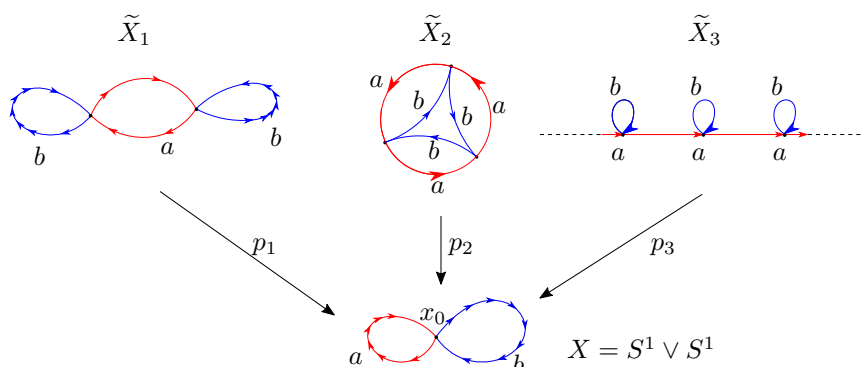
**Corollary 3.1.** For each [path](#)  $\gamma: I \rightarrow X$  in  $X$ ,  $\tilde{x}_0 \in p^{-1}(\gamma(0))$  such that there exists a unique [lift](#)  $\tilde{\gamma}$  starting at  $\tilde{x}_0$ .

And for each [path homotopy](#)  $I \times I \rightarrow X$ , there exists a unique [path homotopy](#)  $\tilde{\gamma}: I \times I \rightarrow \tilde{X}$  starting at  $\tilde{x}_0$ .



**Example.** Let see some examples.

1. Covers of  $S^1 \vee S^1$ .



Note that in each cover (those three on the top), the black dot is the preimage of  $\{x_0\}$ , namely  $p_i^{-1}(\{x_0\})$ .

**Remark.** We see that for each  $p_i^{-1}(\{x_0\})$ , there are exactly

- one  $a$  edge goes out
- one  $b$  edge goes out
- one  $a$  edge goes in
- one  $b$  edge goes in

It turns out that there are much more covers of  $S^1 \vee S^1$ , as long as this main property is satisfied.

**Proposition 3.2.** Let

$$p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a **covering map**. Then

1.  $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$  is injective.
2.  $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0) = \{[\gamma] \mid \text{Lift } \tilde{\gamma} \text{ starting at } \tilde{x}_0 \text{ is a loop.}\}.$

*Proof.* We prove this one by one.

1. Suppose  $\tilde{\gamma} \in \pi_1(\tilde{X}, \tilde{x}_0)$  is in  $\ker(p_*)$ . Then

$$[\gamma] = p_*([\tilde{\gamma}]) = [p \circ \tilde{\gamma}].$$

Let  $\gamma_t$  be a **nullhomotopy** from  $\gamma$  to the constant loop  $c_{x_0} \text{ rel } \{0, 1\}$ . We can then **lift**  $\gamma_t$  to  $\tilde{\gamma}_t$  where  $\tilde{\gamma}_0 = \tilde{\gamma}$ . Now, we claim that

- $\tilde{\gamma}$  is a **homotopy rel**  $\{0, 1\}$ .
- $\tilde{\gamma}_1$  is the constant loop  $c_{\tilde{x}_0}$ .

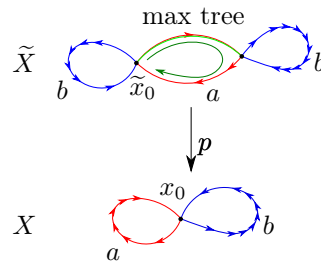
$$\begin{array}{ccc}
 & \tilde{X} & \\
 \tilde{\gamma} \nearrow & \downarrow p & \\
 I & \xrightarrow{\gamma} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \tilde{X} & \\
 \tilde{\gamma}_t \nearrow & \downarrow p & \\
 I \times I & \xrightarrow{\gamma_t} & X
 \end{array}$$

We see that the above diagrams prove the first claim, since we know that the left and right edge of  $I \times I$  maps to  $x_0$  under  $\gamma_t$ , and  $c_{\tilde{x}_0}$  **lifts** this, so by uniqueness  $t \mapsto \tilde{\gamma}_t(0)$  and  $t \mapsto \tilde{\gamma}_t(1)$  must be constant **paths** at  $\tilde{x}_0$  as desired.

Then the **lift**  $\tilde{\gamma}_t$  is a **homotopy of paths** to the constant loop, so  $[\tilde{\gamma}] = 1$ .

- Let see an example to show the idea of the proof.

**Example.** Given



Then

$$p_*\pi_1 = \langle b, a^2, ab\bar{a} \rangle \subseteq \pi_1(X) = \langle a, b \mid \rangle.$$

■

**Proposition 3.3 (Lifting criterion).** Let

$$p: (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$$

be **covering map**. Given

- $f: (X, x_0) \rightarrow (Y, y_0)$ ;
- $X$  is **path-connected**, locally **path-connected**,

then a **lift**

$$\tilde{f}: (X, x_0) \rightarrow (\tilde{Y}, \tilde{y}_0)$$

exists if and only if

$$f_* (\pi_1(X, x_0)) \subseteq p_* (\pi_1(\tilde{Y}, \tilde{y}_0)).$$

$$\begin{array}{ccc} & (\tilde{Y}, \tilde{y}_0) & \\ \exists \tilde{f} \nearrow & \downarrow p & \\ (X, x_0) & \xrightarrow{f} & (Y, y_0) \end{array} \quad \begin{array}{ccc} & \pi_1(\tilde{Y}, \tilde{y}_0) & \\ \tilde{f}_* \nearrow & \downarrow p_* & \\ \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \end{array}$$

## Lecture 15: Lifting

9 Feb. 10:00

Before proving **Proposition 3.3**, we first see an application.

**Example.** Prove that every continuous map  $f: \mathbb{R}P^2 \rightarrow S^1$  is **nullhomotopic**.

*Proof.* If we can show that there is a **lift**  $\tilde{f}: \mathbb{R}P^2 \rightarrow \mathbb{R}$  of  $f$ , then we're done since we can apply the **straight line nullhomotopy** on  $\mathbb{R}$  since

$$\begin{array}{ccc} & \mathbb{R} & \\ \tilde{f} \nearrow & \downarrow p & \\ \mathbb{R}P^2 & \xrightarrow{f} & S^1 \end{array}$$

and consider  $f = p \circ \tilde{f}$  compose **nullhomotopy** with  $p$ , so  $f \simeq$  constant map. Specifically, since  $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$  and  $\pi_1(S^1) = \mathbb{Z}$ , hence

$$f_* (\pi_1(\mathbb{R}P^2)) = 0$$

since  $\mathbb{Z}$  has no (nonzero) torsion. So it **lifts** by **Proposition 3.3**. ■

Now we can proof **Proposition 3.3**.

*Proof.* We prove two directions as follows.

**Necessary.** We see that we can **factorize**  $f_*$  as

$$f_* = p_* \circ \tilde{f}_*$$

follows from the **functoriality** of  $\pi_1$ .

**Sufficient.** Let  $x \in X$ . Choose a path  $\gamma$  from  $x_0$  to  $x$  by the assumption that  $X$  is path-connected. Then,  $f\gamma$  has a unique lift starting at  $\tilde{y}_0$ , denote by  $\tilde{f}\gamma$ . Now, define

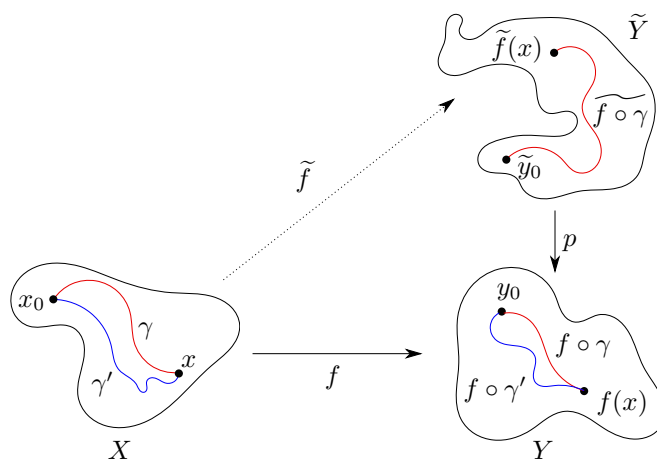
$$\tilde{f}(x) = \tilde{f}\gamma(1).$$

Then, we need to check

1.  $\tilde{f}$  is well-defined. Suppose  $\gamma, \gamma'$  are paths in  $X$  from  $x_0$  to  $x$ . We want to show

$$\tilde{f}\gamma'(1) = \tilde{f}\gamma(1).$$

Since  $\gamma \cdot \overline{\gamma'}$  is a loop in  $X$  at  $x_0$ , we know that  $[(f\gamma) \cdot (f\overline{\gamma'})]$  is a class of loops in  $Y$  in  $\text{Im}(f_*)$ . By hypothesis, this class of loops is in  $\text{Im}(p_*)$ . It lifts to a loop which is based at  $\tilde{y}_0$ . By uniqueness of lifts, this loop lifting  $(f\gamma) \cdot (f\overline{\gamma'})$  to  $\tilde{Y}$  must be equal to the lifts  $\tilde{f}\gamma \cdot \overline{\tilde{f}\gamma'}$  with a common value at  $t = 1/2$ . Hence,  $\tilde{f}\gamma(1) = \tilde{f}\gamma'(1)$  as desired, namely the endpoints agree.

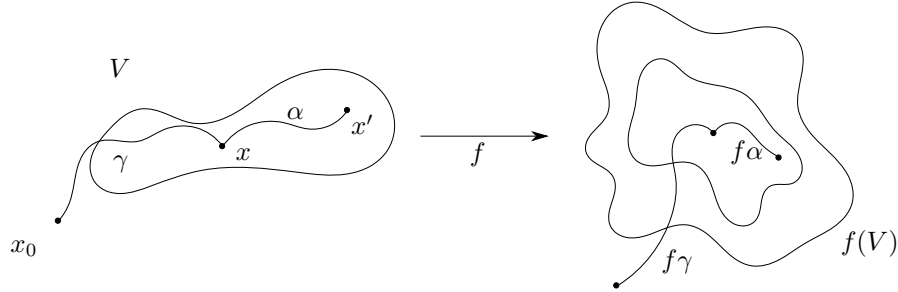


## Lecture 16: Proving Proposition 3.3

11 Feb. 10:00

2.  $\tilde{f}$  is continuous. Choose  $x \in X$  and a neighborhood  $\tilde{U}$  of  $\tilde{f}(x)$  in  $\tilde{Y}$ . Note that we can choose  $\tilde{U}$  small enough to  $p|_{\tilde{U}}$  is homeomorphism to  $U$  in  $Y$ . Now, there exists a neighborhood  $V$  of  $x$  in  $X$  with  $f(V) \subseteq U$ .





The goal is  $\tilde{f}(V) \subseteq \tilde{U}$ . Without loss of generality, we can assume that  $V$  is [path](#)-connected. Then,

$$\widetilde{f\gamma} \cdot \widetilde{f\alpha} = \widetilde{[f\gamma \cdot f\alpha]}.$$

Hence,

$$\widetilde{f\alpha} = (p|_{\tilde{U}})^{-1} \circ f \circ \alpha,$$

where  $(p|_{\tilde{U}})^{-1}$ 's image is in  $\tilde{U}$ , so

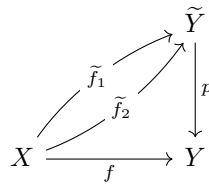
$$\tilde{f}(x') = f\gamma \cdot f\alpha(1) \in \tilde{U},$$

which implies

$$\tilde{f}(V) \subseteq \tilde{U}.$$

■

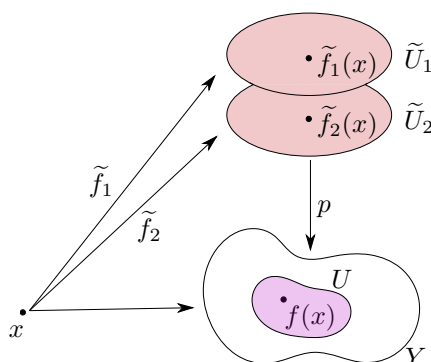
**Proposition 3.4.** Let  $p: \tilde{Y} \rightarrow Y$  be a [covering map](#) with  $X$  is a connected space. If two [lifts](#)  $\tilde{f}_1, \tilde{f}_2$  of the same map  $f$  agree at a single point, then they agree everywhere.



*Proof.* Let  $S$  being

$$S := \{x \in X \mid \tilde{f}_1(x) = \tilde{f}_2(x)\}.$$

We want to show that  $S$  is both closed and open, so if  $S$  is nonempty,  $S = X$ .



We see that  $\tilde{U}_1$  and  $\tilde{U}_2$  are slices of  $p^{-1}(U)$ , where  $U$  is evenly covered neighborhood of  $f(x)$ .

1. If  $\tilde{f}_1(x) \neq \tilde{f}_2(x)$ . Then  $\tilde{U}_1, \tilde{U}_2$  are disjoint. Since  $\tilde{f}_1, \tilde{f}_2$  are continuous, there exists a neighborhood  $N$  of  $x$  with

$$\tilde{f}_1(N) \subseteq \tilde{U}_1, \quad \tilde{f}_2(N) \subseteq \tilde{U}_2,$$

with the fact that they're disjoint, so  $x$  is an interior point of  $S^c$ .

2. If  $\tilde{f}_1(x) = \tilde{f}_2(x)$ . Then  $\tilde{U}_1 = \tilde{U}_2$ . Choose  $N$  as before, then we have

$$\tilde{f}_1(n) = (p|_{\tilde{U}_1})^{-1}(f(n)) = \tilde{f}_2(n),$$

hence  $x \in \text{int}(S)$ .

■

### 3.2 Deck Transformation

We now want to introduce a special kind of transformation.

**Definition 3.3 (Isomorphism of Covers).** Given covering maps

$$p_1: \tilde{X}_1 \rightarrow X, \quad p_2: \tilde{X}_2 \rightarrow X,$$

an *isomorphism of covers* is a homeomorphism

$$f: \tilde{X}_1 \rightarrow \tilde{X}_2$$

such that  $p_1 = p_2 \circ f$ .

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{f} & \tilde{X}_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

**Exercise.** This defines equivalent relation on [covers](#) of  $X$ .

**Definition 3.4 (Deck transformation).** Given a [covering map](#)  $p: \tilde{X} \rightarrow X$ , the [isomorphisms of covers](#)  $\tilde{X} \rightarrow \tilde{X}$  are called *Deck transformation*. Furthermore, we'll let  $G(\tilde{X})$  denotes the *set of deck transformations*.

**Note.** Note that we've suppressed the data of  $p$  in the notation, but this data is essential to what a [deck transformation](#) is, when this is unclear we write  $G(\tilde{X}, p)$ .

## Lecture 17: Deck Transformation

14 Feb. 10:00

**Example.** Let's see some examples.

1.  $p: \mathbb{R} \rightarrow S^1$ .
  - [Deck maps](#): translation by  $n \in \mathbb{Z}$ .
  - Group:  $\mathbb{Z}$
2.  $p: S^1 \rightarrow S^1$  be an  $N$ -sheeted cover.
  - [Deck maps](#): rotation by  $2\pi/N$ .
  - Group:  $\mathbb{Z} / N\mathbb{Z}$

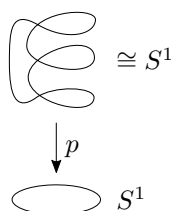


Figure 16:  $p: S^1 \rightarrow S^1$  be an  $N$ -sheeted cover, where  $N = 3$ .

**Exercise.** Given  $X, \tilde{X}$  are [path-connected](#), locally [path-connected](#), [deck map](#) is determined by the image of any one point.

**Answer.**

$$\begin{array}{ccc}
 & & \tilde{X} \\
 & \nearrow f & \downarrow p \\
 \tilde{X} & \xrightarrow{p} & X
 \end{array}$$

**Definition 3.5 (Regular, Normal).** A [covering space](#)  $p: \tilde{X} \rightarrow X$  is *regular* or *normal* if  $\forall x_0 \in X, \forall \tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$ , there exists a [deck map](#)

$$\tilde{x}_0 \mapsto \tilde{x}_1.$$



Figure 17: Covers of  $S^1 \vee S^1$ . The left one is **regular**, while the right one is not since there is no automorphism from  $\tilde{x}_0$  to  $\tilde{x}_1$  or  $\tilde{x}_2$ .

**Definition 3.6 (Normalizer).** Given  $G$  as a group,  $H \subseteq G$  is a subgroup of  $G$ . Then the *normalizer* of  $H$ , denoted by  $N(H)$ , is defined as

$$N(H) := \{g \in G \mid gH = Hg\}.$$

**Example.** Let's see some examples.

1.  $N(H)$  is a subgroup.
2.  $H \leq N(H)$ .
3.  $H$  is normal in  $N(H)$ .
4. If  $H \leq G$  is normal,  $N(H) = G$ .
5.  $N(H)$  is the largest subgroup (under containment) of  $G$  containing  $H$  as normal subgroup.

**Proposition 3.5.** Given  $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  be a **cover**, and  $\tilde{X}, X$  are **path-connected**, locally **path-connected**. Let

$$H = p_* \left( \pi_1(\tilde{X}, \tilde{x}_0) \right) \subseteq \pi_1(X, x_0).$$

Then

1.  $p$  is **normal** if and only if  $H \subset \pi_1(X, x_0)$  is **normal**.
2. We have

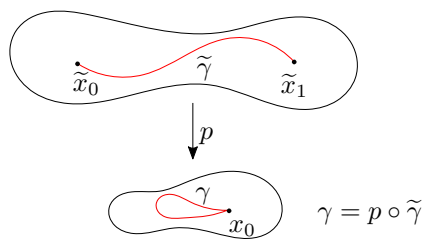
$$G(\tilde{X}) \cong N(H) / H,$$

where  $G(\tilde{X})$  are **Deck maps**, and  $N(H)$  is the **normalizer** of  $H$  in  $\pi_1(X, x_0)$ .

**Remark.** A fact is worth noting is the following. Let  $\tilde{\gamma}$  be a **path**  $\tilde{x}_1$  to  $\tilde{x}_0$ . Then

$$p_* \left( \pi_1(\tilde{X}, \tilde{x}_0) \right) = [\gamma]H[\gamma^{-1}]$$

where  $H \in \pi_1(\tilde{X}, \tilde{x}_1)$ .



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## Appendix

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## References

- [HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: <https://books.google.com/books?id=BjKs86kosqC>.