

MATH602
Real Analysis II

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Abstract

This is a graduate level functional analysis taught by [Joseph Conlon](#). The prerequisites include linear algebra, complex analysis and also [real analysis](#). We'll use Peter Lax[[Lax02](#)] and Reed-Simon[[RS80](#)] as textbooks.

This course is taken in Fall 2022, and the date on the covering page is the last updated time.

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Chapter 1

Introduction

Lecture 1: Introduction

We first briefly review different kinds of vector spaces.

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1.1 Linear Space

Definition 1.1.1 (Linear vector space). A set with operations of addition and multiplication (by a scalar) is called a *linear vector space*.

Example. Denote the multiplicative scalar by λ , then

- $\lambda \in \mathbb{R} \Rightarrow$ real vector space.
- $\lambda \in \mathbb{C} \Rightarrow$ complex vector space

Lemma 1.1.1. Given E a linear vector space, if $v, w \in E$, $\lambda, \mu \in \mathbb{R}$ (or \mathbb{C}), then $\lambda v + \mu w \in E$.

we also have usual rules of associativity and commutativity.

Example. \mathbb{R}^n a n dimensional linear vector space, \mathbb{C}^n a n dimensional complex linear vector space.

We concentrate on ∞ dimensional linear vector space.

Example. Let K is a compact Hausdorff space, then

$$E = \{f: K \rightarrow \mathbb{R} \mid f(\cdot) \text{ is continuous}\}.$$

We then see that E is an ∞ dimensional real linear vector space.

1.2 Quotient Space

Observe that a linear vector space can have many subspaces. Say E is a linear vector space, and $E_1 \subset E$ where E_1 is a proper subspace, i.e., $E_1 \neq E$.

Definition 1.2.1 (Quotient Space). The *quotient space* E/E_1 is the set of equivalence classes of vectors in E where equivalence is given by $x \sim y$ if $x - y \in E_1$. Additionally, denote $[x]$ as the equivalence class of $x \in E$, i.e., $[x] = x + E_1$.

Note that E/E_1 is a linear vector space since if $x_1 + x_2 \in E$, $[x_1] + [x_2] = [x_1 + x_2]$, and also, $\lambda[x] = [\lambda x]$ for $\lambda \in \mathbb{R}$ or \mathbb{C} , i.e., $v, w \in E/E_1$, $\lambda, \mu \in \mathbb{R}$ or \mathbb{C} implies $\lambda v + \mu w \in E$.

Definition 1.2.2 (Codimension). If E / E_1 has finite dimension, then the dimension of E / E_1 is called the *codimension* of E_1 in E .

Example. There exists the case that $\dim(E) = \infty$, $\dim(E_1) < \infty$ where $\dim(E / E_1) < \infty$.

Proof. Let $E = \{f: K \rightarrow \mathbb{R} \mid f(\cdot) \text{ continuous}\}$, and $E_1 = \{f \in E: f(k_1) = 0\}$ where $k_1 \in K$ is fixed. We see that the dimension of E / E_1 is exactly 1 since E / E_1 is the set of constant functions. \circledast

Theorem 1.2.1. If E is finite dimensional, then $\text{codim}(E_1) + \dim(E_1) = \dim(E)$

Definition 1.2.3 (Linear operator). A map $T: E \rightarrow F$ between 2 linear spaces is a *linear operator* if it preserves the properties of addition and multiplication by a scalar, i.e., $T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$ for $v, w \in E$ and $\lambda, \mu \in \mathbb{R}$ or \mathbb{C} .

Definition. Given a linear operator $T: E \rightarrow F$ we have the following.

Definition 1.2.4 (Kernel). The *kernel* of T is the subspace $\ker(T) = \{x \in E \mid Tx = 0\}$.

Definition 1.2.5 (Image). The *image* of T is the subspace $\text{Im}(T) = \{Tx \in F \mid x \in E\}$.

1.3 Normed Spaces

We review some basic notions.

Definition 1.3.1 (Norm). Let E be a linear vector space. A *norm* $\|\cdot\|: E \rightarrow \mathbb{R}$ on E is a function from E to \mathbb{R} with the properties:

- (a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.
- (b) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbb{R}$ or \mathbb{C} .
- (c) $\|x + y\| \leq \|x\| + \|y\|$.

Notation (Dilation). We say that the second condition is the *dilation* property.

Definition 1.3.2 (Normed vector space). A linear vector space E equipped with a norm $\|\cdot\|$ is called a *normed vector space*.

Remark (Induced metric space). A normed vector space E induces a *metric space* with metric $d(x, y) = \|x - y\|$, where the metric has properties

- (a) $d(x, y) \geq 0$. Also, $d(x, x) = 0$ and $d(x, y)$ implies $x = y$.
- (b) $d(x, y) = d(y, x)$.
- (c) $d(x, z) \leq d(x, y) + d(y, z)$.

Example (Bounded sequences ℓ_∞). Let ℓ_∞ be the space of bounded sequences $x = (x_1, x_2, \dots)$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \dots$. Then we define $\|x\| = \|x\|_\infty = \sup_{i \geq 1} |x_i|$.

Example (Absolutely summable sequences ℓ_1). Let ℓ_1 be the space of absolutely summable sequences $x = (x_1, x_2, \dots)$ and $\sum_{i=1}^{\infty} |x_i| < \infty$. Then we define $\|x\| = \|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$.

Example (Continuous functions $C(k)$). The space $C(k)$ of continuous functions $f: K \rightarrow \mathbb{R}$ where K is compact Hausdorff. Then we define $\|f\| = \|f\|_{\infty} = \sup_{x \in K} |f(x)|$.

1.3.1 Geometry of Normed Spaces

Definition 1.3.3 (Ball). A (closed) *ball* centered at a point $x_0 \in E$ with radius $r > 0$ is the set $B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\}$.

Definition 1.3.4 (Sphere). The *sphere* centered at x_0 with radius $r > 0$ is the set $S(x_0, r) = \{x \in E \mid \|x - x_0\| = r\}$.

Remark. We see that $S(x_0, r)$ is the **boundary** of $B(x_0, r)$, i.e., $S(x_0, r) = \partial B(x_0, r)$.

Note (Nonequivalency in infinite dimensional spaces). We know that in finite dimensional, all **norms** are equivalent, which is not true for infinite dimensional vector spaces.

This has something to do with the geometry of **balls**.

Explicitly, **balls** can have different geometries depending on the properties of the **norms**. We see that an $\|\cdot\|_{\infty}$ can have multiple supporting hyperplane at the corner, while for an $\|\cdot\|_2$ can have only one at each point.

Also, unit **balls** for $\|\cdot\|_1$ is also a **square**, where we have

$$B(0, 1) = \{x = (x_1, x_2, \dots) \mid -1 < y_{\epsilon} < 1 \forall \epsilon\}$$

such that $y_{\epsilon} = \sum_{i=1}^{\infty} \epsilon_i x_i$, $\epsilon_i = \pm 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \dots)$.

We see that different **norms** give different geometry, but they have important common features, most notably, convexity properties.

Definition 1.3.5 (Convex set). Given E a **linear vector space**, a set $K \subset E$ is *convex* if $x, y \in K$ and $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda)y \in K$.

Definition 1.3.6 (Convex function). Given E a **linear vector space**, a function $f: E \rightarrow \mathbb{R}$ is called *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for $x, y \in E$, $0 \leq \lambda \leq 1$.

Remark. If $f: E \rightarrow \mathbb{R}$ is a **convex function**, then for any $M \in \mathbb{R}$ the set $\{x \in E \mid f(x) \leq M\}$ is **convex**.

The upshot is that **norms** are **convex**, and the unit **balls** are **convex** as well.

Lecture 2: Banach Spaces and Completion

Let's first see a proposition.

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Proposition 1.3.1. Let $\{E, \|\cdot\|\}$ be a **normed linear space**. Then the norm is **convex** and continuous.

Proof. Let $f: E \rightarrow \mathbb{R}$ be $f(x) = \|x\|$. Then $f(x) - f(y) = \|x\| - \|y\| \leq \|x - y\|$, which implies $|f(x) - f(y)| \leq \|x - y\|$ for $x, y \in E$, i.e., f is Lipschitz continuous. For **convexity**, let $0 < \lambda < 1$,

we have

$$f(\lambda x + (1 - \lambda)y) = \|\lambda x + (1 - \lambda)y\| \leq \|\lambda x\| + \|(1 - \lambda)y\| = \lambda \|x\| + (1 - \lambda) \|y\| = \lambda f(x) + (1 - \lambda)f(y).$$

■

Note. Note that $f(\cdot)$ is continuous implies the closed ball

$$B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\} = \{x \in E \mid f(x - x_0) \leq r\}$$

is closed in topology of E . Also, $f(\cdot)$ is **convex** implies $B(x_0, r)$ is **convex**.

Remark. If $f: E \rightarrow \mathbb{R}$ is **convex**, then the sets $\{x \in E \mid f(x) \leq M\}$ is also **convex**. However, it's possible to have non-**convex functions** f such that all sets $\{x \in E \mid f(x) \leq M\}$ are **convex**.

Example. Take $f(x) = |x|^p$ for $x \in \mathbb{R}$ and $p > 0$. We see that f is **convex** if $p > 1$, and non-**convex** if $p < 1$. The sets $\{x \in \mathbb{R} \mid f(x) \leq M\}$ all **convex** since it's independent of p .

Lemma 1.3.1. Suppose $x \mapsto \|x\|$ satisfies

- (a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.
- (b) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbb{R}$ or \mathbb{C} .
- (c) The unit ball $B(0, 1)$ is **convex**.

Then $f(x) = \|x\|$ satisfies the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$.

Proof. We see that if the third condition is true, then for $u, v \in B(0, 1)$ and $0 < \lambda < 1$, we have $\lambda u + (1 - \lambda)v \in B(0, 1)$. Let $x, y \in E$, and

$$\lambda = \frac{\|x\|}{\|x\| + \|y\|} \Rightarrow 1 - \lambda = \frac{\|y\|}{\|x\| + \|y\|}.$$

By letting $u = x / \|x\|$, $v = y / \|y\|$ we see that

$$\lambda u + (1 - \lambda)v = \frac{\|x\|}{\|x\| + \|y\|} \frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|} \frac{y}{\|y\|} \in B(0, 1) \Rightarrow \left\| \frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|} \right\| \leq 1.$$

From the second condition, it follows that $\|x + y\| \leq \|x\| + \|y\|$, which is the triangle inequality. ■

Remark. If $x \mapsto \|x\|$ satisfies the first two condition and is a **convex**, then it satisfies the triangle inequality.

Proof. Since $\frac{1}{2} \|x + y\| = \left\| \frac{x}{2} + \frac{y}{2} \right\| \leq \frac{1}{2} \|x\| + \frac{1}{2} \|y\|$. ⊗

Now, given a **quotient space** E / E_1 , the question is can we try to define a **norm**?

Problem 1.3.1. On E / E_1 , is $\|[x]\| := \inf_{y \in E_1} \|x + y\|$ a **norm**?

Answer. We see that if $x \in \overline{E_1} \setminus E_1$, then $\|[x]\| = 0$ but $[x] \neq 0 \in E / E_1$. ⊗

Note. Notice the difference from finite dimensional situation. All finite dimensional spaces E_1 are closed but not in general if E_1 has ∞ dimensions.

Example. Let $\ell_1(\mathbb{R})$ be the sequence of x_n for $n \geq 1$ in \mathbb{R} such that $\sum_{i=1}^{\infty} |x_i| \leq \infty$. Define

$$\|x\|_1 := \sum_{i=1}^{\infty} |x_i|,$$

and let E_1 be all sequences with finite number of the x_n are nonzero. We see that $\overline{E_1} = \ell_1(\mathbb{R})$ is infinite dimensional.

Proposition 1.3.2. Let $\{E, \|\cdot\|\}$ be a **normed space** and $E_1 \subseteq E$, E_1 is closed. Then

$$\|\cdot\| : E/E_1 \rightarrow \mathbb{R}, \quad \|[x]\| = \inf_{y \in E_1} \|x + y\|$$

is a **norm** on E/E_1 .

Proof. If $\|[x]\| = 0$, then $\inf_{y \in E_1} \|x - y\| = 0$, which implies $x \in E_1$ since E_1 is closed, so $[x] = 0$. Also, since

$$\|\lambda[x]\| = \inf_{y \in E_1} \|\lambda x + y\| = \inf_{z \in E_1} \|\lambda x + \lambda z\| = |\lambda| \inf_{z \in E_1} \|x + z\| = |\lambda| \|[x]\|,$$

the dilation property is satisfied. Finally, for triangle inequality, we have

$$\|[x] + [y]\| = \inf_{x_1, y_1 \in E_1} \|x + y + x_1 + y_1\| \leq \inf_{x_1 \in E_1} \|x + x_1\| + \inf_{y_1 \in E_1} \|y + y_1\| = \|[x]\| + \|[y]\|.$$

■

Remark. This shows that the only obstacle for this kind of **norm** being an actual **norm** is the closeness of E_1 .

Chapter 2

Banach Spaces

2.1 Introduction

Definition 2.1.1 (Banach space). A **linear normed space** is a *Banach space* if it's complete, i.e., every Cauchy sequence converges.

Note. If $x_n \in E$, $n \geq 1$ is a sequence with property such that $\lim_{m \rightarrow \infty} \sup_{n \geq m} \|x_n - x_m\| > 0$, then $\exists x_\infty \in E$ such that $\lim_{n \rightarrow \infty} \|x_n - x_m\| = 0$.

Example. The spaces ℓ_1 , ℓ_∞ and $C(K)$ are **Banach spaces**.

We want to give a different criterion for showing $\{E, \|\cdot\|\}$ is **Banach**. Let E be a **linear normed space** and $\{x_\ell \mid \ell \geq 1\}$ a sequence in E .

Definition 2.1.2 (Absolutely summable). A sequence is *absolutely summable* if $\sum_{i=1}^{\infty} \|x_i\| < \infty$.

Theorem 2.1.1 (Criterion for completeness). A **normed space** $(E, \|\cdot\|)$ is a **Banach space** if and only if every absolutely summable series in E converges.

Proof. We need to prove two directions.

(\Rightarrow) Suppose E is a **Banach space** and $\{x_k \mid k \geq 1\}$ an **absolutely summable** series. Set $s_n = \sum_{k=1}^n x_k$, $n \geq 1$, we want to show s_n is Cauchy, and if this is the case, completeness of E implies $\exists s_\infty$ and $\lim_{n \rightarrow \infty} \|s_n - s_\infty\| = 0$. Let $n > m$, we see that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \leq \sum_{k=m+1}^{\infty} \|x_k\|.$$

Observe that $\lim_{m \rightarrow \infty} \sum_{k=m+1}^{\infty} \|x_k\| = 0$, we see that the sequence $\{s_n\}$ is Cauchy.

(\Leftarrow) Conversely, suppose E is **not** complete. Then there exists a Cauchy sequence $\{x_n \mid n \geq 1\}$ which does not converge. Furthermore, no subsequence of $\{x_n \mid n \geq 1\}$ converges.^a We now construct an **absolutely summable** series which does not converge.

Define $n(1) \geq 1$ such that $\|x_n - x_{n(1)}\| \leq \frac{1}{2}$ if $n \geq n(1)$, similarly, let $n(2) > n(1)$ be such that $\|x_n - x_{n(2)}\| \leq \frac{1}{2^2}$ if $n > n(2)$. In all, we have $n(1) < n(2) < n(3) < \dots$ such that $\|x_n - x_{n(k)}\| \leq \frac{1}{2^k}$

if $n > n(k)$. Define $w_j := x_{n(j+1)} - x_{n(j)}$ for $j = 1, 2, \dots$. We see that

$$x_{n(m)} = x_{n(1)} + \sum_{j=1}^m w_j$$

for $m = 1, 2, \dots$, and $\{x_{n(m)}\}$ does not converge, hence so does the series $\sum_{j=1}^{\infty} w_j$. However, $\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$, which implies $\{w_j\}$ is **absolutely summable**. ■

^aOtherwise, the whole sequence converges by the fact that it's Cauchy.

2.2 Completion of Normed Space to Banach Space

Theorem 2.2.1. Suppose E is a **normed space**. Then there exists a **Banach space** \hat{E} called a completion of E with the following properties:

- (a) There exists a linear map $i: E \rightarrow \hat{E}$ such that $\|ix\| = \|x\|$.^a
- (b) $\text{Im}(i)$ is dense in \hat{E} , and \hat{E} is the smallest **Banach space** containing image of E .

^aThis is called an *isometric embedding* of E into \hat{E} .

Lecture 3: Banach, Inner Product Spaces

Example (Banach spaces). We already showed spaces ℓ_1 and ℓ_{∞} are Banach spaces.

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We now want to generalize to ℓ_p with $1 < p < \infty$. For $x = \{x_n, n \geq 1\} \in \ell_p$ and if $\sum_{n=1}^{\infty} |x_n|^p < \infty$, for $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$, we want to show that $x \rightarrow \|x\|_p$ satisfies properties of a **norm**. The first two properties of a **norm** is easy check. As for triangle inequality, we have the following.

Lemma 2.2.1 (Minkowski inequality). Let $1 \leq p < \infty$, for $x, y \in \ell_p$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof. Recall that from **Lemma 1.3.1**, we only need to show that $B(0, 1)$ is **convex**, where

$$B(0, 1) = \left\{ x = \{x_n : n \geq 1\} \mid f(x) = \sum_{n=1}^{\infty} |x_n|^p \leq 1 \right\}.$$

But $f(x)$ is **convex** since $x \mapsto |x|^p$, $x \in \mathbb{R}$ is **convex** if $p \geq 1$, we're done. Hence, $\|x + y\|_p \leq \|x\|_p + \|y\|_p$, i.e.,

$$\left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p}.$$

■

Lemma 2.2.2 (Hölder's inequality). Let $1 < p < \infty$, for $x \in \ell_p$, $y \in \ell_q$, we have

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q$$

where $1/p + 1/q = 1$.

Proof. Note first that we can assume without loss of generality, $\sum_{j=1}^{\infty} |x_j|^p = \sum_{j=1}^{\infty} |y_j|^q = 1$.

Then, result follows from the **Young's inequality**,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

for $x, y > 0, x, y \in \mathbb{R}$.

Remark (Legendre transform and the inequality). **Young's inequality** is a special case of the inequality

$$xy \leq f(x) + \mathcal{L}f(y)$$

where $\mathcal{L}f(\cdot)$ is the **Legendre transform** of $f(\cdot)$, i.e., $\mathcal{L}f(y) = \sup_x [xy - f(x)]$.

If f is **convex**, then the function $xy \mapsto xy - f(x)$ is concave so has unique maximum. And $\mathcal{L}f(\cdot)$ always **convex** even if $f(\cdot)$ is not. In particular, if $f(x) = x^p/p$, then $\mathcal{L}f(y) = y^q/q$.

■

Note. **Minkowski inequality** is usually proved via the **Hölder's inequality**. To show this, since

$$\sum_{j=1}^{\infty} |x_j + y_j|^p \leq \sum_{j=1}^{\infty} |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^{\infty} |y_j| |x_j + y_j|^{p-1}.$$

Then **Holder inequality** implies

$$\sum_{j=1}^{\infty} |x_j| |x_j + y_j|^{p-1} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |x_j + y_j|^{(p-1)q} \right)^{1/q},$$

and we're done.^a

^aNote that $(p-1)q = p$.

Remark. The above argument applies to more general spaces of p integrable functions. Let (Ω, Σ, μ) be a measure space and $L_p(\Omega, \Sigma, \mu)$ where all Σ measure functions $f: \Omega \rightarrow \mathbb{R}$ (or \mathbb{C}) such that $\int_{\Omega} |f|^p d\mu < \infty$. Then, $L_p(\Omega, \Sigma, \mu)$ is a **normed space** with **norm**

$$\|f\|_p = \left(\int_{\Omega} |f|^p d\mu \right)^{1/p}.$$

It's more tricky to show that L^p is a **Banach space**, but it's indeed still the case.

Theorem 2.2.2. The p -integrable space $L_p(\Omega, \Sigma, \mu)$ is a **Banach space**.

Proof. Let $\{f_n: n \geq 1\}$ be an **absolutely summable** sequence in L^p . Then the **norm** satisfies

$$\left\| \sum_{k=1}^N f_k \right\|_p \leq \sum_{k=1}^N \|f_k\|_p \leq C.$$

Hence, $\int_{\Omega} \left| \sum_{k=1}^N f_k \right|^p d\mu \leq C^p$.

- Assume all f_k are non-negative. From **monotone convergence theorem**, we have

$$\lim_{N \rightarrow \infty} \int_{\Omega} \left(\sum_{k=1}^N f_k \right)^p d\mu = \int_{\Omega} \left(\sum_{k=1}^{\infty} f_k \right)^p d\mu \leq C^p.$$

Hence, $g = \sum_{k=1}^{\infty} f_k \in L_p$. We now want to show that $\sum_{k=1}^N f_k \rightarrow g$ in L_p . Set $r_n =$

$\sum_{k=n+1}^{\infty} f_k$ where r_n is a decreasing sequence where $r_n \rightarrow 0$ a.e. and also

$$\int_{\Omega} r_1^p d\mu < \infty.$$

This means that $\lim_{n \rightarrow \infty} \|r_n\|_p = 0$ by **dominate convergence theorem**.

- For arbitrary $f_k: \Omega \rightarrow \mathbb{R}$, write $f_k = f_k^+ + f_k^-$ where $f_k^+ = \sup(f_k, 0)$ and $f_k^- = \inf(f_k, 0)$. The sequence $\{f_k^+: k \geq 1\}$ are **absolutely summable**, and we just proceed as before. Similarly, if $f_k: \Omega \rightarrow \mathbb{C}$.

■

Chapter 3

Hilbert Spaces

3.1 Inner Product Spaces

Definition 3.1.1 (Inner product). Let E be a linear space over \mathbb{C} . An *inner product* $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{C}$ is a function which has the following properties:

- (a) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.
- (b) $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$ for $a, b \in \mathbb{C}$.
- (c) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

Remark (Real inner product). We can also define inner products of spaces over \mathbb{R} with no extra conjugation in the last property.

Definition 3.1.2 (Inner product space). An *inner product space* is a linear space E with an inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{C}$.

Definition 3.1.3 (Orthogonal). Given a linear space E , $x, y \in E$ are *orthogonal* if $\langle x, y \rangle = 0$, denote as $x \perp y$.

Theorem 3.1.1 (Cauchy-Schwarz inequality). Let $x, y \in E$ and an inner product $\langle \cdot, \cdot \rangle$, then

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}.$$

Proof. Define $Q(t)$ by $Q(t) = \langle x + ty, x + ty \rangle = \langle y, y \rangle t^2 + 2t \operatorname{Re} \langle x, y \rangle + \langle x, x \rangle$ if $t \in \mathbb{R}$. Then we see that $Q(t) \geq 0$ with $t \in \mathbb{R}$ and the equation $Q(t) = 0$ has no real roots, implying $(\operatorname{Re} \langle x, y \rangle)^2 \leq \langle x, x \rangle \langle y, y \rangle$. Finally, the result follows by choosing $\theta \in \mathbb{R}$ such that

$$\langle x, y \rangle = \operatorname{Re} \langle x e^{i\theta}, y \rangle.$$

■

Corollary 3.1.1. The function $x \mapsto \|x\| := \langle x, x \rangle^{\frac{1}{2}}$ is a norm on E .

Proof. The triangle inequality is a consequence of Theorem 3.1.1 such that

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2 \stackrel{!}{\leq} \|x\|^2 + 2 \|x\| \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

■

Example. The space ℓ_2 of square summable sequences $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$,

$$\langle x, y \rangle := \sum_{j=1}^{\infty} x_j \bar{y}_j.$$

Example. The space $L_2(\Omega, \Sigma, \mu)$ of $f, g \in L_2(\Omega, \Sigma, \mu)$,

$$\langle f, g \rangle = \int_{\Omega} f(x) \bar{g}(x) d\mu(x).$$

Example. The space of $m \times n$ matrices $A = (a_{ij})$, $1 \leq i \leq m, 1 \leq j \leq n$. Then

$$\langle A, B \rangle = \text{Tr } AB^*,$$

where B^* is the **Hermitian adjoint** of B , i.e., for $B = (b_{ij})$, then $B^* = (b_{ij}^*)$ for $b_{ij}^* = \bar{b}_{ji}$.

Remark (Hilbert-Schmidt norm). Specifically, the **norm** corresponding to this **inner product** is

$$\|A\|_{\text{HS}} := \sum_{i,j}^{\infty} \left(|a_{ij}|^2 \right)^{1/2},$$

which is known as the **Hilbert-Schmidt norm**.

For an **inner product space**, the **inner product** can be expressed in terms of the **norm**. This is because both **parallelogram law** and **polarization identity** hold.

Lemma 3.1.1 (Parallelogram law). Given E an **inner product space**, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Lemma 3.1.2 (Polarization identity). Given E an **inner product space**, we have

$$\langle x, y \rangle = \frac{1}{4} \left\{ \|x + y\|^2 - \|x - y\|^2 + i \|x + iy\|^2 - i \|x - iy\|^2 \right\}$$

Lecture 4: Orthogonality and Projection

As previously seen. Recall the **parallelogram law** and **polarization identity**. The proof is just to expand the right-hand side in terms of **inner product**.

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Check it!

Remark. **Polarization identity** shows that the function $x \mapsto \|x\|^2$ determines the **inner product**.

3.2 Hilbert Spaces

Definition 3.2.1 (Hilbert space). A complete **inner product space** is called a *Hilbert space*.

Example. We have seen that ℓ_2 and $L^2(\Omega, \Sigma, \mu)$ are complete, hence are **Hilbert space**.

We'll soon see that the key notion in **Hilbert space** theory is orthogonality.

Definition 3.2.2 (Orthogonal complement). Let $A \subseteq \mathcal{H}$ where \mathcal{H} is a Hilbert space. Then the orthogonal complement A^\perp of A is

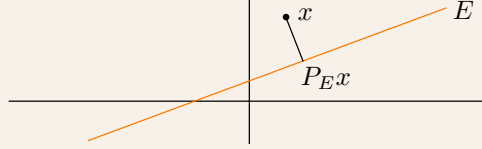
$$A^\perp := \{x \in \mathcal{H} : \langle x, y \rangle = 0 \text{ for } y \in A\}.$$

Remark. A^\perp is also a Hilbert space.

Proof. Since A^\perp is closed linear subspace of \mathcal{H} , where the closure follows from the continuity of the function $x \mapsto \langle x, y \rangle$ for $x \in \mathcal{H}$ by looking at the inverse image of $\{0\}$. \circledast

Theorem 3.2.1 (Orthogonality principle). Assume $E \subseteq \mathcal{H}$ is a closed linear subspace of the Hilbert space \mathcal{H} and $x \in \mathcal{H}$. Then we have the following.

- (a) Then there exists a unique closest point $y = P_E x \in E$ to x , i.e., $\|x - P_E x\| = \inf_{y' \in E} \|x - y'\|$.
- (b) The point $y = P_E x \in E$ is the unique vector such that $x - y \in E^\perp$.



Proof. Note that the function $y' \mapsto \|x - y'\|$ for $y' \in E$ is convex. We expect a minimizer y' .

Note. To show this exists, we typically need

- 1. Compactness properties
- 2. Non-degeneracy properties for uniqueness

- (a) Here by using parallelogram law, we don't need compactness. Let $y_n \in E$ for $n = 1, 2, \dots$ be a minimizing sequence, i.e.,

$$\lim_{n \rightarrow \infty} \|x - y_n\| = \inf_{y' \in E} \|x - y'\| = d.$$

From parallelogram law, we have

$$\|y_n - y_m\|^2 + 4 \left\| x - \frac{1}{2}(y_n + y_m) \right\|^2 = 2 \|x - y_n\|^2 + 2 \|x - y_m\|^2.$$

As $n, m \rightarrow \infty$, the right-hand side goes to $4d^2$. But since $\frac{1}{2}(y_n + y_m) \in E$, we have $\|x - \frac{1}{2}(y_n + y_m)\| \geq d$, so

$$\lim_{m \rightarrow \infty} \sup_{n \geq m} \|y_n - y_m\|^2 = 0,$$

which further implies $\{y_n\}$ is a Cauchy sequence. As \mathcal{H} is complete, we see that $y_n \rightarrow y_\infty \in E$, with $\|x - y_\infty\| = d$.

Now, with the fact that E is closed, we set $y_\infty = P_E x$ where y_∞ is unique since if $\|x - y_\infty\| = \|x - y'_\infty\| = d$, again by the parallelogram law where we now plug in y_∞ and y'_∞ instead of y_n and y_m as above, we see that $\|y_\infty - y'_\infty\| = 0$. In all, $P_E x \in E$ is uniquely defined.

- (b) We now show $P_E x$ is the unique vector $y \in E$ such that $x - y \perp E$, i.e., $x - y \in E^\perp$. Let $y' \in E$ and let $Q(t)$ be the quadratic

$$Q(t) := \langle x - P_E x + ty', x - P_E x + ty' \rangle = \|x - P_E x + ty'\|^2.$$

Since $t \mapsto Q(t)$ has a **strict** minimum at $t = 0$, which implies $Q'(0) = 0$, i.e., $\operatorname{Re}(x - P_E x, y') = 0$ for all $y' \in E$, which further implies $\langle x - P_E x, y' \rangle = 0$ for all $y' \in E$. This shows that $x - P_E x \in E^\perp$. Finally, we need to show $P_E x \in E$ is the unique vector such $x - P_E x \in E^\perp$. This can be seen from $Q(t) = \|x - P_E x\|^2 + t^2 \|y'\|^2$ for any $y' \in E$. ■

Remark. Theorem 3.2.1 shows that the minimizer for the function $y' \mapsto \|x - y'\|$ for $y' \in E$ is characterized by the orthogonality condition, i.e., $x - y \perp E$ for some $y \in E$.

Definition 3.2.3 (Orthogonal projection). Let \mathcal{H} be a Hilbert space and let $E \subseteq \mathcal{H}$ be a closed subspace. The *orthogonal projection operator* $P_E: \mathcal{H} \rightarrow E$ is given by $x \mapsto P_E x$ where $P_E x$ is defined uniquely via $x - P_E x \in E^\perp$.

Definition 3.2.4 (Bounded linear map). Given a mapping $A: \mathcal{B} \rightarrow \mathcal{B}$ on a Banach space \mathcal{B} , we say it's a *bounded linear map* if it's **bounded** and **linear**.

Definition 3.2.5 (Linear map). The operator A is *linear* if for $x, y \in \mathcal{B}$, $a, b \in \mathbb{C}$,

$$A(ax + by) = aA(x) + bA(y).$$

Definition 3.2.6 (Bounded map). The operator A is *bounded* if

$$\|A\| := \sup_{\|x\|=1} \|Ax\| < \infty.$$

Remark. Note that $\|Ax\| \leq \|A\| \|x\|$ for $x \in \mathcal{B}$.

We see that $P_E x$ is a **bounded linear operator** $P_E: \mathcal{H} \rightarrow E$ with the properties $P_E^2 = P_E$ and $\|x\|^2 = \|P_E x\|^2 + \|(I - P_E)x\|^2$ since $(I - P_E)x \perp P_E x$. The latter property shows that

$$\|P_E\| \leq 1, \quad \|(I - P_E)\| \leq 1,$$

and fact, $\|P_E\| = \|I - P_E\| = 1$. Also, $I - P_E$ is also an **orthogonal projection** onto E^\perp .

3.2.1 Orthogonal Systems

We first give the definition.

Definition 3.2.7 (Orthogonal system). A sequence $\{x_k: k \geq 1\}$ of non-zero vectors in a Hilbert space \mathcal{H} is *orthogonal* if $\langle x_k, x_\ell \rangle = 0$ for all $\ell \neq k$.

Definition 3.2.8 (Orthonormal system). A **orthogonal system** is called an *orthonormal system* if in addition, we have $\|x_k\| = 1$ for $k = 1, 2, \dots$

Remark (Equivalence definition of orthonormal system). $\{x_k: k \geq 1\}$ is **orthonormal** if $\langle x_k, x_\ell \rangle = \delta_{k,\ell}$ where δ is the **Kronecker delta**.

We now see some examples.

Example. $x_k = (0, 0, \dots, \delta, 0, \dots, 0) \in \ell_2$ for $k = 1, 2, \dots$ is **orthonormal sequence** in ℓ_2 .

Example (Fourier basis). For $L_2([-\pi, \pi])$,

$$e_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$$

for $k \in \mathbb{R}$ is **orthonormal**. In addition, this is the Fourier basis associated with the Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \frac{1}{\sqrt{2\pi}} e^{ikt}$$

where

$$a_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

Remark. We can further generalize Fourier series to any **Hilbert space** by letting $\{x_k : k \geq 1\}$ be an **orthonormal** set in \mathcal{H} . For $n = 1, 2, \dots$, we define $S_n : \mathcal{H} \rightarrow E_n$ such that

$$S_n(x) = \sum_{k=1}^n \langle x, x_k \rangle x_k$$

for $x \in \mathcal{H}$ where $E_n = \text{span}\{x_1, \dots, x_n\}$. We see that S_n is a **linear operator** and $S_n = P_{E_n}$ is the **orthogonal projection** onto E_n since $\langle x - S_n(x), x_k \rangle = 0$ for $k = 1, \dots, n$ and $S_n(x) \in E_n$, $x - S_n(x) \perp E_n$.

Remark (Bessel's inequality). Additionally,

$$\|S_n(x)\|^2 = \sum_{k=1}^n |\langle x, x_k \rangle|^2,$$

with $S_n = P_{E_n}$ and $\|P_{E_n}x\|^2 \leq \|x\|^2$, we have

$$\sum_{k=1}^n |\langle x, x_k \rangle|^2 \leq \|x\|^2$$

for $x \in \mathcal{H}$. This is the so-called *Bessel's inequality*.

Theorem 3.2.2. Let $\{x_k : k \geq 1\}$ be an **orthonormal** sequence in a **Hilbert space** \mathcal{H} . Then the corresponding Fourier expansion $S_n(x) = \sum_{k=1}^n \langle x, x_k \rangle x_k$ converges, i.e., $\lim_{n \rightarrow \infty} S_n(x) = S_\infty(x)$ exists for $x \in \mathcal{H}$. Furthermore, $S_n = P_{E_n}$ for every n where E_n is the space spanned by $\{x_i\}_{i=1}^n$.^a

^aThis includes $n = \infty$, where E_∞ is the **closure** of the space spanned by $\{x_i\}_i$.

Proof. We show that the sequence $S_n(x)$ for $n = 1, 2, \dots$ is Cauchy. This is because

$$\|S_n(x) - S_m(x)\|^2 = \sum_{k=m+1}^n |\langle x, x_k \rangle|^2,$$

and **Bessel's inequality** implies $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \leq \|x\|^2$. Hence, for any $\epsilon > 0$, there exists $m(\epsilon)$ such that

$$\sum_{k=m(\epsilon)+1}^{\infty} |\langle x, x_k \rangle|^2 < \epsilon,$$

which implies $\|S_n(x) - S_m(x)\|^2 < \epsilon$ if $n > m(\epsilon)$, hence $\{S_n(x) : n \geq 1\}$ is Cauchy, and $\lim_{n \rightarrow \infty} S_n(x) = S_\infty(x) \in \mathcal{H}$. Also, $S_\infty = P_{E_\infty}$ where E_∞ is the closure of the **linear space** generated by the sequence $\{x_k : k \geq 1\}$. ■

Remark. Note that the closeness of E_∞ makes sense since the self-dual of a set's orthogonal complement is itself if it's closed in the first place.

Lecture 5: Abstract Fourier Series

Let's start with a definition.

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Definition 3.2.9 (Complete system). A system of vector $\{x_k : k \geq 1\}$ in Hilbert space \mathcal{H} is *complete* if the space spanned by $\{x_k : k \geq 1\}$ is **dense** in \mathcal{H} .

Example (Fourier inversion formula). If an orthogonal set $\{x_k : k \geq 1\}$ is **complete**, then $E_\infty = \mathcal{H}$, $P_{E_\infty} = I$. This implies

$$x = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k$$

for $x \in \mathcal{H}$. This is **Fourier inversion formula**.

Remark (Parseval's identity). We have $\|x\|^2 = \|P_{E_n}x\|^2 + \|(I - P_{E_n})x\|^2$. By letting $n \rightarrow \infty$, we have

$$\|x\|^2 = \lim_{n \rightarrow \infty} \|P_{E_n}x\|^2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n |\langle x, x_k \rangle|^2 = \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2.$$

Definition 3.2.10 (Separable). A metric space is *separable* if it contains a countable dense subset.

Remark (Banach space). For Banach space, **separability** follows from finding a countable set of vectors $\{x_k : k \geq 1\}$ such that the span of $\{x_k : k \geq 1\}$ is dense in E .

3.3 Gram-Schmidt Orthogonalization

Suppose $x_1, x_2, \dots \in \mathcal{H}$ is a set of vectors and $E_n = \text{span}(\{x_1, \dots, x_n\})$. Then we can find an **orthonormal set** $\{y_k \in \mathcal{H} : k \geq 1\}$ such that $E_n = \text{span}(\{y_1, y_2, \dots, y_{m(n)}\})$ where $m(n) \leq n$.

Firstly, set $y_1 = x_1 / \|x_1\|$, and

$$y_n = \frac{(I - P_{E_{n-1}})x_n}{\|(I - P_{E_{n-1}})x_n\|}$$

if $x_n \notin E_{n-1}$, i.e., E_{n-1} is properly contained in E_n .

Remark. Proving **completeness** of a set of vectors $\{x_k : k \geq 1\}$ in \mathcal{H} can be **non-trivial**.

Example (Haar basis). We consider the *Haar basis* for $L^2([0, 1])$. Let $h : (0, 1) \rightarrow \mathbb{R}$ where

$$h(t) = \begin{cases} 1, & \text{if } 0 < t < 1/2; \\ -1, & \text{if } 1/2 < t < 1. \end{cases}$$

Extend $h(\cdot)$ by zero outside $(0, 1)$, we get $h : \mathbb{R} \rightarrow \mathbb{R}$, $h(t) = 0$ if $t \notin (0, 1)$, otherwise it's the same as above. The function $t \mapsto h(2^k t)$ has support in interval $0 < t < 2^{-k}$. Move the support to interval $\ell 2^{-k} < t < (\ell + 1)2^{-k}$ by translation. Set

$$h_{k,\ell}(t) = h(2^k t - \ell), \quad \ell = 0, 1, \dots, 2^k - 1.$$

The constant function plus functions $h_{k,\ell}$, $k = 0, 1, 2, \dots$, $0 \leq \ell \leq 2^k - 1$ are a **complete**

orthogonal set for $\mathcal{H} = L^2([0, 1])$.

Proof. The span of the Haar functions includes characteristics functions χ_F for all dyadic intervals $[2^{-k}\ell, 2^{-k}(\ell + 1)]$ for $\ell = 0, 1, \dots, 2^k - 1$, $k = 0, 1, \dots$. If the set is **not complete**, then there exists $f \in L^2([0, 1])$ such that

$$\int_F f \, dt = 0$$

for all dyadic intervals F . Since we can approximate any measurable set $E \subseteq (0, 1)$ by a union of dyadic intervals.

Intuition. an easy way to see this is to consider

$$\left\{ F \in \mathcal{B} : \int_F f \, dt = 0 \right\},$$

which is the Borel subalgebra of \mathcal{B} , which indeed is a Borel algebra on $(0, 1)$. Then observe that dyadic intervals generate all open intervals.

Hence, we see that $\int_F f \, dt = 0$ for all measurable $F \subseteq (0, 1)$. Let $F = \{t \in (0, 1) : f(t) > 0\}$, if $m(F) > 0$, then

$$\int_F f \, dt > 0.$$

Hence, a contradiction, so $m(F) = 0$. *

Example (Fourier basis). Consider the Fourier basis $e_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$ for $k \in \mathbb{Z}$, $-\pi < t < \pi$. This is **complete** in $L^2([-\pi, \pi])$.

Proof. We use **Stone-Weierstrass theorem** and apply it to Fourier basis. All $e_k(\cdot)$ are in $C[-\pi, \pi]$, i.e., continuous functions $f : [-\pi, \pi] \rightarrow \mathbb{C}$. We know that $C([-\pi, \pi])$ is a **Banach space** with supremum norm $\|f\| := \sup_{t \in [-\pi, \pi]} |f(t)|$. Stone-Weierstrass theorem implies density of the space spanned by $e_k(\cdot)$, $k \in \mathbb{Z}$ in $C([-\pi, \pi])$, hence the completeness in $L^2([-\pi, \pi])$ follows from the density of continuous functions in $L^2([-\pi, \pi])$. *

3.4 Bounded Linear Functional

Definition. Let E be a **linear space** over \mathbb{R} or \mathbb{C} .

Definition 3.4.1 (Linear functional). A *linear functional* on E is a linear operator $f : E \rightarrow \mathbb{R}$ or \mathbb{C} such that

$$f(ax + by) = af(x) + bf(y)$$

for $x, y \in E$, $a, b \in \mathbb{R}$ or \mathbb{C} .

Definition 3.4.2 (Bounded linear functional). We say a **linear functional** $f(\cdot)$ is a *bounded linear functional* if

$$\|f\| := \sup_{\|x\|=1} |f(x)| < \infty$$

by dilation and additive.

Remark. The boundedness of $f(\cdot)$ implies $|f(x - y)| \leq \|f\| \|x - y\|$ for $x, y \in E$. Hence, $f(\cdot)$ is continuous and in fact Lipschitz continuous.

Remark. Conversely, if a [linear functional](#) is continuous then it is bounded.

Proof. Suppose $f(\cdot)$ is not bounded, then there exists a sequence $x_n \in E$ such that $|f(x_n)| \geq n \|x_n\|$ for $n = 1, 2, \dots$. By linearity,

$$\left| f\left(\frac{x_n}{n \|x_n\|}\right) \right| \geq 1, \quad n = 1, 2, \dots$$

But we know $\lim_{n \rightarrow \infty} \frac{x_n}{n \|x_n\|} = 0$ and $f(0) = 0$, hence $f(\cdot)$ is not continuous at 0. *

Definition 3.4.3 (Dual space). Let E be a [normed space](#). The space of all [bounded linear functionals](#) $f(\cdot)$ on E is known as the *dual space* E^* of E .

Remark. The [dual space](#) is also a [normed space](#) with [norm](#) $\|f\| := \sup_{\|x\|=1} |f(x)|$, which is in fact a [Banach space](#). And it is a [Banach space](#) even if the original E is not.

Definition 3.4.4 (Hyperplane). Let E be a [linear space](#) and $H \subseteq E$ is a subspace. Say H is a *hyperplane* if $\text{codim}(H) = 1$, i.e., $\dim(E/H) = 1$.

The goal is to make an equivalence between [bounded linear functionals](#) on E and *closed hyperplanes* in E .

Problem 3.4.1. Does there exist a **non-closed** [hyperplane](#)?

Answer. We know that this is not the case in finite dimension. And this question is analogous to asking *does there exist a subset $F \subseteq \mathbb{R}$ which is **not** Lebesgue measurable?* The answer to this is yes in both cases. However, construction uses [axiom of choice](#). *

Proposition 3.4.1. Let E be a [linear space](#).

- (a) For every [linear functional](#) on E , $\ker(f)$ is a [hyperplane](#) in E .
- (b) If $f, g \neq 0$ are [linear functionals](#) on E such that $\ker(f) = \ker(g)$, then $f = ag$ for some $a \neq 0$.
- (c) For every [hyperplane](#) $H \subseteq E$, there exists a [linear functional](#) $f \neq 0$ on E such that $\ker(f) = H$.
- (d) If E is a [Banach space](#), then $f(\cdot)$ is bounded if and only if $\ker(f) = H$ is closed.

Proof. ■

Appendix

Appendix A

Additional Proofs

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