

MATH602
Real Analysis II

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Abstract

Additionally, we'll use .

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Chapter 1

Introduction

Lecture 1: Introduction

We first briefly review different kinds of vector spaces.

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1.1 Linear Space

Definition 1.1.1 (Linear vector space). A set with operations of addition and multiplication (by a scalar) is called a *linear vector space*.

Example. Denote the multiplicative scalar by λ , then

- $\lambda \in \mathbb{R} \Rightarrow$ real vector space.
- $\lambda \in \mathbb{C} \Rightarrow$ complex vector space

Lemma 1.1.1. Given E a linear vector space, if $v, w \in E$, $\lambda, \mu \in \mathbb{R}$ (or \mathbb{C}), then $\lambda v + \mu w \in E$.

we also have usual rules of associativity and commutativity.

Example. \mathbb{R}^n a n dimensional linear vector space, \mathbb{C}^n a n dimensional complex linear vector space.

We concentrate on ∞ dimensional linear vector space.

Example. Let K is a compact Hausdorff space, then

$$E = \{f: K \rightarrow \mathbb{R} \mid f(\cdot) \text{ is continuous}\}.$$

We then see that E is an ∞ dimensional real linear vector space.

1.2 Quotient Space

Observe that a linear vector space can have many subspaces. Say E is a linear vector space, and $E_1 \subset E$ where E_1 is a proper subspace, i.e., $E_1 \neq E$.

Definition 1.2.1 (Quotient Space). The *quotient space* E/E_1 is the set of equivalence classes of vectors in E where equivalence is given by $x \sim y$ if $x - y \in E_1$. Additionally, denote $[x]$ as the equivalence class of $x \in E$, i.e., $[x] = x + E_1$.

Note that E/E_1 is a linear vector space since if $x_1 + x_2 \in E$, $[x_1] + [x_2] = [x_1 + x_2]$, and also, $\lambda[x] = [\lambda x]$ for $\lambda \in \mathbb{R}$ or \mathbb{C} , i.e., $v, w \in E/E_1$, $\lambda, \mu \in \mathbb{R}$ or \mathbb{C} implies $\lambda v + \mu w \in E$.

Definition 1.2.2 (Codimension). If E / E_1 has finite dimension, then the dimension of E / E_1 is called the *codimension* of E_1 in E .

Example. There exists the case that $\dim(E) = \infty$, $\dim(E_1) < \infty$ where $\dim(E / E_1) < \infty$.

Proof. Let $E = \{f: K \rightarrow \mathbb{R} \mid f(\cdot) \text{ continuous}\}$, and $E_1 = \{f \in E: f(k_1) = 0\}$ where $k_1 \in K$ is fixed. We see that the dimension of E / E_1 is exactly 1 since E / E_1 is the set of constant functions. \circledast

Theorem 1.2.1. If E is finite dimensional, then $\text{codim}(E_1) + \dim(E_1) = \dim(E)$

Definition 1.2.3 (Linear operator). A map $T: E \rightarrow F$ between 2 linear spaces is a *linear operator* if it preserves the properties of addition and multiplication by a scalar, i.e., $T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$ for $v, w \in E$ and $\lambda, \mu \in \mathbb{R}$ or \mathbb{C} .

Definition. Given a linear operator $T: E \rightarrow F$ we have the following.

Definition 1.2.4 (Kernel). The *kernel* of T is the subspace $\ker(T) = \{x \in E \mid Tx = 0\}$.

Definition 1.2.5 (Image). The *image* of T is the subspace $\text{Im}(T) = \{Tx \in F \mid x \in E\}$.

1.3 Normed Spaces

We review some basic notions.

Definition 1.3.1 (Norm). Let E be a linear vector space. A *norm* $\|\cdot\|: E \rightarrow \mathbb{R}$ on E is a function from E to \mathbb{R} with the properties:

- (a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.
- (b) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbb{R}$ or \mathbb{C} .
- (c) $\|x + y\| \leq \|x\| + \|y\|$.

Notation (Dilation). We say that the second condition is the *dilation* property.

Definition 1.3.2 (Normed vector space). A linear vector space E equipped with a norm $\|\cdot\|$ is called a *normed vector space*.

Remark (Induced metric space). A normed vector space E induces a *metric space* with metric $d(x, y) = \|x - y\|$, where the metric has properties

- (a) $d(x, y) \geq 0$. Also, $d(x, x) = 0$ and $d(x, y)$ implies $x = y$.
- (b) $d(x, y) = d(y, x)$.
- (c) $d(x, z) \leq d(x, y) + d(y, z)$.

Example (Bounded sequences ℓ_∞). Let ℓ_∞ be the space of bounded sequences $x = (x_1, x_2, \dots)$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \dots$. Then we define $\|x\| = \|x\|_\infty = \sup_{i \geq 1} |x_i|$.

Example (Absolutely summable sequences ℓ_1). Let ℓ_1 be the space of absolutely summable sequences $x = (x_1, x_2, \dots)$ and $\sum_{i=1}^{\infty} |x_i| < \infty$. Then we define $\|x\| = \|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$.

Example (Continuous functions $C(k)$). The space $C(k)$ of continuous functions $f: K \rightarrow \mathbb{R}$ where K is compact Hausdorff. Then we define $\|f\| = \|f\|_{\infty} = \sup_{x \in K} |f(x)|$.

1.3.1 Geometry of Normed Spaces

Definition 1.3.3 (Ball). A (closed) *ball* centered at a point $x_0 \in E$ with radius $r > 0$ is the set $B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\}$.

Definition 1.3.4 (Sphere). The *sphere* centered at x_0 with radius $r > 0$ is the set $S(x_0, r) = \{x \in E \mid \|x - x_0\| = r\}$.

Remark. We see that $S(x_0, r)$ is the **boundary** of $B(x_0, r)$, i.e., $S(x_0, r) = \partial B(x_0, r)$.

Note (Nonequivalency in infinite dimensional spaces). We know that in finite dimensional, all **norms** are equivalent, which is not true for infinite dimensional vector spaces.

This has something to do with the geometry of **balls**.

Explicitly, **balls** can have different geometries depending on the properties of the **norms**. We see that an $\|\cdot\|_{\infty}$ can have multiple supporting hyperplane at the corner, while for an $\|\cdot\|_2$ can have only one at each point.

Also, unit **balls** for $\|\cdot\|_1$ is also a **square**, where we have

$$B(0, 1) = \{x = (x_1, x_2, \dots) \mid -1 < y_{\epsilon} < 1 \forall \epsilon\}$$

such that $y_{\epsilon} = \sum_{i=1}^{\infty} \epsilon_i x_i$, $\epsilon_i = \pm 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \dots)$.

We see that different **norms** give different geometry, but they have important common features, most notably, convexity properties.

Definition 1.3.5 (Convex set). Given E a **linear vector space**, a set $K \subset E$ is *convex* if $x, y \in K$ and $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda)y \in K$.

Definition 1.3.6 (Convex function). Given E a **linear vector space**, a function $f: E \rightarrow \mathbb{R}$ is called *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for $x, y \in E$, $0 \leq \lambda \leq 1$.

Remark. If $f: E \rightarrow \mathbb{R}$ is a **convex function**, then for any $M \in \mathbb{R}$ the set $\{x \in E \mid f(x) \leq M\}$ is **convex**.

The upshot is that **norms** are **convex**, and the unit **balls** are **convex** as well.

Lecture 2: Banach Spaces and Completion

Let's first see a proposition.

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Proposition 1.3.1. Let $\{E, \|\cdot\|\}$ be a **normed linear space**. Then the norm is **convex** and continuous.

Proof. Let $f: E \rightarrow \mathbb{R}$ be $f(x) = \|x\|$. Then $f(x) - f(y) = \|x\| - \|y\| \leq \|x - y\|$, which implies $|f(x) - f(y)| \leq \|x - y\|$ for $x, y \in E$, i.e., f is Lipschitz continuous. For **convexity**, let $0 < \lambda < 1$,

we have

$$f(\lambda x + (1 - \lambda)y) = \|\lambda x + (1 - \lambda)y\| \leq \|\lambda x\| + \|(1 - \lambda)y\| = \lambda \|x\| + (1 - \lambda) \|y\| = \lambda f(x) + (1 - \lambda)f(y).$$

■

Note. Note that $f(\cdot)$ is continuous implies the closed ball

$$B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\} = \{x \in E \mid f(x - x_0) \leq r\}$$

is closed in topology of E . Also, $f(\cdot)$ is **convex** implies $B(x_0, r)$ is **convex**.

Remark. If $f: E \rightarrow \mathbb{R}$ is **convex**, then the sets $\{x \in E \mid f(x) \leq M\}$ is also **convex**. However, it's possible to have non-**convex functions** f such that all sets $\{x \in E \mid f(x) \leq M\}$ are **convex**.

Example. Take $f(x) = |x|^p$ for $x \in \mathbb{R}$ and $p > 0$. We see that f is **convex** if $p > 1$, and non-**convex** if $p < 1$. The sets $\{x \in \mathbb{R} \mid f(x) \leq M\}$ all **convex** since it's independent of p .

Lemma 1.3.1. Suppose $x \mapsto \|x\|$ satisfies

- (a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.
- (b) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbb{R}$ or \mathbb{C} .
- (c) The unit ball $B(0, 1)$ is **convex**.

Then $f(x) = \|x\|$ satisfies the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$.

Proof. We see that if the third condition is true, then for $u, v \in B(0, 1)$ and $0 < \lambda < 1$, we have $\lambda u + (1 - \lambda)v \in B(0, 1)$. Let $x, y \in E$, and

$$\lambda = \frac{\|x\|}{\|x\| + \|y\|} \Rightarrow 1 - \lambda = \frac{\|y\|}{\|x\| + \|y\|}.$$

By letting $u = x / \|x\|$, $v = y / \|y\|$ we see that

$$\lambda u + (1 - \lambda)v = \frac{\|x\|}{\|x\| + \|y\|} \frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|} \frac{y}{\|y\|} \in B(0, 1) \Rightarrow \left\| \frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|} \right\| \leq 1.$$

From the second condition, it follows that $\|x + y\| \leq \|x\| + \|y\|$, which is the triangle inequality. ■

Remark. If $x \mapsto \|x\|$ satisfies the first two condition and is a **convex**, then it satisfies the triangle inequality.

Proof. Since $\frac{1}{2} \|x + y\| = \left\| \frac{x}{2} + \frac{y}{2} \right\| \leq \frac{1}{2} \|x\| + \frac{1}{2} \|y\|$. ⊗

Now, given a **quotient space** E / E_1 , the question is can we try to define a **norm**?

Problem 1.3.1. On E / E_1 , is $\|[x]\| := \inf_{y \in E_1} \|x + y\|$ a **norm**?

Answer. We see that if $x \in \overline{E_1} \setminus E_1$, then $\|[x]\| = 0$ but $[x] \neq 0 \in E / E_1$. ⊗

Note. Notice the difference from finite dimensional situation. All finite dimensional spaces E_1 are closed but not in general if E_1 has ∞ dimensions.

Example. Let $\ell_1(\mathbb{R})$ be the sequence of x_n for $n \geq 1$ in \mathbb{R} such that $\sum_{i=1}^{\infty} |x_i| \leq \infty$. Define

$$\|x\|_1 := \sum_{i=1}^{\infty} |x_i|,$$

and let E_1 be all sequences with finite number of the x_n are nonzero. We see that $\overline{E_1} = \ell_1(\mathbb{R})$ is infinite dimensional.

Proposition 1.3.2. Let $\{E, \|\cdot\|\}$ be a **normed space** and $E_1 \subseteq E$, E_1 is closed. Then

$$\|\cdot\| : E/E_1 \rightarrow \mathbb{R}, \quad \|[x]\| = \inf_{y \in E_1} \|x + y\|$$

is a **norm** on E/E_1 .

Proof. If $\|[x]\| = 0$, then $\inf_{y \in E_1} \|x - y\| = 0$, which implies $x \in E_1$ since E_1 is closed, so $[x] = 0$. Also, since

$$\|\lambda[x]\| = \inf_{y \in E_1} \|\lambda x + y\| = \inf_{z \in E_1} \|\lambda x + \lambda z\| = |\lambda| \inf_{z \in E_1} \|x + z\| = |\lambda| \|[x]\|,$$

the dilation property is satisfied. Finally, for triangle inequality, we have

$$\|[x] + [y]\| = \inf_{x_1, y_1 \in E_1} \|x + y + x_1 + y_1\| \leq \inf_{x_1 \in E_1} \|x + x_1\| + \inf_{y_1 \in E_1} \|y + y_1\| = \|[x]\| + \|[y]\|.$$

■

Remark. This shows that the only obstacle for this kind of **norm** being an actual **norm** is the closeness of E_1 .

Chapter 2

Banach Spaces

Definition 2.0.1 (Banach space). A **linear normed space** is a *Banach space* if it's complete, i.e., every Cauchy sequence converges.

Note. If $x_n \in E$, $n \geq 1$ is a sequence with property such that $\lim_{m \rightarrow \infty} \sup_{n \geq m} \|x_n - x_m\| > 0$, then $\exists x_\infty \in E$ such that $\lim_{n \rightarrow \infty} \|x_n - x_m\| = 0$.

Example. The spaces ℓ_1 , ℓ_∞ and $C(K)$ are **Banach spaces**.

We want to give a different criterion for showing $\{E, \|\cdot\|\}$ is **Banach**. Let E be a **linear normed space** and $\{x_\ell \mid \ell \geq 1\}$ a sequence in E .

Definition 2.0.2 (Absolutely summable). A sequence is *absolutely summable* if $\sum_{i=1}^{\infty} \|x_i\| < \infty$.

Theorem 2.0.1 (Criterion for completeness). A **normed space** $\{E, \|\cdot\|\}$ is a **Banach space** if and only if every series in E converges.

Proof. We need to prove two directions.

(\Rightarrow) Suppose E is a **Banach space** and $\{x_k \mid k \geq 1\}$ an **absolutely summable** series. Set $s_n = \sum_{k=1}^n x_k$, $n \geq 1$, we want to show s_n is Cauchy, and if this is the case, completeness of E implies $\exists s_\infty$ and $\lim_{n \rightarrow \infty} \|s_n - s_\infty\| = 0$. Let $n > m$, we see that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \leq \sum_{k=m+1}^{\infty} \|x_k\|.$$

Observe that $\lim_{m \rightarrow \infty} \sum_{k=m+1}^{\infty} \|x_k\| = 0$, we see that the sequence $\{s_n\}$ is Cauchy.

(\Leftarrow) Conversely, suppose E is **not** complete. Then there exists a Cauchy sequence $\{x_n \mid n \geq 1\}$ which does not converge. Furthermore, no subsequence of $\{x_n \mid n \geq 1\}$ converges.^a We now construct an **absolutely summable** series which does not converge.

Define $n(1) \geq 1$ such that $\|x_n - x_{n(1)}\| \leq \frac{1}{2}$ if $n \geq n(1)$, similarly, let $n(2) > n(1)$ be such that $\|x_n - x_{n(2)}\| \leq \frac{1}{2^2}$ if $n \geq n(2)$. In all, we have $n(1) < n(2) < n(3) < \dots$ such that $\|x_n - x_{n(k)}\| \leq \frac{1}{2^k}$ if $n \geq n(k)$. Define $w_j := x_{n(j+1)} - x_{n(j)}$ for $j = 1, 2, \dots$. We see that

$$x_{n(m)} = x_{n(1)} + \sum_{j=1}^m w_j$$

for $m = 1, 2, \dots$, and $\{x_{n(m)}\}$ does not converge, hence so does the series $\sum_{j=1}^{\infty} w_j$. However, $\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$, which implies $\{w_j\}$ is **absolutely summable**. ■

^aOtherwise, the whole sequence converges by the fact that it's Cauchy.

2.1 Completion of Normed Space to Banach Space

Theorem 2.1.1. Suppose E is a **normed space**. Then there exists a **Banach space** \hat{E} called a completion of E with the following properties:

- (a) There exists a linear map $i: E \rightarrow \hat{E}$ such that $\|ix\| = \|x\|$.^a
- (b) $\text{Im}(i)$ is dense in \hat{E} , and \hat{E} is the smallest **Banach space** containing image of E .

^aThis is called an *isometric embedding* of E into \hat{E} .

Appendix

Appendix A

Additional Proofs