MATH597
Analysis II

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\overset{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition we're using to deduce that particular equality.

Notice that there are some proofs is **intended** left as assignments, and for completeness, I put them in Appendix A, use it in your **own risks**! You'll lose the chance to practice and really understand the materials.

Additionally, we'll use [FF99] as our main text, while using [Tao13] and [Axl19] as supplementary references.

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1   Measure

Example. Before we start, we first see some examples.

1. Let $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) \coloneqq \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which is the *power set* of X. We see that

$$\#X = n \implies \#\mathcal{P}(X) = 2^n$$

for $n < \infty$.

2. If $n = \infty$, say $X = \mathbb{N}$, then
$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while $\mathbb{N}$ is a countable set. We can see this as follows. Consider
$$\phi \colon \mathcal{P}(\mathbb{N}) \to [0, 1], \quad A \mapsto 0.a_1 a_2 a_3 \ldots (\text{base } 2),$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example, A can be $A = \{2, 3, 6, \ldots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have
$$\#\mathcal{P}(\mathbb{N}) \geq \#\,[0, 1].$$

But since $[0, 1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X. Hence, we are intriguing to define a map μ such that
$$\mu \colon \mathcal{P}(X) \to [0, \infty].$$

Example. We first see some examples.

1. Let $X = \{0, 1, 2\}$. Then we want to define $\mu \colon \mathcal{P}(X) \to [0, \infty]$, we can have

   - $\mu(A) = \#A$. Then we have

     - $\mu(\{0, 1\}) = 2$
     - $\mu(\{0\}) = 1$

   - $\mu(A) = \sum\limits_{i \in A} 2^i$. Then we have

     - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$

2. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$, we can have

   - $\mu(A) = \#A$. Then we have

     - $\mu(\{2, 3, 4, 5, \ldots\}) = \infty = \mu(\{\text{even numbers}\})$

   - $\mu(A) = e^{-1} \sum\limits_{i \in A} \frac{1}{i!}$. Then we have

     - $\mu(\{0, 2, 4, 6, \ldots\}) = e^{-1} \left(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right)$

   - $\mu(A) = \sum\limits_{i \in A} a_i$

3. Let $X = \mathbb{R}$. Then we want to define $\mu \colon \mathcal{P}(\mathbb{R}) \to [0, \infty]$, we can have

   - $\mu(A) = \#A$
   - $\mu((a, b)) = b - a$.

Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

Answer. No!

- $\mu((a, b)) = e^b - e^a$.

Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

Answer. No!

We immediately see the problems. To extend our native measure method into $\mathbb{R}$ is hard and will cause something counter-intuitive![1] Hence, rather than define measurement on *all* subsets in the power set of X, we only focus on *some* subsets. In other words, we want to define

$$\mu \colon \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \to [0, \infty].$$

1.1 σ-algebras

We start from the definition of the most fundamental element in measure theory.

Definition 1.1 (σ-algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra on X* if

- $\varnothing \in \mathcal{A}$.

- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.

- $\mathcal{A}$ is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\varnothing}_{\in \mathcal{A}}$ and $\mathcal{A}$ is closed under complement.

- $\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c$, namely $\mathcal{A}$ is <u>closed under countable intersections</u>.

- $A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \varnothing \cup \varnothing \cup \ldots$, hence $\mathcal{A}$ is closed under finite unions and intersections.

An immediate definition can be given. We now define so-called *Borel set*.

Definition 1.2 (Borel set). Given a topological space X, a *Borel set* is any set in X that can be formed from open sets through the operations of countable union, countable intersection and relative complement.

Lecture 2: Measure
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Example. Again, we first see some examples.

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[1] https://en.wikipedia.org/wiki/Banach-Tarski_paradox

1. Let $\mathcal{A} = \mathcal{P}(X)$, which is the power σ-algebra.

2. Let $\mathcal{A} = \{\varnothing, X\}$, which is a trivial σ-algebra.

3. Let $B \subset X$, $B \neq \varnothing$, $B \neq X$. Then we see that $\mathcal{A} = \{\varnothing, B, B^c, X\}$ is a σ-algebra.

Lemma 1.1. Let $\mathcal{A}_\alpha$, $\alpha \in I$, be a family of σ-algebra on X. Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is a σ-algebra on X.

Remark. Notice that I may be an uncountable intersection.

Proof. A simple proof can be made as follows. Firstly, $\varnothing \in \mathcal{A}_\alpha$ for every α clearly. Moreover, closure under complement and countable unions for every $\mathcal{A}_\alpha$ implies the same must be true for $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$. Hence, $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ-algebra. ■

The above allows us to give the following definition.

Definition 1.3 (Generation of σ-algebra). Given $\mathcal{E} \subset \mathcal{P}(X)$, where $\mathcal{E}$ is not necessarily a σ-algebra. Let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X containing $\mathcal{E}$, then we call $\langle \mathcal{E} \rangle$ the *σ-algebra generated by* $\mathcal{E}$.

Remark. Clearly, $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$, and it is unique. To check the uniqueness, we suppose there are two different $\langle \mathcal{E} \rangle_1$ and $\langle \mathcal{E} \rangle_2$ generated from $\mathcal{E}$. It's easy to show

$$\langle \mathcal{E} \rangle_1 \subseteq \langle \mathcal{E} \rangle_2,$$

and by symmetry, they are equal.

Example. We see that $\{\varnothing, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle$.

Lemma 1.2. We have

1. Given $\mathcal{A}$ a σ-algebra, $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \mathcal{A}$

2. $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$

Proof. We'll see that after proving the first claim, the second follows smoothly.

1. The first claim is trivial, since we know that $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$, then if $\mathcal{E} \subset \mathcal{A}$, we clearly have $\langle \mathcal{E} \rangle \subset \mathcal{A}$ by the definition.

2. The second claim is also easy. From the first claim and the definition, we have

$$\mathcal{E} \subset \mathcal{F} \subset \langle \mathcal{F} \rangle \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle.$$

∎

At this point, we haven't put any specific structure on X. Now we try to describe those spaces with good structure, which will give the space some nice properties.

> **Definition 1.4 (Borel σ-algebra).** For a topological space X, the *Borel σ-algebra on X*, denotes as $\mathcal{B}(X)$, is the σ-algebra generated by the collection of all open sets in X.

Example. We see that $\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a,b) \mid a < b; a,b \in \mathbb{R}\}$.

- $\mathcal{E}_2 = \{[a,b] \mid a < b; a,b \in \mathbb{R}\}$ since $[a,b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.

- $\mathcal{E}_3 = ((a,b] \mid a < b; a,b \in \mathbb{R})$ since $(a,b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$.

- $\mathcal{E}_4 = ([a,b) \mid a < b; a,b \in \mathbb{R})$ since $[a,b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b)$.

- $\mathcal{E}_5 = ((a,\infty) \mid a \in \mathbb{R})$ since $(a,\infty) = \bigcup_{n=1}^{\infty} (a, a + n)$.

- $\mathcal{E}_6 = ([a,\infty) \mid a \in \mathbb{R})$ since $[a,\infty) = \bigcup_{n=1}^{\infty} [a, a + n)$.

- $\mathcal{E}_7 = ((-\infty,b) \mid b \in \mathbb{R})$ since $(-\infty,b) = \bigcup_{n=1}^{\infty} (b - n, b)$.

- $\mathcal{E}_8 = ((-\infty,b] \mid a \in \mathbb{R})$ since $(-\infty,b] = \bigcup_{n=1}^{\infty} (b - n, b]$.

> **Proposition 1.1.** $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \ldots, 8$.

Proof. Firstly, we see that $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \implies \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$ by Lemma 1.2. Secondly, by definition, $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$ where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R}\}.$$

It's enough to show $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$ since if so, $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$, and clearly $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$, then we will have $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$. Let $O \subset \mathbb{R}$ be an open set, i.e., $O \in \mathcal{E}$. We claim that every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.[2]

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

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[2] https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals

where I_j open interval with the form of $(a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$.

For example, $\mathcal{E}_1$ is trivially true, and

$$(a, b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{[a + \frac{1}{n}, b - \frac{1}{n}]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for $\mathcal{E}_2$ and

$$(a, \infty) = \bigcup_{k=1}^{\infty} (a, a + k)$$

shows the case for $\mathcal{E}_5$. It's now straightforward to check open intervals are in $\langle \mathcal{E}_i \rangle$ for every i. ∎

Now, to put a structure on a space, we define the following.

Definition 1.5 (Measurable space, $\mathcal{A}$-measurable set). A *measurable space* or *Borel space* is a tuple of a set X and a σ-algebra $\mathcal{A}$ on X, denoted by $(X, \mathcal{A})$.

Definition 1.6 (Measurable set). Given a measurable space $(X, \mathcal{A})$, every $E \in \mathcal{A}$ is a so-called $\mathcal{A}-measurable\ set$.

1.2 Measures

With the definition of measurable space, we now can refine our measure function μ as follows.

Definition 1.7 (Measure, Measure space). Given a measurable space on $(X, \mathcal{A})$, a *measure* is a function μ such that

$$\mu \colon \mathcal{A} \to [0, \infty]$$

with

1. $\mu(\varnothing) = 0$

2. $\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_1, A_2, \ldots \in \mathcal{A}$ are **disjoint**. We call this *Countable additivity*.

We denote $(X, \mathcal{A}, \mu)$ a *measure space*.

Notation. We denote $[0, \infty] := [0, \infty) \cup \{\infty\}$.

Remark. The motivation of why we only want countable additivity but not uncountable additivity can be seen by the following example. We'll consider the most intuitive measure on $\mathbb{R}, \mathcal{B}(\mathbb{R})$.

Since we have

$$(0, 1] = (1/2, 1] \cup (1/4, 1/2] \cup (1/8, 1/4] \cup \ldots$$

and also

$$(0, 1] = \bigcup_{x \in (0,1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2}, 1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4}, \frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8}, \frac{1}{4}])} + \ldots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0,1]} 0$$

since $\mu(x) = 0$ for $x \in \mathbb{R}$, which is clearly not what we want.

Example. We see some examples.

1. For any $(X, \mathcal{A})$, we let $\mu(A) \coloneqq \#A$. This is called *counting measure*.

2. Let $x_0 \in X$. For any $(X, \mathcal{A})$, the *Dirac-Delta measure at* x_0 is

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A. \end{cases}$$

3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$,

$$\mu(A) = \sum_{i \in A} a_i,$$

where $a_1, a_2, \ldots \in [0, \infty)$.

Lecture 3: Construct a Measure
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Note. If $A, B \in \mathcal{A}$ and $A \subset B$, then

$$\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

Theorem 1.1. Given $(X, \mathcal{A}, \mu)$ be a measure space.

1. Monotonicity.

$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

2. Countable subadditivity.

$$A_1, A_2, \ldots \in \mathcal{A} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

3. Continuity from below/ monotone convergence theorem (MCT) for sets.

$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \ldots \end{cases} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

4. Continuity from above.

$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \ldots \\ \mu(A_1) < \infty \end{cases} \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

Proof. We prove this theorem one by one.

1. Since $A \subset B$, hence we have

$$\mu(B) = \mu\Big(\underbrace{(B \setminus A)}_{\text{disjoint}} \cup \underline{A}\Big) \overset{!}{=} \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A).$$

2. This should be trivial from countable additivity with the fact that $\mu(A) \geq 0$ for all A.
   > DIY!

3. Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$, then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

are a disjoint union and $B_i \in \mathcal{A}$, hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i).$$

With $\mu\left(\bigcup_{i=1}^{n} B_i\right) = \mu(A_n)$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n \to \infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n \to \infty} \mu(A_n).$$

4. Let $E_i = A_1 \setminus A_i \implies E_i \in \mathcal{A}$, $E_1 \subset E_2 \subset \ldots$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i \right),$$

which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i \right) \implies \mu \left(\bigcap_{i=1}^{\infty} A_i \right) = \mu(A_1) - \mu \left(\bigcup_{i=1}^{\infty} E_i \right)$$

since $\mu \left(\bigcup_{i=1}^{\infty} E_i \right) \leq \mu(A_1) < \infty$. Then from continuity from below, we further have

$$\mu \left(\bigcap_{i=1}^{\infty} A_i \right) = \mu(A_1) - \lim_{n \to \infty} \mu(E_n) = \mu(A_1) - \lim_{n \to \infty} (\mu(A_1) - \mu(A_n)).$$

From monotonicity, we see that $\mu(A_n) \leq \mu(A_1) < \infty$, hence we can split the limit and further get

$$\mu \left(\bigcap_{i=1}^{\infty} A_i \right) = \mu(A_1) - \mu(A_1) + \lim_{n \to \infty} \mu(A_n) = \lim_{n \to \infty} \mu(A_n).$$

$\blacksquare$

Example. Given $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting measure})$. Then we see

- $A_n = \{n, n+1, n+2, \ldots\} \implies \mu(A_n) = \infty$
- $A_1 \supset A_2 \supset A_3 \supset \ldots$
- $\bigcap_{i=1}^{\infty} A_i = \varnothing \implies \mu \left(\bigcap_{i=1}^{\infty} A_i \right) = 0$

Remark. We see that in this case, since $\mu(A_1) \not< \infty$, hence continuity from above doesn't hold.

We now try to characterize some properties of a measure space.

Definition 1.8 (μ-null, μ-subnull, Complete measure space). Given $(X, \mathcal{A}, \mu)$,

- $A \subset X$ is a μ-null set if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- $A \subset X$ is a μ-subnull set if $\exists \mu$-null set B such that $A \subset B$.
- $(X, \mathcal{A}, \mu)$ is a complete measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Note. We see that for a μ-subnull set, it's not necessary $\mathcal{A}$-measurable.

There are some useful terminologies we'll use later relating to μ-null.

Definition 1.9 (Almost everywhere). Given $(X, \mathcal{A}, \mu)$, a statement $P(x)$, $x \in X$ holds μ-*almost everywhere (a.e.)* if the set

$$\{x \in X \colon P(x) \text{ does not hold}\}$$

is μ-null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

Definition 1.10 (Finite measure). , Given $(X, \mathcal{A}, \mu)$

- μ is a *finite measure* if $\mu(X) < \infty$.

- μ is a σ-*finite measure* if $X = \bigcup\limits_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.

Exercise. Every measure space can be **completed**. Namely, we can always find a bigger σ-algebra to complete the space.

1.3 Outer Measures

We start by giving a definition.

Definition 1.11 (Outer measure). An *outer measure* on X is a map

$$\mu^* \colon \mathcal{P}(X) \to [0, \infty]$$

such that

- $\mu^*(\varnothing) = 0$

- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$

- (countable subadditivity) $\mu^* \left(\bigcup\limits_{i=1}^{\infty} A_i \right) \leq \sum\limits_{i=1}^{\infty} \mu^*(A_i)$ for every $A_i \subset X$.

Example. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \colon \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

is an outer measure due to the Proposition 1.2 we're going to show.

Remark. We see that an outer measure need not be a measure. Check Definition 1.7.

Proposition 1.2. Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\varnothing, X \in \mathcal{E}$. Let

$$\rho \colon \mathcal{E} \to [0, \infty]$$

such that $\rho(\varnothing) = 0$. Then

$$\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \ \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

is an outer measure on X.

Note. Recall the Tonelli's Theorem[3] for series:

If $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Specifically, in [Tao13] Theorem 0.0.2.

Lecture 4: Carathéodory extension Theorem   12 Jan. 11:00

As previously seen. We now prove the Proposition 1.2.

Proof. We need to prove

- μ^* is well-defined. i.e., inf is taken over a non-empty set. This is trivial since $X \in \mathcal{E}$ and $X \supset A$ for any $A \in \mathcal{E}$.

- $\mu^*(\varnothing) = 0$. Since $\varnothing \in \mathcal{E}$ and

$$\mu^*(\varnothing) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \varnothing \right\} = 0$$

  since $\rho(\varnothing) = 0$ for all i and further, by Squeeze Theorem[4], we see that $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \rho(\varnothing) = 0$.

- $A \subset B \implies \mu^*(A) \leq \mu^*(B)$. We simply show this by contradiction. Suppose $A \subset B$ and $\mu^*(A) > \mu^*(B)$, then by definition of μ^*, we have

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

$$> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B \right\} = \mu^*(B).$$

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[3]https://en.wikipedia.org/wiki/Fubini%27s_theorem
[4]https://en.wikipedia.org/wiki/Squeeze_theorem

Now, let $B =: (B \setminus A) \cup A$, then we have

$$
\begin{aligned}
\mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\} \\
&> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B \setminus A) \cup A \right\} = \mu^*(B).
\end{aligned}
$$

Now, since $B \setminus A \supseteq \varnothing$, then this inequality can't hold, hence a contradiction $\lightning$.

- Countable subadditivity. Let $A_1, A_2, \ldots \in X$. If one of $\mu^*(A_n) = \infty$, then result holds. So we may assume $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Now, fix any $\epsilon > 0$, we will show that

$$
\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon.
$$

For each $n \in \mathbb{N}$, $\exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ such that

$$
\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n
$$

and

$$
\mu^*(A_n) + \frac{\epsilon}{2^n}{}^{5} \geq \sum_{k=1}^{\infty} \rho(E_{n,k}).
$$

Then we see that

$$
\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k) \in \mathbb{N}^2} E_{k,n},
$$

which implies

$$
\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{(n,k) \in \mathbb{N}^2} \rho\left(E_{k,n}\right) \overset{!}{=} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{k,n}) \leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right)
$$

from the inequality just derived. Now, since the last term is just

$$
\sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon,
$$

hence we finally have

$$
\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon
$$

for arbitrarily small fixed $\epsilon > 0$, hence the subadditivity is proved.

∎

---

[5]This is an important trick!!

Definition 1.12 (Carathéodory measurable). Let μ^* be an outer measure on X. We say $A \subset X$ is *Carathéodory measurable (C-measurable) with respect to μ^** if

$$\forall E \subset X, \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Lemma 1.3. Let μ^* be an outer measure on X. Suppose $B_1, \ldots, B_N$ are disjoint C-measurable sets. Then,

$$\forall E \subset X, \ \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) = \sum_{i=1}^{N} \mu^*(E \cap B_i).$$

Proof. Since we have

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) = \mu^*(E' \cap B_1) + \mu^*(E' \setminus B_1)^{\,6}$$

$$= \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i \cap B_1\right)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right) \cap B_1^c\right)$$

$$= \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_{i=2}^{N} B_i\right)\right)$$

where the equality comes from the fact that B_1 is C-measurable and disjoint from B_i, $i \neq 1$. Then, we simply iterate this argument and have the result. ∎

Remark. This implies that if we restrict an outer measure on a C-measurable set, then it becomes <u>finite additive</u>.

Theorem 1.2 (Carathéodory extension Theorem). Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets (with respect to μ^*). Then,

1. $\mathcal{A}$ is a σ-algebra on X.

2. $\mu = \mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.

3. $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We divide the proof in several steps.

1. We show $\mathcal{A}$ is a σ-algebra by showing

   (a) $\varnothing \in \mathcal{A}$. To show this, we simply check that $\varnothing$ is C-measurable. We see that

   $$\underset{E \subset X}{\forall} \ \mu^*(E) = \mu^*(E \cap \varnothing) + \mu^*(E \setminus \varnothing) = \mu^*(E),$$

---

^{6}Here, $E' := E \cap \left(\bigcup_{i=1}^{N} B_i\right)$ for the simplicity of notation.

which just shows $\varnothing \in \mathcal{A}$.

(b) $\mathcal{A}$ closed under complements. This is equivalent to say that if A is C-measurable, so is A^c. We see that if A is C-measurable, then for every $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Observing that $E \cap A = E \setminus A^c$ and $E \setminus A = E \cap A^c$, hence

$$\mu^*(E) = \mu^*(E \setminus A^c) + \mu^*(E \cap A^c).$$

We immediately see that above implies $A^c \in \mathcal{A}$.

(c) $\mathcal{A}$ closed under countable unions.

Note. To show $\mathcal{A}$ closed under countable unions, we show that $\mathcal{A}$ is closed under:

finite unions $\overset{\text{then}}{\implies}$ countable $\underline{\text{disjoint}}$ unions $\overset{\text{then}}{\implies}$ countable unions.

- We show $\mathcal{A}$ is closed under finite unions.

  Claim. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.
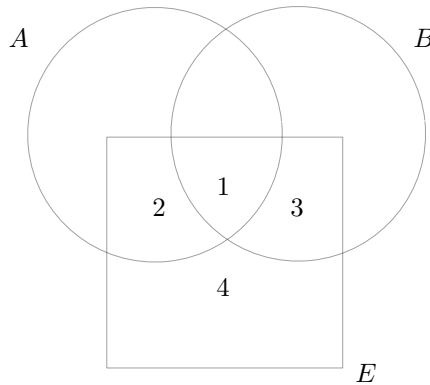
  Fix $E \subset X$ arbitrary. We need to show that

  $$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)),$$

  i.e.,
  $$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

  given $A, B \in \mathcal{A}$.

  - Since A is C-measurable,
    * $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$
    * $\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$

- Since B is C-measurable,

  * $\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$

Hence, we have

$$
\begin{aligned}
\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\
&= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4) \\
&= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).
\end{aligned}
$$

- We show $\mathcal{A}$ is closed under countable $\underline{\text{disjoint}}$ unions.

  Let $A_1, A_2, \ldots \in \mathcal{A}$ and $\underline{\text{disjoint}}$. Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

  $$
  \mu^*(E) \leq \mu^* \left(E \cap \bigcup_{i=1}^{\infty} A_i \right) + \mu^* \left(E \setminus \bigcup_{i=1}^{\infty} A_i \right),
  $$

  hence we only need to show another way around.

  Fix $N \in \mathbb{N}$, we have $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$ since N is finite, and

  $$
  \begin{aligned}
  \mu^*(E) &= \mu^* \left(E \cap \left(\bigcup_{n=1}^{N} A_n \right) \right) + \mu^* \left(E \setminus \left(\bigcup_{n=1}^{N} A_n \right) \right) \\
  &\geq \underbrace{\sum_{n=1}^{N} \mu^*(E \cap A_n)}_{\overset{!}{=} \mu^* \left(E \cap \left(\bigcup_{n=1}^{N} A_n \right) \right)} + \underbrace{\mu^* \left(E \setminus \bigcup_{n=1}^{\infty} A_n \right)}_{\leq \mu^* \left(E \setminus \left(\bigcup_{n=1}^{N} A_n \right) \right)}.
  \end{aligned}
  $$

  Now, take $N \to \infty$ then we are done.

- We show $\mathcal{A}$ is closed under countable unions. DIY

The proof will be continued...

Lecture 5: Hahn-Kolmogorov Theorem

Firstly, we see a stronger version of Lemma 1.3 we have seen before.

Lemma 1.4. Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots$ are $\underline{\text{disjoint}}$ C-measurable sets. Then,

$$
\forall E \subset X, \ \mu^* \left(E \cap \left(\bigcup_{i=1}^{\infty} B_i \right) \right) = \sum_{i=1}^{\infty} \mu^* \left(E \cap B_i \right).
$$

Proof.

$$
\sum_{n=1}^{\infty} \mu^*(E \cap B_i) \geq \mu^* \left(E \cap \bigcup_{n=1}^{\infty} B_n \right) \geq \mu^* \left(E \cap \left(\bigcup_{n=1}^{N} B_n \right) \right) \overset{!}{=} \sum_{n=1}^{N} \mu^* \left(E \cap B_n \right).
$$

Now, we just take $N \to \infty$ (or note that $N \in \mathbb{N}$ is arbitrary, we then get the result according to Squeeze Theorem[7]). $\blacksquare$

Let's continue the proof of Theorem 1.2.

2. Since from Definition 1.7, we need to show

   - $\mu(\varnothing) = 0$. This means that we need to show $\mu^*|_{\mathcal{A}}(\varnothing) = 0$. Since $\varnothing \in \mathcal{A}$ and μ^* is an outer measure, hence from the property of outer measure, it clearly holds.

   - Countable additivity of μ^* on $\mathcal{A}$ follows from the Lemma 1.4 with $E = X$

3. The proof is given in Theorem A.1.

$\blacksquare$

1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets $\mathcal{E}$ and any ρ such that it assigns measure on $\mathcal{E}$, then it induces an outer measure by Proposition 1.2, finally complete the outer measure by Theorem 1.2.

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.2}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to σ-algebra we are working on till now.

Definition 1.13 (Algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on X* if

- $\varnothing \in \mathcal{A}$.

- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.

- $\mathcal{A}$ is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an algebra and a σ-algebra is whether they closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an outer measure.

---

[7] https://en.wikipedia.org/wiki/Squeeze_theorem

Definition 1.14 (Pre-measure). Let $\mathcal{A}_0$ be an algebra on X. We say

$$\mu_0 \colon \mathcal{A}_0 \to [0, \infty]$$

is a *pre-measure* if

1. $\mu_0(\varnothing) = 0$

2. (finite additivity) $\mu_0 \left(\bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \mu_0(A_i)$ if $A_1, \dots, A_n \in \mathcal{A}_0$ are $\underline{\text{disjoint}}$.

3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, $\underline{\text{disjoint}}$, then

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 1.5. (1) + (3) $\implies$ (2) in Definition 1.14.

Proof. It's easy to see that since μ_0 is monotone. ∎

Theorem 1.3 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in Proposition 1.2. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^*, then $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$. i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts.

- We first show $\mathcal{A} \supset \mathcal{A}_0$. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is C-measurable, i.e.,

$$\forall E \subset X \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We first fix an $E \subset X$. From countable subadditivity of μ^*, we have

$$\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

Fix $\epsilon > 0$. By the Proposition 1.2 of μ^*, $\exists B_1, B_2, \dots \in \mathcal{A}_0$, $\bigcup_{n=1}^{\infty} B_n \supset E$ such that

$$\mu^*(E) + \epsilon \overset{!}{\ge} \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left(\mu_0(\underbrace{B_n \cap A}_{\in \mathcal{A}_0}) + \mu_0(\underbrace{B_n \cap A^c}_{\in \mathcal{A}_0}) \right)$$

by the finite additivity of μ_0. Note that

$$
\begin{cases}
\displaystyle\bigcup_{n=1}^{\infty}(B_n \cap A) \supset E \cap A \\[2mm]
\displaystyle\bigcup_{n=1}^{\infty}(B_n \cap A^c) \subset E \cap A^c
\end{cases}
\implies \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)
$$

since

$$
\mu^*(E \cap A) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n \cap A)
$$

and

$$
\mu^*(E \cap A^c) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A^c)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n \cap A^c).
$$

We then see that for any $\epsilon > 0$, the inequality

$$
\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)
$$

holds, hence so does

$$
\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),
$$

which implies $\mathcal{A} \supset \mathcal{A}_0$.

The proof will be

Lecture 6: Hahn-Kolmogorov Theorem and Extension. 18 Jan. 11:00

Let's continue the proof of Theorem 1.3.

- Let $A \in \mathcal{A}_0$, we want to show that

$$
\mu(A) = \mu_0(A).
$$

  – Firstly, let

$$
B_i = \begin{cases} A, & \text{if } i = 1 \\ \varnothing, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,
$$

  hence $\displaystyle\bigcup_{i=1}^{\infty} B_i = A$, then we see that

$$
\mu^*(A) \leq \sum_{i=1}^{\infty}\mu_0(B_i) = \mu_0(A)
$$

  from the definition of μ^* and countable additivity within the algebra of μ_0.

- Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A \cap B_1 \in \mathcal{A}_0$, $C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right) \in \mathcal{A}_0$ for $i \geq 2$ since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

  are disjoint countable unions, by countable additivity within the algebra, we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i) \implies \mu_0(A) \leq \mu^*(A)$$

  by taking the infimum from the definition of μ^*.

Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every $A \in \mathcal{A}_0$, which implies

$$\mu(A) = \mu_0(A)$$

for every $A \in \mathcal{A}_0$ from Theorem 1.2, where we extend μ^* to μ respect to $\mathcal{A}_0$.

∎

Definition 1.15 (HK extension). $(\mathcal{A}, \mu)$ obtained from Theorem 1.3 is the *Hahn-Kolmogorov extensions* of $(\mathcal{A}_0, \mu_0)$.

We can show the uniqueness of HK extension.

Theorem 1.4 (Uniqueness of HK extension). Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$. Let $(\mathcal{A}, \mu)$ be the HK extension of $(\mathcal{A}_0, \mu_0)$. Let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$. Then if μ_0 is σ-finite, $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Note. Notice that $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$ since they both extend $\mathcal{A}_0$.

Proof. Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing the arbitrary cover of A and using the definition of μ^*.

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) < \infty$, and fix $\epsilon > 0$. Then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup\limits_{i=1}^{\infty} B_i \supset A$ such that

$$\mu(A) + \epsilon = \mu^*(A) + \epsilon \overset{!}{\geq} \sum_{i=1}^{\infty} \mu_0(B_i) \overset{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

This implies that
$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$
where the first equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

$$\mu(B) = \lim_{N \to \infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) \overset{8}{=} \lim_{N \to \infty} \mu'\left(\bigcup_{i=1}^{N} B_i\right) = \mu'(B),$$

hence,

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \overset{9}{\leq} \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon$$

for arbitrary ϵ, so we conclude $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is σ-finite, so we know $X = \bigcup\limits_{n=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ such that
$$\mu_0(X_n) < \infty.$$
Replacing X_n by $X_1 \cup \ldots \cup X_n \in \mathcal{A}_0$, we may assume that

$$X_1 \subset X_2 \subset \ldots.$$

Then,
$$\underset{n \in \mathbb{N}}{\forall} \; \mu(A \cap X_n) < \infty \overset{!}{\implies} \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

From the continuity of measure, we then have

$$\mu(A) = \lim_{n \to \infty} \mu(A \cap X_n) \leq \lim_{n \to \infty} \mu'(A \cap X_n) = \mu'(A).$$

∎

---

[8] $\mu = \mu'$ on $\mathcal{A}_0$.
[9] From the first part.

Corollary 1.1. Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then

$$\exists! \text{ measure } \mu \text{ on } \langle \mathcal{A}_0 \rangle \text{ that extends } \mathcal{A}_0.$$

Furthermore,

- The completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.

-
$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \underset{i \in \mathbb{N}}{\forall} \bigcup_{i=1}^{\infty} B_i \supset A \right\}$$

  for all $A \in \overline{\langle \mathcal{A}_0 \rangle}$.

Lecture 7: Borel Measures

1.5 Borel Measures on $\mathbb{R}$

We first introduce so-called *distribution function*.

Definition 1.16 (Distribution function). An increasing[a] function

$$F \colon \mathbb{R} \to \mathbb{R}$$

and right-continuous. F is then a *distribution function*.

---
[a]Here, increasing means $F(x) \leq F(y)$ for $x < y$.

Example. Here are some examples of right-continuous functions.

1. $F(x) = x$.

2. $F(x) = e^x$.

3. Define
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

4. Let $\mathbb{Q} \coloneqq \{r_1, r_2, \ldots\}$. Define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$

   and

$$F(x) \coloneqq \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$

   Then F is a distribution function (hence right-continuous). This is shown in Lemma A.1.

Note. If F is increasing, and

$$F(\infty) \coloneqq \lim_{x \nearrow \infty} F(x), \quad F(-\infty) \coloneqq \lim_{x \searrow \infty} F(x)$$

exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.[10]

Now, we can define a *Borel measure* on $(X, \mathcal{B}(\mathbb{R}))$.

Definition 1.17 (Borel mesaure). A *Borel measure* is any measure μ defined on the σ-algebra of Borel sets.

Definition 1.18 (Locally finite). Let X be a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Note. Some authors will require a Borel measure equipped with the locally finite property. But formally, this is not so common.

Lemma 1.6. Let μ be a locally finite Borel measure on $\mathbb{R}$, then

$$F_\mu(x) = \begin{cases} \mu\left((0, x]\right), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu\left((x, 0]\right), & \text{if } x < 0 \end{cases}$$

is a distribution function.

Proof. To show F_μ is increasing, consider $x < y$ such that

$$F_\mu(x) \leq F_\mu(y)$$

by considering

- $x > 0$: Then $F_\mu(x) = \mu((0, x])$ and

$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$

- $x = 0$: Then $F_\mu(x) = 0$ and

$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$

since $y > 0$.

- $x < 0$: Follows the same argument with $x > 0$.

---

[10]There are distributions [FF99] Ch9., but these are different from distribution functions.

Now, we need to show F_μ is right-continuous. Firstly, assume that $x \geq 0$, then we see that

$$F_\mu(x) = \mu((0, x]) = \mu((0, x^+])$$

from the fact that a measure is right-continuous.[11] Now, if $x \leq 0$, the same argument follows since multiplying -1 will not change the fact that a measure is continuous. ∎

Definition 1.19 (Half intervals). We call

$$\varnothing, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

half-intervals.

Lemma 1.7. Let $\mathcal{H}$ be the collection of finite disjoint unions of half-intervals. Then, $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. We see that

- $\varnothing \in \mathcal{H}$. Clearly.

- To show $\mathcal{H}$ is closed under complements, we have

  - $\varnothing^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$.
  - $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$.[12]
  - $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$.
  - $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$.
  - $(-\infty, \infty)^c = \varnothing \in \mathcal{H}$.

- $\mathcal{H}$ is closed under finite unions, clearly.

∎

---

[11] Actually, a measure is always continuous.

[12] Since it's a two disjoint union of half intervals.

Proposition 1.3 (Distribution function defines a pre-measure). Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function. For a half interval I, define

$$\ell(I) := \ell_F(I) = \begin{cases} 0, & \text{if } I = \varnothing; \\ F(b) - F(a), & \text{if } I = (a, b]; \\ F(\infty) - F(a), & \text{if } I = (a, \infty]; \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b]; \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 := \mu_{0,F}$ as

$$\mu_{0,F}\colon \mathcal{H} \to [0, \infty]$$

by

$$\mu_0(A) = \sum_{k=1}^{N} \ell(I_k) \text{ if } A = \bigcup_{k=1}^{N} I_k,$$

where A is a finite disjoint union of half intervals $I_1, \ldots, I_N$. Then, μ_0 is a pre-measure on $\mathcal{H}$.

Proof. We see that

1. μ_0 is well-defined.

2. $\mu_0(\varnothing) = 0$.

3. μ_0 is finite additive.

4. μ_0 is countable additivity within $\mathcal{H}$.

   Suppose $A \in \mathcal{H}$ where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.[13]

   Focus on the case $I = (a, b]$. Let

   $$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

   which is a disjoint union. Then we only need to check

   $$F(b) - F(a) = \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) \right).$$

   • Since $(a, b] \supset \bigcup_{n=1}^{N} (a_n, b_n]$ for any fixed $N \in \mathbb{N}$, hence

   $$\mathop{\forall}_{N \in \mathbb{N}} F(b) - F(a) \geq \sum_{n=1}^{N} \left(F(b_n) - F(a_n) \right).$$

---

[13]Since $\mathcal{H}$ is only a collection of *finite* disjoint half intervals, hence after considering $A = I$, we can apply the same argument iteratively and stop in finite steps. Formally, we can consider $H \in \mathcal{H}$, $H = \bigcup_{i=1}^{\infty} A^i$, where A^i being a half interval. Then by the above argument, we have $A^i = I^i$ and so on.

By letting $N \to \infty$, we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) \right).$$

- Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

$$F(a') - F(a) < \epsilon.$$

For each $n \in \mathbb{N}$, $\exists b_n' > b_n$ such that

$$F(b_n') - F(b_n) < \frac{\epsilon}{2^n}.$$

Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b_n'),$$

hence

$$\underset{N \in \mathbb{N}}{\exists} \ [a', b] \subset \bigcup_{n=1}^{N} (a_n, b_n'), \ ^{14}$$

which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^{N} F(b_n') - F(a_n).$$

Finally, we see that

$$F(b) - F(a) \leq F(b) - F(a') + \epsilon$$

$$\leq \sum_{n=1}^{\infty} \left(F(b_n') - F(a_n) \right) + \epsilon$$

$$\leq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\epsilon}{2^n} \right) + \epsilon$$

$$= \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) \right) + 2\epsilon$$

for any fixed $\epsilon > 0$, hence

$$F(b) - F(a) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

Combine these two inequalities, we have

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

as we desired.

---

[14] This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.

∎

Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8: Lebesgue-Stieltjes Measure on $\mathbb{R}$

To classify all measures, we now see this last theorem to complete the task.

Theorem 1.5 (Locally finite Borel measures on $\mathbb{R}$). We have

1. $F\colon \mathbb{R} \to \mathbb{R}$ a distribution function, then there exists a **unique** *locally finite Borel measure* μ_F on $\mathbb{R}$ satisfying

$$\mu_F((a,b]) = F(b) - F(a)$$

   for every $a < b$.

2. Suppose $F, G\colon \mathbb{R} \to \mathbb{R}$ are distribution functions. Then,

$$\mu_F = \mu_G$$

   on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. ∎ HW.

Remark. Theorem 1.5 simply states that given a distribution function, if we restrict our attention on locally finite measures on $\mathbb{R}$ following our usual convention, then it defines the measure on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

1.6 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that

$$F \text{ distribution function} \overset{!}{\implies} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 1.20 (Lebesgue-Stieltjes measure). Given a distribution function F, we say μ_F on $\mathcal{A}_{\mu_F}$ is called the *Lebesgue-Stieltjes measure* corresponding to F.

Definition 1.21 (Lebesgue measure, Lebesgue σ-algebra). From Definition 1.20, if $F(x) = x$, then the induced $(\mathcal{A}_{\mu_F}, \mu_F)$ is denoted as $(\mathcal{L}, m)$, where $\mathcal{L}$ is called *Lebesgue σ-algebra*, and m is called *Lebesgue measure*.

Remark. Recall that $\mathcal{L}$ is induced by Theorem 1.2, namely given m, for all $A \subset \mathbb{R}$, we have

$$\mathcal{L} := \left\{ A \subset \mathbb{R} \mid \underset{E \subset \mathbb{R}}{\forall}\ m(A) = m(A \cap E) + m(A \setminus E) \right\}$$

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).^{[15]}$$

Example. We first see some examples.

1. $\mu_F((a, b]) = F(b) - F(a)$. Then

   - $\mu_F(\{a\}) = F(a) - F(a^-)$

   - $\mu_F([a, b]) = F(b) - F(a^-)$

   - $\mu_F((a, b)) = F(b^-) - F(a)$

   This is so-called *discrete measure*.

2. We define
   $$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

   Then

   - $\mu_F(\{0\}) = 1$

   - $\mu_F(\mathbb{R}) = 1$

   - $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. This is easy to see since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, hence

   $$\begin{aligned} \mu_F(\mathbb{R} \setminus \{0\}) &= \mu_F((-\infty, 0) \cup (0, \infty)) \\ &= \underbrace{\mu_F((-\infty, 0))}_{0-0^{[16]}} + \underbrace{\mu_F((0, \infty))}_{1-1^{[17]}} = 0. \end{aligned}$$

   We call that μ_F is the *Dirac measure* at 0.

3. Denote $\mathbb{Q} = \{r_1, r_2, \dots\}$, and we define

   $$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \quad \text{where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r. \end{cases}$$

   Then

   - $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.

---

[15]Some text will use $x-$ and $x+$ instead of x^- and x^+, respectively.

[16]It follows from $F(0^-) - F(-\infty) = 0 - 0 = 0$.

[17]It follows from $F(\infty) - F(0) = 1 - 1 = 0$.

- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$.

  This is shown in Lemma A.2.

4. If F is continuous at a, then $\mu_F(\{a\}) = 0$.

5. $F(x) = x$, then recall that we denote $\mu_F := m$, and we have

   - $m((a, b]) = m((a, b)) = m([a, b]) = b - a$.

6. $F(x) = e^x$

   - $\mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$.

Example (Middle thirds Cantor set). Let $C := \bigcap\limits_{n=1}^{\infty} K_n$, where we have

$$K_0 := [0, 1]$$

$$K_1 := K_0 \setminus \left(\frac{1}{3}, \frac{2}{3} \right)$$

$$K_2 := K_1 \setminus \left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right)$$

$$\vdots$$

$$K_n := K_{n-1} \setminus \bigcup_{k=1}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

We see that C is uncountable and with $m(C) = 0$. And observe that $x \in C$ if and only if $x = \sum\limits_{n=1}^{\infty} \frac{a_n}{3}$ for some $a_n \in \{0, 2\}$. Hence, we can instead formulate K_n by

$$K_n = \bigcup_{\substack{a_i \in \{0,2\} \\ 1 \leq i \leq n}} \left[\sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n} \right].$$
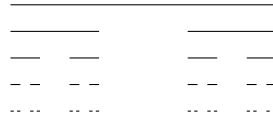

Figure 1: The top line corresponds to K_0, and then K_1, etc.

The proof of $m(C) = 0$ is given in Lemma A.3.

1.6.1 Cantor Function

Consider F as follows. We define a function F to be 0 to the left of 0, and 1 to the right of 1. Then, define F to be $\frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3} \right)$, $\frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9} \right)$, $\frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9} \right)$ and so on. This is so-called *Cantor Function*. We can show F is continuous and increasing, which makes F a distribution function. Also, we see that the measure this F induced is called *Cantor measure*.
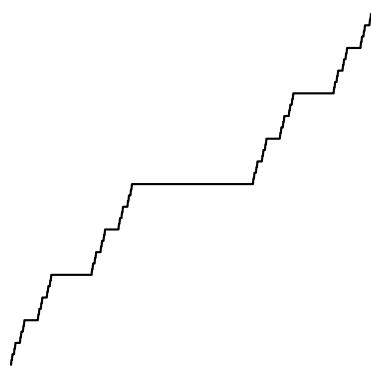
Figure 2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

Cantor Measure μ_F		Lebesgue Measure m
$\mu_F(\mathbb{R} \setminus C) = 0$		$m(\mathbb{R} \setminus C) = \infty > 0$
$\mu_F(C) = 1$	$\Longleftrightarrow$	$m(C) = 0$
$\mu_F(\{a\}) = 0$		$m(\{a\}) = 0$

Remark. μ_F and m are said to be **singular** to each other.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 1.8. Let μ be Lebesgue-Stieltjes measure on $\mathbb{R}$. Then we have

$$\mu(A) \stackrel{!}{=} \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supset A \right\}$$

$$= \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

for every $A \in \mathcal{A}_\mu$

Proof. The second equality follows from the continuity of the measure. ∎

Remark. This is similar to

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - 1/n], \quad (a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n].$$

Lecture 9: Properties of Lebesgue-Stieltjes measure

26 Jan. 11:00

As previously seen. Let $X \subset [0, \infty]$. Recall that

- Finite supremum.

$$\alpha = \sup X < \infty \iff \begin{cases} \underset{x \in X}{\forall} \ \alpha \geq x \\ \underset{\epsilon > 0}{\forall} \ \underset{x \in X}{\exists} \ x + \epsilon \geq \alpha. \end{cases}$$

- Infinite supremum.

$$\alpha = \sup X = \infty \iff \underset{L > 0}{\forall} \ \underset{x \in X}{\exists} \ x \geq L.$$

This should be useful latter on.

Theorem 1.6 (Regularity). Let μ be Lebesgue-Stieltjes measure. Then, for all $A \in \mathcal{A}_\mu$,

1. (outer regularity) $\mu(A) = \inf\{\mu(O) \mid O \supset A, O \text{ is open}\}$

2. (inner regularity) $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$

Proof. We check them separately.

1. _____ DIY

2. Let $s := \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$, then by monotonicity, we have $\mu(A) \geq s$. To show the other direction, we consider

   - A is a bounded set.

     Then $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, $\overline{A}$ is also bounded $\implies \mu(\overline{A}) < \infty$. Fix $\epsilon > 0$, then by outer regularity, there exists an open $O \supset \overline{A} \setminus A$, and $\mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \epsilon$. Let $K := \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$, we

     show that
     $$\mu(K) \geq \mu(A) - \epsilon.$$

   _____ DIY

   - A is an unbounded set with $\mu(A) < \infty$.

     Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \ldots$, then
     $$\lim_{n \to \infty} \mu(A_n) = \mu(A) < \infty.$$

   - A is an unbounded set with $\mu(A) = \infty$.

     We can show that
     $$\lim_{n \to \infty} \mu(A_n) = \mu(A) = \infty.$$

     Fix $L > 0$, then $\exists N$ such that $\mu(A_N) \geq L$.

   ∎

Definition 1.22 (G_δ-set, F_σ-set). Let X be a topological space. Then

- A G_δ-set is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open.

- A F_σ-set is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.7. Let μ be a Lebesgue-Stieltjes measure. Then *TFAE*[a]:

1. $A \in \mathcal{A}_\mu$
2. $A = G \setminus M$, G is a G_δ-set, M is a μ-null set.
3. $A = F \setminus N$, F is a F_σ-set, N is a μ-null set.

---

[a] *TFAE*: The following are equivalent.

Proof. We see that (2.) $\implies$ (1.) and (3.) $\implies$ (1.) are clear.

- (1.) $\implies$ (3.)

  - Assume $\mu(A) < \infty$. From the inner regularity, we have

  $$\forall n \in \mathbb{N} \ \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

  Let $F = \bigcup_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ-null. Check!

  - Assume $\mu(A) = \infty$. Let $A = \bigcup_{k\in\mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

  $$\forall k \in \mathbb{Z} \ A_k = F_k \cup N_k, \ A = \underbrace{\left(\bigcup_k F_k\right)}_{F_\sigma\text{-set}} \cup \underbrace{\left(\bigcup_k N_k\right)}_{\mu\text{-null}}.$$

- (1.) $\implies$ (2.)

  We see that
  $$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

  ∎

Proposition 1.4. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0 \ \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using outer regularity and the fact that every open set is $\bigcup\limits_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. $\blacksquare$ DIY

We now see some properties of Lebesgue measure.

Theorem 1.8. Let $A \in \mathcal{L}$, then we have $A + s \in \mathcal{L}$, $rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,
$$m(A + s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. $\blacksquare$ DIY

Example. We now see some examples.

1. Let $\mathbb{Q} =: \{r_i\}_{i=1}^{\infty}$ which is dense in $\mathbb{R}$. Let $\epsilon > 0$, and
$$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i} \right).$$

   We see that O is open and dense[18] in $\mathbb{R}$. But we see
$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$

   Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$

2. There exists uncountable set A with $m(A) = 0$.

3. There exists A with $m(A) > 0$ but A contains no non-empty open intervals.

4. There exists $A \notin \mathcal{L}$. e.g. Vitali set.[19]

5. There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

Lecture 10: Integration

26 Jan. 11:00

2 Integration

2.1 Measurable Function

We start with a definition.

Definition 2.1 (Measurable function). Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$ are measurable spaces. Then we say $f \colon X \to Y$ is $(\mathcal{A}, \mathcal{B})$-*measurable* if
$$\underset{B \in \mathcal{B}}{\forall} f^{-1}(B) \in \mathcal{A}.$$

Remark. If $\mathcal{A}$ and $\mathcal{B}$ are given, we'll sometimes say f is measurable if it'll not cause any confusions.

---

[18]https://en.wikipedia.org/wiki/Dense_set
[19]https://en.wikipedia.org/wiki/Vitali_set

Lemma 2.1. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, and suppose $\mathcal{B} = \langle \mathcal{E} \rangle$ for some $\mathcal{E} \subset Y$. Then,

$$f \colon X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \underset{E \in \mathcal{E}}{\forall} \, f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part ($\implies$) is clear. On the other direction, we consider the following. Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $\mathcal{E} \subset \mathcal{D}$ by assumption
- $\mathcal{D}$ is a σ-algebra   Check!

hence, we see that $\langle \mathcal{E} \rangle = \mathcal{B} \subset \mathcal{D}$ from Lemma 1.2. The result then follows from the definition of $(\mathcal{A}, \mathcal{B})$-measurable. ∎

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup_{\alpha} E_{\alpha} \right) = \bigcup_{\alpha} f^{-1}(E_{\alpha})$

Definition 2.2 ($\mathcal{A}$-measurable). Let $(X, \mathcal{A})$ be a measurable space. Then,

$$\left.\begin{array}{l} f \colon X \to \mathbb{R} \\ f \colon X \to \overline{\mathbb{R}} \\ f \colon X \to \mathbb{C} \end{array}\right\} \text{ is } \mathcal{A}\text{-measurable if } \begin{cases} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \operatorname{Re} f, \operatorname{Im} f \colon X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{cases}$$

Notation. Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$
- $\mathcal{B}(\overline{\mathbb{R}}) = \left\{ E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R}) \right\}$.
- $\operatorname{Re} f$ is the real part of f, while $\operatorname{Im} f$ is the imaginary part of f.

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \implies$ Every function is $\mathcal{A}$-measurable.
- $\mathcal{A} = \{\varnothing, X\} \implies$ The only $\mathcal{A}$-measurable functions are constant functions.

Definition 2.3 (Lebesgue measurable). A *Lebesgue measurable function* f is a measurable function

$$f \colon (\mathbb{R}, \mathcal{L}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})).$$

Lemma 2.2. Given $f \colon X \to \mathbb{R}$, *TFAE*.

1. f is $\mathcal{A}$-measurable

2. $\forall a \in \mathbb{R},\ f^{-1}((a, \infty)) \in \mathcal{A}$

3. $\forall a \in \mathbb{R},\ f^{-1}([a, \infty)) \in \mathcal{A}$

4. $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a)) \in \mathcal{A}$

5. $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from Lemma 2.1 we just saw. ∎

Remark (Operations preserve $\mathcal{A}$-measurability). Given $f, g \colon X \to \mathbb{R}$ and is $\mathcal{A}$-measurable, then

1. $\phi \colon \mathbb{R} \to \mathbb{R}$, $\mathcal{A}$-measurable[20], then

$$\phi \circ f \colon X \to \mathbb{R}$$

    is $\mathcal{A}$-measurable.

2. $-f$, $3f$, f^2, $|f|$ are all $\mathcal{A}$-measurable, and $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) \neq 0, \forall x \in X$.

3. $f + g$ is $\mathcal{A}$-measurable. We see this from

$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty)) \right)$$

    with Lemma 2.2.

4. $f \cdot g$ is $\mathcal{A}$-measurable. We see this from

$$f(x)g(x) = \frac{1}{2} \left((f(x) + g(x))^2 - f(x)^2 - g(x)^2 \right).$$

5. We see that

$$(f \vee g)(x) := \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) := \min\{f(x), g(x)\}$$

    are $\mathcal{A}$-measurable.

6. Let $f_n \colon X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable. Then

$$\sup_{n \in \mathbb{N}} f_n,\ \inf_{n \in \mathbb{N}} f_n,\ \limsup_{n \to \infty} f_n,\ \liminf_{n \to \infty} f_n$$

    are $\mathcal{A}$-measurable.

    Proof. Consider $\sup_{n \in \mathbb{N}} f_n =: g$, then

$$g^{-1}((a, \infty]) = \bigcup_{n \in \mathbb{N}} f_n^{-1}((a, \infty])$$

for $\sup\limits_n f_n(x) = g(x) > a$. A similar argument can prove the case of $\inf\limits_{n \in \mathbb{N}} f_n$. check

And notice that $\limsup\limits_{n \to \infty} f_n = \inf\limits_{k \in \mathbb{N}} \sup\limits_{n \geq k} f_n$, then the similar argument also proves this case. ∎

7. If $\lim\limits_{n \to \infty} f_n(x)$ converges for every $x \in X$, then f is $\mathcal{A}$-measurable.

8. If $f \colon \mathbb{R} \to \mathbb{R}$ is continuous

   $\implies$ f is Borel measurable

   $\implies$ f is Lebesgue measurable

   since the preimage of an open set of a continuous function is open, then we consider $f^{-1}((a, \infty))$.

Definition 2.4 (Support). The *support* of function $f \colon X \to \overline{\mathbb{R}}$ is

$$\mathrm{supp} f \coloneqq \{x \in X \mid f(x) \neq 0\}.$$

Definition 2.5 (Positive and Negative part). For $f \colon X \to \overline{\mathbb{R}}$, let $f^+ \coloneqq f \vee 0$ and $f^- \coloneqq (-f) \vee 0$,[a] where we call f^+ the *positive part* of f while f^- the *negative part* of f.

[a] i.e., $f^+(x) = \max\{f(x), 0\}, \quad f^-(x) = \max\{-f(x), 0\}$

Remark. If $\mathrm{supp} f^+ \cap \mathrm{supp} f^- = \varnothing$ and $f(x) = f^+(x) - f^-(x)$, then

$$f \text{ is } \mathcal{A}\text{-measurable} \iff f^+, f^- \text{ are } \mathcal{A}\text{-measurable}.$$

Definition 2.6 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator) function* of E is

$$\mathcal{X}_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is $\mathcal{A}$-measurable $\iff E \in \mathcal{A}$.

Definition 2.7 (Simple function). Let $(X, \mathcal{A})$ be a measurable space. Then a *simple function* $\phi \colon X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

Remark. We see that if

$$\phi(X) = \{c_1, \ldots, c_N\},$$

---

[20] In this case, we also call it *Borel measurable*.

then

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \implies \phi = \sum_{i=1}^{N} \underbrace{c_i}_{\neq \pm\infty} \underbrace{\mathbb{1}_{E_i}}_{\in \mathcal{A}}.$$

Lecture 11: Integration of nonnegative functions

31 Jan. 11:00

As previously seen. For a simple function ϕ, c_i can actually be in $\mathbb{C}$.

Theorem 2.1. Given a measurable space $(X, \mathcal{A})$ and let $f \colon X \to [0, \infty]$, the followings are equivalent.

1. f is a $\mathcal{A}$-measurable function.

2. There exists simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ such that
$$\mathop{\forall}_{x \in X} \lim_{n \to \infty} \phi_n(x) = f(x)$$
   i.e., f is a pointwise upward limit of simple functions.

Proof. We'll prove both directions.

- It's clear that $(2.) \implies (1.)$ from the fact that $f(x) = \sup_n \phi_n(x)$ and the remark.

- We want to show that $(1.) \implies (2.)$. Assume f is $\mathcal{A}$-measurable, and fix $n \in \mathbb{N}$.

  Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. Also, for $0 \leq k \leq 2^{2n} - 1$, $E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}$.

  Then, define ϕ_n be

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n},$$

  we have

  - $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ for every $x \in X$
  - $\forall x \in X \setminus F_n$, we have $0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$

  Furthermore, we see that

$$F_1 \supset F_2 \supset \ldots, \quad \bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\}),$$

  then

  - $x \in f^{-1}([0, \infty]) = X \setminus \bigcap_{n=1}^{\infty} F_n \implies \lim_{n \to \infty} \phi_n(x) = f(x)$

  - $x \in f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} F_n \implies f_n(x) \geq 2^n \implies \lim_{n \to \infty} \phi_n(x) = \infty = f(x)$

■

Corollary 2.1. If f is bounded on a set $A \subset \mathbb{R}$, i.e., $\exists L > 0$ such that

$$\underset{x \in A}{\forall} \ |f(x)| \le L,$$

then there exists a sequence of simple functions $\{\phi_n\}$ such that $\phi_n \to f$ uniformly on A.

Proof. ■ DIY

Corollary 2.2. If $f \colon X \to \mathbb{C}$ is a measurable function if and only if there exists simple functions $\phi_n \colon X \to \mathbb{C}$ such that

$$0 \le |\phi_1(x)| \le |\phi_2(x)| \le \ldots \le |f(x)|$$

with

$$\underset{x \in X}{\forall} \ \lim_{n \to \infty} \phi_n(x) = f(x).$$

Proof. ■ DIY

2.2 Integration of Nonnegative Functions

We start with our first definition about integral.

Definition 2.8 (Integration of nonnegative funciton). Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\phi \colon X \to [0, \infty]$ such that

$$\phi = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}$$

be a simple function. Define

$$\int \phi = \int \phi \, \mathrm{d}\mu = \int_X \phi \, \mathrm{d}\mu = \sum_{i=1}^{N} c_i \mu(E_i).$$

Furthermore, for $A \in \mathcal{A}$,

$$\int_A \phi = \int_A \phi \, \mathrm{d}\mu = \int \phi \mathbb{1}_A \, \mathrm{d}\mu.$$

Note. Note that

- In the expression $\sum_{i=1}^{N} c_i \mu(E_i)$, we're using the convention $0 \cdot \infty = 0$.

- The function $\phi \mathbb{1}_A$ is also a simple function since both ϕ and $\mathbb{1}_A$ are simple function.

Proposition 2.1. Suppose we have $\phi, \psi \geq 0$ be two simple functions. Then,

- Definition 2.8 is well-defined.
- $\int c\phi = c \int \phi$ for $c \in [0, \infty)$.
- $\int \phi + \psi = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x)$ for all $x \implies \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi \, d\mu$ is a measure on $(X, \mathcal{A})$.

Proof. ■ ⟦DIY⟧

Definition 2.9 (Generatlization of Integration of nonnegative function). Given $(X, \mathcal{A}, \mu)$ with $f \colon X \to [0, \infty]$ be $\mathcal{A}$-measurable. Define

$$\int f = \int f \, d\mu = \sup \left\{ \int \phi \colon 0 \leq \phi \leq f \text{ such that } \phi \text{ is simple} \right\}.$$

Note. Note that

- If f is a simple function, the Definition 2.8 and Definition 2.9 of $\int f$ are the same.
- $\int cf = c \int f$ for $c \in [0, \infty)$.
- If $f \geq g \geq 0 \implies \int f \geq \int g$.
- But $\int f + g = \int f + \int g$ is not trivial.

Theorem 2.2 (Monotone Convergence Theorem). Given $(X, \mathcal{A}, \mu)$ be a measure space. Then if

- $f_n \colon X \to [0, \infty]$ be $\mathcal{A}$-measurable for every $n \in \mathbb{N}$;
- $0 \leq f_1(x) \leq f_2(x) \leq \dots$ for every $x \in X$;
- $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$,

we have

$$\lim_{n \to \infty} \int f_n = \int f.$$

Proof. Note that if $\lim_{n \to \infty} \int f_n$ exists, then it's equal to $\sup_n \int f_n$.

Then

- $f_n \leq f \implies \int f_n \leq \int f \implies \lim_{n \to \infty} \int f_n \leq \int f$.

- Fix a simple function $0 \leq \phi \leq f$, then it's enough to show $\lim\limits_{n\to\infty} \int f_n \geq \int \phi$.

  We first fix $\alpha = (0,1)$, then it's also enough to show

  $$\lim_{n\to\infty} \int f_n \geq \alpha \int \phi.$$

  Let $A_n \coloneqq \{x \in X \mid f_n(x) \geq \alpha\phi(x)\}$, then since f_n is measurable,

    - $A_n \in \mathcal{A}$
    - $A_1 \subset A_2 \subset A_3 \subset \ldots$
    - $\bigcup\limits_{n=1}^{\infty} A_n = X$ \hrulefill $\boxed{\text{Check!}}$

  We then have

  $$\int f_n \geq \int f_n \mathbb{1}_{A_n} \geq \int \alpha\phi\mathbb{1}_{A_n} = \alpha \int_{A_n} \phi = \alpha\nu(A_n)$$

  where $\nu(A) = \int_A \phi$ is a measure. This implies

  $$\lim_{n\to\infty} \int f_n \geq \alpha \lim_{n\to\infty} \nu(A_n) \overset{21}{=} \alpha\nu(X) = \alpha \int \phi.$$

  $\blacksquare$

Corollary 2.3 (Linearity of nonnegative integral). Let $f, g \geq 0$ be measurable, then
$$\int f + g = \int f + \int g.$$

Proof. There exists simple functions ϕ_n and ψ_n such that

- $0 \leq \phi_1 \leq \phi_2 \leq \ldots$ and $\phi_n \to f$ pointwise
- $0 \leq \psi_1 \leq \psi_2 \leq \ldots$ and $\psi_n \to g$ pointwise

Then,

$$\int (f + g) \overset{!}{=} \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \int \phi_n + \int \psi_n \overset{!}{=} \int f + \int g.$$

$\blacksquare$

Lecture 12: Fatou's Lemma
2 Feb. 11:00

We start with a useful corollary.

---

[21] This follows from the continuity of measure from below

Corollary 2.4 (Tonelli's theorem for nonnegative series and integrals). Given $g_n \geq 0$ for every $n \in \mathbb{N}$ and let g_n be measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Remark. Recall that we have seen two series case before. We'll later see two integrals cases.

Proof. Let $f_N := \sum_{n=1}^{N} g_n$ such that $\lim\limits_{N \to \infty} f_N \sum_{n=1}^{\infty} g_n =: f$, then since $g_n \geq 0$, we have $0 \leq f_1 \leq f_2 \leq \dots$ with

$$\lim_{N \to \infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

By Theorem 2.2, we have

$$\lim_{N \to \infty} \int \underbrace{\sum_{n=1}^{N} g_n}_{f_N} = \int \underbrace{\sum_{n=1}^{\infty} g_n}_{f}.$$

Now, since the terms in the limit on the left-hand side is just a finite sum, by Corollary 2.3, we have

$$\underbrace{\lim_{N \to \infty} \sum_{n=1}^{N} \int g_n}_{\sum_{n=1}^{\infty}} = \lim_{N \to \infty} \int \sum_{n=1}^{N} g_n = \int \sum_{n=1}^{\infty} g_n,$$

hence

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

$\blacksquare$

Theorem 2.3 (Fatou's Lemma). Suppose $f_n \geq 0$ and measurable, then

$$\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n.$$

Remark. Recall that

$$\liminf_{n \to \infty} f_n := \lim_{k \to \infty} \inf_{n \geq k} f_n = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f_n$$

and

$$\exists \lim_{n \to \infty} a_n \iff \limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n.$$

Proof. Let $g_k = \inf_{n \geq k} f_n$, then g_k is measurable and $0 \leq g_1 \leq g_2 \leq \ldots$. Now, from Theorem 2.2, we have

$$\int \lim_{k \to \infty} g_k = \lim_{k \to \infty} \int g_k.$$

Notice that the left-hand side is just $\int \liminf_{n \to \infty} f_n$, while the right-hand side is just $\lim_{k \to \infty} \int \inf_{n \geq k} f_n$, i.e.,

$$\int \liminf_{n \to \infty} f_n = \lim_{k \to \infty} \int \inf_{n \geq k} f_n.$$

We see that we want to take the inf outside the integral on the right-hand side. Observe that

$$\underset{m \geq k}{\forall} \inf_{n \geq k} f_n \leq f_m \implies \underset{m \geq k}{\forall} \int \inf_{n \geq k} f \leq \int f_m \implies \int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m.$$

Then, we have

$$\int \liminf_{n \to \infty} f_n = \lim_{k \to \infty} \int \inf_{n \geq k} f_n \leq \lim_{k \to \infty} \inf_{m \geq k} \int f_m = \liminf_{m \to \infty} \int f_m.$$

■

Example. Given $(\mathbb{R}, \mathcal{L}, m)$.

1. **Escape to horizontal infinity**. Let $f_n := \mathbb{1}_{(n,n+1)}$. We immediately see that

   - $f_n \to 0$ pointwise
   - $\int f_n = 1$ for every n
   - $\int f = 0$

   From Theorem 2.3, we have a strict inequality

   $$0 = \int \liminf_{n \to \infty} f_n, \liminf_{n \to \infty} \int f_n = 1.$$

2. **Escape to width infinity**. Let $f_n := \frac{1}{n} \mathbb{1}_{(0,n)}$.

3. **Escape to vertical infinity**. Let $f_n := n \mathbb{1}_{(0,\frac{1}{n})}$.

Lemma 2.3 (Markov's inequiality). Let $f \geq 0$ be measurable. Then

$$\underset{c \in (0,\infty)}{\forall} \mu(\{x \mid f(x) \geq c\}) \leq \frac{1}{c} \int f.$$

Proof. Denote $\{x \mid f(x) \geq c\} =: E$, then

$$f(x) \geq c \mathbb{1}_E(x) \implies \int f \geq c \int \mathbb{1}_E = c \cdot \mu(E).$$

■

Remark. Notice that $E = f^{-1}([c, \infty])$, hence E is measurable.

Proposition 2.2. Let $f \geq 0$ be measurable. Then,

$$\int f = 0 \iff f = 0 \ a.e..$$

i.e.,

$$\int f \, \mathrm{d}\mu = 0 \iff \mu(A) = 0$$

where $A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$.

Proof. Firstly, assume that $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}$$

where E_i are disjoint and $c_i \in (0, \infty)$. Then,

$$\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0$$

$$\iff \mu(E_1) = \ldots = \mu(E_N) = 0$$

$$\iff \mu(A) = 0, \ A = \bigcup_{i=1}^{N} E_i.$$

Now, assume that f is a general function where $f \geq 0$ is the only constraint.

1. Assume $\mu(A) = 0$ (i.e., $f = 0$ a.e.). Let $0 \leq \phi \leq f$, where ϕ is simple. Then

$$\underset{x \in A^c}{\forall} \ \phi(x) = 0$$

since $f(x) = 0, \forall x \in A^c$. This implies that $\phi = 0$ a.e. since $\mu(A) = 0$, so $\int \phi = 0$. We then have

$$\int f = 0$$

from Definition 2.9.

2. Assume $\int f = 0$. Let $A_n = f^{-1}\left(\left[\frac{1}{n}, \infty\right]\right)$. Then we see that

- $A_1 \subset A_2 \subset \ldots$

- $\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0, \infty)) = A.$

We then have

$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \overset{!}{\leq} n \int f = 0,$$

which further implies

$$\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0$$

from the continuity of measure from below.

∎

Corollary 2.5. If $f, g \geq 0$ are both measurable and $f = g$ a.e., then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \neq g(x)\}^{22}$. Then by assumption, $\mu(A) = 0$, hence

$$f \mathbb{1}_A = 0 \ a.e., \quad g \mathbb{1}_A = 0 \ a.e..$$

This further implies that

$$
\begin{aligned}
\int f &= \int f(\mathbb{1}_A + \mathbb{1}_{A^c}) \\
&\stackrel{!}{=} \int f \mathbb{1}_A + \int f \mathbb{1}_{A^c} \\
&= \int f \mathbb{1}_{A^c} = \int g \mathbb{1}_{A^c} \\
&= \int g \mathbb{1}_{A^c} + \int g \mathbb{1}_A = \int g.
\end{aligned}
$$

∎

Corollary 2.6. Let $f_n \geq 0$ be measurable. Then

1.
$$
\left. \begin{array}{l} 0 \leq f_1 \leq f_2 \leq \ldots \leq f \ a.e. \\ \lim_{n \to \infty} f_n = f \ a.e. \end{array} \right\} \implies \lim_{n \to \infty} \int f_n = \int f.
$$

2. $\lim_{n \to \infty} f_n = f \ a.e. \implies \int f \leq \liminf_{n \to \infty} \int f_n$.

Proof. ∎ DIY

Remark. Almost all the theorems we've proved can be replaced by theorems dealing with almost everywhere condition.

Lecture 13: Integration of Complex Functions

4 Feb. 11:00

2.3 Integration of Complex Functions

As usual, we start with a definition.

---

[22] A is measurable indeed.

Definition 2.10 (Integrable). Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f\colon X \to \overline{\mathbb{R}}$ and $g\colon X \to \mathbb{C}$ be measurable.[a]

Then f, g are called *integrable* if $\int |f| < \infty$ and $\int |g| < \infty$, and we define

$$\int f = \int f^+ - \int f^-, \quad \int g = \int \operatorname{Re} g + i \int \operatorname{Im} g.$$

Furthermore, for $f\colon X \to \overline{\mathbb{R}}$, we define

$$\int f = \begin{cases} \infty, & \text{if } \int f^+ = \infty, \int f^- < \infty; \\ -\infty, & \text{if } \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

---

[a]Recall that for a complex-valued function like g, this means that both $\operatorname{Re} g$ and $\operatorname{Im} g$ are measurable.

We now see a lemma.

Lemma 2.4. Let $f, g\colon X \to \overline{\mathbb{R}}$ or $\mathbb{C}$ integrable. Assume that $f(x) + g(x)$ is well-defined for all $x \in X$.[a]

Then we have

1. $f + g$, cf for all $c \in \mathbb{C}$ are integrable.
2. $\int f + g = \int f + \int g$. This is not trivial since $(f + g)^+ \neq f^+ + g^+$.
3. $\left| \int f \right| \leq \int |f|$.

---

[a]That is, we never see $\infty + (-\infty)$ or $(-\infty) + \infty$.

Proof. Check [FF99] page 53. ∎

Lemma 2.5. Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be an integrable function on X. Then

1. f is finite a.e.. i.e., $\{x \in X \mid |f(x)| = \infty\}$ is a null set.
2. The set $\{x \in X \mid f(x) \neq 0\}$ is σ-finite.

Proof. ∎

HW 5 Q8 by Lemma 2.3

Proposition 2.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, then

1. If h is integrable on X, then

$$\underset{E \in \mathcal{A}}{\forall} \int_E h = 0 \iff \int |h| = 0 \iff h = 0 \ a.e.$$

2. If f, g are integrable on X, then

$$\underset{E \in \mathcal{A}}{\forall} \int_E f = \int_E g \iff f = g \ a.e.$$

Proof. We prove this one by one.

1. We see that the second equivalence is done in Proposition 2.2, hence we prove the first equivalence only. Since we have

$$\int |h| = 0 \implies \left| \int_E h \right| \le \int_E |h| \le \int |h| = 0,$$

   which shows one implication. Now assume that $\int_E h = 0$ for all $E \in \mathcal{A}$, then we can write h as

$$h = u + iv = (u^+ - u^-) + i(v^+ - v^-).$$

   Let $B := \{x \in X \mid u^+(x) > 0\}$, then by assumption, we have

$$0 = \int_B h = \operatorname{Re} \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+,$$

   hence $u^+ = 0$ almost everywhere. Similarly, we have u^-, v^+, v^- are all zero almost everywhere. This gives us that h is zero almost everywhere as desired.

2. ⬚DIY

∎

Theorem 2.4 (Dominated Convergence Theorem). Let $(X, \mathcal{A}, \mu)$ be a measure space, and

- Let f_n be integrable on X.
- $\lim_{n \to \infty} f_n(x) = f(x)$ almost everywhere.
- There is a $g \colon X \to [0, \infty]$ such that g is integrable and

$$\underset{n \in \mathbb{N}}{\forall} |f_n(x)| \le g(x) \ a.e.$$

Then we have

$$\lim_{n \to \infty} \int f_n = \int f = \int \lim_{n \to \infty} f_n.$$

Proof. Let F be the countable union of null set on which the three conditions may fail. Then we see that after modifying the definition of f_n, f and g on F, we may assume that all three conditions hold everywhere since modifying on a null set does not change the integral.

We now consider the $\overline{\mathbb{R}}$-valued case only. Note that the second and the third conditions imply that f is integrable since $|f| \leq g(x)$. We then see that $g+f_n \geq 0$ and $g - f_n \geq 0$ because $-g \leq f_n \leq g$. From Theorem 2.3, we have

$$\int g + f \leq \liminf_{n\to\infty} \int g + f_n, \quad \int g - f \leq \liminf_{n\to\infty} \int g - f_n.$$

From the linearity of integral, we have

$$\int g + \int f \leq \int g + \liminf_{n\to\infty} \int f_n, \quad \int g - \int f \leq \int g - \liminf_{n\to\infty} \int f_n.$$

Now, since $\int g < \infty$, we can cancel it, which gives

$$\int f \leq \liminf_{n\to\infty} \int f_n, \quad -\int f \leq \liminf_{n\to\infty} \int -f_n = -\limsup_{n\to\infty} \int f_n,$$

which implies

$$\int f \leq \liminf_{n\to\infty} f_n \leq \limsup_{n\to\infty} \int f_n \leq \int f.$$

This shows that the limit exists, and the desired result indeed holds. ∎

Corollary 2.7 (Tonelli's theorem for series and integrals). Suppose f_n are integrable functions such that

$$\sum_{n=1}^{\infty} \int |f_n| < \infty,$$

then we have

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

Proof. Take $G(x)$ to be

$$G(x) := \sum_{n=1}^{\infty} |f_n(x)|,$$

then we see

$$G(x) \geq |F_N(x)|$$

where

$$F_N(x) := \sum_{n=1}^{N} f_n(x).$$

By Corollary 2.4, we have

$$\int G(x) = \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Lastly, from Theorem 2.4, the result follows. ∎

Remark. Compare to Corollary 2.4, we see that we further generalize the result!

Lecture 14: L^1 Space

2.4 L^1 Space

We now introduce another space called L^p *spaces*, which are function spaces defined using a natural generalization of the p-norm for finite-dimensional vector spaces. We sometimes call it Lebesgue spaces also.

Before we start, we need to define a *norm*.

Definition 2.11 (Seminorm). Let V be a vector space over filed $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* on V is
$$\|\cdot\| : V \to [0, \infty)$$
such that

- $\|cv\| = |c| \, \|v\|$ for every $v \in V$ and every scalar c.
- $\|v + w\| \le \|v\| + \|w\|$ for every $v, w \in V$.

Definition 2.12 (Norm). A *norm* is a seminorm with

- $\|v\| = 0 \iff v = 0$.

Lemma 2.6. A normed vector space is a metric space with metric
$$\rho(v, w) = \|v - w\| \, .$$

Proof. ∎ DIY

Example (p-norm). $V = \mathbb{R}^d$ with
$$\|x\|_p = \begin{cases} \left(\displaystyle\sum_{i=1}^{d} |x_i|^p \right)^{1/p}, & \text{if } p \in [0, \infty); \\ \displaystyle\max_{1 \le i \le d} |x_i|, & \text{if } p = \infty \end{cases}$$

is a normed vector space. The unit ball
$$\{x \in \mathbb{R}^d \mid \|x\|_p \le 1\}$$

for different p has the following figures.

Remark. All $\lVert \cdot \rVert_p$ norms induce the same topology. i.e., if U is open in p-norm, it is open in p'-norm as well.

Note. Recall that we say f is integrable means

$$\int |f| < \infty,$$

and if $f = g$ a.e., then

$$\int f = \int g$$

Definition 2.13 (L^1 Space). Given $(X, \mathcal{A}, \mu)$,

$$f \in L^1(X, \mathcal{A}, \mu) \left(= L^1(X, \mu) = L^1(X) = L^1(\mu) \right)$$

means that f is an integrable function on X.

Lemma 2.7. $L^1(X, \mathcal{A}, \mu)$ is a vector space with seminorm

$$\lVert f \rVert_1 = \int |f|.$$

Proof. ∎

> Check this is indeed a seminorm.

Definition 2.14 (L^1 Space with equivalence class). Define $f \sim g$ if $f = g$ a.e., then

$$L^1(X, \mathcal{A}, \mu) \Big/_{\sim} = L^1(X, \mathcal{A}, \mu),$$

i.e., we simply denote the collection of equivalence classes by itself.[a]

---
[a]By some abusing of notation of L^1.

Remark. We have

- With Definition 2.14, $L^1(X, \mathcal{A}, \mu)$ is a normed vector space.
- We say that the L^1-*metric* $\rho(f, g)$ is simply

$$\rho(f, g) = \int |f - g|.$$

2.4.1 Dense Subsets of L^1

Note. Recall the definition of a *dense set*[23].

Definition 2.15 (Step function). A *step function* on $\mathbb{R}$ is

$$\psi = \sum_{i=1}^{N} c_i \mathbb{1}_{I_i},$$

where I_i is an <u>interval</u>.

Theorem 2.5. We have the following.

1. {integrable simple functions} is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1-metric).

2. $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$, where μ is a Lebesgue-Stieltjes-measure. Then {integrable simple functions} is dense in $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu)$.

3. $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

Notation. We denote the collection of continuous functions <u>with compact</u> support by $C_c(\mathbb{R})$.

Proof. We prove this one by one.

1. Since there exists simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$, where $\phi_n \to f$ pointwise. Then by Theorem 2.4, we have

$$\lim_{n \to \infty} \int \underbrace{|f_n - f|}_{\leq |\phi_n| + |f| \leq 2|f|} = 0$$

where $2|f|$ is in L^1.

2. Let $\mathbb{1}_E$ <u>approximate</u> by $\sum_{i=1}^{\infty} c_i \mathbb{1}_{I_i}$. From Theorem 1.6 for Lebesgue-Stieltjes-measure,

$$\forall \epsilon' > 0 \, \exists I = \bigcup_{i=1}^{N} I_i \text{ such that } \mu(E \triangle I) \leq \epsilon'.$$

---
[23]https://en.wikipedia.org/wiki/Dense_set

3. To approximate $\mathbb{1}_{(a,b)}$, we simply consider function $g \in C_c(\mathbb{R})$ such that

$$\int \left| \mathbb{1}_{(a,b)} - g \right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

∎

Lecture 15: Riemann Integral

2.5 Riemann Integrability

We are now working in $(\mathbb{R}, \mathcal{L}, m)$. Let's first revisit the definition of Riemann Integral. Let P be a partition of $[a,b]$ as

$$P = \{a = t_0 < t_1 < \ldots < t_k = b\}.$$

Then the *lower Riemann sum* of f using P is equal to L_P, which is defined as

$$L_P = \sum_{i=1}^{K} \left(\inf_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1}),$$

and the *upper Riemann sum* of f using P is equal to U_P, which is defined as

$$U_P = \sum_{i=1}^{K} \left(\sup_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1}).$$

Then we call

- *Lower Riemann integral* of $f = \underline{I} = \sup_P L_P$
- *Upper Riemann integral* of $f = \overline{I} = \inf_P U_P$

Definition 2.16 (Riemann (Darboux) integrable). A <u>bounded</u> function $f \colon [a,b] \to \mathbb{R}$ is called *Riemann (Darboux) integrable* if

$$\underline{I} = \overline{I}$$

If so, then $\underline{I} = \overline{I} = \int_a^b f(x) \, \mathrm{d}x$.

Note. We see that

- If $P \subset P'$, then
$$L_P \leq L_{P'}, \quad U_{P'} \leq U_P.$$

- Recall that continuous functions on $[a,b]$ are Reimann integrable on $[a,b]$.

Theorem 2.6. Let $f\colon [a,b] \to \mathbb{R}$ be a <u>bounded</u> function. Then

1. If f is Reimann integrable, then f is Lebesgue measurable, thus Lebesgue integrable. Further,

$$\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathrm{d}m.$$

2. If f is Reimann integrable $\iff$ f is continuous Lebesgue a.e.[a]

---

[a]Here, we mean that the set where f is discontinuous is a null set under Lebesgue measure.

Proof. There exists $P_1 \subset P_2 \subset \ldots$ such that $L_{P_n} \nearrow \underline{I}$ and $U_{P_n} \searrow \overline{I}$.[24] Now, define simple (step) functions

- $\phi_n = \sum\limits_{i=1}^{K} \left(\inf\limits_{[t_{i-1}, t_i]} f \right) \mathbb{1}_{(t_{i-1}, t_i]}$

- $\psi_n = \sum\limits_{i=1}^{K} \left(\sup\limits_{[t_{i-1}, t_i]} f \right) \mathbb{1}_{(t_{i-1}, t_i]}$

if $P_n = \{a = t_0 < t_1 < \ldots < t_K\}$. Let $\phi := \sup_n \phi_n$ and $\psi := \inf_n \psi_n$. We then see that ϕ, ψ are Lebesgue (Borel) measurable function.

Note. Note that

- $\exists M > 0$ such that $\underset{n \in \mathbb{N}}{\forall}\ |\phi_n|, |\psi_n| \le M \mathbb{1}_{[a,b]}$

- $\int \phi_n\,\mathrm{d}m = L_{P_n}$, $\int \psi_n\,\mathrm{d}m = U_{P_n}$

By Theorem 2.4 and the fact that $M\mathbb{1}_{[a,b]} \in L^1(\mathbb{R}, \mathcal{L}, m)$, we have

$$\underline{I} = \lim_{n \to \infty} \int \phi_n\,\mathrm{d}m = \int \phi\,\mathrm{d}m, \quad \overline{I} = \int \psi\,\mathrm{d}m.$$

Thus,

$$f \text{ is Riemann integrable} \iff \int \phi = \int \psi$$
$$\iff \int (\psi - \phi) = 0$$
$$\iff \psi = \phi \text{ Lebesgue a.e.}$$

$\blacksquare$

2.6 Modes of Convergence

As we should already see, there are different *modes* of convergence. Let's formalize them.

---

[24]Here, we took refinements of P_n if needed.

Definition 2.17 (Pointwise, Uniformly convergence). Let

$$f_n, f \colon X \to \mathbb{C},$$

and $S \subset X$. Then we say

- $f_n \to f$ *pointwise* on S if

$$\underset{x \in S}{\forall} \ \underset{\epsilon > 0}{\forall} \ \underset{N \in \mathbb{N}}{\exists} \ \underset{n \geq N}{\forall} \ |f_n(x) - f(x)| < \epsilon.$$

- $f_n \to f$ *uniformly* on S if

$$\underset{\epsilon > 0}{\forall} \ \underset{N \in \mathbb{N}}{\exists} \ \underset{x \in S}{\forall} \ \underset{n \geq N}{\forall} \ |f_n(x) - f(x)| < \epsilon.$$

Remark. We see that we can replace $\forall \epsilon > 0$ by $\forall k \in \mathbb{N}$ with ϵ changing to $\frac{1}{k}$.

Lemma 2.8. Let $B_{n,k}$ be

$$B_{n,k} := \left\{ x \in X \mid |f_n(x) - f(x)| < \frac{1}{k} \right\}.$$

Then

1. $f_n \to f$ pointwise on S if and only if

$$S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$

2. $f_n \to f$ uniformly on S if and only if $\exists N_1, N_2, \ldots \in \mathbb{N}$ such that

$$S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Proof. This essentially follows from Definition 2.17. ∎

Definition 2.18 (Converges a.e., Converges in L^1). Let $(X, \mathcal{A}, \mu)$ be a measure space. Assuming that f_n, f are measurable function, then

1. $f_n \to f$ *almost everywhere* means

$$\exists \text{ null set } E \text{ such that } f_n \to f \text{ pointwise on } E^c.$$

2. $f_n \to f$ *in L^1* means

$$\lim_{n \to \infty} \|f_n - f\| = 0.$$

Example. Given $(\mathbb{R}, \mathcal{L}, m)$ and let $f = 0$. We see the followings.

1. $f_n = \mathbb{1}_{(n, n+1)}$

2. $f_n = \frac{1}{n}\mathbb{1}_{(0,n)}$

3. $f_n = n\mathbb{1}_{(0,\frac{1}{n})}$

4. **Typewriter functions**.

Typewriter Sequence:
$f_n(x) = 1_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}$,
$\forall\, k \geq 0$ & $2^k \leq n < 2^{k+1}$

Lecture 16: Modes of Convergence

11 Feb. 11:00

Let's start with a proposition.

Proposition 2.4 (Fast L^1 convergence leads to a.e. convergence).
Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n, f are all measurable functions on X. Then
$$\sum_{n=1}^{\infty} \|f_n - f\|_1 < \infty \implies f_n \to f \quad \text{a.e.}$$

Proof. Let
$$E \coloneqq \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c = \{x \in X \mid f_n(x) \nrightarrow f(x)\}.$$

By Lemma 2.3, we see that

$$\underset{k}{\forall}\ \underset{N}{\forall}\ \mu\left(B_{n,k}^c\right) \le k\int|f_n - f| \implies \underset{k}{\forall}\ \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) \le \sum_{n=N}^{\infty} k\left\|f_n - f\right\|_1 \to 0$$

as $N \to \infty$. Now, by continuity of measure from above,

$$\underset{k}{\forall}\ \mu\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = \lim_{N\to\infty}\mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \mu(E) = 0$$

since $f_n \to f$ pointwise on E^c. ∎

Corollary 2.8. Given $\{f_n\}_n$ such that $f_n \to f$ in L^1, there exists a subsequence $\{f_{n_j}\}_{n_j}$ where $f_{n_j} \to f$ a.e.

Proof. Since

$$\underset{j\in\mathbb{N}}{\forall}\ \underset{n_j\in\mathbb{N}}{\forall}\ \left\|f_{n_j} - f\right\|_1 \le \frac{1}{j^2}.$$

Then,

$$\sum_{j=1}^{\infty}\left\|f_{n_j} - f\right\|_1 < \infty.$$

From Proposition 2.4, we have the desired result. ∎

Definition 2.19 (Converge in measure). Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. Then $f_n \to f$ *in measure* means

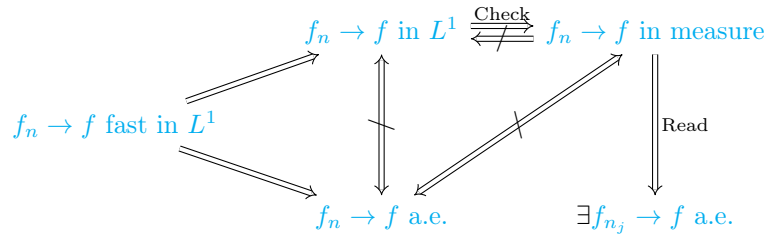$$\underset{\epsilon>0}{\forall}\ \lim_{n\to\infty}\mu\left(\{x \in X \mid |f_n(x) - f(x)| \ge \epsilon\}\right) = 0.$$

Example. Let $f_n = n\mathbb{1}_{(0,\frac{1}{n})}$ and $f = 0$. We see that

$$\forall \epsilon > 0\ \ \{x \in X \mid |f_n(x) - f(x)| > \epsilon\} = \left(0, \frac{1}{n}\right),$$

$f_n \to 0$ in measure. (Recall that $f_n \nrightarrow 0$ in L^1)

Remark. We see that

Definition 2.20 (Uniformly a.e., Almost uniformly). Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$.

1. $f_n \to f$ *uniformly almost everywhere* means $\exists$null set F such that $f_n \to f$ uniformly on F^c.

2. $f_n \to f$ *almost uniformly* means $\forall \epsilon > 0 \; \exists F \in \mathcal{A}$ such that $\mu(F) < \epsilon$, $f_n \to f$ uniformly on F^c.

Lemma 2.9. We have

$$f_n \to f \text{ uniformly on } S \iff \exists N_1, N_2, \ldots \in \mathbb{N} \; S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Theorem 2.7 (Egorov's Theorem). Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. Suppose $\mu(X) < \infty$, then

$$f_n \to f \text{ a.e.} \iff f_n \to f \text{ almost uniformly}.$$

Proof. We prove two directions.

- $\impliedby$ DIY

- $\implies$ Fix $\epsilon > 0$. We see that

$$f_n \to f \text{ a.e.} \implies \mu\left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0$$

$$\implies \underset{k}{\forall} \; \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0.$$

From continuity of measure from above and $\mu(X) < \infty$, we further have

$$\underset{k}{\forall} \; \lim_{N \to \infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0 \implies \underset{k}{\forall} \; \underset{N_k \in \mathbb{N}}{\exists} \; \mu\left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c \right) < \frac{\epsilon}{2^k}.$$

Now, let

$$F := \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c,$$

we see that $\mu(F) < \epsilon$, hence $f_n \to f$ uniformly.

$\blacksquare$

3   Product Measure

3.1   Product σ-algebra

Before we start, we see the setup.

- Product space.

$$X = \prod_{\alpha \in I} X_\alpha$$

  where $x = (x_\alpha)_{\alpha \in I} \in X$.

- Coordinate map.

$$\pi_\alpha \colon X \to X_\alpha.$$

Now we see the formal definition.

Definition 3.1 (Product σ-algebra). Let $(X_\alpha, \mathcal{A}_\alpha)$ be a measurable space for all $\alpha \in I$. Then a *product σ-algebra* on $X = \prod\limits_{\alpha \in I} X_\alpha$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha \in I} \pi_\alpha^{-1} (\mathcal{A}_\alpha) \right\rangle,$$

where

$$\pi_\alpha^{-1} (\mathcal{A}_\alpha) = \left\{ \pi_\alpha^{-1}(E) \mid E \in \mathcal{A}_\alpha \right\}.$$

Notation. We denote $I = \{1, \ldots, d\} \implies X = \prod\limits_{i=1}^{d} X_i, x = (x_1, \ldots, x_d)$. Also,

$$\bigotimes_{i=1}^{d} \mathcal{A}_i = \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_d.$$

Lemma 3.1. If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \underset{i}{\forall} \; E_i \in \mathcal{A}_i \right\} \right\rangle.$$

Proof. If $E_i \in \mathcal{A}_i$, then $\pi_i^{-1}(E_i) = \prod\limits_{j=1}^{\infty} E_j$, where $E_j = X$ for $j \neq i$. On the other hand, since

$$\prod_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} \pi_i^{-1}(E_i),$$

from Lemma 1.2, the result follows. $\blacksquare$

Lecture 17: Product Measure

We now see a lemma.

Lemma 3.2. Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle$ for every $\alpha \in I$. Then

1. $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$

2. $\bigotimes_\alpha \mathcal{A}_\alpha = \langle \bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$

3. If I is countable, then

$$\bigotimes_{i=1}^\infty \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^\infty E_i \mid \underset{i}{\forall} \, E_i \in \mathcal{E}_i \right\} \right\rangle$$

Proof. We prove this one by one.

1. Note that for $f \colon Y \to Z$, and $\mathcal{B}$ be a σ-algebra on Z, then $f^{-1}(\mathcal{B})$ is also a σ-algebra.[25] Hence, π_α^{-1} is a σ-algebra on X, i.e.,

$$\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \overset{!}{\implies} \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha).$$

To show the other direction, let $\mathcal{M}$ being

$$\mathcal{M} = \left\{ B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \right\}.$$

We now check

- $\mathcal{M}$ is a σ-algebra. ⟶ Check (Easy)!
- $\mathcal{E}_\alpha \subset \mathcal{M}$. This is true by definition of $\mathcal{M}$.

Thus, $\langle \mathcal{E}_\alpha \rangle = \mathcal{A}_\alpha \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha$, $E \in \mathcal{M}$, implying

$$\pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle,$$

i.e., $\mathcal{A}_\alpha \subset \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

2. ⟶ DIY

3. ⟶ DIY

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[25]Since $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(B)^c = f^{-1}(B^c)$, and $\bigcup_n f^{-1}(B_n) = f^{-1}(\bigcup_n B_n)$.

Theorem 3.1. Suppose $X_1, \ldots, X_d$ are metric spaces. Let $X = \prod\limits_{i=1}^{d} X_i$ with product metric defined as

$$\rho(x, y) = \sum_{i=1}^{d} \rho_i(x_i, y_i).$$

Then,

1. $\bigotimes\limits_{i=1}^{d} \mathcal{B}(X_i) \subset \mathcal{B}(X)$

2. If in addition, each X_i has a countable dense subset,

$$\bigoplus_{i=1}^{d} \mathcal{B}(X_i) = \mathcal{B}(X).$$

Proof. ∎ ⟨DIY⟩

Remark. We see that

- $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$

- let $f = u + iv \colon X \to \mathbb{C}$, and $\mathcal{A}$ be a σ-algebra on X. Then

$$\underset{E \in \mathcal{B}(\mathbb{R})}{\forall} u^{-1}(E), v^{-1}(E) \in \mathcal{A} \iff f^{-1}(F) \in \mathcal{A}, \forall\, F \in \mathcal{B}(\mathbb{C})$$

  with $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Definition 3.2 (x-section, y-section). Let X, Y be two sets. Then

1. For $E \subset X \times Y$,

$$E_x = \{y \in Y \mid (x, y) \in E\}, \quad E^y = \{x \in X \mid (x, y) \in E\}.$$

2. For $f \colon X \times Y \to \mathbb{C}$, define

$$f_x \colon Y \to \mathbb{C}, \quad f^y \colon X \to \mathbb{C}$$

by

$$f_x(y) = f(x, y) = f^y(x).$$

Example. We see that

$$(\mathbb{1}_E)_x = \mathbb{1}_{E_x}$$

and

$$(\mathbb{1}_E)^y = \mathbb{1}_{E^y}.$$

Proposition 3.1. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, then

1. If $E \in \mathcal{A} \otimes \mathcal{B}$, then

$$\underset{x \in X}{\forall} \ \underset{y \in Y}{\forall} \ E_x \in \mathcal{B}, E^y \in \mathcal{A}.$$

2. If $f \colon X \times Y \to \mathbb{C}$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then

$$\underset{x \in X}{\forall} \ \underset{y \in Y}{\forall} \ f_x \text{ is } \mathcal{B}\text{-measurable}, f^y \text{ is } \mathcal{A}\text{-measurable}.$$

Proof. We prove this one by one.

1. Let $\mathcal{F} := \left\{ E \subset X \times Y \mid \underset{x \in X}{\forall} \ \underset{y \in Y}{\forall} \ E_x \in \mathcal{B}, E^y \in \mathcal{A} \right\}$, then

    - $\mathcal{F}$ is a σ-algebra.
        - $\varnothing \in \mathcal{F}$.
        - $(E^c)_x = E_x^c$.
        - $\left(\overset{\infty}{\underset{j=1}{\bigcup}} E_j \right)_x = \overset{\infty}{\underset{j=1}{\bigcup}} (E_j)_x$.

    And the same is true for y.

    - Let $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$, which is again easy to show from definition.

    Hence, we see that $\langle R_0 \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{F}$.

2. Since
$$(f_x)^{-1}(B) = (f^{-1}(B))_x$$
    and
$$(f^y)^{-1}(B) = (f^{-1}(B))^y,$$
    the result follows from 1.

∎

3.2 Product Measures

We start with the definition.

Definition 3.3 (Rectangle). Given two measurable spaces, a *(measurable) rectangle* is $R = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, we let
$$\mathcal{R}_0 := \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$
and
$$\mathcal{R} := \left\{ \overset{N}{\underset{i=1}{\bigcup}} R_i \mid N \in \mathbb{N}, R_1, \dots, R_N \text{ disjoint rectangles} \right\}.$$

Note. Whenever we're talking about rectangle, they're always measurable.

Lemma 3.3. $\mathcal{R}$ is an algebra, and

$$\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}.$$

Proof. Simply observe that

$$(A \times B)^c = (A^c \times Y) \cup (A \times B)$$

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Lecture 18: Monotone Class

16 Feb. 11:00

Let's start with a theorem.

Theorem 3.2. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces. Then

1. There is a measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

   for every $A \in \mathcal{A}, B \in \mathcal{B}$.

2. If μ, ν are σ-finite , then $\mu \times \nu$ is unique.

Proof. We prove this one by one.

1. Define $\mu \colon \mathcal{R} \to [0, \infty]$ by $\mu(A \times B) = \mu(A)\nu(B)$, and extending linearly, we have
$$\pi(A \times B) = \mu(A)\nu(B),$$
   hence
$$\pi\left(\coprod_{i=1}^{N} A_i \times B_i \right) = \sum_{i=1}^{n} \pi(A_i \times B_i).$$

   We claim that π is a pre-measure. To show this, it's enough to check that $\pi(A \times B) = \sum_{n=1}^{\infty} \pi(A_n \times B_n)$ if $A \times B = \coprod_{n} A_n \times B_n$. Since $A_n \times B_n$ are disjoint, so
$$\mathbb{1}_{A \times B}(x, y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n \times B_n}(x, y).$$

   Thus,
$$\mathbb{1}_A(x)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y).$$

   Integrating with respect to x, and applying Proposition 1.3, we have
$$\int_X \mathbb{1}_A(x)\mathbb{1}_B(y)\,\mathrm{d}\mu(x) = \sum_{n=1}^{\infty} \int_x \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y)\,\mathrm{d}\mu(x),$$

3 PRODUCT MEASURE

which implies

$$\mu(A)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mu(A_n)\mathbb{1}_{B_n}(y)$$

for every y. We can then integrate again with respect to y and apply Proposition 1.3, we have

$$\int_Y \mu(A)\mathbb{1}_B(y)\,\mathrm{d}\nu(y) = \sum_{n=1}^{\infty} \int_Y \mu(A_n)\mathbb{1}_{B_n}(y)\,\mathrm{d}\nu(y),$$

which gives us

$$\mu(A)\nu(B) = \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n).$$

Hence, we see that μ is indeed a pre-measure, so Theorem 1.3 gives $\mu \times \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on $\mathcal{R}$.

2. If μ, ν are σ-finite, then π is σ-finite on $\mathcal{R}$, then Theorem 1.4 applies. Moreover, we have that

$$(\mu \times \nu)(E) = \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \mid E \subset \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\}.$$

$\blacksquare$

3.3   Monotone Class Lemma

Let's start with a definition.

Definition 3.4 (Monotone Class). If X is a set, and $C \subset \mathcal{P}(X)$, we say that C is a *monotone class* on X if

- C is closed under countable increasing unions.

- C is closed under countable decreasing intersections.

Example. We see that

1. Every σ-algebra is a monotone class.

2. If C_α are (arbitrarily many) monotone classes on a set X, then $\bigcap_\alpha C_\alpha$ is a monotone class. Furthermore, if $\mathcal{E} \subset \mathcal{P}(X)$, there is a unique smallest monotone class containing $\mathcal{E}$, denoted by $\langle \mathcal{E} \rangle$, which follows the same idea as in Definition 1.3.

Theorem 3.3 (Monotone Class Lemma). Suppose $\mathcal{A}_0$ is an algebra on X. Then $\langle \mathcal{A}_0 \rangle^a$ is the monotone class generated by $\mathcal{A}_0$.

---
[a] $\langle \mathcal{A}_0 \rangle$ is the σ-algebra generated by $\mathcal{A}_0$ by Definition 1.3.

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$ and let $\mathcal{C}$ be the monotone class generated by $\mathcal{A}_0$. Since $\mathcal{A}$ is a σ-algebra, it's a monotone class. Note that it contains $\mathcal{A}_0$, hence $\mathcal{A} \supset \mathcal{C}$.

To show $\mathcal{C} \supset \mathcal{A}$, it's enough to show that $\mathcal{C}$ is a σ-algebra. We check that

1. $\varnothing \in \mathcal{A}_0 \subseteq \mathcal{C}$.

2. Let $\mathcal{C}' = \{E \subset X \mid E^c \in \mathcal{C}\}$.

   - $\mathcal{C}'$ is a monotone class.

   - $\mathcal{A}_0 \subset \mathcal{C}'$ because if $E \in \mathcal{A}_0$, then $E^c \in \mathcal{A}_0$, so $E^c \in \mathcal{C}$, thus $E \in \mathcal{C}'$.

   We see that $\mathcal{C}' \subset \mathcal{C}'$, so $\mathcal{C}$ is closed under complements.

3. For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.

   - $\mathcal{D}(E) \subset \mathcal{C}$.

   - $\mathcal{D}(E)$ is a monotone class.

   - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. We see this by picking $F \in \mathcal{A}_0$, then $E \cup F \in \mathcal{A}_0 \supset \mathcal{C}$.

   Hence, $C = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.

4. Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\}$. That is $\mathcal{D} = \{E \in \mathcal{C} \mid E \cup F, \forall F \in \mathcal{C}\}$. Then we have

   - $A_0 \subset \mathcal{D}$ by 3.

   - $\mathcal{D}$ is a monotone class.

   - $\mathcal{D} \subset \mathcal{C}$ by definition.

   Thus, $\mathcal{D} = \mathcal{C}$, so if $E, F \in \mathcal{C}$, then $E \cup F \in \mathcal{C}$. This implies that $\mathcal{C}$ is closed under finite unions.

5. Now to show that $\mathcal{C}$ is closed under countable unions, let $E_1, E_2, \ldots \in \mathcal{C}$. We may then define

$$F_N = \bigcup_{n=1}^{N} E_n \in \mathcal{C}.$$

   Then we see that $F_1 \subset F_2 \subset \ldots$, hence $\bigcup_N F_N \in \mathcal{C}$. But this simply implies

$$\bigcup_N F_N = \bigcup_n E_n,$$

   so we're done.

$\blacksquare$

Lecture 19: Fubini-Tonelli's Theorem

As previously seen. If $E \in A \otimes B \implies E_x \in \mathcal{B}, E^y \in \mathcal{A} \ \forall x \in X, \forall y \in Y$. Note that the reverse is not true.

3.4   Fubini-Tonelli Theorem

We start with a theorem.

> **Theorem 3.4 (Tonelli's theorem for characteristic functions).** Given $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure space. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$, then
>
> 1. $\alpha(x) \coloneqq \nu(E_x)\colon X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.
> 2. $\beta(x) \coloneqq \mu(E^y)\colon Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.
> 3. $(\mu \times \nu)(E) = \int_X \nu(E_x) \, \mathrm{d}\mu(x) = \int_Y \mu(E^y) \, \mathrm{d}\nu(y)$.

Proof. We prove this one by one.

1. Assume that μ, ν are finite measure. Let

$$C \coloneqq \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{ Conditions 1., 2., 3., hold}\} \, .$$

   It's enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset C$. We further observe that from the Theorem 3.3 and the fact that $\mathcal{R}$ is an algebra, it's also enough to show that

   - $\mathcal{R} \subset C$.
   - C is a monotone class.

   From condition 1.,

$$\alpha(x) = \nu\left((A \times B)_x\right) = \begin{cases} \nu(B), & \text{if } x \in A; \\ 0, & \text{if } x \notin A \end{cases} = \nu(B)\mathbb{1}_A.$$

   And from condition 2.,

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

   and

$$\int_X \nu((A \times B)_x) \, \mathrm{d}\mu(x) = \nu(B)\mu(A).$$

   Let $E_n \in C$, $E_1 \subset E_2 \subset \dots$. We need to show $E = \bigcup\limits_{n=1}^{\infty} E_n \in C$. We now see that

$$E_x = \bigcup_{n=1}^{\infty} (E_n)_x, (E_1)_x \subset (E_2)_x \subset \dots$$

$$\implies \alpha(x) = \nu(E_n)_x \overset{!}{=} \lim_{n \to \infty} \nu((E_n)_x) \; \forall x \in X.$$

   This implies that 1. is proved.

   For 3., we see that

$$(\mu \times \nu)(E) \overset{!}{=} \lim_{n \to \infty} (\mu \times \nu)(E_n)$$

$$= \lim_{n \to \infty} \int_X \nu((E_n)_x) \, \mathrm{d}\mu(x)$$

$$\overset{!}{=} \int_X \nu(E_x) \, \mathrm{d}\mu(x).$$

Now let $F_n \in C$, $F_1 \supset F_2 \supset \ldots$. We need to show that $F = \bigcap_{n=1}^{\infty} F_n \in C$. Instead of using Theorem 2.2, we now want to use Theorem 2.4, which is applicable since $\mu(X), \nu(Y) < \infty$ by assumption. Then assume that μ, ν are σ-finite, then

$$X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n), \begin{cases} X_1 \subset X_2 \subset \ldots, & \mu(X_k) < \infty \\ Y_1 \subset Y_2 \subset \ldots, & \nu(Y_k) < \infty. \end{cases}$$

> DIY

$\blacksquare$

Theorem 3.5 (Fubini-Tonelli's Theorem). Given two σ-finite measure space $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$, we have the following two versions.

(Tonelli) If $f \colon X \times Y \to [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then

1. $g(x) := \int_Y f(x, y) \, d\nu(y)$, $X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.

2. $h(x) := \int_X f(x, y) \, d\mu(x)$, $Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.

3. We have

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\nu(y).$$

(Fubini) If $f \in L^1(X \times Y, \mu \times \nu)$, then

1. $f_x \in L^1(Y, \nu)$ for μ-a.e. x, and $g(x) \in L^1(X, \mu)$ defined μ-a.e.

2. $f^y \in L^1(X, \mu)$ for ν-a.e. y, and $h(x) \in L^1(Y, \nu)$ defined μ-a.e.

3. The iterated integral formulas hold. Namely, we have

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\nu(y).$$

Proof. Read [FF99]. $\blacksquare$

Lecture 20: Lebesgue Measure on $\mathbb{R}^d$

3.5 Lebesgue Measure on $\mathbb{R}^d$

Example. We first see some examples.

1. $(\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m)$ is not complete.

   - Let $A \in \mathcal{L}$, $A \neq \varnothing$, $m(A) = 0$.

   - Let $B \subset [0, 1]$, $B \notin \mathcal{L}$ (Vital set for example).

   - Let $E = A \times B$, $F = A \times [0, 1]$.

We see that $E \subset F$, $F \in \mathcal{L} \otimes \mathcal{L}$, $(m \times m)(F) = m(A)m([0,1]) = 0$, i.e., F is a null set. But E is **not** $\mathcal{L} \otimes \mathcal{L}$-measurable-function since otherwise, its sections are all measurable.

Definition 3.5. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the *completion* of

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \ldots \times m),$$

which is <u>same</u> as the *completion* of

$$(\mathbb{R}^d, \mathcal{L} \otimes \ldots \otimes \mathcal{L}, m \times \ldots m).$$

Remark. We see that

$$\mathcal{L}^d \supsetneq \mathcal{L} \otimes \ldots \otimes \mathcal{L} = \left\langle \left\{ \prod_{i=1}^d E_i \mid E_i \in \mathcal{L} \right\} \right\rangle.$$

Definition 3.6 (General rectangle). A *rectangle* in $\mathbb{R}^d$ is $R = \prod_{i=1}^d E_i$ where $E_i \in \mathcal{B}(\mathbb{R})$.

Definition 3.7. We let

$$m^d(E) := \inf \left\{ \sum_{k=1}^\infty m^d(R_k) \mid E \subset \bigcup_{k=1}^\infty R_k, R_k \text{ is rectangles} \right\}.$$

Theorem 3.6. Let $E \subset \mathcal{L}^d$. Then

1. $m^d(E) = \inf \left\{ m^d(0) \mid \text{open } O \supset E \right\} = \sup \left\{ m^d(K) \mid \text{compact } K \subset E \right\}$.

2. $E = A_1 \cup N_1 = A_2 \setminus N_2$, where A_1 is F_σ, A_2 is G_δ , and N_i are null.

3. If $m^d(E) < \infty$, $\forall \epsilon > 0$, $\exists R_1, \ldots, R_m$ rectangles whose sides are <u>intervals</u> such that

$$m^d \left(E \triangle \left(\bigcup_{i=1}^m R_i \right) \right) < \epsilon.$$

Proof. Similar to $d = 1$ case. ∎

Theorem 3.7. Integrable step functions and $C_c(\mathbb{R}^d)$, the collection of continuous functions, are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$

Proof. See [FF99]. ∎

> **Theorem 3.8.** Lebesgue measure in $\mathbb{R}^d$ is translation-invariant.

Proof. See [FF99]. ∎

> **Theorem 3.9 (Effect of linear transformation on Lebesgue measure).** If $T \in \mathrm{GL}(\mathbb{R}^d)$, $e \in \mathcal{L}^d$, then $T(E)$ is measurable and
>
> $$m(T(E)) = |\det T| \cdot m(E).$$

Proof. See [FF99]. ∎

4 Differentiation on Euclidean Space

As previously seen. Given $f \colon [a,b] \to \mathbb{R}$, there are two versions of **fundamental theorem of calculus**:

1.
$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a).$$

2.
$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x),$$

   which follows from

$$\lim_{r \to 0^+} \frac{1}{r} \int_x^{x+r} f(t)\,\mathrm{d}t = f(x) = \lim_{r \to 0^+} \frac{1}{r} \int_{x-r}^x f(t)\,\mathrm{d}t.$$

Remark. We see that

$$\lim_{r \to 0^+} \frac{1}{r} \int_x^{x+r} (f(t) - f(x))\,\mathrm{d}t = 0 = \lim_{r \to 0^+} \frac{1}{r} \int_{x-r}^x (f(t) - f(x))\,\mathrm{d}t,$$

where we have

$$f(x) = \frac{1}{r} \int_x^{x+r} f(t)\,\mathrm{d}t.$$

This generalized to $f \colon \mathbb{R}^d \to \mathbb{R}$, namely

$$\lim_{r \to 0^+} \frac{1}{\mathrm{vol}\,(B(x,r))} \int_{B(x,r)} (f(t) - f(x)) \underbrace{\mathrm{d}t}_{\mathbb{R}^d} \overset{?}{=} 0.$$

4.1 Hardy-Littlewood Maximal Function

We first see our notation.

Notation. Given a(n) (open) ball in $\mathbb{R}^d$, $B = B(a,r)$, denote $cB = B(a,cr)$ for $c > 0$.

> **Lemma 4.1 (Vitali-type covering lemma).** Let $B_1, \ldots, B_k$ be a finite collection of open balls in $\mathbb{R}^d$. Then there exists a sub-collection $B_1', \ldots, B_m'$ of <u>disjoint</u> open balls such that
> $$\bigcup_{i=1}^{m} (3B_j') \supset \bigcup_{i=1}^{k} B_i.$$

Proof. Greedy Algorithm. ∎

Lecture 21: Hardy-Littlewood Maximal Function and Inequality

Notation. We let
$$\int_E f \, dm = \int_E f(x) \, dx.$$

The problem we're working on is
$$\frac{1}{m(B(w,r))} \int_{B(x,r)} f(y) \, dy \overset{r \to 0}{\underset{?}{\longrightarrow}} f(x).$$

> **Definition 4.1 (Locally integrable).** Given $f \colon \mathbb{R}^d \to \mathbb{C}$ be Lebesgue measurable function. Then we say f is *locally integrable* if for every compact $K \subset \mathbb{R}^d$,
> $$\int_K |f| \, dm < \infty.$$
> We write $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

> **Definition 4.2 (Hardy-Littlewood maximal function).** Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, the *Hardy-Littlewood maximal function* for f is defined as
> $$\mathrm{H}f(x) \coloneqq \sup \{ \mathrm{A}_r(x) \mid r > 0 \},$$
> where
> $$\mathrm{A}_r(x) \coloneqq \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| \, dy.$$

Note. We note that $\mathrm{A}_r(\cdot)$ means *averaging function*.

> **Lemma 4.2.** Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then
> 1. $\mathrm{A}_r(x)$ is jointly continuous for $(x, r) \in \mathbb{R}^d \times (0, \infty)$.
> 2. $\mathrm{H}f(x)$ is Borel measurable.

Proof. We outline the proof.

1. Let $(x, r) \to (x^*, r^*) \implies A_r(x) \to A_{r^*}(x^*)$. Let (x_n, r_n) be any sequence which converges to x^*, r^*, then we consider $\lim_{n \to \infty} A_{r_n}(x_n)$ and we can calculate
$$\int \underbrace{|f(y)| \, \mathbb{1}_{B(x_n, r_n)}(y)}_{:= h_n(y)},$$
then we apply Theorem 2.4 to h_n.

2. Observe that
$$(\mathrm{H}f)^{-1}(\underbrace{(a, \infty)}_{\text{open}}) = \bigcup_{r > 0} A_r^{-1}((a, \infty))$$
is open, since $A_r^{-1}((a, \infty))$ is open from the 1. Note that the equality comes from the fact that $\mathrm{H}f = \sup_r A_r$.

∎

Theorem 4.1 (Hardy-Littlewood maximal inequality). There exists $C_d > 0$ such that for every $f \in L^1(\mathbb{R}^d)$,

$$\underset{\alpha > 0}{\forall} \; m\left(\left\{x \in \mathbb{R}^d \mid \mathrm{H}f(x) > \alpha\right\}\right) \leq \frac{C_d}{\alpha} \int |f(x)| \, \mathrm{d}x.$$

Proof. We first fix $f \in L^1$ and $\alpha > 0$. We define
$$E := \{x \mid \mathrm{H}f(x) > \alpha\},$$

which is a Borel measurable set by Lemma 4.2. Then

$$x \in E \implies \underset{r_x > 0}{\exists} \; A_{r_x}(x) > \alpha \implies m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x, r_x)} |f(y)| \, \mathrm{d}y.$$

From inner regularity, we have

$$m(E) = \sup\left\{m(K) \mid \text{compact } K \subset E\right\}.$$

Let $K \subset E$ be compact, then

$$K \subset \bigcup_{x \in K} B(x, r_x) \overset{K \text{ compact}}{\implies} K \subset \bigcup_{i=1}^{N} B_i \overset{!}{\implies} K \subset \bigcup_{i=1}^{m} \{3B_j'\}.$$

From here, we further have

$$m(K) \leq \sum_{i=1}^{m} m(3B_j') = 3^d \sum_{j=1}^{m} m(B_j') \leq \frac{3^d}{\alpha} \sum_{j=1}^{m} \int_{B_j'} |f(y)| \, \mathrm{d}y.$$

Now, since $B_i', \ldots, B_m'$ are disjoint, hence we finally have

$$m(K) \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \, \mathrm{d}y.$$

∎

Lecture 22: Lebesgue Differentiation Theorem

We should compare the Hardy-Littlewood maximal inequality to Markov's inequality. Namely, there exists $C_d > 0$ (can take 3^d) such that for all $f \in L^1(\mathbb{R}^d)$, $\alpha > 0$, we have

$$
\begin{cases}
m(\{x \mid \mathrm{H}f(x) > \alpha\}) \leq \dfrac{C_d}{\alpha} \displaystyle\int |f| \,; \\[2ex]
m(\{x \mid |f(x)| > \alpha\}) \leq \dfrac{1}{\alpha} \displaystyle\int |f| \,.
\end{cases}
$$

4.2 Lebesgue Differentiation Theorem

We start with a theorem!

> **Theorem 4.2 (Lebesgue Differentiation Theorem).** Let $f \in L^1$, then
>
> $$
> \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y = 0
> $$
>
> for a.e. x.

Proof. The result holds for $f \in C_c(\mathbb{R}^d)$, namely for those continuous functions with **compact** support. This is because for any $\epsilon > 0$, if r is small and $|f(y) - f(x)| < \epsilon$, then

$$
\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y < \epsilon.
$$

Now, let $f \in L^1(\mathbb{R}^d)$ and fix $\epsilon > 0$. By density, there exists $g \in C_c(\mathbb{R}^d)$ with $\|f - g\|_1 < \epsilon$. We then have

$$
\int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y \leq \int_{B(x,r)} |f(y) - g(y)| \, \mathrm{d}y
$$
$$
+ \int_{B(x,r)} |g(y) - g(x)| \, \mathrm{d}y
$$
$$
+ \int_{B(x,r)} |g(x) - f(x)| \, \mathrm{d}y.
$$

Divide all of these by $m(B(x,r))$, and take $\limsup_{r \to \infty}$, we need to understand the error terms, namely

$$
\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - g(x)| \, \mathrm{d}y = |g(x) - f(x)|
$$

and

$$
\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g(y)| \, \mathrm{d}y \leq (\mathrm{H}(f-g))(x).
$$

We define

$$
Q(x) := \limsup_{r \to \infty} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y.
$$

We want to show $m(\{x \in X \mid Q(x) > 0\}) = 0$. Let $E_\alpha = \{x \in X \mid Q(x) > \alpha\}$. It is enough to show $m(E_\alpha) = 0$ for all $\alpha > 0$ because $\{x \in X \mid Q(x) > 0\} = \bigcup_n E_{\frac{1}{n}}$. We know by the above that

$$Q(x) \leq (\mathrm{H}(f - g))(x) + 0 + |g(x) - f(x)|.$$

Therefore,

$$E_\alpha \subset \{x \in X \mid (\mathrm{H}(f - g))(x) > \alpha/2\} \cup \{x \in X \mid |g(x) - f(x)| > \alpha/2\}.$$

By the Hardy-Littlewood maximal inequality and Markov's inequality, we have

$$\begin{cases} m(\{x \mid (\mathrm{H}(f - g))(x) > \alpha/2\}) \leq \dfrac{2C_d}{\alpha} \int |f - g|\,; \\ m(\{x \mid |g(x) - f(x)| > \alpha/2\}) \leq \dfrac{2}{\alpha} \int |f - g|\,. \end{cases}$$

Thus,

$$0 \leq m(E_\alpha) \leq \frac{2C_d}{\alpha} \|f - g\|_1 + \frac{2}{\alpha} \|f - g\|_1 \leq \frac{2(C_d + 1)}{\alpha}\epsilon.$$

Taking $\epsilon \to 0$, $m(E_\alpha)$ does not depend on ϵ and g, hence $m(E_\alpha) = 0$. ∎

Corollary 4.1. Theorem 4.2 also holds for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Proof. Using the fact that m^d is σ-finite, and apply Theorem 4.2. Specifically, partition $\mathbb{R}^d$ into countably many compact sets K_i and apply Theorem 4.2 to $f\mathbb{1}_{K_i}$ for all i. ∎

Corollary 4.2. For $f \in L^1_{\mathrm{loc}}$, we have

$$\lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} f(y)\,\mathrm{d}y = f(x)$$

for a.e. x.

Proof. . Use that

$$f(x) = \frac{1}{m(B(x, r))} \int_{B(x,r)} f(x)\,\mathrm{d}y$$

and the triangle inequality. ∎

DIY

Definition 4.3 (Lebesgue point). Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, the point $x \in \mathbb{R}^d$ is called a *Lebesgue point of f* if

$$\lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} |f(y) - f(x)|\,\mathrm{d}y = 0.$$

Remark. Corollary 4.1 tells us that almost all points in $\mathbb{R}^d$ in $\mathbb{R}^d$ are Lebesgue points for f.

Definition 4.4 (Shrink nicely). We say that $\{E_r\}_{r>0}$ *shrinks nicely* to x as $r \to 0$ if $E_r \subset B(x,r)$ and

$$\underset{c>0}{\exists} \; c \cdot m(B(x,r)) \leq m(E_r).$$

Corollary 4.3. Suppose E_r shrink nicely to 0, and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, and x is a Lebesgue point. Then

$$\begin{cases} \displaystyle \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r + x} |f(y) - f(x)| \, \mathrm{d}y = 0; \\ \displaystyle \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r + x} |f(y)| \, \mathrm{d}y = f(x). \end{cases}$$

Corollary 4.4. If $f \in L^1_{\text{loc}}(\mathbb{R})$, then $F(x) = \int_0^x f(y) \, \mathrm{d}y$ is differentiable and $F'(x) = f(x)$ almost everywhere.

Lecture 23: Metric, normed and L^p Spaces

5 Normed Vector Space

5.1 Metric Spaces and Normed Spaces

We have seen the definition of a norm before, now we formally introduce the concept of *metric*.

Definition 5.1 (Metric). Let Y be a set, a function $\rho \colon Y \times Y \to [0, \infty)$ is a *metric* on Y if

- $\rho(x, y) = \rho(y, x)$ for all $x, y \in Y$.
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in Y$.
- $\rho(x, y) = 0$ if and only if $x = y$.

Note. The followings make sense in a metric space.

1. Open/closed balls.

2. Open/closed sets.

3. Convergence sequences ($x_n \to x$ with respect to ρ if and only if $\displaystyle \lim_{n \to \infty} \rho(x_n, x) = 0$).

4. Continuous functions.

Example. We have the following metric spaces.

1. $\mathbb{Q}$ with $\rho(x, y) = |x - y|$.

2. $\mathbb{R}$ with $\rho(x, y) = |x - y|$.

3. $\mathbb{R}_+$ with $\rho(x, y) = |\ln(y/x)|$.

4. $\mathbb{R}^d$ with

$$\rho_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{1/p}$$

and

$$\rho_\infty(x, y) = \max_{1 \le i \le d} |x_i - y_i|.$$

These all give the same open sets, hence they are topologically equivalent.

5. $C([0, 1])$ with

$$\rho_p(f, g) = \left(\int_0^1 |f - g|^p \right)^{1/p}$$

and

$$\rho_\infty(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|.$$

6. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Let Y be the set of measurable functions on X, then

$$\rho(f, g) = \int \min \{|f(x) - g(x)|, 1\} \, \mathrm{d}\mu(x)$$

is a metric and $f_n \to f$ in ρ if and only if $f_n \to f$ in measure.

Let V be a vector space over scalar field $K = \mathbb{R}$ or $K = \mathbb{C}$.

As previously seen (Metric induced by a norm). Recall the definition of seminorm and norm. We see that a norm induces a metric

$$\rho(v, w) := \|v - w\|,$$

and we have

$$v_n \to v \iff \lim_{n \to \infty} \|v_n - v\| = 0.$$

Example. We first see some common examples of normed vector space.

1. $L^1(X, \mathcal{A}, \mu)$ with $\|f\|_1 := \int |f| \, \mathrm{d}\mu$.

2. $C([0, 1])$ with $\|f\|_1 := \int_0^1 |f(x)| \, \mathrm{d}x$, $\|f\|_\infty := \max_{0 \le x \le 1} |f(x)|$.

3. For $\mathbb{R}^d$ and $0 < p < \infty$, we have

$$\|x\|_p := \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p}, \qquad \|x\|_\infty := \max_{1 \le i \le d} |x_i|.$$

5.2 L^p Space

It turns out that we can generalize L^1 into L^p.

Definition 5.2 (L^p space). Given a measure space $(X, \mathcal{A}, \mu)$ and a measurable function f and p such that $0 < p < \infty$, we define a seminorm $\|\cdot\|_p$ such that

$$\|f\|_p := \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{1/p},$$

which induces the so-called L^p *space* $L^p(X, \mathcal{A}, \mu)$, where

$$L^p(X, \mathcal{A}, \mu) := \left\{ f \mid \|f\|_p < \infty \right\}.$$

Remark. Note that $\|\cdot\|_p$ is only a seminorm. But if we identity functions which are equal almost everywhere, then it's indeed a norm.

Example. $(\mathbb{R}, \mathcal{L}, m)$ has $f(x) = x^{-\alpha} \mathbb{1}_{(1,\infty)}(x) \in L^p$ if and only if $\alpha p > 1$. In contrast, $g(X) = x^{-\beta} \mathbb{1}_{(0,1)}(x) \in L^p$ if and only if $\beta p < 1$.

Similar to Definition 5.2, we have the following.

Definition 5.3 (ℓ^p space). If $(X, \mathcal{P}(X), \nu)$ is equipped with the counting measure, then we say it's an ℓ^p *space* such that

$$\ell^p(X) := L^p(X, \mathcal{P}(X), \nu).$$

Remark. We are interested in $\ell^p(\mathbb{N})$ in particular. We have

$$\ell^p := \ell^p(\mathbb{N}) = \left\{ a = (a_1, a_2, \ldots) \mid \|a\|_p = \left(\sum_{i=1}^\infty |a_i|^p \right)^{1/p} < \infty \right\}.$$

Lemma 5.1. $L^p(X, \mathcal{A}, \nu)$ is a vector space for all $p \in (0, \infty)$.

Proof. We verify the following.

- $c \cdot f \in L^p(X, \mathcal{A}, \mu)$ for $c \in \mathbb{R}$. Indeed, since

$$\|cf\|_p = \left(\int |cf|^p \, \mathrm{d}\mu \right)^{1/p} = |c| \, \|f\|_p < \infty \iff \|f\|_p < \infty,$$

  which implies $c \cdot f \in L^p(X, \mathcal{A}, \mu)$.

- $f + g \in L^p(X, \mathcal{A}, \mu)$. Indeed, since for any real numbers α, β, we have

$$(\alpha + \beta)^p \le (2 \cdot \max\{|\alpha|, |\beta|\})^p = 2^p \cdot \max\{|\alpha|^p, |\beta|^p\} \le 2^p \left(|\alpha|^p + |\beta|^p\right),$$

which implies that for $f, g \in L^p(X, \mathcal{A}, \mu)$, we have

$$\|f + g\|_p < \infty \iff \|f + g\|_p^p = \int |f + g|^p \, \mathrm{d}\mu \leq 2^p \int (|f|^p + |g|^p) < \infty.$$

This further implies

$$\|f + g\|_p < \infty \iff \|f\|_p, \|g\|_p < \infty,$$

which is what we want.

∎

We see that in the above derivation, it doesn't give us the triangle inequality, namely

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

hence we need some new results.

Theorem 5.1 (Hölder's inequality). Let $1 < p < \infty$, and let $q := p/(p-1)$ so that $1/p + 1/p = 1$. Then we have

$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. We prove this in steps.

1. Note that
$$t \leq \frac{t^p}{p} + 1 - \frac{1}{p} = \frac{t^p}{p} + \frac{1}{q}$$

   for all $t \geq 0$. Hence, by taking $F(t) := t - t^p/p$ and $t \geq 0$, we see that the maximum of F implies the above inequality.

2. <u>Young's Inequality.</u>[26] We have
$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

   for $\alpha, \beta > 0$. This follows by taking $t := \alpha/\beta^{q-1}$ in the first inequality we obtained.

3. Without loss of generality, we can assume that $0 < \|f\|_p, \|g\|_q < \infty$. Now, consider $F(x) = f(x)/\|f\|_p$, $G(x) = g(x)/\|g\|_q$. We know that $\|F\|_p = 1 = \|G\|_q$. Then by Young's Inequality, we have

$$\int |F(x)G(x)| \, \mathrm{d}\mu \leq \int \frac{|F(x)|^p}{p} + \frac{|G(x)|^q}{q} \implies \frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

   which implies our desired result.

∎

---

[26]https://en.wikipedia.org/wiki/Young's_inequality_for_products

Example. For $p = q = 2$, $X = \{1, \ldots, d\}$ with μ being the counting measure, then for any $x, y \in \mathbb{R}^d$, we have

$$\sum_{i=1}^{d} |x_i y_i| \leq \sqrt{\sum_{i=1}^{d} x_i^2} \sqrt{\sum_{i=1}^{d} y_i^2}$$

We now see how we can obtain the desired triangle inequality.

Theorem 5.2 (Minkowski's Inequality). Let $1 \leq p < \infty$, then for $f, g \in L^p$,
$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. For $p = 1$, it's easy since it's just triangle inequality. Now, we assume that $1 < p < \infty$, and we may assume also that $\|f + g\| \neq 0$ without loss of generality. Then

$$\int |f(x) + g(x)|^p \leq \int |f(x) + g(x)|^{p-1} \left(|f(x)| + |g(x)|\right)$$

$$\leq \left(\int |f + g|^{(p-1)q}\right)^{1/q} \left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right]$$

$$\leq \left(\int |f + g|^p\right)^{1/q} \left(\|f\|_p + \|g\|_p\right).$$

We then see that

$$\underbrace{\left(|f(x) + g(x)|^p\right)^{1 - 1/q}}_{(|f(x)+g(x)|^p)^{1/p}} \leq \|f\|_p + \|g\|_p,$$

which is just $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. ∎

Lecture 24: Embedding L^p Space   11 Mar. 11:00

Definition 5.4 (Essential supremum). For a measurable function f on $(X, \mathcal{A}, \mu)$, we define

$$S := \{\alpha \geq 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\}$$
$$= \{\alpha \geq 0 \mid |f(x)| \leq \alpha \text{ a.e.}\}.$$

Then, we say that the *essential supremum of* f, denoted as $\|f\|_\infty$, is defined as
$$\|f\|_\infty := \begin{cases} \inf S, & \text{if } S \neq \varnothing; \\ \infty, & \text{if } S = \varnothing. \end{cases}$$

Definition 5.5 (L^∞ space). Let $L^\infty(X, \mathcal{A}, \mu)$ be

$$L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\}.$$

Definition 5.6 (ℓ^∞ space). We let ℓ^∞ be defined as

$$\ell^\infty = L^\infty(\mathcal{N}, \mathcal{P}(\mathcal{N}), \nu),$$

where ν is the counting measure.

Example. Consider $(\mathbb{R}, \mathcal{L}, m)$. Then

$$f(x) = \frac{1}{x} \mathbb{1}_{(0,\infty)}(x) \notin L^\infty;$$

$$g(x) = x \mathbb{1}_\mathbb{Q}(x) + \frac{1}{1+x^2} \in L^\infty.$$

If f is continuous on $(\mathbb{R}, \mathcal{L}, m)$, then $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$. For $a \in \ell^\infty$, we have $\|a\|_\infty = \sup_{i \in \mathbb{N}} |a_i|$, and sequences in ℓ^∞ are exactly the bounded sequences.

Lemma 5.2. We have the following.

1. Suppose $f \in L^\infty(X, \mathcal{A}, \mu)$. Then,

$$\begin{cases} \mu(\{x \mid |f(x)| > \alpha\}) = 0, & \text{if } \alpha \geq \|f\|_\infty; \\ \mu(\{x \mid |f(x)| > \alpha\}) > 0, & \text{if } \alpha < \|f\|_\infty. \end{cases}$$

2. $|f(x)| \leq \|f\|_\infty$ almost everywhere.

3. $f \in L^\infty$ if and only if there exists a bounded measurable function g such that $f = g$ almost everywhere.

Proof. . ▮ DIY

Theorem 5.3. We have the following.

1. $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.
2. $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
3. $f_n \to f$ in L^∞ if and only if $f_n \to f$ uniformly almost everywhere.

Remark. The motivation for 1. is that

$$\frac{1}{1} + \frac{1}{\infty} = 1,$$

and we want to have the similar result as in Theorem 5.1.

Proof. . We'll do one implication in 3. Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$. Let $A = \bigcup_n A_n$, we see that $\mu(A) = 0$ as well. DIY

For $x \in A^c$ and for every n, we have

$$|f_n(x) - f(x)| \leq \|f_n - f\|_\infty.$$

Given $\epsilon > 0$, there is an N so that

$$\|f_n - f\| < \epsilon$$

for all $n \geq N$. But then for all $x \in A^c$, $|f_n(x) - f(x)| < \epsilon$ as well. ∎

Proposition 5.1. We have the following.

1. For $1 \leq p < \infty$, the collection of simple functions with finite measure support is dense in $L^p(X, \mathcal{A}, \mu)$.

2. For $1 \leq p < \infty$, the collection of step functions with finite measure support is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$, so is $C_c(\mathbb{R})$.

3. For $p = \infty$, the collection of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.

Remark. Note that $C_c(\mathbb{R})$ is **not** dense in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

Proof. . ∎ DIY

5.3 Embedding Properties of L^p Spaces

Definition 5.7 (Equivalent norm). Two norms $\|\cdot\|, \|\cdot\|'$ on V are *equivalent* if there exists $c_1, c_2 > 0$, such that

$$c_1 \|v\| \leq \|v\|' \leq c_2 \|v\|$$

for all $v \in V$.

Note. We see that

1. These norms gives the same topological properties (open sets, closed sets, convergence, etc.).

2. Definition 5.7 is an equivalence relation on norms.

Example. For $\mathbb{R}^d$ we have the norms $\|\cdot\|_p$ for $1 \leq p \leq \infty$. All of these are equivalent. We see that for $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p} \leq (d \|x\|_\infty^p)^{1/p} = d^{1/p} \|x\|_\infty.$$

Also,

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p} \geq (\|x\|_\infty^p)^{1/p} = \|x\|_\infty.$$

Thus, $\|\cdot\|_p$ is equivalent to $\|\cdot\|_\infty$ for every $1 \leq p < \infty$, and transitivity gives that they are all equivalent.

Another way of thinking of this, by assuming $v \neq 0$, and scaling by some t, we may assume v lies on the unit circle in one of the norms. Then we are squeezing

a unit circle in $\|\cdot\|'$ between two circles of radius c_1, c_2 in $\|\cdot\|$. In picture, we have to show that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent, we have
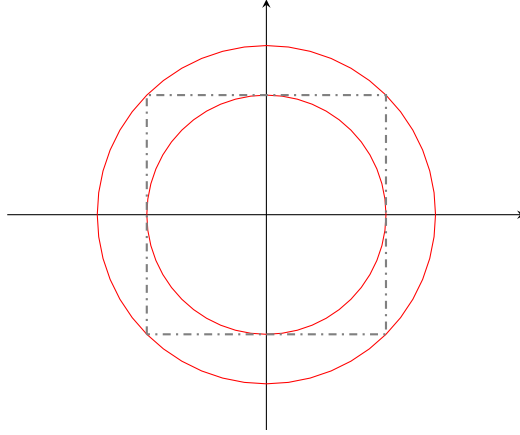
since the circles in $\|\cdot\|_\infty$ are squares.

Example. Fro $1 \leq p, q \leq \infty$, we have $L^p(\mathbb{R}, m)$-norm and $L^q(\mathbb{R}, m)$-norm are not equivalent, even worse, we have that

$$L^p(\mathbb{R}, m) \nsubseteq L^1(\mathbb{R}, m), \quad L^p(\mathbb{R}, m) \nsupseteq L^1(\mathbb{R}, m).$$

Lecture 25: Banach Spaces

14 Mar. 11:00

> **Proposition 5.2.** Suppose $\mu(X) < \infty$, then for every $0 < p < q \leq \infty$, $L^q \subseteq L^p$.

Proof. Suppose $q < \infty$, then

$$\int |f|^p \leq \left(\int \left(|f|^p \right)^{q/p} \right)^{p/q} \left(\int 1^{q/(q-p)} \right)^{1-p/q} = \left(\int |f|^q \right)^{p/q} \mu(x)^{1-p/q} < \infty$$

where we split $\int |f|^p$ into $\int |f|^p \cdot 1$. From Hölder's inequality with $q/p > 1$, we have

$$\|f\|_p \leq \|f\|_q \, \mu(X)^{1/p - 1/q} < \infty.$$

The case that $q = \infty$ is left as an exercise. ∎ DIY

> **Proposition 5.3.** If $0 < p < q \leq \infty$, then $\ell^p \subseteq \ell^q$.

Proof. When $q = \infty$, we have

$$\|a\|_\infty^p = \left(\sup_i |a_i| \right)^p = \sup_i |a_i|^p \leq \sum_{i=1}^\infty |a_i|^p.$$

Thus $\|a\|_\infty \leq \|a\|_p$.

When $q < \infty$, we see that

$$\sum_{i=1}^{\infty} |a_i|^q = \sum_{i=1}^{\infty} |a_i|^p \cdot |a_i|^{q-p} \leq \|a\|_{\infty}^{q-p} \sum_{i=1}^{\infty} |a_i|^p \leq \|a\|_p^{q-p} \cdot \|a\|_p^p = \|a\|_p^q.$$

Therefore,

$$\|a\|_q \leq \|a\|_p.$$

∎

Proposition 5.4. For all $0 < p < q < r \leq \infty$, $L^p \cap L^r \subseteq L^q$.

Proof. ∎ DIY

5.4   Banach Spaces

Let's start with a definition.

Definition 5.8 (Cauchy sequence). Let Y, ρ be a metric space. We call x_n a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\rho(x_n, x_m) < \epsilon$.

Note. Convergent sequence are Cauchy.

Definition 5.9 (Complete). A metric space (Y, ρ) is called *complete* if every Cauchy sequence in Y converges.

Example. We first see some examples.

1. We see that $\mathbb{Q}$ with $\rho(x, y) = |x - y|$ is **not** complete, but $\mathbb{R}$ with the same metric is complete.

2. $C([0,1])$ with $\rho(f, g) = \|f - g\|_{\infty}$ is complete, but with $\rho(f, g) = \int |f - g|$ is not.

Definition 5.10 (Banach space). A *Banach space* is a complete normed vector space.

Remark. Namely, a vector space equipped with a norm whose metric induced by the norm is complete.

Theorem 5.4. Let $(V, \|\cdot\|)$ be a normed space. Then,

V is complete $\iff$ every absolutely convergent series is convergent.

i.e., if $\sum_{i=1}^{\infty} \|v_i\| < \infty$, then $\left\{ \sum_{i=1}^{N} v_i \right\}_{N \in \mathbb{N}}$ converges to some $s \in V$.

Before we prove Theorem 5.4, we first see one of the result based on this theorem.[27]

Theorem 5.5 (Riesz-Fischer theorem). For every $1 \leq p \leq \infty$, we have $L^p(X, \mathcal{A}, \mu)$ is complete, hence a Banach space.

Proof. We prove this in steps.

1. We first prove this for $1 \leq p < \infty$. Suppose $f_n \in L^p$ and $\sum\limits_{n=1}^{\infty} \|f_n\|_p < \infty$.

   We need to show that there is an $F \in L^p$ such that $\left\| \sum\limits_{n=1}^{N} f_n - F \right\|_p \to 0$ as $N \to \infty$. i.e., we need to show

   (a) $\sum\limits_{n=1}^{\infty} f_n(x)$ is convergent a.e. In fact, we can show $\int \sum\limits_{n=1}^{\infty} |f_n(x)| < \infty$.

   Let $G(x) = \sum\limits_{n=1}^{\infty} |f_n(x)| = \sup_N \sum\limits_{n=1}^{N} |f_n(x)|$, $G \colon X \to [0, \infty]$. Also, let $G_N(x) = \sum\limits_{n=1}^{N} |f_n(x)|$. Then, we have

   $$0 \leq G_1 \leq G_2 \leq \ldots \leq G,$$

   and $G_N \to G$. Furthermore,

   $$0 \leq G_1^p \leq G_2^p \leq \ldots \leq G^p,$$

   and $G_N^p \to G^p$. From monotone convergence theorem,

   $$\int G^p = \lim_{N \to \infty} \int G_N^p.$$

   From Minkowski inequality, we further have

   $$\|G_N\|_p \leq \sum_{n=1}^{N} \|f_n\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p := B < \infty.$$

   Thus,

   $$\int G(x)^p = \lim_{N \to \infty} \int G_N^p = \lim_{N \to \infty} \|G_N\|_p^p \leq B^p < \infty.$$

   We see that G is finite a.e. as desired. This implies that $\sum\limits_{n=1}^{\infty} |f_n(x)| < \infty$ a.e., so $\sum\limits_{n=1}^{\infty} f_n(x)$ converges a.e. Now, we simply let

   $$F(x) = \begin{cases} \sum\limits_{n=1}^{\infty} f_n(x), & \text{if it converges;} \\ 0, & \text{otherwise.} \end{cases}$$

---

[27]The proof can be found in here.

(b) $F \in L^p$, where $F(x) := \sum\limits_{n=1}^{\infty} f_n(x)$ a.e. and say is zero elsewhere.

This is clear since

$$|F(x)| \le G(x) \implies \int |F|^p \le \int G^p < \infty,$$

hence $F \in L^p$.

(c) $\left\| \sum\limits_{n=1}^{N} f_n - F \right\|_p \to 0$ as $N \to \infty$.

We now see that

$$\left| \sum_{n=1}^{N} f_n(x) - F(x) \right|^p \le \left(\sum_{n=1}^{\infty} |f_n(x)| + |F(x)| \right)^p \le (2G(x))^p.$$

Since $2G \in L^p$, so $2G^p \in L^1$. Thus, by dominated convergence theorem, we have

$$\lim_{N \to \infty} \int \left| \sum_{n=1}^{N} f_n(x) - F(x) \right|^p \, \mathrm{d}x = 0.$$

This implies

$$\left\| \sum_{n=1}^{N} f_n - F \right\|_p \to 0$$

as $N \to \infty$.

2. The case that $1 \le p \le \infty$ is left as an exercise. DIY

$\blacksquare$

Lecture 26: Bounded Linear Transformations   16 Mar. 11:00

We now prove Theorem 5.4, completing the proof of Theorem 5.5 since the latter relies on this result.

Proof. We prove it by proving two directions.

($\implies$) Suppose V is complete, and fix an absolutely convergent series $\sum_n v_n$. Define $s_N = \sum_{n=1}^{N} v_n$. It suffices to show the partial sums are a Cauchy Sequence.

Fix $\epsilon > 0$, then because $\sum_{n=1}^{\infty} \|v_n\| < \infty$, there is a $K \in \mathbb{N}$ so that

$$\sum_{n=K}^{\infty} \|v_n\| < \epsilon.$$

Now let $M > N > K$, we see that

$$\|s_M - s_N\| = \left\| \sum_{n=N+1}^{M} v_n \right\| \le \sum_{n=N+1}^{M} \|v_n\| \le \sum_{n=N}^{\infty} \|v_n\| < \epsilon,$$

so this is Cauchy.

($\Longleftarrow$) Now suppose $v_n, n \in \mathbb{N}$ is a Cauchy sequence. For all $j \in \mathbb{N}$, there exists an $N_j \in \mathbb{N}$ such that

$$\|v_n - v_m\| < \frac{1}{2^j}$$

for all $n, m \geq N_j$. Without loss of generality, we may assume $N_1 < N_2 < \ldots$.

Let $w_1 = v_{N_1}$, $w_j = v_{N_j} - v_{N_j-1}$ for $j \geq 2$. Therefore,

$$\sum_{j=1}^{\infty} \|w_j\| \leq \|v_{N_1}\| + \sum_{j=2}^{\infty} \frac{1}{2^{j-1}} < \infty.$$

Thus, $\sum_{j=1}^{k} w_j \to s \in V$ as $k \to \infty$. But by telescoping, we have

$$v_{N_k} = \sum_{j=1}^{k} w_j \to s.$$

Now we claim that since v_n is Cauchy, so $v_n \to s$.

Explicitly, take $\epsilon > 0$, and let k be large enough so that $\|v_{N_k} - s\| < \epsilon$ and $1/2^k < \epsilon$. Then if $n > N_k$ then

$$\|v_n - s\| \leq \|v_n - v_{N_k}\| + \|v_{N_k} - s\| < \epsilon + \epsilon = 2\epsilon.$$

Thus, $v_n \to s$. ∎

5.5 Bounded Linear Transformations

Definition 5.11 (Bounded linear transformation). Given two normed vector spaces $(V, \|\cdot\|)$, $(W, \|\cdot\|')$, a linear map $T \colon V \to W$ is called a *bounded map* if there exists $c \geq 0$ such that

$$\|Tv\|' \leq c \|v\|$$

for all $v \in V$.

Proposition 5.5. Suppose $T \colon (V, \|\cdot\|) \to (W, \|\cdot\|')$ is a linear map. Then the followings are equivalent.

1. T is continuous.

2. T is continuous at 0.

3. T is a bounded map.

Proof. 1. $\Longrightarrow$ 2. is clear. For 2. $\Longrightarrow$ 3., take $\epsilon = 1$, then there exists a $\delta > 0$ such that $\|Tu\|' < 1$ if $\|u\| < \delta$.

Now take an arbitrary $\|v\| \in V$, $v \neq 0$. Let $u = \frac{\delta}{2\|v\|}v$. Then $\|u\| < \delta$. Therefore,

$$\|Tu\|' < 1 \implies \frac{\delta}{2\|v\|} \|Tv\|' < 1 \implies \|Tv\|' < \frac{2}{\delta} \|v\|.$$

Then $2/\delta$ is our constant.

For 3. $\implies$ 1., fix $v_0 \in V$. Then for some constant c

$$\|Tv - Tv_0\|' = \|T(v - v_0)\|' \le c \|v - v_0\|.$$

Thus, T is continuous, as when $v \to v_0$ the right-hand side goes to zero, and so $Tv \to Tv_0$. ∎

Example. Let's see some examples.

1. We can look at

$$T \colon \ell^1 \to \ell^1$$
$$(a_1, a_2, \ldots) \mapsto (a_2, a_3, \ldots).$$

   Then clearly $\|Ta\|_1 \le \|a\|_1$, so T is a bounded linear transformation.

2. We can also look at $S \colon (C([-1,1]), \|\cdot\|_1) \to \mathbb{C}$, where $Sf = f(0)$. S is not a bounded linear transformation, because we can make

$$\begin{cases} \|Sf\| & = |f(0)| = n \\ \|f\|_1 & = 1 \end{cases}$$

   for every $n \in \mathbb{N}$ (take f's graph to be a skinny triangle shooting up to n at 0).

3. But $U \colon (C([-1,1]), \|\cdot\|_\infty) \to \mathbb{C}$ defined by $Uf = f(0)$ is a bounded linear transformation, because $|f(0)| \le \|f\|_\infty$.

4. Let A be an $n \times m$ matrix. Then $T \colon \mathbb{R}^m \to \mathbb{R}^n$ defined by $v \mapsto Av$ is a bounded linear transformation.

   Explicitly this is

$$(Tv)_i = (Av)_i = \sum_{j=1}^{m} A_{ij} v_j.$$

5. Let $K(x,y)$ be a continuous function on $[0,1] \times [0,1]$. We'll define

$$T \colon (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$$

   by

$$(Tf)(x) = \int_0^1 K(x,y) f(y) \, \mathrm{d}y.$$

   This is an analogue of matrix multiplication (K is like a continuous matrix). This is a bounded linear transformation.

6. Let us look at $T \colon L^1(\mathbb{R}) \to (C(\mathbb{R}), \|\cdot\|_\infty)$ defined by

$$(Tf)(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, \mathrm{d}x$$

   that is the Fourier transform of f.

7. $T\colon (C^\infty[0,1], \|\cdot\|_\infty) \to (C^\infty[0,1], \|\cdot\|_\infty)$. Define

$$(Tf)(x) = f'(x).$$

This is not a bounded linear transformation. In contrast, S, defined on the same spaces

$$(Sf)(x) = \int_0^x f(t)\,\mathrm{d}t$$

is bounded.

Definition 5.12 (Operator norm). Let $L(V,W)$ be defined as a vector space such that

$$L(V,W) := \{T\colon V \to W \mid T \text{ is a bounded linear transformation}\}.$$

Then for $T \in L(V,W)$, the *operator norm* of T is

$$\|T\| := \inf\{c \geq 0 \mid \|Tv\|'' \leq c\,\|v\|' \text{ for all } v \in V\}$$

$$= \sup\left\{\frac{\|Tv\|''}{\|v\|'} \mid v \neq 0, v \in V\right\}$$

$$= \sup\left\{\|Tv\|'' \mid \|v\|' = 1, v \in V\right\}.$$

Lemma 5.3. We have that

1. The three definitions of $\|T\|$ above are all equal.
2. $(L(V,W), \|\cdot\|)$ is indeed a normed space.

Proof. ■ DIY

Lecture 27: Dual Space

18 Mar. 11:00

As previously seen. From Definition 5.12, we have that

$$\|Tv\|'' \leq \|T\|\,\|v\|'.$$

Remark. Notice that this Definition 5.12 is only for bounded linear transformation.

Theorem 5.6. If W is complete, then $L(V,W)$ is complete.

Proof. Suppose T_n is a Cauchy sequence in $L(V,W)$. Fix $v \in V$ and let $w_n := T_n v \in W$, we then have

$$\|w_n - w_m\| = \|T_n v - T_m v\| = \|(T_n - T_m)v\| \leq \underbrace{\|T_n - T_m\|}_{\to 0}\,\underbrace{\|v\|}_{\substack{\text{fixed}\\\text{value}}}.$$

Thus, w_n is Cauchy, so it converges since W is complete. We call its unique limit Tv. This makes $T\colon V \to W$ into a function. We must show it is a bounded linear transformation and that $\|T_n - T\| \to 0$. ■ DIY

5.6 Dual of L^p Spaces

Example. Let $w \in \mathbb{R}^d$, and denote the inner product between w and $v \in \mathbb{R}^d$ by

$$v \cdot w := \langle v, w \rangle .$$

Then we can consider

$$\max\{v \cdot w \mid \|v\|_2 = 1\} = \|w\|_2 .$$

If $w \in \mathbb{C}^d$, this is similar since

$$\max\{|v \cdot w| \mid \|v\|_2 = 1\} = \|w\|_2 .$$

These maximums are achieved by $v = \frac{\overline{w}}{\|w\|}$ if $w \neq 0$.

Proposition 5.6. Let $1/p + 1/q = 1$ with $1 \leq q < \infty$. For every $g \in L^q$,

$$\|g\|_q = \sup \left\{ \left| \int fg \right| \mid \|f\|_p = 1 \right\} .$$

Suppose μ is σ-finite. Then the result also holds for $q = \infty$, $p = 1$.

As previously seen. For $\alpha \in \mathbb{C}$, $\mathrm{sgn}(\alpha) := e^{i\theta}$ where $\alpha = |\alpha| e^{i\theta}$.

Proof. By Hölder's inequality, we know that

$$\left| \int fg \right| \leq \int |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q = \|g\|_q .$$

Thus, the supremum is less or equal to $\|g\|_q$.

1. Let
$$f(x) = \frac{|g(x)|^{q-1} \cdot \overline{\mathrm{sgn}(g(x))}}{\|g\|_q^{q-1}}$$

   Then $\int |f|^p = 1$, and $\int fg = \|g\|_q$.   [Check]

2. The case that μ is σ-finite and $q = \infty, p = 1$ can be shown.   [DIY]

$\blacksquare$

Remark. One could use the above to prove Minkowski's inequality (as it only uses Hölder's inequality).

Definition 5.13 (Dual space). For a normed space $(V, \|\cdot\|)$, its *dual space* is $V^* = L(V, \mathbb{R})$ or $V^* = L(V, \mathbb{C})$.

Remark. Namely, the dual space of V contains bounded linear transformations with codomain being the scalar field.

Definition 5.14 (Linear functional). Given a normed space $(V, \|\cdot\|)$, $\ell \in V^*$ is called a *linear functional* on V. i.e.,

- $\ell \colon V \to \mathbb{R}$ (or $\mathbb{C}$).
- ℓ is linear.
- There exists a $c \geq 0$ such that $|\ell(v)| = c\,\|v\|$.

Note. V^* is always a Banach space (even if V is not complete).

Corollary 5.1. We have the followings.

1. Let $1/p + 1/q = 1, 1 \leq q < \infty$. For $g \in L^q$ define $\ell_g \in L^p \to \mathbb{C}$ by

$$\ell_g(f) = \int fg.$$

   Then $\ell_g \in (L^p)^*$. Furthermore, $\|\ell_g\| = \|g\|_q$.

2. If μ is σ-finite then this also holds for $q = \infty, p = 1$.

Proof. ℓ_g is clearly linear in f because the integral is linear. Then Proposition 5.6 gives in both 1. and 2. that

$$\|g\|_q = \sup\{|\ell_g(f)| \mid \|g\|_p = 1\} = \|\ell_g\|$$

and so ℓ_g is a bounded linear transformation with the desired properties. ∎

Theorem 5.7. We have the followings.

1. Let $1/p + 1/q = 1$, $1 \leq q < \infty$. The map $T \colon L^q \to (L^p)^*$ given by $Tg = \ell_g$ is an isometric[a] linear isomorphism.

   This means that

   - T is a bounded linear transformation.
   - T is bijective.
   - T is norm-preserving.

2. If μ is σ-finite then this also holds for $q = \infty, p = 1$.

   [a]A map T is called isometric if for a given g, $\|Tg\| = \|g\|$.

Note. Even if μ is σ-finite we might not have $L^1 \cong (L^\infty)^*$.

Also note that $L^2 \cong (L^2)^*$, and for all $1 < p < \infty$ we have $(L^p)^{**} \cong L^p$.

Proof. We have already proved this is isometric in Corollary 5.1, it is clearly linear, and isometry implies injectivity.

We will prove that it is surjective later. ∎ Fix!!!

Lecture 28: Signed Measure

6 Signed and Complex Measures

As previously seen. Suppose $f\colon X \to [0,\infty]$ is a measurable function on $(X, \mathcal{A}, \mu)$.

We can define $\nu(E) = \int_E f \, \mathrm{d}\mu$ for $E \in \mathcal{A}$, and ν is a measure on $(X, \mathcal{A})$.

This gives a map from the set of non-negative measurable functions on X to measures on X. This is injective if we identify functions which are equal almost everywhere. But it is not necessarily surjective. We can then think of measures as a generalization of functions.

For an example, think of a Dirac-Delta measure on $\mathbb{R}$. This is not the Lebesgue integral of any non-negative measurable function.

What if instead we took $f\colon X \to \mathbb{R}, \overline{\mathbb{R}}$ or $\mathbb{C}$. We could take the same construction to get $\nu(E) = \int_E f \, \mathrm{d}\mu$, but this is no longer a measure as it can take $\mathbb{R}, \overline{\mathbb{R}}$ or $\mathbb{C}$ values.

6.1 Signed Measures

> **Definition 6.1.** Let $(X, \mathcal{A})$ be a measurable space. A *signed measure* is $\nu\colon \mathcal{A} \to [-\infty, \infty)$ or $\nu\colon \mathcal{A} \to (-\infty, \infty]$ such that
>
> - $\nu(\varnothing) = 0$.
> - If $A_1, A_2, \ldots \in \mathcal{A}$ are disjoint then
>
> $$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$$
>
> where the series on the right-hand side converges absolutely if
>
> $$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \in (-\infty, \infty).$$
>
> This means the series does not depend on rearrangement.

Example. Consider

1. ν is a positive measure (i.e., measure), then ν is a signed measure.

2. If we have positive measures μ_1, μ_2 such that either $\mu_1(X) < \infty$ or $\mu_2(X) < \infty$, then $\nu = \mu_1 - \mu_2$ is a signed measure.

3. If $f\colon X \to \bar{\mathbb{R}}$ on a measure space $(X, \mathcal{A}, \mu)$ such that $\int_X f^+ \, \mathrm{d}\mu < \infty$ or $\int_X f^- \, \mathrm{d}\mu < \infty$, we can define

$$\nu(E) = \int_E f \, \mathrm{d}\mu$$

and this will be a signed measure.

Note. The following weird things happen with signed measures.

1. $A \subseteq B$ does not imply $\nu(A) \leq \nu(B)$, as $\nu(B) = \nu(A) + \nu(B \setminus A)$, and $\nu(B \setminus A)$ may be negative.

2. If $A \subseteq B$ and $\nu(A) = \infty$, then $\nu(B) = \infty$, because $\nu(B \setminus A) \in (-\infty, \infty]$.

3. Similarly, if $A \subseteq B$ and $\nu(A) = -\infty$ then $\nu(B) = -\infty$.

Lemma 6.1. If ν is a signed measure on $(X, \mathcal{A})$, then we have

1. Continuity from below. If $E_n \in \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \cdots$ then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} \nu(E_N).$$

2. Continuity from above. If $E_n \in \mathcal{A}$, $E_1 \supseteq E_2 \supseteq \cdots$, and $-\infty < \nu(E_1) < \infty$ then

$$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} \nu(E_N).$$

Proof. Read [FF99]. ∎

Definition 6.2 (Positive, negative, null set for a signed measure).
Let ν be a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$, then we say that

1. E is *positive for ν* if for all $F \subseteq E$, $\nu(F) \geq 0$.

2. E is *negative for ν* if for all $F \subseteq E$, $\nu(F) \leq 0$.

3. E is *null for ν* if for all $F \subseteq E$, $\nu(F) = 0$.

Note. We see that

1. If E is a positive set, $F \subseteq E$, then $\nu(F) \leq \nu(E)$.

2. If E is a negative set, $F \subseteq E$, then $\nu(F) \geq \nu(E)$.

Lemma 6.2. Let ν be a signed measure on $(X, \mathcal{A})$, then

1. If E is positive, $G \subseteq E$ is measurable, then G is positive.

2. If E is negative, $G \subseteq E$ is measurable, then G is negative.

3. If E is null, $G \subseteq E$ is measurable, then G is positive.

4. $E_1, E_2, \ldots$ are positive sets, then $\bigcup_{i=1}^{\infty} E_i$ is positive.

Proof. ∎ DIY

Lemma 6.3. Suppose that ν is a signed measure with $\nu \colon \mathcal{A} \to [-\infty, \infty)$. Suppose $E \in \mathcal{A}$ and $0 < \nu(E) < \infty$, then there exists a measurable $A \subseteq E$ such A is a positive set and $\nu(A) > 0$.

Proof. If E is positive, we're done. Otherwise, there exist measurable subsets with negative measure. Let $n_1 \in \mathbb{N}$ be the least such n_1 such that there exists $E_1 \subseteq E$ with $\nu(E_1) < -1/n_1$.

If $E \setminus E_1$ is positive, we're done. Else we can inductively define $n_2, n_3, \ldots$ as well as $E_2, E_3, \ldots$.

Explicitly, if $E \setminus \bigcup_{i=1}^{k-1} E_i$ is not positive, let n_k be the least such that there exists $E_k \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ with $\nu(E_k) < -1/n_k$.

Note then that if $n_k \geq 2$, for all $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$ we have that $\nu(B) \geq -\frac{1}{n_k - 1}$.

Now let $A = E \setminus \bigcup_{i=1}^{\infty} E_i$. Since $E = A \cup (\bigcup_i E_i)$ we have by countable additivity that

$$0 < \nu(E) = \nu(A) + \sum_{k=1}^{\infty} \nu(E_k) < \nu(A).$$

Furthermore, $\nu(E), \nu(A)$ are both in $(0, \infty)$, and we see that

$$0 < \nu(E) \leq \nu(A) - \sum_{k=1}^{\infty} \frac{1}{n_k}.$$

Therefore, the sum on the right-hand side must converge, meaning that $1/n_k \to 0$ as $k \to \infty$. That is $\lim_{k \to \infty} n_k = \infty$.

Now if $B \subseteq A$, then $B \subseteq E \setminus \bigcup_{i=1}^{\infty} E_i$. Therefore, $B \subseteq E \setminus \bigcup_{i=1}^{k-1} E_i$. By the note above, for large enough k such that $n_k \geq 2$ we have
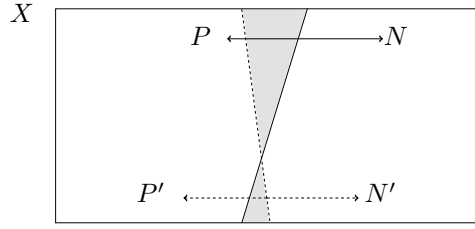
$$\nu(B) \geq \frac{-1}{n_k - 1},$$

then taking $k \to \infty$ we have $\nu(B) \geq 0$, and so A is a positive set as desired. ∎

Theorem 6.1 (Hahn decomposition theorem). If ν is a signed measure on $(X, \mathcal{A})$, then there exist $P, N \in \mathcal{A}$ such that

$$P \cap N = \varnothing, \quad P \cup N = X,$$

where P is positive for ν, N is negative for ν.

Furthermore, if P', N' are another such pair, then $P \triangle P' (= N \triangle N')$ is null for ν.

Lecture 29: Hahn and Jordan Decomposition Theorem

We now prove Theorem 6.1.

Proof. We first show the uniqueness. We see that $P \setminus P' \subseteq P, P \setminus P' \subseteq N'$. Thus, $P \setminus P' \subseteq P \cap N'$ is both positive and negative, hence $P \setminus P'$ is null.

Similarly, for $P' \setminus P$, and then their union $P \triangle P'$ is null as well.

To show the existence, without loss of generality suppose $\nu \colon \mathcal{A} \to [-\infty, \infty)$. If not, consider $-\nu$. Let

$$s := \sup\{\nu(E) \mid E \in \mathcal{A} \text{ is a positive set}\},$$

which is a nonempty supremum because $\varnothing$ is positive. Then there exist $P_1, P_2, \dots$ positive sets such that $\lim_{n \to \infty} \nu(P_n) = s$.

Then we have that $P = \bigcup_n P_n$ is positive by Lemma 6.2. We then have $\nu(P) \leq s$ and $\nu(P) = \nu(P_n) + \nu(P \setminus P_n) \geq \nu(P_n)$. Thus,

$$\nu(P) \geq \lim_{n \to \infty} \nu(P_n) = s.$$

Hence, $\nu(P) = s$ and the supremum is in fact a max. We then know that $s = \nu(P) < \infty$ because ν does not attain the value infinity.

Now let $N = X \setminus P$. We claim that N is negative. If not then there exists a measurable $E \subseteq N$ with $\nu(E) > 0$. By assumption, $\nu(E) < \infty$. Then $0 < \nu(E) < \infty$, so by Lemma 6.3 there exists a measurable $A \subseteq E$ such that A is positive and $\nu(A) > 0$.

But we then know that

$$\nu(P \cup A) = \nu(P) + \nu(A) > \nu(P)$$

which is a contradiction since $P \cup A$ is a positive set, and $\nu(P)$ is maximal. Therefore, N is negative, and the theorem holds.

Finally, if P', N' is another pair of sets as in the statement of the theorem, we have
$$P \setminus P' \subset P, \quad P \setminus P' \subset N',$$
so that $P \setminus P'$ is both positive and negative, hence null; likewise for $P' \setminus P$. ∎

Definition 6.3 (Singular). If μ, ν are signed measures on $(X, \mathcal{A})$, then we say μ and ν are *singular to each other*, denoted as $\mu \perp \nu$, if there exists $E, F \in \mathcal{A}$ such that $E \cap F = \varnothing, E \cup F = X$, F is null for μ, E is null for ν.

Example. Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

1. The Lebesgue measure m.
2. The Cantor measure μ_C defined by the Cantor function.
3. The discrete measure $\mu_D = \delta_1 + 2\delta_{-1}$.

We then see that

1. Take $E = \mathbb{R} \setminus \{-1, 1\}, F = \{1, -1\}$ to see that $m \perp \mu_D$.
2. Take $E = \mathbb{R} \setminus K$ and $F = K$ where K is the Cantor set to see that $m \perp \mu_C$.
3. We can also see that $\mu_C \perp \mu_D$.

Theorem 6.2 (Jordan decomposition theorem). Let ν be a signed measure on $(X, \mathcal{A})$. Then there exists unique positive measures ν^+, ν^- on $(X, \mathcal{A})$ such that for all $E \in \mathcal{A}$ we have
$$\nu(E) = \nu^+(E) - \nu^-(E)$$
and
$$\nu^+ \perp \nu^-.$$

Proof. For existence, we take $\nu^+(E) := \nu(E \cap P), \nu^-(E) := -\nu(E \cap N)$ where P, N is the Hahn decomposition of X.

If there exists μ^+, μ^- such that $\nu = \mu^+ + \mu^-$ and $\mu^+ \perp \mu^-$, let $E, F \in \mathcal{A}$ be such that $E \cap F = \varnothing$, $E \cup F = X$, and $\mu^+(F) = \mu^-(E) = 0$. Then we have that $X = E \cup F$ is another Hahn decomposition for ν, so $P \triangle E$ is ν-null. Therefore, for any $A \in \mathcal{A}$, $\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A)$, hence $\mu^+ = \nu^+$. Likewise, we have $\nu^- = \mu^-$. ∎

Lecture 30: Absolutely Continuous Measures

25 Mar. 11:00

Example. For an example of Theorem 6.2, let $(X, \mathcal{A}, \mu)$ be a measure space, $f \colon X \to \overline{\mathbb{R}}$, and $\nu(E) = \int_E f \, \mathrm{d}\mu$. Then
$$\nu^+(E) = \int_E f^+ \, \mathrm{d}\mu, \quad \nu^-(E) = \int_E f^- \, \mathrm{d}\mu.$$

> **Definition 6.4 (Positive, negative variation).** Given a signed measure ν on $(X, \mathcal{A})$ and its Jordan decomposition $\nu = \nu^+ - \nu^-$, we call ν^+ the *positive variation* of ν, and ν^- the *negative variation* of ν.

> **Definition 6.5 (Total variation).** Let ν be a signed measure on $(X, \mathcal{A})$. The *total variation measure of ν* is $|\nu| := \nu^+ + \nu^-$.

Remark. This is a positive measure on X.

Example. In the above example, $|\nu|(E) = \int_E |f| \, \mathrm{d}\mu$.

> **Lemma 6.4.** We have the following
>
> 1. $|\nu(E)| \leq |\nu|(E)$.
>
> 2. E is ν-null if and only if E is $|\nu|$-null.
>
> 3. If κ is another signed measure, then $\kappa \perp \nu$ if and only if $\kappa \perp |\nu|$ if and only if $\kappa \perp \nu^+$ and $\kappa \perp \nu^-$.

Proof. ∎ DIY

> **Definition 6.6 (Finite signed measure).** A signed measure ν is *finite* if $|\nu|$ is a finite measure, and similarly for σ-finite.

Remark. This holds if and only if ν^+, ν^- are both finite (resp. σ-finite) measures.

6.2 Absolutely Continuous Measures

> **Definition 6.7 (Absolutely continuous).** Let μ be a positive measure, ν be a signed measure, both on $(X, \mathcal{A})$. We say that ν is *absolutely continuous with respect to μ*, denoted as $\nu \ll \mu$, provided that for all $E \in \mathcal{A}$, $\mu(E) = 0$ implies $\nu(E) = 0$.

Remark. This is equivalent to every μ-null set being ν-null.

Example. If $(X, \mathcal{A}, \mu)$, $f \colon X \to \overline{\mathbb{R}}$, $\nu(E) = \int_E f \, \mathrm{d}\mu$, then $\nu \ll \mu$.

Notation. $\mathrm{d}\nu = f \, \mathrm{d}\mu$ means ν is a signed measure defined by

$$\nu(E) = \int_E f \, \mathrm{d}\mu.$$

Lemma 6.5. If μ is a positive measure, ν is a signed measure on $(X, \mathcal{A})$, then

1. $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll< \mu$.

2. $\nu \ll \mu$ and $\nu \perp \mu$ implies $\nu = 0$.

Proof. DIY 1.

For 2., write $X = A \cup B$, $A \cap B = \varnothing$, A μ-null, B ν-null. Then

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap A).$$

Then $E \cap A \subseteq A$, so $\nu(E \cap A) = 0$. By absolute continuity, $\nu(E \cap A) = 0$, thus $\nu(E) = 0$. ∎

Theorem 6.3 (Radon-Nikodym theorem). Suppose μ is a σ-finite positive measure, ν is a σ-finite signed measure, and suppose $\nu \ll \mu$. Then there exists $f \colon X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\nu = f \, \mathrm{d}\mu$, in other words $\nu(E) = \int_E f \, \mathrm{d}\mu$.

If g is another such function with $\mathrm{d}\nu = g \, \mathrm{d}\mu$ then $f = g$ μ-a.e..

Proof. We'll prove a more general form called Lebesgue Radon Nikodym theorem, which is a more general theorem compare to Theorem 6.3. ∎

Definition 6.8 (Randon-Nikodym derivative). Suppose $\nu \ll \mu$. The *Radon-Nikodym derivative of ν with respect to μ* is a function

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \colon X \to \overline{\mathbb{R}}$$

such that

$$\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \, \mathrm{d}\mu$$

for all $E \in \mathcal{A}$.

Remark. i.e. we have $\mathrm{d}\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \, \mathrm{d}\mu$.

Note. By Theorem 6.3, such a function exists and is unique up to equivalence μ-a.e. in the σ-finite case.

Example. Say $F(X) = e^{2x} \colon \mathbb{R} \to \mathbb{R}$. This is continuous and strictly increasing, so we may define a Lebesgue-Stieltjes measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

This is defined to be the unique locally finite measure satisfying $\mu_F([a, b]) = F(b) - F(a) = e^{2b} - e^{2a}$. Then one can check that

$$\mu_F(E) = \int_E 2e^{2x} \, \mathrm{d}x$$

by uniqueness and the classical <u>fundamental theorem of calculus</u>, since the right-hand side is a locally finite Borel measure, and $\kappa([a,b]) = e^{2b} - e^{2a}$, thus $\mu_F = \kappa$.

Therefore, $\mu_F \ll m$ and $\frac{\mathrm{d}\mu_F}{\mathrm{d}m} = 2e^{2x} = \frac{\mathrm{d}F}{\mathrm{d}x}$.

Example. Let $C(X) \colon \mathbb{R} \to \mathbb{R}$ be the Cantor function. Then $C'(x) = 0$ outside the Cantor set. But we don't always have

$$\mu_C(E) \neq \int_E 0 \, \mathrm{d}x.$$

So the candidate derivative is 0, but this fails. In particular,

$$C(b) - C(a) \neq \int_a^b C'(x) \, \mathrm{d}x.$$

In fact, $\mu_C \not\ll m$ because $\mu_C \perp m$ and $\mu_C \neq 0$.

Thus, the existence of a derivative almost everywhere and continuity is not enough to guarantee a version of the <u>fundamental theorem of calculus</u> holds.

Lecture 31: Lebesgue-Radon-Nikodym Theorem

Lemma 6.6. Let μ, ν be finite positive measures on $(X, \mathcal{A})$. Then either

1. $\nu \perp \mu$.

2. There exists an $\epsilon > 0$, an $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for the measure $\nu - \epsilon\mu$, i.e., for all $G \subseteq F$, $\nu(G) \geq \epsilon\mu(G)$.

Proof. Let $\kappa_n = \nu - (1/n)\mu$. By Theorem 6.1 we have $X = P_n \cup N_n$ for P_n positive, N_n negative for κ_n. Also, we let $P = \bigcup_n P_n, N = \bigcap_n N_n = X \setminus P$, then $X = P \cup N$.

We see that for any n we have $\kappa_n(N) \leq 0$ because $N \subseteq N_n$. Thus,

$$0 \leq \nu(N) \leq \frac{1}{n}\mu(N),$$

which implies $\nu(N) = 0$. Because ν is positive for any $N' \subseteq N$ we have $0 \leq \nu(N') \leq \nu(N)$, and thus $\nu(N') = 0$. This shows N is null for ν. Now, we see that

- If $\mu(P) = 0$, then $\nu \perp \mu$.

- If $\mu(P) \neq 0$, then we have $\mu(P) > 0$ hence $\mu(P_n) > 0$ for some n. With $F = P_n$ and $\epsilon = 1/n$, then F is a positive set for $\kappa_n = \nu - (1/n)\mu$ as desired.

$\blacksquare$

Theorem 6.4 (Lebesgue-Radon-Nikodym theorem). Let μ be a σ-finite positive measure, ν a σ-finite signed measure on $(X, \mathcal{A})$. Then there are unique σ-finite signed measures λ, ρ on $(X, \mathcal{A})$ such that

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \nu = \lambda + \rho.$$

Furthermore, there exists a measurable function $f\colon X \to \overline{\mathbb{R}}$ such that $\mathrm{d}\rho = f\,\mathrm{d}\mu$.[a] And if there is another g such that $\mathrm{d}\rho = g\,\mathrm{d}\mu$, then $f = g$ μ-a.e.

---
[a]That is for all $E \in \mathcal{A}$, $\rho(E) = \int_E f\,\mathrm{d}\mu$.

Remark. Notationally, we may write $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}\mu$, where $\mathrm{d}\lambda$ and $\mathrm{d}\mu$ are singular to each other.

Proof. We prove it step by step.

1. Assume μ, ν are finite positive measures. We first prove the existence of λ, f, and $\mathrm{d}\rho = f = \mathrm{d}\mu$.

   Let

   $$\mathscr{F} = \left\{ g\colon X \to [0, \infty] \mid \int_E g\,\mathrm{d}\mu \le \nu(E), \forall E \in \mathcal{A} \right\}$$
   $$= \{ g\colon X \to [0, \infty] \mid \mathrm{d}\nu - g\,\mathrm{d}\mu \text{ is a positive measure} \}.$$

   This set is nonempty since $g = 0 \in \mathscr{F}$. Let $s = \sup\{ \int_X g\,\mathrm{d}\mu \mid g \in \mathscr{F} \}$.

   Claim. There is an $f \in \mathscr{F}$ such that $s = \int_X f\,\mathrm{d}\mu$.

   Proof. If $g, h \in \mathscr{F}$, we can define $u(x) = \max\{g(x), h(x)\}$, then $u \in \mathscr{F}$. This can be seen by letting $A = \{x \mid g(x) \ge h(x)\}$, then

   $$\int_E u\,\mathrm{d}\mu = \int_{E \cap A} g\,\mathrm{d}\mu + \int_{E \cap A^c} h\,\mathrm{d}\mu$$
   $$\le \nu(E \cap A) + \nu(E \cap A^c) = \nu(E).$$

   There exist measurable functions $g_1, g_2, \ldots \in \mathscr{F}$ such that

   $$\lim_{n \to \infty} \int_X g_n\,\mathrm{d}\mu = s.$$

   We can replace g_2 by $\max(g_1, g_2)$, g_3 by $\max(g_1, g_2, g_3)$. Generally,

   $$g_n \leftarrow \max(g_1, g_2, \ldots, g_n),$$

   so that we may assume $0 \le g_1 \le g_2 \le \ldots$.

   Then we still know that $\lim_{n \to \infty} \int_X g_n\,\mathrm{d}\mu = s$, as all the relevant integrals are bounded above by s. Now let $f(x) = \sup_n g_n(x) =$

$\lim_{n \to \infty} g_n(x)$, by Monotone convergence theorem,

$$\int_E f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_E g_n \, \mathrm{d}\mu \le \nu(E).$$

Thus, $f \in \mathscr{F}$, and when $E = X$ we get $\int_X f \, \mathrm{d}\mu = s$ as desired. $\blacksquare$

Let $\rho(E) := \int_E f \, \mathrm{d}\mu$, then we of course have $\rho \ll \mu$, and also, we know

$$0 \le \rho(X) = \int_X f \, \mathrm{d}\mu \le \nu(X) < \infty.$$

Thus, ρ is a finite positive measure, so we can define $\lambda(E) := \nu(E) - \rho(E)$, then

$$\lambda(E) = \nu(E) - \int_E f \, \mathrm{d}\mu \ge 0$$

because $f \in \mathscr{F}$. Thus, λ is also a positive measure, and $\lambda(X) \le \nu(X) < \infty$. It remains to show the following.

Claim. $\lambda \perp \mu$.

Proof. Suppose not, by Lemma 6.6, there exists $\epsilon > 0$, $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for $\lambda - \epsilon\mu$.

Then this says that $\mathrm{d}\lambda - \epsilon\mathbb{1}_F \, \mathrm{d}\mu$ is a positive measure, that is,

$$\mathrm{d}\nu - f \, \mathrm{d}\mu - \epsilon\mathbb{1}_F \, \mathrm{d}\mu$$

is a positive measure. But, this will break maximality of f, specifically, let $g(x) = f(x) + \epsilon\mathbb{1}_F(x)$. Then for all $E \in \mathcal{A}$ we have

$$\begin{aligned}
\int_E g \, \mathrm{d}\mu &= \int_E f \, \mathrm{d}\mu + \epsilon\mu(E \cap F) \\
&= \nu(E) - \lambda(E) + \epsilon\mu(E \cap F) \\
&\le \nu(E) - \lambda(E \cap F) + \epsilon\mu(E \cap F) \le \nu(E)
\end{aligned}$$

since $\lambda(E \cap F) - \epsilon\mu(E \cap F) \ge 0$. Thus, $g \in \mathscr{F}$. We then see that

$$s \ge \int_X g \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu + \int_X \epsilon\mathbb{1}_F \, \mathrm{d}\mu = s + \epsilon\mu(F) > s,$$

which is a contradiction. $\blacksquare$

We see that the existence of λ, f, and $\mathrm{d}\rho = f \, \mathrm{d}\mu$ is proved. As for uniqueness, if there are λ' and f' such that $\mathrm{d}\nu = \mathrm{d}\lambda' + f' \, \mathrm{d}\mu$, we then have

$$\mathrm{d}\lambda - \mathrm{d}\lambda' = (f' - f) \, \mathrm{d}\mu.$$

But we see that $\lambda - \lambda' \perp \mu$ while $(f' - f) \, \mathrm{d}\mu \ll \mathrm{d}\mu$, hence $\boxed{\text{Check!}}$

$$\mathrm{d}\lambda - \mathrm{d}\lambda' = (f' - f) \, \mathrm{d}\mu = 0,$$

so $\lambda = \lambda'$ and $f = f'$ μ-a.e. by Proposition 2.3.

2. Suppose that μ and ν are σ-finite measures. Then X is a countable disjoint union of μ-finite sets and a countable disjoint union of ν-finite sets. By taking intersections of these we obtain a disjoint sequence $\{A_j\} \subset \mathcal{A}$ such that $\mu(A_j)$ and $\nu(A_j)$ are finite for all j and $X = \bigcup_j A_j$. Define $\mu_j(E) = \mu(E \cap A_j)$ and $\nu_j(E) = \nu(E \cap A_j)$, then by the reasoning above, for each j we have

$$\mathrm{d}\nu_j = \mathrm{d}\lambda_j + f_j \, \mathrm{d}\mu_j$$

   where $\lambda_j \perp \mu_j$. Since $\mu_j(A_j^c) = \nu_j(A_j^c) = 0$, we have

$$\lambda_j(A_j^c) = \nu_j(A_j^c) - \int_{A_j^c} f \, \mathrm{d}\mu_j = 0,$$

   and we may assume that $f_j = 0$ on A_j^c. Let $\lambda = \sum_j \lambda_j$ and $f = \sum_j f_j$, we then have

$$\mathrm{d}\nu = \mathrm{d}\lambda + f \, \mathrm{d}\mu, \quad \lambda \perp \mu,$$

   and $\mathrm{d}\lambda$ and $f \, \mathrm{d}\mu$ are σ-finite, as desired. As for uniqueness, it's the same as for the first case.

3. We now consider the general case. If ν is a signed measure, we apply the preceding argument to ν^+ and ν^- and subtract the results.

$\blacksquare$

Lecture 32: Lebesgue Differentiation Theorem for Regular Borel Measures

We now do an example to illustrate Theorem 6.4.

Example. Let $\mu = m$, $\nu = \mu_F$ (Lebesgue-Stieltjes measure for F). We'll define $F(x)$ by

$$F(x) = \begin{cases} e^{3x}, & \text{if } x \leq 0; \\ 1, & \text{if } 0 < x < 1; \\ 5, & \text{if } x \geq 1. \end{cases}$$

Then we will have that

$$\mu_F(E) = \int_{E \cap \mathbb{R}_{<0}} 3e^{3x} \, \mathrm{d}x + 4\delta_1(E).$$

It is enough to check on $(-\infty, x]$ because these are locally finite Borel measures on $\mathbb{R}$.

Then we have $\mu_F = \mathrm{d}\rho + \mathrm{d}\lambda = f \, \mathrm{d}m + \mathrm{d}\lambda$ where $f = \mathbb{1}_{\mathbb{R}_{<0}} 3e^{3x}$ and $\lambda = 4\delta_1$, $\lambda \perp m$.

Specifically, we call such a decomposition *Lebesgue decomposition* of ν with respect to μ. Now, with the condition $\nu \ll \mu$, Theorem 6.4 implies that $\mathrm{d}\nu = f \, \mathrm{d}\mu$ for some f, which is exactly the statement of Theorem 6.3. And, it should be clear now that the definition of Radon Nikodym derivative of ν with respect to μ, denoted as $\mathrm{d}\nu/\mathrm{d}\mu$, is just f in this case.

As previously seen. If $\nu = \nu^+ - \nu^-$, we defined the total variation $|\nu| = \nu^+ + \nu^-$. Then we have $|\nu(E)| \leq |\nu|(E)$.

6.3 Lebesgue Differentiation Theorem for Regular Borel Measures

> **Definition 6.9 (Regular).** A Borel signed measure ν on $\mathbb{R}^d$ is called *regular* if
>
> 1. (compact finite) $|\nu|(K) < \infty$ for all compact K.
>
> 2. (outer regularity) We have outer regularity
>
> $$|\nu|(E) = \inf\{|\nu|(U) \mid \text{ open } U \supseteq E\}$$
>
> for every Borel set E.

Example. We see that

1. Any Lebesgue-Stieltjes measure on $\mathbb{R}$ has this property from Theorem 1.6, so is the difference between two of them (at least if one of them is finite).

2. The Lebesgue measure on $\mathbb{R}^d$ is regular.

Note. From compact finiteness, if ν is regular then it is σ-finite.

> **Lemma 6.7.** $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ if and only if $\mathrm{d}\nu = f \, \mathrm{d}m$ is regular.

Proof. We prove this in two directions.

- Suppose $\mathrm{d}\nu = f \, \mathrm{d}m$ is regular. Then

$$|\nu|(K) = \int_K |f| \, \mathrm{d}m < \infty$$

for all compact K, thus $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

- Suppose $f \in L^1_{\text{loc}(\mathbb{R}^d)}$. This condition is clearly equivalent to compact finiteness. If this holds, then the outer regularity may be verified directly as follows. Suppose that E is a bounded Borel set. Given $\delta > 0$, by Theorem 3.6, there is a bounded open $U \supset E$ such that $m(U) < m(E) + \delta$ and hence $m(U \setminus E) < \delta$. But then, given $\epsilon > 0$, there is[28] an open $U \supset E$ such that

$$\int_{U \setminus E} f \, \mathrm{d}m < \epsilon$$

and hence

$$\int_U f \, \mathrm{d}m < \int_E f \, \mathrm{d}m + \epsilon.$$

---

[28]This follows from [FF99] Corollary 3.6.

The case of unbounded E follows easily by writing

$$E = \bigcup_{j=1}^{\infty} E_j$$

where E_j is bounded and finding an open $U_j \supset E_j$ such that

$$\int_{U_j \setminus E_j} f \, \mathrm{d}m < \epsilon 2^{-j}.$$

∎

As previously seen. Recall the Lebesgue differentiation theorem, here we had that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ implies that for Lebesgue almost every x,

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) \, \mathrm{d}y = f(x)$$

for any $\{E_r\}$ shrinks nicely to x.

Corollary 6.1. Let ρ be a regular signed Borel measure on $\mathbb{R}^d$. Suppose $\rho \ll m$. Then $\mathrm{d}\rho = f \, \mathrm{d}m$ for some $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, So then for Lebesgue almost every x we have

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) \, \mathrm{d}y = f(x).$$

Writing this nicely, using established notation, this is

$$\lim_{r \to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for every $\{E_r\}$ shrinks nicely to x.

Proposition 6.1. Let λ be a regular positive Borel measure on $\mathbb{R}^d$. Suppose $\lambda \perp m$. Then for Lebesgue almost every x, we have

$$\lim_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for every $\{E_r\}$ shrinking to x nicely (equivalently, shrinking to 0 nicely).

Proof. It is enough to consider $E_r = B(x, r)$. We wish to prove that

$$G := \left\{ x \mid \limsup_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} \neq 0 \right\} = \bigcup_{n=1}^{\infty} G_n$$

where

$$G_n := \left\{ x \mid \limsup_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} > \frac{1}{n} \right\}$$

such that $m(G) = 0$. We see that it suffices to show $m(G_n) = 0$ for all n. Since $\lambda \perp m$, so we know there exists A, B such that $\mathbb{R}^d = A \cup B$ disjoint with $\lambda(A) = 0$, $m(B) = 0$. Thus, it suffices to show $m(G_n \cap A) = 0$.[29]

Fix $\epsilon > 0$, since λ is regular, there exists an open set $U \supseteq A$ such that $\lambda(U) \leq \lambda(A) + \epsilon = \epsilon$. We see that for every $x \in G_n \cap A$, there is an $r_x > 0$ such that $\lambda(B(x, r_x))/m(B(x, r_x)) > 1/n$ where $B(x, r_x) \subseteq U$.

Let $K \subseteq G_n \cap A$, compact. Then $K \subseteq \bigcup_{x \in K} B(x, r_x)$. By compactness, we can take a finite sub-cover, and then use Lemma 4.1 to find disjoint $B_1, B_2, \ldots, B_N$ such that each of B_i is in the form of $B(x_i, r_{x_i})$ and $K \subseteq \bigcup_i 3B_i$. Therefore,

$$m(K) \leq 3^d \sum_{i=1}^N m(B_i) \leq 3^d n \sum_{i=1}^N \lambda(B_i) = 3^d n \lambda \left(\bigcup_{i=1}^N B_i \right) \leq 3^d n \lambda(U) = 3^d n \epsilon.$$

By inner regularity, $m(G_n \cap A) \leq 3^d n \epsilon$ for any $\epsilon > 0$. Taking $\epsilon \to 0$ yields $m(G_n \cap A) = 0$, so then $m(G_n) = 0$ as desired. ∎

Lecture 33: Monotone Differentiation Theorem

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As previously seen. We have that if $\rho \ll m$ is regular then

$$\lim_{r \to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$$

for Lebesgue almost every x, where $\{E_r\}$ shrinks nicely to x. Likewise, if $\lambda \perp m$ regular (positive measure) then

$$\lim_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

for Lebesgue almost every x, where $\{E_r\}$ shrinks nicely to x.

From this, we can easily deduce the following important result.

Theorem 6.5 (Lebesgue differentiation theorem for regular measures). Let ν be a regular Borel signed measure on $\mathbb{R}^d$. Then $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}m$, $\lambda \perp m$ by Theorem 6.4. Then for Lebesgue almost every x,

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every $\{E_r\}$ shrinks nicely to x.

Proof. It must be checked that ν regular implies $\lambda, f\,\mathrm{d}m$ are regular. Check!

In particular, since $f \in L^1_{\mathrm{loc}}$, so from Theorem 4.2 and its corollary (Corollary 4.1, Corollary 4.2), we see that it suffices to show that if λ is regular and $\lambda \perp m$, then for Lebesgue a.e. x,

$$\lim_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} \to 0$$

---

[29]Alternatively, we can simply define G_n over A instead of $\mathbb{R}^d$, as in Folland[FF99].

when $\{E_r\}$ shrinks nicely to x. It also suffices to take $E_r = B(r, x)$ and to assume that λ is positive, since for some $\alpha > 0$, we have

$$\left| \frac{\lambda(E_r)}{m(E_r)} \right| \leq \frac{|\lambda|\,(E_r)}{m(E_r)} \leq \frac{|\lambda|\,(B(r,x))}{m(E_r)} \leq \frac{|\lambda|\,(B(r,x))}{\alpha m(B(r,x))}.$$

Therefore, if $|\lambda|\,(E_r)/m(E_r) \to 0$, so does $|\lambda(E_r)/m(E_r)|$, hence $\lambda(E_r)/m(E_r)$. We see that the result then follows directly from Proposition 6.1. ∎

6.4 Monotone Differentiation Theorem

We first formalize one ambiguous notation we used long time ago with discussing distribution function. Namely, $F(x^+), F(x^-)$.

Definition 6.10 $\big(F(x^+), F(x^-)\big)$**.** For $F \colon \mathbb{R} \to \mathbb{R}$ that is monotonically increasing, we denote

$$F(x^+) = \lim_{y \to x^+} F(y), \quad F(x^-) = \lim_{y \to x^-} F(y).$$

Remark. We see that if F is monotonically increasing, then $F(x^+), F(x^-)$ exist and are

$$\inf_{y > x} F(y), \quad \sup_{y < x} F(y)$$

respectively since it's bounded below/above respectively by $F(x)$.

Lemma 6.8. If $F \colon \mathbb{R} \to \mathbb{R}$ is monotonically increasing, then

$$D = \big\{ x \in \mathbb{R} \mid F \text{ is discontinuous at } x \big\}$$

is a countable set.

Proof. $x \in D$ if and only if $F(x^+) > F(x^-)$. For each $x \in D$, let $I_x = (F(x^-), F(x^+))$, not empty. Also, if $x, y \in D$, $x \neq y$, then I_x, I_y are disjoint. Now, for $|x| < N$, I_x lie in the interval $(F(-N), F(N))$. Hence,

$$\sum_{|x| < N} \big[F(x^+) - F(x^-) \big] \leq F(N) - F(-N) < \infty,$$

which implies that

$$D \cap (-N, N) = \big\{ x \in (-N, N) \mid F(x^+) \neq F(x^-) \big\}$$

is countable. Since this is true for all N, the result follows. ∎

> **Theorem 6.6 (Monotone Differentiation Theorem).** Let F be an increasing function, then
>
> - F is differentiable Lebesgue almost everywhere.
>
> - $G(x) := F(x^+)^a$ is differentiable almost everywhere.
>
> - $G' = F'$ almost everywhere
>
> ---
> ^{a}Observe that G is increasing and right-continuous.

Proof. Start with G, which is increasing and right-continuous on $\mathbb{R}$. There is then a Lebesgue-Stieltjes measure μ_G on $\mathbb{R}$, thus it is regular on $\mathbb{R}$. We see

$$\frac{G(x+h) - G(x)}{h} = \begin{cases} \dfrac{\mu_G((x, x+h])}{m((x, x+h])}, & \text{if } h > 0; \\ \dfrac{\mu_G((x+h, x])}{m((x+h, x])}, & \text{if } h < 0. \end{cases}$$

Note that both $\{(x, x+h]\}$ and $\{(x+h, x]\}$ shrink nicely to x as $|h| \to 0$. By Theorem 6.5 (since these shrink nicely), we then know that these both converge for Lebesgue almost every x to some common limit $f(x)$. Hence, G' exists Lebesgue almost everywhere. We now show that by defining $H := G - F$, H' exists and equals zero a.e.

Observe that $H(x) = G(x) - F(x) \geq 0$, and we see that

$$\{x \mid H(x) > 0\} \subseteq \{x \mid F \text{ is discontinuous at } x\}.$$

The latter set is then countable by Lemma 6.8, hence we can write $\{x \mid H(x) > 0\} = \{x_n\}$. Then let

$$\mu := \sum_n H(x_n)\delta_{x_n}.$$

This is a Borel measure, so we check if it is locally finite. Indeed, since

$$\mu((-N, N)) = \sum_{-N < x_n < N} H(x_n) \leq G(N) - F(-N) < \infty,$$

where checking the inequality just consists of seeing that the intervals $(F(x_n), G(x_n))$ are disjoint and is a subset of $(F(-N), G(N))$, so

$$\sum_{-N < x_n < N} H(x_n) = \mu\left(\bigcup_n (F(x_n), G(x_n))\right) \leq \mu((F(-N), G(N))).$$

Thus, μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, so it is regular.

Remark. Special to $\mathbb{R}$, we have

$$\text{that locally finite Borel} \implies \text{Lebesgue-Stieltjes}$$
$$\implies \text{regular}$$
$$\implies \text{outer regularity}.$$

Also, we have $\mu \perp m$ since $m(E) = \mu(E^c) = 0$ where $E = \{x_n\}$. Then we have that

$$\left| \frac{H(x+h) - H(x)}{h} \right| \leq \frac{H(x+h) + H(x)}{|h|} \leq \frac{\mu((x - 2h, x + 2h))}{|h|},$$

which goes to 0 for Lebesgue almost every x by Theorem 6.5 and that $\mu \perp m$.

Thus, H is differentiable almost everywhere and $H' = 0$ almost everywhere, which implies F is differentiable almost everywhere and $F' = G'$ almost everywhere. ∎

Proposition 6.2. Suppose F is an increasing function, then F' exists almost everywhere and is measurable, then

$$\int_a^b F'(x) \, \mathrm{d}x \leq F(b) - F(a).$$

Example. Take $F(x)$ to be 0 on $x \leq 0$, 1 on $x > 0$. Then $F'(x) = 0$ almost everywhere. So

$$\int_{-1}^1 F'(x) \, \mathrm{d}x = 0 < 1 = F(1) - F(-1).$$

Even if F is continuous we might not have equality. Take $F(x)$ to be the Cantor function. Then $F'(x) = 0$ almost everywhere, but

$$\int_0^1 F'(x) \, \mathrm{d}x = 0 < 1 = F(1) - F(0).$$

Lecture 34: Functions of Bounded Variation

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Proof of Proposition 6.2. Let

$$G(x) := \begin{cases} F(a), & \text{if } x < a; \\ F(x), & \text{if } a \leq x \leq b; \\ F(b), & \text{if } x > b. \end{cases}$$

Then G is increasing. We define

$$g_n(x) = \frac{G(x + 1/n) - G(x)}{1/n} \to F'(x)$$

for almost every $x \in [a, b]$. We note that $g_n(x) \geq 0$.

Theorem 2.3 tells us that

$$\int_a^b F'(x) \, \mathrm{d}x = \int_a^b \liminf_{n \to \infty} g_n(x) \, \mathrm{d}x \leq \liminf_{n \to \infty} \int_a^b g_n(x) \, \mathrm{d}x.$$

We then evaluate

$$\int_a^b g_n(x) = n\left(\int_{a+1/n}^{b+1/n} G(x)\,\mathrm{d}x - \int_a^b G(x)\,\mathrm{d}x\right)$$

$$= n\left(\int_b^{b+1/n} G(x)\,\mathrm{d}x - \int_a^{a+1/n} G(x)\,\mathrm{d}x\right)$$

$$\leq n\left(G\left(b+\frac{1}{n}\right)\cdot\frac{1}{n} - G(a)\cdot\frac{1}{n}\right)$$

$$= F(b) - F(a).$$

Therefore,

$$\int_a^b F'(x)\,\mathrm{d}x \leq F(b) - F(a).$$

∎

6.5   Functions of Bounded Variation

Definition 6.11 (Total variation function). For $F\colon \mathbb{R} \to \mathbb{R}$, the *total variation function of F* is $T_F\colon \mathbb{R} \to [0,\infty]$ defined by

$$T_F(x) = \sup\left\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| \mid -\infty < x_0 < x_1 < \ldots < x_n = x\right\}$$

where $n \in \mathbb{N}$.

Lemma 6.9. $T_F(b)$ is equal to

$$T_F(a) + \sup\left\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| \mid a = x_0 < x_1 < \ldots < x_n = b\right\}$$

where $n \in \mathbb{N}$ if $a < b$.

Proof. The idea is that the sums in the Definition 6.5 of T_F are made bigger if the additional subdivision points x_j are added. Hence, if $a < b$, $T_F(b)$ is unaffected if we assume that a is always one of the subdivision points. ∎

Remark. T_F is increasing.

Definition 6.12 (Bounded variation). We say that F is of *bounded variation*, denoted as $F \in BV$, provided that

$$T_F(\infty) = \lim_{x \to \infty} T_F(x) < \infty.$$

Similarly, $F \in BV([a,b])$ means that

$$\sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b \right\} < \infty.$$

Remark. We see the following.

1. If F is of bounded variation, then F is bounded.

2. $F(x) = \sin x$ is not of bounded variation, but it is of bounded variation over any $[a,b]$.

3. For $F(x)$ defined as

$$F(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0 \end{cases}$$

   is not of bounded variation of $[a,b]$ if $a < 0 < b$ because the harmonic series does not converge.

Before we see more properties of bounded variation function, we introduce a useful characterization of a function.

Definition 6.13 (Lipschitz). A function $F \colon [a,b] \to \mathbb{C}$ is called *Lipschitz* provided that there exists an $M \geq 0$ such that $|F(x) - F(y)| \leq M |x - y|$.

Remark. We have the following.

1. If F, G are of bounded variation, $\alpha F + \beta G$ are of bounded variation.

2. If F is increasing and bounded, then F is a function of bounded variation.

3. If F is Lipschitz on $[a,b]$, then $F \in BV([a,b])$.

4. If F is differentiable, and F' is bounded on $[a,b]$, then F is Lipschitz (mean value theorem), so it is in $BV([a,b])$.

5. If $F(x) = \int_{-\infty}^{x} f(t) \, dt$ for $f \in L^1(\mathbb{R})$, then $F \in BV$.

Namely,

$$
\begin{aligned}
\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t)\, \mathrm{d}t \right| \\
&\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\, \mathrm{d}t \\
&= \int_{x_0}^{x_n} |f(t)|\, \mathrm{d}t \\
&\leq \int_{-\infty}^{\infty} |f(t)|\, \mathrm{d}t < \infty.
\end{aligned}
$$

Lemma 6.10. If $F \in BV$, then T_F is bounded, increasing, $T_F(-\infty) = 0$.

Proof. ∎ DIY

Lemma 6.11. $F \in BV$, then $T_F \pm F$ are increasing and bounded and.

Proof. Let $x < y$ and fix $\epsilon > 0$, then there are points $x_0 < x_1 < \cdots < x_n = x$ such that
$$
T_F(x) \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \epsilon.
$$
Furthermore,

$$
T_F(y) \geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)|.
$$

Then, since $\pm(F(y) - F(x)) \leq |F(y) - F(x)|$, we have

$$
T_F(y) \pm (F(y) - F(x)) \geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \geq T_F(x) - \epsilon,
$$

hence
$$
T_F(y) \pm F(y) \geq T_F(x) \pm F(x) - \epsilon.
$$
Taking $\epsilon \to 0$ yields the result. ∎

Remark. Thus, any $F \in BV$ can be written as

$$
F = \frac{T_F + F}{2} - \frac{T_F - F}{2}
$$

which is a difference of increasing and bounded functions.

Theorem 6.7. F is of bounded variation if and only if $F = F_1 - F_2$ for F_1, F_2 increasing and bounded.

Proof. The forward implication is given by the Lemma 6.11. The other direction follows from the examples we gave. ∎ check!

Corollary 6.2 (Bounded Variation Differentiation). $F \in BV$ implies that F is differentiable almost everywhere. Furthermore,

1. $F(x^+), F(x^-)$ exist for all x as do $F(-\infty), F(\infty)$.

2. The set of discontinuities of F is countable.

3. $G(x) = F(x^+)$ is differentiable and $G' = F'$ almost everywhere.

4. $F' \in L^1(\mathbb{R}, m)$ (i.e. $F \in L^1_{\text{loc}}(\mathbb{R})$) for every $a < b$.

Proof. ∎ DIY

Appendix

A Additional Proofs

A.1 Measure

This section gives all additional proofs in Section 1.

> **Theorem A.1 (Theorem 1.2 3.).** Under the setup of Theorem 1.2, $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We see this in two parts.

Claim. If $A \subset X$ satisfies $\mu^*(A) = 0$, then A is Carathéodory measurable with respect to μ^*.

> *Proof.* If $A \subset X$ and $\mu^*(A) = 0$, where μ^* is an outer measure on X, we'll show that A is Carathéodory measurable with respect to μ^*.
>
> Equivalently, we want to show that for any $E \subset X$,
>
> $$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$
>
> Firstly, noting that $(E \cap A) \subset A$, and by monotonicity of μ^*, we see that
>
> $$\mu^*(E \cap A) \leq \mu^*(A) = 0,$$
>
> and since $\mu^* \geq 0$, hence $\mu^*(E \cap A) = 0$. Now, we only need to show that
>
> $$\mu^*(E) = \mu^*(E \setminus A).$$
>
> Since $E \setminus A = E \cap A^c$, and hence we have $E \cap A^c \subset E$, so
>
> $$\mu^*(E) \geq \mu^*(E \setminus A).$$
>
> To show another direction, we note that
>
> $$\mu^*(E) \leq \mu^*(E \cup A) = \mu^*((E \setminus A) \cup A) \leq \mu^*(E \setminus A),$$
>
> hence we conclude that A is Carathéodory measurable with respect to μ^* if $\mu^*(A) = 0$. ∎

Claim. If A is μ-subnull, then $A \in \mathcal{A}$.

> *Proof.* Let $\mathcal{A}$ denotes the Carathéodory σ-algebra, and $\mu := \mu^*|_{\mathcal{A}}$. We want to show if A is μ-subnull, then $A \in \mathcal{A}$.
>
> Firstly, if A is μ-subnull, then there exists a $B \in \mathcal{A}$ such that $A \subset B$ and

$\mu(B) = 0$. But since from the monotonicity of μ^*, we further have

$$0 = \mu(B) = \mu^*(B) \geq \mu^*(A),$$

hence $\mu^*(A) = 0$.

From the first claim, we immediately see that A is Carathéodory measurable with respect to μ^*, which implies A is in Carathéodory σ-algebra, hence $A \in \mathcal{A}$. ∎

We see that the second claim directly proves that $(X, \mathcal{A}, \mu)$ is a complete measure space. ∎

Lemma A.1. The function F defined in this example is a distribution function

---

Proof. We define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n \end{cases}$$

where $\{r_1, r_2, \ldots\} = \mathbb{Q}$, and

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} = \sum_{n; r_n \leq x} \frac{1}{2^n}$$

is both increasing and right-continuous.

- Increasing. Consider $x < y$. We see that

$$F(x) = \sum_{n; r_n \leq x} \frac{1}{2^n} \leq \sum_{n; r_n \leq y} \frac{1}{2^n} = F(y)$$

  clearly.[30]

- Right-continuous. We want to show $F(x^+) = F(x)$. Let $x^+(\epsilon) := x + \epsilon$ with $\epsilon > 0$, we'll show that

$$\lim_{\epsilon \to 0} F(x^+(\epsilon)) = \lim_{\epsilon \to 0} F(x + \epsilon) = F(x).$$

  Firstly, we have

$$F(x^+(\epsilon)) - F(x) = \sum_{n; r_n \leq x + \epsilon} \frac{1}{2^n} - \sum_{n; r_n \leq x} \frac{1}{2^n} = \sum_{n; x < r_n \leq x + \epsilon} \frac{1}{2^n},[31]$$

  and we want to show

$$\lim_{\epsilon \to 0} F(x^+(\epsilon)) - F(x) = \lim_{\epsilon \to 0} \sum_{n; x < r_n \leq x + \epsilon} \frac{1}{2^n} = 0.$$

---

[30]This is trivial since we're always going to sum more strictly positive terms in $F(y)$ than in $F(x)$.

[31]The strict is crucial to show the result, since if $x = r_k$ for some fixed k, then we can't make the summation arbitrarily small.

Before we show how we choose ϵ,[32] we see that

$$\sum_{n=k}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

Now, we observe that

$$\sum_{n;\, x < r_n \leq x+\epsilon} \frac{1}{2^n} \leq \sum_{n=\arg\min_k \; x < r_k \leq x+\epsilon}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

With this observation, it should be fairly easy to see that we can choose ϵ based on how small we want to make 2^{1-k} be,[33] and we indeed see that

$$\lim_{k \to \infty} 2^{1-k} = 0,$$

which implies that F is right-continuous by squeeze theorem.

∎

Lemma A.2. The function F defined in this example satisfies

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$

given in this example.

Proof. We prove them one by one. And notice that F is indeed a distribution function as we proved in Lemma A.1.

1. To show $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$, we first note that

$$\{r\} = \bigcap_{a-1 \leq x < r} (x, r].$$

Then, we see that

$$\mu_F(\{r\}) = \mu_F\left(\bigcap_{a-1 \leq x < a} (x, r] \right),$$

where each $(x, r] \in \mathcal{A}$ and $(x, r] \supset (y, r]$ whenever $r - 1 \leq x \leq y < r$. Notice that we implicitly assign the order of the index by the order of x. Then, we see that $\mu_F(r-1, r] < \infty$.[34] Then, from continuity from above, we see that

$$\mu_F(\{r\}) = \lim_{i \to \infty} \mu_F((x_i, r]),$$

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[32]To be precise, how ϵ depends on r_n.
[33]We're referring to $\epsilon - \delta$ proof approach.
[34]This will be $\mu(A_1)$ in the condition of continuity from above. Furthermore, since $\mathbb{Q}$ is countable, hence $F(x) < \infty$ is promised.

where we again implicitly assign an order to x as the usual order on $\mathbb{R}$ by given index i. It's then clear that as $i \to \infty$, $x_i \to r$. From the definition of F, we see that

$$F((x_i, r]) = F(r) - F(x_i) = \sum_{n; r_n \leq r} \frac{1}{2^n} - \sum_{n; r_n \leq x_i} \frac{1}{2^n} = \sum_{n; x_i < r_n \leq r} \frac{1}{2^n}.$$

It's then clear that since $r \in \mathbb{Q}$, there exists an i' such that $r_{i'} = r$. Then, we immediately see that no matter how close $x_i \to r$, this sum is at least

$$\frac{1}{2^{i'}}$$

for a fixed i'. Hence, we conclude that $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$.

2. Now, we show $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$. Firstly, we claim that

$$\mu_F(\mathbb{Q}) = 1$$

and

$$\mu_F(\mathbb{R}) = 1$$

as well. Since $\mu_F(\mathbb{Q}) = 1$ is clear, we note that the second one essentially follows from the fact that we can write

$$\mathbb{R} = \lim_{N \to \infty} \bigcup_{i=1}^{N} (a - i, a + i]$$

for any $a \in \mathbb{R}$, say 0. From continuity from below, we have

$$\mu_F\left(\bigcup_{i=1}^{\infty} (-i, +i]\right) = \lim_{n \to \infty} \mu_F((-n, n]) = \sum_{n; r_n \in \mathbb{Q}} \frac{1}{2^n} = 1.$$

Given the above, from countable additivity of μ_F, we have

$$\mu_F(\mathbb{R} \setminus \mathbb{Q}) + \underbrace{\mu_F(\mathbb{Q})}_{1} = \underbrace{\mu_F(\mathbb{R})}_{1} \implies \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$$

as we desired.

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Lemma A.3 (Cantor set has measure 0**).** Let C denotes the middle thirds Cantor set, then the Lebesgue measure of C is 0. i.e.,

$$m(C) = 0.$$

Proof. Since we're removing $\frac{1}{3}$ of the whole interval at each n, we see that the measure of those removing parts, denoted by r, is

$$m(r) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 1.$$

Then, by countable additivity of m, we see that

$$m(C) = m([0,1]) - m(r) = 1 - 1 = 0.$$

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A.2 Integration

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