MATH592
Introduction to Algebraic Topology

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Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

Contents

Lecture 5: Operation on Spaces

14 Jan. 10:00

0.1 Operations on CW Complexes

0.1.1 Products

We can consider the product of two CW complexes given by a CW complex structure. Namely, given X and Y two CW complexes, we can take two cells e^n_α from X and e^m_β from Y and form the product space $e^n_\alpha \times e^m_\beta$, which is homeomorphic to an $n+m$-cell. We then take these products as the cells for $X \times Y$.

Specifically, given X, Y are CW complexes, then $X \times Y$ has a cell structure

$$\{e^m_\alpha \times e^n_\alpha \colon e^m_\alpha \text{ is a } m\text{-cell on } X, e^n_\alpha \text{ is a } n\text{-cell on } Y\}.$$

Remark. The product topology may not agree with the weak topology on the $X \times Y$. However, they do agree if X or Y is locally compact <u>or</u> if X and Y both have at most countably many cells.

Note. Notice that if the product is wild enough, then the product topology may not agree with the weak topology.

0.1.2 Wedge Sum

Given X, Y are CW complexes, and $x_0 \in X^0$, $y_0 \in Y^0$ (only points). Then we define

$$X \vee Y = X \coprod Y$$

with quotient topology.

Remark. $X \vee Y$ is a CW complex.

0.1.3 Quotients

Let X be a CW complex, and $A \subseteq X$ subcomplex (closed union of cells), then

$$X \big/ A$$

is a quotient space collapse A to one point and inherits a CW complex structure.

Remark. $X \,/\, A$ is a CW complex.

0-skeleton

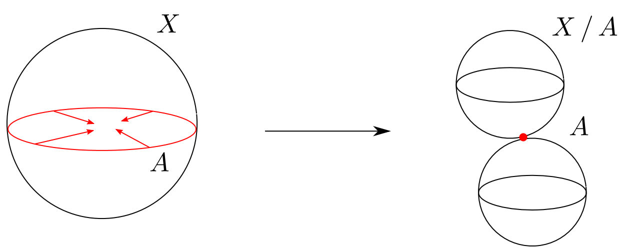
$$(X^0 - A^0) \coprod *$$

where $*$ is a point for A. Each cell of $X - A$ is attached to $(X \,/\, A)^n$ by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n \,/\, A^n$$

Example. Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a wedge of two spheres.



2. We can take the torus and squish down a ring around the hole.

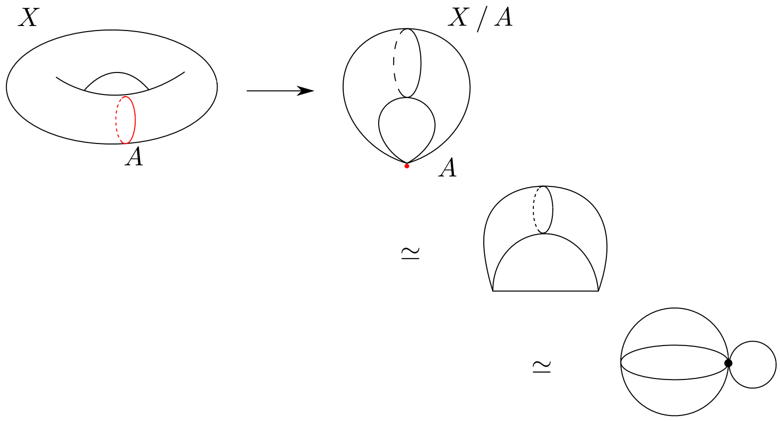


Figure 1: We see that $X \,/\, A$ is homotopy equivalent to a 2-sphere wedged with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other point, forming a circle.

Lecture 6: A Foray into Category Theory

19 Jan. 10:00

1 Category Theory

We start with a definition.

Definition 1.1 (Object, Morphism). A category $\mathscr{C}$ is 3 pieces of data

- A class of objects $\mathrm{Ob}(\mathscr{C})$
- $\forall X, Y \in \mathrm{Ob}(\mathscr{C})$ a class of morphisms or arrows, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- $\forall X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, there exists a composition law

$$\begin{aligned} \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) &\to \mathrm{Hom}(X, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

and 2 axioms

- Associativity. $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms f, g, h where composites are defined.
- Identity. $\forall X \in \mathrm{Ob}(\mathscr{C})\ \exists \mathrm{id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g$$

for all f, g where this makes sense.

Let's see some examples.

Example. We introduce some common category.

$\mathcal{C}$	$\mathrm{Ob}(\mathcal{C})$	$\mathrm{Mor}(\mathcal{C})$
$\underline{\text{set}}$	Sets X	All maps of sets
$\underline{\text{fset}}$	Finite sets	All maps
$\underline{\text{Gp}}$	Groups	Group Homomorphisms
$\underline{\text{Ab}}$	Abelian groups	Group Homomorphisms
$\underline{k\text{−vect}}$	Vector spaces over k	k-linear maps
$\underline{\text{Rng}}$	Rings	Ring Homomorphisms
$\underline{\text{Top}}$	Topological spaces	Continuous maps
$\underline{\text{Haus}}$	Hausdorff Spaces	Continuous maps
$\underline{\text{hTop}}$	Topological spaces	Homotopy classes of continuous maps
$\underline{\text{Top}^*}$	Based topological spaces[1]	Based maps[2]

Remark. Any **diagram** plus composition law.

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[r] & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd}.$$

Definition 1.2 (monic, epic). A morphism $f\colon M \to N$ is *monic* if

$$\forall g_1, g_2 \; f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$\begin{tikzcd} A \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

Dually, f is *epic* if

$$\forall g_1, g_2 \; g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & B \end{tikzcd}$$

Lemma 1.1. In $\underline{\text{set}}$, $\underline{\text{Ab}}$, $\underline{\text{Top}}$, $\underline{\text{Gp}}$, a map is monic if and only if f is injective, and epic if and only if f is surjective.

Proof. In $\underline{\text{set}}$, we prove that f is monic if and only if f is injective. Suppose $f \circ g_1 = f \circ g_2$ and f is injective, then for any a,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

[1]Topological spaces with a distinguished base point $x_0 \in X$
[2]Continuous maps that presence base point $f\colon (x, x_0) \to (y, y_0)$ such that

$$f\colon X \to Y, \quad f(x_0) = y_0$$

is continuous.

hence $g_1 = g_2$.

Now we prove another direction, with contrapositive. Namely, we assume that f is not injective and show that f is not monic. Suppose $f(a) = f(b)$ and $a \neq b$, we want to show such g_i exists. This is easy by considering

$$g_1 : * \mapsto a, \quad g_2 : * \mapsto b.$$

■

1.1 Functor

After introducing category, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

Definition 1.3 (Functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A (covariant) *functor*

$$F : \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F \colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)) \\ [f \colon X \to Y] &\mapsto [F(f) \colon F(X) \to F(Y)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(f) \circ F(g)$

Lecture 7: Functors

21 Jan. 10:00

As previously seen. Assume that we initially have a commutative diagram in $\mathscr{C}$ as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

After applying F, we'll have

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)=F(g)\circ F(f)"'] & F(Y) \arrow[d, "F(g)"] \\ & F(Z) \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

We can also have a so-called contravariant functor.

Definition 1.4 (Contravariant functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A contravariant *functor*

$$F : \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F \colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X)) \\ [f \colon X \to Y] &\mapsto [F(f) \colon F(Y) \to F(X)] \end{aligned}$$

such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
- $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a contravariant functor F, the diagram becomes

$$\begin{tikzcd} F(X) & F(Y) \arrow[l, "F(f)"'] \\ & F(Z) \arrow[ul, "F(g\circ f)=F(f)\circ F(g)"] \arrow[u, "F(g)"'] \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

Example. Let see some examples.

1. Identity functor.

$$I : \mathscr{C} \to \mathscr{C}.$$

2. Forgetful functors.

   -
$$\begin{aligned} F \colon \underline{\mathrm{Gp}} &\to \underline{\mathrm{set}} \\ G &\mapsto G^{3} \\ [f \colon G \to H] &\mapsto [f \colon G \to H] \end{aligned}$$

   -
$$\begin{aligned} F \colon \underline{\mathrm{Top}} &\to \underline{\mathrm{set}} \\ X &\mapsto X^{4} \\ [f \colon X \to Y] &\mapsto [f \colon X \to Y] \end{aligned}$$

[3] G is now just the underlying set of the group G.

3. Free functors.

$$\begin{aligned} \underline{\text{set}} &\to \underline{k\text{−vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

   i.e., vector space with basis s

$$[f\colon A \to B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

4.

$$\begin{aligned} \underline{k\text{−vect}} &\to \underline{k\text{−vect}} \\ V &\mapsto V^* = \mathrm{Hom}_k(V, k) \end{aligned}$$

   If we are working in a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functors.

1.

$$\begin{aligned} \underline{\text{Top}^*} &\to \underline{\text{Gp}} \\ (X, x_0) &\mapsto \Pi_1(X, x_0) \end{aligned}$$

   where Π_1 is so-called *fundamental group.*

2.

$$\begin{aligned} \underline{\text{Top}} &\to \underline{\text{Ab}} \\ X &\mapsto \text{Hp}(X) \end{aligned}$$

   where Hp is so-called p^{th} *homology.*

Let see the formal definition.

2 Free Groups

Definition 2.1 (Free group). Given a set S, the *free group* is a group F_S on S with a map $S \to F_S$ satisfying the universal property.

If G is any group, $f\colon S \to G$ is any map of sets, f extends uniquely to group homomorphism $\overline{f}\colon F_S \to G$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\overline{f}\colon \text{gp hom}"] \\ & G \end{tikzcd}$$

[4] X is now just the underlying set of the topological space X.

Note. This defines a *natural bijection*

$$\mathrm{Hom}_{\underline{\text{set}}}(S, \mathscr{U}(G)) \cong \mathrm{Hom}_{\underline{\text{Gp}}}(F_S, G),$$

where $\mathscr{U}(G)$ is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are adjoint; specifically that the free functor is the left adjoint (appears on the left in the Hom's above).

> **Definition 2.2 (Adjoints functor).** A <u>free</u> and <u>forgetful</u> functors are *adjoints.*

Remark. Whenever we state a universal property for an object (plus a map), an object (plus a map) <u>may or may not</u> exist. If such object exists, then it defines the object **uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the object (plus a map).

> **Lemma 2.1.** Universal property defines F_S (plus a map $S \to F(S)$) uniquely up to unique isomorphism.

Proof. Fix S. Suppose

$$S \to F_S, \quad S \to \widetilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, dashed, "\exists!\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

We'll show φ and ψ are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{tikzcd} & F_S \arrow[dd, "\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[dd, "\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & \widetilde{F}_S \end{tikzcd}$$

Hence, we see that

$$\begin{tikzcd} & F_S \arrow[d, "\psi"] \arrow[dd, bend left, "\varphi\circ\psi=\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, "\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[d, "\varphi"] \arrow[dd, bend left, "\psi\circ\varphi=\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, "\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \mathrm{id}_{F_S}, \qquad \psi \circ \varphi = \mathrm{id}_{\widetilde{F}_S},$$

so φ and ψ are inverses (thus group isomorphism). ■

Lecture 8

24 Jan. 10:00

Example. In category Ab free Abelian group on a set S is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on S** .

2.1 Constructing the free group F_S

Fix a set S, and we define a word as a finite sequence (possibly $\varnothing$) in the formal symbols

$$\left\{s, s^{-1} \mid s \in S\right\}.$$

Then we see that elements in F_S are equivalence classes of words with the equivalence relation being

- delete ss^{-1} or $s^{-1}s$. i.e.,

  $$\begin{aligned} vs^{-1}sw &\sim vw \\ vss^{-1}w &\sim vw \end{aligned}$$

  for every word $v, w, s \in S$,

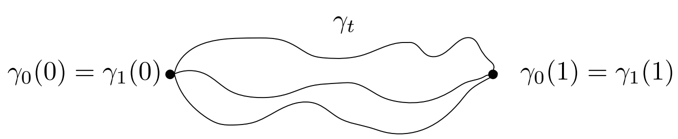
with the group operation being concatenation.

Exercise. Check that F_S satisfies the universal property.

3 Fundamental Group

We start with the definition.

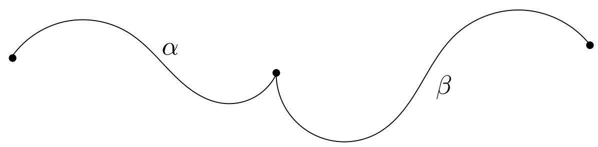
Definition 3.1 (Path). A *path* in a space X is a continuous map $\gamma\colon I \to X$, $I = [0,1]$. A *homotopy of paths* γ_0, γ_1 is a homotopy rel$\{0,1\}$.



Example. Fix $x_1, x_0 \in X$, then $\exists$ homotopy of paths is an equivalence relation on paths from x_0 to x_1 (i.e., γ with $\gamma(0) = x_0, \gamma(1) = x_1$).

Definition 3.2. For paths α, β in X with $\alpha(1) = \beta(0)$, the composition $\alpha \cdot \beta$ is

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$



Definition 3.3 (reparameterization). Let $\gamma\colon I \to X$ be a path, then a *reparameterization* of γ is a path

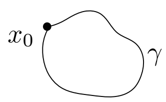
$$\gamma'\colon I \xrightarrow{\phi} I \xrightarrow{\gamma} X$$

where ϕ is continuous and

$$\phi(0) = 0, \quad \phi(1) = 1.$$

Definition 3.4 (Fundamental Group). Let X denotes the space and let $x_0 \in X$ be the base point. The fundamental group $\pi_1(X, x_0)$ of X based at x_0 is a group with

- elements: homotopy classes of paths $[\gamma]$ where γ is a loop with $\gamma(0) = \gamma(1) = x_0$



- operation: composition of paths
- constant loop:
$$\gamma\colon I \to X, \quad t \mapsto x_0$$
- inverses.
$$\overline{\gamma}(t) = \gamma(1-t).$$

Proof. We need to prove that the above define a group. ■ HW.

Theorem 3.1. If X is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Remark. We often write $\pi_1(X)$.

Proof. ■ HW.

Appendix

References

[HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com/books?id=BjKs86kosqgC.