MATH592
Introduction to Algebraic Topology

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Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

Contents

Lecture 7: Functors

21 Jan. 10:00

As previously seen. Assume that we initially have a commutative diagram in $\mathscr{C}$ as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

After applying F, we'll have

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)=F(g)\circ F(f)"'] & F(Y) \arrow[d, "F(g)"] \\ & F(Z) \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

We can also have a so-called contravariant functor.

Definition 0.1 (Contravariant functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A <u>contravariant</u> *functor*

$$F\colon \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F\colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X)) \\ [f\colon X \to Y] &\mapsto [F(f)\colon F(Y) \to F(X)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a contravariant functor F, the diagram becomes

$$\begin{tikzcd} F(X) & F(Y) \arrow[l, "F(f)"'] \\ & F(Z) \arrow[u, "F(g)"'] \arrow[ul, "F(g\circ f)=F(f)\circ F(g)"] \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

Example. Let see some examples.

1. Identity functor.

$$I\colon \mathscr{C} \to \mathscr{C}.$$

2. Forgetful functors.

   -
$$\begin{aligned} F\colon \underline{\mathrm{Gp}} &\to \underline{\mathrm{set}} \\ G &\mapsto G^{1} \\ [f\colon G \to H] &\mapsto [f\colon G \to H] \end{aligned}$$

   -
$$\begin{aligned} F\colon \underline{\mathrm{Top}} &\to \underline{\mathrm{set}} \\ X &\mapsto X^{2} \\ [f\colon X \to Y] &\mapsto [f\colon X \to Y] \end{aligned}$$

[1] G is now just the underlying set of the group G.
[2] X is now just the underlying set of the topological space X.

3. Free functors.

$$\begin{aligned} \underline{\text{set}} &\to \underline{k\text{−vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

   i.e., vector space with basis s

$$[f\colon A \to B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

4.

$$\begin{aligned} \underline{k\text{−vect}} &\to \underline{k\text{−vect}} \\ V &\mapsto V^* = \mathrm{Hom}_k(V, k) \end{aligned}$$

   If we are working on a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functors.

1.

$$\begin{aligned} \underline{\text{Top}^*} &\to \underline{\text{Gp}} \\ (X, x_0) &\mapsto \Pi_1(X, x_0) \end{aligned}$$

   where Π_1 is so-called *fundamental group.*

2.

$$\begin{aligned} \underline{\text{Top}} &\to \underline{\text{Ab}} \\ X &\mapsto \text{Hp}(X) \end{aligned}$$

   where Hp is so-called p^{th} *homology.*

Let see the formal definition.

0.1 Free Groups

Definition 0.2 (Free group). Given a set S, the *free group* is a group F_S on S with a map $S \to F_S$ satisfying the universal property.

If G is any group, $f\colon S \to G$ is any map of sets, f extends uniquely to group homomorphism $\overline{f}\colon F_S \to G$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists! \overline{f}\colon \text{gp hom}"] \\ & G \end{tikzcd}$$

Note. This defines a *natural bijection*

$$\mathrm{Hom}_{\underline{\text{set}}}(S, \mathscr{U}(G)) \cong \mathrm{Hom}_{\underline{\text{Gp}}}(F_S, G),$$

where $\mathscr{U}(G)$ is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are adjoint; specifically that the free functor is the left adjoint (appears on the left in the Hom's above).

Definition 0.3 (Adjoints functor). A <u>free</u> and <u>forgetful</u> functors are *adjoints.*

Remark. Whenever we state a universal property for an object (plus a map), an object (plus a map) <u>may or may not</u> exist. If such object exists, then it defines the object **uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the object (plus a map).

Lemma 0.1. Universal property defines F_S (plus a map $S \to F(S)$) uniquely up to unique isomorphism.

Proof. Fix S. Suppose

$$S \to F_S, \quad S \to \widetilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, dashed, "\exists!\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

We'll show φ and ψ are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{tikzcd} & F_S \arrow[dd, "\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[dd, "\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & \widetilde{F}_S \end{tikzcd}$$

Hence, we see that

$$\begin{tikzcd} & F_S \arrow[d, "\psi"] \arrow[dd, bend left=50, "\varphi\circ\psi=\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, "\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[d, "\varphi"] \arrow[dd, bend left=50, "\psi\circ\varphi=\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, "\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \mathrm{id}_{F_S}, \qquad \psi \circ \varphi = \mathrm{id}_{\widetilde{F}_S},$$

so φ and ψ are inverses (thus group isomorphism). ■

Lecture 8: The Fundamental Group π_1

24 Jan. 10:00

Example. In category Ab free Abelian group on a set S is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on S** .

0.1.1 Constructing the Free Groups F_S

Proposition 0.1. The free group defined by the universal property exists.

Proof. We'll just give a construction below. First, we see the definition.

Definition 0.4. Fix a set S, and we define a word as a finite sequence (possibly $\varnothing$) in the formal symbols

$$\left\{s, s^{-1} \mid s \in S\right\}.$$

Then we see that elements in F_S are equivalence classes of words with the equivalence relation being

- delete ss^{-1} or $s^{-1}s$. i.e.,

  $$vs^{-1}sw \sim vw$$
  $$vss^{-1}w \sim vw$$

  for every word $v, w, s \in S$,

with the group operation being concatenation. ■

Example. Given words ab^{-1}, bba, their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

Exercise. There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that F_S satisfies the universal property with respect to the map

$$S \to F_S, \quad s \mapsto s.$$

1 The Fundamental Group π_1

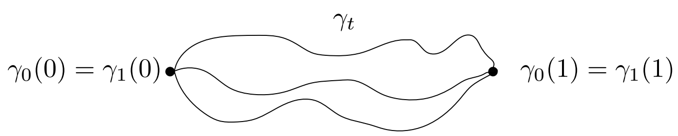
We start with the definition.

Definition 1.1 (Path). A *path* in a space X is a continuous map

$$\gamma \colon I \to X$$

where $I = [0, 1]$.

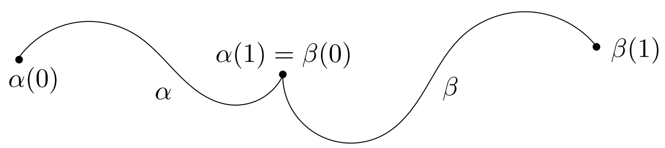
Definition 1.2 (Homotopy path). A *homotopy of paths* γ_0, γ_1 is a homotopy from γ_0 to γ_1 rel$\{0, 1\}$.



Example. Fix $x_1, x_0 \in X$, then $\underline{\exists \text{ homotopy of paths}}$ is an equivalence relation on paths from x_0 to x_1 (i.e., γ with $\gamma(0) = x_0, \gamma(1) = x_1$).

Definition 1.3 (Path composition). For paths α, β in X with $\alpha(1) = \beta(0)$, the *composition*[a] $\alpha \cdot \beta$ is

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



[a] Also named *product*, *concatenation*.

Remark. By the pasting lemma, this is continuous, hence $\alpha \cdot \beta$ is actually a path from $\alpha(0)$ to $\beta(1)$.

Definition 1.4 (Reparameterization). Let $\gamma\colon I \to X$ be a path, then a *reparameterization* of γ is a path

$$\gamma'\colon I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where φ is continuous and

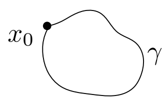
$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

Exercise. A path γ is homotopic rel$\{0,1\}$ to all of its reparameterizations. HW

Exercise. Fix $x_1, x_1 \in X$. Then Homotopy of paths (relative $\{0,1\}$) is an equivalence relation on paths from x_0 to x_1.

Definition 1.5 (Fundamental Group). Let X denotes the space and let $x_0 \in X$ be the base point. The *fundamental group of X based at* x_0, denoted by $\pi_1(X, x_0)$, is a group such that

- Elements: Homotopy classes rel$\{0,1\}$ of paths $[\gamma]$ where γ is a **loop** with $\gamma(0) = \gamma(1) = x_0$[a]

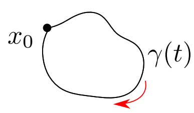


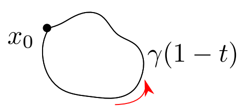
- Operation: Composition of paths.
- Identity: Constant loop γ based at x_0 such that

$$\gamma\colon I \to X, \quad t \mapsto x_0$$

- Inverses: The inverse $[\gamma]^{-1}$ of $[\gamma]$ is represented by the loop $\overline{\gamma}$ such that

$$\overline{\gamma}(t) = \gamma(1-t).$$





[a] We say γ is **based** at x_0.

Proof. We need to prove that the above define a group. ■ HW.

Theorem 1.1. If X is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Remark. We often write $\pi_1(X)$ up to isomorphism.

Proof. ■ HW.

Exercise. Composition of paths is well-defined on homotopy classes rel$\{0,1\}$.

Exercise. If X is a contractible space, then X is path connected and $\pi_1(X)$ is trivial.

Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

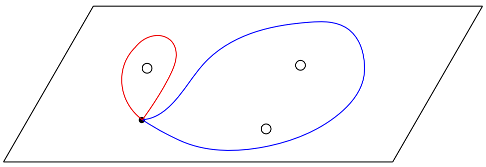


Figure 1: Fundamental Group is basically a *hole detector*!

Theorem 1.2. Given (X, x_0) and (Y, y_9), then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\begin{bmatrix} r\colon I \to X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{bmatrix} \mapsto (r_X, r_Y),$$

where γ is continuous $\iff$ f_X, f_Y are continuous.

Proof. Let $Z \to X \times Y$ with $z \mapsto (f_X(z), f_Y(z))$. Then we have

$$\text{continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply to

- $I \to X \times Y$.
- $I \times I \to X \times Y$.

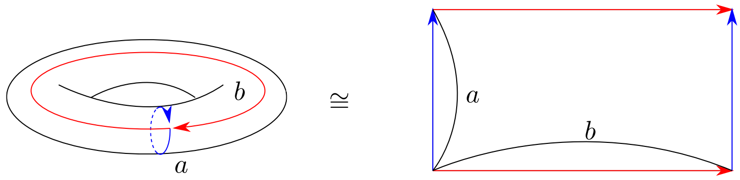
■

Corollary 1.1. Torus $T \cong S^1 \times S^1$. Additionally,

$$\pi_1\left((S^1)^k\right) \cong \mathbb{Z}^k.$$

Proof. Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

Example. We now see some examples.

1. $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2. $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$ since

$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R}.$$

Theorem 1.3. Let π_1 is a functor

$$\begin{aligned}\pi_1 \colon \underline{\mathrm{Top}_*} &\to \underline{\mathrm{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0).\end{aligned}$$

A map $f \colon X \to Y$ taking base point x_0 to y_0 induces a map

$$\begin{aligned}f_* \colon \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma]\end{aligned}$$

i.e.,

$$[f \colon X \to Y] \mapsto [f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))].$$

Notation. We usually write f_* if it's a covariant functor, while writing f^* if it's an contravariant functor.

Proof. We need to check

- well-defined on path homotopy classes.

- f_* is a group homomorphism.

$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \dfrac{1}{2}\right] \\ f(\beta(1-2s)), & \text{if } s \in \left[\dfrac{1}{2}, 1\right] \end{cases}.$$

- $\left(\mathrm{id}_{(X,x_0)}\right)_* = \mathrm{id}_{\pi_1(X,x_0)}$
- $(f_* \circ g_*) = (f \circ g)_*$

$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \implies f_*(\gamma_*(\gamma)).$$

DIY

$\blacksquare$

Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

We first introduce a definition.

Definition 1.6 (Free product with amalgamation). Given some collections of groups $\{G_\alpha\}_\alpha$, the *free product*, denoted by $\underset{\alpha}{*} G_\alpha$ is a group such that

- Elements: Words in $\{g \colon g \in G_\alpha \text{ for any } \alpha\}$ modulo by the equivalence relation generated by

  $$wg_ig_jv \sim w(g_ig_j)v$$

  for every word w, v, $g_i, g_j \in G_\alpha$. Also, for the identity element e_α for $\mathrm{id} = e_\alpha \in G_\alpha$ for any α such that

  $$we_\alpha v \sim wv.$$

- Operation: Concatenation of words.

Furthermore, the *free product with amalgamation* ${}_\alpha *_S G_\alpha$ is defined as

$$\underset{\alpha}{*} G_\alpha$$

modulo the normal subgroup generated by

$$\left\{ i_\alpha(s) i_\beta(s)^{-1} \mid s \in S \right\}$$

satisfies the universal property

$$\begin{tikzcd}
S \arrow[r, "i_\alpha"] \arrow[d, "i_\beta"'] & G_\alpha \arrow[d] \arrow[ddr, bend left] & \\
G_\beta \arrow[r] \arrow[drr, bend right] & G_\alpha *_S G_\beta \arrow[dr, dashed, "\exists!"'] & \\
& & X
\end{tikzcd}$$

Remark. Aside, in <u>Top</u>, the same universal property defines union

$$\begin{tikzcd}
A \cap B \arrow[r, "i_A"] \arrow[d, "i_\beta"'] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \cup B \arrow[dr, dashed, "\exists!"'] & \\
& & X
\end{tikzcd}$$

for A, B are open subsets and the inclusion of intersection.

Theorem 1.4 (Seifert-Van Kampen Theorem). Given (X, x_0) such that $X = \bigcup_\alpha A_\alpha$ with

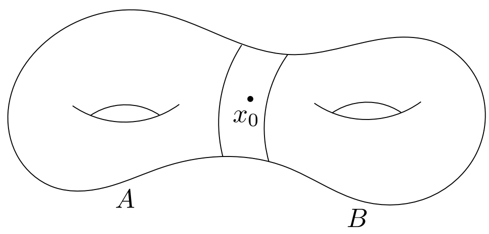
- A_α are open and path-connected $\forall \alpha\ x_0 \in A_\alpha$
- $A_\alpha \cap A_\beta$ is path-connected for all α, β.

Then there exists a surjective group homomorphism

$$\mathop{*}_\alpha\colon \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0).$$

If $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected for every α, β, γ, then

$$\pi_1(X, x_0) \cong \mathop{*}_{\pi_1(A_\alpha \cap A_\beta, x_0)} \pi_1(A_\alpha, x_0).$$



Example.

Appendix

References

[HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com/books?id=BjKs86kosqgC.