MATH597
Analysis II

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\stackrel{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition.

Additionally, we'll use [FF99] as our main text, while using [Tao13] and [Axl19] as supplementary references.

Contents

Lecture 7: Borel Measures

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0.1 Borel Measures on $\mathbb{R}$

We first introduce so-called *distribution function.*

Definition 0.1 (Distribution function). An increasing[a] function

$$F\colon \mathbb{R}\to\mathbb{R}$$

and right-continuous. F is then a *distribution function.*

[a]Here, increasing means $F(x)\leq F(y)$ for $x<y$.

Example. Here are some examples of right-continuous functions.

1. $F(x)=x$.
2. $F(x)=e^x$.

3. Define
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

4. Let $\mathbb{Q} \coloneqq \{r_1, r_2, \ldots\}$. Define
$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$
and
$$F(x) \coloneqq \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$
Then F is a distribution function (hence right-continuous).

Note. If F is increasing, and
$$F(\infty) \coloneqq \lim_{x \nearrow \infty} F(x), \quad F(-\infty) \coloneqq \lim_{x \searrow \infty} F(x)$$
exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.[1]

Definition 0.2 (Locally finite). Let X be a topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Lemma 0.1. Let μ be a locally finite Borel measure on $\mathbb{R}$, then
$$F_\mu(x) = \begin{cases} \mu\left((0, x]\right), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu\left((x, 0]\right), & \text{if } x < 0 \end{cases}$$
is a distribution function.

Proof. To show F_μ is increasing, consider $x < y$ such that
$$F_\mu(x) \leq F_\mu(y)$$
by considering

- $x > 0$: Then $F_\mu(x) = \mu((0, x])$ and
$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$

- $x = 0$: Then $F_\mu(x) = 0$ and
$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$
since $y > 0$.

[1] There are distributions [FF99] Ch9., but these are different from distribution functions.

- $x < 0$: Follows the same argument with $x > 0$.

Now, we need to show F_μ is right-continuous. ■

DIY, use continuity of measure

Definition 0.3 (Half intervals). We call

$$\varnothing, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

half-intervals.

Lemma 0.2. Let $\mathcal{H}$ be the collection of finite disjoint unions of half-intervals. Then, $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. We see that

- $\varnothing \in \mathcal{H}$. Clearly.
- To show $\mathcal{H}$ is closed under complements, we have
  - $\varnothing^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$.
  - $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$.[2]
  - $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$.
  - $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$.
  - $(-\infty, \infty)^c = \varnothing \in \mathcal{H}$.
- $\mathcal{H}$ is closed under finite unions, clearly.

■

[2]Since it's a two disjoint union of half intervals.

Proposition 0.1 (Distribution function defines a pre-measure). Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function. For a half-interval I, define

$$\ell(I) \coloneqq \ell_F(I) = \begin{cases} 0, & \text{if } I = \varnothing \\ F(b) - F(a), & \text{if } I = (a, b] \\ F(\infty) - F(a), & \text{if } I = (a, \infty] \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b] \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 \coloneqq \mu_{0,F}$ as

$$\mu_{0,F}\colon \mathcal{H} \to [0, \infty]$$

by

$$\mu_0(A) = \sum_{k=1}^{N} \ell(I_k) \text{ if } A = \bigcup_{k=1}^{N} I_k,$$

where A is a finite disjoint union of half-intervals $I_1, \ldots, I_N$. Then, μ_0 is a pre-measure on $\mathcal{H}$.

Proof. We see that

1. μ_0 is well-defined.
2. $\mu_0(\varnothing) = 0$.
3. μ_0 is finite additive.
4. μ_0 is countable additive within $\mathcal{H}$.

   Suppose $A \in \mathcal{H}$ where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.[3]

   Focus on the case $I = (a, b]$. Let

   $$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

   which is a disjoint union. Then we only need to check

   $$F(b) - F(a) = \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right).$$

   - Since $(a, b] \supset \bigcup_{n=1}^{N} (a_n, b_n]$ for any fixed $N \in \mathbb{N}$, hence

     $$\underset{N\in\mathbb{N}}{\forall}\; F(b) - F(a) \geq \sum_{n=1}^{N} \left(F(b_n) - F(a_n)\right).$$

[3] why?

By letting $N \to \infty$, we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right).$$

- Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

$$F(a') - F(a) < \epsilon.$$

For each $n \in \mathbb{N}$, $\exists b_n' > b_n$ such that

$$F(b_n') - F(b_n) < \frac{\epsilon}{2^n}.$$

Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b_n'),$$

hence

$$\exists_{N \in \mathbb{N}} [a', b] \subset \bigcup_{n=1}^{N} (a_n, b_n'),^{4}$$

which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^{N} F(b_n') - F(a_n).$$

Finally, we see that

$$\begin{aligned} F(b) - F(a) &\leq F(b) - F(a') + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n') - F(a_n)\right) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\epsilon}{2^n}\right) + \epsilon \\ &= \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right) + 2\epsilon \end{aligned}$$

for any fixed $\epsilon > 0$, hence

$$F(b) - F(a) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

Combine these two inequalities, we have

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

as we desired.

[4]This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.

■

Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8: Lebesgue-Stieltjes Measure on $\mathbb{R}$

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To classify all measures, we now see this last theorem to complete the task.

Theorem 0.1 (Locally finite Borel measures on $\mathbb{R}$). We have

1. $F\colon \mathbb{R} \to \mathbb{R}$ a distribution function, then there exists a **unique** locally finite Borel measure μ_F on $\mathbb{R}$ satisfying
$$\mu_F((a,b]) = F(b) - F(a)$$
for every $a < b$.
2. Suppose $F, G\colon \mathbb{R} \to \mathbb{R}$ are distribution functions. Then,
$$\mu_F = \mu_G$$
on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. ■

HW.

Remark. Theorem 0.1 simply states that given a distribution function, if we restrict our attention on locally finite measures on $\mathbb{R}$ following our usual convention, then it defines the measure on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

0.2 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that

$$F \text{ distribution function} \overset{!}{\implies} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we actually have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 0.4 (Lebesgue-Stieltjes measure). Given a distribution function F, we define

- μ_F on $\mathcal{A}_{\mu_F}$ is called the *Lebesgue-Stieltjes measure* corresponding to F.
- Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{L}, m)$, where $\mathcal{L}$ is called *Lebesgue σ-algebra*, and m is called *Lebesgue measure*.

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).$$[5]

Example. We first see some examples.

1. $\mu_F((a,b]) = F(b) - F(a)$. Then
   - $\mu_F(\{a\}) = F(a) - F(a^-)$
   - $\mu_F([a,b]) = F(b) - F(a^-)$
   - $\mu_F((a,b)) = F(b^-) - F(a)$
2. We define
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$
   Then
   - $\mu_F(\{0\}) = 1$
   - $\mu_F(\mathbb{R}) = 1$
   - $\mu_F(\mathbb{R} \setminus \{0\}) = 0$.

   We call that μ_F is the *Dirac measure* at 0.
3. Denote $\mathbb{Q} = \{r_1, r_2, \ldots\}$, and we define
$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r. \end{cases}$$
   Then HW
   - $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
   - $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$
4. If F is continuous at a, then $\mu_F(\{a\}) = 0$.
5. $F(x) = x$
   - $m((a,b]) = m((a,b)) = m([a,b]) = b - a$.
6. $F(x) = e^x$
   - $\mu_F((a,b]) = \mu_F((a,b)) = e^b - e^a$.

Remark. We see that the first two examples are *discrete measures*.

Example (Middle thirds Cantor set). Let $C := \bigcap_{n=1}^{\infty} K_n$.

[5]Some text will use $x-$ and $x+$ instead of x^- and x^+, respectively.

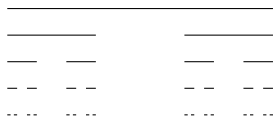

Figure 1: The top line corresponds to K_1, and then K_2, etc.

Since C is uncountable set, hence $m(C) = 0$. And notice that

$$x \in C \iff x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \ a_n \in \{0, 2\}.$$

0.2.1 Cantor Function

Consider F as follows.

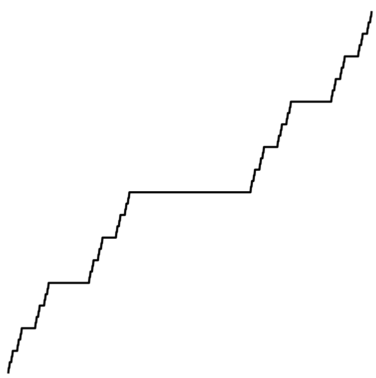

Figure 2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

$$\begin{array}{lcl} \mu_F(\mathbb{R} \setminus C) = 0 & & m(\mathbb{R} \setminus C) = \infty > 0 \\ \mu_F(C) = 1 & \iff & m(C) = 0 \\ \mu_F(\{a\}) = 0 & & m(\{a\}) = 0 \end{array}$$

Remark. μ_F and m are said to be **singular** to each other.

0.3 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 0.3. Let μ be Lebesgue-Stieltjes measure on $\mathbb{R}$. Then we have

$$\begin{aligned}\mu(A) &\overset{!}{=} \inf\left\{\sum_{i=1}^{\infty}\mu((a_i,b_i]) \mid \bigcup_{i=1}^{\infty}(a_i,b_i]\supset A\right\}\\ &= \inf\left\{\sum_{i=1}^{\infty}\mu((a_i,b_i)) \mid \bigcup_{i=1}^{\infty}(a_i,b_i)\supset A\right\}\end{aligned}$$

for every $A\in\mathcal{A}_\mu$

Proof. The second equality follows from the continuity of the measure. ∎

Lecture 9: Properties of Lebesgue-Stieltjes measure

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As previously seen. Let $X\subset[0,\infty]$. Recall that

- $$\alpha=\sup X<\infty \iff \begin{cases}\underset{x\in X}{\forall}\ \alpha\geq x\\ \underset{\epsilon>0}{\forall}\ \underset{x\in X}{\exists}\ \text{such that } x+\epsilon\geq\alpha.\end{cases}$$
- $$\alpha=\sup X=\infty \iff \underset{L>0}{\forall}\ \underset{x\in X}{\exists}\ x\geq L.$$

This should be useful latter on.

Theorem 0.2. Let μ be Lebesgue-Stieltjes measure. Then, for every $a\in\mathcal{A}_\mu$,

1. (outer regularity) $\mu(A)=\inf\{\mu(O)\mid O\supset A, O \text{ is open}\}$
2. (inner regularity) $\mu(A)=\sup\{\mu(K)\mid K\subset A, K \text{ is compact}\}$

Proof. We check them separately.

1. DIY
2. Let $s:=\sup\{\mu(K)\mid K\subset A, K \text{ is compact}\}$, then by monotonicity, we have $\mu(A)\geq s$. To show the other direction,
   - Assume A is a bounded set. Then $\overline{A}\in\mathcal{B}(\mathbb{R})\subset\mathcal{A}_\mu$, $\overline{A}$ is also bounded $\implies \mu(\overline{A})<\infty$. Fix $\epsilon>0$, then by outer regularity, there exists an open $O\supset \overline{A}\setminus A$, and $\mu(O)-\mu(\overline{A}\setminus A)=\mu(O\setminus(\overline{A}\setminus A))\leq\epsilon$. Let $K:=\underbrace{A\setminus O}_{K\subset A}=\underbrace{\overline{A}\setminus O}_{\text{compact}}$, we show that
   $$\mu(K)\geq\mu(A)-\epsilon.$$
   DIY

- Assume A is an unbounded set with $\mu(A) < \infty$. Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \dots$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) < \infty.$$

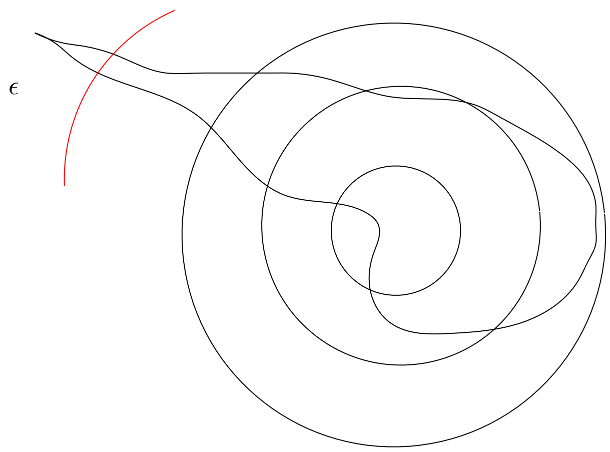

Figure 3

- Assume A is an unbounded set with $\mu(A) = \infty$. We can show that

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) = \infty.$$

Fix $L > 0$, then $\exists N$ such that $\mu(A_N \geq L)$.

■

Definition 0.5. Let X be a topological space. Then

- A G_δ-set is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open.
- F_σ-set is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 0.3. Let μ be a Lebesgue-Stieltjes measure. Then the following are equivalent:

1. $A \in \mathcal{A}_\mu$
2. $A = G \setminus M$, G is a G_δ-set, M is a μ-null.
3. $A = F \setminus N$, F is a F_σ-set, N is a μ-null.

Proof. We see that (2.) $\implies$ (1.) and (3.) $\implies$ (1.) are clear.

- (1.) $\implies$ (3.)
  - Assume $\mu(A) < \infty$. From the inner regularity, we have

$$\forall n \in \mathbb{N}\ \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

  Let $F = \bigcup_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ-null. Check!

  - Assume $\mu(A) = \infty$. Let $A = \bigcup_{k\in\mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

$$\forall k \in \mathbb{Z}\ A_k = F_k \cup N_k,\ \ A = \underbrace{\left(\bigcup_k F_k\right)}_{F_\sigma} \cup \underbrace{\left(\bigcup_k N_k\right)}_{\mu\text{-null}}.$$

- (1.) $\implies$ (2.) We see that

$$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

■

Proposition 0.2. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0\ \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using outer regularity and every open set is $\bigcup_{i=1}^{\infty} I_i$. ■ DIY

We now see some properties of Lebesgue measure.

Theorem 0.4. Let $A \in \mathcal{L} \implies A + s \in \mathcal{L}, rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,

$$m(A+s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. ■ DIY

Example. We now see some examples.

1. Let $\{r_i\}_{i=1}^{\infty}$ which is dense in $\mathbb{R}$. Let $\epsilon > 0$, and
$$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i} \right).$$
We see that O is open and dense in $\mathbb{R}$. But we see
$$m(O) = \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$
Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$
2. There exists uncountable set A with $m(A) = 0$.
3. There exists A with $m(A) > 0$ but A contains no non-empty open intervals.
4. There exists $A \notin \mathcal{L}$. e.g. Vitali set.[6]
5. There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

Lecture 10: Integration

26 Jan. 11:00

1 Integration

1.1 Measurable Function

[Measurable function] We start with a definition.

Definition 1.1. Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$ are measurable spaces. Then we say $f \colon X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable if
$$\forall B \in \mathcal{B} \; f^{-1}(B) \in \mathcal{A}.$$

Lemma 1.1. Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then,
$$f \colon X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall E \in \mathcal{E} \; f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part ($\implies$) is clear. On the other direction, we consider the following.

Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $E \subset \mathcal{D}$ by assumption
- $\mathcal{D}$ is a σ-algebra

check!

$\implies \langle \mathcal{E} \rangle \subset \mathcal{D}$. ■

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$

[6] https://en.wikipedia.org/wiki/Vitali_set

- $f^{-1}(\bigcup\limits_{\alpha} E_\alpha) = \bigcup\limits_{\alpha} f^{-1}(E_\alpha)$

Definition 1.2. Let $(X, \mathcal{A})$ be a measurable space. Then,

$$\left.\begin{array}{l} f\colon X \to \mathbb{R} \\ f\colon X \to \overline{\mathbb{R}} \\ f\colon X \to \mathbb{C} \end{array}\right\} \text{ is } \mathcal{A}\text{-measurable if } \begin{cases} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f\colon X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{cases}$$

Notation. Notice that $\overline{\mathbb{R}}$ is equal to $[-\infty, \infty]$.

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \implies$ every function is $\mathcal{A}$-measurable.
- $\mathcal{A} = \{\varnothing, X\} \implies$ only $\mathcal{A}$-measurable functions are constant functions.

Lemma 1.2. Given $f\colon X \to \mathbb{R}$, the following are equivalent.

1. f is $\mathcal{A}$-measurable
2. $\forall a \in \mathbb{R},\ f^{-1}((a, \infty)) \in \mathcal{A}$
3. $\forall a \in \mathbb{R},\ f^{-1}([a, \infty)) \in \mathcal{A}$
4. $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a)) \in \mathcal{A}$
5. $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from the lemma we just saw. ■

Property. Given $f, g\colon X \to \mathbb{R}$ and is $\mathcal{A}$-measurable, then

1. $\phi\colon \mathbb{R} \to \mathbb{R}$, $\mathcal{A}$-measurable (i.e. Borel measurable), then
$$\phi \circ f\colon X \to \mathbb{R}$$
is $\mathcal{A}$-measurable.
2. $-f,\ 3f,\ f^2,\ |f|$ are all $\mathcal{A}$-measurable, and $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) \neq 0, \forall x \in X$.
3. $f + g$ is $\mathcal{A}$-measurable. We see this from
$$(f+g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty)) \right).$$
4. $f \cdot g$ is $\mathcal{A}$-measurable. We see this from
$$f(x)g(x) = \frac{1}{2}\left((f(x) + g(x))^2 - f(x)^2 - g(x)^2 \right).$$

5. We see that

$$(f \vee g)(x) \coloneqq \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) \coloneqq \min\{f(x), g(x)\}$$

   are $\mathcal{A}$-measurable.

6. Let $f_n \colon X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable. Then

$$\sup_{n\in\mathbb{N}} f_n,\ \inf_{n\in\mathbb{N}} f_n,\ \limsup_{n\to\infty} f_n,\ \liminf_{n\to\infty} f_n$$

   are $\mathcal{A}$-measurable.

   Proof. Consider $\sup\limits_{n\in\mathbb{N}} f_n =: g$, then

$$g^{-1}((a,\infty]) = \bigcup_{n\in\mathbb{N}} f_n^{-1}((a,\infty])$$

   for $\sup\limits_n f_n(x) = g(x) > a$. A similar argument can prove the case of $\inf\limits_{n\in\mathbb{N}} f_n$. check

   And notice that $\limsup\limits_{n\to\infty} f_n = \inf\limits_{k\in\mathbb{N}} \sup\limits_{n\geq k} f_n$, then the similar argument also proves this case. ■

7. If $\lim\limits_{n\to\infty} f_n(x)$ converges for every $x \in X$, then f is $\mathcal{A}$-measurable.

   Example. If $f \colon \mathbb{R} \to \mathbb{R}$ is continuous

   $\implies$ f is Borel measurable.

   $\implies$ f is Lebesgue measurable.

   (Considering $f^{-1}((a,\infty))$.)

Definition 1.3. For $f \colon X \to \overline{\mathbb{R}}$, let $f^+ \coloneqq f \vee 0$ and $f^- \coloneqq (-f) \vee 0$.[a]

[a] i.e.,

$$f^+(x) = \max\{f(x), 0\}, \quad f^-(x) = \min\{-f(x), 0\}$$

Remark. If $\operatorname{supp} f^+ \cap \operatorname{supp} f^- = \varnothing$ and $f(x) = f^+(x) - f^-(x)$, then

$$f \text{ is } \mathcal{A}\text{-measurable} \iff f^+, f^- \text{ are measurable.}$$

Notation. $\operatorname{supp} f$ means the support of f, which is the set of domain which makes f being non-zero.

Definition 1.4 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator)* function of E is

$$\mathcal{X}_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is $\mathcal{A}$-measurable $\iff E \in \mathcal{A}$.

Definition 1.5 (Simple function). Let $(X, \mathcal{A})$ be a measurable space. Then a *simple function* $\phi\colon X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only <u>finitely many values</u>.

Remark. We see that if

$$\phi(X) = \{c_1, \ldots, c_N\},$$

and

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \implies \phi = \sum_{i=1}^{N} \underbrace{c_i}_{\neq \pm\infty} \mathbb{1}_{\underbrace{E_i}_{\in \mathcal{A}}}.$$

Appendix

References

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