MATH597
Analysis II

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\overset{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition.

Additionally, we'll use [FF99] as our main text, while using [Tao13] and [Axl19] as supplementary references.

Contents

Lecture 5: Hahn-Kolmogorov Theorem

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Firstly, we see a stronger version of **??** we have seen before.

Lemma 0.1. Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots$ are <u>disjoint</u> C-measurable sets. Then,

$$\forall E \subset X,\ \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \sum_{i=1}^{\infty} \mu^*\left(E \cap B_i\right).$$

Proof.

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_i) \geq \mu^*\left(E \cap \bigcup_{n=1}^{\infty} B_n\right) \geq \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} B_n\right)\right) \overset{!}{=} \sum_{n=1}^{N} \mu^*\left(E \cap B_n\right).$$

Now, we just take $N \to \infty$ (or note that $N \in \mathbb{N}$ is arbitrary, we then get the result according to Squeeze Theorem[1]). ■

[1] https://en.wikipedia.org/wiki/Squeeze_theorem

Let's continue the proof of **??**.

2. Since from **??**, we need to show
   - $\mu(\varnothing) = 0$. This means that we need to show $\mu^*|_{\mathcal{A}}(\varnothing) = 0$. Since $\varnothing \in \mathcal{A}$ and μ^* is an outer measure, hence from the property of outer measure, it clearly holds.
   - Countable additivity of μ^* on $\mathcal{A}$ follows from the Lemma 0.1 with $E = X$
3. Hw.

■

0.1 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets $\mathcal{E}$ and any ρ such that it assigns measure on $\mathcal{E}$, then induces an outer measure by **??**, finally complete the outer measure by **??**.

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{??} (\mathcal{P}(X), \mu^*) \xrightarrow{??} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to σ-algebra we are working on till now.

Definition 0.1 (Algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on X* if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an algebra and a σ-algebra is whether they closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an outer measure.

Definition 0.2 (Pre-measure). Let $\mathcal{A}_0$ be an algebra on X. We say

$$\mu_0 \colon \mathcal{A}_0 \to [0, \infty]$$

is a *pre-measure* if

1. $\mu_0(\varnothing) = 0$
2. (finite additivity) $\mu_0\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu_0(A_i)$ if $A_1, \dots, A_n \in \mathcal{A}_0$ are disjoint.
3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, disjoint, then

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 0.2. (1) + (3) $\implies$ (2) in Definition 0.2.

Proof. It's easy to see that since μ_0 is monotone. ■

Theorem 0.1 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in **??**. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^*, then $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$. i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts.

- We first show $\mathcal{A} \supset \mathcal{A}_0$. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is C-measurable, i.e.,

$$\forall E \subset X \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

  We first fix an $E \subset X$. From countable subadditivity of μ^*, we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

  Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

  Fix $\epsilon > 0$. By the **??** of μ^*, $\exists B_1, B_2, \dots \in \mathcal{A}_0$, $\bigcup_{n=1}^{\infty} B_n \supset E$ such that

$$\mu^*(E) + \epsilon \overset{!}{\geq} \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left(\mu_0(\underbrace{B_n \cap A}_{\in \mathcal{A}_0}) + \mu_0(\underbrace{B_n \cap A^c}_{\in \mathcal{A}_0}) \right)$$

by the finite additivity of μ_0. Note that

$$\begin{cases} \bigcup_{n=1}^{\infty}(B_n \cap A) \supset E \cap A \\ \bigcup_{n=1}^{\infty}(B_n \cap A^c) \subset E \cap A^c \end{cases} \implies \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

since

$$\mu^*(E \cap A) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A)\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A)$$

and

$$\mu^*(E \cap A^c) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A^c)\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A^c).$$

We then see that for any $\epsilon > 0$, the inequality

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

holds, hence so does

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

which implies $\mathcal{A} \supset \mathcal{A}_0$.

The proof will be continued...

Lecture 6: Hahn-Kolmogorov Theorem and Extension.

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Let's continue the proof of Theorem 0.1.

- Let $A \in \mathcal{A}_0$, we want to show that

$$\mu(A) = \mu_0(A).$$

  - Firstly, let

$$B_i = \begin{cases} A, & \text{if } i = 1 \\ \varnothing, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,$$

    hence $\bigcup_{i=1}^{\infty} B_i = A$, then we see that

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu_0(B_i) = \mu_0(A)$$

    from the definition of μ^* and countable additivity within the algebra of μ_0.

- Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup\limits_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A \cap B_1 \in \mathcal{A}_0$, $C_i = A \cap B_i \setminus \left(\bigcup\limits_{j=1}^{i-1} B_j\right) \in \mathcal{A}_0$ for $i \geq 2$ since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

are disjoint countable unions, by countable additivity within the algebra, we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i) \implies \mu_0(A) \leq \mu^*(A)$$

by taking the infimum from the definition of μ^*.

Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every $A \in \mathcal{A}_0$, which implies

$$\mu(A) = \mu_0(A)$$

for every $A \in \mathcal{A}_0$ from **??**, where we extend μ^* to μ respect to $\mathcal{A}_0$.

■

Definition 0.3 (HK extension). $(\mathcal{A}, \mu)$ obtained from Theorem 0.1 is the *Hahn-Kolmogorov extensions* of $(\mathcal{A}_0, \mu_0)$.

We can actually show the uniqueness of HK extension.

Theorem 0.2 (Uniqueness of HK extension). Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$. Let $(\mathcal{A}, \mu)$ be the HK extension of $(\mathcal{A}_0, \mu_0)$. Let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$. Then if μ_0 is σ-finite, $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Note. Notice that $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$ since they both extend $\mathcal{A}_0$.

Proof. Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing the arbitrary cover of A and using the definition of μ^*.

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) < \infty$, and fix $\epsilon > 0$. Then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup_{i=1}^{\infty} B_i \supset A$ such that

$$\mu(A) + \epsilon = \mu^*(A) + \epsilon \overset{!}{\geq} \sum_{i=1}^{\infty} \mu_0(B_i) \overset{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

This implies that

$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$

where the first equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

$$\mu(B) = \lim_{N\to\infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) =^{2} \lim_{N\to\infty} \mu'\left(\bigcup_{i=1}^{N} B_i\right) = \mu'(B),$$

hence,

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B\setminus A) \leq {}^{3}\mu'(A) + \mu(B\setminus A) \leq \mu'(A) + \epsilon$$

for arbitrary ϵ, so we conclude $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is σ-finite, so we know $X = \bigcup_{n=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ such that

$$\mu_0(X_n) < \infty.$$

Replacing X_n by $X_1 \cup \ldots \cup X_n \in \mathcal{A}_0$, we may assume that

$$X_1 \subset X_2 \subset \ldots.$$

Then,

$$\forall_{n\in\mathbb{N}}\ \mu(A \cap X_n) < \infty \overset{!}{\Longrightarrow} \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

From the continuity of measure, we then have

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) \leq \lim_{n\to\infty} \mu'(A \cap X_n) = \mu'(A).$$

■

Corollary 0.1. Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then

$$\exists! \text{ measure } \mu \text{ on } \langle \mathcal{A}_0 \rangle \text{ that extends } \mathcal{A}_0.$$

Furthermore,

- The completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.
- $$\mu(A) = \inf\left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \forall_{i\in\mathbb{N}} \bigcup_{i=1}^{\infty} B_i \supset A \right\}$$

  for all $A \in \langle \bar{\mathcal{A}}_0 \rangle$.

[2] $\mu = \mu'$ on $\mathcal{A}_0$.
[3] From the first part.

Lecture 7: Borel Measures

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0.2 Borel Measures on $\mathbb{R}$

We first introduce so-called *distribution function*.

Definition 0.4 (Distribution function). An increasing[a] function

$$F\colon \mathbb{R} \to \mathbb{R}$$

and right-continuous. F is then a *distribution function*.

[a]Here, increasing means $F(x) \leq F(y)$ for $x < y$.

Example. Here are some examples of right-continuous functions.

1. $F(x) = x$.
2. $F(x) = e^x$.
3. Define
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$
4. Let $\mathbb{Q} \coloneqq \{r_1, r_2, \ldots\}$. Define
$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$
and
$$F(x) \coloneqq \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$
Then F is a distribution function (hence right-continuous).

Note. If F is increasing, and

$$F(\infty) \coloneqq \lim_{x \nearrow \infty} F(x), \quad F(-\infty) \coloneqq \lim_{x \searrow \infty} F(x)$$

exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.[4]

Definition 0.5 (Locally finite). Let X be a topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

[4]There are distributions [FF99] Ch9., but these are different from distribution functions.

Lemma 0.3. Let μ be a locally finite Borel measure on $\mathbb{R}$, then

$$F_\mu(x) = \begin{cases} \mu\left((0, x]\right), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu\left((x, 0]\right), & \text{if } x < 0 \end{cases}$$

is a distribution function.

Proof. ■

DIY, use continuity of measure

Definition 0.6 (Half intervals). We call

$$\varnothing, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

half-intervals.

Lemma 0.4. Let $\mathcal{H}$ be the collection of finite disjoint unions of half-intervals. Then, $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. ■

DIY

Proposition 0.1 (Distribution function defines a pre-measure). Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function. For a half-interval I, define

$$\ell(I) \coloneqq \ell_F(I) = \begin{cases} 0, & \text{if } I = \varnothing \\ F(b) - F(a), & \text{if } I = (a, b] \\ F(\infty) - F(a), & \text{if } I = (a, \infty] \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b] \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 \coloneqq \mu_{0,F}$ as

$$\mu_{0,F}\colon \mathcal{H} \to [0, \infty]$$

by

$$\mu_0(A) = \sum_{k=1}^{N} \ell(I_k) \text{ if } A = \bigcup_{k=1}^{N} I_k,$$

where A is a finite disjoint union of half-intervals $I_1, \ldots, I_N$. Then, μ_0 is a pre-measure on $\mathcal{H}$.

Proof. We see that

1. μ_0 is well-defined.
2. $\mu_0(\varnothing) = 0$.
3. μ_0 is finite additive.

4. μ_0 is countable additive within $\mathcal{H}$.

   Suppose $A \in \mathcal{H}$ where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.[5]

   Focus on the case $I = (a, b]$. Let

$$(a,b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

   which is a disjoint union. Then we only need to check

$$F(b) - F(a) = \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right).$$

   - Since $(a,b] \supset \bigcup_{n=1}^{N} (a_n, b_n]$, hence

$$\underset{N\in\mathbb{N}}{\forall}\ F(b) - F(a) \geq \sum_{n=1}^{N} \left(F(b_n) - F(a_n)\right).$$

     By letting $N \to \infty$, we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right).$$

   - Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

$$F(a') - F(a) < \epsilon.$$

     For each $n \in \mathbb{N}$, $\exists b'_n > b_n$ such that

$$F(b'_n) - F(b_n) < \frac{\epsilon}{2^n}.$$

     Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b'_n),$$

     hence

$$\underset{N\in\mathbb{N}}{\exists}\ [a', b] \subset \bigcup_{n=1}^{N} (a_n, b'_n),$$

     which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^{N} F(b'_n) - F(a_n).$$

[5] why?

Finally, we see that

$$\begin{aligned} F(b) - F(a) &\leq F(b) - F(a') + \epsilon \\ &\leq \sum_{n=1}^{\infty} (F(b_n') - F(a_n)) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\epsilon}{2^n} \right) + \epsilon. \end{aligned}$$

■

Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8

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To classify all measure, we now see this last theorem to complete the task.

Theorem 0.3 (Locally finite Borel measures on $\mathbb{R}$). We have

1. $F\colon \mathbb{R} \to \mathbb{R}$ a distribution function, then there exists a unique locally finite Borel measure μ_F on $\mathbb{R}$ satisfying $\mu_F((a,b]) = F(b) - F(a)$ for every $a < b$.
2. Suppose $F, G\colon \mathbb{R} \to \mathbb{R}$ are distribution functions. Then,

$$\mu_F = \mu_G$$

on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. ■ HW.

0.3 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that

$$F \text{ distribution function } \overset{!}{\Longrightarrow} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Actually, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 0.7 (Lebesgue-Stieltjes measure). We define

- μ_F on $\mathcal{A}_{\mu_F}$ is called the *Lebesgue-Stieltjes* measure corresponding to F.
- Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{L}, m)$, where $\mathcal{L}$ is called Lebesgue σ-algebra, and m is called Lebesgue measure.

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+),$$

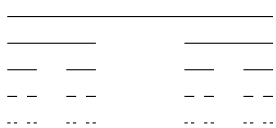
Example. We first see some examples.

1. $\mu_F((a,b]) = F(b) - F(a)$. Then
   - $\mu_F(\{a\}) = F(a) - F(a^-)$
   - $\mu_F([a,b]) = F(b) - F(a^-)$
   - $\mu_F((a,b)) = F(b^-) - F(a)$
2. We define
   $$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$
   Then
   - $\mu_F(\{0\}) = 1$
   - $\mu_F(\mathbb{R}) = 1$
   - $\mu_F(\mathbb{R} \setminus \{0\}) = 0$.

   We call that μ_F is the Dirac measure at 0.
3. Denote $\mathbb{Q} = \{r_1, r_2, \ldots\}$, and we define
   $$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r. \end{cases}$$
   Then HW
   - $\mu_F(\{x\}) > 0$ for all $r_i \in \mathbb{Q}$.
   - $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$
4. If F is continuous at a, then $\mu_F(\{a\}) = 0$.
5. $F(x) = x$
   - $m((a,b]) = m((a,b)) = m([a,b]) = b - a$.
6. $F(x) = e^x$
   - $\mu_F((a,b]) = \mu_F((a,b)) = e^b - e^a$.

Remark. We see that the first two examples are *discrete measures*.

Example (Middle thirds Cantor set). Let $C := \bigcap_{n=1}^{\infty} K_n$.



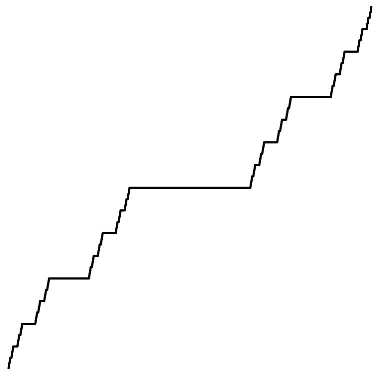


Figure 1: Cantor Function (Devil's Staircase).

Since C is uncountable set, hence $m(C) = 0$. And notice that

$$x \in C \iff x = \sum_{n=1}^{\infty} \frac{a_n}{3^n},\ a_n \in \{0, 2\}.$$

0.3.1 Cantor Function

Consider F as follows.

We see that F is *continuous* and increasing. Furthermore,

$$\begin{array}{lcl} \mu_F(\mathbb{R} \setminus C) = 0 & & m(\mathbb{R} \setminus C) = \infty > 0 \\ \mu_F(C) = 1 & \iff & m(C) = 0 \\ \mu_F(\{a\}) = 0 & & m(\{a\}) = 0 \end{array}$$

Remark. μ_F and m are said to be **singular** to each other.

0.4 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 0.5. Let μ be Lebesgue-Stieltjes measure on $\mathbb{R}$. Then we have

$$\begin{aligned} \mu(A) &\overset{!}{=} \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supset A \right\} \\ &=[6] \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

for every $A \in \mathcal{A}_\mu$

[6] This follows from the continuity of measure.

Appendix

References

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