

MATH592  
Introduction to Algebraic Topology

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### **Abstract**

This course will use Hatcher[HPM02] as the main text, but the order may differ here and there. Enjoy this fun course! In particular, I add some extra content which is not covered in lectures, things like [groupoid](#), [fibered coproduct](#), feel free to skip these content.

Note that I reference all definitions in the text as much as possible, but I may still miss some.

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# Chapter 1

## Foundation of Algebraic Topology

### Lecture 1: Homotopies of Maps

#### 1.1 Homotopy

05 Jan. 10:00

We start with the most important and fundamental concept, [homotopy](#).

**Definition 1.1.1** (Homotopy, homotopic, nullhomotopic). Let  $X, Y$  be topological spaces. Let  $f, g: X \rightarrow Y$  continuous maps. Then a *homotopy* from  $f$  to  $g$  is a 1-parameter family of maps that continuously deforms  $f$  to  $g$ , i.e., it's a continuous function  $F: X \times I \rightarrow Y$ , where  $I = [0, 1]$ , such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x).$$

We often write  $F_t(x)$  for  $F(x, t)$ .

If a homotopy exists between  $f$  and  $g$ , we say they are *homotopic* and write

$$f \simeq g.$$

If  $f$  is homotopic to a constant map, we call it *nullhomotopic*.

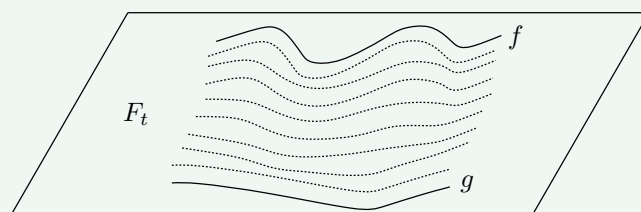


Figure 1.1: The continuous deforming from  $f$  to  $g$  described by  $F_t$

**Remark.** Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

**Example** (Straight line homotopy). Any two (continuous) maps with specification

$$f, g: X \rightarrow \mathbb{R}^n$$

are [homotopic](#) by considering

$$F_t(x) = (1 - t)f(x) + tg(x).$$

We call it *the straight line homotopy*.

**Example.** Let  $S^1$  denote the unit circle in  $\mathbb{R}^2$ , and  $D^2$  denote the unit disk in  $\mathbb{R}^2$ . Then the inclusion  $f: S^1 \hookrightarrow D^2$  is **nullhomotopic** by considering

$$F_t(x) = (1-t)f(x) + (t \cdot 0).$$

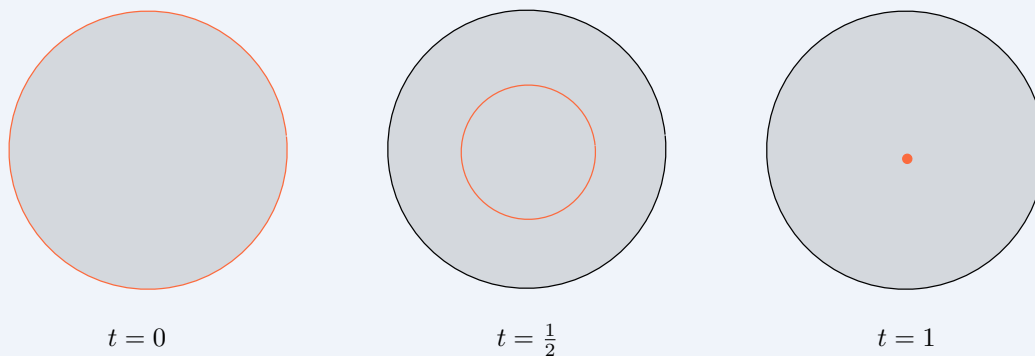


Figure 1.2: The illustration of  $F_t(x)$

We see that there is a **homotopy** from  $f(x)$  to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a **nullhomotopy**.

**Example.** The maps

$$\begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not homotopy**.

**Remark.** It will essentially **flip** the orientation, hence we can't deform one to another continuously.

**Exercise.** A subset  $S \subseteq \mathbb{R}^n$  is star-shaped if  $\exists x_0 \in S$  s.t.  $\forall x \in S$ , the line from  $x_0$  to  $x$  lies in  $S$ . Show that  $\text{id}: S \rightarrow S$  is **nullhomotopic**.

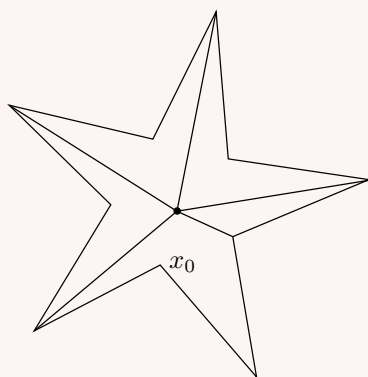


Figure 1.3: Star-shaped illustration

**Answer.** Consider

$$F_t(x) := (1-t)x + tx_0,$$

which essentially just concentrates all points  $x$  to  $x_0$ . ■

**Exercise.** Suppose

$$X \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_0} \end{array} Y \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_0} \end{array} Z$$

where

$$f_0 \simeq_{F_t} f_1, \quad g_0 \simeq_{G_t} g_1.$$

Show

$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

**Answer.** Consider  $I \times X \rightarrow Z$ , where

$$\begin{array}{ccccc} X \times I & \rightarrow & Y \times I & \rightarrow & Z \\ (x, t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

**Remark.** Noting that if one wants to be precise, you need to check the continuity of this construction. ■

**Exercise.** How could you show 2 maps are **not** homotopic?

**Answer.** We'll see! ■

## Lecture 2: Homotopy Equivalence

07 Jan. 10:00

**As previously seen.** Two maps  $f, g: X \rightarrow Y$  is **homotopy** if there exists a map

$$F_t(x): X \times I \rightarrow Y$$

with the properties

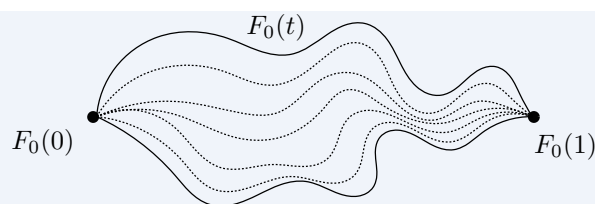
1. Continuous
2.  $F_0(x) = f(x)$
3.  $F_1(x) = g(x)$

**Remark.** The continuity of  $F_t$  is an even stronger condition for the continuity of  $F_t$  for a fixed  $t$ .

We now introduce another concept.

**Definition 1.1.2 (Homotopy relative).** Given two spaces  $X, Y$ , and let  $B \subseteq X$ . Then a **homotopy**  $F_t(x): X \rightarrow Y$  is called *homotopy relative  $B$*  (denotes  $\text{rel}B$ ) if  $F_t(b)$  is independent of  $t$  for all  $b \in B$ .

**Example.** Given  $X$  and  $B = \{0, 1\}$ . Then the **homotopy** of paths from  $[0, 1] \rightarrow X$  is  $\text{rel}\{0, 1\}$ .



## 1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

**Definition 1.2.1** (Homotopy equivalence, homotopy inverse). A map  $f: X \rightarrow Y$  is a *homotopy equivalence* if  $\exists g: Y \rightarrow X$  such that

$$f \circ g \simeq \text{id}_Y, \quad g \circ f \simeq \text{id}_X.$$

We say that  $X, Y$  are *homotopy equivalent*, and  $g$  is called *homotopy inverse* of  $f$ .

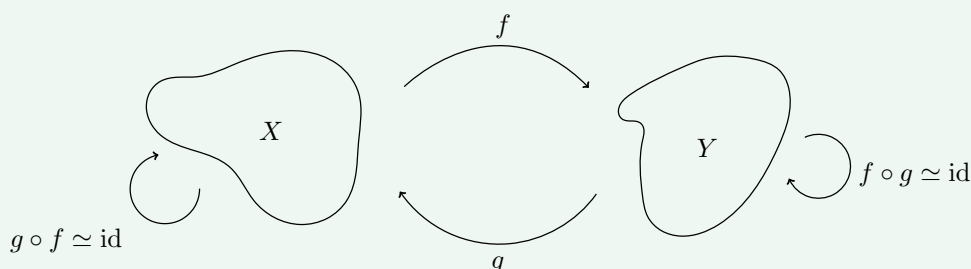
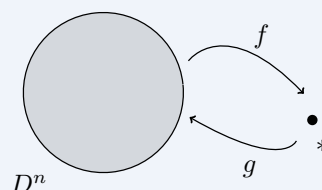


Figure 1.4: Homotopy Equivalence

**Definition 1.2.2** (Homotopy type). If  $X, Y$  are *homotopy equivalent*, then we say that they have the same *homotopy type*.

**Notation.** We denote a closed  $n$ -disk as  $D^n$ .

**Example.**  $D^n$  is *homotopy equivalent* to a point.



**Proof.** We see that  $f \circ g = \text{id}_*$  and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is *homotopic* to  $\text{id}_{D^n}$  by *straight line homotopy*  $F_t(x) = tx$ . Specifically, we see that this holds for any convex set. ■

**Definition 1.2.3** (Contractible). We say that a space  $X$  is *contractible* if  $X$  is *homotopy equivalent* to a point.

The following proposition is added much after, which may use some concepts not yet covered.

**Proposition 1.2.1.** The followings are equivalent.

1.  $X$  is [contractible](#).
2.  $\forall x \in X, \text{id}_X \simeq c_x$ .
3.  $\exists x \in X, \text{id}_X \simeq c_x$ .

**Remark.** Note that the above notation  $c_x$  is introduced [here](#).

**Proof.** We see that 2.  $\Rightarrow$  3. is obvious. We consider 3.  $\Rightarrow$  2. This follows from the following general lemma.

**Lemma 1.2.1.** Given a topological space  $X$  such that  $\exists x \in X, \text{id}_X \simeq c_x$ , with  $f, g: Y \rightarrow X$ , then  $f \simeq g$ .

**Proof.** Let  $x \in X$  such that  $\text{id}_X \simeq c_x$ . Then

$$f = \text{id}_X \circ f \simeq c_x \circ f = c_x \circ g \simeq \text{id}_X \circ g = g.$$

■

Then, from this [Lemma 1.2.1](#), we see that assuming  $x_0 \in X$  such that  $\text{id}_X \simeq c_{x_0}$ , then consider  $c_x$  for all  $x \in X$ , then from [Lemma 1.2.1](#), we see that  $c_x \simeq \text{id}_X$ .

To show 3.  $\Rightarrow$  1., we let  $x_0 \in X$  such that  $\text{id}_X \simeq c_{x_0}$ .

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} \{*\}$$

Since  $g(*) = x_0$ , and

$$\begin{aligned} g \circ f: X &\rightarrow X \\ x &\mapsto x_0, \end{aligned}$$

which is just  $c_{x_0}$ , from the assumption we're done.

Now, we show 1.  $\Rightarrow$  3. Let

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} \{*\}$$

be a [homotopy equivalent](#), let  $g(*) = x_0$ . We see that  $c_{x_0} \simeq \text{id}_X$  since

$$g \circ f = c_{x_0} \simeq \text{id}_X.$$

■

Before doing exercises, we introduce two new concepts.

**Definition 1.2.4 (Retraction, retract).** Given  $B \subseteq X$ , a *retraction* from  $X$  to  $B$  is a map  $f: X \rightarrow X$  (or  $X \rightarrow B$ ) such that  $\forall b \in B, f(b) = b$ , namely  $r|_B = \text{id}_B$ . Or one can see this from

$$\begin{array}{ccc} B & \xrightarrow{i} & X & \xrightarrow{r} & B \\ & & \searrow r \circ i & & \nearrow \end{array}$$

where  $r$  is a retraction if and only if  $r \circ i = \text{id}_B$ , where  $i$  is an inclusion identity.

If  $r$  exists,  $B$  is a *retract* of  $X$ .



**Definition 1.2.5** (Deformation retraction). Given  $X$  and  $B \subseteq X$ , a (strong) deformation retraction  $F_t: X \rightarrow X$  onto  $B$  is a homotopy  $\text{rel} B$  from the  $\text{id}_X$  to a retraction from  $X$  to  $B$ . i.e.,

$$\begin{aligned} F_0(x) &= x & \forall x \in X \\ F_1(x) &\in B & \forall x \in X \\ F_t(b) &= b & \forall t \forall b \in B. \end{aligned}$$

**Exercise.** Let  $X \simeq Y$ . Show  $X$  is path-connected if and only if  $Y$  is.

**Answer.** Suppose  $X$  is path-connected. Then we see that given two points  $x_1$  and  $x_2$  in  $X$ , there exists a path  $\gamma(t)$  with

$$\gamma: [0, 1] \rightarrow X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Since  $X \simeq Y$ , then there exists a pair of  $f$  and  $g$  such that  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  with

$$f \circ g \underset{F}{\simeq} \text{id}_Y, \quad g \circ f \underset{G}{\simeq} \text{id}_X.$$

(Notice the abuse of notation)

For any two  $y_1$  and  $y_2 \in Y$ , we want to construct a path  $\gamma'(t)$  such that

$$\gamma': [0, 1] \rightarrow Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

Firstly, we let  $g(y_1) =: x_1$  and  $g(y_2) =: x_2$ . From the argument above, we know there exists such a  $\gamma$  starting at  $x_1 = g(y_1)$  ending at  $x_2 = g(y_2)$ . Now, consider  $f(\gamma(t)) = (f \circ \gamma)(t)$  such that

$$f \circ \gamma: I \rightarrow Y, \quad f \circ \gamma(0) = y'_1, \quad f \circ \gamma(1) = y'_2,$$

we immediately see that  $y'_1$  and  $y'_2$  is path connected. Now, we claim that  $y_1$  and  $y'_1$  are path connected in  $Y$ , hence so are  $y_2$  and  $y'_2$ . To see this, note that

$$f \circ g \underset{F}{\simeq} \text{id}_Y,$$

which means that there exists  $F: Y \times I \rightarrow Y$  such that

$$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y'_1 \\ F(y_1, 1) = \text{id}_Y(y_1) = y_1. \end{cases}$$

Since  $F$  is continuous in  $I$ , we see that there must exist a path connects  $y_1$  and  $y'_1$ . The same argument applies to  $y_2$  and  $y'_2$ . Now, we see that the path

$$y_1 \rightarrow y'_1 \rightarrow y'_2 \rightarrow y_2$$

is a path in  $Y$  for any two  $y_1$  and  $y_2$ , which shows  $Y$  is path-connected.

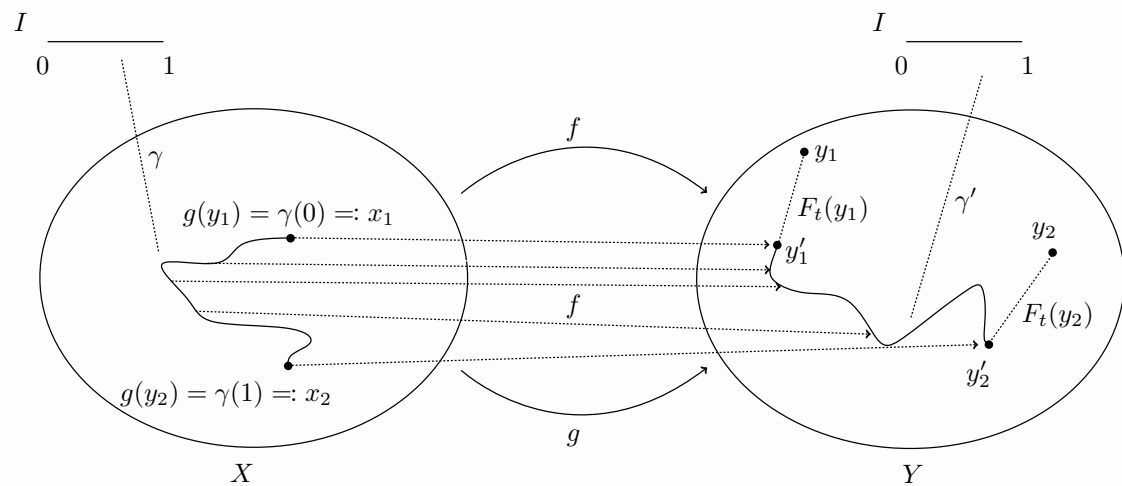


Figure 1.5: Demonstration of the proof.

**Challenge:** One can further show that the connectedness is also preserved by any [homotopy equivalence](#). ■

**Corollary 1.2.1.** A [contractible](#) space is [path-connected](#).

**Exercise.** Show that if there exists [deformation retraction](#) from  $X$  to  $B \subseteq X$ , then  $X \simeq B$ .

## Lecture 3: Deformation Retraction

**As previously seen.** A [deformation retraction](#) is a [homotopy](#) of maps  $\text{rel} B \ X \rightarrow X$  from  $\text{id}_X$  to a [retraction](#) from  $X$  to  $B$ . Then  $B$  is a [deformation retract](#).

**Example.**  $S^1$  is a [deformation retraction](#) of  $D^2 \setminus \{0\}$ .

**Proof.** Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$

10 Jan. 10:00

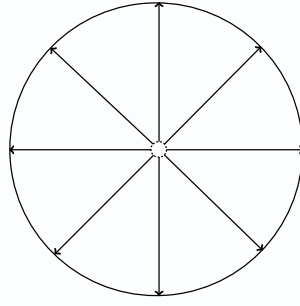


Figure 1.6: The **deformation retraction** of  $D^2 \setminus \{0\}$  is just to *enlarge* that hole and push all the interior of  $D^2$  to the boundary, which is  $S^1$ .

**Example.**  $\mathbb{R}^n$  **deformation retracts** to 0.

**Proof.** Indeed, since

$$F_t(x) = (1 - t)x.$$

This implies that  $\mathbb{R}^n \simeq *$ , hence we see that

- dimension
- compactness
- etc.

are not **homotopy** invariants.

**Example.**  $S^1$  is a **deformation retract** of a cylinder and a Möbius band.

**Proof.** For a cylinder, consider  $X \times I \rightarrow X$ . Define **homotopy** on a closed rectangle, then verify it induces map on quotient.

For a Möbius band, we define a **homotopy** on a closed rectangle, then verify that it respect the equivalence relation.

Finally, we use the universal property of quotient topology to argue that we get a **homotopy** on Möbius band.

**Upshot:** Möbius band  $\simeq S^1 \simeq$  cylinder, hence the orientability is not **homotopy** invariant.

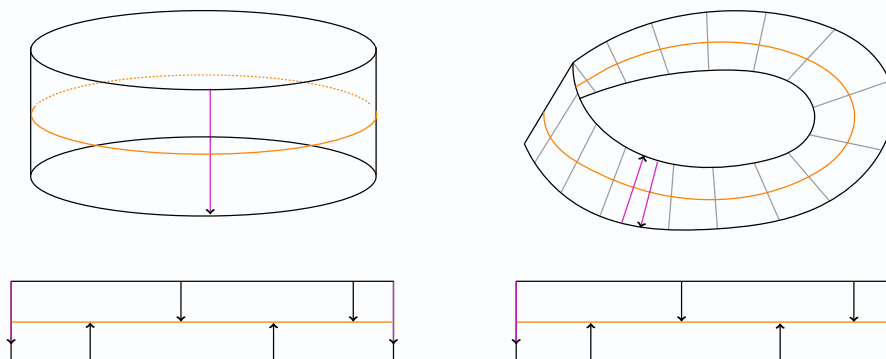


Figure 1.7: The **deformation retraction** for Cylinder and Möbius band

## Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

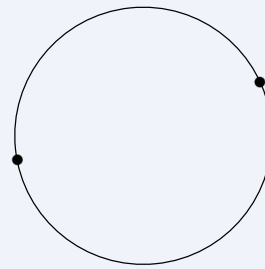
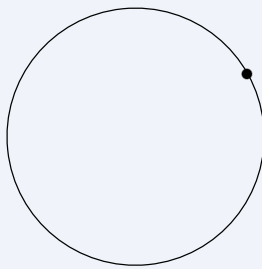
As previously seen. We saw that

- homotopy equivalence
- homotopy invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

### 1.3 CW Complexes

**Example (Constructing spheres).** We now see how to construct  $S^1$  and  $S^2$  from ground up.

- $S^1$  (up to homeomorphism<sup>a</sup>)



- $S^2$ 
  - glue boundary of 2-disk to a point
  - glue 2 disks onto a circle

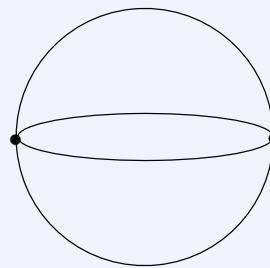
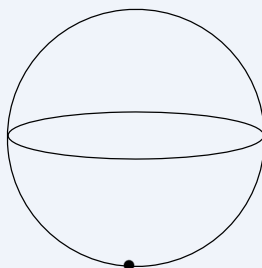
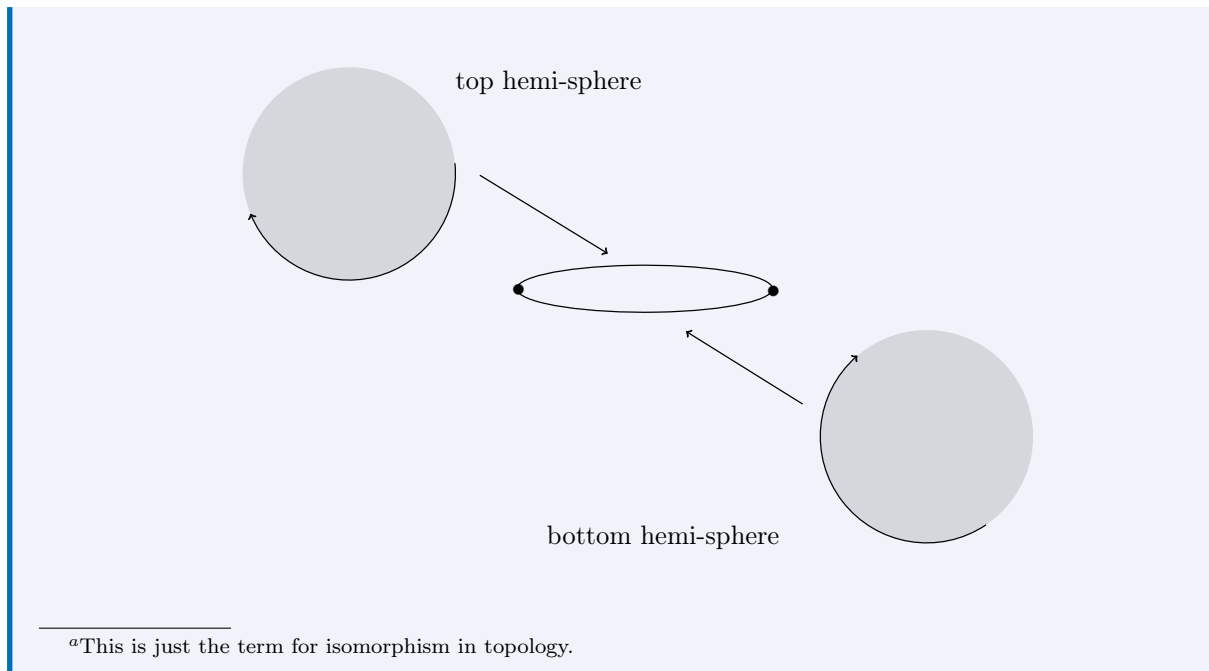
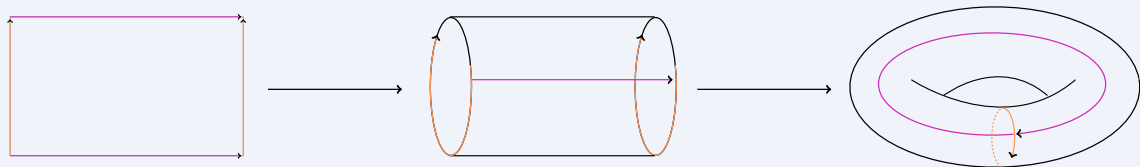


Figure 1.8: **Left:** Glue a 2-disk to a point along its boundary. **Right:** Glue 2 disks to  $S^1$ .

The gluing instruction to construct  $S^2$  in the right-hand side can be demonstrated as follows.



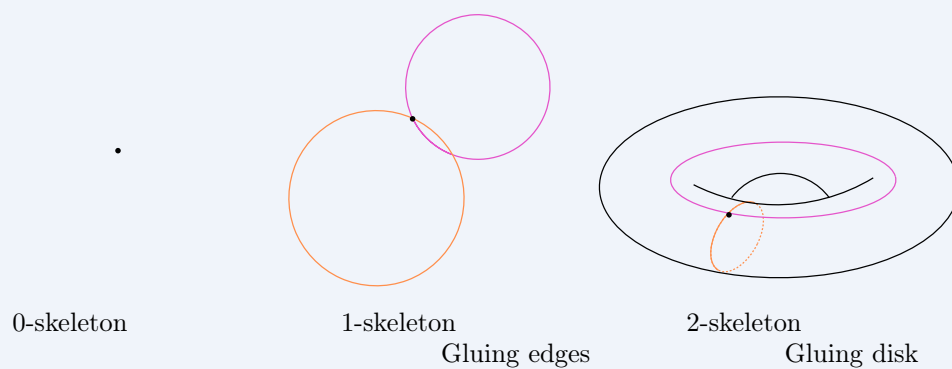
**Example (constructing torus).** A torus  $T$  is just  $T = S^1 \times S^1$ .



view as gluing instructions

vertex + 2 edges + 2-disks.

Specifically, we have



**Notation.** Let  $D^n$  denotes a closed  $n$ -disk (or  $n$ -ball)

$$D^n \simeq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

And let  $S^n$  denotes an  $n$ -sphere

$$S^n \simeq \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

Lastly, we call a point as a  $0$ -cell, and the interior of  $D^n$   $\text{int}(D^n)$  for  $n \geq 1$  as a  $n$ -cell.

Then, formally, we have the following definition.

**Definition 1.3.1 (CW Complex).** A *CW Complex* is a topological space constructed inductively as

1.  $X^0$  (the 0-skeleton) is a set of discrete points.
2. We inductively construct the  $n$ -skeleton  $X^n$  from  $X^{n-1}$  by attaching  $n$ -cells  $e_\alpha^n$ , where  $\alpha$  is the index.

The gluing instructions glued by an attaching map is that  $\forall \alpha, \exists$  continuous map  $\varphi_\alpha$

$$\varphi_\alpha: \partial D_\alpha^n \rightarrow X^{n-1},$$

then

$$X^n = \left( X^{n-1} \amalg \coprod_\alpha D_\alpha^n \right) / x \sim \varphi_\alpha(x)$$

with identification  $x \sim \varphi_\alpha(x)$  for all  $x \in \partial D_\alpha^n$  with quotient topology.

3. We let  $X$  be defined as

$$X = \bigcup_{n=0} X^n,$$

and let  $\bar{w}$  denotes weak topology such that

$$u \subseteq X \text{ is open} \Leftrightarrow \forall n \ u \cap X^n \text{ is open}.$$

If all cells have dimension less than  $N$  and a  $\exists N$ -cell, then  $X = X^N$  and we call it  $N$ -dimensional *CW complex*.

**Remark.** We write  $X^{(n)}$  for  $n$ -skeleton if we need to distinguish from the Cartesian product.

**Example.** Let's look at some examples.

1. 0-dim **CW complex** is a discrete space.
2. 1-dim **CW complex** is a graph.
3. A **CW complex**  $X$  is finite if it has finitely many cells.

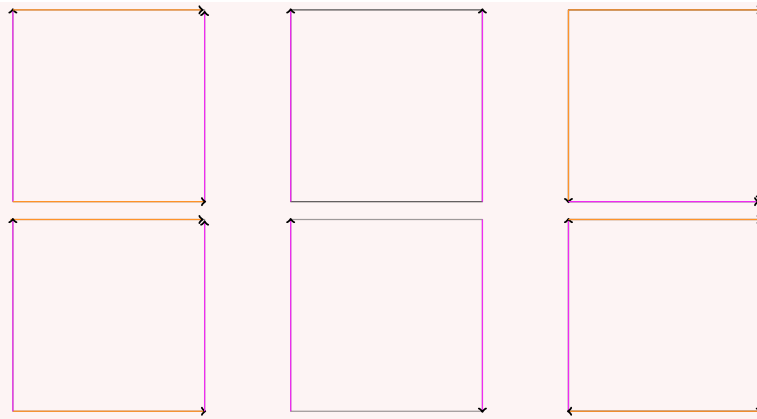
**Definition 1.3.2 (CW subcomplex).** A *CW subcomplex*  $A \subseteq X$  is a closed subset equal to a union of cells

$$e_\alpha^n = \text{int}(D_\alpha^n).$$

**Remark.** This inherits a **CW complex** structure.

**Exercise.** Given the following gluing instruction:

Check the images of attaching maps.



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere,  $\mathbb{R}P$ .

**Notation.** Notice that we call the real projection space as  $\mathbb{R}P$ , and we also have so-called complex projection space, denote as  $\mathbb{C}P$ .

**Answer.** We see that

1. Torus
2. Cylinder
3. 2-sphere
4. Klein bottle
5. Möbius band
6.  $\mathbb{R}P$

■

## Lecture 5: Operation on Spaces

### 1.4 Operations on CW Complexes

14 Jan. 10:00

#### 1.4.1 Products

We can consider the product of two **CW complex** given by a **CW complex** structure. Namely, given  $X$  and  $Y$  two **CW complexes**, we can take two cells  $e_\alpha^n$  from  $X$  and  $e_\beta^m$  from  $Y$  and form the product space  $e_\alpha^n \times e_\beta^m$ , which is homeomorphic to an  $(n+m)$ -cell. We then take these products as the cells for  $X \times Y$ .

Specifically, given  $X, Y$  are **CW complexes**, then  $X \times Y$  has a cell structure

$$\{e_\alpha^m \times e_\alpha^n: e_\alpha^m \text{ is a } m\text{-cell on } X, e_\alpha^n \text{ is an } n\text{-cell on } Y\}.$$

**Remark.** The product topology may not agree with the weak topology on the  $X \times Y$ . However, they do agree if  $X$  or  $Y$  is locally compact or if  $X$  and  $Y$  both have at most countably many cells.

#### 1.4.2 Wedge Sum

Given  $X, Y$  are **CW complexes**, and  $x_0 \in X^0, y_0 \in Y^0$  (only points). Then we define

$$X \vee Y = X \amalg Y$$

with quotient topology.

**Remark.**  $X \vee Y$  is a **CW complex**.

### 1.4.3 Quotients

Let  $X$  be a **CW complex**, and  $A \subseteq X$  **subcomplex** (closed union of cells), then

$$X / A$$

is a quotient space collapse  $A$  to one point and inherits a **CW complex** structure.

**Remark.**  $X / A$  is a **CW complex**.

0-skeleton

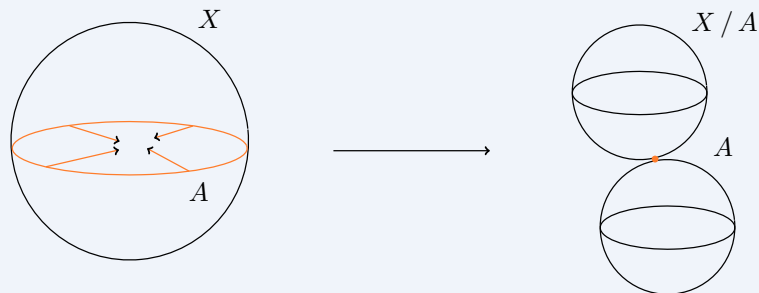
$$(X^0 - A^0) \coprod *$$

where  $*$  is a point for  $A$ . Each cell of  $X - A$  is attached to  $(X / A)^n$  by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n / A^n$$

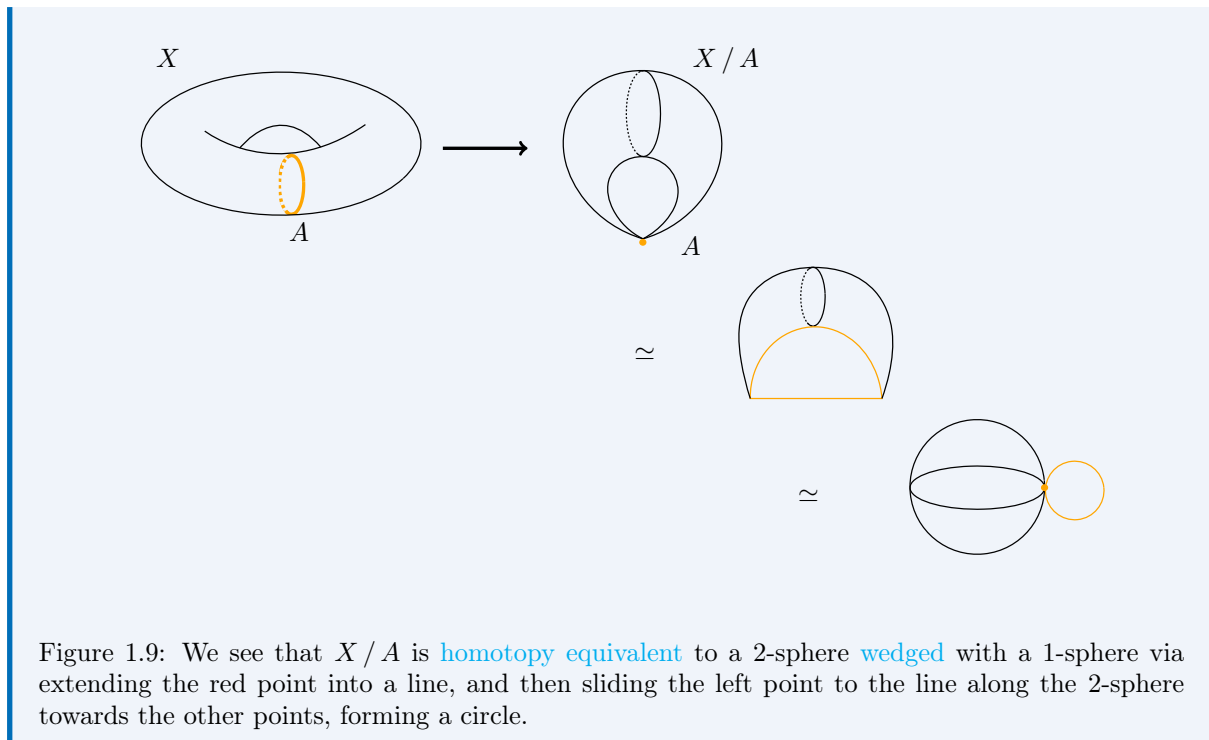
**Example.** Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a **wedge** of two spheres.



2. We can take the torus and squish down a ring around the hole.





## Lecture 6: A Foray into Category Theory

### 1.5 Category Theory

19 Jan. 10:00

We start with a definition.

**Definition 1.5.1 (Category, object, morphism).** A *category*  $\mathcal{C}$  is 3 pieces of data

- A class of *objects*  $\text{Ob}(\mathcal{C})$
- $\forall X, Y \in \text{Ob}(\mathcal{C})$  a class of *morphisms* or arrows,  $\text{Hom}_{\mathcal{C}}(X, Y)$ .
- $\forall X, Y, Z \in \text{Ob}(\mathcal{C})$ , there exists a composition law

$$\text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z), \quad (f, g) \mapsto g \circ f$$

and 2 axioms

- Associativity.  $(f \circ g) \circ h = f \circ (g \circ h)$  for all **morphisms**  $f, g, h$  where composites are defined.
- Identity.  $\forall X \in \text{Ob}(\mathcal{C}) \exists \text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$  such that

$$f \circ \text{id}_X = f, \quad \text{id}_X \circ g = g$$

for all  $f, g$  where this makes sense.

Let's see some examples.

**Example.** We introduce some common **category**.

$\mathcal{C}$	$\text{Ob}(\mathcal{C})$	$\text{Mor}(\mathcal{C})$
$\underline{\text{set}}$	Sets $X$	All maps of sets
$\underline{\text{fset}}$	Finite sets	All maps
$\underline{\text{Gp}}$	Groups	Group Homomorphisms
$\underline{\text{Ab}}$	Abelian groups	Group Homomorphisms
$\underline{k\text{-vect}}$	Vector spaces over $k$	$k$ -linear maps
$\underline{\text{Rng}}$	Rings	Ring Homomorphisms
$\underline{\text{Top}}$	Topological spaces	Continuous maps
$\underline{\text{Haus}}$	Hausdorff Spaces	Continuous maps
$\underline{\text{hTop}}$	Topological spaces	Homotopy classes of continuous maps
$\underline{\text{Top}}^*$	Based topological spaces <sup>a</sup>	Based maps <sup>b</sup>

<sup>a</sup>Topological spaces with a distinguished base point  $x_0 \in X$

<sup>b</sup>Continuous maps that preserve base point  $f: (x, x_0) \rightarrow (y, y_0)$  such that

$$f: X \rightarrow Y, \quad f(x_0) = y_0$$

is continuous.

**Remark.** Any **diagram** plus composition law.

$$\text{id}_A \hookrightarrow A \longrightarrow B \rightrightarrows \text{id}_B.$$

**Definition 1.5.2** (Monic, epic). A **morphism**  $f: M \rightarrow N$  is *monic* if

$$\forall g_1, g_2 \quad f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2.$$

$$A \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} M \xrightarrow{f} N$$

Dually,  $f$  is *epic* if

$$\forall g_1, g_2 \quad g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2.$$

$$M \xrightarrow{f} N \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} B$$

**Lemma 1.5.1.** In  $\underline{\text{set}}, \underline{\text{Ab}}, \underline{\text{Top}}, \underline{\text{Gp}}$ , a map is **monic** if and only if  $f$  is injective, and **epic** if and only if  $f$  is surjective.

**Proof.** In  $\underline{\text{set}}$ , we prove that  $f$  is **monic** if and only if  $f$  is injective. Suppose  $f \circ g_1 = f \circ g_2$  and  $f$  is injective, then for any  $a$ ,

$$f(g_1(a)) = f(g_2(a)) \Rightarrow g_1(a) = g_2(a),$$

hence  $g_1 = g_2$ .

Now we prove another direction, with contrapositive. Namely, we assume that  $f$  is not injective and show that  $f$  is not **monic**. Suppose  $f(a) = f(b)$  and  $a \neq b$ , we want to show such  $g_i$  exists. This is easy by considering

$$g_1: * \mapsto a, \quad g_2: * \mapsto b.$$

■

### 1.5.1 Functor

After introducing the **category**, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

**Definition 1.5.3 (Functor).** Given  $\mathcal{C}, \mathcal{D}$  be two [categories](#). A (covariant) *functor*  $F: \mathcal{C} \rightarrow \mathcal{D}$  is

1. a map on [objects](#)

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of [morphisms](#)

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(X) \rightarrow F(Y)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(f) \circ F(g)$

## Lecture 7: Functors

21 Jan. 10:00

As previously seen. Assume that we initially have a commutative diagram in  $\mathcal{C}$  as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & Z \end{array}$$

After applying  $F$ , we'll have

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ & \searrow F(g \circ f) = F(g) \circ F(f) & \downarrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in  $\mathcal{D}$ .

We can also have a so-called contravariant functor.

**Definition 1.5.4 (Contravariant functor).** Given  $\mathcal{C}, \mathcal{D}$  be two [categories](#). A *contravariant functor*

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is

1. a map on [objects](#)

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of [morphisms](#)

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(Y), F(X)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(Y) \rightarrow F(X)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a **contravariant functor**  $F$ , the diagram becomes

$$\begin{array}{ccc} F(X) & \xleftarrow{F(f)} & F(Y) \\ & \nwarrow & \uparrow F(g) \\ & & F(Z) \end{array} \quad F(g \circ f) = F(f) \circ F(g)$$

which is a commutative diagram in  $\mathcal{D}$ .

**Example (Identity functor).** Define  $I$  as  $I: \mathcal{C} \rightarrow \mathcal{C}$  such that it just send an object  $c \in \mathcal{C}$  to itself.

**Example (Forgetful functor).** We see two examples.

- Define  $F$  as  $F: \underline{\mathbf{Gp}} \rightarrow \underline{\mathbf{set}}$  such that  $G \mapsto G$ .<sup>a</sup> Specifically,

$$[f: G \rightarrow H] \mapsto [f: G \rightarrow H].$$

- Define  $F$  as  $F: \underline{\mathbf{Top}} \rightarrow \underline{\mathbf{set}}$  such that  $X \mapsto X$ .<sup>b</sup> Specifically,

$$[f: X \rightarrow Y] \mapsto [f: X \rightarrow Y].$$

<sup>a</sup> $G$  is now just the underlying set of the group  $G$ .

<sup>b</sup> $X$  is now just the underlying set of the topological space  $X$ .

**Example (Free functor).** Define a **functor** as

$$\begin{array}{l} \underline{\mathbf{set}} \rightarrow \underline{\mathbf{k-vect}} \\ s \mapsto \text{"free" } k\text{-vector space on } s \end{array}$$

i.e., vector space with basis  $s$  such that

$$[f: A \rightarrow B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

**Example.**

$$\begin{array}{l} \underline{\mathbf{k-vect}} \rightarrow \underline{\mathbf{k-vect}} \\ V \mapsto V^* = \text{Hom}_k(V, k) \end{array}$$

If we are working on a basis, then we have

$$A \mapsto A^T.$$

**Remark.** Specifically, we care about two **functors**.

1.

$$\begin{array}{l} \pi_1: \underline{\mathbf{Top}}^* \rightarrow \underline{\mathbf{Gp}} \\ (X, x_0) \mapsto \pi_1(X, x_0) \end{array}$$

where  $\pi_1$  is so-called *fundamental group*.

2.

$$\begin{array}{l} H_p: \underline{\mathbf{Top}} \rightarrow \underline{\mathbf{Ab}} \\ X \mapsto H_p(X) \end{array}$$

where  $H_p$  is so-called  $p^{\text{th}}$  *homology*.

Let's see the formal definition.

## 1.6 Free Groups

**Definition 1.6.1 (Free group).** Given a set  $S$ , the *free group* is a group  $F_S$  on  $S$  with a map  $S \rightarrow F_S$  satisfying the universal property.

If  $G$  is any group,  $f: S \rightarrow G$  is any map of sets,  $f$  extends uniquely to group homomorphism  $\bar{f}: F_S \rightarrow G$ .

$$\begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \bar{f}: \text{gp hom} \\ & & G \end{array}$$

**Note.** This defines a *natural bijection*

$$\text{Hom}_{\text{set}}(S, \mathcal{U}(G)) \cong \text{Hom}_{\text{Grp}}(F_S, G),$$

where  $\mathcal{U}(G)$  is the **forgetful functor** from the **category** of groups to the **category** of sets. This is the statement that the **free functor** and the forgetful functor are **adjoint**; specifically that the **free functor** is the left **adjoint** (appears on the left in the Hom above).

**Definition 1.6.2 (Adjoint functor).** A **free** and **forgetful functor** is *adjoints*.

**Remark.** Whenever we state a universal property for an **object** (plus a map), an **object** (plus a map) may or may not exist. If such **object** exists, then it defines the **object uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the **object** (plus a map).

**Lemma 1.6.1.** Universal property defines  $F_S$  (plus a map  $S \rightarrow F(S)$ ) uniquely up to unique isomorphism.

**Proof.** Fix  $S$ . Suppose

$$S \rightarrow F_S, \quad S \rightarrow \tilde{F}_S$$

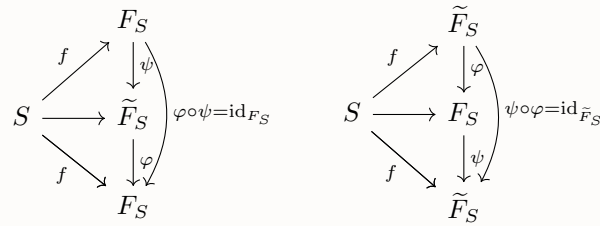
both satisfy the unique property. By universal property, there exist maps such that

$$\begin{array}{ccc} S & \longrightarrow & \tilde{F}_S \\ & \searrow f & \downarrow \exists! \varphi \\ & & F_S \end{array} \quad \begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \psi \\ & & \tilde{F}_S \end{array}$$

We'll show  $\varphi$  and  $\psi$  are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \text{id}_{F_S} & \searrow f \\ S & & F_S \end{array} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \text{id}_{\tilde{F}_S} & \searrow f \\ S & & \tilde{F}_S \end{array}$$

Hence, we see that



where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \text{id}_{F_S}, \quad \psi \circ \varphi = \text{id}_{\tilde{F}_S},$$

so  $\varphi$  and  $\psi$  are inverses (thus group isomorphism). ■

## Lecture 8: The Fundamental Group $\pi_1$

**Example.** In category **Ab** free Abelian group on a set  $S$  is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field** on  $S$ .

24 Jan. 10:00

### 1.6.1 Constructing the Free Groups $F_S$

**Proposition 1.6.1.** The **free group** defined by the universal property exists.

**Proof.** We'll just give a construction below. First, we see the definition.

**Definition 1.6.3 (Word).** Fix a set  $S$ , and we define a *word* as a finite sequence (possibly  $\emptyset$ ) in the formal symbols

$$\{s, s^{-1} \mid s \in S\}.$$

Then we see that elements in  $F_S$  are equivalence classes of **words** with the equivalence relation being

- deleted  $ss^{-1}$  or  $s^{-1}s$ . i.e.,

$$vs^{-1}sw \sim vw$$

$$vss^{-1}w \sim vw$$

for every **word**  $v, w, s \in S$ ,

with the group operation being concatenation. ■

**Example.** Given **words**  $ab^{-1}, bba$ , their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

**Exercise.** There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of **words** has a unique *reduced form*, namely the representation.

3. Check that  $F_S$  satisfies the universal property with respect to the map

$$S \rightarrow F_S, \quad s \mapsto s.$$

# Chapter 2

## The Fundamental Group

### 2.1 Path

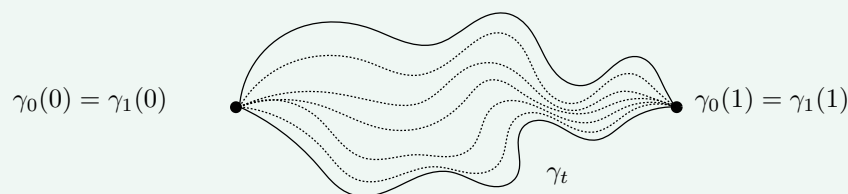
We start with the definition.

**Definition 2.1.1 (Path).** A *path* in a space  $X$  is a continuous map

$$\gamma: I \rightarrow X$$

where  $I = [0, 1]$ .

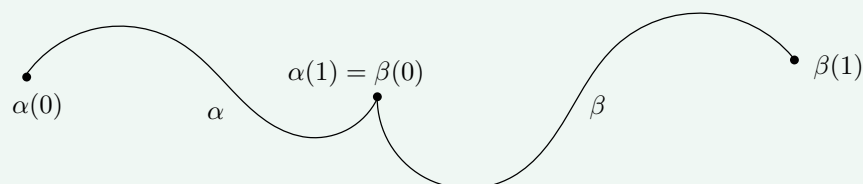
**Definition 2.1.2 (Homotopy path).** A *homotopy of paths*  $\gamma_0, \gamma_1$  is a [homotopy](#) from  $\gamma_0$  to  $\gamma_1$  rel $\{0, 1\}$ .



**Example.** Fix  $x_1, x_0 \in X$ , then  $\exists$  [homotopy](#) of [paths](#) is an equivalence relation on [paths](#) from  $x_0$  to  $x_1$  (i.e.,  $\gamma$  with  $\gamma(0) = x_0, \gamma(1) = x_1$ ).

**Definition 2.1.3 (Path composition).** For [paths](#)  $\alpha, \beta$  in  $X$  with  $\alpha(1) = \beta(0)$ , the *composition*<sup>a</sup>  $\alpha \cdot \beta$  is

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



<sup>a</sup>Also named *product*, *concatenation*.



**Remark.** By the pasting lemma, this is continuous, hence  $\alpha \cdot \beta$  is actually a **path** from  $\alpha(0)$  to  $\beta(1)$ .

**Definition 2.1.4 (Reparameterization).** Let  $\gamma: I \rightarrow X$  be a **path**, then a *reparameterization* of  $\gamma$  is a **path**

$$\gamma': I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where  $\varphi$  is continuous and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

**Exercise.** A **path**  $\gamma$  is **homotopic rel** $\{0, 1\}$  to all of its **reparameterizations**.

**Answer.** We show that  $\gamma$  and  $\gamma \circ \phi$  are **homotopic rel** $\{0, 1\}$  by showing that there exists a continuous  $F_t$  such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since  $\phi$  is continuous, so we define

$$F_t(x) = (1-t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all  $x, t \in I$ .

Now, we check that  $F_t$  really gives us a **homotopic rel** $\{0, 1\}$ . We have

$$\begin{aligned} F_t(0) &= (1-t)\gamma(0) + t \cdot \gamma \circ \phi(0) = (1-t)\gamma(0) + t \cdot \underbrace{\gamma(\phi(0))}_0 = \gamma(0), \\ F_t(1) &= (1-t)\gamma(1) + t \cdot \gamma \circ \phi(1) = (1-t)\gamma(1) + t \cdot \underbrace{\gamma(\phi(1))}_1 = \gamma(1), \end{aligned}$$

which shows that 0 and 1 are independent of  $t$ , hence  $\gamma$  and  $\gamma \circ \phi$  are **homotopic rel** $\{0, 1\}$ . ■

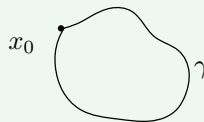
**Exercise.** Fix  $x_1, x_1 \in X$ . Then **homotopy of paths** (**relative**  $\{0, 1\}$ ) is an equivalence relation on **paths** from  $x_0$  to  $x_1$ .

## 2.2 Fundamental Group and Groupoid

### 2.2.1 Fundamental Group

**Definition 2.2.1 (Fundamental Group).** Let  $X$  denotes the space and let  $x_0 \in X$  be the base point. The *fundamental group of  $X$  based at  $x_0$* , denoted by  $\pi_1(X, x_0)$ , is a group such that

- Elements: **Homotopy** classes **rel** $\{0, 1\}$  of **paths**  $[\gamma]$  where  $\gamma$  is a **loop** with  $\gamma(0) = \gamma(1) = x_0$ <sup>a</sup>



- Operation: **Composition of paths**.

- Identity: Constant loop  $\gamma$  based at  $x_0$  such that

$$\gamma: I \rightarrow X, \quad t \mapsto x_0$$

- Inverses: The inverse  $[\gamma]^{-1}$  of  $[\gamma]$  is represented by the loop  $\bar{\gamma}$  such that

$$\bar{\gamma}(t) = \gamma(1 - t).$$



<sup>a</sup>We say  $\gamma$  is **based** at  $x_0$ .

**Proof.** We actually need to prove that the defined  $\pi_1$  actually is a group.

We prove that

**Associativity.**  $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$ . We break this down into

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t), & t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t - 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t - 2), & t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t - 3), & t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t - 1), & t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, we define  $\phi: I \rightarrow I$  such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \frac{1}{2}\right], & t \in \left[0, \frac{1}{4}\right]; \\ t + \frac{1}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right], & t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{t+1}{2} \in \left[\frac{3}{4}, 1\right], & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and  $\phi(t)$  is continuous and satisfied  $\phi(0) = 0$  and  $\phi(1) = 1$ , which implies that the associativity holds.

**Identity.** We want to show that  $[\gamma \cdot c] = [\gamma]$ . Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & t \in \left[0, \frac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, consider  $\phi: I \rightarrow I$  such that

$$\phi(t) = \begin{cases} 2t, & t \in \left[0, \frac{1}{2}\right]; \\ 1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and  $\phi(t)$  is continuous and satisfied  $\phi(0) = 0$  and  $\phi(1) = 1$ .

**Inverses.** We want to show that  $\gamma \cdot \bar{\gamma} \simeq c$ , where  $\bar{\gamma}(t) = \gamma(1-t)$ . Firstly, we have

$$(\gamma \cdot \bar{\gamma})(t) = \begin{cases} \gamma(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2t), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider  $F_t$  given by

$$F_t(x) = \begin{cases} \gamma(2xt), & x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2xt), & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

If  $t = 0$ , we have

$$F_0(x) = \begin{cases} \gamma(0), & x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1), & x \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0$$

for all  $x \in I$ , namely  $F_0 = c$ , while when  $t = 1$ , we have

$$F_1(x) = \begin{cases} \gamma(2x), & x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2x), & x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \bar{\gamma})(x),$$

and we see that  $F_t$  is continuous since at  $x = \frac{1}{2}$ , we have

$$\gamma(2x) = \gamma(1) = \bar{\gamma}(0) = \bar{\gamma}(1-2x),$$

hence we see that  $F_t$  is the [homotopy](#) between  $\gamma \cdot \bar{\gamma}$  and  $c$ .

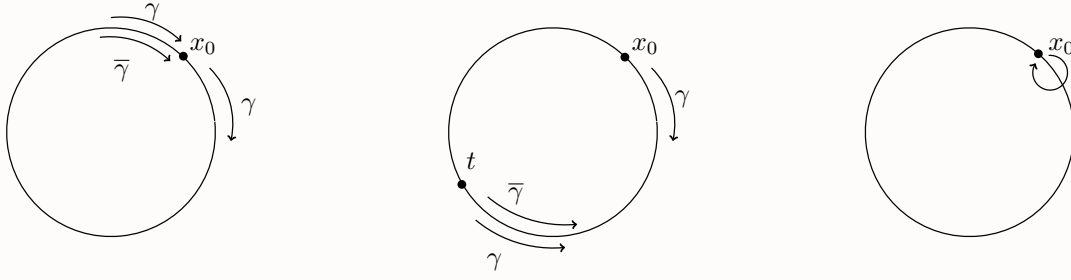


Figure 2.1: Illustration of  $F_t$ . Intuitively, the path  $\gamma \cdot \bar{\gamma}$  is  $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\bar{\gamma}} x_0$ . But now,  $F_t$  is  $x_0 \xrightarrow{\gamma} t \xrightarrow{\bar{\gamma}} x_0$ . We can think of this homotopy as *pulling back* the turning point along the original path.

**Theorem 2.2.1.** If  $X$  is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

**Proof.** To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if  $[h \cdot \gamma \cdot \bar{h}] = [h \cdot \gamma' \cdot \bar{h}]$ , then since we know that  $h^{-1} = \bar{h}$ , hence in the fundamental group  $\pi_1(X, x_0)$ , we see that

$$\bar{h} \cdot h \cdot \gamma \cdot \bar{h} \cdot h = \bar{h} \cdot h \cdot \gamma' \cdot \bar{h} \cdot h. \Rightarrow \gamma = \gamma'$$

as we desired.

- onto. We see that for every  $\alpha \in \pi_1(X, x_0)$ , there exists a  $\gamma \in \pi_1(X, x_0)$  such that

$$\gamma = \bar{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)$$

since  $h \cdot \gamma \cdot \bar{h} = \alpha$ .<sup>a</sup>

We then see that the fundamental group of  $X$  does not depend on the choice of basepoint, only on the choice of the path component of the basepoint. If  $X$  is path-connected, it now makes sense to refer to *the* fundamental group of  $X$  and write  $\pi_1(X)$  for the abstract group (up to isomorphism).

<sup>a</sup>Notice that this is indeed the case, one can verify this by the fact that  $h: x_0 \rightarrow x_1$  and  $\bar{h}: x_1 \rightarrow x_0$ .

**Remark.** We see that we can write  $\pi_1(X)$  up to isomorphism given this result.

**Exercise.** Composition of paths is well-defined on homotopy classes  $\text{rel}\{0, 1\}$ .

**Exercise.** If  $X$  is a contractible space, then  $X$  is path-connected and  $\pi_1(X)$  is trivial.

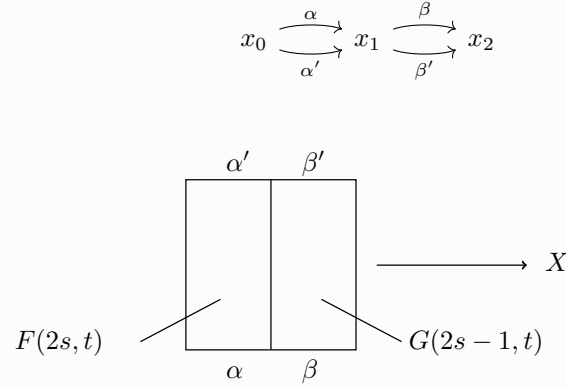
The followings are the properties about homotopy path. They are useful when we introduce fundamental groupoid.

**Lemma 2.2.1.** Given  $x_0, x_1, x_2 \in X$ ,  $\alpha, \alpha'$  are two paths from  $x_0$  to  $x_1$ , and  $\beta, \beta'$  are two paths from  $x_1$  to  $x_2$ . If  $\langle \alpha \rangle = \langle \alpha' \rangle$ ,  $\langle \beta \rangle = \langle \beta' \rangle$ , then  $\langle \alpha \cdot \beta \rangle = \langle \alpha' \cdot \beta' \rangle$ .

**Proof.** Given  $\alpha \underset{F}{\simeq} \alpha' \text{ rel}\{0, 1\}$ ,  $\beta \underset{G}{\simeq} \beta' \text{ rel}\{0, 1\}$ , then we want to prove

$$\alpha \cdot \beta \simeq \alpha' \cdot \beta' \text{ rel}\{0, 1\}.$$

This is done by using [homotopy](#)  $H: I \times I \rightarrow X$  such that it combines  $F(2s, t)$  and  $G(2s - 1, t)$ .

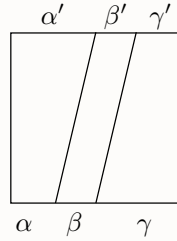


■

**Lemma 2.2.2.** Let  $x_0, x_1, x_2, x_3 \in X$ ,  $\alpha$  is a [path](#) from  $x_0$  to  $x_1$ ,  $\beta$  is a [path](#) from  $x_1$  to  $x_2$ ,  $\gamma$  is a [path](#) from  $x_2$  to  $x_3$ . Then

$$\langle (\alpha \cdot \beta) \cdot \gamma \rangle = \langle \alpha \cdot (\beta \cdot \gamma) \rangle.$$

**Proof.** We can write out the [homotopy](#) by the following diagram.

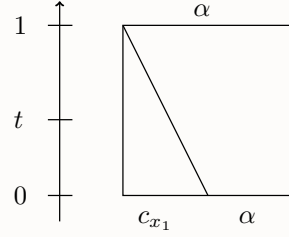


■

**Lemma 2.2.3.** Let  $X$  be a topological space, and  $x_0 \in X$ . Then for every [path homotopy](#)  $\langle \alpha \rangle$  from  $x_1$  to  $x_2$ , we have

$$\langle c_{x_1} \cdot \alpha \rangle = \langle \alpha \rangle = \langle \alpha \cdot c_{x_2} \rangle.$$

**Proof.** We only need to prove  $c_{x_1} \cdot \alpha \simeq \alpha \text{ rel}\{0, 1\}$ . The [homotopy](#) can be written out explicitly by the following diagram.

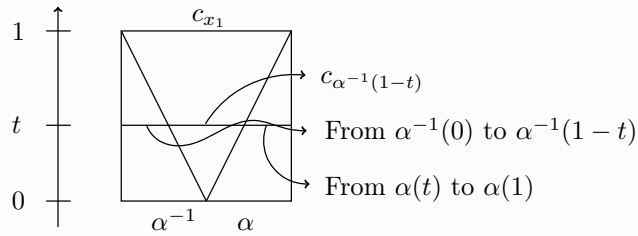


■

**Lemma 2.2.4.** For every path homotopy  $\langle \alpha \rangle$  from  $x_1$  to  $x_2$ , then

$$\langle \alpha \cdot \alpha^{-1} \rangle = \langle c_{x_1} \rangle, \quad \langle \alpha^{-1} \cdot \alpha \rangle = \langle c_{x_2} \rangle.$$

**Proof.** For the first case, we have the following diagram.



The second case follows similarly.

■

## 2.2.2 Fundamental Groupoid

This section is not covered in class, but it's a useful concept. The idea is that after giving Definition 2.2.1, we see that we actually create a **fundamental group** at **every** point in  $X$ , furthermore, when we use Theorem 2.2.1 if  $X$  is **path-connected**, we actually **lose** some information about this space. Here is how we can store all the information.

**Notation** (Constant loop). We denote  $c_x$ , where  $x \in X$  such that

$$\begin{aligned} c_x &: [0, 1] \rightarrow X \\ t &\mapsto x \end{aligned}$$

as a *constant loop*.

**Definition 2.2.2** (Groupoid). A **category**  $\mathcal{C}$  is a *groupoid* if any **morphisms** in  $\mathcal{C}$  is and isomorphism.

**Remark.** We'll soon see that for any topological space  $x$ , Definition 2.2.1 defines a **groupoid**, denoted by  $\Pi(X)$ .

**Definition 2.2.3** (Fundamental groupoid). Let  $X$  denotes the space, then the **category**  $\Pi(X)$  is a *fundamental groupoid* of  $X$  such that

- $\text{Ob}(\Pi(X)) := X$

- $\text{Hom}(\Pi(X)) : \forall p, q \in \text{Ob}(\Pi(X)) = X$ ,

$$\text{Hom}_{\Pi(X)}(p, q) := \{\text{Paths from } p \text{ to } q\} / \sim.$$

- Composition: For every  $p, q, r \in \text{Ob}(\Pi(X)) = X$ ,

$$\begin{aligned} \circ : \text{Hom}_{\Pi(X)}(p, q) \times \text{Hom}_{\Pi(X)}(q, r) &\rightarrow \text{Hom}_{\Pi(X)}(p, r) \\ (\langle \alpha \rangle, \langle \beta \rangle) &\mapsto \langle \beta \rangle \circ \langle \alpha \rangle := \langle \alpha \cdot \beta \rangle. \end{aligned}$$

- Identity: For every  $p \in \text{Ob}(\Pi(X)) = X$ , we define  $1_p := \langle c_p \rangle \in \text{Hom}_{\Pi(X)}(p, p)$  be the constant loop based at  $p$  such that for every  $\langle \alpha \rangle \in \text{Hom}_{\Pi(X)}(p, q)$ ,

$$\langle \alpha \rangle \circ \text{id}_p = \text{id}_q \circ \langle \alpha \rangle = \langle \alpha \rangle.$$

- Associativity: Given  $p, q, r, s \in \text{Ob}(\Pi(X)) = X$ , with the paths

$$p \xrightarrow{\langle \alpha \rangle} q \xrightarrow{\langle \beta \rangle} r \xrightarrow{\langle \gamma \rangle} s$$

Then

$$\langle \gamma \rangle \circ (\langle \beta \rangle \circ \langle \alpha \rangle) = (\langle \gamma \rangle \circ \langle \beta \rangle) \circ \langle \alpha \rangle.$$

**Proof.** Note that in [Definition 2.2.3](#), we need to show some of the definitions is indeed well-defined, and we also need to show that  $\Pi(X)$  is actually a [groupoid](#).

- Composition: Since if  $\alpha \simeq \alpha', \beta \simeq \beta'$ , we have

$$\alpha \cdot \beta \simeq \alpha' \cdot \beta'$$

from [Lemma 2.2.1](#).

- Identity: It follows that

$$\langle \alpha \rangle \circ \text{id}_p = \langle c_p \cdot \alpha \rangle = \langle \alpha \rangle$$

from [Lemma 2.2.3](#). The left identity can be shown similarly.

- Associativity: It's trivial in the sense that all the [homotopy](#) can be easily derived from [Lemma 2.2.2](#).

Additionally, from [Lemma 2.2.4](#), we see that given  $\alpha$  is a [path](#) from  $p$  to  $q$ , then

$$\begin{cases} \langle \alpha^{-1} \cdot \alpha \rangle &= \langle c_q \rangle =: \text{id}_q \\ \langle \alpha \cdot \alpha^{-1} \rangle &= \langle c_p \rangle =: \text{id}_p. \end{cases}$$

Furthermore, since  $\langle \alpha^{-1} \cdot \alpha \rangle = \langle \alpha \rangle \circ \langle \alpha^{-1} \rangle$  and  $\langle \alpha \cdot \alpha^{-1} \rangle = \langle \alpha^{-1} \rangle \circ \langle \alpha \rangle$ , hence this means  $\Pi(X)$  is indeed a [groupoid](#). ■

**Remark.** Assume  $\mathcal{C}$  is a [groupoid](#), then for every  $x \in \text{Ob}(\mathcal{C})$ , we can define

$$\cdot : \text{Hom}_{\mathcal{C}}(x, x) \times \text{Hom}_{\mathcal{C}}(x, x) \rightarrow \text{Hom}_{\mathcal{C}}(x, x)$$

such that

$$(f, g) \mapsto f \cdot g := g \circ f.$$

We can prove that

$$(\text{Hom}_{\mathcal{C}}(x, x), \cdot)$$

defines a group  $\text{Aut}_{\mathcal{C}}(x)$  called the *isotropy group* of  $\mathcal{C}$  at  $x$ .

**Exercise.** For every  $x, y \in \text{Ob}(\mathcal{C})$ , if there exists  $f \in \text{Hom}_{\mathcal{C}}(x, y)$ , then  $f$  induces

$$f_*: \text{Aut}_{\mathcal{C}}(x) \xrightarrow{\cong} \text{Aut}_{\mathcal{C}}(y),$$

where  $f_*$  is a group homomorphism.

**Remark.** For every  $p \in X = \text{Ob}(\Pi(X))$ , we have

$$\text{Aut}_{\Pi(X)}(p) = \pi_1(X, p).$$

Firstly, since they're the same in the sense of **set**:

$$\text{Aut}_{\Pi(X)}(p) = \text{Hom}_{\Pi(X)}(p, p) = \{\text{Loops in } X \text{ based at } p\} / \sim = \pi_1(X, p).$$

Hence, we only need to verify their group composition agrees. But this is trivial, since for every two  $\langle \alpha \rangle, \langle \beta \rangle \in \text{Aut}_{\Pi(X)}(p)$ ,

$$\underbrace{\langle \alpha \rangle \cdot \langle \beta \rangle}_{\text{Composition from } \text{Aut}_{\Pi(X)}} = \langle \beta \rangle \circ \langle \alpha \rangle = \underbrace{\langle \alpha \cdot \beta \rangle}_{\text{Composition from } \pi_1}.$$

This implies that [Theorem 2.2.1](#) is just a particular example as a **groupoid**.

## Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

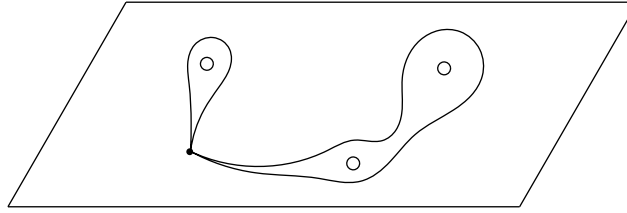


Figure 2.2: [Fundamental Group](#) is basically a *hole detector*!

### 2.3 Calculations with $\pi_1(S^n)$

Let's start with a basic but important theorem.

**Theorem 2.3.1** (The fundamental group of  $S^1$ ). The fundamental group of  $S^1$  is

$$\pi_1(S^1) \cong \mathbb{Z},$$

and this identification is given by the [paths](#)

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))].$$

**Proof.** With the help of [covering spaces](#) and the theorems build around which, we can define

$$\begin{aligned} p: \mathbb{R} &\rightarrow S^1, & x &\mapsto e^{2\pi i x}, \\ \varphi: \mathbb{Z} &\rightarrow \pi_1(S^1, 1), & n &\mapsto \langle p \circ \gamma_n \rangle, \end{aligned}$$

where  $p$  defined above is a [covering map](#). We need to show that this is well-defined.

From the definition of  $\varphi$ , we see that it's a homomorphism. But we also need to show



- $\varphi$  is a surjection. This is shown by [Corollary 3.1.1](#), specifically in the case of [path](#).
- $\varphi$  is an injection. This is shown by [Corollary 3.1.1](#), specifically in the case of [homotopy of paths](#).

■

**Remark.** Intuitively, this winds around  $S^1$   $n$  times. The key to this proof was to understand  $S^1$  via the [covering space](#)  $\mathbb{R} \rightarrow S^1$ . We will talk about [covering spaces](#) much later.

**Theorem 2.3.2.** Given  $(X, x_0)$  and  $(Y, y_0)$ , then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\left[ \begin{array}{l} r: I \rightarrow X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{array} \right] \mapsto (r_X, r_Y).$$

**Proof.** Let  $Z \xrightarrow{f} X \times Y$  with  $z \mapsto (f_X(z), f_Y(z))$ . Then we have

$$f \text{ continuous} \Leftrightarrow f_X, f_Y \text{ are continuous.}$$

Now, apply above to

- [Paths](#)  $I \rightarrow X \times Y$ .
- [Homotopies of paths](#)  $I \times I \rightarrow X \times Y$ .

■

**Corollary 2.3.1** (The fundamental group of  $S^k$ ). The torus  $T \cong S^1 \times S^1$  has [fundamental group](#)  $\pi_1(T) \cong \mathbb{Z}^2$ . Additionally, for a  $k$ -torus

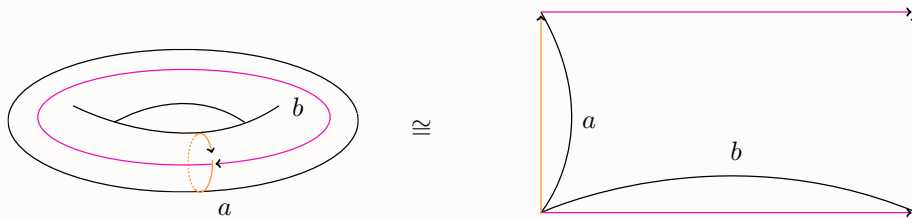
$$\underbrace{S^1 \times S^1 \times \dots \times S^1}_{k \text{ times}} = (S^1)^k,$$

the [fundamental group](#) is then  $\mathbb{Z}^k$ , i.e.

$$\pi_1((S^1)^k) \cong \mathbb{Z}^k.$$

**Proof.** Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

**Remark.** One way to think of the  $k$ -torus is as a  $k$ -dimensional cube with opposite  $(k-1)$ -dimensional faces identified by translation.

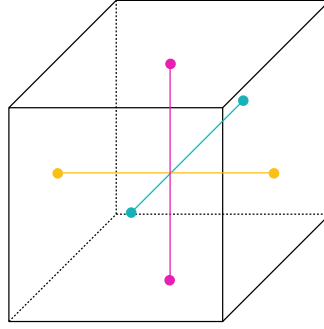


Figure 2.3: 3-torus with cube identified with parallel sides.

**Lemma 2.3.1.** Let  $f, g: X \rightarrow Y$  such that  $f \simeq_F g$ . Let  $x_0 \in X$ , then given

$$\begin{aligned} f_*: \pi_1(X, x_0) &\rightarrow \pi_1(Y, f(x_0)) \\ g_*: \pi_1(X, x_0) &\rightarrow \pi_1(Y, g(x_0)) \end{aligned}$$

with  $\gamma: [0, 1] \rightarrow Y$ ,  $t \mapsto F(x_0, t)$ ,

$$\begin{aligned} \gamma_*: \pi_1(Y, f(x_0)) &\rightarrow \pi_1(Y, g(x_0)) \\ \langle \alpha \rangle &\mapsto \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle, \end{aligned}$$

the following diagram commutes.

$$\begin{array}{ccc} & \pi_1(Y, f(x_0)) & \\ f_* \nearrow & \downarrow \gamma_* & \\ \pi_1(X, x_0) & & \pi_1(Y, g(x_0)) \\ g_* \searrow & & \end{array}$$

**Proof.** We want to prove that for any  $\langle \alpha \rangle \in \pi_1(X, x_0)$ , we have

$$\gamma_* \circ f_*(\langle \alpha \rangle) = g_*(\langle \alpha \rangle).$$

The left-hand side is just

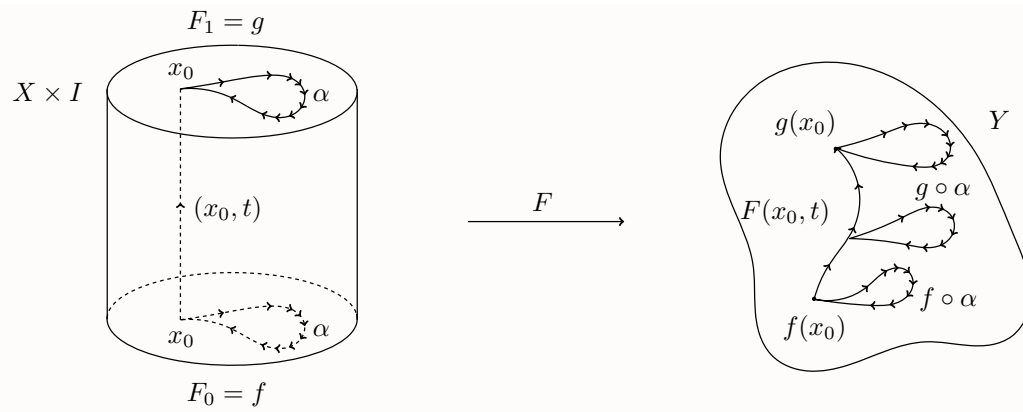
$$\gamma_* \circ f_*(\langle \alpha \rangle) = \gamma_*(\langle f \circ \alpha \rangle) = \langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \rangle,$$

while the right-hand side is just

$$g_*(\langle \alpha \rangle) = \langle g \circ \alpha \rangle.$$

That is, we now want to show

$$\langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \rangle = \langle g \circ \alpha \rangle.$$



We see that we can obtain a [homotopy](#)  $G: I \times I \rightarrow Y$  such that

$$G := F \circ (\alpha \times \text{id}),$$

where we define  $\alpha \times \text{id}$  by

$$\alpha \times \text{id}: I \times I \rightarrow X \times I, \quad (s, t) \mapsto (\alpha(s), t).$$

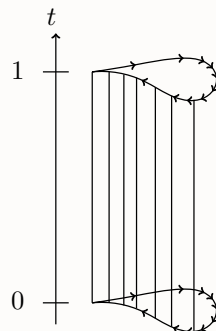
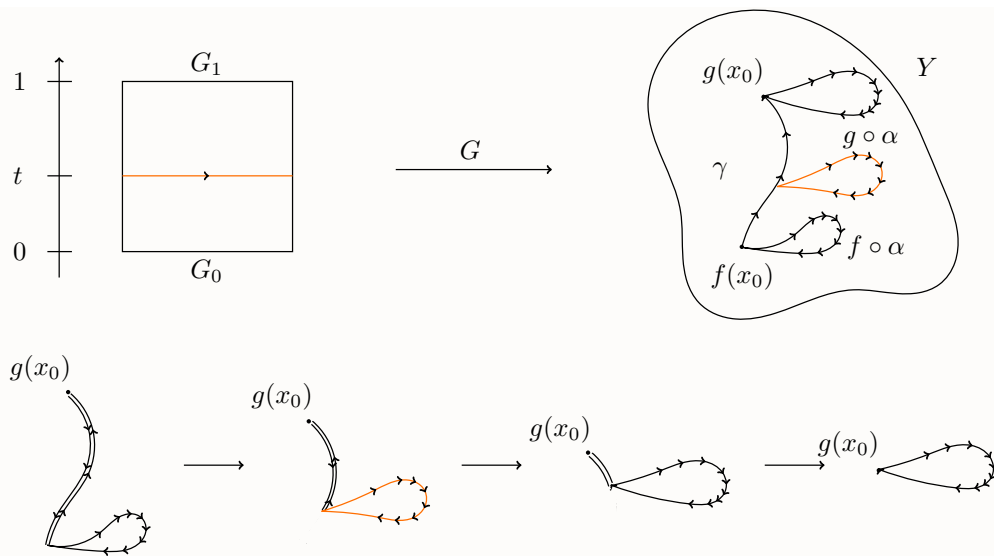
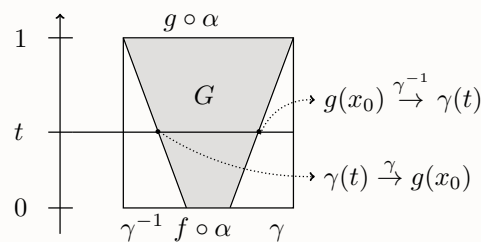


Figure 2.4:  $\alpha \times \text{id}$ 's image.

We see that by defining such  $G$ , we have the following.



To write out this [homotopy](#) explicitly, we see the following diagram.



**Theorem 2.3.3** (Fundamental group is a homotopy invariant). If  $X, Y$  are [homotopy equivalent](#), then their [fundamental groups](#) are isomorphic.

**Proof.**

HW.

**Remark.** This gives us a powerful tool to calculate  $\pi_1$ .

**Example.** We now see some examples.

1.  $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2.  $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong 0 \times \mathbb{Z} = \mathbb{Z}$  since

$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R},$$

which means that the generators are just loops around the hole intuitively.

## 2.4 Fundamental Group and Groupoid Define Functors

**Theorem 2.4.1** (Fundamental group defines a functor).  $\pi_1$  is a **functor** such that

$$\begin{aligned}\pi_1: \underline{\text{Top}}_* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0).\end{aligned}$$

While on a map  $f: X \rightarrow Y$  taking base point  $x_0$  to  $y_0$ ,  $\pi_1$  induces a map

$$\begin{aligned}f_*: \pi_1(X, x_0) &\rightarrow \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma]\end{aligned}$$

i.e.,

$$[f: X \rightarrow Y] \mapsto [f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))].$$

**Proof.** We need to check

- well-defined on **path homotopy** classes.
- $f_*$  is a group homomorphism.

$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \frac{1}{2}\right] \\ f(\beta(1-2s)), & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

- $(\text{id}_{(X, x_0)})_* = \text{id}_{\pi_1(X, x_0)}$
- $(f_* \circ g_*) = (f \circ g)_*$

$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \Rightarrow f_*(\gamma_*(\gamma)).$$

DIY

$$\begin{array}{ccc} (X, x_0) & \rightsquigarrow & \pi_1(X, x_0) \\ f \downarrow & & \downarrow f_* \\ (Y, y_0) & \rightsquigarrow & \pi_1(Y, y_0) \end{array}$$

■

**Remark.** We usually write  $f_*$  if it's a **covariant functor**, while writing  $f^*$  if it's a **contravariant functor**.

**Remark.** We see that the construction of **fundamental group** is actually constructing a **functor**. Specifically,

$$\pi_1: \underline{\text{Top}}_* \rightarrow \underline{\text{Gp}}$$

such that

- on **objects**:

$$\forall (X, x_0) \in \text{Ob}(\underline{\text{Top}}_*), \quad \pi_1(X, x_0) = \text{fundamental group based at } x_0.$$

- on **morphisms**:

$$\forall f: (X, x_0) \rightarrow (Y, y_0), \quad \pi_1(f) = f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).$$

Our initial motivation is to construct a topological invariant, but we see that using  $\pi_1$ , we need an additional **base point**. But as you already imagined, the **fundamental groupoid** actually is a **functor** as well.

Before we proceed further, we need to see the **category of groupoid**, denoted by  $\underline{\text{Gpd}}$ .

**Definition 2.4.1** (Category of groupoid). The *category of groupoid*, denoted as  $\underline{\text{Gpd}}$ , contains the following data.

- $\text{Ob}(\underline{\text{Gpd}})$ : **groupoids**.
- $\text{Hom}(\underline{\text{Gpd}})$ : **functors** between **groupoids**.
- Composition: For every  $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \in \text{Ob}(\underline{\text{Gpd}})$ ,

$$\mathfrak{X} \xrightarrow{F} \mathfrak{Y} \xrightarrow{G} \mathfrak{Z}$$

then  $G \circ F: \mathfrak{X} \rightarrow \mathfrak{Z}$  is a **functor** defined as

- on **objects**:  $\forall X \in \text{Ob}(\mathfrak{X})$ ,  

$$G \circ F(X) := G(F(X)).$$
- on **morphisms**:  $\forall X, Y \in \text{Ob}(\mathfrak{X})$  and  $f: X \rightarrow Y$ ,  

$$G \circ F(f) := G(F(f)).$$

- Identity. For every **groupoid**  $\mathfrak{X}$ , we define  $\text{id}_{\mathfrak{X}}: \mathfrak{X} \rightarrow \mathfrak{X}$ , where
  - $\forall X \in \text{Ob}(\mathfrak{X})$ ,  $\text{id}_{\mathfrak{X}}(X) = X$
  - $\forall f \in \text{Hom}(\mathfrak{X})$ ,  $\text{id}_{\mathfrak{X}}(f) = f$ .
- Associativity. Since the composition is defined based on two **functors**,<sup>a</sup> this holds trivially.

<sup>a</sup>For example, given  $\mathfrak{X} \xrightarrow{F} \mathfrak{Y} \xrightarrow{G} \mathfrak{Z}$ .

**Proof.** We need to show that the composition is well-defined. Specifically, we need to check

- $G \circ F(\text{id}_X) = \text{id}_{G \circ F(X)}$ , since

$$G \circ F(\text{id}_X) = G(F(\text{id}_X)) = G(\text{id}_{F(X)}) = \text{id}_{G(F(X))} = \text{id}_{G \circ F(X)}.$$

- Given  $X_1, X_2, X_3 \in \text{Ob}(\mathfrak{X})$  and

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

we want to show  $G \circ F(g \circ f) = G \circ F(g) \circ G \circ F(f)$ . Firstly, since  $G$  is a **functor**, hence

$$G \circ F(g) \circ G \circ F(f) = G(F(g)) \circ G(F(f)) = G(F(g) \circ F(f)).$$

Again, since  $F$  is a functor, so we further have

$$G \circ F(g) \circ G \circ F(f) = G(F(g \circ f)) = G \circ F(g \circ f).$$

■

**Theorem 2.4.2** (Fundamental groupoid defines a functor).  $\Pi$  is a **functor** such that

$$\Pi: \underline{\text{Top}} \rightarrow \underline{\text{Gpd}},$$

where

- on **objects**: For every  $X \in \text{Ob}(\underline{\text{Top}})$ ,

$$X \mapsto \Pi(X).$$

- on **morphisms**: for every  $X, Y \in \text{Ob}(\underline{\text{Top}})$ ,  $f: X \rightarrow Y$ , define a **functor**

$$\Pi(f): \Pi(X) \rightarrow \Pi(Y)$$

such that

- on **objects**: For every  $p \in \text{Ob}(\Pi(X)) = X$ ,  $\Pi(f)(p) = f(p)$ . i.e.,

$$\Pi(f): \underbrace{\text{Ob}(\Pi(X))}_X \rightarrow \underbrace{\text{Ob}(\Pi(Y))}_Y.$$

- on **morphisms**: For every  $\langle \alpha \rangle \in \text{Hom}_{\Pi(X)}(p, q)$ , define

$$\Pi(f)(\langle \alpha \rangle) := \langle f \circ \alpha \rangle \in \text{Hom}_{\Pi(Y)}(f(p), f(q)).$$

**Proof.** We need to check that the defined **functor**  $\Pi(f)$  satisfies

- $\Pi(f)(\text{id}_p) = \text{id}_{f(p)}$ . Indeed, since

$$\Pi(f)(\text{id}_p) = \Pi(f)(\langle c_p \rangle) = \langle f \circ c_p \rangle = \langle c_{f(p)} \rangle = \text{id}_{f(p)}.$$

- For every  $p, q, r \in X = \text{Ob}(\Pi(X))$ ,

$$p \xrightarrow{\langle \alpha \rangle} q \xrightarrow{\langle \beta \rangle} r$$

we want to show  $\Pi(f)(\langle \beta \rangle \circ \langle \alpha \rangle) = \Pi(f)(\langle \beta \rangle) \circ \Pi(f)(\langle \alpha \rangle)$ . Indeed, since

$$\Pi(f)(\langle \beta \rangle \circ \langle \alpha \rangle) = \Pi(f)(\langle \alpha \cdot \beta \rangle) = \langle f \circ (\alpha \cdot \beta) \rangle,$$

and

$$\Pi(f)(\langle \beta \rangle) \circ \Pi(f)(\langle \alpha \rangle) = \langle f \circ \beta \rangle \circ \langle f \circ \alpha \rangle = \langle (f \circ \alpha) \cdot (f \circ \beta) \rangle.$$

Since  $\langle f \circ (\alpha \cdot \beta) \rangle = \langle (f \circ \alpha) \cdot (f \circ \beta) \rangle$ , hence  $\Pi(f)$  is well-defined.

Now, we need to prove the same thing for  $\Pi$ , namely  $\Pi$  satisfies

- $\Pi(\text{id}_X) = \text{id}_{\Pi(X)}$  for all  $X \in \text{Ob}(\underline{\text{Top}})$ . This is trivial since

$$\Pi(\text{id}_X): \Pi(X) \rightarrow \Pi(X),$$

- on **objects**:  $p \mapsto \text{id}_X(p) = p$ .
- on **morphisms**:  $p \xrightarrow{\langle \alpha \rangle} q \mapsto \langle \text{id}_X \circ \alpha \rangle = \langle \alpha \rangle$ .

- For all  $X, Y, Z \in \text{Ob}(\underline{\text{Top}})$ ,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

then  $\Pi(g \circ f) = \Pi(g) \circ \Pi(f)$ . The diagrams are as follows.

$$\Pi(g \circ f): \Pi(X) \rightarrow \Pi(Z)$$

and

$$\Pi(X) \xrightarrow{\Pi(f)} \Pi(Y) \xrightarrow{\Pi(g)} \Pi(Z)$$

We see that this equality is in the sense of **functor**, hence we consider

- on **objects**: For every  $p \in \text{Ob}(\Pi(X)) = X$ ,  $\Pi(g \circ f)(p) = g \circ f(p)$  and

$$\Pi(g) \circ \Pi(f)(p) = \Pi(g)(\Pi(f)(p)) = \Pi(g)(f(p) = g(f(p))),$$

hence they're the same.

- on **morphisms**: For all  $\langle \alpha \rangle \in \text{Hom}_{\Pi(X)}(p, q)$ ,
  - \*  $\Pi(g \circ f)(\langle \alpha \rangle) = \langle (g \circ f) \circ \alpha \rangle$ .
  - \*  $\Pi(g) \circ \Pi(f)(\langle \alpha \rangle) = \Pi(g) \left( \underbrace{\Pi(f)(\langle \alpha \rangle)}_{\langle f \circ \alpha \rangle} \right) = \langle g \circ (f \circ \alpha) \rangle$ .

We see that they're the same. ■

## Lecture 10: Seifert-Van Kampen Theorem

The goal is to compute  $\pi_1(X)$  where  $X = A \cup B$  using the data

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$$\pi_1(A), \pi_1(B), \pi_1(A \cap B).$$

## 2.5 Free Product

### 2.5.1 Free Product

We first introduce a definition.

**Definition 2.5.1 (Free product).** Given some collections of groups  $\{G_\alpha\}_\alpha$ , the *free product*, denoted by  $*_\alpha G_\alpha$  is a group such that

- Elements: **Words** in  $\{g : g \in G_\alpha \text{ for any } \alpha\}$  modulo by the equivalence relation generated by

$$wg_i g_j v \sim w(g_i g_j) v$$

when both  $g_i, g_j \in G_\alpha$ . Also, for the identity element  $\text{id} = e_\alpha \in G_\alpha$  for any  $\alpha$  such that

$$we_\alpha v \sim wv.$$

Specifically,

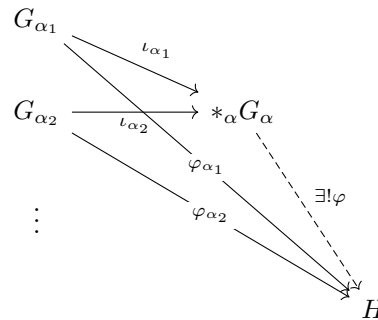
$$*_\alpha G_\alpha := \{\text{words in } \{G_\alpha\}_\alpha\} / \sim.$$

- Operation: Concatenation of **words**.

**Remark.** In particular, we have the following universal property of  $*_\alpha G_\alpha$ . For every  $\alpha$ , there is a  $\iota_\alpha$  such that

$$\iota_\alpha : G_\alpha \rightarrow *_\alpha G_\alpha, \quad g \mapsto \bar{g},$$

where  $\iota_\alpha$  is a group homomorphism obviously. Further,  $(*_\alpha G_\alpha, \iota_\alpha)$  satisfies the following property: For every group  $H$  and a group homomorphism  $\varphi_\alpha : G_\alpha \rightarrow H$  for all  $\alpha$ , there exists a unique group homomorphism  $\varphi : *_\alpha G_\alpha \rightarrow H$  such that  $\varphi \circ \iota_\alpha = \varphi_\alpha$ , i.e., the following diagram commutes.





**Proof.** The proof is straightforward. Firstly, we define  $w = \overline{g_1 g_2 \dots g_n} \in *_\alpha G_\alpha$ ,  $g_i \in G_{\alpha_i}$ ,

$$\varphi(w) := \varphi_{\alpha_1}(g_1) \dots \varphi_{\alpha_n}(g_n).$$

Now, we just need to check

- It's well-defined, since  $\varphi_\alpha$  is a group homomorphism.
- $\varphi$  is a group homomorphism.
- $\varphi \circ \iota_\alpha = \varphi_\alpha$ .
- Such  $\varphi$  is unique. Suppose there exists another  $\psi: *_\alpha G_\alpha \rightarrow H$ , then

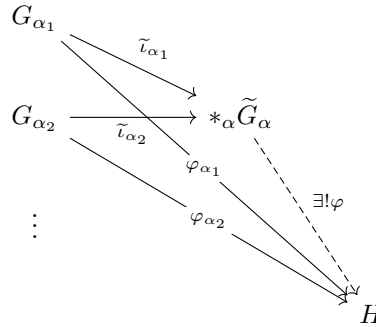
$$\psi \circ \iota_\alpha = \varphi_\alpha \Rightarrow \forall_{g \in G_\alpha} \psi(\bar{g}) = \varphi_\alpha(g),$$

But then for every  $w = \overline{g_1 g_2 \dots g_n} \in *_\alpha G_\alpha$ ,  $g_i \in G_{\alpha_i}$ , we have

$$\psi(w) = \psi(\overline{g_1 \dots g_n}) = \psi(\bar{g}_1) \dots \psi(\bar{g}_n) = \varphi_{\alpha_1}(\bar{g}_1) \dots \varphi_{\alpha_n}(\bar{g}_n),$$

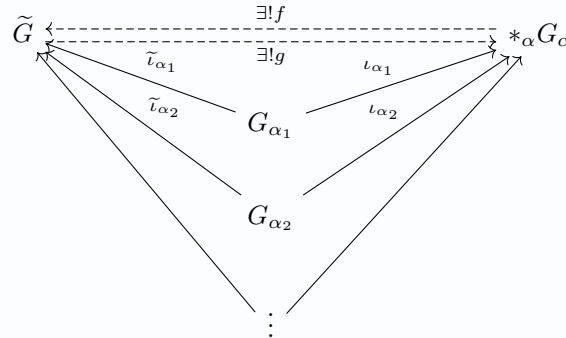
which is just  $\varphi$ . ■

**Remark.** We further claim that this universal property determines such [free product](#) uniquely. i.e., assume there are another group  $\tilde{G}$  and  $\tilde{\iota}_\alpha: G_\alpha \rightarrow \tilde{G}$ . Assume  $(\tilde{G}, \tilde{\iota}_\alpha)$  also satisfies the following property: For every group  $H$  and group homomorphism  $\varphi_\alpha: G_\alpha \rightarrow H$ , then there exists a unique group homomorphism  $\varphi: \tilde{G} \rightarrow H$  such that the following diagram commutes.



Then,  $\tilde{G} \cong *_\alpha G_\alpha$ .

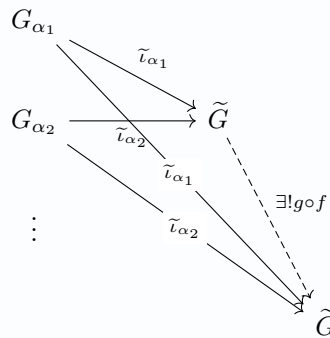
**Proof.** Assume  $(\tilde{G}, \tilde{\iota}_\alpha)$  satisfies the universal property mentioned above. Then from the universal property and viewing  $\tilde{G}$  and  $*_\alpha G_\alpha$  as  $H$  separately, we obtain the following diagram.



We claim that

$$g \circ f = \text{id}, \quad f \circ g = \text{id}.$$

To see this, we simply apply the same observation, for example,



where  $g \circ f$  comes from the previous diagram. But notice that  $\text{id}$  let the diagram commutes also, and since it's unique, hence  $g \circ f = \text{id}$ . Similarly, we have  $f \circ g = \text{id}$ . ■

If you're careful enough, you may find out that all we're doing is just writing out a specific example of [Lemma 1.6.1](#)! Indeed, this is exactly the construction of a [free group](#).

**Definition 2.5.2 (Fibered coproduct).** Given a [category](#)  $\mathcal{C}$ , let  $f: Z \rightarrow X$ ,  $g: Z \rightarrow Y$ . The *fibered coproduct between  $f$  and  $g$*  is the data  $(W, p_1, p_2)$ , where  $W \in \text{Ob}(\mathcal{C})$ ,  $p_1: X \rightarrow W$ ,  $p_2: Y \rightarrow W$  satisfy the following.

- The diagram commutes.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \downarrow p_1 \\ Y & \xrightarrow{p_2} & W \end{array}$$

- For every  $u: X \rightarrow U$ ,  $v: Y \rightarrow U$  such that the following diagram commutes

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \downarrow p_1 \\ Y & \xrightarrow{p_2} & W \end{array} \quad \begin{array}{c} \searrow u \\ \vdots \\ \xrightarrow{\exists! h} \\ \searrow v \end{array} \quad \begin{array}{c} \\ \\ \\ U \end{array}$$

there exists a unique  $h: W \rightarrow U$  such that  $h \circ p_1 = u$ ,  $h \circ p_2 = v$ .

We say

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & W \end{array}$$

is a *Cocartesian* diagram.

**Exercise.** Prove that in a [category](#)  $\mathcal{C}$ , if the [fibered coproduct](#) of  $f$  and  $g$  exists

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \\ Y & & \end{array}$$

then such [fibered coproduct](#) is unique up to isomorphism.

**Remark.** If we reverse all the directions of [morphism](#), then we have so-called *fibred product*.

**Example.** Let  $\mathcal{C} = \underline{\text{Top}}$ , and let  $X \in \text{Ob}(\underline{\text{Top}})$ . Given  $X_0, X_1 \in X$ , and  $\text{int}(X_0) \cup \text{int}(X_1) = X$ , if we have

$$\begin{aligned} i_0: X_0 &\hookrightarrow X, & i_1: X_1 &\hookrightarrow X \\ j_0: X_0 \cap X_1 &\hookrightarrow X_0, & j_1: X_0 \cap X_1 &\hookrightarrow X_1, \end{aligned}$$

then

$$\begin{array}{ccc} X_0 \cap X_1 & \xrightarrow{j_0} & X_0 \\ j_1 \downarrow & & \downarrow i_0 \\ X_1 & \xrightarrow{i_1} & X \end{array}$$

is a [cocartesian](#) diagram.

**Proof.** All we need to show is that given a topological space  $Y \in \underline{\text{Top}}$  and  $f: X_0 \rightarrow Y, g: X_1 \rightarrow Y$  in  $\underline{\text{Top}}$ , we have

$$f \circ j_0 = g \circ j_1.$$

$$\begin{array}{ccccc} X_0 \cap X_1 & \xrightarrow{j_0} & X_0 & & \\ j_1 \downarrow & & \downarrow i_0 & \searrow f & \\ X_1 & \xrightarrow{i_1} & X & \xrightarrow{\exists! h} & Y \\ & \searrow g & & & \end{array}$$

We simply define  $h: X \rightarrow Y, x \mapsto h(x)$  such that

$$h(x) = \begin{cases} f(x), & \text{if } x \in X_0; \\ g(x), & \text{if } x \in X_1. \end{cases}$$

$h$  is clearly well-defined since the diagram commutes, so if  $x \in X_0 \cap X_1$ , then  $f(x) = g(x)$ . The only thing we need to show is that  $h$  is continuous. But this is obvious too since  $X = \text{int}(X_0) \cup \text{int}(X_1)$ , and

$$h|_{\text{int}(X_0)} = f|_{\text{int}(X_0)}, \quad h|_{\text{int}(X_1)} = g|_{\text{int}(X_1)}.$$

The uniqueness is trivial, hence this is indeed a [cocartesian](#) diagram. ■

**Example.** Let  $\mathcal{C} = \underline{\text{Top}}_*$ . Given  $p \in X_0 \cap X_1$ , where all other data are the same with the above example, we see that

$$\begin{array}{ccc} (X_0 \cap X_1, p) & \xrightarrow{j_0} & (X_0, p) \\ j_1 \downarrow & & \downarrow i_0 \\ (X_1, p) & \xrightarrow{i_1} & (X, p) \end{array}$$

is a [cocartesian](#) diagram.

**Example.** Let  $\mathcal{C} = \underline{\text{Gp}}$ . Given  $P, G, H \in \text{Ob}(\underline{\text{Gp}})$ , we claim that the [fibred coproduct](#) of  $i$  and  $j$  exists.

$$\begin{array}{ccc} P & \xrightarrow{i} & G \\ j \downarrow & & \\ & & H \end{array}$$

Consider  $G * H$  be the [free product](#) between  $G$  and  $H$ , with two inclusions

$$\iota_1: G \hookrightarrow G * H, \quad \iota_2: H \hookrightarrow G * H.$$

$$\begin{array}{ccc}
 P & \xrightarrow{i} & G \\
 j \downarrow & & \downarrow \iota_1 \\
 H & \xrightarrow{\iota_2} & G * H
 \end{array}$$

Let

$$N := \langle \{ \iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1} \mid x \in P \} \rangle,$$

we define

$$G *_p H = G * H / N.$$

$$\begin{array}{ccccc}
 P & \xrightarrow{i} & G & & \\
 j \downarrow & & \downarrow \iota_1 & \searrow \tau & \\
 H & \xrightarrow{\iota_2} & G * H & \xrightarrow{\pi} & G *_p H \\
 & \searrow \nu & & & \nearrow
 \end{array}$$

We claim that

$$\begin{array}{ccc}
 P & \xrightarrow{i} & G \\
 j \downarrow & & \downarrow \tau \\
 H & \xrightarrow{\nu} & G *_p H
 \end{array}$$

is a [cocartesian](#) diagram in  $\underline{\mathbf{Gp}}$ .

**Proof.** Firstly, since it's just an outer diagram from above, hence it commutes. So we only need to prove this diagram satisfies the second diagram. Given any group  $K$ , for every  $f: G \rightarrow K$ ,  $g: H \rightarrow K$  such that the following diagram commutes.

$$\begin{array}{ccc}
 P & \xrightarrow{i} & G \\
 j \downarrow & & \downarrow \tau \\
 H & \xrightarrow{\nu} & G *_p H \\
 & \searrow g & \nearrow f \\
 & & K
 \end{array}$$

(Note: A dashed arrow  $h$  from  $G *_p H$  to  $K$  is also shown in the original image, representing the map to be constructed.)

We want to prove that there exists a unique  $h: G *_p H \rightarrow K$  such that this diagram still commutes. The idea is simple, from the universal property of  $G * H$ , we see that there exists a unique  $\tilde{h}: G * H \rightarrow K$  such that

$$\tilde{h} \circ \iota_1 = f, \quad \tilde{h} \circ \iota_2 = g.$$

$$\begin{array}{ccccc}
 P & \xrightarrow{i} & G & & \\
 j \downarrow & & \downarrow \iota_1 & \searrow \tau & \\
 & & G * H & \xrightarrow{\pi} & G *_p H \\
 & \nearrow \iota_2 & \downarrow \nu & & \nearrow f \\
 H & \xrightarrow{\nu} & G *_p H & \xrightarrow{h} & G *_p H
 \end{array}$$

(Note: The diagram shows the factorization of  $\tilde{h}$  through  $\pi$  to get  $h$ . The label  $\exists! \tilde{h}$  is placed near the arrow from  $G * H$  to  $K$ .)

We see that we can actually factor  $\tilde{h}$  through  $\pi$ , as long as  $\ker(\tilde{h}) \supset \ker(\pi)$ . Now, since

$$\ker(\pi) = \langle \{ \iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1} \mid x \in P \} \rangle,$$

we see that the kernel of  $\pi$  is indeed in the kernel of  $\tilde{h}$  since for every  $x \in P$ ,

$$\tilde{h}(\iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1}) = \underbrace{\tilde{h} \circ \tau_1}_{f} \circ i(x) \cdot \underbrace{\tilde{h} \circ \iota_2}_{g} \circ j(x^{-1}) = 1,$$

which implies  $\ker(\tilde{h}) \supset \ker(\pi)$ .

$$\begin{array}{ccc} G * H & \xrightarrow{\pi} & K \\ \tilde{h} \downarrow & & \\ G *_p H & & \end{array}$$

We then see that there exists a unique  $h: G *_p H \rightarrow K$  such that the above diagram commutes. ■

## 2.5.2 Free Product with Amalgamation

After seeing the above examples, the following definition should make sense.

**Definition 2.5.3 (Free product with amalgamation).** If two groups  $G_\alpha$  and  $G_\beta$  have a common subgroup  $S_{\{\alpha,\beta\}}$ <sup>a</sup>, given two inclusion maps<sup>b</sup>  $i_{\alpha\beta}: S_{\{\alpha,\beta\}} \rightarrow G_\alpha$  and  $i_{\beta\alpha}: S_{\{\alpha,\beta\}} \rightarrow G_\beta$ , the *free product with amalgamation*  ${}_\alpha *_S G_\alpha$  is defined as  $*_\alpha G_\alpha$  modulo the normal subgroup generated by

$$\{i_{\alpha\beta}(s_{\{\alpha,\beta\}})i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \mid s_{\{\alpha,\beta\}} \in S_{\{\alpha,\beta\}}\},$$

Namely,<sup>c</sup>

$${}_\alpha *_S G_\alpha = {}^*_\alpha G_\alpha / \langle i_{\alpha\beta}(s_{\{\alpha,\beta\}})i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \rangle$$

and satisfies the universal property

$$\begin{array}{ccc} S & \xrightarrow{i_{\alpha\beta}} & G_\alpha \\ i_{\beta\alpha} \downarrow & & \downarrow \\ G_\beta & \longrightarrow & G_\alpha *_S G_\beta \\ & \searrow & \downarrow \exists! \\ & & X \end{array}$$

<sup>a</sup>In general, we don't need  $S_{\{\alpha,\beta\}}$  to be a subgroup.

<sup>b</sup>We don't actually need  $i_{\alpha\beta}, i_{\beta\alpha}$  to be inclusive as well.

<sup>c</sup>i.e.,  $i_{\alpha\beta}(s)$  and  $i_{\beta\alpha}(s)$  will be identified in the quotient.

**Remark.** We see that

- We can then write out words such as  $g_\alpha \cdot s \cdot g_\beta$  for  $s \in S$ , and view  $s$  as an element of  $G_\alpha$  or  $G_\beta$ . In fact, we can do this construction even when  $i_\alpha$  and  $i_\beta$  are not injective, though this means we are not working with a subgroup.
- Aside, in Top, the same universal property defines union

$$\begin{array}{ccc} A \cap B & \xleftarrow{i_\alpha} & A \\ i_\beta \downarrow & & \downarrow \\ B & \longrightarrow & A \cup B \\ & \searrow & \downarrow \exists! \\ & & X \end{array}$$

for  $A, B$  are open subsets and the inclusion of intersection.

## 2.6 Seifert-Van Kampen Theorem

With [Definition 2.5.3](#), we can now see the important theorem.

**Theorem 2.6.1 (Seifert-Van Kampen Theorem).** Given  $(X, x_0)$  such that  $X = \bigcup_{\alpha} A_{\alpha}$  with

- $A_{\alpha}$  are open and [path](#)-connected and  $\forall \alpha \ x_0 \in A_{\alpha}$
- $A_{\alpha} \cap A_{\beta}$  is [path](#)-connected for all  $\alpha, \beta$ .

Then there exists a surjective group homomorphism

$$*_\alpha: \pi_1(A_{\alpha}, x_0) \rightarrow \pi_1(X, x_0).$$

If we additionally have  $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$  where they are all [path](#)-connected for every  $\alpha, \beta, \gamma$ , then

$$\pi_1(X, x_0) \cong *_\alpha \pi_1(A_{\alpha} \cap A_{\beta}, x_0) \pi_1(A_{\alpha}, x_0)$$

associated to all maps  $\pi_a(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha}), \pi_1(A_{\beta})$  induced by inclusions of spaces. i.e.,  $\pi_1(X, x_0)$  is a quotient of the [free product](#)  $*_{\alpha} \pi_1(A_{\alpha})$  where we have

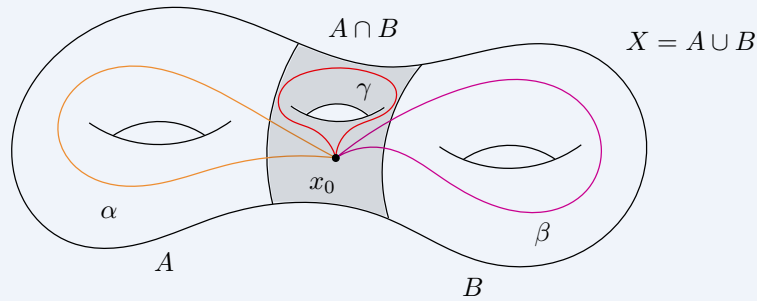
$$(i_{\alpha\beta})_*: \pi_1(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha})$$

which is induced by the inclusion  $i_{\alpha\beta}: A_{\alpha} \cap A_{\beta} \rightarrow A_{\alpha}$ . We then take the quotient by the normal subgroup generated by

$$\{(i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_{\alpha} \cap A_{\beta})\}.$$

We'll defer the [proof](#) of [Theorem 2.6.1](#) until we get familiar with this theorem.

**Example.** We first see a great visualization of the [Theorem 2.6.1](#).



Intuitively we see the [fundamental group](#) of  $X$ , which is built by gluing  $A$  and  $B$  along their intersection. As the [fundamental group](#) of  $A$  and  $B$  glued along the [fundamental group](#) of their intersection. In essence,  $\pi_1(X, x_0)$  is the quotient of  $\pi_1(A) * \pi_1(B)$  by relations to impose the condition that loops like  $\gamma$  lying in  $A \cap B$  can be viewed as elements of either  $\pi_1(A)$  or  $\pi_1(B)$ .

**Remark.** We can use a more abstract way to describe [Theorem 2.6.1](#). Specifically, in the case that  $n = 2$ , i.e.,  $X = \bigcup_{i=1}^2 A_i$ , we let  $A_i =: X_i$ , then we have the following. The [functor](#)  $\pi_1: \underline{\text{Top}}_* \rightarrow \underline{\text{Gp}}$

maps the [cocartesian](#) diagram in  $\underline{\text{Top}}_*$  to a [cocartesian](#) diagram in  $\underline{\text{Gp}}$  as follows.

$$\begin{array}{ccccc}
 (X_0 \cap X_1, x_0) & \xrightarrow{j_0} & (X_0, x_0) & & \pi_1(X_0 \cap X_1, x_0) & \xrightarrow{(j_0)_*} & \pi_1(X_0, x_0) \\
 j_1 \downarrow & & \downarrow i_0 & \xrightarrow{\pi_1} & (j_1)_* \downarrow & & \downarrow (i_0)_* \\
 (X_1, x_0) & \xrightarrow{i_1} & (X, x_0) & & \pi_1(X_1, x_0) & \xrightarrow{(i_1)_*} & \pi_1(X, x_0)
 \end{array}$$

Then, simply from the property of [cocartesian](#) diagram, we see that

$$\pi_1(X, x_0) \cong \pi_1(X_0, x_0) *_{\pi_1(X_0 \cap X_1, x_0)} \pi_1(X_1, x_0).$$

Additionally, there is a more general version of [Theorem 2.6.1](#), which is defined on [groupoid](#). The theorem is stated in [Appendix A.1](#) with the proof.

With this more general version and the proof of which, we can apply it to [Theorem 2.6.1](#). But one question is that, the above proof works in  $\underline{\text{Gpd}}$  rather than in  $\underline{\text{Gp}}$ . We now see how to generalize a group to a [groupoid](#).

For any group  $G$ , we can define a [groupoid](#), denoted as  $G$  also, as follows.

- $\text{Ob}(G) = \{\text{pt}\}$ , a one point set.
- $\text{Hom}(G) = \{g \in G\}$ .
- Composition: We define

$$g \circ h := h \cdot g.$$

We see that the associativity of group elements implies the associativity of composition defined above, and since there is an identity element in  $G$ , hence we also have an identity [morphism](#), these two facts ensure that  $G$  is an [category](#).

Furthermore, since for every  $g \in G$ , there is a  $g^{-1} \in G$ , hence every [morphism](#) is an isomorphism, which implies  $G$  is a [groupoid](#).

With this, we see that we can view the following diagram in the [category](#) of [groupoid](#)  $\underline{\text{Gpd}}$ .

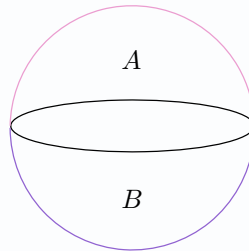
$$\begin{array}{ccc}
 \pi_1(X_0 \cap X_1, x_0) & \xrightarrow{(j_0)_*} & \pi_1(X_0, x_0) \\
 (j_1)_* \downarrow & & \downarrow (i_0)_* \\
 \pi_1(X_1, x_0) & \xrightarrow{(i_1)_*} & \pi_1(X, x_0)
 \end{array}$$

And to prove [Theorem 2.6.1](#), we only need to show this diagram is [cocartesian](#). This version of proof is given in [Appendix A.2](#).

## Lecture 11: Group Presentations

**Example** (Fundamental group of  $S^2$ ). We can use [Seifert Van Kampen Theorem](#) to compute the [fundamental group](#) of  $S^2$ .

**Proof.** We have the following [CW complex](#) structure on  $S^2$  as follows.



We see that  $\pi_1(S^2)$  must be a quotient of  $\pi_1(A) * \pi_1(B)$ , but since  $A, B \simeq D^2$ , we know that  $\pi_1(A)$  and  $\pi_1(B)$  are both zero groups, thus  $\pi_1(A) * \pi_1(B)$  is the zero group, and  $\pi_1(S^2)$  is also the

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zero group.

**Remark.** Note that the inclusion of  $A \cap B \rightarrow A$  induces the zero map  $\pi_1(A \cap B) \rightarrow \pi_1(A)$ , which cannot be an injection. In fact, we know that  $\pi_1(A \cap B) \cong \mathbb{Z}$  since  $A \cap B \simeq S^1$ .

■

**Example** (Fundamental group of torus.). We now find the [fundamental group](#) of a torus.

**Proof.** In the case of torus, consider the following [CW complex](#) structure.

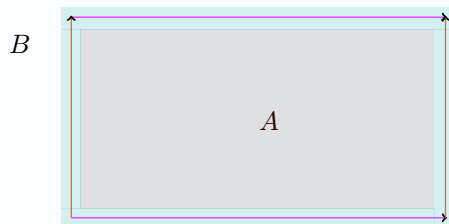


Figure 2.5:  $A$  is the interior, while  $B$  is the neighborhood of the boundary.

Now note that  $A \simeq D^2$  and  $B \simeq S^1 \vee S^1$ , and since it's a thickening of the two loops around the torus in both ways, this suggests the question of how do we find  $\pi_1(B)$ ? We grab a bit of knowledge from [Seifert Van Kampen Theorem](#) before we continue.

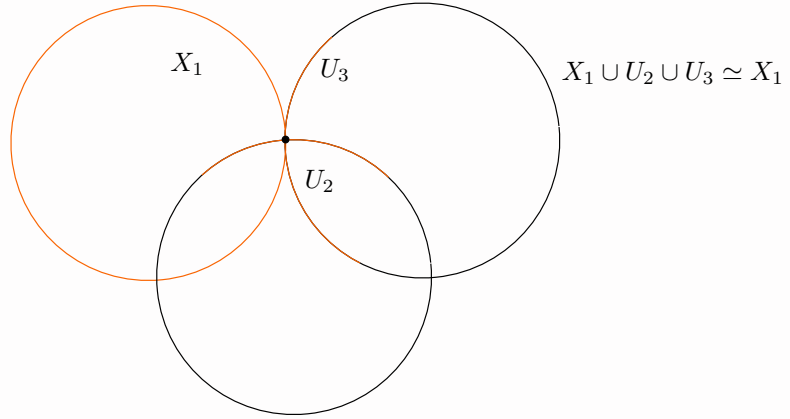
**Exercise.** Suppose we have [path-connected](#) spaces  $(X_\alpha, x_\alpha)$ , and we take their [wedge sum](#)  $\bigvee_\alpha X_\alpha$  by identifying the points  $x_\alpha$  to a single point  $x$ . We also suppose a mild condition for all  $\alpha$ , the point  $x_\alpha$  is a [deformation retract](#) of some neighborhood of  $x_\alpha$ .

For example, this doesn't work if we choose the *bad point* on the Hawaiian earring. Then we can use [Seifert Van Kampen Theorem](#) to show that

$$\pi_1 \left( \bigvee_\alpha X_\alpha, x \right) \cong \ast_\alpha \pi_1 (X_\alpha, x_\alpha).$$



**Answer.** If we denote



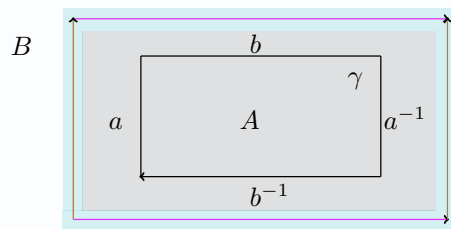
as  $C_n$ , then  $\pi_1(C_n) \cong F_n$ . Then we apply [Theorem 2.6.1](#) to  $A_\alpha = X_\alpha \cup_\beta U_\beta$ . Specifically, take  $A_\alpha = X_\alpha \cup_\beta U_\beta \simeq X_\alpha$ , where  $U_\beta$  is a neighborhood of  $x_\beta$  which [deformation retracts](#) to  $x_\beta$ . This makes  $A_\alpha$  open as desired. ■

**Corollary 2.6.1.** The [wedge sum](#) of circles  $\pi_1(\bigvee_{\alpha \in A} S^1) = *_\alpha \mathbb{Z}$  is a [free group](#) on  $A$ . In particular, when  $A$  is finite, the [fundamental group](#) of a bouquet of circles is the [free group](#) on  $|A|$ .

Returning to the [example of torus](#), we see that

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$

Further, we know that  $\pi_1(A \cap B) \rightarrow \pi_1(A)$  is the zero map. We need to understand  $\pi_1(A \cap B) \rightarrow \pi_1(B)$ . To do so we need to understand how we're able to identify  $\pi_1(S^1 \vee S^1)$  with  $F_2$  and how we identify  $\pi_1(S^1)$  with  $\mathbb{Z}$ . We update our [Figure 2.5](#) to talk about this.



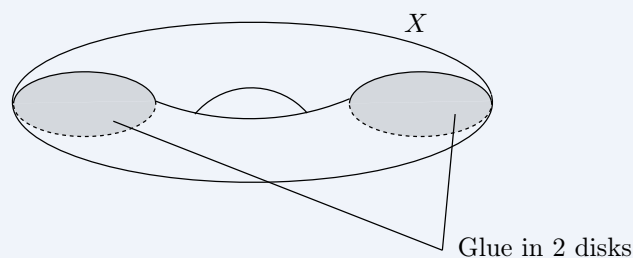
From this, we have

$$\begin{aligned} \pi_1(A \cap B) &\rightarrow \pi_1(B) \cong F_{a,b} \\ \gamma &\mapsto aba^{-1}b^{-1}. \end{aligned}$$

By [Seifert Van Kampen Theorem](#), we identify the image of  $\gamma$  in  $\pi_1(B)[aba^{-1}b^{-1}]$  with its image in  $\pi_1(A)$ , which is just trivial. Therefore, we have

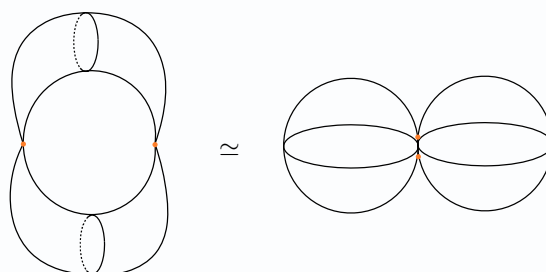
$$\pi_1(T^2) = F_{a,b} / \langle aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2. \quad \blacksquare$$

**Example.** Let's see the last example which illustrate the power of [Seifert Van Kampen Theorem](#). Start with a torus, and we glue in two disks into the hollow inside.

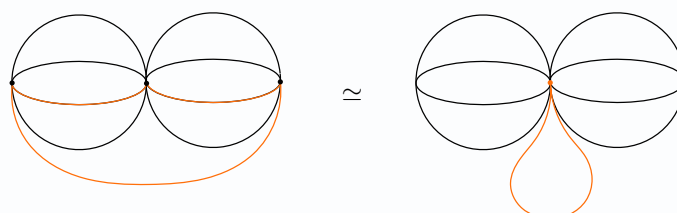


We'll call this space  $X$ , and our goal is to find  $\pi_1(X)$ .

**Proof.** We can place a [CW complex](#) structure on this space so that each disk is a [subcomplex](#). Then, we take [quotient](#) of each disk to a point without changing the [homotopy type](#), hence  $X$  is [homotopy](#) to



By the same property, we can expand one of those points into an interval, and then contract the [red path](#) as follows.



This is exactly  $S^2 \vee S^2 \vee S^1$ . With [Seifert Van Kampen Theorem](#), we have

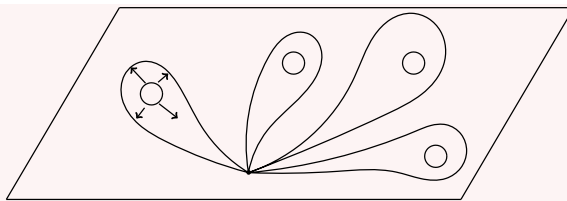
$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}.$$

■

**Exercise.** Consider  $\mathbb{R}^2 \setminus \{x_1, \dots, x_n\}$ , that is the plane punctured at  $n$  points. Then  $X \simeq \bigvee_n S^1$ , so then

$$\pi_1(X) \simeq F_n.$$

One way to do this is to convince yourself that you can do a [deformation retract](#) the plane onto the following [wedge](#).

Figure 2.6: Deformation retract  $X$  onto wedge.

## 2.7 Group Presentation

In order to go further, we introduce the concept of *group presentation*.

**Definition 2.7.1** (Group presentation). A *presentation*  $\langle S \mid R \rangle$  of a group  $G$  is

- $S$ : set of *generators*
- $R$ : set of *relators* (*words* in a generator and inverses)

such that

$$G \cong F_S / \langle R \rangle,$$

where  $\langle R \rangle$  is a subgroup normally generated by the elements of  $R$ .

**Definition 2.7.2** (Finite presentation). If  $S$  and  $R$  are both finite, then  $G = \langle S \mid R \rangle$  is a *finite presentation* if  $S, R$  are, and we say that  $G$  is *finitely presented*.

**Note.** One way to think about whether  $G$  is *finitely presented* is that if  $r$  is a *word* in  $R$  then  $r = 1$ , where  $1$  is the identity of  $G$ .

**Example.** We see that

1.  $F_2 = \langle a, b \mid \rangle$
2.  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \rangle / \overline{\{aba^{-1}b^{-1}\}}$
3.  $\mathbb{Z}/3\mathbb{Z} = \langle a \mid a^3 \rangle$
4.  $S_3 = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$

**Theorem 2.7.1.** Any group  $G$  has a *presentation*.

**Proof.** We first choose a generating set  $S$  for  $G$ . Notice that we can even choose  $S = G$  directly. From the universal property of *free group*, we see that there exists a surjective map  $\varphi: F_S \rightarrow G, s \mapsto s$ . Now, let  $R$  be the generating set for  $\ker(\varphi)$ , by the first isomorphism theorem<sup>a</sup>,  $G \cong F_S / \ker \varphi$ . In fact, we have  $G = \langle S \mid R \rangle$ .

Specifically,  $i: S \rightarrow G$  with  $\iota: S \rightarrow F_S$ , we have  $\varphi \circ \iota = i$ .

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F_S \\ & \searrow i & \downarrow \exists! \varphi \\ & & G \end{array}$$

<sup>a</sup>[https://en.wikipedia.org/wiki/Isomorphism\\_theorems](https://en.wikipedia.org/wiki/Isomorphism_theorems)

**Remark.** The advantages of using [group presentation](#) are that given  $G = \langle S \mid R \rangle$ , it's now easy to define a homomorphism  $\psi: G \rightarrow H$  given a map  $\varphi: S \rightarrow H$ ,  $\psi$  extends to a group homomorphism  $G \rightarrow H$  if and only if  $\psi$  vanishes on  $R$ , i.e.,  $\psi(r) = 0$  for all  $r \in R$ . We see an example to illustrate this.

**Example.** If we have  $G = \langle a, b \mid aba \rangle$ , a map  $\varphi: \{a, b\} \rightarrow H$  gives a group homomorphism if and only if

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H.$$

This essentially uses the universal property of quotients.

**Remark.** It's sometimes easy to calculate  $G^{\text{Ab}}$

$$G^{\text{Ab}} = \langle S \mid R, \text{commutators in } S \rangle.$$

**Example.** Suppose all relations in  $R$  are commutators, so  $R \subseteq [G, G]$ . Then,

$$G^{\text{Ab}} = (F_S)^{\text{Ab}} = \bigoplus_S \mathbb{Z}.$$

**Remark.** The disadvantages are that this is computationally **very difficult**.

**Example.** Given  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ , let

$$\psi: \{a, b\} \rightarrow H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1_H \in H.$$

Namely, this is a [presentation](#) of the trivial group, but this is entirely unclear.

## Lecture 12: Presentations for $\pi_1$ of CW Complexes

Let's first see an exercise.

2 Feb. 10:00

**Exercise.** Consider  $G_1 = \langle S_1 \mid R_1 \rangle$  and  $G_2 = \langle S_2 \mid R_2 \rangle$ . Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\} \rangle$
- $G_1 *_H G_2$  where  $f_1: H \rightarrow G_1$  and  $f_2: H \rightarrow G_2$ . Then we have

$$G_1 *_H G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{f_1(h)f_2(h)^{-1} \mid h \in H\} \rangle.$$

### 2.7.1 Presentations for $\pi_1$ of CW Complexes

For  $X$  a [CW complex](#), we have

1. A 1-dimensional [CW complex](#) has free  $\pi_1$  (call its generators as  $a_1, \dots, a_n$ ).
2. Gluing a 2-disk by its boundary along a word  $w$  in the generators *kills*  $w$  in  $\pi_1$ . We then get a [presentation](#) for  $\pi_1(X^2)$  given by

$$\langle a_1, \dots, a_n \mid w \text{ for each 2-cell in } X_2 \rangle.$$

3. Gluing in any higher dimensional cells along their boundary will not change  $\pi_1$ . That is, in a **CW complex**, we have  $\pi_1(X) = \pi_1(X^2)$ .

**Remark.** We can write the above more precise.

1. Find free generators  $\{a_i\}_{i \in I}$  for  $\pi_1(X^1)$ .
2. For each 2-disk  $D_\alpha^2$ , write attaching map as word  $w_\alpha$  in  $a_i$ . i.e.,

$$\pi_1(X^2) = \langle a_i \mid w_\alpha \rangle.$$

3.  $\pi_1(X) = \pi_1(X^2)$ .

**Example.** Given  $G = \mathbb{Z} / n\mathbb{Z} = \langle a \mid a^n \rangle$ , then we take a loop and then wind a 2-disk around the loop  $a$  for  $n$  times.

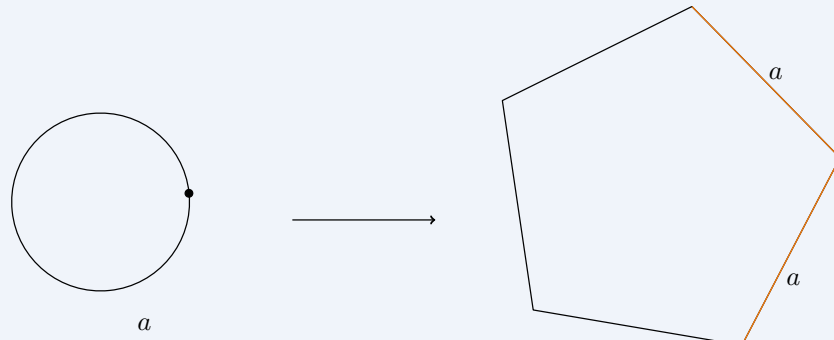


Figure 2.7: For  $G = \mathbb{Z} / n\mathbb{Z} = \langle a \mid a^n \rangle$ , we wind the boundary around  $a$  for  $n$  times.

We then see that given a group  $G$  with **presentation**  $\langle S \mid R \rangle$ , one can construct a 2-dimensional **CW complex** with  $\pi_1 = G$  by

- Set  $X^1 = \bigvee_{s \in S} S^1$
- For each relation  $r \in R$ , glue in a 2-disk along loops specified by the **word**  $r$ .

Every group is then  $\pi_1$  of some space.

**Theorem 2.7.2.** If  $X$  is a **CW complex** and  $\iota_1: X^1 \hookrightarrow X$  and  $\iota_2: X^2 \hookrightarrow X$ , then  $(\iota_1)_*$  surjects onto  $\pi_1$  and  $(\iota_2)_*$  is an isomorphism on  $\pi_1$ .

**Proof.** ■

HW

**Definition 2.7.3** (Graph, subgraph, tree, maximal tree). We import some topological definitions of graph theoretic concepts.

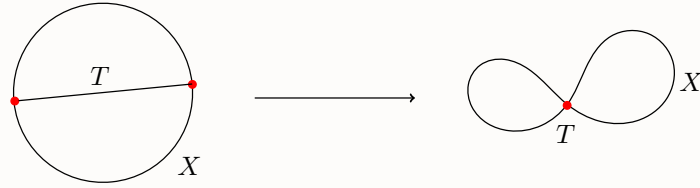
- A *graph* is a 1-dimensional **CW complex**.
- A *subgraph* is a **subcomplex**.
- A *tree* is a contractible **graph**.
- A **tree** in **graph**  $X$  (necessarily a **subgraph**) is *maximal* or *spanning* if it contains all the vertices.

**Theorem 2.7.3.** Every connected graph has a maximal tree. Every tree is contained in a maximal tree.

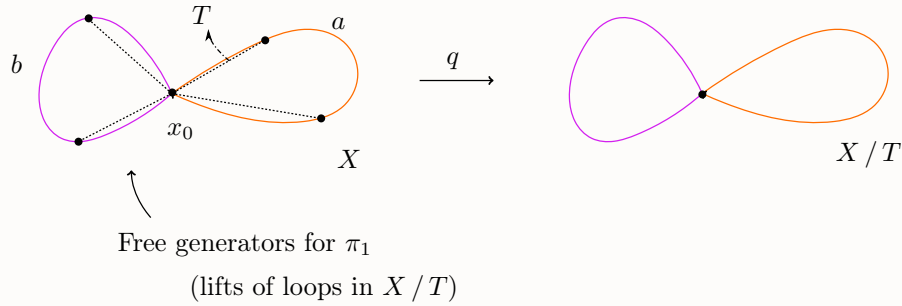
**Corollary 2.7.1.** Suppose  $X$  is a connected graph with basepoint  $x_0$ . Then  $\pi_1(X, x_0)$  is a free group.

Furthermore, we can give a presentation for  $\pi_1(X, x_0)$  by finding a spanning tree  $T$  in  $X$ . The generators of  $\pi_1$  will be indexed by cells  $e_\alpha \in X - T$ , and  $e_\alpha$  will correspond to a loop that passes through  $T$ , traverses  $e_\alpha$  once, then returns to the basepoint  $x_0$  through  $T$ .

**Proof.** The idea is simple.  $X$  is homotopy equivalent to  $X/T$  via previous work on the homework,  $T$  contains all the vertices, so the quotient has a single vertex. Thus, it is a wedge of circles, and each  $e_\alpha$  projects to a loop in  $X/T$ .



The current plan is to calculate the fundamental group of CW complexes. For now, we need to see that the fundamental group of a 1-skeleton (a graph) can be found by taking a maximal tree, and then quotienting out the space by the tree to get a wedge of circles.



We now prove that the maximal trees exist. Recall that  $X$  is a quotient of

$$X^0 \coprod_{\alpha} I_{\alpha}.$$

Each subset  $U$  is open if and only if it intersects each edge  $\bar{e}_\alpha$  in an open subset. A map  $X \rightarrow Y$  if and only if its restriction to each edge  $\bar{e}_\alpha$  is continuous. Now, take  $X_0$  to be a subgraph. Our goal is to construct a subgraph  $Y$  with

- $X_0 \subset Y \subset X$
- $Y$  deformation retracts to  $X_0$
- $Y$  contains all vertices of  $X$ .

So if we take  $X_0$  to be a vertex, then  $Y$  is our tree and we're done!

Our strategy now is to build a sequence  $X_0 \subset X_1 \subset \dots$  and correspondingly,  $Y_0 \subset Y_1 \subset \dots$ . We start with  $X_0$  and inductively define

$$X_i := X_{i-1} \bigcup \text{all edges } \bar{e}_\alpha \text{ with one or both vertices in } X_{i-1}.$$

We then see that  $X = \bigcup_i X_i$ .<sup>a</sup> Now, let  $Y_0 = X_0$ . By induction, we'll assume that  $Y_i$  is a **subgraph** of  $X_i$  such that

Check.

- $Y_i$  contains all vertices of  $X_i$ .
- $Y_i$  **deformation retracts** to  $Y_{i-1}$ .

We can then construct  $Y_{i+1}$  by taking  $Y_i$  and adding to it one edge to adjoin every vertex of  $X_{i+1}$ , namely

$$Y_{i+1} := Y_i \bigcup \text{one edge to adjoin every vertex of } X_i,$$

which is possible if we assume Axiom of Choice.

We then see that  $Y_{i+1}$  **deformation retracts** to  $Y_i$  by just smashing down each edge. Now, we can show that  $Y$  **deformation retracts** to  $Y_0 = X_0$  by performing the **deformation retraction** from  $Y_i$  to  $Y_{i-1}$  during the time interval  $[1/2^i, 1/2^{i-1}]$ . ■

<sup>a</sup>Hatcher[HPM02] do this by arguing the union on the right is both open and closed.

**Example.** Let

- $S^n$ : decompose into 2 open disks
- $A_1$ : neighborhood of top hemisphere
- $A_2$ : neighborhood of lower hemisphere

We see that  $A_1 \cap A_2 \simeq S^{n-1}$ , where we need  $n \geq 2$  to let  $S^{n-1}$  be connected. We then have

$$\pi_1(S^n) \cong 0 \underset{\pi_1(A_1 \cap A_2)}{*} 0 = 0.$$

On the other hand, if  $n \geq 3$ , then we see that

$$S^n = D^n \cup * / \sim.$$

Since 2-skeleton is a point, thus  $\pi_1(S^n) = 0$ .

## Lecture 13: Proof of Seifert-Van-Kampen Theorem

### 2.8 Proof of Seifert-Van-Kampen Theorem

4 Feb. 10:00

Let's start to prove Theorem 2.6.1.

**Proof.** The outline of the proof is the following. Let  $X = \bigcup_\alpha A_\alpha$  where  $A_\alpha$  are open, **path-connected** and contain the bluepoint  $x_0$ . We also must guarantee that  $A_\alpha \cap A_\beta$  is **path-connected**.

1. Since we have a map induced by the inclusions:

$$\Phi: \underset{\alpha}{*} \pi_1(A_\alpha, x_0) \rightarrow \pi_1(X, x_0).$$

We want to show that  $\phi$  is surjective. Take some  $\gamma: I \rightarrow X$ , then by the compactness of the interval  $I$ , we can show that there is a partition  $I$  with  $s_1 < \dots < s_n$  so that

$$\alpha|_{s_i, s_{i+1}} =: \alpha_i$$

has image in  $A_{\alpha_i}$  for some  $\alpha_i$ .<sup>a</sup> Specifically, since

- $A_\alpha$  is open for all  $\alpha$
- $I$  is compact,

then for all  $i$ , we choose a [path](#)  $h_i$  from  $x_0$  to  $\gamma(s_i)$  in  $A_{\sigma_{i-1}} \cap A_{\alpha_i}$ , using [path](#)-connectedness of the pairwise intersections. Now, take  $\gamma$  and write it as

$$\gamma = (\gamma_1 \cdot \bar{h}_1) \cdot (\bar{h}_1 \cdot \gamma_2) \cdot \dots \cdot (\gamma_{n-1} \cdot \bar{h}_{n-1}) \cdot (h_{n-1} \cdot \gamma_n).$$

Observe that each of these [paths](#) is fully contained in  $A_{\alpha_i}$ , so this implies that  $\gamma \in \text{Im}(\Phi)$ , therefore  $\Phi$  is surjective.

2. For the next step, we'll show that the second part of [Theorem 2.6.1](#). Assume that our triple intersections are [path](#)-connected. We want to show that  $\ker(\Phi)$  is generated by

$$(i_{\alpha\beta})_*(\omega)(i_{\beta\alpha})_*(\omega)^{-1},$$

where

$$i_{\alpha\beta}: A_\alpha \cap A_\beta \hookrightarrow A_\alpha$$

for all loops  $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$ .

Before we go further, we'll need some definition.

**Definition 2.8.1 (Factorization).** A *factorization* of a [homotopy](#) class  $[f] \in \pi_1(X, x_0)$  is a formal product

$$[f_1][f_2] \dots [f_\ell]$$

with  $[f_i] \in \pi_1(A_{\alpha_i}, x_0)$  such that

$$f \simeq f_1 \cdot f_2 \cdot \dots \cdot f_\ell.$$

<sup>a</sup>This is a good exercise for point-set topology.

We showed that every  $[f]$  has a [factorization](#) in step 1 already. Now we want to show that two [factorizations](#)

$$[f_1] \cdot \dots \cdot [f_\ell] \text{ and } [f'_1] \cdot \dots \cdot [f'_{\ell'}]$$

of  $[f]$  must be related by two moves:

- (a)  $[f_i] \cdot [f_{i+1}] = [f_i \cdot f_{i+1}]$  if  $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$ . Namely, the relation defining the [free product](#) of groups.
- (b)  $[f_i]$  can be viewed as an element of  $\pi_1(A_\alpha, x_0)$  or  $\pi_1(A_\beta, x_0)$  whenever

$$[f_i] \in \pi_1(A_\alpha \cap A_\beta, x_0).$$

This is the relation defining the [amalgamated free product](#).

Now, let  $F_t: I \times I \rightarrow X$  be a [homotopy](#) from  $f_1 \dots f_\ell$  to  $f'_1 \dots f'_{\ell'}$ , since they both represent  $[f]$ . We subdivide  $I \times I$  into rectangles  $R_{ij}$  so that

$$F(R_{ij}) \subseteq A_{\alpha_{ij}} =: A_{ij}$$

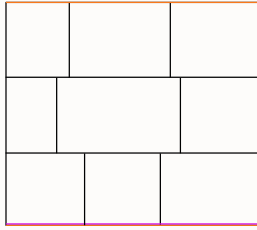
for some  $\alpha_{ij}$  using compactness. We also argue that we can perturb the corners of the squares so that a corner lies only in three of the  $A_\alpha$ 's indexed by adjacent rectangles.



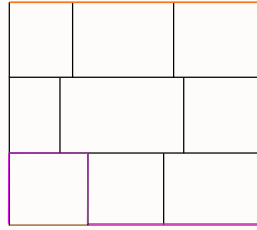
$A_{31}$	$A_{32}$	$A_{33}$
$A_{21}$	$A_{22}$	$A_{23}$
$A_{11}$	$A_{12}$	$A_{13}$

We also argue that we can set up our subdivision so that the partition of the top and bottom intervals must correspond with the two [factorizations](#) of  $[f]$ . We then perform our [homotopy](#) one rectangle at a time.

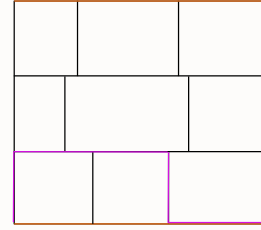
ending loop  $f'_1 \cdot \dots \cdot f'_{\ell'}$



starting loop  $f_1 \cdot \dots \cdot f_\ell$



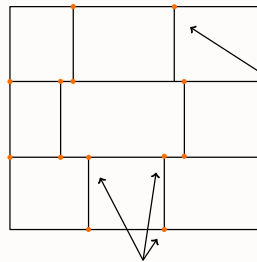
step 1



step 2

**Idea:** Argue that [homotoping](#) over a single rectangle has the effect of using allowable moves to modify the [factorization](#).

At each triple intersection, choose a [path](#) from  $f$  (corner) to  $x_0$  which lies in the triple intersection, so we use the assumption that the triple intersections are [path](#)-connected.



Choose [path](#)  $h$  from image of this corner to  $x_0$

In each intersection, choose a [path](#) from  $f$  (corner) to  $x_0$

Along the top and bottom, we make choices compatible with the two [factorizations](#). It's now an exercise to check that these choices result in [homotoping](#) across a rectangle gives a new [factorization](#) related by an allowable move.

■

# Chapter 3

## Covering Spaces

### Lecture 14: Covering Spaces Theory

#### 3.1 Lifting Properties

As always, we start with a definition.

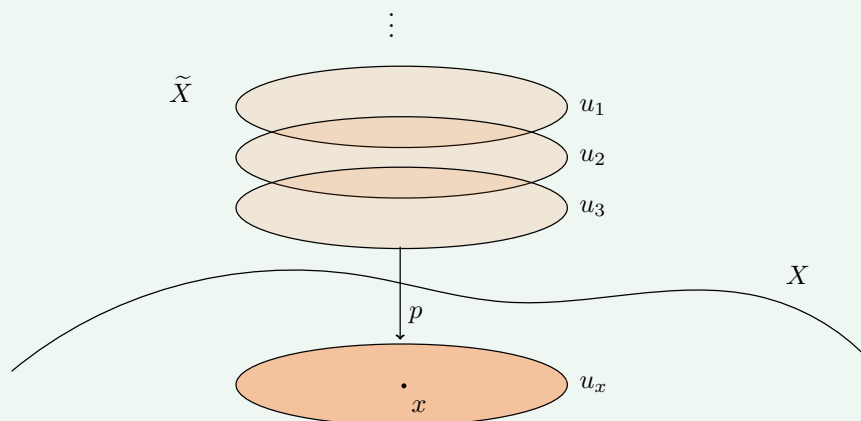
**Definition 3.1.1 (Covering space).** A *covering space*  $\tilde{X}$  of  $X$  is a space  $\tilde{X}$  and a map  $p: \tilde{X} \rightarrow X$  such that  $\forall x \in X \exists$  neighborhood  $u_x$  with  $p^{-1}(u_x)$  the disjoint union of open sets

$$\coprod_{\alpha} u_{\alpha}$$

such that

$$p|_{u_{\alpha}} : u_{\alpha} \rightarrow u_x$$

is a homeomorphism for every  $\alpha$ .



We sometimes call  $p$  as *covering map*.

Although we already investigate into [covering spaces](#) quite a lot in homework, but a terminology is still worth mentioning.

**Definition 3.1.2 (Evenly covered).** Let  $p: \tilde{X} \rightarrow X$  be a continuous map of spaces. Then an open subset  $U \subseteq X$  is called *evenly covered by  $p$*  if

$$p|_{V_i} : V_i \rightarrow U$$

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content...  
Things are  
in Hw.  
7 Feb. 10:00

is a homeomorphism.

We call the parts  $V_i$  of the partition  $\coprod_i V_i$  of  $p^{-1}(U)$  *slices*.

**Remark.** We see that  $p$  is a **covering map** if and only if every point  $x \in X$  has a neighborhood which is **evenly covered**.

We immediately have the following proposition.

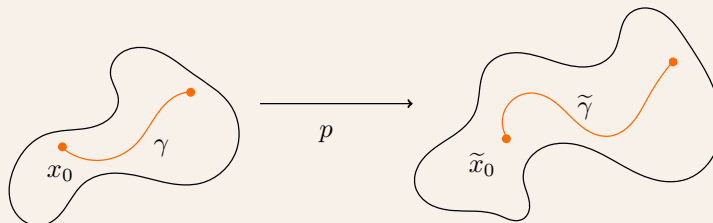
**Proposition 3.1.1** (Homotopy lifting property). The **covering spaces** satisfy the **homotopy lifting property** such that the following diagram commutes.

$$\begin{array}{ccc}
 X \times \{0\} & \xrightarrow{\tilde{F}_0} & \tilde{Y} \\
 \downarrow & \nearrow \exists! \tilde{F}_t & \downarrow p \\
 X \times I & \xrightarrow{F_t} & Y
 \end{array}$$

**Proof.** We already proved this in homework! ■

**Corollary 3.1.1** (Path lifting property). For each **path**  $\gamma: I \rightarrow X$  in  $X$ ,  $\tilde{x}_0 \in p^{-1}(\gamma(0))$  such that there exists a unique **lift**  $\tilde{\gamma}$  starting at  $\tilde{x}_0$ .

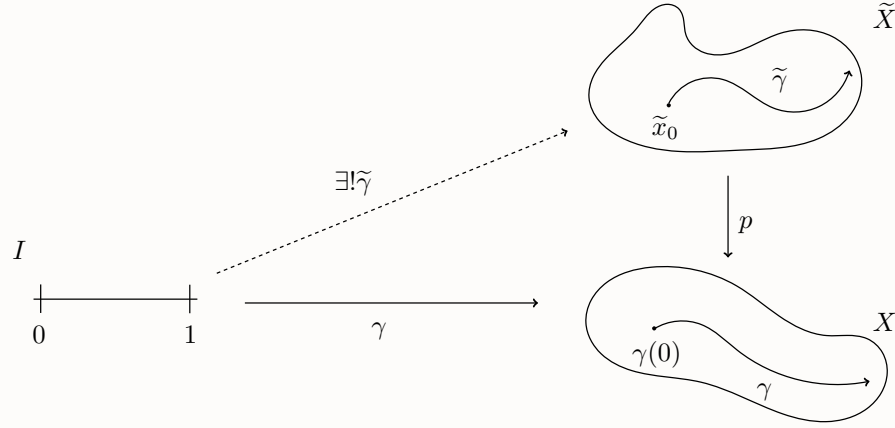
And for each **path homotopy**  $I \times I \rightarrow X$ , there exists a unique **path homotopy**  $\tilde{\gamma}: I \times I \rightarrow \tilde{X}$  starting at  $\tilde{x}_0$ .



**Note.** Though we can directly use **Proposition 3.1.1** to prove this, but we can see some insight by directly proving this.

We prove them separately.

**Lifting a path.** Assume that we have the following lift.



We first prove that a [path](#) will be [lifted](#) uniquely to a [path](#)  $\tilde{\gamma}$  from  $\tilde{x}_0$ . For every  $x \in X$ , there exists an open neighborhood  $U_x$  such that

$$p^{-1}(U_x) = \coprod_{\alpha} U_{x_{\alpha}},$$

where for every  $\alpha$ ,

$$p|_{U_{x_{\alpha}}} : U_{x_{\alpha}} \rightarrow U_x$$

is a homeomorphism. We see that  $\{U_x \mid x \in X\}$  is an open cover of  $X$ , hence

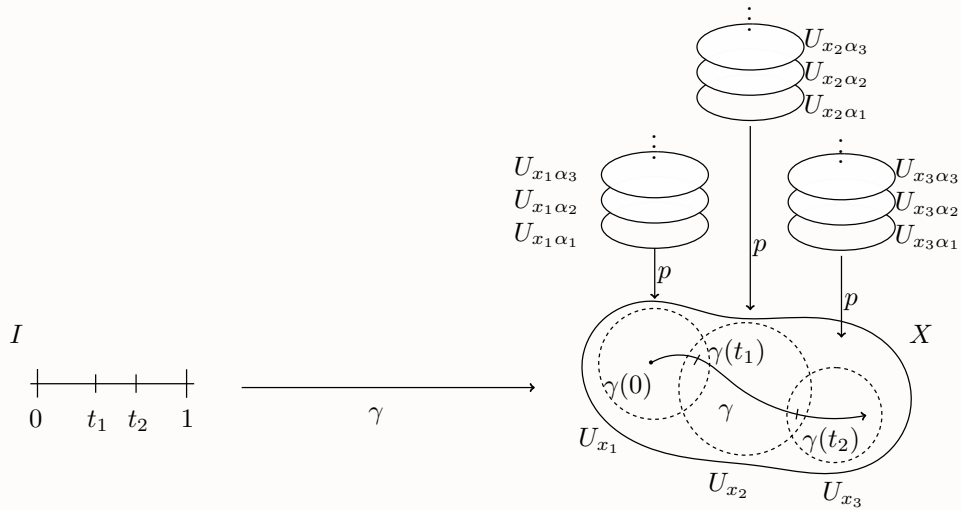
$$\{\gamma^{-1}(U_x) \mid x \in X\}$$

is an open cover of  $[0, 1]$ . Note that since  $[0, 1]$  is a compact metric space, from Lebesgue Lemma<sup>a</sup>, there exists a partition of  $[0, 1]$  such that

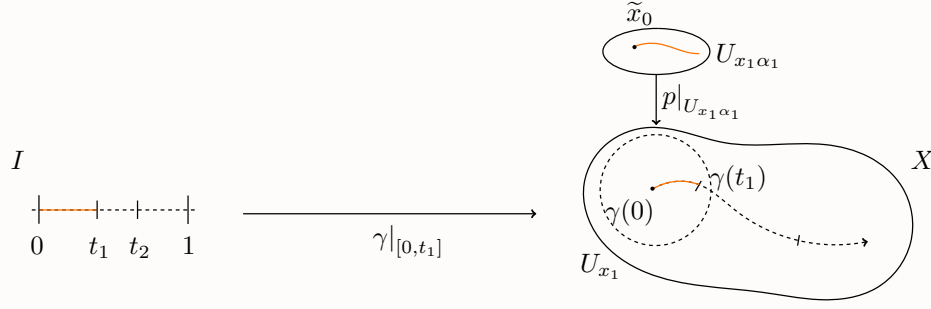
$$0 = t_0 < t_1 < \dots < t_k = 1$$

such that for every  $i$ ,  $[t_i, t_{i+1}] \subset \gamma^{-1}(U_x)$  for some  $x$ . Without loss of generality, we assume that  $[t_i, t_{i+1}] \subset \gamma^{-1}(U_{x_i})$ , i.e.,

$$\gamma([t_i, t_{i+1}]) \subset U_{x_i}.$$



Now, since  $p(\tilde{x}_0) = \gamma(0)$  for  $\gamma_0 \in U_{x_1}$  and  $\tilde{x}_0 \in p^{-1}(U_{x_1})$ , we may assume  $\tilde{x}_0 \in U_{x_1\alpha_1}$ . Consider [lifting](#) the first segment, namely  $\gamma([0, t_1])$ .



Specifically, let  $\tilde{\gamma}_1(t) = \left(p|_{U_{x_1 \alpha_1}}\right)^{-1} \circ \gamma(t)$  for  $0 \leq t \leq t_1$ , we see that

$$\tilde{\gamma}_1: [0, t_1] \rightarrow \tilde{X}$$

is a **lift** of  $\gamma|_{[0, t_1]}$  from  $\tilde{x}_0$ . We claim that this **lift** is unique. Consider there exists another **lift** from  $\tilde{x}_0$   $\tilde{\gamma}_1: [0, t_1] \rightarrow \tilde{X}$ , then since

**Proof.**  $\tilde{\gamma}_1(0) = \tilde{x}_0$

- $\tilde{\gamma}_1$  is continuous
- $\tilde{x}_0 \in U_{x_1 \alpha_1}$ ,

we see that  $\tilde{\gamma}_1([0, t_1]) \subset U_{x_1 \alpha_1}$ , which implies

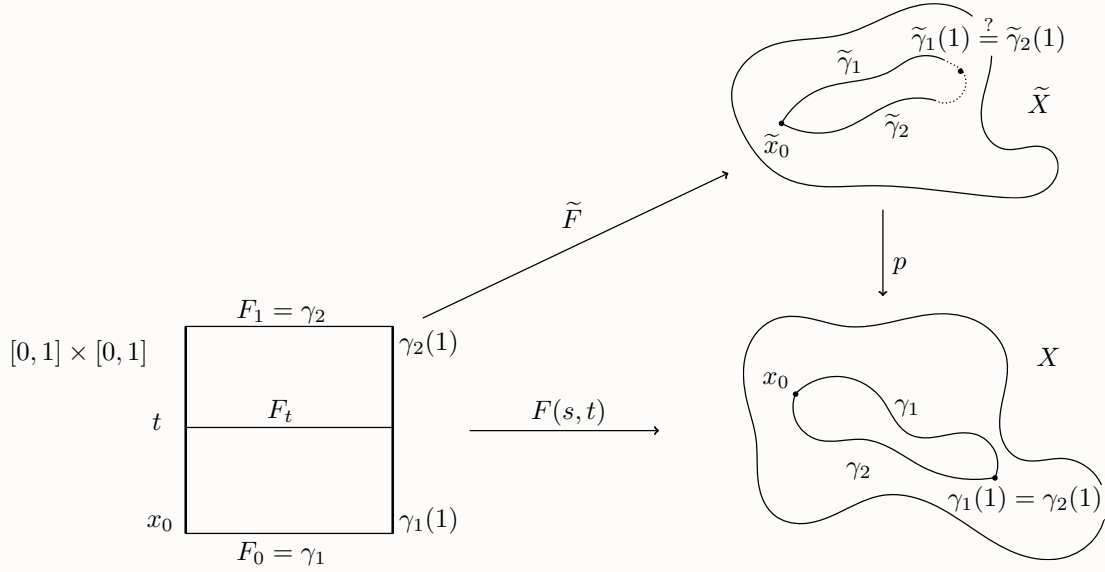
$$\begin{array}{ccc} [0, t_1] & \xrightarrow{\tilde{\gamma}_1} & U_{x_1 \alpha_1} \\ & \searrow \gamma|_{[0, t_1]} & \downarrow p|_{U_{x_1 \alpha_1}} \\ & & U_{x_1} \end{array} \Rightarrow \tilde{\gamma}_1 = \left(p|_{U_{x_1 \alpha_1}}\right)^{-1} \circ \gamma|_{[0, t_1]} = \tilde{\gamma}_1,$$

hence this **lift** is unique. Now, we see that we can simply repeat this argument, namely replacing  $t_i$  by  $t_{i+1}$ ,  $\tilde{\gamma}_i(t_i)$  by  $\tilde{\gamma}_{i+1}(t_{i+1})$  and so on. Since this partition is finite, hence in finitely many steps, we obtain a unique **path homotopy**  $\tilde{\gamma}$  by concatenating all  $\tilde{\gamma}_i$  starting at  $\tilde{x}_0$ .

**Lifting a path homotopy.** We now consider lifting a **path homotopy**. Consider

$$\gamma_1 \simeq_{\tilde{F}} \gamma_2 \text{ rel } \{0, 1\}$$

we'll show that  $\tilde{\gamma}_1 \simeq_{\tilde{F}} \tilde{\gamma}_2 \text{ rel } \{0, 1\}$  where  $p \circ \tilde{F} = F$ . Firstly, we denote  $x_0 := \gamma_1(0) = \gamma_2(0)$ , such that



We claim that it's sufficient to show that there exists a continuous  $\tilde{F}: I \times I \rightarrow \tilde{X}$  such that  $p \circ \tilde{F} = F$ , and  $\tilde{F}(\{0\} \times I) = \tilde{x}_0$ . It's because

$$p \circ \tilde{F}_0 = F_0 = \gamma_1, \quad p \circ \tilde{F}_1 = F_1 = \gamma_2$$

where  $\tilde{F}_0, \tilde{F}_1$  is  $\gamma_1, \gamma_2$ 's **lifting** starting at  $\tilde{x}_0$ , respectively. And since  $p \circ \tilde{F} = F$ , we have

$$p\left(\tilde{F}(\{1\} \times I)\right) = x_1 \Rightarrow \tilde{F}(\{1\} \times I) \subset p^{-1}(\{x_1\}),$$

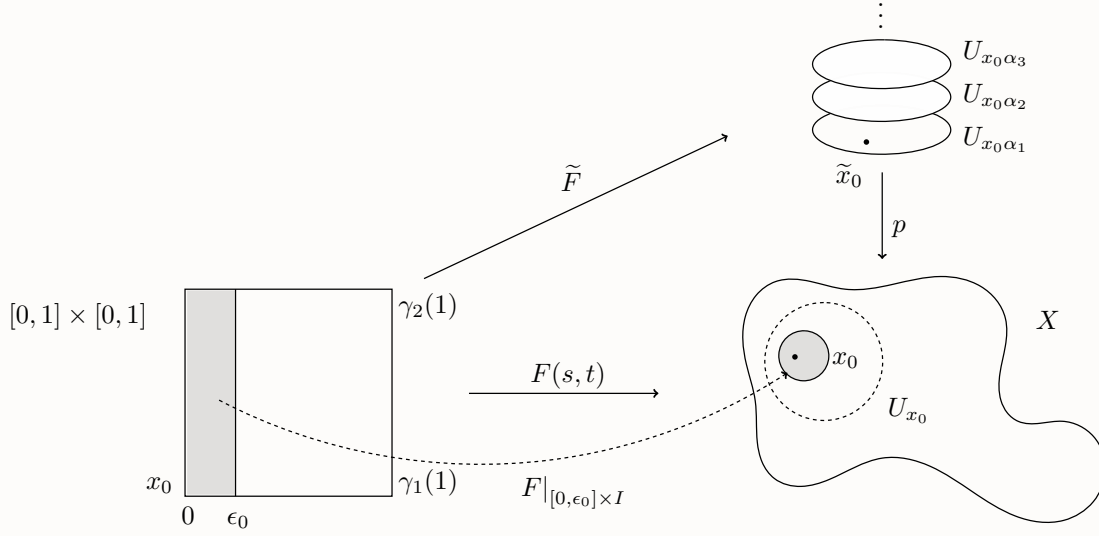
which implies  $\exists \tilde{x}_1 \in p^{-1}(\{x_1\})$  such that  $\tilde{F}(\{1\} \times I) = \tilde{x}_1$  since we know that  $p^{-1}(\{x_1\})$  is a discrete points set and  $\tilde{F}$  is assumed to be continuous, and  $\{1\} \times I$  is connected. We now show  $\tilde{F}$  exists.

We define

$$\begin{aligned} \tilde{F}: I \times I &\rightarrow \tilde{X} \\ (s, t) &\mapsto \tilde{F}_t(s), \end{aligned}$$

where  $\tilde{F}_t: [0, 1] \rightarrow \tilde{X}$  is a **lift** starting at  $\tilde{x}_0$  of  $F_t: [0, 1] \rightarrow X, s \mapsto F(s, t)$ . Obviously,  $p \circ \tilde{F} = F$  from the uniqueness of the **lift** of a path, and also,  $\tilde{F}(\{0\} \times I) = \tilde{x}_0$  holds trivially, hence we only need to show  $\tilde{F}$  is continuous.

1. We show that  $\exists \epsilon_0 > 0$  such that  $\tilde{F}|_{[0, \epsilon_0] \times I}$  is continuous.



Since  $F$  is continuous, we see that there exists an open neighborhood  $U_{x_0}$  of  $x_0$  such that  $p^{-1}(U_{x_0}) = \coprod_{\alpha} U_{x_0 \alpha}$ , where

$$p|_{U_{x_0 \alpha}} : U_{x_0 \alpha} \xrightarrow{\cong} U_{x_0}.$$

Since  $F^{-1}(U_{x_0})$  is an open set contain  $\{0\} \times I$ , there exists a  $\epsilon_0 > 0$  such that  $[0, \epsilon_0] \times I \subset F^{-1}(U_{x_0})$ ,<sup>b</sup> which implies

$$F([0, \epsilon_0] \times I) \subset U_{x_0}.$$

Note that  $x_0 \in U_{x_0}$  and  $p(\tilde{x}_0) = x_0$ , we may assume  $\tilde{x}_0 \in U_{x_0 \alpha_1}$ . Consider  $(p|_{U_{x_0 \alpha_1}})^{-1} \circ F|_{[0, \epsilon_0] \times I}$ , which is a [lift](#) of  $F|_{[0, \epsilon_0] \times I}$ . We claim that

$$(p|_{U_{x_0 \alpha_1}})^{-1} \circ F|_{[0, \epsilon_0] \times I} = \tilde{F}|_{[0, \epsilon_0] \times I}.$$

This is because for every  $t \in I$ ,

$$s \mapsto (p|_{U_{x_0 \alpha_1}})^{-1} \circ F|_{[0, \epsilon_0] \times I}(s, t)$$

is a [lift](#) starting at  $\tilde{x}_0$ ; also, for every  $t \in I$ ,

$$s \mapsto \tilde{F}|_{[0, \epsilon_0] \times I}(s, t)$$

is a [lift](#) of  $F_t$  starting at  $\tilde{x}_0$ . From the uniqueness of the [lift](#) of [paths](#), we see that they're equal. Note that this implies  $\tilde{F}$  is now continuous at  $[0, \epsilon_0] \times I$ , since  $F$  is continuous and  $p|_{U_{x_0 \alpha_1}}$  is a homeomorphism, hence continuous, then from

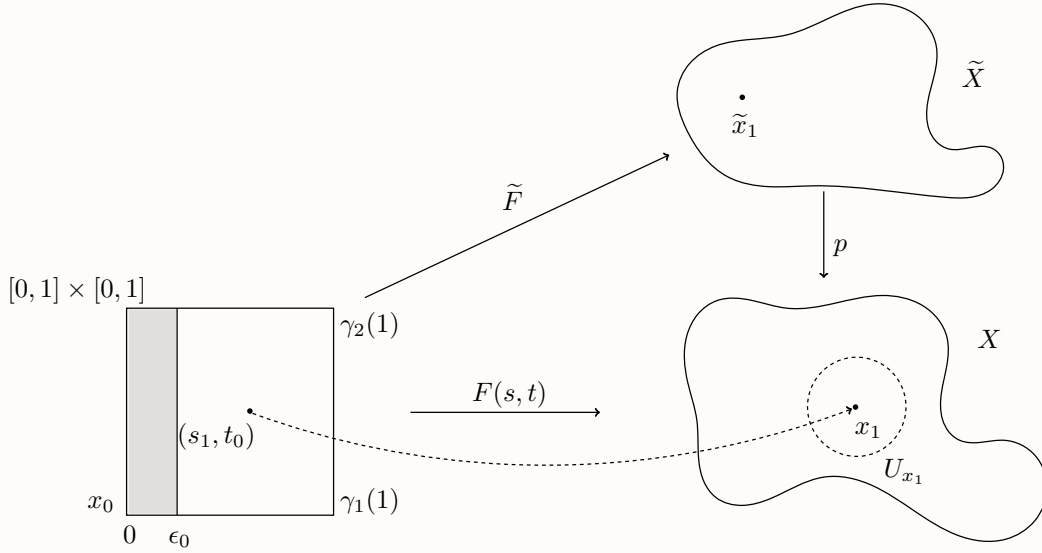
$$\tilde{F}|_{[0, \epsilon_0] \times I} = \underbrace{(p|_{U_{x_0 \alpha_1}})^{-1}}_{\text{continuous}} \circ \underbrace{F|_{[0, \epsilon_0] \times I}}_{\text{continuous}},$$

we see that  $\tilde{F}$  is indeed continuous at  $[0, \epsilon_0] \times I$ .

2. We now prove that  $\tilde{F}: I \times I \rightarrow \tilde{X}$  is continuous. Assume there exists  $(s_0, t_0) \in I \times I$  such that  $\tilde{F}$  is discontinuous at  $(s_0, t_0)$ . Then consider

$$0 < \epsilon_0 \leq \inf \underbrace{\left\{ s \mid \tilde{F} \text{ is discontinuous at } s, t_0 \right\}}_{\ni s_0 \Rightarrow \neq \emptyset} =: s_1,$$

where the first inequality is from the first step.



Let  $x_1 := F(s_1, t_0)$ ,  $\tilde{x}_1 := \tilde{F}(s_1, t_0)$ , then there exists an open neighborhood  $U_{x_1}$  in  $X$  such that  $x_1 \in U_{x_1} = \coprod_{\alpha} U_{x_1\alpha}$ , where

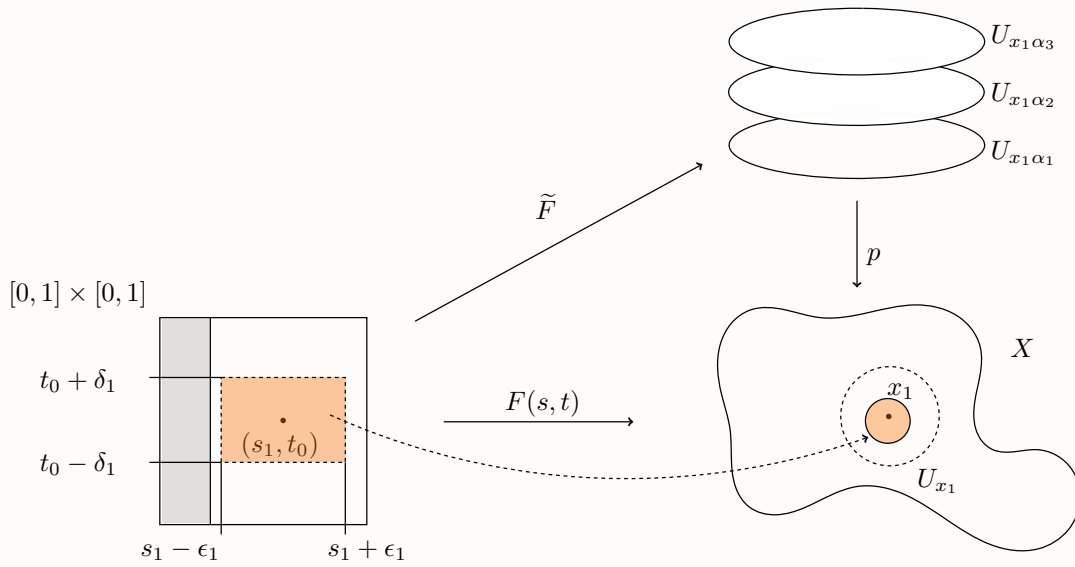
$$p|_{U_{x_1\alpha}} : U_{x_1\alpha} \xrightarrow{\cong} U_{x_1}.$$

Since  $F$  is continuous, there exists an  $\epsilon_1 > 0$ ,  $\delta_1 > 0$  such that

$$F((s_1 - \epsilon_1, s_1 + \epsilon_1) \times (t_0 - \delta_1, t_0 + \delta_1)) \subset U_{x_1}.$$

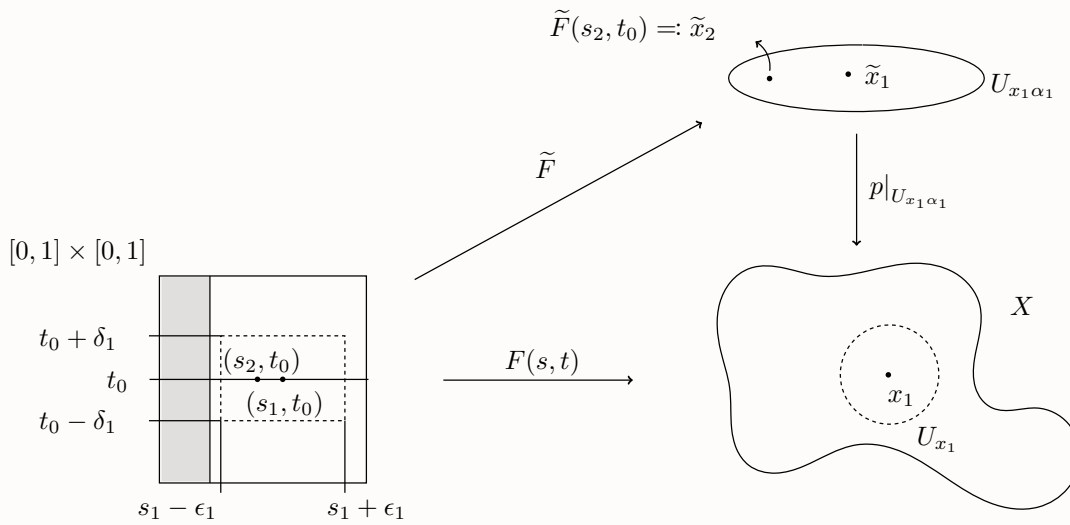
Notice that here we're considering **open** box.





We may assume  $\tilde{x}_1 \in U_{x_1\alpha_1}$ . Then, we see that  $\tilde{F}_{t_0}$  is a **lift** of  $F_{t_0}$ , which means  $\tilde{F}_{t_0}$  is continuous, hence there exists an  $s_2$  such that  $s_1 - \epsilon_1 < s_2 < s_1$  such that

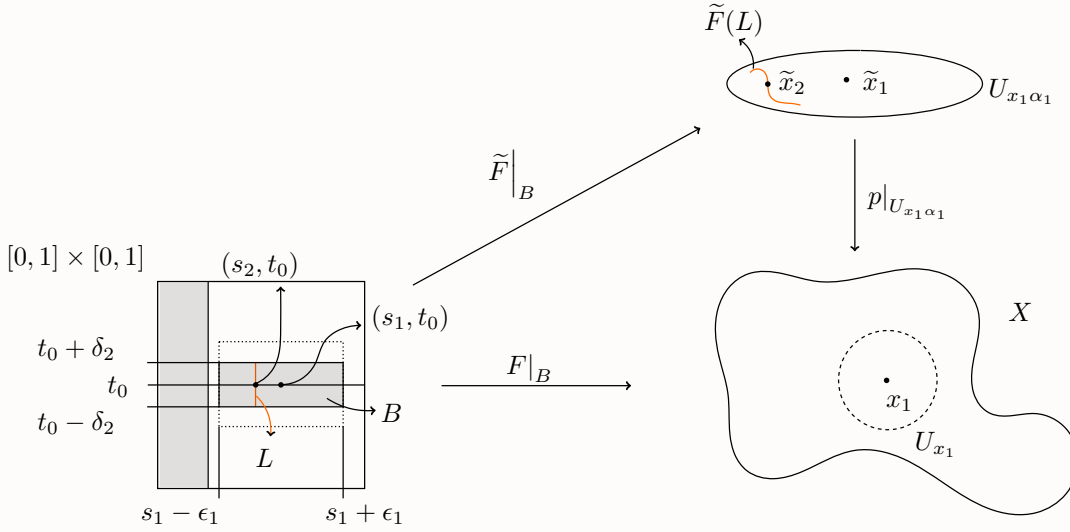
$$\tilde{F}(s_2, t_0) \in U_{x_1\alpha_1}.$$



We see that  $\tilde{F}$  is continuous at  $(s_2, t_0)$ , hence there exists a  $\delta_2 > 0$  such that

$$\tilde{F}(\{s_2\} \times (t_0 - \delta_2, t_0 + \delta_2)) \subset U_{x_1\alpha_1}.$$

Note that here we can also consider a closed interval, which matches what we're going to do. Namely, we're going to construct a **closed** box  $B$ . But this is just a technical detail.



Now, observe that  $\tilde{F}(B) \subset U_{x_1 \alpha_1}$ . To see this, consider a fixed  $t \in (t_0 + \delta_2, t_0 - \delta_2)$ , then the map  $\tilde{F}$  is

$$[s_1 - \epsilon_1, s_1 + \epsilon_1] \rightarrow \tilde{X}, \quad s \mapsto \tilde{F}(s, t) = \tilde{F}_t(s).$$

Specifically,

$$\tilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1]) \subset p^{-1}(U_{x_1}) = \coprod_{\alpha} U_{x_1 \alpha},$$

with the fact that  $\tilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1])$  is connected, and  $\tilde{F}_t(s_2) \in U_{x_1 \alpha_1}$  with  $\tilde{F}_t$  is a [lift](#) of  $F_t$ , hence continuous, so

$$\tilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1]) \subset U_{x_1 \alpha_1}.$$

This is true for every  $t \in [t_0 - \delta_2, t_0 + \delta_2]$ , hence  $\tilde{F}|_B \subset U_{x_1 \alpha_1}$ . Now, since

$$p|_{U_{x_1 \alpha_1}} \circ \tilde{F}|_B = F|_B,$$

and

$$\left( p|_{U_{x_1 \alpha_1}} \right)^{-1} \circ F|_B : B \rightarrow U_{x_1 \alpha_1},$$

so

$$p|_{U_{x_1 \alpha_1}} \circ \left( \left( p|_{U_{x_1 \alpha_1}} \right)^{-1} \circ F|_B \right) = F|_B$$

obviously. Since  $p|_{U_{x_1 \alpha_1}}$  is a homeomorphism, we have

$$\tilde{F}|_B = \underbrace{\left( p|_{U_{x_1 \alpha_1}} \right)^{-1}}_{\text{continuous}} \circ \underbrace{F|_B}_{\text{continuous}},$$

hence we have  $\tilde{F}|_B$  is continuous, which leads to a contradiction since

$$s_1 = \inf \left\{ s \mid \tilde{F} \text{ is discontinuous at } s, t_0 \right\},$$

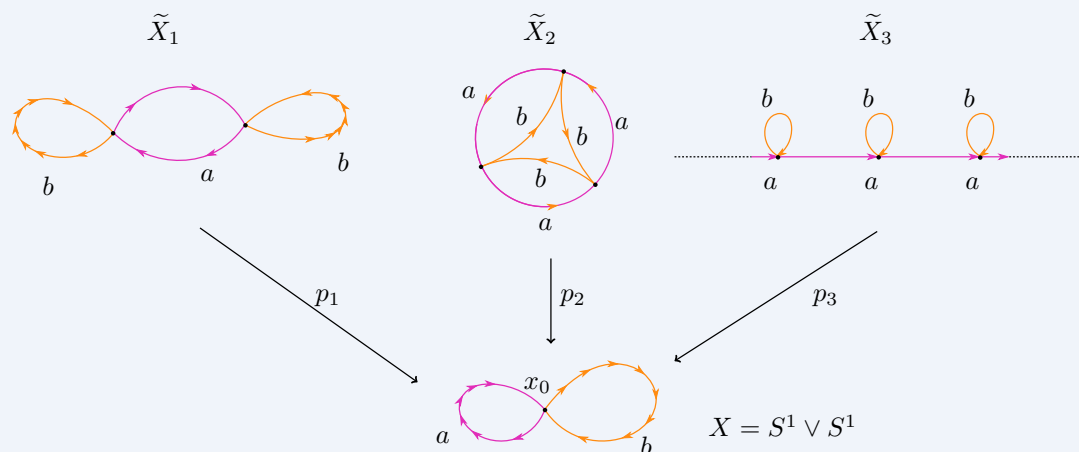
while  $\tilde{F}$  is continuous for all  $B$ , hence we see that  $\tilde{F} : I \times I \rightarrow \tilde{X}$  is continuous.<sup>c</sup>

<sup>a</sup>[https://en.wikipedia.org/wiki/Lebesgue%27s\\_number\\_lemma](https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma)

<sup>b</sup>Notice that we're working on product topology here.

<sup>c</sup>There is a tricky situation, namely while  $s_1 = 1$ . But this can be considered also.

**Example (Covers of  $S^1 \vee S^1$ ).** We have the following covers of  $S^1 \vee S^1$ .



Note that in each cover (those three on the top), the black dot is the preimage of  $\{x_0\}$ , namely  $p_i^{-1}(\{x_0\})$ .

**Remark.** We see that for each  $p_i^{-1}(\{x_0\})$ , there are exactly

- one  $a$  edge goes out
- one  $b$  edge goes out
- one  $a$  edge goes in
- one  $b$  edge goes in

It turns out that there are much more covers of  $S^1 \vee S^1$ , as long as this main property is satisfied.

**Proposition 3.1.2.** Let

$$p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a covering map. Then

1.  $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$  is injective.
2.  $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$ , which picks out the subset

$$\{[\gamma] \mid \text{Lift } \tilde{\gamma} \text{ starting at } \tilde{x}_0 \text{ is a loop.}\}.$$

**Proof.** We prove this one by one.

1. Suppose  $\tilde{\gamma} \in \pi_1(\tilde{X}, \tilde{x}_0)$  is in  $\ker(p_*)$ . Then

$$[\gamma] = p_*([\tilde{\gamma}]) = [p \circ \tilde{\gamma}].$$

Let  $\gamma_t$  be a **nullhomotopy** from  $\gamma$  to the constant loop  $c_{x_0} \text{ rel}\{0, 1\}$ . We can then **lift**  $\gamma_t$  to  $\tilde{\gamma}_t$  where  $\tilde{\gamma}_0 = \tilde{\gamma}$ . Now, we claim that

- $\tilde{\gamma}$  is a **homotopy rel** $\{0, 1\}$ .
- $\tilde{\gamma}_1$  is the constant loop  $c_{\tilde{x}_0}$ .

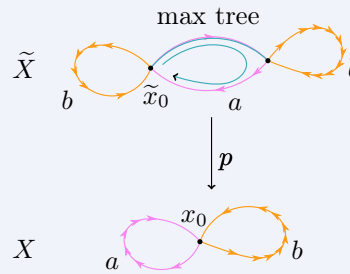
$$\begin{array}{ccc} & \tilde{X} & \\ \tilde{\gamma} \nearrow & \downarrow p & \\ I & \xrightarrow{\gamma} & X \end{array} \quad \begin{array}{ccc} & \tilde{X} & \\ \tilde{\gamma}_t \nearrow & \downarrow p & \\ I \times I & \xrightarrow{\gamma_t} & X \end{array}$$

We see that the above diagrams prove the first claim, since we know that the left and right edge of  $I \times I$  maps to  $x_0$  under  $\gamma_t$ , and  $c_{\tilde{x}_0}$  **lifts** this, so by uniqueness  $t \mapsto \tilde{\gamma}_t(0)$  and  $t \mapsto \tilde{\gamma}_t(1)$  must be constant **paths** at  $\tilde{x}_0$  as desired.

Then the **lift**  $\tilde{\gamma}_t$  is a **homotopy of paths** to the constant loop, so  $[\tilde{\gamma}] = 1$ .

2. Let see an example to show the idea of the proof.

**Example.** Given



Then

$$p_*\pi_1 = \langle b, a^2, ab\bar{a} \rangle \subseteq \pi_1(X) = \langle a, b \mid \rangle.$$

**Proposition 3.1.3** (Lifting criterion). Let  $p: (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$  be a **covering map**. Given

- $f: (X, x_0) \rightarrow (Y, y_0)$ ;
- $X$  is **path**-connected, locally **path**-connected,

then a **lift**

$$\tilde{f}: (X, x_0) \rightarrow (\tilde{Y}, \tilde{y}_0)$$

exists if and only if

$$f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(\tilde{Y}, \tilde{y}_0)).$$

In diagram, we have

$$\begin{array}{ccc} & (\tilde{Y}, \tilde{y}_0) & \\ \exists \tilde{f} \nearrow & \downarrow p & \\ (X, x_0) & \xrightarrow{f} & (Y, y_0) \end{array} \quad \begin{array}{ccc} & \pi_1(\tilde{Y}, \tilde{y}_0) & \\ \tilde{f}_* \nearrow & \downarrow p_* & \\ \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \end{array}$$

## Lecture 15: Lifting

Before proving **Proposition 3.1.3**, we first see an application.

9 Feb. 10:00

**Example.** Prove that every continuous map  $f: \mathbb{R}P^2 \rightarrow S^1$  is **nullhomotopic**.

**Proof.** If we can show that there is a **lift**  $\tilde{f}: \mathbb{R}P^2 \rightarrow \mathbb{R}$  of  $f$ , then we're done since we can apply the **straight line nullhomotopy** on  $\mathbb{R}$ , i.e.,

$$\begin{array}{ccc} & & \mathbb{R} \\ & \nearrow \tilde{f} & \downarrow p \\ \mathbb{R}P^2 & \xrightarrow{f} & S^1 \end{array}$$

and consider  $f = p \circ \tilde{f}$  compose **nullhomotopy** with  $p$ , so  $f \simeq$  constant map. Specifically, since  $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$  and  $\pi_1(S^1) = \mathbb{Z}$ , hence

$$f_*(\pi_1(\mathbb{R}P^2)) = 0$$

since  $\mathbb{Z}$  has no (nonzero) torsion. So it **lifts** by **Proposition 3.1.3**. ■

Now we can proof **Proposition 3.1.3**.

**Proof.** We prove two directions as follows.

**Necessary.** We see that we can **factorize**  $f_*$  as

$$f_* = p_* \circ \tilde{f}_*$$

follows from the **functoriality** of  $\pi_1$ .

**Sufficient.** Let  $x \in X$ . Choose a **path**  $\gamma$  from  $x_0$  to  $x$  by the assumption that  $X$  is **path-connected**. Then,  $f\gamma$  has a unique **lift** starting at  $\tilde{y}_0$ , denote by  $\tilde{f}\gamma$ . Now, define

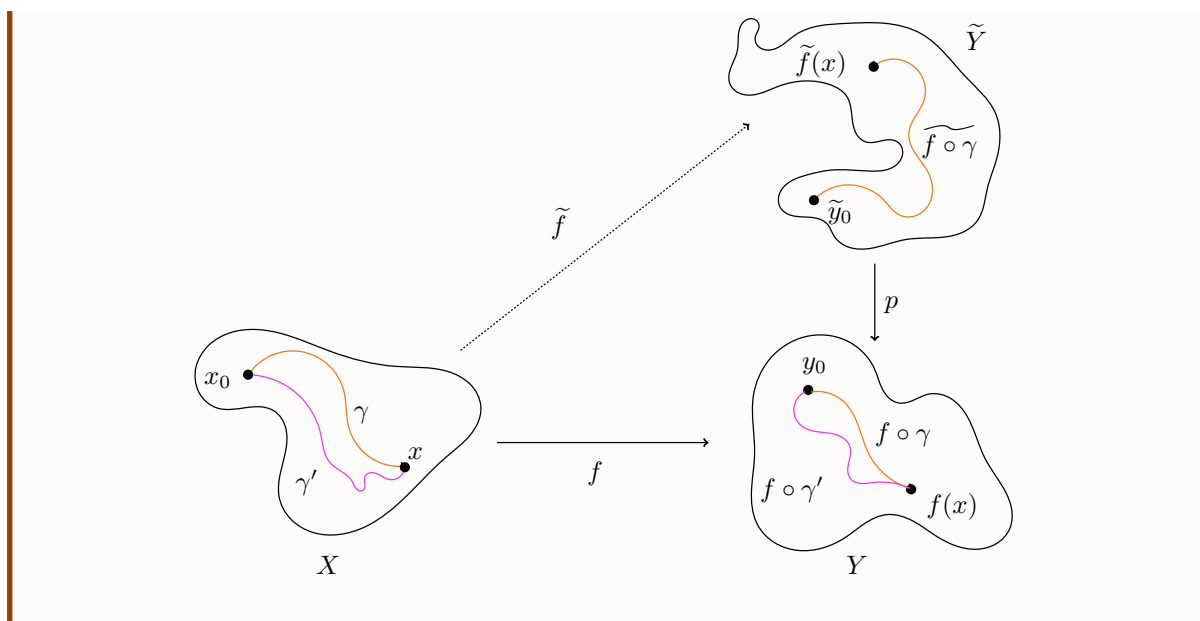
$$\tilde{f}(x) = \tilde{f}\gamma(1).$$

Then, we need to check

1.  $\tilde{f}$  is well-defined. Suppose  $\gamma, \gamma'$  are **paths** in  $X$  from  $x_0$  to  $x$ . We want to show

$$\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1).$$

Since  $\gamma \cdot \overline{\gamma'}$  is a loop in  $X$  at  $x_0$ , we know that  $[(f\gamma) \cdot (f\overline{\gamma'})]$  is a class of loops in  $Y$  in  $\text{Im}(f_*)$ . By hypothesis, this class of loops is in  $\text{Im}(p_*)$ . It **lifts** to a loop which is based at  $\tilde{y}_0$ . By uniqueness of **lifts**, this loop lifting  $(f\gamma) \cdot (f\overline{\gamma'})$  to  $\tilde{Y}$  must be equal to the **lifts**  $\widetilde{f\gamma} \cdot \widetilde{f\overline{\gamma'}}$  with a common value at  $t = 1/2$ . Hence,  $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$  as desired, namely the endpoints agree.

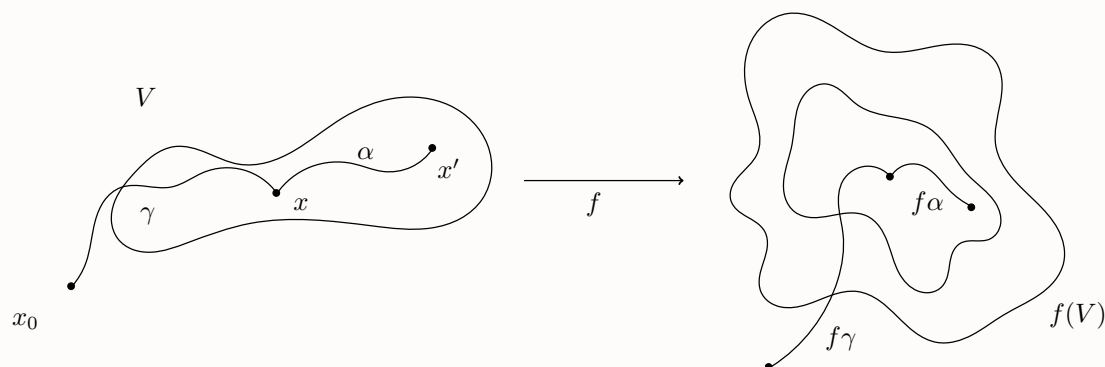


## Lecture 16: Proving Proposition 3.1.3

We now continue our proof of Proposition 3.1.3.

11 Feb. 10:00

**Proof.** 2.  $\tilde{f}$  is continuous. Choose  $x \in X$  and a neighborhood  $\tilde{U}$  of  $\tilde{f}(x)$  in  $\tilde{Y}$ . Note that we can choose  $\tilde{U}$  small enough to  $p|_{\tilde{U}}$  is homeomorphism to  $U$  in  $Y$ . Now, there exists a neighborhood  $V$  of  $x$  in  $X$  with  $f(V) \subseteq U$ .



The goal is  $\tilde{f}(V) \subseteq \tilde{U}$ . Without loss of generality, we can assume that  $V$  is path-connected. Then,

$$\tilde{f}\gamma \cdot \tilde{f}\alpha = [\tilde{f}\gamma \cdot \tilde{f}\alpha].$$

Hence,

$$\tilde{f}\alpha = (p|_{\tilde{U}})^{-1} \circ f \circ \alpha,$$

where  $(p|_{\tilde{U}})^{-1}$ 's image is in  $\tilde{U}$ , so

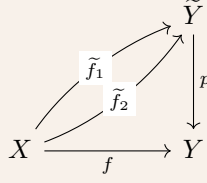
$$\tilde{f}(x') = f\gamma \cdot f\alpha(1) \in \tilde{U},$$

which implies

$$\tilde{f}(V) \subseteq \tilde{U}.$$

■

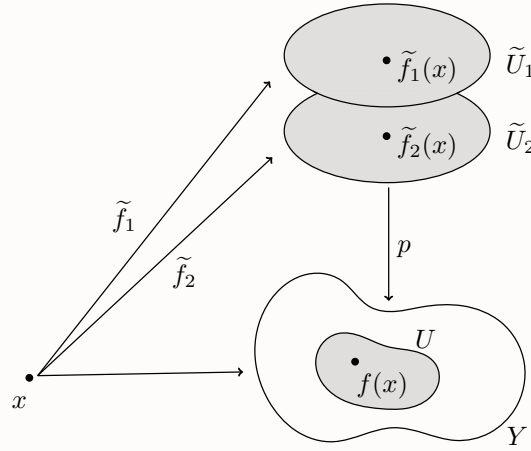
**Proposition 3.1.4** (Uniqueness of lifts). Let  $p: \tilde{Y} \rightarrow Y$  be a covering map with  $X$  is a connected space. If two lifts  $\tilde{f}_1, \tilde{f}_2$  of the same map  $f$  agree at a single point, then they agree everywhere.



**Proof.** Let  $S$  being

$$S := \{x \in X \mid \tilde{f}_1(x) = \tilde{f}_2(x)\}.$$

We want to show that  $S$  is both closed and open, so if  $S$  is nonempty,  $S = X$ .



We see that  $\tilde{U}_1$  and  $\tilde{U}_2$  are slices of  $p^{-1}(U)$ , where  $U$  is evenly covered neighborhood of  $f(x)$ .

1. If  $\tilde{f}_1(x) \neq \tilde{f}_2(x)$ . Then  $\tilde{U}_1, \tilde{U}_2$  are disjoint. Since  $\tilde{f}_1, \tilde{f}_2$  are continuous, there exists a neighborhood  $N$  of  $x$  with

$$\tilde{f}_1(N) \subseteq \tilde{U}_1, \quad \tilde{f}_2(N) \subseteq \tilde{U}_2,$$

with the fact that they're disjoint, so  $x$  is an interior point of  $S^c$ .

2. If  $\tilde{f}_1(x) = \tilde{f}_2(x)$ . Then  $\tilde{U}_1 = \tilde{U}_2$ . Choose  $N$  as before, then we have

$$\tilde{f}_1(n) = (p|_{\tilde{U}_1})^{-1}(f(n)) = \tilde{f}_2(n),$$

hence  $x \in \text{int}(S)$ .

■

## 3.2 Deck Transformation

We now want to introduce a special kind of transformation.

**Definition 3.2.1** (Isomorphism of covers). Given covering maps

$$p_1: \tilde{X}_1 \rightarrow X, \quad p_2: \tilde{X}_2 \rightarrow X,$$

an *isomorphism of covers* is a homeomorphism

$$f: \tilde{X}_1 \rightarrow \tilde{X}_2$$

such that  $p_1 = p_2 \circ f$ .

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{f} & \tilde{X}_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

**Exercise.** This defines equivalent relation on **covers** of  $X$ .

**Definition 3.2.2** (Deck transformation). Given a **covering map**  $p: \tilde{X} \rightarrow X$ , the **isomorphisms of covers**  $\tilde{X} \rightarrow \tilde{X}$  are called *deck transformation*.

Furthermore, we'll let  $G(\tilde{X})$  denotes the *set of deck transformations*.

**Note.** Note that we've suppressed the data of  $p$  in the notation, but this data is essential to what a **deck transformation** is, when this is unclear we write  $G(\tilde{X}, p)$ .

## Lecture 17: Deck Transformation

**Example.** Let's see some examples.

1. **Deck transformations**  $G(\tilde{X})$  are a subgroup of the group of homeomorphisms of  $\tilde{X}$ .
2. Given the **cover**  $p: \mathbb{R} \rightarrow S^1$ .
  - **Deck maps**: translation by  $n \in \mathbb{Z}$  units.
  - $G(\mathbb{R}) \cong \mathbb{Z}$
3. Given the **cover**  $p_n: S^1 \rightarrow S^1$  be an  $n$ -sheeted cover.
  - **Deck maps**: rotation by  $2\pi/n$ .
  - $G(S^1, p_n) \cong \mathbb{Z}/n\mathbb{Z}$

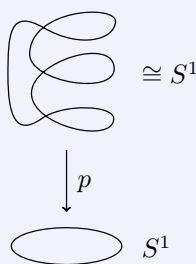


Figure 3.1:  $p_n: S^1 \rightarrow S^1$  be an  $n$ -sheeted **cover**, where  $n = 3$ .

**Exercise** (Deck Transformation is determined by the image of one point). Given  $X, \tilde{X}$  are **path**-connected, locally **path**-connected, **deck map** is determined by the image of any one point.

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**Answer.**

$$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow f & \downarrow p \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

■

**Corollary 3.2.1.** If a **deck transformation** has a fixed point, it is the identity transformation.

**Exercise.** Let  $X$  be connected. Given a **deck transformation**  $\tau: \tilde{X} \rightarrow \tilde{X}$ ,  $\tau$  defines a permutation of  $p^{-1}(\{x_0\})$ . If this permutation has a fixed point, then it is the identity.

**Definition 3.2.3** (Regular (normal) cover). A **covering space**  $p: \tilde{X} \rightarrow X$  is *regular* or *normal* if  $\forall x_0 \in X, \forall \tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$ , there exists a **deck transformation** such that

$$\tilde{x}_0 \mapsto \tilde{x}_1.$$

**Example** (Regular and non-regular cover of  $S^1 \vee S^1$ ). Given the following **covers** of  $S^1 \vee S^1$ , determine which cover is **regular**.



**Proof.** The left one is **regular**, while the right one is not since there is no automorphism from  $\tilde{x}_0$  to  $\tilde{x}_1$  or  $\tilde{x}_2$ . ■

**Remark.** A **regular cover** is *as symmetric as possible*.

**Exercise.** **Regular** means that the group  $G(\tilde{X})$  acts transitively on  $p^{-1}(\{x_0\})$ . Explain why we cannot ask for more than this:

$G(\tilde{X})$  cannot induce the full symmetric group on  $p^{-1}(\{x_0\})$  provided that  $|p^{-1}(\{x_0\})| > 2$ .

**Answer.** The key is uniqueness. ■

**Definition 3.2.4** (Normalizer). Given  $G$  as a group,  $H \subseteq G$  is a subgroup of  $G$ . Then the *normalizer* of  $H$ , denoted by  $N(H)$ , is defined as

$$N(H) := \{g \in G \mid gH = Hg\}.$$

**Exercise.** We can prove the followings.

1.  $N(H)$  is a subgroup.
2.  $H \leq N(H)$ .
3.  $H$  is normal in  $N(H)$ .
4. If  $H \leq G$  is normal,  $N(H) = G$ .

5.  $N(H)$  is the largest subgroup (under containment) of  $G$  containing  $H$  as normal subgroup.

**Proposition 3.2.1.** Given  $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  be a **cover**, and  $\tilde{X}, X$  are **path**-connected, locally **path**-connected. Let

$$H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0).$$

Then

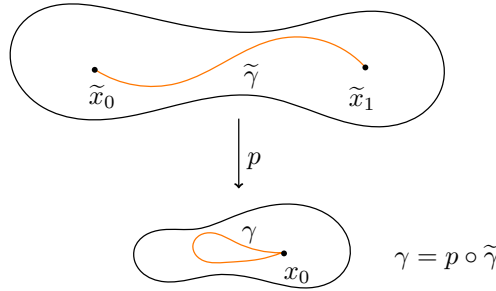
1.  $p$  is **normal** if and only if  $H \subseteq \pi_1(X, x_0)$  is **normal**.
2. We have

$$G(\tilde{X}) \cong N(H) / H,$$

where  $G(\tilde{X})$  are **deck maps**, and  $N(H)$  is the **normalizer** of  $H$  in  $\pi_1(X, x_0)$ .

**Remark.** A fact is worth noting is the following. Let  $\tilde{\gamma}$  be a **path**  $\tilde{x}_1$  to  $\tilde{x}_0$ . Then

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = [\gamma]p_*(\pi_1(\tilde{X}, \tilde{x}_1))[\gamma^{-1}].$$



## Lecture 18: Proving Proposition 3.2.1

Now let's prove Proposition 3.2.1

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**Proof.** Let  $X, x_0$  be the base space and  $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$  where  $p: \tilde{X} \rightarrow X$  is a **covering map**. Further, let  $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ .

In homework, given  $(X, x_0), \tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$  if we change the basepoint from  $\pi_1(\tilde{X}, \tilde{x}_0)$  to  $\pi_1(\tilde{X}, \tilde{x}_1)$ , then we have the induced subgroups of the base spaces **fundamental group** are conjugate by some loop  $[\gamma] \in \pi_1(X, x_0)$ , i.e.,

$$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = [\gamma] \cdot p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cdot [\gamma]^{-1}$$

where  $\gamma$  is **lifted** to a **path** from  $\tilde{x}_0$  to  $\tilde{x}_1$ .

Therefore,  $[\gamma] \in N(H)$  if and only if  $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ , and this holds if and only if there is a **deck transformation** taking  $\tilde{x}_0$  to  $\tilde{x}_1$  by the classification of based **covering spaces** in the homework.<sup>a</sup> This shows that  $p$  is a **normal cover** if and only if  $H$  is normal, which proves the first claim.

We then define a map  $\Phi$  such that

$$\Phi: N(H) \rightarrow G(\tilde{X})[\gamma], \quad \cdot \mapsto \tau$$

where  $\tau$  **lifts** to a **path** from  $\tilde{x}_0$  to  $\tilde{x}_1$  and  $\tau$  is a **deck transformation** mapping  $\tilde{x}_0$  to  $\tilde{x}_1$ , which will be uniquely defined by the uniqueness of **lifts** with specified base points. We now need to check

1.  $\Phi$  is surjective.

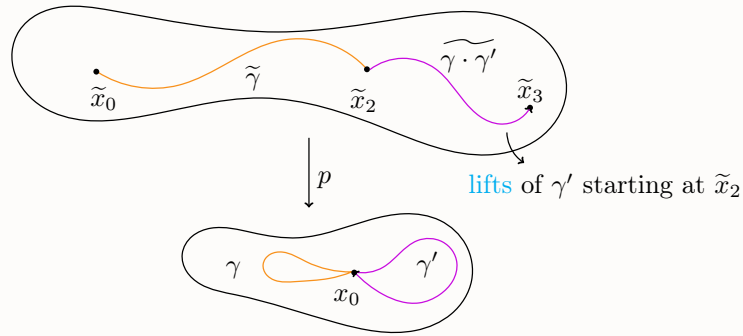
2.  $\ker(\Phi) = H$ .
3.  $\Phi$  is a group homomorphism.

If we can prove all the above, then, from the result follows directly from the first isomorphism theorem.<sup>b</sup>

1. We've proved that  $\Phi$  is surjective before in our work above.
2.  $\Phi([\gamma])$  is the identity if and only if  $\tau$  sends  $\tilde{x}_0$  to  $\tilde{x}_0$ , meaning that  $[\gamma]$  **lifts** to a loop. Then by our characterization of the **fundamental group** downstairs:

$$\ker(\Phi) = \{[\gamma] \mid [\gamma] \text{ lifts to a loop}\} = H.$$

3. Suppose we have loops  $[\gamma_1] \xrightarrow{\Phi} \tau_1$  and  $[\gamma_2] \xrightarrow{\Phi} \tau_2$ . We claim that  $\gamma_1 \cdot \gamma_2$  **lifts** to  $\tilde{\gamma}_1 \cdot \tau(\tilde{\gamma}_2)$ .



It's an exercise to check that the **lift** of  $\gamma_2$  starting at  $\tilde{x}_1$  is exactly  $\phi_1(\tilde{\gamma}_2)$ , where  $\tilde{\gamma}_2$  is a **lift** starting at  $\tilde{x}_0$ .

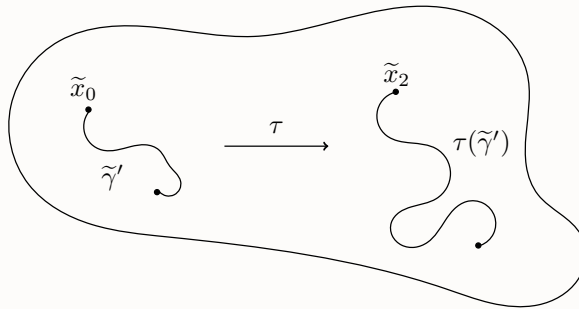


Figure 3.2: Must be **lift** of  $\gamma'$  starting at  $\tilde{x}_2$

The idea is that by uniqueness of **lifts** we'll have the desired claim. We then just observe that this **path**  $\tilde{\gamma}_1 \cdot \tau_1(\tilde{\gamma}_2)$  is a **path** from  $\tilde{x}_0$  to  $\gamma_1(\tilde{\gamma}_2(1)) = \tau_1(\tau_2(\tilde{x}_0))$ , so the image must be a **deck transformation** sending  $\tilde{x}_0$  to  $\tau_1(\tau_2(\tilde{x}_0))$ . But then  $\tau_1 \circ \tau_2$  maps  $\tilde{x}_0$  to this same point, and from **this exercise**, we know that the **deck transformations** are determined by where they send a single point, hence we're done.

<sup>a</sup>Alternatively, we can use the **lifting criterion**.

<sup>b</sup>[https://en.wikipedia.org/wiki/Isomorphism\\_theorems](https://en.wikipedia.org/wiki/Isomorphism_theorems)

---

**Corollary 3.2.2.** If  $p$  is a normal covering, then  $G(\tilde{X}) \cong \pi_1(X, x_0) / H$ .

**Definition 3.2.5** (Universal covering). A cover  $p: \tilde{X} \rightarrow X$  is called a *universal covering* if  $\tilde{X}$  is simply connected.

**Corollary 3.2.3.** If  $\tilde{X}$  is the universal cover, then  $G(\tilde{X}) \cong \pi_1(X, x_0)$ .

**Exercise.** Whether  $\text{Im}(p_*)$  is normal is independent of the basepoint in  $\tilde{X}$  and  $X$ .

So,  $p$  is normal if and only if  $G(\tilde{X})$  is transitive on  $p^{-1}(x_0)$  for at least one  $x_0 \in X$ .

**Exercise.** Let  $\Sigma_g$  be the genus  $g$  surface. Prove that  $\Sigma_g$  has a normal  $n$ -sheeted path-connected cover for every  $n$ .

# Chapter 4

## Homology

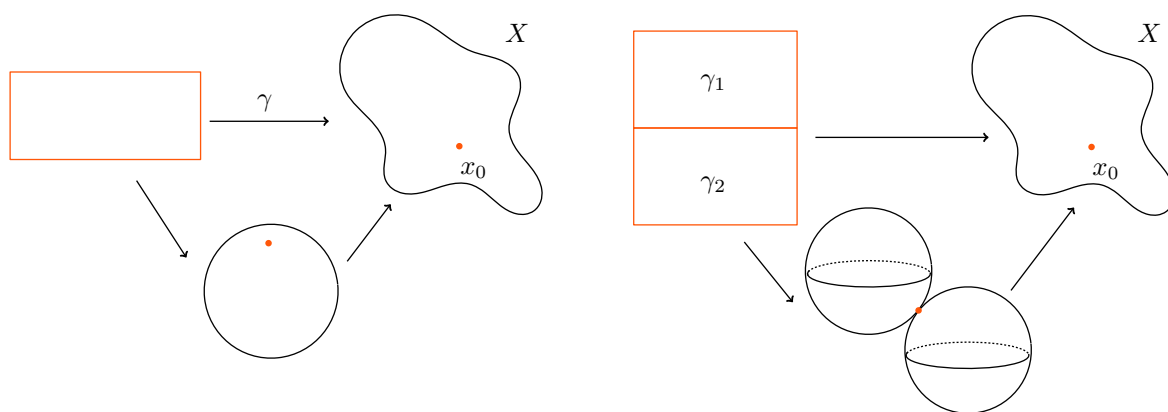
### Lecture 19: Simplex

#### 4.1 Motivation for Homology

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Informally, the higher [homotopy](#) groups is defined as

$$\pi_n(X, x_0): I_*^n \rightarrow (X, x_0), \quad \partial I^n \mapsto x_0.$$



We see that it's extremely hard to compute higher [fundamental group](#). Hence instead, we will study the higher dimensional structure of  $X$  via *homology*.

- **Cons.**

- The definition is more opaque at first encounter.

- **Pros.**

- Lots of computational tools
- Functional
- Abelian Groups

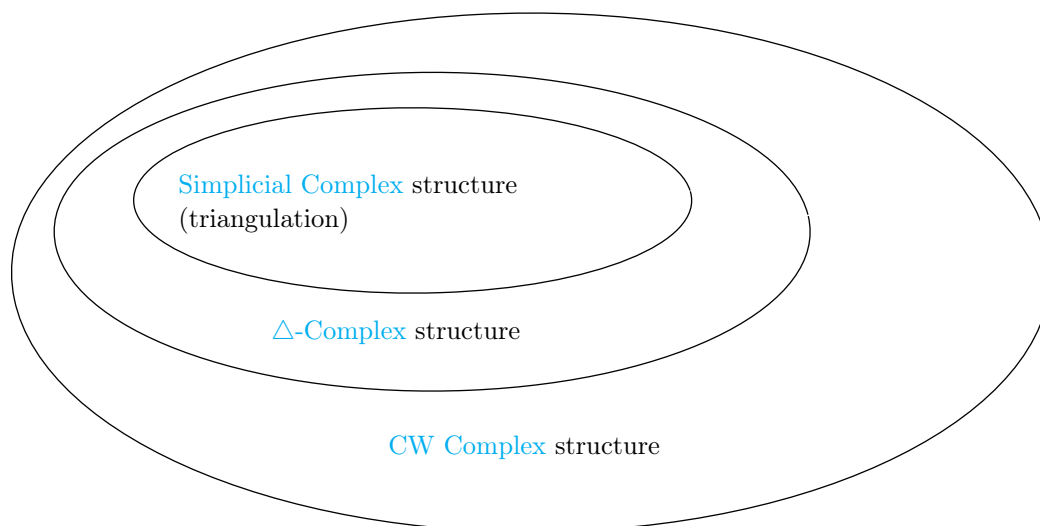
**Remark.** More like  $\pi_n$  for  $n > 1$ .

- No basepoints
- Can compute using [CW](#) structure.
- Good properties. For example,  $H_n = 0$  if  $n > \dim X$

## 4.2 Simplicial Homology

### 4.2.1 $\Delta$ -Simplex

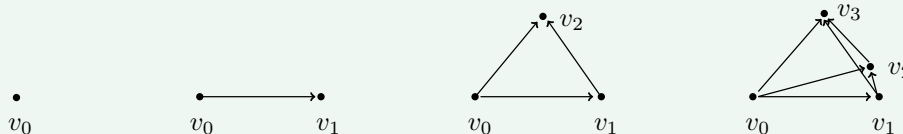
This is a stricter version of a [CW complex](#) which allows us to decompose our spaces into cells. In terms of how things fit together, we have the following diagram.



Now we try to give the definition.

**Definition 4.2.1 (Simplex).** We see that

- 0-simplex. A point.
- 1-simplex. Interval.
- 2-simplex. Triangle.
- 3-simplex. Tetrahedron.
- $n$ -simplex. The convex hull of  $(n + 1)$ -points position in  $\mathbb{R}^n$ .



**Remark.** We see that

- The top of which is the 2-disk and remember cell structure (edges and vertices) and remember orientation (ordering on vertices).
- The top of which is the 3-disk and cells and the orientation.

Further,

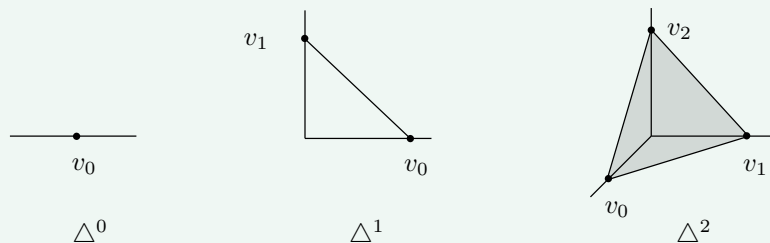
- We can view [simplices](#) as both *combinatorial* and *topological* objects.

An alternative definition can be done.

**Definition 4.2.2 (Standard simplex).** We say that an  $n$ -dimensional *standard simplex*, denoted by  $\Delta^n$  is

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1 \right\}.$$

We'll call such a simplex as *standard  $n$ -simplex*.



**Remark.** In our definition, the *simplices* will implicitly come with a choice of ordering of the vertices as

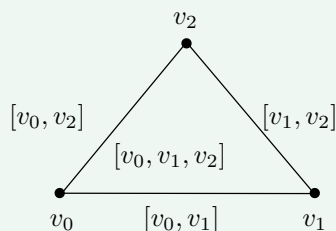
$$\Delta^n = [v_0, v_1, \dots, v_n]$$

such that the convex hull of these points is taken with this ordering.

## Lecture 20: Simplicial Complex

**Definition 4.2.3 (Subsimplex).** A *subsimplex* of a *simplex*  $\sigma$  combinatorially, it's a subset of the vertices; while topologically, it's the convex hull of the subset of vertices.

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**Definition 4.2.4 (Face).** A *face* of a *simplex*  $\Delta^n$  is a *subsimplex* of 1 dimensional lower than  $\Delta^n$  (codimension 1).

**Definition 4.2.5 (Boundary).** The *boundary*  $\partial\sigma$  of a *simplex*  $\sigma$  is the union of its *faces*.

**Definition 4.2.6 (Open simplex).** The *open simplex* of  $\Delta$  is defined as

$$\mathring{\Delta}^n := \Delta^n - \partial\Delta^n.$$

**Definition 4.2.7 ( $\Delta$ -Complex).** A  $\Delta$ -*complex* structure on  $X$  is a collection of maps

$$\sigma_\alpha: \Delta^n \rightarrow X$$

such that

1.  $\sigma_\alpha|_{\mathring{\Delta}^n}$  injective, each point of  $X$  is in the image of exactly one such map.

2. Each restriction of  $\sigma_\alpha$  to a **face** coincides with a map

$$\sigma_\beta: \Delta^{n-1} \rightarrow X.$$

3. A set  $A \subseteq X$  is open if and only if  $\sigma_\alpha^{-1}(A)$  is open in  $\mathring{\Delta}^n$  for all  $\sigma_\alpha$ , i.e.,  $X$  is a **quotient**

$$\coprod_{n,\alpha} \Delta_\alpha^n \xrightarrow{\coprod \sigma_\alpha} X.$$

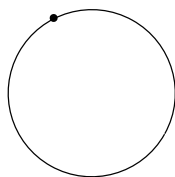
**Exercise.** A  **$\Delta$ -complex**  $X$  is a **CW complex**  $W$  with characteristic maps  $\sigma_\alpha$  with extra constraints on the attaching maps.

**Note.** We see that the second condition of **Definition 4.2.7** implies that attaching maps injective on interior of **faces**.

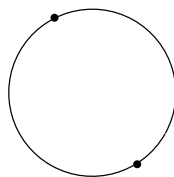
**Definition 4.2.8** (Simplicial complex). A *simplicial complex* is a  **$\Delta$ -complex** such that

- $\sigma_\alpha$  must map every **face** to a different  $(n-1)$ -simplex.
- Every **simplex** is uniquely determined by its vertex set.
- Any  $(n+1)$  vertices in  $X^0$  is the vertex set of at most 1  $n$ -simplex.

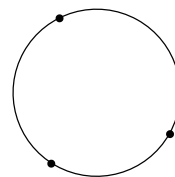
**Remark.** With **Definition 4.2.8**, we see the followings.



**$\Delta$ -simplex**  
not simplicial

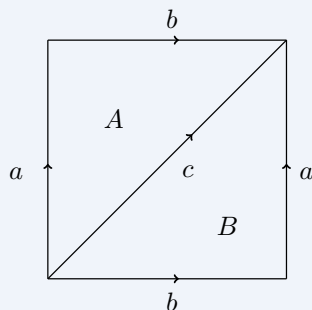


**$\Delta$ -simplex**  
not simplicial



**$\Delta$ -simplex**  
is simplicial

**Example.** The torus with the following edges,  $a, b, c$  and the gluing in triangles  $A$  and  $B$  can be seen as follows.



For this  **$\Delta$ -complex**, notice that we've glued down a triangle whose vertices are all identified. This is not allowed in a **simplicial complex** / triangulation.



**Remark.** The minimum number of triangles in a [simplicial complex](#) structure is **14**.

## Lecture 21: Simplicial Homology

To demonstrate how the definition of homology arise, we first see the idea behind it. Fix a space  $X$  which equips with the  [\$\Delta\$ -complex](#) structure. Then, we define  $C_n(X)$  to be the [free Abelian group](#) on the  $n$ -[simplices](#) of  $X$ . That is,

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$$C_n(X) = \left\{ \text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha: \Delta^n \rightarrow X \right\}.$$

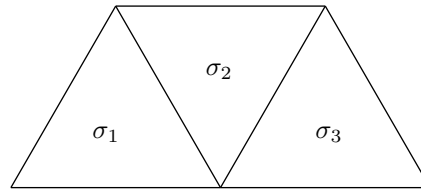
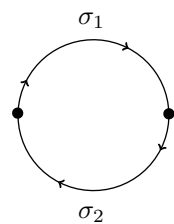


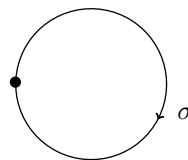
Figure 4.1:  $C_2(X) = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \oplus \mathbb{Z}\sigma_3$ .

Then, the  $n$ -th homology group will be a [subquotient](#) of  $C_n(X)$ , where the heuristic/imprecise idea is

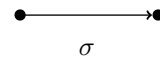
- Take subgroup of  $C_n$  of *cycles*. These are sums of [simplices](#) satisfying a combinatorial condition on the boundary gluing maps to ensure that they *close up*, i.e., they have no [boundary](#).



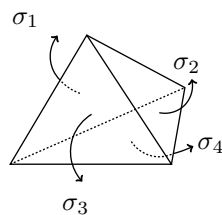
$\sigma_1 + \sigma_2$  cycles



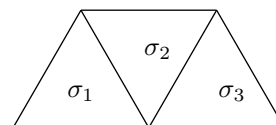
$\sigma$  cycle



$\sigma$  not a cycle

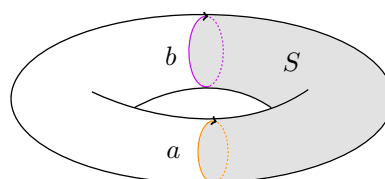


$\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$  cycles

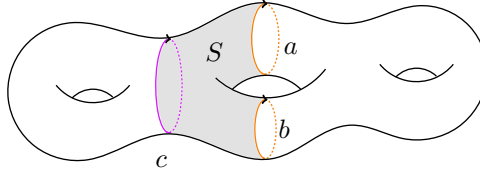


$\sigma_1 + \sigma_2 + \sigma_3$  not a cycles

- To take the [quotient](#), we consider two cycles to be equivalent if their difference is a [boundary](#). For example, in the case of torus,  $a$  is homologous to  $b$  since  $a - b$  is the [boundary](#) of the shaded subsurface  $S$  on of the torus below.



In fact,  $a$  and  $b$  are **homotopic** (which will imply they're homologous essentially), but two loops do not need to be **homotopic** to be homologous. For example, in the figure below,  $a + b$  is homologous to  $c$ , since  $a + b - c$  is the **boundary** of  $S$  ( $a + b^1$  and  $c$  are not **homotopic**).



Let's now see the formal definition.

**Definition 4.2.9** (Simplicial chain group). We define the *simplicial chain group*  $C_n(X)$  of order  $n$  to be the **free Abelian group** on the  $n$ -**simplices** of  $X$  such that

$$C_n(X) := \left\{ \text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha: \Delta^n \rightarrow X \right\}.$$

**Definition 4.2.10** (Cycles). Given any chain group  $C_n(X)$ , a *cycle* of  $C_n(X)$  is those chains  $\sum m_\alpha \sigma_\alpha$  with no **boundaries**.

**Definition 4.2.11** (Boundary homomorphism). A map  $\partial_n: C_n(X) \rightarrow C_{n-1}(X)$  is called a *boundary homomorphism* such that

$$\begin{aligned} \partial_n: C_n(X) &\rightarrow C_{n-1}(X) \\ [\sigma_\alpha] &\mapsto \sum_{i=1}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}, \end{aligned}$$

which defines the map on the basis, and we extend it linearly.

**Remark.** We see that the definition of **boundary homomorphism** indeed coincides with the definition of **boundary** when considering either  **$\Delta$ -complex** or **simplicial complex** structure.

**Example.** We give some lower dimensions examples of **Definition 4.2.11** to motivate the general definition.

- For  $n = 1$ ,  $\partial_1: C_1(X) \rightarrow C_0(X)$  such that

$$[\sigma_\alpha: [v_0, v_1] \rightarrow X] \mapsto \sigma_\alpha|_{[v_1]} - \sigma_\alpha|_{[v_0]}.$$

- For  $n = 2$ ,  $\partial_2: C_2(X) \rightarrow C_1(X)$  such that

$$[\sigma_\alpha: [v_0, v_1, v_2] \rightarrow X] \mapsto \sigma_\alpha|_{[v_1, v_2]} - \sigma_\alpha|_{[v_0, v_2]} + \sigma_\alpha|_{[v_0, v_1]}.$$

**Lemma 4.2.1.** For any  $n \geq 2$ , we have

$$\begin{array}{ccccc} C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \xrightarrow{\partial_{n-1}} & C_{n-2}(X) \\ & \searrow & \xrightarrow{\partial_{n-1} \circ \partial_n = 0} & & \end{array}$$

<sup>1</sup>Which isn't even a loop

**Proof.** Since all  $C_i$  are **free Abelian group**, hence we only need to consider  $\partial_{n-1} \circ \partial_n(\sigma) = 0$  for a generator  $\sigma$ . Given a generator  $\sigma$ , the result follows from directly applying the **definition** and with some calculation. ■

**Definition 4.2.12 (Chain complex).** A *chain complex*  $(C_*, d_*)$  is a collection of maps such that

$$\dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

of **Abelian groups** and group homomorphism such that

$$d_{n-1} \circ d_n = 0.$$

We call  $C_n$  the *n-th chain group* and  $d_n$  the *n-th differential*.

**Note.** Note that **Definition 4.2.12** is purely *abstract*, namely we can put different chain group structure on  $C_n$ . We'll see what this means later.<sup>a</sup> But for now,  $C_n$  can be equipped with the definition we gave for **simplicial chain group**.

<sup>a</sup>Spoiler: It just means we can give different definition about the map  $\sigma$ .

**Remark.** We see that

- **Lemma 4.2.1** guarantees that our **simplicial chain groups** form a **chain complex**.
- **Definition 4.2.12** means that  $\ker(d_n)$  contains  $\text{Im}(d_{n+1})$ , since  $d_n \circ d_{n+1} = 0$ .

**Definition 4.2.13 (Exact).** We say that the sequence is *exact at  $C_n$*  provided that  $\ker(d_n) = \text{Im}(d_{n+1})$ . A **chain complex** is *exact* if it is *exact at each point*.

**Definition 4.2.14 (Homology group).** The  $n^{\text{th}}$  *homology group* of a chain complex  $(C_*, d_*)$ , denoted as  $H_n$  or  $H_n(C_*)$ , is the quotient

$$H_n := \ker(d_n) / \text{Im}(d_{n+1}).$$

**Remark.** The **homology group** measures how far the **chain complex** is from being **exact** at  $C_n$ .

With what we have just defined, it's natural to define **homology groups** of space  $X$  with a  **$\Delta$ -complex** structure.

**Definition 4.2.15 (Homology class).** We say  $\ker(\partial_n)$  is the subgroup of **cycles** in  $C_n(X)$ , and  $\text{Im}(\partial_{n+1})$  is the subgroup of **boundaries** in  $C_n(X)$ . We then set

$$H_n(X) := \ker(\partial_n) / \text{Im}(\partial_{n+1}) = \{\text{cycles}\} / \{\text{boundaries}\}.$$

In other words, it's the **homology** of the **chain complex**

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

where we take it to be 0 in all negative indices, namely

$$\dots \xrightarrow{\partial_3} C_{n+1} \xrightarrow{\partial_2} C_n \xrightarrow{\partial_1} C_{n-1} \xrightarrow{\partial_0} 0$$

We then call the elements of  $H_n(X)$  as *homology classes*.

**Definition 4.2.16** (Simplicial homology group). By considering the chain complex with simplicial chain group, we have so-called *simplicial homology group* induced by Definition 4.2.14.

## Lecture 22: Calculation of Homology

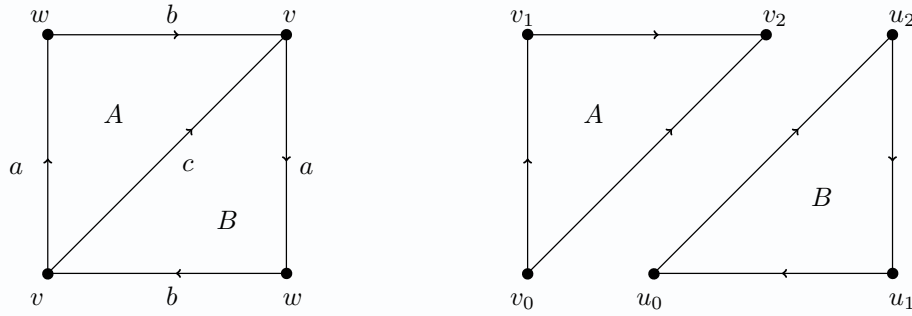
### 4.2.2 Calculation of Homology

25 Feb. 10:00

We start from some calculation about homology group of some spaces.

**Example** (Homology group of  $\mathbb{R}P^2$ ). Calculate the homology group by Definition 4.2.14 with the chain complex being the simplicial chain complex.

**Proof.** Let  $X = \mathbb{R}P^2$ .



We see that we have

- $C_0 = \mathbb{Z}\langle v, w \rangle$
- $C_1 = \mathbb{Z}\langle a, b, c \rangle$
- $C_2 = \mathbb{Z}\langle A, B \rangle = \mathbb{Z}A \oplus \mathbb{Z}B$

The chain complex is then

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Where we let  $A = [v_0, v_1, v_2]$  and  $B = [u_0, u_1, u_2]$ , then

$$\partial_2: \begin{cases} A \mapsto b - c + a \\ B \mapsto -a - c - b \end{cases}, \quad \partial_1: \begin{cases} a \mapsto w - v \\ b \mapsto v - w \\ c \mapsto v - v = 0 \end{cases}$$

We can also calculate the image and the kernel at  $C_i$ , i.e.,

$$\begin{aligned} C_2: \operatorname{Im} \partial_3 &= 0, & \ker \partial_2 &= 0, \\ C_1: \operatorname{Im} \partial_2 &= \langle 2c, b - c + a \rangle, & \ker \partial_1 &= \langle b + a, c \rangle, \\ C_0: \operatorname{Im} \partial_1 &= \langle v - w \rangle, & \ker \partial_0 &= \langle v, w \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} H_0 &\cong \mathbb{Z}\langle v, w \rangle / \mathbb{Z}\langle v - w \rangle \cong \mathbb{Z} \\ H_1 &\cong \mathbb{Z}\langle b + a, c \rangle / \mathbb{Z}\langle 2c, b + a - c \rangle \cong \mathbb{Z}\langle b + a - c, c \rangle / \mathbb{Z}\langle 2c, b + a - c \rangle \cong \mathbb{Z} / 2\mathbb{Z} \\ H_2 &= 0 \end{aligned}$$

**Remark.** Given a basis for a **free Abelian group**  $\langle b_1, \dots, b_n \rangle$  we can replace  $b_i$  with

$$b_i \pm m_1 b_1 \pm \dots \pm \widehat{m_i b_i} \pm \dots \pm m_n b_n.$$

**Remark.** Warning! Care is needed when doing *change of bases* over  $\mathbb{Z}$ . For example, if  $b_1, b_2$  is a basis for  $A \subseteq \mathbb{Z}^n$ , then  $b_1 - b_2, b_1 + b_2$  is not a basis, it is an index-2 subgroup. The key to this is that  $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  has determinant  $-2$  (not unit in  $\mathbb{Z}$ ).

We can transform a basis for a **free group** into a different basis by applying a matrix of determinant  $\pm 1$ . If we apply a matrix of determinant  $D$  we will obtain generators for a subgroup of index  $|D|$ .

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \pm m_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \pm m_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \pm m_{i-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \pm m_{i+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \pm m_n & 0 & \cdots & 1 \end{bmatrix}$$

As a summary, we have the following procedures to compute  $H_n(X)$ .

1. Choose  **$\Delta$ -complex** structure on  $X$ . (We will prove  $H_*(X)$  is independent of the choice of  **$\Delta$ -complex** structure)
2. Choose orientations on each **simplex** (Any choice is okay, but you must commit to a choice, or you will make a sign error!)
3. For each  **$n$ -simplex**  $\sigma$  compute  $\partial_n(\sigma)$  (careful with signs!)
4.  $\text{Im } \partial_n = \langle \partial_n(\sigma) \mid \sigma \text{ an } n\text{-simplex} \rangle$ . Use linear algebra to compute  $\ker(\partial_n)$ .
5. For each  $n$  compute  $H_n(X) = \ker \partial_n / \text{Im } \partial_{n+1}$ . Be careful that any change-of-variables map you apply is invertible over  $\mathbb{Z}$ .

## Lecture 23: Singular Homology

### 4.3 Singular Homology

07 Mar. 10:00

As we noted before, we can give a different structure of **chain complex**, which shall induce a different **homology group** compare to **simplicial homology group**.

We now see one abstract way to define  $\sigma$ , which will give us so-called **singular homology group**.

**Definition 4.3.1** (Singular simplex). A *singular  $n$ -simplex* in a space  $X$  is a continuous map

$$\sigma: \Delta^n \rightarrow X.$$

**Definition 4.3.2** (Singular chain). Let  $C_n(X)$  be the **free group** on **singular  $n$ -simplices** in  $X$ , which we call it the *singular  $n$ -chains*.

**Definition 4.3.3** (Singular chain complex). The singular chains with boundary maps

$$\partial_n: C_n(X) \rightarrow C_{n-1}(X)$$

$$\sigma \mapsto \sum_{i=1}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]}$$

induces a *singular chain complex*.

**Definition 4.3.4** (Singular homology group). The *singular homology groups* are the homology groups of this singular chain complex given as

$$H_n(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}.$$

**Remark.** We now see that from the definition of homology group, we can put different structure on which. But the idea is the same, namely we are taking  $H_n(X)$  being

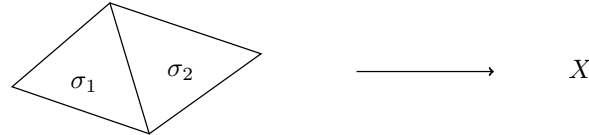
$$H_n(X) := \ker \partial_n / \operatorname{Im} \partial_{n+1},$$

where the difference is what structure we put on  $X$  which induces different chain complex  $C_n(X)$ . In this case, we have singular homology group since we are considering singular chain complex, while we can also have simplicial homology group.

Since the generating sets for  $C_n(X)$  when considering singular chain complex are almost always hugely uncountable from its definition, it's almost impossible to compute with these. However, it does give us a definition that does not depend on any other structure than the topology of  $X$ , making it useful for developing theory.

**Note.** The heuristic is that, we interpret a chain  $\sigma_1 \pm \sigma_2 \pm \dots \pm \sigma_k$  as a map from a  $\Delta$ -complex to  $X$ .

For example, with  $\sigma_1 + \sigma_2$  as below,



where we've glued  $[v_1, v_2]$  of  $\sigma_1$  to  $[v_0, v_2]$  of  $\sigma_2$  if  $\sigma_1|_{[v_1, v_2]}$  and  $\sigma_2|_{[v_0, v_2]}$  are the same singular  $n$ -chain with opposite signs.

With what we have defined, we now have some *goals*.

- Singular homology is a homotopy invariant. (Theorem 4.4.2)
- Singular and simplicial homology groups are isomorphic. (Theorem 4.5.6)

**Exercise.** We see some exercises.

1. Check that if  $X$  has path components  $\{X_\alpha\}$  then

$$H_n(X) \cong \bigoplus_{\alpha} H_n(X_\alpha).$$

2. If  $X = \{*\}$ , then

$$H_n(X) = \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$

3. If  $X$  is [path](#)-connected, then  $H_0(X) \cong \mathbb{Z}$ .

## 4.4 Functoriality and Homotopy Invariance

**Definition 4.4.1** (Induced map on chains). For a given continuous map  $f: X \rightarrow Y$ , we can consider the map  $f_\#$  induced by [chains](#) as

$$\begin{aligned} f_\# : C_n(X) &\rightarrow C_n(Y) \\ [\sigma : \Delta^n \rightarrow X] &\mapsto [f \circ \sigma : \Delta^n \rightarrow Y]. \end{aligned}$$

**Remark.** We see that the functoriality doesn't depend on any kind of [Δ-complex](#) structure.

**Definition 4.4.2** (Chain map). Given two [chain complexes](#)  $(C_*, \partial_*)$  and  $(D_*, \delta_*)$ , a *chain map* between them is a collection of group homomorphisms  $f_n : C_n \rightarrow D_n$  such that the following diagram commutes.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \xrightarrow{\partial_{n-1}} \dots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \dots & \xrightarrow{\delta_{n+2}} & D_{n+1} & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} \xrightarrow{\delta_{n-1}} \dots \end{array}$$

i.e. we have that  $\delta_n \circ f_n = f_{n-1} \circ \partial_n$ .

**Exercise.** We see that

1. We have that  $f_\# \partial = \partial f_\#$ . In other words,  $f_\#$  is a [chain map](#). Thus, by the homework  $f_\#$  induces a group homomorphism on the [homology groups](#). We write this as  $f_* : H_n(X) \rightarrow H_n(Y)$  for all  $n$ .
2. We have [functoriality](#), i.e.  $(f \circ g)_* = f_* \circ g_*$  and  $(\text{id}_X)_* = \text{id}_{H_n(X)}$ .

**Theorem 4.4.1** (Homology group defines a functor). The  $n$ -th [homology group](#)  $H_n : X \mapsto H_n(X)$  gives a [functor](#) from [Top](#) to [Ab](#).

**Proof.** This follows from the two exercises above. ■

**Theorem 4.4.2** (Functoriality is homotopy invariant). If  $f, g : X \rightarrow Y$  are [homotopic](#), then they will induce the same map on [homology](#)

$$f_* = g_* : H_n(X) \rightarrow H_n(Y).$$

The proof of [Theorem 4.4.2](#) can be found [here](#).

**Exercise.** [Theorem 4.4.1](#) and [Theorem 4.4.2](#) imply that  $H_n$  is a [homotopy](#) invariant.

## Lecture 24: Chain Homotopy

To prove [Theorem 4.4.2](#), we introduce some [homological algebra](#).

09 Mar. 10:00

**Definition 4.4.3 (Chain homotopy).** Given chain complexes  $(A_*, \partial_*^A)$  and  $(B_*, \partial_*^B)$  and chain maps  $f_\#, g_\#: A_* \rightarrow B_*$ . A chain homotopy from  $f_\#$  to  $g_\#$  is a sequence of group homomorphisms  $\psi_n: A_n \rightarrow B_{n+1}$  such that

$$f_n - g_n = \partial_{n+1}^B \circ \psi_n + \psi_{n-1} \circ \partial_n^A.$$

In diagram, letting  $h_n := f_n - g_n$ , we have the following.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_{n+2}^A} & A_{n+1} & \xrightarrow{\partial_{n+1}^A} & A_n & \xrightarrow{\partial_n^A} & A_{n-1} & \xrightarrow{\partial_{n-1}^A} & \dots \\ & & \downarrow h_{n+1} & \swarrow \psi_n & \downarrow h_n & \swarrow \psi_{n-1} & \downarrow h_{n-1} & & \\ \dots & \xrightarrow{\partial_{n+2}^B} & B_{n+1} & \xrightarrow{\partial_{n+1}^B} & B_n & \xrightarrow{\partial_n^B} & B_{n-1} & \xrightarrow{\partial_{n-1}^B} & \dots \end{array}$$

This diagram does **not** commute, however, the **red** map is the sum of the **blue** maps composed up, so it shows everything that is going on.

**Theorem 4.4.3.** If there is a chain homotopy  $\psi$  from  $f_\#$  to  $g_\#$ , then the induced maps  $f_*, g_*$  on homology are equal.

**Proof.** Let  $\sigma \in A_n$  be an  $n$ -cycle, i.e.  $\partial_n^A \sigma = 0$ . Then we compute that:

$$(f_n - g_n)(\sigma) = \partial_{n+1}^B(\psi_n(\sigma)) + \psi_{n-1}(\partial_n^A(\sigma)) = \partial_{n+1}^B(\psi_n(\sigma)) \in \text{Im}(\partial_{n+1}^B).$$

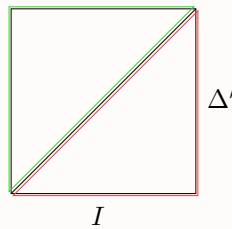
This tells us that  $(f_n - g_n)(\sigma)$  is a **boundary**, and so  $(f_n - g_n)(\sigma) = 0$  when considered as an element of the **homology group** (with degree  $n$ ). Thus,  $f_n(\sigma) = g_n(\sigma)$  in the **homology group**, and so  $f, g$  induce the same map as desired. ■

We now sketch the proof of Theorem 4.4.2 given in Hatcher[HPM02]. From this point in the course many of the theorems require much more algebraic work than we are interested in. We instead want to learn how to use the computational tools.

**Proof.** Suppose we have some homotopy  $F: I \times X \rightarrow Y$  from  $f$  to  $g$ . The most difficulty in this proof is the combinatorial difficulty involved in the fact that the product of a **simplex** in  $X$  and  $I$  is not a **simplex**.

We now consider

1. Subdivide  $\Delta^n \times I$  into  $(n+1)$ -dimensional **subsimplices**.<sup>a</sup>



2. We define the prism operator:

$$P_n: C_n(X) \rightarrow C_{n+1}(Y)$$

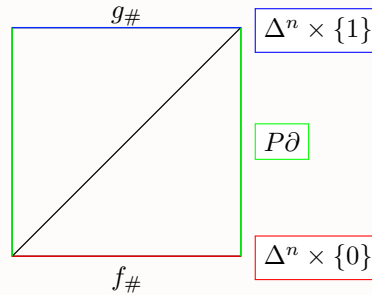
$$[\sigma: \Delta^n \rightarrow X] \mapsto \left[ \begin{array}{l} \text{alternating sums of restrictions} \\ \Delta^n \times I \xrightarrow{\sigma \times \text{id}} X \times I \xrightarrow{F} Y \\ \text{to each simplex in our subdivision} \end{array} \right]$$



3. We now need to check that

$$\partial_{n+1}^Y P_n = g_{\#} - f_{\#} - P_{n-1} \partial_n^X.$$

We have the following diagram.



Thus  $P$  is a [chain homotopy](#), and we're done. ■

<sup>a</sup>We want to do this since the product between two [simplices](#) is not a [simplex](#), as we just note.

## Lecture 25: Relative Homology

We are now interested in the relationship between  $H_n(X), H_n(A), H_n(X/A)$ .

11 Mar. 10:00

### 4.5 Relative Homology

**Definition 4.5.1** (Reduced homology group). The *reduced homology groups*  $\tilde{H}_n(X) = H_n(X)$  when  $n > 0$ . When  $n = 0$  we have that:

$$\tilde{H}_0(X) \oplus \mathbb{Z} = H_0(X).$$

**Remark.** The usefulness of this is that for [path-connected](#) space  $X$  we have  $\tilde{H}_0(X) = 0$ , and for [contractible](#) spaces  $X$  we have  $\tilde{H}_n(X) = 0$ .

**Definition 4.5.2** (Good pair). Let  $X$  be a space, and  $A \subseteq X$ . Then  $(X, A)$  is a *good pair* if  $A$  is closed and nonempty, and also it is a [deformation retract](#) of a neighborhood in  $X$ .

**Example.** Let's see some examples.

1. If  $X$  is a [CW complex](#) and  $A$  is a nonempty [subcomplex](#), then  $(X, A)$  is a [good pair](#). The proof is given in the Appendix of Hatcher[HPM02] and requires some point-set topology.
2. If  $M$  is a smooth manifold, and  $N \subseteq M$  is a smooth submanifold which is nonempty, then  $(M, N)$  is a [good pair](#).
3. (Hawaiian earring, bad point) is not a [good pair](#).
4.  $(\mathbb{R}^n, \text{proper open set})$  is not a [good pair](#).

**Theorem 4.5.1** (Long exact sequence of a good pair). If  $(X, A)$  is a [good pair](#), then there exists a long [exact](#) sequence ([exact](#) at every  $n$ ) on [reduced homology groups](#) given by the following commutative

diagram.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \tilde{H}_n(A) & \xrightarrow{i_*} & \tilde{H}_n(X) & \xrightarrow{j_*} & \tilde{H}_n(X/A) \\
 & & & & \searrow \delta & & \\
 & & \tilde{H}_{n-1}(A) & \xrightarrow{i_*} & \tilde{H}_{n-1}(X) & \xrightarrow{j_*} & \tilde{H}_{n-1}(X/A) \\
 & & & & \searrow \delta & & \\
 \dots & \xleftarrow{i_*} & \tilde{H}_0(X) & \xrightarrow{j_*} & \tilde{H}_0(X/A) & \longrightarrow & 0
 \end{array}$$

where  $i: A \hookrightarrow X$  is the inclusion and  $j: X \rightarrow X/A$  is the quotient map.

We see that both  $i_*$  and  $j_*$  is naturally induced, but not for  $\delta$ . In fact, we'll construct  $\delta$  in the proof! Specifically, we'll see that [Theorem 4.5.1](#) is just a special case of [Theorem 4.5.3](#), hence rather than proof [Theorem 4.5.1](#) directly, we will prove [Theorem 4.5.3](#) instead later.

**Remark.** The fact that this sequence is [exact](#) often means that if we know the [homology groups](#) of two of the spaces we can compute the [homology](#) of the remaining space.

Before we see the proof of [Theorem 4.5.1](#), we see one application.

**Proposition 4.5.1.** We have that:

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & \text{if } i = n; \\ 0, & \text{if } i \neq n. \end{cases}$$

**Proof.** Some facts we need:

- $(D^n, \partial D^n)$  is a [good pair](#) (since it is a [CW complex](#) and a [subcomplex](#))
- $D^n / \partial D^n \cong S^n$ .
- $\tilde{H}_n(D^n) = 0$  for all  $n$  since  $D^n$  is [contractible](#).
- $\partial D^n \cong S^{n-1}$ .

We then proceed by induction on  $n$ . To start with, we need to verify the following.

**Exercise.** Verify [Proposition 4.5.1](#) in the case  $n = 0$ , so  $S^0$  is just 2 points.

Now, using the long [exact sequence](#), we have

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \tilde{H}_n(\partial D^n) & \xrightarrow{i_*} & \tilde{H}_n(D^n) & \xrightarrow{j_*} & \tilde{H}_n(S^n) \\
 & & & & \searrow \delta & & \\
 & & \tilde{H}_{n-1}(\partial D^n) & \xrightarrow{i_*} & \tilde{H}_{n-1}(D^n) & \xrightarrow{j_*} & \tilde{H}_{n-1}(S^n) \\
 & & & & \searrow \delta & & \\
 \dots & \xleftarrow{i_*} & \tilde{H}_0(D^n) & \xrightarrow{j_*} & \tilde{H}_0(S^n) & \longrightarrow & 0
 \end{array}$$

By induction, we have  $\tilde{H}_{n-1}(\partial D^n) = \tilde{H}_{n-1}(S^{n-1}) = \mathbb{Z}$ , hence we can fill in some of these groups

as follows.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & 0 & \xrightarrow{i_*} & 0 & \xrightarrow{j_*} & \tilde{H}_n(S^n) \\
 & & & & & \nearrow \delta & \\
 & & \mathbb{Z} & \xleftarrow{i_*} & 0 & \xrightarrow{j_*} & \tilde{H}_{n-1}(S^n) \\
 & & & & & \nearrow \delta & \\
 & & \dots & \xleftarrow{i_*} & 0 & \xrightarrow{j_*} & \tilde{H}_0(S^n) \longrightarrow 0
 \end{array}$$

In all, we have an **exact** sequence:

$$0 \longrightarrow \tilde{H}_n(S^n) \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0$$

By **exactness**,  $\delta$  is an isomorphism, thus  $\tilde{H}_n(S^n) \cong \mathbb{Z}$ . Now we must verify  $\tilde{H}_i(S^n) = 0$  when  $i \neq n$ . In that case the **exact** sequence looks like:

$$\begin{array}{ccccccc}
 \longrightarrow & \tilde{H}_i(D^n) & \longrightarrow & \tilde{H}_i(S^n) & \longrightarrow & \tilde{H}_{i-1}(\partial D^n) \\
 & & & & & & \\
 \longrightarrow & 0 & \longrightarrow & \tilde{H}_i(S^n) & \longrightarrow & 0
 \end{array}$$

**Exactness** then tells us that  $\tilde{H}_i(S^n) = 0$ . ■

**Theorem 4.5.2** (Brouwer's fixed point theorem).  $\partial D^n$  is not a **retract** of  $D^n$ . Hence, every continuous map  $f: D^n \rightarrow D^n$  has a fixed point.

**Proof.** If  $r: D^n \rightarrow \partial D^n$  were a **retraction**, then by definition this would give us

$$\begin{array}{ccccc}
 \partial D^n & \xrightarrow{i} & D^n & \xrightarrow{r} & \partial D^n \\
 & & \searrow \text{id}_{\partial D^n} & & \nearrow
 \end{array}$$

**Functoriality** of **homology** implies

$$\begin{array}{ccccc}
 \tilde{H}_{n-1}(\partial D^n) & \xrightarrow{i_*} & \tilde{H}_{n-1}(D^n) & \xrightarrow{r_*} & \tilde{H}_{n-1}(\partial D^n) \\
 & & \searrow \text{id} & & \nearrow
 \end{array}$$

So then:

$$\begin{array}{ccccc}
 \mathbb{Z} & \xrightarrow{i_*} & 0 & \xrightarrow{r_*} & \mathbb{Z} \\
 & & \searrow \text{id} & & \nearrow
 \end{array}$$

which is impossible since the map  $\text{id}_{\mathbb{Z}}$  can't be factored through 0. ■

**Exercise.** As with  $D^2$ , if  $f: D^n \rightarrow D^n$  had no fixed point, we could build a **retraction**.

In order to proof **Theorem 4.5.1**, we introduce the concept of *diagram chase*.

**Lemma 4.5.1** (The short five lemma). Suppose we have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 0 & \longrightarrow & A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' \longrightarrow 0
 \end{array}$$

so that the rows are **exact**. Then:

1. If  $\alpha, \gamma$  are injective then  $\beta$  is injective.

2. If  $\alpha, \gamma$  are surjective then  $\beta$  is surjective.
3. If  $\alpha, \gamma$  are isomorphisms then  $\beta$  is an isomorphism

**Proof.** 1. and 2. imply 3. We leave 2. as an exercise. We fix  $b \in B$  such that  $\beta(b) = 0$ . We want to show that  $\beta = 0$ . Well, we draw a diagram chase as

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \bullet & \xrightarrow{\psi} & b & \xrightarrow{\varphi} & \varphi(b) \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 0 & \longrightarrow & \bullet & \xrightarrow{\psi'} & 0 & \xrightarrow{\varphi'} & 0 \longrightarrow 0
 \end{array}$$

And thus by injectivity of  $\gamma$  we know  $\varphi(b) = 0$ . By [exactness](#),  $b \in \text{Im } \psi$ . We then may write for some  $a \in A$  such that the following diagram commutes.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & a & \xrightarrow{\psi} & b & \xrightarrow{\varphi} & 0 \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 0 & \longrightarrow & \alpha(a) & \xrightarrow{\psi'} & 0 & \xrightarrow{\varphi'} & 0 \longrightarrow 0
 \end{array}$$

Therefore  $\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) = 0$  by commutativity. By [exactness](#) of the bottom row we know that  $\psi'$  is an injection.

Thus,  $\alpha(a) = 0$ , so since  $\alpha$  is injective,  $a = 0$ . With this  $b = \psi(a) = \psi(0) = 0$ . Great! With this  $\ker(\beta) = 0$ , and  $\beta$  injects. ■

## Lecture 26: Continue on Relative Homology

We start from a definition.

14 Mar. 10:00

**Definition 4.5.3 (Relative chain complex).** Let  $X$  be a space and let  $A \subseteq X$  be a subspace. Then we define the *relative chain complex*

$$C_n(X, A) = C_n(X) / C_n(A),$$

which is a quotient of [Abelian groups](#) of the [singular chain groups](#).

**Remark.** We can indeed adapt [Definition 4.5.3](#) by either [singular chain complex](#) structure or [simplicial chain complex](#) structure.

**Exercise.** Since  $\partial_n^*(C_n(A)) \subseteq C_{n-1}(A)$ , hence there exists a well-defined map

$$\partial_n: C_n(X) / C_n(A) \rightarrow C_{n-1}(X) / C_{n-1}(A).$$

We can verify that  $\partial^2 = 0$ . Then, since  $\partial^2 = 0$  we can conclude that these groups will in fact form a [chain complex](#)  $(C_*(X, A), \partial)$ .

**Definition 4.5.4 (Relative homology).** The [homology groups](#) of the [chain complex](#)  $(C_*(X, A), \partial)$  are denoted by  $H_n(X, A)$ , and they are called *relative homology groups*.

**(Relative cycle).** Elements in  $\ker \partial_n$  are called *relative  $n$ -cycles*. These are elements  $\alpha \in C_n(X)$  such that  $\partial_n \alpha \in C_{n-1}(A)$ .

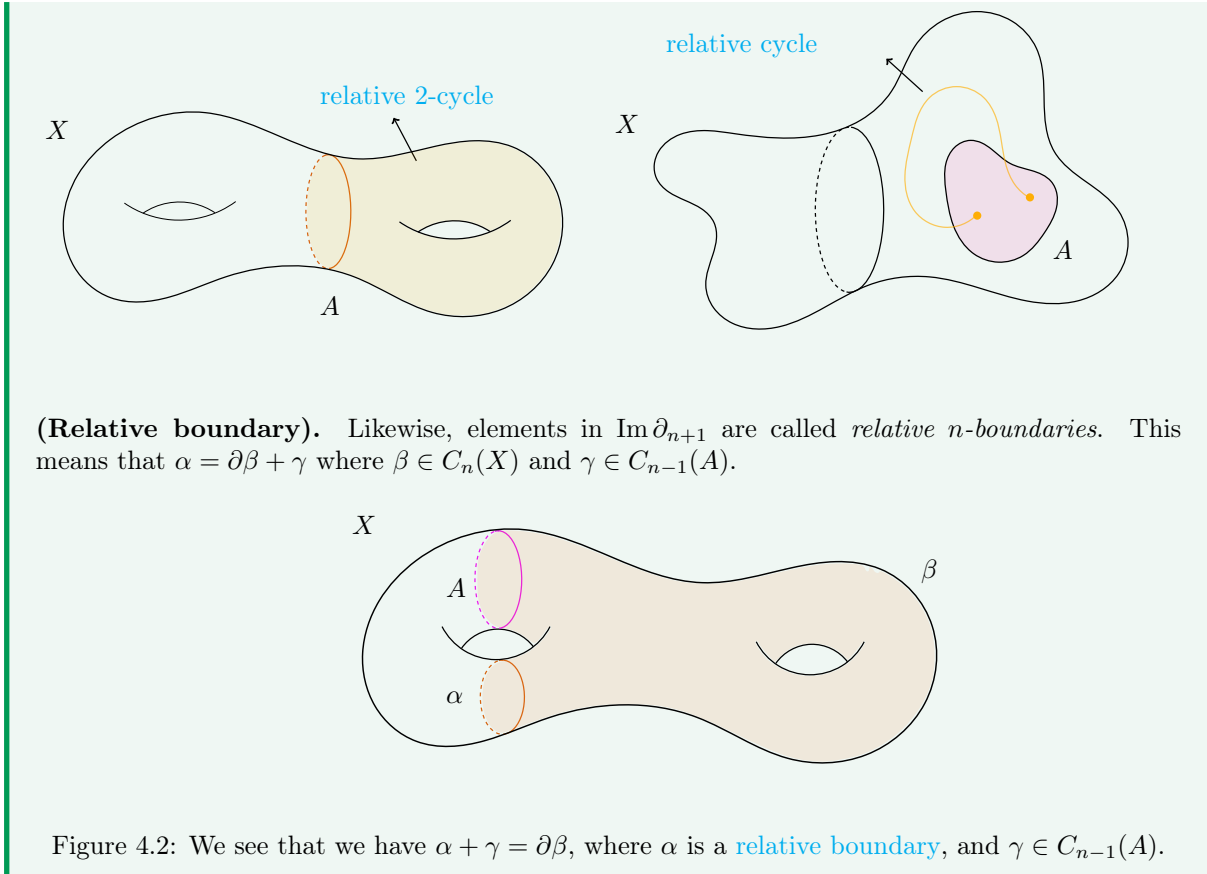


Figure 4.2: We see that we have  $\alpha + \gamma = \partial\beta$ , where  $\alpha$  is a **relative boundary**, and  $\gamma \in C_{n-1}(A)$ .

**Theorem 4.5.3** (Long exact sequence of a pair). Let  $A \subseteq X$  be spaces, then there exists a long **exact** sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & \tilde{H}_n(A) & \xrightarrow{i_*} & \tilde{H}_n(X) & \xrightarrow{q} & \tilde{H}_n(X, A) \\ & & & & & \nearrow \partial & \\ & & \tilde{H}_{n-1}(A) & \xleftarrow{i_*} & \dots & \xrightarrow{q} & \tilde{H}_0(X, A) \longrightarrow 0 \end{array}$$

where  $i_*$  is induced by  $A \hookrightarrow X$ , and  $q$  is induced by  $C_n(X) \twoheadrightarrow C_n(X) / C_n(A)$ .

We will prove that when  $(X, A)$  is a **good pair**, then  $H_n(X, A) \cong \tilde{H}_n(X / A)$ . Then **Theorem 4.5.1** is a special case of **Theorem 4.5.3**. The key to the proof of **Theorem 4.5.3** above is the following slogan.

**Remark.** A **short exact sequence** of **chain complexes** gives rise to a long **exact** sequence of **homology groups**. Namely, given a **short exact sequence** of **chain complexes**  $(A_*, \partial^A), (B_*, \partial^B), (C_*, \partial^C)$  such that

$$0 \longrightarrow A_* \xrightarrow{\iota} B_* \xrightarrow{q} C_* \longrightarrow 0$$

where  $\iota, q$  are **chain maps** such that

$$0 \longrightarrow A_n \xrightarrow{\iota_n} B_n \xrightarrow{q_n} C_n \longrightarrow 0$$

is **exact** for all  $n$ . Then **Theorem 4.5.1** will follow from a **short exact sequence**

$$0 \longrightarrow \tilde{C}_*(A) \longrightarrow \tilde{C}_*(X) \longrightarrow \tilde{C}_*(X, A) \longrightarrow 0$$

where  $\tilde{C}_*$  denotes the *augmented chain complex* (the one with  $\mathbb{Z}$  after it).

**Exercise.** If  $A$  is a single point in  $X$ , then  $H_n(X, A) = \tilde{H}_n(X / A) = \tilde{H}_n(X)$ .

## Lecture 27: Excision

Let's start with a theorem.

16 Mar. 10:00

**Theorem 4.5.4 (Excision).** Suppose we have subspace  $Z \subseteq A \subseteq X$  such that  $\overline{Z} \subseteq \text{Int}(A)$ . Then the inclusion

$$(X - Z, A - Z) \hookrightarrow (X, A)$$

induces isomorphisms

$$H_n(X - Z, A - Z) \xrightarrow{\cong} H_n(X, A).$$

**Remark.** Equivalently, for subspaces  $A, B \subseteq X$  whose interiors cover  $X$ , the inclusion

$$(B, A \cap B) \hookrightarrow (X, A)$$

induces an isomorphism

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

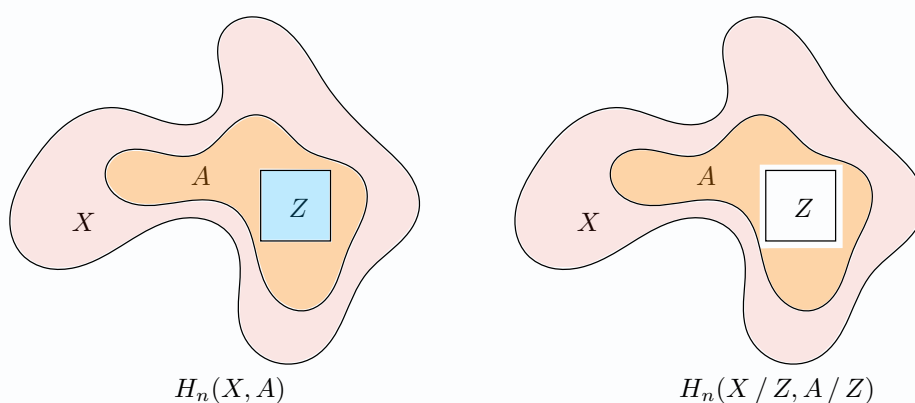
**Proof.** We see that this follows from

$$B := X \setminus Z, \quad Z = X \setminus B,$$

then we see that  $A \cap B = A - Z$  and the condition requires from [Theorem 4.5.4](#),  $\overline{Z} \subseteq \text{Int}(A)$  is then equivalent to

$$X = \text{Int}(A) \cup \text{Int}(B)$$

since  $X \setminus \text{Int}(B) = \overline{Z}$ .



We now prove the equivalent formulation of [Theorem 4.5.4](#) we derived above.

**Proof.** We sketch the proof here, which is notorious for being hairy.

- Given a [relative cycle](#)  $x$  in  $(X, A)$ , subdivide the [simplices](#) to make  $x$  a linear combination of chains on *smaller simplices*, each contained in  $\text{Int}(A)$  or  $X \setminus Z$ .


 Figure 4.3:  $\Delta^n \rightarrow X$  subdivide into subsimplices with images in.

This means  $x$  is homologous to sum of subsimplices with images in  $\text{Int}(A)$  or  $X \setminus Z$ . One of the things we use is that subsimplices are compact, so this process takes finite time.

The key is that the Subdivision operator is chain homotopic to the identity.

- Since we are working relative to  $A$ , the chains with image in  $A$  are zero, thus we have a relative cycle homologous to  $x$  with all subsimplices contained in  $X \setminus Z$ .

■

**Exercise.** Show that  $H_*(Y, y_0) \cong \tilde{H}(Y)$ .

**Theorem 4.5.5.** For good pairs  $(X, A)$ , the quotient map  $q: (X, A) \rightarrow (X/A, A/A)$  induces isomorphisms

$$q_*: H_n(X, A) \xrightarrow{\cong} H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$$

for all  $n$ .

**Proof.** Let  $A \subseteq V \subseteq X$  where  $V$  is a neighborhood of  $A$  that deformation retracts onto  $A$ . Using excision, we obtain a commutative diagram

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\cong} & H_n(X, V) & \xleftarrow{\cong} & H_n(X - A, V - A) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(X/A, A/A) & \xrightarrow{\cong} & H_n(X/A, V/A) & \xleftarrow{\cong} & H_n(X/A - A/A, V/A - A/A) \end{array}$$

Done if we can prove all the colored isomorphisms.

- $\cong$  is an isomorphism by excision.
- $\cong$  is an isomorphism by direct calculation (since  $q$  is a homeomorphism on the complement of  $A$ ).
- $\cong$  on Homework, since  $V$  deformation retracts to  $A$ .

**Remark.** The last equality is from the exercise since  $A/A = \{*\}$ .

■

## Lecture 28: Singular Homology v.s. Simplicial Homology

**Remark.** If  $M$  is a smooth manifold and  $N$  is an embedded smooth closed submanifold, then  $(M, N)$  is a good pair. Why? Well this follows from the tubular neighborhood theorem, which should be proven in a course like 591. We will only use the result in obvious cases, and simply assert that certain pairs are good pairs.

With pairs like  $(\mathbb{R}^{n+1}, S^n)$ , you can just assert that this is a good pair (and do not need to prove that  $S^n$  is a smooth submanifold of  $\mathbb{R}^{n+1}$ ). Another good example is manifolds and their boundary always

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form a [good pair](#).

**Theorem 4.5.6** (Singular homology agrees with simplicial homology). Let  $X$  be a [Δ-complex](#). We use  $\Delta_n(X)$  to represent the [simplicial chain groups](#) on  $X$ , and  $C_n(X)$  to denote the [singular chain groups](#). Likewise, we denote

$$\Delta_n(X, A) = \Delta_n(X) / \Delta_n(A)$$

and

$$C_n(X, A) = C_n(X) / C_n(A).$$

The inclusion  $\Delta_*(X, A) \hookrightarrow C_*(X, A)$  given by

$$[\sigma : \Delta^n \rightarrow X] \mapsto [\sigma : \Delta^n \rightarrow X]$$

induces an isomorphism on [homology](#) such that

$$H_n^\Delta(X, A) \cong H_n(X, A).$$

If we consider the case that  $A = \emptyset$ , we recover the case of [absolute homology](#)

$$H_n^\Delta(X) \cong H_n(X).$$

The proof of [Theorem 4.5.6](#) uses the following lemma.

**Lemma 4.5.2** (The five lemma). If we have a commutative diagram with [exact](#) rows as following,

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E' \end{array}$$

If  $\alpha, \beta, \delta, \epsilon$  are isomorphisms, then so is  $\gamma$ .

**Proof.** Diagram chase! ■

## Lecture 29: Proof of Theorem 4.5.6

We now give a proof sketch for [Theorem 4.5.6](#).

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**Proof.** The idea is as follows.

- We can use the long [exact sequence](#) of a pair and the [Lemma 4.5.2](#) to reduce to proving the result for [absolute homology groups](#) (and we will recover the general result).
- Because the image  $\Delta^n \rightarrow X$  is *compact*, it is contained in some finite skeleton  $X^k$ . Use this to reduce the proof to the finite skeleton  $X^k$  of  $X$ , namely we can use induction.

From the long [exact sequence](#) of a [pair](#) we get

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^k, X^{k-1}) & \longrightarrow & H_n^\Delta(X^{k-1}) & \longrightarrow & H_n^\Delta(X^k) & \longrightarrow & H_n^\Delta(X^k, X^{k-1}) & \longrightarrow & H_{n-1}^\Delta(X^{k-1}) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X^k) & \longrightarrow & H_n(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1}) \end{array}$$

The Goal is to prove  $\gamma$  is an isomorphism using the [Lemma 4.5.2](#).

We assume that  $\beta, \epsilon$  are isomorphisms by induction, checking the case manually for  $X^0$  (which will be a discrete set of points). It remains to show that  $\alpha, \delta$  are isomorphisms.



1. If the dimension is  $\leq d$ , then  $H_n(X) = 0$  for all  $n > d$ .
2. If  $\overline{X}$  has no cells of dimension  $p$ , then  $H_p(X) = 0$ .
3. If  $\overline{X}$  has no cells of dimension  $p$ , then  $H_{p-1}(X)$  is [free Abelian](#).

## 4.6 Degree

From group theory, this map must be multiplication by some integer  $d \in \mathbb{Z}$ , which we call it as the *degree*, denotes as  $\deg(f)$  of  $f$ .

1.  $\deg(\text{id}_{S^n}) = 1$  since  $(\text{id}_{S^n})_* = \text{id}_{\mathbb{Z}}$ .
2. If  $f: S^n \rightarrow S^n$ ,  $n \geq 0$  is not surjective, then  $\deg(f) = 0$ . To see this, we know that  $f_*$  factors as

And since the middle group is zero,  $f_* = 0$ .

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**Note.** The converse is true! We'll see this later.

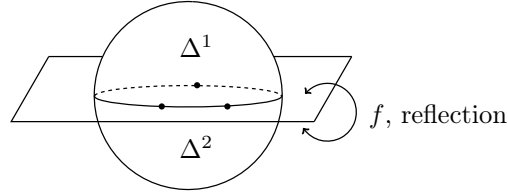
4.  $(f \circ g)_* = f_* \circ g_*$ , and so  $\deg(f \circ g) = \deg(f) \deg(g)$ .

Consequently, if  $f$  is a [homotopy equivalence](#) then  $\deg f = \pm 1$ .

**Exercise.** It is possible to put a [Δ-complex](#) structure with 2  $n$ -cells,  $\Delta_1$  and  $\Delta_2$  glued together along their [boundary](#) ( $\cong S^{n-1}$ ), and

$$H_n(S^n) = \langle \Delta_1 - \Delta_2 \rangle.$$

If  $f$  is a reflection fixing the equator, and swapping the 2-cells, then  $\deg f = -1$ .



5. We now have the following linear algebra exercise.

**Exercise.** The map  $S^{n+1} \rightarrow S^{n+1}$  given by  $x \mapsto -x$  is the composite of  $(n+1)$  reflections.

So the antipodal map  $S^n \rightarrow S^n$  given by  $x \mapsto -x$  has degree which is the product of  $n+1$  copies of  $(-1)$ , and so it has [degree](#)  $(-1)^{n+1}$ . (i.e., since the  $(n+1) \times (n+1)$  scalar matrix  $(-1)$  is composition of  $(n+1)$  reflections.)

6. We see the following.

**Exercise.** If  $f: S^n \rightarrow S^n$  has no fixed points, then we can homotope  $f$  to the antipodal map via

$$f_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}.$$

Therefore,  $\deg f = (-1)^{n+1}$ .

## Lecture 30: Degree

With the definition of [degree](#) and some of its [properties](#), we have the following theorems.

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**Theorem 4.6.1 (Hairy ball theorem).** The sphere  $S^n$  admits a nonvanishing continuous tangent vector field if and only if  $n$  is odd.

**Proof.** Recall that a [tangent vector field](#) to the unit sphere  $S^n \subseteq \mathbb{R}^{n+1}$  is a continuous map

$$v: S^n \rightarrow \mathbb{R}^{n+1}$$

such that  $v(x)$  is tangent to  $S^n$  at  $x$ , i.e.,  $v(x)$  is perpendicular to the vector  $x$  for each  $x$ . Let  $v(x)$  be a nonvanishing tangent vector field on the sphere  $S^n$ , then we define

$$f_t(x) := \cos(\pi t) + \sin(\pi t) \left( \frac{v(x)}{\|v(x)\|} \right),$$

which is a [homotopy](#) from the identity map  $\text{id}_{S^n}: S^n \rightarrow S^n$  to the antipodal map  $-\text{id}_{S^n}: S^n \rightarrow S^n$ .

This simply follows from varying  $t$  from 0 to 1, where we have

$$f_0(x) = \cos(0)x + \sin(0) \left( \frac{v(x)}{\|v(x)\|} \right) = x \Rightarrow f_0 = \text{id}_{S^n},$$

while

$$f_1(x) = \cos(\pi)x + \sin(\pi) \left( \frac{v(x)}{\|v(x)\|} \right) = -x \Rightarrow f_1 = -\text{id}_{S^n}.$$

The last thing needs to be verified is that  $f_t(x)$  is continuous, but this is trivial.

From the [property of degree](#), we know that it's a [homotopy](#) invariant, hence

$$\deg(-\text{id}_{S^n}) = \deg(\text{id}_{S^n}),$$

which implies

$$(-1)^{n+1} = 1,$$

so  $n$  must be odd.

Conversely, if  $n$  is odd, say  $n = 2k-1$ , we can define  $v(x_1, x_2, \dots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$ . Then  $v(x)$  is orthogonal to  $x$ , so  $v$  is a tangent vector field on  $S^n$ , and  $|v(x)| = 1$  for all  $x \in S^n$ . ■

**Theorem 4.6.2** (Groups acting on  $S^{2n}$ ). If  $G$  acts on  $S^{2n}$  [freely](#), then

$$G = \mathbb{Z}/2\mathbb{Z} \text{ or } 1.$$

**Proof.** There exists a homomorphism given by

$$\begin{aligned} G &\rightarrow \{\pm 1\} \\ g &\mapsto \deg(\tau_g) \end{aligned}$$

Where  $\tau_g$  is the action of  $g \in G$  on  $S^{2n}$  as a map  $S^{2n} \rightarrow S^{2n}$ . We know this map is well-defined since  $\tau_g$  is invertible (simply take  $\tau_{g^{-1}}$ ) for each  $g \in G$ . Our note on composites shows this is a homomorphism.

We want to show that the kernel is trivial, since then by the first isomorphism theorem  $G \cong \text{Im}$ , and the image is either trivial or  $\mathbb{Z}/2\mathbb{Z}$ . Suppose that  $g$  is a nontrivial element of  $G$ , then since  $G$  acts [freely](#) we know that  $\tau_g$  has no fixed points. With this in mind we have

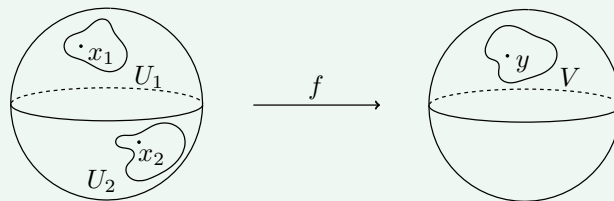
$$\deg \tau_g = (-1)^{2n+1} = -1.$$

Thus,  $g \notin \ker$ , hence the kernel is trivial as desired. ■

**Corollary 4.6.1.**  $S^{2n}$  has only the trivial cover  $S^{2n} \rightarrow S^{2n}$  or [degree 2](#) cover (for example,  $S^{2n} \rightarrow \mathbb{R}P^{2n}$ ).

**Proof.** This follows since any [covering space](#) action acts [freely](#). ■

**Definition 4.6.2** (Local degree). Let  $f: S^n \rightarrow S^n$  ( $n > 0$ ). Suppose there exists  $y \in S^n$  such that  $f^{-1}(y)$  is finite, say,  $\{x_1, \dots, x_m\}$ . Then let  $U_1, \dots, U_m$  be disjoint neighborhoods of  $x_1, \dots, x_m$  that are mapped by  $f$  to some neighborhood  $V$  of  $y$ .



The *local degree* of  $f$  at  $x_i$ , denote as  $\deg f|_{x_i}$ , is the **degree** of the map

$$f_*: \mathbb{Z} \cong H_n(U_i, U_i - \{x_i\}) \rightarrow H_n(V, V - \{y\}) \cong \mathbb{Z}.$$

**Remark.** The homomorphism  $f_*$  is a multiplication by an integer, which is the **local degree** as we just defined, arises from the following natural diagram.

$$\begin{array}{ccccc} & & H_n(U_i, U_i - \{x_i\}) & \xrightarrow{f_*} & H_n(V, V - \{y\}) \\ & \nwarrow \cong & \downarrow k_i & & \downarrow \cong \\ H_n(S^n, S^n - \{x_i\}) & \xleftarrow{p_i} & H_n(S^n, S^n - f^{-1}(y)) & \xrightarrow{f_*} & H_n(S^n, S^n - \{y\}) \\ & \nwarrow \cong & \uparrow j & & \uparrow \cong \\ & & H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \end{array}$$

The two isomorphisms in the upper half come from **excision**, and the lower two isomorphisms come from **exact sequences of pairs**.

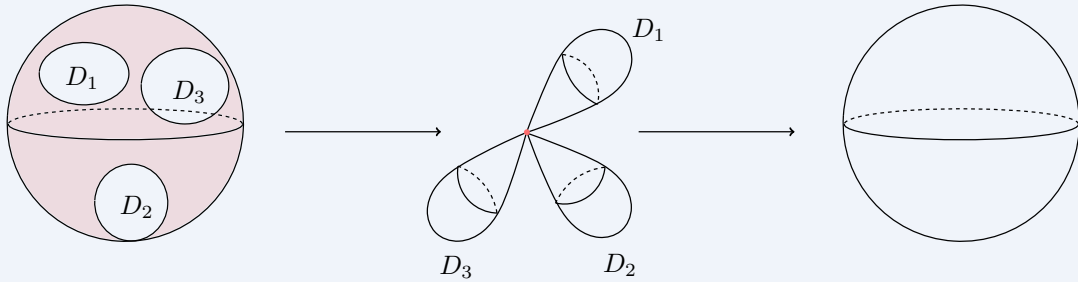
**Theorem 4.6.3.** Let  $f: S^n \rightarrow S^n$  with  $f^{-1}(y) = \{x_1, \dots, x_m\}$  as in **Definition 4.6.2**, then we have

$$\deg f = \sum_{i=1}^m \deg f|_{x_i}.$$

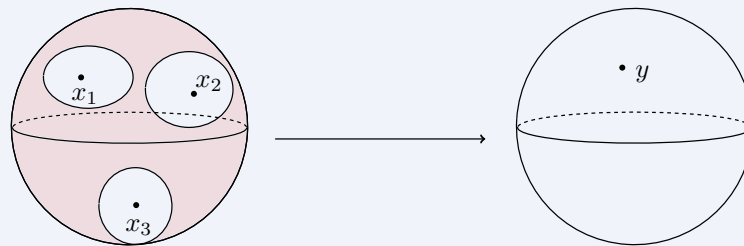
**Remark.** Thus, we can compute the **degree** of  $f$  by computing these **local degrees**.

Let's work with some examples for our edification.

**Example.** Consider  $S^n$  and choose  $m$  disks in  $S^n$ . Namely, we first collapse the complement of the  $m$  disks to a point, and then we identify each of the **wedged  $n$ -spheres** with the  $n$ -sphere itself.



The result will be a map of **degree  $m$** . We can see this by computing **local degree**.



By choosing a good point in the codomain, we get one point for each disk in the preimage, and the map is a local homeomorphism around these points which is orientation preserving. We could

likewise compose the maps to  $S^n$  from the wedge with a reflection to construct a map of degree  $-m$ .

**Remark.** We see that from the above construction, we can produce a map  $S^n \rightarrow S^n$  in any degree.

## Lecture 31: Local Degree and Local Homology

We first see another example of the application of Theorem 4.6.3.

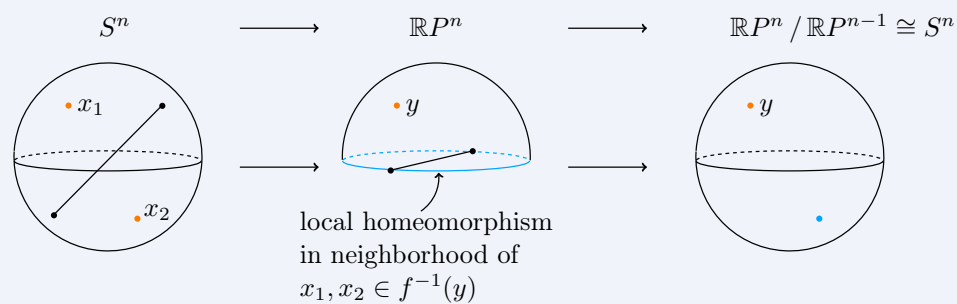
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**Example.** Consider the composition of the quotient maps below

$$S^n \xrightarrow{\quad} \mathbb{R}P^n \xrightarrow{\quad} \mathbb{R}P^n / \mathbb{R}P^{n-1} \cong S^n$$

$f$

We want to compute the degree of this map.



Note that this restricts to a homeomorphism on each component of  $S^n \setminus \text{equator}$  as a map to  $\mathbb{R}P^n / \mathbb{R}P^{n-1}$ . Suppose we've oriented our copies of  $S^n$  in such a way that the homeomorphism on the top hemisphere is orientation-preserving. The homeomorphism on the bottom hemisphere is given by taking the antipodal map and composing with the homeomorphism of the top hemisphere

$$\deg = \deg(\text{id}) = \deg(\text{antipodal}) = 1 + (-1)^{n+1} = \begin{cases} 0, & \text{if } n \text{ even;} \\ 2, & \text{if } n \text{ odd.} \end{cases}$$

We can now prove Theorem 4.6.3.

**Proof.** If  $f: S^n \rightarrow S^n$  and we have some  $y \in S^n$  with  $f^{-1}(\{y\}) = \{x_1, \dots, x_m\}$ , then we have a

nice commutative diagram as follows.

$$\begin{array}{ccc}
 H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\
 \text{LES of pair} \downarrow & & \downarrow \text{LES of a pair} \\
 \bigoplus_{i=1}^m \mathbb{Z} = H_n(S^n, S^n - \{x_1, \dots, x_m\}) & & \cong H_n(S^n, S^n - \{y\}) \\
 \text{excision} \uparrow \cong & & \downarrow \cong \\
 H_n\left(\coprod_{i=1}^m U_i, \coprod_{i=1}^m (U_i - \{x_i\})\right) & & H_n(S^n, S^n - \{y\}) \\
 \text{homology of disjoint union} \uparrow \cong & & \uparrow \text{excision} \\
 \bigoplus_i H_n(U_i, U_i - \{x_i\}) & \longrightarrow & H_n(V, V - \{y\})
 \end{array}$$
  

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & \deg(f_*) \\
 \downarrow & & \downarrow \\
 (1, 1, \dots, 1) & \xrightarrow{\quad} & \deg f = \sum \deg f|_{x_i}
 \end{array}$$

where we trace around the outside of the diagram at the bottom, which just proves the result. ■

With **degree**, we have a very efficient way for computing the **homology groups** of **CW complexes**, which is so-called **cellular homology**. But before we dive into this, we first grab some intuition about the essential of which, namely,

*what really is local homology?*

By **excision**, there is an isomorphism  $H_n(S^n, S^n \setminus \{x_i\}) \cong H_n(U, U \setminus \{x_i\})$  for any open neighborhood  $U$  of  $x_i$ .

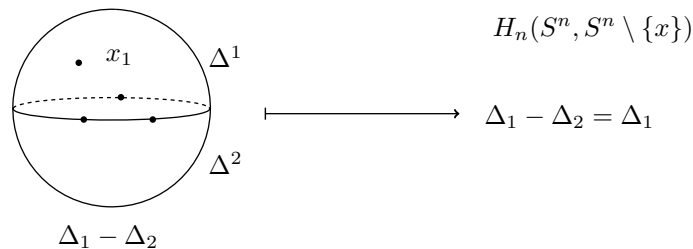
The long **exact sequence** of a **pair** also gives us

$$\dots \rightarrow H_k(S^n \setminus \{x_i\}) \rightarrow H_k(S^n) \rightarrow H_k(S^n, S^n \setminus \{x_i\}) \rightarrow H_{k-1}(S^n \setminus \{x_i\}) \rightarrow \dots$$

Since  $S^n \setminus \{x_i\}$  is homeomorphic to an open  $n$ -ball, we see that  $H_k(S^n \setminus \{x_i\}) = H_{k-1}(S^n \setminus \{x_i\}) = 0$ . With this in mind,  $j_*$  is an isomorphism.

We want to think about what  $j_*$  does when  $k = n$ , i.e., when this is an isomorphism  $\mathbb{Z} \cong H_n(S^n) \rightarrow H_n(S^n, S^n \setminus \{x_i\}) \cong \mathbb{Z}$ .

We see that  $\Delta_1 - \Delta_2$  generate  $H_n(S^n)$ , where  $\Delta_1, \Delta_2$  are the top and bottom hemisphere indicated below.



We then understand that  $j_*(\Delta_1 - \Delta_2) = \Delta_1 - \Delta_2 = \Delta_1$  since  $\Delta_2 = 0$  in  $C_n(S^n)/C_n(S^n \setminus \{x_i\})$ .

The upshot is that  $H_n(S^n, S^n \setminus \{x\})$  is generated by an  **$n$ -simplex** with  $x$  in its interior.

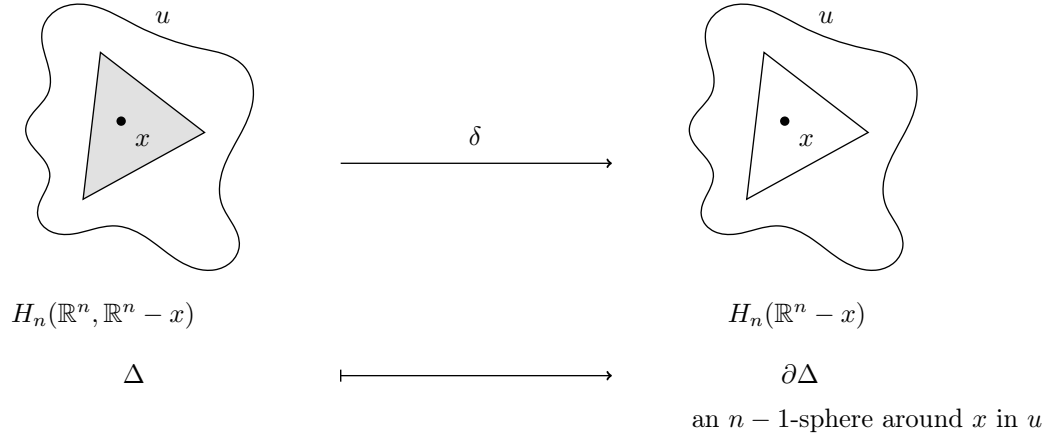
Suppose  $M$  is an  $n$ -manifold. Then  $H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$ , where  $U$  is a small ball around  $x$ . Because  $U$  is a ball homeomorphic to  $\mathbb{R}^n$ , we see that

$$H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}).$$

By the [long exact sequence of a pair](#)

$$0 = H_n(\mathbb{R}^n) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \rightarrow H_{n-1}(\mathbb{R}^n \setminus \{x\}) \rightarrow H_{n-1}(\mathbb{R}^n) = 0$$

And since  $\mathbb{R}^n \setminus \{x\}$  is [homotopy equivalent](#) to an  $n-1$  sphere, this means that  $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}$ . By homework, this connecting homomorphism is given by taking the [boundary](#) of a [relative cycle](#) as below.



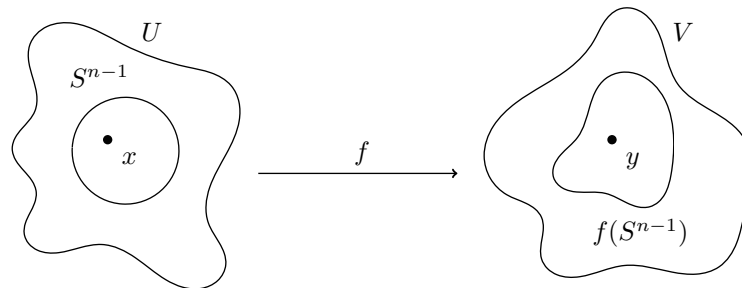
We intuitively want to use this idea to compute [degree](#) using this idea. We use naturality of the long [exact sequence](#), namely the fact that where  $f: (U_i, U_i \setminus \{x_i\}) \rightarrow (V, y)$  is a map of [pairs](#), then the following diagram commutes.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & H_n(U_i, U_i \setminus \{x_i\}) & \longrightarrow & H_{n-1}(U_i, U_i \setminus \{x_i\}) & \longrightarrow & \dots \\
 & & \downarrow f_* & & \downarrow f_* & & \\
 \dots & \longrightarrow & H_n(V, V \setminus \{y\}) & \longrightarrow & H_{n-1}(V, V \setminus \{y\}) & \longrightarrow & \dots
 \end{array}$$

By naturality of the long [exact sequence](#) and the isomorphism discussed above, we can compute the [local degree](#) of a map  $S^n \rightarrow S^n$  at a point  $x$  by computing the [degree](#) of the map

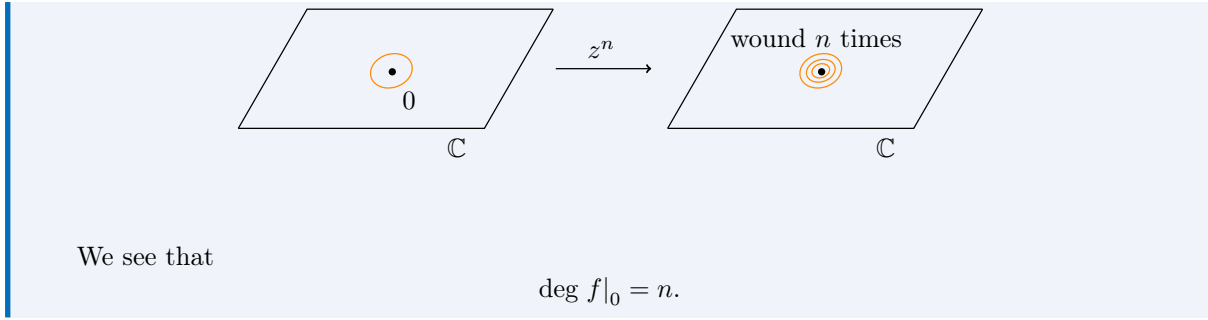
$$H_{n-1}(U \setminus \{x\}) \longrightarrow H_{n-1}(V \setminus \{y\})$$

In fact the [local degree](#) will be the [degree](#) restricted to a small  $S^{n-1}$  in the neighborhood  $U$ .



**Example.** Consider

$$\begin{aligned}
 \hat{\mathbb{C}} &\rightarrow \hat{\mathbb{C}} \\
 z &\mapsto z^n.
 \end{aligned}$$



## Lecture 32: Cellular Homology

### 4.7 Cellular Homology

28 Mar. 10:00

Suppose that  $X$  is a **CW complex**, then  $(X^n, X^{n-1})$  is a **good pair** for all  $n > 1$ , and  $X^n / X^{n-1}$  is a **wedge of  $n$ -spheres**, one for each  $n$ -cell  $e_\alpha^n$ . Hence,

$$H_k(X^n, X^{n-1}) \cong \begin{cases} 0, & \text{if } k \neq n; \\ \langle e_\alpha^n \mid e_\alpha^n \text{ is an } n\text{-cell} \rangle, & \text{if } k = n. \end{cases}$$

**Definition 4.7.1** (Cellular chain complex). The *cellular chain complex* on  $X$ , denoted as  $\bar{w}$ , has

(Chain groups). The chain groups  $C_n(X)$  are defined as

$$C_n(X) := \mathbb{Z} \langle e_\alpha^n \mid e_\alpha^n \text{ an } n\text{-cell of } X \rangle (\cong H_n(X^n, X^{n-1}))$$

with  $X^{-1} = \emptyset$ .

(Boundary maps). For  $n = 0$ , we have

$$\begin{aligned} \partial_1: C_1(X) &\rightarrow C_0(X) \\ \langle 1\text{-cells} \rangle &\rightarrow \langle 0\text{-cells} \rangle, \end{aligned}$$

which is the usual **simplicial boundary map**.<sup>a</sup> For  $n > 1$ , the **boundary map**  $\partial_n$  are defined as

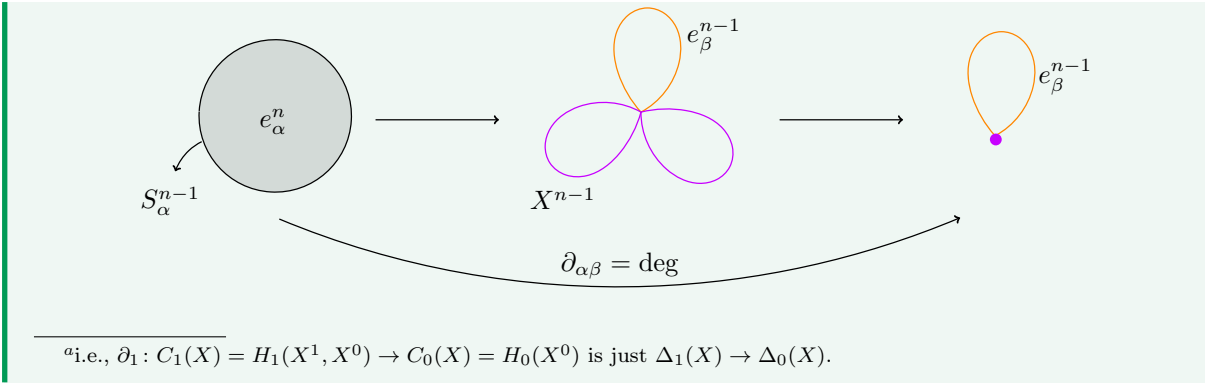
$$\partial_n(e_\alpha^n) = \sum_{\beta} \partial_{\alpha\beta} e_\beta^{n-1}$$

where  $\partial_{\alpha\beta}$  is the **degree** of the map

$$\partial e_\alpha^n = S_\alpha^{n-1} \xrightarrow[\text{map}]{\text{attaching}} X^{n-1} \xrightarrow[X^{n-1} \setminus e_\beta^{n-1}]{\text{quotient by}} S_\beta^{n-1}$$

In pictures, this is given as the following.





**Remark.** We see that

$$C_n(X) \cong H_n(X^n, X^{n-1})$$

since  $(X^n, X^{n-1})$  is a **good pair**, so  $H_n(X^n, X^{n-1}) \cong H_n(X^n / X^{n-1})$ , which is just the **wedge** of 1  $n$ -sphere for each  $n$ -cell of  $X$ .

Furthermore, the orientations on spheres are defined by identifying the domains of characteristic maps  $D_\alpha^n \rightarrow X$  with an (oriented) disk in  $\mathbb{R}^n$ . i.e., we need to choose a generator of

$$H_{n-1}(\partial D_\alpha^n) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

**Note.** In Hatcher[HPM02], the approach of the definition of **cellular chain complex** is a bit different, especially for how we define the boundary maps. Here we simply define  $\partial_n(e_\alpha^n) := \sum_\beta \partial_{\alpha\beta} e_\beta^{n-1}$ , where this is so-called *cellular boundary formula* in Hatcher[HPM02]. Here, we just defined  $\partial_n$  in this way instead, but we should still check that this is well-defined of this definition. The proof is given in [Appendix A.3](#).

**Definition 4.7.2** (Cellular homology group). We define the so-called *cellular homology group* by **cellular chain complex** in our usual way of defining **homology group**.

**Remark.** We sometimes denote the **cellular homology group** as  $H_n^{\text{CW}}(X)$  if it causes confusion.

**Theorem 4.7.1.** [Definition 4.7.1](#) indeed forms a **chain complex**.

**Proof.** We need to check two things, namely the chain group  $H_n(X^n, X^{n-1})$  defined in [Definition 4.7.1](#) is indeed **free Abelian** with basis in each  $n$ -cell. But this is trivial since we have an one-to-one correspondence with the  $n$ -cells of  $X$  as we have shown, and we can think of elements of  $H_n(X^n, X^{n-1})$  as linear combinations of  $n$ -cells of  $X$ .

The fact that the **boundary map** defined in [Definition 4.7.1](#) has the property  $\partial^2 = 0$  will be proved in [Theorem 4.7.2](#). ■

**Theorem 4.7.2** (Cellular homology agrees with singular homology). The **cellular homology groups** coincide with the **singular homology groups**, i.e.,

$$H_n^{\text{CW}}(X) \cong H_n(X).$$

**Note.** i.e., the isomorphism commutes  $\bar{\omega}f_*$  for all continuous  $f: X \rightarrow Y$ .

[Theorem 4.7.2](#) implies the following.

**Corollary 4.7.1.** We have the followings.

- $H_n(X) = 0$  if  $X$  has a **CW complex** structure with no  $n$ -cells.
- If  $X$  has a **CW complex** with  $k$   $n$ -cells, then  $H_n(X)$  is generated by at most  $k$  elements.
- If  $H_n(X)$  is a group with a minimum of  $k$  generators, then any **CW complex** structure on  $X$  must have at least  $k$   $n$ -cells.
- If  $X$  has a **CW complex** with no  $n$ -cells, then

$$H_{n-1}(X) = \ker(\partial_{n-1}),$$

which is **free Abelian**.

- If  $X$  has a **CW complex** with no cells in consecutive dimensions, then all  $\partial_n = 0$ . Its **homology** are **free Abelian** on its  $n$ -cells, namely the **cellular chain groups**.

**Example.** The last point in **Corollary 4.7.1** is quite useful, as the following examples will show.

1.  $S^n, n \geq 2$ . Since if we have  $S^n$  with  $n \geq 2$ , using the **CW complex** structure of  $e^n$  attached to a single point  $x_0$ . The **cellular chain complex** is given as

$$0 \longrightarrow 0 \longrightarrow \langle e^n \rangle \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \langle x_0 \rangle \longrightarrow 0$$

So then all the boundary maps are zero, and we see that

$$H_k(S^n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

2.  $\mathbb{C}P^n, \forall n$ . In this case, we can let  $\mathbb{C}P^n$  equipped with a **CW complex** structure with one cell of each even dimension  $2k \leq 2n$ , thus

$$H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \dots, 2n; \\ 0, & \text{otherwise.} \end{cases}$$

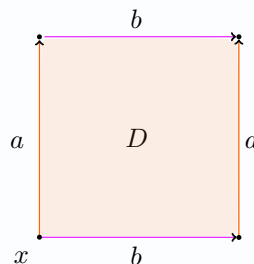
3.  $S^n \times S^n, n > 1$ . We let  $S^n \times S^n$  has the **product CW structure** consisting of a 0-cell, two  $n$ -cells, and a  $2n$ -cell.

**Exercise.** Redo this calculation with other **CW complex** structure on  $S^n$ , e.g. glue 2  $n$ -cells onto  $S^{n-1}$  and proceed inductively.

## Lecture 33: Cellular Homology Examples

**Example** (Cellular homology group of torus). Calculate the **cellular homology group** of a torus.

**Proof.** Let the torus equips with the following **CW complex** structure.



30 Mar. 10:00

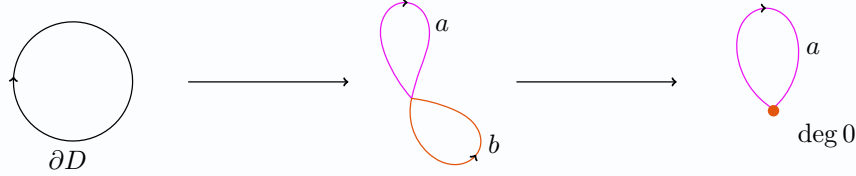
The **cellular chain complex** looks like

$$0 \longrightarrow \langle D \rangle \longrightarrow \langle a, b \rangle \longrightarrow \langle x \rangle \longrightarrow 0$$

where we choose  $x$  as a base point (i.e. the 0-cell).

For  $\partial_1$ , since this is defined as the same as the usual **simplicial boundary map**, hence by  $a \mapsto x - x = 0$  and  $b \mapsto x - x = 0$ , we have  $\partial_1 = 0$ .

Now for  $\partial_2$ , since  $D$  is glued along  $aba^{-1}b^{-1}$ , so we look at the composed up maps



We wind forwards then backwards around  $a$ ,<sup>a</sup> so the **degree** is zero. The same thing happens for  $b$ , so

$$\partial_2 D = \underbrace{0 \cdot a}_{\partial_{\alpha\beta_a} a} + \underbrace{0 \cdot b}_{\partial_{\alpha\beta_b} b} = 0,$$

where we assume that  $\alpha$  is the index of  $D$ , and  $\beta_a$  is the index of  $a$  and same for  $b$ .

This gives a nice **principle**, namely if a 2-cell  $D$  is glued down via some **words**  $w$  (this only makes sense for 2-cells), then the coefficient<sup>b</sup> to a letter  $a$  in  $\partial_2 D$  is the sum of the exponents of  $a$  in  $w$ . In this case, for both  $a$  and  $b$ , the coefficients for are both  $1 + (-1) = 0$ .

Now we just have that the **homology groups** are equal to the **chain groups** because the boundary maps are all zero. Hence, we have

$$H_k(T) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2; \\ \mathbb{Z}^2, & \text{if } k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

■

<sup>a</sup>Intuitively, since we **quotient** out  $b$ , hence the gluing map is **homotopy** to constant maps.

<sup>b</sup>i.e.  $\partial_{\alpha\beta}(a)$  where  $\alpha$  is the index of  $a$ .

**Example** (Cellular homology group of  $\Sigma_g$ ). Calculate the **cellular homology group** of a genus  $g$  surface  $\Sigma_g$ .

**Proof.** A genus  $g$  surface  $\Sigma_g$  has the **CW complex** structure as

- 1 0-cell  $x$ .
- $2g$  1-cells  $a_1, b_1, a_2, b_2, \dots$
- 1 2-cell  $D$  glued along  $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g]$  (a product of commutators)

For  $\partial_1$ , we have

$$\partial_1(a_i) = \partial_1(b_i) = x - x = 0.$$

Furthermore, by the principle discussed above, we know that every 1-cell appears once in the **word**, and its inverse appears once, so all the coefficients of 1-cells in  $\partial_2(D)$  are zero, so  $\partial_2(D) = 0$ . This means we have a **chain complex**

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

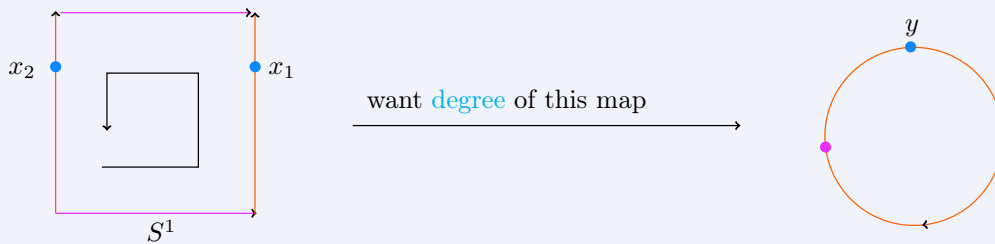
And so then we have that

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2; \\ \mathbb{Z}^{2g}, & \text{if } k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

■

**Exercise.** Calculate the cellular homology group of  $\mathbb{R}P^n$ .

**Example** (Torus example:  $\partial_2$  in more detail). We're going to work through this example a bit more carefully.



Let's zoom in on these two preimage points and use *local homology* to compute this:

Fill this up!

## Lecture 34: Proof of Theorem 4.7.2

We're now going to work towards proving that cellular homology agrees with singular homology. First we need some nontrivial preliminaries. 1 Apr. 10:00

**Lemma 4.7.1.** We have that

1.  $H_k(X^n, X^{n-1}) = \begin{cases} 0, & \text{if } k \neq n; \\ \langle n\text{-cells} \rangle, & \text{if } k = n. \end{cases}$
2.  $H_k(X^n) = 0$  for all  $k > n$ . If  $X$  is finite dimensional, then  $H_k(X^n) = 0$  for all  $k > \dim X$ .
3. The inclusion  $X^n \hookrightarrow X$  induces  $H_k(X^n) \rightarrow H_k(X)$ . Then this map is
  - an isomorphism for  $k < n$
  - surjective for  $k = n$
  - zero for  $k > n$ .

**Proof.** For 1., we see that

$$X^n / X^{n-1} \cong \text{wedge of one } n\text{-sphere for each } n\text{-cell.}$$

The result then follows from Theorem 4.5.5 and its immediately corollary, namely

$$\bigoplus_{\alpha} \tilde{H}_n(X_{\alpha}) \cong \tilde{H}_n \left( \bigvee_{\alpha} X_{\alpha} \right)$$

provided that the wedge sum is formed at basepoints  $x_{\alpha} \in X_{\alpha}$  such that  $(X_{\alpha}, x_{\alpha})$  are good, and then we simply consider  $(X, A) = (\coprod_{\alpha} X_{\alpha}, \coprod_{\alpha} \{x_{\alpha}\})$ .

Now we prove 2. and 3., We consider the long exact sequence of a pair for fixed  $n$ ,

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_{k+1}(X^n, X^{n-1}) & \longrightarrow & H_k(X^{n-1}) & & \\ & & & \searrow \cong & & & \\ & & H_k(X^n) & \longrightarrow & H_k(X^n, X^{n-1}) & \longrightarrow & \dots \end{array}$$

When  $k+1 < n$  or  $k > n$  then  $H_{k+1}(X^n, X^{n-1}) = 0$  and  $H_k(X^n, X^{n-1}) = 0$ , so the above map  $H_k(X^{n-1}) \rightarrow H_k(X^n)$  is an isomorphism. We also get sequences telling us the injective and surjective maps when  $k = n$  or  $k = n-1$ ,

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 = H_{n+1}(X^n, X^{n-1}) & \longrightarrow & H_n(X^{n-1}) & \longrightarrow & H_n(X^n) \\ & & & & \swarrow & & \\ & & H_n(X^n, X^{n-1}) & \longrightarrow & H_{n-1}(X^{n-1}) & \longrightarrow & H_{n-1}(X^n) \\ & & & & \swarrow & & \\ & & H_{n-1}(X^n, X^{n-1}) = 0 & \longrightarrow & \dots & & \end{array}$$

So the maps  $H_n(X^{n-1}) \rightarrow H_n(X^n)$  is injective, and the map  $H_{n-1}(X^{n-1}) \rightarrow H_{n-1}(X^n)$  is surjective.

Fix  $k$ , then we get a pile of maps induced by the inclusions  $X^n \hookrightarrow X^{n+1}$

$$\begin{array}{ccccccc} H_k(X^0) & \xrightarrow{\cong} & H_k(X^1) & \xrightarrow{\cong} & H_k(X^2) & \xrightarrow{\cong} & \dots \\ & & & & \searrow \cong & & \\ & & H_k(X^{k-1}) & \xrightarrow{\text{inj.}} & H_k(X^k) & \xrightarrow{\text{surj.}} & H_k(X^{k+1}) \\ & & & & \searrow \cong & & \\ & & H_k(X^{k+2}) & \xrightarrow{\cong} & H_k(X^{k+3}) & \xrightarrow{\cong} & \dots \end{array}$$

**Note.** This sequence is not exact. Descriptions of maps (in red) follow from our analysis of the long exact sequence of a pair above.

To prove 2.,

- $k = 0$ , we do this by hand.
- $k \geq 1$ , then  $H_k(X^0) = 0$ , so we have that  $H_k(X^0), \dots, H_k(X^{k-1})$  are all zero from the isomorphisms above. That is the  $k$ -th homology  $H_k(X^n) = H_k(X^n)$  is zero for every  $n$ -skeleton where  $n < k$ , just as desired.

We also have the following collection of maps for fixed  $k$

$$H_k(X^k) \xrightarrow{\text{surj.}} H_k(X^{k+1}) \xrightarrow{\cong} H_k(X^{k+2}) \xrightarrow{\cong} \dots$$

This implies 3. when  $X$  is finite dimensional. For general  $X$ , we use the fact that every simplex has image contained in some finite skeleton (since image is compact). ■

**Exercise.** Check 2. and 3. in Lemma 4.7.1 directly in the case that the CW complex structure is a  $\Delta$ -complex structure using simplicial chains.

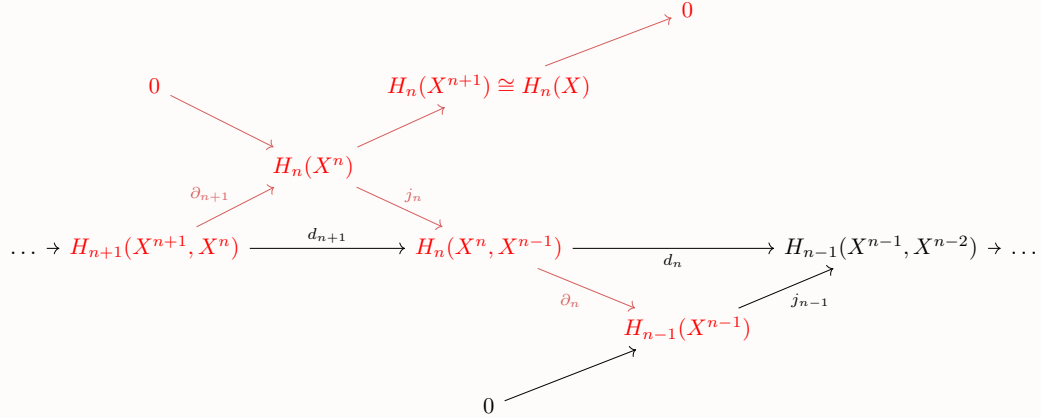
We now prove Theorem 4.7.2.

**Proof.** We get some [exact sequences](#) from our [preliminaries](#),

$$0 = H_{n+1}(X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^n, X^{n+1}) \longrightarrow H_{n-1}(X^n, X^{n+1})$$

$$H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) = 0$$

These come from the [long exact sequences of a pair](#) combined with the things we've deduced in the [preliminaries](#). We can paste these together into a diagram, we have



Hatcher[HPM02] tells us this diagram commutes, and what we've done here tells us that the two red diagonal pieces crossing at  $H_n(X^n)$  are [exact](#). We also have [exactness](#) of the bottom right diagonal by just going down a degree.

Then the horizontal row has to at least be a [chain complex](#) since the diagram commutes, and we have

$$d_n \circ d_{n+1} = (j_{n-1} \circ \underbrace{\partial_n \circ j_n}_{0}) \circ \partial_{n+1} = 0,$$

hence we see that  $d^2 = 0$ .<sup>a</sup>

By [exactness](#), we know that if  $\iota_*: H_n(X^n) \rightarrow H_n(X^{n+1})$ , then using the first isomorphism theorem,

$$H_n(X) \cong H_n(X^{n+1}) = \text{Im } \iota_* \cong H_n(X^n) / \ker \iota_* = H_n(X^n) / \text{Im } \partial_{n+1}.$$

Since  $j_n$  injects by [exactness](#),

$$\begin{aligned} j_n : H_n(X^n) &\xrightarrow{\cong} j_n(H_n(X^n)) \\ \text{Im } \partial_{n+1} &\xrightarrow{\cong} \text{Im}(j_n \circ \partial_{n+1}) = \text{Im } d_{n+1}, \end{aligned}$$

so  $j_{n-1}$  must also inject by [exactness](#), and by applying [exactness](#), we have

$$\ker d_n = \ker \partial_n = \text{Im } j_n.$$

Then we just do some group theory, the  $n$ -th [cellular homology group](#) is

$$\ker d_n / \text{Im } d_{n+1} \cong \text{Im } j_n / \text{Im}(j_n \circ \partial_{n+1}) \cong H_n(X^n) / \text{Im } \partial_{n+1} \cong H_n(X).$$

There is one thing left to show, namely commutativity of this map. We claim that the differentials  $d_n = j_n \circ \partial_{n+1}$  satisfy the formula (in terms of degree) that we stated. This is done by direct analysis of definitions of maps; details in Hatcher[HPM02]. ■

<sup>a</sup>This is the missing part of the proof of [Theorem 4.7.1](#).

## Lecture 35: Eilenberg-Steenrod Axioms

## 4.8 The Formal Viewpoint: Eilenberg-Steenrod Axioms

We can approach the homology theory in an **axiomatic** way. Specifically, we're interested in the Eilenberg-Steenrod axioms. To start with, we first see some definitions.

**Definition 4.8.1** (Natural transformation). Given two **functors**

$$F, G: \mathcal{C} \rightarrow \mathcal{D},$$

a *natural transformation*  $\eta: F \rightarrow G$  is a collection of maps  $\eta_X: F(X) \rightarrow G(X)$  lying in  $\mathcal{D}$  for every  $X \in \mathcal{C}$  so that for any map  $f: X \rightarrow Y$ , we have a commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

**Definition 4.8.2** (Homology theory). A *homology theory* is a sequence of **functors**

$$H_n: \text{pairs } (X, A) \text{ of spaces} \rightarrow \text{Abelian groups}$$

equipped with **natural transformations**  $\partial: H_n(X, A) \rightarrow H_{n-1}(A)$ , where  $H_{n-1}(A) := H_{n-1}(A, \emptyset)$ , is called the boundary map.

Naturality here means that for any map  $f: (X, A) \rightarrow (Y, B)$  we have a commutative diagram

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A) \\ f_* \downarrow & & \downarrow f_* \\ H_n(Y, B) & \xrightarrow{\partial} & H_{n-1}(B) \end{array}$$

These must satisfy the following 5 axioms.

1. (Homotopy) If  $f, g: (X, A) \rightarrow (Y, B)$  and  $f \simeq g$ , then  $f_* = g_*$ .
2. (Excision) If  $U \subseteq A \subseteq X$  such that  $\overline{U} \subseteq \text{Int}(A)$ , then

$$\iota: (X \setminus U, A \setminus U) \hookrightarrow (X, A)$$

induces isomorphisms on  $H_n$ .

3. (Dimension)  $H_n(*) = 0$  for all  $n \neq 0$ .
4. (Additivity)  $H_n(\coprod_{\alpha} X_{\alpha}) = \bigoplus_{\alpha} H_n(X_{\alpha})$ .
5. (Exactness) If we have an inclusion  $\iota: A \hookrightarrow X^a$  and  $j: X \rightarrow (X, A)$  induces a long **exact sequence**

$$\dots \rightarrow H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \dots$$

<sup>a</sup>Note that we use  $X := (X, \emptyset)$  for every space  $X$ .

**Definition 4.8.3** (Extraordinary homology theory). If  $H_*$  satisfies all **axioms** but dimension, it is called an *extraordinary homology theory*.

**Example.** Topological  $K$ -theory, bordism, and cobordism.<sup>a</sup>

<sup>a</sup><https://en.wikipedia.org/wiki/Cobordism>

**Theorem 4.8.1.** If  $H_n : \text{CW pairs} \rightarrow \underline{\text{Ab}}$  is a homology theory and  $H_0(*) = \mathbb{Z}$ , then  $H_n$  are exactly the singular homology functors up to a natural isomorphism of functors.

More generally, if  $H_0(*) = G$ , then  $H_n$  are exactly the singular homology functors with coefficients in the Abelian group  $G$ .

**Proof.** Given  $H_*$ , reconstruct the cellular chain groups  $H_n(X^n, X^{n-1})$  using the axioms.

- Show the homology of this chain complex are the cellular homology groups of  $X$ .
- Show these agree with  $H_n(X^n, X^{n-1})$ . The exact same argument in Theorem 4.7.2 applies.

We then check that the cellular homology groups we just constructed satisfies the degree formula as in our last step. This is a bit more difficult, but we won't get into it. ■



# Chapter 5

## Lefschetz Fixed Point Theorem

### 5.1 Lefschetz Fixed Point Theorem

**Definition 5.1.1 (Trace).** Let  $\varphi: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  be a group homomorphism, we may represent this with a matrix  $A = [a_{ij}]_{i,j}$  with *trace* being

$$\text{tr } A := a_{11} + \dots + a_{nn}.$$

For a group homomorphism  $\varphi: M \rightarrow M$  where  $M$  is a [finitely generated Abelian group](#), we define the *trace* of  $\varphi$  to be the *trace* of the induced map  $\bar{\varphi}: M/M_T \rightarrow M/M_T$ , where  $M_T$  is the [torsion subgroup](#) of  $M$ .

**Exercise.** We have

1.  $\text{tr}(AB) = \text{tr}(BA)$ .
2.  $\text{tr}(A) = \text{tr}(BAB^{-1})$ .

Thus, [trace](#) is independent of change of basis of  $\mathbb{Z}^n$ .

### Lecture 36: Lefschetz Fixed Point Theorem

**Definition 5.1.2 (Lefschetz number).** Let  $X$  be a space with the assumption that  $\bigoplus_k H_k(X)$  is finitely generated.<sup>a</sup> Then the *Lefschetz number*  $\tau(f)$  of a map  $f: X \rightarrow X$  is

$$\tau(f) := \sum_k (-1)^k \text{tr}(f_*: H_k(X) \rightarrow H_k(X)).$$

<sup>a</sup>That is, each [homology group](#) is finitely generated, and there are finitely many nonzero [homology groups](#). For example  $X$  could be a finite [CW complex](#).

**Example.** When  $f \simeq \text{id}_X$ . Then  $f_* = \text{id}_{H_k(X)}$  for all  $k$ . Then  $\text{tr}(f_*: H_k(X) \rightarrow H_k(X)) = \text{rank}(H_k(X))$ . Therefore,

$$\tau(f) = \sum_k \text{rank}(H_k(X)) = \chi(X),$$

where  $\chi(X)$  is the *Euler characteristic*.

**Theorem 5.1.1 (Lefschetz Fixed Point Theorem).** Suppose  $X$  admits a finite triangulation,<sup>a</sup> or more generally,  $X$  is a [retract](#) of a finite [simplicial complex](#).

Then if  $f: X \rightarrow X$  is a map with  $\tau(f) \neq 0$ , then  $f$  has a fixed point.

<sup>a</sup>i.e. a finite [simplicial complex](#) structure

6 Apr. 10:00

**Note.** Note that the converse does not hold.

**Theorem 5.1.2.** If  $X$  is a compact, locally contractible space that can be embedded in  $\mathbb{R}^n$  for some  $n$ , then  $X$  is a retract of a finite simplicial complex.

**Remark.** This includes

- Compact Manifolds.
- Finite CW complexes.

**Definition 5.1.3.** Let  $\mathbb{F}$  be a field, and let  $H_k(X; \mathbb{F})$  be the  $k$ -th homology of  $X$  with coefficients in  $\mathbb{F}$ . Then  $H_k(X; \mathbb{F})$  is always a vector space over  $\mathbb{F}$ . Define  $\tau^{\mathbb{F}}(X)$  be

$$\sum_k (-1)^k \operatorname{tr}(f_* : H_k(X; \mathbb{F}) \rightarrow H_k(X; \mathbb{F})).$$

**Remark.** The Lefschetz fixed point theorem still holds if we replace  $\tau(x) \neq 0$  with  $\tau^{\mathbb{F}} \neq 0$ .

**Example.** Let  $f : S^n \rightarrow S^n$  be a degree  $d$  map. Then  $\tau(f)$  is

$$(-1)^0 \operatorname{tr}(f_* : H_0(S^n) \rightarrow H_0(S^n)) + (-1)^n \operatorname{tr}(f_* : H_n(S^n) \rightarrow H_n(S^n)).$$

Then  $f_* : H_0(S^n) \rightarrow H_0(S^n)$  is the identity, and  $f_* : H_n(S^n) \rightarrow H_n(S^n)$  is given by the  $1 \times 1$  matrix with entry  $d$ . And then we have

$$\tau(f) = 1 + (-1)^n d.$$

**Corollary 5.1.1.**  $f$  has a fixed point whenever  $1 + (-1)^n d \neq 0$ . Namely, whenever  $d \neq (-1)^{n+1}$ . That is  $f$  has a fixed point if its degree is not equal to the degree of the antipodal map.

**Exercise.** If  $f : X \rightarrow X$ , then  $\operatorname{tr}(f_* : H_0(X) \rightarrow H_0(X))$  is equal to the # of path-components of  $X$  mapped to themselves.

**Exercise.** If  $X$  is contractible, then its homology is concentrated in degree zero, so  $\tau(f) = 1$ .

If  $X$  is a compact manifold or finite CW complex, every  $f$  has a fixed point (in particular, this recovers Brouwer's Fixed Point Theorem).

**Example.** If we consider the map  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by translation by  $x \neq 0$ , then  $\tau(f) = 1$ , but  $f$  does not have a fixed point. The key here is that  $\mathbb{R}$  is not compact.

**Example (Qual, May 2016).** Let  $X$  be a finite, connected CW complex.  $\tilde{X}$  is its universal cover, and  $\tilde{X}$  is compact. Show that  $\tilde{X}$  cannot be contractible unless  $X$  is contractible.

**Proof.** We actually have two different approaches.

1. By homework, we then know that, since  $\tilde{X}$  is contractible and  $\tilde{X}$  has finitely many sheets  $d$  over  $X$ ,

$$1 = \chi(\tilde{X}) = d \cdot \chi(X).$$

Therefore,  $\chi(X) = d = 1$ , and so  $p : \tilde{X} \rightarrow X$  is a 1-sheeted cover, so it is a homeomorphism. Therefore,  $X$  is contractible.

2. Since  $\tilde{X}$  is **contractible**,  $\tau(f) = 1$  for all  $f: \tilde{X} \rightarrow \tilde{X}$ . Furthermore, because  $\tilde{X}$  is compact and covers a finite **CW complex**, it is a finite **CW complex**. Therefore, the **Lefschetz fixed point theorem** applies, so any such map has a fixed point. If  $f$  is a **deck map**, then that means that  $f = \text{id}_{\tilde{X}}$  from our **covering space** theory.

We have proved then that  $X \cong \tilde{X} / G(\tilde{X})$  because  $p: \tilde{X} \rightarrow X$  is **normal**, but then the **deck group**  $G(\tilde{X})$  is trivial, so  $X \cong \tilde{X}$ , and we are done. ■

**Exercise.** A 1-sheeted **cover** is always injective and surjective. Furthermore, it's a local homeomorphism. This suffices to show that a 1-sheeted cover is a homeomorphism.

**Theorem 5.1.3.** If  $X$  is a finite **CW complex**, with **cellular chain groups**  $H_n(X^n, X^{n-1})$ . If we have a cellular map  $f: X \rightarrow X$ , so  $f$  induces maps  $f_*: H_n(X^n, X^{n-1}) \rightarrow H_n(X^n, X^{n-1})$ . Then

$$\tau(f) = \sum_n (-1)^n \text{tr}(f_*: H_n(X^n, X^{n-1}) \rightarrow H_n(X^n, X^{n-1})).$$

**Proof.** Do some algebra! This is a purely algebraic fact

**Exercise.** Given a commutative diagram with **exact** rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \end{array}$$

then  $\text{tr}(\beta) = \text{tr}(\alpha) + \text{tr}(\gamma)$ .

Using the above result, the theorem follows by an argument analogous to the argument for Euler Characteristic in Homework. ■

# Appendix

# Appendix A

## Additional Proofs

### A.1 Seifert-Van Kampen Theorem on Groupoid

**Theorem A.1.1** (Seifert-Van Kampen Theorem on groupoid). Given  $X_0, X_1, X$  as topological spaces with  $X_0 \cup X_1 = X$ . Then the functor  $\Pi: \underline{\text{Top}} \rightarrow \underline{\text{Gpd}}$  maps the [cocartesian](#) diagram in  $\underline{\text{Top}}_*$  to a [cocartesian](#) diagram in  $\underline{\text{Gp}}$  as follows.

$$\begin{array}{ccccc} (X_0 \cap X_1, x_0) & \xrightarrow{j_0} & (X_0, x_0) & & \Pi(X_0 \cap X_1) \xrightarrow{\Pi(j_0)} \Pi(X_0) \\ j_1 \downarrow & & \downarrow i_0 & \xrightarrow{\Pi} & \Pi(j_1) \downarrow \quad \quad \downarrow \Pi(i_0) \\ (X_1, x_0) & \xrightarrow{i_1} & (X, x_0) & & \Pi(X_1) \xrightarrow{\Pi(i_1)} \Pi(X) \end{array}$$

**Note.** Notice that  $X_0, X_1, X$  don't need to be [path](#)-connected in particular.

Surprisingly, the proof of [Appendix A.1](#) is much more elegant with the elementary proof of [Theorem 2.6.1](#), hence we give the proof here.

**Proof.** Let  $\mathcal{G} \in \text{Ob}(\underline{\text{Gpd}})$  a [groupoid](#), and given [functors](#)

$$F: \Pi(X_0) \rightarrow \mathcal{G}, \quad G: \Pi(X_1) \rightarrow \mathcal{G}$$

such that

$$\begin{array}{ccc} \Pi(X_0 \cap X_1) & \xrightarrow{\Pi(j_0)} & \Pi(X_0) \\ \Pi(j_1) \downarrow & & \downarrow \Pi(i_0) \\ \Pi(X_1) & \xrightarrow{\Pi(i_1)} & \Pi(X) \end{array} \quad \begin{array}{c} \xrightarrow{F} \\ \searrow \exists! K \\ \xrightarrow{G} \end{array} \mathcal{G}$$

We now only need to prove that there exists a unique [functor](#)  $K: \Pi(X) \rightarrow \mathcal{G}$  such that the above diagram commutes.

We can define  $K$  as

- on [objects](#): For all  $x \in \text{Ob}(\Pi(X)) = X$ ,

$$K(x) = \begin{cases} F(x), & \text{if } x \in X_0; \\ G(x), & \text{if } x \in X_1. \end{cases}$$

This is well-defined since the diagram (without  $K$ ) commutes.

- on **morphisms**: For every  $p, q \in X$ ,  $\langle \gamma \rangle : p \rightarrow q$  in  $\text{Hom}_{\Pi(X)}(p, q)$ , we need to define  $K(\langle \gamma \rangle) \in \text{Hom}_{\mathcal{G}}(K(p), K(q))$ . Our strategy is for every path  $\gamma$  from  $p$  to  $q$ , we define  $\tilde{K}(\gamma) \in \text{Hom}_{\mathcal{G}}(K(p), K(q))$ . Then if we also have  $\tilde{K}(\gamma) = \tilde{K}(\gamma')$  for  $\gamma \simeq \gamma' \text{ rel}\{0, 1\}$ , then we can just let

$$K(\langle \gamma \rangle) := \tilde{K}(\gamma).$$

Now we start to construct  $\tilde{K}$ .

Given a path  $\gamma : [0, 1] \rightarrow X$ ,  $\gamma(0) = p, \gamma(1) = q$ . Since  $\text{int}(X_0) \cup \text{int}(X_1) = X$ , we see that

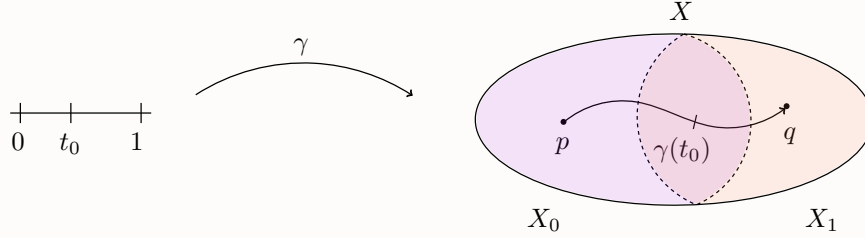
$$\gamma^{-1}(\text{int}(X_0)) \cup \gamma^{-1}(\text{int}(X_1)) = [0, 1].$$

From Lebesgue Lemma<sup>a</sup>, there exists a finite partition

$$0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1$$

such that for every  $i$ ,

$$\gamma([t_{i-1}, t_i]) \subset \text{int}(X_0) \text{ or } \text{int}(X_1).$$



Now, let  $\gamma_i : [0, 1] \rightarrow X, t \mapsto \gamma((1-t)t_{i-1} + t \cdot t_i)$ , we see that  $\gamma_i$  is either a **path** in  $X_0$  or  $X_1$ . We then define  $\tilde{K}(\gamma) := \tilde{K}(\gamma_m) \circ \tilde{K}(\gamma_{m-1}) \circ \dots \circ \tilde{K}(\gamma_1) \in \text{Hom}_{\mathcal{G}}(K(P), K(q))$  such that

$$\tilde{K}(\gamma_i) = \begin{cases} F(\langle \gamma_i \rangle), & \text{if } \gamma_i \subset X_0; \\ G(\langle \gamma_i \rangle), & \text{if } \gamma_i \subset X_1. \end{cases}$$

We need to prove that  $\tilde{K}(\gamma)$  does not depend on the partition. It's sufficient to prove that for any partition

$$0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1,$$

we consider any **finer** partition

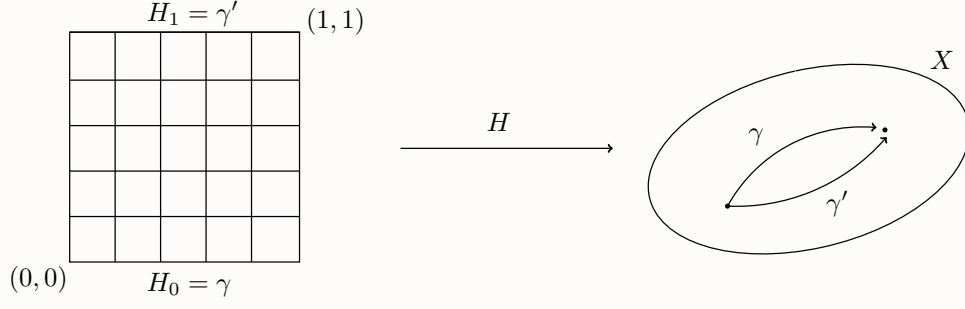
$$0 = t_0 = t_{10} < t_{11} < \dots < t_{1K_1} = t_1 = t_{20} < t_{21} < \dots < t_{mK_m} = t_m = 1.$$

As before, we denote  $\gamma_{ij} : [0, 1] \rightarrow X, t \mapsto \gamma((1-t)t_{ij-1} + t \cdot t_{ij})$ . It's clear that as long as

$$\tilde{K}(\gamma_i) = \tilde{K}(\gamma_{iK_i}) \circ \tilde{K}(\gamma_{iK_i-1}) \circ \dots \circ \tilde{K}(\gamma_{i0}),$$

then our claim is proved. But this is immediate since  $F$  and  $G$  are **functor** and for any  $i$ , we only use either  $F$  or  $G$  all the time.

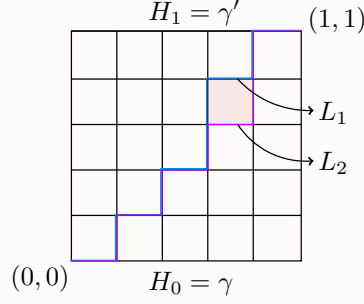
Now we prove  $\gamma \simeq \gamma' \text{ rel}\{0, 1\}$ , then  $\tilde{K}(\gamma) = \tilde{K}(\gamma')$ . This is best shown by some diagram.



The left-hand side represents a partition  $\mathcal{P}$  of  $[0, 1] \times [0, 1]$  such that every small square's image in  $X$  under  $H$  is either entirely in  $X_0$  or in  $x_1$ . Consider all paths from  $(0, 0)$  to  $(1, 1)$  such that it only goes right or up. We see that for any such path  $L$ , consider

$$\gamma_L: [0, 1] \rightarrow L, \quad t \mapsto \gamma_L(t).$$

We let  $\Gamma_L: H|_L \circ \gamma_L: [0, 1] \rightarrow X$ , we see that  $\Gamma_L$  is a **path** from  $p$  to  $q$ . Now, if for two paths  $L_1$  and  $L_2$  such that they only differ from a square.



We claim that  $\gamma_{L_1}, \gamma_{L_2}$  are two **paths** from  $p$  to  $q$ , and  $\tilde{K}(\Gamma_{L_1}) = \tilde{K}(\Gamma_{L_2})$ . Now, we denote  $\Gamma_0$  and  $\Gamma_1$  as follows.

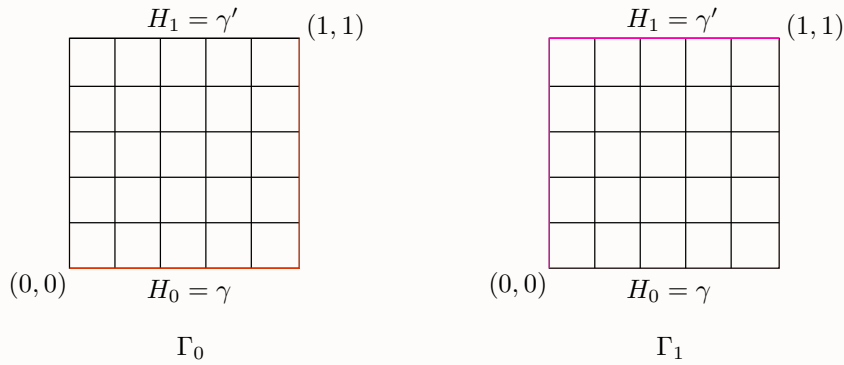


Figure A.1: The definition of  $\Gamma_0$  and  $\Gamma_1$ .

It's clearly that by only finitely many steps, we can transform  $\Gamma_0$  to  $\Gamma_1$ , hence

$$\tilde{K}(\Gamma_0) = \tilde{K}(\Gamma_1).$$

Finally, we observe that

$$\tilde{K}(\gamma_0) = \tilde{K}(\Gamma_0) = \tilde{K}(\Gamma_1) = \tilde{K}(\gamma_1).$$

If we now define  $K(\langle \gamma \rangle) = \tilde{K}(\gamma)$ , then  $K: \text{Mor}(\Pi(X)) \rightarrow \text{Mor}(\mathcal{G})$ , then it's well-defined.

We now prove  $K: \Pi(X) \rightarrow \mathcal{G}$  is indeed a **functor**. But this is immediate from the definition of  $K$ , namely it'll send identity to identity and the composition associates.

Also, we need to prove that the following diagram commutes.

$$\begin{array}{ccc} \Pi(X_0 \cap X_1) & \xrightarrow{\Pi(j_0)} & \Pi_1(X_0) \\ \Pi(j_1) \downarrow & & \downarrow \Pi(i_0) \\ \Pi_1(X_1) & \xrightarrow{\Pi(i_1)} & \Pi_1(X) \\ & \searrow G & \downarrow K \\ & & \mathcal{G} \end{array}$$

*(Note: A curved arrow labeled F points from \Pi(X\_0) to \mathcal{G})*

But this is again trivial.

Finally, we need to show that such  $K$  is unique. This is the same as the proof of [Lemma 1.6.1](#), hence the proof is done. ■

<sup>a</sup>[https://en.wikipedia.org/wiki/Lebesgue%27s\\_number\\_lemma](https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma)

## A.2 An alternative proof of Seifert Van-Kampen Theorem

**Theorem A.2.1.** We claim that the diagram

$$\begin{array}{ccc} \pi_1(X_0 \cap X_1, x_0) & \xrightarrow{(j_0)_*} & \pi_1(X_0, x_0) \\ (j_1)_* \downarrow & & \downarrow (i_0)_* \\ \pi_1(X_1, x_0) & \xrightarrow{(i_1)_*} & \pi_1(X, x_0) \end{array}$$

is **cocartesian**.

**Proof.** The basic idea is that, for this diagram,

$$\begin{array}{ccc} \Pi(X_0 \cap X_1) & \longrightarrow & \Pi(X_0) \\ \downarrow & & \downarrow \\ \Pi(X_1) & \longrightarrow & \Pi(X) \end{array}$$

we want to construct a **morphism**  $r: \Pi(Z) \rightarrow \pi_1(Z, p)$  in  $\underline{\text{Gpd}}$  such that  $Z = X_0 \cap X_1, X_0, X_1, X$ . For every  $x \in Z$ , we fix a **path**  $\gamma_x$  such that it connects  $p$  and  $x$  and satisfies

1. If  $x \in X_0 \cap X_1$ , then  $\text{Im}(\gamma_x) \subset X_0 \cap X_1$
2. If  $x \in X_0$ , then  $\text{Im}(\gamma_x) \subset X_0$
3. If  $x \in X_1$ , then  $\text{Im}(\gamma_x) \subset X_1$
4.  $\gamma_p = c_p$

The proof is given in [https://www.bilibili.com/video/BV1P7411N7fW?p=38&spm\\_id\\_from=pageDriver](https://www.bilibili.com/video/BV1P7411N7fW?p=38&spm_id_from=pageDriver). ■

If have time.



### A.3 Cellular Boundary Formula in Definition 4.7.1

**Theorem A.3.1.** For  $n > 1$ , the boundary maps  $\partial_n$  of cellular chain complex given by

$$\partial_n(e_\alpha^n) = \sum_{\beta} \partial_{\alpha\beta} e_\beta^{n-1}$$

is well-defined.

**Proof.** Here we are identifying the cells  $e_\alpha^n$  and  $e_\beta^{n-1}$  with generators of the corresponding summands of the cellular chain groups, namely  $C_n(X)$ . The summation in the formula contains only finitely many terms since the attaching map of  $e_\alpha^n$  has compact image, so this image meets only finitely many cells  $e_\beta^{n-1}$ . To derive the cellular boundary formula, consider the following commutative diagram.

$$\begin{array}{ccccc} H_n(D_\alpha^n, \partial D_\alpha^n) & \xrightarrow[\cong]{\partial} & \tilde{H}_{n-1}(\partial D_\alpha^n) & \xrightarrow{\Delta_{\alpha\beta*}} & \tilde{H}_{n-1}(S_\beta^{n-1}) \\ \downarrow \Phi_{\alpha*} & & \downarrow \varphi_{\alpha*} & & \uparrow q_{\beta*} \\ H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & \tilde{H}_{n-1}(X^{n-1}) & \xrightarrow{q*} & \tilde{H}_{n-1}(X^{n-1} / X^{n-2}) \\ & \searrow d_n & \downarrow j_{n-1} & & \downarrow \cong \\ & & H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{\cong} & H_{n-1}(X^{n-1} / X^{n-2}, X^{n-2} / X^{n-2}) \end{array}$$

where

- $\Phi_\alpha$  is the characteristic map of the cell  $e_\alpha^n$  and  $\varphi_\alpha$  is its attaching map.
- $q: X^{n-1} \rightarrow X^{n-1} / X^{n-2}$  is the quotient map.
- $q_\beta: X^{n-1} / X^{n-2} \rightarrow S_\beta^{n-1}$  collapses the complement of the cell  $e_\beta^{n-1}$  to a point, the resulting quotient sphere being identified with  $S_\beta^{n-1}$ <sup>a</sup> via the characteristic map  $\Phi_\beta$ .
- $\Delta_{\alpha\beta}: \partial D_\alpha^n \rightarrow S_\beta^{n-1}$  is the composition  $q_\beta q \varphi_\alpha$ , i.e., the attaching map of  $e_\alpha^n$  followed by the quotient map  $X^{n-1} \rightarrow S_\beta^{n-1}$  collapsing the complement of  $e_\beta^{n-1}$  in  $X^{n-1}$  to a point.

The map  $\Phi_{\alpha*}$  takes a chosen generator  $[D_\alpha^n] \in H_n(D_\alpha^n, \partial D_\alpha^n)$  to a generator of the  $\mathbb{Z}$  summand of  $H_n(X^n, X^{n-1})$  corresponding to  $e_\alpha^n$ . Letting  $e_\alpha^n$  denote this generator, commutativity of the left half of the diagram then gives

$$\partial_n(e_\alpha^n) = j_{n-1} \varphi_{\alpha*} \partial[D_\alpha^n].$$

In terms of the basis for  $H_{n-1}(X^{n-1}, X^{n-2})$  corresponding to the cells  $e_\beta^{n-1}$ , the map  $q_{\beta*}$  is the projection of  $\tilde{H}_{n-1}(X^{n-1} / X^{n-2})$  onto its  $\mathbb{Z}$  summand corresponding to  $e_\beta^{n-1}$ . Commutativity of the diagram then yields the formula for  $\partial_n$  given above. ■

<sup>a</sup>Which is just  $D_\beta^{n-1} / \partial D_\beta^{n-1}$ .

## Appendix B

# Abelian Group

This section aims to give some reference about [Abelian groups](#), specifically for [free Abelian group](#), which is used heavily when discuss homology.

### B.1 Abelian Group

**Definition B.1.1 (Abelian group).** A group  $(G, \cdot)$  is an *Abelian group* if for every  $a, b \in G$ , we have

$$a \cdot b = b \cdot a.$$

We often denote  $\cdot$  as  $+$  if  $(G, \cdot)$  is a [Abelian group](#).

**Definition B.1.2 (Product of groups).** Given two groups  $(G, \cdot), (H, \cdot)$ , the *product of  $G$  and  $H$* , denoted by  $G \times H$  is defined as

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

and

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot g_2, h_1 \cdot h_2).$$

**Notation.** For simplicity, given an index set  $I$ , we'll denote the order pair  $(g_{\alpha_1}, g_{\alpha_2}, \dots)$  as  $(g_{\alpha})_{\alpha \in I}$ . Note that the latter notation can handle the case that  $I$  is either countable or uncountable, while the former can only handle the countable case.

**Definition B.1.3 (Direct product).** Given  $(G_{\alpha}, +), \alpha \in I$  as a collection of [Abelian group](#), we define their *direct product* as

$$\left( \prod_{\alpha \in I} G_{\alpha}, + \right),$$

where

$$\prod_{\alpha \in I} G_{\alpha} = \{(g_{\alpha})_{\alpha \in I} \mid g_{\alpha} \in G_{\alpha}\}$$

and  $\forall (g_{\alpha}), (h_{\alpha}) \in \prod_{\alpha \in I} G_{\alpha}$

$$(g_{\alpha}) + (h_{\alpha}) := g_{\alpha} + h_{\alpha}$$

for all  $\alpha \in I$ .

Specifically, if  $I$  is finite, namely there are only finitely many [Abelian groups](#)  $(G_1, +), \dots, (G_n, +)$ , and  $\left( \prod_{i=1}^n G_i, + \right)$  can be denoted as

$$(G_1 \times \dots \times G_n, +).$$

**Definition B.1.4 (External direct sum).** Given a collection of Abelian groups  $\{G_\alpha\}_{\alpha \in I}$ , the *external direct sum* of them, denoted as  $(\bigoplus_{\alpha \in I} G_\alpha, +)$  as

$$\bigoplus_{\alpha \in I} G_\alpha := \left\{ (g_\alpha)_{\alpha \in I} \mid \forall_{\alpha \in I} g_\alpha \in G_\alpha, \# \text{ non-zero elements in } g_\alpha < \infty \right\}.$$

And for every  $(g_\alpha), (h_\alpha) \in \bigoplus_{\alpha \in I} G_\alpha$ ,

$$(g_\alpha) + (h_\alpha) := g_\alpha + h_\alpha$$

for all  $\alpha \in I$ .<sup>a</sup>

<sup>a</sup>This may not be the best notation: What we're really trying to say is  $(g_\alpha)_{\alpha \in I} + (h_\alpha)_{\alpha \in I} := g_i + h_i$  for all  $i \in I$ .

**Note.** We see that

$$\bigoplus_{\alpha \in I} G_\alpha \subset \prod_{\alpha \in I} G_\alpha.$$

Additionally, we also have

$$\left( \bigoplus_{\alpha \in I} G_\alpha, + \right) < \left( \prod_{\alpha \in I} G_\alpha, + \right).$$

**Remark.** We see that the operation  $+$  is indeed closed since the sum of  $g, g' \in \bigoplus_{\alpha \in I} G_\alpha$  will have only finitely non-zero elements if  $g, g'$  both have only finitely many non-zero elements.

We see that if  $I$  is a finite index set, given a collection of Abelian group  $\{G_\alpha\}_{\alpha \in I}$ , then

$$G_1 \times \dots \times G_n = G_1 \oplus \dots \oplus G_n.$$

**Definition B.1.5 (Internal direct sum).** Given an Abelian group  $G$ , and a collection of the subgroups  $\{G_\alpha\}_{\alpha \in I}$  of  $G$ , we say  $G$  is an *internal direct sum* of  $\{G_\alpha\}_{\alpha \in I}$  if for any  $g \in G$ , we can write

$$g = \sum_{\alpha \in I} g_\alpha$$

**uniquely**, where  $g_\alpha \in G_\alpha$  has only finitely many non-zero elements. In this case, we denote

$$G = \bigoplus_{\alpha \in I} G_\alpha.$$

Intuitively, the **external direct sum** is to build a new group based on the given collection of groups  $\{G_\alpha\}_{\alpha \in I}$ , while the internal direct sum is to express an **already known** group  $G$  with an **already known** collection of groups  $\{G_\alpha\}_{\alpha \in I}$ .

**Remark (Relation between Internal and External direct sum).** Given an Abelian group  $G$  and its internal direct sum decomposition  $\bigoplus_{\alpha \in I} G_\alpha$ ,  $G$  is isomorphic to the external direct sum of  $\{G_\alpha\}_{\alpha \in I}$ . We see this from the following group homomorphism:

$$\forall_{g \in G} g = \sum_{\alpha \in I} g_\alpha \mapsto (g_\alpha)_{\alpha \in I}.$$

Conversely, given a collection of Abelian group  $\{G_\alpha\}_{\alpha \in I}$ , and let  $G = \bigoplus_{\alpha \in I} G_\alpha$  as the external direct sum of  $\{G_\alpha\}$ , denote  $i_{\alpha_0}: G_{\alpha_0} \rightarrow \bigoplus_{\alpha \in I} G_\alpha$  as a canonical embedding

$$g_{\alpha_0} \mapsto i_{\alpha_0}(g_{\alpha_0}) = (h_\alpha)_{\alpha \in I},$$

where

$$h_\alpha = \begin{cases} g_{\alpha_0}, & \text{if } \alpha_0 = \alpha; \\ 0, & \text{if } \alpha_0 \neq \alpha \end{cases}$$

given  $\alpha_0$ . Then

$$G'_{\alpha_0} := i_{\alpha_0}(G_{\alpha_0}) < \bigoplus_{\alpha \in I} G_\alpha$$

and  $G$  is the **internal direct sum** of  $G'_{\alpha_0}$ ,  $\alpha_0 \in I$ . This is because  $\forall g = (g_\alpha)_{\alpha \in I} \in G (= \bigoplus_{\alpha \in I} G_\alpha)$ , we have

$$g = \sum_{\alpha \in I} i_\alpha(g_\alpha).$$

Note that the above sum is well-defined since there are only finitely many non-zero elements for each  $g_\alpha$ . And additionally, we can see the uniqueness of this decomposition by defining  $\pi_{\alpha_0}$  such that

$$\pi_{\alpha_0} : \bigoplus_{\alpha \in I} G_\alpha \rightarrow G_{\alpha_0}, \quad (g_\alpha)_{\alpha \in I} \mapsto g_{\alpha_0},$$

then  $\pi_\alpha \circ i_\alpha = \text{id}_{G_\alpha}$ ,  $\pi_\alpha \circ i_\beta = 0$  for all  $\beta \neq \alpha$  and

$$\pi_\beta(g) = \pi_\beta \left( \sum_{\alpha \in I} i_\alpha(g_\alpha) \right) = \sum_{\alpha \in I} \pi_\beta \circ i_\alpha(g_\alpha) = \pi_\beta \circ i_\beta(g_\beta) = g_\beta$$

for all  $\beta \in I$ , where the second equality is because this summation is finite. Hence, we have

$$g = \sum_{\alpha \in I} i_\alpha(\pi_\alpha(g)).$$

**Definition B.1.6.** Given two **Abelian groups**  $G, H$ , we define  $\text{Hom}(G, H)$  as

$$\text{Hom}(G, H) := \{f : G \rightarrow H \mid f \text{ is a group homomorphism}\},$$

then we can define

$$\begin{aligned} + : \text{Hom}(G, H) \times \text{Hom}(G, H) &\rightarrow \text{Hom}(G, H) \\ (\varphi, \psi) &\mapsto \varphi + \psi, \end{aligned}$$

where

$$(\varphi + \psi)(g) := \varphi(g) + \psi(g).$$

**Remark (Relation between direct sum and direct product).** Given a collection of **Abelian groups**  $\{G_\alpha\}_{\alpha \in I}$ , and another **Abelian group**  $H$ , there exists a  $\varphi$  such that

$$\begin{aligned} \varphi : \text{Hom} \left( \bigoplus_{\alpha \in I} G_\alpha, H \right) &\rightarrow \prod_{\alpha \in I} \text{Hom}(G_\alpha, H) \\ f &\mapsto \varphi(f) := (f_\alpha)_{\alpha \in I} \end{aligned}$$

where  $f_\alpha = f \circ i_\alpha$ , where  $i_\alpha$  is the canonical embedding from  $G_\alpha$  to  $\bigoplus_{\alpha \in I} G_\alpha$ . We claim that  $\varphi$  is an isomorphism.

- $\varphi$  is injective. This is obvious since  $\ker(\varphi) = 0$  from the fact that if  $\varphi(f) = 0$ , then  $f_\alpha = 0$  for all  $\alpha$ , hence  $f$  is 0.

- $\varphi$  is surjective. For every  $(f_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} \text{Hom}(G_\alpha, H)$ , we define

$$f: \bigoplus_{\alpha \in I} G_\alpha \rightarrow H$$

$$\sum_{\alpha \in I} g_\alpha \mapsto \sum_{\alpha \in I} f_\alpha(g_\alpha).$$

We see that  $f \in \text{Hom}(\bigoplus_{\alpha \in I} G_\alpha, H)$  and  $\varphi(f) = (f_\alpha)_{\alpha \in I}$ .

This shows that

$$\text{Hom}\left(\bigoplus_{\alpha \in I} G_\alpha, H\right) \cong \prod_{\alpha \in I} \text{Hom}(G_\alpha, H).$$

**Exercise.** We can show that

$$\text{Hom}\left(H, \prod_{\alpha \in I} G_\alpha\right) \cong \prod_{\alpha \in I} \text{Hom}(H, G_\alpha).$$

Note the order in the Hom matters.

## B.2 Free Abelian Group

**Definition B.2.1** (Free Abelian group). Given an Abelian group  $(G, +)$ , we say  $G$  is a *free Abelian group* if there exists a collection of elements  $\{g_\alpha\}_{\alpha \in J}$  in  $G$  such that  $\{g_\alpha\}_{\alpha \in J}$  forms a **basis** of  $G$ , i.e., for all  $g \in G$ ,  $\exists! n_\alpha \in \mathbb{Z}$  for all  $\alpha \in J$  such that

$$g = \sum_{\alpha \in J} n_\alpha g_\alpha$$

with finitely many non-zero  $n_\alpha$ .

**Remark.** If  $G$  is a free Abelian group, and  $\{g_\alpha\}_{\alpha \in J}$  is a basis, then for every  $\alpha \in J$ ,  $\langle g_\alpha \rangle$  is an infinite cyclic group since

$$n \cdot g_\alpha = 0 = 0 \cdot g_\alpha \Rightarrow n = 0.$$

And from Definition B.2.1, we have

$$G = \bigoplus_{\alpha \in J} \langle g_\alpha \rangle.$$

Conversely, assume there are a collection of infinite cyclic group  $\langle g_\alpha \rangle$  for  $\alpha \in I$  in  $G$  such that

$$G = \bigoplus_{\alpha \in I} \langle g_\alpha \rangle,$$

then  $\{g_\alpha\}_{\alpha \in I}$  is a basis of  $G$ , hence  $G$  is a free Abelian group.

**Proposition B.2.1.** If  $G$  is an Abelian group, then the following are equivalent.

1.  $G$  is a free Abelian group.
2.  $G$  is an internal direct sum of some infinite cyclic groups.
3.  $G$  is isomorphic to the external direct sum of some additive groups of integers  $\mathbb{Z}$ .

**Proof.** We see that 1.  $\Leftrightarrow$  2. is already proved. And for 2.  $\Leftrightarrow$  3., this follows directly from the relation between internal and external direct sum. ■

Now, consider  $G$  as a [free Abelian group](#), then

$$u: G \xrightarrow{\cong} \bigoplus_{\alpha \in I} \mathbb{Z}$$

for some  $I$ . Denote  $e_\alpha := i_\alpha(1) \in \bigoplus_{\alpha \in I} \mathbb{Z}$ , where  $i_\alpha: \mathbb{Z} \rightarrow \bigoplus_{\alpha \in I} \mathbb{Z}$  is the canonical embedding, i.e.,  $e_\alpha = (g_\alpha)_{\alpha \in I} \in \bigoplus_{\alpha \in I} \mathbb{Z}$ , where

$$g_\beta = \begin{cases} 1, & \text{if } \beta = \alpha; \\ 0, & \text{if } \beta \neq \alpha. \end{cases}$$

Moreover, denote  $\epsilon_\alpha$  as the image of  $e_\alpha$  under the isomorphism  $u$ , namely  $\epsilon_\alpha = u^{-1}(e_\alpha)$ , then  $\{\epsilon_\alpha\}_{\alpha \in I}$  is a basis of  $G$ .

Now, for every [Abelian group](#)  $H$ , we have

$$\begin{array}{ccc} \text{Hom}(G, H) & \xleftarrow[\cong]{\circ u} & \text{Hom}\left(\bigoplus_{\alpha \in I} \mathbb{Z}, H\right) \\ & \searrow \cong & \downarrow \varphi \\ & & \prod_{\alpha \in I} \text{Hom}(\mathbb{Z}, H) \\ & & \downarrow \cong \\ & & \prod_{\alpha \in I} H \end{array} \quad \begin{array}{ccc} f & \xrightarrow{\quad} & f \circ u^{-1} \\ & \nwarrow & \downarrow \\ & & (f \circ u^{-1} \circ i_\alpha)_{\alpha \in I} \\ & \nwarrow & \downarrow \\ & & (f \circ u^{-1} \circ i_\alpha(1))_{\alpha \in I} \end{array}$$

where  $\varphi$  is the homeomorphism defined in [here](#), and the homeomorphism

$$\prod_{\alpha \in I} \text{Hom}(\mathbb{Z}, H) \xrightarrow{\cong} \prod_{\alpha \in I} H$$

is trivial since every  $f \in \prod_{\alpha \in I} \text{Hom}(\mathbb{Z}, H)$  corresponds to  $f(1) \in H$  uniquely. We see that

$$f \circ u^{-1} \circ i_\alpha(1) = f \circ u^{-1}(e_\alpha) = f(\epsilon_\alpha).$$

In other words, for all [Abelian group](#)  $H$ , a morphism from the set  $\{\epsilon_\alpha\}_{\alpha \in I}$  to  $H$  can be uniquely extended to the group a homomorphism from  $G$  to  $H$ .

**Remark.** This means, to determine  $\text{Hom}(G, H)$ , we only need to determine where each base element in  $G$  will map to in  $H$ , and this is why it's *free*.

We now want to generate [free Abelian group](#) by a set. Roughly speaking, given a set  $S$ , we can generate a [free Abelian group](#)  $Z$  by defining

$$Z := \left\{ \sum_{x \in S} n_x x \mid n_x \in \mathbb{Z}, \# \text{ non-zero elements in } n_x < \infty \right\}$$

with the naturally defined  $+$ . Formally, we have the following.

**Definition B.2.2** (Free Abelian group generated by a set). Given a set  $S$ , the [free Abelian group generated by  \$S\$](#)   $(Z, +)$  is defined as

$$Z := \{f: S \rightarrow \mathbb{Z} \mid \text{only finitely many } x \in S \text{ such that } f(x) \neq 0\},$$

with

$$\begin{aligned} +: Z \times Z &\rightarrow Z \\ (f, g) &\mapsto f + g. \end{aligned}$$

**Remark.**  $\{\phi_x \mid x \in S\}$  forms a basis of  $Z$ , where  $\phi_x: S \rightarrow \mathbb{Z}$  such that

$$y \mapsto \phi_x(y) = \begin{cases} 1, & \text{if } y = x; \\ 0, & \text{if } y \neq x \end{cases}$$

is the characteristic function at  $x$ . We see this by for all  $f \in S$ ,  $f = \sum_{x \in S} f(x)\phi_x$ , which is uniquely defined. Hence,  $(Z, +)$  is a [free Abelian group](#).

**Note.** Note that

$$\begin{aligned} S &\xleftrightarrow{1:1} \{\phi_x \mid x \in S\} \\ x &\mapsto \phi_x. \end{aligned}$$

Hence, we often denote the element  $\sum_{x \in S} \underbrace{n_x}_{f(x)} \phi_x$  in  $Z$  as

$$\sum_{x \in S} n_x \cdot x.$$

**Theorem B.2.1** (The universal property of free Abelian group generated by a set). Denote a canonical embedding  $i: S \rightarrow Z$ ,  $x \mapsto \phi_x$ . Then for all [Abelian group](#)  $H$  and  $f: S \rightarrow H$ , there exists a unique group homomorphism

$$\tilde{f}: Z \rightarrow H$$

such that  $\tilde{f} \circ i = f$ .

**Proof.** We define

$$\tilde{f}\left(\sum_{x \in S} n_x \cdot x\right) := \sum_{x \in S} n_x f(x),$$

and the uniqueness is obvious. ■

Note that we can use the above [universal property](#) to describe a [free Abelian group](#) since we have the following.

**Proposition B.2.2.** Given  $Z'$  as another [Abelian group](#) and  $i': S \rightarrow Z'$  as another canonical embedding such that for all [Abelian group](#)  $H$  and  $f: S \rightarrow H$ , there exists a unique group homomorphism  $\tilde{f}: Z' \rightarrow H$  such that  $\tilde{f} \circ i' = f$ , then

$$Z' \cong Z.$$

Namely, we can describe a [free Abelian group](#) by its [universal property](#) uniquely up to isomorphism.

**Theorem B.2.2.** Assume  $G$  is a [free Abelian group](#). Assume there exists a finite basis  $\{g_1, \dots, g_n\}$  of  $G$ , and also assume that there exists another basis  $\{h_\alpha\}_{\alpha \in I}$ . Then we have

$$\text{card}(I) < \infty,$$

specifically, we have

$$\text{card}(I) = n.$$

**Proof.** Firstly, we observe that if we can show

$$\text{card}(I) \leq n,$$

then by swapping  $\{h_\alpha\}_{\alpha \in I}$  and  $\{g_\alpha\}_{\alpha \in I}$ , we will have  $\text{card}(I) = n$ .

Suppose  $I$  is an infinite set, then we can find  $h_{\alpha_1}, \dots, h_{\alpha_m}$  such that  $m > n$  and  $h_{\alpha_i} \neq h_{\alpha_j}$  for

$i \neq j$ . Then since  $\{g_\alpha\}_{\alpha \in I}$  is a basis, we have

$$h_{\alpha_i} = \sum_{j=1}^n k_i^j g_j, \forall i = 1, \dots, m.$$

Specifically, we have

$$\begin{pmatrix} h_{\alpha_1} \\ \vdots \\ h_{\alpha_m} \end{pmatrix} = \underbrace{\begin{pmatrix} k_1^1 & k_1^2 & \dots & k_1^n \\ \vdots & \vdots & \ddots & \vdots \\ k_m^1 & k_m^2 & \dots & k_m^n \end{pmatrix}}_{K \in M_{m \times n}(\mathbb{Z}) \subset M_{m \times n}(\mathbb{Q})} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix},$$

where  $k_i^j \in \mathbb{Z}$ . From linear algebra, we know that there exists  $(r_1, \dots, r_m) \in \mathbb{Q}^m \setminus \{\vec{0}\}$  such that

$$(r_1, \dots, r_m)K = (0, \dots, 0).$$

Multiplying both sides with the common multiple of the denominator of  $r_i$ , we see that there exists  $(\ell_1, \dots, \ell_m) \in \mathbb{Z}^m \setminus \{\vec{0}\}$  such that

$$\begin{aligned} & (\ell_1, \dots, \ell_m)K = (0, \dots, 0) \\ \Rightarrow & (\ell_1, \dots, \ell_m) \begin{pmatrix} h_{\alpha_1} \\ \vdots \\ h_{\alpha_m} \end{pmatrix} = (\ell_1, \dots, \ell_m)K \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = (0, \dots, 0) \\ \Rightarrow & \sum_{i=1}^m \ell_i h_{\alpha_i} = \vec{0} \text{ for } (\ell_1, \dots, \ell_m) \in \mathbb{Z}^m \setminus \{\vec{0}\} \\ \Rightarrow & \text{card}(I) < \infty. \end{aligned}$$

From the same argument, we see that  $\text{card}(I) \leq n \Rightarrow \text{card}(I) = n$ . ■

**Remark.** Furthermore, one can prove that if  $G$  is a [free Abelian group](#), then we can prove that any two bases of  $G$  are equinumerous, which handle the case that the basis is an infinite set.

This induces the following definition.

**Definition B.2.3 (Rank).** Let  $G$  be a [free Abelian group](#), the *rank* of  $G$  is the cardinality of any basis of  $G$ .

## B.3 Finitely Generated Abelian Group

Since we're going to encounter some group as

$$\mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z},$$

so it's useful to look into those finitely generated [Abelian group](#).

Let's start with a definition.

**Definition B.3.1 (Torsion subgroup).** Given an [Abelian group](#)  $G$ , we say that  $g \in G$  has finite order if  $\exists n \in \mathbb{Z}$  such that  $n \cdot g = 0$ . Specifically, we say that

$$T := \{g \in G \mid g \text{ has finite order}\}$$

is a *torsion subgroup*.

If  $T = 0$  given  $G$ , we say that  $G$  is *torsion free*.



**Note.** Note that  $T$  is indeed a subgroup, since for any  $g_1, g_2 \in T$ ,  $g_1 + g_2 \in T$  from the fact that it still has finite order.

**Remark.** If  $G$  is a **free Abelian group**, then  $G$  is **torsion free**. Conversely, if  $G$  is **torsion free**, we can't deduce  $G$  is a **free Abelian group**. We see this from  $(\mathbb{Q}, +)$ . Firstly, we see that  $\mathbb{Q}$  is **torsion free**. Now, suppose  $\mathbb{Q}$  is a **free Abelian group**, then there exists a basis  $\{r_\alpha\}_{\alpha \in I}$  of  $\mathbb{Q}$  such that  $|I| > 1$ . Now, consider  $\alpha_1, \alpha_2 \in I$  such that  $\alpha_1, \alpha_2 \in I$ , for  $r_{\alpha_1}, r_{\alpha_2}$ , there exists  $n, m \in \mathbb{Z}$  and  $n, m \neq 0$  such that

$$nr_{\alpha_1} + mr_{\alpha_2} = 0 \Rightarrow n = m = 0 \nmid$$

### B.3.1 Classification of Finitely generated Abelian Group

Given a finitely generated **Abelian group**  $G$ , we may assume its generators are  $g_1, \dots, g_n$ . Let  $F$  be

$$F := \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n \text{ times}}$$

then there are a natural surjective homomorphism

$$\varphi: F \rightarrow G, \quad e_i \mapsto g_i$$

where  $e_i = (0, \dots, 0, \underset{i^{th}}{1}, 0, \dots, 0)$ . Now, let  $K := \ker \varphi$ , we have

$$G \cong F / K.$$

Then we have the following lemma.

**Lemma B.3.1.**  $K$  is a finitely generated **Abelian group**.

**Proof.**

$\mathbb{Z}$  is Noetherian,  $F$  is a finitely generated  $\mathbb{Z}$ -module  
 $\Rightarrow F$  is Noetherian module  
 $\Rightarrow K$  as a submodule of  $F$  is a finitely generated  $\mathbb{Z}$ -module  
 $\Rightarrow K$  is a finitely generated **Abelian group**.

Please refer all the concepts above from [AM94]. ■

Hence, we may assume the generators of  $K$  as  $b_1, \dots, b_m$ . From the definition of  $K$ , we can further express  $b_i$  as

$$b_i = (b_{i1}, b_{i2}, \dots, b_{in}) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}_{n \times n}$$

for all  $i = 1, \dots, m$ . Denote all such row vectors  $b_i$  in a matrix  $B$ , namely

$$B := \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix} \in M_{m \times n}(\mathbb{Z}),$$

then we have

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = B \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

**Multiply a matrix on the right-hand side.** Now, consider a  $p \in \text{GL}(n; \mathbb{Z})$ , then

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = B \cdot \underbrace{P P^{-1}}_{\text{new basis}} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = (BP) \cdot \begin{pmatrix} e'_1 \\ \vdots \\ e'_n \end{pmatrix},$$

where

$$P^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} =: \begin{pmatrix} e'_1 \\ \vdots \\ e'_n \end{pmatrix}.$$

We see that  $B \cdot P$  is the coefficient matrix of generators  $b_1, \dots, b_m$  under the new basis  $e'_1, \dots, e'_n$ .

**Multiply a matrix on the left-hand side.** For a  $A \in \text{GL}(m; \mathbb{Z})$ , then

$$\begin{pmatrix} b'_1 \\ \vdots \\ b'_m \end{pmatrix} = Q \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = QB \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix},$$

since  $Q$  is invertible, hence  $b'_1, \dots, b'_m$  are also generators of  $K$ . We see that  $QB$  is the coefficient matrix of new generators  $b'_1, \dots, b'_m$  under basis  $e_1, \dots, e_n$ .

**Generally**  $Q \cdot B \cdot P$  is the matrix representation of a particular set of  $F$ 's generators under a particular basis.

**Proposition B.3.1.** There exists  $P \in \text{GL}(n; \mathbb{Z})$  and  $Q \in \text{GL}(m; \mathbb{Z})$  such that

$$Q \cdot B \cdot P = \begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

where  $d_i \in \mathbb{Z}^+$  and  $d_1 \mid d_2 \mid \dots \mid d_k$ .

**Proof.** In fact,  $P, Q$  can be taken as the multiplication of the following three types of square matrices:

- $P_{ij}$ :

$$P_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1_{(ij)} \\ & & & \ddots & \\ & 1_{(ji)} & & 0 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

where the effect of multiplying  $P_{ij}$  from the right is *swapping column  $i, j$* .

- $P_i(c)$ , where  $c$  is the identity in  $\mathbb{Z}$ , i.e.,  $c = \pm 1$ :

$$P_i(c) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c_{(ii)} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

where the effect of multiplying  $P_i(c)$  from the right is *multiplying  $c$  to column  $i$* .

- $P_{ij}(a)$ ,  $a \in \mathbb{Z}$ :

$$P_{ij} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & a_{(ij)} & \\ & & & \ddots \\ & & & & 1 \end{pmatrix},$$

where the effect of multiplying  $P_{ij}(a)$  from the right is *adding  $a$  times column  $i$  to column  $j$* .

We see that these are *elementary column transformations* in linear algebra. In particular, if we multiply these matrices from the left, then it's called *elementary row transformations*.

That is to say, we're going to show

$$B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix}$$

can become

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 \\ & & & & \ddots \end{pmatrix},$$

$d_i \in \mathbb{Z}^+$ ,  $d_1 \mid d_2 \mid \dots \mid d_k$  from *elementary column/row transformations*.

We now show the steps to make this happens.

- **Step 1.** Using elementary transformations, we make  $b_{11} > 0$ .
- **Step 2.** Using elementary transformations, we make  $b_{11}$  become a divisor of all elements in the first column and row.

We see that if  $b_{11} \nmid b_{1i}$  for  $i \neq 1$ , we have  $b_{1i} = r \cdot b_{11} + s$  where  $0 < s < b_{11}$ . Then we add  $(-r)$  times the  $1^{th}$  column to the  $i^{th}$  column and swapping the  $1^{th}$  and the  $i^{th}$  column, which makes  $B$  becomes

$$\begin{pmatrix} s & \dots \\ \vdots & \ddots \end{pmatrix},$$

for  $0 < s < b_{11}$ . Since  $\text{card}(\{n \in \mathbb{Z} \mid 0 < n < b_{11}\}) < \infty$ , hence in finitely many steps we can make  $B$  becomes

$$\begin{pmatrix} d_1 & \dots \\ \vdots & \ddots \end{pmatrix},$$

where  $d_1$  is a divisor of all other elements in the first column and row.

- **Step 3.** Using elementary transformations, we can multiply the first row by a proper integer and add it to the other rows, do the same but for columns also, then we can make  $B$  becomes

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B_1 & \\ 0 & & & \end{pmatrix}.$$

- **Step 4.** We iteratively apply Step 1. to step 3., we make  $B$  into

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 \\ & & & & \ddots \end{pmatrix},$$

where  $d_i \in \mathbb{Z}^+$ .

- **Step 5.** Using elementary transformations, by swapping columns and rows, we may assume  $d_1 \leq d_2 \leq \dots \leq d_k$ .
- **Step 6.** Using elementary transformations, we can make  $B$  into

$$\begin{pmatrix} d'_1 & & & \\ & \ddots & & \\ & & d'_\ell & \\ & & & 0 \\ & & & & \ddots \end{pmatrix}$$

such that  $0 < d'_1 \leq \dots \leq d'_\ell$ ,  $d'_1 \mid d'_2 \mid \dots \mid d'_\ell$  since if  $d_1 \nmid d_i$  for some  $i \in \{2, \dots, k\}$ , then

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 \\ & & & & \ddots \end{pmatrix} \rightarrow \begin{pmatrix} d_1 & d_i & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 \\ & & & & \ddots \end{pmatrix},$$

then from Step 2., we have

$$\begin{pmatrix} s & \dots \\ \vdots & \ddots \end{pmatrix}$$

where  $0 < s < d_1$  and  $s$  is a divisor of all other elements in the first row and column. Now, we repeat Step 3. to Step 5., we obtain

$$\begin{pmatrix} \tilde{d}_1 & & & \\ & \ddots & & \\ & & \tilde{d}_j & \\ & & & 0 \\ & & & & \ddots \end{pmatrix}$$

where  $\tilde{d}_1 \leq \dots \leq \tilde{d}_j$  such that  $\tilde{d}_1 < d_1$ . Since there are only finitely many integers which is smaller than  $d_1$ , we see that by repeating these steps, we can always make

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 \\ & & & & \ddots \end{pmatrix}$$

into

$$\begin{pmatrix} \tilde{d}_1 & & & \\ & \ddots & & \\ & & \tilde{d}_p & \\ & & & 0 \\ & & & & \ddots \end{pmatrix}$$

such that  $d'_1 \mid d'_i$  for all  $i \neq 1$  and  $d'_1 \leq d'_2 \leq \dots \leq d'_p$ . By the same idea of Step 3., we have the desired matrix.

Since all operations are elementary and there are only finitely many of them, hence the result follows.  $\blacksquare$

From the definition of  $Q \cdot B \cdot P$  and [Proposition B.3.1](#), there exists a basis  $e'_1, \dots, e'_n$  of  $F$  such that  $K$  has finitely many generators  $d_1 e'_1, \dots, d_k e'_k$ , hence

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_k\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n-k \text{ times}}.$$

This leads to the following important theorem.

**Theorem B.3.1** (Fundamental theorem of finitely generated Abelian group). Given a finitely generated Abelian group, either  $G$  is a [free Abelian group](#), or there exists a unique set of  $\{m_i \in \mathbb{Z} \mid m_i > 1, i = 1, \dots, t\}$  such that  $m_1 \mid m_2 \mid \dots \mid m_t$  and a unique non-negative integer  $s$  such that

$$G \cong \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}}.$$

**Proof.** We need to show both uniqueness and existence.

**Existence.** From [Proposition B.3.1](#), we obtain a basis  $e'_1, \dots, e'_n$  of  $F$  and a basis  $d_1 e'_1, \dots, d_k e'_k$  in  $K$  such that  $d_1 \mid \dots \mid d_k$ . Let

$$(d_1, \dots, d_k) = (1, \dots, 1, m_1, \dots, m_t),$$

which implies

$$\begin{aligned} G &\cong F/K \\ &\cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_k\mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \\ &= \mathbb{Z}/1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \\ &= \mathbb{Z}/m_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{\exists! s \text{ times}}. \end{aligned}$$

**Uniqueness.** Under the isomorphism  $\mathbb{Z}/m_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}}$ , we see that

$$\mathbb{Z}/m_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z}$$

corresponds to  $G$ 's [torsion subgroup](#)  $T$ , which implies

$$G/T \cong \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}},$$

which further implies  $G/T$  is a [free Abelian group](#) with

$$\text{rk}\left(\frac{G}{T}\right) = s,$$

which proves the uniqueness of  $s$ .

The proof of the uniqueness of  $m_i$  are long and tedious, we refer to [\[Arm13\]](#). ■

**Definition B.3.2 (Invariant factor).** We call  $m_1, \dots, m_t$  obtained from [Theorem B.3.1](#) the *invariant factor*.

**Lemma B.3.2.** Given a positive integer  $m$  such that

$$m = p_1^{n_1} \cdot \dots \cdot p_s^{n_s}$$

where  $p \in \mathcal{P}$  are all prime and  $p_i \neq p_j$  for  $i \neq j$ , with  $n_i \in \mathbb{Z}^+$  for all  $i$ . Then

$$\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_s^{n_s}\mathbb{Z}.$$

**Proof.** We define  $\phi$  as

$$\begin{aligned} \phi: \mathbb{Z}/m\mathbb{Z} &\rightarrow \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_s^{n_s}\mathbb{Z} \\ \bar{n} &\mapsto (n + \langle p_1^{n_1} \rangle, \dots, n + \langle p_s^{n_s} \rangle). \end{aligned}$$

Then  $\bar{n} \in \ker \phi \Leftrightarrow \forall_i p_i^{n_i} \mid n \Leftrightarrow m \mid n \Leftrightarrow \bar{n} = \bar{0}$ . This means  $\ker \phi = 0$ , hence  $\phi$  is an injection.

We now prove  $\phi$  is a surjection. It's sufficient to prove that for all  $i$ ,

$$(0, \dots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \dots, 0) \in \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_s^{n_s}\mathbb{Z},$$

there exists an  $\bar{n}$  such that

$$\phi(\bar{n}) = (0, \dots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \dots, 0).$$

Notice that for all  $i \neq j$ ,  $\langle p_i^{n_i} \rangle + \langle p_j^{n_j} \rangle \in \mathbb{Z}$ , hence there exists  $u_j \in \langle p_i^{n_i} \rangle$  and  $v_j \in \langle p_j^{n_j} \rangle$  such that  $u_j + v_j = 1$ . Let  $n$  as

$$n = \prod_{i \neq j} (1 - u_j),$$

then

$$n + \langle p_i^{n_i} \rangle = 1 + \langle p_i^{n_i} \rangle, \quad n + \langle p_j^{n_j} \rangle = 0 + \langle p_j^{n_j} \rangle.$$

Above implies

$$\phi(\bar{n}) = (0, \dots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \dots, 0),$$

hence  $\phi$  surjects, so

$$\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_s^{n_s}\mathbb{Z}.$$

■

Combine [Theorem B.3.1](#) and [Lemma B.3.2](#), we see that we now only have

$$G \cong \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}},$$

we can further decompose  $G$  into

$$G \cong \mathbb{Z}/p_1^{s_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_k^{s_k}\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}},$$

where  $p_1, \dots, p_k$  are primes (which may includes repeated terms),  $s_i \in \mathbb{Z}^+$  for all  $i$ .

---

**Definition B.3.3** (Elementary divisors). The set

$$\{p_1^{s_1}, \dots, p_k^{s_k}\}$$

are called *elementary divisors* of  $G$ .

**Theorem B.3.2** (Uniqueness of elementary divisors). [Elementary divisors](#) of a group  $G$  is unique.

**Proof.** Please refer to [\[Arm13\]](#). ■

# Appendix C

## Homological Algebra

### C.1 Exact Sequence

**As previously seen.** Given two [Abelian groups](#)  $A, B$  and the group homomorphism  $\varphi: A \rightarrow B$ , then we have

- $\ker \varphi = \{x \in A \mid \varphi(x) = 0\}$
- $\operatorname{Im} \varphi = \{\varphi(x) \mid x \in A\}$
- $\operatorname{coker} \varphi := B / \operatorname{Im} \varphi$
- $\operatorname{coIm} \varphi := A / \ker \varphi$

Consider a sequence of [Abelian](#) group homomorphism

$$\dots \longrightarrow A_{i-1} \xrightarrow{\phi_{i-1}} A_i \xrightarrow{\phi_i} A_{i+1} \longrightarrow \dots$$

We denote this sequence as  $S$ .

**Definition C.1.1 (Exact).** We say  $S$  is *exact* at  $A_i$  if

$$\operatorname{Im} \phi_{i-1} = \ker \phi_i.$$

**Remark.** [Definition C.1.1](#) is same as [Definition 4.2.13](#).

**Definition C.1.2 (Exact sequence).** We call  $S$  is an *exact sequence* if it's [exact](#) at  $A_i$  for all  $i$ .

**Remark.** Specifically, consider the following two situations.

- We say

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots$$

is an [exact sequence](#) if it's [exact](#) at  $A_i$  for all  $i \geq 1$ .

- We say

$$\dots \longrightarrow A_{-2} \longrightarrow A_{-1} \longrightarrow A_0$$

is an [exact sequence](#) if it's [exact](#) at  $A_i$  for all  $i \leq -1$ .

**Remark.** Denote  $\circ$  as a trivial [Abelian group](#), then

$$A \xrightarrow{\phi} B \longrightarrow \circ \text{ is an } \text{exact sequence} \Leftrightarrow \phi \text{ is a surjective homomorphism;}$$



conversely,

$\circ \longrightarrow B \xrightarrow{\phi} A$  is an **exact sequence**  $\Leftrightarrow \phi$  is an injective homomorphism.

**Definition C.1.3** (Short exact sequence). A *short exact sequence* is an **exact sequence** such that it has the following form

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ.$$

**Remark.** Let  $B \xrightarrow{\psi} C$  as a surjective homomorphism and  $K = \ker \psi$ , and we denote  $K \xrightarrow{i} B$  as an injection. Then

$$\circ \longrightarrow K \xrightarrow{i} B \xrightarrow{\psi} C \longrightarrow \circ$$

is a **short exact sequence**. Conversely, if

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

is a **short exact sequence**, then  $\phi$  is an injective homomorphism since it is **exact** at  $A$ , and  $\psi$  is a surjective homomorphism since it is **exact** at  $C$ , and  $\phi(A) = \ker \psi$  since it is **exact** at  $B$ . This implies  $\phi: A \rightarrow \phi(A) = \ker \psi$  is a group homeomorphism.

**Example.** Given  $A, B$  as **Abelian groups**, then

$$\circ \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\text{Proj}_2} B \longrightarrow \circ$$

$$a \xrightarrow{i} (a, 0)$$

$$(a, b) \xrightarrow{\text{Proj}_2} b$$

is a **short exact sequence**.

**Example.** We see that

$$\circ \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\text{Proj}_2} \mathbb{Z}/n\mathbb{Z} \longrightarrow \circ$$

$$k \longmapsto k \cdot n$$

for  $n \in \mathbb{Z}_{\geq 1}$  is a **short exact sequence**.

**Definition C.1.4** (Isomorphism between sequences). Given  $A_\bullet$  and  $B_\bullet$  defined as two sequences of **Abelian group** homomorphisms

$$A_\bullet: \dots \longrightarrow A_i \xrightarrow{\phi_i} A_{i+1} \longrightarrow \dots$$

and

$$B_\bullet: \dots \longrightarrow B_i \xrightarrow{\psi_i} B_{i+1} \longrightarrow \dots$$

And we say a morphism  $\alpha$  from  $A_\bullet$  to  $B_\bullet$  is a series of group homomorphisms  $\alpha_i: A_i \rightarrow B_i$  for

all  $i \in \mathbb{Z}$  such that the following diagram commutes.

$$\begin{array}{ccccccc} \dots & \longrightarrow & A_i & \xrightarrow{\phi_i} & A_{i+1} & \longrightarrow & \dots \\ & & \downarrow \alpha_i & & \downarrow \alpha_{i+1} & & \\ \dots & \longrightarrow & B_i & \xrightarrow{\psi_i} & B_{i+1} & \longrightarrow & \dots \end{array}$$

Additionally, if for all  $i$ ,  $\alpha_i$  is a group homeomorphism, then we say  $\alpha: A_\bullet \rightarrow B_\bullet$  is a homeomorphism.

**Definition C.1.5** (Split short exact sequence). Given a [short exact sequence](#)

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

we say it is *split* if there exists a group homeomorphism  $\theta: B \rightarrow A \oplus C$  such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus C & \longrightarrow & C \longrightarrow 0 \end{array}$$

is the [isomorphism](#) between these two [short exact sequences](#).

**Remark.** Given [split short exact sequence](#)

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

and  $\theta$  defined in [Definition C.1.5](#), let  $i: A \rightarrow A \oplus C$ ,  $a \mapsto (a, 0)$  and  $j: C \rightarrow A \oplus C$ ,  $c \mapsto (0, c)$  are two canonical embeddings, then we have

$$A \oplus C = i(A) \oplus j(C).$$

Consider  $\theta^{-1}: A \oplus C \xrightarrow{\cong} B$ , then

$$B = \theta^{-1}(i(A)) \oplus \theta^{-1}(j(C)).$$

Since the diagram in [Definition C.1.5](#) commutes, hence

$$\theta^{-1}(i(A)) = \theta^{-1} \circ i(A) = \phi(A),$$

hence

$$B = \phi(A) \oplus \underbrace{\theta^{-1}(j(C))}_D,$$

which implies  $\psi|_D: D \rightarrow C$  is a group homeomorphism. We see that

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

[split](#) implies  $B = \phi(A) \oplus D$  and  $\psi|_D: D \xrightarrow{\cong} C$ .

Conversely, if  $B = \phi(A) \oplus D$  and  $\psi|_D: D \xrightarrow{\cong} C$ , then there exists a  $\theta$

$$\begin{aligned} \theta: B &\rightarrow A \oplus C \\ \phi(a) + d &\mapsto (a, \psi(d)) \end{aligned}$$

for  $a \in A, d \in D$  such that

$$\begin{array}{ccccccc} \circ & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow \circ \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ \circ & \longrightarrow & A & \longrightarrow & A \oplus C & \longrightarrow & C \longrightarrow \circ \end{array}$$

$$\begin{array}{ccc} \phi(a) + d & \longmapsto & \psi(d) \\ \downarrow & & \downarrow \\ (a, \psi(d)) & \longmapsto & \psi(d) \end{array}$$

commutes.

Hence, for a **short exact sequence**  $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$  is **split** if and only if  $B = \phi(A) \oplus D$  and  $\psi|_D : D \xrightarrow{\cong} C$ .

Remarkably, let  $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$  is a **split short exact sequence**, then  $D$  constructed above is not unique. To see this, consider

$$\circ \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\text{Proj}_2} \mathbb{Z} \longrightarrow \circ$$

$$n \longmapsto (n, 0)$$

$$(n, m) \longmapsto m$$

We have  $\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus j(\mathbb{Z})$  where  $j: \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}, n \mapsto (0, n)$ . We see that we can let  $D := j(\mathbb{Z})$ . Meanwhile, we can also let

$$D := \{(n, n) \mid n \in \mathbb{Z}\} < \mathbb{Z} \oplus \mathbb{Z}$$

such that  $\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus D$ .

**Example (Non-split short exact sequence).** We see that

$$\circ \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\text{Proj}_2} \mathbb{Z}/n\mathbb{Z} \longrightarrow \circ$$

$$k \longmapsto k \cdot n$$

is not a **split short exact sequence**, since if it is, then

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} &\cong \mathbb{Z} \\ (0, 1) &\mapsto k, \end{aligned}$$

which is a contradiction since  $\mathbb{Z}$  is **torsion-free** while  $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$  is not.

**Lemma C.1.1** (Splitting lemma). If  $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$  is a **short exact sequence**, then the following are equivalent.

1. This **short exact sequence** **split**.

2.  $\exists p: B \rightarrow A$  such that  $p \circ \phi = \text{id}_A$ .

3.  $\exists q: C \rightarrow B$  such that  $\psi \circ q = \text{id}_C$ .

**Proof.** • 1.  $\Rightarrow$  2. Let  $\theta: B \xrightarrow{\cong} A \oplus C$  such that it's the **isomorphism** which makes the following diagram commutes.

$$\begin{array}{ccccccc} \circ & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow \circ \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ \circ & \longrightarrow & A & \xrightarrow{i} & A \oplus C & \longrightarrow & C \longrightarrow \circ \\ & & & \nwarrow \text{Proj}_1 & & & \end{array}$$

Then we let  $p := \text{Proj}_1 \circ \theta$ , then

$$p \circ \phi = \text{Proj}_1 \circ \theta \circ \phi = \text{Proj}_1 \circ i = \text{id}_A.$$

• 1.  $\Rightarrow$  3. Let  $\theta: B \xrightarrow{\cong} A \oplus C$  such that it's the **isomorphism** which makes the following diagram commutes.

$$\begin{array}{ccccccc} \circ & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow \circ \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ \circ & \longrightarrow & A & \longrightarrow & A \oplus C & \xrightarrow{\text{Proj}_2} & C \longrightarrow \circ \\ & & & & \nwarrow j & & \end{array}$$

Then we let  $q := \theta^{-1} \circ j$ , then for all  $c \in C$ , we have

$$\psi \circ q(c) = \psi(\theta^{-1}(j(c))) = \text{Proj}_2 \circ \theta(\theta^{-1}(j(c))) = \text{Proj}_2(j(c)) = c,$$

hence  $\psi \circ q = \text{id}_C$ .

• 2.  $\Rightarrow$  1. We have

$$\circ \longrightarrow A \xrightleftharpoons[p]{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

where  $p \circ \phi = \text{id}_A$ . We claim that  $B = \phi(A) \oplus \ker(p)$  since for every  $b \in B$ ,  $\phi(p(b)) \in \phi(A)$ , and

$$b = \underbrace{\phi(p(b))}_{\in \phi(A)} + \underbrace{(b - \phi(p(b)))}_{\in \ker(p)}$$

from the fact that

$$p(b - \phi(p(b))) = p(b) - p \circ \phi(p(b)) = p(b) - p(b) = 0.$$

We need to show the uniqueness also. Suppose  $b = \phi(a_1) + d_1 = \phi(a_2) + d_2$ ,  $a_1, a_2 \in A$ ,  $d_1, d_2 \in \ker(p)$ . We see that

$$\phi(a_1 - a_2) = d_2 - d_1 \Rightarrow p(\phi(a_1 - a_2)) = 0 \Rightarrow a_1 = a_2 \Rightarrow d_1 = d_2.$$

Finally, we claim that

$$\psi|_{\ker(p)} : \ker(p) \rightarrow C$$

is a group homeomorphism. But it's obvious that  $\psi|_{\ker(p)}$  are both surjective and injective.

• 3.  $\Rightarrow$  1. We have

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightleftharpoons[q]{\psi} C \longrightarrow \circ$$

---

where  $\psi \circ q = \text{id}_C$ . We claim that  $B = \phi(A) \oplus q(C)$  since for every  $b \in B$ ,

$$b = \underbrace{(b - q(\psi(b)))}_{\in \ker(\psi) = \text{Im}(\phi)} + \underbrace{q(\psi(b))}_{\in q(C)},$$

which implies  $B = \phi(A) + q(C)$ . We can also prove that

$$B = \phi(A) \oplus q(C)$$

similarly. ■

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