MATH597
Analysis II

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February 3, 2022

Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\overset{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition.

Notice that there are some proofs is **intended** left as assignments, and for completeness, I put them in Appendix A, use it in your **own risks**! You'll lose the chance to practice and really understand the materials.

Additionally, we'll use [FF99] as our main text, while using [Tao13] and [Axl19] as supplementary references.

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Lecture 1: σ-algebra   05 Jan. 11:00

1 Measure

Example. Before we start, we first see some examples.

1. Let $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) := \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\},$$

which is the *power set* of X. We see that

$$\#X = n \implies \#\mathcal{P}(X) = 2^n$$

for $n < \infty$.

2. If $n = \infty$, say $X = \mathbb{N}$, then

$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while $\mathbb{N}$ is a countable set. We can see this as follows. Consider

$$\phi\colon \mathcal{P}(\mathbb{N}) \to [0,1], \quad A \mapsto 0.a_1 a_2 a_3 \dots \text{(base 2)},$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example, A can be $A = \{2, 3, 6, \dots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have

$$\#\mathcal{P}(\mathbb{N}) \geq \#\,[0,1].$$

But since $[0,1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X. Hence, we are intriguing to define a map μ such that

$$\mu\colon \mathcal{P}(X) \to [0, \infty].$$

Example. We first see some examples.

1. Let $X = \{0, 1, 2\}$. Then we want to define $\mu\colon \mathcal{P}(X) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$. Then we have
     - $\mu(\{0, 1\}) = 2$
     - $\mu(\{0\}) = 1$
   - $\mu(A) = \sum_{i \in A} 2^i$. Then we have
     - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$

2. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu\colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$. Then we have
     - $\mu(\{2, 3, 4, 5, \dots\}) = \infty = \mu(\{\text{even numbers}\})$
   - $\mu(A) = e^{-1} \sum_{i \in A} \frac{1}{i!}$. Then we have
     - $\mu(\{0, 2, 4, 6, \dots\}) = e^{-1}\left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right)$
   - $\mu(A) = \sum_{i \in A} a_i$

3. Let $X = \mathbb{R}$. Then we want to define $\mu\colon \mathcal{P}(\mathbb{R}) \to [0, \infty]$, we can have

- $\mu(A) = \#A$

- $\mu((a,b)) = b - a.$

    Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

    Answer. No!

- $\mu((a,b)) = e^b - e^a.$

    Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

    Answer. No!

We immediately see the problems. To extend our native measure method into $\mathbb{R}$ is hard and will cause something counter-intuitive![1] Hence, rather than define measurement on *all* subsets in the power set of X, we only focus on *some* subsets. In other words, we want to define

$$\mu \colon \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \to [0, \infty] .$$

1.1 σ-algebras

We start from the definition of the most fundamental element in measure theory.

Definition 1.1 (σ-algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra on X* if

- $\varnothing \in \mathcal{A}$.

- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.

- $\mathcal{A}$ is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\varnothing}_{\in \mathcal{A}}$ and $\mathcal{A}$ is closed under complement.

- $\bigcap\limits_{i=1}^{\infty} A_i = \left(\bigcup\limits_{i=1}^{\infty} A_i^c \right)^c$, namely $\mathcal{A}$ is <u>closed under countable intersections</u>.

- $A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \varnothing \cup \varnothing \cup \ldots$, hence $\mathcal{A}$ is closed under finite unions and intersections.

An immediate definition can be given. We now define so-called *Borel set*.

Definition 1.2 (Borel set). Given a topological space X, a *Borel set* is any set in X that can be formed from open sets through the operations of countable union, countable intersection and relative complement.

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[1] https://en.wikipedia.org/wiki/Banach-Tarski_paradox

Lecture 2: Measure

Example. Again, we first see some examples.

1. Let $\mathcal{A} = \mathcal{P}(X)$, which is the power σ-algebra.

2. Let $\mathcal{A} = \{\varnothing, X\}$, which is a trivial σ-algebra.

3. Let $B \subset X$, $B \neq \varnothing$, $B \neq X$. Then we see that $\mathcal{A} = \{\varnothing, B, B^c, X\}$ is a σ-algebra.

Lemma 1.1. Let $\mathcal{A}_\alpha$, $\alpha \in I$, be a family of σ-algebra on X. Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is a σ-algebra on X.

Remark. Notice that I may be an uncountable intersection.

Proof. A simple proof can be made as follows. Firstly, $\varnothing \in \mathcal{A}_\alpha$ for every α clearly. Moreover, closure under complement and countable unions for every $\mathcal{A}_\alpha$ implies the same must be true for $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$. Hence, $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ-algebra. ∎

The above allows us to give the following definition.

Definition 1.3 (Generation of σ-algebra). Given $\mathcal{E} \subset \mathcal{P}(X)$, where $\mathcal{E}$ is not necessarily a σ-algebra. Let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X containing $\mathcal{E}$, then we call $\langle \mathcal{E} \rangle$ the *σ-algebra* generated by $\mathcal{E}$.

Remark. Clearly, $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$, and it is unique. To check the uniqueness, we suppose there are two different $\langle \mathcal{E} \rangle_1$ and $\langle \mathcal{E} \rangle_2$ generated from $\mathcal{E}$. It's easy to show

$$\langle \mathcal{E} \rangle_1 \subseteq \langle \mathcal{E} \rangle_2,$$

and by symmetry, they are equal.

Example. We see that $\{\varnothing, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle$.

Lemma 1.2. We have

1. Given $\mathcal{A}$ a σ-algebra, $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \mathcal{A}$

2. $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$

Proof. We'll see that after proving the first claim, the second follows smoothly.

1. The first claim is trivial, since we know that $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$, then if $\mathcal{E} \subset \mathcal{A}$, we clearly have $\langle \mathcal{E} \rangle \subset \mathcal{A}$ by the definition.

2. The second claim is also easy. From the first claim and the definition, we have

$$\mathcal{E} \subset \mathcal{F} \subset \langle \mathcal{F} \rangle \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle .$$

$\blacksquare$

At this point, we haven't put any specific structure on X. Now we try to describe those spaces with good structure, which will give the space some nice properties.

> **Definition 1.4 (Borel σ-algebra).** For a topological space X, the *Borel σ-algebra on X*, denotes as $\mathcal{B}(X)$, is the σ-algebra generated by the collection of all open sets in X.

Example. We see that $\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a,b) \mid a < b; a, b \in \mathbb{R}\}$.

- $\mathcal{E}_2 = \{[a,b] \mid a < b; a, b \in \mathbb{R}\}$ since $[a,b] = \bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.

- $\mathcal{E}_3 = ((a,b] \mid a < b; a, b \in \mathbb{R})$ since $(a,b] = \bigcap\limits_{n=1}^{\infty} (a, b + \frac{1}{n})$.

- $\mathcal{E}_4 = ([a,b) \mid a < b; a, b \in \mathbb{R})$ since $[a,b) = \bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b)$.

- $\mathcal{E}_5 = ((a,\infty) \mid a \in \mathbb{R})$ since $(a,\infty) = \bigcup\limits_{n=1}^{\infty} (a, a + n)$.

- $\mathcal{E}_6 = ([a,\infty) \mid a \in \mathbb{R})$ since $[a,\infty) = \bigcup\limits_{n=1}^{\infty} [a, a + n)$.

- $\mathcal{E}_7 = ((-\infty,b) \mid b \in \mathbb{R})$ since $(-\infty,b) = \bigcup\limits_{n=1}^{\infty} (b - n, b)$.

- $\mathcal{E}_8 = ((-\infty,b] \mid a \in \mathbb{R})$ since $(-\infty,b] = \bigcup\limits_{n=1}^{\infty} (b - n, b]$.

> **Proposition 1.1.** $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \ldots, 8$.

Proof. Firstly, we see that $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \implies \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$ by Lemma 1.2. Secondly, by definition, $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$ where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R} \} .$$

It's enough to show $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$ since if so, $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$, and clearly $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$, then we will have $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$. Let $O \subset \mathbb{R}$ be an open set, i.e., $O \in \mathcal{E}$. We claim that every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.[2]

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[2] https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

where I_j open interval with the form of $(a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$.

For example, $\mathcal{E}_1$ is trivially true, and

$$(a, b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{[a + \frac{1}{n}, b - \frac{1}{n}]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for $\mathcal{E}_2$ and

$$(a, \infty) = \bigcup_{k=1}^{\infty} (a, a + k)$$

shows the case for $\mathcal{E}_5$. It's now straightforward to check open intervals are in $\langle \mathcal{E}_i \rangle$ for every i. ∎

Now, to put a structure on a space, we define the following.

Definition 1.5 (Measurable space). $(X, \mathcal{A})$ is called a *measurable space*, and $E \in \mathcal{A}$ is called an $\mathcal{A}-$*measurable set*.

1.2 Measures

With the definition of measurable space, we now can refine our measure function μ as follows.

Definition 1.6 (Measure). Given a measurable space on $(X, \mathcal{A})$, a *measure* is a function μ such that

$$\mu \colon \mathcal{A} \to [0, \infty]$$

with

1. $\mu(\varnothing) = 0$

2. $\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_1, A_2, \ldots \in \mathcal{A}$ are **disjoint**. We call this *Countable additivity*.

We denote $(X, \mathcal{A}, \mu)$ a *measure space*.

Notation. We denote $[0, \infty] \coloneqq [0, \infty) \cup \{\infty\}$.

Remark. The motivation of why we only want *countable additivity* but not uncountable additivity can be seen by the following example. We'll consider the most intuitive measure on $\mathbb{R}, \mathcal{B}(\mathbb{R})$.

Since we have

$$(0,1] = \left(\frac{1}{2}, 1\right] \cup \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left(\frac{1}{8}, \frac{1}{4}\right] \cup \ldots$$

and also

$$(0,1] = \bigcup_{x \in (0,1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2},1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4},\frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8},\frac{1}{4}])} + \ldots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0,1]} 0$$

since $\mu(x) = 0$ for $x \in \mathbb{R}$, which is clearly not what we want.

Example. We see some examples.

1. For any $(X, \mathcal{A})$, we let $\mu(A) := \#A$. This is called *counting measure*.

2. Let $x_0 \in X$. For any $(X, \mathcal{A})$, the *Dirac measure at x_0* is

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A. \end{cases}$$

3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$,

$$\mu(A) = \sum_{i \in A} a_i,$$

where $a_1, a_2, \ldots \in [0, \infty)$.

Lecture 3: Construct a Measure

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Note. If $A, B \in \mathcal{A}$ and $A \subset B$, then

$$\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

Theorem 1.1. Given $(X, \mathcal{A}, \mu)$ be a measure space.

1. (monotonicity) $A, B \in \mathcal{A}$, $A \subset B \implies \mu(A) \leq \mu(B)$.

2. (countable subadditivity) $A_1, A_2, \ldots \in \mathcal{A} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$

3. (continuity from below/ monotone convergence theorem (MCT) for sets)

$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \ldots \end{cases} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

4. (continuity from above)

$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \ldots \\ \mu(A_1) < \infty \end{cases} \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

Proof. We prove this theorem one by one.

1. Since $A \subset B$, hence we have

$$\mu(B) = \mu\Big(\underbrace{(B \setminus A) \cup A}_{\text{disjoint}}\Big) \overset{!}{=} \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A).$$

2. This should be trivial from countable additivity with the fact that $\mu(A) \geq 0$ for all A. ⌐DIY!

3. Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$, then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

are a disjoint union and $B_i \in \mathcal{A}$, hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i).$$

With $\mu\left(\bigcup_{i=1}^{n} B_i\right) = \mu(A_n)$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n \to \infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n \to \infty} \mu(A_n).$$

4. Let $E_i = A_1 \setminus A_i \implies E_i \in \mathcal{A}$, $E_1 \subset E_2 \subset \ldots$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right),$$

which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i \right) \implies \mu\left(\bigcap_{i=1}^{\infty} A_i \right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i \right)$$

since $\mu\left(\bigcup_{i=1}^{\infty} E_i \right) \le \mu(A_1) < \infty$. Then from continuity from below, we further have

$$\mu\left(\bigcap_{i=1}^{\infty} A_i \right) = \mu(A_1) - \lim_{n \to \infty} \mu(E_n) = \mu(A_1) - \lim_{n \to \infty} \left(\mu(A_1) - \mu(A_n) \right).$$

From montonicity, we see that $\mu(A_n) \le \mu(A_1) < \infty$, hence we can split the limit and further get

$$\mu\left(\bigcap_{i=1}^{\infty} A_i \right) = \mu(A_1) - \mu(A_1) + \lim_{n \to \infty} \mu(A_n) = \lim_{n \to \infty} \mu(A_n).$$

$\blacksquare$

Example. Given $(\mathbb{N}, \mathcal{P}(\mathbb{N}),$ counting measure$)$. Then we see

- $A_n = \{n, n+1, n+2, \ldots\} \implies \mu(A_n) = \infty$

- $A_1 \supset A_2 \supset A_3 \supset \ldots$

- $\bigcap_{i=1}^{\infty} A_i = \varnothing \implies \mu\left(\bigcap_{i=1}^{\infty} A_i \right) = 0$

Remark. We see that in this case, since $\mu(A_1) \not< \infty$, hence continuity from above doesn't hold.

We now try to characterize some properties of a measure space.

> **Definition 1.7 (μ-null, μ-subnull, Complete measure space).** Given $(X, \mathcal{A}, \mu)$
>
> - $A \subset X$ is a *μ-null set* if $A \in \mathcal{A}$ and $\mu(A) = 0$.
>
> - $A \subset X$ is a *μ-subnull set* if $\exists \mu$-null set B such that $A \subset B$. Note that A is not necessarily $\mathcal{A}$-measurable.
>
> - $(X, \mathcal{A}, \mu)$ is a *complete* measure space if every μ-subnull set is $\mathcal{A}$-measurable.

There are some useful terminologies we'll use later relating to μ-null.

> **Definition 1.8 (Almost everywhere).** Given $(X, \mathcal{A}, \mu)$, a statement $P(x)$, $x \in X$ holds *μ-almost everywhere (a.e.)* if the set
>
> $$\{x \in X : P(x) \text{ does not hold}\}$$
>
> is μ-null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

Definition 1.9 (finite measure). Given $(X, \mathcal{A}, \mu)$

- μ is a *finite measure* if $\mu(X) < \infty$.

- μ is a *σ-finite measure* if $X = \bigcup\limits_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.

Exercise. Every measure space can be **completed**. Namely, we can always find a bigger σ-algebra to complete the space.

1.3 Outer Measures

We start by giving a definition.

Definition 1.10 (Outer measure). An *outer measure* on X is a map

$$\mu^* \colon \mathcal{P}(X) \to [0, \infty]$$

such that

- $\mu^*(\varnothing) = 0$

- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$

- (countable subadditivity) $\mu^* \left(\bigcup\limits_{i=1}^{\infty} A_i \right) \leq \sum\limits_{i=1}^{\infty} \mu^*(A_i)$ for every $A_i \subset X$.

Example. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \colon \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

is an outer measure due to the Proposition 1.2 we're going to show.

Remark. We see that an outer measure need not be a measure. Check the Definition 1.6 for a measure function.

Proposition 1.2. Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\varnothing, X \in \mathcal{E}$. Let

$$\rho \colon \mathcal{E} \to [0, \infty]$$

such that $\rho(\varnothing) = 0$. Then

$$\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \ \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

is an outer measure on X.

Note. Recall the Tonelli's Theorem[3] for series:

If $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Specifically, in [Tao13] Theorem 0.0.2.

Lecture 4: Carathéodory extension Theorem

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As previously seen. We now prove the Proposition 1.2.

Proof. We need to prove

- μ^* is well-defined. i.e., inf is taken over a non-empty set. This is trivial since $X \in \mathcal{E}$ and $X \supset A$ for any $A \in \mathcal{E}$.

- $\mu^*(\varnothing) = 0$. Since $\varnothing \in \mathcal{E}$ and

$$\mu^*(\varnothing) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \varnothing \right\} = 0$$

  since $\rho(\varnothing) = 0$ for all i and further, by Squeeze Theorem[4], we see that $\lim_{n \to \infty} \sum_{i=1}^{n} \rho(\varnothing) = 0$.

- $A \subset B \implies \mu^*(A) \leq \mu^*(B)$. We simply show this by contradiction. Suppose $A \subset B$ and $\mu^*(A) > \mu^*(B)$, then by definition of μ^*, we have

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$
$$> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B \right\} = \mu^*(B).$$

  Now, let $B =: (B \setminus A) \cup A$, then we have

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$
$$> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B \setminus A) \cup A \right\} = \mu^*(B).$$

  Now, since $B \setminus A \supseteq \varnothing$, then this inequality can't hold, hence a contradiction ↯ .

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[3] https://en.wikipedia.org/wiki/Fubini%27s_theorem
[4] https://en.wikipedia.org/wiki/Squeeze_theorem

- Countable subadditivity. Let $A_1, A_2, \ldots \in X$. If one of $\mu^*(A_n) = \infty$, then result holds. So we may assume $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Now, fix any $\epsilon > 0$, we will show that

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon.$$

For each $n \in \mathbb{N}$, $\exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ such that

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n$$

and

$$\mu^*(A_n) + \frac{\epsilon}{2^n}{}^5 \geq \sum_{k=1}^{\infty} \rho(E_{n,k}).$$

Then we see that

$$\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k) \in \mathbb{N}^2} E_{k,n},$$

which implies

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{(n,k) \in \mathbb{N}^2} \rho\left(E_{k,n}\right) \overset{!}{=} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{k,n}) \leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right)$$

from the inequality just derived. Now, since the last term is just

$$\sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon,$$

hence we finally have

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon$$

for arbitrarily small fixed $\epsilon > 0$, hence the subadditivity is proved.

$\blacksquare$

Definition 1.11 (Carathéodory measurable). Let μ^* be an outer measure on X. We say $A \subset X$ is *Carathéodory measurable (C-measurable) with respect to* μ^* if

$$\forall E \subset X, \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

---

[5]This is an important trick!!

> **Lemma 1.3.** Let μ^* be an outer measure on X. Suppose $B_1, \ldots, B_N$ are disjoint C-measurable sets. Then,
>
> $$\forall E \subset X, \ \mu^* \left(E \cap \left(\bigcup_{i=1}^{N} B_i \right) \right) = \sum_{i=1}^{N} \mu^* \left(E \cap B_i \right).$$

Proof. Since we have

$$\mu^* \left(E \cap \left(\bigcup_{i=1}^{N} B_i \right) \right) = \mu^* \left(E' \cap B_1 \right) + \mu^* \left(E' \setminus B_1 \right) {}^{6}$$

$$= \mu^* \left(E \cap \left(\bigcup_{i=1}^{N} B_i \cap B_1 \right) \right) + \mu^* \left(E \cap \left(\bigcup_{i=1}^{N} B_i \right) \cap B_1^c \right)$$

$$= \mu^* (E \cap B_1) + \mu^* \left(E \cap \left(\bigcup_{i=2}^{N} B_i \right) \right)$$

where the equality comes from the fact that B_1 is C-measurable and disjoint from B_i, $i \neq 1$. Then, we simply iterate this argument and have the result. ■

Remark. This implies that if we restrict an outer measure on a C-measurable set, then it becomes finite additive.

> **Theorem 1.2 (Carathéodory extension Theorem).** Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets (with respect to μ^*). Then,
>
> 1. $\mathcal{A}$ is a σ-algebra on X.
>
> 2. $\mu = \mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.
>
> 3. $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We divide the proof in several steps.

1. We show $\mathcal{A}$ is a σ-algebra by showing

   (a) $\varnothing \in \mathcal{A}$. To show this, we simply check that $\varnothing$ is C-measurable. We see that

   $$\underset{E \subset X}{\forall} \ \mu^*(E) = \mu^*(E \cap \varnothing) + \mu^*(E \setminus \varnothing) = \mu^*(E),$$

   which just shows $\varnothing \in \mathcal{A}$.

   (b) $\mathcal{A}$ closed under complements. This is equivalent to say that if A is C-measurable, so is A^c. We see that if A is C-measurable, then for every $E \subset X$,

   $$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

---

[6] Here, $E' := E \cap \left(\bigcup_{i=1}^{N} B_i \right)$ for the simplicity of notation.

Observing that $E \cap A = E \setminus A^c$ and $E \setminus A = E \cap A^c$, hence
$$\mu^*(E) = \mu^*(E \setminus A^c) + \mu^*(E \cap A^c).$$

We immediately see that above implies $A^c \in \mathcal{A}$.

(c) $\mathcal{A}$ closed under countable unions.

Note. To show $\mathcal{A}$ closed under countable unions, we show that $\mathcal{A}$ is closed under:

finite unions $\overset{\text{then}}{\Longrightarrow}$ countable <u>disjoint</u> unions $\overset{\text{then}}{\Longrightarrow}$ countable unions.

- We show $\mathcal{A}$ is closed under finite unions.

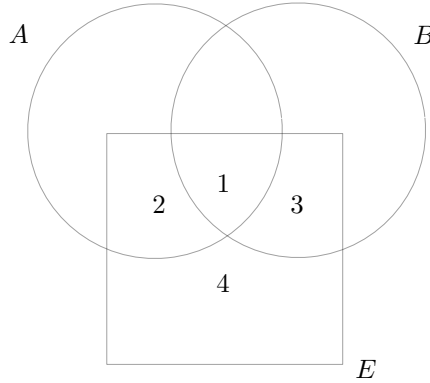 **Claim.** $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

  Fix $E \subset X$ arbitrary. We need to show that
  $$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)),$$
  i.e.,
  $$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$
  given $A, B \in \mathcal{A}$.

  - Since A is C-measurable,
    * $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$
    * $\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$
  - Since B is C-measurable,
    * $\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$

  Hence, we have
  $$\begin{aligned}
  \mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\
  &= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4) \\
  &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).
  \end{aligned}$$

- We show $\mathcal{A}$ is closed under countable <u>disjoint</u> unions.

  Let $A_1, A_2, \ldots \in \mathcal{A}$ and <u>disjoint</u>. Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

  $$\mu^*(E) \leq \mu^* \left(E \cap \bigcup_{i=1}^{\infty} A_i \right) + \mu^* \left(E \setminus \bigcup_{i=1}^{\infty} A_i \right),$$

  hence we only need to show another way around.

  Fix $N \in \mathbb{N}$, we have $\bigcup\limits_{n=1}^{N} A_n \in \mathcal{A}$ since N is finite, and

  $$\mu^*(E) = \mu^* \left(E \cap \left(\bigcup_{n=1}^{N} A_n \right) \right) + \mu^* \left(E \setminus \left(\bigcup_{n=1}^{N} A_n \right) \right)$$

  $$\geq \underbrace{\sum_{n=1}^{N} \mu^*(E \cap A_n)}_{\overset{!}{=} \mu^* \left(E \cap \left(\bigcup\limits_{n=1}^{N} A_n \right) \right)} + \underbrace{\mu^* \left(E \setminus \bigcup_{n=1}^{\infty} A_n \right)}_{\leq \mu^* \left(E \setminus \left(\bigcup\limits_{n=1}^{N} A_n \right) \right)}.$$

  Now, take $N \to \infty$ then we are done.

- We show $\mathcal{A}$ is closed under countable unions. DIY

The proof will be continued...

Lecture 5: Hahn-Kolmogorov Theorem

14 Jan. 11:00

Firstly, we see a stronger version of Lemma 1.3 we have seen before.

Lemma 1.4. Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots$ are <u>disjoint</u> C-measurable sets. Then,

$$\forall E \subset X, \ \mu^* \left(E \cap \left(\bigcup_{i=1}^{\infty} B_i \right) \right) = \sum_{i=1}^{\infty} \mu^* \left(E \cap B_i \right).$$

Proof.

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_i) \geq \mu^* \left(E \cap \bigcup_{n=1}^{\infty} B_n \right) \geq \mu^* \left(E \cap \left(\bigcup_{n=1}^{N} B_n \right) \right) \overset{!}{=} \sum_{n=1}^{N} \mu^* \left(E \cap B_n \right).$$

Now, we just take $N \to \infty$ (or note that $N \in \mathbb{N}$ is arbitrary, we then get the result according to Squeeze Theorem[7]). ∎

Let's continue the proof of Theorem 1.2.

2. Since from Definition 1.6, we need to show

---

[7] https://en.wikipedia.org/wiki/Squeeze_theorem

- $\mu(\varnothing) = 0$. This means that we need to show $\mu^*|_{\mathcal{A}}(\varnothing) = 0$. Since $\varnothing \in \mathcal{A}$ and μ^* is an outer measure, hence from the property of outer measure, it clearly holds.

- Countable additivity of μ^* on $\mathcal{A}$ follows from the Lemma 1.4 with $E = X$

3. The proof is given in Theorem A.1.

∎

1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets $\mathcal{E}$ and any ρ such that it assigns measure on $\mathcal{E}$, then induces an outer measure by Proposition 1.2, finally complete the outer measure by Theorem 1.2.

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.2}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to σ-algebra we are working on till now.

Definition 1.12 (Algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on X* if

- $\varnothing \in \mathcal{A}$.

- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.

- $\mathcal{A}$ is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup\limits_{i=1}^{n} A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an algebra and a σ-algebra is whether they closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an outer measure.

Definition 1.13 (Pre-measure). Let $\mathcal{A}_0$ be an algebra on X. We say

$$\mu_0 \colon \mathcal{A}_0 \to [0, \infty]$$

is a *pre-measure* if

1. $\mu_0(\varnothing) = 0$

2. (finite additivity) $\mu_0 \left(\bigcup\limits_{i=1}^{n} A_i \right) = \sum\limits_{i=1}^{n} \mu_0(A_i)$ if $A_1, \ldots, A_n \in \mathcal{A}_0$ are $\underline{\text{disjoint}}$.

3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and $A = \bigcup\limits_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, $\underline{\text{disjoint}}$, then

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 1.5. $(1) + (3) \implies (2)$ in Definition 1.13.

Proof. It's easy to see that since μ_0 is monotone. ∎

Theorem 1.3 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in Proposition 1.2. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^*, then $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$. i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts.

- We first show $\mathcal{A} \supset \mathcal{A}_0$. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is C-measurable, i.e.,

$$\forall E \subset X \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We first fix an $E \subset X$. From countable subadditivity of μ^*, we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

Fix $\epsilon > 0$. By the Proposition 1.2 of μ^*, $\exists B_1, B_2, \ldots \in \mathcal{A}_0$, $\bigcup\limits_{n=1}^{\infty} B_n \supset E$ such that

$$\mu^*(E) + \epsilon \overset{!}{\geq} \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left(\mu_0(\underbrace{B_n \cap A}_{\in \mathcal{A}_0}) + \mu_0(\underbrace{B_n \cap A^c}_{\in \mathcal{A}_0}) \right)$$

by the finite additivity of μ_0. Note that

$$
\begin{cases}
\displaystyle\bigcup_{n=1}^{\infty}(B_n \cap A) \supset E \cap A \\[3mm]
\displaystyle\bigcup_{n=1}^{\infty}(B_n \cap A^c) \subset E \cap A^c
\end{cases}
\implies \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)
$$

since

$$
\mu^*(E \cap A) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n \cap A)
$$

and

$$
\mu^*(E \cap A^c) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A^c)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n \cap A^c).
$$

We then see that for any $\epsilon > 0$, the inequality

$$
\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)
$$

holds, hence so does

$$
\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),
$$

which implies $\mathcal{A} \supset \mathcal{A}_0$.

The proof will be

Lecture 6: Hahn-Kolmogorov Theorem and Extension. 18 Jan. 11:00

Let's continue the proof of Theorem 1.3.

- Let $A \in \mathcal{A}_0$, we want to show that

$$
\mu(A) = \mu_0(A).
$$

  – Firstly, let

$$
B_i = \begin{cases} A, & \text{if } i = 1 \\ \varnothing, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,
$$

  hence $\displaystyle\bigcup_{i=1}^{\infty} B_i = A$, then we see that

$$
\mu^*(A) \leq \sum_{i=1}^{\infty}\mu_0(B_i) = \mu_0(A)
$$

  from the definition of μ^* and countable additivity within the algebra of μ_0.

– Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A \cap B_1 \in \mathcal{A}_0$, $C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right) \in \mathcal{A}_0$ for $i \geq 2$ since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

are disjoint countable unions, by countable additivity within the algebra, we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i) \implies \mu_0(A) \leq \mu^*(A)$$

by taking the infimum from the definition of μ^*.

Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every $A \in \mathcal{A}_0$, which implies

$$\mu(A) = \mu_0(A)$$

for every $A \in \mathcal{A}_0$ from Theorem 1.2, where we extend μ^* to μ respect to $\mathcal{A}_0$.

∎

Definition 1.14 (HK extension). $(\mathcal{A}, \mu)$ obtained from Theorem 1.3 is the *Hahn-Kolmogorov extensions* of $(\mathcal{A}_0, \mu_0)$.

We can show the uniqueness of HK extension.

Theorem 1.4 (Uniqueness of HK extension). Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$. Let $(\mathcal{A}, \mu)$ be the HK extension of $(\mathcal{A}_0, \mu_0)$. Let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$. Then if μ_0 is σ-finite, $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Note. Notice that $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$ since they both extend $\mathcal{A}_0$.

Proof. Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing the arbitrary cover of A and using the definition of μ^*.

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) < \infty$, and fix $\epsilon > 0$. Then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup\limits_{i=1}^{\infty} B_i \supset A$ such that

$$\mu(A) + \epsilon = \mu^*(A) + \epsilon \overset{!}{\geq} \sum_{i=1}^{\infty} \mu_0(B_i) \overset{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

  This implies that
$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$

  where the first equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

$$\mu(B) = \lim_{N \to \infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) \overset{8}{=} \lim_{N \to \infty} \mu'\left(\bigcup_{i=1}^{N} B_i\right) = \mu'(B),$$

  hence,

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \overset{9}{\leq} \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon$$

  for arbitrary ϵ, so we conclude $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is σ-finite, so we know $X = \bigcup\limits_{n=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ such that
$$\mu_0(X_n) < \infty.$$

  Replacing X_n by $X_1 \cup \ldots \cup X_n \in \mathcal{A}_0$, we may assume that

$$X_1 \subset X_2 \subset \ldots.$$

  Then,
$$\underset{n \in \mathbb{N}}{\forall} \mu(A \cap X_n) < \infty \overset{!}{\implies} \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

  From the continuity of measure, we then have

$$\mu(A) = \lim_{n \to \infty} \mu(A \cap X_n) \leq \lim_{n \to \infty} \mu'(A \cap X_n) = \mu'(A).$$

■

---

[8] $\mu = \mu'$ on $\mathcal{A}_0$.
[9] From the first part.

Corollary 1.1. Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then

$$\exists! \text{ measure } \mu \text{ on } \langle \mathcal{A}_0 \rangle \text{ that extends } \mathcal{A}_0.$$

Furthermore,

- The completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.

-
$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \underset{i \in \mathbb{N}}{\forall} \bigcup_{i=1}^{\infty} B_i \supset A \right\}$$

  for all $A \in \langle \bar{\mathcal{A}}_0 \rangle$.

Lecture 7: Borel Measures

1.5 Borel Measures on $\mathbb{R}$

We first introduce so-called *distribution function.*

Definition 1.15 (Distribution function). An <u>increasing</u>[a] function

$$F \colon \mathbb{R} \to \mathbb{R}$$

and <u>right-continuous</u>. F is then a *distribution function.*

---
[a]Here, increasing means $F(x) \leq F(y)$ for $x < y$.

Example. Here are some examples of right-continuous functions.

1. $F(x) = x$.

2. $F(x) = e^x$.

3. Define
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

4. Let $\mathbb{Q} \coloneqq \{r_1, r_2, \ldots\}$. Define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$

and

$$F(x) \coloneqq \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$

Then F is a distribution function (hence right-continuous). This is shown in Lemma A.1.

Note. If F is increasing, and

$$F(\infty) := \lim_{x \nearrow \infty} F(x), \quad F(-\infty) := \lim_{x \searrow \infty} F(x)$$

exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.[10]

Now, we can define a *Borel measure* on $(X, \mathcal{B}(\mathbb{R}))$.

Definition 1.16 (Borel mesaure). A *Borel measure* is any measure μ defined on the σ-algebra of Borel sets.

Definition 1.17 (Locally finite). Let X be a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Note. Some authors will require a Borel measure equipped with the locally finite property. But formally, this is not so common.

Lemma 1.6. Let μ be a locally finite Borel measure on $\mathbb{R}$, then

$$F_\mu(x) = \begin{cases} \mu\left((0, x]\right), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu\left((x, 0]\right), & \text{if } x < 0 \end{cases}$$

is a distribution function.

Proof. To show F_μ is increasing, consider $x < y$ such that

$$F_\mu(x) \leq F_\mu(y)$$

by considering

- $x > 0$: Then $F_\mu(x) = \mu((0, x])$ and

$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$

- $x = 0$: Then $F_\mu(x) = 0$ and

$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$

  since $y > 0$.

- $x < 0$: Follows the same argument with $x > 0$.

---

[10]There are distributions [FF99] Ch9., but these are different from distribution functions.

Now, we need to show F_μ is right-continuous. Firstly, assume that $x \geq 0$, then we see that

$$F_\mu(x) = \mu((0, x]) = \mu((0, x^+])$$

from the fact that a measure is right-continuous.[11] Now, if $x \leq 0$, the same argument follows since multiplying -1 will not change the fact that a measure is continuous. ∎

Definition 1.18 (Half intervals). We call

$$\varnothing, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

half-intervals.

Lemma 1.7. Let $\mathcal{H}$ be the collection of finite disjoint unions of half-intervals. Then, $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. We see that

- $\varnothing \in \mathcal{H}$. Clearly.

- To show $\mathcal{H}$ is closed under complements, we have

  - $\varnothing^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$.
  - $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$.[12]
  - $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$.
  - $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$.
  - $(-\infty, \infty)^c = \varnothing \in \mathcal{H}$.

- $\mathcal{H}$ is closed under finite unions, clearly.

∎

---

[11] Actually, a measure is always continuous.
[12] Since it's a two disjoint union of half intervals.

Proposition 1.3 (Distribution function defines a pre-measure). Let $F \colon \mathbb{R} \to \mathbb{R}$ be a distribution function. For a half interval I, define

$$\ell(I) \coloneqq \ell_F(I) = \begin{cases} 0, & \text{if } I = \varnothing; \\ F(b) - F(a), & \text{if } I = (a, b]; \\ F(\infty) - F(a), & \text{if } I = (a, \infty]; \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b]; \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 \coloneqq \mu_{0,F}$ as

$$\mu_{0,F} \colon \mathcal{H} \to [0, \infty]$$

by

$$\mu_0(A) = \sum_{k=1}^{N} \ell(I_k) \text{ if } A = \bigcup_{k=1}^{N} I_k,$$

where A is a finite disjoint union of half intervals $I_1, \ldots, I_N$. Then, μ_0 is a pre-measure on $\mathcal{H}$.

Proof. We see that

1. μ_0 is well-defined.

2. $\mu_0(\varnothing) = 0$.

3. μ_0 is finite additive.

4. μ_0 is countable additive within $\mathcal{H}$.

   Suppose $A \in \mathcal{H}$ where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.[13]

   Focus on the case $I = (a, b]$. Let

   $$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

   which is a disjoint union. Then we only need to check

   $$F(b) - F(a) = \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) \right).$$

   - Since $(a, b] \supset \bigcup_{n=1}^{N} (a_n, b_n]$ for any fixed $N \in \mathbb{N}$, hence

   $$\underset{N \in \mathbb{N}}{\forall} F(b) - F(a) \geq \sum_{n=1}^{N} \left(F(b_n) - F(a_n) \right).$$

---

[13] Since $\mathcal{H}$ is only a collection of *finite* disjoint half intervals, hence after considering $A = I$, we can apply the same argument iteratively and stop in finite steps. Formally, we can consider $H \in \mathcal{H}$, $H = \bigcup_{i=1}^{\infty} A^i$, where A^i being a half interval. Then by the above argument, we have $A^i = I^i$ and so on.

By letting $N \to \infty$, we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) \right).$$

- Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

$$F(a') - F(a) < \epsilon.$$

For each $n \in \mathbb{N}$, $\exists b_n' > b_n$ such that

$$F(b_n') - F(b_n) < \frac{\epsilon}{2^n}.$$

Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b_n'),$$

hence

$$\underset{N \in \mathbb{N}}{\exists} \, [a', b] \subset \bigcup_{n=1}^{N} (a_n, b_n'), {}^{14}$$

which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^{N} F(b_n') - F(a_n).$$

Finally, we see that

$$F(b) - F(a) \leq F(b) - F(a') + \epsilon$$

$$\leq \sum_{n=1}^{\infty} \left(F(b_n') - F(a_n) \right) + \epsilon$$

$$\leq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\epsilon}{2^n} \right) + \epsilon$$

$$= \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) \right) + 2\epsilon$$

for any fixed $\epsilon > 0$, hence

$$F(b) - F(a) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

Combine these two inequalities, we have

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

as we desired.

---

[14] This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.

■

Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8: Lebesgue-Stieltjes Measure on $\mathbb{R}$

24 Jan. 11:00

To classify all measures, we now see this last theorem to complete the task.

Theorem 1.5 (Locally finite Borel measures on $\mathbb{R}$). We have

1. $F \colon \mathbb{R} \to \mathbb{R}$ a distribution function, then there exists a **unique** *locally finite Borel measure* μ_F on $\mathbb{R}$ satisfying

$$\mu_F((a,b]) = F(b) - F(a)$$

   for every $a < b$.

2. Suppose $F, G \colon \mathbb{R} \to \mathbb{R}$ are distribution functions. Then,

$$\mu_F = \mu_G$$

   on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. ■ HW.

Remark. Theorem 1.5 simply states that given a distribution function, if we restrict our attention on locally finite measures on $\mathbb{R}$ following our usual convention, then it defines the measure on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

1.6 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that

$$F \text{ distribution function } \overset{!}{\implies} \mu_F \text{ on Carathéodory } \sigma\text{-algebra}\,\mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 1.19 (Lebesgue-Stieltjes measure). Given a distribution function F, we define

- μ_F on $\mathcal{A}_{\mu_F}$ is called the *Lebesgue-Stieltjes measure* corresponding to F.

- Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{L}, m)$, where $\mathcal{L}$ is called *Lebesgue σ-algebra*, and m is called *Lebesgue measure*.

Remark. Recall that $\mathcal{L}$ is induced by Theorem 1.2, namely given m, for all $A \subset \mathbb{R}$, we have

$$\mathcal{L} := \left\{ A \subset \mathbb{R} \mid \underset{E \subset \mathbb{R}}{\forall} \; m(A) = m(A \cap E) + m(A \setminus E) \right\}$$

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).[15]$$

Example. We first see some examples.

1. $\mu_F((a,b]) = F(b) - F(a)$. Then

   - $\mu_F(\{a\}) = F(a) - F(a^-)$

   - $\mu_F([a,b]) = F(b) - F(a^-)$

   - $\mu_F((a,b)) = F(b^-) - F(a)$

2. We define
   $$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

   Then

   - $\mu_F(\{0\}) = 1$

   - $\mu_F(\mathbb{R}) = 1$

   - $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. This is easy to see since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, hence
     $$\mu_F(\mathbb{R} \setminus \{0\}) = \mu_F((-\infty, 0) \cup (0, \infty))$$
     $$= \underbrace{\mu_F((-\infty, 0))}_{0-0[16]} + \underbrace{\mu_F((0, \infty))}_{1-1[17]} = 0.$$

   We call that μ_F is the *Dirac measure* at 0.

3. Denote $\mathbb{Q} = \{r_1, r_2, \ldots\}$, and we define

   $$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r. \end{cases}$$

   Then

   - $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.

   - $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$.

---

[15]Some text will use $x-$ and $x+$ instead of x^- and x^+, respectively.
[16]It follows from $F(0^-) - F(-\infty) = 0 - 0 = 0$.
[17]It follows from $F(\infty) - F(0) = 1 - 1 = 0$.

This is shown in Lemma A.2.

4. If F is continuous at a, then $\mu_F(\{a\}) = 0$.

5. $F(x) = x$, then recall that we denote $\mu_F := m$, and we have

  - $m((a, b]) = m((a, b)) = m([a, b]) = b - a$.

6. $F(x) = e^x$

  - $\mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$.

Remark. We see that the first two examples are *discrete measures*.

Example (Middle thirds Cantor set). Let $C := \bigcap\limits_{n=1}^{\infty} K_n$, where we have

$$K_0 := [0, 1]$$
$$K_1 := K_0 \setminus \left(\frac{1}{3}, \frac{2}{3} \right)$$
$$K_2 := K_1 \setminus \left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right)$$
$$\vdots$$
$$K_n := K_{n-1} \setminus \bigcup_{k=1}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3^{k+2}}{3^{n+1}} \right).$$

We see that C is uncountable and with $m(C) = 0$. Firstly, since $x \in C$ if and only if $x = \sum\limits_{n=1}^{\infty} \frac{a_n}{3}$ for some $a_n \in \{0, 2\}$. Hence, we can instead formulate K_n by

$$K_n = \bigcup_{\substack{a_i \in \{0,2\} \\ 1 \le i \le n}} \left[\sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n} \right].$$
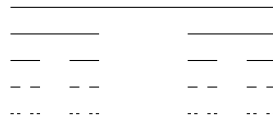

Figure 1: The top line corresponds to K_0, and then K_1, etc.

1.6.1 Cantor Function

Consider F as follows. We define a function F to be 0 to the left of 0, and 1 to the right of 1. Then, define F to be $\frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3} \right)$, $\frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9} \right)$, $\frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9} \right)$ and so on. This is so-called *Cantor Function*. We can show F is continuous and increasing, which makes F a distribution function. The proof is given in **??**. Also, we see that the measure this F induced is called *Cantor measure*.
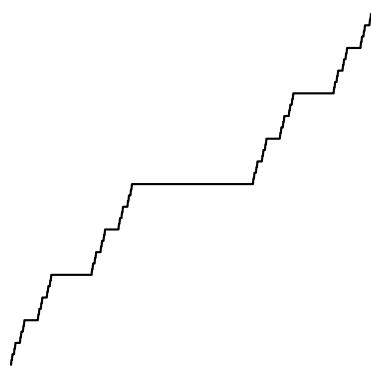
Figure 2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

Cantor Measure μ_F		Lebesgue Measure m
$\mu_F(\mathbb{R} \setminus C) = 0$		$m(\mathbb{R} \setminus C) = \infty > 0$
$\mu_F(C) = 1$	$\Longleftrightarrow$	$m(C) = 0$
$\mu_F(\{a\}) = 0$		$m(\{a\}) = 0$

Remark. μ_F and m are said to be **singular** to each other.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 1.8. Let μ be Lebesgue-Stieltjes measure on $\mathbb{R}$. Then we have

$$\mu(A) \overset{!}{=} \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supset A \right\}$$

$$= \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

for every $A \in \mathcal{A}_\mu$

Proof. The second equality follows from the continuity of the measure. ∎

Remark. This is similar to

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - 1/n], \quad (a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n].$$

Lecture 9: Properties of Lebesgue-Stieltjes measure 26 Jan. 11:00

As previously seen. Let $X \subset [0, \infty]$. Recall that

- $$\alpha = \sup X < \infty \iff \begin{cases} \underset{x \in X}{\forall} \ \alpha \geq x \\ \underset{\epsilon > 0}{\forall} \ \underset{x \in X}{\exists} \ x + \epsilon \geq \alpha. \end{cases}$$

- $$\alpha = \sup X = \infty \iff \underset{L > 0}{\forall} \ \underset{x \in X}{\exists} \ x \geq L.$$

This should be useful latter on.

Theorem 1.6. Let μ be Lebesgue-Stieltjes measure. Then, for all $A \in \mathcal{A}_\mu$,

1. (outer regularity) $\mu(A) = \inf\{\mu(O) \mid O \supset A, O \text{ is open}\}$

2. (inner regularity) $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$

Proof. We check them separately.

1. _____ DIY

2. Let $s := \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$, then by monotonicity, we have $\mu(A) \geq s$. To show the other direction, we consider

   - A is a bounded set.

     Then $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, $\overline{A}$ is also bounded $\implies \mu(\overline{A}) < \infty$. Fix $\epsilon > 0$, then by outer regularity, there exists an open $O \supset \overline{A} \setminus A$, and $\mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \epsilon$. Let $K := \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$, we

     show that
     $$\mu(K) \geq \mu(A) - \epsilon.$$
     _____ DIY

   - A is an unbounded set with $\mu(A) < \infty$.

     Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \ldots$, then
     $$\lim_{n \to \infty} \mu(A_n) = \mu(A) < \infty.$$

   - A is an unbounded set with $\mu(A) = \infty$.

     We can show that
     $$\lim_{n \to \infty} \mu(A_n) = \mu(A) = \infty.$$

     Fix $L > 0$, then $\exists N$ such that $\mu(A_N) \geq L$.

     $\blacksquare$

Definition 1.20 (G_δ-set, F_σ-set). Let X be a topological space. Then

- A G_δ-*set* is $G = \bigcap\limits_{i=1}^{\infty} O_i$, O_i open.

- A F_σ-*set* is $F = \bigcup\limits_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.7. Let μ be a Lebesgue-Stieltjes measure. Then *TFAE[a]*:

1. $A \in \mathcal{A}_\mu$

2. $A = G \setminus M$, G is a G_δ-set, M is a μ-null set.

3. $A = F \setminus N$, F is a F_σ-set, N is a μ-null set.

---
   [a]*TFAE*: The following are equivalent.

Proof. We see that (2.) $\implies$ (1.) and (3.) $\implies$ (1.) are clear.

- (1.) $\implies$ (3.)

    - Assume $\mu(A) < \infty$. From the inner regularity, we have

    $$\forall n \in \mathbb{N}\ \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

    Let $F = \bigcup\limits_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ-null. _____ ⌐ Check!

    - Assume $\mu(A) = \infty$. Let $A = \bigcup\limits_{k \in \mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

    $$\forall k \in \mathbb{Z}\ A_k = F_k \cup N_k,\ \ A = \underbrace{\left(\bigcup_k F_k \right)}_{F_\sigma\text{-set}} \cup \underbrace{\left(\bigcup_k N_k \right)}_{\mu\text{-null}}.$$

- (1.) $\implies$ (2.)

    We see that
    $$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

    ∎

Proposition 1.4. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0\ \exists I = \bigcup\limits_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using outer regularity and the fact that every open set is $\bigcup\limits_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. $\blacksquare$ DIY

We now see some properties of Lebesgue measure.

Theorem 1.8. Let $A \in \mathcal{L}$, then we have $A + s \in \mathcal{L}$, $rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,
$$m(A + s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. $\blacksquare$ DIY

Example. We now see some examples.

1. Let $\mathbb{Q} =: \{r_i\}_{i=1}^{\infty}$ which is dense in $\mathbb{R}$. Let $\epsilon > 0$, and
$$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i} \right).$$

   We see that O is open and dense in $\mathbb{R}$. But we see
   $$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$

   Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$

2. There exists uncountable set A with $m(A) = 0$.

3. There exists A with $m(A) > 0$ but A contains no non-empty open intervals.

4. There exists $A \notin \mathcal{L}$. e.g. Vitali set.[18]

5. There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

Lecture 10: Integration

26 Jan. 11:00

2 Integration

2.1 Measurable Function

We start with a definition.

Definition 2.1 (Measurable space). A *measurable space* or *Borel space* is a tuple of a set X and a σ-algebra $\mathcal{A}$ on X, denoted by $(X, \mathcal{A})$.

Definition 2.2 (Measurable function). Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$ are measurable spaces. Then we say $f \colon X \to Y$ is $(\mathcal{A}, \mathcal{B})$-*measurable* if
$$\forall B \in \mathcal{B} \ f^{-1}(B) \in \mathcal{A}.$$

---

[18]https://en.wikipedia.org/wiki/Vitali_set

> **Lemma 2.1.** Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then,
>
> $$f\colon X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall E \in \mathcal{E} \ f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part ($\implies$) is clear. On the other direction, we consider the following. Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $E \subset \mathcal{D}$ by assumption
- $\mathcal{D}$ is a σ-algebra ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯ Check!

hence, we see that $\langle \mathcal{E} \rangle \subset \mathcal{D}$. ∎

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup\limits_{\alpha} E_\alpha \right) = \bigcup\limits_{\alpha} f^{-1}(E_\alpha)$

> **Definition 2.3 ($\mathcal{A}$-measurable).** Let $(X, \mathcal{A})$ be a measurable space. Then,
>
> $$\left.\begin{array}{l} f\colon X \to \mathbb{R} \\ f\colon X \to \overline{\mathbb{R}} \\ f\colon X \to \mathbb{C} \end{array}\right\} \text{ is } \mathcal{A}\text{-}measurable \text{ if } \begin{cases} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f \colon X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{cases}$$

Notation. Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$
- $\mathcal{B}(\overline{\mathbb{R}}) = \left\{ E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R}) \right\}$.

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \implies$ every function is $\mathcal{A}$-measurable.
- $\mathcal{A} = \{\varnothing, X\} \implies$ only $\mathcal{A}$-measurable functions are constant functions.

> **Lemma 2.2.** Given $f\colon X \to \mathbb{R}$, *TFAE*.
>
> 1. f is $\mathcal{A}$-measurable
> 2. $\forall a \in \mathbb{R}, \ f^{-1}((a, \infty)) \in \mathcal{A}$
> 3. $\forall a \in \mathbb{R}, \ f^{-1}([a, \infty)) \in \mathcal{A}$
> 4. $\forall a \in \mathbb{R}, \ f^{-1}((-\infty, a)) \in \mathcal{A}$
> 5. $\forall a \in \mathbb{R}, \ f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from the lemma we just saw. ∎

Property. Given $f, g\colon X \to \mathbb{R}$ and is $\mathcal{A}$-measurable, then

1. $\phi\colon \mathbb{R} \to \mathbb{R}$, $\mathcal{A}$-measurable[19], then

$$\phi \circ f\colon X \to \mathbb{R}$$

   is $\mathcal{A}$-measurable.

2. $-f$, $3f$, f^2, $|f|$ are all $\mathcal{A}$-measurable, and $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) \neq 0, \forall x \in X$.

3. $f + g$ is $\mathcal{A}$-measurable. We see this from

$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty)) \right).$$

4. $f \cdot g$ is $\mathcal{A}$-measurable. We see this from

$$f(x)g(x) = \frac{1}{2} \left((f(x) + g(x))^2 - f(x)^2 - g(x)^2 \right).$$

5. We see that

$$(f \vee g)(x) := \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) := \min\{f(x), g(x)\}$$

   are $\mathcal{A}$-measurable.

6. Let $f_n\colon X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable. Then

$$\sup_{n \in \mathbb{N}} f_n, \ \inf_{n \in \mathbb{N}} f_n, \ \limsup_{n \to \infty} f_n, \ \liminf_{n \to \infty} f_n$$

   are $\mathcal{A}$-measurable.

   Proof. Consider $\sup_{n \in \mathbb{N}} f_n =: g$, then

$$g^{-1}((a, \infty]) = \bigcup_{n \in \mathbb{N}} f_n^{-1}((a, \infty])$$

   for $\sup_n f_n(x) = g(x) > a$. A similar argument can prove the case of $\boxed{\text{check}}$
   $\inf_{n \in \mathbb{N}} f_n$.

   And notice that $\limsup_{n \to \infty} f_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n$, then the similar argument also proves this case. ∎

7. If $\lim_{n \to \infty} f_n(x)$ converges for every $x \in X$, then f is $\mathcal{A}$-measurable.

   Example. If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous

   $\implies$ f is Borel measurable

   $\implies$ f is Lebesgue measurable

   since the preimage of an open set of a continuous function is open, then we consider $f^{-1}((a, \infty))$.

---

[19]In this case, we also call it *Borel measurable*.

Definition 2.4 (f^+, f^-). For $f\colon X \to \overline{\mathbb{R}}$, let $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$.[a]

---

[a]i.e., $f^+(x) = \max\{f(x), 0\}$, $\quad f^-(x) = \min\{-f(x), 0\}$

Remark. If $\operatorname{supp} f^+ \cap \operatorname{supp} f^- = \varnothing$ and $f(x) = f^+(x) - f^-(x)$, then

$$f \text{ is } \mathcal{A}\text{-measurable} \iff f^+, f^- \text{ are } \mathcal{A}\text{-measurable}.$$

Notation. $\operatorname{supp} f$ means the support of f, which is the set of domain which makes f being non-zero.

Definition 2.5 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator) function* of E is

$$\mathcal{X}_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is $\mathcal{A}$-measurable $\iff E \in \mathcal{A}$.

Definition 2.6 (Simple function). Let $(X, \mathcal{A})$ be a measurable space. Then a *simple function* $\phi\colon X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

Remark. We see that if

$$\phi(X) = \{c_1, \dots, c_N\},$$

and

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \implies \phi = \sum_{i=1}^{N} \underbrace{c_i}_{\neq \pm\infty} \underbrace{\mathbb{1}_{E_i}}_{\in \mathcal{A}}.$$

Lecture 11: Integration of nonnegative functions

31 Jan. 11:00

As previously seen. For a simple function ϕ, c_i can actually be in $\mathbb{C}$.

Theorem 2.1. Given $(X, \mathcal{A})$ be a measurable space, and let $f\colon X \to [0, \infty]$, the following is equivalent.

1. f is $\mathcal{A}$-measurable function.

2. There exists simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$ such that
$$\forall_{x \in X} \ \lim_{n \to \infty} \phi_n(x) = f(x)$$
   i.e., f is a pointwise upward limit of simple functions.

Proof. It's clear that (2.) $\implies$ (1.) from the fact that $f(x) = \sup_n \phi_n(x)$. Now, we want to show that (1.) $\implies$ (2.). Assume f is $\mathcal{A}$-measurable, and fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. Then for $0 \leq k \leq 2^{2n} - 1$, $E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}$.

Then, define ϕ_n be

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n},$$

we have

- $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ for every $x \in X$

- $\forall x \in X \setminus F_n$, we have $0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$

Furthermore, we see that

$$F_1 \supset F_2 \supset \ldots, \quad \bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\}),$$

then

- $x \in f^{-1}([0, \infty]) = X \setminus \bigcap_{n=1}^{\infty} F_n \implies \lim_{n \to \infty} \phi_n(x) = f(x)$

- $x \in f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} F_n \implies f_n(x) \geq 2^n \implies \lim_{n \to \infty} \phi_n(x) = \infty = f(x)$

∎

Corollary 2.1. If f is bounded on a set $A \subset \mathbb{R}$, i.e., $\exists b > 0$ such that

$$\forall_{x \in A} |f(x)| \leq L,$$

then $\phi_n \to f$ <u>uniformly on A</u>.

Proof. _____ ∎ DIY

Corollary 2.2. If $f \colon X \to \mathbb{C}$ is a measurable function if and only if there exists simple functions $\phi_n \colon X \to \mathbb{C}$ such that

$$0 \leq |\phi_1(x)| \leq |\phi_2(x)| \leq \ldots \leq |f(x)|$$

with

$$\forall_{x \in X} \lim_{n \to \infty} \phi_n(x) = f(x).$$

Proof. _____ ∎ DIY

2.2 Integration of nonnegative functions

We start with a definition.

> **Definition 2.7 (Integration of nonnegative funciton).** Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\phi \colon X \to [0, \infty]$ such that
>
> $$\phi = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}$$
>
> be a simple function. Define
>
> $$\int \phi \, dx = \int \phi \, d\mu = \int_X \phi \, d\mu = \sum_{i=1}^{N} c_i \mu(E_i).$$
>
> Furthermore, for $A \in \mathcal{A}$,
>
> $$\int_A \phi = \int_A \phi \, d\mu = \int \phi \mathbb{1}_A \, d\mu$$

Note. Notice that in the expression $\sum_{i=1}^{N} c_i \mu(E_i)$ we're using the convention such that $0 \cdot \infty = 0$, and in the function $\phi \mathbb{1}_A$ is also a simple function since both ϕ and $\mathbb{1}_A$ are simple function.

> **Proposition 2.1.** Suppose we have $\phi, \psi \geq 0$ be two simple functions. Then,
>
> - Definition 2.7 is well-defined.
> - $\int c\phi = c \int \phi$ for $c \in [0, \infty)$.
> - $\int \phi + \psi = \int \phi + \int \psi$.
> - $\phi(x) \geq \psi(x)$ for all $x \implies \int \phi \geq \int \psi$.
> - $\nu(A) = \int_A \phi \, d\mu$ is a measure on $(X, \mathcal{A})$.

Proof. ∎ DIY

> **Definition 2.8 (Generatlization of Integration of nonnegative function).** Given $(X, \mathcal{A}, \mu)$ with $f \colon X \to [0, \infty]$ be $\mathcal{A}$-measurable. Define
>
> $$\int f = \int f \, d\mu = \sup \left\{ \int \phi \mid 0 \leq \phi \leq f \text{ such that } \phi \text{ is simple} \right\}.$$

- If f is a simple function, the Definition 2.7 and Definition 2.8 of $\int f$ are the same.
- $\int cf = c \int f$ for $c \in [0, \infty)$.

- If $f \geq g \geq 0 \implies \int f \geq \int g$.

- But $\int f + g = \int f + \int g$ is not trivial.

Theorem 2.2 (Monotone Convergence Theorem (MCT)). Given $(X, \mathcal{A}, \mu)$ be a measure space. Then if

- $f_n \colon X \to [0, \infty]$ be $\mathcal{A}$-measurable for every $n \in \mathbb{N}$

- $0 \leq f_1(x) \leq f_2(x) \leq \ldots$ for every $x \in X$

- $\lim\limits_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$,

we have
$$\lim_{n \to \infty} \int f_n = \int f.$$

Proof. Note that $\lim\limits_{n \to \infty} f_n(x)$ converges for every $x \in X$ and $\lim\limits_{n \to \infty} \int f_n$ converges. Then

- $f_n \leq f \implies \int f_n \leq \int f \implies \lim\limits_{n \to \infty} \int f_n \leq \int f$.

- Fix a simple function $0 \leq \phi \leq f$, then it's enough to show $\lim\limits_{n \to \infty} \int f_n \geq \int \phi$.

  We first fix $\alpha = (0, 1)$, then it's also enough to show
  $$\lim_{n \to \infty} \int f_n \geq \alpha \int \phi.$$

  Let $A_n \coloneqq \{x \in X \mid f_n(x) \geq \alpha \phi(x)\}$, then since f_n is measurable,

  - $A_n \in \mathcal{A}$

  - $A_1 \subset A_2 \subset A_3 \subset \ldots$

  - $\bigcup\limits_{n=1}^{\infty} A_n = X$ ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯ `Check!`

  We then have
  $$\int f_n \geq \int f_n \mathbb{1}_{A_n} \geq \int \alpha \phi \mathbb{1}_{A_n} = \alpha \nu(A_n)$$

  where $\nu(A) = \int_A \phi$ is a measure. This implies
  $$\lim_{n \to \infty} \int f_n \geq \alpha \lim_{n \to \infty} \nu(A_n) \overset{20}{=} \alpha \nu(X) = \alpha \int \phi.$$

∎

Corollary 2.3. Let $f, g \geq 0$ be measurable, then
$$\int f + g = \int f + \int g.$$

---

[20]This follows from the continuity of measure from below

Proof. There exists simple functions ϕ_n and ψ_n such that

- $0 \leq \phi_1 \leq \phi_2 \leq \ldots$ and $\phi_n \to f$ pointwise

- $0 \leq \psi_1 \leq \psi_2 \leq \ldots$ and $\psi_n \to f$ pointwise

Then,

$$\int (f+g) \overset{!}{=} \lim_{n \to \infty} (\phi_n + \psi_n) = \lim_{n \to \infty} \int \phi_n + \int \psi_n \overset{!}{=} \int f + \int g.$$

$\blacksquare$

Lecture 12

We start with a useful corollary.

Corollary 2.4 (Tonelli's theorem for series and integrals). Given $g_n \geq 0$ for every $n \in \mathbb{N}$ and let g_n be measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Remark. Recall that we have seen two series case before. We'll later see two integrals case.

Proof. Let $f_N := \sum_{n=1}^{N} g_n$ such that $\lim_{N \to \infty} f_N \sum_{n=1}^{\infty} g_n =: f$, then since $g_n \geq 0$, we have $0 \leq f_1 \leq f_2 \leq \ldots$ with

$$\lim_{N \to \infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

By Theorem 2.2, we have

$$\lim_{N \to \infty} \int \underbrace{\sum_{n=1}^{N} g_n}_{f_N} = \int \underbrace{\sum_{n=1}^{\infty} g_n}_{f}.$$

Now, since the terms in the limit on the left-hand side is just a finite sum, by Corollary 2.3, we have

$$\underbrace{\lim_{N \to \infty} \sum_{n=1}^{N} \int g_n}_{\sum_{n=1}^{\infty}} = \lim_{N \to \infty} \int \sum_{n=1}^{N} g_n = \int \sum_{n=1}^{\infty} g_n,$$

hence

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

$\blacksquare$

Theorem 2.3 (Fatou's Lemma). Suppose $f_n \geq 0$ and measurable, then

$$\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n.$$

Remark. Recall that

$$\liminf_{n \to \infty} f_n := \lim_{k \to \infty} \inf_{n \geq k} f_n = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f_n$$

and

$$\exists \lim_{n \to \infty} a_n \iff \limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n.$$

Proof. Let $g_k = \inf_{n \geq k} f_n$, then g_k is measurable and $0 \leq g_1 \leq g_2 \leq \ldots$. Now, from Theorem 2.2, we have

$$\int \lim_{k \to \infty} g_k = \lim_{k \to \infty} \int g_k.$$

Notice that the left-hand side is just $\int \liminf_{n \to \infty} f_n$, while the right-hand side is just $\lim_{k \to \infty} \int \inf_{n \geq k} f_n$, i.e.,

$$\int \liminf_{n \to \infty} f_n = \lim_{k \to \infty} \int \inf_{n \geq k} f_n.$$

We see that we want to take the inf outside the integral on the right-hand side. Observe that

$$\underset{m \geq k}{\forall} \inf_{n \geq k} f_n \leq f_m \implies \underset{m \geq k}{\forall} \int \inf_{n \geq k} f \leq \int f_m \implies \int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m.$$

Then, we have

$$\int \liminf_{n \to \infty} f_n = \lim_{k \to \infty} \int \inf_{n \geq k} f_n \leq \lim_{k \to \infty} \inf_{m \geq k} \int f_m = \liminf_{m \to \infty} \int f_m.$$

∎

Example. Given $(\mathbb{R}, \mathcal{L}, m)$.

1. **Escape to horizontal infinity**. Let $f_n := \mathbb{1}_{(n, n+1)}$. We immediately see that

   - $f_n \to 0$ pointwise
   - $\int f_n = 1$ for every n
   - $\int f = 0$

2. **Escape to width infinity**. Let $f_n := \frac{1}{n} \mathbb{1}_{(0,n)}$.

3. **Escape to vertical infinity**. Let $f_n := n \mathbb{1}_{(0, \frac{1}{n})}$.

Lemma 2.3 (Markov's inequiality). Let $f \geq 0$ be measurable. Then

$$\forall_{c \in (0,\infty)} \mu \left(\{ x \mid f(x) \geq c \} \right) \leq \frac{1}{c} \int f.$$

Proof. Denote $\{ x \mid f(x) \geq c \} =: E$, then

$$f(x) \geq c \mathbb{1}_E(x) \implies \int f \geq c \int \mathbb{1}_E = c \cdot \mu(E).$$

∎

Remark. Notice that $E = f^{-1}([c, \infty])$, hence E is measurable.

Proposition 2.2. Let $f \geq 0$ be measurable. Then,

$$\int f = 0 \iff f = 0 a.e..$$

i.e.,

$$\int f \, \mathrm{d}\mu = 0 \iff \begin{cases} \mu(A) = 0 \\ A = \{ x \mid f(x) > 0 \} = f^{-1}((0, \infty]). \end{cases}$$

Proof. Firstly, assume that $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}$$

where E_i are disjoint and $c_i \in (0, \infty)$. Then,

$$\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0$$
$$\iff \mu(E_1) = \ldots = \mu(E_N) = 0$$
$$\iff \mu(A) = 0, \ A = \bigcup_{i=1}^{N} E_i.$$

Now, assume that f is a general function where $f \geq 0$ is the only constraint.

1. Assume $\mu(A) = 0$ (i.e., $f = 0$ a.e.). Let $0 \leq \phi \leq f$, where ϕ is simple. Then

$$\forall_{x \in A^c} \phi(x) = 0$$

since $f(x) = 0$, $\forall x \in A^c$. This implies that $\phi = 0$ a.e. since $\mu(A) = 0$, so $\int \phi = 0$. We then have

$$\int f = 0$$

from Definition 2.8.

2. Assume $\int f = 0$. Let $A_n = f^{-1}\left(\left[\frac{1}{n}, \infty\right]\right)$. Then we see that

- $A_1 \subset A_2 \subset \ldots$

- $\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0, \infty)) = A.$

We then have

$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \overset{!}{\leq} n \int f = 0,$$

which further implies

$$\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0$$

from the continuity of measure from below.

∎

Corollary 2.5. If $f, g \geq 0$ are both measurable and $f = g$ a.e., then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \neq g(x)\}$[21]. Then by assumption, $\mu(A) = 0$, hence

$$f \mathbb{1}_A = 0 \ a.e., \quad g \mathbb{1}_A = 0 \ a.e..$$

This further implies that

$$\int f = \int f(\mathbb{1}_A + \mathbb{1}_{A^c})$$

$$\overset{!}{=} \int f \mathbb{1}_A + \int f \mathbb{1}_{A^c}$$

$$= \int f \mathbb{1}_{A^c} = \int g \mathbb{1}_{A^c}$$

$$= \int g \mathbb{1}_{A^c} + \int g \mathbb{1}_A = \int g.$$

∎

Corollary 2.6. Let $f_n \geq 0$ be measurable. Then

1.
$$\left.\begin{array}{l} 0 \leq f_1 \leq f_2 \leq \ldots \leq f \ a.e. \\ \lim_{n \to \infty} f_n = f \ a.e. \end{array}\right\} \implies \lim_{n \to \infty} \int f_n = \int f.$$

2. $\lim_{n \to \infty} f_n = f \ a.e. \implies \int f \leq \liminf_{n \to \infty} \int f_n.$

Proof. ∎ DIY

---

[21] A is measurable indeed.

Appendix

A   Additional Proofs

A.1   Measure

This section gives all additional proofs in Section 1.

> **Theorem A.1 (Theorem 1.2 3.).** Under the setup of Theorem 1.2, $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We see this in two parts.

1. **Claim:** If $A \subset X$ satisfies $\mu^*(A) = 0$, then A is Carathéodory measurable with respect to μ^*.

   Proof. If $A \subset X$ and $\mu^*(A) = 0$, where μ^* is an outer measure on X, we'll show that A is Carathéodory measurable with respect to μ^*.

   Equivalently, we want to show that for any $E \subset X$,
   $$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

   Firstly, noting that $(E \cap A) \subset A$, and by monotonicity of μ^*, we see that
   $$\mu^*(E \cap A) \leq \mu^*(A) = 0,$$
   and since $\mu^* \geq 0$, hence $\mu^*(E \cap A) = 0$. Now, we only need to show that
   $$\mu^*(E) = \mu^*(E \setminus A).$$

   Since $E \setminus A = E \cap A^c$, and hence we have $E \cap A^c \subset E$, so
   $$\mu^*(E) \geq \mu^*(E \setminus A).$$

   To show another direction, we note that
   $$\mu^*(E) \leq \mu^*(E \cup A) = \mu^*((E \setminus A) \cup A) \leq \mu^*(E \setminus A),$$

   hence we conclude that A is Carathéodory measurable with respect to μ^* if $\mu^*(A) = 0$. ∎

2. **Claim:** If A is μ-subnull, then $A \in \mathcal{A}$.

   Proof. Let $\mathcal{A}$ denotes the Carathéodory σ-algebra, and $\mu \coloneqq \mu^*|_{\mathcal{A}}$. We want to show if A is μ-subnull, then $A \in \mathcal{A}$.

   Firstly, if A is μ-subnull, then there exists a $B \in \mathcal{A}$ such that $A \subset B$ and $\mu(B) = 0$. But since from the monotonicity of μ^*, we further have
   $$0 = \mu(B) = \mu^*(B) \geq \mu^*(A),$$

   hence $\mu^*(A) = 0$.

   From the first claim, we immediately see that A is Carathéodory measurable with respect to μ^*, which implies A is in Carathéodory σ-algebra, hence $A \in \mathcal{A}$. ∎

We see that the second claim directly proves that $(X, \mathcal{A}, \mu)$ is a complete measure space. ∎

Lemma A.1. The function F defined in this example is a distribution function

Proof. We define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n \end{cases}$$

where $\{r_1, r_2, \ldots\} = \mathbb{Q}$, and

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} = \sum_{n; r_n \leq x} \frac{1}{2^n}$$

is both increasing and right-continuous.

- Increasing. Consider $x < y$. We see that

$$F(x) = \sum_{n; r_n \leq x} \frac{1}{2^n} \leq \sum_{n; r_n \leq y} \frac{1}{2^n} = F(y)$$

clearly.[22]

- Right-continuous. We want to show $F(x^+) = F(x)$. Let $x^+(\epsilon) \coloneqq x + \epsilon$ with $\epsilon > 0$, we'll show that

$$\lim_{\epsilon \to 0} F(x^+(\epsilon)) = \lim_{\epsilon \to 0} F(x + \epsilon) = F(x).$$

Firstly, we have

$$F(x^+(\epsilon)) - F(x) = \sum_{n; r_n \leq x+\epsilon} \frac{1}{2^n} - \sum_{n; r_n \leq x} \frac{1}{2^n} = \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n},{}^{[23]}$$

and we want to show

$$\lim_{\epsilon \to 0} F(x^+(\epsilon)) - F(x) = \lim_{\epsilon \to 0} \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} = 0.$$

Before we show how we choose ϵ,[24] we see that

$$\sum_{n=k}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

---

[22]This is trivial since we're always going to sum more strictly positive terms in $F(y)$ than in $F(x)$.

[23]The strict is crucial to show the result, since if $x = r_k$ for some fixed k, then we can't make the summation arbitrarily small.

[24]To be precise, how ϵ depends on r_n.

Now, we simply observe that

$$\sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} \leq \sum_{n=\arg\min_k x < r_k \leq x+\epsilon}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

With this observation, it should be fairly easy to see that we can choose ϵ based on how small we want to make 2^{1-k} be,[25] and we indeed see that

$$\lim_{k \to \infty} 2^{1-k} = 0,$$

which implies that F is right-continuous by squeeze theorem.

∎

Lemma A.2. The function F defined in this example satisfies

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$

given in this example.

Proof. We prove them one by one. And notice that F is indeed a distribution function as we proved in Lemma A.1.

1. To show $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$, we first note that $\{r\} = \bigcap_{a-1 \leq x < r} (x, r]$.

   Then, we see that

   $$\mu_F(\{r\}) = \mu_F \left(\bigcap_{a-1 \leq x < a} (x, r] \right),$$

   where each $(x, r] \in \mathcal{A}$ and $(x, r] \supset (y, r]$ whenever $r - 1 \leq x \leq y < r$. Notice that we implicitly assign the order of the index by the order of x. Then, we see that $\mu_F(r-1, r] < \infty$.[26] Then, from continuity from above, we see that

   $$\mu_F(\{r\}) = \lim_{i \to \infty} \mu_F((x_i, r]),$$

   where we again implicitly assign an order to x as the usual order on $\mathbb{R}$ by given index i. It's then clear that as $i \to \infty$, $x_i \to r$. From the definition of F, we see that

   $$F((x_i, r]) = F(r) - F(x_i) = \sum_{n; r_n \leq r} \frac{1}{2^n} - \sum_{n; r_n \leq x_i} \frac{1}{2^n} = \sum_{n; x_i < r_n \leq r} \frac{1}{2^n}.$$

   It's then clear that since $r \in \mathbb{Q}$, there exists an i' such that $r_{i'} = r$. Then, we immediately see that no matter how close $x_i \to r$, this sum is at least

   $$\frac{1}{2^{i'}}$$

   for a fixed i'. Hence, we conclude that $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$.

---

[25]We're referring to $\epsilon - \delta$ proof approach.

[26]This will be $\mu(A_1)$ in the condition of continuity from above. Furthermore, since $\mathbb{Q}$ is countable, hence $F(x) < \infty$ is promised.

2. Now, we show $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$. Firstly, we claim that

$$\mu_F(\mathbb{Q}) = 1$$

and

$$\mu_F(\mathbb{R}) = 1$$

as well. Since $\mu_F(\mathbb{Q}) = 1$ is clear, we note that the second one essentially follows from the fact that we can write

$$\mathbb{R} = \lim_{N \to \infty} \bigcup_{i=1}^{N} (a - i, a + i]$$

for any $a \in \mathbb{R}$, say 0. From continuity from below, we have

$$\mu_F \left(\bigcup_{i=1}^{\infty} (-i, +i] \right) = \lim_{n \to \infty} \mu_F((-n, n]) = \sum_{n; r_n \in \mathbb{Q}} \frac{1}{2^n} = 1.$$

Given the above, from countable additivity of μ_F, we have

$$\mu_F(\mathbb{R} \setminus \mathbb{Q}) + \underbrace{\mu_F(\mathbb{Q})}_{1} = \underbrace{\mu_F(\mathbb{R})}_{1} \implies \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$$

as we desired.

■

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