CS598
Topics in Graph Algorithms

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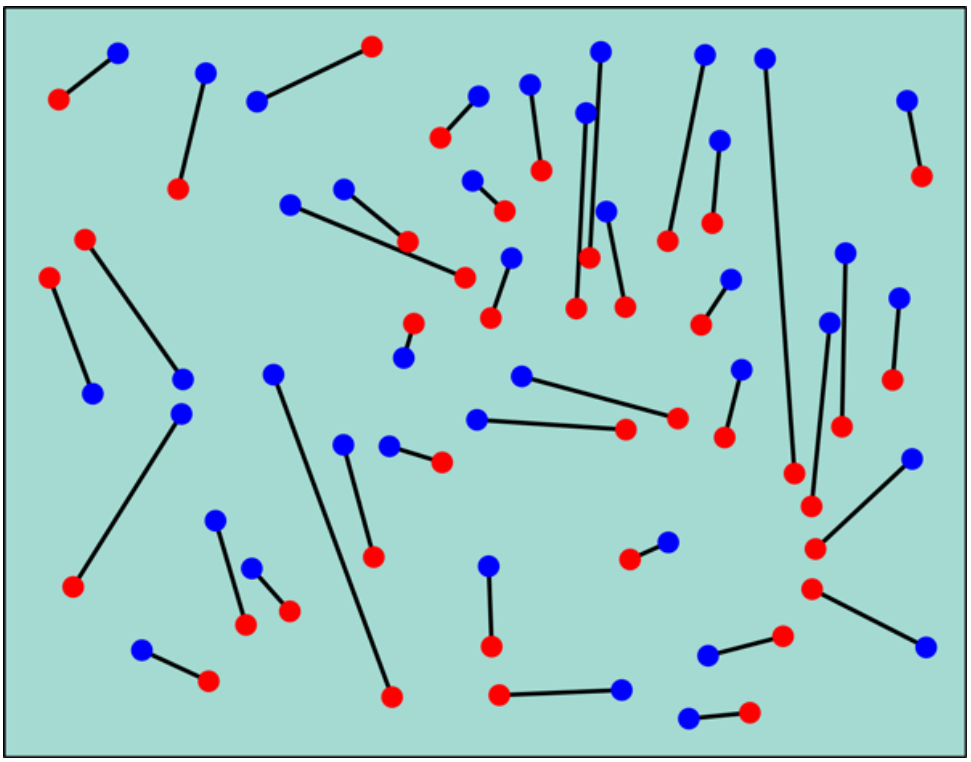
Abstract

This is an advanced graduate-level graph algorithm course taught by Chandra Chekuri at University of Illinois Urbana-Champaign.



This course is taken in Fall 2024, and the date on the cover page is the last updated time.

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Chapter 1

Introduction

Lecture 1: Overview

Throughout the course, we consider a graph $G = (V, E)$ such that $n := |V|$ and $m := |E|$. Let's see some examples about the recent breakthroughs. 27 Aug. 11:00

Example (Shortest paths with negative length). The classical algorithm runs in $O(mn)$. In 2022, [BNW] came up with an algorithm $O(m \log^3 nC)$, where C is the largest absolute value of the *integer* length.

This is not a strongly polynomial time algorithm. In 2024 [Fineman] come up with $\widetilde{O}(mn^{8/9})$, and soon after 2024 [HJQ] improve this to $\widetilde{O}(mn^{4/5})$.

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Example (s-t maxflow). The tradition running time is $O(mn \log m/n)$, and it's later improved to be $O(m\sqrt{n} \log nC)$. Recently, [Chen et-al] improve to $O(m^{1+o(1)})$, which is almost-linear.[a]

cite

[a] This can be also applied to min-cost flow and quadratic-cost flow.

1.1 Minimum Spanning Tree

Finding the minimum cost spanning tree (MST) in a connected graph is a basic algorithmic problem that has been long-studied. We introduce the problem formally.

Definition 1.1.1 (Spanning tree). A *spanning tree* T of a connected graph $G = (V, E)$ is an induced subgraph of G which spans G, i.e., $V(T) = V$ and $E(T) \subseteq E$.

Then, the problem can be formalized as follows.

Problem 1.1.1 (Minimum spanning tree). Given a connected graph $G = (V, E)$ with edge weight $c \colon E \to \mathbb{R}$, find the min-cost spanning tree.

Remark. The edge costs need not be positive, but we can make them positive by adding a large number without affecting correctness.

Standard algorithm that are covered in most undergraduate courses are Kruskal's algorithm, Jarnik-Prim's (JP) algorithm,[1] and (sometimes) Borůvka's algorithm. There are many algorithms for MST and their correctness relies on two simple rules (structural properties). The first one is about cuts:

Lemma 1.1.1 (Cut rule). If e is a minimum cost edge in a cut $\delta(S)$ for some $S \subseteq V$, then e is in some MST. In particular, if e is the unique minimum cost edge in the cut, then e is in every MST.

[1] This is typically attributed usually to Prim but first described by Jarnik

Definition 1.1.2 (Light). An edge e is *light* or *safe* if there exists a cut $\delta(S)$ such that e is the cheapest cost edge crossing the cut. We also say that e is *light* w.r.t. a set of edges $F \subseteq E$ if e is light in (V, F).

Another one is about cycles:

Lemma 1.1.2 (Cycle rule). If e is the highest cost edge in a cycle C, then there exists an MST that does not contain e. In particular, if e is the unique highest cost edge in C, then e cannot be in any MST.

Definition 1.1.3 (Heavy). An edge e is *heavy* or *unsafe* if there exists a cycle C such that e is the highest cost edge in C. We also say that e is *heavy* w.r.t. a set of edges $F \subseteq E$ if e is heavy in (V, F).

Corollary 1.1.1. Suppose the edge costs are unique and G is connected. Then the MST is unique and consists of the set of all light edges.

Remark. Without loss of generality, we can assume that the cost are unique by, e.g., perturbation or consistent tie-breaking rule.

1.1.1 Standard Algorithms

Let's review the basic algorithms, the data structures they use, and the run-times that they yield.

Kruskal's Algorithm

Intuitively speaking, Kruskal's algorithm sorts the edges in increasing cost order and greedily inserts edges in this order while maintaining a maximal forest F at each step. When considering the i^{th} edge e_i, the algorithm needs to decide if $F + e_i$ is a forest or whether adding e creates a cycle.

Algorithm 1.1: Kruskal's algorithm

Data: A connected graph $G = (V, E)$ with edge weight $c\colon E \to \mathbb{R}$
Result: A MST $T = (V, F)$

```
1 Sort the edges such that c(e_1) ≤ c(e_2) ≤ ··· ≤ c(e_m)        // O(m log n)
2 F ← ∅                                                          // Initialize the tree
3 for i = 1, ..., m do
4     if e_i + F has no cycle then
5         F ← F + e_i
6 return (V, F)
```

Theorem 1.1.1. Kruskal's algorithm takes $O(m \log n)$.

Proof. Sorting takes $O(m \log n)$ time. The standard solution for line 4 is to use a union-find data structure. Union-find data structure with path compression yields a total run time, after sorting, of $O(m\alpha(m, n))$ where $\alpha(m, n)$ is inverse Ackerman function which is extremely slowly growing. Thus, the bottleneck is sorting, and the run-time is $O(m \log n)$. ■

Jarnik-Prim's Algorithm

Jarnik-Prim's algorithm grows a tree starting at some arbitrary root vertex r while maintaining a tree T rooted at r. In each iteration it adds the cheapest edge leaving T until T becomes spanning. Thus, the Jarnik-Prim's algorithm takes $n - 1$ iterations.

Algorithm 1.2: Jarnik-Prim's algorithm

Data: A connected graph $G = (V, E)$ with edge weight $c\colon E \to \mathbb{R}$
Result: A MST $T = (V, F)$

```
1 r ← uniform(V)                                          // Sample a root
2 V' ← {r}, F ← ∅                                         // Initialize the tree
3 while V' ≠ V do
4     e ← arg min_{e=(u,v)∈δ(V'),u∈V'} c(e)
5     F ← F + e, V' ← V' + v                              // Update the tree
6 return (V, F)
```

Theorem 1.1.2. Jarnik-Prim's algorithm takes $O(m + n \log n)$.

Proof. To find the cheapest edge leaving T (line 4), one typically uses a priority queue where we maintain vertices not yet in the tree with a key for v equal to the cost of the cheapest edge from v to the current tree. When a new vertex v is added to T the algorithm scans the edges in $\delta(v)$ to update the keys of neighbors of v. Thus, one sees that there are a total of $O(m)$ `decrease-key` operations, $O(n)$ `extract-min` operations, and initially we set up an empty queue. Standard priority queues implement `decrease-key` and `extract-min` in $O(\log n)$ time each, so the total time is $O(m \log n)$. However, Fibonacci heaps and related data structures show that one can implement `decrease-key` in amortized $O(1)$ time which reduces the total run time to $O(m + n \log n)$. ■

Remark. The Jarnik-Prim's algorithm runs in linear-time for moderately dense graphs!

Borůvka's Algorithm

Borůvka's algorithm seems to be the first MST algorithm, which has very nice properties and essentially uses no data structures. The algorithm works in phases. We describe it recursively to simplify the description, while refer to Algorithm 1.3 for the real implementation. In the first phase the algorithm finds, for each vertex v the cheapest edge in $\delta(v)$. By the cut rule this edge is in every MST.

Note. An edge $e = uv$ may be the cheapest edge for both u and v.

The algorithm collects all these edges, say F, and adds them to the tree. It then shrinks the connected components induced by F and recurses on the resulting graph $H = (V', E')$. It's easy to see that Borůvka's algorithm can be parallelized, unlike the other two algorithms.

Algorithm 1.3: Borůvka's algorithm

Data: A connected graph $G = (V, E)$ with edge weight $c\colon E \to \mathbb{R}$
Result: A MST $T = (V, F)$

```
1 F = ∅                                                   // Initialize the tree
2 S ← {S_v = {v}}                                         // Collection of all sets
3 while |S| > 1 do
4     for S ∈ S do
5         e_S = (u, v) ← arg min_{e∈δ(S)} c(e)
6         S ← S − {S_u, S_v} + S_u ∪ S_v                  // Merge (i.e., shrink)
7         F ← F + e_S                                     // Update the tree
8 return (V, F)
```

Notation. In line 6, S_u and S_v both refer to $S := S_u \cup S_v$ later in the algorithm.

Theorem 1.1.3. Borůvka's algorithm takes $O(m \log n)$.

Proof. The first phase needs $O(m)$ from a linear scan of the adjacency lists, and also computing H (i.e., shrinking) can be done in $O(m)$ time. The main observation is that $|V'| \leq |V|/2$ since each vertex v is in a connected component of size at least 2 as we add an edge leaving v to F. Thus, the algorithm terminates in $O(\log n)$ phases for a total of $O(m \log n)$ time. ■

1.1.2 Faster Algorithms

A natural question is whether there is a linear-time, i.e., $O(m)$, MST algorithm. The following is the history of fast MST algorithms:

- Very early on, Yao, in 1975, obtained an algorithm that ran in $O(m \log\log n)$[Yao75], which leverages the idea developed in 1974 for the linear-time Selection algorithm.
- In 1987, Fredman and Tarjan [FT87] developed the Fibonacci heaps and give an MST algorithm which runs in $O(m \log^* n)$.[2] This was further improved to $O(m \log\log^* n)$ [Gab+86].
- Karger, Klein, and Tarjan [KKT95] obtained a linear time randomized algorithm that will be the main topic of this lecture.
- Chazelle's algorithm [Cha00] that runs in $O(m\alpha(m, n))$ is the fastest known deterministic algorithm.

Note. Pettie and Ramachandran gave an optimal deterministic algorithm in the comparison model without known what its actual running time is [PR02]!

Perhaps an easier question is the following.

Problem 1.1.2 (MST verification). Given a graph G and a tree T, decide T is an MST of G or not.

One can always use an MST algorithm to solve the verification problem, but not necessarily the other way around. Interestingly, there is indeed a linear-time MST verification algorithm based on several non-trivial ideas and data structures and was first developed in the RAM model by Dixon, Rauch, and Tarjan [DRT92] with insights from Komlós [Kom85]. Simplification is done by King [Kin97].

Note (RAM model). The RAM model allows bit-wise operation on $O(\log n)$ bit words in $O(1)$ time.

Theorem 1.1.4 (MST verification). There is a linear-time MST verification algorithm in the RAM model. In fact, the algorithm is based on a more general result that we will need: Given a graph $G = (V, E)$ with edge costs and a spanning tree $T = (V, F)$, there is an $O(m)$-time algorithm that outputs all the F-heavy edge of G.

Proof. The original complicated algorithm has been simplified over the years. See lecture notes of Gupta and Assadi for accessible explanation, also the MST surveys [Eis97; Mar08]. ■

Fredman-Tarjan's Algorithm

Here we briefly describe Fredman and Tarjan's algorithm [FT87; Mar08] via Fibonacci heaps, which is reasonably simple to describe and analyze modulo a few implementation details that we will gloss over for the sake of brevity. First, we develop a simple $O(m \log\log n)$ time algorithm by combining Borůvka's algorithm and Jarnik-Prim's algorithm.

As previously seen. Jarnik-Prim's algorithm takes $O(m + n \log n)$ time via Fibonacci heaps where the bottleneck is when $m = o(n \log n)$. On the other hand, Borůvka's algorithm starts with a graph on n nodes and after i^{th} phases, reduces the number of nodes to $n/2^i$; each phase takes $O(m)$ times.

[2] Formally, it runs in $O(m\beta(m, n))$, where $\beta(m, n)$ is the minimum value of i such that $\log^{(i)} n \leq m/n$, where $\log^{(i)} n$ is the logarithmic function iterated i times. Since $m \leq n^2$, $\beta(m, n) \leq \log^* n$.

Intuition. Suppose we run Borůvka's algorithm for k phases and then run Jarnik-Prim's algorithm once the number of nodes is reduced. We can see that the total run time is $O(mk)$ for the k phases of Borůvka's algorithm, and $O(m + n/2^k \log n/2^k)$ for the Jarnik-Prim's algorithm on the reduced graph. Thus, if we choose $k = \log \log n$, we obtain a total run-time of $O(m \log \log n)$.

Tarjan and Fredman obtained a more sophisticated scheme based on the Jarnik-Prim's algorithm, but the basic idea is to reduce the number of vertices. The algorithm runs again in phases. We describe the first phase here.

Intuition (First phase). Start growing the tree. If the heap gets too big, we stop.

Consider an integer parameter t such that $1 < t \leq n$. Pick an arbitrary root r_1 and grow a tree T_1 via Jarnik-Prim's algorithm with a Fibonacci heap. We stop the tree growth when the heap size exceeds t for the first time or if we run out of vertices. All the vertices in the tree are marked as visited. Now pick an arbitrary, unmarked vertex as root $r_2 \in V - T$ and grow a tree T_2, and we stop growing T_2 if it touches T_1, in which case it merges with it, or if the heap size exceeds t or if we run out of vertices. The algorithm proceeds in this fashion by picking new roots and growing them until all nodes are marked.

Note. While growing T_2, the heap may contain previously marked vertices. It is only when the algorithm finds one of the marked vertices as the cheapest neighbor of the current tree that we merge the trees and stop.

It's easy to see that the first phase of Fredman-Tarjan algorithm correctly adds a set of MST edges F. After this, we simply shrink these trees and recurse on the smaller graph.

Algorithm 1.4: Fredman-Tarjan's algorithm

Data: A connected graph $G = (V, E)$ with edge weight $c\colon E \to \mathbb{R}$
Result: A MST $T = (V, F)$

```
V' ← V, F ← ∅                                              // Initialize the tree
while |V| > 1 do
    T ← Grow(G)                                            // First phase
    F ← F ∪ E(T)                                           // Update the tree
    Shrink G w.r.t. T, update V and E                      // Second phase
return (V', F)

Grow(G):
    V' ← ∅, F ← ∅, T ← (V', F)                             // Initialize the forest
    while V' ≠ V do
        r ← uniform(V − V')                                // Pick an unmarked vertex
        T' ← ({r}, ∅)                                      // Initialize a tree
        while |N(T')| < t or V(T') ∩ V' ≠ ∅ do
            Run one more step of Jarnik-Prim(r, T')        // Starting at r, maintaining T'
        V' ← V' ∪ V(T')                                    // Mark
        F ← F ∪ E(T')                                      // Update the forest by merging the tree
    return (V, F)                                          // Return a forest of G
```

Note. This can be seen as a parameterized version of Borůvka's algorithm.

The difficult part is to determine its runtime. We have the following.

Theorem 1.1.5. Fredman-Tarjan's algorithm takes $O(m\beta(m, n))$.

Proof. Firstly, the total time to scan edges and insert vertices into heaps and do `decrease-key` is $O(m)$ since an edge is only visited twice, once from each end point. Since each heap is not allowed to grow to more than size t, the total time for all the `extract-min` operations take $O(n \log t)$. With the fact that the initialization of each data structure is easy as it starts as an empty one, hence, the

first phase takes $O(m + n \log t)$. We claim that it also reduces the number of vertices to $2m/t$.

Claim. The number of connected components induced by F is $\leq 2m/t$ after the first phase.

Proof. Let $C_1, \ldots, C_h$ be the connected components of F. If for every C_i, $\sum_{v \in C_i} \deg(v) \geq t$,

$$2m = \sum_{v \in V} \deg(v) = \sum_{i=1}^{h} \sum_{v \in C_i} \deg(v) \geq ht \Rightarrow h \leq \frac{2m}{t}.$$

To see why the assumption holds, consider the growth of a tree T' in line 14:

- If we stop T' because heap size $|N(T')|$ exceeds t, then each of the vertex in the heap is a witness to a unique edge incident to T', hence the property holds.
- If T' merged with a previous tree, then the property holds because the previous tree already had the property and adding vertices can only increase the total degree of the component.

The only reason the property may not hold is if line 17 terminates a tree because all vertices are already included in it, but then that phase finishes the algorithm. ⊛

The question reduces to choosing t.

Intuition. We want linear time in the first phase, i.e., $n \log t$ to be no more than $O(m)$, leading to $t = 2^{2m/n}$. If we do this in every iteration, then this leads to $O(m)$ time per iteration.

We now bound the number of iteration. Consider $t_1 := 2^{2m/n}$ and $t_i := 2^{2m/n_i}$,[a] where n_i and m_i are the number of vertices and edges at the beginning of the i^{th} iteration, with $m_1 = m$ and $n_1 = n$. From the previous claim, $n_{i+1} \leq 2m_i/t_i$, which gives

$$t_{i+1} = 2^{2m/n_{i+1}} \geq 2^{\frac{2m}{2m_i/t_i}} \geq 2^{t_i}.$$

Thus, t_i is a power of twos with $t_1 = 2^{2m/n}$, and the Fredman-Tarjan's algorithm stops if $t_i \geq n$ since it will grow a single tree and finish. Thus, the algorithm needs at most $\beta(m, n)$ iterations, giving the total time $O(m\beta(m, n))$. ■

[a] Technically, we need to choose $t_i := 2^{\lceil 2m/n_i \rceil}$, but we will be aa bit sloppy and ignore the ceilings here.

Linear-Time Randomized Algorithm

Lemma 1.1.3 (Sampling lemma). Suppose $G = (V, E)$ is a graph. Let $E' \subseteq E$ be obtained by sampling each edge e with probability $1/2$. Let F be a minimum spanning tree in E'. Then the expected number of F-light edge in G is less than $2n$.

Lecture 2

29 Aug. 11:00

Appendix

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