MATH597
Analysis II

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\stackrel{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition.

Contents

Lecture 1: σ-algebra

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1 Measure

Example. Before we start, we first see some examples.

1. Let $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) \coloneqq \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which is the *power set* of X. We see that

$$\#X = n \implies \#\mathcal{P}(X) = 2^n$$

for $n < \infty$.

2. If $n = \infty$, say $X = \mathbb{N}$, then

$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while $\mathbb{N}$ is a countable set. We can see this as follows. Consider

$$\phi\colon \mathcal{P}(\mathbb{N}) \to [0, 1], \quad A \mapsto 0.a_1a_2a_3\ldots (\text{base } 2),$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example, A can be $A = \{2, 3, 6, \ldots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have

$$\#\mathcal{P}(\mathbb{N}) \geq \# [0,1] .$$

But since $[0,1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X. Hence, we are intriguing to define a map μ such that

$$\mu \colon \mathcal{P}(X) \to [0, \infty] .$$

Example. We first see some examples.

1. Let $X = \{0, 1, 2\}$. Then we want to define $\mu \colon \mathcal{P}(X) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$. Then we have
     - $\mu(\{0, 1\}) = 2$
     - $\mu(\{0\}) = 1$
   - $\mu(A) = \sum\limits_{i \in A} 2^i$. Then we have
     - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$
2. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$. Then we have
     - $\mu(\{2, 3, 4, 5, \ldots\}) = \infty = \mu(\{\text{even numbers}\})$
   - $\mu(A) = e^{-1} \sum\limits_{i \in A} \frac{1}{i!}$. Then we have
     - $\mu(\{0, 2, 4, 6, \ldots\}) = e^{-1} \left(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right)$
   - $\mu(A) = \sum\limits_{i \in A} a_i$
3. Let $X = \mathbb{R}$. Then we want to define $\mu \colon \mathcal{P}(\mathbb{R}) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$
   - $\mu((a, b)) = b - a$.

     Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

     Answer. No!
   - $\mu((a, b)) = e^b - e^a$.

     Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

     Answer. No!

We immediately see the problems. To extend our native measure method into $\mathbb{R}$ is hard and will cause something counter-intuitive![1] Hence, rather than define measurement on *all* subsets in the power set of X, we only focus on *some* subsets. In other words, we want to define

$$\mu \colon \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \to [0, \infty] .$$

1.1 σ-algebras

Definition 1.1 (σ-algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra on X* if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\varnothing}_{\in \mathcal{A}}$ and $\mathcal{A}$ is closed under complement.
- $\bigcap\limits_{i=1}^{\infty} A_i = \left(\bigcup\limits_{i=1}^{\infty} A_i^c \right)^c$, namely $\mathcal{A}$ is closed under countable intersections.
- $A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \varnothing \cup \varnothing \cup \ldots$, hence $\mathcal{A}$ is closed under finite unions and intersections.

Lecture 2: Measure

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Example. Again, we first see some examples.

1. Let $\mathcal{A} = \mathcal{P}(X)$, which is the power σ-algebra.
2. Let $\mathcal{A} = \{\varnothing, X\}$, which is a trivial σ-algebra.
3. Let $B \subset X$, $B \neq \varnothing$, $B \neq X$. Then we see that $\mathcal{A} = \{\varnothing, B, B^c, X\}$ is a σ-algebra.

Lemma 1.1. Let $\mathcal{A}_\alpha$, $\alpha \in I$, be a family of σ-algebra on X. Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is a σ-algebra on X.

Remark. Notice that I may be an uncountable intersection.

Proof. A simple proof can be made as follows. Firstly, $\varnothing \in \mathcal{A}_\alpha$ for every α clearly. Moreover, closure under complement and countable unions for every

[1] https://en.wikipedia.org/wiki/Banach-Tarski_paradox

$\mathcal{A}_\alpha$ implies the same must be true for $\bigcap\limits_{\alpha\in I} \mathcal{A}_\alpha$. Therefore, $\bigcap\limits_{\alpha\in I} \mathcal{A}_\alpha$ is a σ-algebra. ■

The above allows us to give the following definition.

Definition 1.2 (Generation of σ-algebra). Given $\mathcal{E} \subset \mathcal{P}(X)$, where $\mathcal{E}$ is not necessarily a σ-algebra. Let $\langle\mathcal{E}\rangle$ be the intersection of all σ-algebras on X containing $\mathcal{E}$, then we call $\langle\mathcal{E}\rangle$ the *σ-algebra generated by $\mathcal{E}$.*

Remark. Clearly, $\langle\mathcal{E}\rangle$ is the smallest σ-algebra containing $\mathcal{E}$, and it is unique. To check the uniqueness, we suppose there are two different $\langle\mathcal{E}\rangle_1$ and $\langle\mathcal{E}\rangle_2$ generated from $\mathcal{E}$. It's easy to show

$$\langle\mathcal{E}\rangle_1 \subseteq \langle\mathcal{E}\rangle_2\,,$$

and by symmetry, they are equal.

Example. We see that $\{\varnothing, B, B^c, X\} = \langle\{B\}\rangle = \langle\{B^c\}\rangle$.

Lemma 1.2. We have

1. Given $\mathcal{A}$ a σ-algebra, $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \implies \langle\mathcal{E}\rangle \subset \mathcal{A}$
2. $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \implies \langle\mathcal{E}\rangle \subset \langle\mathcal{F}\rangle$

Proof. We'll see that after proving the first claim, the second follows smoothly.

1. The first claim is trivial, since we know that $\langle\mathcal{E}\rangle$ is the smallest σ-algebra containing $\mathcal{E}$, then if $\mathcal{E} \subset \mathcal{A}$, we clearly have $\langle\mathcal{E}\rangle \subset \mathcal{A}$ by the definition.
2. The second claim is also easy. From the first claim and the definition, we have
$$\mathcal{E} \subset \mathcal{F} \subset \langle\mathcal{F}\rangle \implies \langle\mathcal{E}\rangle \subset \langle\mathcal{F}\rangle\,.$$

■

At this point, we haven't put any specific structure on X. Now we try to describe those spaces with good structure, which will give the space some nice properties.

Definition 1.3 (Borel σ-algebra). For a topological space X, the *Borel σ-algebra on X*, denotes as $\mathcal{B}(X)$, is the σ-algebra generated by the collection of all open sets in X.

Example. We see that $\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a,b) \mid a < b; a, b \in \mathbb{R}\}$.
- $\mathcal{E}_2 = \{[a,b] \mid a < b; a, b \in \mathbb{R}\}$ since $[a,b] = \bigcap\limits_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$.

- $\mathcal{E}_3 = ((a,b] \mid a < b; a,b \in \mathbb{R})$ since $(a,b] = \bigcap_{n=1}^{\infty}(a, b+\frac{1}{n})$.
- $\mathcal{E}_4 = ([a,b) \mid a < b; a,b \in \mathbb{R})$ since $[a,b) = \bigcap_{n=1}^{\infty}(a-\frac{1}{n}, b)$.
- $\mathcal{E}_5 = ((a,\infty) \mid a \in \mathbb{R})$ since $(a,\infty) = \bigcup_{n=1}^{\infty}(a, a+n)$.
- $\mathcal{E}_6 = ([a,\infty) \mid a \in \mathbb{R})$ since $[a,\infty) = \bigcup_{n=1}^{\infty}[a, a+n)$.
- $\mathcal{E}_7 = ((-\infty,b) \mid b \in \mathbb{R})$ since $(-\infty,b) = \bigcup_{n=1}^{\infty}(b-n, b)$.
- $\mathcal{E}_8 = ((-\infty,b] \mid a \in \mathbb{R})$ since $(-\infty,b] = \bigcup_{n=1}^{\infty}(b-n, b]$.

Proposition 1.1. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \dots, 8$.

Proof. Firstly, we see that $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \implies \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$ by Lemma 1.2. Secondly, by definition, $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$ where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R}\}.$$

It's enough to show $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$ since if so, $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$, and clearly $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$, then we will have $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$. Let $O \subset \mathbb{R}$ be an open set, i.e., $O \in \mathcal{E}$. We claim that every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.[2]

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

where I_j open interval with the form of $(a,b), (-\infty,b), (a,\infty), (-\infty,\infty)$.

For example, $\mathcal{E}_1$ is trivially true, and

$$(a,b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{[a+\frac{1}{n}, b-\frac{1}{n}]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for $\mathcal{E}_2$ and

$$(a,\infty) = \bigcup_{k=1}^{\infty}(a, a+k)$$

shows the case for $\mathcal{E}_5$. It's now straightforward to check open intervals are in $\langle \mathcal{E}_i \rangle$ for every i. ■

Now, to put a structure on a space, we define the following.

[2] https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals

Definition 1.4 (Measurable space). $(X, \mathcal{A})$ is called a *measurable space*, and $E \in \mathcal{A}$ is called a $\mathcal{A}-$*measurable set*.

1.2 Measures

With the definition of measurable space, we now can refine our measure function μ as follows.

Definition 1.5 (Measure). Given a measurable space on $(X, \mathcal{A})$, a *measure* is a function μ such that

$$\mu \colon \mathcal{A} \to [0, \infty]$$

with

1. $\mu(\varnothing) = 0$
2. $\mu\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty} \mu(A_i)$ if $A_1, A_2, \ldots \in \mathcal{A}$ are **disjoint**. We call this *Countable additivity.*

We denote $(X, \mathcal{A}, \mu)$ a *measure space.*

Notation. We denote $[0, \infty] \coloneqq [0, \infty) \cup \{\infty\}$.

Remark. The motivation of why we only want *countable additivity* but not uncountable additivity can be seen by the following example. We'll consider the most intuitive measure on $\mathbb{R}, \mathcal{B}(\mathbb{R})$.

Since we have

$$(0, 1] = (\frac{1}{2}, 1] \cup (\frac{1}{4}, \frac{1}{2}] \cup (\frac{1}{8}, \frac{1}{4}] \cup \ldots$$

and also

$$(0, 1] = \bigcup_{x \in (0,1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2}, 1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4}, \frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8}, \frac{1}{4}])} + \ldots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0,1]} 0$$

since $\mu(x) = 0$ for $x \in \mathbb{R}$, which is clearly not what we want.

Example. We see some examples.

1. For any $(X, \mathcal{A})$, we let $\mu(A) \coloneqq \#A$. This is called *counting measure.*

2. Let $x_0 \in X$. For any $(X, \mathcal{A})$, the *Dirac measure at x_0* is

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A. \end{cases}$$

3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$,

$$\mu(A) = \sum_{i \in A} a_i,$$

where $a_1, a_2, \ldots \in [0, \infty)$.

Lecture 3: Construct a Measure

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Note. If $A, B \in \mathcal{A}$ and $A \subset B$, then

$$\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

Theorem 1.1. Given $(X, \mathcal{A}, \mu)$ be a measure space.

1. (monotonicity) $A, B \in \mathcal{A},\ A \subset B \implies \mu(A) \leq \mu(B)$.
2. (countable subadditivity) $A_1, A_2, \ldots \in \mathcal{A} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$
3. (continuity from below/ monotone convergence theorem (MCT) for sets)

$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \ldots \end{cases} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

4. (continuity from above)

$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \ldots \\ \mu(A_1) < \infty \end{cases} \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

Proof. We prove this theorem one by one.

1. Since $A \subset B$, hence we have

$$\mu(B) = \mu\Big(\underbrace{(B \setminus A) \cup A}_{\text{disjoint}}\Big) \overset{!}{=} \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A).$$

2. This should be trivial from countable additivity with the fact that $\mu(A) \geq 0$ for all A. DIY!
3. Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$, then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

is a disjoint union and $B_i \in \mathcal{A}$, hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i).$$

With $\mu\left(\bigcup_{i=1}^{n} B_i\right) = \mu(A_n)$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu(A_n).$$

4. Let $E_i = A_1 \setminus A_i \implies E_i \in \mathcal{A}$, $E_1 \subset E_2 \subset \ldots$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right),$$

which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i\right) \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$$

since $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu(A_1) < \infty$. Then from continuity from below, we further have

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right).$$

From montonicity, we see that $\mu(A_n) \leq \mu(A_1) < \infty$, hence we can split the limit and further get

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu(A_1) + \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(A_n).$$

∎

Example. Given $(\mathbb{N}, \mathcal{P}(\mathbb{N})$, counting measure). Then we see

- $A_n = \{n, n+1, n+2, \ldots\} \implies \mu(A_n) = \infty$
- $A_1 \supset A_2 \supset A_3 \supset \ldots$
- $\bigcap_{i=1}^{\infty} A_i = \varnothing \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = 0$

Remark. We see that in this case, since $\mu(A_1) \not< \infty$, hence continuity from above doesn't hold.

We now try to characterize some properties of a measure space.

Definition 1.6. Given $(X, \mathcal{A}, \mu)$

- $A \subset X$ is a *μ-null set* if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- $A \subset X$ is a *μ-subnull set* if $\exists \mu$-null set B such that $A \subset B$. Note that A is not necessarily $\mathcal{A}$-measurable.
- $(X, \mathcal{A}, \mu)$ is a *complete* measure space if every μ-subnull set is $\mathcal{A}$-measurable.

There are some useful terminologies we'll use later relating to μ-null.

Definition 1.7 (Almost everywhere). Given $(X, \mathcal{A}, \mu)$, a statement $P(x)$, $x \in X$ holds *μ-almost everywhere (a.e.)* if the set

$$\{x \in X : P(x) \text{ does not hold}\}$$

is μ-null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

Definition 1.8 (finite measure). Given $(X, \mathcal{A}, \mu)$

- μ is a *finite measure* if $\mu(X) < \infty$.
- μ is a *σ-finite measure* if $X = \bigcup\limits_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.

Exercise. Every measure space can be **completed**. Namely, we can always find a bigger σ-algebra to complete the space.

1.3 Outer Measures

We start by giving a definition.

Definition 1.9 (Outer measure). An *outer measure* on X is a map

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that

- $\mu^*(\varnothing) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$
- (countable subadditivity) $\mu^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) \leq \sum\limits_{i=1}^{\infty} \mu^*(A_i)$ for every $A_i \subset X$.

Example. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \colon \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

is an outer measure due to the Proposition 1.2 we're going to show.

Remark. We see that an outer measure need not be a measure. Check the Definition 1.5 for a measure function.

Proposition 1.2. Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\varnothing, X \in \mathcal{E}$. Let

$$\rho \colon \mathcal{E} \to [0, \infty]$$

such that $\rho(\varnothing) = 0$. Then

$$\mu^*(A) \coloneqq \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E},\ \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

is an outer measure on X.

Note. Recall the Tonelli's Theorem[3] for series:

If $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Lecture 4: Carathéodory extension Theorem

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As previously seen. We now prove the Proposition 1.2.

Proof. We need to prove

- μ^* is well-defined. i.e., inf is taken over a non-empty set. This is trivial since $X \in \mathcal{E}$ and $X \supset A$ for any $A \in \mathcal{E}$.
- $\mu^*(\varnothing) = 0$. Since $\varnothing \in \mathcal{E}$ and

$$\mu^*(\varnothing) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \varnothing \right\} = 0$$

  since $\rho(\varnothing) = 0$ for all i and further, by Squeeze Theorem[4], we see that $\lim_{n \to \infty} \sum_{i=1}^{n} \rho(\varnothing) = 0$.

[3] https://en.wikipedia.org/wiki/Fubini%27s_theorem
[4] https://en.wikipedia.org/wiki/Squeeze_theorem

- $A \subset B \implies \mu^*(A) \leq \mu^*(B)$. We simply show this by contradiction. Suppose $A \subset B$ and $\mu^*(A) > \mu^*(B)$, then by definition of μ^*, we have

$$\begin{aligned}\mu^*(A) &= \inf\left\{\sum_{i=1}^{\infty}\rho(E_i)\colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A\right\}\\ &> \inf\left\{\sum_{i=1}^{\infty}\rho(E_i)\colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B\right\} = \mu^*(B).\end{aligned}$$

Now, let $B =: (B \setminus A) \cup A$, then we have

$$\begin{aligned}\mu^*(A) &= \inf\left\{\sum_{i=1}^{\infty}\rho(E_i)\colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A\right\}\\ &> \inf\left\{\sum_{i=1}^{\infty}\rho(E_i)\colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B\setminus A)\cup A\right\} = \mu^*(B).\end{aligned}$$

Now, since $B\setminus A \supseteq \varnothing$, then this inequality can't hold, hence a contradiction ↯ .

- Countable subaddivity. Let $A_1, A_2, \ldots \in X$. If one of $\mu^*(A_n) = \infty$, then result holds. So we may assume $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Now, fix any $\epsilon > 0$, we will show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\mu^*(A_n) + \epsilon.$$

For each $n \in \mathbb{N}$, $\exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ such that

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n$$

and

$$\mu^*(A_n) + \frac{\epsilon}{2^n}{}^{5} \geq \sum_{k=1}^{\infty}\rho(E_{n,k}).$$

Then we see that

$$\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{k,n},$$

which implies

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{(n,k)\in\mathbb{N}^2}\rho\left(E_{k,n}\right) \overset{!}{=} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\rho(E_{k,n}) \leq \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right)$$

from the inequality just derived. Now, since the last term is just

$$\sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) = \sum_{n=1}^{\infty}\mu^*(A_n) + \epsilon,$$

[5] This is an important trick!!

hence we finally have

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon$$

for arbitrarily small fixed $\epsilon > 0$, hence the subadditivity is proved.

■

Definition 1.10 (Carathéodory measurable). Let μ^* be an outer measure on X. We say $A \subset X$ is *Carathéodory measurable (C-measurable) with respect to* μ^* if

$$\forall E \subset X,\ \mu^*(E) = \mu^*\left(E \cap A\right) + \mu^*\left(E \setminus A\right).$$

Lemma 1.3. Let μ^* be an outer measure on X. Suppose $B_1, \ldots, B_N$ are disjoint C-measurable sets. Then,

$$\forall E \subset X,\ \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) = \sum_{i=1}^{N} \mu^*\left(E \cap B_i\right).$$

Proof. Since we have

$$\begin{aligned}
\mu^*\left(E \cap \left(\bigcup_{i=1} B_i\right)\right) &= \mu^*\left(E' \cap B_1\right) + \mu^*\left(E' \setminus B_1\right)^{6} \\
&= \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i \cap B_1\right)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right) \cap B_1^c\right) \\
&= \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_{i=2}^{N} B_i\right)\right)
\end{aligned}$$

where the equality comes from the fact that B_1 is C-measurable and disjoint from B_i, $i \neq 1$. Then, we simply iterate this argument and have the result. ■

Remark. This implies that if we restrict an outer measure on C-measurable set, then it becomes finite additive.

Theorem 1.2 (Carathéodory extension Theorem). Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets (with respect to μ^*). Then,

1. $\mathcal{A}$ is a σ-algebra on X.
2. $\mu = \mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.
3. $(X, \mathcal{A}, \mu)$ is a complete measure space.

[6] Here, $E' := E \cap \left(\bigcup_{i=1}^{N} B_i\right)$ for the simplicity of notation.

Proof. We divide the proof in several steps.

1. We show $\mathcal{A}$ is a σ-algebra by showing

   (a) $\varnothing \in \mathcal{A}$. To show this, we simply check that $\varnothing$ is C-measurable. We see that

   $$\underset{E\subset X}{\forall}\ \mu^*(E) = \mu^*(E \cap \varnothing) + \mu^*(E \setminus \varnothing) = \mu^*(E),$$

   which just shows $\varnothing \in \mathcal{A}$.

   (b) $\mathcal{A}$ closed under complements. This is equivalent to say that if A is C-measurable, so is A^c. We see that if A is C-measurable, then for every $E \subset X$,

   $$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

   Observing that $E \cap A = E \setminus A^c$ and $E \setminus A = E \cap A^c$, hence

   $$\mu^*(E) = \mu^*(E \setminus A^c) + \mu^*(E \cap A^c).$$

   We immediately see that above implies $A^c \in \mathcal{A}$.

   (c) $\mathcal{A}$ closed under countable unions.

   Note. To show $\mathcal{A}$ closed under countable unions, we show that $\mathcal{A}$ is closed under:

   finite unions $\overset{\text{then}}{\Longrightarrow}$ countable $\underline{\text{disjoint}}$ unions $\overset{\text{then}}{\Longrightarrow}$ countable unions.

   - We show $\mathcal{A}$ is closed under finite unions.

     Claim. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

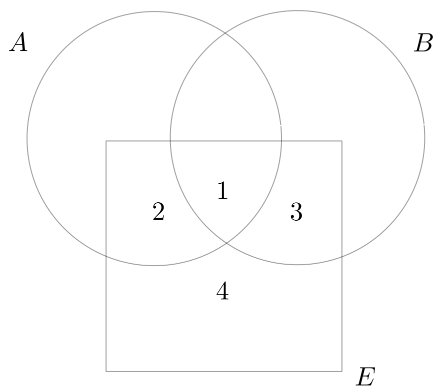
     Fix $E \subset X$ arbitrary. We need to show that

     $$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)),$$

     i.e.,

     $$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

     given $A, B \in \mathcal{A}$.



- Since A is C-measurable,
  - $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$
  - $\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$
- Since B is C-measurable,
  - $\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$

Hence, we have

$$\begin{aligned}\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4)\\ &= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4)\\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).\end{aligned}$$

- We show $\mathcal{A}$ is closed under countable disjoint unions.

  Let $A_1, A_2, \ldots \in \mathcal{A}$ and disjoint. Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^{\infty} A_i\right),$$

  hence we only need to show another way around.

  Fix $N \in \mathbb{N}$, we have $\bigcup\limits_{n=1}^{N} A_n \in \mathcal{A}$ since N is finite, and

$$\begin{aligned}\mu^*(E) &= \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} A_n\right)\right) + \mu^*\left(E \setminus \left(\bigcup_{n=1}^{N} A_n\right)\right)\\ &\geq \underbrace{\sum_{n=1}^{N} \mu^*(E \cap A_n)}_{\overset{!}{=}\mu^*\left(E \cap \left(\bigcup_{n=1}^{N} A_n\right)\right)} + \underbrace{\mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right)}_{\leq \mu^*\left(E \setminus \left(\bigcup_{n=1}^{N} A_n\right)\right)}.\end{aligned}$$

  Now, take $N \to \infty$ then we are done.

- We show $\mathcal{A}$ is closed under countable unions. DIY

The proof will be continued...

Lecture 5: Hahn-Kolmogorov Theorem

14 Jan. 11:00

Firstly, we see a stronger version of Lemma 1.3 we have seen before.

Lemma 1.4. Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots$ are disjoint C-measurable sets. Then,

$$\forall E \subset X,\ \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i).$$

Proof.

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_i) \geq \mu^*\left(E \cap \bigcup_{n=1}^{\infty} B_n\right) \geq \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} B_n\right)\right) \overset{!}{=} \sum_{n=1}^{N} \mu^*\left(E \cap B_n\right).$$

Now, we just take $N \to \infty$ (or note that $N \in \mathbb{N}$ is arbitrary, we then get the result according to Squeeze Theorem[7]). ■

Let's continue the proof of Theorem 1.2.

2. Since from Definition 1.5, we need to show
   - $\mu(\varnothing) = 0$. This means that we need to show $\mu^*|_{\mathcal{A}}(\varnothing) = 0$. Since $\varnothing \in \mathcal{A}$ and μ^* is an outer measure, hence from the property of outer measure, it clearly holds.
   - Countable additivity of μ^* on $\mathcal{A}$ follows from the Lemma 1.4 with $E = X$
3. Hw.

■

1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets $\mathcal{E}$ and any ρ such that it assign measure on $\mathcal{E}$, then induces an outer measure by Proposition 1.2, finally complete the outer measure by Theorem 1.2.

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.2}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to σ-algebra we are working on till now.

Definition 1.11 (Algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on* X if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an algebra and a σ-algebra is whether they closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an outer measure.

[7] https://en.wikipedia.org/wiki/Squeeze_theorem

Definition 1.12 (Pre-measure). Let $\mathcal{A}_0$ be an algebra on X. We say

$$\mu_0 \colon \mathcal{A}_0 \to [0, \infty]$$

is a *pre-measure* if

1. $\mu_0(\varnothing) = 0$
2. (finite additivity) $\mu_0\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu_0(A_i)$ if $A_1, \dots, A_n \in \mathcal{A}_0$ are $\underline{\text{disjoint}}$.
3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, $\underline{\text{disjoint}}$, then
$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 1.5. (1) + (2) $\implies$ (3) in Theorem 1.2.

Proof. ■ DIY

Theorem 1.3 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in Proposition 1.2. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^*, then $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$. i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts.

- We first show $\mathcal{A} \supset \mathcal{A}_0$. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is C-measurable, i.e.,
$$\forall E \subset X \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$
We first fix an $E \subset X$. From countable subadditivity of μ^*, we have
$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$
Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

  Fix $\epsilon > 0$. By the Proposition 1.2 of μ^*, $\exists B_1, B_2, \ldots \in \mathcal{A}_0$, $\bigcup_{n=1}^{\infty} B_n \supset E$ such that
$$\mu^*(E) + \epsilon \geq \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left(\mu_0(\underbrace{B_n \cap A}_{\in \mathcal{A}_0}) + \mu_0(\underbrace{B_n \cap A^c}_{\in \mathcal{A}_0}) \right)$$

by the finite additivity of μ_0.

Note that

$$\begin{cases} \bigcup_{n=1}^{\infty}(B_n \cap A) & \supset E\cap A \\ \bigcup_{n=1}^{\infty}(B_n \cap A^c) & \subset E\cap A^c \end{cases} \implies \mu^*(E)+\epsilon \geq \mu^*(E\cap A)+\mu^*(E\cap A^c)$$

since

$$\mu^*(E\cap A) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n\cap A)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n\cap A)$$

and

$$\mu^*(E\cap A^c) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n\cap A^c)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n\cap A^c).$$

We then see that for any $\epsilon > 0$, the inequality

$$\mu^*(E)+\epsilon \geq \mu^*(E\cap A)+\mu^*(E\cap A^c)$$

holds, hence so does

$$\mu^*(E) \geq \mu^*(E\cap A)+\mu^*(E\cap A^c),$$

which implies $\mathcal{A} \supset \mathcal{A}_0$.

The proof will be continued...

Lecture 6

Jan. 11:00

Let's continue the proof of Theorem 1.3.

- Let $A \in \mathcal{A}_0$, we want to show that

$$\mu(A) = \mu_0(A).$$

  - Firstly, let

$$B_i = \begin{cases} A, & \text{if } i = 1 \\ \varnothing, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0$$

    and $\bigcup_{i=1}^{\infty} B_i \supset A$, then we see that

$$\mu^*(A) \leq \sum_{i=1}^{\infty}\mu_0(B_i) = \mu_0(A).$$

  - Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A\cap B_1 \in \mathcal{A}_0$, $C_i = A\cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j\right) \in \mathcal{A}_0$ since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

are disjoint countable unions, by countable additivity within the algebra, we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \implies \mu_0(A) \leq \sum_{i=1}^{\infty} \mu(B_i) \implies \mu_0(A) \leq \mu^*(A).$$

■

Definition 1.13 (HK extension). $(\mathcal{A}, \mu)$ is the *Hahn-Kolmogorov extensions* of $(\mathcal{A}_0, \mu_0)$.

Theorem 1.4 (uniqueness of HK extension). Let $\mathcal{A}$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$. Let $\mathcal{A}, \mu$ be the HK extension of $(\mathcal{A}_0, \mu_0)$. Let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$. Then if μ_0 is σ-finite, $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Note. Notice that $\mathcal{A} \subset \mathcal{A} \cup \mathcal{A}'$ since they both extend $\mathcal{A}$.

Proof. Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing arbitrary cover of A and using the definition of μ^*.

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) < \infty$. Fix $\epsilon > 0$. Then

  $$\exists B_i \in \mathcal{A}_0, \forall_{i \in \mathbb{N}} \bigcup_{i=1}^{\infty} B_i \supset A,$$

  and

  $$\mu(A) + \epsilon = \mu^*(A) + \epsilon \geq \sum_{i=1}^{\infty} \mu_0(B_i) = \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) =: \mu(B).$$

  This implies that

  $$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$

  where the first equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

  $$\mu(B) = \lim_{N \to \infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) =^{8} \lim_{N \to \infty} \mu'\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu'(B),$$

  hence

  $$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq^{9} \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon.$$

[8] $\mu = \mu'$ on $\mathcal{A}_0$.

So, $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is σ-finite, hence

$$X = \bigcup_{n=1}^{\infty} X_n, \quad X_n \in \mathcal{A}_0, \quad \mu_0(X_n) < \infty.$$

Replacing X_n by $X_1 \cup \ldots \cup X_n \in \mathcal{A}_0$, we may assume that

$$X_1 \cup X_2 \cup \ldots.$$

Then,

$$\underset{n\in\mathbb{N}}{\forall} \mu(A \cap X_n) < \infty \implies \mu(A \cap X_n) \leq \mu'(A \cap X_n) \implies \mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) \leq \lim_{n\to\infty} \mu'(A \cap X_n) = \mu'(A$$

■

Corollary 1.1. Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then

$$\exists! \text{ measure } \mu \text{ on } \langle \mathcal{A}_0 \rangle \text{ that extends } \mathcal{A}_0.$$

Furthermore,

- The completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.
- $$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \underset{i\in\mathbb{N}}{\forall} \bigcup_{i=1}^{\infty} B_i \supset A \right\}$$

  for all $A \in \langle \bar{\mathcal{A}}_0 \rangle$.

[9]From the first part.