

STAT575
Large Sample Theory

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Abstract

This is a graduate-level theoretical statistics course taught by [Georgios Fellouris](#) at University of Illinois Urbana-Champaign, aiming to provide an introduction to asymptotic analysis of various statistical methods, including weak convergence, Lindeberg-Feller CLT, asymptotic relative efficiency, etc.

We list some references of this course, although we will not follow any particular book page by page: *Asymptotic Statistics* [[Vaa98](#)], *Asymptotic Theory of Statistics and Probability* [[Das08](#)], *A course in Large Sample Theory* [[Fer17](#)], *Approximation Theorems of Mathematical Statistics* [[Ser09](#)], and *Elements of Large-Sample Theory* [[Leh04](#)].



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Chapter 1

Introduction

Lecture 1: Introduction to Large Sample Theory

Say we first collect n data points $x_1, \dots, x_n \in \mathbb{R}^d$, where we may treat x_i as a realization of a random vector X_i on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this course, we will primarily consider the case that X_i 's are i.i.d., i.e., independent and identically distributed from a distribution function, or the *cumulative density function* (cdf) F such that

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$$X = (X^1, \dots, X^d) \sim F(x_1, \dots, x_d) \equiv \mathbb{P}(X^1 \leq x_1, \dots, X^d \leq x_d)$$

for all $x_i \in \mathbb{R}$. If we have access to F , we can compute the corresponding *probability density function* (pdf) f , and then have access to $\mathbb{P}(X \in A)$ for all (measurable) $A \subseteq \mathbb{R}^d$ of interest.

Notation. In the measure-theoretic sense, the measure \mathbb{P} in $(\Omega, \mathcal{F}, \mathbb{P})$ is the **Lebesgue-Stieltjes measure** μ_F induced by the distribution function F . When doing integration, we will often denote

$$d\mu_F(x) = d\mathbb{P}(x) =: F(dx) =: dF(x) =: f(x)dx$$

Remark. If we know any of the above, we know every thing about the population.

Hence, the goal is to compute this by collecting data x_i 's, which is a statistical inference problem. Notably, *large sample theory* concerns with the limiting theory as $n \rightarrow \infty$.

1.1 Parametrized Approach

There are various ways of doing this task, one way is the so-called parametrized approach. By postulating a family of cdfs $\{F_\theta, \theta \in \Theta\}$ where Θ is often a subset of \mathbb{R}^m for some m (generally $\neq n$), the goal is to select a member of this family that is the “closest”, or the “best fit” to the truth, i.e., F , based on the data.

Note. To emphasize that this depends on the data, we sometimes write the function we found as $\hat{\theta}_n(x_1, \dots, x_n)$ so that $F_{\hat{\theta}_n(x_1, \dots, x_n)}$ is our proxy for F .

Now, assume that the family is initially given, the problem is then how to select $\hat{\theta}_n$.

Example. Fisher suggested that we should look at the maximum likelihood estimator (MLE).

The justification for MLE is not about finite n , but about its asymptotic behavior when $n \rightarrow \infty$. Specifically, we have the following theorem due to Fisher (informally stated).

Theorem 1.1.1 (Fisher). If $F \in \{F_\theta: \theta \in \Theta\}$, i.e., if $F = F_{\theta^*}$ for some $\theta^* \in \Theta$, then under certain conditions, $\hat{\theta}_n$ will be “close” to θ^* as $n \rightarrow \infty$. Under some other conditions, $\sqrt{n}(\hat{\theta}_n - \theta)$ is approximately Gaussian with variance being the “best possible” in some sense.

On the other hand, in the misspecified case, i.e., $F \notin \{F_\theta, \theta \in \Theta\}$, we can still compute the MLE, which leads to another justification for MLE since even in this case, $\hat{\theta}_n$ will still be “close” to θ^* such that F_{θ^*} is, in some sense, the “closest” to F among all possible F_θ (minimizing divergence, to be precise).

1.2 Hypothesis Testing

We will also develop theory for hypothesis testing for some hypothesis we’re interested in, e.g., whether the data we collect is really i.i.d., or whether our proposed family is reasonable enough. Say now X_i ’s are scalar random variable with $\mathbb{E}[X] = \mu$, and we want to test the null hypothesis $H_0: \mu = 0$.

Example. Consider a controlled group Z and a treatment group Y , and we observe Z_1, \dots, Z_n , and Y_1, \dots, Y_n , respectively, and compute $X_i = Z_i - Y_i$ for all i . Testing H_0 on the distribution of X will show the effect of the treatment.

To do this, a well-known, elementary, and fundamental method is the so-called t -test.

Definition 1.2.1 (t -statistic). Given a sample X_1, \dots, X_n , the t -statistic is defined as

$$T_n = \frac{\bar{X}_n}{s_n/\sqrt{n}},$$

where s_n is the sample standard derivation.

Note. As long as X is Gaussian, $T_n \sim t_{n-1}$, i.e., the t -distribution with $n - 1$ degrees of freedom.

Hence, one can reject H_0 when T_n is too large (or small) when $X \sim \mathcal{N}$ as we know the exact distribution T_n will follow. What if X is not an Gaussian? We will show that even if X is not Gaussian, this result is “approximately valid” when n is “large enough” as long as $\text{Var}[X] < \infty$.

Remark (Sample Size). When we say n is “large enough”, what we mean really depends on how fast the underlying distribution will approach Gaussian as n grows. Hence, if we can say more about the underlying population, we can say more about when does n is “large enough”; otherwise such a limiting theory might be completely useless in practice.

What if $\text{Var}[X]$ doesn’t exit for some heavy tailed distribution like the Cauchy?

Example (Cauchy distribution). Cauchy distribution doesn’t have finite moment of order greater than 1.

In this case, other tests are needed. A simple test would be looking at the sign of X_i .

Example (Sign test). We might reject H_0 if $\sum_{i=1}^n \mathbb{1}_{X_i > 0}$ is large. Note that under H_0 , $\sum_{i=1}^n \mathbb{1}_{X_i > 0} \sim \text{Bin}(n, 1/2)$, and this test is valid even if expectation doesn’t exist.

We see that without saying anything about F , the sign test is valid even for $n = 3$ or 5 as the sum is exactly binomial distribution under H_0 . Although simple and have good property, only looking at the sign of X_i might be too weak. A natural idea is to look at the absolute value of X_i .

Example (Wilcoxon’s rank-sum test). Let $R_{i,n}$ to be the rank of $|X_i|$, then consider the so-called *Wilcoxon’s rank-sum test*

$$\sum_{i=1}^n \mathbb{1}_{X_i > 0} R_{i,n}.$$

As one can imagine, the closed form of the above sum will be complicated; however, asymptotically, the above statics will follow Gaussian again, such that the rate of convergence doesn’t depend on the underlying population.

Finally, we also ask how can we compare these different tests? This will also be addressed in this course.

Chapter 2

Modes of Convergence

Lecture 2: Modes of Convergence

2.1 Different Modes of Convergence

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Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider a sequence of d -dimensional random vectors (X_n) and a random vector X , i.e., $X_n, X: \Omega \rightarrow \mathbb{R}^d$. We now discuss different modes of convergence for (X_n) .

Definition 2.1.1 (Point-wise converge). (X_n) *point-wise converges* to X , denoted as $X_n \rightarrow X$, if $X_n(\omega) \rightarrow X(\omega)$ for all $\omega \in \Omega$.^a

^aI.e., for every $\epsilon > 0$, there exists $n_0(\omega) \in \mathbb{N}$ such that for every $n \geq n_0$, $\|X_n(\omega) - X(\omega)\|_2 < \epsilon$.

Since we don't care about measure zero sets, we may instead consider the following.

Definition 2.1.2 (Converge almost-surely). (X_n) *converges almost-surely* to X , denoted as $X_n \xrightarrow{\text{a.s.}} X$, if $\mathbb{P}(X_n \rightarrow X) = 1$.^a

^aI.e., $X_n(\omega) \rightarrow X(\omega)$ for all $\omega \in \Omega \setminus N$ where $\mathbb{P}(N) = 0$.

However, this might still be too strong.

Definition 2.1.3 (Converge in probability). (X_n) *converges in probability* to X , denoted as $X_n \xrightarrow{p} X$, if for every $\epsilon > 0$, $\mathbb{P}(\|X_n - X\| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$.

Remark. $X_n \rightarrow X$ if and only if $\|X_n - X\| \rightarrow 0$. The same also holds for \xrightarrow{p} and $\xrightarrow{\text{a.s.}}$.

A related notion is the following, where we now sum over n .

Definition 2.1.4 (Converge completely). (X_n) *converges completely* to X , denoted as $X_n \xrightarrow{\text{comp}} X$, if for every $\epsilon > 0$, $\sum_{n=1}^{\infty} \mathbb{P}(\|X_n - X\| > \epsilon) < \infty$.

Finally, we have the following.

Definition 2.1.5 (Converge in L^p). (X_n) *converges in L^p* to X for some $p > 0$, denoted as $X_n \xrightarrow{L^p} X$, if $\mathbb{E}[\|X_n - X\|^p] \rightarrow 0$ as $n \rightarrow \infty$.

2.1.1 Connection Between Modes of Convergence

We have the following connections between different modes of convergence.

$$\text{completely} \implies \text{almost-surely} \implies \text{in probability} \longleftarrow \text{in } L^p$$

To show the above, the following characterization for [almost-surely convergence](#) is useful.

Proposition 2.1.1. For a sequence of random vectors (X_n) and a random vector X , we have

$$\begin{aligned} X_n \xrightarrow{\text{a.s.}} X &\Leftrightarrow \mathbb{P}(\|X_k - X\| > \epsilon \text{ for some } k \geq n) \xrightarrow{n \rightarrow \infty} 0 \\ &\Leftrightarrow \mathbb{P}(\|X_n - X\| > \epsilon \text{ for infinitely many } n\text{'s}) = 0 \\ &\Leftrightarrow \mathbb{P}(\limsup_{n \rightarrow \infty} \|X_n - X\| > \epsilon) = 0, \end{aligned}$$

where the above holds for every $\epsilon > 0$.

From [Proposition 2.1.1](#), it's clear that $\xrightarrow{\text{a.s.}}$ implies \xrightarrow{P} since

$$\mathbb{P}(\|X_k - X\| > \epsilon \text{ for some } k \geq n) \geq \mathbb{P}(\|X_n - X\| > \epsilon),$$

hence if the former goes to 0, so does the latter. On the other hand, $\xrightarrow{\text{comp}}$ implies $\xrightarrow{\text{a.s.}}$ follows from the third equivalence. Lastly, the [convergence in \$L^p\$](#) implies the [convergence in probability](#) since

$$\mathbb{P}(\|X_n - X\| > \epsilon) \leq \frac{1}{\epsilon^p} \mathbb{E}[\|X_n - X\|^p]$$

from Markov's inequality. However, the converse is not always true.

Theorem 2.1.1 (Dominated convergence theorem). If $X_n \xrightarrow{P} X$ and $\|X_n - X\| \leq Z$ for all $n \geq 1$ where $\mathbb{E}[\|Z\|^p] < \infty$, then $X_n \xrightarrow{L^p} X$.

Theorem 2.1.2 (Scheffé's theorem). Let $X_n \xrightarrow{P} X$. If $\mathbb{E}[\|X_n\|^p] \rightarrow \mathbb{E}[\|X\|^p] < \infty$, then $X_n \xrightarrow{L^p} X$. In particular, with [Fatou's lemma](#), $\limsup_{n \rightarrow \infty} \mathbb{E}[\|X_n\|^p] \leq \mathbb{E}[\|X\|^p] < \infty$ suffices.

2.1.2 Consistent Estimator

Let $(X_n) \stackrel{\text{i.i.d.}}{\sim} F$ where F is a distribution function. Say we're interested in some aspect of F , for example, some parameter $\theta = T(F) \in \mathbb{R}^m$. By collecting data X_1, \dots, X_n , we estimate θ by computing an estimator $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$ of θ . There are some properties we might want for $\hat{\theta}_n$.

Definition 2.1.6 (Consistent). $\hat{\theta}_n$ is *consistent* of θ if $\hat{\theta}_n \xrightarrow{P} \theta$.

Definition 2.1.7 (Strongly consistent). $\hat{\theta}_n$ is *strongly consistent* of θ if $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta$.

Definition 2.1.8 (Converge in mean squared error). $\hat{\theta}_n$ converges to θ in mean squared error if $\hat{\theta}_n \xrightarrow{L^2} \theta$.

Remark. When $d = 1$, $\mathbb{E}[(\hat{\theta}_n - \theta)^2] = \text{Var}[\hat{\theta}_n] + (\mathbb{E}[\hat{\theta}_n - \theta])^2$. Therefore, $\hat{\theta}_n$ [converges in mean squared error](#) to θ if and only if $\mathbb{E}[\hat{\theta}_n] \rightarrow \theta$ and $\text{Var}[\hat{\theta}_n] \rightarrow 0$.

Let's first see the most well-known estimation problem, the mean estimation.

Example (Mean estimation). Suppose $d = 1$, and let X be non-negative. Say we're interested in $\theta = \mathbb{E}[X]$. It's standard that in this case, we can compute $\mathbb{E}[X]$ by

$$\theta = \mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > t) dt = \int_0^\infty (1 - F(t)) dt.$$

If X has a pmf f , then $\mathbb{E}[X] = \sum_x x f(x) = \sum_x x \Delta F(x)$ where $f(x) = \Delta F(x) \equiv F(x) - F(x^-)$; if

X has a pdf f , then

$$\mathbb{E}[X] = \int_0^\infty xf(x) dx = \int_0^\infty xF(dx).$$

Now, let $\hat{\theta}_n$ to be the sample mean, i.e., $\hat{\theta}_n = \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. From the [strong law of large number](#), $\bar{X}_n \xrightarrow{\text{a.s.}} \mathbb{E}[X]$, which implies that $\hat{\theta}_n$ is a [strongly consistent estimator](#) of θ .

On the other hand, if $\text{Var}[X] < \infty$, then $\bar{X}_n \xrightarrow{L^2} \mathbb{E}[X]$, which further implies $\bar{X}_n \xrightarrow{p} \mathbb{E}[X]$, hence $\hat{\theta}_n$ is [consistent](#).^a

^aThe latter is true even when $\text{Var}[X] = \infty$ as we expect.

Proof. We show the last statement. Since $\text{Var}[X] < \infty$, then

$$\frac{\text{Var}[X]}{n} = \text{Var}[\bar{X}_n] = \mathbb{E}[(\bar{X}_n - \mathbb{E}[X])^2] \rightarrow 0$$

as $n \rightarrow \infty$, which implies $\bar{X}_n \xrightarrow{p} \mathbb{E}[X]$. *

Another interesting problem is the supremum estimation.

Example (Supremum estimation). Suppose $d = 1$ and there is a $\theta \in \mathbb{R}$ and a distribution function F such that $F(\theta - \epsilon) < 1 = F(\theta)$ for all $\epsilon > 0$, i.e., $\theta = \sup_{\omega} X(\omega)$ since $\mathbb{P}(X \leq \theta - \epsilon) = F(\theta - \epsilon)$ and $F(\theta) = \mathbb{P}(X \leq \theta)$.^a Then $\hat{\theta}_n = \max_{1 \leq i \leq n} X_i$ is indeed a [strongly consistent estimator](#) of θ .

^aSuch a distribution exists, for example, $\mathcal{U}(0, \theta)$.

Proof. We see that for any $\epsilon > 0$,

$$\begin{aligned} \mathbb{P}(|\hat{\theta}_n - \theta| > \epsilon) &= \mathbb{P}(\hat{\theta}_n > \theta + \epsilon) + \mathbb{P}(\hat{\theta}_n < \theta - \epsilon) \\ &= \mathbb{P}\left(\bigcup_{i=1}^n \{X_i > \theta + \epsilon\}\right) + \mathbb{P}\left(\bigcap_{i=1}^n \{X_i < \theta - \epsilon\}\right) \\ &\leq \sum_{i=1}^n \underbrace{\mathbb{P}(X_i > \theta + \epsilon)}_0 + \prod_{i=1}^n \mathbb{P}(X_i < \theta - \epsilon) = (\mathbb{P}(X_1 < \theta - \epsilon))^n \leq (F(\theta - \epsilon))^n \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ since $F(\theta - \epsilon) < 1$. This shows that $\hat{\theta}_n$ is indeed [consistent](#). Moreover, since $\mathbb{P}(|\hat{\theta}_n - \theta| > \epsilon)$ decays exponentially, so this is absolutely summable, hence it's also [strongly consistency](#). *

Proving convergence of $\hat{\theta}_n$ is useful, but this might not be enough.

Example. Consider any deterministic sequence (a_n) in \mathbb{R} which converges to 0. Adding a_n to $\hat{\theta}_n$ will not change the convergence of $\hat{\theta}_n$.

The above suggests that we should look at the *distribution* of $\hat{\theta}_n - \theta$ in order to say how does $\hat{\theta}_n \rightarrow \theta$.

Example (Mean estimation for Gaussian). Suppose $X \sim \mathcal{N}(\theta, 1)$. Then $\hat{\theta}_n = \bar{X}_n \sim \mathcal{N}(\theta, 1/n)$, i.e., $\sqrt{n}(\hat{\theta}_n - \theta) \sim \mathcal{N}(0, 1)$, i.e., we can write down a confidence interval such as $\hat{\theta}_n \pm 1.96/\sqrt{n}$ with 95% confidence level for θ .

Doing this for other kind of estimators and F is not that straightforward and will be challenging.

Remark. Let (X_n) and X be d -dimensional random vectors, $h: \mathbb{R}^d \rightarrow \mathbb{R}^m$, and $c \in \mathbb{R}^d$ constant.

(a) If $X_n \rightarrow c$, then $h(X_n) \rightarrow h(c)$ if h is continuous at c .^a This also holds for $\xrightarrow{\text{a.s.}}$ and \xrightarrow{p} .

(b) If $X_n \rightarrow X$, then $h(X_n) \rightarrow h(X)$ if h is continuous. This also holds for $\xrightarrow{\text{a.s.}}$ and \xrightarrow{p} .

^aThis is an if and only if condition if this holds for any h .

Let's see some examples.

Example. If $d = 1$, and $X_n \rightarrow \theta \neq 0$. Then $1/X_n \rightarrow 1/\theta$ where

$$h(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0; \\ c, & \text{if } x = 0 \end{cases}$$

for any $c \in \mathbb{R}$. The same holds for $\xrightarrow{\text{a.s.}}$ and \xrightarrow{p} .

Example. If $X_n \rightarrow X$ and $Y_n \rightarrow Y$, then $(X_n, Y_n) \rightarrow (X, Y)$.^a The same holds for $\xrightarrow{\text{a.s.}}$ and \xrightarrow{p} .

^aThe converse is also true since projections are continuous.

Proof. $\|(X_n, Y_n) - (X, Y)\| \rightarrow 0$ since $\|(X_n, Y_n) - (X, Y)\| \leq \|X_n - X\| + \|Y_n - Y\|$ for all $n \geq 1$.^a The latter two terms go to 0 (in whatever sense) by assumption. \circledast

^aThis can be seen from $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.

Lecture 3: Weak Convergence Portmanteau Theorem

2.2 Weak Convergence

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The convergences we have seen are not “distribution-wise” since to evaluate $\|X_n - X\|$, X_n and X need to be defined on the same probability space. If all we care about is distribution, consider probability spaces $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ (and $(\Omega, \mathcal{F}, \mathbb{P})$) for which X_n (and X) is defined on.

2.2.1 Convergence in Total Variation

Definition 2.2.1 (Total variation). The *total variation* distance between X and Y on Ω is defined as

$$\text{TV}(X, Y) = \sup_{B \in \mathcal{F}} |\mathbb{P}(X \in B) - \mathbb{P}(Y \in B)|$$

The above makes sense even if X and Y are defined on different probability spaces, e.g., in our situation, consider a sequence of random variables (X_n) and a random variable X .

Definition 2.2.2 (Converge in total variation). (X_n) converges in total variation to X , denoted as $X_n \xrightarrow{\text{TV}} X$, if $\text{TV}(X_n, X) \rightarrow 0$ as $n \rightarrow \infty$.

Note. Specifically, $X_n \xrightarrow{\text{TV}} X$ if $\mathbb{P}_n(X_n \in B) \rightarrow \mathbb{P}(X \in B)$ for all $B \in \mathcal{B}(\mathbb{R}^d)$.

Remark. If X_n has density f_n and X has density f , then $\text{TV}(X_n, X) = \frac{1}{2} \int |f_n - f|$, hence $f_n \rightarrow f$ implies $X_n \xrightarrow{\text{TV}} X$ from [Scheffé's theorem](#).

Example. If $X_n \sim \text{Bin}(n, p_n)$ such that $np_n \rightarrow \lambda \in \mathbb{R}$, then $X_n \sim \text{Bin}(n, p_n) \xrightarrow{\text{TV}} X \sim \text{Pois}(\lambda)$.

Example. Let $X_n \sim f_{\theta_n}$ where $f_{\theta}(x) = f(x)e^{\theta x - \psi(\theta)}$ for some $\theta \in \Theta$. If $\theta_n \rightarrow \theta$, then $X_n \xrightarrow{\text{TV}} X \sim f_{\theta}$. For example, if $X_n \sim \text{Pois}(\theta_n)$ and $\theta_n \rightarrow \theta$, then $X_n \xrightarrow{\text{TV}} X \sim \text{Pois}(\theta)$.

2.2.2 Weak Convergence

However, [convergence in total variation](#) might be too strong to work with.

Example. Let $X_n \sim \mathcal{U}\{0, 1/n, \dots, (n-1)/n\}$, which should be converging to $X \sim \mathcal{U}(0, 1)$. However, this doesn't happen in total variation distance as we can take B to be \mathbb{Q} .

This suggests that we should look at something weaker.

Definition 2.2.3 (Converge weakly). (X_n) converges weakly to X , denoted as $X_n \xrightarrow{w} X$, if for all bounded continuous $g: \mathbb{R}^d \rightarrow \mathbb{R}$, $\mathbb{E}_n[g(X_n)] \rightarrow \mathbb{E}[g(X)]$.

To see how is [weak convergence](#) compared to [convergence in total variation](#), we revisit the above.

Example. Let $X_n \sim \mathcal{U}\{0, 1/n, \dots, (n-1)/n\}$, which should be converging to $X \sim \mathcal{U}(0, 1)$. We have

$$\mathbb{E}_n[g(X_n)] = \sum_{k=0}^{n-1} g(k/n) \left(\frac{k+1}{n} - \frac{k}{n} \right) \rightarrow \int_0^1 g(x) dx = \mathbb{E}[g(X)]$$

as g is bounded and continuous on $[0, 1]$, hence Riemann integrable.

2.2.3 Portmanteau Theorem

The following is our main tool of proving [weak convergence](#).

Theorem 2.2.1 (Portmanteau theorem). The following are equivalent.

- (a) $X_n \xrightarrow{w} X$.
- (b) $\mathbb{E}_n[g(X_n)] \rightarrow \mathbb{E}[g(X)]$ for all bounded Lipschitz $g: \mathbb{R}^d \rightarrow \mathbb{R}$.
- (c) $\mathbb{P}(X \in A) \leq \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A)$ for all $A \subseteq \mathbb{R}^d$ open.
- (d) $\mathbb{P}(X \in A) \geq \limsup_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A)$ for all $A \subseteq \mathbb{R}^d$ closed.
- (e) $\mathbb{P}_n(X_n \in A) \rightarrow \mathbb{P}(X \in A)$ for all $A \in \mathcal{F}$ such that $\mathbb{P}(X \in \partial A) = 0$.

Before we prove [Portmanteau theorem](#), we should note that all our discussion can be extended to metric spaces from Euclidean spaces. Let's first recall some basic results for metric spaces.

Claim. Given a metric space (S, ρ) , $\rho(\cdot, A)$ is Lipschitz for any $A \subseteq S$, i.e., for any $x, y \in S$,

$$|\rho(x, A) - \rho(y, A)| \leq \rho(x, y).$$

Proof. Since for any $z \in S$, $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, hence $\rho(x, A) - \rho(y, A) \leq \rho(x, y)$ by taking the infimum over $z \in A$. Interchanging x and y gives another inequality. \circledast

Claim. Given a metric space (S, ρ) , for any $A \subseteq S$, $x \in \overline{A} \Leftrightarrow \rho(x, A) = 0$.

Proof. If $x \in \overline{A}$, there exists (x_n) in A such that $\rho(x_n, x) \rightarrow 0$. Then for any $z \in A$, $\rho(x, z) \leq \rho(x, x_n) + \rho(x_n, z)$, implying

$$\rho(x, A) \leq \rho(x, x_n) + \rho(x_n, A) \rightarrow 0,$$

hence $\rho(x, A) = 0$. On the other hand, suppose $\rho(x, A) = 0$. As $\rho(x, A) = \inf_{y \in A} \rho(x, y)$, there exists (y_n) in A such that $\rho(x, y_n) \rightarrow \rho(x, A) = 0$, i.e., $x \in \overline{A}$. \circledast

The crucial lemma we're going to use to prove [Portmanteau theorem](#) is the following.

Lemma 2.2.1. Given a metric space (S, ρ) and let $A \subseteq S$ be a closed subset. Then there exists bounded Lipschitz $g_k: S \rightarrow \mathbb{R}$, decreasing in k such that $g_k(x) \searrow \mathbb{1}_A(x)$.

Proof. To motivate, since A is closed, $A = \bar{A}$ and

$$\mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x \in A \Leftrightarrow \rho(x, A) = 0; \\ 0, & \text{if } x \notin A \Leftrightarrow \rho(x, A) > 0. \end{cases}$$

Then, consider

$$g_k(x) = \begin{cases} 0, & \text{if } \rho(x, A) > \frac{1}{k}; \\ 1 - k\rho(x, A), & \text{otherwise;} \end{cases} = 1 - (k\rho(x, A) \wedge 1).$$

We see that

- if $x \in A$: $\mathbb{1}_A(x) = 1$, and $g_k(x) = 1$ since $\rho(x, A) = 0$;
- if $x \notin A$: $\mathbb{1}_A(x) = 0$, and $\rho(x, A) > 0$ since A closed, and $g_k(x) = 0$ for all large enough k .

Finally, it's clear that $g_k(x)$ takes values in $[0, 1]$, and we now show it's Lipschitz. We have

$$|g_k(x) - g_k(y)| = |(k\rho(x, A) \wedge 1) - (k\rho(y, A) \wedge 1)| \leq k\rho(x, y)$$

for all $x, y \in S$. ■

Then we can prove the [Portmanteau theorem](#).

Proof of Theorem 2.2.1. (a) \Rightarrow (b) is clear, and we start by proving (c) \Leftrightarrow (d).

Claim. (c) \Leftrightarrow (d).

Proof. We first prove that (d) \Rightarrow (c). Since when A is open,

$$\begin{aligned} \mathbb{P}(X \in A) &= 1 - \mathbb{P}(X \in A^c) \leq 1 - \limsup_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A^c) \\ &= 1 - \limsup_{n \rightarrow \infty} (1 - \mathbb{P}_n(X_n \in A)) = \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A). \end{aligned} \tag{d}$$

(c) \Rightarrow (d) is exactly the same, hence (c) \Leftrightarrow (d). ⊗

Next, we prove (b) \Rightarrow (d), which gives us (a) \Rightarrow (b) \Rightarrow (d) \Leftrightarrow (c).

Claim. (b) \Rightarrow (d).

Proof. From [Lemma 2.2.1](#), there exists bounded Lipschitz $g_k \searrow \mathbb{1}_A$ such that for all closed A ,

$$\mathbb{P}_n(X_n \in A) = \mathbb{E}_n[\mathbb{1}_A(X_n)] \leq \mathbb{E}_n[g_k(X_n)].$$

This is true for every k and n since $g_k \geq \mathbb{1}_A$, and by taking the limit as $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A) \leq \limsup_{n \rightarrow \infty} \mathbb{E}_n[g_k(X_n)] = \mathbb{E}[g_k(X)]$$

from our assumption (b). Finally, as $k \rightarrow \infty$, it goes to $\mathbb{E}[\mathbb{1}_A(X)] = \mathbb{P}(X \in A)$ as desired. ⊗

The proof will be continued...

Lecture 4: Continuous Mapping Theorem

Before finishing the proof of [Portmanteau theorem](#), we need one additional tool.

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Lemma 2.2.2. If $\{A_i\}_{i \in I}$ are pairwise disjoint events, then $\{i \in I: \mathbb{P}(A_i) > 0\}$ is countable.^a

^aNote that I can be uncountable.

Proof. It suffices to show $|I_k| < \infty$ where $I_k := \{i \in I : \mathbb{P}(A_i) \geq 1/k\}$ for any $k \geq 1$ since

$$\{i \in I : \mathbb{P}(A_i) > 0\} = \bigcup_{k=1}^{\infty} \left\{ i \in I : \mathbb{P}(A_i) \geq \frac{1}{k} \right\} =: \bigcup_{k=1}^{\infty} I_k.$$

We show $|I_k| \leq k$ for any k . Suppose not, then there exists a countable $J_k \subseteq I_k$ such that $|J_k| > k$,

$$\mathbb{P}\left(\bigcup_{i \in J_k} A_i\right) = \sum_{i \in J_k} \mathbb{P}(A_i) \geq \frac{|J_k|}{k} > 1,$$

which is a contradiction. ■

We now finish the proof of [Portmanteau theorem](#).

Proof of Theorem 2.2.1 (cont.) We already proved $(a) \Rightarrow (b) \Rightarrow (d) \Leftrightarrow (c)$.

Claim. $(c) + (d) \Rightarrow (e)$.

Proof. We see that for any A , $A^\circ \subseteq A \subseteq \bar{A}$, and from (c) ,

$$\begin{aligned} \mathbb{P}(X \in A^\circ) &\leq \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A^\circ) \leq \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A) \\ &\leq \limsup_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A) \leq \limsup_{n \rightarrow \infty} \mathbb{P}_n(X_n \in \bar{A}) \leq \mathbb{P}(X \in \bar{A}) \end{aligned}$$

where the last step follows from (d) . Finally, since

$$\mathbb{P}(X \in \bar{A}) - \mathbb{P}(X \in A^\circ) = \mathbb{P}(\{X \in \bar{A}\} \setminus \{X \in A^\circ\}) = \mathbb{P}(X \in (\bar{A} \setminus A^\circ)) = \mathbb{P}(X \in \partial A),$$

which is 0 by our assumption, i.e., inequalities above are all equalities. In particular, since

$$\liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A) \leq \lim_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A) \leq \limsup_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A)$$

$$\text{and } \mathbb{P}(X \in A^\circ) \leq \mathbb{P}(X \in A) \leq \mathbb{P}(X \in \bar{A}), \mathbb{P}(X \in A) = \lim_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A). \quad \textcircled{*}$$

Finally, we prove the following.

Claim. $(e) \Rightarrow (a)$.

Proof. For every $g: \mathbb{R}^d \rightarrow \mathbb{R}$ bounded and continuous, we want to show $\mathbb{E}_n[g(X_n)] \rightarrow \mathbb{E}[g(X)]$. Suppose $g \geq 0$,^a and let $K \geq g(x)$ for every $x \in \mathbb{R}^d$ (which exists since g is bounded), then

$$\mathbb{E}_n[g(X_n)] = \int_0^K \mathbb{P}_n(g(X_n) > t) dt, \quad \mathbb{E}[g(X)] = \int_0^K \mathbb{P}(g(X) > t) dt,$$

so we just need to prove the convergence of the above two integrals. From [bounded convergence theorem](#), it suffices to show that for almost every $t \in [0, K]$,

$$\mathbb{P}_n(g(X_n) > t) \rightarrow \mathbb{P}(g(X) > t).$$

Observe that $\mathbb{P}_n(g(X_n) > t) = \mathbb{P}_n(X_n \in \{g > t\})$ and $\mathbb{P}(g(X) > t) = \mathbb{P}(X \in \{g > t\})$, so from (e) with $A := \{g > t\}$, it suffices to show $\mathbb{P}(X \in \partial\{g > t\}) = 0$ for almost all t . Firstly,

$$\mathbb{P}(X \in \partial\{g > t\}) = \mathbb{P}(X \in \overline{\{g > t\}} \setminus \{g > t\}^\circ) = \mathbb{P}(X \in \overline{\{g \geq t\}} \setminus \{g > t\}) = \mathbb{P}(g(X) = t).$$

Moreover, consider the events $\{g(X) = t\}_{t \in [0, K]}$, which are pairwise disjoint, hence [Lemma 2.2.2](#) implies $\mathbb{P}(g(X) = t) = 0$ for all but countably many t 's, exactly what we want to show. \textcircled{*}

^aOtherwise, we consider $g = g^+ - g^-$ where $g^+ = \max(g, 0)$ and $g^- = \max(-g, 0)$, and everything follows.

This finishes the proof. ■

2.2.4 Continuous Mapping Theorem

A common scenario is that given a nice function h (in terms of continuity), if $X_n \xrightarrow{w} X$, we want to know when will $h(X_n) \xrightarrow{w} h(X)$. To develop the theorem of this, we need some more facts about metric spaces.

As previously seen. Given two metric spaces (S, ρ) , (S', ρ') , $g: S \rightarrow S'$ is continuous if $x_n \xrightarrow{\rho} x$ implies $g(x_n) \xrightarrow{\rho'} g(x)$, or for open $A \subseteq S'$, $g^{-1}(A) \subseteq S$ is open.

Notation. We sometimes write $g^{-1}(A) =: \{g \in A\}$.

It's clear that the following holds.

Note. If $g: S \rightarrow S'$ is continuous and $A \subseteq S'$ is closed, then $\overline{\{g \in A\}} = \{g \in \overline{A}\}$.

However, when g is not continuous and A is not closed, the situation is a bit more complicated. But at least we can first look at the set where g is continuous.

Notation (Continuous set). For any $g: S \rightarrow S'$, we denote the *continuous set* as $C_g := \{x \in S: g \text{ is continuous at } x\}$.

Then we have the following.

Proposition 2.2.1. Given $g: S \rightarrow S'$ between metric spaces and $A \subseteq S'$,

$$C_g \cap \overline{\{g \in A\}} \subseteq \{g \in \overline{A}\}.$$

Proof. Let $x \in C_g \cap \overline{\{g \in A\}}$. Since $x \in \overline{\{g \in A\}}$, there exists $(x_n) \in \{g \in A\}$ such that $x_n \xrightarrow{\rho} x$. Moreover, $x \in C_g$ implies g is continuous at x , hence $g(x_n) \xrightarrow{\rho'} g(x)$, i.e., $g(x) \in \overline{A}$. ■

This allows us to prove the following theorem, which answers our main question in this section.

Theorem 2.2.2 (Continuous mapping theorem). Consider $X_n \xrightarrow{w} X$ and $h: \mathbb{R}^d \rightarrow \mathbb{R}^m$. If $\mathbb{P}(X \in C_h) = 1$, then $h(X_n) \xrightarrow{w} h(X)$.

Proof. Let $A \subseteq \mathbb{R}^m$ be a closed set. Then from [Portmanteau theorem \(d\)](#), we need to show

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n(h(X_n) \in A) \leq \mathbb{P}(h(X) \in A).$$

Since $\limsup_{n \rightarrow \infty} \mathbb{P}_n(h(X_n) \in A) = \limsup_{n \rightarrow \infty} \mathbb{P}_n(X_n \in \{h \in A\})$, implying

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n(h(X_n) \in A) \leq \limsup_{n \rightarrow \infty} \mathbb{P}_n(X_n \in \overline{\{h \in A\}}) \leq \mathbb{P}(X \in \overline{\{h \in A\}}),$$

where the last inequality follows again from [Portmanteau theorem \(d\)](#) since $\overline{\{h \in A\}}$ is clearly closed and $X_n \xrightarrow{w} X$. Finally, as $\mathbb{P}(X \in C_h) = 1$,

$$\mathbb{P}(X \in \overline{\{h \in A\}}) = \mathbb{P}(X \in \overline{\{h \in A\}} \cap C_h) \leq \mathbb{P}(X \in \{h \in \overline{A}\})$$

from [Proposition 2.2.1](#), i.e.,

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n(h(X_n) \in A) \leq \mathbb{P}(X \in \{h \in \overline{A}\}) = \mathbb{P}(X \in \{h \in A\}) = \mathbb{P}(h(X) \in A)$$

since A is closed, hence we're done. ■

Example. Let $d = 1$ and $X_n \xrightarrow{w} X$ where X is continuous. Then $1/X_n \xrightarrow{w} 1/X$ and $X_n^2 \xrightarrow{w} X^2$.

Proof. For $X_n^2 \xrightarrow{w} X^2$, [continuous mapping theorem](#) applies with $h(x) = x^2$. For $1/X_n \xrightarrow{w} 1/X$,

$$h(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$$

is suitable with $C_h = \mathbb{R} \setminus \{0\}$. To apply [continuous mapping theorem](#), we show $\mathbb{P}(X \in C_h) = 1$. Observe that this is the same as asking $\mathbb{P}(X = 0) = 0$, which is true when X is continuous.^a \circledast

^aEven if X is not continuous, as long as this is true we can conclude the same thing.

2.2.5 Slutsky's Theorem

Another useful theorem for proving [weak convergence](#) is the following.

Theorem 2.2.3 (Converging together). Let $X_n \xrightarrow{w} X$, and if Y_n on the same probability space as X_n such that $\|X_n - Y_n\| \xrightarrow{p} 0$, i.e., for all $\epsilon > 0$, $\mathbb{P}_n(\|X_n - Y_n\| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$. Then, $Y_n \xrightarrow{w} X$.

The following corollary draws connections between [weak convergence](#) and [convergence in probability](#).

Corollary 2.2.1. If $Y_n \xrightarrow{p} X$, then $Y_n \xrightarrow{w} X$. The converse holds if $\mathbb{P}(X = c) = 1$ for a constant c .

Proof. By considering $X_n = X$ for all n , [converging together](#) implies that if $Y_n \xrightarrow{p} X$, $Y_n \xrightarrow{w} X$. Conversely, if $Y_n \xrightarrow{w} c$, from [Portmanteau theorem \(c\)](#), for any fixed $\epsilon > 0$,^a

$$1 = \mathbb{P}(c \in B(c, \epsilon)) \leq \liminf_{n \rightarrow \infty} \mathbb{P}_n(Y_n \in B(c, \epsilon)),$$

implying $\mathbb{P}_n(Y_n \in B(c, \epsilon)) \rightarrow 1$, i.e., $\mathbb{P}_n(\|Y_n - c\| < \epsilon) \rightarrow 1$. \blacksquare

^aRecall that $B(c, \epsilon)$ is the open ball centered at c with radius ϵ .

Remark. [Weak convergence](#) doesn't give [convergence in probability](#) even if $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n) = (\Omega, \mathcal{F}, \mathbb{P})$.

Example. Let $X \sim \mathcal{N}(0, 1)$, $Y_n = -X$ for all $n \geq 1$. Then, $Y_n \xrightarrow{w} X$, but clearly not [in probability](#).

Lecture 5: Convergence in Distribution and Weak Convergence

Now we prove [converging together](#).

Proof of Theorem 2.2.3. From [Portmanteau theorem \(b\)](#), we want to prove that $\mathbb{E}_n[g(Y_n)] \rightarrow \mathbb{E}[g(X)]$ for all bounded and Lipschitz $g: \mathbb{R}^d \rightarrow \mathbb{R}$. Specifically, let $|g(x)| \leq C$ for all $x \in \mathbb{R}^d$ and $|g(x) - g(y)| \leq K\|x - y\|$ for all $x, y \in \mathbb{R}^d$. From triangle inequality,

$$|\mathbb{E}_n[g(Y_n)] - \mathbb{E}[g(X)]| \leq |\mathbb{E}_n[g(Y_n)] - \mathbb{E}_n[g(X_n)]| + |\mathbb{E}_n[g(X_n)] - \mathbb{E}[g(X)]|.$$

Since $X_n \xrightarrow{w} X$, the second term goes to 0. As for the first term, we see that

$$\begin{aligned} |\mathbb{E}_n[g(Y_n)] - \mathbb{E}_n[g(X_n)]| &= |\mathbb{E}_n[g(Y_n) - g(X_n)]| \\ &\leq \mathbb{E}_n[|g(Y_n) - g(X_n)|] \\ &= \mathbb{E}_n[|g(Y_n) - g(X_n)| \cdot \mathbb{1}_{\|X_n - Y_n\| > \epsilon}] + \mathbb{E}_n[|g(Y_n) - g(X_n)| \cdot \mathbb{1}_{\|X_n - Y_n\| \leq \epsilon}] \\ &\leq 2C\mathbb{P}_n(\|X_n - Y_n\| > \epsilon) + K\epsilon\mathbb{P}_n(\|X_n - Y_n\| \leq \epsilon) \\ &\leq 2C\mathbb{P}_n(\|X_n - Y_n\| > \epsilon) + K\epsilon. \end{aligned}$$

As $n \rightarrow \infty$, $\limsup_{n \rightarrow \infty} |\mathbb{E}_n[g(Y_n)] - \mathbb{E}[g(X)]| \leq K\epsilon$ for all $\epsilon > 0$, by letting $\epsilon \rightarrow 0$, we're done. \blacksquare

Another characterization regards the difference between marginal and joint [weak convergence](#).

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As previously seen. $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ if and only if $(X_n, Y_n) \xrightarrow{P} (X, Y)$. Same for $\xrightarrow{a.s.}$.

However, even if $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n) = (\Omega, \mathcal{F}, \mathbb{P})$, the marginal and joint **weak convergences** are not equivalent. Specifically, in the case of **weak convergence**, from **continuous mapping theorem**, if $(X_n, Y_n) \xrightarrow{w} (X, Y)$, then $X_n \xrightarrow{w} X$ and $Y_n \xrightarrow{w} Y$. However, the converse needs not be true.

Example. Let $X_n = X$, $Y_n = -X$ for all $n \geq 1$. If $X \sim \mathcal{N}(0, 1)$, we see that $\mathbb{P}(X \in A) = \mathbb{P}(-X \in A)$ for all $A \subseteq \mathbb{R}^d$, implying $X_n \xrightarrow{w} X$ and $Y_n \xrightarrow{w} X$.

However, this does not imply $(X_n, Y_n) \xrightarrow{w} (X, X)$ since otherwise, by **continuous mapping theorem**, $X_n + Y_n \xrightarrow{w} X + X = 2X$, which is not true since $X_n + Y_n = 0$.

But in the case of Y is a constant, the converse is actually true, and the result is quite useful.

Theorem 2.2.4 (Slutsky's theorem). If $X_n \xrightarrow{w} X$ in \mathbb{R}^d and $Y_n \xrightarrow{P} c$ in \mathbb{R}^m ,^a then $(X_n, Y_n) \xrightarrow{w} (X, c)$.

^aRecall that from **Corollary 2.2.1**, for a constant c , **weak convergence** is equivalent to **convergence in probability**.

Proof. Firstly, we show that $(X_n, c) \xrightarrow{w} (X, c)$. Indeed, since for every continuous and bounded $g: \mathbb{R}^{d+m} \rightarrow \mathbb{R}$, from $X_n \xrightarrow{w} X$ with $g(\cdot, c)$ being continuous and bounded, $\mathbb{E}_n[g(X_n, c)] \rightarrow \mathbb{E}[g(X, c)]$.

Secondly, we show that $\|(X_n, Y_n) - (X_n, c)\| \xrightarrow{P} 0$. This is easy since

$$\|(X_n, Y_n) - (X_n, c)\| \leq \|X_n - X_n\| + \|Y_n - c\| = \|Y_n - c\|,$$

which goes to 0 **in probability**. Combining the above with **converging together** gives the result. ■

Revisiting the **counter-example**, we see that now it's not the case when Y is a constant.

Corollary 2.2.2. If $X_n \xrightarrow{w} X$ and $Y_n \xrightarrow{P} c$ in \mathbb{R}^d , $X_n \pm Y_n \xrightarrow{w} X \pm c$, $X_n \cdot Y_n \xrightarrow{w} X \cdot c$. If $d = 1$ and $c \neq 0$, then $X_n/Y_n \xrightarrow{w} X/c$.

Proof. This follows directly from **Slutsky's theorem** and **continuous mapping theorem**. ■

2.3 Convergence in Distribution

The convergences we have been talking about applies to general probability space, not necessarily \mathbb{R}^d . However, compared to **weak convergence**, \mathbb{R}^d is considered first in terms of distributional convergence.

Intuition. There's a conical ordering available in \mathbb{R}^d to define F_X and F_{X_n} .

Definition 2.3.1 (Converge in distribution). Let (X_n) and X be random vectors in \mathbb{R}^d . Then (X_n) *converges in distribution* to X , denoted as $X_n \xrightarrow{D} X$, if for all $(t_1, \dots, t_d) \in C_{F_X}$,

$$F_{X_n}(t_1, \dots, t_d) \rightarrow F_X(t_1, \dots, t_d).$$

Specifically, to see how this relates to what we have seen, recall that

$$F_{X_n}(t_1, \dots, t_d) = \mathbb{P}_n(X_n^i \leq t_i, \forall 1 \leq i \leq d) = \mathbb{P}_n(X_n \in (-\infty, t_1] \times \dots \times (-\infty, t_d]),$$

same for F_X . So this reduces to the form we're familiar with, i.e., $\mathbb{P}_n(X_n \in A)$ for some A . Let's make some remarks for this new notion of convergence.

Remark. $X_n \xrightarrow{TV} X$ implies $X_n \xrightarrow{D} X$.

Proof. Since $X_n \xrightarrow{TV} X$ means $\mathbb{P}_n(X_n \in A) \rightarrow \mathbb{P}(X \in A)$ uniformly in A , but $X_n \xrightarrow{D} X$ only requires the above holds for A in the form of $(-\infty, t_1] \times \dots \times (-\infty, t_d]$, which is weaker. ⊛

There are more classical results that are worth mentioning.

Remark (De Moivre's central limit theorem). Let $X_n \sim \text{Bin}(n, p)$, then for every $t \in \mathbb{R}$, as $n \rightarrow \infty$,

$$\mathbb{P}\left(\frac{X_n - np}{\sqrt{np(1-p)}} \leq t\right) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du = \Phi(t).$$

Proposition 2.3.1. Let X_n and X be in \mathbb{Z} such that f_n and f are their corresponding pmf's, then

$$f_n \rightarrow f \Leftrightarrow X_n \xrightarrow{\text{TV}} X \Leftrightarrow X_n \xrightarrow{D} X.$$

Proof. The forward implications are clear, so we just need to show $X_n \xrightarrow{D} X$ implies $f_n \rightarrow f$. Since for every $t \in \mathbb{Z}$, since X_n and X are discrete in \mathbb{Z} , for some $\epsilon > 0$ small enough,

$$f_n(t) = \mathbb{P}_n(X_n = t) = \mathbb{P}_n(X_n \leq t + \epsilon) - \mathbb{P}_n(X_n \leq t - \epsilon).$$

Since $t \pm \epsilon \in C_X$, $X_n \xrightarrow{D} X$ implies $\mathbb{P}_n(X_n \leq t + \epsilon) \rightarrow \mathbb{P}(X \leq t + \epsilon)$. The same holds for $t - \epsilon$, hence

$$\begin{aligned} f_n(t) &= \mathbb{P}_n(X_n = t) = \mathbb{P}_n(X_n \leq t + \epsilon) - \mathbb{P}_n(X_n \leq t - \epsilon) \\ &\rightarrow \mathbb{P}(X \leq t + \epsilon) - \mathbb{P}(X \leq t - \epsilon) = \mathbb{P}(X = t) = f(t). \end{aligned}$$

As this holds for every $t \in \mathbb{Z}$, we're done. ■

One important remark is the following.

Remark. It's necessary to not require the condition for all $t \in \mathbb{R}^d$, but only $t \in C_{F_X}$.

Proof. Consider for $d = 1$ with $X = c \in \mathbb{R}$, i.e., F_X is the step function at c . To show $X_n \xrightarrow{D} c$, we don't have to show $\mathbb{P}_n(X_n \leq c) \rightarrow \mathbb{P}(X \leq c) = 1$. Otherwise, if we need to show this for all t , in particular, c , $X_n = c + 1/n$ would not satisfy this. ⊛

In terms of continuity, if $X_n \xrightarrow{D} X$ and X is continuous, then F_{X_n} converges to F_X not only point-wise, but uniformly. Specifically, we have the following.

Remark (Pólya's theorem). If F_X is continuous, $X_n \xrightarrow{D} X$ is equivalent as

$$\sup_{t \in \mathbb{R}^d} |F_{X_n}(t) - F_X(t)| \rightarrow 0.$$

2.3.1 Equivalency of Convergence in Distribution and Weak Convergence

Surprisingly, [convergence in distribution](#) is actually just a renaming of [weak convergence](#) in \mathbb{R}^d .

Theorem 2.3.1. Given (X_n) and X in \mathbb{R}^d , $X_n \xrightarrow{w} X$ if and only if $X_n \xrightarrow{D} X$.

Proof. We prove for the case of $d = 1$, then it's easy to see the same holds for $d \geq 1$. For the forward direction, we want to show that for all $t \in C_{F_X}$, $\mathbb{P}_n(X_n \leq t) \rightarrow \mathbb{P}(X \leq t)$. Note that

$$\mathbb{P}(X \leq t) = \mathbb{P}(X \in (-\infty, t]), \text{ and } \mathbb{P}_n(X_n \leq t) = \mathbb{P}_n(X_n \in (-\infty, t]),$$

hence, from [Portmanteau theorem \(e\)](#) with $A = (-\infty, t]$, $X_n \xrightarrow{w} X$ is equivalently to $\mathbb{P}_n(X_n \leq t) \rightarrow \mathbb{P}(X \leq t)$ if $\mathbb{P}(X \in \partial A) = 0$, i.e.,

$$\mathbb{P}(X \in \partial(-\infty, t]) = \mathbb{P}(X \in \{t\}) = \mathbb{P}(X = t) = 0,$$

which is true since $t \in C_{F_X}$.

To show the backward direction, we need the following lemma.

Lemma 2.3.1. $X_n \xrightarrow{D} X$ if and only if for all $x \in \mathbb{R}^d$,

$$F_X(x^-) \leq \liminf_{n \rightarrow \infty} F_{X_n}(x^-) \leq \liminf_{n \rightarrow \infty} F_{X_n}(x) \leq \limsup_{n \rightarrow \infty} F_{X_n}(x) \leq F_X(x).$$

Proof. The backward direction is clear, so we prove the forward direction. When $x \in C_{F_X}$, we're clearly done, so consider $x \notin C_{F_X}$. Firstly, note that $|C_{F_X}^c|$ is countable, so there exists $(x_k) \nearrow x$ and $(y_k) \searrow x$, both in C_{F_X} . Hence, for all $n \geq 1$ and $k \geq 1$,

$$F_{X_n}(x_k) \leq F_{X_n}(x) \leq F_{X_n}(y_k)$$

as F_{X_n} is increasing. We now have for every $k \geq 1$,

$$\begin{aligned} F_X(x_k) &= \lim_{n \rightarrow \infty} F_{X_n}(x_k) && x_k \in C_{F_X} \\ &\leq \liminf_{n \rightarrow \infty} F_{X_n}(x^-) \\ &\leq \liminf_{n \rightarrow \infty} F_{X_n}(x) && F_{X_n} \text{ is increasing} \\ &\leq \limsup_{n \rightarrow \infty} F_{X_n}(x) \\ &\leq \limsup_{n \rightarrow \infty} F_{X_n}(y_k) = F_X(y_k). && y_k \in C_{F_X} \end{aligned}$$

By taking $k \rightarrow \infty$, $F_X(x_k) \rightarrow F_X(x^-)$, while $F_X(y_k) \rightarrow F_X(x)$,^a and we're done.

^aRecall that the distribution function is always right-continuous.

The proof will be *continued*...

Lecture 6: Stochastic Boundedness and Delta Theorem

Before we finish the proof of [Theorem 2.3.1](#), we recall one important characterization of \liminf .

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As previously seen. Given two real sequence x_n and y_n ,

$$\liminf_{n \rightarrow \infty} (x_n + y_n) \geq \liminf_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n,$$

where the equality holds when either x_n or y_n converges (not if and only if).

We can then finish the proof of [Theorem 2.3.1](#).

Proof of Theorem 2.3.1 (cont.) Now we can prove the backward direction. From [Portmanteau theorem \(c\)](#), it suffices to show that for every open $A \subseteq \mathbb{R}$, we have

$$\mathbb{P}(X \in A) \leq \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A).$$

From the elementary analysis, we see that it suffices to show when $A = (a, b)$ since when $A \subseteq \mathbb{R}$ is open, one can write $A = \bigcup_{k=1}^{\infty} (a_k, b_k)$ where (a_k, b_k) 's disjoint, and have

$$\begin{aligned} \mathbb{P}(X \in A) &= \sum_{k=1}^{\infty} \mathbb{P}(X \in (a_k, b_k)) \\ &\leq \sum_{k=1}^{\infty} \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in (a_k, b_k)) && \text{assume true for each } (a_k, b_k) \\ &\leq \liminf_{n \rightarrow \infty} \sum_{k=1}^{\infty} \mathbb{P}_n(X_n \in (a_k, b_k)) = \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A), \end{aligned}$$

where the last inequality follows from an induction on $\liminf_{n \rightarrow \infty} (x_n + y_n) \geq \liminf_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n$. Now, we show that $\mathbb{P}(X \in A) \leq \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in A)$ when $A = (a, b)$.

Claim. $\mathbb{P}(X \in (a, b)) \leq \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in (a, b))$.

Proof. Observe that $\mathbb{P}(X \in (a, b)) = F_X(b^-) - F_X(a)$, with [Lemma 2.3.1](#), we further have

$$\begin{aligned} \mathbb{P}(X \in (a, b)) &= F_X(b^-) - F_X(a) \\ &\leq \liminf_{n \rightarrow \infty} F_{X_n}(b^-) - \left(\limsup_{n \rightarrow \infty} F_{X_n}(a) \right) \\ &\leq \liminf_{n \rightarrow \infty} F_{X_n}(b^-) + \liminf_{n \rightarrow \infty} (-F_{X_n}(a)) \\ &\leq \liminf_{n \rightarrow \infty} (F_{X_n}(b^-) - F_{X_n}(a)) = \liminf_{n \rightarrow \infty} \mathbb{P}_n(X_n \in (a, b)), \end{aligned}$$

which proves the claim. ⊗

This proves the case of $d = 1$. ■

[Theorem 2.3.1](#) means that when talking about random vectors, we can use every result we have proved for the case of [weak convergence](#). Let's see one application.

Proposition 2.3.2. If $X_n \xrightarrow{D} X$ and $t_n \rightarrow t \in C_{F_X}$, then $\mathbb{P}_n(X_n \leq t_n) \rightarrow \mathbb{P}(X \leq t)$.

Proof. We see that from [Corollary 2.2.2](#), $X_n - t_n \xrightarrow{w} X - t$, i.e., $X_n - t_n \xrightarrow{D} X - t$. Hence,

$$\mathbb{P}_n(X_n \leq t_n) = \mathbb{P}_n(X_n - t_n \leq 0) = F_{X_n - t_n}(0) \rightarrow F_{X - t}(0) = \mathbb{P}(X - t \leq 0)$$

as long as $0 \in C_{F_{X-t}}$, i.e., $\mathbb{P}(X - t = 0) = \mathbb{P}(X = t) = 0$, which is just $t \in C_{F_X}$ as we assumed. ■

2.4 Stochastic Boundedness

So far we have been talking about the notion of convergence, now we switch the gear a bit and consider boundedness. In this section, let $(X_i)_{i \in I}$ be a family of d -dimensional random vectors defined on probability spaces $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, with the non-empty index set I , which can be either finite or infinite.

Definition 2.4.1 (Bounded in probability). $(X_i)_{i \in I}$ is said to be *bounded in probability* if for every $\epsilon > 0$, there exists an $M > 0$ such that for every $i \in I$,

$$\mathbb{P}_i(\|X_i\| \geq M) < \epsilon.$$

In other words, for every $\epsilon > 0$, there is an $M > 0$ such that $\mathbb{P}_i(\|X_i\| < M) \geq 1 - \epsilon$ for every $i \in I$.

Intuition. For any arbitrary large probability close to 1 we want, one can find an upper-bound M on $\|X_i\|$ uniformly for all $i \in I$.

Note. When $X_i = X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ for every $i \in I$, $(X_i)_{i \in I}$ is trivially [bounded in probability](#).

Proof. Since if not, there exists $\epsilon > 0$, for every $M > 0$, $\mathbb{P}(\|X\| \geq M) \geq \epsilon$. Then as $M \rightarrow \infty$, $\mathbb{P}(\|X\| = \infty) \geq \epsilon$, which is a contradiction since $\|X\| = \infty$. ⊗

Remark. When I is finite, $(X_i)_{i \in I}$ is also trivially [bounded in probability](#). On the other hand, when I is infinite, by considering $X_n = n$ (deterministic), which is not [bounded in probability](#) anymore.

2.4.1 Sufficient Conditions for Stochastic Boundedness

We now provide some sufficient conditions for being [bounded in probability](#).

Proposition 2.4.1. If $(X_i)_{i \in I}$ is bounded in L^p for some $p > 0$, i.e., $\sup_{i \in I} \mathbb{E}_i [\|X_i\|^p] < \infty$, then $(X_i)_{i \in I}$ is **bounded in probability**.

Proof. Denote $K := \sup_{i \in I} \mathbb{E}_i [\|X_i\|^p] < \infty$. Since for any $\epsilon > 0$, from Markov's inequality,

$$\mathbb{P}_i(\|X_i\| > M) \leq \frac{\mathbb{E}_i [\|X_i\|^p]}{M^p} \leq \frac{K}{M^p} =: \epsilon$$

for $M := \sqrt[p]{K/\epsilon}$. Hence, we're done. \blacksquare

We can generalize some relations between convergence and boundedness from the elementary analysis.

As previously seen. If a deterministic sequence in \mathbb{R} converges, then it's bounded.

In our context, we might expect something like “if $X_n \xrightarrow{p} X$, then (X_n) is **bounded in probability**.” In fact, we have the following “stronger” result where we only require **convergence in distribution**.

Proposition 2.4.2. If $X_n \xrightarrow{D} X$, then (X_n) is **bounded in probability**.

Proof. Fix an $\epsilon > 0$. There is an $M > 0$ such that $\mathbb{P}(\|X\| \geq M) < \epsilon$ since this is a single random vector. To relate this back to X_n , from **Portmanteau theorem (d)**,

$$\epsilon > \mathbb{P}(\|X\| \geq M) = \mathbb{P}(X \in B^c(0, M)) \geq \limsup_{n \rightarrow \infty} \mathbb{P}_n(X_n \in B^c(0, M)) = \limsup_{n \rightarrow \infty} \mathbb{P}_n(\|X_n\| \geq M).$$

In other words, $\liminf_{n \rightarrow \infty} \mathbb{P}_n(\|X_n\| < M) > 1 - \epsilon$, hence there exists an n_0 such that for every $n \geq n_0$, $\mathbb{P}_n(\|X_n\| < M) \geq 1 - \epsilon$. As for those $n < n_0$, since $\{X_n : n < n_0\}$ is a finite family, we can find $M' > 0$ such that $\mathbb{P}_n(\|X_n\| < M') > 1 - \epsilon$ for every $n < n_0$. Finally, by considering $M'' := \max(M, M')$, we have $\mathbb{P}_n(\|X_n\| < M'') > 1 - \epsilon$, i.e., $\mathbb{P}_n(\|X_n\| \geq M'') < \epsilon$ as desired. \blacksquare

A kind of converse theorem is called **Prokhorov's theorem**, but we won't prove it here right now. We now see another useful characterization that generalizes our intuition in \mathbb{R} . Recall the following.

As previously seen. In \mathbb{R} , if $a_n \rightarrow 0$ and b_n is bounded, $a_n b_n \rightarrow 0$.

The generalization is the following.

Proposition 2.4.3. Let $d = 1$ such that (X_n) and (Y_n) are defined on the same probability space. If $X_n \xrightarrow{p} 0$ and Y_n is **bounded in probability**, then $X_n Y_n \xrightarrow{p} 0$.

Proof. Fix an $\epsilon > 0$. We want to show that $\mathbb{P}_n(|X_n Y_n| > \epsilon) \rightarrow 0$. This is because

$$\begin{aligned} \mathbb{P}_n(|X_n Y_n| > \epsilon) &= \mathbb{P}_n(|X_n Y_n| > \epsilon, |Y_n| > M) + \mathbb{P}_n(|X_n Y_n| > \epsilon, |Y_n| \leq M) \\ &\leq \mathbb{P}_n(|Y_n| > M) + \mathbb{P}_n(|X_n Y_n| > \epsilon, |Y_n| \leq M) \leq \mathbb{P}_n(|Y_n| > M) + \mathbb{P}_n(|X_n| > \epsilon/M) \end{aligned}$$

for any M . Now, we see that

- since Y_n is **bounded in probability**, there's an $M > 0$ such that $\mathbb{P}_n(|Y_n| > M) < \epsilon$ for all n ;
- since $X_n \xrightarrow{p} 0$, for the M (depends on the fixed ϵ) above, $\mathbb{P}_n(|X_n| > \epsilon/M) \rightarrow 0$ as $n \rightarrow \infty$.

We see that the second term always goes to 0, while the first term can always be upper-bounded by ϵ . Hence, by letting $\epsilon \rightarrow 0$, we're done. \blacksquare

We often write the following.

Notation. We write $X_n = o_p(1)$ for $X_n \xrightarrow{p} 0$, and $X_n = O_p(1)$ when (X_n) is **bounded in probability**.

Remark. Proposition 2.4.3 means $o_p(1) \times O_p(1) = o_p(1)$.

2.4.2 Delta Method

Let's see one important application which combines the above. Consider an estimator T_n of θ , and a deterministic sequence b_n which goes to ∞ . In this case, we often have

$$b_n(T_n - \theta) \xrightarrow{D} Y.$$

Example. When $X_n \sim \text{Bin}(n, p)$, then for $b_n = \sqrt{n/p(1-p)} \rightarrow \infty$, $T_n = X_n/n$, and $\theta = p$, we have

$$\frac{X_n - np}{\sqrt{np(1-p)}} = \sqrt{\frac{n}{p(1-p)}} \left(\frac{X_n}{n} - p \right) = b_n(T_n - \theta) \rightarrow Y \sim \mathcal{N}(0, 1).$$

This allows us to compute the rate of convergence and the limiting distribution. But what can we say when we care about $g(T_n)$ for a function g ?

Theorem 2.4.1 (Delta method). Let $\theta \in \mathbb{R}^d$, (T_n) and Y be random vectors in \mathbb{R}^d , and $b_n \rightarrow \infty$ be a positive deterministic sequence. If $b_n(T_n - \theta) \xrightarrow{D} Y$, then $T_n \xrightarrow{P} \theta$. Moreover, if $g: \mathbb{R}^d \rightarrow \mathbb{R}^m$ is differentiable at θ , $b_n(g(T_n) - g(\theta)) \xrightarrow{D} \nabla g(\theta)Y$, where $\nabla g \in \mathbb{R}^{d \times m}$ is the Jacobian of g .

Proof. We first observe that $\|b_n(T_n - \theta)\| \in O_p(1)$ since $b_n(T_n - \theta) \xrightarrow{D} Y$, with [continuous mapping theorem](#) and the fact that $\|\cdot\|$ is continuous, $\|b_n(T_n - \theta)\| \xrightarrow{P} \|Y\|$, so $\|b_n(T_n - \theta)\| \in O_p(1)$ by [Proposition 2.4.2](#). With this, as $b_n \rightarrow \infty$, $T_n \xrightarrow{P} \theta$ since

$$\|T_n - \theta\| = \frac{1}{b_n} \|b_n(T_n - \theta)\| = o(1)O_p(1) \xrightarrow{P} 0$$

as $o(1)O_p(1) = o_p(1)$ from [Proposition 2.4.3](#). For the second claim, since g is differentiable at θ ,

$$\frac{g(x) - g(\theta) - \nabla g(\theta)(x - \theta)}{\|x - \theta\|} \rightarrow 0$$

when $x \rightarrow \theta$. Let $r(x) := g(x) - g(\theta) - \nabla g(\theta)(x - \theta)$ for $x \in \mathbb{R}^d$ be the remainder, and consider

$$h(x) = \begin{cases} 0, & \text{if } x = \theta; \\ \frac{r(x)}{\|x - \theta\|}, & \text{if } x \neq \theta, \end{cases}$$

which is continuous at θ . Rewriting everything, we have

$$r(x) = g(x) - g(\theta) - \nabla g(\theta)(x - \theta) = h(x)\|x - \theta\|$$

for every $x \in \mathbb{R}^d$. Now, let $x = T_n$, multiply both sides by b_n , and take the norm, we see that

$$\|b_n(g(T_n) - g(\theta)) - \nabla g(\theta)b_n(T_n - \theta)\| = \|h(T_n)\| \|b_n(T_n - \theta)\|.$$

We observe the following.

Claim. It suffices to show that the right-hand sides goes to 0 [in probability](#).

Proof. Since it implies that $b_n(g(T_n) - g(\theta))$ has the same weak limit as $\nabla g(\theta)b_n(T_n - \theta)$ from [converging together](#), i.e., $\nabla g(\theta)Y$ from our assumption with [continuous mapping theorem](#). \otimes

It's enough to show $\|h(T_n)\| = o_p(1)$ since we know that $\|b_n(T_n - \theta)\| = O_p(1)$ and $o_p(1)O_p(1) = o_p(1)$ from [Proposition 2.4.3](#). Indeed, as $T_n \xrightarrow{P} \theta$, $h(T_n) \xrightarrow{P} h(\theta) = 0$ again by [continuous mapping theorem](#) with h being continuous at θ . This further implies $\|h(T_n)\| \xrightarrow{P} 0$ as we desired.^a Combining the above, the result follows. \blacksquare

^aThis involves [continuous mapping theorem](#) and [Corollary 2.2.1](#) since $h(\theta) = 0$, a constant (so does its norm).

Hence, we see that the answer to our original question is rather simple: as $b_n(T_n - \theta) \xrightarrow{D} Y$,

$$b_n(g(T_n) - g(\theta)) \xrightarrow{D} \nabla g(\theta) \cdot Y$$

for any differentiable g at θ .

Lecture 7: Skorohod's Representation Theorem

2.5 Skorohod's Representation Theorem

6 Feb. 9:30

So far, we have seen the following.



Now, we show an interesting result that one might not expect.

Theorem 2.5.1 (Skorohod's representation theorem). If $X_n \xrightarrow{D} X$, there exists $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ on which we can define random vectors (Y_n) and Y such that $Y_n \stackrel{D}{=} X_n$ for all n and $Y \stackrel{D}{=} X$, and $\tilde{\mathbb{P}}(Y_n \rightarrow Y) = 1$.

Intuition. We have [convergence in distribution](#) “implies” [almost surely convergence](#).

2.5.1 Quantile Function

We want to prove [Skorohod's representation theorem](#) for $d = 1$. To start, say $X \sim F$ on $(\Omega, \mathcal{F}, \mathbb{P})$. We will consider $F^{-1}(p)$, which exists if there exists a unique $t \in \mathbb{R}$ such that $F(t) = p$, then $F^{-1}(p) = t$. However, this is not really practical since in the discrete case, the preimage might not exist; and even if in the continuous F , when F flats out (at $p = 1$), the preimage is not unique.

Definition 2.5.1 (Quantile). A p^{th} quantile of X is defined as any $t \in \mathbb{R}$ such that

$$\mathbb{P}(X \leq t) \geq p \geq \mathbb{P}(X < t).$$

Now, we can define $F^{-1}(p)$ as the smallest [quantile](#).

Definition 2.5.2 (Quantile function). The *quantile function* of $X \sim F$ is defined as

$$F^{-1}(p) = \inf\{t \in \mathbb{R} : F(t) \geq p\}.$$

We sometimes also call F^{-1} as the *generalized inverse* of F .

Remark. $t \geq F^{-1}(p)$ if and only if $F(t) \geq p$; in other words, $t < F^{-1}(p)$ if and only if $F(t) < p$.

One application of F^{-1} is that given any cdf F , we can construct a corresponding random variable.

Remark (Construction of random variable). Let $U \sim \mathcal{U}(0, 1)$ be a uniform random variable on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Then, $F^{-1}(U) =: Y$ is a random variable with cdf F .

Proof. Since for any $t \in \mathbb{R}$,

$$\tilde{\mathbb{P}}(Y \leq t) = \tilde{\mathbb{P}}(F^{-1}(U) \leq t) = \mathbb{P}(U \leq F(t)) = F(t).$$

⊛

2.5.2 Proof of Skorohod's representation theorem

Now we can prove [Skorohod's representation theorem](#).

Proof of Theorem 2.5.1. Consider $\tilde{\Omega} = (0, 1)$, and $\tilde{\mathbb{P}}((a, b)) = b - a$ for all $a < b$. Then, we can define $U(p) = p$ for all $p \in \tilde{\Omega}$, i.e., $U \sim \mathcal{U}(0, 1)$. Define $Y_n = F_{X_n}^{-1}(U)$ and $Y = F_X^{-1}(U)$ from the [quantile functions](#). Denote Φ be the cdf of $\mathcal{N}(0, 1)$, and let $Z = \Phi^{-1}(U)$.

It's clear that $Y_n \stackrel{D}{=} X_n$ and $Y \stackrel{D}{=} X$, so we just need to show $\tilde{\mathbb{P}}(Y_n \rightarrow Y) = 1$.

Claim. It's equivalent to $\tilde{\mathbb{P}}(F_{X_n}(Z) < p) \rightarrow \tilde{\mathbb{P}}(F_X(Z) < p)$ for almost all p 's.

Proof. Observe further that $Y_n(p) = F_{X_n}^{-1}(p)$, $Y(p) = F_X^{-1}(p)$, and $Z(p) = \Phi^{-1}(p)$ for all $p \in (0, 1)$. Since for almost all p 's, $Y_n(p) \rightarrow Y(p)$ if and only if $\Phi(Y_n(p)) \rightarrow \Phi(Y(p))$ as Φ is strictly increasing and continuous, or equivalently,

$$\Phi(Y_n(p)) = \tilde{\mathbb{P}}(Z \leq Y_n(p)) \rightarrow \tilde{\mathbb{P}}(Z \leq Y(p)) = \Phi(Y(p)).$$

As Z is continuous, this is equivalent to $\tilde{\mathbb{P}}(Z < Y_n(p)) \rightarrow \tilde{\mathbb{P}}(Z < Y(p))$, i.e.,

$$\tilde{\mathbb{P}}(Z < F_{X_n}^{-1}(p)) \rightarrow \tilde{\mathbb{P}}(Z < F_X^{-1}(p)),$$

which holds if and only if $\tilde{\mathbb{P}}(F_{X_n}(Z) < p) \rightarrow \tilde{\mathbb{P}}(F_X(Z) < p)$.^a ⊗

^aFollows from [the remark](#). Explicitly, firstly, it's equivalent to $\tilde{\mathbb{P}}(Z \geq F_{X_n}^{-1}(p)) \rightarrow \tilde{\mathbb{P}}(Z \geq F_X^{-1}(p))$, and with $\tilde{\mathbb{P}}(Z \geq F_{X_n}^{-1}(p)) = \tilde{\mathbb{P}}(F_{X_n}(Z) \geq p)$ and $\tilde{\mathbb{P}}(Z \geq F_X^{-1}(p)) = \tilde{\mathbb{P}}(F_X(Z) \geq p)$, the result follows.

Now we show $\tilde{\mathbb{P}}(F_{X_n}(Z) < p) \rightarrow \tilde{\mathbb{P}}(F_X(Z) < p)$ for almost all p 's. Since $X_n \xrightarrow{D} X$ means $F_{X_n}(t) \rightarrow F_X(t)$, from [Lemma 2.3.1](#), it further implies $F_{X_n}(t^-) \rightarrow F_X(t^-)$ for all $t \in C_{F_X}$. Note that $\tilde{\mathbb{P}}(Z \in C_{F_X}) = 1$ since there can be only countably many discontinuities of F_X . Hence,

$$\tilde{\mathbb{P}}(F_{X_n}(Z) \rightarrow F_X(Z)) = 1,$$

i.e., [converges almost surely](#), which implies $F_{X_n}(Z) \xrightarrow{D} F_X(Z)$, i.e., for all $p \in C_{F_X}(Z)$

$$\tilde{\mathbb{P}}(F_{X_n}(Z) \leq p) \rightarrow \tilde{\mathbb{P}}(F_X(Z) \leq p),$$

and also $\tilde{\mathbb{P}}(F_{X_n}(Z) < p) \rightarrow \tilde{\mathbb{P}}(F_X(Z) < p)$ from [Lemma 2.3.1](#). Again, as F_X can have only countably many discontinuities, this holds for almost all p 's, which is what we want to show. ■

We now see some applications of [Skorohod's representation theorem](#), where we can obtain relatively simple proofs for several theorems, such as [Theorem 2.3.1](#).

Remark. [Theorem 2.3.1](#) can be proved from [Skorohod's representation theorem](#).

Proof. If $X_n \xrightarrow{D} X$, from [Skorohod's representation theorem](#), we can obtain $Y_n \xrightarrow{a.s.} Y$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that $X_n \stackrel{D}{=} Y_n$ and $X \stackrel{D}{=} Y$. Then for any bounded and continuous g ,

$$\mathbb{E}[g(X_n)] = \tilde{\mathbb{E}}[g(Y_n)] \rightarrow \tilde{\mathbb{E}}[g(Y)] = \mathbb{E}[g(X)]$$

by the [bounded convergence theorem](#), which proves $X_n \xrightarrow{w} X$. ⊗

Another application is to generalize [Fatou's lemma](#).

Proposition 2.5.1 (Fatou's lemma). Let $X_n \xrightarrow{D} X^a$ and $g: \mathbb{R}^d \rightarrow [0, \infty)$ continuous. Then

$$\mathbb{E}[g(X)] \leq \liminf_{n \rightarrow \infty} \mathbb{E}_n[g(X_n)].$$

^aCan be on different probability spaces.

Proof. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, from [Skorohod's representation theorem](#), we can construct $Y_n \stackrel{D}{=} X_n$, $Y \stackrel{D}{=} X$, and $Y_n \xrightarrow{\text{a.s.}} Y$, which implies $g(Y_n) \xrightarrow{\text{a.s.}} g(Y)$. From [Fatou's lemma](#) in $d = 1$, $\tilde{\mathbb{E}}[g(Y)] \leq \liminf_{n \rightarrow \infty} \tilde{\mathbb{E}}[g(Y_n)]$. The result then follows directly from

$$\mathbb{E}[g(X)] = \tilde{\mathbb{E}}[g(Y)] \leq \liminf_{n \rightarrow \infty} \tilde{\mathbb{E}}[g(Y_n)] = \liminf_{n \rightarrow \infty} \mathbb{E}_n[g(X_n)].$$

The following is well-known from real analysis [dominated convergence theorem](#).

Theorem 2.5.2. If $X_n \xrightarrow{\text{a.s.}} X$, $g: \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous and $(g(X_n))$ is uniformly integrable^a if and only if $\mathbb{E}_n[g(X_n)] \rightarrow \mathbb{E}[g(X)]$.

^aI.e., $\lim_{t \rightarrow \infty} \sup_{n \geq 1} \mathbb{E}[|g(X_n)| \mathbb{1}_{|g(X_n)| \geq t}] = 0$.

If $X_n \xrightarrow{w} X$, then from the definition, we will have $\mathbb{E}_n[g(X_n)] \rightarrow \mathbb{E}[g(X)]$ if g is continuous and bounded. We can indeed relax both continuity and boundedness as follows.

Proposition 2.5.2. If $X_n \xrightarrow{w} X$ and $\mathbb{P}(X \in C_g) = 1$ where $g: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $(g(X_n))$ is uniformly integrable, then $\mathbb{E}_n[g(X_n)] \rightarrow \mathbb{E}[g(X)]$.

Proof. From $\mathbb{P}(X \in C_g) = 1$ and $X_n \xrightarrow{w} X$, from [continuous mapping theorem](#), $g(X_n) \xrightarrow{w} g(X)$, hence $\mathbb{E}_n[g(X_n)] \rightarrow \mathbb{E}[g(X)]$. ■

Remark. [Proposition 2.5.2](#) can be proved with [Skorohod's representation theorem](#) also.

2.6 Characteristic Function

It turns out that [convergence in distribution](#) has a very neat characterization. To motivate the idea, consider the problem of proving $X_n \xrightarrow{D} X$, which is usually inefficient if we start from the [definition](#). To get some intuition for potential proof strategies, consider a deterministic sequence (x_n) in a metric space (S, ρ) .

Theorem 2.6.1. $(x_n) \rightarrow x$ if and only if every subsequence of (x_n) has a subsequence that converges to the same limit x .

Proof. The forward direction is clear. For the backward direction, if not, there exists (x_{n_k}) and $\epsilon > 0$ such that $\rho(x_{n_k}, x) \geq \epsilon$ for every $k \geq 1$. But if there exists a subsubsequence $(x_{n_{k_\ell}})$ that converges to x , this is clearly a contradiction. ■

In the same vein, with the same argument, we have the following.

Theorem 2.6.2. $X_n \xrightarrow{w} X$ if and only if every subsequence of (X_n) has a subsequence that [converges weakly](#), and all [weakly convergent](#) subsequences have the same limit X .

Proof. Mimicking the proof as in [Theorem 2.6.1](#). ■

Lecture 8: Characteristic Functions

We see other similar theorems apart from [Theorem 2.6.2](#).

Theorem 2.6.3. If $X_n \xrightarrow{w} X$ and $X_n \xrightarrow{w} Y$, then $X \stackrel{D}{=} Y$. More generally, if $X_n \xrightarrow{w} X$ and $Y_n \xrightarrow{w} Y$, with $X_n \stackrel{D}{=} Y_n$ for all $n \geq 1$, $X \stackrel{D}{=} Y$.

in L_1 ?

Seems no need of $(g(X_n))$ being u.i.

the in L_1 version?

Proof. For every $n \geq 1$, $\mathbb{E}_n[g(X_n)] = \mathbb{E}_n[g(Y_n)]$ for all $g: \mathbb{R}^d \rightarrow \mathbb{R}$. If g is bounded and continuous, $\mathbb{E}_n[g(X_n)] \rightarrow \mathbb{E}[g(X)]$ and $\mathbb{E}_n[g(Y_n)] \rightarrow \mathbb{E}[g(Y)]$. To show that $X \stackrel{D}{=} Y$, we want to show $F_X = F_Y$, or $\mathbb{P}(X \in B) = \mathbb{P}(Y \in B)$ for all $B \in \mathcal{F} = \mathcal{B}(\mathbb{R}^d)$. In fact, it's enough to show this for closed B . With [Lemma 2.2.1](#), there exists $(g_k) \searrow \mathbb{1}_B$ for closed B and bounded, Lipschitz g_k , i.e.,

$$\begin{aligned} \mathbb{E}[\mathbb{1}_B(X)] &= \lim_{k \rightarrow \infty} \mathbb{E}[g_k(X)] = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E}_n[g_k(X_n)] \\ &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E}_n[g_k(Y_n)] = \lim_{k \rightarrow \infty} \mathbb{E}[g_k(Y)] = \mathbb{E}[\mathbb{1}_B(Y)], \end{aligned}$$

where the third equality follows from the fact that $X_n \stackrel{D}{=} Y_n$. ■

One question is that, if we don't have things like [weak convergent](#) but just some moment information (i.e., when $g(x) = x^k$ when computing $\mathbb{E}[g(X)]$), can we conclude the same thing?

Problem (Method of Moments). If $\mathbb{E}[X^k] = \mathbb{E}[Y^k] < \infty$ for all $k \geq 1$, does $X \stackrel{D}{=} Y$?

Answer. Not in general. We will discuss this more in the assignment. ⊛

2.6.1 Characteristic Function

To answer the question left above, we will see that it actually suffices to show only for $g(x) = \cos(t \cdot x)$ or $\sin(t \cdot x)$ for $t, x \in \mathbb{R}^d$. This leads to the so-called [characteristic functions](#).

Definition 2.6.1 (Characteristic function). The *characteristic function* of a d -dimensional random vector X is defined as $\phi_X: \mathbb{R}^d \rightarrow \mathbb{C}$ where $t \in \mathbb{R}^d$ such that

$$\phi_X(t) = \mathbb{E}[\cos(t \cdot X)] + i\mathbb{E}[\sin(t \cdot X)] = \mathbb{E}[e^{i(t \cdot X)}].$$

Notation. We sometimes drop the inner product, i.e., write $t \cdot X =: tX$.

If we write ϕ_X explicitly, we have

$$\phi_X(t) = \mathbb{E}[e^{itX}] = \int e^{itx} f_X(x) dx = \int e^{itx} F_X(dx).$$

Remark. [Characteristic functions](#) are bounded.

Proof. Since

$$|\phi_X(t)| = \sqrt{(\mathbb{E}[\cos(tX)])^2 + (\mathbb{E}[\sin(tX)])^2} \leq \sqrt{\mathbb{E}[\cos^2(tX)] + \mathbb{E}[\sin^2(tX)]} = 1.$$

⊛

This implies that ϕ_X is meaningful for any random vector X , unlike the moment generating function.

Remark. If X and Y are independent, $\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$.

We make one more remark for future reference.

Remark. If X, Y are discrete, $f_{X+Y}(x) = \sum_y f_Y(x-y)f_X(y)$. More generally, if X, Y have pdfs,

$$f_{X+Y}(x) = \int f_Y(x-y)f_X(y) dy = \int f_Y(x-y)F_X(dy).$$

Furthermore, even if X doesn't have pdf, as long as Y does, the above still holds.

2.6.2 Uniqueness Theorem

Now we can prove the following uniqueness theorem, which states that indeed, it suffices to check only $\sin(tx)$ and $\cos(tx)$ when proving [weak convergence](#).

Theorem 2.6.4 (Uniqueness). If $\phi_X(t) = \phi_Y(t)$ for all $t \in \mathbb{R}^d$, then $X \stackrel{D}{=} Y$. The converse is trivial.

Proof. Consider $d = 1$. Observe that if we can write F_X in terms of only ϕ_X , then $\phi_X = \phi_Y$ implies $F_X = F_Y$. To do this, consider the following.

Claim. For $Z, Z' \sim \mathcal{N}(0, 1)$ (independent of X and Y), if one can write $F_{X+\sigma Z}$ for all $\sigma > 0$ in terms of only ϕ_X , $\phi_X = \phi_Y$ implies $X \stackrel{D}{=} Y$.

Proof. Fix some $\sigma > 0$. In this case, if we can write $F_{X+\sigma Z}$ in terms of only ϕ_X , $\phi_X = \phi_Y$ implies $F_{X+\sigma Z} = F_{Y+\sigma Z'}$. This implies $X + \sigma Z \stackrel{D}{=} Y + \sigma Z'$. Now, for $\sigma = 1/k$, $k \in \mathbb{N}$,

$$X + \frac{1}{k}Z \stackrel{D}{=} Y + \frac{1}{k}Z'.$$

With [Corollary 2.2.2](#), since $Z/k \xrightarrow{P} 0$ (and also $Z'/k \xrightarrow{P} 0$), we have $X + Z/k \xrightarrow{D} X$ and $Y + Z'/k \xrightarrow{D} Y$, which implies $X \stackrel{D}{=} Y$ from [Theorem 2.6.3](#). \otimes

Hence, our goal now is to write $F_{X+\sigma Z}$ in terms of ϕ_X . Firstly, for all $t \in \mathbb{R}$,

$$\phi_Z(t) = \int e^{itz} F_Z(dz) = \int e^{itz} f_Z(z) dz = \int e^{itz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = e^{-t^2/2}. \quad (2.1)$$

Now, consider $f_{X+\sigma Z}(x)$ instead, which exists since Z has a pdf from the [remark](#). We see that

$$\begin{aligned} f_{X+\sigma Z}(x) &= \int f_{\sigma Z}(x-y) F_X(dy) \\ &= \int \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-y)^2/2\sigma^2} F_X(dy), \end{aligned}$$

by replacing $e^{-(x-y)^2/2\sigma^2}$ from [Equation 2.1](#) with $t = (x-y)/\sigma$,

$$\begin{aligned} &= \int \frac{1}{\sigma\sqrt{2\pi}} \int e^{i\frac{y-x}{\sigma}z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz F_X(dy). \\ &= \frac{1}{2\pi} \iint e^{i(y-x)u} e^{-\sigma^2 u^2/2} du F_X(dy), \quad z/\sigma =: u \\ &= \frac{1}{2\pi} \int e^{-ixu - \sigma^2 u^2/2} \underbrace{\int e^{iyu} F_X(dy)}_{\phi_X(u)} du, \end{aligned}$$

where we interchange the order of integrals with [Fubini's theorem](#) (justified by [Tonelli's theorem](#)) when integrands are absolute integrable. This implies that $F_{X+\sigma Z}(dx)$ can be written in terms of ϕ_X where with no other dependencies, hence we're done. \blacksquare

Note. Now showing $X \stackrel{D}{=} Y$ reduces to calculus.

2.6.3 Continuity Theorem

One immediate consequence of the [uniqueness theorem](#) is that it's enough to have the [characteristic functions](#) converging to some function (not necessarily a [characteristic functions](#) of some X) for us to conclude that the subsequences of (X_n) have the same weak limit. To do this, we need to prove [Prokhorov's theorem](#).

Theorem 2.6.5 (Prokhorov's theorem). If $(X_n) = O_p(1)$, then there exists a **weakly convergent** subsequence of (X_n) .

Proof. Based on **Helly's selection theorem**, $F_{X_n}(t) \rightarrow F(t)$ for all $t \in C_F$, there exists an increasing F , right continuous, $F(+\infty) \leq 1$ and $F(-\infty) \geq 0$ (called the *defective cdf*). Consider $d = 1$, we show that this F is indeed a cdf when $X_n = O_p(1)$.

Fix $\epsilon > 0$, then there exists $M_\epsilon > 0$ in C_F such that

$$F_{X_n}(M_\epsilon) = \mathbb{P}_n(X_n \leq M_\epsilon) \geq \mathbb{P}_n(|X_n| \leq M_\epsilon) \geq 1 - \epsilon$$

for all $n \geq 1$. Since $M_\epsilon \in C_F$, $F_{X_n}(M_\epsilon) \rightarrow F(M_\epsilon)$. We then see that for all $\epsilon > 0$, there exists $M_\epsilon > 0$ such that $F(+\infty) \geq F(M_\epsilon) \geq 1 - \epsilon$. As $\epsilon \rightarrow 0$, $F(+\infty) = 1$. Similarly, $F(-\infty) = 0$. ■

We now state the theorem.

Theorem 2.6.6 (Lévy-Cramér continuity theorem). If $\phi_{X_n}(t) \rightarrow \phi(t)$ for all $t \in \mathbb{R}^d$, then all **weakly convergent** subsequences of (X_n) have the same weak limit. Furthermore, if also ϕ is continuous at 0, then there exists X such that $\phi = \phi_X$ and $X_n \xrightarrow{D} X$.

Proof. For the first claim, suppose $Y_n \xrightarrow{w} Y$ and $Z_n \xrightarrow{w} Z$ are two subsequences of X_n such that $Y \neq Z$. But since $\phi_{Y_n}(t) \rightarrow \phi_Y(t)$ and $\phi_{Z_n}(t) \rightarrow \phi_Z(t)$, with the fact that $(\phi_{Y_n}(t))$ and $(\phi_{Z_n}(t))$ are subsequences of $(\phi_{X_n}(t))$ for every t , as $\phi_{X_n}(t) \rightarrow \phi(t)$, both subsequences need to converge to the same limit, i.e., $\phi_Y(t) = \phi(t) = \phi_Z(t)$ for all $t \in \mathbb{R}^d$. From the **uniqueness theorem**, $Y \stackrel{D}{=} Z$.

For the second claim, we just need to prove the following.

Claim. It's enough to show that if ϕ is continuous at 0, $(X_n) = O_p(1)$.

Proof. Since if $(X_n) = O_p(1)$, **Prokhorov's theorem** implies there exists a **weakly convergent** subsequence of (X_n) . With the first claim, we can find the weak limit X . ⊗

The proof will be **continued**...

Lecture 9: Proof of Lévy-Cramér Continuity Theorem

We now finish the proof of **Lévy-Cramér continuity theorem**.

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Proof of Theorem 2.6.6 (cont.) Fix $\epsilon > 0$. Then there exists $\delta > 0$ such that for all $|t| < \delta$,

$$|\phi(t) - \phi(0)| = |\phi(t) - 1| < \frac{\epsilon}{4}$$

since for any $n \geq 1$, $\phi_{X_n}(0) = 1$, so is $\phi(0)$. Hence, we have

$$\frac{\epsilon}{2} = \frac{1}{\delta} \int_{-\delta}^{\delta} \frac{\epsilon}{4} dt > \frac{1}{\delta} \int_{-\delta}^{\delta} |\phi(t) - 1| dt.$$

We claim that we can find an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\mathbb{P}_n(|X_n| \geq 2/\delta) < \epsilon$.^a To bound $|X_n|$ with ϕ_{X_n} , firstly, for all x , $|\sin x| \leq |x|$. This bound is good only when x is close to 0. If it's not the case, then we can use $|\sin x/x| \leq 1/|x| \leq 1/2$ if $|x| \geq 2$. Hence, in general, for $x \neq 0$,

$$\frac{\sin x}{x} \leq \left| \frac{\sin x}{x} \right| \leq \frac{1}{2} \cdot \mathbb{1}_{|x| \geq 2} + 1 \cdot \mathbb{1}_{|x| < 2} = 1 - \frac{1}{2} \mathbb{1}_{|x| \geq 2} \Rightarrow \mathbb{1}_{|x| \geq 2} \leq 2 \left(1 - \frac{\sin x}{x} \right)$$

as $\mathbb{1}_{|x| < 2} = 1 - \mathbb{1}_{|x| \geq 2}$. Plug in δx , for any $x \neq 0$, we have

$$\mathbb{1}_{|\delta x| \geq 2} \leq 2 \left(1 - \frac{\sin(\delta x)}{\delta x} \right) = \frac{1}{\delta} \left(2\delta - 2 \frac{\sin(\delta x)}{x} \right) = \frac{1}{\delta} \int_{-\delta}^{\delta} 1 - \cos(tx) dt.$$

Indeed, the above is true for all $x \in \mathbb{R}$ by manually checking. Finally, by replacing x by X_n and

take the expectation on the both sides,

$$\mathbb{P}_n(|\delta X_n| \geq 2) \leq \frac{1}{\delta} \int_{-\delta}^{\delta} 1 - \mathbb{E}_n[\cos(tX_n)] dt = \frac{1}{\delta} \int_{-\delta}^{\delta} \operatorname{Re}(1 - \phi_{X_n}(t)) dt \leq \frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_{X_n}(t)| dt,$$

where we pass the expectation (i.e., limit) inside the integral from **Fubini's theorem** since $\cos(tX_n)$ is bounded. It remains to show that there is some $\delta > 0$ such that the right-hand side is less than ϵ for all $n \geq n_0$. As $\phi_{X_n}(t) \rightarrow \phi(t)$ for all t , we have $|1 - \phi_{X_n}(t)| \rightarrow |1 - \phi(t)|$ point-wise, hence by the **bounded convergence theorem**,

$$\frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_{X_n}(t)| dt \rightarrow \frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi(t)| dt < \frac{\epsilon}{2}$$

from our assumption. Putting everything together, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\mathbb{P}(|\delta X_n| \geq 2) = \mathbb{P}(|X_n| \geq 2/\delta) \leq \frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_{X_n}(t)| dt < \frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi(t)| dt + \frac{\epsilon}{2} < \epsilon,$$

where the second-last inequality follows from the point-wise convergence of $\frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_{X_n}(t)| dt$ to $\frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi(t)| dt$ being $\epsilon/2$ -close for n large enough, i.e., when $n \geq n_0$ for some n_0 . ■

^aIf this is the case, then we can handle the $n < n_0$ case easily as usual by taking the maximum over all $n < n_0$.

2.6.4 Inversion Theorem

On the other hand, another way to prove **Lévy-Cramér continuity theorem** is to directly calculate the pdf of X , given ϕ_X . It's follows the same vein of the proof of **uniqueness theorem**.

Intuition. In the proof of **uniqueness theorem**, we only obtain a pdf for $X + \sigma Z$. Imposing constraints on ϕ_X and calculate $\mathbb{E}[g(X)]$ in terms of ϕ_X will tell us which condition should we add.

Theorem 2.6.7 (Feller's inversion formula). Let X be a d -dimensional random vector with the **characteristic function** ϕ_X .

(a) If g has a bounded support and $\mathbb{P}(X \in C_g) = 1$, then

$$\mathbb{E}[g(X)] = \lim_{\sigma \searrow 0} \frac{1}{2\pi} \iint g(x) e^{-iux - \sigma^2 u^2/2} du dx.$$

(b) For any $a, b \in C_{F_X}$,

$$F_X(b) - F_X(a) = \lim_{\sigma \searrow 0} \frac{1}{2\pi} \int_a^b \int e^{-iux - \sigma^2 u^2/2} \phi_X(u) du dx.$$

(c) If further, ϕ_X is absolute integrable, then X has a pdf

$$f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \phi_X(u) du.$$

Proof. The proof is based on **uniqueness theorem**.

(a) In the **uniqueness theorem**, $\sigma \searrow 0$ such that $X + \sigma Z \xrightarrow{D} X$, which implies $g(X + \sigma Z) \xrightarrow{D} g(X)$ when $\mathbb{P}(X \in C_g) = 1$. Since now g is also bounded, by the **bounded convergence theorem**,

$$\mathbb{E}[g(X)] = \lim_{\sigma \searrow 0} \mathbb{E}[g(X + \sigma Z)].$$

We now calculate $\mathbb{E}[g(X + \sigma Z)]$. Since $g: \mathbb{R} \rightarrow \mathbb{R}$ has bounded support, the same calculation

from the proof of [uniqueness theorem](#) gives

$$\mathbb{E}[g(X + \sigma Z)] = \lim_{\sigma \searrow 0} \frac{1}{2\pi} \int g(x) \int e^{-ixu - \sigma^2 u^2/2} \phi_X(u) du dx.$$

It remains to change the order of integration, which is justified by [Tonelli's theorem](#) as $\mathbb{E}[|g(X + \sigma Z)|] < \infty$ for all $\sigma > 0$, hence we obtain the result for the first part.

- (b) Given $a, b \in C_{F_X}$, consider $g(x) = \mathbb{1}_{(a,b)}(x)$, which implies $\mathbb{P}(X \in C_g) = 1$ (and trivially g has a bounded support), hence the result above applies.
- (c) Finally, if ϕ_X is absolute integrable, our goal now is to pass the limit $\sigma \searrow 0$ inside the integral for $F_X(b) - F_X(a)$ given $a, b \in C_{F_X}$, i.e., to get

$$F_X(b) - F_X(a) = \frac{1}{2\pi} \int_a^b \int \lim_{\sigma \searrow 0} e^{-ixu - \sigma^2 u^2/2} \phi_X(u) du dx = \frac{1}{2\pi} \int_a^b \int e^{-ixu} \phi_X(u) du dx.$$

Since cdfs are characterized by values in C_{F_X} , i.e., if the above holds for $a, b \in C_{F_X}$, the same holds for $a, b \in \mathbb{R}$, and we're done. To do so, [dominated convergence theorem](#) states that

$$\int_a^b \int \sup_{\sigma > 0} |e^{-ixu - \sigma^2 u^2/2} \phi_X(u)| du dx < \infty$$

is the right condition. We see that the left-hand side is less than

$$\int_a^b \int_{\mathbb{R}} |\phi_X(u)| \sup_{\sigma > 0} |e^{-\sigma^2 u^2/2}| du dx \leq \int_a^b \int_{\mathbb{R}} |\phi_X(u)| du dx$$

which is finite since $\int |\phi_X(u)| du < \infty$. ■

Corollary 2.6.1. Given (X_n) and X such that ϕ_X and ϕ_{X_n} for every n are integrable. If $\phi_{X_n} \xrightarrow{L^1} \phi_X$, i.e., $\int_{\mathbb{R}} |\phi_{X_n}(t) - \phi_X(t)| dt \rightarrow 0$, then $X_n \xrightarrow{TV} X$.

Proof. It suffices to prove that $|f_{X_n}(x) - f_X(x)| \rightarrow 0$, where these pdfs exist due to [Feller's inversion formula \(c\)](#). We see that

$$|f_{X_n}(x) - f_X(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |e^{-iux}| \cdot |\phi_{X_n}(u) - \phi_X(u)| du, \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\phi_{X_n}(u) - \phi_X(u)| du$$

with the assumption the right-hand side goes to 0. ■

2.6.5 Properties of Characteristic Function

Finally, we see the following characterizations of ϕ_X . The first one is that it's uniformly continuous.

Proposition 2.6.1. For any random vector X , ϕ_X is uniformly continuous, i.e.,

$$\lim_{h \rightarrow 0} \sup_{t \in \mathbb{R}^d} |\phi_X(t+h) - \phi_X(t)| = 0.$$

Proof. We see that for any h ,

$$|\phi_X(t+h) - \phi_X(t)| = |\mathbb{E}[e^{i(t+h)X}] - \mathbb{E}[e^{itX}]| \leq \mathbb{E}[|e^{itX}| |e^{ihX} - 1|] \leq \mathbb{E}[|e^{ihX} - 1|],$$

which goes to 0 as $h \rightarrow 0$ since $|e^{ihX} - 1| \leq 2$ with [bounded convergence theorem](#). ■

The next theorem gives us a way to calculate the derivatives of ϕ_X and its connection to moments.

Theorem 2.6.8. If $X \in L^p$ for any $p \in \mathbb{N}$, then the p^{th} derivative of $\phi_X(t)$ is given by

$$\phi_X^{(p)}(t) = \mathbb{E}[(iX)^p e^{itX}]$$

for every t . In particular, $\phi_X^{(p)}(0) = i^p \mathbb{E}[X^p]$ and $\sup_t |\phi_X^{(p)}(t)| \leq \mathbb{E}[|X|^p] < \infty$.

Proof. Consider $p = 1$ since for $p > 1$, it can be shown by induction. It's enough to prove

$$\lim_{h \rightarrow 0} \left| \frac{\phi_X(t+h) - \phi_X(t)}{h} - \mathbb{E}[iX e^{itX}] \right| = 0$$

Writing the ϕ_X explicitly, by Jensen's inequality, for any $h \neq 0$, the left-hand side is

$$\begin{aligned} \left| \frac{\mathbb{E}[e^{i(t+h)X}] - \mathbb{E}[e^{itX}] - \mathbb{E}[ihX e^{itX}]}{h} \right| &\leq \frac{\mathbb{E}[|e^{i(t+h)X} - e^{itX} - ihX e^{itX}|]}{|h|} \\ &= \frac{\mathbb{E}[|e^{itX}| |e^{ihX} - 1 - ihX|]}{|h|} \leq \frac{\mathbb{E}[|e^{ihX} - 1 - ihX|]}{|h|}. \end{aligned}$$

Let $G(h) = e^{ihX}$, then $G'(h) = iX e^{ihX}$, and the right-hand side is equal to

$$\frac{\mathbb{E}[|G(h) - G(0) - G'(0)h|]}{|h|}.$$

Since G is differentiable, $G(h) - G(0) = \int_0^h G'(y) dy$, hence

$$G(h) - G(0) - G'(0)h = \int_0^h G'(y) - G'(0) dy = h \int_0^1 G'(uh) - G'(0) du = h \int_0^1 iX e^{iuhX} - iX du$$

where we let $y = uh$. Plugging in, we have

$$\begin{aligned} \mathbb{E} \left[\frac{|e^{ihX} - 1 - ihX|}{|h|} \right] &\leq \mathbb{E} \left[\int_0^1 |G'(uh) - G'(0)| du \right] \\ &= \mathbb{E} \left[\int_0^1 |iX e^{iuhX} - iX| du \right] \leq \mathbb{E} \left[|X| \int_0^1 |e^{iuhX} - 1| du \right]. \end{aligned}$$

Finally, taking the limit as $h \rightarrow 0$, with the fact that $\mathbb{E}[|X|] < \infty$ and $\int_0^1 |e^{ihuX} - 1| du \leq 2$, we see that $|X| \int_0^1 |e^{ihuX} - 1| du \leq 2|X|$, and the latter is integrable since $\mathbb{E}[|X|] < \infty$, hence **dominated convergence theorem** applies, i.e., we can pass the limit into the expectation,

$$\lim_{h \rightarrow 0} \mathbb{E} \left[|X| \int_0^1 |e^{ihuX} - 1| du \right] = \mathbb{E} \left[|X| \lim_{h \rightarrow 0} \int_0^1 |e^{ihuX} - 1| du \right] = 0$$

since $\lim_{h \rightarrow 0} \int_0^1 |e^{ihuX} - 1| du = 0$, again from the **bounded convergence theorem**. ■

Corollary 2.6.2. If $X \in L^p$ for some $p \in \mathbb{N}$, then $\phi_X^{(p)}$ is uniformly continuous.^a

^aThis is a generalization of [Proposition 2.6.1](#).

Proof. To show that $\phi_X^{(p)}$ is uniformly continuous, we show that $\sup_{t \in \mathbb{R}} |\phi_X^{(p)}(t+h) - \phi_X^{(p)}(t)| \rightarrow 0$ as $h \rightarrow 0$. But this is clear since for any $h \in \mathbb{R}$, with [Theorem 2.6.8](#),

$$\sup_{t \in \mathbb{R}} |\phi_X^{(p)}(t+h) - \phi_X^{(p)}(t)| \leq \mathbb{E}[|X|^p |e^{ihX} - 1|],$$

which goes to 0 as $h \rightarrow 0$ from the **dominated convergence theorem**. ■

Chapter 3

Fundamental Theorems of Probability

Lecture 10: WLLN and CLT, and Applications to Inferences

With the tools we developed, we can now prove the fundamental theorems of probability and see some applications to inferences. 15 Feb. 9:30

3.1 Law of Large Number and Central Limit Theorem

In this section, we will study the [weak law of large number](#) and the [central limit theorem](#).

3.1.1 Weak Law of Large Number

The first result, the [weak law of large number](#), states that the sample mean [converges](#) to the mean.

Theorem 3.1.1 (Khinchin's weak law of large number). Let X and (X_n) be i.i.d. random vectors with $X \in L^1$, i.e., $\mathbb{E}[|X|] < \infty$. Then $\bar{X}_n \xrightarrow{P} \mathbb{E}[X]$.

Proof. Since $c := \mathbb{E}[X]$ is a constant, it suffices to show that $\phi_{\bar{X}_n}(t) \rightarrow \phi_c(t) = e^{itc}$ for all t from [Corollary 2.2.1](#). Firstly, let $\bar{X}_n = S_n/n$, we have

$$\phi_{\bar{X}_n}(t) = \mathbb{E}[e^{itS_n/n}] = \phi_{S_n}(t/n) = \prod_{i=1}^n \phi_{X_i}(t/n) = (\phi(t/n))^n$$

where we let $\phi_{X_i} =: \phi$ since (X_n) are i.i.d. From the fundamental theorem of calculus, with the fact that the first moment of X exists, ϕ is differentiable such that

$$(\phi(t/n))^n = \left(1 + \frac{t}{n} \int_0^1 \phi'(ut/n) du\right)^n.$$

Since $(1 + a_n)^n \rightarrow e^c$ if $na_n \rightarrow c$, it remains to show $\int_0^1 \phi'(ut/n) du \rightarrow ic$. First, $\phi'(t)$ is continuous at 0 from [Corollary 2.6.2](#),^a as $n \rightarrow \infty$

$$\phi'(ut/n) \rightarrow \phi'(0) = i\mathbb{E}[X] = ic.$$

With the fact that $\sup_t |\phi'(t)| \leq \mathbb{E}[|X|]$, the [bounded convergence theorem](#) implies

$$\int_0^1 \phi'(ut/n) du \rightarrow \int_0^1 ic du = ic$$

since we can now pass the limit inside the integral. ■

^aWe see that assuming ϕ is differentiable at 0 such that $\phi'(0) = ic$ is enough.

Although we will not show, but the stronger version holds, i.e., it [converges almost surely](#).

Theorem 3.1.2 (Strong law of large number). Let X and (X_n) be i.i.d. random vectors with $X \in L^1$. Then $\bar{X}_n \xrightarrow{\text{a.s.}} \mathbb{E}[X]$.

3.1.2 Central Limit Theorem

In terms of the distributional result, we need higher-order moments. In particular, if the second moment exists, then we can generalize we have done as in the proof of [Theorem 2.6.8](#).

As previously seen. If g is continuously differentiable at 0, then for x around 0,

$$g(x) = g(0) + g'(0)x + x \int_0^1 g'(ux) - g'(0) du.$$

Note. If in addition, g' is also continuously differentiable at 0, then for x around 0,

$$\begin{aligned} g(x) &= g(0) + g'(0)x + x \int_0^1 \int_0^{ux} g''(y) dy du \\ &= g(0) + g'(0)x + x^2 \int_0^1 \int_0^1 g''(xuv) u dv du. \end{aligned} \quad y = uxv, dy = uxdv$$

We now state the theorem.

Theorem 3.1.3 (Lindeberg-Lévy central limit theorem). Let (X_n) be i.i.d. random variables (i.e., $d = 1$) with $\mathbb{E}[X_i] =: \mu$, $\text{Var}[X_i] =: \sigma^2 < \infty$ for all $1 \leq i \leq n$. Then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1).$$

Proof. Without loss of generality, let $\mu = 0$, $\sigma = 1$. Since $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$, it's enough to show that $\phi_{S_n/\sqrt{n}}(t) \rightarrow e^{-t^2/2}$ for any $t \in \mathbb{R}$ from [Lévy-Cramér continuity theorem](#) and [Equation 2.1](#). Firstly,

$$\phi_{S_n/\sqrt{n}}(t) = \mathbb{E}[e^{itS_n/\sqrt{n}}] = \phi_{S_n}(t/\sqrt{n}) = (\phi(t/\sqrt{n}))^n$$

where we let $\phi_{X_n} =: \phi$ since (X_n) are i.i.d. By applying the above [note](#), we further have

$$\begin{aligned} (\phi(t/\sqrt{n}))^n &= \left(\phi(0) + \phi'(0)\frac{t}{\sqrt{n}} + \frac{t^2}{n} \int_0^1 \int_0^1 u\phi''(uvt/\sqrt{n}) du dv \right)^n \\ &= \left(1 + \frac{t^2}{n} \int_0^1 \int_0^1 u\phi''(uvt/\sqrt{n}) du dv \right)^n \end{aligned}$$

since $\phi(0) = 1$ and $\phi'(0) = i\mu = 0$. It remains to show that the double integral converges to $-1/2$ since it'll imply $(\phi(t/\sqrt{n}))^n \rightarrow e^{-t^2/2}$. We see that as $n \rightarrow \infty$, the integrand

$$u\phi''(uvt/\sqrt{n}) \rightarrow u\phi''(0) = u(i^2\mathbb{E}[X^2]) = -u(\text{Var}[X] + (\mathbb{E}[X])^2) = -u(1 + 0) = -u.$$

Hence, from the [bounded convergence theorem](#),

$$\int_0^1 \int_0^1 u\phi''(ut/\sqrt{n}) du dv \rightarrow \int_0^1 \int_0^1 -u du dv = -\frac{1}{2},$$

which shows the result. ■

Remark. From the [central limit theorem](#), we can indeed deduce the [weak law of large number](#). But since the former requires more conditions, hence [weak law of large number](#) still has its own merit.

3.2 Inference for Population Mean and Variance

We now apply what we have proved to one of the most basic problems, inference for mean and variance.

3.2.1 Population Mean

Firstly, let's consider the applications for mean estimation. Let X, X_1, \dots, X_n be i.i.d. samples such that $\mathbb{E}[X] = \mu < \infty$, $\text{Var}[X] = \sigma^2$. If, also, X_i 's are Gaussian, $\bar{X}_n \sim \mathcal{N}(\mu, \sigma^2/n)$, i.e.,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1),$$

In this case, a natural estimator of μ is \bar{X}_n , and we have the distribution of $\bar{X}_n - \mu$, i.e., we know how our estimator perform in terms of distribution, which can in turn provides a confidence interval.

Intuition. We want to make the distribution, specifically, its variance (denominator at the left-hand side) independent of parameters to get a corresponding confidence interval.

Right now, our confidence interval depends on σ . To solve this, consider replacing σ by the sample standard deviation s_n , then

$$T_n := \frac{\bar{X}_n - \mu}{s_n/\sqrt{n}} \sim t_{n-1} \xrightarrow{\text{TY}} \mathcal{N}(0, 1)$$

as $n \rightarrow \infty$, where T_n follows t -distribution with $n - 1$ degrees of freedom.

Notation. We let $s_n^2 := \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $\hat{\sigma}_n^2 := \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2$.

We see that when X is Gaussian, an asymptotically valid $100(1 - \alpha)\%$ confidence interval for μ is

$$\bar{X} \pm Z_{\alpha/2} \frac{s_n}{\sqrt{n}}.$$

Notation. For any $\alpha > 0$, we define Z_α by $\alpha = \mathbb{P}(Z > Z_\alpha)$, **not** $\alpha = \mathbb{P}(Z < Z_\alpha)$.

The first question we will address is “what if X_i 's are not Gaussian, and can we replace s_n by $\hat{\sigma}_n$.”

Proposition 3.2.1. If $X \in L^2$, then both $\hat{\sigma}_n^2$ and s_n^2 are **consistent estimators** of σ^2 . Furthermore, $T_n \xrightarrow{D} \mathcal{N}(0, 1)$, and the same holds if s_n is replaced by $\hat{\sigma}_n$ in the definition of T_n .

Proof. Indeed, by letting $Y_i := X_i - \mu$ for all i (and also $Y = X - \mu$), as $n \rightarrow \infty$,

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y}_n)^2 = \frac{1}{n} \sum_{i=1}^n Y_i^2 - (\bar{Y}_n)^2 \xrightarrow{P} \sigma^2 + 0$$

since $\frac{1}{n} \sum_{i=1}^n Y_i^2 \xrightarrow{P} \mathbb{E}[Y^2] = \text{Var}[X] = \sigma^2 < \infty$ as $X \in L^2$, and $(\bar{Y}_n)^2 \xrightarrow{P} (\mathbb{E}[Y])^2 = 0$, both from **weak law of large number**. This implies that s_n^2 is also a **consistent estimator** of σ^2 since

$$s_n^2 = \frac{n}{n-1} \hat{\sigma}_n^2 \xrightarrow{P} 1 \cdot \sigma^2 = \sigma^2,$$

again from **Slutsky's theorem**. The distributional result follows directly from **central limit theorem** for $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1)$ and **Slutsky's theorem**. ■

Proposition 3.2.1 says that for mean estimation, even if the data is not Gaussian, we're fine.

Corollary 3.2.1. If $X \in L^2$, then $\bar{X}_n \pm Z_{\alpha/2} s_n / \sqrt{n}$ and $\bar{X}_n \pm Z_{\alpha/2} \hat{\sigma}_n / \sqrt{n}$ are both asymptotically valid $100(1 - \alpha)\%$ confidence intervals for μ .

3.2.2 Population Variance

Next, let's consider variance estimation and further assume that $\sigma^2 < \infty$. Again, let X, X_1, \dots, X_n be i.i.d. samples. If they are Gaussian,

$$(n-1) \frac{s_n^2}{\sigma^2} \stackrel{D}{=} \sum_{i=1}^{n-1} Z_i^2 \sim \chi_{n-1}^2$$

where $(Z_{n-1}) \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. Firstly, since $\mathbb{E}[Z_i^2] = \text{Var}[Z_i] + (\mathbb{E}[Z_i])^2 = 1$, and $\text{Var}[Z_i^2] = \mathbb{E}[Z_i^4] - (\mathbb{E}[Z_i^2])^2 = 3 - 1 = 2$, standardizing, from the normal approximation to the chi-square distribution,

$$\frac{(n-1) \frac{s_n^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} \stackrel{D}{=} \frac{\sum_{i=1}^{n-1} Z_i^2 - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{D} \mathcal{N}(0, 1),$$

i.e., as $n \rightarrow \infty$,

$$\sqrt{n-1} \left(\frac{s_n^2}{\sigma^2} - 1 \right) \xrightarrow{D} \mathcal{N}(0, 2) \Leftrightarrow \sqrt{n} \left(\frac{s_n^2}{\sigma^2} - 1 \right) \xrightarrow{D} \mathcal{N}(0, 2) \Leftrightarrow \sqrt{n}(s_n^2 - \sigma^2) \xrightarrow{D} \mathcal{N}(0, 2\sigma^4),$$

and an asymptotically valid $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\frac{s_n^2}{1 \pm Z_{\alpha/2} \sqrt{2/n}}.$$

Let's again ask what will happen when X_i 's are not Gaussian anymore.

Proposition 3.2.2. If $X \in L^2$, then the following hold when $\hat{\sigma}_n^2$ is replaced by s_n^2 . Firstly,

$$\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i^2 - \sigma^2) + o_p(1).$$

Moreover, if $X \in L^4$ and $\mathbb{E}[(X - \mu)/\sigma]^4 > 1$, then $\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{D} \mathcal{N}(0, \mathbb{E}[(X - \mu)^4] - \sigma^4)$.

Proof. We see that from the same calculation as above, with $Y_i := X_i - \mu$ (and also $Y = X - \mu$),

$$\begin{aligned} \hat{\sigma}_n^2 &= \frac{1}{n} \sum_{i=1}^n Y_i^2 - \bar{Y}_n^2 \Rightarrow \hat{\sigma}_n^2 - \sigma^2 = \frac{1}{n} \sum_{i=1}^n (Y_i^2 - \sigma^2) - \bar{Y}_n^2 \\ &\Rightarrow \sqrt{n}(\hat{\sigma}_n^2 - \sigma^2) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i^2 - \sigma^2) - \frac{(\sqrt{n}\bar{Y}_n)^2}{\sqrt{n}}. \end{aligned}$$

As $n \rightarrow \infty$, since $(\sqrt{n}\bar{Y}_n)^2$ converges in distribution from the central limit theorem for $\sqrt{n}\bar{Y}_n$ (as $X \in L^2$) and continuous mapping theorem, $(\sqrt{n}\bar{Y}_n)^2 = O_p(1)$ from Proposition 2.4.2, hence

$$\frac{(\sqrt{n}\bar{Y}_n)^2}{\sqrt{n}} = o(1)O_p(1) = o_p(1),$$

proving the first claim. Now, if further $\text{Var}[Y_i^2] < \infty$ from $X \in L^4$, central limit theorem gives

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i^2 - \sigma^2) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i^2 - \mathbb{E}[Y_i^2]) \xrightarrow{D} \mathcal{N}(0, \text{Var}[Y_i^2]),$$

implying $\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{D} \mathcal{N}(0, \text{Var}[Y^2])$ from the first claim and Slutsky's theorem, where

$$\text{Var}[Y^2] = \mathbb{E}[(X - \mu)^4] - (\mathbb{E}[(X - \mu)^2])^2 = \sigma^4 \mathbb{E} \left[\left(\frac{X - \mu}{\sigma} \right)^4 \right] - \sigma^4 = \sigma^4 \left(\mathbb{E} \left[\left(\frac{X - \mu}{\sigma} \right)^4 \right] - 1 \right),$$

which proves the second claim. Finally, we note that

$$\sqrt{n}(\hat{\sigma}_n^2 - s_n^2) = \frac{\sqrt{n}}{n-1} \hat{\sigma}_n^2 \xrightarrow{P} 0 \cdot \sigma^2 = 0,$$

hence the same results above hold for replacing $\hat{\sigma}_n^2$ by s_n^2 from [Slutsky's theorem](#). ■

The quantity (and a related one) in our assumption deserves a special name.

Definition 3.2.1 (Kurtosis). The *Kurtosis* of a random variable X is defined as $\mathbb{E}[(X - \mu)/\sigma]^4$.

Definition 3.2.2 (Skewness). The *skewness* of a random variable X is defined as $\mathbb{E}[(X - \mu)/\sigma]^3$.

Example (Kurtosis for Gaussian). The [Kurtosis](#) of the standard Gaussian is 3.

Let $Z = (X - \mu)/\sigma$, we note that [Proposition 3.2.2](#) requires $\mathbb{E}[Z^4] > 1$. However, from Jensen's inequality, $\mathbb{E}[Z^4] \geq (\mathbb{E}[Z^2])^2 \geq 1$, hence indeed, the assumption might not be true in general.

Example. If $\mathbb{E}[Z^4] = 1$,

$$\text{Var}[Y^2] = 0 \Leftrightarrow \mathbb{P}(Y^2 = \mathbb{E}[Y^2]) = 1 \Leftrightarrow \mathbb{P}(Y = \pm\sigma) = 1 \Leftrightarrow \mathbb{P}(X = \mu \pm \sigma) = 1,$$

i.e., the violation might happen for X being concentrated on two points.

The takeaway is when X is not a normal (or when the [Kurtosis](#) of X is different from 3), then the distribution of $\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2)$ is different. Specifically, if the [Kurtosis](#) exists and is not equal to 1, then an asymptotically valid $100(1 - \alpha)\%$ confidence interval for σ^2 is

$$\frac{\hat{\sigma}_n^2}{1 \pm Z_{\alpha/2} \sqrt{(\mathbb{E}[(X - \mu)/\sigma]^4 - 1)/n}}.$$

However, if we don't know the [Kurtosis](#) of X , we can't say anything about the confidence interval.

Intuition. By [Slutsky's theorem](#), if we have a [consistent estimator](#) of the [Kurtosis](#), we can then use it instead and get a desired asymptotic confidence interval.

Lecture 11: Sample Standardized Central Moments

Following the intuition, let's find such [consistent](#) estimators. Let $Y := X - \mu = X - \mathbb{E}[X]$ (and also $Y_i = X_i - \mu$ as usual), $\mu_k := \mathbb{E}[Y^k] = \mathbb{E}[(X - \mu)^k]$ for all $k \geq 2$, and finally $\tilde{\mu}_k = \mu_k/\sigma^k = \mathbb{E}[(X - \mu)^k/\sigma^k]$. 20 Feb. 9:30

As previously seen. In this notation, [Proposition 3.2.2](#) gives $\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2) \rightarrow \mathcal{N}(0, (\tilde{\mu}_4 - 1)\sigma^4)$, i.e.,

$$\frac{\sqrt{n}}{\sqrt{\tilde{\mu}_4 - 1}} \left(\frac{\hat{\sigma}_n^2}{\sigma^2} - 1 \right) \rightarrow \mathcal{N}(0, 1).$$

The task is then the following.

Problem. How to estimate $\tilde{\mu}_4$, or more generally, how to estimate $\tilde{\mu}_k$ [consistently](#)?

Answer. Consider the k^{th} *sample central moment*

$$M_k := \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^k.$$

Let's also define the k^{th} *sample standardized central moment* as $\tilde{M}_k := M_k/\hat{\sigma}_n^k$. ⊛

The above essentially is motivated from the following observation.

Intuition. If we know μ , then $\frac{1}{n} \sum_{i=1}^n (X_i - \mu)^k \xrightarrow{P} \mu_k$ by the [weak law of large number](#). However, since we don't know μ , we need to use \bar{X}_n .

We now show that this still yields a [consistent](#) estimator.

Proposition 3.2.3. If $X \in L^k$ for $k > 2$, then $M_k \xrightarrow{P} \mathbb{E}[Y^k] = \mu_k$. Same for \widetilde{M}_k and $\widetilde{\mu}_k$.

Proof. Let's denote $\bar{X}_n =: \bar{X}$ and $\bar{Y}_n =: \bar{Y}$. Then

$$M_k = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^k = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^k = \frac{1}{n} \sum_{i=1}^n \sum_{\ell=0}^k \binom{k}{\ell} Y_i^\ell (-\bar{Y})^{k-\ell} = \sum_{\ell=0}^k \binom{k}{\ell} (-\bar{Y})^{k-\ell} \frac{1}{n} \sum_{i=1}^n Y_i^\ell.$$

Let $\frac{1}{n} \sum_{i=1}^n Y_i^\ell =: \bar{Y}^\ell$, then we further get

$$M_k = \sum_{\ell=0}^k \binom{k}{\ell} (-\bar{Y})^{k-\ell} \bar{Y}^\ell = \bar{Y}^k + \sum_{\ell=0}^{k-1} \binom{k}{\ell} (-\bar{Y})^{k-\ell} \bar{Y}^\ell. \quad (3.1)$$

By the [weak law of large number](#), $\bar{Y}^k \xrightarrow{P} \mathbb{E}[Y^k] = \mu_k$ and $(-\bar{Y})^{k-\ell} \xrightarrow{P} 0$ for $\ell < k$ from $-\bar{Y} \xrightarrow{P} 0$ (by [weak law of large number](#)) and [continuous mapping theorem](#), hence $M_k \xrightarrow{P} \mu_k$ by [Slutsky's theorem](#). The [consistency](#) of $\hat{\sigma}_n$ implies $\widetilde{M}_k \xrightarrow{P} \widetilde{\mu}_k$ clearly. ■

This gives the following useful confidence interval for σ^2 .

Corollary 3.2.2. If the [Kurtosis](#) of X exists and is not equal to 1, then an asymptotically valid $100(1 - \alpha)\%$ confidence interval for σ^2 is

$$\frac{\hat{\sigma}_n^2}{1 \pm Z_{\alpha/2} \sqrt{(\widetilde{M}_4 - 1)/n}}.$$

Proof. This directly follows from [Proposition 3.2.2](#) and [Proposition 3.2.3](#). ■

3.3 Testing for Normality

As we will soon see, it's natural to extend what we just discussed to the problem of hypothesis testing, and specifically, testing for normality.

3.3.1 Asymptotic Distribution of Sample Central Moments

It turns out that asking for the asymptotic distribution of M_k , i.e., $\sqrt{n}(M_k - \mu_k)$ is quite valuable, although the motivation is not so clear right now. Anyway, we have the following.

Theorem 3.3.1. If $X \in L^k$ for some $k > 2$, then

$$\sqrt{n}(M_k - \mu_k) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i^k - \mu_k - k\mu_{k-1}Y_i) + o_p(1).$$

Moreover, if $X \in L^{2k}$ and $v_k > 0$ where

$$v_k := \text{Var}[Y^k - k\mu_{k-1}Y] = \mu_{2k} - \mu_k^2 + k^2\mu_{k-1}^2\sigma^2 - 2k\mu_{k-1}\mu_{k+1},$$

then $\sqrt{n}(M_k - \mu_k) \xrightarrow{D} \mathcal{N}(0, v_k)$.

Proof. Firstly, if $X \in L^k$, from [Equation 3.1](#),

$$\sqrt{n}(M_k - \mu_k) = \sqrt{n}(\overline{Y^k} - \mu_k) + \sum_{\ell=0}^{k-1} \binom{k}{\ell} (-\overline{Y})^{k-\ell} \overline{Y^\ell} \sqrt{n} = \sqrt{n}(\overline{Y^k} - \mu_k) + \sum_{\ell=0}^{k-1} \binom{k}{\ell} \frac{(-\overline{Y} \sqrt{n})^{k-\ell}}{\sqrt{n}^{k-\ell-1}} \overline{Y^\ell}.$$

We see that from [Proposition 2.4.2](#), for $\ell < k-1$,

- $(-\overline{Y} \sqrt{n})^{k-\ell} = O_p(1)$ from [central limit theorem](#) and [continuous mapping theorem](#);
- $\overline{Y^\ell} = O_p(1)$ since $\overline{Y^\ell} \xrightarrow{p} \mathbb{E}[Y^\ell]$ from [weak law of large number](#);
- $1/\sqrt{n}^{k-\ell-1} = o(1)$.

Combining, every term in the summation is $O_p(1)O_p(1)o(1) = o_p(1)$ except for $\ell = k-1$, hence

$$\begin{aligned} \sqrt{n}(M_k - \mu_k) &= \sqrt{n}(\overline{Y^k} - \mu_k) - \binom{k}{k-1} \overline{Y^{k-1}} \sqrt{n} \overline{Y} + \sum_{\ell=0}^{k-2} \binom{k}{\ell} o_p(1) \\ &= \sqrt{n}(\overline{Y^k} - \mu_k) - k \overline{Y^{k-1}} \sqrt{n} \overline{Y} + o_p(1) \end{aligned}$$

while $\sqrt{n} \overline{Y} = O_p(1)$, $\overline{Y^{k-1}}$ is not $O_p(1)$. By replacing $\overline{Y^{k-1}}$ by $\overline{Y^{k-1}} - \mu_{k-1} + \mu_{k-1}$,

$$\begin{aligned} &= \sqrt{n}(\overline{Y^k} - \mu_k) - k \left(\overline{Y^{k-1}} - \mu_{k-1} \right) \sqrt{n} \overline{Y} - k \mu_{k-1} \sqrt{n} \overline{Y} + o_p(1) \\ &= \sqrt{n}(\overline{Y^k} - \mu_k) - k \mu_{k-1} \sqrt{n} \overline{Y} + o_p(1) \end{aligned}$$

since $\overline{Y^{k-1}} - \mu_{k-1} \xrightarrow{p} 0$ from the [weak law of large number](#), finally,

$$\begin{aligned} &= \sqrt{n} \frac{1}{n} \sum_{i=1}^n (Y_i^k - \mu_k) - k \mu_{k-1} \sqrt{n} \frac{1}{n} \sum_{i=1}^n Y_i + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i^k - \mu_k - k \mu_{k-1} Y_i) + o_p(1), \end{aligned}$$

proving the first claim. Moreover, since $Y_i^k - \mu_k - k \mu_{k-1} Y_i$'s are i.i.d., it [converges in distribution](#) to $\mathcal{N}(0, v_k) = \mathcal{N}(0, \text{Var}[Y^k - \mu_k - k \mu_k Y])$ by [central limit theorem](#) and [Slutsky's theorem](#), where

$$\begin{aligned} v_k &:= \text{Var}[Y^k - \mu_k - k \mu_{k-1} Y] = \text{Var}[Y^k - k \mu_{k-1} Y] \\ &= \text{Var}[Y^k] + k^2 \mu_{k-1}^2 \text{Var}[Y] - 2k \mu_{k-1} \text{Cov}[Y, Y^k] \\ &= \mu_{2k} - \mu_k^2 + k^2 \mu_{k-1}^2 \sigma^2 - 2k \mu_{k-1} \mu_{k+1} \end{aligned}$$

since $\text{Cov}[Y, Y^k] = \mathbb{E}[Y \cdot Y^k] - \mathbb{E}[Y] \mathbb{E}[Y^k] = \mathbb{E}[Y^{k+1}] = \mu_{k+1}$ and $\mu_{2k} < \infty$ from $X \in L^{2k}$. ■

Note. [Theorem 3.3.1](#) doesn't give an asymptotic distribution of $\widetilde{M}_k = M_k / \hat{\sigma}_n^k$ since it requires the joint distribution of $\hat{\sigma}_n^k$ and M_k .

However, it turns out that when k is odd and the distribution is symmetric, [Theorem 3.3.1](#) does give an asymptotic distribution for \widetilde{M}_k .

3.3.2 Testing Normality with Odd Moments

To motivate why we want to have an asymptotic distribution for \widetilde{M}_k , consider the problem of testing normality, i.e., let $H_0: X \sim \mathcal{N}(\mu, \sigma^2)$ for some μ, σ^2 .

Intuition. The idea is that to reject H_0 if $|\widetilde{M}_k| = |M_k / \hat{\sigma}_n^k|$ deviates significantly.

In this regard, [Theorem 3.3.1](#) is not enough since it's only for M_k , but we really need \widetilde{M}_k .

Problem. What is the asymptotic distribution of $\widetilde{M}_k = M_k / \hat{\sigma}_n^k$?

First observe that if X_i 's are Gaussian, as Gaussian is symmetric, $\mu_k = 0$ (and hence $\tilde{\mu}_k = 0$) for all odd k . It turns out that this property allows us to bypass the joint if we focus on odd k . Formally, suppose k is odd, and $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mu_k = 0$, hence [Theorem 3.3.1](#) gives

$$\sqrt{n}(M_k - \mu_k) = \sqrt{n}M_k \xrightarrow{D} \mathcal{N}(0, \text{Var}[Y^k - k\mu_{k-1}Y]) \Rightarrow \sqrt{n}\frac{M_k}{\sigma^k} \xrightarrow{D} \mathcal{N}(0, \sigma^{-2k} \text{Var}[Y^k - k\mu_{k-1}Y]).$$

Then, by [Slutsky's theorem](#), $\sqrt{n}M_k/\hat{\sigma}_n^k$ also [converges](#) to this normal. Since all we use is the fact that $\mu_k = 0$ for odd k and [Theorem 3.3.1](#), let's write this general result as a corollary.

Corollary 3.3.1. If $X \in L^{2k}$ for some odd $k > 2$ such that $\mu_k = 0$ and $\tilde{v}_k := v_k/\sigma^{2k} > 0$, then $\sqrt{n}M_k/\hat{\sigma}_n^k = \sqrt{n}\tilde{M}_k \xrightarrow{D} \mathcal{N}(0, \tilde{v}_k)$.

Remark. We get the asymptotic distribution of $M_k/\hat{\sigma}_n^k$ without computing the joint of M_k and $\hat{\sigma}_n^k$.

Example. Consider $k = 3$, under $H_0: X \sim \mathcal{N}(\mu, \sigma^2)$,

$$\sqrt{\frac{n}{6}} \frac{M_3}{\hat{\sigma}_n^3} \xrightarrow{D} \mathcal{N}(0, 1).$$

Proof. From the symmetry of normal distribution, [Corollary 3.3.1](#) gives

$$\sqrt{n}\frac{M_3}{\hat{\sigma}_n^3} \xrightarrow{D} \mathcal{N}(0, \sigma^{-6} \text{Var}[Y^3 - 3\sigma^2 Y]) = \mathcal{N}(0, \sigma^{-6} (\text{Var}[Y^3] + 9\sigma^4 \sigma^2 - 6\sigma^2 \mathbb{E}[Y^4]))$$

where $\mu_2 = \sigma^2$ and $\text{Cov}[Y^3, Y] = \mathbb{E}[Y^4] - \mathbb{E}[Y]\mathbb{E}[Y^3] = \mathbb{E}[Y^4]$. Hence, by plugging $\text{Var}[Y^3] = \mu_{2 \times 3} = \mu_6$,^a the variance of the normal is further equal to

$$\frac{\mu_6 + 9\sigma^6 - 6\sigma^2 \mu_4}{\sigma^6} = \tilde{\mu}_6 + 9 - 6\tilde{\mu}_4 = 15 + 9 - 6 \times 3 = 6,$$

which provides the result. *

^aMore generally, $\text{Var}[Y^k] = \mathbb{E}[Y^{2k}] - (\mathbb{E}[Y^k])^2 = \mathbb{E}[Y^{2k}] = \mu_{2k}$ since $(\mathbb{E}[Y^k])^2 = \mu_k^2 = 0$.

For even k or odd k but $\mu_k \neq 0$, we really need to work out the joint. Since we know the asymptotic distribution of both M_k and $\hat{\sigma}_n^2$, the joint can be obtained by the [delta method](#) with $g(M_k, \hat{\sigma}_n^2) = M_k/\hat{\sigma}_n^k = \tilde{M}_k$ and the “multivariate” version of [central limit theorem](#).

3.3.3 Multivariate Central Limit Theorem

As mentioned above, we now prove the [multivariate central limit theorem](#), i.e., the high dimensional generalization of [central limit theorem](#). We first need the following tool.

Theorem 3.3.2 (Cramér-Wold device). Let (X_n) be a sequence of random vectors and X be a random vector in \mathbb{R}^d . Then $X_n \xrightarrow{D} X$ if and only if $t \cdot X_n \xrightarrow{D} t \cdot X$ for every $t \in \mathbb{R}^d$.

Proof. The forward direction is clear from [continuous mapping theorem](#) for the linear functional induced from t . For the backward direction, assume that $t \cdot X_n \xrightarrow{D} t \cdot X$. Then

$$\phi_{X_n}(t) = \mathbb{E}[e^{it \cdot X_n}] = \phi_{t \cdot X_n}(1) \rightarrow \phi_{t \cdot X}(1) = \mathbb{E}[e^{it \cdot X}] = \phi_X(t),$$

which implies $X_n \xrightarrow{D} X$ by the [Lévy-Cramér continuity theorem](#). ■

Remark. Proving $X_n \xrightarrow{D} X$ reduces to proving something in the scalar case.

Lecture 12: Asymptotic Joint Distribution by Multivariate CLT

Theorem 3.3.3 (Multivariate central limit theorem). Let (X_n) be i.i.d. random vectors in \mathbb{R}^d with $\mathbb{E}[X_i] = \mu \in \mathbb{R}^d$, $\text{Var}[X_i] = \Sigma \in \mathbb{R}^{d \times d}$ for all $1 \leq i \leq n$. Then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - \mu) \xrightarrow{D} \mathcal{N}(0, \Sigma).$$

Proof. Set $\mu = 0$, and from [Cramér-Wold device](#), it suffices to show that for any $t \in \mathbb{R}^d$,

$$t \cdot \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \right) \xrightarrow{D} t \cdot Z \sim \mathcal{N}(0, t^\top \Sigma t)$$

where $Z \sim \mathcal{N}(0, \Sigma)$. Indeed, since from the [univariate central limit theorem](#), the left-hand side converges to $\mathcal{N}(0, \text{Var}[t \cdot X_i])$, with $\text{Var}[t \cdot X] = t^\top \text{Var}[X_i] t = t^\top \Sigma t = \text{Var}[t \cdot Z]$, we're done. ■

3.3.4 Testing Normality with General Moments

With [multivariate central limit theorem](#), we can now generalize [Corollary 3.3.1](#), i.e., finding the asymptotic distribution of $\tilde{M}_k = M_k / \hat{\sigma}_n^k$ for general k . Recall the setup, where we let (X_n) and X be i.i.d. random variable, $Y_i = X_i - \mu$ (and $Y = X - \mu$), $\sigma^2 = \text{Var}[X]$, $\mu_k = \mathbb{E}[Y^k]$, and $\tilde{\mu}_k = \mu_k / \sigma^k$. Let's start with $k = 1$, i.e., compute the asymptotic law of $\bar{X}_n / \hat{\sigma}_n$. In this case, we have proved the following.

As previously seen. From [Proposition 3.2.1](#) and [Proposition 3.2.2](#),

- $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} \mathcal{N}(0, \sigma^2)$ from $\sqrt{n}(\bar{X}_n - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i$, assuming $X \in L^2$;
- $\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{D} \mathcal{N}(0, \mu_4 - \sigma^4)$ from $\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i^2 - \sigma^2) + o_p(1)$, assuming $X \in L^4$ and $\tilde{\mu}_4 > 1$.^a

^aThe latter representation result needs only the assumption of $X \in L^2$.

This together with [multivariate central limit theorem](#) and [Slutsky's theorem](#) give the following.

Proposition 3.3.1. If $X \in L^2$,

$$\sqrt{n} \left(\begin{pmatrix} \bar{X}_n \\ \hat{\sigma}_n^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} Y_i \\ Y_i^2 - \sigma^2 \end{pmatrix} + o_p(1).$$

Moreover, if $X \in L^4$ and $\tilde{\mu}_4 = \mu_4 / \sigma^4 > 1$, then the above converge in distribution to $\mathcal{N}(0, \Sigma)$ where

$$\Sigma = \text{Var} \left[\begin{pmatrix} Y \\ Y^2 \end{pmatrix} \right] = \begin{pmatrix} \text{Var}[Y] & \text{Cov}[Y, Y^2] \\ \text{Cov}[Y, Y^2] & \text{Var}[Y^2] \end{pmatrix} = \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}.$$

Remark (Asymptotically independent). We know that when X is Gaussian, \bar{X}_n and s_n^2 are independent. Related back to [Corollary 3.3.1](#), when their skewness is 0, \bar{X}_n and $\hat{\sigma}_n^2$ (or s_n^2) are asymptotically independent, which is again confirmed by [Proposition 3.3.1](#) here.

[Proposition 3.3.1](#) gives an asymptotic distribution of \bar{X}_n and $\hat{\sigma}_n^2$, but not $\hat{\sigma}_n$. This is fine since we can further apply the [delta method](#) with $g(\bar{X}_n, \hat{\sigma}_n^2) := \bar{X}_n / \hat{\sigma}_n$ to get the distribution of $\bar{X}_n / \hat{\sigma}_n$. However, let's leave the application of the [delta method](#) to the general k . We note the following.

Note. The actual characterization of \bar{X}_n and $\hat{\sigma}_n^2$ right before applying [central limit theorem](#) is much more useful than the final asymptotic distributions.

Next, we compute the asymptotic law of $\tilde{M}_k = M_k / \hat{\sigma}_n^k$ for general $k > 2$. Following a similar calculation, for $\hat{\sigma}_n^k$, we can again use the result from [Proposition 3.2.2](#) for $\hat{\sigma}_n^2$.

As previously seen. From [Theorem 3.3.1](#), if $X \in L^k$,

$$\sqrt{n}(M_k - \mu_k) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i^k - \mu_k - k\mu_{k-1}Y_i) + o_p(1),$$

and $\sqrt{n}(M_k - \mu_k) \rightarrow \mathcal{N}(0, \text{Var}[Y^k - k\mu_{k-1}Y])$ if $X \in L^{2k}$ and the variance is strictly positive.

This implies that for $X \in L^k$ for any $k > 2$,¹

$$Y := \sqrt{n} \left(\begin{pmatrix} \hat{\sigma}_n^2 \\ M_k \end{pmatrix} - \begin{pmatrix} \sigma^2 \\ \mu_k \end{pmatrix} \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} Y_i^2 - \sigma^2 \\ Y_i^k - \mu_k - k\mu_{k-1}Y_i \end{pmatrix} + o_p(1), \quad (3.2)$$

which converges to $\mathcal{N}(0, \Sigma)$ from [multivariate central limit theorem](#) when $X \in L^{2k}$, where

$$\Sigma = \begin{pmatrix} \text{Var}[Y^2] & \text{Cov}[Y^2, Y^k - k\mu_{k-1}Y] \\ \text{Cov}[Y^2, Y^k - k\mu_{k-1}Y] & \text{Var}[Y^k - k\mu_{k-1}Y] \end{pmatrix}.$$

Remark. In general k , if $\mu_\ell = 0$ for all odd ℓ , then M_k and $\hat{\sigma}_n^2$ are asymptotically independent. This is why we get a simplification for odd case in [Corollary 3.3.1](#).

Putting everything together formally, we have the following result for general k .

Theorem 3.3.4. Let $X \in L^k$ for some $k > 2$. Then for $Z = (X - \mu)/\sigma = Y/\sigma$,

$$\sqrt{n}(\tilde{M}_k - \tilde{\mu}_k) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(-\frac{k}{2} \tilde{\mu}_k (Z_i^2 - 1) + (Z_i^k - \tilde{\mu}_k - k\tilde{\mu}_{k-1}Z_i) \right) + o_p(1).$$

Moreover, if $X \in L^{2k}$ and $\tilde{v}_k := \text{Var} \left[-\frac{k}{2} \tilde{\mu}_k Z^2 + Z^k - k\tilde{\mu}_{k-1}Z \right] > 0$, then $\sqrt{n}(\tilde{M}_k - \tilde{\mu}_k) \xrightarrow{D} \mathcal{N}(0, \tilde{v}_k)$.

Proof. Since [Proposition 3.2.2](#) is for $\hat{\sigma}_n^2$ but not $\hat{\sigma}_n^k$, we need to use [delta method](#) by considering $\tilde{M}_k = M_k/\hat{\sigma}_n^k = g(\hat{\sigma}_n^2, M_k)$ where $g(x, y) := y/x^{k/2}$ for $x > 0$, $y \in \mathbb{R}$. We see that

$$\nabla g(\sigma^2, \mu_k) = \begin{pmatrix} -\frac{k}{2} \mu_k \sigma^{-k-2} & \sigma^{-k} \end{pmatrix} = \begin{pmatrix} -\frac{k}{2} \tilde{\mu}_k \sigma^{-2} & \sigma^{-k} \end{pmatrix}$$

since $\tilde{\mu}_k = \mu_k/\sigma^k$, $\partial g/\partial x = -k y x^{-k/2-1}/2$, and $\partial g/\partial y = x^{-k/2}$. From [delta method](#) and [Equation 3.2](#) with $X \in L^k$, with $\tilde{\mu}_k = g(\sigma^2, \mu_k)$, we get $\sqrt{n}(g(\hat{\sigma}_n^2, M_k) - g(\sigma^2, \mu_k)) \xrightarrow{D} \nabla g Y$, i.e.,

$$\begin{aligned} \sqrt{n}(\tilde{M}_k - \tilde{\mu}_k) &= \nabla g(\sigma^2, \mu_k) \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} Y_i^2 - \sigma^2 \\ Y_i^k - \mu_k - k\mu_{k-1}Y_i \end{pmatrix} + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(-\frac{k}{2} \tilde{\mu}_k \frac{1}{\sigma^2} (Y_i^2 - \sigma^2) + \frac{1}{\sigma^k} (Y_i^k - \mu_k - k\mu_{k-1}Y_i) \right) + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(-\frac{k}{2} \tilde{\mu}_k (Z_i^2 - 1) + (Z_i^k - \tilde{\mu}_k - k\tilde{\mu}_{k-1}Z_i) \right) + o_p(1) \end{aligned}$$

by letting $Z_i := (X_i - \mu)/\sigma = Y_i/\sigma$, proving the first claim. Then by the [multivariate central limit theorem](#) and [Slutsky's theorem](#), the above further converges in distribution to $\mathcal{N}(0, \tilde{v}_k)$ when

$$\tilde{v}_k := \text{Var} \left[-\frac{k}{2} \tilde{\mu}_k (Z^2 - 1) + (Z^k - \tilde{\mu}_k - k\tilde{\mu}_{k-1}Z) \right] = \text{Var} \left[-\frac{k}{2} \tilde{\mu}_k Z^2 + Z^k - k\tilde{\mu}_{k-1}Z \right] > 0,$$

as we assumed. ■

Compared to [Corollary 3.3.1](#) for odd k and $\mu_k = 0$, there we only get an asymptotic distribution, not an explicit decomposition. With this explicit formula, we can do more. Consider the following example.

¹This “Y” will be used in the [delta method](#) later, although this is not exact since Y should be the random vector corresponding the asymptotic distribution. But this is fine in the end from [Slutsky's theorem](#).

Example. Consider using both \widetilde{M}_3 and \widetilde{M}_4 to test $H_0: X \sim \mathcal{N}$. We see that under H_0 ,

$$\left(\sqrt{\frac{n}{\widetilde{v}_3}}\widetilde{M}_3\right)^2 + \left(\sqrt{\frac{n}{\widetilde{v}_4}}(\widetilde{M}_4 - \widetilde{\mu}_4)\right)^2 \xrightarrow{D} \chi_2^2.$$

Proof. One can write down $\sqrt{n}(\widetilde{M}_\ell - \widetilde{\mu}_\ell)$ for even ℓ , and also $\sqrt{n}(\widetilde{M}_k - \widetilde{\mu}_k) = \sqrt{n}\widetilde{M}_k$ for odd k , and see that while they both converge to $\mathcal{N}(0, 1)$, their covariance is 0, i.e., asymptotically independent, so the square of them add up to χ_2^2 . \circledast

Generalizing the above example, for any X with $k > 1$ odd and $\ell > 2$ even, such that every odd central moments vanish with $\widetilde{v}_k, \widetilde{v}_\ell < \infty$,

$$\frac{n}{\widetilde{v}_k}\widetilde{M}_k^2 + \frac{n}{\widetilde{v}_\ell}(\widetilde{M}_\ell - \widetilde{\mu}_\ell)^2 \xrightarrow{D} \chi_2^2.$$

3.4 A Quick Detour

We take a slight detour discussing how to asymptotically compare two estimators and how to make the confidence interval (when it depends on too many estimators) more stable.

3.4.1 Asymptotic Relative Efficiency

First, consider the following illustrative example.

Example. Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Pois}(\theta)$. To estimate θ , as $\theta = \mathbb{E}[X] = \text{Var}[X]$, two natural estimators are \overline{X}_n and $\hat{\sigma}_n^2$. To compare them, we see that

- $\sqrt{n}(\overline{X}_n - \theta) \xrightarrow{D} \mathcal{N}(0, \sigma^2)$;
- $\sqrt{n}(\hat{\sigma}_n^2 - \theta) \xrightarrow{D} \mathcal{N}(0, \mu_4 - \sigma^4)$.

As $\sigma^2 = \theta$ and $\mu_4 = 3\theta^2 + \theta$, we see that \overline{X}_n is better since its variance is smaller.

To further quantify how much better is it, we ask how many data we need to we get a similar precision: consider the problem of estimating a scalar parameter θ such that for two estimators T_n^1 and T_n^2 ,

$$\sqrt{n}(T_n^i - \theta) \xrightarrow{D} \sigma_i(\theta)Z \sim \mathcal{N}(0, \sigma_i^2(\theta))$$

Our goal is to find a single number which compares these two estimators. Firstly, for n large enough,

$$\mathbb{P}(\theta \in I_n^i) := \mathbb{P}\left(\theta \in T_n^i \pm Z_{\alpha/2} \frac{\sigma_i(\theta)}{\sqrt{n}}\right) \cong 1 - \alpha$$

where $I_n^i := T_n^i \pm Z_{\alpha/2} \sigma_i(\theta)/\sqrt{n}$. Let $n_i(\gamma)$ be the value of n such that $|I_n^i| = \gamma$, for γ small enough,

$$\gamma \cong 2Z_{\alpha/2} \frac{\sigma_i(\theta)}{\sqrt{n_i(\gamma)}} \Rightarrow n_i(\gamma) \cong \left(\frac{2Z_{\alpha/2} \sigma_i(\theta)}{\gamma}\right)^2,$$

i.e., $n_1(\gamma)/n_2(\gamma) \cong \sigma_1^2(\theta)/\sigma_2^2(\theta)$. We called this the **asymptotic relative efficiency** $\text{ARE}_\theta(T^1, T^2)$.

Definition 3.4.1 (Asymptotic relative efficiency for estimator). The *asymptotic relative efficiency* between two estimators T_n^1 and T_n^2 for θ such that $\sqrt{n}(T_n^i - \theta) \xrightarrow{D} \mathcal{N}(0, \sigma_i^2(\theta))$ is defined as

$$\text{ARE}_\theta(T^1, T^2) = \frac{\sigma_1(\theta)^2}{\sigma_2(\theta)^2}.$$

Intuition. We can read $\text{ARE}_\theta(T^1, T^2) < 1$ as $n_1 < n_2$ and infer T^1 is better than T^2 .

Note. Definition 3.4.1 is different from the convention, where we usually define the *asymptotic relative efficiency* of T^1 w.r.t. T^2 as $\text{ARE}_\theta(T^1, T^2) = (\sigma_2(\theta)/\sigma_1(\theta))^2$. But it's just the convention.

3.4.2 Variance Stabilizing Transformation

Continuing on the previous [example](#), say we use \bar{X}_n as the estimator of θ . We have

$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{D} \sqrt{\theta}Z \sim \sqrt{\theta}\mathcal{N}(0, 1) = \mathcal{N}(0, \theta).$$

As the asymptotic distribution depends on θ , we don't directly get a confidence interval.

As previously seen. We will usually write $\sqrt{n}/\sqrt{\theta}(\bar{X}_n - \theta) \xrightarrow{D} Z \sim \mathcal{N}(0, 1)$, replace $\sqrt{\theta}$ by $\sqrt{\bar{X}_n}$, and apply [continuous mapping theorem](#) and [Slutsky's theorem](#) to get a confidence interval.

We see that our usual approach relies on ([consistently](#)) estimating the variance of the asymptotic distribution, which is potentially “unstable” for small n . To get around this, observe that from the [delta method](#) with some $g: \mathbb{R} \rightarrow \mathbb{R}$ differentiable at θ and $g'(\theta) \neq 0$,

$$\sqrt{n}(g(\bar{X}_n) - g(\theta)) \xrightarrow{D} g'(\theta)\sqrt{\theta}Z.$$

This suggests that if we can select g such that $g'(\theta)\sqrt{\theta} = c > 0$ is some constant for every $\theta > 0$, our goal is achieved since now we have

$$\frac{\sqrt{n}}{c}(g(\bar{X}_n) - g(\theta)) \xrightarrow{D} \mathcal{N}(0, 1).$$

In this case, we get an asymptotic confidence interval for $g(\theta)$ with confidence level $1 - \alpha$ as

$$\left(g(\bar{X}_n) - Z_{\alpha/2} \frac{c}{\sqrt{n}}, g(\bar{X}_n) + Z_{\alpha/2} \frac{c}{\sqrt{n}} \right),$$

and hence an asymptotic confidence interval for θ with confidence level $1 - \alpha$ is just

$$\left(g^{-1} \left(g(\bar{X}_n) - Z_{\alpha/2} \frac{c}{\sqrt{n}} \right), g^{-1} \left(g(\bar{X}_n) + Z_{\alpha/2} \frac{c}{\sqrt{n}} \right) \right),$$

This is the so-called *variance stabilizing transformation*.

Claim. For $c = 1/2$, $g(\theta) = \sqrt{\theta}$ suffices. Hence, in this case, $g^{-1}(u) = u^2$.

Proof. Since for $g'(\theta) = \frac{1}{2\sqrt{\theta}}$, we have $g(\theta) = \sqrt{\theta}$. ⊗

Lastly, we note that the above can be easily generalized.

Remark. Consider estimating a scalar parameter θ in some open interval Θ , where we replace:

- $\sqrt{\theta}$ by $h(\theta)$, a positive function;^a
- \sqrt{n} by b_n , a positive divergent strictly increasing sequence;
- \bar{X}_n by T_n , a [consistent](#) estimator of θ .

In this way, letting $g'(\theta)h(\theta) = c > 0$ for all $\theta \in \Theta$ asserts $g'(\theta) > 0$ for all $\theta \in \Theta$, hence g is strictly increasing and its usual inverse g^{-1} is well-defined.

^aWe don't need continuity since we don't need $h(T_n)$ when doing the variance stabilizing transformation.

Note. Variance stabilizing transformation doesn't have any theoretical guarantees. Rather, it's just our guess that by making the variance independent of n , it'll become better in terms of stability.

Lecture 13: Bahadur's Representation for Quantiles

3.5 Inference for Population Quantiles

27 Feb. 9:30

Let $X, X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F$ for some distribution function F , and let θ_p for some $p \in (0, 1)$ be the p^{th} quantile, which we recall is defined as $F^{-1}(p) = \inf\{t \in \mathbb{R}: F(t) \geq p\}$.

Intuition. Since $F^{-1}(p)$ depends on F , if we have an estimation of F itself, then we can have an estimation of $F^{-1}(p)$.

Specifically, to estimate F , consider the empirical cdf $\hat{F}_n(t)$ such that for all $t \in \mathbb{R}$,

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{X_i \leq t}$$

Now, from $\hat{F}_n(t)$, we estimate $\theta_p = F^{-1}(p)$ by the p^{th} -sample quantile

$$\hat{\theta}_p := \hat{F}_n^{-1}(p) := \inf\{t \in \mathbb{R}: \hat{F}_n(t) \geq p\}.$$

Remark. If F is continuous, then apart from a null set we have

$$\hat{\theta}_p = \inf\{X_{(i)}: \hat{F}_n(X_{(i)}) = i/n \geq p\} = \inf\left\{t \in \mathbb{R}: \sum_{i=1}^n \mathbb{1}_{X_i \leq t} \geq \lceil np \rceil\right\} = X_{(\lceil np \rceil)}.$$

Proof. Since F is continuous, with probability 1 there are no ties among X_i 's, hence $\hat{F}_n(t)$ has jumps of size $1/n$ at every order statistic $X_{(i)}$. Finally, the ceiling can be taken since $\sum_{i=1}^n \mathbb{1}_{X_i \leq t} \in \mathbb{N}$. *

3.5.1 Consistency

Firstly, \hat{F}_n is a consistent estimator of F since by weak law of large number, $\hat{F}_n(t) \xrightarrow{P} \mathbb{P}(X \leq t) = F(t)$. In fact, the convergence is exponentially fast in n by observing the following.

Note. By fixing t , $\mathbb{1}_{X \leq t}$ is $\text{Ber}(F(t))$, hence $\sqrt{n}(\hat{F}_n(t) - F(t)) \xrightarrow{D} \mathcal{N}(0, F(t)(1 - F(t)))$.

This implies that $\hat{F}_n(t)$ is an average of i.i.d. Bernoulli random variables, hence Hoeffding's inequality implies that the convergence is exponentially fast, i.e., for all $n \in \mathbb{N}$, $t \in \mathbb{R}$, and $\epsilon > 0$,

$$\mathbb{P}(|\hat{F}_n(t) - F(t)| > \epsilon) \leq 2 \exp(-n\epsilon^2/2).$$

We now show the consistency of $\hat{\theta}_p$ when the corresponding θ_p is unique. Recall the following.

As previously seen. $t \geq F^{-1}(p) \Leftrightarrow F(t) \geq p$ and $t < F^{-1}(p) \Leftrightarrow F(t) < p$. This is also true for \hat{F}_n .

Theorem 3.5.1. If $F(\theta_p + \epsilon) > F(\theta_p) \geq p$ for any $\epsilon > 0$, then $\hat{\theta}_p \xrightarrow{P} \theta_p$. More generally, if $p_n \rightarrow p$, then $\hat{\theta}_{p_n} \xrightarrow{P} \theta_p$.

Proof. We want to show that for any $\epsilon > 0$, $\mathbb{P}(|\hat{\theta}_{p_n} - \theta_p| > \epsilon) \rightarrow 0$. We see that

$$\mathbb{P}(|\hat{\theta}_{p_n} - \theta_p| > \epsilon) = \mathbb{P}(\hat{\theta}_{p_n} > \theta_p + \epsilon) + \mathbb{P}(\hat{\theta}_{p_n} < \theta_p - \epsilon).$$

For the first term, $\hat{\theta}_{p_n} = \hat{F}_n^{-1}(p_n) > \theta_p + \epsilon$, hence $p_n > \hat{F}_n(\theta_p + \epsilon)$, which gives

$$p_n - p + p - F(\theta_p + \epsilon) > \hat{F}_n(\theta_p + \epsilon) - F(\theta_p + \epsilon).$$

Since $p < F(\theta_p + \epsilon)$, let $-\delta := p - F(\theta_p + \epsilon)$ for some $\delta > 0$, then

$$\hat{F}_n(\theta_p + \epsilon) - F(\theta_p + \epsilon) < p_n - p - \delta < \frac{\delta}{2} - \delta = -\frac{\delta}{2}$$

for large enough n such that $|p_n - p| < \delta/2$, which implies $|\hat{F}_n(\theta_p + \epsilon) - F(\theta_p + \epsilon)| > \delta/2$, i.e.,

$$\mathbb{P}(\hat{\theta}_{p_n} > \theta + \epsilon) \leq \mathbb{P}(|\hat{F}_n(\theta_p + \epsilon) - F(\theta_p + \epsilon)| > \delta/2),$$

which goes to 0 as $n \rightarrow \infty$ from the [consistency](#) of \hat{F}_n . The second term can be proved similarly. ■

Note. The convergence in [Theorem 3.5.1](#) is also exponentially fast in n .

3.5.2 Bahadur's Representation Theorem

If F is differentiable, we can establish the asymptotic normality of $\hat{\theta}_{p_n}$.

Theorem 3.5.2 (Bahadur's representation). If $F'(\theta_p) =: f(\theta_p) > 0$ and $\sqrt{n}(p_n - p) = O(1)$, then

$$\sqrt{n}(\hat{\theta}_{p_n} - \theta_p) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{p_n - \mathbb{1}_{X_i \leq \theta_p}}{f(\theta_p)} + o_p(1).$$

Let's postpone the [proof](#) and discuss its implication first.

Corollary 3.5.1. If $F'(\theta_p) =: f(\theta_p) > 0$ and $\sqrt{n}(p_n - p) \rightarrow c \in [0, \infty)$, then

$$\sqrt{n}(\hat{\theta}_{p_n} - \theta_p) \xrightarrow{P} \frac{c}{f(\theta_p)}$$

and

$$\sqrt{n}(\hat{\theta}_{p_n} - \theta_p) \xrightarrow{D} \mathcal{N}\left(\frac{c}{f(\theta_p)}, \frac{p(1-p)}{f^2(\theta_p)}\right).$$

Proof. From [Bahadur's representation](#) shows

$$\sqrt{n}(\hat{\theta}_{p_n} - \theta_p) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{p - \mathbb{1}_{X_i \leq \theta_p}}{f(\theta_p)} + \frac{\sqrt{n}(p_n - p)}{f(\theta_p)} + o_p(1),$$

implying the first claim. For the second claim, firstly, if $\sqrt{n}(p_n - p) \rightarrow 0$, from [central limit theorem](#),

$$\sqrt{n}(\hat{\theta}_{p_n} - \theta_p) \xrightarrow{D} \mathcal{N}\left(0, \frac{F(\theta_p)(1 - F(\theta_p))}{f^2(\theta_p)}\right) = \mathcal{N}\left(0, \frac{p(1-p)}{f^2(\theta_p)}\right).$$

Now for $\sqrt{n}(p_n - p) \rightarrow c$, we first look at $\hat{\theta}_{p_n}$ and $\hat{\theta}_p$ instead, which gives

$$\sqrt{n}(\hat{\theta}_{p_n} - \hat{\theta}_p) = \sqrt{n}\left((\hat{\theta}_{p_n} - \theta_p) - (\hat{\theta}_p - \theta_p)\right) = \sqrt{n} \frac{p_n - p}{f(\theta_p)} + o_p(1) \xrightarrow{P} \frac{c}{f(\theta_p)}.$$

Moreover, from [central limit theorem](#) and [Slutsky's theorem](#),

$$\sqrt{n}(\hat{\theta}_{p_n} - \theta_p) = \sqrt{n}(\hat{\theta}_{p_n} - \hat{\theta}_p) + \sqrt{n}(\hat{\theta}_p - \theta_p) \xrightarrow{D} \mathcal{N}\left(\frac{c}{f(\theta_p)}, \frac{p(1-p)}{f^2(\theta_p)}\right),$$

where the variance calculation is the same as the case of $c = 0$ above. ■

Intuition. This is expected since if the density is low, then we don't have many data to evaluate θ_p in the first place, hence the precision will be low (large variance).

3.5.3 Confidence Intervals

When $c = 0$, [Corollary 3.5.1](#) gives an asymptotically valid $100(1 - \alpha)\%$ confidence interval for θ_p as

$$\hat{\theta}_{p_n} \pm Z_{\alpha/2} \frac{\sqrt{p(1-p)}}{\sqrt{n}f(\theta_p)}.$$

However, to implement this confidence interval, we need to estimate $f(\theta_p)$ [consistently](#). To avoid this, consider a sequence of intervals $(\hat{\theta}_{\ell_n}, \hat{\theta}_{u_n})$ for some $\ell_n < p_n < u_n$ such that

$$\hat{\theta}_{\ell_n} \xrightarrow{p} \hat{\theta}_p - Z_{\alpha/2} \frac{\sqrt{p(1-p)}}{\sqrt{n}f(\theta_p)} \text{ and } \hat{\theta}_{u_n} \xrightarrow{p} \hat{\theta}_p + Z_{\alpha/2} \frac{\sqrt{p(1-p)}}{\sqrt{n}f(\theta_p)}.$$

This will also give us an asymptotically valid $100(1 - \alpha)\%$ confidence interval for θ_p . The upshot is that, this is easy to construct without estimating $f(\theta_p)$ explicitly.

Example. Consider $\ell_n = p - Z_{\alpha/2} \sqrt{p(1-p)}/\sqrt{n}$, and similarly, $u_n = p + Z_{\alpha/2} \sqrt{p(1-p)}/\sqrt{n}$.

The above construction works due to the following.

Proposition 3.5.1. Let $c = Z_{\alpha/2} \sqrt{p(1-p)}$, and let ℓ_n and u_n such that $\sqrt{n}(\ell_n - p) \rightarrow -c$ and $\sqrt{n}(u_n - p) \rightarrow c$. If $F'(\theta_p) =: f(\theta_p) > 0$, then $\mathbb{P}(\hat{\theta}_{\ell_n} \leq \theta_p \leq \hat{\theta}_{u_n}) \rightarrow 1 - \alpha$.

Proof. First, consider ℓ_n . Since $\sqrt{n}(\ell_n - p) \rightarrow -c$, then $\hat{\theta}_{\ell_n}$ defined above is guaranteed from [Corollary 3.5.1](#) since it's equivalent to

$$\sqrt{n}(\hat{\theta}_{\ell_n} - \hat{\theta}_p) \xrightarrow{p} \frac{-c}{f(\theta_p)} = -Z_{\alpha/2} \frac{\sqrt{p(1-p)}}{f(\theta_p)}.$$

The same holds for u_n , hence we're done. ■

Remark. We can construct $(\hat{\theta}_{\ell_n}, \hat{\theta}_{u_n})$ without assuming knowledge or having to estimate $f(\theta_p)$.

3.5.4 Estimating the Center of a Distribution

Another implication is comparing the sample mean and the sample [median](#) as estimators of the center of a symmetric distribution.

Definition 3.5.1 (Median). When $p = 1/2$, $\theta_{1/2}$ is called the *median*.

Firstly, for $p = 1/2$, if $F'(\theta_{1/2}) =: f(\theta_{1/2}) > 0$, from [Corollary 3.5.1](#) we have

$$\sqrt{n}(\hat{\theta}_{1/2} - \theta_{1/2}) \xrightarrow{D} \mathcal{N}\left(0, \frac{1}{4f^2(\theta_{1/2})}\right).$$

Suppose further, $\theta_{1/2} = \mu$ and $\text{Var}[X] = \sigma^2 < \infty$. Then both $\hat{\theta}_{1/2}$ and \bar{X}_n are two possible estimators of μ , and in this case, we might want to look at the [asymptotic relative efficiency](#). Specifically,

$$\text{ARE}(\bar{X}_n, \hat{\theta}_{1/2}) = \frac{\sigma^2}{\frac{1}{4f^2(\theta_{1/2})}} = 4\sigma^2 f^2(\theta_{1/2}).$$

Let's summarize the above in the following.

Proposition 3.5.2. Suppose $\mu = \mathbb{E}[X]$ exists and $\sigma^2 = \text{Var}[X] < \infty$ such that $\mu = \theta_{1/2}$. If $F'(\mu) =: f(\mu) > 0$, then $\text{ARE}(\bar{X}_n, \hat{\theta}_{1/2}) = (2\sigma f(\mu))^2$.

The following two examples suggest that the sample [median](#) is asymptotically better than the sample mean when X has heavy tails.

Example. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then \bar{X}_n is a better estimator of μ than $\hat{\theta}_{1/2}$.

Proof. Since $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, hence $f(\mu) = 1/\sigma\sqrt{2\pi}$, i.e.,

$$\text{ARE}(\bar{X}_n, \hat{\theta}_{1/2}) = 4\sigma^2 \frac{1}{\sigma^2 2\pi} = \frac{2}{\pi} < 1,$$

which means \bar{X}_n is a better estimator of μ than $\hat{\theta}_{1/2}$. \ast

Example. If $X \sim \text{Laplace}(\mu, b)$ where $\sigma^2 = 2b^2$, then $\hat{\theta}_{1/2}$ is a better estimator of μ than \bar{X}_n .

Proof. Since $f(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}} = \frac{1}{\sigma\sqrt{2}} e^{-\frac{|x-\mu|}{\sigma/\sqrt{2}}}$, hence $f(\mu) = 1/\sqrt{2}\sigma$, i.e.,

$$\text{ARE}(\bar{X}_n, \hat{\theta}_{1/2}) = 4\sigma^2 \frac{1}{2\sigma^2} = 2 > 1,$$

which means $\hat{\theta}_{1/2}$ is a better estimator of μ than \bar{X}_n . \ast

One might want to consider $c\bar{X}_n + (1-c)\hat{\theta}_{1/2}$ for any $c \in [0, 1]$. In this case, by [Bahadur's representation](#) and [delta method](#), one can have

$$\sqrt{n} \left(c\bar{X}_n + (1-c)\hat{\theta}_{1/2} - \mu \right) \xrightarrow{D} \mathcal{N}(0, V)$$

where

$$V = c^2 \text{Var}[X] + (1-c)^2 \frac{1}{4f^2(\mu)} + 2c(1-c) \text{Cov} \left[X - \mu, \frac{1/2 - \mathbb{1}_{X \leq \mu}}{f(\mu)} \right].$$

Lecture 14: Proof of Bahadur's Representation Theorem

3.5.5 Proof of Bahadur's Representation Theorem

29 Feb. 9:30

Now we prove the [Bahadur's representation theorem](#). Recall the statement.

As previously seen. Given $F'(\theta_p) =: f(\theta_p) > 0$ and $\sqrt{n}(p_n - p) = O(1)$, we want to prove that

$$\sqrt{n}(\hat{\theta}_{p_n} - \theta_p) - \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{p - \mathbb{1}_{X_i \leq \theta_p}}{f(\theta_p)} - \sqrt{n} \frac{p_n - p}{f(\theta_p)} = o_p(1).$$

We now start the proof.

Proof of Theorem 3.5.2. Firstly, we write

$$W_n := \sqrt{n}(\hat{\theta}_{p_n} - \theta_p) - \sqrt{n} \frac{p_n - p}{f(\theta_p)},$$

and from $p = F(\theta_p)$,

$$U_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{p - \mathbb{1}_{X_i \leq \theta_p}}{f(\theta_p)} = \frac{\sqrt{n}(p - \hat{F}_n(\theta_p))}{f(\theta_p)} = \frac{\sqrt{n}(F(\theta_p) - \hat{F}_n(\theta_p))}{f(\theta_p)},$$

so we want to show $W_n - U_n = o_p(1)$. Consider the following lemma.

Lemma 3.5.1. Given two sequences of random variable $(W_n), (U_n)$ such that one of them is $O_p(1)$ and for every $\epsilon > 0$ and every $t \in \mathbb{R}$,

$$\mathbb{P}(W_n \leq t, U_n \geq t + \epsilon) + \mathbb{P}(U_n \leq t, W_n \geq t + \epsilon) \rightarrow 0,$$

then $U_n - W_n \xrightarrow{P} 0$.

Proof. Without loss of generality, suppose $W_n = O_p(1)$, and we show that for every $\epsilon > 0$, $\mathbb{P}(|W_n - U_n| > \epsilon) \rightarrow 0$. Firstly, observe that for every fixed $\epsilon > 0$, if $b - a < \epsilon/2$,

$$\mathbb{P}(a \leq W_n \leq b, |W_n - U_n| > \epsilon) \rightarrow 0$$

since the left-hand side is equal to

$$\begin{aligned} & \mathbb{P}(a \leq W_n \leq b, U_n > W_n + \epsilon) + \mathbb{P}(a \leq W_n \leq b, U_n < W_n - \epsilon) \\ & \leq \mathbb{P}(W_n \leq b, U_n > a + \epsilon) + \mathbb{P}(a \leq W_n, U_n < b - \epsilon) \\ & \leq \mathbb{P}(W_n \leq b, U_n > a + (2b - 2a)) + \mathbb{P}(a \leq W_n, U_n < b - (2b - 2a)) \\ & = \mathbb{P}(W_n \leq b, U_n > b + (b - a)) + \mathbb{P}(a \leq W_n, U_n < a + (a - b)), \end{aligned}$$

which goes to 0 from our assumption. Furthermore, fix any $\delta > 0$, since $W_n = O_p(1)$, there exists $M > 0$ such that $\mathbb{P}(|W_n| \leq M) \geq 1 - \delta$ for every $n \geq 1$. Then,

$$\begin{aligned} \mathbb{P}(|U_n - W_n| > \epsilon) & \leq \mathbb{P}(|W_n| > M) + \mathbb{P}(|W_n| \leq M, |U_n - W_n| > \epsilon) \\ & \leq \delta + \mathbb{P}(-M \leq W_n \leq M, |U_n - W_n| > \epsilon). \end{aligned}$$

The second term is like the first observation, but now we have a larger interval $[-M, M]$ rather than some $[a, b]$ with $b - a < \epsilon/2$. To compensate this, consider pair-wise disjoint intervals (a_i, b_i) for $i \in I$ with $|I| < \infty$ such that $b_i - a_i < \epsilon/2$ for all $i \in I$ and $\bigcup_{i \in I} [a_i, b_i] \supseteq [-M, M]$,

$$\mathbb{P}(-M \leq W_n \leq M, |U_n - W_n| > \epsilon) \leq \sum_{i \in I} \mathbb{P}(a_i \leq W_n \leq b_i, |U_n - W_n| > \epsilon).$$

Since I is finite, together with the first observation, implies $\limsup_{n \rightarrow \infty} \mathbb{P}(|U_n - W_n|) \leq \delta$. As δ is arbitrary, letting $\delta \rightarrow 0$ completes the proof. ■

Clearly, $U_n = O_p(1)$ since it [converges in distribution](#), so we can try to apply [Lemma 3.5.1](#). First, we study the numerator of U_n , i.e., $Z_n(t) := \sqrt{n}(F(t) - \hat{F}_n(t))$. We have seen that $\mathbb{E}[Z_n(t)] = 0$ and $\text{Var}[Z_n(t)] = F(t)(1 - F(t))$, and $Z_n(t) \xrightarrow{D} \mathcal{N}(0, F(t)(1 - F(t)))$ by [central limit theorem](#).

Claim. For any $t, s \in \mathbb{R}$, $\text{Var}[Z_n(t) - Z_n(s)] = \mathbb{E}[(Z_n(t) - Z_n(s))^2] \leq |F(t) - F(s)|$. Hence, if $s_n \rightarrow s$ and F is continuous at s , $Z_n(s_n) - Z_n(s) \xrightarrow{L^2} 0$, hence $Z_n(s_n) - Z_n(s) \xrightarrow{P} 0$.

Proof. Observe that $\text{Var}[Z_n(t) - Z_n(s)] = \text{Var}[\mathbb{1}_{X \leq t} - \mathbb{1}_{X \leq s}] \leq \mathbb{E}[|\mathbb{1}_{X \leq t} - \mathbb{1}_{X \leq s}|^2]$ where

$$|\mathbb{1}_{X \leq t} - \mathbb{1}_{X \leq s}| = \begin{cases} 1, & \text{if } s < X \leq t \text{ or } t < X \leq s; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, as $|\mathbb{1}_{X \leq t} - \mathbb{1}_{X \leq s}| = |\mathbb{1}_{X \leq t} - \mathbb{1}_{X \leq s}|^2$,

$$\begin{aligned} \mathbb{E}[|\mathbb{1}_{X \leq t} - \mathbb{1}_{X \leq s}|^2] & = \mathbb{P}(s < X \leq t) + \mathbb{P}(t < X \leq s) \\ & = (F(t) - F(s))^+ + (F(s) - F(t))^+ = |F(t) - F(s)|, \end{aligned}$$

i.e., $|\mathbb{1}_{X \leq t} - \mathbb{1}_{X \leq s}| \sim \text{Ber}(|F(t) - F(s)|)$. ⊛

From [Lemma 3.5.1](#), it suffices to show $\mathbb{P}(W_n \leq t, U_n \geq t + \epsilon) \rightarrow 0$ and $\mathbb{P}(U_n \leq t, W_n \geq t + \epsilon) \rightarrow 0$ for every $t \in \mathbb{R}$ and $\epsilon > 0$. Let's show the first one only. Fix $t \in \mathbb{R}$ and $\epsilon > 0$, then

$$\begin{aligned} W_n \leq t & \Leftrightarrow \sqrt{n}(\hat{\theta}_{p_n} - \theta_p) - \sqrt{n} \frac{p_n - p}{f(\theta_p)} \leq t \\ & \Leftrightarrow \hat{\theta}_{p_n} = \hat{F}_n^{-1}(p_n) \leq \theta_p + \frac{t}{\sqrt{n}} + \frac{p_n - p}{f(\theta_p)} =: \theta_p + \delta_n \quad \delta_n := \frac{t}{\sqrt{n}} + \frac{p_n - p}{f(\theta_p)} \end{aligned}$$

From the property of \hat{F}_n^{-1} , $p_n \leq \hat{F}_n(\theta_p + \delta_n)$,

$$\Leftrightarrow \sqrt{n}(p_n - F(\theta_p + \delta_n)) \leq \sqrt{n}(\hat{F}_n(\theta_p + \delta_n) - F(\theta_p + \delta_n)) = -Z_n(\theta_p + \delta_n),$$

which can be written as

$$Z_n(\theta_p + \delta_n) \leq \sqrt{n}(F(\theta_p + \delta_n) - p_n) \Leftrightarrow \frac{Z_n(\theta_p + \delta_n)}{f(\theta_p)} \leq \frac{\sqrt{n}(F(\theta_p + \delta_n) - p_n)}{f(\theta_p)} =: t_n.$$

Putting everything together, with $U_n = Z_n(\theta_p)/f(\theta_p)$, we have

$$\begin{aligned} \mathbb{P}(W_n \leq t, U_n \geq t + \epsilon) &= \mathbb{P}(Z_n(\theta_p + \delta_n) \leq t_n f(\theta_p), Z_n(\theta_p) \geq f(\theta_p)(t + \epsilon)) \\ &\leq \mathbb{P}(Z_n(\theta_p + \delta_n) - Z_n(\theta_p) \leq (t_n - t - \epsilon)f(\theta_p)) \\ &= \mathbb{P}\left(\frac{Z_n(\theta_p + \delta_n) - Z_n(\theta_p)}{f(\theta_p)} - (t_n - t) \leq -\epsilon\right), \end{aligned}$$

which goes to 0 as $n \rightarrow \infty$ if $t_n \rightarrow t$ since from the previous [claim](#):

- let $s_n := \theta_p + \delta_n$, $s := \theta_p$, with F being continuous at s and $\delta_n \rightarrow 0$, $Z_n(\theta_p + \delta_n) - Z_n(\theta_p) \xrightarrow{p} 0$;
- if further, $t_n \rightarrow t$, the left-hand side goes to 0, and the inequality tends to be vacuous.

Claim. Indeed, $t_n \rightarrow t$.

Proof. We want to show that

$$t_n = \frac{F(\theta_p + \delta_n) - p_n}{f(\theta_p)/\sqrt{n}} \rightarrow t.$$

By assumption, as $\delta_n \rightarrow 0$ and $F'(\theta_p) = f(\theta_p)$,

$$\frac{F(\theta_p + \delta_n) - F(\theta_p)}{\delta_n} \rightarrow f(\theta_p) \Leftrightarrow \frac{F(\theta_p + \delta_n) - F(\theta_p) - \delta_n f(\theta_p)}{\delta_n} \rightarrow 0,$$

i.e., $F(\theta_p + \delta_n) = F(\theta_p) + \delta_n f(\theta_p) + o(\delta_n)$. Since $F(\theta_p) = p$ and $\delta_n = t/\sqrt{n} + (p_n - p)/f(\theta_p)$,

$$F(\theta_p + \delta_n) = p + \left(\frac{t}{\sqrt{n}} + \frac{p_n - p}{f(\theta_p)}\right) f(\theta_p) + o(\delta_n) = p + \frac{t}{\sqrt{n}} f(\theta_p) + (p_n - p) + o(\delta_n).$$

Rearranging, with $o(\delta_n) \cdot \sqrt{n}/f(\theta_p) = \sqrt{n}o(\delta_n)$ from $f(\theta_p) > 0$, we have

$$t_n = \frac{F(\theta_p + \delta_n) - p_n}{f(\theta_p)/\sqrt{n}} = t + \sqrt{n}o(\delta_n).$$

Finally, since $o(\delta_n) = \delta_n o(1)$, with

$$\sqrt{n}\delta_n = \sqrt{n}\left(\frac{t}{\sqrt{n}} + \frac{p_n - p}{f(\theta_p)}\right) = t + \frac{\sqrt{n}(p_n - p)}{f(\theta_p)} = O(1)$$

from our assumption, we have $\sqrt{n}o(\delta_n) = O(1)o(1) = o(1)$, hence $t_n = t + o(1) \rightarrow t$. ⊗

The second claim $\mathbb{P}(U_n \leq t, W_n \geq t + \epsilon) \rightarrow 0$ can be proved similarly, hence we're done. ■

3.6 Inference for Distribution Function

Since we estimate F by \hat{F}_n when estimating θ_p by $\hat{\theta}_p$, one might just as well focus on the former task.

3.6.1 Consistency

Since $\sqrt{n}(\hat{F}_n(t) - F(t)) \xrightarrow{D} \mathcal{N}(0, F(t)(1 - F(t)))$ for any fixed t , so given $t_1, \dots, t_m \in \mathbb{R}$, we have

$$\begin{pmatrix} \sqrt{n}(\hat{F}_n(t_1) - F(t_1)) \\ \vdots \\ \sqrt{n}(\hat{F}_n(t_m) - F(t_m)) \end{pmatrix} = \begin{pmatrix} Z_n(t_1) \\ \vdots \\ Z_n(t_m) \end{pmatrix} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} \mathbb{1}_{X_i \leq t_1} - F(t_1) \\ \vdots \\ \mathbb{1}_{X_i \leq t_m} - F(t_m) \end{pmatrix} \xrightarrow{D} \mathcal{N}(0, \Sigma)$$

from [multivariate central limit theorem](#) where

$$\Sigma_{ij} = \text{Cov}[\mathbb{1}_{X \leq t_i}, \mathbb{1}_{X \leq t_j}] = \mathbb{P}(X \leq t_i \wedge X \leq t_j) - \mathbb{P}(X \leq t_i)\mathbb{P}(X \leq t_j).$$

Lecture 15: Inference for Cumulative Density Function

Surprisingly, this [consistency](#) property holds uniformly over t .

5 Mar. 9:30

Theorem 3.6.1 (Glivenko-Cantelli). Given a cdf F , $\|\hat{F}_n - F\|_\infty \xrightarrow{\text{a.s.}} 0$ as $n \rightarrow \infty$.

Proof. Fix some $\epsilon > 0$, and let $\epsilon > 2/k$ for some $k \in \mathbb{N}$. Then, for finitely many t_1, \dots, t_{k-1} , $\hat{F}_n(t_i) \xrightarrow{\text{a.s.}} F(t_i)$ and $\hat{F}_n(t_i^-) \xrightarrow{\text{a.s.}} F(t_i^-)$ for every $1 \leq i \leq k-1$ from the [strong law of large number](#). This means that there exists n_0 such that for $n \geq n_0$, for every $\omega \notin N$ such that $\mathbb{P}(N) = 0$,

$$|\hat{F}_n(t_i, \omega) - F(t_i, \omega)| < \frac{1}{k}, \text{ and } |\hat{F}_n(t_i^-, \omega) - F(t_i^-, \omega)| < \frac{1}{k},$$

so when $t \in \{t_i\}_{i=1}^{k-1}$ for some finite k , the desired bound is established. In particular, consider $t_i = \inf\{t \in \mathbb{R} : F(t) > i/k\}$ for $1 \leq i \leq k-1$, and $t_0 := -\infty$, $t_k = \infty$. Then for any $t \in \mathbb{R} \setminus \{t_i\}_{i=1}^{k-1}$, there exists a unique i such that $t \in (t_{i-1}, t_i)$, and furthermore, for all $n \geq n_0$,

$$\hat{F}_n(t) - F(t) \leq \hat{F}_n(t_i^-) - F(t_{i-1}) = \hat{F}_n(t_i^-) - F(t_i^-) + F(t_i^-) - F(t_{i-1}) \leq \frac{1}{k} + \frac{i}{k} - \frac{i-1}{k} = \frac{2}{k} < \epsilon$$

Similarly, we can show $\hat{F}_n(t) - F(t) > -\epsilon$ for all $t \in \mathbb{R} \setminus \{t_i\}_{i=1}^{k-1}$, which completes the proof. ■

3.6.2 Donsker's Theorem

On the other hand, for distributional result, first recall the *empirical process*

$$Z_n(t) := \sqrt{n}(F(t) - \hat{F}_n(t))$$

for $t \in \mathbb{R}$ introduced in the [proof](#) of [Bahadur representation theorem](#). Recall the following.

As previously seen. We have seen that

$$Z_n(t) := \sqrt{n}(F(t) - \hat{F}_n(t)) \xrightarrow{D} B_F(t) := \mathcal{N}(0, F(t)(1 - F(t))).$$

Furthermore, for any $t_1, \dots, t_m \in \mathbb{R}$,

$$(Z_n(t_1), \dots, Z_n(t_m)) \xrightarrow{D} (B_F(t_1), \dots, B_F(t_m)) \sim \mathcal{N}(0, \Sigma_F(t_1, \dots, t_m))$$

where for $1 \leq i \leq j \leq m$,

$$\text{Cov}[B_F(t_i), B_F(t_j)] = \text{Cov}[\mathbb{1}_{X \leq t_i}, \mathbb{1}_{X \leq t_j}] = F(t_i \wedge t_j) - F(t_i)F(t_j).$$

We now ask the same question, i.e., does the convergence hold uniformly over t ?

Intuition. As the theory of [weak convergence](#) applies to sequences of random elements that take values in general metric spaces, it's reasonable to conjecture that (Z_n) [converges weakly](#) to some random process B_F with index set \mathbb{R} , i.e., $B_F = \{B_F(t)\}_{t \in \mathbb{R}}$, such that for every $t, s \in \mathbb{R}$,

$$\mathbb{E}[B_F(t)] = 0 \text{ and } \text{Cov}[B_F(t), B_F(s)] = F(t \wedge s) - F(t)F(s).$$

The conjecture is indeed correct, and it's an extension of [Donsker's theorem](#).

Note. The formal setup is to view each Z_n as a random element that takes values on the space \mathcal{D} of right continuous functions with left limits with the norm $\|\cdot\|_\infty$.

Example (Brownian bridge). A Brownian bridge is $B := B_F$ when $F(t) = t$, i.e., $F \sim \mathcal{U}([0, 1])$.

Note. For any cdf F , $B_F(t)$ is just B index at $F(t)$ for any $t \in \mathbb{R}$, i.e., $B_F(t) = B(F(t))$.

Taking this convergence as granted (and work with continuous F), we have

$$(Z_n) := (t \mapsto Z_n(t))_{n \geq 1} \xrightarrow{w} B_F := \{t \mapsto B(t)\}_{t \in \mathbb{R}}.$$

One immediate consequence is the following.

Proposition 3.6.1. If $T: \mathcal{D} \rightarrow \mathbb{R}$ is continuous,^a then $T(Z_n) \xrightarrow{D} T(B_F)$.

^aI.e., $T(G_n) \rightarrow T(F)$ if $(G_n), F \in \mathcal{D}$ such that $\|G_n - F\|_\infty \rightarrow 0$.

Proof. Since $Z_n \xrightarrow{w} B_F$, [continuous mapping theorem](#) proves the result. ■

3.6.3 Confidence Bands

One immediate application of [Proposition 3.6.1](#) is the following.

Corollary 3.6.1. If F is continuous, then $\sqrt{n}\|\hat{F}_n - F\|_\infty \xrightarrow{D} \|B_F\|_\infty$.

Proof. From [Proposition 3.6.1](#), we just note that $\|\cdot\|_\infty$ is continuous since it's a norm. ■

In particular, [Corollary 3.6.1](#) allows us to do inference.

Example (Confidence band). For any $\alpha \in (0, 1)$, let d_α to be defined as $\alpha =: \mathbb{P}(\|B_F\|_\infty > d_\alpha)$, then if F is continuous, $\mathbb{P}(\sqrt{n}\|F - \hat{F}_n\|_\infty \geq d_\alpha) \rightarrow \alpha$, i.e.,

$$\mathbb{P}\left(\forall t \in \mathbb{R}: \hat{F}_n(t) - \frac{d_\alpha}{\sqrt{n}} \leq F(t) \leq \hat{F}_n(t) + \frac{d_\alpha}{\sqrt{n}}\right) \rightarrow 1 - \alpha.$$

Another application of [Proposition 3.6.1](#) is the following.

Corollary 3.6.2. If F is continuous, then

$$n \int_{\mathbb{R}} \left(\hat{F}_n(t) - F(t)\right)^2 F(dt) \xrightarrow{D} \int_{\mathbb{R}} B_F^2(t) F(dt).$$

Proof. From [Proposition 3.6.1](#), it suffices to show that $G \mapsto \int_{\mathbb{R}} G^2 dF$ is continuous for $G \in \mathcal{D}$. Firstly, let $(G_n), G \in \mathcal{D}$ such that $\|G_n - G\|_\infty \rightarrow 0$. Then

$$\begin{aligned} |T(G_n) - T(G)| &= \left| \int_{\mathbb{R}} G_n^2 - G^2 dF \right| \\ &\leq \int_{\mathbb{R}} |G_n^2 - G^2| dF \leq \|G_n - G\|_\infty \int_{\mathbb{R}} |G_n(t) + G(t)| F(dt) \leq 2\|G_n - G\|_\infty \end{aligned}$$

since $\|G_n - G\|_\infty = \sup_t |G_n(t) - G(t)|$. As $\|G_n - G\|_\infty \rightarrow 0$, we're done. ■

Note. We can also obtain a confidence band from [Corollary 3.6.2](#) as in the [previous example](#).

3.6.4 Goodness of Fit Tests

We now consider the problem of testing the null hypothesis $H_0: F = F_0$ given F_0 , i.e., the *goodness of fit test*. Suppose F is continuous, then [Corollary 3.6.1](#) and [Corollary 3.6.2](#) suggest we can test H_0 using the following two statistics.

Example (Kolmogorov-Smirnov statistic). Consider the *Kolmogorov-Smirnov statistic*

$$K_n := \|\hat{F}_n - F_0\|_\infty.$$

For any $\alpha \in (0, 1)$, we reject H_0 when $\sqrt{n}K_n > d_\alpha$, where d_α is defined as $\alpha = \mathbb{P}(\|B_F\|_\infty > d_\alpha)$.

Example (Cramér-von Mises statistic). Consider the *Cramér-von Mises statistic*

$$C_n := \int_{\mathbb{R}} \left(\hat{F}_n(t) - F_0(t) \right)^2 F_0(dt).$$

For any $\alpha \in (0, 1)$, we reject H_0 when $nC_n > c_\alpha^2$, where c_α is defined as $\alpha = \mathbb{P}(\int_0^1 B_F^2(t) dt \geq c_\alpha^2)$.

In particular, [Corollary 3.6.1](#) and [Corollary 3.6.2](#) shows the following.

Remark. Under H_0 , the above two tests will only reject with probability approaching α .

On the other hand, when $F \neq F_0$, we will reject with probability approaching 1:

Proposition 3.6.2. Suppose F is continuous. Consider testing $H_0: F = F_0$ using the [Kolmogorov-Smirnov statistic](#), then for any $F \neq F_0$, $\mathbb{P}_F(\text{reject}) = \mathbb{P}(\sqrt{n}K_n > d_\alpha) \rightarrow 1$ as $n \rightarrow \infty$.

Proof. For any metric d , we have

$$\mathbb{P}_F(\sqrt{nd}(\hat{F}_n, F_0) \geq d_\alpha) \geq \mathbb{P}_F(\sqrt{nd}(d(\hat{F}_n, F) - d(F, F_0)) \geq d_\alpha) = \mathbb{P}_F(\sqrt{nd}(\hat{F}_n, F) \geq d_\alpha + \sqrt{nd}(F, F_0)).$$

As $F \neq F_0$, $d(F, F_0) > 0$ is a fixed number, so $\sqrt{nd}(F, F_0) \rightarrow \infty$. Hence, from [Corollary 3.6.1](#), $\sqrt{n}\|\hat{F}_n - F\|_\infty = O_p(1)$, with $d(x, y) := \|x - y\|_\infty$, the right-hand side goes to 1. ■

The same result can be obtained when the [Cramér-von Mises statistic](#) is used as follows.

Proposition 3.6.3. Suppose F is continuous. Consider testing $H_0: F = F_0$ using the [Cramér-von Mises statistic](#), then for any $F \neq F_0$, $\mathbb{P}_F(\text{reject}) = \mathbb{P}(nC_n > c_\alpha^2) \rightarrow 1$ as $n \rightarrow \infty$.

We omit the proof of [Proposition 3.6.3](#), instead, we focus on a related result regarding the representation of C_n , and the proof of [Proposition 3.6.3](#) can be done similarly. First, recall the following.

As previously seen. From [Corollary 3.6.2](#), under $H_0: F = F_0$,

$$nC_n = n \int_{\mathbb{R}} (\hat{F}_n - F_0)^2 dF_0 \xrightarrow{D} \int_{\mathbb{R}} B_F^2 dF_0.$$

Problem. What is the distribution in the case of $F \neq F_0$?

Answer. Intuitively, consider

$$h(F) := \int_{\mathbb{R}} (F - F_0)^2 dF_0.$$

Since h is continuous, $C_n = h(\hat{F}_n) \rightarrow h(F)$. ⊛

Formally, for a distributional result, we have the following.

Proposition 3.6.4. There is a function g so that $\mathbb{E}[g(X)] = 0$ and

$$\sqrt{n} \left(C_n - \int_{\mathbb{R}} (F - F_0)^2 dF_0 \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n g(X_i) + o_p(1).$$

If we further have $F \neq F_0$, then the above converges to $\mathcal{N}(0, \text{Var}[g(X)])$.

Proof. We first note that the left-hand side is just $\sqrt{n}(h(\hat{F}_n) - h(F))$. Now, since

$$\begin{aligned} h(\hat{F}_n) &= C_n = \int_{\mathbb{R}} (\hat{F}_n - F_0)^2 dF_0 \\ &= \int_{\mathbb{R}} (\hat{F}_n - F + F - F_0)^2 dF_0 \\ &= \int_{\mathbb{R}} (\hat{F}_n - F)^2 dF_0 + \underbrace{\int_{\mathbb{R}} (F - F_0)^2 dF_0}_{h(F)} + 2 \int_{\mathbb{R}} (\hat{F}_n - F)(F - F_0) dF_0, \end{aligned}$$

we have

$$\sqrt{n} \left(h(\hat{F}_n) - h(F) \right) = \sqrt{n} \int_{\mathbb{R}} (\hat{F}_n - F)^2 dF_0 + 2\sqrt{n} \int_{\mathbb{R}} (\hat{F}_n - F)(F - F_0) dF_0.$$

As $n \int_{\mathbb{R}} (\hat{F}_n - F)^2 dF_0 \xrightarrow{w} \int_{\mathbb{R}} B_F^2 dF_0$, [Proposition 2.4.2](#) implies

$$\sqrt{n} \int_{\mathbb{R}} (\hat{F}_n - F)^2 dF_0 = \frac{n}{\sqrt{n}} \int_{\mathbb{R}} (\hat{F}_n - F)^2 dF_0 = \frac{O_p(1)}{\sqrt{n}} = o_p(1),$$

which gives

$$\sqrt{n} \left(h(\hat{F}_n) - h(F) \right) = 2\sqrt{n} \int_{\mathbb{R}} (\hat{F}_n - F)(F - F_0) dF_0 + o_p(1) =: \frac{1}{\sqrt{n}} \sum_{i=1}^n g(X_i) + o_p(1)$$

where we define the function $g: \mathbb{R} \rightarrow \mathbb{R}$ as

$$g(x) := 2 \int_{\mathbb{R}} (\mathbb{1}_{x \leq t} - F(t))(F(t) - F_0(t)) F_0(dt)$$

since

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{i=1}^n g(X_i) &= \frac{2}{\sqrt{n}} \sum_{i=1}^n \int_{\mathbb{R}} (\mathbb{1}_{X_i \leq t} - F(t))(F(t) - F_0(t)) F_0(dt) \\ &= \frac{2}{\sqrt{n}} \int_{\mathbb{R}} \sum_{i=1}^n (\mathbb{1}_{X_i \leq t} - F(t))(F(t) - F_0(t)) F_0(dt) \\ &= \frac{2}{\sqrt{n}} \int_{\mathbb{R}} \sum_{i=1}^n \mathbb{1}_{X_i \leq t} (F(t) - F_0(t)) - nF(t)(F(t) - F_0(t)) F_0(dt) \\ &= \frac{2}{\sqrt{n}} \int_{\mathbb{R}} n \left(\frac{1}{n} \sum_{i=1}^n \mathbb{1}_{X_i \leq t} (F(t) - F_0(t)) - F(t)(F(t) - F_0(t)) \right) F_0(dt) \\ &= 2\sqrt{n} \int_{\mathbb{R}} (\hat{F}_n - F)(F - F_0) dF_0. \end{aligned}$$

To show $\mathbb{E}[g(X)] = 0$, as $F(t), F_0(t), \mathbb{1}_{x \leq t}$ are all bounded by 1, [Fubini's theorem](#) gives

$$\mathbb{E}[g(X)] = 2 \int_{\mathbb{R}} (\mathbb{P}(X \leq t) - F(t))(F(t) - F_0(t)) F_0(dt) = 0$$

since $\mathbb{P}(X \leq t) = F(t)$. Finally, when $F \neq F_0$, $0 < \mathbb{E}[g^2(X)] < \infty$ follows from the same calculation, hence [central limit theorem](#) gives the distributional result. ■

Chapter 4

Lindeberg Central Limit Theorem

Lecture 16: Lindeberg Central Limit Theorem

We extend the [central limit theorem](#) to the case that (X_n) are only independent but not identically distributed. In particular, consider the case that $S_n = X_1 + \dots + X_n$'s are a sum of independent, but not necessarily identically distributed, random variables whose distribution may vary with n .

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The following examples show that in this more general case, assuming finite variance doesn't suffice.

Example. If $X_i \sim \text{Pois}(1/i^2)$ for all $i \geq 1$ and are independent to each other, then

$$S_n \sim \text{Pois}\left(\sum_{i=1}^n \frac{1}{i^2}\right) \xrightarrow{\text{TV}} \text{Pois}\left(\sum_{i=1}^{\infty} \frac{1}{i^2}\right) = \text{Pois}\left(\frac{\pi^2}{6}\right),$$

which does not go to normal as we expected since X_i are not identically distributed.

On the other hand, something trickier can happen when X_i are "seemingly" identically distributed.

Example. Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Pois}(1/n)$ for every $n \geq 1$. But since $S_n \sim \text{Pois}(1)$ for all $n \geq 1$,

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\text{Var}[S_n]}} \not\xrightarrow{D} \mathcal{N}(0, 1).$$

This does not contradict to [central limit theorem](#) since $\text{Pois}(1/n)$ depends on n .

In general, for any $n \geq 1$, let $k_n \nearrow \infty$ be the number of independent random variables in the sequence $(X_{nk_n}) = (X_{n1}, \dots, X_{nk_n})$ with $\text{Var}[X_{nj}] < \infty$ for all $1 \leq j \leq k_n$ and n . Again, we define $S_n = X_{n1} + \dots + X_{nk_n}$. In picture, we have something like a *triangular array* of random variables:

$$\begin{aligned} n = 1: & \quad X_{11}, \dots, X_{1k_1}; \\ n = 2: & \quad X_{21}, X_{22}, \dots, X_{2k_2}; \\ & \quad \vdots \\ n: & \quad X_{n1}, X_{n2}, X_{n3}, \dots, X_{nk_n}. \end{aligned}$$

Example. As a special case, we previously have $X_{nj} = X_j$ for all $1 \leq j \leq n$, i.e., $k_n = n$.

Remark. For different n , (X_{nk_n}) can be defined on different probability space.

4.1 Lindeberg Central Limit Theorem

The goal of this section is to establish the following.

Theorem 4.1.1 (Lindeberg central limit theorem). For every $n \geq 1$, let (X_{nk_n}) be a sequence of independent variables with $k_n \nearrow \infty$ and let $Y_{nj} := (X_{nj} - \mathbb{E}[X_{nj}])/\sqrt{\text{Var}[S_n]}$ for every $1 \leq j \leq k_n$. If the [Lindeberg condition](#) holds, then

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\text{Var}[S_n]}} = \sum_{j=1}^{k_n} \frac{X_{nj} - \mathbb{E}[X_{nj}]}{\sqrt{\text{Var}[S_n]}} =: \sum_{j=1}^{k_n} Y_{nj} \xrightarrow{D} \mathcal{N}(0, 1).$$

Note. In the above notation, for all $n \geq 1$, $\mathbb{E}[Y_{nj}] = 0$ for all $1 \leq j \leq k_n$ and $\sum_{j=1}^{k_n} \text{Var}[Y_{nj}] = 1$.

4.1.1 Lindeberg Condition

We first explain the sufficient condition stated in the [Lindeberg central limit theorem](#), i.e., the [Lindeberg condition](#). Firstly, a weaker but more intuitive notion one might consider is the following.

Definition 4.1.1 (Uniform asymptotic negligibility). Given a (family of) sequence (X_{nk_n}) , we say it satisfies the *uniform asymptotic negligibility* (UAN), if as $n \rightarrow \infty$,

$$\frac{\max_{1 \leq j \leq k_n} \text{Var}[X_{nj}]}{\text{Var}[S_n]} \rightarrow 0.$$

However, as we have seen in the [second examples](#), UAN doesn't suffice for the [Lindeberg central limit theorem](#) to hold since in this case, $\max_{1 \leq j \leq k_n} \text{Var}[X_{nj}] = 1/n \rightarrow 0$, but we know that [Lindeberg central limit theorem](#) fail. Hence, we consider the following stronger notion.

Definition 4.1.2 (Lindeberg condition). Given a (family of) sequence (X_{nk_n}) , let $Y_{nj} := (X_{nj} - \mathbb{E}[X_{nj}])/\sqrt{\text{Var}[S_n]}$ for every $1 \leq j \leq k_n$ and every $n \geq 1$. Then we say (X_{nk_n}) satisfies the *Lindeberg condition* if for every $\epsilon > 0$, as $n \rightarrow \infty$,

$$\sum_{j=1}^{k_n} \mathbb{E}[Y_{nj}^2 \cdot \mathbb{1}_{|Y_{nj}| > \epsilon}] \rightarrow 0.$$

Indeed, [Lindeberg condition](#) is stronger than [uniform asymptotic negligibility](#).

Proposition 4.1.1. The [Lindeberg condition](#) implies [uniform asymptotic negligibility](#).

Proof. We want to prove that $\max_{1 \leq j \leq k_n} \text{Var}[Y_{nj}] \rightarrow 0$ as $n \rightarrow \infty$. Firstly, for any n and every $1 \leq j \leq k_n$, by splitting up the expectation, for every $\epsilon > 0$, we have $\text{Var}[Y_{nj}] \leq \mathbb{E}[Y_{nj}^2 \cdot \mathbb{1}_{|Y_{nj}| > \epsilon}] + \epsilon^2$. Then with the [Lindeberg condition](#), we have

$$\max_{1 \leq j \leq k_n} \text{Var}[Y_{nj}] \leq \sum_{j=1}^{k_n} \mathbb{E}[Y_{nj}^2 \cdot \mathbb{1}_{|Y_{nj}| > \epsilon}] + \epsilon^2 \rightarrow \epsilon^2,$$

i.e., $\limsup_{n \rightarrow \infty} \max_{1 \leq j \leq k_n} \text{Var}[Y_{nj}] \leq \epsilon^2$. By letting $\epsilon \rightarrow 0$, we complete the proof. \blacksquare

While the UAN is insufficient, it's also not necessary.

Remark. Let (X_n) be a sequence of independent Gaussian variables with $\text{Var}[X_i] = 1/i^2$ for all $i \geq 1$. Then indeed, $S_n \sim \mathcal{N}$. However, the [uniform asymptotic negligibility](#) does not hold since

$$\frac{\max_{1 \leq i \leq n} \text{Var}[X_i]}{\text{Var}[S_n]} = \frac{1}{\sum_{i=1}^n 1/i^2} \rightarrow \frac{6}{\pi^2} > 0.$$

4.1.2 Proof of Lindeberg Central Limit Theorem

Now, to prove the [Lindeberg central limit theorem](#), we again turn to the [characteristic function](#) and use the [uniqueness theorem](#), as what we have done for the usual [central limit theorem](#). Since this turns the problem into calculus, we will need a series of lemmas that provide some bounds.

Lemma 4.1.1. For any $w_1, \dots, w_n, z_1, \dots, z_n \in \mathbb{C}$ such that $|w_i|, |z_i| \leq 1$ for all $1 \leq i \leq n$, we have

$$\left| \prod_{i=1}^n z_i - \prod_{i=1}^n w_i \right| \leq \sum_{i=1}^n |w_i - z_i|.$$

It turns out that we will also need to bound $e^{ix} - (1 + ix)$, i.e., the remainder of the second order Taylor expansion of e^{ix} . Let's first see two uniform bounds for this.

Lemma 4.1.2. For any $x \in \mathbb{R}$, $|e^{ix} - (1 + ix)| \leq x^2/2$ and $|e^{ix} - 1 - ix - (ix)^2/2| \leq x^2$.

Proof. Recall the specific form of [Taylor expansion](#) we used before, which gives

$$e^{ix} = 1 + ix + (ix)^2 \int_0^1 \int_0^1 e^{iuvx} u \, du \, dv = 1 + ix + \frac{(ix)^2}{2} + (ix)^2 \int_0^1 \int_0^1 (e^{iuvx} - 1) u \, du \, dv,$$

which gives both inequalities by bounding the two integrals differently. ■

On the other hand, when $|z|$ is small enough, we have the following tighter bounds.

Lemma 4.1.3. For any $z \in \mathbb{C}$ such that $|z| \leq \epsilon < 1$, $|e^z - 1 - z - z^2/2| \leq |z|^3/(1 - \epsilon)$.

Proof. Since

$$\left| e^z - 1 - z - \frac{z^2}{2} \right| \leq \left| \sum_{n=3}^{\infty} \frac{z^n}{n!} \right| \leq |z|^3 \sum_{n=0}^{\infty} |z|^n = |z|^3 \cdot \frac{1}{1 - |z|} \leq \frac{|z|^3}{1 - \epsilon},$$

where the series converges from the fact that $|z| < 1$. ■

Lemma 4.1.4. For any $z \in \mathbb{C}$ such that $|z| < \delta/2$ where $\delta \in (0, 1)$, $|e^{iz} - (1 + iz)| \leq \delta|z|$.

Proof. Since

$$|e^{iz} - 1 - iz| = \left| \sum_{n=2}^{\infty} \frac{(iz)^n}{n!} \right| \leq \sum_{n=2}^{\infty} |z|^n = |z|^2 \sum_{n=0}^{\infty} |z|^n = \frac{|z|^2}{1 - |z|} = \frac{|z|}{1 - |z|} |z| < \frac{\delta/2}{2 - \delta} |z| \leq \delta|z|,$$

where the series converges from the fact that $|z| < \delta/2 < 1$. ■

Finally, recall the following.

As previously seen. From [Equation 2.1](#), for $Z \sim \mathcal{N}(0, 1)$, $\phi_Z(t) = e^{-t^2/2}$.

We can now prove the [Lindeberg central limit theorem](#).

Proof of Theorem 4.1.1. Let $\phi_{nj}(t) := \mathbb{E}[e^{itX_{nj}}]$ for $t \in \mathbb{R}$. We want to show that

$$\sum_{j=1}^{k_n} Y_{nj} \xrightarrow{D} \mathcal{N}(0, 1) \Leftrightarrow \prod_{j=1}^{k_n} \phi_{nj}(t) \rightarrow e^{-t^2/2}$$

for every $t \in \mathbb{R}$ from the [uniqueness theorem](#). Fix $t \in \mathbb{R}$, from triangle inequality, it suffices to show

$$\left| \prod_{j=1}^{k_n} \phi_{nj}(t) - \prod_{j=1}^{k_n} e^{\phi_{nj}(t)-1} \right| + \left| \prod_{j=1}^{k_n} e^{\phi_{nj}(t)-1} - e^{-t^2/2} \right| \rightarrow 0.$$

Firstly, consider the first term, and recall what we have shown in the homework.

As previously seen. If ϕ is a [characteristic function](#), so is $e^{\lambda(\phi-1)}$ for any $\lambda > 0$.

Hence, $e^{\phi_{nj}(t)-1}$ is a [characteristic function](#), so both $\phi_{nj}(t)$ and $e^{\phi_{nj}(t)-1}$ are bounded by 1. This means we can apply [Lemma 4.1.1](#) and get

$$\left| \prod_{j=1}^{k_n} \phi_{nj}(t) - \prod_{j=1}^{k_n} e^{\phi_{nj}(t)-1} \right| \leq \sum_{j=1}^{k_n} |\phi_{nj}(t) - e^{\phi_{nj}(t)-1}| = \sum_{j=1}^{k_n} |e^{\phi_{nj}(t)-1} - (\phi_{nj}(t) - 1) - 1|.$$

Let $z_j := \phi_{nj}(t) - 1$, then the above is just $\sum_{j=1}^{k_n} |e^{z_j} - (z_j + 1)|$, suggesting [Lemma 4.1.4](#). Fixing some $\delta \in (0, 1)$, we show that $\max_{1 \leq j \leq k_n} |z_j| < \delta/2$ for large enough n .

Claim. For any $\delta \in (0, 1)$, $\max_{1 \leq j \leq k_n} |\phi_{nj}(t) - 1| \leq \delta/2$ for n large enough.

Proof. As $\mathbb{E}[Y_{nj}] = 0$ for all $1 \leq j \leq k_n$, by using [Lemma 4.1.2](#), we have

$$\begin{aligned} \max_{1 \leq j \leq k_n} |\phi_{nj}(t) - 1| &= \max_{1 \leq j \leq k_n} |\mathbb{E}[e^{itY_{nj}} - 1 - itY_{nj}]| \\ &\leq \max_{1 \leq j \leq k_n} \mathbb{E}[|e^{itY_{nj}} - (1 + itY_{nj})|] \leq \frac{t^2}{2} \max_{1 \leq j \leq k_n} \mathbb{E}[Y_{nj}^2] \end{aligned}$$

From the [Lindeberg condition](#), $\max_{1 \leq j \leq k_n} \mathbb{E}[Y_{nj}^2] \rightarrow 0$, hence we're done. \otimes

Hence, for any $\delta \in (0, 1)$, when n is large enough, [Lemma 4.1.4](#) and the above calculation gives,

$$\left| \prod_{j=1}^{k_n} \phi_{nj}(t) - \prod_{j=1}^{k_n} e^{\phi_{nj}(t)-1} \right| \leq \delta \sum_{j=1}^{k_n} |\phi_{nj}(t) - 1| \leq \delta \cdot \frac{t^2}{2} \sum_{j=1}^{k_n} \mathbb{E}[Y_{nj}^2] = \frac{\delta t^2}{2}$$

since $\sum_{j=1}^{k_n} \mathbb{E}[Y_{nj}^2] = \sum_{j=1}^{k_n} \text{Var}[Y_{nj}] = 1$. By letting $n \rightarrow \infty$, and $\delta \rightarrow 0$, we see that the first term indeed goes to 0 as $n \rightarrow \infty$. As for the second term, it suffices to show that for every $t \in \mathbb{R}$,

$$\sum_{j=1}^{k_n} (\phi_{nj}(t) - 1) \rightarrow -\frac{t^2}{2} \Leftrightarrow \sum_{j=1}^{k_n} (\phi_{nj}(t) - 1) + \frac{t^2}{2} \rightarrow 0 \Leftrightarrow \sum_{j=1}^{k_n} \left[(\phi_{nj}(t) - 1) + \frac{t^2}{2} \text{Var}[Y_{nj}] \right] \rightarrow 0$$

as $\sum_{j=1}^{k_n} \text{Var}[Y_{nj}] = 1$. Since Y_{nj} is centered, we have $\mathbb{E}[Y_{nj}] = 0$ and $\text{Var}[Y_{nj}] = \mathbb{E}[Y_{nj}^2]$, hence

$$\sum_{j=1}^{k_n} \left[(\phi_{nj}(t) - 1) + \frac{t^2}{2} \text{Var}[Y_{nj}] \right] = \sum_{j=1}^{k_n} \mathbb{E} \left[e^{itY_{nj}} - 1 - itY_{nj} - \frac{(itY_{nj})^2}{2} \right].$$

To bound this, we decompose it via the event $|Y_{nj}| > \epsilon$ (for some $\epsilon > 0$ to be determined later) as

$$\sum_{j=1}^{k_n} \mathbb{E} \left[\left(e^{itY_{nj}} - 1 - itY_{nj} - \frac{(itY_{nj})^2}{2} \right) \mathbb{1}_{|Y_{nj}| > \epsilon} \right] + \sum_{j=1}^{k_n} \mathbb{E} \left[\left(e^{itY_{nj}} - 1 - itY_{nj} - \frac{(itY_{nj})^2}{2} \right) \mathbb{1}_{|Y_{nj}| \leq \epsilon} \right].$$

We then see that

- from [Lemma 4.1.2](#), with $x := tY_{nj}$, we can bound the first term as

$$\sum_{j=1}^{k_n} \mathbb{E} \left[\left(e^{itY_{nj}} - 1 - itY_{nj} - \frac{(itY_{nj})^2}{2} \right) \mathbb{1}_{|Y_{nj}| > \epsilon} \right] \leq t^2 \sum_{j=1}^{k_n} \mathbb{E} [Y_{nj}^2 \mathbb{1}_{|Y_{nj}| > \epsilon}] \rightarrow 0$$

as $n \rightarrow \infty$ by the [Lindeberg condition](#);

- from [Lemma 4.1.3](#), with $z := itY_{nj}$ such that $|z| = |tY_{nj}| \leq |t|\epsilon$ under the event. Let ϵ be defined such that $|t|\epsilon < 1$, then we can bound the second term by

$$\frac{1}{1-\epsilon} \sum_{j=1}^{k_n} \mathbb{E}[|tY_{nj}|^3 \cdot \mathbb{1}_{|Y_{nj}| \leq \epsilon}] \leq \frac{|t|^3}{1-\epsilon} \sum_{j=1}^{k_n} \mathbb{E}[|Y_{nj}|^2 \cdot \mathbb{1}_{|Y_{nj}| \leq \epsilon}] \leq \frac{|t|^3}{1-\epsilon} \sum_{j=1}^{k_n} \mathbb{E}[|Y_{nj}|^2] = \frac{|t|^3}{1-\epsilon} \epsilon$$

since again $\sum_{j=1}^{k_n} \mathbb{E}[Y_{nj}^2] = \sum_{j=1}^{k_n} \text{Var}[Y_{nj}] = 1$. By letting $\epsilon \rightarrow 0$, $|t|^3\epsilon/(1-\epsilon) \rightarrow 0$ as desired.

Hence, we see that both terms go to 0 when $n \rightarrow \infty$, so the second term indeed go to 0 \blacksquare

4.1.3 Sufficiency of Lindeberg Condition

To apply [Lindeberg central limit theorem](#), checking the [Lindeberg condition](#) might not be the most efficient way. We now study several sufficient conditions for the [Lindeberg condition](#) to hold.

Corollary 4.1.1 (Hájek-Sidak central limit theorem). If $X_{ni} = c_{ni}Z_i^a$ for all $1 \leq i \leq k_n$ where (Z_{k_n}) are i.i.d. with $\mathbb{E}[Z_i] = \mu$ and $\text{Var}[Z_i] = \sigma^2$. If $\max_{1 \leq i \leq k_n} c_{ni}^2 / \sum_{i=1}^{k_n} c_{ni}^2 \rightarrow 0$ as $n \rightarrow \infty$, then the [Lindeberg condition](#) holds.

^aNote that in this notation, we implicitly assume that when n varies, only c_{ni} varies, but not Z_i .

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Proof. Firstly, we see that $\text{Var}[S_n] = \sum_{j=1}^{k_n} c_{nj}^2 \sigma^2$, hence with our usual notation, we define

$$Y_{nj} := \frac{c_{nj}(Z_j - \mu)}{\sqrt{\sum_{i=1}^{k_n} c_{ni}^2} \sigma} =: \frac{c_{nj}}{\sqrt{\sum_{i=1}^{k_n} c_{ni}^2}} W_j,$$

where $W_j := (Z_j - \mu)/\sigma$. Then, for any $\epsilon > 0$, we can check that [Lindeberg condition](#) as

$$\sum_{j=1}^{k_n} \mathbb{E}[Y_{nj}^2 \mathbb{1}_{|Y_{nj}| > \epsilon}] = \sum_{j=1}^{k_n} \frac{c_{nj}^2}{\sum_{i=1}^{k_n} c_{ni}^2} \mathbb{E} \left[W_j^2 \mathbb{1}_{\frac{|c_{nj}|}{\sqrt{\sum_{i=1}^{k_n} c_{ni}^2}} |W_j| > \epsilon} \right]$$

since the only dependence of j in the expectation is $|c_{nj}|$ in the indicator,

$$\leq \sum_{j=1}^{k_n} \frac{c_{nj}^2}{\sum_{i=1}^{k_n} c_{ni}^2} \mathbb{E} \left[W_j^2 \mathbb{1}_{\max_{1 \leq k \leq k_n} \frac{|c_{nk}|}{\sqrt{\sum_{i=1}^{k_n} c_{ni}^2}} |W_k| > \epsilon} \right]$$

which makes the term in the expectation i.i.d., so we can replace both W_j and W_k by $W := W_1$,

$$\begin{aligned} &= \sum_{j=1}^{k_n} \frac{c_{nj}^2}{\sum_{i=1}^{k_n} c_{ni}^2} \mathbb{E} \left[W^2 \mathbb{1}_{\max_{1 \leq k \leq k_n} \frac{|c_{nk}|}{\sqrt{\sum_{i=1}^{k_n} c_{ni}^2}} |W| > \epsilon} \right] \\ &= \mathbb{E} \left[W^2 \mathbb{1}_{\max_{1 \leq k \leq k_n} \frac{|c_{nk}|}{\sqrt{\sum_{i=1}^{k_n} c_{ni}^2}} |W| > \epsilon} \right]. \end{aligned}$$

Hence, it reduces to show $\mathbb{E}[W^2 \mathbb{1}_{|W| > x}] \rightarrow 0$ as $n \rightarrow \infty$ for $x := \epsilon \sqrt{\sum_{i=1}^{k_n} c_{ni}^2} / \max_{1 \leq k \leq k_n} |c_{nk}|$. From our assumption, for any $\epsilon > 0$, $x \rightarrow \infty$ as $n \rightarrow \infty$, hence the expectation indeed goes to 0 as long as W has finite second moment, which is indeed the case by our assumption. \blacksquare

We see that [Hájek-Sidak central limit theorem](#) is very common in practice.

Intuition. Often time for every n , $c_{ni} = c_n$, the same for every $1 \leq i \leq k_n$. In this case, we may write $X_{ni} = c_n \cdot X_{ni}/c_n =: c_n Z_i$ such that $Z_i := X_{ni}/c_n$ is i.i.d. distributed, ready for checking the [Hájek-Sidak condition](#).

Let's see three examples.

Example (Uniform distribution). For every $n \geq 1$, let $(X_{nk_n}) \stackrel{\text{i.i.d.}}{\sim} \mathcal{U}(-c_n, c_n)$ for some $c_n > 0$. Then we see that $Z_i := X_{ni}/c_n \sim \mathcal{U}(-1, 1)$ is now i.i.d. distributed.

Example (Rademacher distribution). For every $n \geq 1$, let (X_{nk_n}) be i.i.d. such that $\mathbb{P}(X_{ni}/c_n = \pm 1) = 1/2$. Then $Z_i := X_{ni}/c_n$ is now i.i.d. distributed.

Example (Exponential distribution). For every $n \geq 1$, let $(X_{nk_n}) \stackrel{\text{i.i.d.}}{\sim} \text{Exp}(1/c_n)$ for some $c_n > 0$, hence $Z_i := X_{ni}/c_n \sim \text{Exp}(1)$ is now i.i.d. distributed.

On the other hand, if we insist the same setup as in the [Lindeberg central limit theorem](#), it suffices to check a slightly higher moment rather than the truncated one used in the [Lindeberg condition](#).

Corollary 4.1.2 (Lyapunov's central limit theorem). Consider the setup as in the [Lindeberg central limit theorem](#). If $\sum_{j=1}^{k_n} \mathbb{E}[|Y_{nj}|^{2+\delta}] \rightarrow 0$ for some $\delta > 0$, then the [Lindeberg condition](#) holds.

Proof. Fix some $\delta > 0$ such that the assumption holds. Then for any $\epsilon > 0$, we have

$$\sum_{j=1}^{k_n} \mathbb{E}[|Y_{nj}|^2 \mathbb{1}_{|Y_{nj}| > \epsilon}] \leq \sum_{j=1}^{k_n} \mathbb{E} \left[\left(\frac{|Y_{nj}|}{\epsilon} \right)^\delta |Y_{nj}|^2 \mathbb{1}_{|Y_{nj}| > \epsilon} \right] \leq \sum_{j=1}^{k_n} \mathbb{E} \left[\left(\frac{|Y_{nj}|}{\epsilon} \right)^\delta |Y_{nj}|^2 \right] \rightarrow 0$$

by taking $\epsilon^{-\delta}$ out and then the result follows from the assumption. \blacksquare

For bounded random variable, we have a simpler form.

Corollary 4.1.3. Let $|X_{ni}| \leq C_n$ for all $1 \leq i \leq k_n$. Then if $C_n/\sqrt{\text{Var}[S_n]} \rightarrow 0$, the [Lindeberg condition](#) holds. In particular, when $C_n =: C$, it suffices to check $\text{Var}[S_n] \rightarrow \infty$.

Proof. We see that for every $1 \leq j \leq k_n$,

$$|Y_{nj}| = \frac{|X_{nj} - \mathbb{E}[X_{nj}]|}{\sqrt{\text{Var}[S_n]}} \leq \frac{2C_n}{\sqrt{\text{Var}[S_n]}}.$$

In this case, for any $\delta > 0$, recall that $\sum_{j=1}^{k_n} \mathbb{E}[Y_{nj}^2] = 1$, hence

$$\sum_{j=1}^{k_n} \mathbb{E}[|Y_{nj}|^{2+\delta}] \leq \left(\frac{2C_n}{\sqrt{\text{Var}[S_n]}} \right)^\delta \sum_{j=1}^{k_n} \mathbb{E}[Y_{nj}^2] = \left(\frac{2C_n}{\sqrt{\text{Var}[S_n]}} \right)^\delta \rightarrow 0,$$

hence the [Lyapunov's condition](#) holds. \blacksquare

Example (Bernoulli distribution). For every $n \geq 1$, let $X_{ni} \sim \text{Ber}(p_{ni})$ for all $1 \leq i \leq k_n$. Since $X_{ni} \leq 1$, from [Corollary 4.1.3](#), it suffices to check

$$\text{Var}[S_n] = \sum_{i=1}^{k_n} p_{ni}(1 - p_{ni}) \rightarrow \infty.$$

- If $p_{ni} = 1/i$: then $\text{Var}[S_n] = \sum_{i=1}^{k_n} 1/i - \sum_{i=1}^{k_n} 1/i^2 \rightarrow \infty$.
- If $p_{ni} = p_n$: then $\text{Var}[S_n] = k_n p_n(1 - p_n)$. In particular, for $p_n = 1/n$ and $k_n = n$,

$$\text{Var}[S_n] = n \cdot \frac{1}{n} \cdot \frac{n-1}{n} \rightarrow 1 \neq \infty.$$

In general, if $np_n \rightarrow \lambda > 0$, then the sum S_n [converges](#) to $\text{Pois}(\lambda)$.

4.1.4 Testing the I.I.D. Assumption

With this new tool in hand, i.e., the [Lindeberg central limit theorem](#), let's see one illustrative application, i.e., *testing the i.i.d. assumption*.

Problem (Testing the i.i.d. assumption). Consider collecting a sequence of data $X_1, \dots, X_n \sim F$ where F is continuous. We want to test whether X_i 's are i.i.d. from F .

Answer. To do this, consider that for all $i \geq 1$, define the *rank* to be^a

$$R_i = \sum_{j=1}^i \mathbb{1}_{X_j < X_i}.$$

In particular, if $R_i = i$ ($R_i = 1$), then X_i is the largest (smallest) of X_1, \dots, X_i . ⊛

^aWe don't need to worry about the equality in the indicator since F is continuous.

To see how R_i 's help us to decide whether X_i 's are i.i.d., under this hypothesis, we have the following.

Theorem 4.1.2. Let $(X_n) \stackrel{\text{i.i.d.}}{\sim} F$ where F is continuous. Then (R_n) are independent and $\mathbb{P}(R_i = r) = 1/i$ for all $1 \leq r \leq i$, i.e., $R_i \sim \mathcal{U}(\{1, \dots, i\})$. Moreover,

$$\frac{6}{\sqrt{n^3}} \left(\sum_{i=1}^n R_i - \frac{n(n+1)}{4} \right) \xrightarrow{D} \mathcal{N}(0, 1).$$

Proof. Since X_i 's are i.i.d. and F is continuous (hence no ties),

$$\mathbb{P}(X_{\sigma(1)} < X_{\sigma(2)} < \dots < X_{\sigma(i)}) = \frac{1}{i!}$$

for any permutation σ of $\{1, \dots, i\}$. This implies $\mathbb{P}(R_1 = r_1, \dots, R_i = r_i) = 1/i!$, hence

$$\mathbb{P}(R_i = r) = \sum_{r_1, \dots, r_{i-1}} \mathbb{P}(R_1 = r_1, \dots, R_{i-1} = r_{i-1}, R_i = r) = (i-1)! \cdot \frac{1}{i!} = \frac{1}{i}.$$

This proves the first part. Now, observe that $\mathbb{E}[\sum_{i=1}^n R_i] = \sum_{i=1}^n \frac{i}{2} = n(n+1)/4$ and

$$\text{Var} \left[\sum_{i=1}^n R_i \right] = \sum_{i=1}^n \frac{i^2 - 1}{12} = \frac{n(2n^2 + 3n - 5)}{72} \sim \frac{n^3}{36},$$

hence if the [Lindeberg central limit theorem](#) holds, then we're done. We check [Corollary 4.1.3](#): by noting that $|R_i| \leq n$, we indeed have $n/\sqrt{\text{Var}[\sum_{i=1}^n R_i]} \rightarrow 0$. ■

Remark (Record). We may instead consider the *record* $Z_i = \mathbb{1}_{R_i=i}$.^a Since $R_i \leq i$, $\mathbb{P}(Z_i = 1) = 1/i$, i.e., $Z_i \sim \text{Ber}(1/i)$. Then the [previous example](#) gives asymptotic normality of $\sum_{i=1}^n Z_i$.

^aIntuitive, it indicates when X_i is the largest among X_1, \dots, X_{i-1} .

In either case (R_i or Z_i), the above gives us some hints about how to deal with this kind of problems.

Intuition. Find some statistics whose distribution is *independent* of the underlying F under H_0 .

4.2 Testing for Symmetry

Our next goal is to apply the intuition we get from the i.i.d. problem, and apply it to the problem of [testing for symmetry](#).

Problem 4.2.1 (Testing for symmetry). Let $X, X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F$ where F is a continuous distribution function. The problem of *testing symmetry* is to test the null hypothesis, H_0 , that F is symmetric about 0, i.e., whether $X \stackrel{D}{=} -X$.

4.2.1 Motivation: Two-Sample Problem with no I.I.D. Assumption

Symmetry testing is important since it justifies various tests we use in practice for the classical *two-sample problem*, even if the i.i.d. assumption is violated.

Problem 4.2.2 (Two-sample problem). Given the data of two random samples obtained from a different given population. The *two-sample problem* considers the task of determining whether the difference between these two populations is statistically significant.

One classical example of a *two-sample problem* is the *treatment effect testing*.

Example (Treatment effect). Consider a treatment group and a control group (the two populations), and we observe Y_i^T 's and Y_i^C 's from the treatment and the control group, respectively. The goal is to test whether the treatment takes effect. It's common to pair the data together and get $(Y_1^T, Y_1^C), \dots, (Y_n^T, Y_n^C)$, and let $X_i := Y_i^T - Y_i^C$ for all $1 \leq i \leq n$.

Usually, we assume X_1, \dots, X_n are i.i.d. with mean μ and finite variance, and the null hypothesis we want to test is $H_0: \mu = 0$. As X_i 's are i.i.d., we may use the *t*-test to the *treatment effect problem*, namely, we reject H_0 if the *t*-statistic is too large, i.e.,

$$T_n = \sqrt{n} \frac{\bar{X}_n}{\hat{\sigma}_n} = \frac{\sum_{i=1}^n X_i}{\sqrt{n} \sqrt{\frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X}_n^2}} = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} > Z_\alpha$$

for some $\alpha > 0$. Since $\text{Var}[X_i] < \infty$, $T_n \xrightarrow{D} \mathcal{N}(0, 1)$ under H_0 .

Problem. In reality, X_i 's are never i.i.d., but people still use the *t*-test. Why?

Answer. This is because we usually give the treatment in a “randomized” way. In addition, we can “condition” on the observed data $|X_i|$'s. In this case, we can design a hypothesis testing such that it only depends on statistics that are now i.i.d., and by applying the *central limit theorem*, we will get something like $\sum_{i=1}^n X_i / \sqrt{\sum_{i=1}^n X_i^2}$, pretty much like what we have in the *t*-test. \circledast

Lecture 18: Wilcoxon Signed-Rank Test

Before we explain the above answer, let's point out other potential problems: firstly, since even for i.i.d. X_i 's, the results for T_n is asymptotic, and we might wonder what's the rate of convergence. 26 Mar. 9:30

Problem. How fast does $T_n \xrightarrow{D} \mathcal{N}(0, 1)$ under H_0 ?

Answer. It depends heavily on F , so it's unlikely to get a universal answer. \circledast

On the other hand, what if the underlying distribution has heavy tails?

Problem. What if $\text{Var}[X_i]$, or even $\mathbb{E}[X_i]$, doesn't exist?

Answer. This is something we will solve along the way when we deal with the i.i.d. problem. \circledast

Now, we focus on the non-i.i.d. problem. It turns out that if each unit in each pair is equally likely to be in the treatment and the control group, independently of what happens in the other pairs, then $H_0: \mu = 0$ can be expressed as for all $1 \leq i \leq n$, $\mathbb{P}(X_i = \pm |X_i| \mid |X_j| = |x_j|, 1 \leq j \leq n) = 1/2$, i.e.,

$$H_0: \mathbb{P}(\text{sgn}(X_i) = \pm 1 \mid |X_j| = |x_j|, 1 \leq j \leq n) = \frac{1}{2} \text{ for all } 1 \leq i \leq n.$$

Hence, we can test this “conditionally on the observed absolute values of X_1, \dots, X_n ,” i.e., given $|X_j| = |x_j|$ for $1 \leq j \leq n$. This is exactly the problem of [testing for symmetry](#).

Intuition. We see that now under H_0 , $\text{sgn}(X_i)$ ’s are i.i.d., overcoming the problem of non-i.i.d.

Note. In practice, when doing inference, we usually implicitly condition on $|X_j|$ ’s too: by doing inference on the selected “design points”, we’re essentially condition on those.

Example (*t*-statistic for two-sample problem). Consider writing $\sum_{i=1}^n X_i = \sum_{i=1}^n |X_i| \text{sgn}(X_i) = \sum_{i=1}^n |x_i| \text{sgn}(X_i)$. Then by treating $|x_i|$ ’s as constants, under H_0 ,

$$T_n = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n x_i^2}} \sqrt{\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n (x_i - \bar{x}_n)^2}} = \frac{\sum_{i=1}^n |X_i| \text{sgn}(X_i)}{\sqrt{\sum_{i=1}^n x_i^2}} \sqrt{\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n (x_i - \bar{x}_n)^2}} \xrightarrow{D} \mathcal{N}(0, 1) \quad (4.1)$$

by the [Lindeberg central limit theorem](#) as long as $\max_{1 \leq i \leq n} x_i^2 / \sqrt{\sum_{j=1}^n x_j^2} \rightarrow 0$. We can then use the left-hand side for our test as usual.

Let’s use this example as our running example for the problem of [testing for symmetry](#).

4.2.2 Student’s *t*-Test, Sign Test, and Wilcoxon Signed-Rank Test

Returning to the general problem of [testing for symmetry](#), let $X, X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F$.

As previously seen. Now H_0 is defined to be whether F is symmetric about 0, or $X \stackrel{D}{=} -X$.

As suggested by the [running example](#), one standard way to solve this is still considering the *t*-statistic,

$$T_n = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}},$$

and reject H_0 when T_n is larger than some critical value.

As previously seen. If X has positive and finite variance, $\mathbb{P}_{H_0}(T_n > Z_\alpha) \rightarrow \alpha$ as $n \rightarrow \infty$. However, as we have seen, the rate of such an approximation depends on the unknown F .

An alternative and even more classical way to solve this, which works even if X does not have any moments, is to consider the [sign statistic](#).

Definition 4.2.1 (Sign statistic). Given a sample X_1, \dots, X_n , the *sign statistic* is defined as

$$\text{sign}_n := \sum_{i=1}^n \text{sgn}(X_i) = 2 \sum_{i=1}^n \mathbb{1}_{X_i > 0} - n.$$

Observe that under H_0 , the distribution of sign_n is independent of F since $\sum_{i=1}^n \mathbb{1}_{X_i > 0} \stackrel{H_0}{\sim} \text{Bin}(n, 1/2)$, i.e., we can conduct the hypothesis test non-asymptotically. On the other hand, under H_0 , even if we consider its asymptotic behavior, i.e., from the usual [central limit theorem](#),

$$\frac{1}{\sqrt{n}} \text{sign}_n = \frac{\sum_{i=1}^n \text{sgn}(X_i)}{\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1),$$

the quality of the approximation doesn’t depend on F , and we reject H_0 when $\sum_{i=1}^n \text{sgn}(x_i) / \sqrt{n} \geq Z_\alpha$.

Example (Sign-statistic for two-sample problem). One can also motivate the [sign statistic](#) from the use of *t*-statistic T_n in our [running example](#), i.e., by simply ignoring $|X_i|$ ’s used in T_n to avoid conditioning on them. Hence, we reject H_0 if $\sum_{i=1}^n \text{sgn}(X_i) = \text{sign}_n$ is large.

The sign test with the [sign statistic](#) is simple, but it's questionable that how powerful it is as we're not using $|X_i|$'s at all. In view of the above intuition, let's utilize $|X_i|$'s in other ways.

Intuition. We can utilize $|X_i|$'s by replacing $|X_i|$'s in [Equation 4.1](#) by their “rank” information.

In particular, consider the *rank* R_i for $1 \leq i \leq n$

$$R_i := \sum_{j=1}^n \mathbb{1}_{|X_j| \leq |X_i|}.$$

It's clear that R_i 's are dependent to each other.

As previously seen. This is different from the previous definition $R_i = \sum_{j=1}^i \mathbb{1}_{X_j < X_i}$.

Following the intuition, consider the so-called [Wilcoxon signed-rank statistic](#).

Definition 4.2.2 (Wilcoxon signed-rank statistic). Given a sample X_1, \dots, X_n , the *Wilcoxon signed-rank statistic* is defined as

$$W_n = \sum_{i=1}^n R_i \operatorname{sgn}(X_i).$$

Then, the corresponding Wilcoxon signed-rank test is to reject H_0 if W_n is “large” as usual.

Note. Under H_0 , W_n is again independent of F .

However, it turns out that it's quite hard to get the exact critical value for a general problem of [testing for symmetry](#).

Example (Wilcoxon signed-rank statistic for two-sample problem). If we condition on $|X_j|$'s as in the [running example](#), without loss of generality, $R_i = i$ for every $1 \leq i \leq n$.^a In this case, $W_n = \sum_{i=1}^n i \operatorname{sgn}(X_i)$, hence under H_0 ,

$$\frac{\sum_{i=1}^n i \operatorname{sgn}(X_i)}{\sqrt{\sum_{i=1}^n i^2}} \xrightarrow{D} \mathcal{N}(0, 1) \Rightarrow \frac{W_n}{\sqrt{n^3}} \xrightarrow{D} \mathcal{N}(0, 1/3)$$

by the [Lindeberg central limit theorem](#). We can then use this to test H_0 .

^aThis is doable since $|X_j|$'s are treated as constants.

Without conditioning on $|X_j|$'s, the above doesn't hold. However, some sign tests are still possible.

Intuition (Another sign test). Consider $H_0: \operatorname{sgn}(X_i) = \epsilon_i$ where ϵ_i is a Rademacher random variable, i.e., $\mathbb{P}(X > 0) = \mathbb{P}(X < 0) = 1/2$ if X is symmetric. This doesn't need to assume $\mathbb{E}[X_i] < \infty$.

If we further assume $\operatorname{Var}[X_i] < \infty$, then it's possible to say something about W_n without conditioning on $|X_j|$'s. Let's first see a useful characterization of W_n to get started.

Proposition 4.2.1. Let $h(x_1, x_2) := \mathbb{1}_{x_1 + x_2 \geq 0}$. Then we have

$$W_n = \sum_{i=1}^n \operatorname{sgn}(X_i) + \sum_{i \neq j} \left(h(X_i, X_j) - \frac{1}{2} \right).$$

Proof. Consider $W_n =: W_n^+ - W_n^-$ where

$$W_n^+ = \sum_{i=1}^n R_i \mathbb{1}_{X_i > 0}, \text{ and } W_n^- = \sum_{i=1}^n R_i \mathbb{1}_{X_i < 0}.$$

Some calculation gives the following.

Claim. $W_n^+ - \sum_{i=1}^n \mathbb{1}_{X_i > 0} = \frac{1}{2} \sum_{i \neq j} \mathbb{1}_{X_i + X_j \geq 0}$ and $W_n^- - \sum_{i=1}^n \mathbb{1}_{X_i < 0} = \frac{1}{2} \sum_{i \neq j} \mathbb{1}_{X_i + X_j < 0}$.

Proof. Let's first focus on W_n^+ . We see that

$$W_n^+ = \sum_{i=1}^n \left(\sum_{j=1}^n \mathbb{1}_{|X_j| \leq |X_i|} \right) \mathbb{1}_{X_i > 0} = \sum_{i=1}^n \mathbb{1}_{X_i > 0} + \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} \mathbb{1}_{|X_j| \leq |X_i|, X_i > 0},$$

Let's abbreviate the argument $1 \leq i, j \leq n: i \neq j$ in the double summation by $i \neq j$, we have

$$\begin{aligned} W_n^+ - \sum_{i=1}^n \mathbb{1}_{X_i > 0} &= \sum_{i \neq j} \mathbb{1}_{|X_j| \leq |X_i|} \\ &= \sum_{i \neq j} \mathbb{1}_{-X_i \leq X_j \leq X_i} \\ &= \underbrace{\sum_{i \neq j} \mathbb{1}_{X_i + X_j \geq 0} \mathbb{1}_{X_j \leq X_i}}_{S_1} = \sum_{i \neq j} \mathbb{1}_{X_i + X_j \geq 0} - \underbrace{\sum_{i \neq j} \mathbb{1}_{X_i + X_j \geq 0} \mathbb{1}_{X_j > X_i}}_{S_2}, \end{aligned}$$

where the last equality can be justified by the fact that $1 = \mathbb{1}_{X_j \leq X_i} + \mathbb{1}_{X_j > X_i}$. Observe that

$$S_1 = \sum_{i \neq j} \mathbb{1}_{X_i + X_j \geq 0} \mathbb{1}_{X_j \leq X_i} = \sum_{i \neq j} \mathbb{1}_{X_j + X_i \geq 0} \mathbb{1}_{X_i \leq X_j} = \sum_{i \neq j} \mathbb{1}_{X_i + X_j \geq 0} \mathbb{1}_{X_j > X_i} = S_2,$$

since F is continuous. Hence, by letting $S := S_1 = S_2$, the above calculation gives

$$W_n^+ - \sum_{i=1}^n \mathbb{1}_{X_i > 0} = S = \sum_{i \neq j} \mathbb{1}_{X_i + X_j \geq 0} - S \Rightarrow S = W_n^+ - \sum_{i=1}^n \mathbb{1}_{X_i > 0} = \frac{1}{2} \sum_{i \neq j} \mathbb{1}_{X_i + X_j \geq 0}.$$

The results for W_n^- follows similarly. ⊗

By subtracting the above, since $\mathbb{1}_{X_i > 0} - \mathbb{1}_{X_i < 0} = \text{sgn}(X_i)$, we have

$$W_n - \sum_{i=1}^n \text{sgn}(X_i) = \frac{1}{2} \sum_{i \neq j} (\mathbb{1}_{X_i + X_j \geq 0} - \mathbb{1}_{X_i + X_j < 0}) = \frac{1}{2} \sum_{i \neq j} (2\mathbb{1}_{X_i + X_j \geq 0} - 1).$$

Plugging the definition of h yields the result. ■

Hence, from [Proposition 4.2.1](#), we should first study $\sum_{i \neq j} h(X_i, X_j)$. First, consider testing

$$H_0: X, X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F \text{ where } F \text{ is continuous such that } X \stackrel{D}{=} -X,$$

i.e., $\mathbb{P}(X \geq x) = \mathbb{P}(X \leq -x)$ for all $x \in \mathbb{R}$. Note that we're not assuming $\mathbb{E}[X_i]$ to exists.

Note. Since X_i 's are i.i.d. under H_0 , $h(X_i, X_j)$'s are identically distributed under H_0 for $i \neq j$.

However, it's clear that $h(X_i, X_j)$'s are not independent. Anyway, recall what we're trying to study.

As previously seen. From [Proposition 4.2.1](#), under H_0 (in particular, F being continuous),

$$W_n - \sum_{i=1}^n \text{sgn}(X_i) = \sum_{i \neq j} \left(h(X_i, X_j) - \frac{1}{2} \right)$$

This is a sum of identically distributed random variables subtracting something.

Intuition. If $1/2$ is the expectation of $h(X_i, X_j)$ for $i \neq j$, we're getting closer to the familiar form.

Indeed, under H_0 , $\mathbb{E}[h(X_i, X_j)] = 1/2$ for $i \neq j$ as

$$\begin{aligned}\mathbb{E}_{H_0}[h(X_i, X_j)] &= \mathbb{E}_{H_0}[h(X_1, X_2)] = \mathbb{P}(X_1 + X_2 \geq 0) \\ &= \mathbb{P}(X_1 \geq -X_2) \\ &= \int_{\mathbb{R}} \mathbb{P}(X_1 \geq -x) F(dx) = \int_{\mathbb{R}} \mathbb{P}(X \leq x) F(dx) = \mathbb{E}[F(X)] = \frac{1}{2}\end{aligned}$$

since F is continuous, and $F(X) \sim \mathcal{U}(0, 1)$ as we have shown in the homework. Hence, under H_0 ,

$$\sum_{i \neq j} \left(h(X_i, X_j) - \frac{1}{2} \right) = 2 \sum_{i < j} \left(h(X_i, X_j) - \frac{1}{2} \right) = 2 \sum_{i < j} (h(X_i, X_j) - \mathbb{E}[h(X_i, X_j)]).$$

By rescaling by the number of terms in the summation, we're looking at

$$\frac{1}{\binom{n}{2}} \left(W_n - \sum_{i=1}^n \text{sgn}(X_i) \right) = \frac{2}{\binom{n}{2}} \sum_{i < j} (h(X_i, X_j) - \mathbb{E}[h(X_i, X_j)]) =: 2 \left(U_n - \frac{1}{2} \right)$$

where we define U_n for some permutation symmetric function h^1 as

$$U_n := \frac{1}{\binom{n}{2}} \sum_{i < j} h(X_i, X_j).$$

This is an **U -statistic**, where U stands for unbiased. We will formally define it later, and we will show that by multiplying \sqrt{n} on the both sides, this **converges** to a standard normal, i.e.,

$$\frac{\sqrt{n}}{2 \binom{n}{2}} \left(W_n - \sum_{i=1}^n \text{sgn}(X_i) \right) = \sqrt{n} \left(U_n - \frac{1}{2} \right) \xrightarrow{D} \mathcal{N}(0, 1/3). \quad (4.2)$$

Note. This will imply an asymptotic distribution for W_n exactly alone since under H_0 ,

$$\frac{\sqrt{n}}{n(n-1)} \sum_{i=1}^n \text{sgn}(X_i) = \frac{1}{n-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \text{sgn}(X_i) \right) \xrightarrow{p} 0.$$

4.3 U -Statistics

In this section, we develop the asymptotic theory for **U -statistics**. While postponing defining **U -statistics**, we first outline what we're going to do. The main tool is the following.

Proposition 4.3.1. Given two sequences (S_n) , (\tilde{S}_n) of random variables in L^2 , if they satisfy $\text{Var}[S_n]/\text{Var}[\tilde{S}_n] \rightarrow 1$ and $\text{Corr}(S_n, \tilde{S}_n) \rightarrow 1$, then

$$\frac{\text{Var}[S_n - \tilde{S}_n]}{\text{Var}[S_n]} \rightarrow 0 \Leftrightarrow \frac{S_n - \mathbb{E}[S_n]}{\sqrt{\text{Var}[S_n]}} - \frac{\tilde{S}_n - \mathbb{E}[\tilde{S}_n]}{\sqrt{\text{Var}[\tilde{S}_n]}} \xrightarrow{L^2} 0.$$

Proof. Since for $n \in \mathbb{N}$, $\text{Var}[S_n - \tilde{S}_n] = \text{Var}[S_n] - \text{Var}[\tilde{S}_n] - 2\sqrt{\text{Var}[S_n] \text{Var}[\tilde{S}_n]} \text{Corr}(S_n, \tilde{S}_n)$, i.e.,

$$\frac{\text{Var}[S_n - \tilde{S}_n]}{\text{Var}[S_n]} = 1 + \frac{\text{Var}[\tilde{S}_n]}{\text{Var}[S_n]} - 2\sqrt{\frac{\text{Var}[\tilde{S}_n]}{\text{Var}[S_n]}} \text{Corr}(S_n, \tilde{S}_n).$$

The right-hand side goes to 0 under the assumptions, proving the result. ■

Hence, to show $(S_n - \mathbb{E}[S_n])/\sqrt{\text{Var}[S_n]} \xrightarrow{D} Y$, we may find (\tilde{S}_n) such that $(\tilde{S}_n - \mathbb{E}[\tilde{S}_n])/\sqrt{\text{Var}[\tilde{S}_n]} \xrightarrow{D} Y$ and apply **Proposition 4.3.1** with **Theorem 2.2.3**. In particular, after defining what's an **U -statistic**, we will then apply the above strategy to which.

¹Indeed, since $\mathbb{E}[h(X_1, X_2)] = \mathbb{P}(X_1 + X_2 \geq 0)$, so $h(X_1, X_2) = h(X_2, X_1)$.

Lecture 19: Projection and U -Statistic

4.3.1 Projection

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A natural approach to find such (\tilde{S}_n) is to “project” (S_n) onto some space, which potentially has nice properties for us to do the analysis. Consider the space $L^2 = \{Y : \|Y\| = \sqrt{\mathbb{E}[Y^2]} < \infty\}$, and let $K \subseteq L^2$ with $Y \in L^2$. The goal is to approximate Y by a sequence in K .

Definition 4.3.1 (Projection). Given a subspace $K \subseteq L^2$, $Y^* \in K$ is a *projection* of $Y \in L^2$ onto K if $\|Y^* - Y\| \leq \|Z - Y\|$ for all $Z \in K$.

The following characterization of the [projection](#) is useful in our analysis.

Proposition 4.3.2. Suppose $K \subseteq L^2$ is a linear subspace. If $Y^* \in K$ and $\mathbb{E}[Y^*Z] = \mathbb{E}[YZ]$ for every $Z \in K$, then Y^* is a [projection](#) of Y onto K . If in addition, $1 \in K$, then $\mathbb{E}[YY^*] = \mathbb{E}[(Y^*)^2]$ and $\text{Cov}[Y, Y^*] = \text{Var}[Y^*]$. In particular, $\text{Corr}(Y, Y^*) = \sqrt{\text{Var}[Y^*]/\text{Var}[Y]}$.

Proof. We see that for all $Z \in K$,

$$\|Y - Z\|^2 = \|Y - Y^*\|^2 + \|Y^* - Z\|^2 + 2\mathbb{E}[(Y - Y^*)(Y^* - Z)]$$

from the assumption, for every $W \in K$, $\mathbb{E}[(Y^* - Y)W] = 0$, with $W := Y^* - Z \in K$,

$$= \|Y - Y^*\|^2 + \|Y^* - Z\|^2,$$

which is greater than $\|Y - Y^*\|^2$. Now, if $1 \in K$, by taking $Z = 1$, $\mathbb{E}[Y^*] = \mathbb{E}[Y]$. On the other hand, for $Z = Y^*$, we have $\mathbb{E}[YY^*] = \mathbb{E}[(Y^*)^2]$. Hence,

$$\text{Cov}[Y, Y^*] = \mathbb{E}[YY^*] - \mathbb{E}[Y]\mathbb{E}[Y^*] = \mathbb{E}[(Y^*)^2] - \mathbb{E}[Y^*]^2 = \text{Var}[Y^*].$$

The correlation is then $\text{Corr}(Y, Y^*) = \text{Var}[Y^*]/\sqrt{\text{Var}[Y]\text{Var}[Y^*]} = \sqrt{\text{Var}[Y^*]/\text{Var}[Y]}$. ■

If we combine the above, our plan is clear now. In particular, we have the following.

Corollary 4.3.1. For each $n \in \mathbb{N}$, let \tilde{S}_n be a [projection](#) of S_n onto a linear subspace K_n of L^2 with $1 \in K_n$. If $\text{Var}[S_n]/\text{Var}[\tilde{S}_n] \rightarrow 1$, then

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\text{Var}[S_n]}} - \frac{\tilde{S}_n - \mathbb{E}[\tilde{S}_n]}{\sqrt{\text{Var}[\tilde{S}_n]}} \xrightarrow{L^2} 0.$$

Proof. Since $\text{Var}[S_n]/\text{Var}[\tilde{S}_n] \rightarrow 1$, from [Proposition 4.3.2](#), $\text{Corr}(Y, Y^*) \rightarrow 1$ is automatically satisfied. The result then follows from [Proposition 4.3.1](#). ■

With [Corollary 4.3.1](#), the goal is to find suitable K_n 's such that $\text{Var}[S_n]/\text{Var}[\tilde{S}_n] \rightarrow 1$, and $\tilde{S}_n - \mathbb{E}[\tilde{S}_n]$ [converges](#) somewhere. However, we should first understand how to find [projections](#) in practice.

Example. Let X be a random vector and $K = \{g(X) \in L^2\}$. Then $Y^* = \mathbb{E}[Y | X]$ is a [projection](#) of $Y \in L^2$ onto K . More generally, let X_1, \dots, X_n be a sequence of random vectors, and let

$$K_n = \{g(X_1, \dots, X_n) \in L^2\}.$$

Then $Y^* = \mathbb{E}[Y | X_1, \dots, X_n]$ is a [projection](#) of $Y \in L^2$ onto K_n .

Proof. For all $g \in K$, from the basic properties for conditional expectation, we have

$$\mathbb{E}[\mathbb{E}[Y | X] \cdot g(X)] = \mathbb{E}[\mathbb{E}[g(X) \cdot Y | X]] = \mathbb{E}[g(X) \cdot Y].$$

Then from [Proposition 4.3.2](#), the result follows. ⊛

A more elaborated example is the following, which turns out to be flexible enough for our purpose.

Example. Let X_1, \dots, X_n be a sequence of independent random vectors, and let

$$K_n = \left\{ \sum_{i=1}^n g_i(X_i) : g_i(X_i) \in L^2 \text{ for all } 1 \leq i \leq n \right\}.$$

In this case, $Y^* - \mathbb{E}[Y] = \sum_{i=1}^n \mathbb{E}[Y - \mathbb{E}[Y] \mid X_i]$, where Y^* is a **projection** of $Y \in L^2$ onto K_n .

Proof. We want to show that for all $\sum_{i=1}^n g_i \in K_n$, $\mathbb{E}[Y^* \sum_{i=1}^n g_i(X_i)] = \mathbb{E}[Y \sum_{i=1}^n g_i(X_i)]$, i.e.,

$$\mathbb{E} \left[(Y^* - \mathbb{E}[Y^*]) \sum_{i=1}^n g_i(X_i) \right] = \mathbb{E} \left[(Y - \mathbb{E}[Y]) \sum_{i=1}^n g_i(X_i) \right]$$

since $\mathbb{E}[Y^*] = \mathbb{E}[Y]$. Expanding the left-hand side, and using $\mathbb{E}[Y^*] = \mathbb{E}[Y]$ again,

$$\begin{aligned} \mathbb{E} \left[(Y^* - \mathbb{E}[Y^*]) \sum_{i=1}^n g_i(X_i) \right] &= \mathbb{E} \left[\left(\sum_{i=1}^n \mathbb{E}[Y - \mathbb{E}[Y] \mid X_i] \right) \sum_{j=1}^n g_j(X_j) \right] \\ &= \sum_{i=1}^n \sum_{j=1}^n \mathbb{E} [\mathbb{E}[(Y - \mathbb{E}[Y]) \mid X_i] g_j(X_j)] \end{aligned}$$

observe that for $i \neq j$, $\mathbb{E}[\mathbb{E}[(Y - \mathbb{E}[Y]) \mid X_i]]$ is independent of $g_j(X_j)$, hence

$$\begin{aligned} &= \sum_{i \neq j} \mathbb{E}[(Y - \mathbb{E}[Y])] \mathbb{E}[g_j(X_j)] + \sum_{i=1}^n \mathbb{E} [\mathbb{E}[Y - \mathbb{E}[Y] \mid X_i] \cdot g_i(X_i)] \\ &= \sum_{i=1}^n \mathbb{E} [\mathbb{E}[(Y - \mathbb{E}[Y]) \cdot g_i(X_i) \mid X_i]] \\ &= \mathbb{E} \left[\sum_{i=1}^n (Y - \mathbb{E}[Y]) g_i(X_i) \right], \end{aligned}$$

where we again use the property of conditional expectation. *

We see that the **projection** in the above example is a sum of independent random variables, so we can apply the **Lindeberg central limit theorem** to establish asymptotic normality.

Remark. In particular, with the help of **Corollary 4.3.1**, we only need to make sure that K_n is a linear subspace that contains constants and $\text{Var}[S_n] / \text{Var}[\tilde{S}_n] \rightarrow 1$ as $n \rightarrow \infty$.

Remark. Since we're using **Lindeberg central limit theorem**, it's possible to generalize the above to the case of triangular array of random variables X_{n1}, \dots, X_{nn} with what we're going to do next.

4.3.2 Asymptotic Distribution of U -Statistic

Now, let's start developing a theory for the **U -statistics**. First, consider the following.

Definition 4.3.2 (U -statistic). Given a sequence of i.i.d. random vectors (X_n) and a permutation symmetric function $h: \mathbb{R}^m \rightarrow \mathbb{R}^a$ with $n \geq m$, the U -statistic is defined as

$$U_n = \frac{1}{\binom{n}{m}} \sum_{\substack{i_1, \dots, i_m \\ \{i_1, \dots, i_m\} \subseteq [n]}} h(X_{i_1}, \dots, X_{i_m}).$$

^aWe refer to h as U_n 's *kernel*.

Notation. Let $[n] := \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$.

Remark. The U -statistic is an unbiased estimator of $\theta = \mathbb{E}[h(X_1, \dots, X_m)]$, and in particular, an average of dependent, but identically distributed random variables.

Note. Since h is permutation symmetric, the order of indices used in the argument doesn't matter.

Example (Wilcoxon signed-rank statistic). When studying the Wilcoxon signed-rank statistic W_n , we see that $(W_n - \sum_{i=1}^n \text{sgn}(X_i)) / \binom{n}{2} = 2U_n - 1$ where U_n is an U -statistic defined as^a

$$U_n = \frac{1}{\binom{n}{2}} \sum_{i < j} h(X_i, X_j).$$

^aNote that $i < j$ is the same as $\{i, j\} \subseteq [n]$. Indeed, one can use $i_1 < \dots < i_m$ without loss of generality.

From Corollary 4.3.1, given the projection U_n^* of U_n onto $K_n = \{\sum_{i=1}^n g_i(X_i)\}$, we want to show

$$\left| \frac{U_n - \mathbb{E}[U_n]}{\sqrt{\text{Var}[U_n]}} - \frac{U_n^* - \mathbb{E}[U_n^*]}{\sqrt{\text{Var}[U_n]}} \right| \xrightarrow{p} 0,$$

and establish the asymptotic normality of U_n . For this, assuming that $U_n \in L^2$ for each $n \geq m$, i.e., $\text{Var}[h(X_1, \dots, X_m)] < \infty$. Let's first find U_n^* .

Proposition 4.3.3. For every $n \geq m$, let $K_n = \{\sum_{i=1}^n g_i(X_i) : g_i(X_i) \in L^2\}$. Furthermore, let $h^*(x) := \mathbb{E}[h(x, X_2, \dots, X_m)]$, then the projection of U_n onto K_n is given by

$$U_n^* = \mathbb{E}[U_n] + \frac{m}{n} \sum_{k=1}^n (h^*(X_k) - \mathbb{E}[U_n]).$$

Proof. Denote $\theta := \mathbb{E}[U_n] = \mathbb{E}[U_n^*]$ (recall Proposition 4.3.2), then from the last example,

$$\begin{aligned} U_n^* - \theta &= \sum_{i=1}^n \mathbb{E}[U_n - \theta \mid X_i] \\ &= \sum_{k=1}^n \frac{1}{\binom{n}{m}} \sum_{\{i_1, \dots, i_m\} \subseteq [n]} \mathbb{E}[h(X_{i_1}, \dots, X_{i_m}) - \theta \mid X_k] \\ &= \sum_{k=1}^n \frac{1}{\binom{n}{m}} \sum_{\{i_1, \dots, i_m\} \subseteq [n]} (\mathbb{E}[h(X_{i_1}, \dots, X_{i_m}) \mid X_k] - \theta). \end{aligned}$$

When $k \neq i_1, \dots, i_m$, the conditional expectation becomes an unconditional one, which is θ , i.e., the only terms survive is when $i_j = k$ for some $1 \leq j \leq m$. Hence,

$$= \sum_{k=1}^n \frac{1}{\binom{n}{m}} \sum_{\{i_2, \dots, i_m\} \subseteq [n] \setminus \{k\}} (\mathbb{E}[h(X_k, X_{i_2}, \dots, X_{i_m}) \mid X_k] - \theta)$$

as X_i 's are i.i.d. and h is permutation symmetry, with h^* , we have

$$\begin{aligned} &= \sum_{k=1}^n \frac{1}{\binom{n}{m}} \sum_{\{i_2, \dots, i_m\} \subseteq [n] \setminus \{k\}} (h^*(X_k) - \theta) \\ &= \sum_{k=1}^n \frac{1}{\binom{n}{m}} \binom{n-1}{m-1} (h^*(X_k) - \theta) \\ &= \frac{m}{n} \sum_{k=1}^n (h^*(X_k) - \theta). \end{aligned}$$

Rearranging the terms gives the result. ■

As $h^*(X_k)$'s are i.i.d., we can apply the central limit theorem if $\text{Var}[h^*(X)] \in (0, \infty)$ and get

$$\sqrt{n}(U_n^* - \theta) \xrightarrow{D} \mathcal{N}(0, m^2 \text{Var}[h^*(X)]).$$

Hence, we just need to show that $\text{Var}[U_n]/\text{Var}[U_n^*] \rightarrow 1$. Firstly, since

$$\text{Var}[U_n^*] = \frac{m^2}{n} \text{Var}[h^*(X_i)],$$

it reduces to show $\text{Var}[U_n] \sim m^2 \text{Var}[h^*(X_i)]/n$. We have the following.

Proposition 4.3.4. For every $n \geq 2m - 1$, we have

$$\binom{n}{m} \text{Var}[U_n] = \sum_{r=1}^m \binom{m}{r} \binom{n-m}{m-r} \xi_{r,m},$$

where for every $r \in [m]$, let $\{i_{r+1}, \dots, i_m\} \cap \{i'_{r+1}, \dots, i'_m\}$ be disjoint subsets of $[n]$ and set

$$\xi_{r,m} := \text{Cov}[h(X_1, \dots, X_r, X_{i_{r+1}}, \dots, X_{i_m}), h(X_1, \dots, X_r, X_{i'_{r+1}}, \dots, X_{i'_m})].$$

Proof. By the definition of the variance, we have

$$\begin{aligned} \binom{n}{m}^2 \text{Var}[U_n] &= \sum_{\{i_1, \dots, i_m\} \subseteq [n]} \text{Var}[h(X_{i_1}, \dots, X_{i_m})] \\ &\quad + \sum_{\substack{\{i_1, \dots, i_m\}, \{i'_1, \dots, i'_m\} \subseteq [n] \\ \{i_1, \dots, i_m\} \neq \{i'_1, \dots, i'_m\}}} \text{Cov}[h(X_{i_1}, \dots, X_{i_m}), h(X_{i'_1}, \dots, X_{i'_m})]. \end{aligned}$$

The variance terms are clear since for every subset, they're all the same, hence

$$\sum_{\{i_1, \dots, i_m\} \subseteq [n]} \text{Var}[h(X_{i_1}, \dots, X_{i_m})] = \binom{n}{m} \text{Var}[h(X_1, \dots, X_m)] = \binom{n}{m} \xi_{m,m}.$$

For the covariance terms, if $\{i_1, \dots, i_m\} \cap \{i'_1, \dots, i'_m\} = \emptyset$, the covariance vanishes from independence. Hence, consider iterate through subsets of i 's and i' 's with $r \geq 1$ common indices.

Intuition. Let $1, \dots, r$ be the common indices between $\{i_1, \dots, i_m\}$ and $\{i'_1, \dots, i'_m\}$, i.e., $[r] = \{i_1, \dots, i_m\} \cap \{i'_1, \dots, i'_m\}$. In this case, the covariance term becomes

$$\xi_{r,m} = \text{Cov}[h(X_1, \dots, X_r, X_{i_{r+1}}, \dots, X_{i_m}), h(X_1, \dots, X_r, X_{i'_{r+1}}, \dots, X_{i'_m})]$$

where $\{i_{r+1}, \dots, i_m\} \cap \{i'_{r+1}, \dots, i'_m\} = \emptyset$. Such a covariance is fixed for every r , regardless of other indices i_j, i'_j for $r+1 \leq j \leq m$ since $X_{i_{r+1}}, \dots, X_{i_m}$ and $X_{i'_{r+1}}, \dots, X_{i'_m}$ are i.i.d.

With this, the second summation for covariance becomes

$$\sum_{\substack{\{i_1, \dots, i_m\}, \{i'_1, \dots, i'_m\} \subseteq [n] \\ \{i_1, \dots, i_m\} \neq \{i'_1, \dots, i'_m\}}} \text{Cov}[h(X_{i_1}, \dots, X_{i_m}), h(X_{i'_1}, \dots, X_{i'_m})] = \sum_{r=1}^{m-1} \binom{n}{m} \binom{m}{r} \binom{n-m}{m-r} \xi_{r,m},$$

where the summation is from $r = 1$ to $m - 1$ since we require $\{i_1, \dots, i_m\} \neq \{i'_1, \dots, i'_m\}$, i.e., $r \neq m$. We note that the counting is calculated as:

1. choose m indices for the first h arbitrarily from n indices;
2. choose r indices to be the common indices between the first and second h from the m indices chosen above;
3. choose $m - r$ indices for the second h from indices different from those are chosen before. In total, $n - m$ of them.

Combining the sum of variances and covariances, dividing both sides by $\binom{n}{m}$ gives the result. ■

Lecture 20: Comparing Different Tests for Symmetry

We can finally establish the asymptotic normality of U_n .

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Theorem 4.3.1. If for every $r \in [m]$, $\xi_{r,m}$ is bounded away from 0 and infinity as $n \rightarrow \infty$, then

$$\sqrt{n}(U_n - \mathbb{E}[U_n]) = \frac{m}{\sqrt{n}} \sum_{k=1}^n (h^*(X_k) - \mathbb{E}[U_n]) + o_p(1).$$

Furthermore, if $\text{Var}[h^*(X)] > 0$, then $\sqrt{n}(U_n - \mathbb{E}[U_n]) \xrightarrow{D} \mathcal{N}(0, m^2 \text{Var}[h^*(X)])$.

Proof. Recall that by [Proposition 4.3.3](#) and [Corollary 4.3.1](#), it suffices to show that $\text{Var}[U_n] \sim \text{Var}[\tilde{U}_n] = m^2/n \cdot \text{Var}[h^*(X_i)]$ for all $n \in \mathbb{N}$. By [Proposition 4.3.4](#), and the Stirling's approximation,^a

$$\text{Var}[U_n] = \frac{1}{\binom{n}{m}} \sum_{r=1}^m \binom{m}{r} \binom{n-m}{m-r} \xi_{r,m} \sim \frac{\binom{m}{1} \binom{n-m}{m-1}}{\binom{n}{m}} \xi_{1,m} \sim \frac{m^2}{n} \xi_{1,m}$$

with some algebra (and our assumptions). It remains to show that $\text{Var}[h^*(X)] = \xi_{1,m}$. Indeed,

$$\begin{aligned} \xi_{1,m} &= \text{Cov}[(X_1, X_2, \dots, X_m), h(X_1, X_{m+1}, \dots, X_{2m-1})] \\ &= \mathbb{E}[(h(X_1, X_2, \dots, X_m) - \theta) \cdot (h(X_1, X_{m+1}, \dots, X_{2m-1}) - \theta)] \\ &= \int \mathbb{E}[h(x, X_2, \dots, X_m) - \theta] \cdot \mathbb{E}[h(x, X_{m+1}, \dots, X_{2m-1})] F(dx) \\ &= \int (h^*(x) - \theta)(h^*(x) - \theta) F(dx) \\ &= \mathbb{E}[(h^*(X) - \theta)^2] \\ &= \text{Var}[h^*(X)], \end{aligned}$$

which is exactly what we want to show. ■

^aRecall that as $n \rightarrow \infty$, for any fixed m , $\binom{n}{m} \sim n^m/m!$ by the Stirling's approximation.

Let's conclude the discussion by looking at our initial motivation.

Example (Wilcoxon signed-rank statistic). For $d = 1$, consider $h(x_1, x_2) = \mathbb{1}_{x_1+x_2>0}$. Then, $\theta = \mathbb{E}[h(X_1, X_2)] = \mathbb{P}(X_1 + X_2 > 0)$ and $h^*(x) = \mathbb{P}(x + X > 0) = 1 - F(-x)$. If X is not trivial, then $\text{Var}[h^*(X)] = \text{Var}[F(-X)] > 0$. For example, under $H_0: X \stackrel{D}{=} -X$ and X is continuous, $\text{Var}[h^*(X)] = \text{Var}[F(X)] = 1/12$ as $F(X) \sim \mathcal{U}(0, 1)$ since F is assumed to be continuous. Moreover,

$$\theta = \mathbb{P}(X_1 > -X_2) = \mathbb{E}[h^*(X)] = \mathbb{E}[1 - F(-X)] = 1 - \frac{1}{2} = \frac{1}{2}.$$

Therefore, $\sqrt{n}(U_n - 1/2) \xrightarrow{D} \mathcal{N}(0, 2^2/12) = \mathcal{N}(0, 1/3)$ from [Theorem 4.3.1](#). Finally, recall the [representation](#) that the [Wilcoxon signed-rank statistic](#) W_n admits under H_0 , i.e.,

$$\frac{\sqrt{n}}{2 \cdot \binom{n}{2}} \left(W_n - \sum_{i=1}^n \text{sgn}(X_i) \right) = \frac{\sqrt{n}}{n(n-1)} \left(W_n - \sum_{i=1}^n \text{sgn}(X_i) \right) = \sqrt{n} \left(U_n - \frac{1}{2} \right) \xrightarrow{D} \mathcal{N}\left(0, \frac{1}{3}\right),$$

where U is a [U-statistic](#) with $h(x_1, x_2) = \mathbb{1}_{x_1+x_2 \geq 0}$, hence the above calculation^a applies, giving the asymptotic normality. This further implies that under H_0 , $W_n/n^{3/2} \xrightarrow{D} \mathcal{N}(0, 1/3)$.^b

^aIgnore the difference between $\mathbb{1}_{x_1+x_2 \geq 0}$ for W_n and $\mathbb{1}_{x_1+x_2 > 0}$ in the above example as now X is continuous.

^bSince from the [previous note](#), the sum of $\text{sgn}(X_i)$'s will vanish.

Note. This is the same guarantee when we condition on $|X_j|$'s as seen in the [previous example](#).

We make one last remark before we move on to the next topic.

Remark. U_n is a function of $X_{(1)}, \dots, X_{(n)}$, i.e., the function of the order statistics, which is an UMVUE since in this non-parametric formulation,^a the order statistic is complete and sufficient.

^aSince we didn't assume anything about F excepts for the continuity.

4.4 Asymptotic Relative Efficiency of Tests

For the [testing for symmetry problem](#), we have three test statistics, i.e., [t-statistic](#) $T_n = \sqrt{n}\bar{X}_n/\hat{\sigma}_n$, the [sign statistic](#) $\text{sign}_n = \sum_{i=1}^n \text{sgn}(X_i)$, and the [Wilcoxon signed-rank statistic](#) $W_n = \sum_{i=1}^n R_i \text{sgn}(X_i)$.

Problem. How does T_n , sign_n , and W_n compared in terms of their efficiency? Can we say something similar for the estimator's efficiency we have developed, i.e., the [asymptotic relative efficiency](#)?

To answer the above question, we'll look into their *powers*. Consider an alternative hypothesis testing, where we have $H_0: \theta = \theta_0$ and $H_1: \theta > \theta_0$.

Example. One example of θ is $\theta = \mathbb{P}(X_1 + X_2 > 0)$.

Let T_n now denote a generic statistic (not necessarily the [t-statistic](#)) such that there exists an increasing $\mu(\theta)$ such that when the true parameter is θ ,

$$\frac{T_n - \mu(\theta)}{\sigma(\theta)/\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1).$$

Then, we reject H_0 whenever T_n is large.

Example. One can write all the [t-statistic](#) " T_n ", the sign test statistic sign_n , and also the [Wilcoxon signed-rank statistic](#) W_n in this form.

Specifically, under this setup, by treating the asymptotic normality as exact, we reject H_0 if

$$T_n \geq \mu(\theta_0) + Z_\alpha \frac{\sigma(\theta_0)}{\sqrt{n}},$$

which gives that $\mathbb{P}_{\theta_0}(\text{reject}) \rightarrow \alpha$ as we usually get.

Note. It's easy to see that we can always control the type-I error.

Since we can always control α , the interesting question might be whether we can control the type-II error β . We see that under H_1 ,

$$\mathbb{P}_\theta(\text{reject}) = \mathbb{P}_\theta \left(\frac{T_n - \mu(\theta)}{\sigma(\theta)/\sqrt{n}} \geq \frac{\mu(\theta_0) - \mu(\theta)}{\sigma(\theta)/\sqrt{n}} + Z_\alpha \frac{\sigma(\theta_0)}{\sigma(\theta)} \right) \xrightarrow{\theta > \theta_0} 1$$

as $n \rightarrow \infty$ since $\mu(\theta_0) - \mu(\theta) < 0$.

Remark. The power always approaches 1, not very interesting.

Hence, we might turn our focus to some non-asymptotic results. One way to look at this problem is by fixing the type-I and type-II error, and see how many samples we need to achieve them.

4.4.1 A Heuristic Approach

Given some fixed α , β , and θ_0 , suppose we want $\mathbb{P}_{\theta^*}(\text{reject}) = 1 - \beta$ for some θ^* with some n . From the above calculation with asymptotic normality we assume for T_n , we have

$$1 - \beta = \mathbb{P}_{\theta^*} \left(\frac{T_n - \mu(\theta^*)}{\sigma(\theta^*)/\sqrt{n}} \geq \frac{\mu(\theta_0) - \mu(\theta^*)}{\sigma(\theta^*)/\sqrt{n}} + Z_\alpha \frac{\sigma(\theta_0)}{\sigma(\theta^*)} \right) \rightarrow 1 - \Phi \left(\frac{\mu(\theta_0) - \mu(\theta^*)}{\sigma(\theta^*)/\sqrt{n}} + Z_\alpha \frac{\sigma(\theta_0)}{\sigma(\theta^*)} \right),$$

where the convergent is not rigorous since the right-hand side still depend on n . Anyway, this leads to²

$$Z_{1-\beta} = -Z_\beta = \frac{\mu(\theta_0) - \mu(\theta^*)}{\sigma(\theta^*)/\sqrt{n}} + Z_\alpha \frac{\sigma(\theta_0)}{\sigma(\theta^*)} \Rightarrow \sqrt{n} \frac{\mu(\theta^*) - \mu(\theta_0)}{\sigma(\theta^*)} = Z_\beta + \frac{\sigma(\theta_0)}{\sigma(\theta^*)} Z_\alpha.$$

Let θ_i be the θ^* for which the above holds when $n = i$, and denote the corresponding sequence as (θ_n) .

Note. Obviously we then have $\theta_n \rightarrow \theta_0$ for (θ_n) .

In this case, by replacing θ^* by θ_n for every $n \in \mathbb{N}$, we have

$$\sqrt{n} \frac{\mu(\theta_n) - \mu(\theta_0)}{\sigma(\theta_n)} = Z_\beta + \frac{\sigma(\theta_0)}{\sigma(\theta_n)} Z_\alpha.$$

If σ is continuous at θ_0 , then as $\sigma(\theta_n) \rightarrow \sigma(\theta_0)$, the right-hand side becomes $Z_\alpha + Z_\beta$. If we further assume that μ is differentiable at θ_0 , with $\sqrt{n}(\mu(\theta_n) - \mu(\theta_0)) = \mu'(\theta_0)\sqrt{n}(\theta_n - \theta_0) + \sqrt{n} \cdot o(\theta_n - \theta_0)$,

$$\sqrt{n}(\theta_n - \theta_0) \rightarrow \frac{Z_\alpha + Z_\beta}{\mu'(\theta_0)/\sigma(\theta_0)}.$$

Let n^* be the n such that $\theta_{n^*} = \theta_n = \theta^*$. Then, we have

$$\sqrt{n^*} \rightarrow \frac{Z_\alpha + Z_\beta}{\frac{\mu'(\theta_0)}{\sigma(\theta_0)}(\theta_{n^*} - \theta_0)}.$$

Note. n^* only depends on $\mu'(\theta_0)/\sigma(\theta_0)$.

Proof. Since $Z_\alpha + Z_\beta$ is fixed while $\theta_{n^*} - \theta_0$ is assumed to be independent of the statistic also since we treat their asymptotic normality as exact, so θ_{n^*} will be the same across different statistics. \otimes

Definition 4.4.1 (Slope). For any statistics T_n with μ and σ , its *slope* is defined as $\mu'(\theta_0)/\sigma(\theta_0)$.

We see that if the analysis can be made formal, then we can compare two statistics T_n and \tilde{T}_n in terms of their required sample sizes to achieve α and β for a fixed θ_0 .

Remark. This analysis relies on the fact that when $\sqrt{n}(\theta_n - \theta_0)$ converges, then

$$\frac{T_n - \mu(\theta_n)}{\sigma(\theta_n)/\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1).$$

Lecture 21: Slope of a Statistics and Pitman (Local) Alternatives

4.4.2 Deriving the Slope

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Formally, let $\xi \geq 0$ such that $\sqrt{n}(\theta_n - \theta_0) \rightarrow \xi$, and suppose there exists $\mu(\theta)$ and $\sigma(\theta)$ such that

$$\sqrt{n} \frac{T_n - \mu(\theta_n)}{\sigma(\theta_n)} \xrightarrow{D} \mathcal{N}(0, 1) \Leftrightarrow \mathbb{P}_{\theta_n} \left(\frac{T_n - \mu(\theta_n)}{\sigma(\theta_n)/\sqrt{n}} \leq x \right) \rightarrow \Phi(x)$$

for all $x \in \mathbb{R}$. Firstly, when $\xi = 0$, then we know that

$$\sqrt{n} \frac{T_n - \mu(\theta_0)}{\sigma(\theta_0)} \xrightarrow{D} \mathcal{N}(0, 1).$$

Hence, we reject H_0 if $T_n > \mu(\theta_0) + \sigma(\theta_0)Z_\alpha/\sqrt{n}$. We see that this happens with probability

$$\mathbb{P}_{\theta_n}(\text{reject}) = \mathbb{P}_{\theta_n} \left(T_n > \mu(\theta_0) + \frac{\sigma(\theta_0)}{\sqrt{n}} Z_\alpha \right) = \mathbb{P}_{\theta_n} \left(\frac{T_n - \mu(\theta_n)}{\sigma(\theta_n)/\sqrt{n}} > \frac{\mu(\theta_0) - \mu(\theta_n)}{\sigma(\theta_n)/\sqrt{n}} + Z_\alpha \frac{\sigma(\theta_0)}{\sigma(\theta_n)} \right).$$

²Recall that Z_α is defined for the right-tail.

If μ is differentiable at θ_0 and σ is continuous at θ_0 , then as $\sqrt{n}(\theta_n - \theta_0) \rightarrow \xi$, the above converges to

$$\Phi\left(-\left(-\frac{\mu'(\theta_0)}{\sigma(\theta_0)}\xi + Z_\alpha\right)\right) = \Phi\left(\frac{\mu'(\theta_0)}{\sigma(\theta_0)}\xi - Z_\alpha\right).$$

Let θ^* to be defined as $\mathbb{P}_{\theta^*}(\text{reject}) = 1 - \beta$ for some $\beta > 0$. Then, denote n^* such that $\theta_{n^*} = \theta^*$, and define $\xi > 0$ such that $\sqrt{n^*}(\theta^* - \theta_0) = \xi$, i.e., $\theta^* = \theta_0 + \xi/\sqrt{n^*}$. Then $\mathbb{P}_{\theta^*}(\text{reject})$ will converge to

$$\Phi\left(\frac{\mu'(\theta_0)}{\sigma(\theta_0)}\xi - Z_\alpha\right) = 1 - \beta \Rightarrow \frac{\mu'(\theta_0)}{\sigma(\theta_0)}\xi - Z_\alpha = Z_\beta \Rightarrow \sqrt{n^*}(\theta^* - \theta_0) = \xi = \frac{Z_\alpha + Z_\beta}{\mu'(\theta_0)/\sigma(\theta_0)},$$

solving w.r.t. $\sqrt{n^*}$ gives

$$\sqrt{n^*} = \frac{Z_\alpha + Z_\beta}{\frac{\mu'(\theta_0)}{\sigma(\theta_0)}(\theta^* - \theta_0)},$$

which confirms our heuristic argument.

Remark. n^* still only depends on $\mu'(\theta_0)/\sigma(\theta_0)$ from the same reason.

4.4.3 Asymptotic Relative Efficiency for Statistics

If we have another statistic \tilde{T} associates with $\tilde{\mu}$, $\tilde{\sigma}$, and \tilde{n}^* , such that it also satisfies all the assumptions, i.e., asymptotic normality, differentiability for $\tilde{\mu}$, and continuity for $\tilde{\sigma}$, then from the same analysis, we can then compare how many samples we need to reach α , β , given θ_0 .

Definition 4.4.2 (Asymptotic relative efficiency for statistic). Given θ_0 , α , and β , the *asymptotic relative efficiency* between two statistics T_n and \tilde{T}_n is defined as

$$\text{ARE}(T, \tilde{T}) = \frac{n^*}{\tilde{n}^*} = \left(\frac{\tilde{\mu}'(\theta_0)/\tilde{\sigma}(\theta_0)}{\mu'(\theta_0)/\sigma(\theta_0)}\right)^2.$$

Note. Same as Definition 3.4.1, Definition 4.4.2 is different from the convention, where we usually define the *asymptotic relative efficiency* of T w.r.t. \tilde{T} as $\text{ARE}_\theta(T, \tilde{T}) = \tilde{n}^*/n^*$.

Let compare the t -test, sign test, and Wilcoxon signed-rank test on the [problem of testing symmetry](#). In particular, consider the following variation of the problem.

Problem. Let $\epsilon, \epsilon_1, \dots, \epsilon_n \stackrel{\text{i.i.d.}}{\sim} F$ where F is continuous and symmetric around 0, i.e., $\epsilon \stackrel{D}{=} -\epsilon$. Furthermore, assuming $X_{n1}, \dots, X_{nn} \stackrel{\text{i.i.d.}}{\sim} \theta_n + \epsilon_i$ such that $\sqrt{n}(\theta_n - \theta_0) = \xi$ for some fixed $\xi \geq 0$. We're interested in testing whether $H_0: X_{n1} \stackrel{D}{=} -X_{n1}$. In other words, $H_0: \theta_0 = 0$.

Let's try the simplest sign test and compute its [slope](#).

Example (Sign test). The [slope](#) of the averaged [sign statistic](#) $\overline{\text{sign}}_n := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{X_{ni} > 0}$ is $2f(0)$.

Proof. We first see that

$$\overline{\text{sign}}_n := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{X_{ni} > 0} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\epsilon_i > -\theta_n}.$$

The mean of $\mathbb{1}_{\epsilon_i > -\theta_n}$ is

$$\mathbb{P}(\epsilon > -\theta_n) = \mathbb{P}(\epsilon \leq \theta_n) = F(\theta_n)$$

since ϵ is symmetric and continuous. Hence, the [Lindeberg central limit theorem](#) gives

$$\frac{\sum_{i=1}^n (\mathbb{1}_{\epsilon_i > -\theta_n} - F(\theta_n))}{\sqrt{n} \sqrt{F(\theta_n)(1 - F(\theta_n))}} = \sqrt{n} \frac{\frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\epsilon_i > -\theta_n} - F(\theta_n)}{\sqrt{F(\theta_n)(1 - F(\theta_n))}} = \sqrt{n} \frac{\overline{\text{sign}}_n - F(\theta_n)}{\sqrt{F(\theta_n)(1 - F(\theta_n))}} \xrightarrow{D} \mathcal{N}(0, 1)$$

by checking the [Lyapunov condition](#), indeed, since we have

$$\text{Var} \left[\sum_{i=1}^n \mathbb{1}_{\epsilon_i > -\theta_n} \right] = nF(\theta_n)(1 - F(\theta_n)) \rightarrow nF(\theta_0)(1 - F(\theta_0)) \rightarrow \infty.$$

We see that

- $\mu(\theta) = F(\theta)$, hence if F is differentiable at 0 with $F'(0) =: f(0)$, then $\mu'(0) = f(0)$;
- $\sigma(\theta) = \sqrt{F(\theta)(1 - F(\theta))}$, so $\sigma(0) = 1/2$ since $F(0) = 1/2$ by the symmetry of F .

We conclude that the [slope](#) of $\overline{\text{sign}}_n$ is $\mu'(0)/\sigma(0) = 2f(0)$. ⊗

Note. For the sign test, we don't need any moment assumption. Additionally, it's expected to be a weak test since it seems only care about the density around 0, which is intuitive.

Now, let's try the t -test. This time, we will need the second moment to exist.

Example (t -test). Suppose $\text{Var}[\epsilon] = \sigma^2 < \infty$, then the [slope](#) of the “normalized” t -statistic $\tilde{T}_n := T_n/\sqrt{n}$ is $1/\sigma$.

Proof. Since $\tilde{T}_n := T_n/\sqrt{n} = \bar{X}_n/\hat{\sigma}_n$, with $\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n} \sum_{i=1}^n (\epsilon_i - \bar{\epsilon}_n)^2$, we have

$$\tilde{T}_n = \frac{\bar{X}_n}{\hat{\sigma}_n} = \frac{\theta_n + \bar{\epsilon}_n}{\hat{\sigma}_n} = \left(\frac{\theta_n}{\hat{\sigma}_n} - \frac{\theta_n}{\sigma} \right) + \left(\frac{\bar{\epsilon}_n}{\hat{\sigma}_n} + \frac{\theta_n}{\sigma} \right) \Rightarrow \sqrt{n} \left(\tilde{T}_n - \frac{\theta_n}{\sigma} \right) = \sqrt{n} \theta_n \left(\frac{1}{\hat{\sigma}_n} - \frac{1}{\sigma} \right) + \sqrt{n} \frac{\bar{\epsilon}_n}{\hat{\sigma}_n}.$$

Since $\theta_0 = 0$ and $\sqrt{n}(\theta_n - \theta_0) = \xi$, we have $\sqrt{n}\theta_n \rightarrow \xi \geq 0$, $1/\hat{\sigma}_n - 1/\sigma \xrightarrow{p} 0$, so the first term goes to 0. On the other hand, by the usual [central limit theorem](#), $\sqrt{n}\bar{\epsilon}_n \xrightarrow{D} \mathcal{N}(0, \sigma^2)$, hence

$$\sqrt{n} \left(\tilde{T}_n - \frac{\theta_n}{\sigma} \right) = \sqrt{n} \frac{\tilde{T}_n - \frac{\theta_n}{\sigma}}{1} \xrightarrow{D} \mathcal{N}(0, 1).$$

We see that

- $\mu(\theta) = \theta/\sigma$, which is clearly differentiable with $\mu'(\theta) = 1/\sigma$;
- $\sigma(\theta) = 1$, so $\sigma(0) = 1$.

We conclude that the [slope](#) of \tilde{T}_n is $\mu'(0)/\sigma(0) = 1/\sigma$. ⊗

This gives us a way to compare $\overline{\text{sign}}_n$ and \tilde{T}_n .

Proposition 4.4.1. Consider the problem of [testing for symmetry](#), the [asymptotic relative efficiency](#) between the [sign statistic](#) and the t -statistic is

$$\text{ARE}(\tilde{T}_n, \overline{\text{sign}}_n) = (2f(0)\sigma)^2.$$

Remark. From [Proposition 3.5.2](#), $\text{ARE}(\bar{X}_n, \hat{\theta}_{1/2}) = (2f(0)\sigma)^2$, exactly the same!

Let's see some actual example, where we can borrow from the previous calculation.³

Example (Gaussian). If $\epsilon \sim \mathcal{N}(\mu, \sigma^2)$, then $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$, so $f(0) = 1/\sigma\sqrt{2\pi}$, hence

$$\text{ARE}(\tilde{T}_n, \overline{\text{sign}}_n) = \left(\frac{2\sigma}{\sqrt{2\pi}\sigma} \right)^2 = \frac{2}{\pi} < 1$$

³The previous examples include [normal](#) and [Laplace](#).

Example (Laplace). If $\epsilon \sim \text{Laplace}(\mu, b)$ with $\sigma^2 = 2b^2$, since $f(0) = 1/\sqrt{2}\sigma$, hence

$$\text{ARE}(\tilde{T}_n, \overline{\text{sign}_n}) = 4\sigma^2 \frac{1}{2\sigma^2} = 2 > 1,$$

Example (Uniform). If $\epsilon \sim \mathcal{U}(-c, c)$ for some c such that $\text{Var}[\epsilon] = \sigma^2$, we have

$$\text{ARE}(\tilde{T}_n, \overline{\text{sign}_n}) = \frac{1}{3}.$$

Proof. We see that since $\sigma^2 = (2c)^2/12 = c^2/3$, c should be $\sqrt{3}\sigma$. Hence, $f(0) = 1/2c = 1/2\sqrt{3}\sigma$. Plugging in $(2f(0)\sigma)^2$ gives $1/3$. \circledast

Finally, let's consider the Wilcoxon signed-rank test. Recall [what we have shown](#).

As previously seen. Under H_0 , i.e., X is continuous and $X \stackrel{D}{=} -X$, with $U_n := \binom{n}{2}^{-1} \sum_{i < j} h(X_i, X_j)$ where $h(x_1, x_2) = \mathbb{1}_{x_1 + x_2 \geq 0}$, we have

$$\frac{\sqrt{n}}{2 \cdot \binom{n}{2}} \left(W_n - \sum_{i=1}^n \text{sgn}(X_i) \right) = \sqrt{n} \left(U_n - \frac{1}{2} \right) \xrightarrow{D} \mathcal{N}(0, 1/3).$$

In our case, since $X_i = \theta_n + \epsilon_i$, we have

$$U_n = \frac{1}{\binom{n}{2}} \sum_{\{i,j\} \subseteq [n]} \mathbb{1}_{X_i + X_j > 0} = \frac{1}{\binom{n}{2}} \sum_{\{i,j\} \subseteq [n]} \mathbb{1}_{\epsilon_i + \epsilon_j > -2\theta_n}.$$

Example (Wilcoxon signed-rank test). The [slope](#) of the corresponding U -statistic U_n of the [Wilcoxon signed-rank statistic](#) W_n is $2\sqrt{3} \cdot \int f^2(x) dx$.

Proof. Let $h_n(\epsilon_1, \epsilon_2) = \mathbb{1}_{\epsilon_1 + \epsilon_2 > -2\theta_n}$ (note that it depends on n), then from [Theorem 4.3.1](#),

$$\sqrt{n}(U_n - \mathbb{E}[U_n]) = \frac{2}{\sqrt{n}} \sum_{k=1}^n (h_n^*(\epsilon_k) - \mathbb{E}[U_n]) + o_p(1)$$

where the 2 = m is the number of arguments of h . Here, $\mathbb{E}[h_n(\epsilon_1, \epsilon_2)] = \mathbb{E}[U_n] = \mathbb{E}[h_n^*(\epsilon)]$ where

$$h_n^*(x) = \mathbb{E}[h(x, \epsilon)] = \mathbb{P}(x + \epsilon > -2\theta_n) = \mathbb{P}(\epsilon > -x - 2\theta_n) = \mathbb{P}(\epsilon \leq x + 2\theta_n) = F(x + 2\theta_n),$$

with ϵ_i 's being i.i.d., by dividing both sides by $2\sqrt{\text{Var}[F(\epsilon + 2\theta_n)]}$ and the [central limit theorem](#),

$$\frac{\sqrt{n}(U_n - \mathbb{E}[F(\epsilon + 2\theta_n)])}{2\sqrt{\text{Var}[F(\epsilon + 2\theta_n)]}} = \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{h_n^*(\epsilon_k) - \mathbb{E}[F(\epsilon + 2\theta_n)]}{\sqrt{\text{Var}[F(\epsilon + 2\theta_n)]}} + o_p(1) \xrightarrow{D} \mathcal{N}(0, 1)$$

as long as $\text{Var}[F(\epsilon + 2\theta_n)] > 0$. We see that

- $\mu(\theta) = \mathbb{E}[F(\epsilon + 2\theta)]$, if we assume f exists, it's just

$$\mu(\theta) = \int_{\mathbb{R}} F(x + 2\theta) F(dx) = \int_{\mathbb{R}} F(x + 2\theta) f(x) dx.$$

If we further assume that we can interchange the derivative and the integral, we then have $\mu'(\theta) = \int_{\mathbb{R}} 2f(x + 2\theta)f(x) dx$, giving $\mu'(0) = 2 \int_{\mathbb{R}} f^2(x) dx$.

- $\sigma(\theta) = 2\sqrt{\text{Var}[F(\epsilon + 2\theta)]}$, hence $\sigma(0) = 2\sqrt{\text{Var}[F(\epsilon)]} = 2 \cdot \sqrt{1/12} = 1/\sqrt{3}$.

Hence, the [slope](#) of U_n is $2\sqrt{3} \cdot \int_{\mathbb{R}} f^2(x) dx$. \circledast

This gives us a way to compare U_n and \tilde{T}_n , i.e.,

$$\text{ARE}(\tilde{T}_n, U_n) = \left(\frac{2\sqrt{3} \int_{\mathbb{R}} f^2(x) dx}{1/\sigma} \right)^2 = 12\sigma^2 \left(\int_{\mathbb{R}} f^2(x) dx \right)^2.$$

Lecture 22: The Theory of Simple Linear Rank Statistics

4.5 Simple Linear Rank Statistics

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Consider $X_1, \dots, X_N, \dots \stackrel{\text{i.i.d.}}{\sim} F$ such that F is continuous, and for $1 \leq i \leq N$, let the *rank* be

$$R_{Ni} := \sum_{j=1}^N \mathbb{1}_{X_j \leq X_i},$$

i.e., R_{Ni} is the rank of the i^{th} observation among the first N . Clearly, as N varies, R_{Ni} is distributed differently. In particular, we have the following relatively easy lemma where we omit the proof.

Lemma 4.5.1. The rank vector $\tilde{R}_N := (R_{N1}, \dots, R_{NN})$ is uniformly distributed on the permutations of $[N]$. Consequently,

- R_{Ni} is uniformly distributed on $[N]$ for every $1 \leq i \leq N$.
- (R_{Ni}, R_{Nj}) is uniformly distributed on $\{(k, \ell) : 1 \leq k \neq \ell \leq N\}$ for every $1 \leq i \neq j \leq N$.

On the other hand, we also note the following.

Remark. Since F is continuous, $U_i := F(X_i) \stackrel{\text{i.i.d.}}{\sim} \mathcal{U}(0, 1)$ for all $i \geq 1$, and we have $R_{Ni} = \sum_{j=1}^N \mathbb{1}_{U_j \leq U_i}$ almost surely.

Now, let's revisit the [two-sample problem](#).

Example (Two-sample problem). Consider two samples $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F$ and $Y_1, \dots, Y_m \stackrel{\text{i.i.d.}}{\sim} G$, and we want to test whether $H_0: F = G$. Set $X_{n+i} = Y_i$ for $1 \leq i \leq m$, which gives

$$X_1, \dots, X_n, X_{n+1} = Y_1, \dots, X_{n+m} = Y_m$$

with $N := n + m$. Under H_0 , $X_1, \dots, X_N \stackrel{\text{i.i.d.}}{\sim} F$. Naively, we will reject H_0 when $\bar{Y}_m - \bar{X}_n$ is “large” (or “small”). Equivalently, we may consider $\sum_{i=1}^m Y_i - \sum_{i=1}^n X_i = \sum_{i=1}^N c_{Ni} X_i$ where

$$c_{Ni} = \begin{cases} 1, & \text{if } n < i \leq N; \\ -1, & \text{if } 1 \leq i \leq n. \end{cases}$$

From our experience, one might replace X_i by R_{Ni} , i.e., consider

$$\sum_{i=1}^N c_{Ni} R_{Ni} = \sum_{i=1}^N (2\tilde{c}_{Ni} - 1) R_{Ni} = 2 \sum_{i=1}^N \tilde{c}_{Ni} R_{Ni} - \sum_{i=1}^N R_{Ni} = 2 \sum_{i=n+1}^N R_{Ni} - \frac{N(N+1)}{2}$$

since $\sum_{i=1}^N R_{Ni} = \sum_{i=1}^N i$, and we define

$$\tilde{c}_{Ni} = \frac{c_{Ni} + 1}{2} = \begin{cases} 1, & \text{if } n < i \leq N; \\ 0, & \text{if } 1 \leq i \leq n. \end{cases}$$

Observe that $\sum_{i=n+1}^N R_{Ni}$ is just a [Wilcoxon two-sample rank statistic](#).

This suggests we look into the following.

Definition 4.5.1 (Simple linear rank statistic). Consider $X_1, \dots, X_N \stackrel{\text{i.i.d.}}{\sim} F$ where F is continuous. The *simple linear rank statistic* is defined as

$$\sum_{i=1}^N c_N(i) \alpha_N(R_{Ni}),$$

where $c_N(i) =: c_{Ni}$ and $\alpha_N(i) =: \alpha_{Ni}$ for $1 \leq i \leq N$ are all constants.

Remark. The distribution of any *simple linear rank statistics* is independent of F !

In fact, *simple linear rank statistic* is very common in practice.

Example (Median statistic). We can also consider $\sum_{i=n+1}^M \mathbb{1}_{R_{Ni} \geq (N+1)/2}$.

Example (Simple random sampling). Given a finite population $\{x_1, \dots, x_N\}$, to estimate the population average $(x_1 + \dots + x_N)/N$, we take a sample of size n and evaluate the sample mean

$$\frac{1}{n} \sum_{i=1}^N x_i \mathbb{1}_{i^{\text{th}} \text{ population is in the sample}}.$$

Consider a *simple random sample*, i.e., all subset of size n from the population is equally likely to be selected. In this case, (R_{N1}, \dots, R_{NN}) is equally likely to be any permutation of $[N]$. Hence, the above indicators are just $\mathbb{1}_{R_{Ni} \leq n}$, which suggests $\alpha_N(R_{Ni}) := \mathbb{1}_{R_{Ni} \leq n}$ in the above notation.

All these examples motivates us to develop a general theory for the *simple linear rank statistic* in the form of $T_N := \sum_{i=1}^N c_{Ni} \alpha_N(R_{Ni})$, in particular, to establish the asymptotic normality of them.

4.5.1 Moments of Linear Rank Statistics

To start, let's first compute the expectation and the variance. We will adopt the following notations.

Notation. We write $\bar{\alpha}_N$ to be the mean of α_{Ni} and \bar{c}_N to be the average of c_{Ni} , i.e.,

$$\bar{\alpha}_N := \frac{1}{N} \sum_{i=1}^N \alpha_N(i), \text{ and } \bar{c}_N := \frac{1}{N} \sum_{i=1}^N c_{Ni}.$$

Moreover, let $\sigma_{N\alpha}^2$ to be the variance of α_{Ni} , and similarly define σ_{Nc}^2 , i.e.,

$$\sigma_{N\alpha}^2 = \frac{1}{N} \sum_{i=1}^N (\alpha_N(i) - \bar{\alpha}_N)^2, \text{ and } \sigma_{Nc}^2 = \frac{1}{N} \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2.$$

These make sense since for any individual R_{Ni} , marginally they're identically distributed from **Lemma 4.5.1**. On the other hand, there's no randomness in c_i 's.

With this notation, we can now compute the first and the second moments.

Claim. For any *simple linear rank statistic* $T_N := \sum_{i=1}^N c_{Ni} \alpha_N(R_{Ni})$, $\mathbb{E}[T_N] = N \bar{\alpha}_N \bar{c}_N$.

Proof. Since marginally, R_{Ni} 's are just uniform over $[N]$, hence the expectation of T_N is

$$\mathbb{E}[T_N] = \sum_{i=1}^N c_{Ni} \mathbb{E}[\alpha_N(R_{Ni})] = \sum_{i=1}^N c_{Ni} \mathbb{E}[\alpha_N(R_{N1})] = \sum_{i=1}^N c_{Ni} \sum_{j=1}^N \frac{\alpha_N(j)}{N} =: N \bar{\alpha}_N \bar{c}_N$$

as $\bar{\alpha}_N = \mathbb{E}[\alpha_N(R_{N1})] = \frac{1}{N} \sum_{i=1}^N \alpha_{Ni}$ and $\bar{c}_N = \frac{1}{N} \sum_{i=1}^N c_{Ni}$. *

Computing the variance is a bit more challenging, but still doable.

Claim. For any simple linear rank statistic $T_N := \sum_{i=1}^N c_{Ni} \alpha_N(R_{Ni})$, $\text{Var}[T_N] = \frac{N^2}{N-1} \sigma_{Nc}^2 \sigma_{N\alpha}^2$.

Proof. Let's first center T_N , which gives

$$T_N - \mathbb{E}[T_N] = \sum_{i=1}^N c_{Ni} \alpha_N(R_{Ni}) - \sum_{i=1}^N c_{Ni} \bar{\alpha}_N = \sum_{i=1}^N (c_{Ni} - \bar{c}_N) \alpha_N(R_{Ni}) + \bar{c}_N \sum_{i=1}^N \alpha_N(R_{Ni}) - \sum_{i=1}^N c_{Ni} \bar{\alpha}_N.$$

Now, observe that $\sum_{i=1}^N \alpha_N(R_{Ni}) = \sum_{i=1}^N \alpha_N(i)$, the last two terms will cancel out, giving $T_N - \mathbb{E}[T_N] = \sum_{i=1}^N (c_{Ni} - \bar{c}_N) \alpha_N(R_{Ni})$. Then, by the definition of variance,

$$\begin{aligned} \text{Var}[T_N] &= \text{Var} \left[\sum_{i=1}^N (c_{Ni} - \bar{c}_N) \alpha_N(R_{Ni}) \right] \\ &= \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 \text{Var}[\alpha_N(R_{Ni})] + \sum_{i \neq j} (c_{Ni} - \bar{c}_N)(c_{Nj} - \bar{c}_N) \text{Cov}[\alpha_N(R_{Ni}), \alpha_N(R_{Nj})]. \end{aligned}$$

The first sum is just $N \sigma_{Nc}^2 \sigma_{N\alpha}^2$, so we focus on the second sum.

Intuition. For $i \neq j$, (R_{Ni}, R_{Nj}) is equally likely to take any value in $\{(i, j) : 1 \leq i \neq j \leq N\}$.

Hence, we can replace $\text{Cov}[\alpha_N(R_{Ni}), \alpha_N(R_{Nj})]$ by $\text{Cov}[\alpha_N(R_{N1}), \alpha_N(R_{N2})]$, and focus only on $\sum_{i \neq j} (c_{Ni} - \bar{c}_N)(c_{Nj} - \bar{c}_j)$. In particular, we have the following.

Note. For any sequence (x_N) , we have $\sum_{i \neq j} (x_i - \bar{x}_N)(x_j - \bar{x}_N) = -\sum_{i=1}^N (x_i - \bar{x}_N)^2$.

Proof. From the identity $(\sum_{i=1}^N x_i)^2 = \sum_{i=1}^N x_i^2 + \sum_{i \neq j} x_i x_j$, hence

$$0 = \left(\sum_{i=1}^N (x_i - \bar{x}_N) \right)^2 = \sum_{i=1}^N (x_i - \bar{x}_N)^2 + \sum_{i \neq j} (x_i - \bar{x}_N)(x_j - \bar{x}_N).$$

Rearranging the terms gives the equality. ⊗

Hence, we see that by using the above identity and the fact that the joint distribution of R_{N1} and R_{N2} is the uniform, with the definition of the covariance,

$$\begin{aligned} & \sum_{i \neq j} (c_{Ni} - \bar{c}_N)(c_{Nj} - \bar{c}_N) \text{Cov}[\alpha_N(R_{Ni}), \alpha_N(R_{Nj})] \\ &= \text{Cov}[\alpha_N(R_{N1}), \alpha_N(R_{N2})] \cdot \sum_{i \neq j} (c_{Ni} - \bar{c}_N)(c_{Nj} - \bar{c}_N) \\ &= \left[\frac{1}{N(N-1)} \sum_{i \neq j} (\alpha_{Ni} - \bar{\alpha}_N)(\alpha_{Nj} - \bar{\alpha}_N) \right] (-N \sigma_{Nc}^2) \\ &= \left[-\frac{1}{N(N-1)} \sum_{i=1}^N (\alpha_{Ni} - \bar{\alpha}_N)^2 \right] (-N \sigma_{Nc}^2) = \frac{N}{N-1} \sigma_{N\alpha}^2 \sigma_{Nc}^2. \end{aligned}$$

Putting everything together, we have

$$\text{Var}[T_N] = N \sigma_{Nc}^2 \sigma_{N\alpha}^2 + \frac{N}{N-1} \sigma_{N\alpha}^2 \sigma_{Nc}^2 = \frac{N^2}{N-1} \sigma_{Nc}^2 \sigma_{N\alpha}^2,$$

which gives the desired result. ⊗

4.5.2 Asymptotic Normality of Linear Rank Statistics

With the above calculation, we now want to establish the asymptotic normality of the [simple linear rank statistic](#). Specifically, we consider a special form of $\alpha_N(i)$ given some $\phi: [0, 1] \rightarrow \mathbb{R}$, i.e., for $1 \leq i \leq N$,

$$\alpha_N(i) = \phi\left(\frac{i}{N+1}\right).$$

It might seem cryptic and mysterious at the first glance why we want to consider $\alpha_N(i)$ in this form.

Intuition. Consider order statistics $U_{N(1)} \leq \dots \leq U_{N(N)}$ for the uniform U_i 's. Then, for $1 \leq i \leq N$, $\mathbb{E}[U_{N(i)}] = i/(N+1)$, implying $\alpha_N(i) = \phi(\mathbb{E}[U_{N(i)}])$.

Example. The [simple linear rank statistic](#) we have seen so far can be written in the above form.

To proceed, one might expect something like $\phi(\mathbb{E}[U_{N(i)}]) \approx \mathbb{E}[\phi(U_{N(i)})]$ to hold, and in fact, while both of them are of our interests, the latter is easier to analyze than $\phi(\mathbb{E}[U_{N(i)}])$.

Intuition. For $\alpha_N(i) = \mathbb{E}[\phi(U_{N(i)})]$, we have $\alpha_N(R_{Ni}) = \mathbb{E}[\phi(U_i) \mid \tilde{R}_N]$.

The above shows that $\alpha_N(i) = \mathbb{E}[\phi(U_{N(i)})]$ is more convenient since we will have

$$T_N = \sum_{i=1}^N c_{Ni} \alpha_N(R_{Ni}) = \sum_{i=1}^N c_{Ni} \mathbb{E}[\phi(U_i) \mid \tilde{R}_N],$$

which is a conditional expectation. We can then apply the theory of [projection](#).

Notation. In what follows, we denote $\alpha_{Ni} := \mathbb{E}[\phi(U_{N(i)})]$ and $\alpha'_{Ni} := \phi(\mathbb{E}[U_{N(i)}])$ for $1 \leq i \leq N$, and refer to the corresponding simple linear rank statistic as T_N and T'_N , respectively.

Lecture 23: Asymptotically Normality of Linear Rank Statistics

We will first work with $\alpha_N(i) = \mathbb{E}[\phi(U_{N(i)})]$, where we recall that $\alpha_N(R_{Ni}) = \mathbb{E}[\phi(U_i) \mid \tilde{R}_N]$.

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Proposition 4.5.1. For every $N \geq 1$, we have $T_N - \mathbb{E}[T_N] = \mathbb{E}[\tilde{T}_N \mid \tilde{R}_N]$ where

$$\tilde{T}_N := \sum_{i=1}^N (c_{Ni} - \bar{c}_N) \phi(U_i).$$

Proof. It'll be convenient to consider

$$T_N = \sum_{i=1}^N (c_{Ni} - \bar{c}_N) \alpha_N(R_{Ni}) + \bar{c}_N \sum_{i=1}^N \alpha_N(R_{Ni}) = \sum_{i=1}^N (c_{Ni} - \bar{c}_N) \alpha_N(R_{Ni}) + N \bar{c}_N \bar{\alpha}_N,$$

with the fact that $\mathbb{E}[T_N] = N \bar{c}_N \bar{\alpha}_N$, we see that

$$T_N - \mathbb{E}[T_N] = \sum_{i=1}^N (c_{Ni} - \bar{c}_N) \alpha_N(R_{Ni})$$

and since $\alpha_N(R_{Ni}) = \mathbb{E}[\phi(U_i) \mid \tilde{R}_N]$, we further have

$$= \sum_{i=1}^N (c_{Ni} - \bar{c}_N) \mathbb{E}[\phi(U_i) \mid \tilde{R}_N] = \mathbb{E} \left[\sum_{i=1}^N (c_{Ni} - \bar{c}_N) \phi(U_i) \mid \tilde{R}_N \right] =: \mathbb{E}[\tilde{T}_N \mid \tilde{R}_N]$$

where we let $\tilde{T}_N := \sum_{i=1}^N (c_{Ni} - \bar{c}_N) \phi(U_i)$. ■

To see how can we obtain asymptotic normality by the theory of [projection](#), observe the following.

Claim. We can easily have asymptotically normality for \tilde{T}_N .

Proof. We see that by the [Hájek-Sidak central limit theorem](#), if $\phi(U_i)$'s are i.i.d. such that

$$0 < \mathbb{E}[\phi^2(U)] = \int_0^1 \phi^2(u) du < \infty, \text{ and } \max_{1 \leq i \leq N} \frac{(c_{Ni} - \bar{c}_N)^2}{\sum_{j=1}^N (c_{Nj} - \bar{c}_N)^2} \rightarrow 0,$$

then $\tilde{T}_N / \sqrt{\text{Var}[\tilde{T}_N]} \xrightarrow{D} \mathcal{N}(0, 1)$ as $\mathbb{E}[\tilde{T}_N] = 0$. ⊗

Applying the [projection](#) theory we have developed, we have the following.

Theorem 4.5.1. Suppose that (c_{Ni}) satisfy $\max_{1 \leq i \leq N} (c_{Ni} - \bar{c}_N)^2 / \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 \rightarrow 0$ as $N \rightarrow \infty$. If $\text{Var}[\phi(U)] \in (0, \infty)$, then

$$\frac{T_N - \mathbb{E}[T_N]}{\sqrt{\text{Var}[T_N]}} \xrightarrow{D} \mathcal{N}(0, 1).$$

Proof. From [Corollary 4.3.1](#) and the above [claim](#), it suffices to show that $\text{Var}[T_N] / \text{Var}[\tilde{T}_N] \rightarrow 1$ as $N \rightarrow \infty$. Recall that $\text{Var}[T_N] = \frac{N^2}{N-1} \sigma_{N\alpha}^2 \alpha_{Nc}^2$, furthermore, $\text{Var}[\tilde{T}_N] = N \sigma_{Nc}^2 \text{Var}[\phi(U_1)]$ since

$$\text{Var}[\tilde{T}_N] = \text{Var}[\phi(U_1)] \cdot \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 = N \sigma_{Nc}^2 \text{Var}[\phi(U)],$$

so it suffices to show $\sigma_{N\alpha}^2 = \text{Var}[\alpha_N(R_{N1})] \rightarrow \text{Var}[\phi(U_1)]$.

Note. To show $\text{Var}[X] \rightarrow \text{Var}[Y]$, it suffices to show $X \xrightarrow{L^2} Y$ since it implies convergence for both the first and second moments, hence the variance

In particular, it reduces to show $\alpha_N(R_{N1}) = \mathbb{E}[\phi(U_1) \mid \tilde{R}_N] \xrightarrow{L^2} \phi(U_1)$. Firstly, we write

$$\mathbb{E}[\phi(U_1) \mid \tilde{R}_N] = \mathbb{E}[\phi(U_1) \mid \tilde{R}_1, \dots, \tilde{R}_N]$$

as the condition on the right-hand side is equivalent to $\tilde{R}_N, U_2, \dots, U_N$, and U_1 is independent of U_2, \dots, U_N . From the martingale limit theorem, we have

$$\mathbb{E}[\phi(U_1) \mid \tilde{R}_1, \dots, \tilde{R}_N] \xrightarrow{\text{a.s.}} \mathbb{E}[\phi(U_1) \mid \tilde{R}_N, N \geq 1],$$

which will equal to $\phi(U_1)$ if $\phi(U_1)$ is a function of the conditions. Hence, it remains to show that U_1 is a measurable function of $\{\tilde{R}_N\}_{N \geq 1}$.

Claim. Knowing the first components of \tilde{R}_N for every $N \geq 1$, i.e., R_{N1} , determines U_1 (in L^2).

Proof. We observe that $\mathbb{E}[U_1 \mid R_{N1}] = R_{N1} / (N+1)$, i.e., the expectation of $U_{(R_{N1})}$. Hence, by [Corollary 4.3.1](#), $U_1 - R_{N1} / (N+1) \xrightarrow{L^2} 0$ if the ratio between variances converges to 1. Indeed,

$$\frac{\text{Var}[U_1]}{\text{Var}\left[\frac{R_{N1}}{N+1}\right]} = \frac{1/12}{\frac{1}{(N+1)^2} \text{Var}[R_{N1}]} = \frac{1/12}{\frac{1}{(N+1)^2} \frac{N^2-1}{12}} = \frac{N+1}{N-1} \rightarrow 1$$

since $R_{N1} \sim \mathcal{U}([N])$, hence we're done. ⊗

Finally, note that the limits in L^2 are unique up to a null set, hence we're done. ■

We now discuss the connection between T_N and T'_N . Firstly, recall that

$$T_N = \sum_{i=1}^N c_{Ni} \alpha_N(R_{Ni}), \text{ and } T'_N = \sum_{i=1}^N c_{Ni} \alpha'_N(R_{Ni})$$

where $\alpha_{Ni} = \mathbb{E}[\phi(U_{N(i)})]$ and $\alpha'_{Ni} := \phi(\mathbb{E}[U_{N(i)}]) = \phi(i/(N+1))$.

Theorem 4.5.2. Suppose that (c_{Ni}) satisfy $\max_{1 \leq i \leq N} (c_{Ni} - \bar{c}_N)^2 / \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 \rightarrow 0$ as $N \rightarrow \infty$. If $\text{Var}[\phi(U)] \in (0, \infty)$, and in addition, if ϕ is almost surely continuous and

$$\limsup_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \phi^2 \left(\frac{i}{N+1} \right) \leq \int_0^1 \phi^2(u) du,$$

then

$$\frac{T'_N - \mathbb{E}[T'_N]}{\sqrt{\text{Var}[T'_N]}} \xrightarrow{D} \mathcal{N}(0, 1).$$

Proof. From [Theorem 4.5.1](#), to show that T'_N is asymptotically normal as T_N , it suffices to show

$$\frac{T_N - \mathbb{E}[T_N]}{\sqrt{\text{Var}[T_N]}} - \frac{T'_N - \mathbb{E}[T'_N]}{\sqrt{\text{Var}[T'_N]}} \xrightarrow{L^2} 0 \Leftrightarrow \frac{\text{Var}[T_N - T'_N]}{\text{Var}[T_N]} \rightarrow 0$$

Firstly, recall that $\text{Var}[T_N] = \frac{N^2}{N-1} \sigma_{Nc}^2 \sigma_{N\alpha}^2$, so $\text{Var}[T_N - T'_N] = \frac{N^2}{N-1} \sigma_{Nc}^2 \sigma_{N(\alpha-\alpha')}^2$ since we can write

$$T_N - T'_N = \sum_{i=1}^N c_{Ni} (\alpha_N - \alpha'_{Ni}) (R_{Ni}),$$

which is again a [simple linear rank statistic](#), so the same calculation applies. Hence, it suffices to show $\sigma_{N(\alpha-\alpha')}^2 / \sigma_{N\alpha}^2 \rightarrow 0$. Recall what we have already shown earlier.

As previously seen. In the proof of [Theorem 4.5.1](#), we have $\sigma_{N\alpha}^2 \rightarrow \text{Var}[\phi(U_1)] > 0$.

Hence, we just need to show that $\sigma_{N(\alpha-\alpha')}^2 = \text{Var}[(\alpha_N - \alpha'_{N1})(R_{N1})] \rightarrow 0$. From the same reason as before, it suffices to show that $(\alpha_N - \alpha'_{N1})(R_{N1}) \xrightarrow{L^2} 0$, i.e.,

$$\mathbb{E}[\phi(U_1) | \tilde{R}_N] - \phi \left(\frac{R_{N1}}{N+1} \right) \xrightarrow{L^2} 0.$$

Let's again recall what we have shown earlier.

As previously seen. In the proof of [Theorem 4.5.1](#), we have $\mathbb{E}[\phi(U_1) | \tilde{R}_N] \xrightarrow{L^2} \phi(U_1)$.

Hence, it reduces to show $\phi(R_{N1}/(N+1)) \xrightarrow{L^2} \phi(U_1)$.

Intuition. If $\phi = \text{id}$, then we have showed that $R_{N1}/(N+1) \xrightarrow{L^2} U_1$ in the [previous claim](#).

Hence, $R_{N1}/(N+1) \xrightarrow{P} U_1$. As ϕ is assumed to be continuous almost surely, we know that

$$\phi \left(\frac{R_{N1}}{N+1} \right) \xrightarrow{P} \phi(U_1) \stackrel{?}{\Rightarrow} \phi \left(\frac{R_{N1}}{N+1} \right) \xrightarrow{L^2} \phi(U_1).$$

The implication can be provided by [Scheffé's theorem](#), i.e., we only need to check

$$\limsup_{N \rightarrow \infty} \mathbb{E} \left[\phi^2 \left(\frac{R_{N1}}{N+1} \right) \right] = \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \phi^2 \left(\frac{i}{N+1} \right) \leq \mathbb{E}[\phi^2(U_1)] = \int_0^1 \phi^2(u) du$$

since $R_{N1} \sim \mathcal{U}([N])$ as we mentioned. This is what we assumed exactly. ■

We end this chapter by noting that the condition required in [Theorem 4.5.2](#), in particular, the integral inequality, is not that hard to satisfy. The following is one example.

Example. The conditions required in [Theorem 4.5.2](#) are satisfied when ϕ is increasing, or, more generally, the difference of two increasing functions.

Proof. Let's only illustrate the case of increasing functions. Observe that we can understand

$$\limsup_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \phi^2(i/(N+1))$$

in the condition of [Theorem 4.5.2](#) as a Riemann integral. In particular,

$$\int_0^1 \phi^2(u) \, du = \lim_{N \rightarrow \infty} \sum_{i=1}^{N+1} \int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}} \phi^2(u) \, du$$

since ϕ is increasing, so is ϕ^2 , hence

$$\geq \lim_{N \rightarrow \infty} \sum_{i=1}^{N+1} \phi^2\left(\frac{i-1}{N+1}\right) \cdot \frac{1}{N+1} \geq \lim_{N \rightarrow \infty} \frac{N}{N+1} \frac{1}{N} \sum_{i=1}^N \phi^2\left(\frac{i}{N+1}\right),$$

which is what we want. ⊗

Chapter 5

M-Estimation

Lecture 24: The Maximum Likelihood Estimator

Consider $X, X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F$, and we would like to say something about F or $T(F)$. To do this, one can we first obtain the empirical cdf \hat{F}_n , and plug in $T(\hat{F}_n)$. This is what we did previously. 16 Apr. 9:30

Example (Quantile). The p^{th} -quantile is defined as $T(F) = \theta_p = F^{-1}(p)$, which is estimated by the sample p^{th} -quantile $T(\hat{F}_n) = \hat{\theta}_p = \hat{F}_n^{-1}(p)$.

Example (Moment). The k^{th} -moment is defined as $T(F) = \mu_k = \int (x - \mu)^k F(dx)$ where $\mu = \int x F(dx)$, which is estimated by the sample k^{th} -central moment $T(\hat{F}_n) = M_k = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^k$.^a

^aNote that we're essentially interpreting $M_k = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^k$ as $\int (x - \bar{X}_n)^k \hat{F}_n(dx)$ with $\bar{X}_n = \int x \hat{F}_n(dx)$.

On the other hand, instead of directly estimating F , we can start by postulating a family of cdfs $\{G_\theta : \theta \in \Theta\}$ where (Θ, ρ) is a metric space, with the goal being to choose $\hat{\theta}_n$ among Θ such that $G_{\hat{\theta}_n}$ approximates F . Our choice, $\hat{\theta}_n$, should be a function of the data X_1, \dots, X_n .

5.1 Maximum Likelihood Estimator

If we assume G_θ has a corresponding density g_θ for all $\theta \in \Theta$, then a natural choice is to select $\hat{\theta}_n \in \Theta$ by maximizing the *likelihood function*, i.e., the well-known **maximum likelihood estimator** (MLE).

Definition 5.1.1 (Maximum likelihood estimator). Given $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F$ and a family of cdfs $\{G_\theta : \theta \in \Theta\}$ for some metric space (Θ, ρ) . The *maximum likelihood estimator* $\hat{\theta}_n$ is the maximizer of the likelihood function, i.e., $\hat{\theta}_n = \arg \max_{\theta \in \Theta} \prod_{i=1}^n g_\theta(X_i)$.

5.1.1 Divergence Minimizing

This is all good, since we're just proposing a method. The true motivating question is the following.

Problem. What does the **maximum likelihood estimator** $\hat{\theta}_n$ estimating?

Observe that $\hat{\theta}_n$ is also a maximizer of $\frac{1}{n} \sum_{i=1}^n \log(g_\theta(X_i))$.¹ As $n \rightarrow \infty$, from the **strong law of large number**, we know that for all $\theta \in \Theta$,

$$\frac{1}{n} \sum_{i=1}^n \log(g_\theta(X_i)) \xrightarrow{\text{a.s.}} \mathbb{E}[\log(g_\theta(X))] =: M(\theta) \in [-\infty, \infty].$$

¹We adapt the convention that $\log(0) := -\infty$ since it happens when $\{G_\theta\}$ is misspecified.

Note. We will need to assume that $M(\theta) > -\infty$ for some θ , otherwise there's no hope.

Since $\hat{\theta}_n$ is maximizing the left-hand side, it's natural to conjecture the following.

Conjecture 5.1.1. $\hat{\theta}_n$ should “converge” to the maximizer of $M(\theta)$.

To see why this might be the case, assuming F has a density f , then we can write

$$M(\theta) = \int_{\mathbb{R}} \log g_{\theta}(x) f(x) dx = \int_{\mathbb{R}} \log \frac{g_{\theta}(x)}{f(x)} f(x) dx + \int_{\mathbb{R}} \log f(x) f(x) dx.$$

The second term is independent of θ , and the first term is just the **Kullback-Leibler divergence** since

$$\int_{\mathbb{R}} \log \frac{g_{\theta}(x)}{f(x)} f(x) dx = - \int_{\mathbb{R}} \log \frac{f(x)}{g_{\theta}(x)} f(x) dx = -\text{KL}(f \| g_{\theta}).$$

Remark. The maximizer of $M(\theta)$ is minimizing the **KL divergence** $\text{KL}(f \| g_{\theta})$.

Let $\theta^* = \arg \max_{\theta \in \Theta} M(\theta)$. Then if we can actually show **Conjecture 5.1.1**, then we can conclude that $\hat{\theta}_n$ is estimating θ^* , i.e., $\hat{\theta}_n$ also tries to minimize the **KL divergence** between f and $g_{\hat{\theta}_n}$.

Example (Supremum estimation). Suppose X_1, \dots, X_n follows $\mathbb{P}(0 \leq X \leq u) = 1$ for some $u > 0$ such that for all $\epsilon > 0$, $F(u - \epsilon) < 1 = F(u)$. Consider the family of uniform distribution $\{G_{\theta} = \mathcal{U}(0, \theta) : \theta > 0\}$. The likelihood function is given by

$$\Theta \ni \theta \mapsto \prod_{i=1}^n \frac{1}{\theta^n} \mathbb{1}_{\max_{1 \leq i \leq n} X_i \leq \theta},$$

hence the maximizer of the likelihood function is given by $\hat{\theta}_n = \max_{1 \leq i \leq n} X_i$.

On the other hand, $M(\theta) = \mathbb{E}[\log g_{\theta}(X)]$ for all $\theta > 0$, and we want to find $\sup_{\theta > 0} M(\theta)$. But if $\theta < u$, $\mathbb{P}(g_{\theta}(x) = 0) > 0$, then $M(\theta) = -\infty$, hence

$$\sup_{\theta > 0} M(\theta) = \sup_{\theta \geq u} M(\theta) = \sup_{\theta \geq u} \log \frac{1}{\theta},$$

which is attained by u . From the very **first example**, indeed, $\hat{\theta}_n \xrightarrow{\text{a.s.}} u$.

Next, for the following examples let's consider the following setup: consider a family $\{G_{\theta}(x) = G(x - \theta)\}$ for some cdf G with a density g . In this case, $\hat{\theta}_n$ maximizes $\prod_{i=1}^n g(X_i - \theta)$, or equivalently,

$$\hat{\theta}_n = \arg \max_{\theta \in \Theta} \sum_{i=1}^n \log(g(X_i - \theta)).$$

The following two examples also confirm **Conjecture 5.1.1**.

Example (Normal). If $G \sim \mathcal{N}(0, 1)$, i.e., $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, then

- $\hat{\theta}_n = \arg \max_{\theta \in \Theta} \sum_{i=1}^n -(X_i - \theta)^2 = \bar{X}_n$;
- $\theta^* = \arg \max_{\theta \in \Theta} \mathbb{E}[-(X - \theta)^2] = \mathbb{E}[X]$.

From **strong law of large number**, $\bar{X}_n \xrightarrow{\text{a.s.}} \mathbb{E}[X]$ assuming it converges.

Example (Laplace). If $G \sim \text{Laplace}(\mu, b)$ where $\sigma^2 = 2b^2$, i.e., $g(x) = \frac{1}{2b} e^{-\frac{|x|}{b}} = \frac{1}{\sigma\sqrt{2}} e^{-\frac{|x-\mu|}{\sigma/\sqrt{2}}}$, then

- $\hat{\theta}_n = \arg \max_{\theta \in \Theta} \sum_{i=1}^n -|X_i - \theta|$, which is the sample median;

- $\theta^* = \arg \max_{\theta \in \Theta} \mathbb{E}[|X - \theta|]$, which is the population median.

Hence, from [Corollary 3.5.1](#), $\hat{\theta}_n \xrightarrow{P} \theta^*$.

Remark. We see that the [MLE](#) will do the right thing: when estimating μ (hence $\theta_{1/2}$), as we have shown, for [normal](#), \bar{X}_n is better than $\hat{\theta}_{1/2}$, while for [Laplace](#) $\hat{\theta}_{1/2}$ is better.

However, sometimes we don't know $\hat{\theta}_n$ and θ^* , hence we don't know whether [Conjecture 5.1.1](#) is true.

Example (Cauchy). If $G \sim \text{Cauchy}$ with location 0 and scale 1, i.e., $g(x) = 1/\pi(1 + x^2)$, then

- $\hat{\theta}_n = \arg \max_{\theta \in \Theta} \sum_{i=1}^n -\log(1 + (X_i - \theta)^2)$;
- $\theta^* = \arg \max_{\theta \in \Theta} \mathbb{E}[-\log(1 + (X - \theta)^2)]$.

However, we don't know what are $\hat{\theta}_n$ and θ^* this time, hence we need a different technique.

5.1.2 M -Estimator

The upshot is that in the above examples, we're dealing with $\hat{\theta}_n = \arg \max_{\theta \in \Theta} \sum_{i=1}^n m(X_i - \theta)$ where $m(x) = -\log(g(x))$. But we're not limited to choosing this specific m .

Example. Consider $m(x) = x^2 \mathbb{1}_{|x| \leq k} + (2k|x| - k^2) \mathbb{1}_{|x| > k}$ for some $k \in \mathbb{R}$.

This kind of general estimator in the form of maximizing some m is called the [M-estimator](#).

Definition 5.1.2 (M -estimator). Given data X_1, \dots, X_n , not necessarily i.i.d., and a metric space (Θ, ρ) , let $m_\theta(x)$ be a function for every $\theta \in \Theta$, and define a function $M_n(\theta)$ of X_i 's as $M_n(\theta) := \frac{1}{n} \sum_{i=1}^n m_\theta(X_i)$. An M -estimator $\hat{\theta}_n$ is a maximizer of M_n , i.e., $\hat{\theta}_n = \arg \max_{\theta \in \Theta} M_n(\theta)$.

We see that if X_i 's are i.i.d., then $M_n(\theta) \rightarrow M(\theta) := \mathbb{E}[m_\theta(X)] \in [-\infty, \infty]^2$ as $n \rightarrow \infty$ for any fixed $\theta \in \Theta$, where the convergence can be either [almost surely](#) or just [in probability](#).

Notation. We will use $\xrightarrow[p]{\text{a.s.}}$ from now on to indicate either $\xrightarrow{\text{a.s.}}$ or \xrightarrow{P} , depending on the assumption.

Hence, the goal of an [M-estimator](#) is to use $(\hat{\theta}_n)$ to approximate $\theta^* = \arg \max_{\theta \in \Theta} M(\theta)$, what we really care about. With $M_n(\theta) \xrightarrow[p]{\text{a.s.}} M(\theta)$, we can also approximate $\max_{\theta \in \Theta} M(\theta)$.

Intuition. We don't need to specify the family $\{G_\theta : \theta \in \Theta\}$ explicitly, allowing more flexibility.

5.2 Consistency

Our first goal is to establish $\hat{\theta}_n \xrightarrow[p]{\text{a.s.}} \theta^*$, i.e., the [consistency](#) of the [M-estimator](#). However, we will see that the *approximate maximizer* $\hat{\theta}_n$ of $\theta \mapsto M_n(\theta)$ suffices for our purpose. In particular, we define $\hat{\theta}_n$ in the way that for a (positive) vanishing sequence $\xi_n \xrightarrow[p]{\text{a.s.}} 0$ as $n \rightarrow \infty$, $\hat{\theta}_n$ satisfies

$$M_n(\hat{\theta}_n) \geq \sup_{\theta \in \Theta} M_n(\theta) - \xi_n. \quad (5.1)$$

Intuition. We don't need to solve the optimization problem $\arg \max_{\theta \in \Theta} M_n(\theta)$ exactly.

²Again, assume that $M(\theta) > -\infty$ for some $\theta \in \Theta$, otherwise such an optimization problem is meaningless.

5.2.1 Assumptions for Consistency

We first consider the most fundamental question:

Problem. Where does $\hat{\theta}_n$ converge, in what sense, if anywhere?

We will need some assumptions around θ^* to answer this. In particular, consider the following.

- (A1) M has a unique maximizer θ^* , and is well-separated, i.e., for all $\epsilon > 0$, $M(\theta^*) > \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta)$.
 (A2) For all $\delta > 0$ and $\epsilon > 0$, $\mathbb{P}(\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) - \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) \geq \delta) \rightarrow 0$.

We can also consider a stronger version of (A2):

- (A3) For all $\delta > 0$ and $\epsilon > 0$, $\mathbb{P}(\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \geq \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) + \delta \text{ i.o.}) = 0$.³

Notation (Open ball). Let $B(\theta^*, \epsilon) = \{\theta \in \Theta: \rho(\theta, \theta^*) < \epsilon\}$ be an open ball around $\theta^* \in \Theta$ with radii $\epsilon > 0$, where d is the metric on Θ .

Let's understand them one-by-one. Firstly, we see that if $\tilde{\theta} \neq \theta^*$, then $\tilde{\theta} \notin B(\theta^*, \epsilon)$ for some $\epsilon > 0$. Hence, (A1) implies $M(\theta^*) > \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) \geq M(\tilde{\theta})$. To satisfy (A1), consider the following.

Definition 5.2.1 (Upper semi-continuous). A function $M: \Theta \rightarrow \mathbb{R}$ is *upper semi-continuous* if for all $\theta \in \Theta$ and $\theta_n \rightarrow \theta$, $M(\theta) \geq \limsup_{n \rightarrow \infty} M(\theta_n)$.

Upper semi-continuity is useful since we have the following result from analysis.

Theorem 5.2.1. If Θ is compact and $M(\theta)$ is upper semi-continuous, then $\sup_{\theta \in \Theta} M(\theta)$ is obtained, i.e., there exists a maximizer θ^* .

A simple application of Theorem 5.2.1 allows us to prove (A1).

Corollary 5.2.1. If Θ is compact, M is upper semi-continuous, and the maximizer θ^* is unique, then (A1) is satisfied.

Proof. Firstly, we can write $\sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) = \sup_{\theta \in (B(\theta^*, \epsilon))^c} M(\theta)$. Since $B(\theta^*, \epsilon)$ is open, $(B(\theta^*, \epsilon))^c$ is closed, hence compact since Θ is compact. Applying Theorem 5.2.1, we see that $\sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta)$ is obtained by a maximizer θ^* , which is unique as assumed. ■

Hence, as long as Θ is compact and $M(\theta)$ is reasonable enough, (A1) can be satisfied. On the other hand, (A2) is a bit more involved. To understand it, observe the following.

Claim. $\sup_{\theta \notin B(\theta^*, \epsilon)} |M_n(\theta) - M(\theta)| \xrightarrow{P} 0$ implies (A2).

Proof. By the triangle inequality, $M_n(\theta) \leq |M_n(\theta) - M(\theta)| + M(\theta)$, hence

$$\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \leq \sup_{\theta \notin B(\theta^*, \epsilon)} |M_n(\theta) - M(\theta)| + \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta),$$

which implies

$$\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) - \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) \leq \sup_{\theta \notin B(\theta^*, \epsilon)} |M_n(\theta) - M(\theta)|.$$

Hence, $\sup_{\theta \notin B(\theta^*, \epsilon)} |M_n(\theta) - M(\theta)| \xrightarrow{P} 0$ implies $\mathbb{P}(\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) - \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) \geq \delta) \rightarrow 0$ for any $\delta > 0$, which is (A2). *

Intuition. (A2) uniformly controls the convergence.

³Here, i.o. means *infinitely often* (in terms of n).

In this regard, by reformulating (A3) as

$$\mathbb{P}\left(\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \geq \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) + \delta \text{ i.o.}\right) = 0 \Leftrightarrow \mathbb{P}\left(\limsup_{n \rightarrow \infty} \sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \leq \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta)\right) = 1,$$

since the i.o. can be expressed as $\mathbb{P}(\limsup_{n \rightarrow \infty} \sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \geq \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) + \delta) = 0$. Then, the same argument shows that $\sup_{\theta \notin B(\theta^*, \epsilon)} |M_n(\theta) - M(\theta)| \xrightarrow{\text{a.s.}} 0$ implies (A3).

Theorem 5.2.2. If (A1) and (A2) holds, then $\hat{\theta}_n \xrightarrow{P} \theta^*$. On the other hand, if we replace (A2) by (A3), then we have $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$.

Lecture 25: Understanding Assumptions for Consistency

We first prove Theorem 5.2.2.

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Proof. From the definition, we have

- $\hat{\theta}_n \xrightarrow{P} \theta^* \Leftrightarrow \mathbb{P}(\rho(\hat{\theta}_n, \theta^*) \geq \epsilon) \rightarrow 0$ for all $\epsilon > 0$;
- $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta^* \Leftrightarrow \mathbb{P}(\limsup_{n \rightarrow \infty} \rho(\hat{\theta}_n, \theta^*) \geq \epsilon) = 0$ for all $\epsilon > 0$.

Hence, we need to show that for all $\epsilon > 0$, $\mathbb{P}(\rho(\hat{\theta}_n, \theta^*) \geq \epsilon) \rightarrow 0$ or $\mathbb{P}(\rho(\hat{\theta}_n, \theta^*) > \epsilon \text{ i.o.}) = 0$. Firstly, suppose $\rho(\hat{\theta}_n, \theta^*) \geq \epsilon$, which implies $\hat{\theta}_n \notin B(\theta^*, \epsilon)$, and with Equation 5.1,

$$\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \geq M_n(\hat{\theta}_n) \geq \sup_{\theta \in \Theta} M_n(\theta) - \xi_n \geq M_n(\theta^*) - \xi_n.$$

This gives $\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) - M_n(\theta^*) + \xi_n \geq 0$, i.e.,

$$\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) - M_n(\theta^*) + \xi_n - \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) + M(\theta^*) \geq M(\theta^*) - \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) =: \delta_\epsilon > 0$$

by (A1). Write $A_n := \sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) - \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta)$, $C_n := M(\theta^*) - M_n(\theta^*)$, we have

$$A_n + C_n + \xi_n \geq \delta_\epsilon.$$

This gives $\{\rho(\hat{\theta}_n, \theta^*) \geq \epsilon\} \subseteq \{A_n + C_n + \xi_n \geq \delta_\epsilon\}$, hence

$$\mathbb{P}(\rho(\hat{\theta}_n, \theta^*) \geq \epsilon) \leq \mathbb{P}\left(A_n \geq \frac{\delta_\epsilon}{3}\right) + \mathbb{P}\left(C_n \geq \frac{\delta_\epsilon}{3}\right) + \mathbb{P}\left(\xi_n \geq \frac{\delta_\epsilon}{3}\right).$$

Now, with (A2), $\mathbb{P}(A_n \geq \delta_\epsilon/3)$ vanishes. Moreover, as $M_n(\theta) \xrightarrow{P} M(\theta)$ for any fixed θ , in this case, θ^* for C_n , $\mathbb{P}(C_n \geq \delta_\epsilon/3)$ also vanishes. Finally, $\mathbb{P}(\xi_n \geq \delta_\epsilon/3)$ vanishes if we require $\xi_n \xrightarrow{P} 0$.

The case for (A3) is the same, but this time we need $M_n(\theta) \xrightarrow{\text{a.s.}} M(\theta)$ and $\xi_n \xrightarrow{\text{a.s.}} 0$. ■

5.2.2 Proving the Assumptions

With Theorem 5.2.2, the only task left is to check (A1) and (A2) or (A3). Although we have briefly seen how to prove (A1), i.e., Corollary 5.2.1, but there are still lots to be discussed. Let's assume that X_1, \dots, X_n are i.i.d. and $M_n(\theta) = \frac{1}{n} \sum_{i=1}^n m_\theta(X_i)$ where $\theta \mapsto m_\theta(x)$ is upper semi-continuous for all x .

Note. $M_n(\theta)$ has a maximizer since $\theta \mapsto M_n(\theta)$ is also upper semi-continuous.

Proof. Since the sum of upper semi-continuous functions is still upper semi-continuous. *

But what about M ? I.e., can we say that the upper semi-continuity is preserved in the limit?

Proposition 5.2.1. If for all θ , there exists $r_\theta > 0$ such that $\mathbb{E}[\sup_{u \in B(\theta, r_\theta)} m_u(X)] < \infty$, then M is upper semi-continuous.

Proof. We want to show that as $\theta_n \rightarrow \theta$, $M(\theta) \geq \limsup_{n \rightarrow \infty} M(\theta_n)$. From the strong law of large number, $M_n(\theta) \xrightarrow{\text{a.s.}} M(\theta) = \mathbb{E}[m_\theta(X)] \in [-\infty, \infty]$. Since $\theta \mapsto m_\theta(x)$ is upper semi-continuous,

$$M(\theta) = \mathbb{E}[m_\theta(X)] \geq \mathbb{E}\left[\limsup_{n \rightarrow \infty} m_{\theta_n}(X)\right] \geq \limsup_{n \rightarrow \infty} \mathbb{E}[m_{\theta_n}(X)] = \limsup_{n \rightarrow \infty} M(\theta_n),$$

where the last inequality is provided by the reverse Fatou's lemma. In particular, it requires $\mathbb{E}[\sup_{n \geq n_0} m_{\theta_n}(X)] < \infty$ for some large enough n_0 . But since for any $\epsilon > 0$, as $\theta_n \rightarrow \theta$, there exists n_0 such that for all $n \geq n_0$, $\theta_n \in B(\theta, \epsilon)$, such a requirement becomes what we assumed. ■

With Proposition 5.2.1, we now have a tool to check upper semi-continuous for M , and with Corollary 5.2.1, we only need to check that M has a unique maximizer.

Example (MLE). For MLE, if the model class $\{g_\theta : \theta \in \Theta\}$ we postulated is identifiable is well-specified, i.e., $g_\theta = g_{\theta'}$ implies $\theta = \theta'$ and the true pdf is in $\{g_\theta\}$, then M has a unique maximizer.

Proof. For MLE, recall that $m_\theta(x) = \log(g_\theta(x))$, and say $X, X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} f = g_{\theta^*}$. In this case,

$$M(\theta) = \mathbb{E}[\log g_\theta(X)] = -\text{KL}(f \| g_\theta) + C \leq C$$

for some constant C as we have seen. As $\theta = \theta^*$, the maximum is achieved. However, suppose there exists $\tilde{\theta}$ such that $M(\tilde{\theta}) = M(\theta^*)$, we have $\text{KL}(g_{\theta^*} \| g_{\tilde{\theta}}) = 0$ implies $g_{\theta^*} = g_{\tilde{\theta}}$ almost surely. Hence, if $g_{\theta^*} = g_{\tilde{\theta}}$ doesn't imply $\theta^* = \tilde{\theta}$, i.e., not identifiable, the maximizer might not be unique. ⊗

In view of this, (A1) can be checked in general. Before we turn our focus to checking either (A2) or (A3), we see one more tool related to upper semi-continuity, which will be useful later.

Proposition 5.2.2. Let M be an upper semi-continuous function on a compact Θ , and let $\theta \in \Theta$ be fixed. Then, as $r \rightarrow 0$, $\sup_{u \in B(\theta, r)} M(u) \rightarrow M(\theta)$.

Proof. Suppose not, then we have the following.

- If $M(\theta) > -\infty$: there exists $\epsilon > 0$ and $(r_n) \rightarrow 0$ such that $\sup_{u \in B(\theta, r_n)} M(u) \geq M(\theta) + \epsilon$.
- If $M(\theta) = -\infty$: there exists $C_0 \in \mathbb{R}$ such that for all $C \leq C_0$, $\sup_{u \in B(\theta, r_n)} M(u) \geq C$.

The above holds for all $n \geq 1$. Combining the above, we see that

$$\sup_{u \in B(\theta, r_n)} M(u) \geq \max(M(\theta) + \epsilon, C) \Rightarrow \sup_{u \in \overline{B(\theta, r_n)}} M(u) \geq \max(M(\theta) + \epsilon, C).$$

Since $\overline{B(\theta, r_n)}$ is also compact, there exists a θ_n in $\overline{B(\theta, r_n)}$ such that $M(\theta_n) \geq \max(M(\theta) + \epsilon, C)$ for all $n \geq 1$, $C \leq C_0$, for some $\epsilon > 0$. Taking the limit, we see that

$$\liminf_{n \rightarrow \infty} M(\theta) \geq \max(M(\theta) + \epsilon, C) \geq \max\left(\limsup_{n \rightarrow \infty} M(\theta_n) + \epsilon, C\right)$$

since M is upper semi-continuous and $\theta_n \rightarrow \theta$ as $r_n \rightarrow 0$. By letting $C \rightarrow -\infty$, we have

$$\liminf_{n \rightarrow \infty} M(\theta_n) \geq \limsup_{n \rightarrow \infty} M(\theta_n) + \epsilon > \limsup_{n \rightarrow \infty} M(\theta_n),$$

which is a contradiction. ■

This concludes the discussion on upper semi-continuity. Now, let's focus on the stronger assumption, i.e., (A3), since as we will soon see, the assumption used in Proposition 5.2.1 suffices to prove (A3).

Explicitly, for (A3), we want to show that for any $\epsilon > 0$,

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \leq \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) \right) = 1.$$

Replacing $\theta \notin B(\theta^*, \epsilon)$ by $\theta \in (B(\theta^*, \epsilon))^c$, where $(B(\theta^*, \epsilon))^c$ is compact as shown before. Now, consider a trivial open cover

$$(B(\theta^*, \epsilon))^c \subseteq \bigcup_{\theta \in (B(\theta^*, \epsilon))^c} B(\theta, r_\theta) \text{ for some } r_\theta > 0 \text{ for all } \theta \in (B(\theta^*, \epsilon))^c,$$

from the compactness of $(B(\theta^*, \epsilon))^c$, there exists a finite subset $K \subseteq (B(\theta^*, \epsilon))^c$ such that

$$(B(\theta^*, \epsilon))^c \subseteq \bigcup_{\theta \in K} B(\theta, r_\theta).$$

This implies that

$$\sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \leq \max_{\theta \in K} \sup_{u \in B(\theta, r_\theta)} M_n(u) \leq \max_{\theta \in K} \frac{1}{n} \sum_{i=1}^n \sup_{u \in B(\theta, r_\theta)} m_u(X_i) \xrightarrow{\text{a.s.}} \max_{\theta \in K} \mathbb{E} \left[\sup_{u \in B(\theta, r_\theta)} m_u(X) \right]$$

by the [strong law of large number](#). However, we need to be careful here by combining $\xrightarrow{\text{a.s.}}$ and $\max_{\theta \in K}$.

Claim. We can indeed interchange the limit and the maximum.

Proof. Since the [convergence](#) happens for every $\theta \in K$ such that

$$\frac{1}{n} \sum_{i=1}^n \sup_{u \in B(\theta, r_\theta)} m_u(X_i(\omega)) \xrightarrow{\text{a.s.}} \mathbb{E} \left[\sup_{u \in B(\theta, r_\theta)} m_u(X(\omega)) \right].$$

In other words, for all $\theta \in K$, there exists a null set N_θ such that the convergence holds for all $\omega \notin N_\theta$. With $|K| < \infty$, $\bigcup_{\theta \in K} N_\theta$ is a *countable* union of null sets, which is still a null set. Hence, K might be countable if this is the only issue. However, this is not the only story.

Consider $|K| = 2$, or in particular, $K = \{\theta_1, \theta_2\}$. In this case, we can say that the maximum of the limits is the limits of the maximum since for these two sequences of convergence sequence, we're saying that for θ_i , the limiting statement involves “there exists n_i such that for all $n \geq n_i$, such that ...” Hence, if K is finite, one can just take $n_0 := \max_{\theta \in K} n_\theta$, which is guaranteed to be finite. If K is countable, n_0 can be unbounded, causing problems. \circledast

Hence, with the above justification, taking limits on both sides we have

$$\limsup_{n \rightarrow \infty} \sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \leq \max_{\theta \in K} \mathbb{E} \left[\sup_{u \in B(\theta, r_\theta)} m_u(X) \right].$$

With this, the upshot is that by using the same assumption used in [Proposition 5.2.1](#), the above can be controlled carefully. Specifically, since $\theta \mapsto m_\theta(x)$ is [upper semi-continuous](#) for all x , we know that as $r \rightarrow 0$, $\sup_{u \in B(\theta, r)} m_u(X) \rightarrow m_\theta(X)$ for every $\theta \in \Theta$, as shown in [Proposition 5.2.2](#). Hence,

$$M(\theta) = \mathbb{E}[m_\theta(X)] = \lim_{r \rightarrow 0} \mathbb{E} \left[\sup_{u \in B(\theta, r)} m_u(X) \right]$$

by the [monotone convergence theorem](#) as the sequence is the supremum over smaller and smaller sets. This implies that for all $\theta \in \Theta$ and $\delta > 0$, there exists $r_\theta > 0$ such that $\mathbb{E}[\sup_{u \in B(\theta, r_\theta)} m_u(X)] < M(\theta) + \delta$. Hence, combining this with the above, by choosing θ_δ for every $\theta \in \Theta$ in this way, we have

$$\limsup_{n \rightarrow \infty} \sup_{\theta \notin B(\theta^*, \epsilon)} M_n(\theta) \leq \max_{\theta \in K} \mathbb{E} \left[\sup_{u \in B(\theta, r_\theta)} m_u(X) \right] \leq \max_{\theta \in K} M(\theta) + \delta \leq \sup_{\theta \notin B(\theta^*, \epsilon)} M(\theta) + \delta$$

almost surely, which is exactly (A3).⁴

⁴We didn't handle the case that $M(\theta) = -\infty$. However, this can be easily addressed as in [Proposition 5.2.2](#).

Intuition. We can use a single assumption to prove both (A1) and (A3)!

Lecture 26: Asymptotic Normality of M -Estimators

Let's summarize what we have done.

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Theorem 5.2.3 (Wald). Given $X, (X_n) \stackrel{\text{i.i.d.}}{\sim} F$, a compact metric space (Θ, ρ) , and $\theta \mapsto m_\theta(x)$ **upper semi-continuous** for all x . Define $M_n(\theta) = \frac{1}{n} \sum_{i=1}^n m_\theta(X_i)$, and let $\hat{\theta}_n$ to be the approximate maximizer of M_n .^a If $M(\theta) = \mathbb{E}[m_\theta(X)]$ has a unique maximizer θ^* and $\mathbb{E}[\sup_{u \in B(\theta, r)} m_u(X)] < \infty$ for some small $r > 0$ and all $\theta \in \Theta$, then $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$.

^aI.e., Equation 5.1: $M_n(\hat{\theta}_n) \geq M_n(\theta^*) - \xi_n$ for some $\xi_n \xrightarrow{\text{a.s.}} 0$.

Proof. From the **strong law of large number**, $M_n(\theta) \xrightarrow{\text{a.s.}} M(\theta) = \mathbb{E}[m_\theta(X)] \in [-\infty, \infty]$ for all $\theta \in \Theta$. Furthermore, since for some $r > 0$, $\mathbb{E}[\sup_{u \in B(\theta, r)} m_u(X)] < \infty$, **Proposition 5.2.1** implies that M is also **upper semi-continuous** as $m_\theta(x)$ is. Hence, as Θ is compact and the maximizer θ^* of M is unique, **Corollary 5.2.1** implies that (A1) is satisfied. Finally, as we discussed above, (A3) is also satisfied by the same set of assumptions, hence **Theorem 5.2.2** proves that $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$. ■

Indeed, **Theorem 5.2.3** helps us find the MLE of Cauchy that we **previously** don't know how to solve.

Example (Cauchy). Let $X \sim F = G_{\theta^*}$ with a family of cdfs $\{G_\theta \sim \text{Cauchy}(\theta, 1) : \theta \in \Theta = \overline{\mathbb{R}}\}$ where the pdf of G_θ is given by $g_\theta(x) = 1/\pi(1 + (x - \theta)^2)$. Consider the setup of MLE, i.e., $m_\theta(x) = \log(g_\theta(x))$, and let $\hat{\theta}_n := \arg \max_{\theta \in \Theta} M_n(\theta)$ and $\theta^* = \arg \max_{\theta \in \Theta} M(\theta)$, then $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$.

Proof. We first check the following.

- $\theta \mapsto m_\theta(x)$ is **upper semi-continuous**: clearly since $\theta \mapsto g_\theta(x)$ is continuous.
- Θ is compact: we consider the extended real line $\overline{\mathbb{R}} = [-\infty, \infty]$. However, we need to be careful to define $m_{\pm\infty}(x) = -\infty$ for all $x \in \mathbb{R}$, which indeed preserves the continuity of $m_\theta(x)$ since $g_\theta(x) \rightarrow 0$ as $\theta \rightarrow \pm\infty$.
- $\mathbb{E}[\sup_{u \in B(\theta, r)} m_u(X)] < \infty$ for small r : indeed, since $m_\theta(x) \leq -\log(\pi)$ for all $\theta \in \Theta$ and $x \in \mathbb{R}$.
- M does have a unique maximizer: since Cauchy distribution is identifiable.

Hence, we can apply **Theorem 5.2.3**, and conclude the result. ⊛

Another example is the following.

Example (Uniform). Let $X \sim F = G_{\theta^*}$ with a family of cdfs $\{G_\theta \sim \mathcal{U}(0, \theta) : \theta > 0\}$. From the **previous example**, the MLE is given by $\hat{\theta}_n = \max_{1 \leq i \leq n} X_i$. Applying **Theorem 5.2.3** gives $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$.

5.3 Asymptotic Distributions

With **Theorem 5.2.3**, under reasonable assumptions, we can establish **consistency** for **M -estimators**. A further natural question is that whether we can obtain some asymptotic distribution for **M -estimators** as well. However, asymptotic normality is impossible in general.

Example (Uniform is not asymptotically normal). From the **previous example**, clearly we don't have $\sqrt{n}(\theta^* - \hat{\theta}_n) \xrightarrow{D} \mathcal{N}$ as we always have $\theta^* - \hat{\theta}_n \geq 0$.

More specifically, for the **uniform example**, the asymptotic distribution is the exponential.

Proposition 5.3.1. If $X_i \sim \mathcal{U}(0, \theta^*)$ with the family $\{\mathcal{U}(0, \theta) : \theta > 0\}$, then $n(\theta^* - \hat{\theta}_n) \xrightarrow{D} \text{Exp}(\theta^*)$.

Proof. We want to show that for all $x > 0$, $\mathbb{P}(n(\theta^* - \hat{\theta}_n) > x) \rightarrow e^{-x/\theta^*}$. As $\hat{\theta}_n = \max_{1 \leq i \leq n} X_i$,

$$\mathbb{P}(n(\theta^* - \hat{\theta}_n) > x) = \mathbb{P}\left(\hat{\theta}_n < \theta^* - \frac{x}{n}\right) = \prod_{i=1}^n \mathbb{P}\left(X_i < \theta^* - \frac{x}{n}\right) = \left(\mathbb{P}\left(X < \theta^* - \frac{x}{n}\right)\right)^n.$$

For n large enough, $x/n < \theta^*$, then $\mathbb{P}(X < \theta^* - x/n) = (\theta^* - x/n)/\theta^*$. This further gives

$$\mathbb{P}(n(\theta^* - \hat{\theta}_n) > x) = \left(\frac{\theta^* - x/n}{\theta^*}\right)^n = \left(1 + \frac{-x/\theta^*}{n}\right)^n \rightarrow e^{-x/\theta^*}$$

as $n \rightarrow \infty$. ■

Remark. In general, we don't have asymptotic normality for M -estimators.

Hence, our next goal is to show that under what conditions we can establish the asymptotic normality. But firstly, we discuss the method of moments estimators, which turns out to be useful for later.

5.3.1 Method of Moments Estimators

Let $X, X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F$ and let $\{G_\theta : \theta \in \Theta \subseteq \mathbb{R}^m\}$ be a family of cdfs. Write $h(x) = (x, \dots, x^m)$, then consider $T(\theta) = \mathbb{E}_\theta[h(X)]$ for $\theta \in \Theta \subseteq \mathbb{R}^m$, i.e.,

$$T(\theta) = (\mathbb{E}_\theta[X], \dots, \mathbb{E}_\theta[X^m]).$$

If T is one-to-one, we can then define the *method of moments (MOM) estimator* $\tilde{\theta}_n := T^{-1}(\overline{h(X)}_n)$ where $\overline{h(X)}_n = \frac{1}{n} \sum_{i=1}^n h(X_i)$.

Note. More generally, consider $h(x) = (h_1(x), \dots, h_m(x))$ and $T(\theta) = (\mathbb{E}_\theta[h_1(X)], \dots, \mathbb{E}_\theta[h_m(X)])$.

Clearly, from the [strong law of large number](#), $\overline{h(X)}_n \xrightarrow{\text{a.s.}} \mathbb{E}_F[h(X)]$ as long as $\mathbb{E}_F[h(X)] < \infty$. Hence, if T^{-1} is continuous at $\mathbb{E}_F[h(X)]$, [continuous mapping theorem](#) implies

$$\tilde{\theta}_n = T^{-1}(\overline{h(X)}_n) \xrightarrow{P} T^{-1}(\mathbb{E}_F[h(X)]).$$

Moreover, since $\sqrt{n}(\overline{h(X)}_n - \mathbb{E}_F[h(X)]) \xrightarrow{D} \mathcal{N}(0, \text{Var}[h(X)])$ by the [central limit theorem](#) as long as the variance is well-defined. Hence, if T^{-1} is differentiable at $\mathbb{E}_F[h(X)]$, by the [Delta method](#),

$$\sqrt{n}(\tilde{\theta}_n - T^{-1}(\mathbb{E}_F[h(X)])) \xrightarrow{D} \mathcal{N}(0, \Sigma).$$

We see the following.

Lemma 5.3.1. If the model is well-specified, i.e., $F = G_{\theta^*}$ for some $\theta^* \in \Theta$, $\tilde{\theta}_n \xrightarrow{P} \theta^*$.

Proof. Since $T^{-1}(\mathbb{E}_F[h(X)]) = T^{-1}(\mathbb{E}_{G_{\theta^*}}[h(X)]) = \theta^*$ from the definition of T^{-1} . ■

On the other hand, what if F is not in $\{G_\theta : \theta \in \Theta\}$?

Example (Uniform). Let $F(u - \epsilon) < 1 = F(u)$ for all $\epsilon > 0$ such that $\mathbb{P}(0 \leq X \leq u) = 1$, i.e., X has support $[0, u]$. Consider the family of cdfs $\{G_\theta \sim \mathcal{U}(0, \theta) : \theta > 0\}$, then $T(\theta) = \mathbb{E}_\theta[X] = \theta/2$ for $\theta > 0$, which is clearly invertible, and we can define T^{-1} to get $\tilde{\theta}_n = 2\overline{X}_n$.

Hence, from the [strong law of large number](#), $\tilde{\theta}_n \xrightarrow{\text{a.s.}} 2\mathbb{E}[X]$, so $\tilde{\theta}_n$ is a [strongly consistent](#) estimator of u as long as $\mathbb{E}[X] = u/2$. Furthermore, by the [central limit theorem](#), we also have $\sqrt{n}(\tilde{\theta}_n - 2\mathbb{E}[X]) \xrightarrow{D} \mathcal{N}(0, 4\text{Var}[X])$.

Comparing the above example and [Proposition 5.3.1](#), we see that while the MOM estimator is asymptotically normal, the MLE is not. Moreover, they converge with different rates: specifically, $n(u - \hat{\theta}_n) \xrightarrow{D} \text{Exp}(u)$ for the MLE and $\sqrt{n}(\tilde{\theta}_n - 2\mathbb{E}[X]) \xrightarrow{D} \mathcal{N}(0, 4\text{Var}[X])$ for the MOM estimator. We see that MLE is better than the MOM estimator since the rate is much faster (n v.s. \sqrt{n}).

Remark (Consistent under symmetry). The above example for MOM holds whenever X is symmetric about $u/2$ since in this case, $\mathbb{E}[X] = u/2$ also. That means, the MOM estimator is also **consistent** and asymptotically normal with a rate \sqrt{n} .

5.3.2 M -Estimators

Again, let $X, X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F$ such that $M_n(\theta) = \frac{1}{n} \sum_{i=1}^n m_\theta(X_i)$ where $\theta \in \Theta \subseteq \mathbb{R}^m$. There are two closely related problems we're going to address in this section.

Problem. How to find the approximate maximizer $\hat{\theta}_n$ of M_n in practice?

Answer. If $\theta^* \in \text{int}(\Theta)$ and $\theta \mapsto M_n(\theta)$ is differentiable, then we can find $\hat{\theta}_n$ by searching the solution set of $\nabla M_n(\theta) =: \Psi_n(\theta) = 0$ via some numerical algorithms, e.g., the classical **Newton-Raphson**. \circledast

As previously seen (Newton-Raphson algorithm). Starting with some $\hat{\theta}_n^{(0)} \in \Theta$. Then for $k \geq 0$, consider updating $\hat{\theta}_n^{(k)}$ by

$$\hat{\theta}_n^{(k+1)} = \hat{\theta}_n^{(k)} - \left(\nabla \Psi_n(\hat{\theta}_n^{(k)}) \right)^{-1} \Psi_n(\hat{\theta}_n^{(k)}).$$

However, we should note that typically, $\Psi_n(\theta)$ has many roots (sometimes even grows linearly with n) and in general, we don't know whether our numerical algorithm gives us the actual maximizer of $M_n(\theta)$. One might want to try different $\hat{\theta}_n^{(0)}$ and run the numerical algorithm multiple times.

Intuition. The up-shot is that we don't need to do this: if $\hat{\theta}_n^{(0)}$ is "good," then we're fine.

If we can find $\hat{\theta}_n$, with **Theorem 5.2.3**, we can have $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$. However, a follow-up question arises:

Problem. Under what condition can we establish asymptotic normality of $\hat{\theta}_n$?

Answer. In general, $\hat{\theta}_n$ is not asymptotically normal, e.g., the **uniform example**. On the other hand, we will show that if $\hat{\theta}_n^{(0)}$ is "good," or more specifically, if $\sqrt{n}(\hat{\theta}_n^{(0)} - \theta^*) = O_p(1)$, then by only one Newton-Raphson update, we can obtain a sequence of estimators $(\hat{\theta}_n^{(1)})$ that is asymptotically normal, hence also **consistent**. The above approach is the so-called *one-step MLE*. \circledast

Lecture 27: Consistency with Computational Consideration

To continue the discussion, we will need to establish a *uniform* version of the **strong law of large number**. 25 Apr. 9:30

In particular, say $X, (X_n) \stackrel{\text{i.i.d.}}{\sim} F$ and $h_\theta(\cdot) \in \mathbb{R}$ is given for every $\theta \in \Theta$. From the **strong law of large number**, $\frac{1}{n} \sum_{i=1}^n h_\theta(X_i) \xrightarrow{\text{a.s.}} \mathbb{E}[h_\theta(X)]$ for any fixed $\theta \in \Theta$. Now, we interested in whether

$$\frac{1}{n} \sum_{i=1}^n h_{\theta_n}(X_i) \xrightarrow{\text{a.s.}} \mathbb{E}[h_{\theta^*}(X)]$$

for some sequence (θ_n) , either deterministic or random, such that $\theta_n \rightarrow \theta^* \in \Theta$ (or $\theta_n \xrightarrow{\text{a.s.}} \theta^*$).

Theorem 5.3.1 (Uniform strong law of large number). Let $X, (X_n) \stackrel{\text{i.i.d.}}{\sim} F$ and $h_\theta(x) \in \mathbb{R}$ be given. Furthermore, let (θ_n) be a sequence such that $\theta_n \rightarrow \theta^*$. If $\theta \mapsto h_\theta(x)$ is continuous for all x and $\mathbb{E}[\sup_{\theta \in \overline{B}(\theta^*, r)} |h_\theta(X)|] < \infty$ for some $r > 0$, then

$$\frac{1}{n} \sum_{i=1}^n h_{\theta_n}(X_i) \xrightarrow{\text{a.s.}} \mathbb{E}[h_{\theta^*}(X)].$$

Proof. The main idea is to apply [Proposition 5.2.1](#) twice with h and $-h$, and conclude

$$\sup_{\theta \in B(\theta^*, r)} \left| \frac{1}{n} \sum_{i=1}^n h_{\theta}(X_i) - \mathbb{E}[h_{\theta}(X)] \right| \rightarrow 0.$$

This will conclude the proof since we can write

$$\left| \frac{1}{n} \sum_{i=1}^n h_{\theta_n}(X_i) - \mathbb{E}[h_{\theta^*}(X)] \right| \leq \left| \frac{1}{n} \sum_{i=1}^n h_{\theta_n}(X_i) - \mathbb{E}[h_{\theta_n}(X)] \right| + |\mathbb{E}[h_{\theta_n}(X)] - \mathbb{E}[h_{\theta^*}(X)]|.$$

Firstly, the second term goes to 0 as $n \rightarrow \infty$ if $\theta \mapsto \mathbb{E}[h_{\theta}(X)]$ is continuous at θ^* . Indeed, since $\theta \mapsto h_{\theta}(x)$ is continuous for all x , $\theta \mapsto \mathbb{E}[h_{\theta}(X)]$ is also continuous at θ^* . On the other hand,

$$\left| \frac{1}{n} \sum_{i=1}^n h_{\theta_n}(X_i) - \mathbb{E}[h_{\theta_n}(X)] \right| \leq \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n h_{\theta}(X_i) - \mathbb{E}[h_{\theta}(X)] \right|,$$

and since we only care about $n \rightarrow \infty$, we may assume that for some $r > 0$,

$$\mathbb{P}(\theta_n \in B(\theta^*, r) \text{ for all } n \text{ large enough}) = 1.$$

Hence, we can replace Θ by $B(\theta^*, r)$, which gives

$$\left| \frac{1}{n} \sum_{i=1}^n h_{\theta_n}(X_i) - \mathbb{E}[h_{\theta_n}(X)] \right| \leq \sup_{\theta \in B(\theta^*, r)} \left| \frac{1}{n} \sum_{i=1}^n h_{\theta}(X_i) - \mathbb{E}[h_{\theta}(X)] \right|$$

for all large enough n with probability 1. This goes to 0 as we have shown in the beginning. \blacksquare

Before resuming our discussion on asymptotic normality of [M-estimators](#), let's recall our motivation.

As previously seen. Let $M_n(\theta) = \frac{1}{n} \sum_{i=1}^n m_{\theta}(X_i) \xrightarrow{\text{a.s.}} M(\theta) = \mathbb{E}[m_{\theta}(X)]$ for all $\theta \in \Theta$, and let $\hat{\theta}_n$ be the maximizer of $\theta \mapsto M_n(\theta)$. From [Theorem 5.2.3](#), $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$ under assumptions:

- Θ is compact;
- $\theta \mapsto m_{\theta}(x)$ is [upper semi-continuous](#) for all x ;
- θ^* is the unique maximizer of $\theta \mapsto M(\theta)$;
- $\mathbb{E}[\sup_{\theta \in B(\theta^*, r)} m_{\theta}(X)] < \infty$ for some $r > 0$,

While $\hat{\theta}_n$ is [strongly consistent](#), we have seen that $\hat{\theta}_n$ doesn't need to be asymptotically normal, e.g., the [uniform example](#). Furthermore, it's also unclear how to compute $\hat{\theta}_n$ in practice.

Toward this end, we will need additional assumptions in order to solve these two problems. Firstly, assume further that $\Theta \subseteq \mathbb{R}^m$ with $\theta^* \in \text{int}(\Theta)$,⁵ and $\theta \mapsto m_{\theta}(x) \in C^2$ for any x .

Notation. The class of second-differentiable functions is denoted as C^2 .

Furthermore, let $\psi_{\theta}(x) := \nabla_{\theta} m_{\theta}(x)$ for all $\theta \in \text{int}(\Theta)$ and assume for $r > 0$ small enough we have

$$\mathbb{E} \left[\sup_{\theta \in B(\theta^*, r)} \|\nabla \psi_{\theta}(X)\|_{\max} \right] < \infty,$$

where the *max norm* is defined as $\|A\|_{\max} = \max_{i,j} |A_{ij}|$ for $A = (A_{ij})$.

Note. We can interchange ∇ and \mathbb{E} from our assumption on $\|\nabla \psi_{\theta}(X)\|_{\max}$.

⁵Before this, for [consistency](#) discussion, we can work with a general metric space.

Since M is differentiable and $\theta^* \in \text{int}(\Theta)$, θ^* is a root of

$$\nabla M(\theta) = \nabla \mathbb{E}[m_\theta(X)] = \mathbb{E}[\nabla m_\theta(X)] = \mathbb{E}[\psi_\theta(X)],$$

implying $\mathbb{E}[\psi_{\theta^*}(X)] = 0$. Finally, assume that $\psi_{\theta^*}(X)$ has a covariance matrix $J_* := \text{Var}[\psi_{\theta^*}(X)]$ and $\theta \mapsto m_\theta(x)$ has a second partial derivative $J(\theta) := \mathbb{E}[-\nabla \psi_\theta(X)]$ of θ , which exists and is continuous in $\text{int}(\Theta)$ such that J_* and $J(\theta^*)$ are both invertible, and in particular, positive definite.

Remark. In particular, the assumption of $\mathbb{E}[\sup_{\theta \in \overline{B(\theta^*, r)}} \|\nabla \psi_\theta(X)\|_{\max}] < \infty$ implies that $\theta \mapsto J(\theta)$ is continuous in a ball around θ^* .

Note. Even under the MLE setup, $J_* \neq J(\theta^*)$ in general if the family is not well-specified.

Example. Under the framework of MLE and with our assumptions, $J(\theta^*) = J_*$. This is something commonly used in the calculation of **Cramér-Rao lower bound**.

With these notations and new assumptions, we answer a slightly different version of the first problem.

As previously seen. In [Theorem 5.2.3](#), we're searching for the (approximate) maximizer $\hat{\theta}_n$ over the entire space Θ , hence it's possible that $\hat{\theta}_n$ isn't the actual root of $\nabla M_n(\theta)$.

In practice, when finding the maximizer $\hat{\theta}_n$ of M_n , we will find the root of the derivative of M_n , i.e.,

$$\Psi_n(\theta) := \nabla M_n(\theta) = \frac{1}{n} \sum_{i=1}^n \nabla m_\theta(X_i) = \frac{1}{n} \sum_{i=1}^n \psi_\theta(X_i)$$

since if $\hat{\theta}_n$ is the maximizer of M_n and is in $\text{int}(\Theta)$, indeed $\Psi_n(\hat{\theta}_n) = 0$. However, we note the following.

Note. Ψ_n may not have a unique root or the root may lie on the boundary of Θ , hence it's in general impossible to guarantee that a root we find by numerical algorithms of Ψ_n is indeed the maximizer.

We then ask that if we restrict our attention to roots of $\Psi_n(\theta)$ anyway, can we establish **consistency**?

Problem. Does there exist a sequence of roots $(\tilde{\theta}_n)$ of $\Psi_n(\theta)$ in $\text{int}(\Theta)$ that is **consistent** with θ^* ?

Answer. For every x , for some $r > 0$, consider for all $\theta \in \overline{B(\theta^*, r)}$ with the **Taylor expansion**

$$m_\theta(x) - m_{\theta^*}(x) = \psi_{\theta^*}(x)(\theta - \theta^*) + (\theta - \theta^*)^\top \left(\int_0^1 \int_0^1 \nabla \psi_{\theta^* + uv(\theta - \theta^*)}(X) u \, du \, dv \right) (\theta - \theta^*).$$

Then, we apply [Theorem 5.2.3](#) with $m_\theta(x) - m_{\theta^*}(x)$ by considering the metric space being $\overline{B(\theta^*, r)}$ instead of Θ , i.e., we only need to check $\mathbb{E}[\sup_{\theta \in \overline{B(\theta^*, r)}} m_\theta(X)] < \infty$, or equivalently,

$$\begin{aligned} & \mathbb{E} \left[\sup_{\theta \in \overline{B(\theta^*, r)}} (m_\theta(X) - m_{\theta^*}(X)) \right] \\ &= \mathbb{E} \left[\sup_{\theta \in \overline{B(\theta^*, r)}} \psi_{\theta^*}(X)(\theta - \theta^*) + (\theta - \theta^*)^\top \left(\int_0^1 \int_0^1 \nabla \psi_{\theta^* + uv(\theta - \theta^*)}(X) uv \, du \, dv \right) (\theta - \theta^*) \right] < \infty. \end{aligned}$$

This is true since $\mathbb{E}[\psi_{\theta^*}(X)] = 0$ and $\mathbb{E}[\sup_{\theta \in \overline{B(\theta^*, r)}} \|\nabla \psi_\theta(X)\|_{\max}] < \infty$. Hence, there exists a maximizer $\tilde{\theta}_n$ of $M_n(\theta) - M_n(\theta^*)$ such that $\tilde{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$ by [Theorem 5.2.3](#).

Claim. $\tilde{\theta}_n$ is a root of $\Psi_n(\theta)$ for large enough n with probability 1.

Proof. As $\tilde{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$, for large n and with probability 1, $\tilde{\theta}_n$ will not be on the boundary of $B(\theta^*, r)$, i.e., it's indeed a root of $\Psi_n(\theta)$ since if $\tilde{\theta}_n$ is in the interior and is a maximizer, the first-order condition gives $\nabla M_n(\tilde{\theta}_n) = \Psi_n(\tilde{\theta}_n) = 0$. \circledast

Since $\tilde{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$ and $\tilde{\theta}_n$'s are roots of $\Psi_n(\theta)$ for large enough n with probability 1, we're done. \circledast

Therefore, we see that there indeed exists a sequence of roots $(\tilde{\theta}_n)$ of $\Psi_n(\theta)$ that is **consistent** of θ^* . Next, we answer the question about asymptotic normality, but for $\tilde{\theta}_n$ this time.

Problem. Let $(\tilde{\theta}_n)$ be a sequence of roots of $\Psi_n(\theta)$ such that $\tilde{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$. Is $\tilde{\theta}_n$ asymptotically normal?

Answer. By the first-order Taylor expansion,

$$\Psi_n(\tilde{\theta}_n) = \frac{1}{n} \sum_{i=1}^n \psi_{\tilde{\theta}_n}(X_i) = \frac{1}{n} \sum_{i=1}^n \psi_{\theta^*}(X_i) + \frac{1}{n} \sum_{i=1}^n \left(\int_0^1 \nabla \psi_{\theta^*+u(\tilde{\theta}_n-\theta^*)}(X_i) du \right) (\tilde{\theta}_n - \theta^*),$$

which is 0 for n large enough since then $\Psi_n(\tilde{\theta}_n) = 0$. Multiplying both sides by \sqrt{n} , we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_{\theta^*}(X_i) = -\sqrt{n}(\tilde{\theta}_n - \theta^*) \cdot \frac{1}{n} \sum_{i=1}^n \left(\int_0^1 \nabla \psi_{\theta^*+u(\tilde{\theta}_n-\theta^*)}(X_i) du \right).$$

By the **multivariate central limit theorem**, the left-hand side **converges to** $\mathcal{N}(0, J_*)$ where $J_* = \text{Var}[\psi_{\theta^*}(X)]$. For the right-hand side, we see the following.

Intuition. If we can ignore $u(\tilde{\theta}_n - \theta^*)$, then by the **strong law of large number**, we have

$$\frac{1}{n} \sum_{i=1}^n \int_0^1 \nabla \psi_{\theta^*}(X_i) du \xrightarrow{\text{a.s.}} \mathbb{E}[\nabla \psi_{\theta^*}(X)] = -J(\theta^*).$$

Indeed, by the **uniform strong law of large number**, with $\tilde{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$, this can be rigorously justified. Combining everything together, we see that

$$\sqrt{n}(\tilde{\theta}_n - \theta^*) \xrightarrow{D} J(\theta^*)^{-1} Z$$

where $Z \sim \mathcal{N}(0, J_*)$. Hence, we conclude that $\tilde{\theta}_n$ is asymptotically normal. \circledast

The question now becomes how to obtain $\tilde{\theta}_n$ in practice. While our numerical algorithm can find roots, we're not guaranteed to get $\tilde{\theta}_n$ since there might be multiple roots. However, if we initialize our algorithm with some $\check{\theta}_n$ "close enough" to θ^* , we're fine.

Claim. Let $(\check{\theta}_n)$ be a sequence such that $\sqrt{n}(\check{\theta}_n - \theta^*) = O_p(1)$. Then with one Newton-Raphson update, i.e., $\hat{\theta}_n := \check{\theta}_n - (\nabla \Psi_n(\check{\theta}_n))^{-1} \Psi_n(\check{\theta}_n)$, we have $\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n) \xrightarrow{p} 0$.

Proof. Firstly, consider writing

$$\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n) = \sqrt{n}(\hat{\theta}_n - \check{\theta}_n) + \sqrt{n}(\check{\theta}_n - \tilde{\theta}_n).$$

By the definition, the first term is $-\sqrt{n}(\nabla \Psi_n(\check{\theta}_n))^{-1} \Psi_n(\check{\theta}_n)$, which can be written as

$$-\sqrt{n}(\nabla \Psi_n(\check{\theta}_n))^{-1} \left[\Psi_n(\tilde{\theta}_n) + \left(\int_0^1 \nabla \Psi_n(\tilde{\theta}_n + u(\check{\theta}_n - \tilde{\theta}_n)) du \right) (\check{\theta}_n - \tilde{\theta}_n) \right],$$

and with $\Psi_n(\tilde{\theta}_n) = 0$, we then have

$$\sqrt{n}(\hat{\theta}_n - \check{\theta}_n) = -\sqrt{n} \underbrace{(\nabla \Psi_n(\check{\theta}_n))^{-1} \left(\int_0^1 \nabla \Psi_n(\tilde{\theta}_n + u(\check{\theta}_n - \tilde{\theta}_n)) du \right)}_{:= B_n} (\check{\theta}_n - \tilde{\theta}_n) =: \sqrt{n} B_n (\check{\theta}_n - \tilde{\theta}_n).$$

Hence, $\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n) = (I - B_n)\sqrt{n}(\check{\theta}_n - \tilde{\theta}_n)$. To show that this is $o_p(1)$, we first see that $\sqrt{n}(\check{\theta}_n - \tilde{\theta}_n) = O_p(1)$ since $\tilde{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$ and $\sqrt{n}(\check{\theta}_n - \theta^*) = O_p(1)$. On the other hand, for $I - B_n$, from the **dominated convergence theorem**,

$$\int_0^1 \nabla \Psi_n(\tilde{\theta}_n + u(\check{\theta}_n - \tilde{\theta}_n)) du \rightarrow \mathbb{E}[\nabla \psi_{\theta^*}(X)],$$

and by the **uniform strong law of large number**,

$$\nabla \Psi_n(\check{\theta}_n) \rightarrow \mathbb{E}[\nabla \Psi_n(\theta^*)] = \mathbb{E}[\nabla \psi_{\theta^*}(X)],$$

which implies $(\nabla \Psi_n(\check{\theta}_n))^{-1} \rightarrow (\mathbb{E}[\nabla \psi_{\theta^*}(X)])^{-1}$ since inverse function is continuous. Combining both, we see that $B_n \rightarrow (\mathbb{E}[\nabla \psi_{\theta^*}(X)])^{-1} \mathbb{E}[\nabla \psi_{\theta^*}(X)] = I$, which gives

$$\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n) = (I - B_n)\sqrt{n}(\check{\theta}_n - \tilde{\theta}_n) = o(1)O_p(1) = o_p(1),$$

i.e., $\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n) \xrightarrow{p} 0$. *

Lecture 28: The End

We now formalize what we have proved so far.

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Theorem 5.3.2. Assuming that the above assumptions holds. Then with probability 1, there exists $(\tilde{\theta}_n)$ in $\text{int}(\Theta)$ where each $\tilde{\theta}_n$ is a function of (X_n) and is a root of $\Psi_n(\theta)$, i.e., $\Psi_n(\tilde{\theta}_n) = 0$, for all large n , such that $\tilde{\theta}_n \xrightarrow{\text{a.s.}} \theta^*$ and

$$\sqrt{n}(\tilde{\theta}_n - \theta^*) \xrightarrow{D} (J(\theta^*))^{-1} Z$$

where $Z \sim \mathcal{N}(0, J_*)$. Furthermore, if $\sqrt{n}(\check{\theta}_n - \theta^*) = O_p(1)$, then

$$\sqrt{n} \left(\check{\theta}_n - (\nabla \Psi_n(\check{\theta}_n))^{-1} \Psi_n(\check{\theta}_n) - \theta_n - \tilde{\theta}_n \right) \xrightarrow{p} 0.$$

Remark (Fisher scoring). Since $\nabla \Psi_n(\theta) = \frac{1}{n} \sum_{i=1}^n \nabla \psi_{\theta}(X_i) \xrightarrow{\text{a.s.}} \mathbb{E}[\nabla \psi_{\theta}(X)] = -J(\theta)$, hence in reality, one can avoid computing $(\nabla \Psi_n(\check{\theta}_n))^{-1}$ by considering $-(J(\check{\theta}_n))^{-1}$.

Example (Well-specified MLE). If $F = G_{\theta^*}$ where the postulated family of cdfs $\{G_{\theta} : \theta \in \Theta\}$ is identifiable, then $J(\theta^*) = J_*$. In this case, **Theorem 5.3.2** implies

$$\sqrt{n}(\tilde{\theta}_n - \theta^*) \xrightarrow{D} \mathcal{N}(0, J_*^{-1}),$$

where J_*^{-1} is the **Cramér-Rao lower bound**. Such an estimator is called *asymptotically efficient*.

Lastly, we make one more remark on the above result, which basically says that attaining **Cramér-Rao lower bound** asymptotically is not the end of the story.

Example (Superefficiency). Suppose $d = 1$ such that $\sqrt{n}(\hat{\theta}_n - \theta^*) \xrightarrow{D} \mathcal{N}(0, 1/J_*)$. Fix a $u \in \Theta$ and a

sequence $b_n \rightarrow \infty$ at a certain rate,^a and define the *Hodges' estimator*

$$\bar{\theta}_n := \begin{cases} \hat{\theta}_n, & \text{if } \sqrt{n}|\hat{\theta}_n - u| > b_n; \\ u, & \text{if } \sqrt{n}|\hat{\theta}_n - u| \leq b_n. \end{cases}$$

If $\theta^* \neq u$, then $\sqrt{n}(\bar{\theta}_n - \theta^*) \xrightarrow{D} \mathcal{N}(0, 1/J_*)$. On the other hand, if $\theta^* = u$, then $\sqrt{n}(\bar{\theta}_n - \theta^*) \xrightarrow{p} 0$.

^aThe rate is not too fast; in particular, it's about $n^{1/4}$.

Proof. It suffices to show that $\mathbb{P}(\hat{\theta}_n \neq \bar{\theta}_n) \rightarrow 0$ for the first case. The second case is more direct. \circledast

Appendix

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