MATH592
Introduction to Algebraic Topology

Pingbang Hu

February 21, 2022

Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

Contents

Lecture 1: Homotopies of Maps

05 Jan. 10:00

1 Foundation of Algebraic Topology

1.1 Homotopy

Definition 1.1 (Homotopy, homotopic, nullhomotopic). Let X, Y be topological spaces. Let $f,\ g\colon X \to Y$ continuous maps. Then a *homotopy* from f to g is a $1-$parameter family of maps that continuously deforms f to g, i.e., it's a continuous function $F\colon X \times I \to Y$, where $I = [0,1]$, such that

$$F(x,0) = f(x), \quad F(x,1) = g(x).$$

We often write $F_t(x)$ for $F(x,t)$.

If a homotopy exists between f and g, we say they are *homotopic* and write

$$f \simeq g.$$

If f is homotopic to a constant map, we call it *nullhomotopic*.

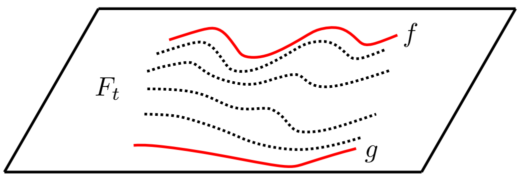


Figure 1: The continuous deforming from f to g described by F_t

Remark. Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

Example. We first see some examples.

1. Any two maps (continuous) with specification
$$f,\ g\colon X \to \mathbb{R}^n$$
are homotopic by considering
$$F_t(x) = (1-t)f(x) + tg(x).$$
We call it *the straight line homotopy.*
2. Let S^1 denotes the unit circle in $\mathbb{R}^2$, and D^2 denotes the unit disk in $\mathbb{R}^2$. Then the inclusion $f\colon S^1 \hookrightarrow D^2$ is nullhomotopic by considering
$$F_t(x) = (1-t)f(x)(+t \cdot 0).$$

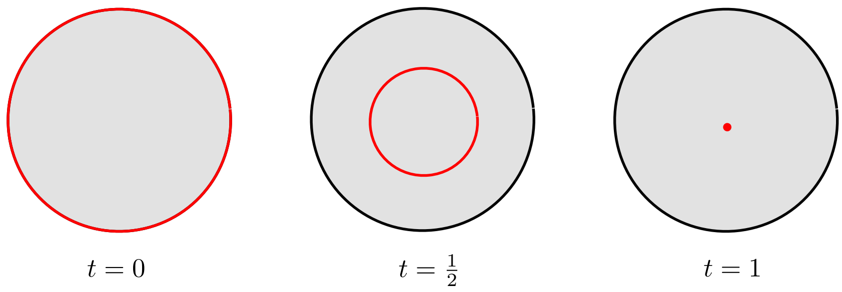


Figure 2: The illustration of $F_t(x)$

We see that there is a homotopy from $f(x)$ to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a nullhomotopy.

3. The maps

$$\begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not** homotopy.

Remark. It will essentially **flip** the orientation, hence we can't deform one to another continuously.

Exercise. We first see some exercises.

1. A subset $S \subseteq \mathbb{R}^n$ is star-shaped if

$$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from x_0 to x lies in S.

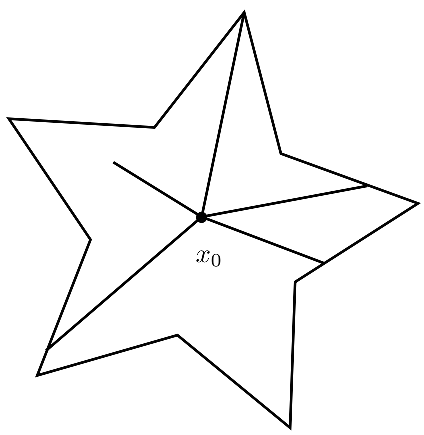


Figure 3: Star-shaped illustration

Show that $\mathrm{id}\colon S \to S$ is nullhomotopic.

Answer. Consider

$$F_t(x) \coloneqq (1-t)x + tx_0,$$

which essentially just concentrates all points x to x_0.

2. Suppose

$$X \underset{f_0}{\overset{f_1}{\longrightarrow}} Y \underset{g_0}{\overset{g_1}{\longrightarrow}} Z$$

where

$$f_0 \underset{F_t}{\simeq} f_1, \quad g_0 \underset{G_t}{\simeq} g_1.$$

Show

$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

Answer. Consider $I \times X \to Z$. Then

$$\begin{array}{ccccc} X \times I & \to & Y \times I & \to & Z \\ (x,t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

Remark. Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** homotopic?

Answer.

Lecture 2: Homotopy Equivalence

07 Jan. 10:00

As previously seen. Two maps $f, g\colon X \to Y$ is homotopy if there exists a map

$$F_t(x)\colon X \times I \to Y$$

with the properties

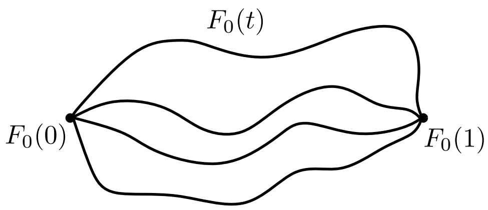
1. Continuous
2. $F_0(x) = f(x)$
3. $F_1(x) = g(x)$

Remark. The continuity of F_t is an even stronger condition for the continuity of F_t for a fixed t.

We now introduce another concept.

Definition 1.2 (Homotopy relative). Given two spaces X, Y, and let $B \subseteq X$. Then a homotopy $F_t(x)\colon X \to Y$ is called *homotopy relative* B (denotes $\mathrm{rel} B$) if $F_t(b)$ is independent of t for all b.

Example. Let $X = [0,1]$ and $B = \{0,1\}$. Then the homotopy of paths from $[0,1] \to X$ is $\mathrm{rel}\{0,1\}$.



1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

Definition 1.3 (Homotopy equivalence, homotopy inverse). A map $f\colon X \to Y$ is a *homotopy equivalence* if $\exists g\colon Y \to X$ such that

$$f \circ g \simeq \mathrm{id}_Y, \quad g \circ f \simeq \mathrm{id}_X.$$

We say that X, Y are *homotopy equivalent*, and g is called *homotopy inverse* of f.

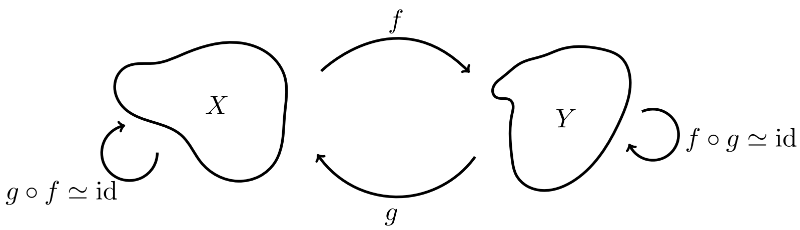
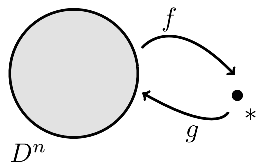


Figure 4: Homotopy Equivalence

If X, Y are homotopy equivalent, then we say that they have the same *homotopy type*.

Notation. We denote a closed n-disk as D^n.

Example. D^n is homotopy equivalent to a point.



We see that $f \circ g = \mathrm{id}_*$ and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is homotopic to id_{D^n} by straight line homotopy $F_t(x) = tx$.

Note. We say that a space is *contractible* if H is homotopy equivalent to a point.

Before doing exercises, we introduce two new concepts.

Definition 1.4 (Retraction, retract). Given $B \subseteq X$, a *retraction* from X to B is a map $f\colon X \to X$ (or $X \to B$) such that $\forall b \in B$ $f(b) = b$, namely $r|_B = \mathrm{id}_B$. Or one can see this from

$$\begin{tikzcd} B \arrow[r, hook, "i"] \arrow[rr, bend right, "r\circ i"'] & X \arrow[r, "r"] & B \end{tikzcd}$$

where r is a retraction if and only if $r \circ i = \mathrm{id}_B$, where i is an inclusion identity. If r exists, B is a *retract* of X.

Definition 1.5 (Deformation retraction). Given X and $B \subseteq X$, a *(strong) deformation retraction* $F_t\colon X \to X$ onto B is a homotopy rel$B$ from the id_X to a *retraction* from X to B. i.e.,

$$\begin{aligned} F_0(x) &= x && \forall x \in X \\ F_1(x) &\in B && \forall x \in X \\ F_t(b) &= b && \forall t\ \forall b \in B. \end{aligned}$$

Exercise. We now see some problems.

1. Let $X \simeq Y$. Show X is path-connected if and only if Y is.

   Answer. Suppose X is path-connected. Then we see that given two points x_1 and x_2 in X, there exists a path $\gamma(t)$ with

   $$\gamma\colon [0,1] \to X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Since $X \simeq Y$, then there exists a pair of f and g such that $f\colon X \to Y$ and $g\colon Y \to X$ with

$$f \circ g \underset{F}{\simeq} \mathrm{id}_Y, \quad g \circ f \underset{G}{\simeq} \mathrm{id}_X.$$

(Notice the abuse of notation)

For any two y_1 and $y_2 \in Y$, we want to construct a path $\gamma'(t)$ such that

$$\gamma'\colon [0,1] \to Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

Firstly, we let $g(y_1) =: x_1$ and $g(y_2) =: x_2$. From the argument above, we know there exists such a γ starting at $x_1 = g(y_1)$ ending at $x_2 = g(y_2)$. Now, consider $f(\gamma(t)) = (f \circ \gamma)(t)$ such that

$$f \circ \gamma\colon I \to Y, \quad f \circ \gamma(0) = y_1', \quad f \circ \gamma(1) = y_2',$$

we immediately see that y_1' and y_2' is path connected. Now, we claim that y_1 and y_1' are path connected in Y, hence so are y_2 and y_2'. To see this, note that

$$f \circ g \underset{F}{\simeq} \mathrm{id}_Y,$$

which means that there exists $F\colon Y \times I \to Y$ such that

$$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y_1' \\ F(y_1, 1) = \mathrm{id}_Y(y_1) = y_1. \end{cases}$$

Since F is continuous in I, we see that there must exist a path connects y_1 and y_1'. The same argument applies to y_2 and y_2'. Now, we see that the path

$$y_1 \to y_1' \to y_2' \to y_2$$

is a path in Y for any two y_1 and y_2, which shows Y is path-connected.

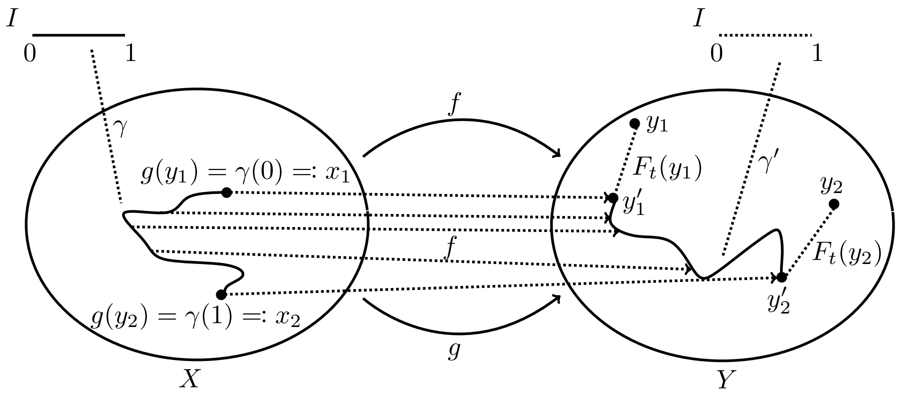


Figure 5: Demonstration of the proof

Challenge: One can further show that the connectedness is also preserved by any homotopy equivalence.

2. Show that if there exists deformation retraction from X to $B \subseteq X$, then $X \simeq B$.

Lecture 3: Deformation Retraction

10 Jan. 10:00

As previously seen. A *deformation retraction* is a homotopy of maps rel$B$ $X \to X$ from id_X to a retraction from X to B. Then B is a *deformation retract*.

Example. We can also show

1. S^1 is a deformation retraction of $D^2 \setminus \{0\}$. Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$

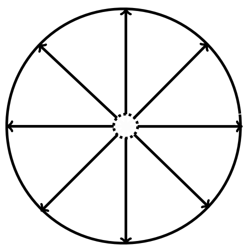


Figure 6: The deformation retraction of $D^{2\setminus\{0\}}$ is just to *enlarge* that hold and push all the interior of D^2 to the boundary, which is S^1

2. $\mathbb{R}^n$ *deformation retracts* to 0. Indeed, since

$$F_t(x). = (1-t)x.$$

This implies that $\mathbb{R}^n \simeq *$, hence we see that

- dimension
- compactness
- etc.

are <u>not</u> homotopy invariants.

3. S^1 is a deformation retract of a cylinder and a Möbius band.

For a cylinder, consider $X \times I \to X$. Define homotopy on a closed rectangle, then verify it induces map on quotient.

For a Möbius band, we define a homotopy on a closed rectangle, then verify that it respect the equivalence relation.

Finally, we use the universal property of quotient topology to argue that we get a homotopy on Möbius band.

Upshot: Möbius band $\simeq S^1 \simeq$ cylinder, hence the orientability is not homotopy invariant.

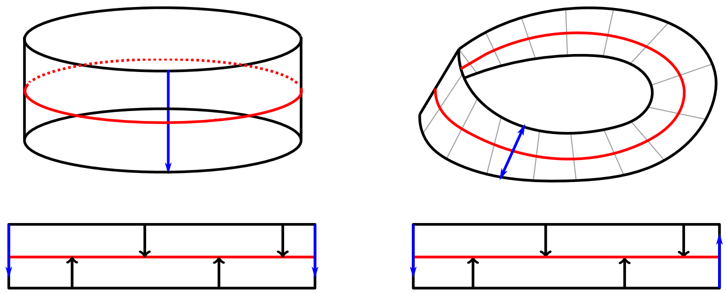


Figure 7: The deformation retraction for Cylinder and Möbius band

Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

As previously seen. We saw that

- homotopy equivalence
- homotopy invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

1.3 CW Complexes

Example. Let's start with a few examples.

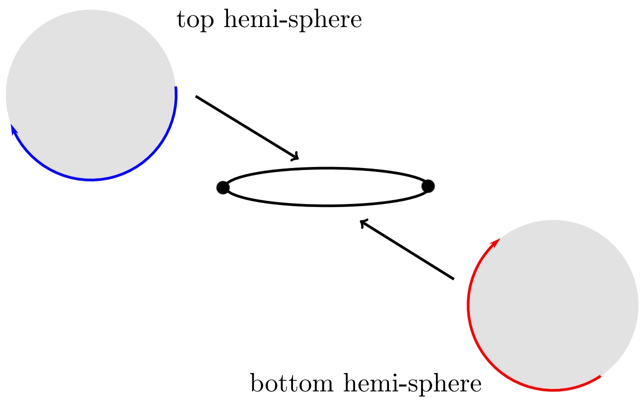
1. Constructing spheres:
   - S^1 (up to homeomorphism[1])

[1] This is just the term for isomorphism in topology.

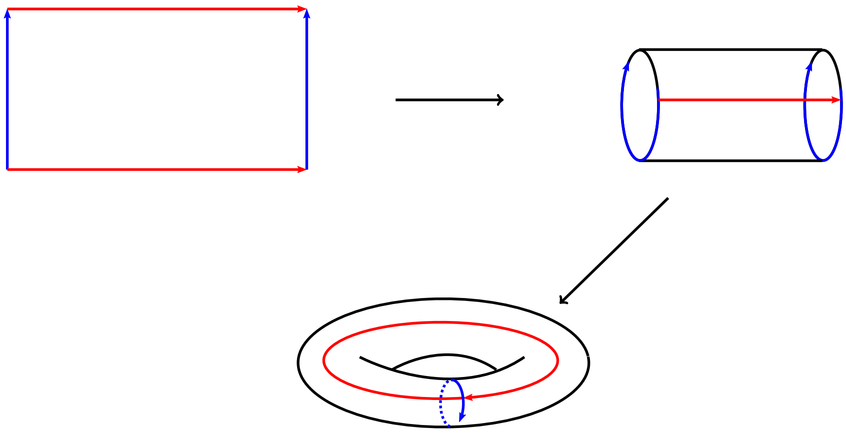
- S^2
  - glue boundary of 2-disk to a point
  - glue 2 disks onto a circle

Figure 8: **Left**: Glue a 2-disk to a point along its boundary. **Right**: Glue 2 disks to S^1.

The gluing instruction to construct S^2 in the right-hand side can be demonstrated as follows.



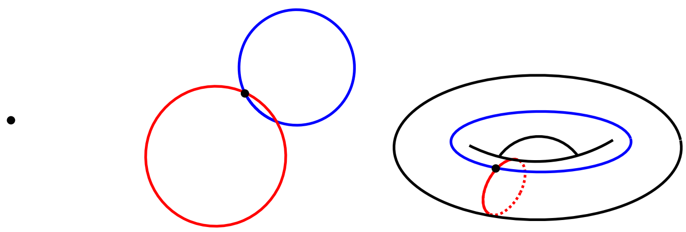
- $T = S^1 \times S^1$



view as gluing instructions

$$\text{vertex } + 2 \text{ edges } + 2\text{-disks.}$$

Specifically, we have



0-skeleton 1-skeleton 2-skeleton

Glue in edges Glue in disk

Formally, we have the following definition.

Notation. Let D^n denotes a closed n-disk (or n-ball)

$$D^n \simeq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

And let S^n denotes an n-sphere

$$S^n \simeq \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\}.$$

Lastly, we call a point as a *0-cell*, and the interior of D^n $\mathrm{int}(D^n)$ for $n \geq 1$ as a *n-cell*.

Definition 1.6 (CW Complex). A *CW Complex* is a topological space constructed inductively as

1. X^0 (the 0-skeleton) is a set of discrete points.
2. We inductively construct the n-skeleton X^n from X^{n-1} by attaching n-cells e_α^n, where α is the index.

   The gluing instructions glued by an attaching map is that $\forall\alpha$, $\exists$ continuous map φ_α
   $$\varphi_\alpha \colon \partial D_\alpha^n \to X^{n-1},$$
   then
   $$X^n = \left(X^{n-1} \coprod_\alpha D_\alpha^n \right) \Big/ _{x \sim \varphi_\alpha(x)}$$
   with identification $x \sim \varphi_\alpha(x)$ for all $x \in \partial D_\alpha^n$ with quotient topology.
3. $$X = \bigcup_{n=0} X^n,$$
   and let $\overline{w}$ denotes weak topology. Then
   $$u \subseteq X \text{ is open } \iff \forall n \; u \cap X^n \text{ is open }.$$
   If all cells have dimension less than N and a $\exists N$-cell, then $X = X^N$ and we call it N-dim CW complex.

Remark. We write $X^{(n)}$ for n-skeleton if we need to distinguish from the Cartesian product.

Example. Let's look at some examples.

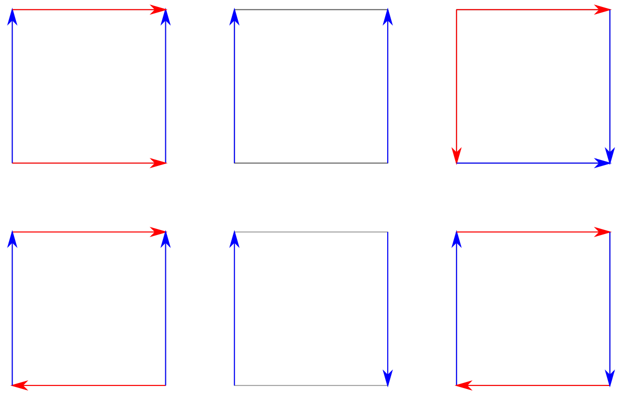
1. 0-dim CW complex is a discrete space.
2. 1-dim CW complex is a graph.
3. A CW complex X is finite if it has finitely many cells.

Definition 1.7 (CW subcomplex). A *CW subcomplex* $A \subseteq X$ is a closed subset equal to a union of cells
$$e_\alpha^n = \operatorname{int}\left(D_\alpha^n\right).$$

Remark. This inherits a CW complex structure.

Check the images of attaching maps.

Exercise. Given the following gluing instruction:



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere, $\mathbb{R}P$.

Answer. We see that

1. Torus	2. Cylinder	3. 2-sphere
4. Klein bottle	5. Möbius band	6. $\mathbb{R}P$

Notation. We call the real projection space as $\mathbb{R}P$, and we also have so-called complex projection space, denote as $\mathbb{C}P$.

Lecture 5: Operation on Spaces

14 Jan. 10:00

1.4 Operations on CW Complexes

1.4.1 Products

We can consider the product of two CW complexes given by a CW complex structure. Namely, given X and Y two CW complexes, we can take two cells e_α^n from X and e_β^m from Y and form the product space $e_\alpha^n \times e_\beta^m$, which is homeomorphic to an $n+m$-cell. We then take these products as the cells for $X \times Y$.

Specifically, given X, Y are CW complexes, then $X \times Y$ has a cell structure

$$\{e_\alpha^m \times e_\alpha^n : e_\alpha^m \text{ is a } m\text{-cell on } X, e_\alpha^n \text{ is an } n\text{-cell on } Y\}.$$

Remark. The product topology may not agree with the weak topology on the $X \times Y$. However, they do agree if X or Y is locally compact or if X and Y both have at most countably many cells.

Note. Notice that if the product is wild enough, then the product topology may not agree with the weak topology.

1.4.2 Wedge Sum

Given X, Y are CW complexes, and $x_0 \in X^0$, $y_0 \in Y^0$ (only points). Then we define

$$X \vee Y = X \coprod Y$$

with quotient topology.

Remark. $X \vee Y$ is a CW complex.

1.4.3 Quotients

Let X be a CW complex, and $A \subseteq X$ subcomplex (closed union of cells), then

$$X \big/ A$$

is a quotient space collapse A to one point and inherits a CW complex structure.

Remark. $X \,/\, A$ is a CW complex.

0-skeleton

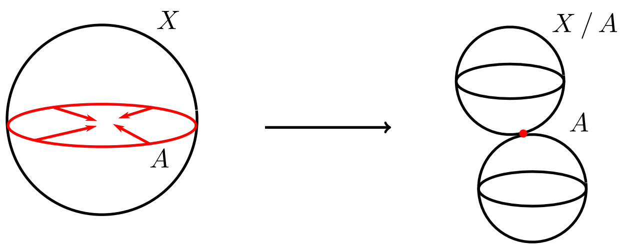
$$(X^0 - A^0) \coprod *$$

where $*$ is a point for A. Each cell of $X - A$ is attached to $(X \,/\, A)^n$ by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n \,/\, A^n$$

Example. Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a wedge of two spheres.



2. We can take the torus and squish down a ring around the hole.

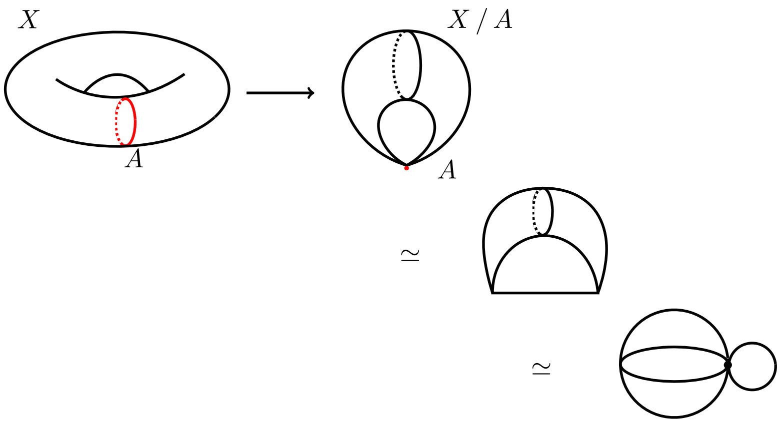


Figure 9: We see that X / A is homotopy equivalent to a 2-sphere wedged with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other points, forming a circle.

Lecture 6: A Foray into Category Theory

19 Jan. 10:00

1.5 Category Theory

We start with a definition.

Definition 1.8 (Category, object, morphism). A *category* $\mathscr{C}$ is 3 pieces of data

- A class of *objects* $\mathrm{Ob}(\mathscr{C})$
- $\forall X, Y \in \mathrm{Ob}(\mathscr{C})$ a class of *morphisms* or arrows, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- $\forall X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, there exists a composition law

$$\begin{aligned}\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) &\to \mathrm{Hom}(X, Z)\\ (f, g) &\mapsto g \circ f\end{aligned}$$

and 2 axioms

- Associativity. $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms f, g, h where composites are defined.
- Identity. $\forall X \in \mathrm{Ob}(\mathscr{C})\ \exists \mathrm{id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g$$

for all f, g where this makes sense.

Let's see some examples.

Example. We introduce some common category.

$\mathcal{C}$	$\text{Ob}(\mathcal{C})$	$\text{Mor}(\mathcal{C})$
set	Sets X	All maps of sets
fset	Finite sets	All maps
Gp	Groups	Group Homomorphisms
Ab	Abelian groups	Group Homomorphisms
$k-$vect	Vector spaces over k	k-linear maps
Rng	Rings	Ring Homomorphisms
Top	Topological spaces	Continuous maps
Haus	Hausdorff Spaces	Continuous maps
hTop	Topological spaces	Homotopy classes of continuous maps
Top*	Based topological spaces[2]	Based maps[3]

Remark. Any **diagram** plus composition law.

$$\begin{tikzcd} A \arrow[loop left, "\text{id}_A"] \arrow[r] & B \arrow[loop right, "\text{id}_B"] \end{tikzcd}.$$

Definition 1.9 (monic, epic). A morphism $f\colon M \to N$ is *monic* if

$$\forall g_1, g_2 \; f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$\begin{tikzcd} A \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

Dually, f is *epic* if

$$\forall g_1, g_2 \; g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & B \end{tikzcd}$$

Lemma 1.1. In set, Ab, Top, Gp, a map is monic if and only if f is injective, and epic if and only if f is surjective.

Proof. In set, we prove that f is monic if and only if f is injective. Suppose $f \circ g_1 = f \circ g_2$ and f is injective, then for any a,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence $g_1 = g_2$.

[2]Topological spaces with a distinguished base point $x_0 \in X$
[3]Continuous maps that presence base point $f\colon (x, x_0) \to (y, y_0)$ such that

$$f\colon X \to Y, \quad f(x_0) = y_0$$

is continuous.

Now we prove another direction, with contrapositive. Namely, we assume that f is not injective and show that f is not monic. Suppose $f(a) = f(b)$ and $a \neq b$, we want to show such g_i exists. This is easy by considering

$$g_1 \colon * \mapsto a, \quad g_2 \colon * \mapsto b.$$

■

1.5.1 Functor

After introducing the category, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

Definition 1.10 (Functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A (covariant) *functor*

$$F \colon \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F \colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)) \\ [f \colon X \to Y] &\mapsto [F(f) \colon F(X) \to F(Y)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(f) \circ F(g)$

Lecture 7: Functors

21 Jan. 10:00

As previously seen. Assume that we initially have a commutative diagram in $\mathscr{C}$ as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

After applying F, we'll have

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)=F(g)\circ F(f)"'] & F(Y) \arrow[d, "F(g)"] \\ & F(Z) \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

We can also have a so-called contravariant functor.

Definition 1.11 (Contravariant functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A *contravariant functor*

$$F : \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F\colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X)) \\ [f\colon X \to Y] &\mapsto [F(f)\colon F(Y) \to F(X)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a contravariant functor F, the diagram becomes

$$\begin{tikzcd} F(X) & F(Y) \arrow[l, "F(f)"'] \\ & F(Z) \arrow[ul, "F(g\circ f)=F(f)\circ F(g)"] \arrow[u, "F(g)"'] \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

Example. Let see some examples.

1. Identity functor.

$$I : \mathscr{C} \to \mathscr{C}.$$

2. Forgetful functor.

   - $$\begin{aligned} F : \underline{\mathrm{Gp}} &\to \underline{\mathrm{set}} \\ G &\mapsto G^{4} \\ [f : G \to H] &\mapsto [f : G \to H] \end{aligned}$$

   - $$\begin{aligned} F\colon \underline{\mathrm{Top}} &\to \underline{\mathrm{set}} \\ X &\mapsto X^{5} \\ [f\colon X \to Y] &\mapsto [f\colon X \to Y] \end{aligned}$$

[4] G is now just the underlying set of the group G.

3. Free functor.

$$\begin{aligned} \underline{\text{set}} &\to \underline{k\text{−vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

i.e., vector space with basis s

$$[f\colon A \to B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

4.

$$\begin{aligned} \underline{k\text{−vect}} &\to \underline{k\text{−vect}} \\ V &\mapsto V^* = \mathrm{Hom}_k(V, k) \end{aligned}$$

If we are working on a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functor.

1.

$$\begin{aligned} \underline{\text{Top}^*} &\to \underline{\text{Gp}} \\ (X, x_0) &\mapsto \Pi_1(X, x_0) \end{aligned}$$

where Π_1 is so-called *fundamental group*.

2.

$$\begin{aligned} \underline{\text{Top}} &\to \underline{\text{Ab}} \\ X &\mapsto \mathrm{Hp}(X) \end{aligned}$$

where Hp is so-called p^{th} *homology*.

Let see the formal definition.

1.6 Free Groups

We start with a definition.

Definition 1.12 (Free group). Given a set S, the *free group* is a group F_S on S with a map $S \to F_S$ satisfying the universal property.

If G is any group, $f\colon S \to G$ is any map of sets, f extends uniquely to group homomorphism $\overline{f}\colon F_S \to G$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\overline{f}\colon \text{gp hom}"] \\ & G \end{tikzcd}$$

[5] X is now just the underlying set of the topological space X.

Note. This defines a *natural bijection*

$$\mathrm{Hom}_{\underline{\mathrm{set}}}(S, \mathscr{U}(G)) \cong \mathrm{Hom}_{\underline{\mathrm{Gp}}}(F_S, G),$$

where $\mathscr{U}(G)$ is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are adjoint; specifically that the free functor is the left adjoint (appears on the left in the Hom above).

> **Definition 1.13 (Adjoints functor).** A free and forgetful functors are *adjoints.*

Remark. Whenever we state a universal property for an object (plus a map), an object (plus a map) may or may not exist. If such object exists, then it defines the object **uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the object (plus a map).

> **Lemma 1.2.** Universal property defines F_S (plus a map $S \to F(S)$) uniquely up to unique isomorphism.

Proof. Fix S. Suppose

$$S \to F_S, \quad S \to \widetilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, dashed, "\exists! \varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists! \psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

We'll show φ and ψ are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{tikzcd} & F_S \arrow[dd, "\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[dd, "\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & \widetilde{F}_S \end{tikzcd}$$

Hence, we see that

$$\begin{tikzcd} & F_S \arrow[d, "\psi"] \arrow[dd, bend left, "\varphi \circ \psi = \mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, "\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[d, "\varphi"] \arrow[dd, bend left, "\psi \circ \varphi = \mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, "\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \mathrm{id}_{F_S}, \qquad \psi \circ \varphi = \mathrm{id}_{\widetilde{F}_S},$$

so φ and ψ are inverses (thus group isomorphism). ■

Lecture 8: The Fundamental Group π_1

24 Jan. 10:00

Example. In category <u>Ab</u> free Abelian group on a set S is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on S** .

1.6.1 Constructing the Free Groups F_S

Proposition 1.1. The free group defined by the universal property exists.

Proof. We'll just give a construction below. First, we see the definition.

Definition 1.14. Fix a set S, and we define a *word* as a finite sequence (possibly $\varnothing$) in the formal symbols

$$\left\{s, s^{-1} \mid s \in S\right\}.$$

Then we see that elements in F_S are equivalence classes of words with the equivalence relation being

- delete ss^{-1} or $s^{-1}s$. i.e.,

  $$\begin{aligned} vs^{-1}sw &\sim vw \\ vss^{-1}w &\sim vw \end{aligned}$$

  for every word $v, w, s \in S$,

with the group operation being concatenation. ■

Example. Given words ab^{-1}, bba, their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

Exercise. There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that F_S satisfies the universal property with respect to the map

$$S \to F_S, \quad s \mapsto s.$$

2 The Fundamental Group

2.1 Definition

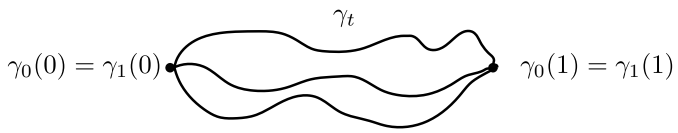
We start with the definition.

Definition 2.1 (Path). A *path* in a space X is a continuous map

$$\gamma \colon I \to X$$

where $I = [0, 1]$.

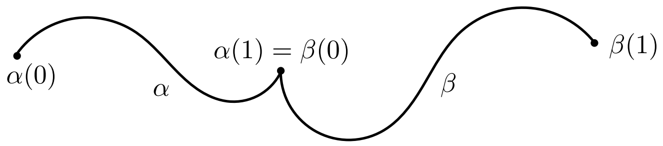
Definition 2.2 (Homotopy path). A *homotopy of paths* γ_0, γ_1 is a homotopy from γ_0 to γ_1 rel$\{0, 1\}$.



Example. Fix $x_1, x_0 \in X$, then $\underline{\exists \text{ homotopy of paths}}$ is an equivalence relation on paths from x_0 to x_1 (i.e., γ with $\gamma(0) = x_0, \gamma(1) = x_1$).

Definition 2.3 (Path composition). For paths α, β in X with $\alpha(1) = \beta(0)$, the *composition*[a] $\alpha \cdot \beta$ is

$$(\alpha \cdot \beta)(t) \coloneqq \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \dfrac{1}{2}\right] \\ \beta(2t - 1), & \text{if } t \in \left[\dfrac{1}{2}, 1\right] \end{cases}.$$



[a] Also named *product*, *concatenation*.

Remark. By the pasting lemma, this is continuous, hence $\alpha \cdot \beta$ is actually a path from $\alpha(0)$ to $\beta(1)$.

Definition 2.4 (Reparameterization). Let $\gamma\colon I \to X$ be a path, then a *reparameterization* of γ is a path

$$\gamma'\colon I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where φ is continuous and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

Exercise. A path γ is homotopic rel$\{0,1\}$ to all of its reparameterizations.

Proof. We show that γ and $\gamma \circ \phi$ are homotopic rel$\{0,1\}$ by showing that there exists a continuous F_t such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since ϕ is continuous, so we define

$$F_t(x) = (1-t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all $x, t \in I$.

Now, we check that F_t really gives us a homotopic rel$\{0,1\}$. We have

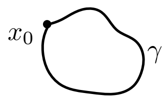
$$F_t(0) = (1-t)\gamma(0) + t \cdot \gamma \circ \phi(0) = (1-t)\gamma(0) + t \cdot \gamma(\underbrace{\phi(0)}_{0}) = \gamma(0),$$
$$F_t(1) = (1-t)\gamma(1) + t \cdot \gamma \circ \phi(1) = (1-t)\gamma(1) + t \cdot \gamma(\underbrace{\phi(1)}_{1}) = \gamma(1),$$

which shows that 0 and 1 are independent of t, hence γ and $\gamma \circ \phi$ are homotopic rel$\{0,1\}$. ■

Exercise. Fix $x_1, x_1 \in X$. Then Homotopy of paths (relative $\{0,1\}$) is an equivalence relation on paths from x_0 to x_1.

Definition 2.5 (Fundamental Group). Let X denotes the space and let $x_0 \in X$ be the base point. The *fundamental group of X based at x_0*, denoted by $\pi_1(X, x_0)$, is a group such that

- Elements: Homotopy classes rel$\{0,1\}$ of paths $[\gamma]$ where γ is a **loop** with $\gamma(0) = \gamma(1) = x_0$[a]

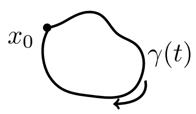


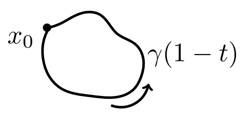
- Operation: Composition of paths.
- Identity: Constant loop γ based at x_0 such that

$$\gamma \colon I \to X, \quad t \mapsto x_0$$

- Inverses: The inverse $[\gamma]^{-1}$ of $[\gamma]$ is represented by the loop $\overline{\gamma}$ such that

$$\overline{\gamma}(t) = \gamma(1-t).$$





[a] We say γ is **based** at x_0.

Proof. We prove that

- Associativity: $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$. We break this down into

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t-2), & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t-3), & \text{if } t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t-1), & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, we define $\phi\colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \dfrac{1}{2}\right], & \text{if } t \in \left[0, \dfrac{1}{4}\right]; \\ t + \dfrac{1}{4} \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right], & \text{if } t \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right]; \\ \dfrac{t+1}{2} \in \left[\dfrac{3}{4}, 1\right], & \text{if } t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$, which implies that the associativity holds.

- Identity: We want to show that $[\gamma \cdot c] = [\gamma]$. Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \dfrac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & \text{if } t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

Now, consider $\phi\colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t, & \text{if } t \in \left[0, \dfrac{1}{2}\right]; \\ 1, & \text{if } t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$.

- Inverses: We want to show that $\gamma \cdot \overline{\gamma} \simeq c$, where $\overline{\gamma}(t) = \gamma(1-t)$. Firstly, we have

$$(\gamma \cdot \overline{\gamma})(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \dfrac{1}{2}\right]; \\ \overline{\gamma}(1-2t), & \text{if } t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

We consider F_t given by

$$F_t(x) = \begin{cases} \gamma(2xt), & \text{if } x \in \left[0, \dfrac{1}{2}\right]; \\ \overline{\gamma}(1-2xt), & \text{if } x \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

If $t = 0$, we have

$$F_0(x) = \begin{cases} \gamma(0), & \text{if } x \in \left[0, \dfrac{1}{2}\right]; \\ \overline{\gamma}(1), & \text{if } x \in \left[\dfrac{1}{2}, 1\right] \end{cases} = x_0$$

for all $x \in I$, namely $F_0 = c$, while when $t = 1$, we have

$$F_1(x) = \begin{cases} \gamma(2x), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2x), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \overline{\gamma})(x),$$

and we see that F_t is continuous since at $x = \frac{1}{2}$, we have

$$\gamma(2x) = \gamma(1) = \overline{\gamma}(0) = \overline{\gamma}(1-2x),$$

hence we see that F_t is the homotopy between $\gamma \cdot \overline{\gamma}$ and c.

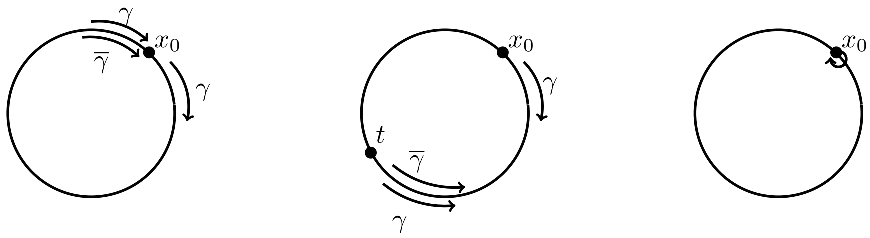


Figure 10: Illustration of F_t. Intuitively, the path $\gamma \cdot \overline{\gamma}$ is $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\overline{\gamma}} x_0$. But now, F_t is $x_0 \xrightarrow{\gamma} t \xrightarrow{\overline{\gamma}} x_0$. We can think of this homotopy is *pulling back* the turning point along the original path.

$\blacksquare$

Theorem 2.1. If X is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Remark. We often write $\pi_1(X)$ up to isomorphism.

Proof. To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if $[h \cdot \gamma \cdot \overline{h}] = [h \cdot \gamma' \cdot \overline{h}]$, then since we know that $h^{-1} = \overline{h}$, hence in the fundamental group $\pi_1(X, x_0)$, we see that

$$\overline{h} \cdot h \cdot \gamma \cdot \overline{h} \cdot h = \overline{h} \cdot h \cdot \gamma' \cdot \overline{h} \cdot h. \implies \gamma = \gamma'$$

  as we desired.

- onto. We see that for every $\alpha \in \pi_1(X, x_0)$, there exists a $\gamma \in \pi_1(X, x_0)$ such that

$$\gamma = \overline{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)^{6}$$

  since $h \cdot \gamma \cdot \overline{h} = \alpha$.

[6] Notice that this is indeed the case, one can verify this by the fact that $h \colon x_0 \to x_1$ and $\overline{h} \colon x_1 \to x_0$.

We then see that the fundamental group of X does not depend on the choice of basepoint, only on the choice of the path component of the basepoint. If X is path-connected, it now makes sense to refer to *the* fundamental group of X and write $\pi_1(X)$ for the abstract group (up to isomorphism). ■

Exercise. Composition of paths is well-defined on homotopy classes $\mathrm{rel}\{0,1\}$.

Exercise. If X is a contractible space, then X is path-connected and $\pi_1(X)$ is trivial.

Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

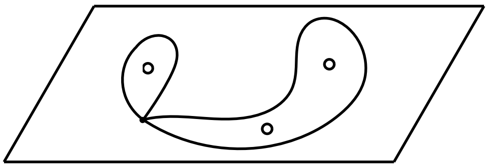


Figure 11: Fundamental Group is basically a *hole detector*!

2.1.1 Calculations with $\pi_1(S^n)$

Let's start with a simple theorem.

Theorem 2.2. $\pi_1(S^1) \cong \mathbb{Z}$, and this identification is given by the paths

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))].$$

Remark. Intuitively, this winds around S^1 n times. The key to this proof was to understand S^1 via the covering space $\mathbb{R} \to S^1$. We will talk about covering spaces more later.

Proof. ■ HW

Theorem 2.3. Given (X, x_0) and (Y, y_0), then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\begin{bmatrix} r\colon I \to X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{bmatrix} \mapsto (r_X, r_Y),$$

where γ is continuous $\iff$ f_X, f_Y are continuous.

Proof. Let $Z \xrightarrow{f} X \times Y$ with $z \overset{f}{\mapsto} (f_X(z), f_Y(z))$. Then we have

$$\text{continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply above to

- Paths $I \to X \times Y$.
- Homotopies of paths $I \times I \to X \times Y$.

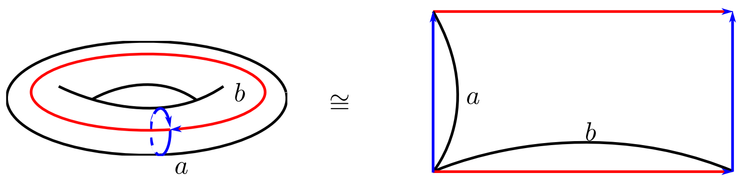
■

Corollary 2.1. The torus $T \cong S^1 \times S^1$ has fundamental group $\pi_1(T) \cong \mathbb{Z}^2$. Additionally, for a k-torus $\underbrace{S^1 \times S^1 \times \ldots \times S^1}_{k \text{ times}} = (S^1)^k$, the fundamental group is then $\mathbb{Z}^k$, i.e.

$$\pi_1\left((S^1)^k\right) \cong \mathbb{Z}^k.$$

Proof. Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

Remark. One way to think of the k-torus is as a k-dimensional cube with opposite $(k-1)$-dimensional faces identified by translation.

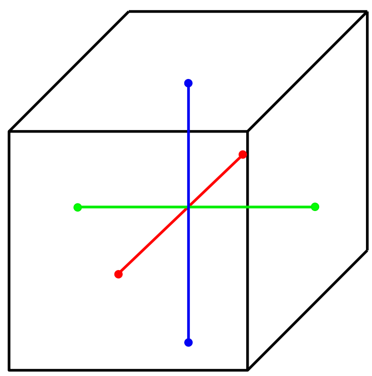


Figure 12: 3-torus with cube identified with parallel sides.

Example. We now see some examples.

1. $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2. $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong 0 \times \mathbb{Z} = \mathbb{Z}$ since

$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R},$$

which means that the generators are just loops around the hold intuitively.

Theorem 2.4. π_1 is a functor such that

$$\begin{aligned} \pi_1 \colon \underline{\mathrm{Top}_*} &\to \underline{\mathrm{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0). \end{aligned}$$

A map $f \colon X \to Y$ taking base point x_0 to y_0 induces a map

$$\begin{aligned} f_* \colon \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

i.e.,

$$[f \colon X \to Y] \mapsto [f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))].$$

Notation. We usually write f_* if it's a covariant functor, while writing f^* if it's a contravariant functor.

Proof. We need to check

- well-defined on path homotopy classes.
- f_* is a group homomorphism.

$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \frac{1}{2}\right] \\ f(\beta(1-2s)), & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

- $\left(\mathrm{id}_{(X, x_0)}\right)_* = \mathrm{id}_{\pi_1(X, x_0)}$
- $(f_* \circ g_*) = (f \circ g)_*$

$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \implies f_*(\gamma_*(\gamma)).$$

DIY

∎

Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

The goal is to compute $\pi_1(X)$ where $X = A \cup B$ using the data

$$\pi_1(A), \pi_1(B), \pi_1(A \cap B).$$

2.2 Seifert-Van Kampen Theorem

2.2.1 Free Product with Amalgamation

We first introduce a definition.

Definition 2.6 (Free product with amalgamation). Given some collections of groups $\{G_\alpha\}_\alpha$, the *free product*, denoted by $\underset{\alpha}{*}G_\alpha$ is a group such that

- Elements: Words in $\{g\colon g \in G_\alpha \text{ for any } \alpha\}$ modulo by the equivalence relation generated by
$$wg_ig_jv \sim w(g_ig_j)v$$
when both $g_i, g_j \in G_\alpha$. Also, for the identity element $\mathrm{id} = e_\alpha \in G_\alpha$ for any α such that
$$we_\alpha v \sim wv.$$
- Operation: Concatenation of words.

Furthermore, if two groups G_α and G_β have a common subgroup $S_{\{\alpha,\beta\}}$[a], given two inclusion maps[b] $i_{\alpha\beta}\colon S_{\{\alpha,\beta\}} \to G_\alpha$ and $i_{\beta\alpha}\colon S_{\{\alpha,\beta\}} \to G_\beta$, the *free product with amalgamation* ${}_\alpha{*}_S G_\alpha$ is defined as $\underset{\alpha}{*}G_\alpha$ modulo the normal subgroup generated by
$$\left\{ i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \mid s_{\{\alpha,\beta\}} \in S_{\{\alpha,\beta\}} \right\},$$
Namely[c],
$${}_\alpha{*}_S\, G_\alpha = \underset{\alpha}{*}G_\alpha \Big/ \left\langle i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \right\rangle$$
and satisfies the universal property
$$\begin{tikzcd}
S \arrow[r, "i_{\alpha\beta}"] \arrow[d, "i_{\beta\alpha}"'] & G_\alpha \arrow[d] \arrow[ddr, bend left] & \\
G_\beta \arrow[r] \arrow[drr, bend right] & G_\alpha *_S G_\beta \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}$$

[a] In general, we don't need $S_{\{\alpha,\beta\}}$ to be a subgroup.
[b] We don't actually need $i_{\alpha\beta}, i_{\beta\alpha}$ to be inclusive as well.
[c] i.e., $i_{\alpha\beta}(s)$ and $i_{\beta\alpha}(s)$ will be identified in the quotient.

Remark. We see that

- We can then write out words such as $g_\alpha \cdot s \cdot g_\beta$ for $s \in S$, and view s as an element of G_α or G_β. In fact, we can do this construction even when i_α and i_β are not injective, though this means we are not working with a subgroup.

- Aside, in Top, the same universal property defines union

$$
\begin{tikzcd}
A \cap B \arrow[r, "i_\alpha"] \arrow[d, "i_\beta"'] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \cup B \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}
$$

for A, B are open subsets and the inclusion of intersection.

Theorem 2.5 (Seifert-Van Kampen Theorem). Given (X, x_0) such that $X = \bigcup_\alpha A_\alpha$ with

- A_α are open and path-connected and $\forall \alpha\ x_0 \in A_\alpha$
- $A_\alpha \cap A_\beta$ is path-connected for all α, β.

Then there exists a surjective group homomorphism

$$\underset{\alpha}{*}\colon \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0).$$

If we additionally have $A_\alpha \cap A_\beta \cap A_\gamma$ where they are all path-connected for every α, β, γ, then

$$\pi_1(X, x_0) \cong_\alpha *_{\pi_1(A_\alpha \cap A_\beta, x_0)} \pi_1(A_\alpha, x_0)$$

associated to all maps $\pi_a(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha), \pi_1(A_\beta)$ induced by inclusions of spaces. i.e., $\pi_1(X, x_0)$ is a quotient of the free product $*_\alpha \pi_1(A_\alpha)$ where we have

$$(i_{\alpha\beta})_*\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A + \alpha)$$

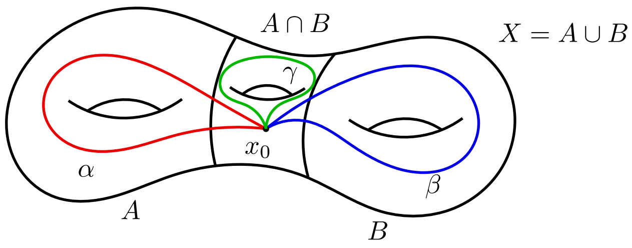
which is induced by the inclusion $i_{\alpha\beta}\colon A_\alpha \cap A_\beta \to A_\alpha$. We then take the quotient by the normal subgroup generated by

$$\left\{ (i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_\alpha \cap A_\beta) \right\}.$$

We'll defer the proof of Theorem 2.5 until we get familiar with this theorem.[7]

Example. We first see a great visualization of the Theorem 2.5.

[7]The proof can be found in Section 2.4.



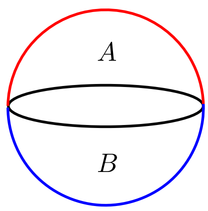
Intuitively we see the fundamental group of X, which is built by gluing A and B along their intersection. As the fundamental group of A and B glued along the fundamental group of their intersection. In essence, $\pi_1(X, x_0)$ is the quotient of $\pi_1(A) * \pi_1(B)$ by relations to impose the condition that loops like γ lying in $A \cap B$ can be viewed as elements of either $\pi_1(A)$ or $\pi_1(B)$.

Lecture 11: Group Presentations

31 Jan. 10:00

Example. We now see some applications of Theorem 2.5.

1. We can use Seifert Van Kampen Theorem to compute the fundamental group of S^2. We see that



   We see that $\pi_1(S^2)$ must be a quotient of $\pi_1(A) * \pi_1(B)$, but since $A, B \simeq D^2$, we know that $\pi_1(A)$ and $\pi_1(B)$ are both zero groups, thus $\pi_1(A) * \pi_1(B)$ is the zero group, and $\pi_1(S^2)$ is also the zero group.

   Remark. Note that the inclusion of $A \cap B \to A$ induces the zero map $\pi_1(A \cap B) \to \pi_1(A)$, which cannot be an injection. In fact, we know that $\pi_1(A \cap B) \cong \mathbb{Z}$ since $A \cap B \simeq S^1$.

2. In the case of torus, consider the following.

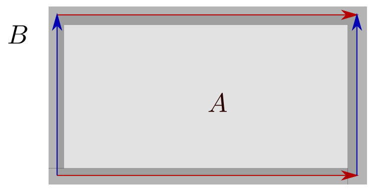


Figure 13: A is the interior, while B is the neighborhood of the boundary.

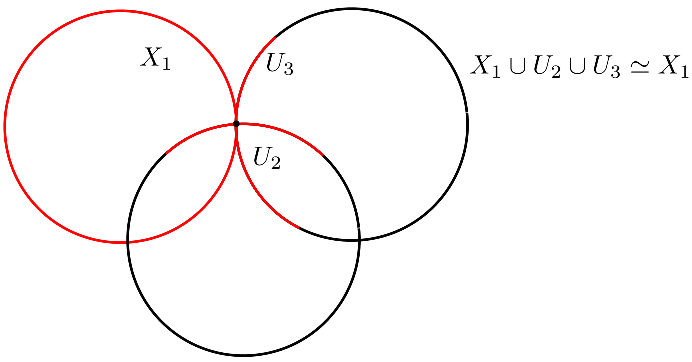
Now note that $A \simeq D^2$ and $B \simeq S^1 \vee S^1$, and since it's a thickening of the two loops around the torus in both ways, this suggests the question of how do we find $\pi_1(B)$? We grab a bit of knowledge from Seifert Van Kampen Theorem before we continue.

Exercise. Suppose we have path-connected spaces (X_α, x_α), and we take their wedge sum $\bigvee_\alpha X_\alpha$ by identifying the points x_α to a single point x. We also suppose a mild condition for all α, the point x_α is a deformation retract of some neighborhood of x_α.

For example, this doesn't work if we choose the *bad point* on the Hawaiian earring. Then we can use Seifert Van Kampen Theorem to show that

$$\pi_1\left(\bigvee_\alpha X_\alpha, x\right) \cong \underset{\alpha}{*}\pi_1\left(X_\alpha, x_\alpha\right).$$

Proof. If we denote



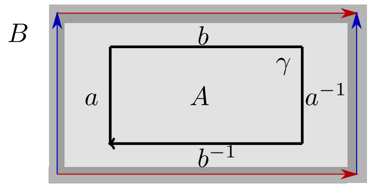
as C_n, then $\pi_1(C_n) \cong F_n$. Then we apply Theorem 2.5 to $A_\alpha = X_\alpha \cup_\beta U_\beta$ Specifically, take $A_\alpha = X_\alpha \cup_\beta U_\beta \simeq X_\alpha$, where U_β is a neighborhood of x_β which deformation retracts to x_β. This makes A_α open as desired. ■

Corollary 2.2. The wedge sum of circles $\pi_1(\bigvee_{\alpha\in A} S^1) = *_\alpha \mathbb{Z}$ is a free group on A. In particular, when A is finite, the fundamental group of a bouquet of circles is the free group on $|A|$.

Returning to the example of torus, we see that

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$

Further, we know that $\pi_1(A \cap B) \to \pi_1(A)$ is the zero map. We need to understand $\pi_1(A \cap B) \to \pi_1(B)$. To do so we need to understand how we're able to identify $\pi_1(S^1 \vee S_1)$ with F_2 and how we identify $\pi_1(S^1)$ with $\mathbb{Z}$. We update our Figure 13 to talk about this.



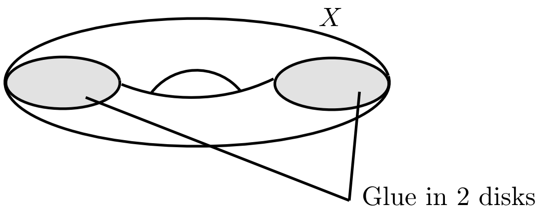
From this, we have

$$\pi_1(A \cap B) \to \pi_1(B) \cong F_{a,b}, \quad \gamma \mapsto aba^{-1}b^{-1}.$$

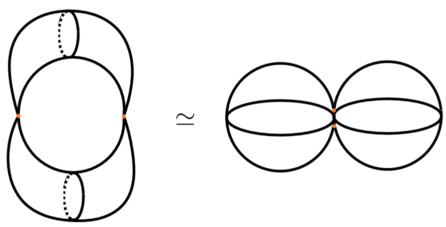
By Seifert Van Kampen Theorem, we identify the image of γ in $\pi_1(B)[aba^{-1}b^{-1}]$ with its image in $\pi_1(A)$, which is just trivial. Therefore, we have

$$\pi_1(T^2) = {}^{F_{a,b}}\Big/_{\langle aba^{-1}b^{-1} \rangle} \cong \mathbb{Z}^2.$$

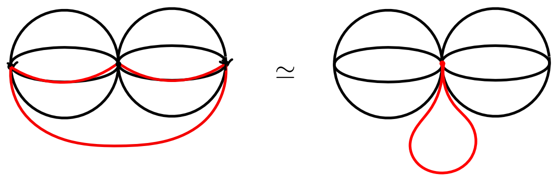
3. Let's see the last example which illustrate the power of Seifert Van Kampen Theorem. Start with a torus, and we glue in two disks into the hollow inside.



We'll call this space X, and out goal is to find $\pi_1(X)$. We can place a CW complex structure on this space so that each disk is a subcomplex. Then, we take quotient of each disk to a point without changing the homotopy type, hence X is homotopy to



By the same property, we can expand one of those points into an interval, and then contract the red path as follows.



This is exactly $S^2 \vee S^2 \vee S^1$. With Seifert Van Kampen Theorem, we have

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}.$$

Exercise. Consider $\mathbb{R}^2 \setminus \{x_1, \dots, x_n\}$, that is the plane punctured at n points. Then $X \simeq \bigvee_n S^1$, so then

$$\pi_1(X) \simeq F_n.$$

One way to do this is to convince yourself that you can do a deformation retract the plane onto the following wedge.

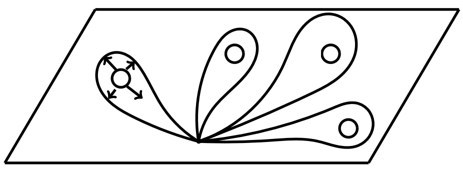


Figure 14: Deformation retract X onto wedge.

2.3 Group Presentation

In order to go further, we introduce the concept of *group presentation.*

Definition 2.7 (Group presentation). A *presentation* $\langle S \mid R \rangle$ of a group G is

- S: set of *generators*
- R: set of *relaters* (words in a generator and inverses)

such that

$$G \cong {}^{F_S}\big/_{\langle R \rangle},$$

where $\langle R \rangle$ is a subgroup normally generated by the elements of R.

Definition 2.8 (Finite presentation). If S and R are both finite, then $G = \langle S \mid R \rangle$ is a *finite presentation* if S, R are, and we say that G is *finitely presented.*

Note. One way to think about whether G is finitely presented is that if r is a word in R then $r = 1$, where 1 is the identity of G.

Example. We see that

1. $F_2 = \langle a, b \mid \ \rangle$
2. $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle$
3. $\mathbb{Z} / 3\mathbb{Z} = \left\langle a \mid a^3 \right\rangle$
4. $S_3 = \left\langle a, b \mid a^2, b^2, (ab)^3 \right\rangle$

Theorem 2.6. Any group G has a presentation.

Proof. We first choose a generating set S for G. Notice that we can even choose $S = G$ directly. From the universal property of free group, we see that there exists a surjective map $\varphi\colon F_S \to G, s \mapsto s$. Now, let R be the generating set for $\ker(\varphi)$, by the first isomorphism theorem[8], $G \cong F_S / \ker\varphi$. In fact, we have $G = \langle S \mid R \rangle$. ∎

Remark. The advantages of using group presentation are that given $G = \langle S \mid R \rangle$, it's now easy to define a homomorphism $\psi\colon G \to H$ given a map $\varphi\colon S \to H$, ψ extends to a group homomorphism $G \to H$ if and only if ψ vanishes on R, i.e., $\phi(r) = 0$ for all $r \in R$. We see an example to illustrate this.

Example. If we have $G = \langle a, b \mid aba \rangle$, a map $\varphi\colon \{a, b\} \to H$ gives a group homomorphism if and only if

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H.$$

This essentially uses the universal property of quotients.

[8] https://en.wikipedia.org/wiki/Isomorphism_theorems

Remark. It's sometimes easy to calculate G^{Ab}

$$G^{\mathrm{Ab}} = \langle S \mid R, \text{commutators in } S\rangle .$$

Example. Suppose all relations in R are commutators, so $R \subseteq [G, G]$. Then,

$$G^{\mathrm{Ab}} = (F_S)^{\mathrm{Ab}} = \bigoplus_S \mathbb{Z}.$$

Remark. The disadvantages are that, this is computationally **very difficult**.

Example. Given $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1}\right\rangle$, let

$$\psi\colon \{a, b\} \to H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1_H \in H.$$

Namely, this is a presentation of the trivial group, but this is entirely unclear.

Lecture 12: Presentations for π_1 of CW Complexes

2 Feb. 10:00

Let's first see an exercise.

Exercise. Consider $G_1 = \langle S_1 \mid R_1\rangle$ and $G_2 = \langle S_2 \mid R_2\rangle$. Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2\rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\}\rangle$
- $G_1 *_H G_2$ where $f_1\colon H \to G_1$ and $f_2\colon H \to G_2$. Then we have

$$G_1 *_H G_2 = \left\langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \left\{f_1(h)f_2(h)^{-1} \mid h \in H\right\}\right\rangle .$$

2.3.1 Presentations for π_1 of CW Complexes

For X a CW complex, we have

1. A 1-dimensional CW complex has free π_1 (call its generators as $a_1, \ldots, a_n$).
2. Gluing a 2-disk by its boundary along a word w in the generators *kills* w in π_1. We then get a presentation for $\pi_1(X^2)$ given by

$$\langle a_1, \ldots, a_n \mid w \text{ for each 2-cell in } X_2\rangle .$$

3. Gluing in any higher dimensional cells along their boundary will not change π_1. That is, in a CW complex, we have $\pi_1(X) = \pi_1(X^2)$.

Remark. We can write the above more precise.

1. Find free generators $\{a_i\}_{i\in I}$ for $\pi_1(X^1)$.
2. For each 2-disk D^2_α, write attaching map as word w_α in a_i. i.e.,
$$\pi_1(X^2) = \langle a_i \mid w_\alpha \rangle .$$
3. $\pi_1(X) = \pi_1(X^2)$.

Example. Given $G = \mathbb{Z} / n\mathbb{Z} = \langle a, a^n \rangle$, then we take a loop and then wind a 2-disk around the loop a for n times.

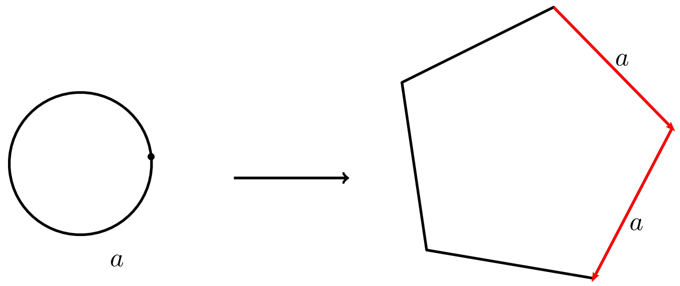


Figure 15: For $G = \mathbb{Z} / n\mathbb{Z} = \langle a \mid a^n \rangle$, we wind the boundary around a for n times.

We then see that given a group G with presentation $\langle S \mid R \rangle$, one can construct a 2-dimensional CW complex with $\pi_1 = G$ by

- Set $X^1 = \bigvee_{s\in S} S^1$
- For each relation $r \in R$, glue in a 2-disk along loops specified by the word r.

Every group is then π_1 of some space.

Theorem 2.7. If X is a CW complex and $\iota_1 \colon X^1 \hookrightarrow X$ and $\iota \colon X^2 \hookrightarrow X$, then $(\iota_1)_*$ surjects onto π_1 and $(\iota_2)_*$ is an isomorphism on π_1.

Proof. ■ HW

Definition 2.9 (Graph, subgraph, tree, maximal tree). We import some topological definitions of graph theoretic concepts.

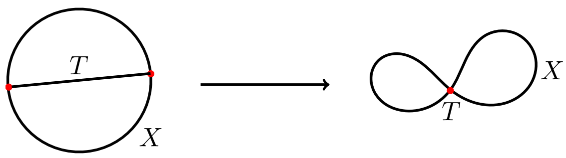
- A *graph* is a 1-dimensional CW complex.
- A *subgraph* is a subcomplex.
- A *tree* is a contractible graph.
- A tree in graph X (necessarily a subgraph) is *maximal* or *spanning* if it contains all the vertices.

Theorem 2.8. Every connected graph has a maximal tree. Every tree is contained in a maximal tree.

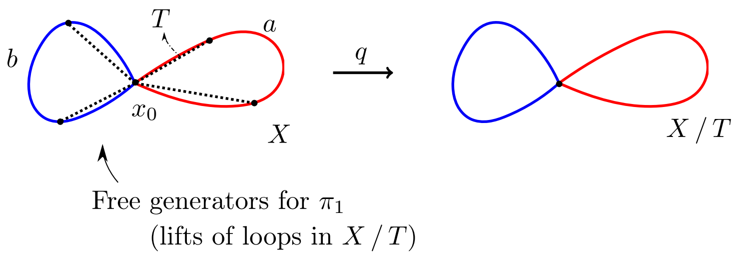
Corollary 2.3. Suppose X is a connected graph with basepoint x_0. Then $\pi_1(X, x_0)$ is a free group.

Furthermore, we can give a presentation for $\pi_1(X, x_0)$ by finding a spanning tree T in X. The generators of π_1 will be indexed by cells $e_\alpha \in X - T$, and e_α will correspond to a loop that passes through T, traverses e_α once, then returns to the basepoint x_0 through T.

Proof. The idea is simple. X is homotopy equivalent to $X\,/\,T$ via previous work on the homework, T contains all the vertices, so the quotient has a single vertex. Thus, it is a wedge of circles, and each e_α projects to a loop in $X\,/\,T$.



The current plan is to calculate the fundamental group of CW complexes. For now, we need to see that the fundamental group of a 1-skeleton (a graph) can be found by taking a maximal tree, and then quotienting out the space by the tree to get a wedge of circles.



We now prove that the maximal trees exist. Recall that X is a quotient of

$$X^0 \coprod_\alpha I_\alpha.$$

Each subset U is open if and only if it intersects each edge $\overline{e}_\alpha$ in an open subset. A map $X \to Y$ if and only if its restriction to each edge $\overline{e}_\alpha$ is continuous. Now, take X_0 to be a subgraph. Our goal is to construct a subgraph Y with

- $X_0 \subset Y \subset X$

- Y deformation retracts to X_0
- Y contains all vertices of X.

So if we take X_0 to be a vertex, then Y is out tree and we're done!

Our strategy now is to build a sequence $X_0 \subset X_1 \subset \ldots$ and correspondingly, $Y_0 \subset Y_1 \subset \ldots$. We start with X_0 and inductively define

$$X_i \coloneqq X_{i-1} \bigcup \text{all edges } \overline{e}_\alpha \text{ with one or both vertices in } X_{i-1} .$$

We then see that $X = \bigcup_i X_i$.[9] Now, let $Y_0 = X_0$. By induction, we'll assume that Y_i is a subgraph of X_i such that

Check.

- Y_i contains all vertices of X_i.
- Y_i deformation retracts to Y_{i-1}.

We can then construct Y_{i+1} by taking Y_i and adding to it one edge to adjoin every vertex of X_{i+1}, namely

$$Y_{i+1} \coloneqq Y_i \bigcup \text{one edge to adjoint every vertex of } X_i{}^{10}$$

We then see that Y_{i+1} deformation retracts to Y_i by just smashing down each edge. Now, we can show that Y deformation retracts to $Y_0 = X_0$ by performing the deformation retraction from Y_i to Y_{i-1} during the time interval $[1/2^i, 1/2^{i-1}]$. ■

Example. Let

- S^n: decompose into 2 open disks
- A_1: neighborhood of top hemisphere
- A_2: neighborhood of lower hemisphere

We see that $A_1 \cap A_2 \simeq S^{n-1}$, where we need $n \geq 2$ to let S^{n-1} be connected. We then have

$$\pi_1(S^n) \cong 0 \underset{\pi_1(A_1 \cap A_2)}{*} 0 = 0.$$

On the other hand, if $n \geq 3$, then we see that

$$S^n = {D^n \cup *}\big/{\sim}.$$

Since 2-skeleton is a point, thus $\pi_1(S^n) = 0$.

Lecture 13: Proof of Seifert-Van-Kampen Theorem

4 Feb. 10:00

2.4 Proof of Seifert-Van-Kampen Theorem

Let's start to prove Theorem 2.5.

[9] [HPM02] do this by arguing the union on the right is both open and closed.
[10] This is possible if we assume Axiom of Choice.

Proof. The outline of the proof is the following. Let $X = \bigcup_\alpha A_\alpha$ where A_α are open, path-connected and contain the bluepoint x_0. We also must guarantee that $A_\alpha \cap A_\beta$ is path-connected.

1. Since we have a map induced by the inclusions:

$$\Phi\colon \mathop{*}_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0).$$

We want to show that ϕ is surjective. Take some $\gamma\colon I \to X$, then by the compactness of the interval I, we can show that there is a partition I with $s_1 < \ldots < s_n$ so that

$$\alpha|_{s_i, s_{i+1}} =: \alpha_i$$

has image in A_{α_i} for some α_i.[11] Specifically, since

- A_α is open for all α
- I is compact,

then for all i, we choose a path h_i from x_0 to $\gamma(s_i)$ in $A_{\sigma_{i-1}} \cap A_{\alpha_i}$, using path-connectedness of the pairwise intersections. Now, take γ and write it as

$$\gamma = (\gamma_1 \cdot \overline{h}_1) \cdot (\overline{h}_1 \cdot \gamma_2) \cdot \ldots \cdot (\gamma_{n-1} \cdot \overline{h}_{n-1}) \cdot (h_{n-1} \cdot \gamma_n).$$

Observe that each of these paths is fully contained in A_{α_i}, so this implies that $\gamma \in \mathrm{Im}(\Phi)$, therefore Φ is surjective.

2. For the next step, we'll show that the second part of Theorem 2.5. Assume that our triple intersections are path-connected. We want to show that $\ker(\Phi)$ is generated by

$$(i_{\alpha\beta})_*(\omega)(i_{\beta\alpha})_*(\omega)^{-1},$$

where

$$i_{\alpha\beta}\colon A_\alpha \cap A_\beta \hookrightarrow A_\alpha$$

for all loops $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$.

Before we go further, we'll need some definition.

> **Definition 2.10 (Factorization).** A *factorization* of a homotopy class $[f] \in \pi_1(X, x_0)$ is a formal product
>
> $$[f_1][f_2]\ldots[f_\ell]$$
>
> with $[f_i] \in \pi_1(A_\alpha, x_0)$ such that
>
> $$f \simeq f_1 \cdot f_2 \cdot \ldots \cdot f_\ell.$$

We showed that every $[f]$ has a factorization in step 1 already. Now we want to show taht two factorizations

$$[f_1] \cdot \ldots \cdot [f_\ell] \text{ and } [f_1'] \cdot \ldots \cdot [f_{\ell'}']$$

of $[f]$ must be related by two moves:

[11] This is a good exercise for point-set topology.

(a) $[f_i]\cdot[f_{i+1}] = [f_i \cdot f_{i+1}]$ if $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$. Namely, the reaction defining the free product of groups.

(b) $[f_i]$ can be viewed as an element of $\pi_1(A_\alpha, x_0)$ or $\pi_1(A_\beta, x_0)$ whenever

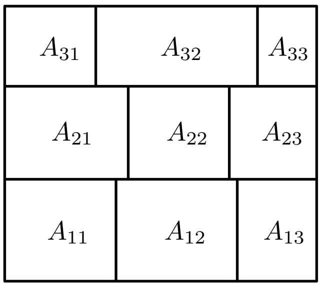
$$[f_i] \in \pi_1(A_\alpha \cap A_\beta, x_0).$$

This is the relation defining the amalgamated free product.

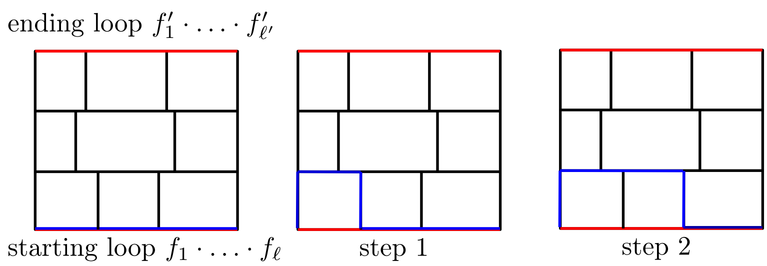
Now, let $F_t \colon I \times I \to X$ be a homotopy from $f_1 \dots f_\ell$ to $f'_1 \dots f'_{\ell'}$, since they both represent $[f]$. We subdivide $I \times I$ into rectangles R_{ij} so that

$$F(R_{ij}) \subseteq A_{\alpha_{ij}} =: A_{ij}$$

for some α_{ij} using compactness. We also argue that we can perturb the corners of the squares so that a corner lies only in three of the A_α's indexed by adjacent rectangles.

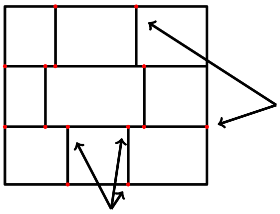


We also argue that we can set up our subdivision so that the partition of the top and bottom intervals must correspond with the two factorizations of $[f]$. We then perform our homotopy one rectangle at a time.



Idea: Argue that homotoping over a single rectangle has the effect of using allowable moves to modify the factorization.

At each triple intersection, choose a path from f (corner) to x_0 which lies in the triple intersection, so we use the assumption that the triple intersections are path-connected.



Along the top and bottom, we make choices compatible with the two factorizations. It's now an exercise to check that these choices result in homotoping across a rectangle gives a new factorization related by an allowable move.

■

Lecture 14: Covering Spaces

7 Feb. 10:00

3 Covering Spaces

3.1 Lifting Properties

As always, we start with a definition.

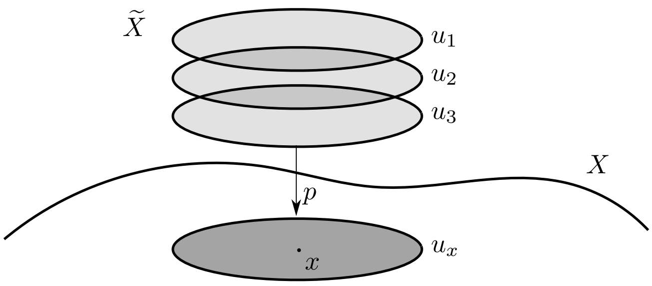
Definition 3.1 (Covering space). A *covering space* $\widetilde{X}$ of X is a space $\widetilde{X}$ and a map $p\colon \widetilde{X} \to X$ such that $\forall x \in X$ $\exists$ neighborhood u_x with $p^{-1}(u_x)$ the disjoint union of open sets

$$\coprod_{\alpha} u_{\alpha}$$

such that

$$p|_{u_{\alpha}} : u_x \to u_x$$

is a homeomorphism for every α.



We sometimes call p as *covering map*.

Although we already investigate into covering spaces quite a lot in homework, but a terminology is still worth mentioning.

Definition 3.2 (Evenly covered). Let $p\colon \widetilde{X} \to X$ be a continuous map of spaces. Then an open subset $U \subseteq X$ is called *evenly covered by p* if

$$p|_{V_i} : V_i \to U$$

is a homeomorphism.

We call the parts V_i of the partition $\coprod_i V_i$ of $p^{-1}(U)$ *slices*.

Remark. We see that p is a covering map if and only if every point $x \in X$ has a neighborhood which is evenly covered.

We immediately have the following proposition.

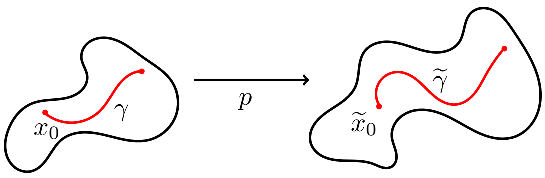
Proposition 3.1 (Homotopy lifting property). The covering spaces satisfy the *homotopy lifting property* such that the following diagram commutes.

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widetilde{F}_0"] \arrow[d, hook] & \widetilde{Y} \arrow[d, "p"] \\
X \times I \arrow[r, "F_t"'] \arrow[ur, dashed, "\exists! \widetilde{F}_t"] & Y
\end{tikzcd}$$

Proof. We already proved this in homework! ■

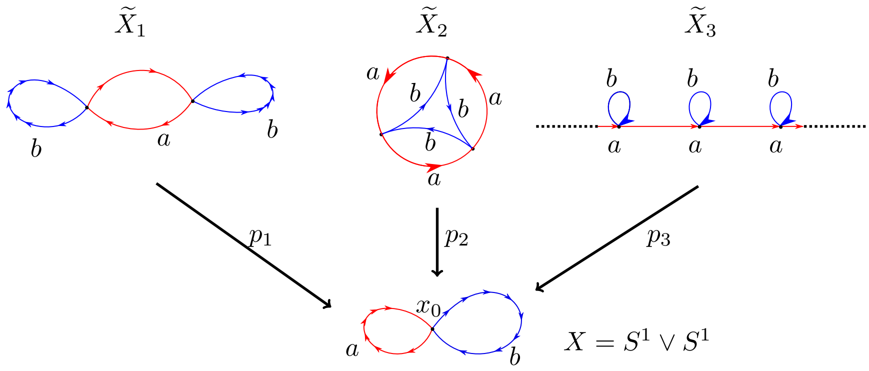
Corollary 3.1. For each path $\gamma\colon I \to X$ in X, $\widetilde{x}_0 \in p^{-1}(\gamma(0))$ such that there exists a unique lift $\widetilde{\gamma}$ starting at $\widetilde{x}_0$.

And for each path homotopy $I \times I \to X$, there exists a unique path homotopy $\widetilde{\gamma}\colon I \times I \to \widetilde{X}$ starting at $\widetilde{x}_0$.



Example. Let see some examples.

1. Covers of $S^1 \vee S^1$.



Note that in each cover (those three on the top), the black dot is the preimage of $\{x_0\}$, namely $p_i^{-1}(\{x_0\})$.

Remark. We see that for each $p_i^{-1}(\{x_0\})$, there are exactly

- one a edge goes out
- one b edge goes out
- one a edge goes in
- one b edge goes in

It turns out that there are much more covers of $S^1 \vee S^1$, as long as this main property is satisfied.

Proposition 3.2. Let

$$p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$$

be a covering map. Then

1. $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.
2. $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0) = \{[\gamma] \mid \text{Lift } \widetilde{\gamma} \text{ starting at } \widetilde{x}_0 \text{ is a loop.}\}$.

Proof. We prove this one by one.

1. Suppose $\widetilde{\gamma} \in \pi_1(\widetilde{X}, \widetilde{x}_0)$ is in $\ker(p_*)$. Then

$$[\gamma] = p_*([\widetilde{\gamma}]) = [p \circ \widetilde{\gamma}]\,.$$

Let γ_t be a nullhomotopy from γ to the constant loop c_{x_0} rel$\{0,1\}$. We can then lift γ_t to $\widetilde{\gamma}_t$ where $\widetilde{\gamma}_0 = \widetilde{\gamma}$. Now, we claim that

- $\widetilde{\gamma}$ is a homotopy rel$\{0,1\}$.
- $\widetilde{\gamma}_1$ is the constant loop $c_{\widetilde{x}_0}$.

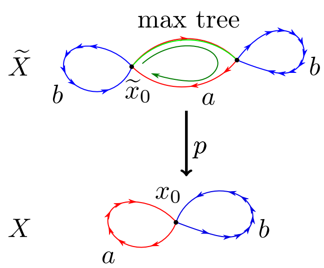
$$
\begin{tikzcd}
 & \widetilde{X} \arrow[d, "p"] \\
I \arrow[ur, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widetilde{X} \arrow[d, "p"] \\
I \times I \arrow[ur, "\widetilde{\gamma}_t"] \arrow[r, "\gamma_t"'] & X
\end{tikzcd}
$$

We see that the above diagrams prove the first claim, since we know that the left and right edge of $I \times I$ maps to x_0 under γ_t, and $c_{\widetilde{x}_0}$ lifts this, so by uniqueness $t \mapsto \widetilde{\gamma}_t(0)$ and $t \mapsto \widetilde{\gamma}_t(1)$ must be constant paths at $\widetilde{x}_0$ as desired.

Then the lift $\widetilde{\gamma}_t$ is a homotopy of paths to the constant loop, so $[\widetilde{\gamma}] = 1$.

2. Let see an example to show the idea of the proof.

Example. Given



Then

$$p_*\pi_1 = \left\langle b, a^2, ab\overline{a} \right\rangle \subseteq \pi_1(X) = \langle a, b \mid \rangle .$$

■

Proposition 3.3 (Lifting criterion). Let

$$p\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$$

be covering map. Given

- $f\colon (X, x_0) \to (Y, y_0)$;
- X is path-connected, locally path-connected,

then a lift

$$\widetilde{f}\colon (X, x_0) \to (\widetilde{Y}, y_0)$$

exists if and only if

$$f_* \left(\pi_1(X, x_0)\right) \subseteq p_* \left(\pi_1(\widetilde{Y}, \widetilde{y}_\alpha)\right).$$

$$\begin{tikzcd} & (\widetilde{Y}, \widetilde{y}_0) \arrow[d, "p"] \\ (X, x_0) \arrow[ur, dashed, "\exists \widetilde{f}"] \arrow[r, "f"'] & (Y, y_0) \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(\widetilde{Y}, \widetilde{y}_0) \arrow[d, "p_*"] \\ \pi_1(X, x_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(Y, y_0) \end{tikzcd}$$

Lecture 15: Lifting

9 Feb. 10:00

Before proving Proposition 3.3, we first see an application.

Example. Prove that every continuous map $f\colon \mathbb{R}P^2 \to S^1$ is nullhomotopic.

Proof. If we can show that there is a lift $\widetilde{f}\colon \mathbb{R}P^2 \to \mathbb{R}$ of f, then we're done since we can apply the straight line nullhomotopy on $\mathbb{R}$ since

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

and consider $f = p \circ \widetilde{f}$ compose nullhomotopy with p, so $f \simeq$ constant map. Specifically, since $\pi_1(\mathbb{R}P^2) = \mathbb{Z} / 2\mathbb{Z}$ and $\pi_1(S^1) = \mathbb{Z}$, hence

$$f_*(\pi_1(\mathbb{R}P^2)) = 0$$

since $\mathbb{Z}$ has no (nonzero) torsion. So it lifts by Proposition 3.3. ∎

Now we can proof Proposition 3.3.

Proof. We prove two directions as follows.

Necessary. We see that we can factorize f_* as

$$f_* = p_* \circ \widetilde{f}_*$$

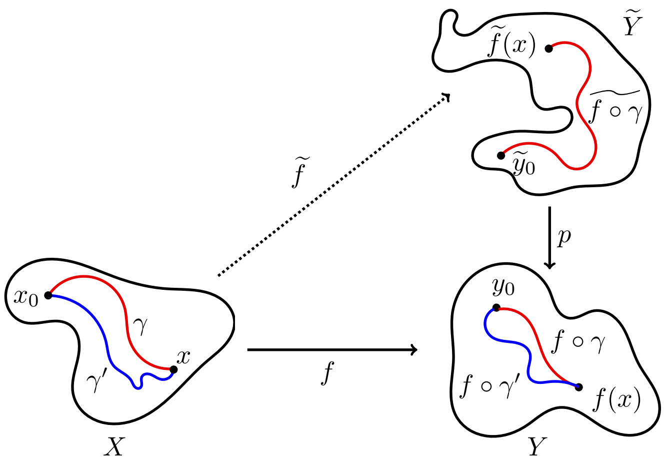
follows from the functoriality of π_1.

Sufficient. Let $x \in X$. Choose a path γ from x_0 to x by the assumption that X is path-connected. Then, $f\gamma$ has a unique lift starting at $\widetilde{y}_0$, denote by $\widetilde{f\gamma}$. Now, define

$$\widetilde{f}(x) = \widetilde{f\gamma}(1).$$

Then, we need to check

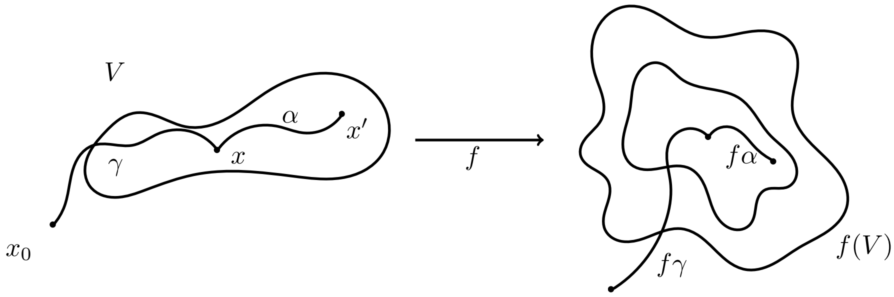
1. $\widetilde{f}$ is well-defined. Suppose γ, γ' are paths in X from x_0 to x. We want to show
$$\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1).$$
Since $\gamma \cdot \overline{\gamma'}$ is a loop in X at x_0, we know that $[(f\gamma) \cdot (\overline{f\gamma'})]$ is a class of loops in Y in $\mathrm{Im}(f_*)$. By hypothesis, this class of loops is in $\mathrm{Im}(p_*)$. It lifts to a loop which is based at $\widetilde{y}_0$. By uniqueness of lifts, this loop lifting $(f\gamma) \cdot \overline{(f\gamma')}$ to $\widetilde{Y}$ must be equal to the lifts $\widetilde{f\gamma} \cdot \widetilde{\overline{f\gamma'}}$ with a common value at $t = 1/2$. Hence, $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$ as desired, namely the endpoints agree.



Lecture 16: Proving Proposition 3.3

11 Feb. 10:00

2. $\widetilde{f}$ is continuous. Choose $x \in X$ and a neighborhood $\widetilde{U}$ of $\widetilde{f}(x)$ in $\widetilde{Y}$. Note that we can choose $\widetilde{U}$ small enough to $p|_{\widetilde{U}}$ is homeomorphism to U in Y. Now, there exists a neighborhood V of x in X with $f(V) \subseteq U$.



The goal is $\widetilde{f}(V) \subseteq \widetilde{U}$. Without loss of generality, we can assume that V is path-connected. Then,

$$\widetilde{f\gamma} \cdot \widetilde{f\alpha} = [\widetilde{f\gamma \cdot f\alpha}].$$

Hence,

$$\widetilde{f\alpha} = (p|_{\widetilde{U}})^{-1} \circ f \circ \alpha,$$

where $(p|_{\widetilde{U}})^{-1}$'s image is in $\widetilde{U}$, so

$$\widetilde{f}(x') = f\gamma \cdot f\alpha(1) \in \widetilde{U},$$

which implies

$$\widetilde{f}(V) \subseteq \widetilde{U}.$$

∎

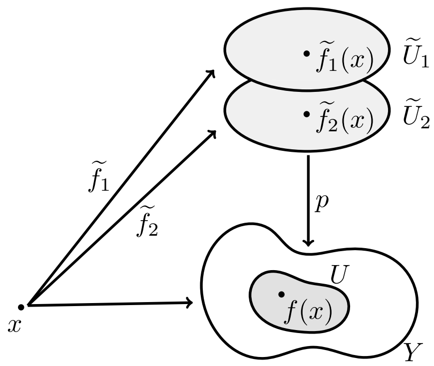
Proposition 3.4. Let $p\colon \widetilde{Y} \to Y$ be a covering map with X is a connected space. If two lifts $\widetilde{f}_1, \widetilde{f}_2$ of the same map f agree at a single point, then they agree everywhere.

$$\begin{tikzcd} & \widetilde{Y} \arrow[d, "p"] \\ X \arrow[ur, bend left, "\widetilde{f}_1"] \arrow[ur, bend right, "\widetilde{f}_2"'] \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. Let S being

$$S \coloneqq \left\{ x \in X \mid \widetilde{f}_1(x) = \widetilde{f}_2(x) \right\}.$$

We want to show that S is both closed and open, so if S is nonempty, $S = X$.



We see that $\widetilde{U}_1$ and $\widetilde{U}_2$ are slices of $p^{-1}(U)$, where U is evenly covered neighborhood of $f(x)$.

1. If $\widetilde{f}_1(x) \neq \widetilde{f}_2(x)$. Then $\widetilde{U}_1, \widetilde{U}_2$ are disjoint. Since $\widetilde{f}_1, \widetilde{f}_2$ are continuous, there exists a neighborhood N of x with
$$\widetilde{f}_1(N) \subseteq \widetilde{U}_1, \quad \widetilde{f}_2(N) \subseteq \widetilde{U}_2,$$
with the fact that they're disjoint, so x is an interior point of S^c.
2. If $\widetilde{f}_1(x) = \widetilde{f}_2(x)$. Then $\widetilde{U}_1 = \widetilde{U}_2$. Choose N as before, then we have
$$\widetilde{f}_1(n) = (p|_{\widetilde{u}_1})^{-1}(f(n)) = \widetilde{f}_2(n),$$
hence $x \in \operatorname{int}(S)$.

■

3.2 Deck Transformation

We now want to introduce a special kind of transformation.

Definition 3.3 (Isomorphism of Covers). Given covering maps
$$p_1 \colon \widetilde{X}_1 \to X, \qquad p_2 \colon \widetilde{X}_2 \to X,$$
an *isomorphism of covers* is a homeomorphism
$$f \colon \widetilde{X}_1 \to \widetilde{X}_2$$
such that $p_1 = p_2 \circ f$.
$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Exercise. This defines equivalent relation on covers of X.

Definition 3.4 (Deck transformation). Given a covering map $p\colon \widetilde{X} \to X$, the isomorphisms of covers $\widetilde{X} \to \widetilde{X}$ are called *Deck transformation.*

Furthermore, we'll let $G(\widetilde{X})$ denotes the *set of deck transformations.*

Note. Note that we've suppressed the data of p in the notation, but this data is essential to what a deck transformation is, when this is unclear we write $G(\widetilde{X}, p)$.

Lecture 17: Deck Transformation 14 Feb. 10:00

Example. Let's see some examples.

1. Deck transformations $G(\widetilde{X})$ are a subgroup of the group of homeomorphisms of $\widetilde{X}$.
2. Given the cover $p\colon \mathbb{R} \to S^1$.
   - Deck maps: translation by $n \in \mathbb{Z}$ units.
   - $G(\mathbb{R}) \cong \mathbb{Z}$
3. Given the cover $p_n\colon S^1 \to S^1$ be an n-sheeted cover.
   - Deck maps: rotation by $2\pi/n$.
   - $G(S^1, p_n) \cong \mathbb{Z} / N\mathbb{Z}$

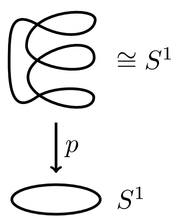


Figure 16: $p\colon S^1 \to S^1$ be an N-sheeted cover, where $N = 3$.

Exercise (Deck Transformation is determined by the image of one point). Given $X, \widetilde{X}$ are path-connected, locally path-connected, deck map is determined by the image of any one point.

Answer.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ \widetilde{X} \arrow[ur, "f"] \arrow[r, "p"'] & X \end{tikzcd}$$

Corollary 3.2. If a deck transformation has a fixed point, it is the identity transformation.

Exercise. Let X be connected. Given a deck transformation $\tau\colon \widetilde{X} \to \widetilde{X}$, τ defines a permutation of $p^{-1}(\{x_0\})$. If this permutation has a fixed point, then it is the identity.

Definition 3.5 (Regular, Normal). A covering space $p\colon \widetilde{X} \to X$ is *regular* or *normal* if $\forall x_0 \in X$, $\forall \widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(\{x_0\})$, there exists a deck transformation such that

$$\widetilde{x}_0 \mapsto \widetilde{x}_1.$$

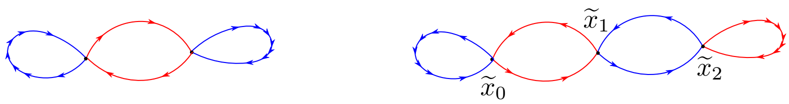


Figure 17: Covers of $S^1 \vee S^1$. The left one is regular, while the right one is not since there is no automorphism from $\widetilde{x}_0$ to $\widetilde{x}_1$ or $\widetilde{x}_2$.

Remark. A regular cover is *as symmetric as possible.*

Exercise. Regular means that the group $G(\widetilde{X})$ acts transitively on $p^{-1}(\{x_0\})$. Explain why we cannot ask for more than this:

$G(\widetilde{X})$ cannot induce the full symmetric group on $p^{-1}(\{x_0\})$ provided that $\left|p^{-1}(\{x_0\})\right| > 2$.

Answer. The key is uniqueness.

Definition 3.6 (Normalizer). Given G as a group, $H \subseteq G$ is a subgroup of G. Then the *normalizer* of H, denoted by $N(H)$, is defined as

$$N(H) \coloneqq \{g \in G \mid gH = Hg\}.$$

Exercise. We can prove the followings.

1. $N(H)$ is a subgroup.
2. $H \leq N(H)$.
3. H is normal in $N(H)$.
4. If $H \leq G$ is normal, $N(H) = G$.
5. $N(H)$ is the largest subgroup (under containment) of G containing H as normal subgroup.

Proposition 3.5. Given $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a cover, and $\widetilde{X}, X$ are path-connected, locally path-connected. Let

$$H = p_* \left(\pi_1(\widetilde{X}, \widetilde{x}_0) \right) \subseteq \pi_1(X, x_0).$$

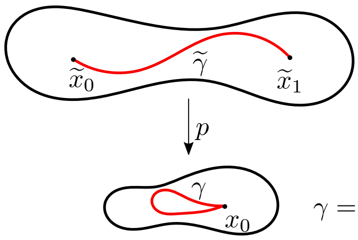
Then

1. p is normal if and only if $H \subset \pi_1(X, x_0)$ is normal.
2. We have
$$G(\widetilde{X}) \cong {}^{N(H)}\big/_{H},$$
where $G(\widetilde{X})$ are Deck maps, and $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

Remark. A fact is worth noting is the following. Let $\widetilde{\gamma}$ be a path $\widetilde{x}_1$ to $\widetilde{x}_0$. Then

$$p_* \left(\pi_1(\widetilde{X}, \widetilde{x}_0) \right) = [\gamma] H [\gamma^{-1}]$$

where $H \in \pi_1(\widetilde{X}, \widetilde{x}_1)$.



Lecture 18: Proving Proposition 3.5

16 Feb. 10:00

Now let's prove Proposition 3.5

Proof. Let X, x_0 be the base space and $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(\{x_0\})$ where $p\colon \widetilde{X} \to X$ is a covering map. Further, let $H \coloneqq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

In homework, given $(X, x_0), \widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(\{x_0\})$ if we change the basepoint from $\pi_1(\widetilde{X}, \widetilde{x}_0)$ to $\pi_1(\widetilde{X}, \widetilde{x}_1)$, then we have the induced subgroups of the base spaces fundamental group are conjugate by some loop $[\gamma] \in \pi_1(X, x_0)$, i.e.,

$$p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = [\gamma] \cdot p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \cdot [\gamma]^{-1}$$

where γ is lifted to a path from $\widetilde{x}_0$ to $\widetilde{x}_1$.

Therefore, $[\gamma] \in N(H)$ if and only if $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, and this holds if and only if there is a deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$ by the classification of based covering spaces in the homework.[12] This shows that p is a normal cover if and only if H is normal, which proves the first claim.

[12] Alternatively, we can use the lifting criterion.

We then define a map Φ such that

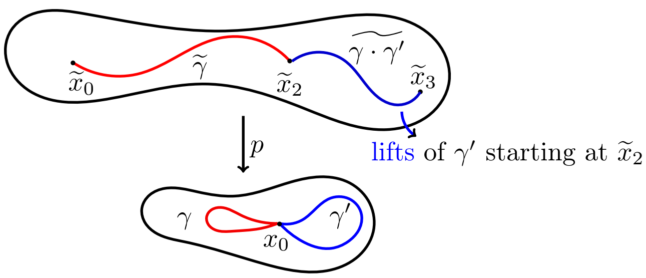
$$\Phi\colon N(H) \to G(\widetilde{X})[\gamma], \quad \cdot \mapsto \tau$$

where τ lifts to a path from $\widetilde{x}_0$ to $\widetilde{x}_1$ and τ is a deck transformation mapping $\widetilde{x}_0$ to $\widetilde{x}_1$, which will be uniquely defined by the uniqueness of lifts with specified base points. We now need to check

1. Φ is surjective.
2. $\ker(\Phi) = H$.
3. Φ is a group homomorphism.

If we can prove all the above, then from the result follows directly from the first isomorphism theorem.[13]

1. We've proved that Φ is surjective before in our work above.
2. $\Phi([\gamma])$ is the identity if and only if τ sends $\widetilde{x}_0$ to $\widetilde{x}_0$, meaning that $[\gamma]$ lifts to a loop. Then by our characterization of the fundamental group downstairs:
$$\ker(\Phi) = \{[\gamma] \mid [\gamma] \text{ lifts to a loop}\} = H.$$
3. Suppose we have loops $[\gamma_1] \overset{\Phi}{\mapsto} \tau_1$ and $[\gamma_2] \overset{\Phi}{\mapsto} \tau_2$. We claim that $\gamma_1 \cdot \gamma_2$ lifts to $\widetilde{\gamma}_1 \cdot \tau(\widetilde{\gamma}_2)$.



It's an exercise to check that the lift of γ_2 starting at $\widetilde{x}_1$ is exactly $\phi_1(\widetilde{\gamma}_2)$, where $\widetilde{\gamma}_2$ is a lift starting at $\widetilde{x}_0$.

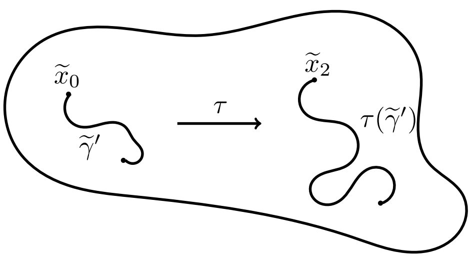


Figure 18: Must be lift of γ' starting at $\widetilde{x}_2$

[13] https://en.wikipedia.org/wiki/Isomorphism_theorems

The idea is that by uniqueness of lifts we'll have the desired claim. We then just observe that this path $\widetilde{\gamma}_1 \cdot \tau_1(\widetilde{\gamma}_2)$ is a path from $\widetilde{x}_0$ to $\gamma_1(\widetilde{\gamma}_2(1)) = \tau_1(\tau_2(\widetilde{x}_0))$, so the image must be a deck transformation sending $\widetilde{x}_0$ to $\tau_1(\tau_2(\widetilde{x}_0))$. But then $\tau_1 \circ \tau_2$ maps $\widetilde{x}_0$ to this same point, and from this exercise, we know that the deck transformations are determined by where they send a single point, hence we're done.

■

Corollary 3.3. If p is a normal covering, then $G(\widetilde{X}) \cong \pi_1(X, x_0) \,/\, H$.

Corollary 3.4. If $\widetilde{X}$ is the universal cover, then $G(\widetilde{X}) \cong \pi_1(X, x_0)$.

Exercise. Whether $\mathrm{Im}(p_*)$ is normal is independent of the basepoint in $\widetilde{X}$ and X.

So, p is normal if and only if $G(\widetilde{X})$ is transitive on $p^{-1}(x_0)$ for at least one $x_0 \in X$.

Exercise. Let Σg be the genus g surface. Prove that Σg has a normal n-sheeted path-connected cover for every n.

Lecture 19: Simplex

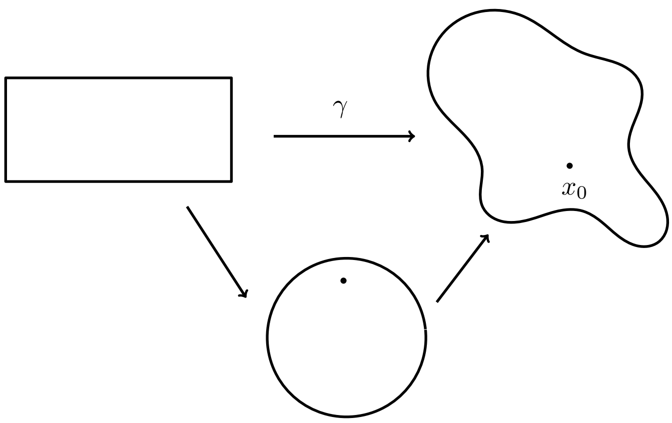
18 Feb. 10:00

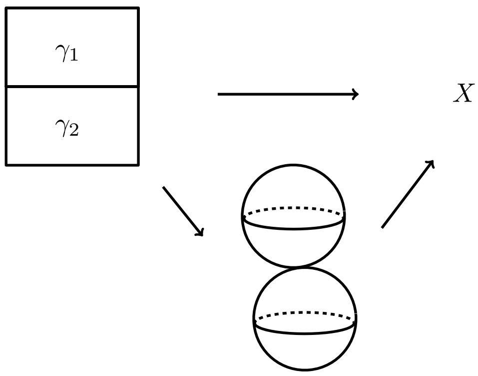
4 Homology

4.1 Motivation for Homology

Informally, the higher homotopy groups is defined as

$$\pi_n(X, x_0)\colon I^n_* \to (X, x_0), \quad \partial I^n \mapsto x_0$$





We see that it's extremely hard to compute higher fundamental group. Hence instead, we will study higher dimension structure of X via *homology*.

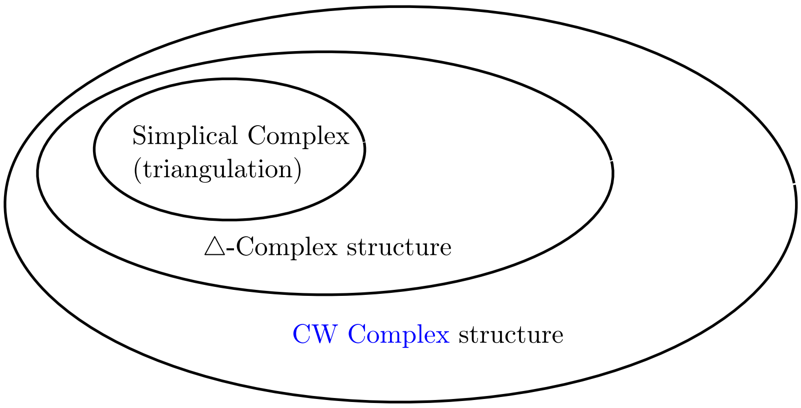
- **Cons.**
  - The definition is more opaque at first encounter.
- **Pros.**
  - Lots of computational tools
  - Functional
  - Abelian Groups

    Remark. More like π_n for $n > 1$.
  - No basepoints
  - Can compute using CW structure.
  - Good properties. For example, $H_n = 0$ if $n > \dim X$

4.2 Simplical Homology

4.2.1 Δ-Simplex

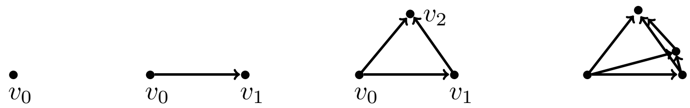
This is a stricter version of a CW complex which allows us to decompose our spaces into cells. In terms of how things fit together, we have the following diagram.



Now we try to give the definition.

Definition 4.1 (Simplex). We see that

- 0-simplex. A point.
- 1-simplex. Interval.
- 2-simplex. Triangle.
- 3-simplex. Tetrahedron.
- n-simplex. The convex hull of $(n+1)$-points position in $\mathbb{R}^n$.



Remark. We see that

- The top of which is the 2-disk and remember cell structure (edges and vertices) and remember orientation (ordering on vertices).
- The top of which is the 3-disk and cells and the orientation.

Further,

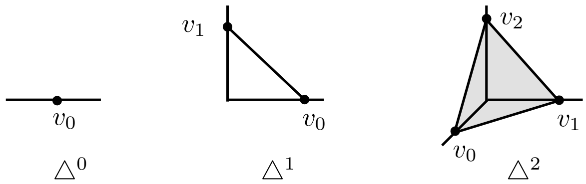
- We can view simplices as both *combinatorial* and *topological* objects.

An alternative definition can be done.

Definition 4.2 (Standard simplex). We say that an n-dimensional *standard simplex*, denoted by Δ^n is

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0,\ \sum_i t_i = 1 \right\}.$$

We'll call such a simplex as *standard n-simplex*.



Remark. In our definition, the simplices will implicitly come with a choice of ordering of the vertices as

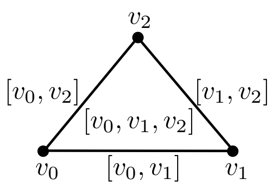
$$\Delta^n = [v_0, v_1, \ldots, v_n]$$

such that the convex hull of these points is taken with this ordering.

Lecture 20: Homology

21 Feb. 10:00

Definition 4.3 (Subsimplex). A *subsimplex* of a simplex σ combinatorially, it's a subset of the vertices; while topologically, it's the convex hull of the subset of vertices.



Definition 4.4 (Face). A *face* of a simplex is a subsimplex of 1 dimensional lower than Δ^n (codimension 1).

Definition 4.5 (Boundary). The *boundary* $\partial\sigma$ of a simplex σ is the union of its faces.

Definition 4.6 (Open simplex). The *open simplex* Δ is defined as

$$\mathring{\Delta}^n = \Delta^n \big/ \partial\Delta^n.$$

Definition 4.7 (Δ-Complex). A Δ-*complex* structure on X is a collection of maps

$$\sigma_\alpha \colon \Delta^n \to X$$

such that

1. $\sigma_\alpha|_{\mathring{\Delta}^n}$ injective, each point of X is in the image of exactly one such map.
2. Each restriction of σ_α to a face coincides with a map

$$\sigma_\beta \colon \Delta^{n-1} \to X.$$

3. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in $\mathring{\Delta}^n$ for all σ_α, i.e., X is a quotient

$$\coprod_{n,\alpha} \Delta_\alpha^n \overset{\coprod \sigma_\alpha}{\to} X.$$

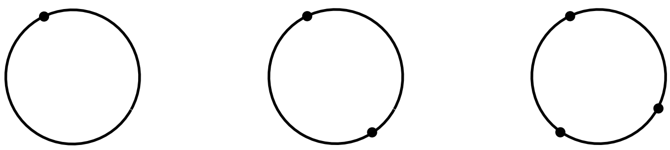
Exercise. A Δ-complex X is a CW complex W characteristic maps σ_α with extra constraints on the attaching maps.

Note. We see that the second condition of Definition 4.7 implies that attaching maps injective on interior of faces.

Definition 4.8 (Simplical complex). A *simplical complex* is a Δ-complex such that

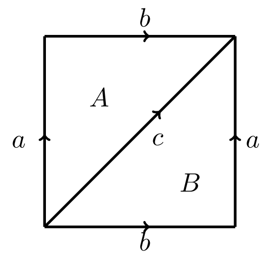
- σ_α must map every face to a different $(n-1)$-simplex.
- Every simplex is uniquely determined by its vertex set.
- Any $(n+1)$ vertices in X^0 is the vertex set of at most 1 simpelx.

Remark. With Definition 4.8, we see the followings.



Δ-simple not Simplical | Δ-simple not Simplical | Δ-simple is Simplical

Example. The torus with the following edges, a, b, c and the gluing in triangles A and B can be seen as follows.



For this Δ-complex, notice that we've glued down a triangle whose vertices are all identified. This is not allowed in a simplical complex / triangulation.

Remark. The minimum number of triangles in a simplical complex structure is **14**.

Appendix

References

[HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com/books?id=BjKs86kosqgC.