MATH602
Real Analysis II

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Abstract

This is a graduate level functional analysis taught by Joseph Conlon at University of Michigan. The prerequisites include linear algebra, complex analysis and also real analysis. We'll use Peter Lax[Lax02] and Reed-Simon[RS80] as textbooks.

The focus of this course is rather standard, including Banach and Hilbert Spaces Theory, Bounded Linear, Compact, and Self-Adjoint Operators Theorem, Representation, Hahn-Banach, Open Mapping Theorem, and Spectral Theory. We also covered some point-set topology along the way.

This course is taken in Fall 2022, and the date on the covering page is the last updated time.

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Chapter 1

Banach and Hilbert Spaces

Lecture 1: Introduction

We first briefly review different kinds of vector spaces. 30 Aug. 14:30

1.1 Linear Spaces

Let's first see the simplest (i.e., without structures) vector space called linear vector space.

Definition 1.1.1 (Linear vector space). A *linear vector space* E over a field $\mathbb{F}$ is a set with operations of addition and multiplication (by a scalar) such that it's closed under operations, and also the addition and scalar multiplication obey

(a) $u + v = v + u$ for $u, v \in E$

(b) $u + (v + w) = (u + v) + w$ for $u, v, w \in E$

(c) $\exists 0 \in E$ such that $0 + u = u + 0 = u$ for $u \in E$

(d) $\forall u \in E$, $\exists - u \in E$ such that $u + (-u) = 0$

(e) $\lambda(u + v) = \lambda u + \lambda v$ for $u, v \in E$, $\lambda \in \mathbb{F}$

(f) $(\lambda + \mu)u = \lambda u + \mu u$ for $u \in E$, $\lambda, \mu \in \mathbb{F}$

(g) $\lambda(\mu u) = (\lambda\mu)u$ for $u \in E$, $\lambda, \mu \in \mathbb{F}$

Remark. If $v, w \in E$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), then $\lambda v + \mu w \in E$.

Notation (Real and complex vector space). If E is over $\mathbb{F} = \mathbb{C}$, we usually call E a *complex vector space*; if $\mathbb{F} = \mathbb{R}$, we say E is a *real vector space*.

Example. Given $n \in \mathbb{N}$, $\mathbb{R}^n$ is an n dimensional real linear vector space.

Example. Given $n \in \mathbb{N}$, $\mathbb{C}^n$ is an n dimensional complex linear vector space.

We concentrate on ∞ dimensional linear vector space.

Example. Let K is a compact Hausdorff space, then

$$E = \{f \colon K \to \mathbb{R} \mid f(\cdot) \text{ is continuous}\}$$

is a ∞ dimensional real linear vector space.

Notation (Subspace). If E is a linear vector space, then we say $E_1 \subseteq E$ is a *subspace* if $E_1 \subseteq E$ and E_1 is itself a linear vector space. Moreover, if $E_1 \subsetneq E$, we say E_1 is a *proper subspace*.

Observe that a linear vector space can have many subspaces.

1.2 Quotient Spaces

Sometimes we don't care about vectors in some directions, suggesting the notion of quotient space.

Definition 1.2.1 (Quotient Space). The *quotient space* E / E_1 of two linear vector spaces E, E_1 such that $E_1 \subseteq E$ is the set of equivalence classes of vectors in E where equivalence is given by $x \sim y$ if $x - y \in E_1$. Additionally, denote $[x]$ as the equivalence class of $x \in E$, i.e., $[x] = x + E_1$.

One can see that quotient space E / E_1 is a linear vector space since if $x_1 + x_2 \in E$, $[x_1] + [x_2] = [x_1 + x_2]$, and also, $\lambda[x] = [\lambda x]$ for $\lambda \in \mathbb{R}$ or $\mathbb{C}$, i.e., $v, w \in E / E_1$, $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$ implies $\lambda v + \mu w \in E$.

The dimension of a quotient space is defined as follows.

Definition 1.2.2 (Codimension). If E / E_1 has finite dimension, then the dimension of E / E_1 is called the *codimension* of E_1 in E, denoted as $\operatorname{codim}(E_1)$.

Definition 1.2.2 is introduced since the way of defining dimensions for finite dimensional vector spaces doesn't work here. Recall Theorem 1.2.1 in the finite dimension case.

Theorem 1.2.1. If E is finite dimensional, then $\operatorname{codim}(E_1) + \dim(E_1) = \dim(E)$

We see that we may encounter something like $\infty - \infty$ if we define $\operatorname{codim}(E_1) := \dim(E) - \dim(E_1)$, and indeed, Definition 1.2.2 is well-defined in this sense.

Example. There exists the case that $\dim(E) = \infty$, $\dim(E_1) < \infty$ where $\dim(E / E_1) < \infty$.

Proof. Let $E = \{f \colon K \to \mathbb{R} \mid f(\cdot) \text{ continuous}\}$ and $E_1 = \{f \in E \colon f(k_1) = 0\}$ for a fixed $k_1 \in K$. We see that the dimension of E / E_1 is exactly 1 since E / E_1 is the set of constant functions. ⊛

Definition 1.2.3 (Linear operator). A map $T \colon E \to F$ between linear spaces E and F is a *linear operator* if it preserves the properties of addition and multiplication by a scalar, i.e., for $v, w \in E$ and $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$,

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w).$$

Definition. Given a linear operator $T \colon E \to F$ we have the following.

Definition 1.2.4 (Kernel). The *kernel* of T is the subspace $\ker(T) = \{x \in E \mid Tx = 0\}$.

Definition 1.2.5 (Image). The *image* of T is the subspace $\operatorname{Im}(T) = \{Tx \in F \mid x \in E\}$.

1.3 Normed Spaces

Given a vector, we want to measure the length of which. This suggests the following definitions.

Definition 1.3.1 (Norm). Let E be a linear vector space. A *norm* $\|\cdot\| : E \to \mathbb{R}$ on E is a function from E to $\mathbb{R}$ with the properties:

(a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Notation (Dilation). We say that the second condition is the *dilation* property.

Definition 1.3.2 (Normed vector space). A linear vector space E equipped with a norm $\|\cdot\|$ is called a *normed vector space*, denoted by $(E, \|\cdot\|)$.

A similar notion called metric is also widely used, though the structure is slightly coarser.

As previously seen (Metric). Given a vector space E, the *metric* $d(\cdot, \cdot)\colon E \times E \to \mathbb{R}$ on E is a function form $E \times E$ to $\mathbb{R}$ with the properties:

(a) $d(x, y) \geq 0$. Also, $d(x, x) = 0$ and $d(x, y)$ implies $x = y$.

(b) $d(x, y) = d(y, x)$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

As one can imagine, if we can measure the length of a vector (by a norm), we can also measure the distance between vectors (by a metric).

Remark (Induced metric space). A normed vector space $(E, \|\cdot\|)$ induces a *metric space* (E, d) with the induced metric $d(x, y) = \|x - y\|$.

Now we give some well-known examples of normed spaces.

Example (Bounded sequences ℓ^∞). Let ℓ_∞ be the space of bounded sequences $x = (x_1, x_2, \dots)$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \dots$. Then we define $\|x\| = \|x\|_\infty = \sup_{i \geq 1} |x_i|$.

Example (Absolutely summable sequences ℓ_1). Let ℓ_1 be the space of absolutely summable sequences $x = (x_1, x_2, \dots)$ and $\sum_{i=1}^{\infty} |x_i| < \infty$. Then we define $\|x\| = \|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$.

Example (Continuous functions $C(k)$). The space $C(k)$ of continuous functions $f\colon K \to \mathbb{R}$ where K is compact Hausdorff. Then we define $\|f\| = \|f\|_\infty = \sup_{x \in K} |f(x)|$.

1.3.1 Geometry of Normed Spaces

Now we can look into the structure of a normed space we're referring to without actually explaining what this really means previously. Intuitively, it's about the geometric properties of the spaces like how do balls, spheres and other shapes look like in that space when defining these shapes with norms.

Definition 1.3.3 (Ball). A (closed) *ball* centered at a point $x_0 \in E$ with radius $r > 0$ is the set

$$B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\}.$$

Definition 1.3.4 (Sphere). The *sphere* centered at x_0 with radius $r > 0$ is the set

$$S(x_0, r) = \{x \in E \mid \|x - x_0\| = r\}.$$

Note. We see that $S(x_0, r)$ is the **boundary** of $B(x_0, r)$, i.e., $S(x_0, r) = \partial B(x_0, r)$.

Let's first look at balls. In finite dimensional, all norms are equivalent, which is not true for infinite dimensional vector spaces. This has something to do with the geometry of balls.

Explicitly, balls can have different geometries depending on the properties of the norms. We see that a $\|\cdot\|_\infty$ can have multiple supporting hyperplane at the corner, while for a $\|\cdot\|_2$ can have only one at each point.

Remark. The unit balls for $\|\cdot\|_1$ looks like **squares**, where we have

$$B(0,1) = \{x = (x_1, x_2, \dots) \mid -1 < y_\epsilon < 1 \text{ for all } \epsilon\}$$

such that $y_\epsilon = \sum_{i=1}^{\infty} \epsilon_i x_i$, $\epsilon_i = \pm 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \dots)$.

We see that different norms give different geometry, but they have important common features, most notably, convexity properties.

Definition 1.3.5 (Convex set). Given E a linear vector space, a set $K \subset E$ is *convex* if for $x, y \in K$ and $0 \leq \lambda \leq 1$,

$$\lambda x + (1-\lambda) y \in K.$$

Definition 1.3.6 (Convex function). Given E a linear vector space, a function $f\colon E \to \mathbb{R}$ is called *convex* if for $x, y \in E$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Remark (Sublevel set). If $f\colon E \to \mathbb{R}$ is a convex function, then for any $M \in \mathbb{R}$ the *sublevel set* $\{x \in E \mid f(x) \leq M\}$ is convex.

The upshot is that norms are convex, and the unit balls are convex as well.

Lecture 2: Banach Spaces and Completion

Let's first see a proposition. 01 Sep. 14:30

Proposition 1.3.1. Let $(E, \|\cdot\|)$ be a normed linear space, then the norm is convex and continuous.

Proof. Let $f\colon E \to \mathbb{R}$ be $f(x) = \|x\|$. Then $f(x) - f(y) = \|x\| - \|y\| \leq \|x - y\|$, which implies $|f(x) - f(y)| \leq \|x - y\|$ for $x, y \in E$, i.e., f is Lipschitz continuous hence continuous. For convexity, let $0 \leq \lambda \leq 1$, we have

$$f(\lambda x + (1-\lambda)y) = \|\lambda x + (1-\lambda)y\| \leq \|\lambda x\| + \|(1-\lambda)y\| = \lambda \|x\| + (1-\lambda)\|y\| = \lambda f(x) + (1-\lambda) f(y).$$

■

Note. Note that $f(\cdot) = \|\cdot\|$ is continuous implies the closed ball

$$B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\} = \{x \in E \mid f(x - x_0) \leq r\}$$

is closed in topology of E. Also, $f(\cdot)$ is convex implies $B(x_0, r)$ is convex.

Remark. If $f\colon E \to \mathbb{R}$ is convex, then the sets $\{x \in E \mid f(x) \leq M\}$ is also convex. However, it's possible to have non-convex functions f such that all sets $\{x \in E \mid f(x) \leq M\}$ are convex.

Proof. Take $f(x) = |x|^p$ for $x \in \mathbb{R}$ and $p > 0$. We see that f is convex if $p > 1$, and non-convex if $p < 1$. However, the sets $\{x \in \mathbb{R} \mid f(x) \leq M\}$ all convex since it's independent of p. ⊛

Lemma 1.3.1. Suppose $x \mapsto \|x\|$ satisfies

(a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) The unit ball $B(0,1)$ is convex.

Then $f(x) = \|x\|$ satisfies the triangle inequality $\|x+y\| \leq \|x\| + \|y\|$.

Proof. We see that if the third condition is true, the for $u, v \in B(0,1)$ and $0 < \lambda < 1$, we have $\lambda u + (1-\lambda)v \in B(0,1)$. Let $x, y \in E$, and

$$\lambda = \frac{\|x\|}{\|x\| + \|y\|} \Rightarrow 1 - \lambda = \frac{\|y\|}{\|x\| + \|y\|}.$$

By letting $u = x / \|x\|$, $v = y / \|y\|$ we see that

$$\lambda u + (1-\lambda)v = \frac{\|x\|}{\|x\| + \|y\|} \frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|} \frac{y}{\|y\|} \in B(0,1) \Rightarrow \left\| \frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|} \right\| \leq 1.$$

From the second condition, it follows that $\|x+y\| \leq \|x\| + \|y\|$, which is the triangle inequality. ■

Remark. If $x \mapsto \|x\|$ satisfies the first two condition and is convex, then it satisfies the triangle inequality.

Proof. Since

$$\frac{1}{2}\|x+y\| = \left\| \frac{x}{2} + \frac{y}{2} \right\| \leq \frac{1}{2}\|x\| + \frac{1}{2}\|y\|.$$

⊛

Now, given a quotient space $E \,/\, E_1$, the question is can we try to define a norm?

Problem 1.3.1. On $E \,/\, E_1$, is $\|[x]\| := \inf_{y \in E_1} \|x+y\|$ a norm?

Answer. No! If $x \in \overline{E}_1 \setminus E_1$, then $\|[x]\| = 0$ but $0 \neq [x] \in E \,/\, E_1$. ⊛

We now see the difference from finite dimensional situation. All finite dimensional spaces E_1 are closed but not in general if E_1 has ∞ dimensions.

Example. Let $\ell_1(\mathbb{R})$ be the sequence of x_n for $n \geq 1$ in $\mathbb{R}$ such that $\sum_{i=1}^{\infty} |x_i| \leq \infty$. Define

$$\|x\|_1 := \sum_{i=1}^{\infty} |x_i|,$$

and let E_1 be all sequences with finite number of the x_n are nonzero. We see that $\overline{E}_1 = \ell_1(\mathbb{R})$ is infinite dimensional.

Proposition 1.3.2. Let $(E, \|\cdot\|)$ be a normed space and $E_1 \subseteq E$, E_1 is closed. Then

$$\|\cdot\| : E \big/ E_1 \to \mathbb{R}, \quad \|[x]\| = \inf_{y \in E_1} \|x+y\|$$

is a norm on $E \,/\, E_1$.

Proof. If $\|[x]\| = 0$, then $\inf_{y \in E_1} \|x - y\| = 0$, which implies $x \in E_1$ since E_1 is closed, so $[x] = 0$. Also, since

$$\|\lambda [x]\| = \inf_{y \in E_1} \|\lambda x + y\| = \inf_{z \in E_1} \|\lambda x + \lambda z\| = |\lambda| \inf_{z \in E_1} \|x + z\| = |\lambda| \, \|[x]\|,$$

the dilation property is satisfied. Finally, for triangle inequality, we have

$$\|[x]+[y]\| = \inf_{x_1,y_1\in E}\|x+y+x_1+y_1\| \leq \inf_{x_1\in E_1}\|x+x_1\| + \inf_{y_1\in E_1}\|y+y_1\| = \|[x]\|+\|[y]\|.$$

■

Remark. This shows that the only obstacle for this kind of norm being an actual norm is whether E_1 is closed.

1.4 Banach Spaces

Turns out that a normed vector space is not enough in general, hence we introduce the following.

Definition 1.4.1 (Banach space). A linear normed space is a *Banach space* if it's complete, i.e., every Cauchy sequence converges.

This implies that given a Banach space $(E, \|\cdot\|)$, if $\{x_n\}_{n\geq 1}$ is a sequence in E with the property such that $\lim\limits_{m\to\infty}\sup\limits_{n\geq m}\|x_n - x_m\| = 0$, then $\exists x_\infty \in E$ such that $\lim\limits_{n\to\infty}\|x_n - x_m\| = 0$ as well.

Example. The spaces ℓ_1, ℓ_∞ and $C(K)$ are Banach spaces.

1.4.1 Completion of Normed Space

We now show an important theorem which characterizes completeness in terms of convergence of series rather than sequences. We first see the definition.

Definition 1.4.2 (Absolutely summable). Let E be a linear normed space and a sequence $\{x_i\}_{i\geq 1}$ in E. Then $\{x_i\}_{i\geq 1}$ is *absolutely summable* if $\sum_{i=1}^\infty \|x_i\| < \infty$.

Then, we have the following.

Theorem 1.4.1 (Criterion for completeness). A normed space $(E, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in E converges.

Proof. We need to prove two directions.

($\Rightarrow$) Suppose E is a Banach space and $\{x_k\}_{k\geq 1}$ an absolutely summable series. Set $s_n = \sum_{k=1}^n x_k$ for $n \geq 1$, we want to show s_n is Cauchy, and if this is the case, completeness of E implies $\exists s_\infty$ and $\lim\limits_{n\to\infty}\|s_n - s_\infty\| = 0$. Let $n > m$, we see that

$$\|s_n - s_m\| = \left\|\sum_{k=m+1}^{n} x_k\right\| \leq \sum_{k=m+1}^{n}\|x_k\| \leq \sum_{k=m+1}^{\infty}\|x_k\|.$$

Observe that $\lim\limits_{m\to\infty}\sum\limits_{k=m+1}^{\infty}\|x_k\| = 0$, we see that the sequence $\{s_n\}$ is Cauchy, hence it converges.

($\Leftarrow$) Conversely, suppose E is **not** complete. Then there exists a Cauchy sequence $\{x_n\}_{n\geq 1}$ which does not converge, implying no subsequence of $\{x_n\}_{n\geq 1}$ converges.[a] We now construct an absolutely summable series which does not converge.

Define $n(1) \geq 1$ such that $\|x_n - x_{n(1)}\| \leq 1/2$ if $n \geq n(1)$, similarly, let $n(2) > n(1)$ be such that $\|x_n - x_{n(2)}\| \leq 1/2^2$ if $n > n(2)$. In all, we have $n(1) < n(2) < n(3) < \dots$ such that

$\|x_n - x_{n(k)}\| \leq 1/2^k$ if $n > n(k)$. Define $w_j \coloneqq x_{n(j+1)} - x_{n(j)}$ for $j = 1, 2, \ldots$. We see that

$$x_{n(m)} = x_{n(1)} + \sum_{j=1}^{m} w_j$$

for $m = 1, 2, \ldots$, and $\{x_{n(m)}\}$ does not converge, hence so does the series $\sum_{j=1}^{\infty} w_j$. However, $\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{j=1}^{\infty} 1/2^j = 1$, which implies $\{w_j\}$ is absolutely summable. ■

[a] Otherwise, the whole sequence converges by the fact that it's Cauchy.

Theorem 1.4.2 (Completion)**.** Suppose E is a normed space. Then there exists a Banach space $\hat{E}$ called *the completion* of E with the following properties:

(a) There exists a linear map $\iota\colon E \to \hat{E}$ such that $\|\iota x\| = \|x\|$.[a]

(b) $\operatorname{Im}(\iota)$ is dense in $\hat{E}$, and $\hat{E}$ is the smallest Banach space containing image of E.

[a] This is called an *isometric embedding* of E into $\hat{E}$.

Lecture 3: Banach, Inner Product Spaces

Notice that ℓ_1 and ℓ_∞ are Banach, and we want to generalize to ℓ_p with $1 < p < \infty$. For $x = \{x_n\}_{n\geq 1}$ in ℓ_p and if $\sum_{n=1}^{\infty} |x_n|^p < \infty$, for $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$, we want to show that $x \mapsto \|x\|_p$ satisfies properties of a norm. The first two properties of a norm is easy check. As for triangle inequality, we have the following. 06 Sep. 14:30

Lemma 1.4.1 (Minkowski inequality)**.** Let $1 \leq p < \infty$, for $x, y \in \ell_p$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p .$$

Proof. Recall that from Lemma 1.3.1, we only need to show that $B(0,1)$ is convex, where

$$B(0,1) = \left\{ x = \{x_n : n \geq 1\} \mid f(x) = \sum_{n=1}^{\infty} |x_n|^p \leq 1 \right\}.$$

But $f(x)$ is convex since $x \mapsto |x|^p$, $x \in \mathbb{R}$ is convex if $p \geq 1$, we're done. ■

Lemma 1.4.2 (Hölder's inequality)**.** Let $1 < p < \infty$, for $x \in \ell_p$, $y \in \ell_q$, we have

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q$$

where $1/p + 1/q = 1$.

Proof. Note first that we can assume without loss of generality, $\sum_{j=1}^{\infty} |x_j|^p = \sum_{j=1}^{\infty} |y_j|^q = 1$. Then, result follows from the Young's inequality,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

for $x, y > 0$, $x, y \in \mathbb{R}$. ■

Remark (Legendre transform and the inequality)**.** Young's inequality is a special case of the inequality

$$xy \leq f(x) + \mathcal{L}f(y)$$

where $\mathcal{L}f(\cdot)$ is the *Legendre transform* of $f(\cdot)$, i.e., $\mathcal{L}f(y) = \sup_x [xy - f(x)]$.

If f is convex, then the function $xy \mapsto xy - f(x)$ is concave so has unique maximum. And $\mathcal{L}f(\cdot)$ always convex even if $f(\cdot)$ is not. In particular, if $f(x) = x^p/p$, then $\mathcal{L}f(y) = y^q/q$.

Note. Minkowski inequality is usually proved via the Hölder's inequality.

Proof. To show this, since

$$\sum_{j=1}^{\infty} |x_j + y_j|^p \leq \sum_{j=1}^{\infty} |x_j| \, |x_j + y_j|^{p-1} + \sum_{j=1}^{\infty} |y_j| \, |x_j + y_j|^{p-1} .$$

Then Hölder's inequality implies

$$\sum_{j=1}^{\infty} |x_j| \, |x_j + y_j|^{p-1} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |x_j + y_j|^{(p-1)q} \right)^{1/q} ,$$

and similarly,

$$\sum_{j=1}^{\infty} |y_j| \, |x_j + y_j|^{p-1} \leq \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |x_j + y_j|^{(p-1)q} \right)^{1/q} .$$

Note that $(p-1)q = p$, hence by combining both, we have

$$\sum_{j=1}^{\infty} |x_j + y_j|^p \leq \left[\left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p} \right] \left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1/q} ,$$

i.e.,

$$\left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1-1/q} = \left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p} ,$$

proving the result. ⊛

Notice that Minkowski inequality and Hölder's inequality also hold for $1 \leq p \leq \infty$, or more generally, both hold for L^p spaces also. Let (Ω, Σ, μ) be a measure space and $L^p(\Omega, \Sigma, \mu)$ where all Σ measurable functions $f \colon \Omega \to \mathbb{R}$ (or $\mathbb{C}$) such that $\int_\Omega |f|^p \, \mathrm{d}\mu < \infty$. Then, $L^p(\Omega, \Sigma, \mu)$ is a normed space with norm

$$\|f\|_p \coloneqq \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{1/p} .$$

It's more tricky to show that L^p is a Banach space, but it's indeed still the case.

Theorem 1.4.3 (Riesz-Fisher)**.** The space $L^p(\Omega, \Sigma, \mu)$ is a Banach space for $1 \leq p < \infty$.

Proof. Toward using Theorem 1.4.1, let $\{f_n\}_{n \geq 1}$ be an absolutely summable sequence in L^p. Then the norm satisfies

$$\left\| \sum_{k=1}^{N} f_k \right\|_p \overset{!}{\leq} \sum_{k=1}^{N} \|f_k\|_p \leq C < \infty \Rightarrow \int_\Omega \left| \sum_{k=1}^{N} f_k \right|^p \, \mathrm{d}\mu \leq C^p .$$

- Assume all f_k are non-negative. From monotone convergence theorem, we have

$$\lim_{N \to \infty} \int_\Omega \left(\sum_{k=1}^{N} f_k \right)^p \, \mathrm{d}\mu = \int_\Omega \left(\sum_{k=1}^{\infty} f_k \right)^p \, \mathrm{d}\mu \leq C^p .$$

Hence, $g = \sum_{k=1}^{\infty} f_k \in L^p$. We now want to show that $\sum_{k=1}^{N} f_k \to g$ in L^p. Set $r_n = \sum_{k=n+1}^{\infty} f_k$ where r_n is a decreasing sequence where $r_n \to 0$ a.e. and also

$$\int_\Omega r_1^p \,\mathrm{d}\mu < \infty.$$

This means that $\lim_{n\to\infty} \|r_n\|_p = 0$ by dominate convergence theorem.

- For arbitrary $f_k\colon \Omega \to \mathbb{R}$, write $f_k = f_k^+ + f_k^-$ where $f_k^+ = \sup(f_k, 0)$ and $f_k^- = \inf(f_k, 0)$. The sequence $\{f_k^+\}_{k\geq 1}$ are absolutely summable, and we just proceed as before. Similarly, if $f_k\colon \Omega \to \mathbb{C}$, we get the same result.

■

1.5 Inner Product Spaces

Indeed, a slightly stronger structure than a normed space equipped is the so-called inner product, since it actually induces a norm.

Definition 1.5.1 (Inner product)**.** Let E be a linear space over $\mathbb{C}$. An *inner product* $\langle\cdot,\cdot\rangle : E\times E \to \mathbb{C}$ is a function which has the following properties:

(a) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

(b) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for $a, b \in \mathbb{C}$.

(c) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

Notation (Real inner product)**.** We can also define inner products of spaces over $\mathbb{R}$ with no extra conjugation in the last property.

Definition 1.5.2 (Inner product space)**.** An *inner product space* is a linear space E with an inner product $\langle\cdot,\cdot\rangle : E\times E \to \mathbb{C}$.

Definition 1.5.3 (Orthogonal)**.** Given a linear space E, $x, y \in E$ are *orthogonal* if $\langle x, y\rangle = 0$, denote as $x \perp y$.

Theorem 1.5.1 (Cauchy-Schwarz inequality)**.** Let $x, y \in E$ and an inner product $\langle\cdot,\cdot\rangle$, then

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}}.$$

Proof. Define $Q(t)$ by $Q(t) = \langle x + ty, x + ty\rangle = \langle y, y\rangle t^2 + 2t\,\mathrm{Re}\langle x, y\rangle + \langle x, x\rangle$ if $t \in \mathbb{R}$. Then we see that $Q(t) \geq 0$ with $t \in \mathbb{R}$, by looking at the discriminant, we have $(\mathrm{Re}\langle x, y\rangle)^2 \leq \langle x, x\rangle\langle y, y\rangle$. Finally, the result follows by choosing $\theta \in \mathbb{R}$ such that $\langle x, y\rangle = \mathrm{Re}\left\langle xe^{i\theta}, y\right\rangle$, we then see that

$$|\langle x, y\rangle| = \left|\mathrm{Re}\left\langle xe^{i\theta}, y\right\rangle\right| = |\mathrm{Re}\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle\langle y, y\rangle},$$

proving the result. ■

Corollary 1.5.1. The function $x \mapsto \|x\| := \langle x, x\rangle^{\frac{1}{2}}$ is a norm on E.

Proof. The first two properties of a norm is easy to verify, and the triangle inequality is a consequence of Cauchy-Schwarz inequality such that

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\operatorname{Re}\langle x,y\rangle + \|y\|^2 \overset{!}{\leq} \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2.$$

■

Remark (Pythagorean theorem). The calculation in Corollary 1.5.1 clearly implies *Pythagorean theorem*, which states that if $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Example (Canonical inner product in ℓ_2). Consider $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \ell_2$, the space of square summable sequences, then

$$\langle x, y\rangle \coloneqq \sum_{j=1}^{\infty} x_j \overline{y}_j$$

defines an inner product.

Example (Canonical inner product in L^2). Consider $f, g \in L^2(\Omega, \Sigma, \mu)$, the space of square integrable functions, then

$$\langle f, g\rangle = \int_\Omega f(x)\overline{g}(x)\,\mathrm{d}\mu(x)$$

defines an inner product. Furthermore, $\|f\|_2 = \langle f, f\rangle^{1/2}$.

Proof. The only non-trivial fact to prove is that $\langle f, g\rangle$ is finite, i.e., $f\overline{g}$ is integrable. Firstly, f^2, $\overline{f}^2$ and $(f+g)^2$ are all integrable since f, $\overline{g}$ and $f+\overline{g}$ are all in L^2, hence $f\overline{g}$ is also integrable. ⊛

Example. Consider A, B in the space of $m \times n$ matrices $A = (a_{ij})$, $1 \leq i \leq m, 1 \leq j \leq n$, then

$$\langle A, B\rangle = \operatorname{tr}(AB^*)$$

defines an inner product, where B^* is the Hermitian adjoint of B, i.e., for $B = (b_{ij})$, then $B^* = (b_{ij}^*)$ for $b_{ij}^* = \overline{b}_{ji}$.

Remark (Hilbert-Schmidt (Frobenius) norm). Specifically, the norm corresponding to this inner product is

$$\|A\|_{\mathrm{HS}} \coloneqq \left(\sum_{i,j}^{\infty} |a_{ij}|^2\right)^{1/2},$$

which is known as the *Hilbert-Schmidt* or *Frobenius* norm.

1.5.1 Geometry of Inner Product Spaces

Indeed, the structure of an inner product space is much more interesting, since we can now consider the notion of angle between vectors.

As previously seen. Recall that in Euclidean space $\mathbb{R}^n$, the inner product can be computed by the formula

$$\langle x, y\rangle = \|x\|\|y\|\cos\theta(x,y)$$

where $\theta(x,y)$ denotes the angle between x and y.

Similarly, we can define the angle between x, y in an inner product space by

$$\cos\theta(x,y) \coloneqq \frac{\langle x, y\rangle}{\|x\|\|y\|} \in [-1, 1]$$

where the range is ensured by Cauchy-Schwarz inequality, so it's well-defined. Though this concept this rarely used anyway. Indeed, the only useful case is when $\cos\theta = 0$, namely when x and y are perpendicular, or orthogonal.

But beyond orthogonality, there are other geometric properties in an inner product space captures by norms. Specifically, both parallelogram law and polarization identity hold, and the result is stated in terms of norm while they actually rely on the property of inner product.

Lemma 1.5.1 (Parallelogram law)**.** Given E an inner product space, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Proof. Recall that $\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$ and similarly, $\|x-y\|^2 = \|x\|^2 - 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$, hence the result follows. ■

Lemma 1.5.2 (Polarization identity)**.** Given E an inner product space, we have

$$\langle x, y\rangle = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right\}$$

Proof. The proof is just to expand the right-hand side in terms of inner product. ■

Remark. Polarization identity shows that the function $x \mapsto \|x\|^2$ determines the inner product.

Lecture 4: Orthogonality and Projection

1.6 Hilbert Spaces

08 Sep. 14:30

Just like the case of normed spaces, the inner product spaces are incomplete in general, hence we define the completed spaces of which, called Hilbert spaces.

Definition 1.6.1 (Hilbert space)**.** A complete inner product space is called a *Hilbert space*.

Example. Both ℓ_2 and $L^2(\Omega, \Sigma, \mu)$ are normed spaces and complete, hence are Hilbert space.

1.6.1 Orthogonality

We'll soon see that the key notion in Hilbert space theory is orthogonality.

Definition 1.6.2 (Orthogonal complement)**.** Let $A \subseteq \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space, then the *orthogonal complement* $A^\perp$ of A is

$$A^\perp := \{x \in \mathcal{H} : \langle x, y\rangle = 0 \text{ for } y \in A\}.$$

Remark. $A^\perp$ is also a Hilbert space, in particular, closed and $A^\perp \cap A \subseteq \{0\}$.

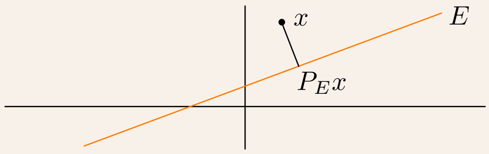
Proof. $A^\perp$ is closed linear subspace of $\mathcal{H}$ where the closure follows from the continuity of the function $x \mapsto \langle x, y\rangle$ for $x \in \mathcal{H}$ by looking at the inverse image of $\{0\}$. Also, for $x \in A^\perp \cap A$, $\langle x, x\rangle = 0$ implies $x = 0$. The reverse inclusion is false since A can be empty. ⊛

The fundamental theory of Hilbert spaces is Theorem 1.6.1.

Theorem 1.6.1 (Orthogonality principle)**.** Assume $E \subseteq \mathcal{H}$ is a closed linear subspace of the Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$. Then we have the following.

(a) There exists a unique closest point $y = P_E x \in E$ to x, i.e., $\|x - P_E x\| = \inf_{y' \in E} \|x - y'\|$.

(b) The point $y = P_E x \in E$ is the unique vector such that $x - y \in E^\perp$.



Proof. Note that the function $y' \mapsto \|x - y'\|$ for $y' \in E$ is convex. We expect a minimizer y'.

(a) Let $y_n \in E$ for $n = 1, 2, \ldots$ be a minimizing sequence, i.e.,

$$\lim_{n \to \infty} \|x - y_n\| = \inf_{y' \in E} \|x - y'\| =: d.$$

From parallelogram law, we have

$$\|y_n - y_m\|^2 + 4\|x - (y_n + y_m)/2\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2.$$

As $n, m \to \infty$, the right-hand side goes to $4d^2$. But since $\frac{1}{2}(y_n + y_m) \in E$, we have $\left\|x - \frac{1}{2}(y_n - y_m)\right\| \geq d$, so

$$\lim_{m \to \infty} \sup_{m \geq n} \|y_n - y_m\|^2 = 0,$$

which implies $\{y_n\}$ is a Cauchy sequence. As $\mathcal{H}$ is complete, we see that $y_n \to y_\infty \in E$, with $\|x - y_\infty\| = d$.

Now, with the fact that E is closed, we set $y_\infty = P_E x$ where y_∞ is unique since if $\|x - y_\infty\| = \|x - y'_\infty\| = d$, again by the parallelogram law where we now plug in y_∞ and y'_∞ instead of y_n and y_m as above, we see that $\|y_\infty - y'_\infty\| = 0$, hence $y_\infty = P_E x \in E$ is well-defined.

(b) We now show $P_E x$ is the unique vector $y \in E$ such that $x - y \perp E$, i.e., $x - y \in E^\perp$. Let $y' \in E$ and let $Q(t)$ be the quadratic

$$Q(t) := \langle x - P_E x + ty', x - P_E x + ty' \rangle = \|x - P_E x + ty'\|^2.$$

Since $t \mapsto Q(t)$ has a **strict** minimum at $t = 0$, which implies $Q'(0) = 0$, i.e., $\operatorname{Re}(x - P_E x, y') = 0$ for all $y' \in E$, which further implies $\langle x - P_E x, y' \rangle = 0$ for all $y' \in E$. This shows that $x - P_E x \in E^\perp$.

Finally, we need to show $P_E x \in E$ is the unique vector such $x - P_E x \in E^\perp$. This can be seen from $Q(t) = \|x - P_E x\|^2 + t^2\|y'\|^2$ for any $y' \in E$.

■

We see that orthogonality principle is actually quite surprising, since to show existence of such a closest point, we typically need

(a) Compactness,

(b) Non-degeneracy for uniqueness.

But here by using parallelogram law and the completeness of $\mathcal{H}$, we don't need these.

Remark. Orthogonality principle shows that the minimizer for the function $y' \mapsto \|x - y'\|$ for $y' \in E$ is characterized by the orthogonality condition, i.e., $x - y \perp E$ for some $y \in E$.

This suggests the following definition.

Definition 1.6.3 (Orthogonal projection). Let $\mathcal{H}$ be a Hilbert space ad let $E \subseteq \mathcal{H}$ be a closed subspace. The *orthogonal projection operator* $P_E \colon \mathcal{H} \to E$ is given by $x \mapsto P_E x$ where $P_E x$ is defined uniquely via $x - P_E x \in E^\perp$.

The orthogonal projection is actually a so-called bounded linear map which is defined below.

Definition 1.6.4 (Bounded linear map). Given a mapping $A\colon E \to E$ on a Banach space E, we say it's a *bounded linear map* if it's bounded and linear.

Definition 1.6.5 (Linear map). The operator A is *linear* if for $x, y \in E$, $a, b \in \mathbb{C}$,

$$A(ax + by) = aA(x) + bB(y).$$

Definition 1.6.6 (Bounded map). The operator A is *bounded* if

$$\|A\| \coloneqq \sup_{\|x\|=1} \|Ax\| < \infty.$$

Remark. Note that $\|Ax\| \leq \|A\| \|x\|$ for $x \in E$.

We see that $P_E x$ is a bounded linear map $P_E\colon \mathcal{H} \to E \subseteq \mathcal{H}$ with the properties $P_E^2 = P_E$ and $\|x\|^2 = \|P_E x\|^2 + \|(I - P_E)x\|^2$ since $(I - P_E)x \perp P_E x$. The latter property shows that

$$\|P_E\| \leq 1, \quad \|I - P_E\| \leq 1,$$

and in fact, $\|P_E\| = \|I - P_E\| = 1$. Also, $I - P_E$ is also an orthogonal projection onto $E^\perp$.

1.7 Fourier Series

Hilbert space gives a geometric framework for studying Fourier series. The classical Fourier analysis studies situations where a function $f\colon [-\pi, \pi] \to \mathbb{C}$ can be expanded as Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{1}{\sqrt{2\pi}} e^{ikt}$$

with the Fourier coefficients

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \,\mathrm{d}t.$$

In order to make Fourier analysis rigorous, we have to understand what functions f can be written as Fourier series, and in what sense the Fourier series converges. To do so, it's of great advantage to depart from this specific situation and carry out Fourier analysis in an abstract Hilbert space. Let $f(t)$ be a vector in the function space $L^2([-\pi, \pi])$, and the exponential functions e^{-ikt} will form a set of orthogonal vectors in this space. Then, Fourier series will become an orthogonal decomposition of a vector f w.r.t. an orthogonal system of coordinates.

1.7.1 Orthogonal Systems

We first give the definition.

Definition 1.7.1 (Orthogonal system). A sequence $\{x_k\}_{k\geq 1}$ of non-zero vectors in a Hilbert space $\mathcal{H}$ is *orthogonal* if $\langle x_k, x_\ell \rangle = 0$ for all $\ell \neq k$.

Definition 1.7.2 (Orthonormal system). An orthogonal system $\{x_k\}_{k\geq 1}$ is an *orthonormal system* if in addition, we have $\|x_k\| = 1$ for all k.

Write it in a more compact way, $\{x_k\}_{k\geq 1}$ is orthonormal if $\langle x_k, x_\ell \rangle = \delta_{k,\ell}$ where δ is the Kronecker delta. Here is an immediate generation the given remark.

Theorem 1.7.1 (Pythagorean theorem)**.** Let $\{x_k\}_{k\geq 1}$ be an orthogonal system in a Hilbert space $\mathcal{H}$. Then for every $n \in \mathbb{N}$,

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2$$

Proof. From orthogonality, we have

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \left\langle \sum_{k=1}^{n} x_k, \sum_{k=1}^{n} x_k \right\rangle = \sum_{k,j=1}^{n} \langle x_k, x_j \rangle = \sum_{k=1}^{n} \langle x_k, x_k \rangle = \sum_{k=1}^{n} \|x_k\|^2.$$

■

We now see some examples.

Example (Canonical basis of ℓ_2)**.** In the space ℓ_2, $x_k = (0, 0, \dots, 1, 0, \dots, 0) \in \ell_2$ for $k = 1, 2, \dots$ is an orthonormal system in ℓ_2 forming a basis.

Example (Fourier basis in L^2)**.** In the space $L^2([-\pi, \pi])$, consider the exponential

$$e_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$$

for $t \in [-\pi, \pi]$. The set $\{e_k\}_{k=-\infty}^{\infty}$ is an orthonormal system in $L^2[-\pi, \pi]$, forming a basis.

1.7.2 Fourier Series

We can further generalize Fourier series to any Hilbert space by letting $\{x_k\}_{k\geq 1}$ be an orthonormal set in $\mathcal{H}$ as follows.

Definition. Consider an orthonormal system $\{x_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ and a vector $x \in \mathcal{H}$.

Definition 1.7.3 (Fourier series)**.** The *Fourier series* of x w.r.t. $\{x_k\}_{k\geq 1}$ is the formal series

$$\sum_{k=1}^{\infty} \langle x, x_k \rangle \, x_k.$$

Definition 1.7.4 (Fourier coefficient)**.** The coefficient $\langle x, x_k \rangle$ in the Fourier series are called *Fourier coefficients* of x.

To understand the convergence of Fourier series, we first focus on the finite case and study the partial sums of Fourier series. For $n = 1, 2, \dots$, we define $S_n \colon \mathcal{H} \to E_n$ such that

$$S_n(x) = \sum_{k=1}^{n} \langle x, x_k \rangle \, x_k$$

for $x \in \mathcal{H}$ where $E_n = \operatorname{span}(\{x_1, \dots, x_n\})$. We see that S_n is a linear operator and $S_n = P_{E_n}$ is the orthogonal projection onto E_n since $\langle x - S_n(x), x_k \rangle = 0$ for $k = 1, \dots, n$, hence $S_n(x) \in E_n$ and $x - S_n(x) \perp E_n$.

Theorem 1.7.2 (Bessel's inequality)**.** Let $\{x_k\}_k$ be an orthogonal system in a Hilbert space $\mathcal{H}$. Then for every $x \in \mathcal{H}$,

$$\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \leq \|x\|^2 .$$

Proof. To estimate the size of $S_n(x)$, consider $x - S_n(x)$ and from Pythagorean theorem,

$$\|S_n(x)\|^2 + \|x - S_n(x)\|^2 = \|x\|^2 \Rightarrow \|S_n(x)\|^2 \leq \|x\|^2 .$$

On the other hand, again by Pythagorean theorem and orthogonality,

$$\|S_n(x)\|^2 = \sum_{k=1}^{n} \|\langle x, x_k\rangle x_k\|^2 = \sum_{k=1}^{n} |\langle x, x_k\rangle|^2 .$$

We see that by combining these two inequalities and let $n \to \infty$, we have the result. ∎

Remark. In particular, we see that $\|S_n(x)\|^2 = \sum_{k=1}^{n} |\langle x, x_k\rangle|^2$, with $S_n = P_{E_n}$ we have $\|P_{E_n} x\|^2 \leq \|x\|^2$ for all $x \in \mathcal{H}$.

This implies the following.

Corollary 1.7.1. Let $\{x_k\}_{k\geq 1}$ be an orthonormal system in a Hilbert space $\mathcal{H}$. Then the Fourier series $\sum_k \langle x, x_k\rangle x_k$ for every $x \in \mathcal{H}$ converges in $\mathcal{H}$.

Proof. This follows directly from Bessel's inequality with the fact that the tail sum is Cauchy, i.e., we have

$$\left\| \sum_{k=n}^{m} x_k \right\|^2 = \sum_{k=n}^{m} \|x_k\|^2 \to 0$$

as $n, m \to \infty$ from Pythagorean theorem. ∎

Corollary 1.7.1 tells us that Fourier series of x converge. However, it needs not converge to x, although we can still compute the point where it converges to by considering Bessel's inequality, and the optimality is guaranteed by orthogonality principle.

Theorem 1.7.3 (Optimality of Fourier series). Let $\{x_k\}_k$ be an orthonormal system in a Hilbert space $\mathcal{H}$. Then the corresponding Fourier series $S_n(x) = \sum_{k=1}^{n} \langle x, x_k\rangle x_k$ converges, i.e., $\lim_{n\to\infty} S_n(x) = S_\infty(x)$ exists for $x \in \mathcal{H}$. Furthermore, $S_n = P_{E_n}$ for every n where E_n is the space spanned by $\{x_i\}_{i=1}^{n}$.[a]

[a] This includes $n = \infty$, where E_∞ is the **closure** of the space spanned by $\{x_k\}_{k\geq 1}$.

Proof. We show that the sequence $S_n(x)$ for $n = 1, 2, \ldots$ is Cauchy. This is because

$$\|S_n(x) - S_m(x)\|^2 = \sum_{k=m+1}^{n} |\langle x, x_k\rangle|^2 ,$$

and Bessel's inequality implies $\sum_{k=1}^{\infty} |\langle x, x_k\rangle|^2 \leq \|x\|^2$. Hence, for any $\epsilon > 0$, there exists $m(\epsilon)$ such that

$$\sum_{k=m(\epsilon)+1}^{\infty} |\langle x, x_k\rangle|^2 < \epsilon,$$

which implies $\|S_n(x) - S_m(x)\|^2 < \epsilon$ if $n > m(\epsilon)$, hence $\{S_n(x)\}_{n\geq 1}$ is Cauchy, implying

$$\lim_{n\to\infty} S_n(x) = S_\infty(x) \in \mathcal{H}.$$

Also, $S_\infty = P_{E_\infty}$ where E_∞ is the closure of the linear space generated by the $\{x_k\}_{k\geq 1}$. ∎

Remark. Orthogonality principle states that among all convergent series of the form $S = \sum_k a_k x_k$, the approximation error $\|x - S\|$ is minimized by the Fourier series of x since it's the projection.

We finally note that the closedness of E_∞ makes sense since the self-dual of a set's orthogonal complement is itself if it's closed in the first place.

Lecture 5: Abstract Fourier Series

1.7.3 Orthonormal Bases

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It should be easy to identify an extra condition which makes the Fourier series of every vector x converges to x.

Definition 1.7.5 (Complete system). A system of vector $\{x_k\}_k$ in Hilbert space $\mathcal{H}$ is *complete* if the space spanned by $\{x_k\}_k$ is dense in $\mathcal{H}$, i.e., $\overline{\operatorname{span}(\{x_k\}_k)} = \mathcal{H}$.

Definition 1.7.6 (Orthonormal basis). A complete orthonormal system in a Hilbert space $\mathcal{H}$ is called an *orthonormal basis* of $\mathcal{H}$.

Then, we have the so-called Fourier expansion, which is sometimes called Fourier inversion formula.

Theorem 1.7.4 (Fourier expansions). Let $\{x_k\}_k$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Then every vector $x \in \mathcal{H}$ can be expanded in its Fourier series

$$x = \sum_k \langle x, x_k \rangle x_k.$$

Proof. If an orthogonal set $\{x_k\}_k$ is complete, then $E_\infty = \mathcal{H}$, $P_{E_\infty} = I$. This implies $x = \sum_{k=1}^\infty \langle x, x_k \rangle x_k$ for $x \in \mathcal{H}$. ∎

Corollary 1.7.2 (Parseval's identity). Let $\{x_k\}_k$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Then

$$\|x\|^2 = \sum_{k=1}^\infty \left|\langle x, x_k \rangle\right|^2.$$

Proof. From Fourier expansion, $\|x\|^2 = \|P_{E_n} x\|^2 + \|I - P_{E_n}\|^2$. By letting $n \to \infty$, we have

$$\|x\|^2 = \lim_{n\to\infty} \|P_{E_n} x\|^2 = \lim_{n\to\infty} \sum_{k=1}^n |\langle x, x_k \rangle|^2 = \sum_{k=1}^\infty |\langle x, x_k \rangle|^2.$$

∎

1.7.4 Gram-Schmidt Orthogonalization

Suppose $x_1, x_2, \dots \in \mathcal{H}$ is a set of vectors and $E_n = \operatorname{span}(\{x_1, \dots, x_n\})$. Then we can find an orthonormal set $\{y_k\}_{k\geq 1}$ in $\mathcal{H}$ such that $E_n = \operatorname{span}(\{y_1, y_2, \dots, y_{m(n)}\})$ where $m(n) \leq n$. Such a procedure is called *Gram-Schmidt orthogonalization*. To do this, firstly, set $y_1 = x_1 / \|x_1\|$, then for $n \geq 2$, we have

$$y_n = \frac{(I - P_{E_{n-1}})x_n}{\left\|(I - P_{E_{n-1}})x_n\right\|}$$

if $x_n \notin E_{n-1}$, i.e., E_{n-1} is properly contained in E_n.

Remark. Proving completeness of a set of vectors $\{x_k\}_{k\geq 1}$ in $\mathcal{H}$ can be **non**-trivial.

Note that we can effectively compute the vectors $P_{E_n}(x_{n+1})$ since we know that $S_n(x)$ is the orthogonal projection of x onto $\operatorname{span}(\{y_k\}_{k\geq 1})$, which is the partial sum of Fourier series

$$S_n(x) = \sum_{k=1}^n \langle x, y_k \rangle y_k.$$

As for $P_n(x)$, we see that it's the orthogonal projection onto the orthogonal complement, i.e.,

$$P_{E_n}(x) = x - S_n(x) = x - \sum_{k=1}^{n} \langle x, y_k \rangle y_k \Rightarrow P_{E_n}(x_{n+1}) = x_{n+1} - \sum_{k=1}^{n} \langle x_{n+1,y_k} \rangle y_k.$$

Let's now see some examples.

Example (Haar basis). We consider the *Haar basis* for $L^2([0,1])$. Let $h\colon (0,1) \to \mathbb{R}$ where

$$h(t) = \begin{cases} 1, & \text{if } 0 < t < 1/2; \\ -1, & \text{if } 1/2 < t < 1. \end{cases}$$

Extend $h(\cdot)$ by zero outside $(0,1)$, we get $h\colon \mathbb{R} \to \mathbb{R}$, $h(t) = 0$ if $t \notin (0,1)$, otherwise it's the same as above. The function $t \mapsto h(2^k t)$ has support in interval $0 < t < 2^{-k}$. Move the support to interval $\ell 2^{-k} < t < (\ell+1)2^{-k}$ by translation. Set

$$h_{k,\ell}(t) = h(2^k t - \ell), \quad \ell = 0, 1, \ldots, 2^k - 1.$$

The constant function plus functions $h_{k,\ell}$, $k = 0, 1, 2, \ldots$, $0 \leq \ell \leq 2^k - 1$ are a complete orthogonal set for $\mathcal{H} = L^2([0,1])$.

Proof. The span of the Haar functions includes characteristics functions χ_F for all dyadic intervals $[2^{-k}\ell, 2^{-k}(\ell+1)]$ for $\ell = 0, 1, \ldots, 2^{k-1}$, $k = 0, 1, \ldots$. If the set is **not** complete, then there exists $f \in L^2([0,1])$ such that

$$\int_F f \,\mathrm{d}t = 0$$

for all dyadic intervals F. Since we can approximate any measurable set $E \subseteq (0,1)$ by a union of dyadic intervals.

Intuition. An easy way to see this is to consider

$$\left\{ F \in \mathcal{B} \colon \int_F f \,\mathrm{d}t = 0 \right\},$$

which is the Borel subalgebra of $\mathcal{B}$, which indeed is a Borel algebra on $(0,1)$. Then observe that dyadic intervals generate all open intervals.

Hence, we see that $\int_F f \,\mathrm{d}t = 0$ for all measurable $F \subseteq (0,1)$. Let $F = \{t \in (0,1) \colon f(t) > 0\}$, if $m(F) > 0$, then

$$\int_F f \,\mathrm{d}t > 0.$$

Hence, a contradiction, so $m(F) = 0$. ⊛

Example (Fourier basis). Consider the Fourier basis $e_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$ for $k \in \mathbb{Z}$, $-\pi < t < \pi$. This is complete in $L^2([-\pi, \pi])$.

Proof. We use Stone-Weierstrass theorem and apply it to Fourier basis. All $e_k(\cdot)$ are in $C([-\pi,\pi])$, i.e., continuous functions $f\colon [-\pi,\pi] \to \mathbb{C}$. We know that $C([-\pi,\pi])$ is a Banach space with supremum norm $\|f\| \coloneqq \sup_{t \in [-\pi,\pi]} |f(t)|$. Stone-Weierstrass theorem implies density of the space spanned by $e_k(\cdot)$, $k \in \mathbb{Z}$ in $C([-\pi,\pi])$, hence the completeness in $L^2([-\pi,\pi])$ follows from the density of continuous functions in $L^2([-\pi,\pi])$. ⊛

Proposition 1.7.1. Let $\{x_k\}_k$ be a linear independent system in a Hilbert space $\mathcal{H}$. Then the system $\{y_k\}_k$ obtained by Gram-Schmidt orthogonalization of $\{x_k\}_k$ is orthonormal in $\mathcal{H}$, and for all $n \in \mathbb{N}$,

$$\operatorname{span}(\{y_k\}_{k=1}^n) = \operatorname{span}(\{x_k\}_{k=1}^n).$$

Proof. The system $\{y_k\}_k$ is orthonormal by construction, and we obviously have the inclusion $\operatorname{span}(\{y_k\}_k) \subseteq \operatorname{span}(\{x_k\}_k)$. Furthermore, since the dimensions of these subspaces both equal n by construction, so they're indeed equal. ■

1.7.5 Existence of Orthogonal Bases

From Proposition 1.7.1, every Hilbert space that is not *too large* has an orthonormal basis. We call this Hilbert space separable.

Definition 1.7.7 (Separable)**.** A metric space is *separable* if it contains a countable dense subset.

For Banach space, separability follows from finding a countable set of vectors $\{x_k\}_k$ such that the span of $\{x_k\}_k$ is dense in E. Formally, we have the following.

Lemma 1.7.1 (Separable spaces)**.** A Banach space E is separable if and only if it contains a system of vectors $\{x_k\}_{k\geq 1}$ whose linear span is dense in E, i.e., $\overline{\operatorname{span}(\{x_k\}_{k\geq 1})} = E$.

Furthermore, we can prove the following.

Theorem 1.7.5. Every separable Hilbert space has an orthonormal basis.

Remark. We developed the theory for countable orthogonal systems and bases. One can generalize it for systems of arbitrary cardinality.

A final remark will be Theorem 1.7.6, which states that all Hilbert spaces of the same cardinality have the same geometry.

Theorem 1.7.6. All infinite-dimensional separable Hilbert spaces are isometric to each other. Precisely, for every such spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, there is a linear bijective map $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ preserving the inner product, i.e., for all $x, y \in \mathcal{H}_1$, $\langle Tx, Ty\rangle = \langle x, y\rangle$.

While leaving out the proof, we note that from Theorem 1.7.6, $\|Tx\| = \|x\|$ for all $x \in \mathcal{H}_1$, hence

$$\|T(x-y)\| = \|Tx - Ty\| = \|x - y\|$$

for all $x, y \in \mathcal{H}_1$, i.e., T preserves all pairwise distances, hence the name *isometry*.

Remark. Every separable Hilbert space is isometric to ℓ_2 and $L^2([0,1])$.

Proof. Since ℓ_2 and $L^2([0,1])$ are separable Hilbert spaces, the result follows from Theorem 1.7.6. ⊛

Chapter 2

Bounded Linear Operators

In this chapter we study certain transformations of Banach spaces. Because these spaces are linear, the appropriate transformations to study will be linear operators. Furthermore, since Banach spaces carry topology, it is most appropriate to study continuous transformations, i.e. continuous linear operators. They are also called bounded linear operators for the reasons that will become clear shortly.

2.1 Bounded Linear Functionals

When the operators' range is $\mathbb{R}$ or $\mathbb{C}$, it is interesting enough already, hence we study this case first.

2.1.1 Continuity and Boundedness

At this moment, the topology does not matter, so we define linear functional on general linear vector spaces.

Definition. Let E be a linear space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1 (Linear functional)**.** A *linear functional* on E is a linear operator $f\colon E \to \mathbb{R}$ or $\mathbb{C}$ such that for $x, y \in E$, $a, b \in \mathbb{R}$ or $\mathbb{C}$,

$$f(ax + by) = af(x) + bf(y).$$

Definition 2.1.2 (Bounded linear functional)**.** A linear functional $f(\cdot)$ is *bounded* if

$$\|f\| \coloneqq \sup_{\|x\|=1} |f(x)| < \infty.$$

Clearly, the boundedness of $f(\cdot)$ implies $|f(x-y)| \leq \|f\| \|x-y\|$ for $x, y \in E$, hence, $f(\cdot)$ is continuous[1] if it's bounded.

Remark. Conversely, if a linear functional is continuous, then it is bounded.

Proof. Suppose $f(\cdot)$ is not bounded, then there exists a sequence $x_n \in E$ such that $|f(x_n)| \geq n \|x_n\|$ for $n = 1, 2, \ldots$. By linearity,

$$\left| f\left(\frac{x_n}{n \|x_n\|} \right) \right| \geq 1, \quad n = 1, 2, \ldots.$$

But we know $\lim\limits_{n\to\infty} \frac{x_n}{n\|x_n\|} = 0$ and $f(0) = 0$, hence $f(\cdot)$ is not continuous at 0. ⊛

[1] In fact, it is Lipschitz continuous.

2.1.2 Dual Spaces and Hyperplanes

Indeed, we have a special name for the space of all bounded linear functionals called dual spaces due to its importance.

Definition 2.1.3 (Dual space)**.** Let E be a normed space, then the space of all bounded linear functionals $f(\cdot)$ on E is called the *dual space* E^* of E.

The dual space is also a normed space with norm

$$\|f\| := \sup_{\|x\|=1} |f(x)|\,,$$

which is in fact a Banach space.

Remark. E^* is a Banach space even if the original E is not.

This definition of $\|\cdot\|_{E^*}$ implies $|f(x)| \leq \|f\|\|x\|$ for $x \in E$, $f \in E^*$, and $\|f\|$ is the smallest number in this inequality that makes it valid for all $x \in X$.

Definition 2.1.4 (Hyperplane)**.** Given a linear space E, a subspace $H \subseteq E$ is a *hyperplane* if $\operatorname{codim}(H) = 1$, i.e., $\dim(E \,/\, H) = 1$.

The following question arises.

Problem 2.1.1. Does there exist a **non**-closed hyperplane?

Answer. We know that this is not the case in finite dimension. And this question is analogous to *does there exist a subset $F \subseteq \mathbb{R}$ which is **not** Lebesgue measurable?* The answer to this is yes in both cases. However, construction uses axiom of choice. ⊛

The goal is to make an equivalence between bounded linear functionals on E and *closed* hyperplanes in E. Turns out that there is a canonical correspondence between the linear functionals and the hyperplanes in E. This is clarified in the following.

Proposition 2.1.1 (Linear functionals and hyperplanes)**.** Let E be a linear space.

(a) For every linear functional f on E, $\ker(f)$ is a hyperplane in E. If E is a Banach space, and $f(\cdot)$ is bounded, then $\ker(f) = H$ is closed.

(b) If $f, g \neq 0$ are linear functionals on E such that $\ker(f) = \ker(g)$, then $f = ag$ for some $a \neq 0$.

(c) For every hyperplane $H \subseteq E$, there exists a linear functional $f \neq 0$ on E such that $\ker(f) = H$. If E is a Banach space and $\ker(f) = H$ is closed, then $f(\cdot)$ is bounded.

Lecture 6: Riesz Representation Theorem

Let's first see the proof of Proposition 2.1.1. 15 Sep. 14:30

Proof of Proposition 2.1.1. We prove them in order.

(a) Let $x, y \notin \ker(f)$, then $f(x), f(y) \neq 0$, meaning that there exists a scalar $\lambda \neq 0$ such that $f(x) = \lambda f(y)$, i.e., $x - \lambda y \in \ker(f)$. Hence, if $[x], [y] \in E \,/\, \ker(f)$, $[x] = \lambda[y]$, implying $\dim(E \,/\, \ker(f)) = 1$.

Now, if f is bounded, then f is continuous, so $\ker(f) = f^{-1}(\{0\})$ is closed.

(b) Consider the induced functionals $\widetilde{f}, \widetilde{g} \colon E \,/\, H \to \mathbb{R}$ or $\mathbb{C}$ where $H = \ker(f) = \ker(g)$. This implies

$$\dim \left({}^{E}\!\big/_{H} \right) = 1 \Rightarrow \widetilde{f} = a\widetilde{g} \text{ for some } a \neq 0 \Rightarrow f = ag.$$

(c) Assume $\dim(E \,/\, H) = 1$, so $E \,/\, H = \{a[x_0] \colon a \in \mathbb{C} \text{ (or } \mathbb{R})\}$ for some $x_0 \in E$. Then, for any $x \in E$, $[x] = a(x)[x_0]$ for some $a(x) \in \mathbb{C}$ or $\mathbb{R}$. Define $f(x) \coloneqq a(x)$, we see that f is linear and $\ker(f) = H$.

Now, if E is a Banach space and H is closed with $\dim(E \,/\, H) = 1$. Recall that $E \,/\, H$ is also a Banach space with norm $\|[x]\| = \inf_{y \in H} \|x + y\|$ for $x \in E$.[a] Let $\widetilde{f}$ be a linear functional on $E \,/\, H$. Since $\dim(E \,/\, H)$ is finite, $\widetilde{f}$ is continuous, implying $|\widetilde{f}([x])| \leq A \|[x]\|$ for all $x \in E$ for some scalar A. Finally, we define $f(x) = \widetilde{f}([x])$ for $x \in E$, then $\ker(f) = H$ and $|f(x)| \leq A \|[x]\| \leq A \|x\|$.

■

[a] We see now why we need the closure: otherwise we'll get a non-zero function with norm 0.

2.2 Representation Theorems

In concrete Banach spaces, the bounded linear functionals usually have a specific and useful form. Generally speaking, all linear functionals on function spaces (such as L^p and $C(K)$) act by integration of the function (with respect to some weight or measure). Similarly, all linear functionals on sequence spaces (such as ℓ_p) act by summation with weights.

2.2.1 Dual of Hilbert Spaces

We now start by characterizing bounded linear functionals on a Hilbert space $\mathcal{H}$.

Theorem 2.2.1 (Riesz representation theorem)**.** Let $\mathcal{H}$ be a Hilbert space. Then we have the following.

(a) For every $y \in \mathcal{H}$, then function $f(x) = \langle x, y \rangle$ for $x \in \mathcal{H}$ is a bounded linear functional on $\mathcal{H}$.

(b) If $f \colon \mathcal{H} \to \mathbb{C}$ or $\mathbb{R}$ is a bounded linear functional on $\mathcal{H}$, then there exists $y \in \mathcal{H}$ such that $f(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$. Hence, the dual $\mathcal{H}^*$ of $\mathcal{H}$ is isometric to $\mathcal{H}$.

Proof. We prove this in order.

(a) $f(x) = \langle x, y \rangle$ is clearly a linear functional. Boundedness follows form Cauchy-Schwarz inequality such that

$$|f(x)| = |\langle x, y \rangle| \leq \|x\| \|y\|,$$

and we can achieve $\|f\| = \|y\|$ by setting $x = y / \|y\|$.

> **Note.** Note that there exists x_f such that $\|x_f\| = 1$ since $\|f\| = \sup_{\|x\|=1} |f(x)| = f(x_f)$, i.e., the supremum is achieved, although we're working on an infinite dimensional space. This property does not always hold for bounded linear functionals on Banach space since the unit ball can be not compact. But this holds for Hilbert space.

(b) Let $f \colon \mathcal{H} \to \mathbb{C}$ or $\mathbb{R}$ be a bounded linear functional on $\mathcal{H}$. Let $H = \ker(f)$, which is closed from Proposition 1.7.1. Let $H^\perp$ be the orthogonal complement of H, i.e., $\mathcal{H} = H \oplus H^\perp$. Then $\dim(\mathcal{H} \,/\, H) = 1 \Rightarrow \dim(H^\perp) = 1$. Choose $y' \in H^\perp$ such that $g(x) = \langle x, y' \rangle$, which is in $\mathcal{H}^*$ from (i). Furthermore, we see that $\ker(g) = \ker(f)$, so from Proposition 1.7.1, f and g are equal up to a constant $\lambda \in \mathbb{C}$ or $\mathbb{R}$, i.e., $f = \lambda g$. It follows that

$$f(x) = \lambda g(x) = \lambda \langle x, y' \rangle = \langle x, \lambda y' \rangle \eqqcolon \langle x, y \rangle$$

for $y \coloneqq \lambda y'$, hence we're done.[a]

■

[a] We can even show that y here is unique.

In a concise form, Riesz representation theorem can be realized as $\mathcal{H}^* = \mathcal{H}$. Given a Hilbert space $\mathcal{H}$, Riesz representation theorem identifies the dual space $\mathcal{H}^*$, which can be used to show Radon-Nikodym theorem.

2.2.2 Proof of Radon-Nikodym Theorem

Riesz representation theorem can be used to give a *soft* proof of Radon-Nikodym theorem. Consider two measures μ, ν on the same σ-algebra.

> **As previously seen** (Absolutely continuous)**.** Recall that ν is called *absolutely continuous* w.r.t. μ, abbreviated as $\nu \ll \mu$, if for measurable sets A, $\mu(A) = 0$ implies $\nu(A) = 0$.

Theorem 2.2.2 (Radon-Nikodym theorem)**.** Let μ, ν be two finite measures[a] such that $v \ll \mu$, then there exists $g \geq 0$ such that g is μ-integrable and

$$\nu(A) = \int_A g \, \mathrm{d}\mu$$

for A measurable.

[a] This can be extended to σ-finite measures by decomposition.

Proof. Consider the linear functional $F\colon L^2(\mu) \to \mathbb{R}$ or $\mathbb{C}$ such that

$$F(f) = \int_\Omega f \, \mathrm{d}\mu.$$

Then we have $\|F(f)\| \leq \|f\|_2 \sqrt{\mu(\Omega)}$, i.e., F is also a bounded linear functional on $L^2(\mu+\nu)$, hence by Riesz representation theorem, there exists $h \in L^2(\mu+\nu)$ such that

$$F(f) = \int_\Omega fh \, \mathrm{d}(\mu+\nu)$$

for $f \in L^2(\mu+\nu)$, i.e.,

$$\int_\Omega f \, \mathrm{d}\mu = \int_\Omega fh \, \mathrm{d}\mu + \int_\Omega fh \, \mathrm{d}\nu \tag{2.1}$$

if $f \in L^2(\mu+\nu)$. This further implies

$$\int_\Omega fh \, \mathrm{d}\nu = \int_\Omega f[1-h] \, \mathrm{d}\mu \tag{2.2}$$

for $f \in L^2(\mu+\nu)$.

Claim. Such h satisfies $0 < h \leq 1$ μ-a.e., moreover, $(\mu+\nu)$-a.e.

Proof. We first note that $\mu(A) = 0 \Leftrightarrow \mu(A) + \nu(A) = 0$. Let $A = \{h \leq 0\}$, $f = \mathbb{1}_A$ be the characteristic function on A. Then Equation 2.1 implies

$$\int_A h(\,\mathrm{d}\mu + \,\mathrm{d}\nu) \leq 0 \Rightarrow \mu(A) = 0 \Rightarrow h > 0 \ \mu \text{ a.e.}$$

But since g is a positive function, so we also need $h \leq 1$. Again, set $B = \{h > 1\}$, $f = \mathbb{1}_B$. Then Equation 2.1 implies

$$\mu(B) = \int_B h\,(\,\mathrm{d}\mu + \,\mathrm{d}\nu) > \mu(B)$$

unless $\mu(B) = 0$. ⊛

Now, by using monotone convergence theorem, we conclude[a] that Equation 2.2 holds for all $f \geq 0$, $f \in L^2(\mu+\nu)$.[b] Finally, let $A \subseteq \Omega$ measurable and $hf = \chi_A$, from Equation 2.2,

$$\nu(A) = \int_A \frac{1-h}{h} \,\mathrm{d}\mu.$$

By letting $g := 1 - h/h \Rightarrow g = \mathrm{d}\nu/\mathrm{d}\mu$, we're done. ■

[a] Consider $f_n(t) := \min(f(t), n)$ and let $n \to \infty$.
[b] Both sides could be ∞.

Notation (Radon-Nikodym derivative)**.** g in Radon-Nikodym theorem is referred to as the *Radon-Nikodym derivative* where $g := \mathrm{d}\nu/\mathrm{d}\mu$.

Note (Uniqueness)**.** The uniqueness of Radon-Nikodym derivatives can be shown via

$$\int_A g \,\mathrm{d}\mu = 0$$

for all μ-measurable A, i.e., $g = 0$ μ-a.e.

2.2.3 Dual of L^p and ℓ_p

Another useful application of Riesz representation theorem is to characterize L^p and ℓ_p spaces and their dual L_p^* and ℓ_p^*. We first see the following.

Remark. Consider spaces $L^p(\Omega, \mu)$ for $1 \leq p \leq \infty$, then we have

$$L^q(\Omega, \Sigma, \mu) \subseteq (L^p(\Omega, \Sigma, \mu))^*$$

where $1/p + 1/q = 1$.

Proof. The easy part is that $g \in L^q$ induces a bounded linear functional on L^p by setting

$$F(f) = \int_\Omega fg \,\mathrm{d}\mu.$$

By Hölder's inequality, $|F(f)| \leq \|f\|_p \|g\|_q$, hence $\|F\| \leq \|g\|_q$. To show the equality and $\sup_{\|f\|_p} |F(f)|$ is attained for $1 < p < \infty$, we choose $f = g^{q-1} \operatorname{sgn}(g)$ since

$$F(f) = \int_\Omega |g|^q \,\mathrm{d}\mu = \|g\|_q^q,$$

and from $1/p + 1/q = 1 \Rightarrow q - 1 = q/p$, we have

$$\|f\|_p^p = \int |f|^p \,\mathrm{d}\mu = \int_\Omega |g|^q \,\mathrm{d}\mu = \|g\|_q^q \Rightarrow \|f\|_p = \|g\|_q^{q/p} = \|g\|_q^{q-1}.$$

This implies

$$F(f) = \int_\Omega |g|^q \,\mathrm{d}\mu \Rightarrow \|g\|_q^q = \|g\|_q \|f\|_p.$$

Note. We see that $\sup_{\|f\|_p=1} |F(f)|$ is attained by taking $f = \operatorname{sgn}(g)$.

⊛

In particular, we have the following.

Theorem 2.2.3 ($L^{p*} = L^q$)**.** Consider the space $L^p = L^p(\Omega, \Sigma, \mu)$ with finite measure of σ-finite measure μ. Then for $1 \leq p < \infty$ and the conjugate exponent q of q.

(a) For every weight function $g \in L^q$, integration with weight

$$G(f) \coloneqq \int_\Omega fg \,\mathrm{d}\mu$$

for $f \in L^p$ is a bounded linear functional on L^p, and its norm is $\|G\| = \|g\|_q$.

(b) Conversely, every bounded linear functional $G \in L^{p*}$ can be represented as integration with weight for some unique weight function $g \in L^q$. Moreover, $\|G\| = \|g\|_q$.

Lecture 7: Hahn-Banach Theorem

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Remark. When $p = 1$, the supremum is not attained necessarily. Consider $g \in L^\infty$, $F(f) \coloneqq \int fg \,\mathrm{d}\mu$ is dual of L^1. If $g(\cdot)$ is continuous on $\mathbb{R}$ with unique maximum, then the supremum $\sup_{\|f\|_1} |F(f)|$ is not attained. In all, for $1 \leq p \leq \infty$, L^q contained in the dual of L^p. If $1 < p \leq \infty$, then $\sup_{\|f\|_p = 1} |F(f)|$ is attained. For $p = 1$, the supremum is not necessarily attained.

Now, we're ready to prove Theorem 2.2.3.

Proof of Theorem 2.2.3. To show that the dual of L^p is L^q if $1 \leq p < \infty$ where $1/p + 1/q = 1$, we use Radon-Nikodym theorem. Suppose $E = L^p(\Omega, \Sigma, \mu)$ with $1 \leq p < \infty$ and $f \in E^*$. Just consider finite measure space, i.e., $\mu(\Omega) < \infty$. We define a measure ν on Σ by $\nu(A) \coloneqq F(\chi_A)$ for $A \in \Sigma$, where χ_A is the characteristic function of A. We see that

$$\mu(A) = 0 \Rightarrow \nu(A) = 0 \Rightarrow \nu \ll \mu,$$

and Radon-Nikodym theorem implies

$$\nu(A) = \int_A g \,\mathrm{d}\mu$$

for some $g \eqqcolon \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^1(\Omega, \Sigma, \mu)$. Note that g may not be in L^q since $q > 1$. Hence, $F(f) = \int_\Omega fg \,\mathrm{d}\mu$ for all simple function f assuming $g \geq 0$. Set $f = g^{q-1}$ with the fact that $|F(f)| \leq \|F\|_p \|f\|_p$. Recall that $q - 1 = q/p$, hence

$$\int g^q \,\mathrm{d}\mu \leq \|F\|_p \left(\int g^q \,\mathrm{d}\mu \right)^{1/p} \Rightarrow \|g\|_q^q \leq \|F\|_p \|g\|_q^{q/p} = \|F\|_p \|g\|_q^{q-1},$$

hence $\|g\|_q \leq \|F\|_p$.

Note. We assume $g \geq 0$ is because ν is a sign measure, then if we have a bounded variation function, we can just break it into $\nu^+ + \nu^-$.

■

Remark. L^1 is a subset of $L^{\infty *}$ but not equal to it. If $F \colon L^\infty(\mu) \to \mathbb{C}$ is a bounded linear functional, then if $\Omega = K$ is a compact Hausdorff space F induces a bounded linear functional on $C(K)$, i.e., the space of continuous functions on K. We see that $C(K) \subseteq L^\infty(K, \Sigma, \mu)$ where Σ is the Borel algebra on K.

2.2.4 Dual of $C(K)$

Finally, we state the following important characterization of bounded linear functionals on $C(K)$.

Theorem 2.2.4 (Riesz representation for $C(K)$). Let $E = C(K)$ be the space of continuous functions on compact Hausdorff space K. Then we have the following.

(a) For every Borel regular signed measure on K, the functional $F(f) = \int_K f \,\mathrm{d}\mu$ is a bounded linear functional on K.

(b) Every bounded linear functional on $C(K)$ can be expressed as $F(f) = \int_K f \,\mathrm{d}\mu$ for some measure μ, and $\|F\| = |\mu|(K)$, i.e., $TV(K)$.

Proof. In this case, the proof is much more difficult, and we put the proof in Section 6.1. ∎

2.3 Hahn-Banach Theorem

Hahn-Banach theorem allows one to extend continuous linear functionals f from a subspace to the whole normed space, while preserving the continuity of f. Hahn-Banach theorem is a major tool in functional analysis. Together with its variants and consequences, this result has applications in various areas of mathematics, computer science, economics and engineering.

Theorem 2.3.1 (Hahn-Banach theorem). Let E_0 be a subspace of a Banach space E. Then every $f_0 \colon E_0 \to \mathbb{R}$ or $\mathbb{C}$ has a continuous extension $f \colon E \to \mathbb{R}$ or $\mathbb{C}$ such that $\|f\| = \|f_0\|$.

Proof. We assume E is separable, otherwise we need transfinite induction. Let $\{x_n\}_{n\geq 1}$ have the property that its span is dense in E.

> **Intuition.** Separability allows us to extend f_0 one dimension at a time. Now, if we can extend f_0 such that
>
> $$E_0 \to E_0 + \operatorname{span}(\{x_1\}) \to E_0 + \operatorname{span}(\{x_1, x_2\}) \to \cdots \to E_0 + \operatorname{span}(\{x_n\}_{n\geq 1}),$$
>
> then $\|f\| = \|f_0\|$, with the final space is dense in E, we can extend f to E by continuity.

Lecture 8: Proof of Hahn-Banach Theorem and Duality

Let's first proceed the proof of Hahn-banach theorem. 20 Sep. 14:30

Proof of Theorem 2.3.1 (Continued). To extend f by 1 dimension, i.e., $E \to E + \operatorname{span}(\{x_1\})$, first note that extension is determined by a single number $\gamma = f(x_1)$ since f is a linear functional. We want that $\|f\| = \|f_0\|$ such that the linear functional $f_0 \colon E_0 \to \mathbb{R}$ extends to $f \colon D_0 + \{x_1\} \to \mathbb{R}$, i.e., we want

$$|f_0(x_0) + \lambda\gamma| \leq \|x_0 + \lambda x_1\|$$

for $x_0 \in E$, $\lambda \in \mathbb{R}$. By dividing the inequality by $\lambda \neq 0$, it's sufficient to find γ such that $|f_0(x_0) + \gamma| \leq \|x_0 + x_1\|$, $x_0 \in E_0$.

- Suppose f_0 is a real-valued function, we need

$$-\|x_0 + x_1\| \leq f_0(x_0) + \gamma \leq \|x_0 + x_1\|$$

for all $x_0 \in E_0$. Such a γ exists, provides $\|x_0 + x_1\| - f_0(x_0) \geq -\|x_0' + x_1\| - f_0(x_0')$ for all $x_0, x_0' \in E_0$. Furthermore, this is equivalent to write

$$f_0(x_0 - x_0') \leq \|x_0 + x_1\| + \|x_0' + x_1\|$$

for all $x_0, x_0' \in E_0$, i.e., $f_0(x_0 - x_0') \leq \|x_0 + x_1\| + \|-x_1 - x_0'\|$ for $x_0, x_0' \in E_0$. Recall that $\|f_0\| = 1$, we have

$$f_0(x_0 - x_0') \leq \|x_0 - x_0'\| \leq \|x_0 + x_1\| + \|-x_1 - x_0'\|.$$

- For complex valued f, consider $f\colon E \to \mathbb{C}$ be a linear functional over $\mathbb{C}$ and let $g(x) = \operatorname{Re} f(x)$. Then $g\colon E \to \mathbb{R}$ is a real-valued linear functional. We see that $f(x) = g(x) - ig(ix)$ for all $x \in E$.[a] Conversely, if $g\colon E \to \mathbb{R}$ is a real linear functional on Banach space E over $\mathbb{C}$, then $f\colon E \to \mathbb{C}$ defined by $f(x) = g(x) - ig(ix)$, $x \in E$ is a complex linear functional on E.

  But we need to be a bit careful since when we extend $f_0\colon E_0 \to \mathbb{C}$, we're extending 2 real dimensions since for $g_0 = \operatorname{Re} f_0$, we need to do

$$E_0 \to E_0 + \operatorname{span}(\{x_1\}) \to E_0 \to \operatorname{span}(\{x_1, x_2\}).$$

  Again, define $f(\cdot) = g(\cdot) - ig(i\cdot)$, we want to show $|f| = \|f_0\|$. We use the fact that for $x \in E_0 + \{\lambda x_0 \colon \lambda \in \mathbb{C}\}$,

$$e^{i\theta} f(x) = f(xe^{i\theta})$$

  for $\theta \in \mathbb{R}$. Choose θ such that $f(xe^{i\theta}) = g(xe^{i\theta})$, and since we already have $\left|g(xe^{i\theta})\right| \leq \|f_0\| \left\|xe^{i\theta}\right\|$, we see that $|f(x)| \leq \|f_0\| \|x\|$ for $x \in E_0 + \{\lambda x_1 \colon \lambda \in \mathbb{C}\}$.

The above shows that we can indeed extend one dimension at a time, and the result follows from the fact that the space is separable. ■

[a] Since $f(ix) = if(x)$, hence $g(ix) = -\operatorname{Im} f(x)$.

2.3.1 Supporting Functionals

Hahn-Banach theorem has a variety of analytic and geometric consequences. One of the basic tools guaranteed by Hahn-Banach theorem is the existence of a supporting functional $f \in X^*$ for every $x \in X$.

Theorem 2.3.2 (Supporting functional)**.** Let E be a Banach space, then for every $x \in E$, there exists $f \in E^*$ such that $\|f\| = 1$, $f(x) = \|x\|$, i.e., $\sup_{\|y\|=1} |f(y)|$ attained at $y = x$.

Proof. Consider dimension 1 space $E_0 = \operatorname{span}(x) = \{tx, t \in \mathbb{R} \text{ or } \mathbb{C}\}$. Define $f_0\colon E_0 \to \mathbb{R}$ or $\mathbb{C}$ such that $f_0(tx) = t\|x\|$, then $\|f_0\| = 1$, and Hahn-Banach theorem implies there exists $f \in E^*$ with $\|f\| = 1$. We see that $f(x) = \|x\|$ explicitly attain the norm and $\|\cdot\|$ is clearly a continuous extension of $\|\cdot\|_{E_0} = f_0$ as required. ■

Notice that we don't have uniqueness (as we don't have it in Hahn-Banach theorem) since a unit ball in L^∞ has corner, which will give multiple hyperplanes.

Remark (Geometric interpretation)**.** Let B be a unit ball $\{x \in E \colon \|x\| \leq 1\}$ in a real Banach space E. Choose $x_0 \in \partial B$ such that $\|x_0\| = 1$. Then there exists $f \in E^*$, $\|f\| = 1$, $f(x) = \|x\|$. Let $H = \ker(f) + x_0$ where H intersects B at x_0, we see that H divides E into 2 disjoint subsets, while B lies in one of which.

Proof. Since $x \in B$ and $\|x\| < 1$ implies $|f(x)| \leq \|x\| < 1$, we have $f(x) < 1$, i.e., $B \subseteq \{x \colon f(x) < 1\}$ and $E = \{x \colon f(x) < 1\} \cup H \cup \{x \colon f(x) > 1\}$. ⊛

supporting-functional theorem states that for every vector x, we indeed attain its norm on some functional $f \in E^*$, i.e., their supporting functional. But recall that the norm of a functional $f \in E^*$ is defined as

$$\|f\| \coloneqq \sup_{x \neq 0} \frac{|f(x)|}{\|x\|},$$

and in general, f will not attain its norm on some vector x. This observation leads to the following.

Corollary 2.3.1. For every vector x in a normed space E,

$$\|x\| = \max_{f \neq 0} \frac{|f(x)|}{\|f\|}$$

where the maximum is taken over all non-zero linear functionals $f \in E^*$.

Hahn-Banach theorem implies that there are enough bounded linear functionals $f \in E^*$ on every space E. One manifestation of this is the following.

Corollary 2.3.2 (Separation of points)**.** For every two vectors $x_1 \neq x_2$ in a normed space E, there exists a functional $f \in E^*$ such that $f(x_1) \neq f(x_2)$.

Proof. The supporting functional $f \in E^*$ of the vector $x = x_1 - x_2$ must satisfy

$$f(x_1 - x_2) = \|x_1 - x_2\| \neq 0,$$

as required. ■

2.3.2 Second Dual Space

Let E be a normed space, then the functionals f are designed to act on vectors $x \in E$ via

$$f\colon x \mapsto f(x).$$

But indeed, we can instead say that *vectors* $x \in E$ *act on functionals* $f \in E^*$ via

$$x\colon f \mapsto f(x).$$

Thus, a vector $x \in E$ can itself be considered as a function from E^* to $\mathbb{R}$, i.e., a functional. Furthermore, this function x is clearly linear, so we may consider x as a linear functional on E^*. Also, the inequality

$$|f(x)| \leq \|x\| \, \|f\|$$

shows that this functional is bounded, so $x \in E^{**}$. We may instead write x as x^{**} for clarity. Note that the norm of x^{**} as a functional is $\|x^{**}\|_{E^{**}} \leq \|x\|$ since

$$\|x^{**}\| = \sup_{\substack{\|f\|=1\\ f\in E^*}} |x^{**}(f)| = \sup_{\substack{\|f\|=1\\ f\in E^*}} |f(x)| \leq \|x\|\,,$$

implying that $\|x^{**}\| \leq \|x\|$ for all $x \in E$. But from supporting functional $f \in E^*$ of x, we actually have

$$\|x^{**}\| = \|x\|\,,$$

i.e., we have a *canonical embedding* of E into E^{**}. The above discussion leads to the second dual space theorem.

Theorem 2.3.3 (Second dual space)**.** Let E be a normed space. Then E can be considered as a linear subspace of E^{**}. For this, a vector $x \in E$ is considered as a bounded linear functional on E^* via the action

$$x\colon f \mapsto f(x), \quad f \in E^*.$$

To characterize the canonical embedding, we have the following definition.

Definition 2.3.1 (Reflexive space)**.** A normed space E is called *reflexive space* if $E = E^{**}$ under the canonical embedding.

Example. L^p spaces for $1 < p < \infty$ are reflexive spaces.

Proof. We know that $L^{p*} = L^q$ where $1 \leq p < \infty$ for q being the conjugate index of p. ⊛

Example. L^p spaces for $p = 1$ or ∞ are not reflexive spaces

Proposition 2.3.1. Let E be a reflexive space, then every linear functional $f \in E^*$ attains its norm on E.

Proof. By reflexivity, the supporting functional of f is a vector $x \in E^{**} = E$, thus $\|x\| = 1$ and $f(x) = \|f\|$, as required. ∎

Remark (James' theorem)**.** The converse of Proposition 2.3.1 is also true, i.e., if every functional $f \in E^*$ on a Banach space E attains its norm, then E is reflexive.

Lecture 9: Hahn-Banach Theorem for Sublinear Functions

From Proposition 2.3.1, we see that to show a Banach space E is not reflexive, it's sufficient to find $f \in E^*$ such that $\sup_{\|x\|=1} |f(x)|$ is not attained. 27 Sep. 14:30

Example. Let $C([0,1])$ be the space of continuous functions $g\colon [0,1] \to \mathbb{C}$ with $\|g\| \coloneqq \sup_{0\leq t\leq 1} |g(t)|$. Then for $f \in E^*$,

$$f(g) = \int_0^1 h(x)g(x)\,\mathrm{d}x$$

for

$$h(x) = \begin{cases} -1, & \text{if } 0 < x < \dfrac{1}{2}; \\ 1, & \text{if } \dfrac{1}{2} < x < 1. \end{cases}$$

Then $\|f\| = 1 = \sup_{\|g\|=1} |f(g)|$, but the supremum is not attained since g needs to be continuous.

2.4 Separation of Convex Sets

In this section, we can extend supporting functional theorem such that we now have it for arbitrary convex sets other than the unit ball. Since supporting functional theorem depends on Hahn-Banach theorem, so we should first generalize Hahn-Banach theorem.

2.4.1 Hahn-Banach Theorem for Sublinear Functions

By looking into the proof of Hahn-Banach theorem, we see that we only used positive homogeneity and triangle inequality of the axiom of norm, which suggests we define the following.

Definition 2.4.1 (Sublinear)**.** Let E be a linear space, a function $\|\cdot\| : E \to [0, \infty)$ is *sublinear* if

(a) $\|\lambda x\| = \lambda \|x\|$ for $\lambda \in \mathbb{R}^+$, $x \in E$.

(b) $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in E$.

Remark (Differences from norm)**.** Note that for a sublinear function to be a norm, we need

(a) $\|-x\| = \|x\|$, $x \in E$

(b) $\|x\| = 0 \Rightarrow x = 0$.

Now, we can then generalize Hahn-Banach theorem to sublinear functions.

Theorem 2.4.1 (Hahn-Banach theorem for sublinear functions)**.** Let E_0 be a subspace of a linear space E over $\mathbb{R}$. Let $\|\cdot\|$ be a sublinear functional on E, and $f_0\colon E_0 \to \mathbb{R}$ be a linear functional on E_0 satisfying $f_0(x) \leq \|x\|$ for $x \in E_0$. Then f_0 admits an extension f to E such that $f(x) \leq \|x\|$ for $x \in E$.

Proof. The idea is the same from Hahn-Banach theorem. ∎

2.4.2 Geometric Properties of Sublinear Functions

We see that by considering sublinear functionals instead of norms offers us more flexibility in geometric applications. In particular, sublinear functionals arise as Minkowski functionals of convex sets.

Definition 2.4.2 (Absorbing)**.** A subset K of a linear vector space is *absorbing* if

$$E = \bigcup_{t \geq 0} tK$$

where $tK \coloneqq \{tk \colon k \in K\}$.

Definition 2.4.3 (Minkowski functional)**.** Let K be an absorbing convex subset of a linear vector space E such that $0 \in K$. Then the *Minkowski functional* $\|\cdot\|_K$ is defined as

$$\|x\|_K \coloneqq \inf \{t > 0 \colon x/t \in K\}.$$

Proposition 2.4.1. Let K be an absorbing convex subset of a linear vector space E such that $0 \in K$. Then Minkowski functional $\|x\|_K$ is a sublinear functional on E. Conversely, let $\|\cdot\|$ be a sublinear functional on a linear vector space E, then the sub-level set

$$K = \{x \in E \colon \|x\| \leq 1\}$$

is an absorbing convex set, and $0 \in K$.

Proof. To prove the forward direction, the main observation is that since $0 \in K$ and K is convex, then $x \in K \Rightarrow tx \in K$ if $0 \leq t < 1$. To show dilation, for $\lambda > 0$,

$$\|\lambda x\| = \inf \left\{t > 0 \colon x \in \frac{t}{\lambda} K\right\} = \lambda \inf \{s > 0 \colon x \in sK\} = \lambda \|x\|.$$

To show triangle inequality, suppose $x \in tK$, $y \in sK$, then $x = tk_1$, $y = sk_2$ for some $k_1, k_2 \in K$. We then have

$$x + y = (t+s)\left(\frac{t}{t+s}k_1 + \frac{s}{t+s}k_2\right) = (t+s)k$$

for some $k \in K$ since K is convex, hence $x + y \in (t+s)K$, we then have $\|x+y\| \leq \|x\| + \|y\|$.

Now, if $\|\cdot\|$ is sublinear, then $K = \{x \in E \colon \|x\| \leq 1\}$ is absorbing, convex and $0 \in K$.[a] ∎

[a] $0 \in K$ since $\|0\| = 0$, while the convexity comes from the triangle inequality.

Remark. If $K \neq -K$, then $\exists x \in E$ with $\|x\| \neq \|-x\|$. If $K = E$, then $\|\cdot\| \equiv 0$.

2.4.3 Separation of Convex Sets

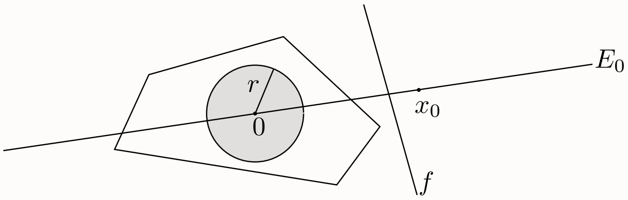
Hahn-Banach theroem has some remarkable geometric implications, which are grouped together under the name of *separation theorems*. Under mild topological requirements, these results guarantee that two convex sets A, B can always be separated by a hyperplane.

Theorem 2.4.2 (Separation of a point from a convex set)**.** Let K be an open convex subset of a normed space E and $x_0 \notin K$. Then there exists a continuous linear functional $f \colon E \to \mathbb{R}$ with $f \neq 0$ and $f(x) < f(x_0)$ for $x \in K$.

Proof. By translation, we can assume without loss of generality that $0 \in K$. Since K is open, it is absorbing. Now, let $\|\cdot\|_K$ be the Minkowski functional, then

$$\|x\|_K \leq \frac{1}{r}\|x\|$$

for $x \in E$ if $B(0,r) \subseteq K$.



Proceed as in supporting functional theorem for unit ball, we define f_0 on $\operatorname{span}(\{x_0\})$ by

$$f_0(tx_0) = t\|x_0\|_K$$

for $t \in \mathbb{R}$. Then if $E_0 = \{\lambda x_0 \colon \lambda \in \mathbb{R}\}$, $f_0(x) \leq \|x\|_K$ for $x \in E_0$ (i.e., $\|\cdot\|_K$ dominates f_0) since for $t \geq 0$,

$$f_0(tx_0) = t\|x_0\|_K = \|tx_0\|_K;$$

while for $t \leq 0$,

$$f_0(tx_0) = t\|x_0\|_K \leq 0 \leq \|tx_0\|_K.$$

Then from Hahn-Banach theorem, we can extend f_0 to $f \colon E \to \mathbb{R}$ such that

$$f(x) \leq \|x\|_K \leq \frac{1}{r}\|x\|$$

for $x \in E$, hence $f \in E^*$. For separation, we see that if $x \in K$ (hence in E),

$$f(x) \leq \|x\|_K \leq 1 \leq \|x_0\|_K = f_0(x_0) = f(x_0),$$

hence $f(x) \leq f(x_0)$. To get a strict separation, since K is open, so $x + tv \in K$ for $x \in K$ and some $t > 0$ and all v with $\|v\| = 1$. Hence, for all $t = t_x > 0$, we have

$$f(x+tv) \leq f(x_0) \Rightarrow f(x) + t \sup_{\|v\|=1} f(v) \leq f(x_0).$$

With the fact that $f \neq 0$, so $\|f\| = \sup_{\|v\|=1} f(v) \neq 0$, we conclude that $f(x) < f(x_0)$. ■

A more general version holds.

Theorem 2.4.3 (Separation of convex sets)**.** Let A, B be disjoint convex subsets of a Banach space E.

(a) If A is open, then there $\exists f \colon E \to \mathbb{R}$ such that $f(a) < f(b)$ for $a \in A$, $b \in B$.

(b) If A, B are closed and B is compact, then there $\exists f \colon E \to \mathbb{R}$ such that $\sup_{a\in A} f(a) < \inf_{b\in B} f(b)$.

Proof. We have the following.

(a) Let $K = A - B = \{a - b \colon a \in A, b \in B\}$, we then see that K is open, convex and $0 \notin K$. Since we can separate a point from a convex set, there exists $f \in E^*$ such that

$$f(a-b) < f(0) = 0$$

for $a \in A$, $b \in B$, hence $f(a) < f(b)$ for $a \in A$, $b \in B$.

(b) Let A be closed, B be compact. Then we have

$$d(A,B) = \inf\{\|x-y\| : x \in A, y \in B\} = r > 0.$$

Define $A_\delta := \{x \in E : d(x, A) < \delta\}$ where A_δ is open. By setting $\delta := r/2$, we have $A_\delta \cap B = 0$. From (a), we see that there exists $f \in E^*$ such that $f(x) < f(y)$ for $x \in A_\delta$, $y \in B$. Then $a \in A$ implies $a + \delta/2v \in A_\delta$ for some v such that $\|v\| = 1$, hence

$$f(a + \delta/2v) < f(b)$$

for $b \in B$. So

$$f(a) + \frac{\delta}{2} f(v) < f(b)$$

for $b \in B$, $\|v\| = 1$. Take the supremum over $\|v\| = 1$, we have $\sup_{\|v\|=1} |f(v)| = \delta > 0$, implying $f(a) < f(b) - \delta$, $a \in A$, $b \in B$. Finally, we have

$$\sup_{a \in A} f(a) < \inf_{b \in B} f(b).$$

■

Lecture 10: Adjoint Operators and Ergodic Theorem

Before ending this section, we have this final characterization of convex sets: they're intersections of half-spaces! 29 Sep. 14:30

Definition 2.4.4 (Half-space). A *half-space* $H \subseteq E$ has the form of

$$H = \{x \in E : f(x) \leq \lambda\}$$

for $f \in E^*$ and $\lambda \in \mathbb{R}$, i.e., it is what lies on one side of a hyperplane.

Corollary 2.4.1. Let $K \subseteq E$ be a closed convex set, then K is the intersection of all half-spaces containing K.

Proof. Firstly, K is trivially contained in the intersection of the half-spaces that contain K. Denote such an intersection as S, then we have $K \subseteq S$. On the other hand, to show $K \supseteq S$, if $x_0 \notin K$, we show that there's a half-space contains K but not x_0, hence $x_0 \notin S$ too, i.e., $S \subseteq K$.

From separation of convex sets theorem with $A = K$ and $B = \{x_0\}$, there exists $f \in E^*$ such that $\lambda := \sup_{k \in K} f(k) < f(x_0)$. We then see that the half-space $\{x \in E : f(x) \leq \lambda\}$ contains K but not x_0. ■

2.5 Bounded Linear Operators

Turns out that we can generalize the notion of linear functionals $f: E \to \mathbb{R}$ or $\mathbb{C}$ by further abstracting out the range by another Banach space. As one can imagine, several results for linear operators will be generalizations of those we have already seen for linear functionals, but there'll be important differences though. For example, a natural extension of Hahn-Banach theorem fails for linear operators.

Firstly, the operator norm is defined as follows, which is a norm on bounded linear operators.

Definition 2.5.1 (Operator norm). Given an operator $T: E \to F$ acting between normed spaces E and F, its *operator norm* is defined as

$$\|T\| := \sup_{\substack{\|x\|=1 \\ x \in E}} \|Tx\|.$$

2.5.1 Continuity and Boundedness

As for Definition 2.1.2, we have the following.

Definition (Bounded linear operator). Let X, Y be two Banach spaces and let T be a linear operator between X and Y. Then we say T is a *bounded linear operator* if $\|T\| < \infty$.

Remark (Bounded operator). We can also talk about boundedness of a(n) (nonlinear) operator T just the same as requiring $\|T\| < \infty$.

As before, given Definition 2.5.1, we always have

$$\|Tx\| \leq \|T\|\|x\|$$

for a linear operator $T\colon X \to Y$, $x \in X$.

Definition 2.5.2 (Lipschitz). The operator T is called *Lipschitz* if for $x, y \in E$,

$$\|Tx - Ty\| \leq \|T\|\|x - y\|.$$

We see that for a bounded linear operator, it's Lipschitz as well.

Remark (Continuity and Boundnedness). Same as linear functionals, the continuity and boundedness of linear operators are equivalent.

2.5.2 Space of Operators

Let X and Y be normed space, and let $\mathcal{L}(X, Y)$ be the space of bounded linear operators $T\colon X \to Y$, then $\mathcal{L}(X, Y)$ is a Banach space under the norm $T \to \|T\|$.

Example. The dual space of E is just $E^* = \mathcal{L}(E, \mathbb{R})$.

Remark. In particular, we have

(a) $\|T\| = 0 \Leftrightarrow T = 0$.

(b) $\|\lambda T\| = |\lambda| \, \|T\|$ for $\lambda \in \mathbb{R}$ or $\mathbb{C}$, $T \in \mathcal{L}(X, Y)$.

(c) $\|T + S\| \leq \|T\| + \|S\|$, $T, S \in \mathcal{L}(X, Y)$.

(d) $\|TS\| \leq \|T\|\|S\|$, $T, S \in \mathcal{L}(X, Y)$.

2.5.3 Adjoint Operators

The concept of adjoint operators is a generalization of matrix transpose in linear algebra. Recall that if $A = (a_{ij})$ is an $n \times n$ matrix with complex entries, then the Hermitian transpose of A is an $n \times n$ matrix $A^* = (\overline{a_{ij}})$. The transpose thus satisfies the identity

$$\langle A^* x, y \rangle = \langle x, Ay \rangle$$

for $x, y \in \mathbb{C}^n$. We now extend this to linear operators.

Definition 2.5.3 (Adjoint operator). Let $T \in \mathcal{L}(X, Y)$, the *adjoint* $T^* \in \mathcal{L}(Y^*, X^*)$ of T is defined as

$$T^* f\colon X \to \mathbb{R} \text{ or } \mathbb{C}$$

for $f \in Y^*$, and $T^* f(x) = f(Tx)$ for $x \in X$.

We should note that T^* is indeed a bounded linear operator since

$$|T^* f(x)| = |f(Tx)| \leq \|f\|\|Tx\| \leq \|f\|\|T\|\|x\|$$

for $x \in X$, hence $T^* f$ is a linear functional where

$$\|T^* f\| = \sup_{\|x\|=1} |T^* f(x)| \leq \sup_{\|x\|=1} \|f\| \|T\| \|x\| = \|f\| \|T\|,$$

so $T^* f \in X^*$ and $\|T^* f\| \leq \|T\| \|f\|$. This implies $T^* \colon Y^* \to X^*$ with T^* being a linear operator and T^* is bounded with

$$\|T^*\| \leq \|T\|.$$

In fact, we can achieve equality, which is shown in Proposition 2.5.1.

Proposition 2.5.1. For every $T \in \mathcal{L}(X, Y)$, the adjoint T^* is in $\mathcal{L}(Y^*, X^*)$ with $\|T^*\| = \|T\|$.

Proof. Since

$$\begin{aligned}\|T^*\| = \sup_{\|f\|_{Y^*}=1} \|T^* f\|_{X^*} &= \sup_{\|f\|_{Y^*}=1} \sup_{\|x\|_X=1} |T^* f(x)| \\ &= \sup_{\|f\|_{Y^*}=1} \sup_{\|x\|_X=1} |f(Tx)| = \sup_{\|x\|_X=1} \sup_{\|f\|_{Y^*}=1} |f(Tx)|.\end{aligned}$$

By choosing f to be a supporting functional of Tx, $\sup_{\|f\|_{Y^*}=1} |f(Tx)| = \|Tx\|_Y$, hence

$$\|T^*\| = \sup_{\|x\|_X=1} \|Tx\|_Y = \|T\|.$$

■

Let's look at some properties of adjoint operators. Let $T, S \in \mathcal{L}(X, Y)$ and $T^*, S^* \in \mathcal{L}(Y^*, X^*)$, then

(a) $(aT + bS)^* = aT^* + bS^*$, $a, b \in \mathbb{R}$ or $\mathbb{C}$. Also, $(aT)^* f(x) = f(aTx) = af(Tx) = aT^* f(x)$.

(b) $(ST)^* = T^* S^*$. This implies that if $T \in \mathcal{L}(X, X)$ is invertible, then $T^* \in \mathcal{L}(X^*, X^*)$ is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Remark (Adjoint operators on Hilbert spaces). Specialize to Hilbert space $\mathcal{H}$, then by Riesz representation theorem, $\mathcal{H}^* \equiv \mathcal{H}$, i.e., $f \in \mathcal{H}^* \Leftrightarrow \exists y \in \mathcal{H}$ such that $f(x) = \langle x, y \rangle$ for $x \in \mathcal{H}$. Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$, and $T^* \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*)$ with

$$T^* f(x) = f(Tx) = \langle Tx, y \rangle$$

for $x, y \in \mathcal{H}$, $f \in \mathcal{H}^*$. By writing $T^* f(x) = \langle x, T^* y \rangle$, which defined $T^* y \colon \mathcal{H} \to \mathcal{H}$, hence $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for $x, y \in \mathcal{H}$. Clearly, T^* is a bounded linear operator on $\mathcal{H}$, i.e., $T^* \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*)$ since

$$\|T^*\| = \sup_{\|y\|=1} \|T^* y\| = \sup_{\|y\|=\|x\|=1} \langle x, T^* y \rangle = \sup_{\|y\|=\|x\|=1} \langle Tx, y \rangle = \|T\|$$

just like Proposition 2.5.1. We see that $T^* \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*) \Rightarrow T^* \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ via Riesz representation. Note that if $T^* \in \mathcal{L}(\mathcal{H}, \mathcal{H})$,

$$(aT)^* = \overline{a} T^*$$

for $a \in \mathbb{C}$, where for T defined on Banach space, $(aT)^* = aT^*$.

Just as with Hilbert space, we have a generalized notion of orthogonality, which we call annihilator.

Definition 2.5.4 (Annihilator). Let $A \subseteq X$ where X is a Banach space, then the *annihilator* $A^\perp$ of A is a subset of X^* defined as

$$A^\perp \coloneqq \{f \in X^* \colon f(x) = 0, x \in A\}.$$

Note. $A^\perp$ is a closed linear subspace of X^*.

Proposition 2.5.2. Given two Banach spaces X and Y, let $T \in \mathcal{L}(X, Y)$ and $T^* \in \mathcal{L}(Y^*, X^*)$. Then $(\operatorname{Im} T)^\perp, \ker(T^*) \subseteq Y^*$ satisfy

$$(\operatorname{Im} T)^\perp = \ker(T^*).$$

Proof. Since $f \in (\operatorname{Im} T)^\perp \Leftrightarrow f(Tx) = 0$ for all $x \in X$, i.e., $T^* f(x) = 0 \Leftrightarrow T^* f = 0 \Leftrightarrow f \in \ker(T^*)$, proving the result. ■

Corollary 2.5.1. Let $\mathcal{H}$ be a Hilbert space, and $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$. Then the orthogonal decomposition holds, i.e.,

$$\mathcal{H} = \overline{\operatorname{Im} T} \oplus \ker(T^*).$$

Proof. By Proposition 2.5.2, $\ker(T^*) = (\operatorname{Im} T)^\perp$. And since $\mathcal{H}$ is Hilbert space, $(\overline{\operatorname{Im} T})^\perp = (\operatorname{Im} T)^\perp$ [a] hence $(\overline{\operatorname{Im} T})^\perp = \ker(T^*)$. Then by using orthogonality principle, the proof is complete. ■

[a] Since if $E \subseteq \mathcal{H}$, $(E^\perp)^\perp = \overline{E}$.

2.5.4 Ergodic Theory

We now see an application on ergodic theorems. Ergodic theorems allow one to compute space averages as time averages. Given a probability space $(\Omega, \mathcal{F}, P)$ with $P(\Omega) = 1$, let $T \colon \Omega \to \Omega$ be a measurable map, i.e., $T^{-1}A \in \mathcal{F}$ if $A \in \mathcal{F}$. Then, we define the following.

Definition 2.5.5 (Measure-preserving)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A transformation $T \colon \Omega \to \Omega$ is called *measure-preserving* if

$$P(T^{-1}A) = P(A)$$

for $A \in \mathcal{F}$, where $T^{-1}A = \{\omega \in \Omega \colon T\omega \in A\}$.

Let's first see some examples which illustrate the so-called *time and space averages*. We start with simple dynamical systems corresponding to rotation.

Example (Rotation)**.** Let $\Omega = [0, 1]$, P be the Lebesgue measure and $\mathcal{F}$ be Borel sets. Given $\lambda \in \mathbb{R}$, define

$$T\omega = \omega + \lambda \bmod 1.$$

This is equivalent to rotation on the unit circle through an angle $2\pi\lambda$, and we see that T is measure-preserving and one-to-one, and T^{-1} exists.

Example (Shift Operator)**.** Let $\Omega = [0, 1]$, P be the Lebesgue measure and $\mathcal{F}$ be Borel sets. Now, let

$$T\omega = 2\omega \bmod 1.$$

Then we see that T is just the shift operator on the binary representation, i.e., given $\omega = \sum_{j=1}^\infty \frac{a_j}{2^j}$ for $a_j = 0$ or 1, then

$$T\omega = \sum_{j=1}^\infty \frac{a_{j+1}}{2^j}.$$

Now, let the *dyadic interval* $I_{n,k}$ be defined as

$$I_{n,k} := \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$$

for $1 \leq k \leq 2^n$, we have $T^{-1}I_{n,k} = I_{n+1,k} \cup I_{n+1,k+2^n}$, hence $P(T^{-1}I_{n,k}) = P(I_{n,k})$ for all dyadic intervals $I_{n,k}$. This implies

$$P(T^{-1}O) = P(O)$$

for all $O \in \mathcal{F}$, hence T is measure-preserving, but not one-to-one. In fact, T is a two-to-one mapping. The action of T is $[0,1/2] \xrightarrow{T} [0,1]$, $[1/2,1] \xrightarrow{T} [0,1]$. We see that T doubles the length of a dyadic interval. To summarize,

- T is measure-preserving since it is two-to-one.
- T is an expanding map, which is called hyperbolic.

Lecture 11: Ergodic Theorem and Open Mapping

Now, we're ready to discuss ergodic theorem formally. Suppose $T\colon \Omega \to \Omega$ is measure-preserving, we can associate operator U on $L^2(\Omega)$ by defining $Uf(\omega) = f(T\omega)$ for $f \in L^2(\Omega)$ and $\omega \in \Omega$. Notice that 4 Oct. 14:30

$$\int_\Omega f(T\omega)\,\mathrm{d}\mu(\omega) = \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega)$$

for all $f \in L^1(\Omega)$,[2] so for $\varphi \in L^2(\Omega)$, $U\varphi(\omega) = \varphi(T\omega)$ and since

$$\langle U\varphi, U\psi\rangle = \int_\Omega \varphi(T\omega)\psi(T\omega)\,\mathrm{d}\mu(\omega) = \int_\Omega \varphi(\omega)\psi(\omega)\,\mathrm{d}\mu(\omega) = \langle \varphi, \psi\rangle$$

for $\varphi, \psi \in L^2(\Omega)$, we see that U is a bounded linear operator on $\mathcal{H} = L^2(\Omega)$ with $\|U\| = 1$, $\|U\varphi\| = \|\varphi\|$, $\varphi \in \mathcal{H}$. In addition, for $\varphi, \psi \in \mathcal{H}$, $\langle U\varphi, U\psi\rangle = \langle \varphi, \psi\rangle$ implies $\langle U^*U\varphi, \psi\rangle = \langle \varphi, \psi\rangle$, which further implies $U^*U = I$, so U is one-to-one. Let's first see one more definition before we proceed.

Definition 2.5.6 (Unitary operator). A *unitary operator* is a bounded linear operator $U\colon \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ such that U is surjective and for all $x, y \in \mathcal{H}$,

$$\langle Ux, Uy\rangle_{\mathcal{H}} = \langle x, y\rangle_{\mathcal{H}}.$$

Notice that U is not necessarily onto. However, if U is indeed onto, then $UU^* = U^*U = I$, implying that U is a unitary operator on $\mathcal{H}$ and invertible.

Note. U is invertible if and only if T is one-to-one.

Proof. Since U just need to be onto for U being invertible, with $U^*\varphi(\omega) = \varphi(T^{-1}\omega)$ for $\omega \in \Omega$, if T is one-to-one then T^{-1} is onto, implying U^* is onto, so is U. ⊛

Remark. $T\colon \Omega \to \Omega$ is one-to-one implies T is almost onto.

Proof. Let A be a set such that $T(\Omega) \subset A$, and hence $T^{-1}A = \Omega$ so $P(T^{-1}A) = P(\Omega) = 1$, implying that $P(A) = 1$, hence $P(\Omega \setminus A) = 0$. ⊛

In the case T is not invertible (e.g. a 2-1 mapping), one might expect a similar formula for U^*. In the shift operator example, $T_1\colon [0,1/2] \to [0,1]$, $T_2\colon [1/2,1] \to [0,1]$, and T_1, T_2 are invertible, we have

$$U^*\varphi(\omega) = \frac{1}{2}\left(\varphi(T_1^{-1}\omega) + \varphi(T_2^{-1}\omega)\right).$$

Definition 2.5.7 (Ergodic transformation). A one-to-one, measure-preserving transformation T is *ergodic* if the only functions $f \in L^2(\Omega, \mathcal{F}, P)$ which satisfy $f(T\omega) = f(\omega)$ for almost all $\omega \in \Omega$ are the constant functions.

Remark (Eigenfunction). Phrasing differently, a measure-preserving mapping $T\colon \Omega \to \Omega$ is ergodic if and only if the only eigenfunction $\varphi \in L^2(\Omega)$ of the corresponding operator U is the constant

[2] This is true by letting $f = \mathbb{1}_A$ and then extend to $L^1(\Omega)$.

function, i.e. $U\varphi = \varphi$ implying φ is a constant.

Lemma 2.5.1. A measure-preserving mapping $T\colon \Omega \to \Omega$ is ergodic if and only if invariant sets of T have probability 0 or 1, i.e. if $A \in \mathcal{F}$ satisfies

$$P((A - T^{-1}A) \cup (T^{-1}A - A)) = 0,$$

then $P(A) = 0$ or $P(A) = 1$.

Proof. Assume T is not ergodic, then there exists $\varphi \in L^2(\Omega)$ such that $U\varphi = \varphi$. Hence, we can find $a, b \in \mathbb{R}$, $a < b$ such that $A = \{\omega \in \Omega \colon a < \varphi(\omega) < b\}$ has $0 < P(A) < 1$. However,

$$T^{-1}A = \{\omega :\ T\omega \in A\} = \{\omega :\ a < \varphi(T\omega) < b\} = \{\omega :\ a < \varphi(\omega) < b\} = A,$$

and thus A is invariant.

Conversely, suppose $A \in \mathcal{F}$, we have $A = T^{-1}A$ up to measure-zero sets and $0 < P(A) < 1$, then $\varphi = \mathbb{1}_A$ satisfies $U\varphi = \varphi \in L^2(\Omega)$ with the fact that φ is not constant., proving the result. ■

Proposition 2.5.3. Suppose $T\colon \Omega \to \Omega$ is measure-preserving and $\varphi \in L^2(\Omega)$, $\mathbb{E}[\varphi] = 0$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \varphi(T^n\cdot) \to 0$$

in $L^2(\Omega)$.

Proof. Note it suffices to assume $\mathbb{E}[\varphi] = 0$. We want to show

$$\lim_{N\to\infty} \frac{1}{N}[I + U + U^2 + \cdots + U^{N-1}]\varphi(\cdot) = 0$$

in $L^2(\Omega)$. If φ is orthogonal to the constant function. Since $\mathbb{E}[\varphi] = 0$, then $\langle \varphi, 1\rangle = 0$. Define a *derivative* operator on $L^2(\Omega)$ such that

$$D\varphi = (U - I)\varphi = \varphi(T\cdot) - \varphi(\cdot).$$

Using the fundamental theorem of calculus argument,

$$[I + U + U^2 + \cdots + U^{N-1}]D\varphi = (U^N - I)\varphi.$$

Hence,

$$\left\| \frac{I + U + U^2 + \cdots + U^{N-1}}{N}\varphi \right\| \leq \frac{2\,\|\psi\|}{N}$$

if $\varphi = D\psi$. In that case that as $N \to \infty$ is zero, i.e. if $\varphi \in \operatorname{Im}(D) \subset \mathcal{H} = L^2(\Omega)$, then we're done. Note also that

$$\left\| \frac{I + U + U^2 + \cdots + U^{N-1}}{N} \right\| \leq 1$$

since $\|U\| = 1$. Hence, converge to zero if $\varphi \in \overline{\operatorname{Im}(D)}$, which implies that there exists $\varphi_\epsilon \in \operatorname{Im}(D)$ such that $\|\varphi_\epsilon - \varphi\| < \epsilon$, i.e.,

$$\left\| \frac{I + U + \cdots + U^{N-1}}{N}(\varphi_\epsilon - \varphi) \right\| < \epsilon.$$

Recall $\overline{\operatorname{Im}(D)} \oplus \ker(D^*) = \mathcal{H} = L^2(\Omega)$. It suffices to show $\ker(D^*)$ is spanned by constant functions. Note T is ergodic implies $\ker(D)$ is spanned by constants, we have $D\varphi = 0 \Leftrightarrow U\varphi = \varphi$, and

$$(D^*\varphi = 0 \Leftrightarrow U^*\varphi = 0) \Rightarrow (\langle \varphi, U^*\varphi, \varphi\rangle = \langle \varphi, \varphi\rangle).$$

Therefore, we have $\langle U\varphi, \varphi\rangle = \langle \varphi, \varphi\rangle$, and also,

$$\int \varphi(T\omega)\varphi(\omega)\,\mathrm{d}P(\omega) = \int \varphi(\omega)^2\,\mathrm{d}\omega = \int \varphi(T\omega)^2\,\mathrm{d}\omega,$$

which implies

$$\frac{1}{2}\int [\varphi(T\omega)^2 + \varphi(\omega)^2]\,\mathrm{d}P(\omega) - \int \varphi(T\omega)\varphi(\omega)\,\mathrm{d}P(\omega) = 0.$$

i.e. $\frac{1}{2}\int [\varphi(T\omega) - \varphi(\omega)]^2\,\mathrm{d}P(\omega) = 0$, which means

$$\varphi(T\omega) = \varphi(\omega),\ \omega \in \Omega.$$

i.e. $\varphi \equiv$ constant by ergodicity. ■

Theorem 2.5.1 (von Newmann ergodic theorem)**.** Suppose $T\colon \Omega \to \Omega$ is measure-preserving, then for any $\varphi \in L^2(\Omega)$, one has

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \varphi(T^n\cdot) = \int_\Omega \varphi(\omega)\,\mathrm{d}P(\omega).$$

Remark. Convergence is in the $L^2(\Omega)$ sense, i.e. mean square.

Chapter 3

Main Principles of Functional Analysis

In this chapter, we'll study three of the fundamental theorems in functional analysis, which together with Hahn-Banach theorem, form the main principles of functional analysis. Those are the open mapping theorem, closed graph theorem and the uniform boundedness principle.

3.1 Open Mapping Theorem

Suppose $T\colon X \to Y$ is a bounded linear operator on Banach spaces, and T is injective and surjective, i.e. $T^{-1}\colon Y \to X$ exists. We'll soon see that the open mapping theorem implies T^{-1} is a bounded operator, where the main argument relies on Baire category theorem.

Definition 3.1.1 (Nowhere dense). A set S in a metric space M is *nowhere dense* if its closure $\overline{S}$ has empty interior.

Example. The Cantor set is a nowhere dense set.

Lecture 12: Open Mapping Theorem

Let's start with Baire category theorem, which essentially states that every complete metric space is a set of second category.[1] 6 Oct. 14:30

Proposition 3.1.1 (Baire category theorem). A complete metric space M is never the union of a countable number of nowhere dense sets.

Proof. We prove this by contradiction. Assume $M = \bigcup_{n=1}^{\infty} A_n$ with each A_n nowhere dense. Since A_1 is nowhere dense, so we can find $x_1 \in M - \overline{A}_1$. Furthermore, since $\overline{A}_1$ is closed, so we can find open ball B_1 centered at x_1 with radius less or equal to 1 such that $B_1 \cap A_1 = \varnothing$.

Similarly, A_2 is nowhere dense, so there exists $x_1 \in B_1 - \overline{A}_2$, with $\overline{A}_2$ closed, we can still find ball B_2 centered at x_2 with radius less or equal to $1/2$ such that

$$x_2 \in B_2 \subseteq \overline{B}_2 \subseteq B_1$$

and $B_2 \cap A_2 = \varnothing$. By induction, we can find a sequence $\{x_n\}_{n=1}^{\infty}$ and open balls B_n such that

$$x_{n+1} \in B_{n+1} \subseteq \overline{B}_{n+1} \subseteq B_n$$

where B_n has radius smaller than $1/2^{n-1}$ and $B_n \cap A_n = \varnothing$.

Now, since the sequence $\{x_n\}$ is Cauchy and M is complete, we know that $x_n \to x_\infty \in M$, so

[1] See Meagre set. Notice that this is nothing to do with the category theory.

$x_\infty \in B_n$ for all n and hence $x_\infty \notin A_n$ for all n. This implies

$$M \neq \bigcup_{n=1}^{\infty} A_n,$$

which is a contradiction $\lightning$ ■

We can now prove the central theorem in functional analysis, the open mapping theorem.

Theorem 3.1.1 (Open mapping theorem). Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Assume T is surjective, i.e., $T(X) = Y$, then T maps open sets in X to open sets in Y.

Proof. Let $B_X \coloneqq \{x \in X \mid \|x\| \leq 1\}$ be a unit ball in X, similarly B_Y be a unit ball in Y.

Claim. It's sufficient to show $T(B_X) \supseteq \epsilon B_Y$ for some $\epsilon > 0$.

Proof. To see this, let $U \subseteq X$ be an open set and $y \in TU$. We need to show TU contains a neighborhood of y. Let $x \in U$ such that $Tx = y$. Since U is open, so there exists $\delta > 0$ such that $U \supseteq x + \delta B_X$, so

$$TU \supseteq T(x + \delta B_X) = y + \delta T(B_X) \supseteq y + \delta' B_Y$$

for some $\delta' = \delta\epsilon > 0$, i.e., TU contains a neighborhood of y. ⊛

We now show $TB_X \supseteq \epsilon B_Y$ for some $\epsilon > 0$. Observe that $X = \bigcup_{n=1}^{\infty} nB_X$, hence

$$Y = TX = \bigcup_{n=1}^{\infty} nT(B_X).$$

From Baire category theorem, we know that there exists $n \geq 1$ such that $\overline{nT(B_X)}$ has non-empty interior, i.e., $\overline{TB_X}$ has non-empty interior too. Hence, there exists $y \in Y$, $\delta > 0$ such that $y + \delta B_Y \subseteq \overline{TB_X}$. With $TX = Y$, there exists $x \in X$ such that $Tx = y$, hence $\delta B_Y \subseteq \overline{T(B_X - \{x\})}$. Since $B_X - \{x\} \subseteq nB_X$ for some $n \geq 1$, meaning that $\delta B_Y \subseteq n\overline{TB_X}$, implying $\overline{TB_X} \supseteq \epsilon B_Y$ for some $\epsilon > 0$. Finally, we show that $\overline{TB_X} \subseteq T(2B_X)$, which will imply

$$TB_X \supseteq \frac{1}{2}\overline{TB_X} \supseteq \frac{\epsilon}{2} B_Y,$$

completes the proof. To see this, we use a scaling argument. Let $y \in \overline{TB_X}$, then there exists $x_1 \in B_X$ such that

$$y - Tx_1 \in \frac{\epsilon}{2} B_Y \subseteq \overline{T\frac{1}{2}B_X}.$$

We can then choose $x_2 \in \frac{1}{2}B_X$ such that

$$y - Tx_1 - Tx_2 \in \frac{\epsilon}{4} B_Y \subseteq \overline{T\frac{1}{2^2}B_X}.$$

By induction, we can construct a sequence $\{x_n\}_{n\geq 1}$ such that

$$x_n \in \frac{1}{2^{n-1}} B_X, \quad y - \sum_{j=1}^{n} Tx_j \in \frac{\epsilon}{2^n} B_Y.$$

Then, $x = \sum_{n=1}^{\infty} x_n \in 2B_X$ where $Tx = y$. ■

3.1.1 Inverse Mapping Theorem

As an immediate consequence of the open mapping theorem, we have the inverse mapping theorem.

Theorem 3.1.2 (Inverse mapping theorem). Let $T \colon X \to Y$ be a bounded linear operator between Banach spaces X and Y which is both injective and surjective. Then T has a bounded inverse $T^{-1} \in \mathcal{L}(Y, X)$.

Proof. Since open mapping theorem states that the preimages of open sets under T^{-1} are open, hence T^{-1} is continuous. ■

Inverse mapping theorem is used to establish stability of solutions of linear equations. Consider a linear equation in x in a Banach space

$$Tx = b$$

for $T \in \mathcal{L}(X, Y)$ and $b \in Y$. Assume that a solution x exists and is unique for every b, then, from inverse mapping theorem, we see that the solution $x = x(b)$ is continuous w.r.t. b. In other words, the solution is stable under perturbations of b. In case T is not injective but is surjective, we can still apply inverse mapping theorem to the injectivization of T as follows.

Corollary 3.1.1 (Surjective operators are essentially quotient maps). Let X, Y be Banach spaces. Then every surjective bounded linear operator $T \in \mathcal{L}(X, Y)$ is a composition of a quotient map and an isomorphism. Specifically,

$$T = \widetilde{T} q,$$

where $q \colon X \to X \,/\, \ker(T)$ is the quotient map, $\widetilde{T} \colon X \,/\, \ker(T) \to Y$ is an isomorphism.

Proof. Let $\widetilde{T}$ be the injectivization of T then by construction, $T = \widetilde{T} q$ and $\widetilde{T}$ is injective. Since T is surjective, $\widetilde{T}$ is also surjective. Hence, by inverse mapping theorem, $\widetilde{T}$ is an isomorphism. ■

3.1.2 Isomorphic Embeddings

Finally, as we know, the kernel of every bounded linear operators $T \in \mathcal{L}(X, Y)$ is always a closed subspace from Proposition 2.1.1, while the image of T may or may not be closed. We can also characterize this.

Proposition 3.1.2 (Isomorphic embedding). Given two Banach spaces X, Y and $T \in \mathcal{L}(X, Y)$, the following are equivalent.

(a) T is injective and $\operatorname{Im}(T)$ is closed.

(b) T is bounded below, i.e., $\exists c > 0$, $\|Tx\| \geq c\|x\|$ for all $x \in X$.

Proof. To show that (a) implies (b), we see that $T^{-1} \colon \operatorname{Im}(T) \to X$ is bounded since $\operatorname{Im}(T)$ is Banach space, from open mapping theorem,

$$\|T^{-1} y\| \leq c^{-1} \|y\|$$

for $y \in \operatorname{Im}(T)$, $c > 0$ being some constant. Set $y \coloneqq Tx$, then $\|Tx\| \geq c\|x\|$ for $x \in X$, we're done.

To show another direction, suppose T is bounded below, then T is injective since $Tx = 0$ implies $x = 0$. To see $\operatorname{Im}(T)$ is closed, let $x_n \in X$ for $n \geq 1$ be a sequence such that $\{Tx_n\}_{n \geq 1}$ is Cauchy such that $\|Tx_n - Tx_m\| \geq c\|x_n - x_m\|$ for all n, m, implying $\{x_n\}_{n \geq 1}$ is Cauchy, hence $x_n \to x_\infty \in X$, i.e., $Tx_n \to Tx_\infty \in \operatorname{Im}(T)$, proving the result. ■

Remark. With a bit more work, one can show that T is an isomorphic embedding is equivalent to two conditions given in Proposition 3.1.2.

3.2 Closed Graph Theorem

We now study the second main theorem in functional analysis, which characterizes the property of the graph of a bounded linear operator.

Definition 3.2.1 (Graph). Let $T \in \mathcal{L}(X, Y)$ for X, Y being Banach spaces. Then the *graph* $\Gamma(T)$ of T is defined as

$$\Gamma(T) \coloneqq \{(x, Tx) \in X \times Y \mid x \in X\}.$$

Clearly, $\Gamma(T)$ is a linear subspace of the normed space $X \oplus Y$.

Definition 3.2.2 (Closed graph). The graph $\Gamma(T)$ of T is *closed* if it is a closed subspace of $X \times Y$.

Comparing the continuity of T and closedness of $\Gamma(T)$, we see that T is continuous if and only if

$$x_n \to x \in X \text{ implies } Tx_n \to Tx;$$

while $\Gamma(T)$ is closed if and only if

$$x_n \to x \in X \text{ and } Tx_n \to y \in Y \text{ imply } y = Tx.$$

Hence, the continuity of T always implies the graph is closed, and indeed, the converse is also true.

Theorem 3.2.1 (Closed graph theorem). Let $T\colon X \to Y$ be a linear operator between Banach spaces X and Y. Then T is bounded (continuous) if and only if $\Gamma(T)$ is closed.

Proof. We have already shown that if T is bounded, then $\Gamma(T)$ is closed. Now assume $\Gamma(T)$ is closed, then we see that $\Gamma(T)$ is a Banach space, so we can now use open mapping theorem. Define a norm on $X \times Y$ by

$$\|(x, y)\| = \|x\| + \|y\|,$$

then $\Gamma(T)$ is a Banach space with this norm. Define $u\colon \Gamma(T) \to X$ by $u(x, Tx) = x$ for $x \in X$, then u is bounded since $\|u\| \leq 1$. From open mapping theorem, we know that u is surjective and injective, implying $u^{-1}\colon X \to \Gamma(T)$ is bounded from inverse mapping theorem. Hence, we have $\|u(x, Tx)\| \geq c\|(x, Tx)\|$ for all $x \in X$ and some $c > 0$, i.e.,

$$\|x\| \geq c\left(\|x\| + \|Tx\|\right) \Rightarrow \|Tx\| \leq \left(\frac{1}{c} - 1\right)\|x\|$$

for all $x \in X$, so T is bounded. ∎

Remark. When trying to prove T is continuous by showing $x_n \to x \in X$ implying $Tx_n \to Tx$, we can always assume Tx_n converges in Y.

Proof. From closed graph theorem, checking $x_n \to x \in X$ implies $Tx_n \to Tx$ is equivalent to check $x_n \to x \in X$ and $Tx_n \to y \in Y$ implies $y = Tx$, so we may just assume the limit exists. ⊛

3.2.1 Symmetric Operators on Hilbert Spaces

One application of closed graph theorem to self-adjoint (symmetric) operator, i.e., $T^* = T$, on Hilbert space is the following.

Theorem 3.2.2 (Hellinger-Toeplitz theorem). Let $T\colon \mathcal{H} \to \mathcal{H}$ be a linear operator on a Hilbert space $\mathcal{H}$. If T is self-adjoint, i.e., $\langle Tx, y\rangle = \langle x, Ty\rangle$ for $x, y \in \mathcal{H}$, then T is bounded.

Proof. From closed graph theorem, it suffices to show that for a self-adjoint operator T, $\Gamma(T)$ is closed. Let $\{x_n\}_{n\geq 1}$ in $\mathcal{H}$ such that $x_n \to x_\infty \in \mathcal{H}$ and $Tx_n \to y_\infty \in \mathcal{H}$, then we need to show $Tx_\infty \to y_\infty$. From the self-adjointness of T and the continuity of an inner product, for all $z \in \mathcal{H}$,

$$\langle z, y_\infty\rangle = \lim_{n\to\infty} \langle z, Tx_n\rangle = \lim_{n\to\infty} \langle Tz, x_n\rangle = \langle Tz, x_\infty\rangle = \langle z, Tx_\infty\rangle.$$

Since this holds for all $z \in \mathcal{H}$, we know that $Tx_\infty = y_\infty$, hence $\Gamma(T)$ is closed, so T is bounded. ∎

Hellinger-Toeplitz theorem identifies the source of considerable difficulties in mathematical physics since many natural operators such as differential, though satisfy the symmetry condition, but are unbounded, and hence Hellinger-Toeplitz theorem declares that such operators *can not be defined everywhere* on the Hilbert space.

Example. There are no useful notions of differentiation that would make all $f \in L^2$ differentiable.

Lecture 13: Principle of Uniform Boundedness

3.3 Principle of Uniform Boundedness

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The final consequence of open mapping theorem is the following, which completes the whole picture of functional analysis. We first see some definitions.

Definition. Let X, Y be Banach spaces and let $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ be a family of bounded linear operators from X to Y.

Definition 3.3.1 (Point-wise bounded)**.** $\mathcal{T}$ is *point-wise bounded* if $\sup_{T\in\mathcal{T}} \|Tx\| < \infty$ for all $x \in X$.

Definition 3.3.2 (Uniformly bounded)**.** $\mathcal{T}$ is *uniformly bounded* if $\sup_{T\in\mathcal{T}} \|T\| < \infty$.

Theorem 3.3.1 (Uniform boundedness theorem)**.** Let X, Y be Banach spaces and let $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ be a family of bounded linear operators from X to Y such that it's point-wise bounded, then it's uniformly bounded.

Proof. Define $M\colon X \to \mathbb{R}$ by $M(x) = \sup_{T\in\mathcal{T}} \|Tx\|$ for $x \in X$, also, let $X_n := \{x \in X \colon M(x) \leq n\}$, we can then write $X = \bigcup_{n=1}^{\infty} X_n$. From Baire category theorem, there exists $n \geq 1$ such that $\overline{X}_n$ has non-empty interior.

Claim. X_n is closed.

Proof. Note that the function $x \mapsto M(x)$ for $x \in X$ is lower semi-continuous, i.e., $M(x) \leq \liminf_{x_n \to x} M(x_n)$ since

$$\|Tx\| \leq \lim_{n\to\infty} \|Tx_n\| \leq \liminf_{n\to\infty} M(x_n),$$

and by taking supremum over x, we have $M(x) \leq \liminf_{n\to\infty} M(x_n)$. Hence, we see that X_n is closed, i.e., $\overline{X}_n = X_n$. ⊛

Hence, X_n itself has non-empty interior, so $X_n \supseteq x_0 + \epsilon B_X$ for some $\epsilon > 0$ and $B_X := \{x \in X \colon \|x\| \leq 1\}$.

Since $M(\cdot)$ is symmetric and convex, i.e., $M(-x) = M(x)$ for $x \in X$ and $M(\lambda x + (1-\lambda)y) \leq \lambda M(x) + (1-\lambda)M(y)$ for $x, y \in X$, $0 < \lambda < 1$, we see that $X_n \supseteq x_0 + \epsilon B_X$. From symmetric, we also have $X_n \supseteq -x_0 + \epsilon B_X$. Then by convexity, we together have $X_n \supseteq \epsilon B_X$, hence

$$\|x\| \leq \epsilon \Rightarrow \sup_{T\in\mathcal{T}} \|Tx\| \leq n \Rightarrow \sup_{T\in\mathcal{T}} \|T\| \leq n/\epsilon < \infty.$$

■

Remark (Completeness)**.** Uniform boundedness theorem still holds if X is a Banach space while Y is only a normed space.

Proof. In the above proof, we only use the completeness of X, not Y. ⊛

Note (Principle of condensation of singularities)**.** The uniform Boundedness theorem is called *principle of condensation of singularities* by Banach and Steinhaus initially.

Proof. Suppose a family $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is not uniformly bounded, then the set of vectors

$$\{Tx \colon x \in B_X, T \in \mathcal{T}\}$$

is unbounded. Then from the uniform boundedness theorem, $\mathcal{T}$ is not even point-wise bounded, so there exists *one* vector $x \in X$ with unbounded trajectory $\{Tx \colon T \in \mathcal{T}\}$. One can say that the unboundedness of the family $\mathcal{T}$ is condensated in a single *singularity* vector x. ⊛

3.3.1 Weak and Strong Boundedness

Principle of uniform boundedness can be used to check whether a given set in a Banach space is bounded in the following way. Firstly, let's see some definitions.

Definition. Let $A \subseteq X$ where X is a Banach space.

Definition 3.3.3 (Weakly bounded)**.** A is *weakly bounded* if $\sup_{f \in X^*} |f(x)| < \infty$ for all $x \in A$.

Definition 3.3.4 (Strongly bounded)**.** A is *strongly bounded* if $\sup_{x \in A} \|x\| < \infty$.

Clearly, strong boundedness implies weak boundedness, moreover, the converse is also true.

Corollary 3.3.1 (Weak boundedness implies strong boundedness)**.** Let $A \subseteq X$ for X being a Banach space. If A is weakly bounded, then A is strongly bounded.

Proof. Firstly, embed A into $A^{**} \subseteq X^{**}$ by considering the conical embedding $X \to X^{**}$, then

$$\sup_{x^{**} \in A^{**}} |x^{**}(f)| < \infty$$

for all $f \in X^*$. From the uniform boundedness theorem, we have $\sup_{x^{**} \in A^{**}} \|x^{**}\| < \infty$, and with Hahn-Banach theorem, we have $\|x^{**}\| = \|x\|$ for all $x \in X$, proving the result. ■

We now review the midterm in Appendix A.1.

Lecture 14: Midterm

Good luck! 13 Oct. 14:30

Lecture 15: Compactness in Banach Spaces

3.3.2 Schauder Bases

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Finally, we introduce a useful basis called Schauder basis. Recall that some bases we have seen is uncountable, making them hard to work with in practice, on the other hand, Schauder basis is countable and with nice properties.

Definition 3.3.5 (Schauder basis)**.** Let X be a separable Banach space. A sequence $\{x_k\}_{k \geq 1}$ is a *Schauder basis* for X if every $x \in X$ can be uniquely represented as a convergent series

$$x = \sum_{k=1}^{\infty} a_k x_k$$

for $a_k \in \mathbb{R}$ or $\mathbb{C}$.

Remark. We can show that only separable spaces can have Schauder basis, which is why we require it directly in Definition 3.3.5.

It's clearly that if $\{x_k\}_{k\geq 1}$ is a Schauder basis, it's linear independent and complete. Surprisingly, Definition 3.3.5 is stronger than these.

Remark. For completeness, given any $\epsilon > 0$, we can find $\{a_k\}_{k=1}^n$ such that

$$\left\| x - \sum_{k=1}^{n} a_k x_k \right\| \leq \epsilon.$$

However, it might be the case that a_k actually depends on ϵ, hence $\lim_{\epsilon\to 0} a_k(\epsilon)$ generally does not exist. But in fact, from the basis property, Definition 3.3.5 guarantees that one can achieve higher accuracy by using more and more terms without changing the previous a_k.

A major property of Schauder bases is the uniform bound on the partial sums.

Theorem 3.3.2 (Partial sums of a Schauder basis)**.** Let $\{x_k\}_{k\geq 1}$ be a Schauder basis for a Banach space X. Then there exists an $M \geq 0$ such that for all $n \geq 1$,

$$\left\| \sum_{k=1}^{n} a_k x_k \right\| \leq M \left\| x \right\| = M \left\| \sum_{k=1}^{\infty} a_k x_k \right\|$$

for $x \in X$.

Proof. Define a sequence space

$$E := \left\{ a = \{a_k\}_{k\geq 1} : \sum_{k=1}^{\infty} a_k x_k \text{ converges in } X \right\}$$

and for $a \in E$, define

$$\|a\| = \sup_{n\geq 1} \left\| \sum_{k=1}^{n} a_k x_k \right\| < \infty.$$

We see that $\|\cdot\|$ is a norm on E since $\|a\| = 0 \Rightarrow a = 0$ follows from the uniqueness property for Schauder basis and the fact that E is a Banach space, so E is complete.

Now, define a linear operator $T\colon E \to X$ by

$$Ta = \sum_{k=1}^{\infty} a_k x_k,$$

we have $\|Ta\| \leq \|a\|$, so T is also bounded, injective and surjective. From open mapping theorem, $T^{-1}\colon X \to E$ is bounded such that $\left\|T^{-1}\right\| \leq M < \infty$, i.e.,

$$\|Ta\| \geq \frac{1}{M} \|a\|$$

for all $a \in E$. This is equivalent to say

$$\sup_{n\geq 1} \left\| \sum_{k=1}^{n} a_k x_k \right\| \leq M \left\| \sum_{k=1}^{\infty} a_k x_k \right\|.$$

■

Notation (Basis constant)**.** The $M \geq 0$ in Theorem 3.3.2 is called the *basis constant*.

Definition 3.3.6 (Biorthogonal functional). Given a Schauder basis $\{x_k\}_{k\geq 1}$ and $x \in X$, we have the set of *biorthogonal functionals* $\{a_k(x)\}_{k\geq 1} =: \{x_k^*(x)\}_{k\geq 1}$ such that $x = \sum_{k=1}^n a_k x_k$.

The reason we call $\{a_k\}_{k\geq 1}$ for a specific x the set of biorthogonal functionals is as follows. We can define a partial sum operators S_n for $n = 1, 2, \ldots$ such that

$$S_n \colon X \to X, \quad S_n(x) = \sum_{k=1}^n a_k x_k$$

for $x = \sum_{i=1}^\infty a_k x_k$, and we have shown that S_n is a bounded linear operator since $\sup_{n\geq 1} \|S_n\| = M < \infty$. Observe that $a_k = a_k(x)$ is a linear functional on X, so the basis expansion of $x \in X$ looks like

$$x = \sum_{k=1}^\infty x_k^*(x) x_k.$$

This resembles the Fourier series with respect to orthogonal bases in a Hilbert space, except now we discuss this in general Banach spaces.

Proposition 3.3.1. The biorthogonal functionals $\{x_k^*\}_{k\geq 1}$ of a Schauder basis $\{x_k\}_{k\geq 1}$ are uniformly bounded, i.e.,

$$\sup_{k\in\mathbb{N}} \|x_k^*\| \|x_k\| < \infty.$$

Proof. To do this, we write

$$x_n^*(x) x_n = S_n(x) - S_{n-1}(x)$$

for $n \geq 1$. From Theorem 3.3.2, we have

$$\|x_n^*(x) x_n\| \leq \|S_n(x)\| + \|S_{n-1}(x)\| \leq 2M \|x\|,$$

hence we conclude that $x_n^* \in X^*$ and $\sup_{n\geq 1} \|x_n^*\| \|x_n\| < \infty$. ■

3.4 Compact Sets in Banach Spaces

Compactness is a useful substitute of finite dimensionality as we'll see. Let's give a brief review.

3.4.1 Compactness

We first review some properties of compactness.

Definition 3.4.1 (Compact). A subset A of a topological space is *compact* if every open cover of A has a finite subcover.

This means, given a cover $A \subseteq \bigcup_\alpha U_\alpha$ for some collection of open sets U_α, then $A \subseteq \bigcup_{k=1}^n U_{\alpha_k}$ for some finite subcollection.

Remark. Properties of compact sets:

(a) Compact sets of a Hausdorff space are closed (Corollary 6.1.1).

(b) Closed subsets of compact sets are compact (Corollary 6.1.2).

(c) The image of a compact set under a continuous function is compact (Theorem 6.1.5).

(d) Continuous functions on compact sets are uniformly continuous and attain their maximum and minimum.

Definition 3.4.2 (Precompact). A set A is *precompact* if its closure $\overline{A}$ is compact.

Definition 3.4.3 (ϵ-net)**.** Let A be a subset of a metric space X. Then a subset $\Omega_\epsilon \subseteq X$ is an *ϵ-net* for A if A can be covered by balls of radius ϵ centered at points of Ω_ϵ, i.e.,

$$A \subseteq \{y \colon d(y,x) < \epsilon \text{ for some } x \in \Omega_\epsilon\}.$$

Theorem 3.4.1. Let A be a subset of a complete metric space X, the following are equivalent.

(a) A is precompact.

(b) Every sequence $\{x_n\}$ in A has a Cauchy subsequence which converges in X.

(c) For every $\epsilon > 0$, there exists a finite ϵ-net for A.

Theorem 3.4.2 (Heine-Borel theorem)**.** A subset A of a finite dimensional normed space X is precompact if and only if A is bounded.

3.4.2 Compactness in Infinite-Dimensional Normed Spaces

We can extend Heine-Borel theorem to infinite dimensional spaces.

Lemma 3.4.1 (Approximation by finite dimensional subspaces)**.** A subspace A of a normed space X is precompact if and only if A is bounded, and for every $\epsilon > 0$, there exists a finite dimensional subspace Y_ϵ of X containing an ϵ-net for A.

Proof. We first prove the necessity. Let A be precompact and $\epsilon > 0$. Then there exists a finite ϵ-net Ω_ϵ for A. Now, take $Y_\epsilon = \operatorname{span}(\Omega_\epsilon)$, which is finite-dimensional.

As for sufficiency, assume A is bounded, so $A \subseteq rB_X$ for some $r > 0$ where B_X is the unit ball $\{x \in X \colon \|x\| \leq 1\}$. Also, given ϵ, we have a finite-dimensional subspace Y_ϵ as an ϵ-net of A. Observe that we can restrict to points of Y_ϵ contained in $(r+\epsilon)B_{Y_\epsilon}$ since

$$A \subseteq \{x \in X \colon d(x, (r+\epsilon)B_{Y_\epsilon}) < \epsilon\},$$

i.e., $(r+\epsilon)B_{Y_\epsilon}$ is also an ϵ-net of A. Since Y is finite-dimensional, from Heine-Borel theorem, $(r+\epsilon)B_{Y_\epsilon}$ is precompact, i.e., we find a precompact ϵ-net of A, therefore A itself is precompact from Theorem 3.4.1. ■

On the other hand, Heine-Borel theorem states that the unit ball B_X of a finite-dimensional normed space X is compact, but this never holds in the infinite-dimensional case.

Theorem 3.4.3 (Riesz's theorem)**.** The unit ball B_X of an infinite dimensional normed space X is never compact.

Proof. Suppose $B_X = \{x \in X \colon \|x\| \leq 1\}$ is compact. Then from Lemma 3.4.1, we can find a finite dimensional subspace Y containing an ϵ-net with $\epsilon = 1/2$ for B_X, i.e., $d(x, Y) \leq 1/2$ for all $x \in B_X$.

Recall that X is infinite dimensional, Y is finite dimensional, hence the quotient space $X \mathbin{/} Y$ is nontrivial. Note that Y is a closed subspace of X,[a] hence, the norm on X induces a norm on $X \mathbin{/} Y$ such that $\|[x]\| = \inf_{y\in Y} \|x - y\|$. We can then find an $x \in X$ and $[x] \in X \mathbin{/} Y$ such that $\|[x]\| = 0.9$, i.e., there's an $\overline{y} \in Y$ such that $\|x - \overline{y}\| \leq 1$. In this case, $x - \overline{y} \in B_X$ and $d(x - \overline{y}, Y) = \|[x]\| = 0.9 > 1/2$, a contradiction. ■

[a] Since it's finite-dimensional.

Lecture 16: Strong Convergence & Weak Topology

Lastly, we see that point-wise convergence of operators implies uniformly convergence on compact sets. 25 Oct. 14:30

Definition. Let X, Y be normed spaces, $\{T_n\}_{n\geq 1}$ be a sequence in $\mathcal{L}(X,Y)$ and $T \in \mathcal{L}(X,Y)$.

Definition 3.4.4 (Point-wise convergence)**.** The sequence $\{T_n\}_{n\geq 1}$ *converges point-wise to* T if $\lim\limits_{n\to\infty} T_n x = Tx$ for all $x \in X$.

Definition 3.4.5 (Uniformly convergence)**.** The sequence $\{T_n\}_{n\geq 1}$ *converges uniformly to* T *on* $A \subseteq X$ if $\lim\limits_{n\to\infty} \|T_n x - Tx\| = 0$ for all $x \in A$.

Equivalently, $\{T_n\}_{n\geq}$ converges uniformly to T on A if

$$\lim_{n\to\infty} \sup_{x\in A} \|T_n x - Tx\| = 0.$$

Theorem 3.4.4 (Convergence on compact set)**.** Let X, Y be Banach spaces, and $\{T_n\}_{n\geq 1}, T \in \mathcal{L}(X,Y)$. Suppose the sequence $\{T_n\}_{n\geq 1}$ converges point-wise to T, then $\{T_n\}_{n\geq 1}$ converges uniformly to T on all precompact subsets $A \subseteq X$.

Proof. Since $\{T_n\}$ converges point-wise, it's point-wise bounded. From uniform boundedness theorem, we know that $\sup_{n\geq 1}\|T_n\| < \infty$, i.e., $\exists M$ such that $\|T_n\| \leq M$ for all $n \geq 1$. Let $\epsilon > 0$ and choose a finite ϵ-net Ω_ϵ for A. Since Ω_ϵ is finite, there exists N_ϵ such that $\|T_n y - Ty\| \leq \epsilon$ for $n \geq N_\epsilon$, $y \in \Omega_\epsilon$, i.e., $T_n \to T$ uniformly on Ω_ϵ. Now, for an arbitrarily $x \in A$, there exists $y \in \Omega_\epsilon$ such that $\|x - y\| < \epsilon$, hence

$$\|T_n x - Tx\| \leq \|T_n y - Ty\| + \|(T_n - T)(x - y)\| \leq \epsilon + (\|T_n\| + \|T\|)\|x - y\| \leq \epsilon + 2M\epsilon$$

if $n \geq N_\epsilon$. This implies $\|T_n x - Tx\| \leq (2M+1)\epsilon$ for $n \geq N_\epsilon$ for all $x \in A$, hence we have uniform convergence on A. ∎

3.4.3 Compactness Criteria in Various Spaces

There is a useful criterion of compactness in spaces with Schauder basis (which covers all classical spaces).

Corollary 3.4.1. Let X be a Banach space with a Schauder basis $\{x_k\}_{k\geq 1}$. A subset $A \subseteq X$ is precompact if and only if A is bounded and the basis expansion of vectors $x \in A$ converges uniformly, i.e.,

$$\lim_{n\to\infty} \sup_{x\in A} \|x - S_n x\| = 0.$$

Proof. From Theorem 3.4.4, since $\{x_k\}_{k\geq 1}$ is a Schauder basis, the projection $S_n \to I$ point-wise, implying uniform convergence since A is precompact.

Conversely, for any $\epsilon > 0$, there exists n such that $\|x - S_n x\| < \epsilon$ for all $x \in A$, and $\operatorname{Im}(S_n)$ is finite dimensional and $\operatorname{Im}(S_n A)$ is bounded. Hence, there exists an ϵ-net Ω_ϵ for $\operatorname{Im}(S_n A)$, so A is covered by a finite 2ϵ-net, i.e., A is precompact from Lemma 3.4.1. ∎

Finally, we state without proof one of the most important compactness criteria in $C([a,b])$. To start with, we introduce the notion of equicontinuous, specifically for real-valued function family.

Definition 3.4.6 (Equicontinuous)**.** A real-valued function family $\mathcal{F}$ is *equicontinuous* if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \leq \epsilon$$

for all $f \in \mathcal{F}$ if $|x - y| \leq \delta$.

Remark (Uniformly Equicontinuous)**.** Definition 3.4.6 is often referred to *uniformly equicontinuous*. There's also a point-wise version of which, but we will not bother introducing it here.

Theorem 3.4.5 (Arzelà-Ascoli theorem). A subset $A \subseteq C([a,b])$ is precompact if and only if A is bounded and equicontinuous.

3.5 Weak Topology

Every normed space X is a metric space with the metric given by $d(x,y) = \|x-y\|$ for $x, y \in X$. This topology on X is called *strong topology*, i.e., a sequence $x_n \to x$ converges strongly in X if $\|x_n - x\| \to 0$. But actually, in additional to the strong topology, X carries a different topology called weak topolgy, as we're going to study it in this section.

3.5.1 Weak Convergence

Let's first formally introduce the definitions.

Definition. Let $\{x_n\}_{n\geq 1}$ be a sequence in a normed space X.

Definition 3.5.1 (Strongly convergence). The sequence $\{x_n\}_{n\geq 1}$ *converges strongly* to $x \in X$ if $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.

Definition 3.5.2 (Weakly convergence). The sequence $\{x_n\}_{n\geq 1}$ *converges weakly* to $x \in X$ if $\lim\limits_{n\to\infty} f(x_n) = f(x)$ for all $f \in X^*$.

Notation. If $\{x_n\}_{n\geq 1}$ is weakly converging to x, we write $x_n \overset{w}{\to} x$.

Remark (Strong and weak). As we have seen before (Definition 3.3.4, Definition 3.3.3 and Definition 3.5.1, Definition 3.5.2), the convention is that *strong* is for norm, while *weak* is for functional.

We see that strongly convergence implies weakly convergence, while not as before, the converse is often not true. Even with this, there are several useful ties between weak and strong properties.

Proposition 3.5.1. If the sequence $\{x_n\}_{n\geq 1}$ converges weakly to $x \in X$, then we have the following.

(a) $\sup_{n\geq 1} \|x_n\| < \infty$.

(b) $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

(c) $x \in \overline{\operatorname{conv}(\{x_n\}_{n\geq 1})}$.[a]

[a] $\operatorname{conv}(A)$ is the convex hull of A, i.e., the smallest closed convex set containing the sequence.

Proof. Let's prove this one by one.

(a) For $y \in X$, let $y^{**} \in X^{**}$ be from the embedding $X \to X^{**}$, $y \mapsto y^{**}$ such that $\|y^{**}\| = \|y\|$. Then for $n \geq 1$, $x_n \in X$, so $x_n^{**} \in X^{**}$, we have $\sup_{n\geq 1} |x_n^{**}(f)| < \infty$ since $f(x_n) \to f(x)$. Then, uniform boundedness theorem implies $\sup_{n\geq 1} \|x_n^{**}\| < \infty$. Now, since $\|x_n\| = \|x_n^{**}\|$, we conclude $\sup_{n\geq 1} \|x_n\| < \infty$.

(b) If $x_n \overset{w}{\to} x$ in X, by supporting functional theorem, there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$. Since $\|f\| = 1$, $f(x_n) \leq \|x_n\|$ for $n \geq 1$. And since $x_n \overset{w}{\to} x$, we have $f(x_n) \to f(x) = \|x\|$, i.e., $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$ as desired.

(c) To show x lies in the closure of the convex hull of $\{x_n\}_{n\geq 1}$, denoted it by K, we first note that K is a closed convex set. If $x \notin K$, by separating hyperplane theorem, there exists $f \in X^*$ such that $\sup_{y\in K} f(y) < f(x)$, and hence $\sup_{n\geq 1} f(x_n) < f(x)$. Since $\lim_{n\to\infty} f(x_n) = f(x)$, we have a contradiction. ■

There are some known criteria of weak convergence in classical normed spaces, one of them is as follows.

Lemma 3.5.1 (Testing weak convergence on a dense set)**.** Let X be a normed space and $A \subseteq X^*$ be a dense set. Then $x_n \overset{w}{\to} x$ if and only if $\{x_k\}_{k\geq 1}$ is bounded and $\lim_{n\to\infty} f(x_n) = f(x)$ for every $f \in A$.

Proof. The necessity follows from Proposition 3.5.1. To show the sufficiency, let $g \in X^*$, we need to show that $\lim_{n\to\infty} g(x_n) = g(x)$. Let $\epsilon > 0$, and A is dense in X^*, so there exists $f \in A$ such that $\|g - f\| < \epsilon$. Then

$$\begin{aligned}\limsup_{n\to\infty} |g(x_n - x)| &\leq \limsup_{n\to\infty} |f(x_n - x)| + \limsup_{n\to\infty} |(g - f)(x_n - x)| \\ &\leq \|g - f\| \limsup_{n\to\infty} (\|x_n\| + \|x\|) \\ &\leq 2M\epsilon\end{aligned}$$

where $M := \sup_{n\geq 1} \|x_n\| + \|x\| < \infty$. We see that since $\epsilon > 0$ is arbitrary, so $\limsup_{n\to\infty} |g(x_n - x)| = 0$ hence $x_n \overset{w}{\to} x$. ■

Note. We see that to show weakly convergence, instead of checking for all $f \in X^*$, it's sufficient to check only $f \in A$ for some dense set $A \subseteq X^*$.

3.5.2 Weak Topology

Weak convergence is just a concept induced from weak topology, so by studying it directly, we can consider other weak properties as we'll soon see.

Definition 3.5.3 (Weak topology)**.** The *weak topology* on a normed space X is the weakest topology such that all maps $f \in X^*$ are continuous.

Note (Strong topology)**.** To distinguish two natural topologies on X, the norm topology is sometimes called *strong topology* on X.

Intuitively, if f is continuous at x_0, the preimage of an ϵ-ball $\{x \in X : |f(x) - f(x_0)| < \epsilon\}$ around $f(x_0)$ is open, i.e., the base of the weak topology are cylinders of the form

$$\{x \in X : |f_k(x - x_0)| < \epsilon, f_k \in X^*, \epsilon > 0, k = 1, 2, \ldots, N\}$$

around $x_0 \in X$. In all, these cylinders form a local base of weak topology at point x_0.

Remark ($\mathbb{R}^\infty$ embedding)**.** Weak topology is induced from the (infinite) products of reals.

Proof. Consider the embedding from X to an infinite product of $\mathbb{R}$ such that

$$X \to \mathbb{R}^\infty, \quad x \mapsto (f_k(x))_{f_k \in X^*} = (f_1(x), f_2(x), \ldots),$$

where we identify $x \in X$ by its value of uncountably many $f_k \in X^*$, i.e., $f_k(x)$. ⊛

Note (Realtion between CW complex)**.** The weak topology in algebraic topology is actually the strongest one, i.e., it's the final topology. See here for a further discussion.

Although there are lots of difference between weak and the corresponding strong properties, some are equivalent.

Proposition 3.5.2 (Weak closedness). Let K be a convex subset of a Banach space. Then K is weakly closed if and only if K is strongly closed.

Proof. Clearly, weak closure implies strong closure.[a] Conversely, if K is a strongly closed convex set, it is the intersection of all hyperplane containing K from Corollary 2.4.1, i.e., $K = \bigcap_{f,a} A_{f,a}$ where $A_{f,a} \coloneqq \{x \in X : f(x) \leq a, f \in X^*\}$. Since it is weakly closed from $A_{f,a} = f^{-1}((-\infty, a])$,[b] the intersection of them is also weakly closed, proving the result. ∎

[a] Notice that this case doesn't involve convexity since we only use the coarser relation between topologies.
[b] Since f is continuous, f^{-1} maps closed set in $\mathbb{R}$ ($(-\infty, a]$) to (weakly) closed set in X with weak topology.

Lecture 17: Weak* Topology

There's another important case regarding $C(K)$. 27 Oct. 14:30

Theorem 3.5.1 (Weak convergence in $C(K)$). Let K be a compact Hausdorff space and $C(K)$ be the space of continuous functions $f\colon K \to \mathbb{R}$ with $\|f\| = \sup_{t\in K} |f(t)|$. Then a sequence $\{x_n\}_{n\geq 1}$ in $C(K)$ converges weakly to $x \in C(K)$ if and only if the sequence $\{x_n\}_{n\geq 1}$ is bounded in $C(K)$ and $\lim\limits_{n\to\infty} x_n(t) = x(t)$ for all $t \in K$.

Proof. Suppose $\{x_n\}_{n\geq 1}$ converges weakly to x, then $\sup_{n\geq 1} \|x_n\| < \infty$. By Proposition 3.5.2, if $t \in K$, then $\delta_t \in C(K)^*$ where $\delta_t(x) = x(t)$. We see that

$$\lim_{n\to\infty} \delta_t(x_n) = \delta_t(x) \Rightarrow \lim_{n\to\infty} x_n(t) = x(t).$$

Conversely, assume $\sup_{n\geq 1} \|x_n\| < \infty$ and $\lim_{n\to\infty} x_n(t) = x(t)$ for all $t \in K$. We now need to show that $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in C(K)^*$. Recall that $C(K)^*$ is the space of bounded signed measures μ on K,[a] and $\|\mu\| = TV(\mu)$, we have

$$\lim_{n\to\infty} \int_K x_n \,\mathrm{d}\mu = \int_K x \,\mathrm{d}\mu,$$

from the dominated convergence theorem, proving the result. ∎

[a] This is just a variant of Riesz representation theorem.

3.6 Weak* Topology

On X^*, there are two natural weaker topologies: one is the weak topology which makes all functionals in X^{**} continuous on X^*; the other topology, called weak* topology, is only concerned with the continuity of functionals that come from $X \subseteq X^{**}$.

3.6.1 Weak* Convergence

We again start from convergence, and in this case, the weak* convergence.

Definition 3.6.1 (Weak* convergence). Let X be a normed space, a sequence $\{f_n\}_{n\geq 1}$ in X^* is *weak* converging* to f in X^* if $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$.

Notation. If $\{f_n\}_{n\geq 1}$ is weak* converging to f, we write $f_n \overset{\mathrm{w}^*}{\to} f$.

We see that if we consider the weak convergence for $f_k \in X^*$, we should test on all the double dual elements $x^{**} \in X^{**}$; but as said, weak* convergence only test on the subset $X \subseteq X^{**}$, so Definition 3.6.1 becomes point-wise convergence.

Remark. The weak and weak* convergence coincides on X^* if X is reflexive, i.e., $X^{**} \equiv X$.

3.6.2 Weak* Topology

Similarly to Definition 3.5.3, we now defined the so-called weak* topology on X^*.

Definition 3.6.2 (Weak* topology). Let X be a normed space. The *weak* topology* on X^* is defined as the weakest topology in which point evaluation maps $f \mapsto f(x)$ from X^* to $\mathbb{R}$ are continuous for all $x \in X$.

Note (Completeness). If $\{f_n\}_{n\geq 1}$ is weak* Cauchy, then uniform boundedness principle implies that there exists $f \in X^*$ such that $f_n \overset{w^*}{\to} f$.

Equivalently, the base of the weak* topology is given by the cylinders

$$\{f \in X^* \colon\ |(f - f_0)(x_k)| < \epsilon, x_k \in X, \epsilon > 0, k = 1, \dots, N\}$$

around $f_0 \in X^*$. So again, these cylinders form a local base of weak* topology at f_0.

Remark ($\mathbb{R}^\infty$ embedding). Weak* topology is again induced from the (infinite) products of reals.

Proof. Consider the embedding from X^* to an infinite product of $\mathbb{R}$ such that

$$X^* \to \mathbb{R}^\infty, \quad f \mapsto (f(x_k))_{x_k \in X \subseteq X^*} = (f(x_1), f(x_2), \dots),$$

where we identify $f \in X^*$ by its value on uncountably many $x \in X \subseteq X^*$, i.e., $f(x_k)$. ⊛

As we mentioned before, since $X \subseteq X^{**}$, weak* topology is weaker than the weak topology on X^*. However, for reflexive spaces, these two topologies are of course equivalent.

The main result on weak* topology is Banach-Alaoglu theorem, which allows us to still have a weaker notion of (weak*) compactness of unit balls even though we know that it's not (strongly) compact from Riesz's theorem. The result depends on Tychonoff's theorem.

Theorem 3.6.1 (Tychonoff's theorem). Let $\{X_\gamma\}_{\gamma\in\Gamma}$ be a collection of any number[a] of topological spaces X_γ. The Cartesian product $\prod_{\gamma\in\Gamma} X_\gamma$ can be equipped with the product topology whose base is formed by the sets of the form

$$\left\{ \prod_{\gamma\in\Gamma} A_\gamma \colon A_\gamma \text{ is open in } X_\gamma\text{; all but finitely many of } A_\gamma \text{ equal } X_\gamma \right\}.$$

Then, if each X_γ is compact, then $\prod_{\gamma\in\Gamma} X_\gamma$ is compact in the product topology.

[a]May be countable or uncountable.

The proof is omitted here, but the upshot is that although the $\mathbb{R}^\infty$ embedding may involve uncountably many $\mathbb{R}$, Tychonoff's theorem ensures that the compactness is still preserved.

Theorem 3.6.2 (Banach-Alaoglu theorem). Let X be a Banach space with dual X^*, then the unit ball $B_{X^*} = \{f \in X^* \colon\ \|f\| \leq 1\}$ in X^* is compact in the weak* topology.

Proof. Since $f \in B_{X^*} \Rightarrow |f(x)| \leq \|x\|$ for $x \in X$, so we can embed B_{X^*} into $\prod_{x\in X}[-\|x\|, \|x\|]$.[a] Note that this is the product of compact spaces, so Tychonoff's theorem implies that this product is compact, and we only need to show B_{X^*} is weak* closed in $\prod_{x\in X}[-\|x\|, \|x\|]$. Observe that

$$B_{X^*} = \bigcap_{\substack{x,y\in X\\ a,b\in\mathbb{R}}} B_{x,y,a,b}, \text{ where } B_{x,y,a,b} = \{f \in K \colon\ f(ax + by) = af(x) + bf(y)\}$$

for $K := \prod_{x\in X}[-\|x\|, \|x\|]$,[b] so it's sufficient to show $B_{x,y,a,b}$ is weak* closed in K. But since $B_{x,y,a,b}$ is the preimage of (a closed set) $\{0\}$ under mapping $f \mapsto f(ax+by) - af(x) - bf(y)$, which is continuous from the definition of weak* topology, hence we know all $B_{x,y,a,b}$ is (weak*) closed as well, so is their intersection B_{X^*}. ■

[a]Recall the $\mathbb{R}^\infty$ embedding where we identify $f \in X^*$ by $(f(x_k))_{x_k\in X}$.

[b]What we're claiming is that B_{X^*} consists of linear functions in K.

Let's see some applications of Banach-Alaoglu's theorem. Consider spaces $L^p(\mu)$ with $1 < p < \infty$, we know that these are reflexive, then the unit ball in $L^p(\mu)$ is compact in the weak topology. Namely, let $f_n \in L^p(\mu)$ for $n \geq 1$ be bounded, then there exists a subsequence f_{n_k} for $k \geq 1$ and $f \in L^p(\mu)$ such that

$$\lim_{k\to\infty} \int f_{n_k} g \,\mathrm{d}\mu = \int fg \,\mathrm{d}\mu$$

for all $g \in L^q(\mu)$ with $1/p + 1/q = 1$.

Another example is that let the Banach space be $C(K)$, the dual $C(K)^*$ is the space of finite signed measure μ on K with $TV(\mu) < \infty$. Let μ_n be a sequence of measures on K such that $\sup_{n\geq 1} TV(\mu_n) < \infty$, then $\{\mu_n\}_{n\geq 1}$ is bounded in $C(K)^*$. We see that there exists a subsequence μ_{n_k} for $k \geq 1$ such that $\mu_{n_k} \overset{\mathrm{w}^*}{\to} \mu \in C(K)^*$, i.e.,

$$\lim_{n\to\infty} \int_K f \,\mathrm{d}\mu_{n_k} = \int_K f \,\mathrm{d}\mu$$

for all $f \in C(K)$.

Note. This is generally referred to as *weak convergence of measures*.

Chapter 4

Compact Operators and Spectral Theory

4.1 Compact Operators

Compact operators form an important class of bounded linear operators. On the one hand, they are *almost* finite rank operators,[1] so they share some properties of finite rank operators, which facilitates their study. On the other hand, the class of compact operators is wide enough to include integral and Hilbert-Schmidt operators, which are important in many applications.

Definition 4.1.1 (Compact operator)**.** Let X, Y be Banach spaces, we say a bounded linear operator $T\colon X \to Y$ is *compact* if it maps bounded sets in X to precompact sets in Y.

Notation. The set of compact operators $T\colon X \to Y$ is denoted as $\mathcal{K}(X, Y)$.

In other words, the closure of $T(B_X)$ is compact for T being compact. Notice that since precompact sets are bounded, compact operators are always bounded, i.e., $\mathcal{K}(X, Y) \subseteq \mathcal{L}(X, Y)$. So indeed, we may remove the bounded condition in Definition 4.1.1.

Remark (Finite rank operator)**.** An operator T is with *finite rank* if $\dim \operatorname{Im} T < \infty$. And we see that every finite rank operator $T \in \mathcal{L}(X, Y)$ is compact.

Proof. Since $T(B_X)$ is a bounded subset of a finite dimensional normed space $\operatorname{Im} T \subseteq Y$, so $T(B_X)$ is precompact by Heine-Borel theorem. ⊛

4.1.1 Properties of Compact Operators

Let's study some basic properties of compact operators.

Proposition 4.1.1 (Peroperties of $\mathcal{K}(X, Y)$)**.** Let X, Y be Banach spaces.

(a) $\mathcal{K}(X, Y)$ is a closed linear subspace of $\mathcal{L}(X, Y)$.

(b) If $T \in \mathcal{K}(X, Y)$ and S is bounded linear, then both TS and ST are compact.[a]

[a] S needs to have proper domain and range, i.e., TS is compact if $S\colon Z \to X$; ST is compact if $S\colon Y \to Z$.

Proof. Let's prove this one by one.

(a) Linearity follows since the sum of two precompact sets is precompact. For closedness, let $T_n \in \mathcal{K}(X, Y)$, $T \in \mathcal{L}(X, Y)$ with $\lim_{n\to\infty} \|T_n - T\| = 0$, we need to show T is compact, which can be done by finding a finite ϵ-net of $T(B_X)$.

[1] In the same way as compact sets are *almost* finite dimensional.

Given any $\epsilon > 0$, choose N such that $\|T_N - T\| < \epsilon/2$, i.e., $\|T_N x - Tx\| < \epsilon/2$ for every $x \in B_X$. This means $T_N(B_X)$ is an $\epsilon/2$-net of $T(B_X)$. Now, since $T_N(B_X)$ is itself precompact, we can find another finite $\epsilon/2$-net $\Omega_{\epsilon/2}$ of $T_N(B_X)$, hence $\Omega_{\epsilon/2}$ is a finite ϵ-net of $T(B_X)$.

(b) Consider TS first. Since S maps unit balls to bounded sets, and T maps bounded sets to precompact sets, hence TS is compact. As for ST, since T maps unit balls to precompact sets, and because S is continuous, S maps precompact sets to precompact sets.

■

Remark (Operator ideal)**.** If $\mathcal{K}(X, Y)$ satisfies the second property in Proposition 4.1.1, we call it an *operator ideal.*

Lecture 18: Hilbert-Schmidt Operators

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Corollary 4.1.1 (Isomorphisms are not compact)**.** Let X be an infinite dimension normed space, then the identity map $I\colon X \to X$ is not compact. More generally, an invertible operator (i.e., isomorphism) $T \in \mathcal{L}(X, Y)$ is not compact.

Proof. I is not compact since Riesz theorem implies that B_X is not compact. And in general, if $T \in \mathcal{L}(X, Y)$ is compact and invertible, Proposition 4.1.1 implies that $T^{-1}T = I\colon X \to X$ is compact since T^{-1} is bounded and T is compact, contradiction. ■

Finally, an important observation is that since we know $\mathcal{K}(X, Y)$ is closed from Proposition 4.1.1, and any finite rank operators are compact, hence for any operator that can be approximated by finite rank operators is also compact, leading to the following.

Corollary 4.1.2 (Almost finite rank operators are compact)**.** Suppose a linear operator $T\colon X \to Y$ can be approximated by finite rank operators $T_n \in \mathcal{L}(X, Y)$ as $\lim_{n\to\infty} \|T_n - T\| = 0$, then T is compact.

4.1.2 Hilbert-Schmidt Operators

This is a most frequently used class of compact operators in Hilbert spaces. As we will see, it covers the class of important operators, i.e., integral operators, motivating us to study this class.

Definition 4.1.2 (Hilbert-Schmidt operators)**.** Let $\mathcal{H}$ be a Hilbert space which is separable and $\{x_k\}_{k\geq 1}$ is an orthonormal basis in $\mathcal{H}$. A linear operator $T\colon \mathcal{H} \to \mathcal{H}$ is *Hilbert-Schmidt* if

$$\sum_{k=1}^{\infty} \|Tx_k\|^2 < \infty.$$

From Definition 4.1.2, the following naturally induced norm is defined.

Definition 4.1.3 (Hilbert-Schmidt norm)**.** The *Hilbert-Schmidt norm* of a linear operator T is

$$\|T\|_{\mathrm{HS}} = \left(\sum_{k=1}^{\infty} \|Tx_k\|^2 \right)^{1/2}.$$

To characterize Hilbert-Schmidt operators, we have the following.

Proposition 4.1.2. Let $T\colon \mathcal{H} \to \mathcal{H}$ be a linear operator on the separable Hilbert space $\mathcal{H}$ with an orthonormal basis $\{x_k\}_{k\geq 1}$.

(a) The Hilbert-Schmidt norm of T is independent of the choice of orthonormal basis.

(b) If T is Hilbert-Schmidt, then T^* is Hilbert-Schmidt and $\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}$.

(c) If T is Hilbert-Schmidt, then T is bounded on $\mathcal{H}$ and $\|T\| \leq \|T\|_{\mathrm{HS}}$.

(d) If T is Hilbert-Schmidt, then T is compact.

Proof. We prove this in the following order.

(c) Let $x \in \mathcal{H}$ such that $x = \sum_{k=1}^{\infty} a_k x_k$, $a_k \in \mathbb{C}$. From the Parseval identity, $\|x\|^2 = \sum_{k=1}^{\infty} |a_k|^2$ where $a_k = \langle x, x_k \rangle$, then

$$\|Tx\| = \left\| \sum_{k=1}^{\infty} a_k T x_k \right\| \leq \sum_{k=1}^{\infty} |a_k| \|Tx_k\| \leq \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} \|Tx_k\|^2 \right)^{1/2} = \|x\| \|T\|_{\mathrm{HS}},$$

so T is bounded with $\|T\| \leq \|T$
$rVert_{\mathrm{HS}}$.[a]

(b) Since

$$\sum_{k=1}^{\infty} \|Tx_k\|^2 = \sum_{k,j} |\langle Tx_k, x_j \rangle|^2 = \sum_{k,j} |\langle x_k, T^* x_j \rangle|^2 = \sum_j \|T^* x_j\|^2 = \|T^*\|_{\mathrm{HS}}^2,$$

hence $\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}$.

(a) Let $\{x_k'\}_{k\geq 1}$ be another separate orthonormal basis, then from the Parseval identity and $\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}$,

$$\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}} = \sum_j \|T^* x_j\|^2 = \sum_{j,k} |\langle x_k', T^* x_j \rangle|^2 = \sum_{j,k} |\langle T x_k', x_j \rangle|^2 = \sum_k \|T x_k'\|^2,$$

i.e., the Hilbert-Schmidt norm is independent of the choice of basis.

(d) From Corollary 4.1.2, we show that T is a limit of finite rank operators. Define for $N \geq 1$, T_N by $T_N x_k = T x_k$ for $k = 1, 2, \ldots, N$, $T_N x_k = 0$ for $k > N$, hence T_N is finite rank. We then have

$$\|T - T_N\|_{\mathrm{HS}}^2 = \sum_{k=N+1}^{\infty} \|Tx_k\|^2 \Rightarrow \lim_{N\to\infty} \|T - T_N\|_{\mathrm{HS}} = 0.$$

Since $\|\cdot\| \leq \|\cdot\|_{\mathrm{HS}}$, $\lim_{N\to\infty} \|T - T_N\| = 0$ as desired.

■

[a] This implies that T^* is well-defined.

One important motivation of studying compact operators is because integral operators are compact, and furthermore, is Hilbert-Schmidt.

Proposition 4.1.3 (Hilbert-Schmidt integral operator)**.** Let $k \in L^2([0,1] \times [0,1])$ such that

$$\int_0^1 \int_0^1 |k(t,s)|^2 \,\mathrm{d}t \,\mathrm{d}s < \infty,$$

and define an integral operator $T \colon L^2([0,1]) \to L^2([0,1])$ by

$$Tf(t) = \int_0^1 k(t,s) f(s) \,\mathrm{d}s$$

for $0 < t < 1$ and $f \in L^2([0,1]) = \mathcal{H}$. Then T is Hilbert-Schmidt on $\mathcal{H}$ and $\|T\|_{\mathrm{HS}} = \|k\|_{L^2([0,1]^2)}$.

Proof. Let $K_t(s) := k(t,s)$ for $0 < s,t < 1$, then $Tf(t) = \langle K_t, f\rangle$. Note that $\|Tf(t)\| \leq \|K_t\|_2 \|f\|_2$ from Cauchy-Schwarz, and also, notice that

$$\int_0^1 \|K_t\|_2^2 \,\mathrm{d}t = \|k\|_2^2 \Rightarrow \|K_t\|_2 < \infty \text{ a.e. } t \in [0,1].$$

Hence, $Tf(t)$ is defined for a.e. t. Let $\{x_k\}_{k\geq 1}$ be an orthonormal basis for $\mathcal{H} = L^2([0,1])$,

$$\|T\|_{\mathrm{HS}}^2 = \sum_{k=1}^{\infty} \|Tx_k\|_2^2 = \sum_{i=1}^{\infty} \int_0^1 |Tx_k(t)|^2 \,\mathrm{d}t = \sum_{k=1}^{\infty} \int_0^1 |\langle K_t, x_t\rangle|^2 \,\mathrm{d}t = \int_0^1 \sum_{k=1}^{\infty} |\langle K_t, x_t\rangle|^2 \,\mathrm{d}t,$$

where the last equality follows from the monotone convergence theorem. Further, from Parseval,

$$\|T\|_{\mathrm{HS}}^2 = \int_0^1 \sum_{k=1}^{\infty} |\langle K_t, x_t\rangle|^2 \,\mathrm{d}t = \int_0^1 \|K_t\|_2^2 \,\mathrm{d}t = \|k\|_2^2$$

by the definition of K_t and Fubini's theorem ■

4.1.3 Compactness of the Adjoint Operators

Recall the basic duality property for bounded linear operators.

As previously seen. If $T \in \mathcal{L}(X,Y)$, then $T^* \in \mathcal{L}(Y^*, X^*)$ and $\|T^*\| = \|T\|$.

Indeed, a similar duality principle holds for compact operators as guaranteed by Schauder's theorem.

Theorem 4.1.1 (Schauder's theorem)**.** Let X, Y be Banach spaces, then if $T \in \mathcal{K}(X,Y)$, $T^* \in \mathcal{K}(Y^*, X^*)$.

Proof. Without loss of generality, we prove that $T^* B_{Y^*}$ is precompact in X^* given $T \in \mathcal{K}(X,Y)$, i.e., consider $f_n \in B_{Y^*}$ for $n \geq 1$ such that $\{T^* f_n\}_{n\geq 1}$ has a convergent subsequence.

Claim. B_{Y^*} is precompact in Y^*, i.e., there is a convergent subsequence $\{f_{n_k}\}_{k\geq 1}$ of $\{f_n\}_{n\geq 1}$.

Proof. This can be done by embedding B_{Y^*} into $C(K)$ with $K := \overline{TB_X}$ by considering $f_n|_K \in C(K)$. Observe that $\{f_n|_K\}_{n\geq 1}$ is bounded and equicontinuity: First, $\{f_n|_K\}_{n\geq 1}$ is bounded since $f_n \in B_{Y^*}$ implying $\|f_n\|_{Y^*} \leq 1$ for all $n \geq 1$, and so is $f_n|_K$; while for equicontinuity, we have

$$|f_n|_K(y) - f_n|_K(y')| = |f(y_1 - y_2)| \leq \|y - y'\| \leq \|f_n\| \|y - y'\|$$

for all $y, y' \in Y$. In all, since K is compact, with Arzelà-Ascoli theorem, $\{f_n|_K\}_{n\geq 1}$ has a convergent subsequence $\{f_{n_k}|_K\}_{k\geq 1}$ in $C(K)$.[a] ⊛

[a]Same as saying B_{Y^*} is precompact as the original statement in Arzelà-Ascoli theorem.

Hence, to show that $\{T^* f_n\}_{n\geq 1}$ has a convergent subsequence, consider $\{T^* f_{n_k}\}_{k\geq 1}$, we have[b]

$$\|T^* f_{n_i} - T^* f_{n_j}\|_{X^*} = \sup_{x\in B_X} |f_{n_i}(Tx) - f_{n_j}(Tx)| = \sup_{y\in K} \left| f_{n_i}|_K(y) - f_{n_j}|_K(y)\right| \to 0$$

as $n_i, n_j \to \infty$ since f_{n_k} converges, i.e., $\{T^* f_{n_k}\}_{k\geq 1}$ is Cauchy in X^* hence converges. ■

[b]The last equality follows from the fact that TB_X is dense in K (i.e., $\overline{TB_X} = K$).

Lecture 19: Fredholm Alternative

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4.2 Fredholm Theory

Fredholm theory studies operators of the form *identity plus compact*. They are conveniently put in the form $I - T$ for I being the identity operator on some Banach space X and $T \in \mathcal{K}(X, X)$.

Furthermore, Fredholm theory is motivated by two applications. One is for solving linear equations $\lambda x - Tx = b$, and in particular, integral equations.[2] Another is in spectral theory, where the spectrum of T consists of numbers λ for which the operator $\lambda I - T$ is not invertible.

4.2.1 Closed Image

Firstly, we see that the image of operators in the form $I - T$ has a closed image.

Lemma 4.2.1. Let X be a Banach space and $T \in \mathcal{K}(X, X)$, then the operator $I - T$ has closed image $\operatorname{Im}(I - T)$.

Proof. Let $A = I - T$, since $\ker(A)$ is closed, we consider the injectivization $\widetilde{A}\colon X / \ker(A) \to X$ be the induced operator $\widetilde{A}([x]) = Ax$ for all $x \in X$. Since $\operatorname{Im}(A) = \operatorname{Im}(\widetilde{A})$, it's sufficient to show $\operatorname{Im}(\widetilde{A})$ is closed, which is equivalent to show that $\widetilde{A}$ is bounded below from Proposition 3.1.2, i.e., $\exists c > 0$ such that

$$\|\widetilde{A}[x]\| \geq c \, \|[x]\|$$

for all $x \in X$. Toward a contradiction, suppose $\widetilde{A}$ is not bounded below, then there exists a sequence $\{[x_k]\}_{k \geq 1}$ in X such that $\|[x_k]\| = 1$ for all $k \geq 1$ and $\|\widetilde{A}[x_k]\| \to 0$ as $k \to \infty$, or $\widetilde{A}([x_k]) \to 0$.

Now, choose $\{x_k \in [x_k]\}_{k \geq 1}$ such that $\|x_k\| \leq 2$, with $\inf_{y \in \ker(A)} \|x_k - y\| = 1$. From the fact that T is compact, $\{Tx_k\}_{k \geq 1}$ has a converging subsequence, so we can assume $Tx_k \to z$ as $k \to \infty$, with $x_k - Tx_k \to 0$, we have $x_k \to z$ as $k \to \infty$ and $Az = 0$, i.e., $z \in \ker(A)$. But from $\inf_{y \in \ker(A)} \|z - y\| = 1$, a contradiction since this distance should be 0 if $z \in \ker(A)$. ■

4.2.2 Fredholm Alternative

We now study a partial case of the so-called Fredholm alternative.

Theorem 4.2.1 (Fredholm alternative). Let X be a Banach space, $T\colon X \to X$ be compact. Then $I - T$ is injective if and only if $I - T$ is surjective.

Proof. Assume that $A := I - T$ is injective but not surjective, to have a contradiction, we just need to find a sequence $\{f_n\}_{n \geq 1}$ in X^* with $\|f_n\| = 1$ for all $n \geq 1$ such that the sequence $\{T^* f_n\}_{n \geq 1}$ has no converging subsequence, since it will contradict the fact that $T^*(B_{X^*})$ is precompact.

Claim. Let $Y_n := \operatorname{Im}(A^n)$ for $n \geq 1$, then $Y_{n+1} \subsetneq Y_n$ for all n.

Proof. Since A is not surjective, $\operatorname{Im}(A) \neq X$, i.e., $Y_1 \subsetneq Y_0 = X$. Consider $y \notin \operatorname{Im}(A)$, and suppose $\operatorname{Im}(A^{n+1}) = \operatorname{Im}(A^n)$, then there exists x such that $A^{n+1}x = A^n y$, i.e., $A^n(Ax - y) = 0$. From the injectivity of A, $\ker(A^n) = 0$, so $Ax - y = 0$, implying $y \in \operatorname{Im}(A)$ ↯ So Y_{n+1} is properly contained in Y_n for all $n \geq 1$. ⊛

Claim. Y_n are all closed.

Proof. Since $Y_n = \operatorname{Im}(A^n) = \operatorname{Im}((I - T)^n)$, where $(I - T)^n = I - ST = I - \widetilde{T}$ for S being bounded and T being compact, hence $\widetilde{T} = ST$ is compact from Proposition 4.1.1. The result follows from Lemma 4.2.1. ⊛

Now, since Y_n / Y_{n+1} is a Banach space, let $\widetilde{f}_n\colon Y_n / Y_{n+1} \to \mathbb{R}$ be a bounded linear functional with $\|\widetilde{f}_n\| = 1$, which can be found by the supporting functional theorem. Define $f_n\colon Y_n \to \mathbb{R}$ by $f_n(y) = \widetilde{f}_n([y])$ for $y \in Y_n$ with $f_n(y) = 0$ when $y \in Y_{n+1}$, implying that $f_n \in Y_{n+1}^{\perp}$ from the Riesz

[2] T being an integral operator.

representation theorem. Finally, we extend f_n to f_n to $f_n \colon X \to \mathbb{R}$ with $\|f_n\| = 1$ by Hahn Banach theorem to avoid any domain issue.

We now show the sequence $\{T^* f_n\}_{n\geq 1}$ has no converging subsequence by shown that for $n > m \geq 1$, $T^* f_n$ and $T^* f_m$ are pairwise separated.

Claim. For $n > m \geq 1$, $(T^* f_n - T^* f_m)(x) = f_n(x)$.

Proof. We have

$$T^* f_n - T^* f_m = T^*(f_n - f_m) = (I - T^*)(f_m - f_n) + (f_n - f_m)$$

where $f_n \in Y_{n+1}^{\perp}$ and $f_m \in Y_{m+1}^{\perp} \subseteq Y_{n+1}^{\perp}$, so $f_n - f_m \in Y_{n+1}^{\perp}$. Now, observe that $(I - T)x = Ax \in Y_{n+1}$, hence $(I - T^*)(f_n - f_m) = (f_n - f_m)(I - T) = 0$. In all, we have that for $x \in Y_n$, $T^*(f_n - f_m)(x) = (f_n - f_m)(x) = f_n(x)$ from $m < n$ ⊛

This implies that

$$\|T^* f_n - T^* f_m\| = \sup_{\|x\|=1} \|[T^* f_n - T^* f_m](x)\| = \sup_{\|x\|=1} \|f_n(x)\| = \|f_n\| = 1,$$

i.e., all terms in the sequence $\{T^* f_n\}$ are pairwise separated as desired.

Conversely, assume $I - T$ is surjective, we want to prove that $I - T$ is injective. Again, let $A := I - T$, we have $\operatorname{Im}(A) = X$ and $\ker(A^*) = (\operatorname{Im} A)^{\perp}$, implying $\ker(A^*) = \{0\}$, i.e., A^* is injective. Since $A^* = I - T^*$, where T^* is compact, the previous result implies A^* is surjective. Note that $(\ker A)^{\perp} \supseteq (\operatorname{Im} A^*) = X^*$, hence $\ker(A) = \{0\}$, so A is injective. ■

We see that the spectrum in infinite dimension is much more complicated compared to the finite dimension case.

Lecture 20: Spectrum Theory

Fredholm alternative does not hold if T is not compact, see the following example. 8 Nov. 14:30

Example (Shift operator). Consider the shift operators on $\ell_2 = \left\{(a_1, a_2, \dots), a_j \in \mathbb{R}, \sum_{j=1}^{\infty} a_j^2 < \infty\right\}$. The right shift T_r is defined as

$$T_r(a_1, a_2, \dots) = (0, a_1, a_2, \dots),$$

which is injective but not surjective; while the left shift T_ℓ is defined as

$$T_\ell(a_1, a_2, \dots) = (a_2, a_3, \dots),$$

which is surjective but not injective.

The name Fredholm alternative is explained by the following application to solving linear equations of the form

$$\lambda x - Tx = b$$

where $T \in \mathcal{K}(X, X)$, $\lambda \in \mathbb{C}$, $b \in X$. One is interested in existence and uniqueness of solution. Fredholm alternative states that exactly one of the following statements holds for every $\lambda \neq 0$:

- *either* the homogeneous equation $\lambda x - Tx = 0$ has a nontrivial solution,
- *or* the inhomogeneous equation $\lambda x - Tx = b$ has a solution for every b, where this solution is automatically unique.

Also, this alternative is particularly useful for studying integral equations, since for the integral operator

$$(Tf)(t) = \int_0^1 k(t, x) f(s) \, \mathrm{d}s,$$

the homogeneous Fredholm equation is

$$\lambda f(t) - \int_0^1 k(t,x) f(s) \, \mathrm{d}s = 0,$$

while the inhomogeneous Fredholm equation (*of second kind*) is

$$\lambda f(t) - \int_0^1 k(t,s) f(s) \, \mathrm{d}s = b(t).$$

4.3 Spectrum of Bounded Linear Operators

Studying linear operators through their spectral properties is a powerful approach in analysis and mathematical physics. Recall from linear algebra that the spectrum of a linear operator T acting on $\mathbb{C}^n$ consists of the *eigenvalues* of T, which are the numbers $\lambda \in \mathbb{C}$ such that $Tx = \lambda x$ for some nonzero vector $x \in \mathbb{C}^n$; such x are called the *eigenvectors* of T.

There are at most n eigenvalues of T, or one can say exactly n counting multiplicities.[3] Eigenvectors corresponding to different eigenvalues are linearly independent. However, the eigenvalues do not need to form a basis of $\mathbb{C}^n$. The dimension of the span of eigenvectors corresponding to a given eigenvalue (the eigenspace) may be strictly less than the multiplicity of that root. This happens, for example, for the Jordan block

$$T = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

An orthonormal basis of eigenvectors exists if and only if T is normal, i.e. $T^*T = TT^*$.

To start our formal discussion, we first formally define the notion of spectrum points.

Definition. Let X be a complex Banach space, and $T\colon X \to X$ be a bounded linear operator.

Definition 4.3.1 (Regular point)**.** A number $\lambda \in \mathbb{C}$ is called a *regular point* of T if the operator $T - \lambda I$ is invertible, i.e., $(T - \lambda I)^{-1} \in \mathcal{L}(X, X)$.

Definition 4.3.2 (Spectrum point)**.** A number $\lambda \in \mathbb{C}$ is called a *spectrum point* of T if it's not a regular point.

Notation. The set of regular points for T is denoted as $\rho(T)$, while the set of spectrum points is denoted as $\sigma(T)$.

Remark (Resolvent point)**.** We sometimes called a regular point as a *resolvent point*.

From definitions, we know that $\sigma(T) = \mathbb{C} - \rho(T)$.

4.3.1 Classification of Spectrum

For operators T acting on a finite dimensional space, the spectrum consists of eigenvalues of T; in infinite dimensions, this is not true, as there are various reason why $T - \lambda I$ may be non-invertible.

Let's first see some examples illustrating the fact that the spectrum is a richer concept in infinite-dimensional spaces than in finite-dimensional spaces.

Example (Uncountable number of eigenvalues)**.** Consider the differential operator $T = \mathrm{d}/\mathrm{d}t$ acting on $C^1(\mathbb{C})$. Suppose λ is an eigenvalue of T with eigenvector $u \in C^1(\mathbb{C})$. This implies that the ordinary differential equation $u' = \lambda u$ holds. The solution has the form $u(t) = Ce^{\lambda t}$, so every $\lambda \in \mathbb{C}$ is an eigenvalue of T.

[3] Eigenvalues always exist by the fundamental theorem of algebra, as they are the roots of the characteristic polynomial $\det(T - \lambda I) = 0$.

Example (No eigenvalues). Consider a multiplication operator on $L^2([0,1])$ acting as

$$(Tf)(t) = tf(t).$$

Suppose λ is an eigenvalue of T with eigenvector $f \in L^2([0,1])$. This implies

$$tf(t) = \lambda f(t)$$

for all $t \in [0,1]$. But this means $f = 0$, so T has no eigenvalues.

These reasons cause the following different types of spectrum.

Definition (Classification of spectrum). Let T be a bounded linear operator on X.

Definition 4.3.3 (Point spectrum). The *point spectrum* $\sigma_p(T)$ contains λ such that $\ker(T-\lambda I) \neq \{0\}$, i.e., $T - \lambda I$ is not injective, i.e., $\lambda \in \sigma_p(T)$ if λ is an eigenvalue of T.

Definition 4.3.4 (Continuous spectrum). The *continuous spectrum* $\sigma_c(T)$ contains λ such that $\ker(T - \lambda I) = \{0\}$ ($\lambda \notin \sigma_p(T)$) and $\operatorname{Im}(T - \lambda I)$ is dense in X, i.e., $\lambda \in \sigma_c(T)$ if λ is not an eigenvalue and $\operatorname{Im}(T - \lambda I) \neq X$ but $\overline{\operatorname{Im}(T - \lambda I)} = X$.[a]

[a]Note that by inverse mapping theorem, if $\operatorname{Im}(T - \lambda I) = X$, then $\lambda \in \rho(T)$.

Definition 4.3.5 (Residual spectrum). The *residual spectrum* $\sigma_r(T)$ is defined as $\sigma_r(T) \coloneqq \sigma(T) - \sigma_p(T) - \sigma_c(T)$, i.e., $\lambda \in \sigma_r(T)$ if λ is not an eigenvalue and $\overline{\operatorname{Im}(T - \lambda I)} \neq X$.

From the above, we have

$$\sigma(T) = \sigma_p(T) \sqcup \sigma_c(T) \sqcup \sigma_r(T).$$

Let's now compute and classify the spectrum of some basic linear operators.

Example (Diagonal operator on ℓ_2). Let $\{\lambda_k\}_{k\geq 1}$ be a sequence in $\mathbb{C} \setminus \{0\}$ such that $\lim_{k\to\infty} \lambda_k = 0$. Define $T\colon \ell_2 \to \ell_2$ by

$$T(\{x_k\}_{k\geq 1}) = \{\lambda_k x_k\}_{k\geq 1}$$

where the sequence $\{\lambda_k\}_{k\geq 1}$ is bounded, hence T is a bounded linear operator. Then, $(T - \lambda I)x = \{(\lambda_k - \lambda)x_k\}_{k\geq 1}$, so given $y = \{y_k\}_{k\geq 1}$,

$$(T - \lambda I)^{-1} y = \left\{ \frac{y_k}{\lambda_k - \lambda} \right\}_{k\geq 1},$$

which implies $(T - \lambda I)^{-1}$ is bounded on ℓ_2 if $\sup_{k\geq 1} \lvert\lambda_k - \lambda\rvert^{-1} < \infty$. Indeed, since $\lim_{k\to\infty} \lambda_k = 0$, $\sup_{k\geq 1} \lvert\lambda_k - \lambda\rvert^{-1} < \infty$ if $\lambda \notin \{\lambda_k\}_{k\geq 1} \cup \{0\}$. Further,

- if $\lambda = \lambda_k$ then $\ker(T - \lambda I) \neq \{0\}$;
- if $\lambda = 0$, then $\operatorname{Im}(T - \lambda I)$ is dense in ℓ_2, and notice that $\ker(T - \lambda I) = \{0\}$,[a] so 0 is in the continuous spectrum.

Hence, $\sigma(T) = \{\lambda_k\}_{k\geq 1} \cup \{0\}$ with $\sigma_p(T) = \{\lambda_k\}_{k\geq 1}$, $\sigma_c(T) = \{0\}$ with $\sigma_r(T) = \varnothing$.

[a]Notice that $\lambda_k \in \mathbb{C} \setminus \{0\}$.

Let's revisit previous example on the multiplication operator on L^2.

Example (Multiplication operator on L^2). Consider the multiplication on $L^2([0,1])$ such that $Tf(t) =$

$tf(t)$ for $0 < t < 1$. Then

$$(T - \lambda I)f(t) = (t - \lambda)f(t) \Rightarrow (T - \lambda I)^{-1}g(t) = \frac{g(t)}{t - \lambda}$$

for $0 < t < 1$. Hence, $T - \lambda I$ is invertible if $\lambda \notin [0,1]$, so $\sigma(T) = [0,1]$. And if $\lambda \in [0,1]$, then $\ker(T - \lambda I) = 0$, i.e., $\sigma_p(T) = \varnothing$. Lastly, since $\operatorname{Im}(T - \lambda I)$ is dense in $L^2([0,1])$ if $\lambda \in [0,1]$, hence $\sigma_c(T) = [0,1]$, so $\sigma_r(T) = \varnothing$.

4.3.2 Properties of Spectrum

In this section, let X denotes a Banach space and $T \in \mathcal{L}(X, X)$. We'll see that studying the spectrum of T is convenient via the so-called resolvent operator.

Definition 4.3.6 (Resolvent operator). To each regular point $\lambda \in \rho(T)$ for $T \in \mathcal{L}(X, X)$, the associated *resolvent operator* $R(\lambda) = (T - \lambda I)^{-1}$ is defined by $R\colon \rho(T) \to \mathcal{L}(X, X)$.[a]

[a] This is the so-called resolvent function.

The resolvent operator can be computed in terms of series expansion involving T. This technique is based on the following simple lemma.

Lemma 4.3.1 (Von Neumann). Let $S \in \mathcal{L}(X, X)$ such that $\|S\| < 1$, then $I - S$ is invertible and $(I - S)^{-1}$ is given by the geometric series

$$(I - S)^{-1} = \sum_{k=0}^{\infty} S^k \text{ and } \|(I - S)^{-1}\| \leq \frac{1}{1 - \|S\|}.$$

Proof. From the inequality $\|S^k\| \leq \|S\|^k$ for all k, $\sum_{k=0}^{\infty} S^k$ converges absolutely. Hence,

$$(I - S)\sum_{k=0}^{\infty} S^k = \sum_{k=0}^{\infty} S^k(I - S) = I$$

by telescoping series. Finally,

$$\|(I - S)^{-1}\| \leq \sum_{k=0}^{\infty} \|S\|^k \leq \frac{1}{1 - \|S\|}.$$

■

Proposition 4.3.1. The resolvent set $\rho(T) \subseteq \mathbb{C}$ is open and containing $\{\lambda \in \mathbb{C} \colon |\lambda| > \|T\|\}$, with $\|R(\lambda)\| \leq 1/(|\lambda| - \|T\|)$.

Proof. Since

$$(T - \lambda I)^{-1} = -\lambda^{-1}(I - \lambda^{-1}T)^{-1} = -\lambda^{-1}(I - S),$$

by letting $S = T/\lambda$, we have $\|S\| < 1$ if $|\lambda| > \|T\|$, hence by Lemma 4.3.1,

$$\|(T - \lambda I)^{-1}\| \leq \frac{1}{|\lambda|}\frac{1}{1 - \|S\|} = \frac{1}{|\lambda|}\frac{1}{1 - |\lambda|^{-1}\|T\|} = \frac{1}{|\lambda| - \|T\|},$$

i.e., if $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$. To show $\rho(T)$ is open, since

$$\frac{1}{x - \lambda} - \frac{1}{x - \mu} = \frac{\lambda - \mu}{(x - \lambda)(x - \mu)}$$

for all $\mu, \lambda \in \mathbb{C}$ and $x \in \mathbb{C}$, we can generalize this to[a]

$$R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu)$$

since $R(\lambda), R(\mu)$ commutes. Hence,

$$R(\mu) = \left[I - (\mu - \lambda)R(\lambda)\right]^{-1} R(\lambda) = (I - S)^{-1} R(\lambda),$$

so $R(\mu)$ is bounded if $\|S\| < 1$, i.e., $|\mu - \lambda| \|R(\lambda)\| < 1$. We see that $\lambda \in \rho(T)$ implies that the disk $D(\lambda, r) \subseteq \rho(T)$ if $r\|R(\lambda)\| < 1$, i.e., $\rho(T)$ is open. ■

[a] This is known as the resolvent identity.

Proposition 4.3.1 implies the following.

Proposition 4.3.2. The spectrum $\sigma(T)$ is a closed bounded set with

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}.$$

Proof. From Proposition 4.3.1, we have shown that $\sigma(T)$ is a closed set since $\rho(T)$ is open and $\mathbb{C} \setminus \rho(T) = \sigma(T)$, together with $\sigma(T)$ being bounded such that $\sigma(T) \subseteq \overline{D(0, \|T\|)}$. ■

4.3.3 Spectral Radius

The spectrum of any operator $T \in \mathcal{L}(X, X)$ is a bounded set by Proposition 4.3.2, and moreover, we have a quantitative bound $|\lambda| \leq \|T\|$ for all $\lambda \in \sigma(T)$. This bound is not always sharp, and we will try to come up with a sharper bound.

To do this, we first introduce the notion of spectral radius.

Definition 4.3.7 (Spectral radius). The *spectral radius* of an operator $T \in \mathcal{L}(X, X)$ is defined as

$$r(T) \coloneqq \max_{\lambda \in \sigma(T)} |\lambda|.$$

From Proposition 4.3.2, $r(T) \leq \|T\|$, but the equality is not always achieved. However, we have the Gelfand's formula, which characterizes the asymptomatic behavior of $r(T)$, essentially stating that

$$\|T^n\| \sim r(T)^n.$$

Theorem 4.3.1 (Gelfand's formula). Let $T \in \mathcal{L}(X, X)$ on a Banach space X, then

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} = \inf_{n \geq 1} \|T^n\|^{1/n}.$$

Proof. Let r be a large integer and $m \geq 1$ an integer with $r = am + b$ for $a \geq 0$, $0 \leq b < m$. Then

$$\|T^r\| = \|T^{am} T^b\| \leq \|T^m\|^a \|T^b\|,$$

hence

$$\|T^r\|^{1/r} \leq \|T^m\|^{a/r} \|T^b\|^{1/r} = \left(\|T^m\|^{1/m}\right)^{\frac{a}{a+b/m}} \|T^b\|^{1/r}.$$

Let $r \to \infty$, we have $\limsup_{r \to \infty} \|T^r\|^{1/r} \leq \|T^m\|^{1/m}$ for all $m \geq 1$ since $a/(a + b/m) \to 1$ and $\|\cdot\|^{1/r} \to 1$, hence

$$\limsup_{r \to \infty} \|T^r\|^{1/r} = \liminf_{r \to \infty} \|T^r\|^{1/r} = \inf_{m \geq 1} \|T^m\|^{1/m}.$$

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Proof of Theorem 4.3.1 (Continued). Now, the next goal is to show that $\sigma(T)$ is nonempty.

Claim. $\sigma(T)$ is nonempty

Proof. Toward a contradiction, assume that $\rho(T) = \mathbb{C}$, then $R(\lambda) = (T - \lambda I)^{-1}$, which is defined for all $\lambda \in \mathbb{C}$ such that

$$\|R(\lambda)\| \leq \frac{1}{|\lambda| - \|T\|}$$

if $|\lambda| > \|T\|$ by Proposition 4.3.1, hence $\lim_{|\lambda|\to\infty}\|R(\lambda)\| = 0$. Let $f(\cdot)$ be a bounded linear functional on $\mathcal{L}(X, X)$, i.e., $f \in \mathcal{L}(X, X)^*$. Set $g(\lambda) = f(R(\lambda))$, for any $\lambda_0 \in \rho(T)$,

$$R(\mu) = \left(\sum_{n=0}^{\infty}(\mu - \lambda)^n R(\lambda)^n\right) R(\lambda),$$

implying that $g(\lambda)$ is a analytic function since it's locally given by a convergent power series. Also, since $\lim_{|\lambda|\to\infty}|g(\lambda)| = 0$, Liouville's theorem implies $g(\cdot) \equiv 0$, so $f(R(\lambda)) \equiv 0$ for all $f \in \mathcal{L}(X, X)^*$. From Hahn-Banach theorem, $R(\lambda) = 0$, which is a contradiction since $R(\lambda)(T - \lambda I) = I$. ⊛

Claim. $r(T) \leq \lim_{n\to\infty}\|T^n\|^{1/n}$.

Proof. To see this, we use the fact that if $\lambda \in \mathbb{C}$, we have $\lambda^n \in \rho(T^n)$, i.e., $(T^n - \lambda^n I)^{-1}$ exists, then $\lambda \in \rho(T)$ because $(T^n - \lambda^n I) = (T - \lambda I)\sum_{j=0}^{n-1}\lambda^j T^{n-1-j}$, we have

$$(T - \lambda I)^{-1} = (T^n - \lambda^n I)^{-1}\sum_{j=0}^{n-1}\lambda^j T^{n-1-j}.$$

Hence, $|\lambda^n| > \|T^n\|$, implying that $\lambda \in \rho(T)$, so $r(T) \leq \|T^n\|^{1/n}$ for all $n \geq 1$. ⊛

Finally, we show the following.

Claim. $r(T) \geq \limsup_{n\geq 1}\|T^n\|^{1/n}$.

Proof. We use the Taylor expansion

$$R(\lambda) = (T - \lambda I)^{-1} = -\sum_{n=0}^{\infty}\lambda^{-(n+1)}T^n.$$

Let $f \in \mathcal{L}(X, X)^*$, set $g(\lambda) = f(R(\lambda))$, we have

$$g(\lambda) = -\sum_{n=0}^{\infty}\lambda^{-(n+1)}f(T^n),$$

where $g(\lambda)$ is analytic for $|\lambda| > r(T)$. Hence, the Laurent series for $g(\lambda)$ converges if $|\lambda| > r(T)$, i.e., $\sup_{n\geq 1}|\lambda^{-n}f(T^n)| < \infty$ if $|\lambda| > r(T)$, which is true for all $f \in \mathcal{L}(X, X)^*$. From the uniform boundedness theorem, $\sup_{n\geq 1}|\lambda|^{-n}\|T^n\| < \infty$ if $|\lambda| > r(T)$, i.e., if $|\lambda| > r(T)$, then $|\lambda| \geq \limsup_{n\to\infty}\|T^n\|^{1/n}$, implying $r(T) \geq \limsup_{n\to\infty}\|T^n\|^{1/n}$. ⊛

In all, we conclude that $r(T) = \lim_{n\to\infty}\|T^n\|^{1/n}$, proving the result. ■

Note. Gelfand's formula is an improvement upon Proposition 4.3.2.

Proof. Clearly, we see that

$$r(T) \leq \|T^n\|^{1/n} \leq \|T\|^{n \cdot 1/n} \leq \|T\|.$$

⊛

4.4 Spectrum of Compact Operators

As compact operators are proxies of finite rank operators, one is able to fully classify their spectrum. First, for every $T \in \mathcal{K}(X, X)$, $0 \in \sigma(T)$ from T is not invertible because isomorphism is not compact.

Theorem 4.4.1 (Point spectrum of compact operators). Let $T \in \mathcal{K}(X, X)$ on a Banach space X. Then for every $\epsilon > 0$, there exists a finite number of linearly independent eigenvectors corresponding to eigenvalues $\lambda \in \mathbb{C}$ with $|\lambda| > \epsilon$.

Proof. Toward a contradiction, we show that we can obtain a sequence $\{y_n\}_{n\geq 1}$ such that $\|y_n\| = 1$ for all $n \geq 1$, but the sequence $\{Ty_n\}_{n\geq 1}$ has no converging subsequence. This contradicts to the compactness of T.

From the assumption, there exists a sequence $\{x_k\}_{k\geq 1}$ in X such that $x_k \neq 0$ and are all linearly independent vectors with $Tx_k = \lambda_k x_k$ and $|\lambda_k| \geq \epsilon > 0$ for all $k \geq 1$. Let E_n be the span of $\{x_1, \dots, x_n\}$, so $E_1 \subseteq E_2 \subseteq \dots$. We can then choose $y_n \in E_n$ with $\|y_n\| = 1$ such that

$$\operatorname{dist}(y_n, E_{n-1}) = \inf_{y \in E_{n-1}} \|y_n - y\| \geq 1/2.$$

Now, to show that $\{Ty_n\}_{n\geq 1}$ contains no Cauchy subsequences, we express y_n as a linear combination

$$y_n = \sum_{k=1}^{n} a_k^{(n)} x_k = a_n^{(n)} x_n + u_{n-1} \text{ for } u_{n-1} \in E_{n-1},$$

then

$$Ty_n = a_n^{(n)} \lambda_n x_n + v_{n-1} \text{ for } v_{n-1} = Tu_{n-1} \in E_{n-1}.$$

Let $n > m$, then

$$\begin{aligned}
\|Ty_n - Ty_m\| &= \|\lambda_n a_n^{(n)} x_n + w_{n-1}\| && \text{where } w_{n-1} \in E_{n-1} \\
&= \|\lambda_n y_n + \widetilde{y}\| && \text{where } \widetilde{y} \in E_{n-1} \\
&= |\lambda_n| \|y_n + \widetilde{y}'\| && \text{where } \widetilde{y}' \in E_{n-1} \\
&\geq \frac{|\lambda_n|}{2} && \|y_n + \widetilde{y}'\| \geq 1/2 \text{ when } \widetilde{y}' \in E_{n-1} \\
&\geq \frac{\epsilon}{2},
\end{aligned}$$

so there are no converging subsequences for $\{Ty_n\}_{n\geq 1}$. We see that if $y_n \in B_X$ for $n \geq 1$, if T is compact, TB_X is precompact, which is a contradiction. ■

Remark. Consequently, from Theorem 4.4.1, the point spectrum $\sigma_p(T)$ is at most countable, and it lies in a sequence that converges to 0. Also, each eigenvalue λ_k of T has finite multiplicity, i.e., $\dim \ker(T - \lambda_k I) < \infty$.

Theorem 4.4.1 implies that there are countable many point spectrums, and the only possible accumulation point is 0, and indeed, for a compact operator, the only spectrum is either in $\sigma_p(T)$ or 0 as guaranteed by Proposition 4.4.1.

Proposition 4.4.1 (Classification of spectrum of compact operators). Let $T \in \mathcal{K}(X, X)$, then $\sigma_p(T)$ is countable and $\sigma(T) = \sigma_p(T) \cup \{0\}$.

Proof. By noncompactness of unit ball from Riesz's theorem, $0 \in \sigma(T)$ as noted at the beginning of the section. Recall that if $\lambda \neq 0$, $(T - \lambda I)$ is not surjective,[a] and hence λ is an eigenvalue by Fredholm alternative, so all others $0 \neq \lambda \in \sigma(T)$ is in $\sigma_p(T)$, i.e., $\sigma(T) = \sigma_p(T) \cup \{0\}$. ■

[a] Otherwise, from inverse mapping theorem, $T - \lambda I$ is invertible so $\lambda \in \rho(T)$, contradiction.

Recall that if $T \in \mathcal{L}(X, X)$, then $T^* \in \mathcal{L}(X^*, X^*)$, and we have $\|T^*\| = \|T\|$. Now, we want to find the relation between $\sigma(T^*)$ and $\sigma(T)$.

Theorem 4.4.2. Let $T \in \mathcal{L}(X, X)$ and $T^* \in \mathcal{L}(X^*, X^*)$, then we have the following.

(a) $\sigma(T^*) = \sigma(T)$.

(b) If $\lambda \in \sigma_r(T)$, then $\lambda \in \sigma_p(T^*)$.

(c) If $\lambda \in \sigma_p(T)$, then $\lambda \in \sigma_p(T^*) \cup \sigma_r(T^*)$.

Proof. We prove this one by one.

(a) We first show that $\rho(T) \subseteq \rho(T^*)$. Suppose $\lambda \in \rho(T)$, i.e., $(T - \lambda I)^{-1}$ exists, so $(T - \lambda I)^*$ is also invertible and

$$[(T - \lambda I)^*]^{-1} = (T^* - \lambda I)^{-1}.$$

Hence, $\lambda \in \rho(T^*)$, so we have $\rho(T) \subseteq \rho(T^*)$. To show that $\rho(T^*) \subseteq \rho(T)$, we need to show that if $S \in \mathcal{L}(X, X)$ and S^* is invertible, then S is invertible. We first show that S is injective, i.e., $\ker(S) = \{0\}$. Suppose not, then there exists $x \neq 0$ such that $Sx = 0$, implying $S^* f(x) = f(Sx) = 0$ for all $f \in X^*$. But since S^* is invertible, i.e., $\operatorname{Im} S^* = X^*$, implying $g(x) = 0$ for all $g \in X^*$. Then from Hahn-Banach theorem, $x = 0$, which is a contradiction. Next, we want to show S is surjective, i.e., $\operatorname{Im}(S) = X$. Since S^* is invertible, there exists $\epsilon > 0$ such that

$$\|S^* f\|_{X^*} \geq \epsilon \|f\|_{X^*}$$

for all $f \in X^*$. We now use this to show that $\overline{SB_X}$ contains a ball of radius ϵ. Let $x_0 \in X$ lies outside $\overline{SB_X}$. Since $\overline{SB_X}$ is closed and convex, the separation theorem states that there exists $f_0 \in X^*$ such that $f_0(x_0) > 1$ and $f_0(y) \leq 1$ for all $y \in \overline{SB_X}$. Note that $S^* f_0(x) = f_0(Sx) \leq 1$ for all $x \in B_X$, we have $\|S^* f_0\|_{X^*} \leq 1$. Hence, we conclude that

$$\epsilon < \epsilon f_0(x_0) \leq \epsilon \|f_0\|_{X^*} \|x_0\|_X \leq \|S^* f_0\|_{X^*} \|x_0\|_X \leq \|x_0\|_X,$$

so $\epsilon B_X \subseteq \overline{SB_X}$. Now, we use the argument from open mapping theorem to conclude that $\overline{SB_X} \subseteq S(2B_X)$. Hence, $S(2B_X) \subseteq \epsilon B_X$, i.e., S is surjective.

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Proof of Theorem 4.4.2 (Continued). We now continue to prove Theorem 4.4.2.

(b) Let $S \in \mathcal{L}(X, X)$, then we already observed that $(\operatorname{Im} S)^\perp = \ker(S^*)$. Assume that $\overline{\operatorname{Im} S} \neq X$, by separation theorem, there exists $f_0 \in X^*$, $f_0 \neq 0$ and $f_0 \in (\operatorname{Im} S)^\perp = \ker(S^*)$.

(c) We need to show that if $S \in \mathcal{L}(X, X)$ and $\ker(X) \neq \{0\}$, then $\operatorname{Im} S$ is not dense in X. Assume $Sx = 0$ and $x \neq 0$, since $\ker S \neq \{0\}$, then

$$S^* f(x) = f(Sx) = 0$$

for all $f \in X^*$. Hence, if $g \in \operatorname{Im} S^*$, then $g(x) = 0$. If $\operatorname{Im} S^*$ is dense in X^*, conclude that $g(x) = 0$ for all g, implying that $x = 0$, contradiction.

■

Remark. The latter two claims in Theorem 4.4.2 can be summarized as

$$\sigma_r(T) \subseteq \sigma_p(T^*) \subseteq \sigma_r(T) \cup \sigma_p(T).$$

4.5 Spectrum of Unitary Operators

In this section, let $\mathcal{H}$ denotes a Hilbert space.

As previously seen (Unitary operator). An operator $U \in \mathcal{L}(\mathcal{H})$ is a *unitary operator* if U is a bijective isometry on $\mathcal{H}$, i.e., $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$.

Lemma 4.5.1. The operator $U \in \mathcal{L}(\mathcal{H})$ is unitary if and only if $U^*U = UU^* = I$.

Proof. Suppose $U^*U = I$, then

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \langle U^*Ux, x\rangle = \langle x, x\rangle = \|x\|^2,$$

so U is an isometry. Conversely, suppose $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$, then this implies $U^*U = I$ since with polarization identity,

$$\begin{aligned}\langle U^*Ux, y\rangle = \frac{1}{4}\left[\langle U^*U(x+y), x+y\rangle - \langle U^*U(x-y), x-y\rangle\right. \\ \left. + i\langle U^*U(x+iy), x+iy\rangle - i\langle U^*U(x-iy), x-iy\rangle\right],\end{aligned}$$

and by the assumption, $\langle U^*Uz, z\rangle = \langle z, z\rangle$ for all $z \in \mathcal{H}$, so $\langle U^*, Ux, y\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$, so $U^*U = I$. ∎

Remark. Lemma 4.5.1 states that U is unitary if and only if U is invertible and $U^{-1} = U^*$.

Proposition 4.5.1. Let $U \in \mathcal{L}(\mathcal{H})$ be unitary, then

$$\sigma(U) \subseteq \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}.$$

Proof. Since $\|U\| = \|U^{-1}\| = 1$, we know that $\|U\| \leq 1$ implies $\rho(U)$ contains $\{\lambda \in \mathbb{C} \colon |\lambda| > 1\}$; and $\|U^{-1}\| \leq 1$ implies that $\rho(U^{-1})$ contains $\{\lambda \in \mathbb{C} \colon |\lambda| > 1\}$, so $\lambda \in \rho(U^{-1}) \Leftrightarrow \lambda^{-1} \in \rho(U)$. So,

$$(U^{-1} - \lambda I)^{-1} = \lambda^{-1}U(\lambda^{-1}I - U)^{-1}.$$

If $\lambda \neq 0$, then we can conclude that $\rho(U)$ contains $\{\lambda \in \mathbb{C} \colon |\lambda| \neq 1\}$. ∎

Finally, let's see an example.

Example (Shift operator). Let T be the left shift operator on ℓ_1, i.e.,

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots).$$

Notice that $\ell_1^* = \ell_\infty$,[a] so T^* is the right shift operator on ℓ_∞, i.e.,

$$T^*(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

Since $\|T\| = \|T^*\| = 1$, hence $\lambda \in \rho(T)$ if $|\lambda| > 1$. Suppose $|\lambda| < 1$, then the vector $x_\lambda = (1, \lambda, \lambda^2, \dots)$ is in ℓ_1 such that $Tx_\lambda = \lambda x_\lambda$, so $\lambda \in \sigma_p(T)$. With $\sigma(T)$ is closed, we have

$$\sigma(T) = \{\lambda \in \mathbb{C} \colon |\lambda| \leq 1\}.$$

Next, we want to show that T^* has no point spectrum. Suppose $a = (a_1, a_2, \dots \in \ell_\infty)$ and $(T \pm \lambda I)a = 0$. Then

$$\lambda a_1 = 0, \quad \lambda a_2 - a_1 = 0, \quad \lambda a_3 - a_2 = 0, \quad \dots$$

We then see that if $\lambda \neq 0$, then $a \equiv 0$; if $\lambda = 0$, then $a \equiv 0$ as well, hence $\sigma_p(T^*)$ is empty. By Theorem 4.4.2, $\lambda \in \sigma_p(T) \Rightarrow \lambda \notin \sigma_c(T^*)$. Now, consider

- $|\lambda| < 1$: we further have $\lambda \in \sigma_p(T)$ and $\lambda \notin \sigma_p(T^*)$, hence $\lambda \in \sigma_r(T^*)$.
- $|\lambda| = 1$: in this case, we want to show $\lambda \in \sigma(T) = \sigma(T^*)$. It's clear that $\lambda \notin \sigma_p(T)$. On the other hand, if $\lambda \in \sigma_r(T)$, then $\lambda \in \sigma_p(T^*)$ by Theorem 4.4.2, with the fact that $\sigma_p(T^*) = \varnothing$, so $\sigma_r(T) = \varnothing$. So we conclude that if $|\lambda| = 1$, $\lambda \in \sigma_c(T)$.

Finally, we show that if $|\lambda| = 1$, then $\lambda \in \sigma_r(T^*)$. To do this, we shall find an open ball disjoint from $\operatorname{Im}(T^* - \lambda I)$. Suppose $a = \{a_n\}_{n\geq 1}$ and $b = \{b_n\}_{n\geq 1}$ are in ℓ_∞ with $a = (\lambda I - T^*)b$, hence

$$a_1 = \lambda b_1, \quad a_2 = \lambda b_2 - b_1, \quad a_3 = \lambda b_3 - b_3, \quad \ldots$$

This is equivalent to write

$$\begin{aligned} b_1 &= \frac{a_1}{\lambda} = \overline{\lambda} a_1, \\ b_2 &= \frac{a_1}{\lambda} + \frac{b_1}{\lambda} = \frac{a_2}{\lambda} + \frac{a_1}{\lambda^2} = \overline{\lambda}^2 (a_1 + \lambda a_2), \\ &\vdots \\ b_n &= \overline{\lambda}^{n+1} \sum_{m=1}^{n} \lambda^m a_m. \end{aligned}$$

Define $c = \{c_n\}_{n\geq 1}$ such that $c_n = \overline{\lambda}^n$. Suppose $d \in \ell_\infty$, $\|d - c\|_\infty < 1/2$. Then

$$\operatorname{Re}\{\lambda^n d_n\} \geq \operatorname{Re}\{\lambda^n c_n\} - \|d - c\|_\infty \geq \frac{1}{2}.$$

If $(\lambda I - T^*)e = d$ for some $e = \ell_\infty$, then

$$e = \overline{\lambda}^{n+1} \sum_{m=1}^{n} \lambda^m d_m,$$

implying that $|e_n| \geq n/2$, i.e., $e \notin \ell_\infty$, contradiction. We then conclude that $\operatorname{Im}(\lambda I - T^*)$ does not intersect ball centered at c with radius $1/2$, hence $\lambda \in \sigma_r(T^*)$.

[a]While $\ell_\infty^* \neq \ell_1$.

Chapter 5

Self-Adjoint Operators on Hilbert Spaces

Throughout this chapter, $\mathcal{H}$ will denote a Hilbert space, and we will study bounded self-adjoint operators on $\mathcal{H}$, i.e., $T \in \mathcal{L}(\mathcal{H}, \mathcal{H}) =: \mathcal{L}(\mathcal{H})$.

5.1 Spectrum of Self-Adjoint Operators

Recall that in a Hilbert space $\mathcal{H}$, given $T \in \mathcal{L}(\mathcal{H})$, the adjoint $T^* \in \mathcal{L}(\mathcal{H})$ is defined by

$$\langle T^* x, y\rangle = \langle x, Ty\rangle$$

for $x, y \in \mathcal{H}$. An interesting situation is the following.

Definition 5.1.1 (Self-adjoint operator)**.** Let $T \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, then T is *self-adjoint* if for all $x, y \in \mathcal{H}$,

$$\langle Tx, y\rangle = \langle x, Ty\rangle .$$

There are lots of examples of self-adjoint operators.

Example (Hermitian matrix)**.** The linear operators on $\mathbb{C}^n$ given by Hermitian matrices $A = (a_{ij})$ is self-adjoint.

Proof. Since $a_{ij} = \overline{a_{ij}}$. ⊛

Example (Integral operator)**.** The integral operators T on $L^2([0,1])$ with Hermitian symmetric kernels $k(s,t)$ given by

$$(Tf)(t) = \int_0^1 k(s,t) f(s) \,\mathrm{d}s$$

is self-adjoint.

Proof. Since $k(s,t) = \overline{k(s,t)}$. ⊛

Also, the orthogonal projection P on $\mathcal{H}$ is self-adjoint.

Remark. Every $A \in \mathcal{L}(\mathcal{H})$ can be represented as $A = T + iS$ with T, S being self-adjoint.

Proof. If $A = T + iS$, then $A^* = T - iS$. Solving these two equations gives

$$T = \frac{A + A^*}{2}, \quad S = \frac{A - A^*}{2i}.$$

⊛

5.1.1 Quadratic Form and Norm of Self-Adjoint Operators

An important object in the study of self-adjoint operator is the quadratic form $f\colon \mathcal{H} \to \mathbb{R}$ where

$$f(x) = \langle Tx, x \rangle$$

for $x \in \mathcal{H}$, where $f(\cdot)$ is real since $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$. Furthermore, $f(\cdot)$ determines T uniquely by the generalized polarization identity[1]

$$\langle Tx, y\rangle = \frac{1}{4}\left[f(x+y) - f(x-y) + if(x+iy) - if(x-iy)\right].$$

Since $f(\cdot)$ determines T uniquely, we should be able to compute properties of T using f. The first property is the following.

Proposition 5.1.1. For every self-adjoint operator $T \in \mathcal{L}(\mathcal{H})$,

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Proof. Firstly, from Cauchy-Schwarz, we have $|\langle Tx, x\rangle| \leq \|Tx\|\|x\|$, implying that

$$\sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \sup_{\|x\|=1} \|Tx\| = \|T\|.$$

To get the equality, we use the polarization identity, where

$$\operatorname{Re}\langle Tx, y\rangle = \frac{1}{4}\left[\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right].$$

Let $M := \sup_{\|x\|=1} |\langle Tx, x\rangle|$, we have

$$\operatorname{Re}\langle Tx, y\rangle \leq \frac{M}{4}\left[\|x+y\|^2 + \|x-y\|^2\right] = \frac{M}{4}\left[2\|x\|^2 + 2\|y\|^2\right],$$

where the last equality follows from the parallelogram law. This implies that

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle| = \sup_{\|x\|=\|y\|=1} \operatorname{Re}\langle Tx, y\rangle \leq M.$$

■

5.1.2 Criterion of Spectrum Points

We now study the spectrum of self adjoint operators $T \in \mathcal{L}(\mathcal{H})$. The goal is to show that the whole spectrum of T is real, i.e., $\sigma(T) \subseteq \mathbb{R}$. Let's start with some basic facts.

Lemma 5.1.1. Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint, then $\sigma_p(T) \subseteq \mathbb{R}$ and $\sigma_r(T) = \varnothing$.

Proof. To show $\sigma_p(T) \subseteq \mathbb{R}$, suppose $Tx = \lambda x$, then

$$\begin{cases} \langle Tx, x\rangle = \langle \lambda x, x\rangle = \lambda\|x\|^2; \\ \langle x, Tx\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\|x\|^2 \end{cases} \Rightarrow (\lambda - \overline{\lambda})\|x\|^2 = 0.$$

Since $x \neq 0$, this implies that $\lambda - \overline{\lambda} = 0$, i.e., $\lambda \in \mathbb{R}$.

To show $\sigma_r(T) = \varnothing$, let $\lambda \in \sigma_r(T)$, then $\ker(T - \lambda I) = \{0\}$ and $\operatorname{Im}(T - \lambda I)$ is not dense in $\mathcal{H}$. Notice that since $\lambda \notin \sigma_p(T)$, then $\overline{\lambda} \notin \sigma_p(T)$ as we just proved. We then have

$$\{0\} = \ker(T - \overline{\lambda} I) = \ker(T - \lambda I)^*,$$

[1] When $T = I$, we get back the usual polarization identity.

with Proposition 2.5.2,

$$\operatorname{Im}(T-\lambda I)^{\perp} = \ker(T-\lambda I)^{*} = \ker(T-\overline{\lambda} I) = \{0\}\,,$$

hence $\operatorname{Im}(T-\lambda I)$ is dense in $\mathcal{H}$, a contradiction. So λ does not exist, proving that $\sigma_r(T) = \varnothing$. ■

Lecture 23: Spectral Theorems, Continuous Functional Calculus

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Lemma 5.1.2 (Invertibility criterion). Let T be a bounded self-adjoint operator on $\mathcal{H}$, then T is invertible if and only if T is bounded below, i.e., for all $x \in \mathcal{H}$, there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$.

Proof. If T is invertible, then T is bounded below with $c = \|T^{-1}\|^{-1}$.

Conversely, if T is bounded below, then T is injective and $\operatorname{Im} T$ is closed by the isomorphic embedding criterion. Since $\sigma_r(T)$ is empty, so $0 \neq \sigma_r(T)$, hence the injectivity of T implies that $\operatorname{Im} T$ is dense in $\mathcal{H}$, so $\operatorname{Im} T = \mathcal{H}$, i.e., T is surjective and injective. With inverse mapping theorem, T is invertible with $\|T^{-1}\| \leq c^{-1}$. ■

We see that by applying the invertibility criterion for the operator $T - \lambda I$, we immediately obtain the following.

Corollary 5.1.1 (Criterion of spectrum points). Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator, then $\lambda \in \sigma(T)$ if and only if the operator $T - \lambda I$ is not bounded below.

Finally, we introduce the following.

Definition. Let $T \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$.

Definition 5.1.2 (Approximate eigenvalue). A number $\lambda \in \sigma(T)$ for which $T - \lambda I$ is not bounded below is called an *approximate eigenvalue* of T.

Definition 5.1.3 (Approximate point spectrum). The set of all approximate eigenvalues of T is called the *approximate point spectrum* of T.

Criterion of spectrum points states that for self-adjoint operators, the whole spectrum is the approximate point spectrum.

The reason for the name *approximate* is the following. If λ is an eigenvalue, then $(T - \lambda I)x = 0$ for some x with $\|x\| = 1$. If λ is an approximate eigenvalue, then $(T - \lambda I)x$ can be made *arbitrarily close* to 0 for some x with $\|x\| = 1$. So, eigenvalues of T form the point spectrum $\sigma_p(T)$ while the approximate eigenvalues of T form the continuous spectrum $\sigma_c(T)$.

Remark. $\lambda \in \sigma(T)$ is an approximate point spectrum if and only if there exists a sequence $\{x_n\}_{n\geq 1}$ in $\mathcal{H}$ and $\|x_n\| = 1$, and

$$\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0.$$

5.1.3 The Spectrum Interval

We now compute the *tightest* interval that contains the spectrum of a self-adjoint operator T. This interval can be computed from the quadratic form of T as follows.

Theorem 5.1.1 (Spectrum interval). Suppose $T \in \mathcal{L}(\mathcal{H})$ for T being a self-adjoint operator. Then

(a) $\sigma(T) \subseteq [m, M]$ for $m := \inf_{\|x\|=1} \langle Tx, x\rangle$, $M := \sup_{\|x\|=1} \langle Tx, x\rangle$.

(b) The endpoints $m, M \in \sigma(T)$.

Proof. We prove this one by one.

(a) Let $\lambda \in \mathbb{C} - [m, M]$, and set d be

$$d := \operatorname{dist}(\lambda, [m, M]) = \inf_{m \leq y \leq M} |\lambda - y| > 0.$$

Then we have

$$\|(T - \lambda I)x\| \geq |\langle (T - \lambda I)x, x\rangle| = |\langle Tx, x\rangle - \lambda| \geq d$$

since $\|x\| = 1$, which implies $T - \lambda I$ is bounded below, hence $\lambda \in \rho(T)$ from Corollary 5.1.1.

(b) Without loss of generality, assume that $0 \leq m \leq M$ by considering $T - mI$ instead of T. Now, choose a sequence $\{x_n\}_{n \geq 1}$ in $\mathcal{H}$ where $\|x_n\| = 1$ such that

$$\lim_{n \to \infty} \langle Tx_n, x_n\rangle = M.$$

By the parallelogram law,

$$\|(T - MI)x_n\|^2 = \langle (T - MI)x_n, (T - MI)x_n\rangle = \|Tx_n\|^2 - 2M \langle Tx_n, x_n\rangle + M^2 \|x_n\|^2.$$

Since we already showed that $\|T\| = M$, i.e., $\|T\| = \sup_{\|x\|=1} \langle Tx, x\rangle$ from $\langle Tx, x\rangle \geq 0$, by letting $n \to \infty$, the right-hand side goes to ≤ 0 since $\|Tx_n\|^2 \leq M^2 \|x_n\|^2$ and $\langle Tx_n, x_n\rangle \to M$, we may conclude that

$$\lim_{n \to \infty} \|(T - MI)x_n\| = 0,$$

and hence $M \in \sigma(T)$.

■

As a consequence, $r(T) = \|T\|$ for T being a self-adjoint operator. This means that Proposition 4.3.2 is tight, while Gelfand's formula is useless for self-adjoint operators! This observation leads to the following.

Corollary 5.1.2 (Spectral radius)**.** Let $T \in \mathcal{L}(\mathcal{H})$ for T being a self-adjoint operator. Then

$$r(T) = \max_{\lambda \in \sigma(T)} |\lambda| = \|T\|.$$

Proof. From the property of spectrum interval, we know that $r(T) = \max(|m|, |M|) = \|T\|$. ■

5.2 Spectral Theorem for Compact Self-Adjoint Operators

Compact self-adjoint operators on a Hilbert space $\mathcal{H}$ are proxies of Hermitian matrices. As we know from linear algebra, every Hermitian matrix has diagonal form in some orthonormal basis of $\mathbb{C}^n$, or equivalently, there exists an orthonormal basis of $\mathbb{C}^n$ consisting of the eigenvectors. In this section, we generalize this fact to infinite dimensions, for all compact self-adjoint operators on $\mathcal{H}$.

5.2.1 Invariant Subspaces

Proposition 5.2.1 (Eigenvectors orthogonal)**.** Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator on $\mathcal{H}$. Then its eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. If $Tx_1 = \lambda_1 x_1$ and $Tx_2 = \lambda_2 x_2$, then

$$\lambda_1 \langle x_1, x_2\rangle = \langle Tx_1, x_2\rangle = \langle x_1, Tx_2\rangle = \lambda_2 \langle x_1, x_2\rangle,$$

so if $\lambda_1 \neq \lambda_2$, then $\langle x_1, x_2\rangle = 0$, proving the result. ■

Definition 5.2.1 (Invariant subspace). A subspace E of $\mathcal{H}$ is an *invariant subspace* of T if $T(E) \subseteq E$.

Example. Every eigenspace of T is invariant. More generally, the linear span of any subset of eigenvectors of T is an invariant subspace.

One of the most well-known open problems in functional analysis is the *invariant subspace problem*. It asks whether every operator $T \in \mathcal{L}(\mathcal{H})$ has a proper invariant subspace, i.e., different from $\{0\}$ and $\mathcal{H}$. While we don't know the answer for this, we have the following characterization.

Proposition 5.2.2. Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator on $\mathcal{H}$. If $E \subseteq \mathcal{H}$ is an invariant subspace of T, then $E^\perp$ is also an invariant subspace of T.

Proof. Let $x \in E^\perp$, and we need to check that $Tx \in E^\perp$ given E is invariant. Let's choose $y \in E$ arbitrarily, then we have

$$\langle Tx, y\rangle = \langle x, Ty\rangle = 0$$

since $x \in E^\perp$ and $y \in E$, hence $Ty \in E$, as required. ■

5.2.2 Spectral Theorem

The following result is known as the Hilbert-Schmidt theorem.

Theorem 5.2.1 (Spectral theorem for compact self-adjoint operator). Let T be a compact self-adjoint operator on a separable $\mathcal{H}$. Then there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of T.

Proof. Let's first prove that T has at least one eigenvector. Firstly, since T is compact, by Proposition 4.4.1,

$$\sigma(T) = \sigma_p(T) \cup \{0\}.$$

So if $\sigma(T) \neq \{0\}$, then $\sigma_p(T) \neq \varnothing$, i.e., T has an eigenvector. Otherwise, if $\sigma(T) = \{0\}$, then from Corollary 5.1.2, $r(T) = \|T\| = 0$, i.e., $T \equiv 0$. In this case, any orthonormal basis gives a basis of eigenvectors,[a] a contradiction.

Now suppose T has an eigenvector with $\sigma_p(T) \neq \varnothing$. The fact that $\mathcal{H}$ is separable, all such basis are at most countable, so from Zorn's lemma, this family has a maximal element $\{\phi_k\}_{k\geq 1}$, so the result follows by showing that

$$E := \overline{\operatorname{span}(\{\phi_k\}_{k\geq 1})} = \mathcal{H}.$$

Suppose $E \neq \mathcal{H}$. Since E is an invariant subspace of T, $E^\perp \neq \{0\}$ is also an invariant subspace of T by Proposition 5.2.2. By using the first part of the proof for the restriction $T|_{E^\perp}$ which is a compact self-adjoint operator on $E^\perp$. It follows that $T|_{E^\perp}$, and thus T itself, has an eigenvector in $E^\perp$. But this contradicts the maximality of $\{\phi_k\}_{k\geq 1}$, so $E = \mathcal{H}$. ■

[a] Since every vector is an eigenvector of T.

Finally, we introduce a new kind of operators called normal operators, where the above result generalizes to which.

Definition 5.2.2 (Normal operator). An operator T is *normal* if $TT^* = T^*T$.

Remark. The spectral theorems for compact self-adjoint operators extend to normal operator.

However, spectrum of normal operators do not have to be real,[2] we only have $\sigma(T) \subseteq \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$.

[2] For example, the unitary operators $U^*U = UU^* = I$.

5.3 Continuous Functional Calculus

In this section, we develop the analogy between numbers and operators by introducing a partial order on the set of self-adjoint operators on $T \in \mathcal{L}(\mathcal{H})$, and we define an operator $f(T) \in \mathcal{L}(\mathcal{H})$ for every continuous function $f : \mathbb{C} \to \mathbb{C}$. This is the so-called *functional calculus* of operators.

5.3.1 Positive Operators

Definition 5.3.1 (Positive operator). A self-adjoint operator $T \in \mathcal{L}(\mathcal{H})$ is *positive* if for all $x \in \mathcal{H}$,

$$\langle Tx, x \rangle \geq 0.$$

Example. T^2 for every self-adjoint $T \in \mathcal{L}(\mathcal{H})$ is positive.

Proof. Since $\langle T^2 x, x \rangle = \langle Tx, Tx \rangle \geq 0$. ⊛

Example. Hermitian matrices, or more generally, compact self-adjoint operators on $\mathcal{H}$ with non-negative eigenvalues are positive.

Note. Positive operators are generalizations of non-negative numbers, which correspond to operators on one-dimensional space $\mathbb{C}$.

Remark (Positive semi-definite). In linear algebra, positive operators are called positive semi-definite. Furthermore, we denote A being positive semi-definite by $A \succeq 0$, and define a partial order between A and B by $A \succeq B \Leftrightarrow A - B \succeq 0$.

Consider the analogous notion for self-adjoint operators, where $T \geq 0$ means T is positive. Then we have the following.

Definition 5.3.2 (Partial order). For self-adjoint operators $S, T \in \mathcal{L}(\mathcal{H})$, we say $S \leq T$ if $T - S \geq 0$.

Definition 5.3.2 defines a partial order on $\mathcal{L}(\mathcal{H})$, and we may restate the spectrum interval theorem with this new notion.

Theorem 5.3.1 (Spectrum interval). Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator, and let m, M be the smallest and the largest numbers such that $mI \leq T \leq MI$. Then $\sigma(T) \subseteq [m, M]$ and $m, M \in \sigma(T)$.

As an immediate corollary, T is positive if and only if its spectrum is positive.

Corollary 5.3.1. Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator, then $T \geq 0$ if and only if $\sigma(T) \subseteq [0, \infty)$.

5.3.2 Polynomials of Operators

We start to develop a functional calculus for self-adjoint operators $T \in \mathcal{L}(\mathcal{H})$ by defining polynomials of T, and then we extend the definition to continuous functions of T by approximation. Working with polynomials is straightforward, and the result of this subsection remain valid for every bounded linear operator T on a general Banach space X.

Definition 5.3.3 (Polynomial operator). Consider a polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n$, then for an operator $T \in \mathcal{L}(\mathcal{H})$, we define

$$p(T) := a_0 I + a_1 T + \cdots + a_n T^n.$$

We first note that if T is self-adjoint, then $p(T)$ is also self-adjoint if p is real since

$$\begin{aligned}\langle p(T)x, y\rangle &= \langle (a_0 I + a_1 T + \cdots + a_n T^n)x, y\rangle \\ &= a_0 \langle x, y\rangle + a_1 \langle Tx, y\rangle + \cdots + a_n \langle T^n x, y\rangle \\ &= \langle x, \overline{a_0} y\rangle + \langle x, \overline{a_1} Ty\rangle + \cdots + \langle x, \overline{a_n} T^n y\rangle \\ &= \langle x, a_0 y\rangle + \langle x, a_1 Ty\rangle + \cdots + \langle x, a_n T^n y\rangle \\ &= \langle x, (a_0 I + a_1 T + \cdots + a_n T^n) y\rangle \\ &= \langle x, p(T)y\rangle .\end{aligned}$$

Moreover, we have the following properties for polynomial operators.

Proposition 5.3.1. Let p, q be complex polynomials and $T \in \mathcal{L}(\mathcal{H})$.

(a) $(ap + bq)(T) = ap(T) + bq(T)$ for $a, b \in \mathbb{C}$.

(b) $(pq)(T) = p(T)q(T)$.

(c) $p(T)^* = \overline{p}(T^*)$.[a]

[a] $\overline{p}$ is the polynomial with coefficients that are complex conjugates of the coefficients of p.

The following example may serve us as a test case for many future results.

Example. Let T be a self-adjoint linear operator on an n-dimensional Hilbert space. In an orthonormal basis of eigenvectors, T can be identified with the $n \times n$ diagonal matrix

$$T = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

where λ_k are the eigenvalues of T. Then for every polynomial $p(t)$, we have

$$p(T) = \operatorname{diag}(p(\lambda_1), \ldots, p(\lambda_n)).$$

This can be generalized for all compact self-adjoint operators T on a general Hilbert space $\mathcal{H}$.

5.3.3 Spectral Mapping Theorem for Polynomial Operators

Let's study some important theorem for the polynomial operators.

Lemma 5.3.1 (Invertibility for polynomial operator)**.** Let $p(t)$ be a polynomial and $T \in \mathcal{L}(\mathcal{H})$, then $p(T)$ is invertible if and only if $p(t) \neq 0$ for all $t \in \sigma(T)$.

Proof. Consider

$$p(t) = a_n(t - t_1)(t - t_2)\ldots(t - t_n)$$

where $t_1, \ldots, t_n$ are zeros of $p(\cdot)$, then

$$p(T) = a_n(T - t_1 I)\ldots(T - t_n I).$$

We see that if $p(t) \neq 0$ for $t \in \sigma(T)$, $p(T)$ is clearly invertible.

Conversely, observe that if $S, R \in \mathcal{L}(\mathcal{H})$ and SR is invertible, then both S and R are invertible.[a] Hence, if $p(T)$ is invertible, then $T - t_k I$ are all invertible, i.e., $t_1, \ldots, t_n \in \rho(T)$. ■

[a] Since if SR is invertible, then $S^{-1} = R(SR)^{-1}$.

The spectrum of a polynomial $p(T)$ can be easily computed from the spectrum of T.

Theorem 5.3.2 (Spectral mapping theorem for polynomial operator)**.** Let $p(t)$ be a polynomial and $T \in \mathcal{L}(\mathcal{H})$. Then

$$\sigma(p(T)) = p(\sigma(T))$$

where $p(\sigma(T)) \coloneqq \{p(t) \colon t \in \sigma(T)\}$.

Proof. Since $\lambda \in \sigma(p(T))$ if and only if $p(T) - \lambda I$ is not invertible, which from Lemma 5.3.1, it is equivalent to say $p(t) - \lambda = 0$ for some $t \in \sigma(T)$, i.e., $\lambda \in p(\sigma(T))$. ∎

Using the spectral mapping theorem for polynomial operator, one can in particular easily compute the norm of polynomial operator.

Corollary 5.3.2 (Operator norm of polynomial operator)**.** Suppose T is a bounded self-adjoint operator on $\mathcal{H}$ and $p(t)$ is a polynomial with real coefficients. Then $p(T)$ is self-adjoint and

$$\|p(T)\| = \sup_{t \in \sigma(T)} |p(t)|.$$

Proof. Firstly, $p(T)$ is self-adjoint since $\overline{p} = p$ as noted in the beginning of the section, with the Corollary 5.1.2 and spectral mapping theorem for polynomial operator,

$$\|p(T)\| = r(p(T)) = \sup_{t \in \sigma(p(T))} |t| = \max_{t \in p(\sigma(T))} |t| = \max_{s \in \sigma(T)} |p(s)|.$$

∎

Corollary 5.3.2 generalizes Corollary 5.1.2, which states that $r(T) = \|T\|$ for the spectral radius of self-adjoint operators T.

Lecture 24: The Universal Spectral Mapping Theorem

5.3.4 Continuous Functions of Operators

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We're now going to study the spectral mapping theorem, which is a generalization of finite dimensions self-adjoint T on $\mathbb{R}^n$, where there exists an orthonormal basis of eigenvectors. In this basis, the action of t is just multiplication by scalars. To generalize this to infinite dimensions, we need measure theory. In particular, we need Weierstrass approximation theorem.

Theorem 5.3.3 (Weierstrass approximation theorem)**.** Let $K \subseteq \mathbb{R}$ be compact and $f \colon K \to \mathbb{R}$ be continuous, then for any $\epsilon > 0$, there exists a polynomial p^ϵ such that

$$\sup_{x \in K} |f(x) - p^\epsilon(x)| < \epsilon.$$

Let T be self-adjoint on $\mathcal{H}$ and $f \colon \sigma(T) \to \mathbb{R}$ be a continuous function, then we want to define a self-adjoint operator $f(T)$. To do this, we use Weierstrass approximation theorem to find polynomials $p^{\epsilon_n}(t)$ such that

$$p^{\epsilon_n}(t) \to f(t)$$

uniformly on $\sigma(T)$. This suggests us to define $f(T)$ as the limit of operator polynomials $p^{\epsilon_n}(T)$.

Definition 5.3.4 (Continuous function operator)**.** The sequence $p^{\epsilon_n}(T)$ converges in $\mathcal{L}(\mathcal{H})$ to a limit that we call $f(T) \in \mathcal{L}(\mathcal{H})$, which is also self-adjoint.

Note (Well-defined)**.** To show that Definition 5.3.4 is well-defined, we need to show that $p^{\epsilon_n}(T)$ indeed converges to $f(T)$, and $f(T)$ is self-adjoint and does not depend on the choice of the approximating polynomials p^{ϵ_n}.

Proof. To construct $f(T)$, choose a sequence $p^{\epsilon_n}(\cdot)$ of polynomial such that $\epsilon_n \to 0$ and

$$\sup_{x \in \sigma(T)} |f(x) - p^{\epsilon_n}(x)| < \epsilon \Rightarrow \sup_{x \in \sigma(T)} |p^{\epsilon_n}(x) - p^{\epsilon_m}(x)| < \epsilon_n + \epsilon_m.$$

From Corollary 5.3.2, $\|p(T)\| = \sup_{t\in\sigma(T)} |p(t)|$, so we have

$$\|p^{\epsilon_n}(T) - p^{\epsilon_m}(T)\| < \epsilon_n + \epsilon_m,$$

so the sequence $\{p^{\epsilon_n}(T)\}_{n\geq 1}$ is in $\mathcal{L}(\mathcal{H})$ is Cauchy, so there exists a limit $f(T)$ with

$$\lim_{n\to\infty} \|p^{\epsilon_n}(T) - f(T)\| = 0,$$

which shows that $f(T) \in \mathcal{L}(\mathcal{H})$ is unique.

Moreover, $f(T)$ is self adjoint since the self adjoint operators form a closed subset of $\mathcal{L}(\mathcal{H})$, and by repeating the above estimation, given two approximating sequences p^{ϵ_n} and q^{ϵ_n}, they will both converge to $f(T)$. ⊛

By passing to the limit in the corresponding properties for polynomial operators as in Proposition 5.3.1, we see that these properties of $f(T)$ inherited from properties of $p(T)$ when $p(\cdot)$ is a polynomial, e.g.,

(a) $(af + bg)(T) = af(T) + bg(T)$;

(b) $(fg)(T) = f(T)g(T)$;

(c) $f(T)^* = \overline{f}(T)$ for $f\colon \sigma(T) \to \mathbb{C}$.

Note. The first two are also true for continuous complex functions $f, g\colon \sigma(T) \to \mathbb{C}$.

Proof. If $f\colon \sigma(T) \to \mathbb{C}$ is complex, we can then write $f = f_1 + if_2$ for $f_1, f_2\colon \sigma(T) \to \mathbb{R}$ such that

$$f(T) \coloneqq f_1(T) + if_2(T).$$

⊛

5.3.5 Spectral Mapping Theorem

We will now generalize the spectral mapping theorem for polynomial operators to continuous functions of an operator. It is based on the straightforward generalization of the invertibility lemma for polynomial operators.

Lemma 5.3.2 (Invertibility)**.** Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator and $f \in C(\sigma(T))$. Then the operator $f(T)$ is invertible if and only if $f(t) \neq 0$ for all $t \in \sigma(T)$.

Proof. ■ Add!

Now the spectral mapping theorem follows from invertibility lemma by the same argument as the corresponding result for polynomial operators, i.e., Theorem 5.3.2.

Theorem 5.3.4 (Spectral mapping theorem)**.** Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint and $f \in C(\sigma(T))$, then

$$\sigma(f(T)) = f(\sigma(T)).$$

This gives a simple way to create positive operators.

Corollary 5.3.3. Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator and $f \in C(\sigma(T))$. If $f(t) \geq 0$ for all $t \in \sigma(T)$, then $f(T) \geq 0$.

Proof. By generalizing Corollary 5.3.1, it suffices to check $\sigma(f(T)) \subseteq [0, \infty)$. From spectral mapping theorem,

$$\sigma(f(T)) = f(\sigma(T)) \geq 0$$

as desired. ■

5.3.6 Square Root of Operators

Consider a positive self-adjoint operator $T \in \mathcal{L}(\mathcal{H})$, then $\sigma(T) \subseteq [0, \infty)$. The function $f(t) = \sqrt{t}$ is continuous on $[0, \infty)$, so we can define $f(T) = \sqrt{T}$. A simple observation leads to the following.

Proposition 5.3.2 (Square root of operator)**.** Let $T \in \mathcal{L}(\mathcal{H})$ be positive self-adjoint, i.e., $\langle Tx, x\rangle \geq 0$ for $x \in \mathcal{H}$. Then there exists a unique positive self-adjoint operator $\sqrt{T} \in \mathcal{L}(\mathcal{H})$ such that

$$(\sqrt{T})^2 = T.$$

Proof. Let $f\colon \sigma(T) \to \mathbb{R}$, $f(t) = \sqrt{t}$ where $\sigma(T) \subseteq \mathbb{R}^+$ since T is positive. Since $f(t) = \sqrt{t}$ is continuous on $\sigma(T)$, so we can define $f(T) = \sqrt{T}$. Furthermore, since $f(t) \geq 0$ for all $t \in \sigma(T)$, Corollary 5.3.3 states that $f(T) = \sqrt{T} \geq 0$. Finally, since $\sqrt{t}^2 = t$, the algebra homomorphism property implies that $(\sqrt{T})^2 = T$ as well.[a] ■

[a] The uniqueness is left as an exercise.

5.3.7 Modulus of Operators

Now, consider an arbitrary operator $T \in \mathcal{L}(\mathcal{H})$, which is not necessarily self-adjoint. Then T^*T is a positive self-adjoint operator, so it has a unique positive square root. This suggests the following definition.

Definition 5.3.5 (Modulus)**.** Let $T \in \mathcal{L}(\mathcal{H})$, then the *modulus* of T is defined as $|T| := \sqrt{T^*T}$.

This generalizes the concept of modulus of complex numbers, i.e., $|z| = \sqrt{\overline{z}z}$ for $z \in \mathbb{C}$.

Lemma 5.3.3. For every operator $T \in \mathcal{L}(\mathcal{H})$ and vector $x \in \mathcal{H}$, one has

$$\||T|x\| = \|Tx\|.$$

Proof. Since

$$\||T|x\|^2 = \langle |T|x, |T|x\rangle = \left\langle |T|^2 x, x\right\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2.$$

■

Lemma 5.3.3 leads to the polar decomposition theorem.

5.3.8 Polar Decomposition

Lemma 5.3.3 motivates us to consider a well-defined linear map

$$U\colon |T|x \mapsto Tx$$

for $x \in \mathcal{H}$, and Lemma 5.3.3 states that $U \in \mathcal{L}(\mathcal{H})$ is an isometry.

Theorem 5.3.5 (Polar decomposition)**.** For every $T \in \mathcal{L}(\mathcal{H})$, there exists a unique bijective linear isometry $U \in \mathcal{L}(\operatorname{Im}(|T|), \operatorname{Im} T)$ such that $T = U|T|$.

Proof. Define U on $\operatorname{Im}(|T|)$ by $U(|T|x) = Tx$ for all $x \in \mathcal{H}$.[a] Then, U is an isometry since $|T| = \sqrt{T^*T}$, and U is injective and U maps onto $\operatorname{Im}(T)$, so U is surjective. ■

[a] Notice that this makes sense since $|Tx| = 0 \Leftrightarrow Tx = 0$.

Polar decomposition generalizes the polar decomposition in the complex plane. The latter states that every $z \in \mathbb{C}$ can be represented as

$$z = e^{i\arg(z)}|z|,$$

where $e^{i\arg(z)}$ is a unit scalar (generalized by U), and $|z|$ is the modulus of z (generalized by $|T|$).

Theorem 5.3.6 (Polar decomposition for invertibile operator). If $T \in \mathcal{L}(\mathcal{H})$ is invertible, then there exists a unique unitary $U \in \mathcal{L}(\mathcal{H})$ such that

$$T = U|T|.$$

Proof. Since T is invertible, T^* is also invertible, hence T^*T is invertible. Finally, we can show that $|T| = \sqrt{T^*T}$ is also invertible, therefore $\mathrm{Im}(T) == \mathrm{Im}(|T|) = \mathcal{H}$, then the claim follows from polar decomposition. ■

5.4 Borel Functional Calculus

We can extend functional calculus to bounded Borel functions of operators. This is done primarily to define the spectral projections, which are indicator functions of an operator. Once we have spectral projections, we formulate the spectral theorem for general (not necessarily compact) self-adjoint operators.

As usual, let $T \in \mathcal{L}(\mathcal{H})$ be a fixed self-adjoint operator on a Hilbert space $\mathcal{H}$. Let's also fix the spectrum interval $[m, M]$.

5.4.1 Borel Functional Calculus

We consider the linear space of bounded Borel complex-valued functions on $[m, M]$, denote this space as $\mathcal{B}([m, M])$. We would like to define $f(T)$ for $f \in \mathcal{B}([m, M])$, so that this extends the definition of $f(T)$ for continuous functions $f \in C([m, M])$. The difficulty is that Borel functions are only point-wise (but not uniform)limits of continuous functions.

Theorem 5.4.1 (Borel functional calculus). Let $\sigma(T) \subseteq [m, M]$ and $\mathcal{B}([m, M])$ be the linear space of bounded Borel measurable functions $f\colon [m, M] \to \mathbb{C}$ such that

$$\|f\|_\infty = \sup_{t\in[m,M]} |f(t)| < \infty.$$

Then we can define a self-adjoint operator $f(T) \in \mathcal{L}(\mathcal{H})$ with the following properties.

(a) If $f(\cdot)$ is real-valued, then $f(T)$ is self-adjoint.

(b) If $f\colon [m, M] \to \mathbb{C}$, $f \in \mathcal{B}([m, M])$, then $\|f(T)\| \leq \|f\|_\infty$.

(c) Suppose $f_n \in \mathcal{B}([m, M])$ for all $n \geq 1$ and $f \in \mathcal{B}([m, M])$ such that $\sup_{n\geq 1}\|f_n\|_\infty < \infty$ and $f_n \to f$ point-wise, i.e., $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in [m, M]$. Then $f_n(T)$ converges strongly to $f(T)$, i.e.,

$$\lim_{n\to\infty} \|f_n(T)x - f(T)x\| = 0$$

for all $x \in \mathcal{H}$.

(d) Suppose $T, S \in \mathcal{L}(\mathcal{H})$ are self-adjoint and commute, i.e., $TS = ST$, and assume further that $\sigma(T), \sigma(S) \subseteq [m, M]$ and $f, g \in \mathcal{B}([m, M])$. Then $f(T)$ and $g(S)$ commute, i.e., $f(T) \cdot g(S) = g(S) \cdot f(T)$.

Lecture 25: Proofs of Borel Functional Calculus Theorem

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Proof. Let's prove this one by one.

(a) We construct $f(T)$ using Riesz representation for $C(K)$ where $K = [m, M]$ is compact Hausdorff. For $x, y \in \mathcal{H}$, define functional $F_{x,y}\colon C(K) \to \mathbb{C}$ by

$$F_{x,y}(f) = \langle f(T)x, y\rangle$$

where $f \in C([m, M])$. Since $F_{x,y} \in C(K)^*$, we hav e

$$|F_{x,y}(f)| \leq \|f(T)\|\|x\|\|y\| = \|f\|_\infty \|x\|\|y\|,$$

so $\|F_{x,y}\|\|x\|\|y\|$. Then Riesz representation implies that there exists a unique Borel measure $\mu_{x,y}$ on $[m, M]$ such that

$$\langle f(T)x, y\rangle = \int_m^M f(\lambda)\, \mathrm{d}\mu_{x,y}(\lambda)$$

for $f \in C([m, M])$. Now, we extend $F_{x,y}$ to all Borel measurable functions $f\colon [m, M] \to \mathbb{R}$ with $\|f\|_\infty < \infty$ by

$$Bf(x, y) = \int_m^M f(\lambda)\, \mathrm{d}\mu_{x,y}(\lambda).$$

Notice that $TV(\mu_{x,y}) \leq \|x\|\|y\|$ by Riesz representation, and hence $|Bf(x, y)| \leq \|f\|_\infty \|x\|\|y\|$. Note that the function $[x, y] \to Bf(x, y)$ is linear in x and sesquilinear[a] in y. Hence, there exists $f(T) \in \mathcal{L}(\mathcal{H})$ such that

$$Bf(x, y) = \langle f(T)x, y\rangle$$

by Riesz representation for Hilbert space, so $f(T)$ is defined uniquely by the sesquilinear property.

(b) $\|f(T)\| \leq \|f\|_\infty$ follows from

$$|Bf(x, y)| \leq \|f\|_\infty \|x\|\|y\|$$

for $x, y \in \mathcal{H}$.

(c) Consider $f_n, f\colon [m, M] \to \mathbb{R}$ such that $\sup_{n\geq 1} \|f_n\|_\infty < \infty$ with $f_n(t) \to f(t)$ for $t \in [m, M]$. Then we have

$$\langle f_n(T)x, y\rangle = \int_m^M f_n(\lambda)\, \mathrm{d}\mu_{x,y}.$$

From the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_m^M f_n(\lambda)\, \mathrm{d}\mu_{x,y} = \int_m^M f(\lambda)\, \mathrm{d}\mu_{x,y},$$

hence

$$\lim_{n\to\infty} \langle f_n(T)x, y\rangle = \langle f(T)x, y\rangle$$

for all $x, y \in \mathcal{H}$, which is saying that $f_n(T)$ converges weakly to $f(T)$. To show strong convergence, note that

$$\begin{aligned}
\|(f_n(T) - f(T))x\|^2 &= \langle (f_n(T) - f(T))x, (f_n(T) - f(T))x\rangle \\
&= \left\langle (f_n(T) - f(T))^2 x, x\right\rangle \\
&= \int_m^M (f_n(T) - f(T))^2\, \mathrm{d}\mu_{x,x} \to 0
\end{aligned}$$

by dominated convergence theorem. Hence, we conclude that

$$\lim_{n\to\infty} \|(f_n(T) - f(T))x\| = 0$$

for all $x \in \mathcal{H}$.

(d) The commutativity follows from two approximation arguments. Firstly, it holds if f, g are polynomials, so by Weierstrass approximation theorem, this holds for continuous $f, g\colon [m, M] \to \mathbb{R}$. Then, if this holds for continuous functions, it also follows for Borel functions by approximating Borel functions with continuous functions.[b]

■

[a]Recall that if $f(T)$ is sesquilinear, then $Bf(x, y) = \overline{Bf(y, x)}$.
[b]This is the so-called Lusin's theorem.

5.4.2 Spectral Measures

From Borel functional calculus, we have

$$\langle Tx, y\rangle = \int_m^M \lambda \, \mathrm{d}\mu_{x,y}(\lambda)$$

for $x, y \in \mathcal{H}$. We can abstract this result to construct integrals with respect to spectral projections.

Let $E \subseteq [m, M]$ be a Borel set, and $\mathbb{1}_E$ be the indicator function such that

$$\mathbb{1}_E(\lambda) = \begin{cases} 1, & \text{if } \lambda \in E; \\ 0, & \text{if } \lambda \notin E. \end{cases}$$

Set $P_E = \mathbb{1}_E(T)$, from $(\mathbb{1}_E)^2 = \mathbb{1}_E$, we have $P_E^2 = P_E$, so P_E is indeed a projection.

Proposition 5.4.1. The spectral projections $E \to P_E$ for $E \in \mathcal{B}([m, M])$ satisfies $P_{[m,M]} = I$. Furthermore, if $E = \bigcup_{k=1}^{\infty} E_k$ and E_k are disjoint for all $k \geq 1$, then

$$P_E = \sum_{k=1}^{\infty} P_{E_k}$$

where convergence is in the strong sense.

Proof. It follows Theorem 5.4.1 where if $f_n \to f$, then $f_n(T) \to f(T)$ strongly. ■

Remark. The mapping $E \to P_E$ for $E \in \mathcal{B}([m, M])$ is an operator valued measures.

Theorem 5.4.2 (Spectral theorem)**.** Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint and $E \to P_E$ be the associated spectral measure. Then

$$T = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}P_\lambda.$$

Proof. This is just a way of interpretation, i.e.,

$$\langle Tx, y\rangle = \int_{-\infty}^{\infty} \lambda \, [\mathrm{d}P_\lambda x, y]$$

for $x, y \in \mathcal{H}$ where $[\mathrm{d}P_\lambda x, y] = d_{\mu_{x,y}}$. ■

Chapter 6

Epilogue

Lastly, we prove some left-out theorems, start with the Riesz representation for $C(K)$.

6.1 Proof of Riesz Representation for $C(K)$

This section is the proof about Riesz representation for $C(K)$. Let's first restate the theorem.

> **As previously seen** (Riesz representation for $C(K)$). Let $E = C(K)$ be the space of continuous functions on compact Hausdorff space K. Then we have the following.
>
> (a) For every Borel regular signed measure on K, the functional $F(f) = \int_K f \,\mathrm{d}\mu$ is a bounded linear functional on K.
>
> (b) Every bounded linear functional on $C(K)$ can be expressed as $F(f) = \int_K f \,\mathrm{d}\mu$ for some measure μ, and $\|F\| = |\mu|(K)$, i.e., $TV(K)$.

Let's first outline the proof. The main theorem we're going to use is the Urysohn's lemma for the construction of continuous functions. Urysohn's lemma allows us to construct means of positive linear functional on $C(X)$ where X is locally compact Hausdorff. Then, let $\Lambda \colon C(X) \to \mathbb{R}$ where Λ is linear and $\Lambda(f) \geq 0$ if $f(x) \geq 0$ for all $x \in X$. Now, we define means of an open set $V \subseteq X$ where

$$\mu(V) = \sup \{\Lambda(f) \colon 0 \leq f \leq \mathbb{1}_V\}.$$

Then, we define μ for any subset $E \subseteq X$,

$$\mu(E) = \inf \{\mu(V) \colon E \subseteq V, V \text{ open}\},$$

where μ is the outer measure, i.e., subadditive.

Lecture 26: Urysohn's Lemma

Now, we build the foundation toward proving the Urysohn's lemma. 1 Dec. 14:30

> **Definition 6.1.1** (Topology). Given a nonempty set X of points, a family $\mathcal{T}$ of subsets (the open sets) is called a *topology* if it satisfies the following.
>
> (a) $\mathcal{T}$ contains X and the empty set $\varnothing$.
>
> (b) If $O_1, O_2 \in \mathcal{T}$, $O_1 \cap O_2 \in \mathcal{T}$.
>
> (c) If $O_d \in \mathcal{T}$ for $d \in \mathcal{F}$, $\bigcup_{d \in \mathcal{F}} O_d \in \mathcal{T}$.

> **Definition 6.1.2** (Topological space). A *topological space* $(X, \mathcal{T})$ is a nonempty set X of points together with a topology $\mathcal{T}$.

Definition 6.1.3 (Hausdorff). A topological space $(X, \mathcal{T})$ is *Hausdorff* if given two distinct points $x_1, x_2 \in X$, there exists disjoint open sets $O_1, O_2 \in \mathcal{T}$ such that $x_1 \in O_1$, $x_2 \in O_2$.

Definition 6.1.4 (Locally compact). A Hausdorff space $(X, \mathcal{T})$ is *locally compact* if for every $x \in X$, there exists open $O \in \mathcal{T}$ such that $x \in O$ and the closure $\overline{O}$ of O is compact.

Remark (Closure). Formally, given a topological space $(X, \mathcal{T})$, the closure $\overline{O}$ of an open set $O \in \mathcal{T}$ is defined as

$$\overline{O} \coloneqq \bigcap_{F \supseteq O} F,$$

where F is closed (complement of some open set).

Theorem 6.1.1. Let $(X, \mathcal{T})$ be a topological space and $K \subseteq X$ being compact, i.e., every open cover of K has a finite subcover. Suppose F is closed and $F \subseteq K$, then F is also compact.

Proof. If $\{V_\alpha\}_{\alpha \in \mathcal{F}}$ is an open cover of F, then $\{V_\alpha\}_{\alpha \in \mathcal{F}}$ and $X \setminus F$ is an open cover of K. Since K is compact, we can find a finite subcover of K, which also covers F since $F \subseteq K$, hence F is compact from definition. ■

Theorem 6.1.2. Suppose X is Hausdorff, $K \subseteq X$ being compact and $p \notin K$. Then there exists disjoint open sets U, W such that $p \in U$, $K \subseteq W$.

Proof. By the Hausdorff property, for every $q \in K$, there exists open sets U_p, V_q such that $p \in U_p$, $q \in V_q$ with U_p, V_q disjoint. Then $\{V_q\}_{q \in K}$ is an open cover of K, so we have a finite subcover $V_{q_1}, \dots, V_{q_n}$ of K. Now, take

$$W = \bigcup_{j=1}^{n} V_{q_j} \supseteq K, \quad U = \bigcap_{j=1}^{n} U_{p_j} \ni p,$$

we have that U, W being open and disjoint. ■

Corollary 6.1.1. Compact subsets of Hausdorff space are closed.

Corollary 6.1.2. If F is closed and K is compact, then $F \cap K$ is compact.

Theorem 6.1.3. If $\{K_\alpha\}_{\alpha \in \mathcal{F}}$ is a collection of compact subsets of a Hausdorff space such that $\bigcap_{\alpha \in \mathcal{F}} K_\alpha$ is empty, then some finite subcollection of K_α, $\alpha \in \mathcal{F}$, has empty intersection.

Proof. Consider any K_{α_0} with $\alpha_0 \in \mathcal{F}$. Then

$$K_{\alpha_0} \subseteq \bigcup_{\substack{\alpha \in \mathcal{F} \\ \alpha \neq \alpha_0}} (X \setminus K_\alpha).$$

This is an open cover of K_{α_0}, so there exists a finite subcovers, i.e.,

$$K_{\alpha_0} \subseteq (X \setminus K_{\alpha_1}) \cup \dots \cup (X \setminus K_{\alpha_n}),$$

leading to the fact that $\bigcap_{j=0}^{n} K_{\alpha_j} = \varnothing$. ■

Theorem 6.1.4. Let X be a locally compact Hausdorff space, and U is open, K is compact and $K \subseteq U$. Then there exists an open set V such that $\overline{V}$ is compact and $K \subseteq V \subseteq \overline{V} \subseteq U$.

Proof. By the locally compactness property, every point of K has an open neighborhood with compact closure. Since K is covered by a finite union of these open neighborhoods, hence $K \subseteq G \subseteq \overline{G}$ and $\overline{G}$ is compact. We see that if $U = X$ we're done since we can take $V = G$. Otherwise, note that for each $p \in X \setminus U$, $p \notin K$, so there exists open W_p and $p \notin \overline{W}_p$. Now, consider the family $(X \setminus U) \cap \overline{G} \cap \overline{W}_p$, $p \in X \setminus U$, with $\overline{G}$ compact, we see that this is a family of compact sets with empty intersection. This means there is a finite number of these $p_1, \dots, p_n$ have empty intersection, so by taking $V = G \cap W_{p_1} \cap \dots \cap W_{p_n}$,

$$K \subseteq V \subseteq \overline{V} \subseteq U.$$

■

Definition. Let X be a topological space and $f \colon X \to \mathbb{R}$.

Definition 6.1.5 (Lower semi-continuous)**.** If $\{x \in X \colon f(x) > \alpha\}$ is open for all $\alpha \in \mathbb{R}$, then f is *lower semi-continuous.*

Definition 6.1.6 (Upper semi-continuous)**.** If $\{x \in X \colon f(x) < \alpha\}$ is open for all $\alpha \in \mathbb{R}$, then f is *upper semi-continuous.*

Remark. A real function $f \colon X \to R$ is continuous if and only if it is both upper and lower semi-continuous.

Remark. The characteristic functions of open sets are lower semi-continuous, while the characteristic functions of closed sets are upper semi-continuous.

Remark. The supremum of a family of lower semi-continuous is lower semi-continuous; while the infimum of a family of upper semi-continuous is upper semi-continuous.

Remark. $f(\cdot)$ is lower semi-continuous if $\{x \in X \colon f(x) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$. For metric spaces X, this is equivalent to

$$f(x) \leq \liminf_{x_n \to x} f(x_n).$$

Definition 6.1.7 (Support)**.** Let $(X, \mathcal{T})$ be a topological space. The *support*of a function $f \colon X \to \mathbb{R}$ is the closure of the set $\{x \in X \colon f(x) \neq 0\}$.

Notation. The collection of all continuous functions on X with compact support is denoted as $C_c(X)$.

Theorem 6.1.5. Let X, Y be topological spaces and $f \colon X \to Y$ be continuous. If $K \subseteq X$ is compact, then $f(K)$ is compact in Y.

Proof. The open cover $\{O_\alpha\}_{\alpha \in \mathcal{F}}$ of $f(K)$ induces an open cover $\left\{f^{-1}(O_\alpha)\right\}_{\alpha \in \mathcal{F}}$ of K. Since K is compact, we can find a finite subcover $f^{-1}(O_{\alpha_1}), \dots, f^{-1}(O_{\alpha_n})$ of K, i.e.,

$$f(K) \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

■

Remark. The range of any $f \in C_c(X)$ is compact, i.e., $f(X)$ is compact.

Theorem 6.1.6 (Urysohn's lemma). Let X be a locally compact Hausdorff space and V open in X, $K \subseteq V$ compact. Then there exists $f \in C_c(X)$ such that

$$\chi_K \leq f \leq \chi_X,$$

i.e., $f(x) = 1$ for $x \in K$, $0 \leq f(y) \leq 1$ for $y \in X$ and $f(y) = 0$ for $y \notin X$.

Proof. Set $r_1 = 0$, $r_2 g1$, and let r_3, r_4 be any enumeration of the ration number r with $0 < r < 1$. By Theorem 6.1.5, we can find open sets V_0, V_1 with

$$K \subseteq V_1 \subseteq \overline{V}_1 \subseteq V_0 \subseteq \overline{V}_0 \subseteq V$$

such that $\overline{V}_0$ is compact. Now, define a sequence V_r for rationals r, $0 < r < 1$. Suppose $n \geq 2$ and $V_{r_1}, V_{r_2}, \ldots, V_{r_n}$ have already been chosen such that if $r_i < r_j$, we have $\overline{V}_{r_j} \subseteq V_{r_i}$ and $\overline{V}_{r_j}$ compact and V_{r_i} open. Let r_{n+1} be the next in enumerations of the rationals. Chose $V_{r_{n+1}}$ open with $\overline{V}_{r_{n+1}}$ compact. One of the

Hence, $\overline{V}_{r_j} \subseteq V_{r_i}$, where $\overline{V}_{r_j}$ is compact and V_{r_i} is open. Let $V_{r_{n+1}}$ be open, $\overline{V}_{r_{n+1}}$ be compact, we have

$$\overline{V}_{r_j} \subseteq V_{r_{n+1}} \subseteq \overline{V}_{r_{n+1}} \subseteq V_{r_i}.$$

Continuing have by induction a countable set V_r, $0 \leq r \leq 1$, r, V_r is open and $\overline{V}_r$ is compact, we have $\overline{V}_r \subseteq V_s$ if $r > s$, r rational. For each rational r, define function f_r

$$f_r(x) = \begin{cases} r, & \text{if } x \in V_r; \\ 0, & \text{otherwise,} \end{cases}$$

so f_r is lower semi-continuous; also, we define g_s

$$g_s(x) = \begin{cases} 1, & \text{if } x \in \overline{V}_s; \\ s, & \text{otherwise,} \end{cases}$$

so g_s is upper semi-continuous. Define $f = \sup_r f_r$, we know that f is lower semi-continuous; also, define $g = \inf_s g_s$, g is upper semi-continuous. Note that $f(x) = 1$ for $x \in K$, $f(x) = 0$ for $x \notin \overline{V}_0$ with $0 \leq f \leq 1$; while $g(x) = 1$ for $x \in K$, $g(x) = 0$ for $x \notin \overline{V}_0$. Suppose $f \equiv g$, hence f is continuous.

Claim. $f = g$

Proof. Suppose $f_r(x) > g_s(x)$, then $r > s$, implying $x \in V_r$. Also, $\overline{V} \subseteq V_s$, so $x \in V_s$ implies $g_s(x) = 1$, contradiction, hence $f \leq g$.

On the other hand, suppose $f(x) < g(x)$, then there exists rational r, s such that $0 < r, s < 1$ such that $f(x) < r < s < g(x)$. Since $f(x) < s$, $x \notin V_r$; and since $g(x) > s$, $x \in \overline{V}_s$, hence $g_s(x) = s$. Also, we have $\overline{V}_s \subseteq V_s$ since $s > 1$, which is a contradiction. ⊛

■

Lecture 27: Riesz Representation Theorem

We will see that the Urysohn's lemma leads to the construction of partition of unity. 6 Dec. 14:30

Theorem 6.1.7. Let $V_1, \ldots, V_n$ be open subsets of a locally compact Hausdorff space X, and $K \subseteq \bigcup_{j=1}^n V_j$ be compact. Then there exists functions h_i for $i = 1, \ldots, n$ such that $h_i \in C_c(X)$, and $0 \leq h_i \leq 1$ for all i, with $\operatorname{supp}(h_i) \subseteq V_i$ for all i. Moreover, we have

$$h_1(x) + h_2(x) + \cdots + h_n(x) = 1$$

for $x \in K$.

Proof. Let $x \in K$, then $x \in V_i$ for some $i = i(x)$. Then there exists open set W_x such that

$$x \in W_x \subseteq \overline{W}_x \subseteq V_{i(x)}$$

by the previous result. Then, $\{W_x\}_{x\in K}$ is an open cover of K, hence there exists a finite subcover

?

$$K \subseteq W_{x_1} \cup W_{x_w} \cup \cdots \cup W_{x_N}.$$

Now, for $1 \leq i \leq n$. Let H_i be the union of the W_{x_j} such that $\overline{W}_{x_j} \subseteq V_i$, so we have $\overline{H}_i \subseteq V_i$ for all $i = 1, \ldots, n$, so $K \subseteq \bigcup_{i=1}^n H_i$. From Urysohn's lemma, there exists g_i such that

$$\chi_{\overline{H}_i} \leq g_i \leq \chi_{V_i}$$

for $i = 1, 2, \ldots, n$. Define the partition of unity $h_1, \ldots, h_n$ by

$$\begin{aligned}
h_1 &= g_1, \\
h_2 &= (1 - g_1)g_2, \\
&\vdots \\
h_n &= (1 - g_1)(1 - g_2)\ldots(1 - g_{n-1})g_n,
\end{aligned}$$

hence $0 \leq h_i \leq \chi_{V_i}$ for all $i = 1, \ldots, n$. Moreover, we have

$$h_1 + h_2 + \cdots + h_n = 1 - (1 - g_1)(1 - g_2)\ldots(1 - g_n).$$

We have $K \subseteq H_1 \cup \cdots \cup H_n$, and when $x \in K$, $g_i(x) = 1$ for some i, i.e., $h_1(x) + \cdots + h_n(x) = 1$. ■

Theorem 6.1.8 (Riesz representation theorem)**.** Let X be locally compact Hausdorff space, and Λ be a positive linear functional on $C_c(X)$, i.e., $\Lambda(\cdot)$ is a linear functional on $C_c(X)$ and $\Lambda(f) \geq 0$ if $f(x) \geq 0$ for $x \in X$. Then there exists a σ-algebra $\mathcal{M}$ in X which contains all Borel sets of X, and there exists a unique positive measure μ on $\mathcal{M}$ which represents Λ in the following sense.

(a) $\Lambda(f) = \int_X f \, \mathrm{d}\mu$ for $f \in C_c(X)$.

(b) $\mu(K) < \infty$ for all compact K.

(c) $\mu(E) = \inf\{\mu(V) \colon E \subseteq V, V \text{ open}\}$.

(d) The relation $\mu(E) = \sup\{\mu(K) \colon K \subseteq E, K \text{ compact}\}$ holds for every open set E, and for every $E \in \mathcal{M}$ with $\mu(E) < \infty$.

Proof. Let's first prove the uniqueness. (c) and (d) imply that the measure μ is determined by its values on compact sets K, so it's sufficient to prove that if μ_1, μ_2 are two such measures, $\mu_1(K) = \mu_2(K)$ for all compact K. From (c), for any compact $K \subseteq V$ and $\epsilon > 0$, there exists an open V such that $K \subseteq V$ with $\mu(V) < \mu(K) + \epsilon$. From Urysohn's lemma, there exists $f \in C_c(X)$ such that $\chi_K \leq f \leq \chi_V$. Note that

$$\mu_1(K) = \int_X \chi_K \, \mathrm{d}\mu_1 \leq \int_X f \, \mathrm{d}\mu_1 = \Lambda(f) = \int_X f \, \mathrm{d}\mu_2 \leq \int_X \chi_V \, \mathrm{d}\mu_2 = \mu_2(V),$$

so $\mu_1(K) \leq \mu_2(V) < \mu_2(K) + \epsilon$. Let $\epsilon \to 0$, $\mu_1(K) \leq \mu_2(K)$, and similarly, $\mu_1(K) \geq \mu_2(K)$, so $\mu_1(K) = \mu_2(K)$.

To construct μ and $\mathcal{M}$, for every open set V in X, we define

$$\mu(V) = \sup\{\Lambda(f) \colon f \in C_c(X), 0 \leq f < \chi_V\}.$$

Then, define $\mu(E)$ for all subsets $E \subseteq X$ such that $\mu(E) = \inf\{\mu(V) \colon E \subseteq V\}$.

Note. Note that these two construction are consistent, i.e., if E is open, $\mu(E)$ is given by the first one.

Proof. Since from the first supremum definition, open sets V_1, V_2 such that $V_1 \subseteq V_2$, it implies $\mu(V_1) \leq \mu(V_2)$. ⊛

To establish additive for the measure μ, we need to restrict to some σ-algebra of subsets of X. This is analogous of first defining outer measure and then the actual measure. To define $\mathcal{M}$, we first define $\widetilde{\mathcal{M}}$ as the class of all subsets $E \subseteq X$ such that $\mu(E) < \infty$, and

$$\mu(E) = \sup\left\{\mu(K) \colon K \subseteq E, K \text{ comapct}\right\}.$$

Then $\mathcal{M}$ is given by the class of all subsets $E \subseteq X$ such that $E \cap K \in \widetilde{\mathcal{M}}$ for every compact K. We now want to show μ is a measure on $\mathcal{M}$.

Note. μ is monotone, i.e., $\mu(A) \leq \mu(B)$ for $A \subseteq B$. Also, $\mu(E) = 0$ implies $E \in \mathcal{M}$.

We then use the monotonicity of $\Lambda(\cdot)$ to prove the remaining properties, i.e., $f \leq g \Rightarrow \Lambda(f) \leq \Lambda(g)$, i.e., $g - f \geq 0 \Rightarrow \Lambda(g - f) = \Lambda(g) - \Lambda(f) \geq 0$.

For subadditivity, let E_n be subsets of X for all $n \geq 1$, then we want

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

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Proof of Theorem 6.1.8 (Cont.) Firstly, we show that $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$ if V_1, V_2 are open. To do this, let $g \in C_c(X)$ and $0 \leq g \leq \chi_{V_1 \cup V_2}$. Recall that $\mu(V) = \sup_{0 \leq g \leq \chi_{V_1 \cup V_2}} \Lambda(g)$. Let $K = \operatorname{supp}(g)$ such that K is compact and $K \subseteq V_1 \cup V_2$. Choose a partition of unity h_1, h_2 such that $0 \leq h_1, h_2 \leq 1$ and $0 \leq h_1 \leq \chi_{V_1}$, $0 \leq h_2 \leq \chi_{V_2}$ with $h_1(x) + h_2(x) = 1$ for $x \in K$ from the Urysohn's lemma. Then $g \coloneqq h_1 g + h_2 g$, we have

$$\Lambda(g) = \Lambda(h_1 g) + \Lambda(h_2 g),$$

with $0 \leq h_1 g \leq \chi_{V_1}$ and $0 \leq h_2 g \leq \chi_{V_2}$, we have $\Lambda(h_1 g) \leq \mu(V_1)$ and $\Lambda(h_2 g) \leq \mu(V_2)$, hence

$$\Lambda(g) \leq \mu(V_1) + \mu(V_2)$$

for all $g \in C_c(X)$ such that $0 \leq g \leq \chi_{V_1 \cup V_2}$. Taking the supremum over g, we then get

$$\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$$

if V_1, V_2 are open. For general E_n, $n \geq 1$, subsets of X, we can assume that $\mu(E_n) < \infty$ for all $n \geq 1$. Hence, there exists open V_n such that $e_n \subseteq V_n$ such that

$$\mu(E_n) \leq \mu(V_n) + \frac{\epsilon}{2^n}$$

for $n \geq 1$. Let $V = \bigcup_{i=1}^{\infty} V_i$, which is open, then $E = \bigcup_{n=1}^{\infty} E_n \subseteq V$, i.e., $\mu(E) \leq \mu(V)$. Let $f \in C_c(X)$ and $0 \leq f < \chi_V$, we have $\mu(V) = \inf\left\{\Lambda(f) \colon 0 \leq f < \chi_V\right\}$. Then let $K = \operatorname{supp}(f)$ such that K is compact, the sets V_j for $j \geq 1$ is an open covering of K, hence there exists a finite subcover $K \subseteq \bigcup_{j=1}^{N} V_j$ for some $N < \infty$. This implies that

$$\Lambda(f) \leq \mu\left(\bigcup_{j=1}^{N} V_j\right) \leq \sum_{j=1}^{N} \mu(V_j).$$

Since $\mu(V_j) \leq \mu(E_j) + \epsilon/2^j$, we have

$$\Lambda(f) \leq \sum_{j=1}^{N} \mu(E_j) + \frac{\epsilon}{2^j} \leq \sum_{j=1}^{\infty} \mu(E_j) + \epsilon.$$

Take supremum over all f with $0 \leq f < \chi v V$, we have $\mu(V) \leq \sum_{j=1}^{\infty} \mu(E_j) + \epsilon$, i.e.,

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_j) + \epsilon.$$

By letting $\epsilon \to 0$, we have subadditivity.

Recall that $\widetilde{\mathcal{M}}$ contains all subsets E of X such that $\mu(E) < \infty$. And

$$\mu(E) = \sup \{\mu(K) \colon K \subseteq E, K \text{ compact}\}.$$

In particular, $\widetilde{\mathcal{M}}$ contains all compact subsets K of X and all open subsets V of X such that $\mu(V) < \infty$. Let $K \subseteq X$ and compact, suppose $f \in C_c(X)$ with $\chi_K \leq f$, with $V = \{x \in X \colon f(x) > 1/2\}$, which is open and $K \subseteq V$. Furthermore, let $g \in C_c(X)$ such that $0 \leq g < \chi_V$, hence $g \leq 2f$, which implies

$$\mu(K) \leq \mu(V) = \sup \{\Lambda(g) \colon 0 \leq g < \chi_V\} \leq \Lambda(2f) < \infty.$$

We then conclude that $\mu(K) < \infty$, i.e., $K \in \widetilde{\mathcal{M}}$. Next, we show that if V is open and $\mu(V) < \infty$, then $\mu(V) = \sup \{\mu(K) \colon K \subseteq V, K \text{ compact}\}$. This is clear if $\mu(V) = 0$, so assume $\mu(V) > 0$. Let α satisfy $0 < \alpha < \mu(V)$, then there exists $f \in C_c(X)$ such that $0 \leq f < \chi_V$, we have $\alpha < \Lambda(f)$. Set $K = \operatorname{supp}(f)$, so K is compact. Then if W is open and $K \subseteq W$ have $\alpha < \Lambda(f) \leq \mu(W)$, so $\alpha < \mu(W)$ for all W containing K. With $\mu(K) = \inf \{\mu(W) \colon K \subseteq W\}$, $\alpha \leq \mu(K)$ for any $\alpha < \mu(V)$. This leads to

$$\mu(V) = \sup \{\mu(K) \colon K \subseteq V, K \text{ compact}\}.$$

Now, for $E \subseteq X$, $E \in \widetilde{\mathcal{M}}$ if $\mu(E) < \infty$ and $\mu(E) = \sup \{\mu(K) \colon K \subseteq E, K \text{ compact}\}$, we have shown that if K is compact, we have $K \in \widetilde{\mathcal{M}}$; while if V is open and $\mu(V) < \infty$, we have $V \in \widetilde{\mathcal{M}}$. We now show the additivity, i.e., let $E_1, E_2, \ldots$ be in $\widetilde{\mathcal{M}}$ and the E_j, $j \geq 1$ disjoint, we want

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j).$$

If in addition $\mu(E) < \infty$, then $E \in \widetilde{\mathcal{M}}$. First we show that if $K_1, K_2, \ldots$ is compact and disjoint, then $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$. Let $K = K_1$, $U = X - K_2$, then $K \subseteq U$. By Theorem 6.1.4, there exists open V_1 such that

$$K \subseteq V_1 \subseteq \overline{V}_1 \subseteq U.$$

Set $V_2 = X - \overline{V}_1$, then we have $K_1 \subseteq V_1$, $K_2 \subseteq V_2$ such that V_1 and V_2 are disjoint. For $\epsilon > 0$, let W be open, $K_1 \cup K_2 \subseteq W$, we have

$$\mu(W) < \mu(K_1 + K_2) + \epsilon.$$

With $W \cap V_1$ and $W \cap V_2$ are disjoint, there exists functions $f_1, f_2 \in C_c(X)$ such that

$$0 \leq f_1 < \chi_{W \cap V_1}, \quad 0 \leq f_2 < \chi_{W \cap V_2}$$

with

$$\Lambda(f_1) > \mu(W \cap V_1) - \epsilon, \quad \Lambda(f_2) > \mu(W \cap V_2) - \epsilon.$$

Then

$$\begin{aligned}
\mu(K_1)+\mu(K_2) &\leq \mu(W\cap V_1)+\mu(W\cap V_2)\\
&< \Lambda(f_1)+\Lambda(f_2)+2\epsilon\\
&= \Lambda(f_1+f_2)+2\epsilon\\
&\leq \mu(W)+2\epsilon\\
&< \mu(K_1\cup K_2)+3\epsilon
\end{aligned}$$

since V_1, V_2 are disjoint, with $0\leq f_1+f_2<\chi_W$. Let $\epsilon\to 0$, we have

$$\mu(K_1)+\mu(K_2)\leq \mu(K_1\cup K_2),$$

and from subadditivity, $\mu(K_1\cup K_2)=\mu(K_1)+\mu(K_2)$. Now, we show $\mu(E)=\sum_{j=1}^{\infty}\mu(E_j)$ where $E=\bigcup_{j=1}^{\infty}E_j$ such that $E_j\in\widetilde{\mathcal{M}}$ are disjoint. We may assume that $\mu(E)<\infty$, i.e., $\mu(E_i)<\infty$ for all i since $E_i\subseteq E$. Since $E_i\in\widetilde{\mathcal{M}}$, there exists a compact $H_i\subseteq E_i$ such that $\mu(H_i)>\mu(E_i)-\epsilon/2^i$ for all $i\geq 1$. Then, form $\bigcup_{i=1}^{n}H_i\subseteq E$, we have

$$\mu(E)\geq\mu\left(\bigcup_{i=1}^{n}H_i\right)=\sum_{i=1}^{n}\mu(H_i)$$

from finite additivity for compact sets. Let $n\to\infty$,

$$\mu(E)\geq\sum_{i=1}^{\infty}\mu(H_i)\geq\sum_{i=1}^{\infty}\mu(E_i)-\frac{\epsilon}{2^i}=\sum_{i=1}^{\infty}\mu(E_i)-\epsilon,$$

by letting $\epsilon\to 0$, we have $\mu(E)=\sum_{i=1}^{\infty}\mu(E_i)$.

Finally, we want to show that if $\mu(E)<\infty$, then

$$\mu(E)=\sup\left\{\mu(K)\colon K\subseteq E, K\text{ compact}\right\},$$

which further implies $E\in\widetilde{\mathcal{M}}$. We have shown that

$$\mu(E)=\sum_{i=1}^{\infty}\mu(E_i),$$

so if $\mu(E)<\infty$, for any $\epsilon>0$, there exists an $N=N(\epsilon)$ such that

$$\sum_{i=1}^{N}\mu(E_i)\geq\mu(E)-\epsilon.$$

Since $\mu(H_i)\geq\mu(E_i)-\epsilon/2^i$ for all $i\geq 1$ for $H_i\subseteq E_i$ being compact, this implies

$$\mu\left(\bigcup_{i=1}^{N}H_i\right)\geq\mu(E)-\epsilon,$$

hence we're done. ■

Appendix

Appendix A

Review

A.1 Midterm Review

A.1.1 Normed Spaces

Recall the normed spaces, and the properties of which. In particular, focus on convexity and note that $x \mapsto \|x\|$ is a convex function.

Example (Normed spaces). The spaces ℓ_p for $1 \leq p \leq \infty$ of sequences and $L^p(\Omega, \mathcal{F}, \mu)$ of integrable functions. Also, the space of continuous functions on compact Hausdorff space with supremum norm $C(K)$. Notice that

$$C(K) \subseteq L^\infty(K, \mathcal{F}).$$

Remark (Legendre transform). Recall the Legendre transform of convex functions. The most general form is that let X be a Banach space and X^* its dual space with a convex function $f\colon X \to \mathbb{R}$ and $f^*\colon X^* \to \mathbb{R}$. We have

$$f^*(y^*) = \sup_{x \in X} \left[y^*(x) - f^*(x)\right].$$

Notice that f^* is convex and lower semi-continuous where $f^*\colon X^* \to \mathbb{R} \cup \{+\infty\}$.

A.1.2 Quotient Spaces

Let X be a normed space and E be a subspace of X. Then $X \,/\, E = \{[x] = x + E \colon x \in X\}$ if E is closed, then $X \,/\, E$ is also a normed space with the norm $\|[x]\| \coloneqq \inf_{y \in E} \|x - y\|$.

Remark. E need to be closed since we need $\|[x]\| = 0 \Rightarrow [x] = 0$.

A.1.3 Banach Spaces

Any normed space E can be completed to a Banach space $\hat{E}$ by Theorem 1.4.2.

Example. ℓ_p and L^p are Banach spaces. For $x \in \ell_p$, $x = \{x_n, n \geq 1\}$ with

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

Notice that Minkowski inequality is the triangle inequality for ℓ_p and L^p, and we can prove this using Hölder's inequality where we have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

for $1/p + 1/q = 1$.

Remark (Proof of completeness of the ℓ_p spacees). This is easy for ℓ_p, but for L^p, we need to use dominated convergence theorem.

A.1.4 Inner Product Spaces and Hilbert Spaces

Notice that the Hilbert spaces are the completion of inner product spaces. Recall the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

and the Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \|y\| .$$

Orthogonality

Recall the orthogonal projection P_E onto a closed subspace $E \subseteq \mathcal{H}$ is $P_E x = x(y)$ where $x(y)$ is the minimizer of $\min_{y\in E} \|x-y\|$.

Remark. P_E is the projection, i.e., $P_E^2 g P_E$, and $I - P_E$ is proaction onto the orthogonal complement $E^{\perp}$ of E in $\mathcal{H}$ such that $\mathcal{H} = E \oplus E^{\perp}$. We see that

$$\|x\|^2 = \|P_E x\|^2 + \|(I-P_E)x\|^2$$

for $x \in \mathcal{H}$.

Consider the orthogonal or orthonormal sets of vectors x_k, $k = 1, 2, \ldots$ in $\mathcal{H}$ with the corresponding Fourier series being

$$S_n(x) \coloneqq \sum_{k=1}^{n} \langle x, x_k\rangle x_k$$

such that

$$\|S_n(x)\|^2 = \sum_{k=1}^{n} |\langle x, x_k\rangle|^2 .$$

If the set $\{x_k\}_{k=1}^{\infty}$ is orthonormal, then $S_n = P_{E_n}$ where E_n is the span of $\{x_1, \ldots, x_n\}$, and

$$\|S_n x\|^2 = \|P_{E_n} x\|^2 \leq \|x\|^2 ,$$

which is the Bessel's inequality.

Remark. $S_n x \to S_\infty x$ in $\mathcal{H}$ where $S_\infty = P_{E_\infty}$ and E_∞ is the closure of spaces E_n, $n \geq 1$.

The orthonormal system $\{x_k\}_{k\geq 1}$ is complete if $E_\infty = \mathcal{H}$. In that case, $\|x\|^2 = \|P_{E_\infty} x\|^2 = \sum_{k=1}^{\infty} |\langle x, x_k\rangle|^2$.

Remark. Proving completeness can be difficult.

Example (Haar basis). The Haar basis for $L^2([0,1])$ is the Fourier basis $e^{2\pi n i x}$, $n \in \mathbb{Z}$ for $L^2([0,1])$.

Proof. Let x_k, $k \geq 1$ be any arbitrary sequence of vectors in $\mathcal{H}$. We can then construct an orthonormal sequence y_k, $k \geq 1$ by applying Gram-Schmidt procedure. ⊛

A.1.5 Bounded Linear Functionals

Consider bounded linear functionals on a Banach space E, $f \in E^*$, $\|f\| = \sup_{\|x\|=1} |f(x)|$ and E^* is a Banach space. Recall that $f(\cdot)$ is essentially defined by $H = \ker(f)$ where H is a closed subspace of E with $\operatorname{codim}(H) = 1$, i.e., $\dim E \,/\, H = 1$ and we have

$$\widetilde{f}\colon {}^{E}\big/_{H} \to \mathbb{R}$$

is defined via $\widetilde{f}([x]) = f(x)$ for $x \in E$, and $\widetilde{f}(a[x]) = ca$ for some constant c.

A.1.6 Representation Theorem

The important representation theorem for bounded linear functionals is the Riesz representation theorem. The easiest case is $E = \mathcal{H}$ being a Hilbert space and $E^* \equiv \mathcal{H}$. This implies Radon-Nikodym theorem, where if we have $\nu \ll \mu$, then

$$\nu(E) = \int_E f \, \mathrm{d}\mu, \quad f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^1(\mu)$$

for ν, μ being finite measures. Furthermore, the Radon-Nikodym theorem implies the Riesz representation theorem for ℓ_p and L^p with $1 \leq p < \infty$.

> **Remark.** We have $E^* = \ell_q$ or L^q with $1/p + 1/q = 1$ for $1 \leq p < \infty$, and remarkably, $\ell_1^* = \ell_\infty$ but $\ell_\infty^* \neq \ell_1$.

> **Remark.** The Riesz representation theorem for $C(K)$ is space of bounded Borel measures where for $g \in C(K)^*$,
>
> $$g(f) = \int_K f \, \mathrm{d}\mu$$
>
> for $f \in C(K)$.

A.1.7 Hahn-Banach Theorem

Let E be a Banach space and E_0 be a subspace such that $f_0 \colon E_0 \to \mathbb{R}$ a bounded linear functional on E_0 such that $\|f_0\| < \infty$. Then there exists an extension f of f_0 to E with $\|f\| = \|f_0\|$.

> **Remark.** f is not necessary unique. Nevertheless, it is unique for Hilbert spaces, or ℓ_p, L^p with $1 < p < \infty$.

Reflexivity

Consider the embedding $E \to E^{**}$ such that $x \mapsto x^{**}$, then E is reflexive if the embedding is surjective. Also, E is reflexive implies that

$$\|f\| = \sup_{\|x\|=1} |f(x)| = f(x_f)$$

for some $x_f \in E$ with $\|x_f\| = 1$ for every $f \in E^*$.

> **Remark.** This is one way of showing some spaces is not reflexive.

Separation Theorem

Recall the separation theorem for convex sets from a point. Given a convex set K and a point $x_0 \notin K$, there is a hyperplane such that $f(x_0) > f(k)$ for all $k \in K$. The Minkowski functional for convex set essentially makes convex sets unit ball for some semi-norm.

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