

MATH597

Analysis II

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\stackrel{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition we're using to deduce that particular equality.

Notice that there are some proofs is **intended** left as assignments, and for completeness, I put them in [Appendix A](#), use it in your **own risks**! You'll lose the chance to practice and really understand the materials.

Additionally, we'll use [\[FF99\]](#) as our main text, while using [\[Tao13\]](#) and [\[Ax19\]](#) as supplementary references.

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Lecture 1: σ -algebra

05 Jan. 11:00

1 Measure

Example. Before we start, we first see some examples.

- Let $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) := \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which is the *power set* of X . We see that

$$\#X = n \implies \#\mathcal{P}(X) = 2^n$$

for $n < \infty$.

- If $n = \infty$, say $X = \mathbb{N}$, then

$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while \mathbb{N} is a countable set. We can see this as follows. Consider

$$\phi: \mathcal{P}(\mathbb{N}) \rightarrow [0, 1], \quad A \mapsto 0.a_1a_2a_3 \dots \text{ (base 2),}$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example, A can be $A = \{2, 3, 6, \dots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have

$$\#\mathcal{P}(\mathbb{N}) \geq \#[0, 1].$$

But since $[0, 1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X . Hence, we are intriguing to define a map μ such that

$$\mu: \mathcal{P}(X) \rightarrow [0, \infty].$$

Example. We first see some examples.

- Let $X = \{0, 1, 2\}$. Then we want to define $\mu: \mathcal{P}(X) \rightarrow [0, \infty]$, we can have

- $\mu(A) = \#A$. Then we have
- $\mu(\{0, 1\}) = 2$

- $\mu(\{0\}) = 1$
- $\mu(A) = \sum_{i \in A} 2^i$. Then we have
 - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$
- 2. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu: \mathcal{P}(\mathbb{N}) \rightarrow [0, \infty]$, we can have
 - $\mu(A) = \#A$. Then we have
 - $\mu(\{2, 3, 4, 5, \dots\}) = \infty = \mu(\{\text{even numbers}\})$
 - $\mu(A) = e^{-1} \sum_{i \in A} \frac{1}{i!}$. Then we have
 - $\mu(\{0, 2, 4, 6, \dots\}) = e^{-1} (1 + \frac{1}{2!} + \frac{1}{3!} + \dots)$
 - $\mu(A) = \sum_{i \in A} a_i$
- 3. Let $X = \mathbb{R}$. Then we want to define $\mu: \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$, we can have
 - $\mu(A) = \#A$
 - $\mu((a, b)) = b - a$.

Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

Answer. No!

- $\mu((a, b)) = e^b - e^a$.

Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

Answer. No!

We immediately see the problems. To extend our native measure method into \mathbb{R} is hard and will cause something counter-intuitive!¹ Hence, rather than define measurement on *all* subsets in the power set of X , we only focus on *some* subsets. In other words, we want to define

$$\mu: \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \rightarrow [0, \infty].$$

1.1 σ -algebras

We start from the definition of the most fundamental element in measure theory.

¹https://en.wikipedia.org/wiki/Banach-Tarski_paradox

Definition 1.1 (σ -algebra). Let X be a set. A collection \mathcal{A} of subsets of X , i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a σ -algebra on X if

- $\emptyset \in \mathcal{A}$.
- \mathcal{A} is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- \mathcal{A} is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\emptyset}_{\in \mathcal{A}}$ and \mathcal{A} is closed under complement.
- $\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c$, namely \mathcal{A} is closed under countable intersections.
- $A_1 \cup A_2 \cup \dots \cup A_n = A_1 \cup A_2 \cup \dots \cup A_n \cup \emptyset \cup \emptyset \cup \dots$, hence \mathcal{A} is closed under finite unions and intersections.

An immediate definition can be given. We now define so-called *Borel set*.

Definition 1.2 (Borel set). Given a topological space X , a *Borel set* is any set in X that can be formed from open sets through the operations of countable union, countable intersection and relative complement.

Lecture 2: Measure

07 Jan. 11:00

Example. Again, we first see some examples.

1. Let $\mathcal{A} = \mathcal{P}(X)$, which is the power σ -algebra.
2. Let $\mathcal{A} = \{\emptyset, X\}$, which is a trivial σ -algebra.
3. Let $B \subset X$, $B \neq \emptyset$, $B \neq X$. Then we see that $\mathcal{A} = \{\emptyset, B, B^c, X\}$ is a σ -algebra.

Lemma 1.1. Let \mathcal{A}_α , $\alpha \in I$, be a family of σ -algebra on X . Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is a σ -algebra on X .

Remark. Notice that I may be an uncountable intersection.

Proof. A simple proof can be made as follows. Firstly, $\emptyset \in \mathcal{A}_\alpha$ for every α clearly. Moreover, closure under complement and countable unions for every \mathcal{A}_α implies the same must be true for $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$. Hence, $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ -algebra. ■

The above allows us to give the following definition.

Definition 1.3 (Generation of σ -algebra). Given $\mathcal{E} \subset \mathcal{P}(X)$, where \mathcal{E} is not necessarily a σ -algebra. Let $\langle \mathcal{E} \rangle$ be the intersection of all σ -algebras on X containing \mathcal{E} , then we call $\langle \mathcal{E} \rangle$ the σ -algebra generated by \mathcal{E} .

Remark. Clearly, $\langle \mathcal{E} \rangle$ is the smallest σ -algebra containing \mathcal{E} , and it is unique. To check the uniqueness, we suppose there are two different $\langle \mathcal{E} \rangle_1$ and $\langle \mathcal{E} \rangle_2$ generated from \mathcal{E} . It's easy to show

$$\langle \mathcal{E} \rangle_1 \subseteq \langle \mathcal{E} \rangle_2,$$

and by symmetry, they are equal.

Example. We see that $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle$.

Lemma 1.2. We have

1. Given \mathcal{A} a σ -algebra, $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \mathcal{A}$
2. $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$

Proof. We'll see that after proving the first claim, the second follows smoothly.

1. The first claim is trivial, since we know that $\langle \mathcal{E} \rangle$ is the smallest σ -algebra containing \mathcal{E} , then if $\mathcal{E} \subset \mathcal{A}$, we clearly have $\langle \mathcal{E} \rangle \subset \mathcal{A}$ by the definition.
2. The second claim is also easy. From the first claim and the definition, we have

$$\mathcal{E} \subset \mathcal{F} \subset \langle \mathcal{F} \rangle \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle.$$

■

At this point, we haven't put any specific structure on X . Now we try to describe those spaces with good structure, which will give the space some nice properties.

Definition 1.4 (Borel σ -algebra). For a topological space X , the *Borel σ -algebra* on X , denoted as $\mathcal{B}(X)$, is the σ -algebra generated by the collection of all open sets in X .

Example. We see that $\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a, b) \mid a < b; a, b \in \mathbb{R}\}$.
- $\mathcal{E}_2 = \{[a, b] \mid a < b; a, b \in \mathbb{R}\}$ since $[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.
- $\mathcal{E}_3 = \{(a, b] \mid a < b; a, b \in \mathbb{R}\}$ since $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$.
- $\mathcal{E}_4 = \{[a, b) \mid a < b; a, b \in \mathbb{R}\}$ since $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b)$.

- $\mathcal{E}_5 = ((a, \infty) \mid a \in \mathbb{R})$ since $(a, \infty) = \bigcup_{n=1}^{\infty} (a, a+n)$.
- $\mathcal{E}_6 = ([a, \infty) \mid a \in \mathbb{R})$ since $[a, \infty) = \bigcup_{n=1}^{\infty} [a, a+n)$.
- $\mathcal{E}_7 = ((-\infty, b) \mid b \in \mathbb{R})$ since $(-\infty, b) = \bigcup_{n=1}^{\infty} (b-n, b)$.
- $\mathcal{E}_8 = ((-\infty, b] \mid b \in \mathbb{R})$ since $(-\infty, b] = \bigcup_{n=1}^{\infty} (b-n, b]$.

Proposition 1.1. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \dots, 8$.

Proof. Firstly, we see that $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \implies \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$ by Lemma 1.2. Secondly, by definition, $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$ where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R}\}.$$

It's enough to show $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$ since if so, $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$, and clearly $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$, then we will have $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$. Let $O \subset \mathbb{R}$ be an open set, i.e., $O \in \mathcal{E}$. We claim that every open set in \mathbb{R} is a countable union of disjoint open intervals.²

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

where I_j open interval with the form of $(a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$.

For example, \mathcal{E}_1 is trivially true, and

$$(a, b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{\left[a + \frac{1}{n}, b - \frac{1}{n}\right]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for \mathcal{E}_2 and

$$(a, \infty) = \bigcup_{k=1}^{\infty} (a, a+k)$$

shows the case for \mathcal{E}_5 . It's now straightforward to check open intervals are in $\langle \mathcal{E}_i \rangle$ for every i . ■

Now, to put a structure on a space, we define the following.

Definition 1.5 (Measurable space, \mathcal{A} -measurable set). A *measurable space* or *Borel space* is a tuple of a set X and a σ -algebra \mathcal{A} on X , denoted by (X, \mathcal{A}) .

Furthermore, for every $E \in \mathcal{A}$ is called an \mathcal{A} -measurable set.

²<https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals>

1.2 Measures

With the definition of [measurable space](#), we now can refine our [measure](#) function μ as follows.

Definition 1.6 (Measure, Measure space). Given a [measurable space](#) on (X, \mathcal{A}) , a *measure* is a function μ such that

$$\mu: \mathcal{A} \rightarrow [0, \infty]$$

with

1. $\mu(\emptyset) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_1, A_2, \dots \in \mathcal{A}$ are **disjoint**. We call this *Countable additivity*.

We denote (X, \mathcal{A}, μ) a *measure space*.

Notation. We denote $[0, \infty] := [0, \infty) \cup \{\infty\}$.

Remark. The motivation of why we only want *countable additivity* but not uncountable additivity can be seen by the following example. We'll consider the most intuitive [measure](#) on $\mathbb{R}, \mathcal{B}(\mathbb{R})$.

Since we have

$$(0, 1] = (1/2, 1] \cup (1/4, 1/2] \cup (1/8, 1/4] \cup \dots$$

and also

$$(0, 1] = \bigcup_{x \in (0, 1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2}, 1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4}, \frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8}, \frac{1}{4}])} + \dots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0, 1]} 0$$

since $\mu(x) = 0$ for $x \in \mathbb{R}$, which is clearly not what we want.

Example. We see some examples.

1. For any (X, \mathcal{A}) , we let $\mu(A) := \#A$. This is called *counting measure*.
2. Let $x_0 \in X$. For any (X, \mathcal{A}) , the *Dirac measure* at x_0 is

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A. \end{cases}$$

3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$,

$$\mu(A) = \sum_{i \in A} a_i,$$

where $a_1, a_2, \dots \in [0, \infty)$.

Lecture 3: Construct a Measure

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Note. If $A, B \in \mathcal{A}$ and $A \subset B$, then

$$\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

Theorem 1.1. Given (X, \mathcal{A}, μ) be a [measure space](#).

1. (monotonicity)

$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

2. (countable subadditivity)

$$A_1, A_2, \dots \in \mathcal{A} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

3. (continuity from below/ monotone convergence theorem (MCT) for sets)

$$\begin{cases} A_1, A_2, \dots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \dots \end{cases} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

4. (continuity from above)

$$\begin{cases} A_1, A_2, \dots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \dots \\ \mu(A_1) < \infty \end{cases} \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Proof. We prove this theorem one by one.

1. Since $A \subset B$, hence we have

$$\mu(B) = \mu\left(\underbrace{(B \setminus A) \cup A}_{\text{disjoint}}\right) \stackrel{!}{=} \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A).$$

2. This should be trivial from [countable additivity](#) with the fact that $\mu(A) \geq 0$ for all A .

DIY!

3. Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$, then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

are a disjoint union and $B_i \in \mathcal{A}$, hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i).$$

With $\mu\left(\bigcup_{i=1}^n B_i\right) = \mu(A_n)$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n B_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

4. Let $E_i = A_1 \setminus A_i \implies E_i \in \mathcal{A}$, $E_1 \subset E_2 \subset \dots$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right),$$

which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i\right) \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$$

since $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu(A_1) < \infty$. Then from [continuity from below](#), we further have

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(E_n) = \mu(A_1) - \lim_{n \rightarrow \infty} (\mu(A_1) - \mu(A_n)).$$

From [monotonicity](#), we see that $\mu(A_n) \leq \mu(A_1) < \infty$, hence we can split the limit and further get

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu(A_1) + \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \mu(A_n).$$

■

Example. Given $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, [counting measure](#). Then we see

- $A_n = \{n, n+1, n+2, \dots\} \implies \mu(A_n) = \infty$
- $A_1 \supset A_2 \supset A_3 \supset \dots$
- $\bigcap_{i=1}^{\infty} A_i = \emptyset \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = 0$

Remark. We see that in this case, since $\mu(A_1) \not< \infty$, hence [continuity from above](#) doesn't hold.

We now try to characterize some properties of a measure space.

Definition 1.7 (μ -null, μ -subnull, Complete measure space). Given (X, \mathcal{A}, μ) ,

- $A \subset X$ is a μ -null set if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- $A \subset X$ is a μ -subnull set if $\exists \mu$ -null set B such that $A \subset B$.
- (X, \mathcal{A}, μ) is a complete measure space if every μ -subnull set is \mathcal{A} -measurable.

Note. We see that for a μ -subnull set, it's not necessary \mathcal{A} -measurable.

There are some useful terminologies we'll use later relating to μ -null.

Definition 1.8 (Almost everywhere). Given (X, \mathcal{A}, μ) , a statement $P(x)$, $x \in X$ holds μ -almost everywhere (a.e.) if the set

$$\{x \in X : P(x) \text{ does not hold}\}$$

is μ -null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

Definition 1.9 (Finite measure). , Given (X, \mathcal{A}, μ)

- μ is a finite measure if $\mu(X) < \infty$.
- μ is a σ -finite measure if $X = \bigcup_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.

Exercise. Every measure space can be completed. Namely, we can always find a bigger σ -algebra to complete the space.

1.3 Outer Measures

We start by giving a definition.

Definition 1.10 (Outer measure). An outer measure on X is a map

$$\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$$

such that

- $\mu^*(\emptyset) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$
- (countable subadditivity) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ for every $A_i \subset X$.

Example. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

is an **outer measure** due to the **Proposition 1.2** we're going to show.

Remark. We see that an **outer measure** need not be a **measure**. Check **Definition 1.6**.

Proposition 1.2. Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\emptyset, X \in \mathcal{E}$. Let

$$\rho: \mathcal{E} \rightarrow [0, \infty]$$

such that $\rho(\emptyset) = 0$. Then

$$\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

is an **outer measure** on X .

Note. Recall the Tonelli's Theorem³ for series:

If $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Specifically, in [Tao13] Theorem 0.0.2.

Lecture 4: Carathéodory extension Theorem

12 Jan. 11:00

As previously seen. We now prove the **Proposition 1.2**.

Proof. We need to prove

- μ^* is well-defined. i.e., inf is taken over a non-empty set. This is trivial since $X \in \mathcal{E}$ and $X \supset A$ for any $A \in \mathcal{E}$.
- $\mu^*(\emptyset) = 0$. Since $\emptyset \in \mathcal{E}$ and

$$\mu^*(\emptyset) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \emptyset \right\} = 0$$

since $\rho(\emptyset) = 0$ for all i and further, by Squeeze Theorem⁴, we see that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(\emptyset) = 0.$$

³https://en.wikipedia.org/wiki/Fubini%27s_theorem

⁴https://en.wikipedia.org/wiki/Squeeze_theorem

- $A \subset B \implies \mu^*(A) \leq \mu^*(B)$. We simply show this by contradiction. Suppose $A \subset B$ and $\mu^*(A) > \mu^*(B)$, then by definition of μ^* , we have

$$\begin{aligned} \mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\} \\ &> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B \right\} = \mu^*(B). \end{aligned}$$

Now, let $B = (B \setminus A) \cup A$, then we have

$$\begin{aligned} \mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\} \\ &> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B \setminus A) \cup A \right\} = \mu^*(B). \end{aligned}$$

Now, since $B \setminus A \supseteq \emptyset$, then this inequality can't hold, hence a contradiction \nexists .

- Countable subadditivity. Let $A_1, A_2, \dots \in X$. If one of $\mu^*(A_n) = \infty$, then result holds. So we may assume $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Now, fix any $\epsilon > 0$, we will show that

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon.$$

For each $n \in \mathbb{N}$, $\exists E_{n,1}, E_{n,2}, \dots \in \mathcal{E}$ such that

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n$$

and

$$\mu^*(A_n) + \frac{\epsilon}{2^n} \geq \sum_{k=1}^{\infty} \rho(E_{n,k}).$$

Then we see that

$$\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k) \in \mathbb{N}^2} E_{k,n},$$

which implies

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{(n,k) \in \mathbb{N}^2} \rho(E_{k,n}) \stackrel{!}{=} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{k,n}) \leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right)$$

from the inequality just derived. Now, since the last term is just

$$\sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon,$$

⁵This is an important trick!!

hence we finally have

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon$$

for arbitrarily small fixed $\epsilon > 0$, hence the subadditivity is proved. ■

Definition 1.11 (Carathéodory measurable). Let μ^* be an **outer measure** on X . We say $A \subset X$ is *Carathéodory measurable* (*C-measurable*) with respect to μ^* if

$$\forall E \subset X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Lemma 1.3. Let μ^* be an **outer measure** on X . Suppose B_1, \dots, B_N are disjoint **C-measurable** sets. Then,

$$\forall E \subset X, \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \right) \right) = \sum_{i=1}^N \mu^*(E \cap B_i).$$

Proof. Since we have

$$\begin{aligned} \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \right) \right) &= \mu^*(E' \cap B_1) + \mu^*(E' \setminus B_1) \text{ }^6 \\ &= \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \cap B_1 \right) \right) + \mu^* \left(E \cap \left(\bigcup_{i=1}^N B_i \right) \cap B_1^c \right) \\ &= \mu^*(E \cap B_1) + \mu^* \left(E \cap \left(\bigcup_{i=2}^N B_i \right) \right) \end{aligned}$$

where the equality comes from the fact that B_1 is **C-measurable** and disjoint from $B_i, i \neq 1$. Then, we simply iterate this argument and have the result. ■

Remark. This implies that if we restrict an **outer measure** on a **C-measurable** set, then it becomes finite additive.

Theorem 1.2 (Carathéodory extension Theorem). Let μ^* be an **outer measure** on X . Let \mathcal{A} be the collection of **C-measurable** sets (with respect to μ^*). Then,

1. \mathcal{A} is a **σ -algebra** on X .
2. $\mu = \mu^*|_{\mathcal{A}}$ is a **measure** on (X, \mathcal{A}) .
3. (X, \mathcal{A}, μ) is a **complete measure space**.

⁶Here, $E' := E \cap \left(\bigcup_{i=1}^N B_i \right)$ for the simplicity of notation.

Proof. We divide the proof in several steps.

1. We show \mathcal{A} is a σ -algebra by showing

(a) $\emptyset \in \mathcal{A}$. To show this, we simply check that \emptyset is \mathbf{C} -measurable. We see that

$$\forall_{E \subset X} \mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E \setminus \emptyset) = \mu^*(E),$$

which just shows $\emptyset \in \mathcal{A}$.

(b) \mathcal{A} closed under complements. This is equivalent to say that if A is \mathbf{C} -measurable, so is A^c . We see that if A is \mathbf{C} -measurable, then for every $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Observing that $E \cap A = E \setminus A^c$ and $E \setminus A = E \cap A^c$, hence

$$\mu^*(E) = \mu^*(E \setminus A^c) + \mu^*(E \cap A^c).$$

We immediately see that above implies $A^c \in \mathcal{A}$.

(c) \mathcal{A} closed under countable unions.

Note. To show \mathcal{A} closed under countable unions, we show that \mathcal{A} is closed under:

finite unions $\xRightarrow{\text{then}}$ countable disjoint unions $\xRightarrow{\text{then}}$ countable unions.

- We show \mathcal{A} is closed under finite unions.

Claim. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

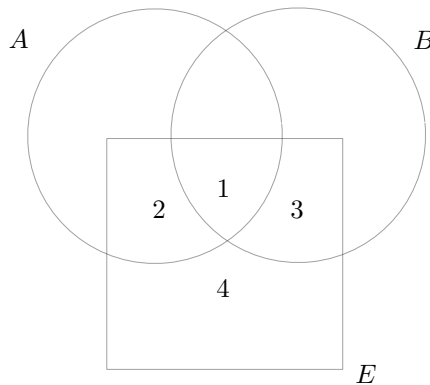
Fix $E \subset X$ arbitrary. We need to show that

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)),$$

i.e.,

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

given $A, B \in \mathcal{A}$.



- Since A is **C-measurable**,
 - * $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$
 - * $\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$
- Since B is **C-measurable**,
 - * $\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$

Hence, we have

$$\begin{aligned} \mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4). \end{aligned}$$

- We show \mathcal{A} is closed under countable disjoint unions.

Let $A_1, A_2, \dots \in \mathcal{A}$ and disjoint. Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^{\infty} A_i\right),$$

hence we only need to show another way around.

Fix $N \in \mathbb{N}$, we have $\bigcup_{n=1}^N A_n \in \mathcal{A}$ since N is finite, and

$$\begin{aligned} \mu^*(E) &= \mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right) + \mu^*\left(E \setminus \left(\bigcup_{n=1}^N A_n\right)\right) \\ &\geq \underbrace{\sum_{n=1}^N \mu^*(E \cap A_n)}_{\stackrel{!}{=} \mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right)} + \underbrace{\mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right)}_{\leq \mu^*\left(E \setminus \left(\bigcup_{n=1}^N A_n\right)\right)}. \end{aligned}$$

Now, take $N \rightarrow \infty$ then we are done.

- We show \mathcal{A} is closed under countable unions.

DIY

The proof will be **continued**...

Lecture 5: Hahn-Kolmogorov Theorem

14 Jan. 11:00

Firstly, we see a stronger version of **Lemma 1.3** we have seen before.

Lemma 1.4. Let μ^* be an **outer measure** on X . Suppose B_1, B_2, \dots are disjoint **C-measurable** sets. Then,

$$\forall E \subset X, \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i).$$

Proof.

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_n) \geq \mu^* \left(E \cap \bigcup_{n=1}^{\infty} B_n \right) \geq \mu^* \left(E \cap \left(\bigcup_{n=1}^N B_n \right) \right) \stackrel{!}{=} \sum_{n=1}^N \mu^*(E \cap B_n).$$

Now, we just take $N \rightarrow \infty$ (or note that $N \in \mathbb{N}$ is arbitrary, we then get the result according to Squeeze Theorem⁷). ■

Let's continue the proof of Theorem 1.2.

2. Since from Definition 1.6, we need to show

- $\mu(\emptyset) = 0$. This means that we need to show $\mu^*|_{\mathcal{A}}(\emptyset) = 0$. Since $\emptyset \in \mathcal{A}$ and μ^* is an outer measure, hence from the property of outer measure, it clearly holds.
- Countable additivity of μ^* on \mathcal{A} follows from the Lemma 1.4 with $E = X$

3. The proof is given in Theorem A.1. ■

1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets \mathcal{E} and any ρ such that it assigns measure on \mathcal{E} , then it induces an outer measure by Proposition 1.2, finally complete the outer measure by Theorem 1.2.

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.2}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to σ -algebra we are working on till now.

Definition 1.12 (Algebra). Let X be a set. A collection \mathcal{A} of subsets of X , i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on X* if

- $\emptyset \in \mathcal{A}$.
- \mathcal{A} is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- \mathcal{A} is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^n A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an algebra and a σ -algebra is whether they closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an outer measure.

⁷https://en.wikipedia.org/wiki/Squeeze_theorem

Definition 1.13 (Pre-measure). Let \mathcal{A}_0 be an algebra on X . We say

$$\mu_0: \mathcal{A}_0 \rightarrow [0, \infty]$$

is a *pre-measure* if

1. $\mu_0(\emptyset) = 0$
2. (finite additivity) $\mu_0\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu_0(A_i)$ if $A_1, \dots, A_n \in \mathcal{A}_0$ are disjoint.
3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, disjoint, then

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 1.5. (1) + (3) \implies (2) in Definition 1.13.

Proof. It's easy to see that since μ_0 is monotone. ■

Theorem 1.3 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra \mathcal{A}_0 on X . Let μ^* be the outer measure induced by (\mathcal{A}_0, μ_0) in Proposition 1.2. Let \mathcal{A} and μ be the Carathéodory σ -algebra and measure for μ^* , then (\mathcal{A}, μ) extends (\mathcal{A}_0, μ_0) . i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts.

- We first show $\mathcal{A} \supset \mathcal{A}_0$. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is C-measurable, i.e.,

$$\forall E \subset X \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We first fix an $E \subset X$. From countable subadditivity of μ^* , we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

Fix $\epsilon > 0$. By the Proposition 1.2 of μ^* , $\exists B_1, B_2, \dots \in \mathcal{A}_0$, $\bigcup_{n=1}^{\infty} B_n \supset E$ such that

$$\mu^*(E) + \epsilon \geq \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left(\underbrace{\mu_0(B_n \cap A)}_{\in \mathcal{A}_0} + \underbrace{\mu_0(B_n \cap A^c)}_{\in \mathcal{A}_0} \right)$$

by the [finite additivity](#) of μ_0 . Note that

$$\left\{ \begin{array}{l} \bigcup_{n=1}^{\infty} (B_n \cap A) \supset E \cap A \\ \bigcup_{n=1}^{\infty} (B_n \cap A^c) \subset E \cap A^c \end{array} \right. \implies \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

since

$$\mu^*(E \cap A) \leq \mu^*\left(\bigcup_{n=1}^{\infty} (B_n \cap A)\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A)$$

and

$$\mu^*(E \cap A^c) \leq \mu^*\left(\bigcup_{n=1}^{\infty} (B_n \cap A^c)\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A^c).$$

We then see that for any $\epsilon > 0$, the inequality

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

holds, hence so does

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

which implies $\mathcal{A} \supset \mathcal{A}_0$.

The proof will be [continued](#)...

Lecture 6: Hahn-Kolmogorov Theorem and Extension.

18 Jan. 11:00

Let's continue the proof of [Theorem 1.3](#).

- Let $A \in \mathcal{A}_0$, we want to show that

$$\mu(A) = \mu_0(A).$$

– Firstly, let

$$B_i = \begin{cases} A, & \text{if } i = 1 \\ \emptyset, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,$$

hence $\bigcup_{i=1}^{\infty} B_i = A$, then we see that

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu_0(B_i) = \mu_0(A)$$

from the [definition](#) of μ^* and [countable additivity within the algebra](#) of μ_0 .

- Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A \cap B_1 \in \mathcal{A}_0$, $C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right) \in \mathcal{A}_0$ for $i \geq 2$ since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

are disjoint countable unions, by [countable additivity within the algebra](#), we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i) \implies \mu_0(A) \leq \mu^*(A)$$

by taking the infimum from the [definition](#) of μ^* .

Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every $A \in \mathcal{A}_0$, which implies

$$\mu(A) = \mu_0(A)$$

for every $A \in \mathcal{A}_0$ from [Theorem 1.2](#), where we extend μ^* to μ respect to \mathcal{A}_0 . ■

Definition 1.14 (HK extension). (\mathcal{A}, μ) obtained from [Theorem 1.3](#) is the *Hahn-Kolmogorov extensions* of (\mathcal{A}_0, μ_0) .

We can show the uniqueness of [HK extension](#).

Theorem 1.4 (Uniqueness of HK extension). Let \mathcal{A}_0 be an [algebra](#) on X , μ_0 be a [pre-measure](#) on \mathcal{A}_0 . Let (\mathcal{A}, μ) be the [HK extension](#) of (\mathcal{A}_0, μ_0) . Let (\mathcal{A}', μ') be another extension of (\mathcal{A}_0, μ_0) . Then if μ_0 is [σ-finite](#), $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Note. Notice that $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$ since they both extend \mathcal{A}_0 .

Proof. Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing the arbitrary cover of A and using the [definition](#) of μ^* .

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) < \infty$, and fix $\epsilon > 0$. Then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup_{i=1}^{\infty} B_i \supset A$ such that

$$\mu(A) + \epsilon = \mu^*(A) + \epsilon \geq \sum_{i=1}^{\infty} \mu_0(B_i) \stackrel{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

This implies that

$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$

where the first equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

$$\mu(B) = \lim_{N \rightarrow \infty} \mu\left(\bigcup_{i=1}^N B_i\right) \stackrel{8}{=} \lim_{N \rightarrow \infty} \mu'\left(\bigcup_{i=1}^N B_i\right) = \mu'(B),$$

hence,

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \stackrel{9}{\leq} \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon$$

for arbitrary ϵ , so we conclude $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is σ -finite, so we know $X = \bigcup_{n=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ such that

$$\mu_0(X_n) < \infty.$$

Replacing X_n by $X_1 \cup \dots \cup X_n \in \mathcal{A}_0$, we may assume that

$$X_1 \subset X_2 \subset \dots$$

Then,

$$\forall_{n \in \mathbb{N}} \mu(A \cap X_n) < \infty \stackrel{!}{\implies} \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

From the continuity of [measure](#), we then have

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A \cap X_n) \leq \lim_{n \rightarrow \infty} \mu'(A \cap X_n) = \mu'(A).$$

■

⁸ $\mu = \mu'$ on \mathcal{A}_0 .

⁹From the first part.

Corollary 1.1. Let μ_0 be a [pre-measure](#) on [algebra](#) \mathcal{A}_0 on X . Suppose μ_0 is [\$\sigma\$ -finite](#), then

$\exists!$ [measure](#) μ on $\langle \mathcal{A}_0 \rangle$ that extends \mathcal{A}_0 .

Furthermore,

- The completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the [HK extension](#) of (\mathcal{A}_0, μ_0) .

-

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \forall_{i \in \mathbb{N}} \bigcup_{i=1}^{\infty} B_i \supset A \right\}$$

for all $A \in \overline{\langle \mathcal{A}_0 \rangle}$.

Lecture 7: Borel Measures

21 Jan. 11:00

1.5 Borel Measures on \mathbb{R}

We first introduce so-called *distribution function*.

Definition 1.15 (Distribution function). An [increasing](#)^a function

$$F: \mathbb{R} \rightarrow \mathbb{R}$$

and [right-continuous](#). F is then a *distribution function*.

^aHere, increasing means $F(x) \leq F(y)$ for $x < y$.

Example. Here are some examples of right-continuous functions.

1. $F(x) = x$.

2. $F(x) = e^x$.

3. Define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

4. Let $\mathbb{Q} := \{r_1, r_2, \dots\}$. Define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$

and

$$F(x) := \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$

Then F is a distribution function (hence right-continuous). This is shown in [Lemma A.1](#).

Note. If F is increasing, and

$$F(\infty) := \lim_{x \nearrow \infty} F(x), \quad F(-\infty) := \lim_{x \searrow -\infty} F(x)$$

exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.¹⁰

Now, we can define a *Borel measure* on $(X, \mathcal{B}(\mathbb{R}))$.

Definition 1.16 (Borel measure). A *Borel measure* is any **measure** μ defined on the **σ -algebra** of **Borel sets**.

Definition 1.17 (Locally finite). Let X be a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Note. Some authors will require a **Borel measure** equipped with the **locally finite** property. But formally, this is not so common.

Lemma 1.6. Let μ be a **locally finite Borel measure** on \mathbb{R} , then

$$F_\mu(x) = \begin{cases} \mu((0, x]), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu((x, 0]), & \text{if } x < 0 \end{cases}$$

is a **distribution function**.

Proof. To show F_μ is increasing, consider $x < y$ such that

$$F_\mu(x) \leq F_\mu(y)$$

by considering

- $x > 0$: Then $F_\mu(x) = \mu((0, x])$ and

$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$

- $x = 0$: Then $F_\mu(x) = 0$ and

$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$

since $y > 0$.

- $x < 0$: Follows the same argument with $x > 0$.

¹⁰There are distributions [FF99] Ch9., but these are different from distribution functions.

Now, we need to show F_μ is right-continuous. Firstly, assume that $x \geq 0$, then we see that

$$F_\mu(x) = \mu((0, x]) = \mu((0, x^+])$$

from the fact that a measure is right-continuous.¹¹ Now, if $x \leq 0$, the same argument follows since multiplying -1 will not change the fact that a [measure](#) is continuous. ■

Definition 1.18 (Half intervals). We call

$$\emptyset, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

half-intervals.

Lemma 1.7. Let \mathcal{H} be the collection of finite disjoint unions of [half-intervals](#). Then, \mathcal{H} is an [algebra](#) on \mathbb{R} .

Proof. We see that

- $\emptyset \in \mathcal{H}$. Clearly.
- To show \mathcal{H} is closed under complements, we have
 - $\emptyset^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$.
 - $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$.¹²
 - $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$.
 - $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$.
 - $(-\infty, \infty)^c = \emptyset \in \mathcal{H}$.
- \mathcal{H} is closed under finite unions, clearly.

■

¹¹Actually, a measure is always continuous.

¹²Since it's a two disjoint union of half intervals.

Proposition 1.3 (Distribution function defines a pre-measure). Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be a [distribution function](#). For a [half interval](#) I , define

$$\ell(I) := \ell_F(I) = \begin{cases} 0, & \text{if } I = \emptyset; \\ F(b) - F(a), & \text{if } I = (a, b]; \\ F(\infty) - F(a), & \text{if } I = (a, \infty]; \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b]; \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 := \mu_{0,F}$ as

$$\mu_{0,F}: \mathcal{H} \rightarrow [0, \infty]$$

by

$$\mu_0(A) = \sum_{k=1}^N \ell(I_k) \text{ if } A = \bigcup_{k=1}^N I_k,$$

where A is a finite disjoint union of [half intervals](#) I_1, \dots, I_N . Then, μ_0 is a [pre-measure](#) on \mathcal{H} .

Proof. We see that

1. μ_0 is well-defined.
2. $\mu_0(\emptyset) = 0$.
3. μ_0 is finite additive.
4. μ_0 is countable additive within \mathcal{H} .

Suppose $A \in \mathcal{H}$ where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.¹³

Focus on the case $I = (a, b]$. Let

$$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

which is a disjoint union. Then we only need to check

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

- Since $(a, b] \supset \bigcup_{n=1}^N (a_n, b_n]$ for any fixed $N \in \mathbb{N}$, hence

$$\forall_{N \in \mathbb{N}} F(b) - F(a) \geq \sum_{n=1}^N (F(b_n) - F(a_n)).$$

¹³Since \mathcal{H} is only a collection of *finite* disjoint [half intervals](#), hence after considering $A = I$, we can apply the same argument iteratively and stop in finite steps. Formally, we can consider $H \in \mathcal{H}$, $H = \bigcup_{i=1}^{\infty} A^i$, where A^i being a [half interval](#). Then by the above argument, we have $A^i = I^i$ and so on.

By letting $N \rightarrow \infty$, we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

- Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

$$F(a') - F(a) < \epsilon.$$

For each $n \in \mathbb{N}$, $\exists b'_n > b_n$ such that

$$F(b'_n) - F(b_n) < \frac{\epsilon}{2^n}.$$

Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b'_n),$$

hence

$$\exists_{N \in \mathbb{N}} [a', b] \subset \bigcup_{n=1}^N (a_n, b'_n),^{14}$$

which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^N F(b'_n) - F(a_n).$$

Finally, we see that

$$\begin{aligned} F(b) - F(a) &\leq F(b) - F(a') + \epsilon \\ &\leq \sum_{n=1}^{\infty} (F(b'_n) - F(a_n)) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\epsilon}{2^n} \right) + \epsilon \\ &= \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) + 2\epsilon \end{aligned}$$

for any fixed $\epsilon > 0$, hence

$$F(b) - F(a) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

Combine these two inequalities, we have

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

as we desired.

¹⁴This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.



Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8: Lebesgue-Stieltjes Measure on \mathbb{R}

24 Jan. 11:00

To classify all **measures**, we now see this last theorem to complete the task.

Theorem 1.5 (Locally finite Borel measures on \mathbb{R}). We have

1. $F: \mathbb{R} \rightarrow \mathbb{R}$ a **distribution function**, then there exists a **unique locally finite Borel measure** μ_F on \mathbb{R} satisfying

$$\mu_F((a, b]) = F(b) - F(a)$$

for every $a < b$.

2. Suppose $F, G: \mathbb{R} \rightarrow \mathbb{R}$ are **distribution functions**. Then,

$$\mu_F = \mu_G$$

on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof.



HW.

Remark. **Theorem 1.5** simply states that given a **distribution function**, if we restrict our attention on **locally finite measures** on \mathbb{R} following our usual convention, then it defines the **measure** on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

1.6 Lebesgue-Stieltjes Measure on \mathbb{R}

We see that

$$F \text{ distribution function} \xRightarrow{!} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 1.19 (Lebesgue-Stieltjes measure). Given a **distribution function** F , we say μ_F on \mathcal{A}_{μ_F} is called the *Lebesgue-Stieltjes measure* corresponding to F .

Definition 1.20 (Lebesgue measure, Lebesgue σ -algebra). From **Definition 1.19**, if $F(x) = x$, then the induced $(\mathcal{A}_{\mu_F}, \mu_F)$ is denoted as (\mathcal{L}, m) , where \mathcal{L} is called *Lebesgue σ -algebra*, and m is called *Lebesgue measure*.

Remark. Recall that \mathcal{L} is induced by [Theorem 1.2](#), namely given m , for all $A \subset \mathbb{R}$, we have

$$\mathcal{L} := \left\{ A \subset \mathbb{R} \mid \forall_{E \subset \mathbb{R}} m(A) = m(A \cap E) + m(A \setminus E) \right\}$$

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).^{15}$$

Example. We first see some examples.

1. $\mu_F((a, b]) = F(b) - F(a)$. Then

- $\mu_F(\{a\}) = F(a) - F(a^-)$
- $\mu_F([a, b]) = F(b) - F(a^-)$
- $\mu_F((a, b)) = F(b^-) - F(a)$

2. We define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Then

- $\mu_F(\{0\}) = 1$
- $\mu_F(\mathbb{R}) = 1$
- $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. This is easy to see since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, hence

$$\begin{aligned} \mu_F(\mathbb{R} \setminus \{0\}) &= \mu_F((-\infty, 0) \cup (0, \infty)) \\ &= \underbrace{\mu_F((-\infty, 0))}_{0-0^{16}} + \underbrace{\mu_F((0, \infty))}_{1-1^{17}} = 0. \end{aligned}$$

We call that μ_F is the *Dirac measure* at 0.

3. Denote $\mathbb{Q} = \{r_1, r_2, \dots\}$, and we define

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n. \end{cases}$$

Then

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$.

¹⁵Some text will use x^- and x^+ instead of x^- and x^+ , respectively.

¹⁶It follows from $F(0^-) - F(-\infty) = 0 - 0 = 0$.

¹⁷It follows from $F(\infty) - F(0) = 1 - 1 = 0$.

This is shown in [Lemma A.2](#).

4. If F is continuous at a , then $\mu_F(\{a\}) = 0$.
5. $F(x) = x$, then recall that we denote $\mu_F := m$, and we have
 - $m((a, b]) = m((a, b)) = m([a, b]) = b - a$.
6. $F(x) = e^x$
 - $\mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$.

Remark. We see that the first two examples are *discrete measures*.

Example (Middle thirds Cantor set). Let $C := \bigcap_{n=1}^{\infty} K_n$, where we have

$$\begin{aligned} K_0 &:= [0, 1] \\ K_1 &:= K_0 \setminus \left(\frac{1}{3}, \frac{2}{3} \right) \\ K_2 &:= K_1 \setminus \left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \\ &\vdots \\ K_n &:= K_{n-1} \setminus \bigcup_{k=1}^{3^{n-1}} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right). \end{aligned}$$

We see that C is uncountable and with $m(C) = 0$. And observe that $x \in C$ if and only if $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for some $a_n \in \{0, 2\}$. Hence, we can instead formulate K_n by

$$K_n = \bigcup_{\substack{a_i \in \{0, 2\} \\ 1 \leq i \leq n}} \left[\sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n} \right].$$



Figure 1: The top line corresponds to K_0 , and then K_1 , etc.

The proof of $m(C) = 0$ is given in [Lemma A.3](#).

1.6.1 Cantor Function

Consider F as follows. We define a function F to be 0 to the left of 0, and 1 to the right of 1. Then, define F to be $\frac{1}{2}$ on $(\frac{1}{3}, \frac{2}{3})$, $\frac{1}{4}$ on $(\frac{1}{9}, \frac{2}{9})$, $\frac{3}{4}$ on $(\frac{7}{9}, \frac{8}{9})$ and so on. This is so-called *Cantor Function*. We can show F is continuous and increasing, which makes F a distribution function. Also, we see that the measure this F induced is called *Cantor measure*.



Figure 2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

| Cantor Measure μ_F | | Lebesgue Measure m |
|-------------------------------------|--------|--|
| $\mu_F(\mathbb{R} \setminus C) = 0$ | | $m(\mathbb{R} \setminus C) = \infty > 0$ |
| $\mu_F(C) = 1$ | \iff | $m(C) = 0$ |
| $\mu_F(\{a\}) = 0$ | | $m(\{a\}) = 0$ |

Remark. μ_F and m are said to be **singular** to each other.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 1.8. Let μ be **Lebesgue-Stieltjes measure** on \mathbb{R} . Then we have

$$\begin{aligned} \mu(A) &\stackrel{!}{=} \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

for every $A \in \mathcal{A}_\mu$

Proof. The second equality follows from the **continuity of the measure**. ■

Remark. This is similar to

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - 1/n], \quad (a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n].$$

Lecture 9: Properties of Lebesgue-Stieltjes measure

26 Jan. 11:00

As previously seen. Let $X \subset [0, \infty]$. Recall that

- Finite supremum.

$$\alpha = \sup X < \infty \iff \begin{cases} \forall_{x \in X} \alpha \geq x \\ \forall_{\epsilon > 0} \exists_{x \in X} x + \epsilon \geq \alpha. \end{cases}$$

- Infinite supremum.

$$\alpha = \sup X = \infty \iff \forall_{L > 0} \exists_{x \in X} x \geq L.$$

This should be useful latter on.

Theorem 1.6 (Regularity). Let μ be [Lebesgue-Stieltjes measure](#). Then, for all $A \in \mathcal{A}_\mu$,

1. (outer regularity) $\mu(A) = \inf\{\mu(O) \mid O \supset A, O \text{ is open}\}$
2. (inner regularity) $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$

Proof. We check them separately.

1.

DIY

2. Let $s := \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$, then by [monotonicity](#), we have $\mu(A) \geq s$. To show the other direction, we consider

- A is a bounded set.

Then $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, \overline{A} is also bounded $\implies \mu(\overline{A}) < \infty$. Fix $\epsilon > 0$, then by [outer regularity](#), there exists an open $O \supset \overline{A} \setminus A$, and $\mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \epsilon$. Let $K := \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$, we

show that

$$\mu(K) \geq \mu(A) - \epsilon.$$

DIY

- A is an unbounded set with $\mu(A) < \infty$.

Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \dots$, then

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) < \infty.$$

- A is an unbounded set with $\mu(A) = \infty$.

We can show that

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) = \infty.$$

Fix $L > 0$, then $\exists N$ such that $\mu(A_N) \geq L$.

■

Definition 1.21 (G_δ -set, F_σ -set). Let X be a topological space. Then

- A G_δ -set is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open.
- A F_σ -set is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.7. Let μ be a Lebesgue-Stieltjes measure. Then *TFAE*^a:

1. $A \in \mathcal{A}_\mu$
2. $A = G \setminus M$, G is a G_δ -set, M is a μ -null set.
3. $A = F \setminus N$, F is a F_σ -set, N is a μ -null set.

^aTFAE: The following are equivalent.

Proof. We see that (2.) \implies (1.) and (3.) \implies (1.) are clear.

- (1.) \implies (3.)

– Assume $\mu(A) < \infty$. From the inner regularity, we have

$$\forall n \in \mathbb{N} \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

Let $F = \bigcup_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ -null.

Check!

– Assume $\mu(A) = \infty$. Let $A = \bigcup_{k \in \mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

$$\forall k \in \mathbb{Z} \ A_k = F_k \cup N_k, \quad A = \underbrace{\left(\bigcup_k F_k \right)}_{F_\sigma\text{-set}} \cup \underbrace{\left(\bigcup_k N_k \right)}_{\mu\text{-null}}.$$

- (1.) \implies (2.)

We see that

$$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

■

Proposition 1.4. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0 \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using [outer regularity](#) and the fact that every open set is $\bigcup_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. ■ [DIY](#)

We now see some properties of [Lebesgue measure](#).

Theorem 1.8. Let $A \in \mathcal{L}$, then we have $A + s \in \mathcal{L}$, $rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,

$$m(A + s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. ■ [DIY](#)

Example. We now see some examples.

1. Let $\mathbb{Q} = \{r_i\}_{i=1}^{\infty}$ which is dense in \mathbb{R} . Let $\epsilon > 0$, and

$$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i} \right).$$

We see that O is open and dense¹⁸ in \mathbb{R} . But we see

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$

Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$

2. There exists uncountable set A with $m(A) = 0$.
3. There exists A with $m(A) > 0$ but A contains no non-empty open intervals.
4. There exists $A \notin \mathcal{L}$. e.g. Vitali set.¹⁹
5. There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

Lecture 10: Integration

26 Jan. 11:00

2 Integration

2.1 Measurable Function

We start with a definition.

Definition 2.1 (Measurable function). Suppose (X, \mathcal{A}) , (Y, \mathcal{B}) are [measurable spaces](#). Then we say $f: X \rightarrow Y$ is $(\mathcal{A}, \mathcal{B})$ -measurable if

$$\forall_{B \in \mathcal{B}} f^{-1}(B) \in \mathcal{A}.$$

Remark. If \mathcal{A} and \mathcal{B} are given, we'll sometimes say f is [measurable](#) if it'll not cause any confusions.

¹⁸https://en.wikipedia.org/wiki/Dense_set

¹⁹https://en.wikipedia.org/wiki/Vitali_set

Lemma 2.1. Given two measurable spaces (X, \mathcal{A}) and (Y, \mathcal{B}) , and suppose $\mathcal{B} = \langle \mathcal{E} \rangle$ for some $\mathcal{E} \subset Y$. Then,

$$f: X \rightarrow Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall_{E \in \mathcal{E}} f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part (\implies) is clear. On the other direction, we consider the following. Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $\mathcal{E} \subset \mathcal{D}$ by assumption
- \mathcal{D} is a σ -algebra

Check!

hence, we see that $\langle \mathcal{E} \rangle = \mathcal{B} \subset \mathcal{D}$ from Lemma 1.2. The result then follows from the definition of $(\mathcal{A}, \mathcal{B})$ -measurable. ■

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup_{\alpha} E_{\alpha}\right) = \bigcup_{\alpha} f^{-1}(E_{\alpha})$

Definition 2.2 (\mathcal{A} -measurable). Let (X, \mathcal{A}) be a measurable space. Then,

$$\left. \begin{array}{l} f: X \rightarrow \mathbb{R} \\ f: X \rightarrow \overline{\mathbb{R}} \\ f: X \rightarrow \mathbb{C} \end{array} \right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{ \begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f: X \rightarrow \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{array} \right.$$

Notation. Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$
- $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$.
- $\Re f$ is the real part of f , while $\Im f$ is the imaginary part of f .

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \implies$ Every function is \mathcal{A} -measurable.
- $\mathcal{A} = \{\emptyset, X\} \implies$ The only \mathcal{A} -measurable functions are constant functions.

Definition 2.3 (Lebesgue measurable). A Lebesgue measurable function f is a measurable function

$$f: (\mathbb{R}, \mathcal{L}) \rightarrow (\mathbb{C}, \mathcal{B}(\mathbb{C})).$$

Lemma 2.2. Given $f: X \rightarrow \mathbb{R}$, *TFAE*.

1. f is \mathcal{A} -measurable
2. $\forall a \in \mathbb{R}, f^{-1}((a, \infty)) \in \mathcal{A}$
3. $\forall a \in \mathbb{R}, f^{-1}([a, \infty)) \in \mathcal{A}$
4. $\forall a \in \mathbb{R}, f^{-1}((-\infty, a)) \in \mathcal{A}$
5. $\forall a \in \mathbb{R}, f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from Lemma 2.1 we just saw. ■

Remark (Operations preserve \mathcal{A} -measurability). Given $f, g: X \rightarrow \mathbb{R}$ and is \mathcal{A} -measurable, then

1. $\phi: \mathbb{R} \rightarrow \mathbb{R}$, \mathcal{A} -measurable²⁰, then

$$\phi \circ f: X \rightarrow \mathbb{R}$$

is \mathcal{A} -measurable.

2. $-f, 3f, f^2, |f|$ are all \mathcal{A} -measurable, and $\frac{1}{f}$ is \mathcal{A} -measurable if $f(x) \neq 0, \forall x \in X$.
3. $f + g$ is \mathcal{A} -measurable. We see this from

$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} (f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty)))$$

with Lemma 2.2.

4. $f \cdot g$ is \mathcal{A} -measurable. We see this from

$$f(x)g(x) = \frac{1}{2} ((f(x) + g(x))^2 - f(x)^2 - g(x)^2).$$

5. We see that

$$(f \vee g)(x) := \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) := \min\{f(x), g(x)\}$$

are \mathcal{A} -measurable.

6. Let $f_n: X \rightarrow \overline{\mathbb{R}}$ be \mathcal{A} -measurable. Then

$$\sup_{n \in \mathbb{N}} f_n, \inf_{n \in \mathbb{N}} f_n, \limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n$$

are \mathcal{A} -measurable.

Proof. Consider $\sup_{n \in \mathbb{N}} f_n =: g$, then

$$g^{-1}((a, \infty]) = \bigcup_{n \in \mathbb{N}} f_n^{-1}((a, \infty])$$

for $\sup_n f_n(x) = g(x) > a$. A similar argument can prove the case of $\inf_{n \in \mathbb{N}} f_n$. check

And notice that $\limsup_{n \rightarrow \infty} f_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n$, then the similar argument also proves this case. ■

7. If $\lim_{n \rightarrow \infty} f_n(x)$ converges for every $x \in X$, then f is \mathcal{A} -measurable.

8. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous

$\implies f$ is Borel measurable

$\implies f$ is Lebesgue measurable

since the preimage of an open set of a continuous function is open, then we consider $f^{-1}((a, \infty))$.

Definition 2.4 (Support). The *support* of function $f: X \rightarrow \overline{\mathbb{R}}$ is

$$\text{supp } f := \{x \in X \mid f(x) \neq 0\}.$$

Definition 2.5 (Positive and Negative part). For $f: X \rightarrow \overline{\mathbb{R}}$, let $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$,^a where we call f^+ the *positive part* of f while f^- the *negative part* of f .

^ai.e., $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$

Remark. If $\text{supp } f^+ \cap \text{supp } f^- = \emptyset$ and $f(x) = f^+(x) - f^-(x)$, then

f is \mathcal{A} -measurable $\iff f^+, f^-$ are \mathcal{A} -measurable.

Definition 2.6 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator) function* of E is

$$\chi_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is \mathcal{A} -measurable $\iff E \in \mathcal{A}$.

Definition 2.7 (Simple function). Let (X, \mathcal{A}) be a measurable space. Then a *simple function* $\phi: X \rightarrow \mathbb{C}$ that is \mathcal{A} -measurable and takes only finitely many values.

Remark. We see that if

$$\phi(X) = \{c_1, \dots, c_N\},$$

²⁰In this case, we also call it *Borel measurable*.

then

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \implies \phi = \sum_{i=1}^N \underbrace{c_i}_{\neq \pm\infty} \mathbb{1}_{\underbrace{E_i}_{\in \mathcal{A}}}.$$

Lecture 11: Integration of nonnegative functions

31 Jan. 11:00

As previously seen. For a [simple function](#) ϕ , c_i can actually be in \mathbb{C} .

Theorem 2.1. Given a [measurable space](#) (X, \mathcal{A}) and let $f: X \rightarrow [0, \infty]$, the followings are equivalent.

1. f is a [mathcal{A}-measurable](#) function.
2. There exists [simple functions](#) $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$ such that

$$\forall_{x \in X} \lim_{n \rightarrow \infty} \phi_n(x) = f(x)$$

i.e., f is a [pointwise upward](#) limit of [simple functions](#).

Proof. We'll prove both directions.

- It's clear that (2.) \implies (1.) from the fact that $f(x) = \sup_n \phi_n(x)$ and [the remark](#).
- We want to show that (1.) \implies (2.). Assume f is [mathcal{A}-measurable](#), and fix $n \in \mathbb{N}$.

Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. Also, for $0 \leq k \leq 2^{2^n} - 1$, $E_{n,k} = f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}]) \in \mathcal{A}$.

Then, define ϕ_n be

$$\phi_n = \sum_{k=0}^{2^{2^n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n},$$

we have

- $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$ for every $x \in X$
- $\forall x \in X \setminus F_n$, we have $0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$

Furthermore, we see that

$$F_1 \supset F_2 \supset \dots, \quad \bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\}),$$

then

- $x \in f^{-1}([0, \infty]) = X \setminus \bigcap_{n=1}^{\infty} F_n \implies \lim_{n \rightarrow \infty} \phi_n(x) = f(x)$
- $x \in f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} F_n \implies f_n(x) \geq 2^n \implies \lim_{n \rightarrow \infty} \phi_n(x) = \infty = f(x)$

■

Corollary 2.1. If f is bounded on a set $A \subset \mathbb{R}$, i.e., $\exists L > 0$ such that

$$\forall_{x \in A} |f(x)| \leq L,$$

then there exists a sequence of [simple functions](#) $\{\phi_n\}$ such that $\phi_n \rightarrow f$ [uniformly](#) on A .

Proof.

■

DIY

Corollary 2.2. If $f: X \rightarrow \mathbb{C}$ is a [measurable function](#) if and only if there exists [simple functions](#) $\phi_n: X \rightarrow \mathbb{C}$ such that

$$0 \leq |\phi_1(x)| \leq |\phi_2(x)| \leq \dots \leq |f(x)|$$

with

$$\forall_{x \in X} \lim_{n \rightarrow \infty} \phi_n(x) = f(x).$$

Proof.

■

DIY

2.2 Integration of Nonnegative Functions

We start with our first definition about integral.

Definition 2.8 (Integration of nonnegative function). Let (X, \mathcal{A}, μ) be a [measure space](#), and $\phi: X \rightarrow [0, \infty]$ such that

$$\phi = \sum_{i=1}^N c_i \mathbb{1}_{E_i}$$

be a [simple function](#). Define

$$\int \phi = \int \phi \, d\mu = \int_X \phi \, d\mu = \sum_{i=1}^N c_i \mu(E_i).$$

Furthermore, for $A \in \mathcal{A}$,

$$\int_A \phi = \int_A \phi \, d\mu = \int \phi \mathbb{1}_A \, d\mu.$$

Note. Note that

- In the expression $\sum_{i=1}^N c_i \mu(E_i)$, we're using the convention $0 \cdot \infty = 0$.

- The function $\phi \mathbb{1}_A$ is also a **simple function** since both ϕ and $\mathbb{1}_A$ are **simple function**.

Proposition 2.1. Suppose we have $\phi, \psi \geq 0$ be two **simple functions**. Then,

- **Definition 2.8** is well-defined.
- $\int c\phi = c \int \phi$ for $c \in [0, \infty)$.
- $\int \phi + \psi = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x)$ for all $x \implies \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi d\mu$ is a **measure** on (X, \mathcal{A}) .

Proof.



DIY

Definition 2.9 (Generalization of Integration of nonnegative function). Given (X, \mathcal{A}, μ) with $f: X \rightarrow [0, \infty]$ be **\mathcal{A} -measurable**. Define

$$\int f = \int f d\mu = \sup \left\{ \int \phi : 0 \leq \phi \leq f \text{ such that } \phi \text{ is simple} \right\}.$$

Note. Note that

- If f is a **simple function**, the **Definition 2.8** and **Definition 2.9** of $\int f$ are the same.
- $\int cf = c \int f$ for $c \in [0, \infty)$.
- If $f \geq g \geq 0 \implies \int f \geq \int g$.
- But $\int f + g = \int f + \int g$ is not trivial.

Theorem 2.2 (Monotone Convergence Theorem (MCT)). Given (X, \mathcal{A}, μ) be a **measure space**. Then if

- $f_n: X \rightarrow [0, \infty]$ be **\mathcal{A} -measurable** for every $n \in \mathbb{N}$;
- $0 \leq f_1(x) \leq f_2(x) \leq \dots$ for every $x \in X$;
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for every $x \in X$,

we have

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

Proof. Note that if $\lim_{n \rightarrow \infty} \int f_n$ exists, then it's equal to $\sup_n \int f_n$.

Then

- $f_n \leq f \implies \int f_n \leq \int f \implies \lim_{n \rightarrow \infty} \int f_n \leq \int f$.

- Fix a **simple function** $0 \leq \phi \leq f$, then it's enough to show $\lim_{n \rightarrow \infty} \int f_n \geq \int \phi$.

We first fix $\alpha = (0, 1)$, then it's also enough to show

$$\lim_{n \rightarrow \infty} \int f_n \geq \alpha \int \phi.$$

Let $A_n := \{x \in X \mid f_n(x) \geq \alpha \phi(x)\}$, then since f_n is **measurable**,

- $A_n \in \mathcal{A}$
- $A_1 \subset A_2 \subset A_3 \subset \dots$
- $\bigcup_{n=1}^{\infty} A_n = X$

Check!

We then have

$$\int f_n \geq \int f_n \mathbb{1}_{A_n} \geq \int \alpha \phi \mathbb{1}_{A_n} = \alpha \int_{A_n} \phi = \alpha \nu(A_n)$$

where $\nu(A) = \int_A \phi$ is a **measure**. This implies

$$\lim_{n \rightarrow \infty} \int f_n \geq \alpha \lim_{n \rightarrow \infty} \nu(A_n) \stackrel{21}{=} \alpha \nu(X) = \alpha \int \phi.$$

■

Corollary 2.3 (Linearity of nonnegative integral). Let $f, g \geq 0$ be **measurable**, then

$$\int f + g = \int f + \int g.$$

Proof. There exists **simple functions** ϕ_n and ψ_n such that

- $0 \leq \phi_1 \leq \phi_2 \leq \dots$ and $\phi_n \rightarrow f$ **pointwise**
- $0 \leq \psi_1 \leq \psi_2 \leq \dots$ and $\psi_n \rightarrow g$ **pointwise**

Then,

$$\int (f + g) \stackrel{!}{=} \lim_{n \rightarrow \infty} \int (\phi_n + \psi_n) = \lim_{n \rightarrow \infty} \int \phi_n + \int \psi_n \stackrel{!}{=} \int f + \int g.$$

■

Lecture 12: Fatou's Lemma

2 Feb. 11:00

We start with a useful corollary.

²¹This follows from the **continuity of measure from below**

Corollary 2.4 (Tonelli's theorem for nonnegative series and integrals). Given $g_n \geq 0$ for every $n \in \mathbb{N}$ and let g_n be measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Remark. Recall that we have seen [two series case](#) before. We'll later see two integrals cases.

Proof. Let $f_N := \sum_{n=1}^N g_n$ such that $\lim_{N \rightarrow \infty} f_N = \sum_{n=1}^{\infty} g_n =: f$, then since $g_n \geq 0$, we have $0 \leq f_1 \leq f_2 \leq \dots$ with

$$\lim_{N \rightarrow \infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

By [Theorem 2.2](#), we have

$$\lim_{N \rightarrow \infty} \underbrace{\int \sum_{n=1}^N g_n}_{f_N} = \underbrace{\int \sum_{n=1}^{\infty} g_n}_f.$$

Now, since the terms in the limit on the left-hand side is just a finite sum, by [Corollary 2.3](#), we have

$$\underbrace{\lim_{N \rightarrow \infty} \sum_{n=1}^N \int g_n}_{\sum_{n=1}^{\infty} \int g_n} = \lim_{N \rightarrow \infty} \int \sum_{n=1}^N g_n = \int \sum_{n=1}^{\infty} g_n,$$

hence

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

■

Theorem 2.3 (Fatou's Lemma). Suppose $f_n \geq 0$ and measurable, then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Remark. Recall that

$$\liminf_{n \rightarrow \infty} f_n := \lim_{k \rightarrow \infty} \inf_{n \geq k} f_n = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f_n$$

and

$$\exists \lim_{n \rightarrow \infty} a_n \iff \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

Proof. Let $g_k = \inf_{n \geq k} f_n$, then g_k is measurable and $0 \leq g_1 \leq g_2 \leq \dots$. Now, from Theorem 2.2, we have

$$\int \lim_{k \rightarrow \infty} g_k = \lim_{k \rightarrow \infty} \int g_k.$$

Notice that the left-hand side is just $\int \liminf_{n \rightarrow \infty} f_n$, while the right-hand side is just $\lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n$, i.e.,

$$\int \liminf_{n \rightarrow \infty} f_n = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n.$$

We see that we want to take the inf outside the integral on the right-hand side. Observe that

$$\forall_{m \geq k} \inf_{n \geq k} f_n \leq f_m \implies \forall_{m \geq k} \int \inf_{n \geq k} f_n \leq \int f_m \implies \int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m.$$

Then, we have

$$\int \liminf_{n \rightarrow \infty} f_n = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n \leq \lim_{k \rightarrow \infty} \inf_{m \geq k} \int f_m = \liminf_{m \rightarrow \infty} \int f_m.$$

■

Example. Given $(\mathbb{R}, \mathcal{L}, m)$.

1. **Escape to horizontal infinity.** Let $f_n := \mathbb{1}_{(n, n+1)}$. We immediately see that

- $f_n \rightarrow 0$ pointwise
- $\int f_n = 1$ for every n
- $\int f = 0$

From Theorem 2.3, we have a strict inequality

$$0 = \int \liminf_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} \int f_n = 1.$$

2. **Escape to width infinity.** Let $f_n := \frac{1}{n} \mathbb{1}_{(0, n)}$.
3. **Escape to vertical infinity.** Let $f_n := n \mathbb{1}_{(0, \frac{1}{n})}$.

Lemma 2.3 (Markov's inequality). Let $f \geq 0$ be measurable. Then

$$\forall_{c \in (0, \infty)} \mu(\{x \mid f(x) \geq c\}) \leq \frac{1}{c} \int f.$$

Proof. Denote $\{x \mid f(x) \geq c\} =: E$, then

$$f(x) \geq c \mathbb{1}_E(x) \implies \int f \geq c \int \mathbb{1}_E = c \cdot \mu(E).$$

■

Remark. Notice that $E = f^{-1}([c, \infty])$, hence E is measurable.

Proposition 2.2. Let $f \geq 0$ be measurable. Then,

$$\int f = 0 \iff f = 0 \text{ a.e.}$$

i.e.,

$$\int f \, d\mu = 0 \iff \mu(A) = 0$$

where $A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$.

Proof. Firstly, assume that $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^N c_i \mathbb{1}_{E_i}$$

where E_i are disjoint and $c_i \in (0, \infty)$. Then,

$$\begin{aligned} \int \phi &= \sum_{i=1}^N c_i \mu(E_i) = 0 \\ \iff \mu(E_1) = \dots = \mu(E_N) &= 0 \\ \iff \mu(A) = 0, \quad A = \bigcup_{i=1}^N E_i. \end{aligned}$$

Now, assume that f is a general function where $f \geq 0$ is the only constraint.

1. Assume $\mu(A) = 0$ (i.e., $f = 0$ a.e.). Let $0 \leq \phi \leq f$, where ϕ is simple. Then

$$\forall_{x \in A^c} \phi(x) = 0$$

since $f(x) = 0, \forall x \in A^c$. This implies that $\phi = 0$ a.e. since $\mu(A) = 0$, so $\int \phi = 0$. We then have

$$\int f = 0$$

from Definition 2.9.

2. Assume $\int f = 0$. Let $A_n = f^{-1}([\frac{1}{n}, \infty])$. Then we see that

- $A_1 \subset A_2 \subset \dots$
- $\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} [\frac{1}{n}, \infty]\right) = f^{-1}((0, \infty)) = A$.

We then have

$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \stackrel{!}{\leq} n \int f = 0,$$

which further implies

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n) = 0$$

from the [continuity of measure from below](#).

■

Corollary 2.5. If $f, g \geq 0$ are both [measurable](#) and $f = g$ [a.e.](#), then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \neq g(x)\}$ ²². Then by assumption, $\mu(A) = 0$, hence

$$f \mathbb{1}_A = 0 \text{ [a.e.](#), } g \mathbb{1}_A = 0 \text{ [a.e.](#)..}$$

This further implies that

$$\begin{aligned} \int f &= \int f(\mathbb{1}_A + \mathbb{1}_{A^c}) \\ &\stackrel{!}{=} \int f \mathbb{1}_A + \int f \mathbb{1}_{A^c} \\ &= \int f \mathbb{1}_{A^c} = \int g \mathbb{1}_{A^c} \\ &= \int g \mathbb{1}_{A^c} + \int g \mathbb{1}_A = \int g. \end{aligned}$$

■

Corollary 2.6. Let $f_n \geq 0$ be [measurable](#). Then

1.
$$\left. \begin{array}{l} 0 \leq f_1 \leq f_2 \leq \dots \leq f \text{ [a.e.](#) } \\ \lim_{n \rightarrow \infty} f_n = f \text{ [a.e.](#) } \end{array} \right\} \implies \lim_{n \rightarrow \infty} \int f_n = \int f.$$
2. $\lim_{n \rightarrow \infty} f_n = f \text{ [a.e.](#) } \implies \int f \leq \liminf_{n \rightarrow \infty} \int f_n.$

Proof.

■

DIY

Remark. Almost all the theorems we've proved can be replaced by theorems dealing with [almost everywhere](#) condition.

Lecture 13: Integration of Complex Functions

4 Feb. 11:00

2.3 Integration of Complex Functions

As usual, we start with a definition.

²² A is [measurable](#) indeed.

Definition 2.10 (Integrable). Let (X, \mathcal{A}, μ) be a **measure space** and let $f: X \rightarrow \overline{\mathbb{R}}$ and $g: X \rightarrow \mathbb{C}$ be **measurable**.^a

Then f, g are called *integrable* if $\int |f| < \infty$ and $\int |g| < \infty$, and we define

$$\int f = \int f^+ - \int f^-, \quad \int g = \int \Re g + i \int \Im g.$$

Furthermore, for $f: X \rightarrow \overline{\mathbb{R}}$, we define

$$\int f = \begin{cases} \infty, & \text{if } \int f^+ = \infty, \int f^- < \infty; \\ -\infty, & \text{if } \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

^aRecall that for a complex-valued function like g , this means that both $\Re g$ and $\Im g$ are **measurable**.

We now see a lemma.

Lemma 2.4. Let $f, g: X \rightarrow \overline{\mathbb{R}}$ or \mathbb{C} **integrable**. Assume that $f(x) + g(x)$ is well-defined for all $x \in X$.^a

Then we have

1. $f + g, cf$ for all $c \in \mathbb{C}$ are **integrable**.
2. $\int f + g = \int f + \int g$. This is not trivial since $(f + g)^+ \neq f^+ + g^+$.
3. $|\int f| \leq \int |f|$.

^aThat is, we never see $\infty + (-\infty)$ or $(-\infty) + \infty$.

Proof. Check [FF99] page 53. ■

Lemma 2.5. Let (X, \mathcal{A}, μ) be a **measure space** and let f be an **integrable** function on X . Then

1. f is finite **a.e.** i.e., $\{x \in X \mid |f(x)| = \infty\}$ is a **null set**.
2. The set $\{x \in X \mid f(x) \neq 0\}$ is **σ -finite**.

Proof. _____ ■

HW 5
Q8 by
Lemma 2.3

Proposition 2.3. Let (X, \mathcal{A}, μ) be a **measure space**, then

1. If h is **integrable** on X , then

$$\forall_{E \in \mathcal{A}} \int_E h = 0 \iff \int |h| = 0 \iff h = 0 \text{ a.e.}$$

2. If f, g are **integrable** on X , then

$$\forall_{E \in \mathcal{A}} \int_E f = \int_E g \iff f = g \text{ a.e.}$$

Proof. We prove this one by one.

1. We see that the second equivalence is done in **Proposition 2.2**, hence we prove the first equivalence only. Since we have

$$\int |h| = 0 \implies \left| \int_E h \right| \leq \int_E |h| \leq \int |h| = 0,$$

which shows one implication. Now assume that $\int_E h = 0$ for all $E \in \mathcal{A}$, then we can write h as

$$h = u + iv = (u^+ - u^-) + i(v^+ - v^-).$$

Let $B := \{x \in X \mid u^+(x) > 0\}$, then by assumption, we have

$$0 = \int_B h = \Re \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+,$$

hence $u^+ = 0$ **almost everywhere**. Similarly, we have u^-, v^+, v^- are all zero **almost everywhere**. This gives us that h is zero **almost everywhere** as desired.

2. _____

DIY

■

Theorem 2.4 (Dominated Convergence Theorem). Let (X, \mathcal{A}, μ) be a **measure space**, and

- Let f_n be **integrable** on X .
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ **almost everywhere**.
- There is a $g: X \rightarrow [0, \infty]$ such that g is **integrable** and

$$\forall_{n \in \mathbb{N}} |f_n(x)| \leq g(x) \text{ a.e.}$$

Then we have

$$\lim_{n \rightarrow \infty} \int f_n = \int f = \int \lim_{n \rightarrow \infty} f_n.$$

Proof. Let F be the countable union of [null set](#) on which the three conditions may fail. Then we see that after modifying the definition of f_n, f and g on F , we may assume that all three conditions hold everywhere since modifying on a [null set](#) does not change the integral.

We now consider the \mathbb{R} -valued case only. Note that the second and the third conditions imply that f is [integrable](#) since $|f| \leq g(x)$. We then see that $g + f_n \geq 0$ and $g - f_n \geq 0$ because $-g \leq f_n \leq g$. From [Theorem 2.3](#), we have

$$\int g + f \leq \liminf_{n \rightarrow \infty} \int g + f_n, \quad \int g - f \leq \liminf_{n \rightarrow \infty} \int g - f_n.$$

From the [linearity of integral](#), we have

$$\int g + \int f \leq \int g + \liminf_{n \rightarrow \infty} \int f_n, \quad \int g - \int f \leq \int g - \liminf_{n \rightarrow \infty} \int f_n.$$

Now, since $\int g < \infty$, we can cancel it, which gives

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n, \quad -\int f \leq \liminf_{n \rightarrow \infty} \int -f_n = -\limsup_{n \rightarrow \infty} \int f_n,$$

which implies

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n \leq \limsup_{n \rightarrow \infty} \int f_n \leq \int f.$$

This shows that the limit exists, and the desired result indeed holds. \blacksquare

Corollary 2.7 (Tonelli's theorem for series and integrals). Suppose f_n are [integrable](#) functions such that

$$\sum_{n=1}^{\infty} \int |f_n| < \infty,$$

then we have

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

Proof. Take $G(x)$ to be

$$G(x) := \sum_{n=1}^{\infty} |f_n(x)|,$$

then we see

$$G(x) \geq |F_N(x)|$$

where

$$F_N(x) := \sum_{n=1}^N f_n(x).$$

By [Corollary 2.4](#), we have

$$\int G(x) = \sum_{n=1}^{\infty} \int |f_n(x)| < \infty.$$

Lastly, from [Theorem 2.4](#), the result follows. \blacksquare

Check
C-valued
case

Remark. Compare to [Corollary 2.4](#), we see that we further generalize the result!

Lecture 14: L^1 Space

7 Feb. 11:00

2.4 L^1 Space

We now introduce another space called L^p spaces, which are function spaces defined using a natural generalization of the [p-norm](#) for finite-dimensional vector spaces. We sometimes call it Lebesgue spaces also.

Before we start, we need to define a *norm*.

Definition 2.11 (Seminorm). Let V be a vector space over field \mathbb{R} or \mathbb{C} . A *seminorm* on V is

$$\|\cdot\| : V \rightarrow [0, \infty)$$

such that

- $\|cv\| = |c| \|v\|$ for every $v \in V$ and every scalar c .
- $\|v + w\| \leq \|v\| + \|w\|$ for every $v, w \in V$.

Definition 2.12 (Norm). A *norm* is a [seminorm](#) with

- $\|v\| = 0 \iff v = 0$.

Lemma 2.6. A [normed](#) vector space is a metric space with metric

$$\rho(v, w) = \|v - w\|.$$

Proof.

■ **DIY**

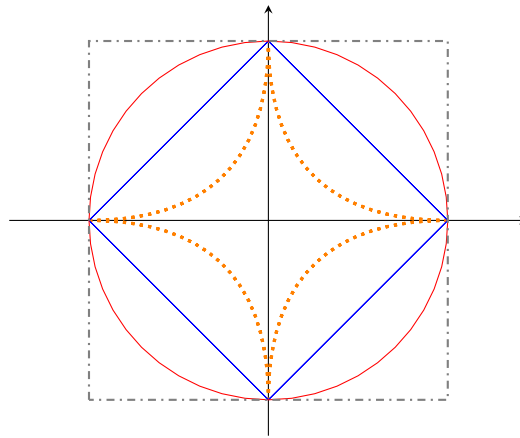
Example (p -norm). $V = \mathbb{R}^d$ with

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^d |x_i|^p \right)^{1/p}, & \text{if } p \in [0, \infty); \\ \max_{1 \leq i \leq d} |x_i|, & \text{if } p = \infty \end{cases}$$

is a [normed](#) vector space. The unit ball

$$\{x \in \mathbb{R}^d \mid \|x\|_p \leq 1\}$$

for different p has the following figures.



Remark. All $\|\cdot\|_p$ norms induce the same topology. i.e., if U is open in p -norm, it is open in p' -norm as well.

Note. Recall that we say f is **integrable** means

$$\int |f| < \infty,$$

and if $f = g$ **a.e.**, then

$$\int f = \int g$$

Definition 2.13 (L^1 Space). Given (X, \mathcal{A}, μ) ,

$$f \in L^1(X, \mathcal{A}, \mu) (= L^1(X, \mu) = L^1(X) = L^1(\mu))$$

means that f is an **integrable** function on X .

Lemma 2.7. $L^1(X, \mathcal{A}, \mu)$ is a vector space with **seminorm**

$$\|f\|_1 = \int |f|.$$

Proof.



Check this
is indeed a
seminorm.

Definition 2.14 (L^1 Space with equivalence class). Define $f \sim g$ if $f = g$ **a.e.**, then

$$L^1(X, \mathcal{A}, \mu) / \sim = L^1(X, \mathcal{A}, \mu),$$

i.e., we simply denote the collection of equivalence classes by itself.^a

^aBy some abusing of notation of L^1 .

Remark. We have

- With Definition 2.14, $L^1(X, \mathcal{A}, \mu)$ is a **normed** vector space.
- We say that the L^1 -metric $\rho(f, g)$ is simply

$$\rho(f, g) = \int |f - g|.$$

2.4.1 Dense Subsets of L^1

Note. Recall the definition of a *dense set*²³.

Definition 2.15 (Step function). A *step function* on \mathbb{R} is

$$\psi = \sum_{i=1}^N c_i \mathbb{1}_{I_i},$$

where I_i is an interval.

Theorem 2.5. We have the following.

1. {integrable simple functions} is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1 -metric).
2. $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$, where μ is a Lebesgue-Stieltjes-measure. Then {integrable simple functions} is dense in $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu)$.
3. $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

Notation. We denote the collection of continuous functions with compact support by $C_c(\mathbb{R})$.

Proof. We prove this one by one.

1. Since there exists simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|$, where $\phi_n \rightarrow f$ pointwise. Then by Theorem 2.4, we have

$$\lim_{n \rightarrow \infty} \int \underbrace{|f_n - f|}_{\leq |\phi_n| + |f| \leq 2|f|} = 0$$

where $2|f|$ is in L^1 .

2. Let $\mathbb{1}_E$ approximate by $\sum_{i=1}^{\infty} c_i \mathbb{1}_{I_i}$. From Theorem 1.6 for Lebesgue-Stieltjes-measure,

$$\forall \epsilon' > 0 \exists I = \bigcup_{i=1}^N I_i \text{ such that } \mu(E \triangle I) \leq \epsilon'.$$

²³https://en.wikipedia.org/wiki/Dense_set

3. To approximate $\mathbb{1}_{(a,b)}$, we simply consider function $g \in C_c(\mathbb{R})$ such that

$$\int |\mathbb{1}_{(a,b)} - g| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

■

Lecture 15: Riemann Integral

9 Feb. 11:00

2.5 Riemann Integrability

We are now working in $(\mathbb{R}, \mathcal{L}, m)$. Let's first revisit the definition of Riemann Integral. Let P be a partition of $[a, b]$ as

$$P = \{a = t_0 < t_1 < \dots < t_k = b\}.$$

Then the *lower Riemann sum* of f using P is equal to L_P , which is defined as

$$L_P = \sum_{i=1}^K \left(\inf_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1}),$$

and the *upper Riemann sum* of f using P is equal to U_P , which is defined as

$$U_P = \sum_{i=1}^K \left(\sup_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1}).$$

Then we call

- *Lower Riemann integral* of $f = \underline{I} = \sup_P L_P$
- *Upper Riemann integral* of $f = \bar{I} = \inf_P U_P$

Definition 2.16 (Riemann (Darboux) integrable). A bounded function $f: [a, b] \rightarrow \mathbb{R}$ is called *Riemann (Darboux) integrable* if

$$\underline{I} = \bar{I}$$

If so, then $\underline{I} = \bar{I} = \int_a^b f(x) dx$.

Note. We see that

- If $P \subset P'$, then

$$L_P \leq L_{P'}, \quad U_{P'} \leq U_P.$$

- Recall that continuous functions on $[a, b]$ are [Riemann integrable](#) on $[a, b]$.

Theorem 2.6. Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function. Then

1. If f is Riemann integrable, then f is Lebesgue measurable, thus Lebesgue integrable. Further,

$$\int_a^b f(x) dx = \int_{[a,b]} f dm.$$

2. If f is Riemann integrable $\iff f$ is continuous Lebesgue a.e.^a

^aHere, we mean that the set where f is discontinuous is a null set under Lebesgue measure.

Proof. There exists $P_1 \subset P_2 \subset \dots$ such that $L_{P_n} \nearrow \underline{I}$ and $U_{P_n} \searrow \bar{I}$.²⁴ Now, define simple (step) functions

$$\begin{aligned} \bullet \phi_n &= \sum_{i=1}^K \left(\inf_{[t_{i-1}, t_i]} f \right) \mathbb{1}_{(t_{i-1}, t_i]} \\ \bullet \psi_n &= \sum_{i=1}^K \left(\sup_{[t_{i-1}, t_i]} f \right) \mathbb{1}_{(t_{i-1}, t_i]} \end{aligned}$$

if $P_n = \{a = t_0 < t_1 < \dots < t_K\}$. Let $\phi := \sup_n \phi_n$ and $\psi := \inf_n \psi_n$. We then see that ϕ, ψ are Lebesgue (Borel) measurable function.

Note. Note that

$$\begin{aligned} \bullet \exists M > 0 \text{ such that } \forall_{n \in \mathbb{N}} |\phi_n|, |\psi_n| &\leq M \mathbb{1}_{[a,b]} \\ \bullet \int \phi_n dm &= L_{P_n}, \int \psi_n dm = U_{P_n} \end{aligned}$$

By Theorem 2.4 and the fact that $M \mathbb{1}_{[a,b]} \in L^1(\mathbb{R}, \mathcal{L}, m)$, we have

$$\underline{I} = \lim_{n \rightarrow \infty} \int \phi_n dm = \int \phi dm, \quad \bar{I} = \int \psi dm.$$

Thus,

$$\begin{aligned} f \text{ is Riemann integrable} &\iff \int \phi = \int \psi \\ &\iff \int (\psi - \phi) = 0 \\ &\iff \psi = \phi \text{ Lebesgue a.e.} \end{aligned}$$

■

2.6 Modes of Convergence

As we should already see, there are different *modes* of convergence. Let's formalize them.

²⁴Here, we took refinements of P_n if needed.

Definition 2.17 (Pointwise, Uniformly convergence). Let

$$f_n, f: X \rightarrow \mathbb{C},$$

and $S \subset X$. Then we say

- $f_n \rightarrow f$ *pointwise* on S if

$$\forall_{x \in S} \forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{n \geq N} |f_n(x) - f(x)| < \epsilon.$$

- $f_n \rightarrow f$ *uniformly* on S if

$$\forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{x \in S} \forall_{n \geq N} |f_n(x) - f(x)| < \epsilon.$$

Remark. We see that we can replace $\forall \epsilon > 0$ by $\forall k \in \mathbb{N}$ with ϵ changing to $\frac{1}{k}$.

Lemma 2.8. Let $B_{n,k}$ be

$$B_{n,k} := \left\{ x \in X \mid |f_n(x) - f(x)| < \frac{1}{k} \right\}.$$

Then

1. $f_n \rightarrow f$ *pointwise* on S if and only if

$$S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$

2. $f_n \rightarrow f$ *uniformly* on S if and only if $\exists N_1, N_2, \dots \in \mathbb{N}$ such that

$$S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Proof. This essentially follows from [Definition 2.17](#). ■

Definition 2.18 (Converges a.e., Converges in L^1). Let (X, \mathcal{A}, μ) be a *measure space*. Assuming that f_n, f are *measurable function*, then

1. $f_n \rightarrow f$ *almost everywhere* means

$$\exists \text{ null set } E \text{ such that } f_n \rightarrow f \text{ pointwise on } E^c.$$

2. $f_n \rightarrow f$ *in L^1* means

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0.$$

Example. Given $(\mathbb{R}, \mathcal{L}, m)$ and let $f = 0$. We see the followings.

1. $f_n = \mathbb{1}_{(n, n+1)}$
2. $f_n = \frac{1}{n} \mathbb{1}_{(0, n)}$
3. $f_n = n \mathbb{1}_{(0, \frac{1}{n})}$
4. **Typewriter functions.**



Lecture 16: Modes of Convergence

11 Feb. 11:00

Let's start with a proposition.

Proposition 2.4 (Fast L^1 convergence leads to a.e. convergence).
 Let (X, \mathcal{A}, μ) be a **measure space**, and f_n, f are all **measurable** functions on X . Then

$$\sum_{n=1}^{\infty} \|f_n - f\|_1 < \infty \implies f_n \rightarrow f \text{ a.e.}$$

Proof. Let

$$E := \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c = \{x \in X \mid f_n(x) \not\rightarrow f(x)\}.$$

By Lemma 2.3, we see that

$$\forall_k \forall_N \mu(B_{n,k}^c) \leq k \int |f_n - f| \implies \forall_k \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) \leq \sum_{n=N}^{\infty} k \|f_n - f\|_1 \rightarrow 0$$

as $N \rightarrow \infty$. Now, by continuity of measure from above,

$$\forall_k \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = \lim_{N \rightarrow \infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \mu(E) = 0$$

since $f_n \rightarrow f$ pointwise on E^c . ■

Corollary 2.8. Given $\{f_n\}_n$ such that $f_n \rightarrow f$ in L^1 , there exists a subsequence $\{f_{n_j}\}_{n_j}$ where $f_{n_j} \rightarrow f$ a.e.

Proof. Since

$$\forall_{j \in \mathbb{N}} \forall_{n_j \in \mathbb{N}} \|f_{n_j} - f\|_1 \leq \frac{1}{j^2}.$$

Then,

$$\sum_{j=1}^{\infty} \|f_{n_j} - f\|_1 < \infty.$$

From Proposition 2.4, we have the desired result. ■

Definition 2.19 (Converge in measure). Let f_n, f be measurable functions on (X, \mathcal{A}, μ) . Then $f_n \rightarrow f$ in measure means

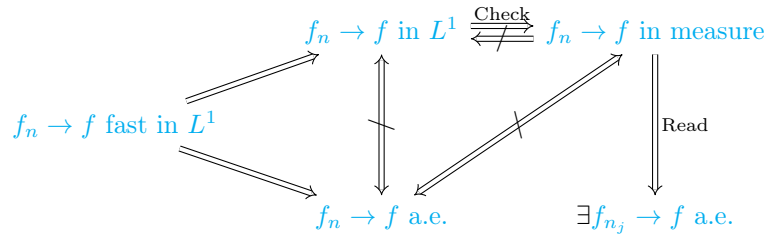
$$\forall_{\epsilon > 0} \lim_{n \rightarrow \infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

Example. Let $f_n = n \mathbb{1}_{(0, \frac{1}{n})}$ and $f = 0$. We see that

$$\forall \epsilon > 0 \quad \{x \in X \mid |f_n(x) - f(x)| > \epsilon\} = \left(0, \frac{1}{n}\right),$$

$f_n \rightarrow 0$ in measure. (Recall that $f_n \not\rightarrow 0$ in L^1)

Remark. We see that



Definition 2.20 (Uniformly a.e., Almost uniformly). Let f_n, f be measurable functions on (X, \mathcal{A}, μ) .

1. $f_n \rightarrow f$ *uniformly almost everywhere* means \exists null set F such that $f_n \rightarrow f$ *uniformly* on F^c .
2. $f_n \rightarrow f$ *almost uniformly* means $\forall \epsilon > 0 \exists F \in \mathcal{A}$ such that $\mu(F) < \epsilon$, $f_n \rightarrow f$ *uniformly* on F^c .

Lemma 2.9. We have

$$f_n \rightarrow f \text{ uniformly on } S \iff \exists N_1, N_2, \dots \in \mathbb{N} \ S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Theorem 2.7 (Egorov's Theorem). Let f_n, f be measurable functions on (X, \mathcal{A}, μ) . Suppose $\mu(X) < \infty$, then

$$f_n \rightarrow f \text{ a.e.} \iff f_n \rightarrow f \text{ almost uniformly.}$$

Proof. We prove two directions.

• \Leftarrow

DIY

• \Rightarrow Fix $\epsilon > 0$. We see that

$$\begin{aligned} f_n \rightarrow f \text{ a.e.} &\implies \mu \left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0 \\ &\implies \forall_k \mu \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0. \end{aligned}$$

From *continuity of measure from above* and $\mu(X) < \infty$, we further have

$$\forall_k \lim_{N \rightarrow \infty} \mu \left(\bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0 \implies \forall_k \exists_{N_k \in \mathbb{N}} \mu \left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c \right) < \frac{\epsilon}{2^k}.$$

Now, let

$$F := \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c,$$

we see that $\mu(F) < \epsilon$, hence $f_n \rightarrow f$ *uniformly*.

■

3 Product Measure

3.1 Product σ -algebra

Before we start, we see the setup.

- Product space.

$$X = \prod_{\alpha \in I} X_{\alpha}$$

where $x = (x_{\alpha})_{\alpha \in I} \in X$.

- Coordinate map.

$$\pi_{\alpha}: X \rightarrow X_{\alpha}.$$

Now we see the formal definition.

Definition 3.1 (Product σ -algebra). Let $(X_{\alpha}, \mathcal{A}_{\alpha})$ be a measurable space for all $\alpha \in I$. Then a *product σ -algebra* on $X = \prod_{\alpha \in I} X_{\alpha}$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_{\alpha} = \left\langle \bigcup_{\alpha \in I} \pi_{\alpha}^{-1}(\mathcal{A}_{\alpha}) \right\rangle,$$

where

$$\pi_{\alpha}^{-1}(\mathcal{A}_{\alpha}) = \{\pi_{\alpha}^{-1}(E) \mid E \in \mathcal{A}_{\alpha}\}.$$

Notation. We denote $I = \{1, \dots, d\} \implies X = \prod_{i=1}^d X_i, x = (x_1, \dots, x_d)$. Also,

$$\bigotimes_{i=1}^d \mathcal{A}_i = \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_d.$$

Lemma 3.1. If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \forall_i E_i \in \mathcal{A}_i \right\} \right\rangle.$$

Proof.



DIY

Lecture 17: Product Measure

14 Feb. 11:00

We now see a lemma.

Lemma 3.2. Suppose $\mathcal{A}_{\alpha} = \langle \mathcal{E}_{\alpha} \rangle$ for every $\alpha \in I$. Then

1. $\pi_{\alpha}^{-1}(\mathcal{A}_{\alpha}) = \langle \pi_{\alpha}^{-1}(\mathcal{E}_{\alpha}) \rangle$

2. $\bigotimes_{\alpha} \mathcal{A}_{\alpha} = \left\langle \bigcup_{\alpha} \pi_{\alpha}^{-1}(\mathcal{E}_{\alpha}) \right\rangle$

3. If I is countable, then

$$\bigotimes_i \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \forall_i E_i \in \mathcal{E}_i \right\} \right\rangle$$

Proof. We prove this one by one.

1. Note that for $f: Y \rightarrow Z$, and \mathcal{B} be a σ -algebra on Z , then $f^{-1}(\mathcal{B})$ is also a σ -algebra.²⁵ Hence, π_α^{-1} is a σ -algebra on X , i.e.,

$$\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \stackrel{!}{\implies} \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha).$$

To show the other direction, let \mathcal{M} being

$$\mathcal{M} = \{B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle\}.$$

We now check

- \mathcal{M} is a σ -algebra.
- $\mathcal{E}_\alpha \subset \mathcal{M}$. This is true by definition of \mathcal{M} .

Thus, $\langle \mathcal{E}_\alpha \rangle = \mathcal{A}_\alpha \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha$, $E \in \mathcal{M}$, implying

$$\pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle,$$

i.e., $\mathcal{A}_\alpha \subset \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

2.

3.

Check
(Easy)!

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■

Theorem 3.1. Suppose X_1, \dots, X_d are metric spaces. Let $X = \prod_{i=1}^d X_i$ with product metric defined as

$$\rho(x, y) = \sum_{i=1}^d \rho_i(x_i, y_i).$$

Then,

1. $\bigotimes_{i=1}^d \mathcal{B}(X_i) \subset \mathcal{B}(X)$
2. If in addition, each X_i has a countable dense subset,

$$\bigoplus_{i=1}^d \mathcal{B}(X_i) = \mathcal{B}(X).$$

Proof.

DIY

Remark. We see that

$$\bullet \mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \dots \otimes \mathcal{B}(\mathbb{R})$$

²⁵Since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(B)^c = f^{-1}(B^c)$, and $\bigcup_n f^{-1}(B_n) = f^{-1}(\bigcup_n B_n)$.

- let $f = u + iv: X \rightarrow \mathbb{C}$, and \mathcal{A} be a σ -algebra on X . Then

$$\bigvee_{E \in \mathcal{B}(\mathbb{R})} u^{-1}(E), v^{-1}(E) \in \mathcal{A} \iff f^{-1}(F) \in \mathcal{A}, \forall F \in \mathcal{B}(\mathbb{C})$$

with $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Definition 3.2 (x -section, y -section). Let X, Y be two sets. Then

1. For $E \subset X \times Y$,

$$E_x = \{y \in Y \mid (x, y) \in E\}, \quad E^y = \{x \in X \mid (x, y) \in E\}.$$

2. For $f: X \times Y \rightarrow \mathbb{C}$, define

$$f_x: Y \rightarrow \mathbb{C}, \quad f^y: X \rightarrow \mathbb{C}$$

by

$$f_x(y) = f(x, y) = f^y(x).$$

Example. We see that

$$(\mathbb{1}_E)_x = \mathbb{1}_{E_x}$$

and

$$(\mathbb{1}_E)^y = \mathbb{1}_{E^y}.$$

Proposition 3.1. Given two measurable spaces (X, \mathcal{A}) and (Y, \mathcal{B}) , then

1. If $E \in \mathcal{A} \otimes \mathcal{B}$, then

$$\bigvee_{x \in X} \bigvee_{y \in Y} E_x \in \mathcal{B}, E^y \in \mathcal{A}.$$

2. If $f: X \times Y \rightarrow \mathbb{C}$ is $\mathcal{A} \otimes \mathcal{B}$ -measurable, then

$$\bigvee_{x \in X} \bigvee_{y \in Y} f_x \text{ is } \mathcal{B}\text{-measurable, } f^y \text{ is } \mathcal{A}\text{-measurable.}$$

Proof. We prove this one by one.

1. Let $\mathcal{F} := \left\{ E \subset X \times Y \mid \bigvee_{x \in X} \bigvee_{y \in Y} E_x \in \mathcal{B}, E^y \in \mathcal{A} \right\}$, then

- \mathcal{F} is a σ -algebra (Easy to show since $(E^c)_x = E_x^c, \dots$)
- Let $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$, which is again easy to show from definition.

Hence, we see that $\langle \mathcal{R}_0 \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{F}$.

2. DIY

■

3.2 Product Measures

We start with the definition.

Definition 3.3 (Rectangle). Given two measurable spaces, a (*measurable*) *rectangle* is $R = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, we let

$$\mathcal{R}_0 := \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$

and

$$\mathcal{R} := \left\{ \bigcup_{i=1}^N R_i \mid N \in \mathbb{N}, R_1, \dots, R_N \text{ disjoint rectangles} \right\}.$$

Note. Whenever we're talking about *rectangle*, they're always *measurable*.

Lemma 3.3. \mathcal{R} is an *algebra*, and

$$\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}.$$

Proof. Simply observe that

$$(A \times B)^c = (A^c \times Y) \cup (A \times B^c)$$

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Lecture 18: Monotone Class

16 Feb. 11:00

Let's start with a theorem.

Theorem 3.2. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be *measure spaces*. Then

1. There is a measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

for every $A \in \mathcal{A}, B \in \mathcal{B}$.

2. If μ, ν are *σ -finite*, then $\mu \times \nu$ is unique.

Proof. We prove this one by one.

1. Define $\mu: \mathcal{R} \rightarrow [0, \infty]$ by $\mu(A \times B) = \mu(A)\nu(B)$, and extending linearly, we have

$$\pi(A \times B) = \mu(A)\nu(B),$$

hence

$$\pi\left(\prod_{i=1}^N A_i \times B_i\right) = \sum_{i=1}^n \pi(A_i \times B_i).$$

We claim that π is a *pre-measure*. To show this, it's enough to check that

$$\pi(A \times B) = \sum_{n=1}^{\infty} \pi(A_n \times B_n) \text{ if } A \times B = \prod_n A_n \times B_n. \text{ Since } A_n \times B_n \text{ are}$$

disjoint, so

$$\mathbb{1}_{A \times B}(x, y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n \times B_n}(x, y).$$

Thus,

$$\mathbb{1}_A(x) \mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(x) \mathbb{1}_{B_n}(y).$$

Integrating with respect to x , and applying [Proposition 1.3](#), we have

$$\int_X \mathbb{1}_A(x) \mathbb{1}_B(y) d\mu(x) = \sum_{n=1}^{\infty} \int_X \mathbb{1}_{A_n}(x) \mathbb{1}_{B_n}(y) d\mu(x),$$

which implies

$$\mu(A) \mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mu(A_n) \mathbb{1}_{B_n}(y)$$

for every y . We can then integrate again with respect to y and apply [Proposition 1.3](#), we have

$$\int_Y \mu(A) \mathbb{1}_B(y) d\nu(y) = \sum_{n=1}^{\infty} \int_Y \mu(A_n) \mathbb{1}_{B_n}(y) d\nu(y),$$

which gives us

$$\mu(A) \nu(B) = \sum_{n=1}^{\infty} \mu(A_n) \nu(B_n).$$

Hence, we see that μ is indeed a [pre-measure](#), so [Theorem 1.3](#) gives $\mu \times \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on \mathcal{R} .

2. If μ, ν are [\$\sigma\$ -finite](#), then π is [\$\sigma\$ -finite](#) on \mathcal{R} , then [Theorem 1.4](#) applies. Moreover, we have that

$$(\mu \times \nu)(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) \mid E \subset \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\}.$$

■

3.3 Monotone Class Lemma

Let's start with a definition.

Definition 3.4 (Monotone Class). If X is a set, and $C \subset \mathcal{P}(X)$, we say that C is a *monotone class* on X if

- C is closed under countable increasing unions.
- C is closed under countable decreasing intersections.

Example. We see that

1. Every [\$\sigma\$ -algebra](#) is a [monotone class](#).

2. If C_α are (arbitrarily many) **monotone classes** on a set X , then $\bigcap_{\alpha} C_\alpha$ is a **monotone class**. Furthermore, if $\mathcal{E} \subset \mathcal{P}(X)$, there is a unique smallest **monotone class** containing \mathcal{E} .

Theorem 3.3 (Monotone Class Lemma). Suppose \mathcal{A}_0 is an **algebra** on X . Then $\langle \mathcal{A}_0 \rangle^a$ is the **monotone class** generated by \mathcal{A}_0 .

$\langle \mathcal{A}_0 \rangle^a$ is the **σ -algebra** generated by \mathcal{A}_0 by Definition 1.3.

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$ and let \mathcal{C} be the **monotone class** generated by \mathcal{A}_0 . Since \mathcal{A} is a **σ -algebra**, it's a **monotone class**. Note that it contains \mathcal{A}_0 , hence $\mathcal{A} \supset \mathcal{C}$.

To show $\mathcal{C} \supset \mathcal{A}$, it's enough to show that \mathcal{C} is a **σ -algebra**. We check that

1. $\emptyset \in \mathcal{A}_0 \subseteq \mathcal{C}$.
2. Let $\mathcal{C}' = \{E \subset X \mid E^c \in \mathcal{C}\}$.
 - \mathcal{C}' is a **monotone class**.
 - $\mathcal{A}_0 \subset \mathcal{C}'$ because if $E \in \mathcal{A}_0$, then $E^c \in \mathcal{A}_0$, so $E^c \in \mathcal{C}$, thus $E \in \mathcal{C}'$.

We see that $\mathcal{C}' \subset \mathcal{C}'$, so \mathcal{C} is closed under complements.

3. For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.
 - $\mathcal{D}(E) \subset \mathcal{C}$.
 - $\mathcal{D}(E)$ is a **monotone class**.
 - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. We see this by picking $F \in \mathcal{A}_0$, then $E \cup F \in \mathcal{A}_0 \subset \mathcal{C}$.

Hence, $\mathcal{C} = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.

4. Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\}$. That is $\mathcal{D} = \{E \in \mathcal{C} \mid E \cup F, \forall F \in \mathcal{C}\}$. Then we have
 - $\mathcal{A}_0 \subset \mathcal{D}$ by 3.
 - \mathcal{D} is a **monotone class**.
 - $\mathcal{D} \subset \mathcal{C}$ by definition.

Thus, $\mathcal{D} = \mathcal{C}$, so if $E, F \in \mathcal{C}$, then $E \cup F \in \mathcal{C}$. This implies that \mathcal{C} is closed under finite unions.

5. Now to show that \mathcal{C} is closed under countable unions, let $E_1, E_2, \dots \in \mathcal{C}$. We may then define

$$F_N = \bigcup_{n=1}^N E_n \in \mathcal{C}.$$

Then we see that $F_1 \subset F_2 \subset \dots$, hence $\bigcup_N F_N \in \mathcal{C}$. But this simply implies

$$\bigcup_N F_N = \bigcup_n E_n,$$

so we're done.



Lecture 19: Fubini-Tonelli's Theorem

18 Feb. 11:00

As previously seen. If $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A} \forall x \in X, \forall y \in Y$.
Note that the reverse is not true.

3.4 Fubini-Tonelli Theorem

We start with a theorem.

Theorem 3.4 (Tonelli for characteristic functions). Given (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be σ -finite measure space. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$, then

1. $\alpha(x) := \nu(E_x): X \rightarrow [0, \infty]$ is a \mathcal{A} -measurable function.
2. $\beta(y) := \mu(E^y): Y \rightarrow [0, \infty]$ is a \mathcal{B} -measurable function.
3. $(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$.

Proof. We prove this one by one.

1. Assume that μ, ν are finite measure. Let

$$C := \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{Conditions 1., 2., 3., hold}\}.$$

It's enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset C$. We further observe that from the monotone class lemma and the fact that \mathcal{R} is an algebra, it's also

citation

- $\mathcal{R} \subset C$.
- C is a monotone class.

From condition 1.,

$$\alpha(x) = \nu((A \times B)_x) = \begin{cases} \nu(B), & \text{if } x \in A; \\ 0, & \text{if } x \notin A \end{cases} = \nu(B) \mathbb{1}_A.$$

And from condition 2.,

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

and

$$\int_X \nu((A \times B)_x) d\mu(x) = \nu(B)\mu(A).$$

Let $E_n \in C$, $E_1 \subset E_2 \subset \dots$. We need to show $E = \bigcup_{n=1}^{\infty} E_n \in C$. We now see that

$$\begin{aligned} E_x &= \bigcup_{n=1}^{\infty} (E_n)_x, (E_1)_x \subset (E_2)_x \subset \dots \\ \implies \alpha(x) &= \nu(E_x) \stackrel{!}{=} \lim_{n \rightarrow \infty} \nu((E_n)_x) \quad \forall x \in X. \end{aligned}$$

This implies that 1. is proved.

For 3., we see that

$$\begin{aligned} (\mu \times \nu)(E) &\stackrel{!}{=} \lim_{n \rightarrow \infty} (\mu \times \nu)(E_n) \\ &= \lim_{n \rightarrow \infty} \int_X \nu((E_n)_x) d\mu(x) \\ &\stackrel{!}{=} \int_X \nu(E_x) d\mu(x). \end{aligned}$$

Now let $F_n \in C$, $F_1 \supset F_2 \supset \dots$. We need to show that $F = \bigcap_{n=1}^{\infty} F_n \in C$. Instead of using [Theorem 2.2](#), we now want to use [Theorem 2.4](#), which is applicable since $\mu(X), \nu(Y) < \infty$ by assumption. Then assume that μ, ν are σ -finite, then

$$X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n), \begin{cases} X_1 \subset X_2 \subset \dots, & \mu(X_k) < \infty \\ Y_1 \subset Y_2 \subset \dots, & \nu(Y_k) < \infty. \end{cases}$$

DIY

Theorem 3.5 (Fubini-Tonelli's Theorem). We have the following two versions.

Tonelli's ver. If $f: X \times Y \rightarrow [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$ -measurable, then

1. $g(x) := \int_Y f(x, y) d\nu(y)$, $X \rightarrow [0, \infty]$ is a \mathcal{A} -measurable function.
2. $h(x) := \int_X f(x, y) d\mu(x)$, $Y \rightarrow [0, \infty]$ is a \mathcal{B} -measurable function.
3. We have

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y).$$

Fubini's ver. If $f \in L^1(X \times Y, \mu \times \nu)$, then

1. $f_x \in L^1(Y, \nu)$ for μ -a.e. x , and $g(x) \in L^1(X, \mu)$ defined μ -a.e.
2. $f_y \in L^1(X, \mu)$ for ν -a.e. y , and $h(y) \in L^1(Y, \nu)$ defined ν -a.e.
3. The iterated integral formulas hold.

Proof. Read [\[F99\]](#).

Lecture 20: Lebesgue Measure on \mathbb{R}^d

21 Feb. 11:00

3.5 Lebesgue Measure on \mathbb{R}^d

Example. We first see some examples.

1. $(\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m)$ is not **complete**.

- Let $A \in \mathcal{L}$, $A \neq \emptyset$, $m(A) = 0$.
- Let $B \subset [0, 1]$, $B \notin \mathcal{L}$ (Vital set for example).
- Let $E = A \times B$, $F = A \times [0, 1]$.

We see that $E \subset F$, $F \in \mathcal{L} \otimes \mathcal{L}$, $(m \times m)(F) = m(A)m([0, 1]) = 0$, i.e., F is a **null** set. But E is **not** $\mathcal{L} \otimes \mathcal{L}$ -measurable since otherwise, its sections are all **measurable**.

Definition 3.5. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the *completion* of

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \dots \times m),$$

which is same as the *completion* of

$$(\mathbb{R}^d, \mathcal{L} \otimes \dots \otimes \mathcal{L}, m \times \dots \times m).$$

Remark. We see that

$$\mathcal{L}^d \supsetneq \mathcal{L} \otimes \dots \otimes \mathcal{L} = \left\langle \left\{ \prod_{i=1}^d E_i \mid E_i \in \mathcal{L} \right\} \right\rangle.$$

Definition 3.6 (General rectangle). A *rectangle* in \mathbb{R}^d is $R = \prod_{i=1}^d E_i$ where $E_i \in \mathcal{B}(\mathbb{R})$.

Definition 3.7. We let

$$m^d(E) := \inf \left\{ \sum_{k=1}^{\infty} m^d(R_k) \mid E \subset \bigcup_{k=1}^{\infty} R_k, R_k \text{ is rectangles} \right\}.$$

Theorem 3.6. Let $E \subset \mathcal{L}^d$. Then

1. $m^d(E) = \inf \{m^d(O) \mid \text{open } O \supset E\} = \sup \{m^d(K) \mid \text{compact } K \subset E\}$.
2. $E = A_1 \cup N_1 = A_2 \setminus N_2$, where A_1 is F_σ , A_2 is G_δ , and N_i are **null**.
3. If $m^d(E) < \infty$, $\forall \epsilon > 0$, $\exists R_1, \dots, R_m$ **rectangles** whose sides are intervals such that

$$m^d \left(E \triangle \left(\bigcup_{i=1}^m R_i \right) \right) < \epsilon.$$

Proof. Similar to $d = 1$ case. ■

Theorem 3.7. Integrable step functions and $C_c(\mathbb{R}^d)$, the collection of continuous functions, are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$

Theorem 3.8. Lebesgue measure in \mathbb{R}^d is translation-invariant.

Theorem 3.9. Effect of linear transformation on Lebesgue measure.

4 Differentiation on Euclidean Space

As previously seen. Given $f: [a, b] \rightarrow \mathbb{R}$, there are two versions of **fundamental theorem of calculus**:

1.

$$\int_a^b f'(x) dx = f(b) - f(a).$$

2.

$$\frac{d}{dx} \int_a^x f(t) dt = f(x),$$

which follows from

$$\lim_{r \rightarrow 0^+} \frac{1}{r} \int_x^{x+r} f(t) dt = f(x) = \lim_{r \rightarrow 0^+} \frac{1}{r} \int_{x-r}^x f(t) dt.$$

Remark. We see that

$$\lim_{r \rightarrow 0^+} \frac{1}{r} \int_x^{x+r} (f(t) - f(x)) dt = 0 = \lim_{r \rightarrow 0^+} \frac{1}{r} \int_{x-r}^x (f(t) - f(x)) dt,$$

where we have

$$f(x) = \frac{1}{r} \int_x^{x+r} f(t) dt.$$

This generalized to $f: \mathbb{R}^d \rightarrow \mathbb{R}$, namely

$$\lim_{r \rightarrow 0^+} \frac{1}{\text{vol}(B(x, r))} \int_{B(x, r)} (f(t) - f(x)) \underbrace{dt}_{\mathbb{R}^d} \stackrel{?}{=} 0.$$

4.1 Hardy-Littlewood Maximal Function

We first see our notation.

Notation. Given a(n) (open) ball in \mathbb{R}^d , $B = B(a, r)$, denote $cB = B(a, cr)$ for $c > 0$.

Lemma 4.1 (Vitali-type covering lemma). Let B_1, \dots, B_k be a finite collection of open balls in \mathbb{R}^d . Then there exists a sub-collection B'_1, \dots, B'_m of disjoint open balls such that

$$\bigcup_{i=1}^m (3B'_i) \supset \bigcup_{i=1}^k B_i.$$

Proof. Greedy Algorithm. ■

Lecture 21

25 Feb. 11:00

Notation. We let

$$\int_E f \, dm = \int_E f(x) \, dx.$$

The problem we're working on is

$$\frac{1}{m(B(w, r))} \int_{B(x, r)} f(y) \, dy \xrightarrow[r]{r \rightarrow 0} f(x).$$

Definition 4.1 (Locally integrable). Given $f: \mathbb{R}^d \rightarrow \mathbb{C}$ be [Lebesgue measurable](#) function. Then we say f is *locally integrable* if

$$\int_K |f| \, dm < \infty, \forall \text{ compact } K \subset \mathbb{R}^d.$$

We write $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Definition 4.2 (Hardy-Littlewood maximal function). Given $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the *Hardy-Littlewood maximal function* for f is defined as

$$\text{Hf}(x) := \sup \{A_r(x) \mid r > 0\},$$

where

$$A_r(x) := \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y)| \, dy.$$

Note. We note that $A_r(\cdot)$ means *averaging function*.

Lemma 4.2. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then

1. $A_r(x)$ is jointly continuous for $(x, r) \in \mathbb{R}^d \times (0, \infty)$.
2. $\text{Hf}(x)$ is [Borel measurable](#).

Proof. We outline the proof.

1. Let $(x, r) \rightarrow (x^*, r^*) \implies A_r(x) \rightarrow A_{r^*}(x^*)$. Let (x_n, r_n) be any sequence which converges to x^*, r^* , then we consider $\lim_{n \rightarrow \infty} A_{r_n}(x_n)$ and we can calculate

$$\int \underbrace{|f(y)| \mathbb{1}_{B(x_n, r_n)}(y)}_{:=h_n(y)},$$

then we apply [Theorem 2.4](#) to h_n .

2. Observe that

$$(\text{Hf})^{-1}(\underbrace{(a, \infty)}_{\text{open}}) = \bigcup_{r>0} A_r^{-1}((a, \infty))$$

is open, since $A_r^{-1}((a, \infty))$ is open from the 1. Note that the equality comes from the fact that $\text{Hf} = \sup_r A_r$.

■

Theorem 4.1 (Hardy-Littlewood maximal inequality). There exists $C_d > 0$ such that for every $f \in L^1(\mathbb{R}^d)$,

$$\forall_{\alpha>0} m(\{x \in \mathbb{R}^d \mid \text{Hf}(x) > \alpha\}) \leq \frac{C_d}{\alpha} \int |f(x)| \, dx.$$

Proof. We first fix $f \in L^1$ and $\alpha > 0$. We define

$$E := \{x \mid \text{Hf}(x) > \alpha\},$$

which is a [Borel measurable set](#) by [Lemma 4.2](#). then

$$x \in E \implies \exists_{r_x>0} A_{r_x}(x) > \alpha \implies m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x, r_x)} |f(y)| \, dy.$$

From [inner regularity](#), we have

$$m(E) = \sup \{m(K) \mid \text{compact } K \subset E\}.$$

Let $K \subset E$ be compact, then

$$K \subset \bigcup_{x \in K} B(x, r_x) \xrightarrow{K \text{ compact}} K \subset \bigcup_{i=1}^N B_i \xrightarrow{!} K \subset \bigcup_{i=1}^m \{3B'_i\}.$$

From here, we further have

$$m(K) \leq \sum_{i=1}^m m(3B'_i) = 3^d \sum_{j=1}^m m(B'_j) \leq \frac{3^d}{\alpha} \sum_{j=1}^m \int_{B'_j} |f(y)| \, dy.$$

Now, since B'_1, \dots, B'_m are disjoint, hence we finally have

$$m(K) \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \, dy.$$

■

4.2 Lebesgue Differentiation Theorem

We start with a theorem!

Theorem 4.2 (Lebesgue Differentiation Theorem). Let $f \in L^1$, then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dy = 0$$

for a.e. x .

Corollary 4.1. Theorem 4.2 also holds for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Proof. Using the fact that m^d is σ -finite, and apply Theorem 4.2. ■

Corollary 4.2. For $f \in L^1_{\text{loc}}$, we have

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) \, dy = f(x)$$

for a.e. x .

Proof. ■

DIY

Definition 4.3 (Shrink nicely). We say that $\{E_r\}_{r>0}$ *shrinks nicely* to x as $r \rightarrow 0$ if $E_r \subset B(x, r)$ and

$$\exists_{c>0} c \cdot m(B(x, r)) \leq m(E_r).$$

Appendix

A Additional Proofs

A.1 Measure

This section gives all additional proofs in [Section 1](#).

Theorem A.1 ([Theorem 1.2 3.](#)). Under the setup of [Theorem 1.2](#), (X, \mathcal{A}, μ) is a [complete measure space](#).

Proof. We see this in two parts.

1. **Claim:** If $A \subset X$ satisfies $\mu^*(A) = 0$, then A is [Carathéodory measurable](#) with respect to μ^* .

Proof. If $A \subset X$ and $\mu^*(A) = 0$, where μ^* is an outer measure on X , we'll show that A is [Carathéodory measurable](#) with respect to μ^* .

Equivalently, we want to show that for any $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Firstly, noting that $(E \cap A) \subset A$, and by [monotonicity](#) of μ^* , we see that

$$\mu^*(E \cap A) \leq \mu^*(A) = 0,$$

and since $\mu^* \geq 0$, hence $\mu^*(E \cap A) = 0$. Now, we only need to show that

$$\mu^*(E) = \mu^*(E \setminus A).$$

Since $E \setminus A = E \cap A^c$, and hence we have $E \cap A^c \subset E$, so

$$\mu^*(E) \geq \mu^*(E \setminus A).$$

To show another direction, we note that

$$\mu^*(E) \leq \mu^*(E \cup A) = \mu^*((E \setminus A) \cup A) \leq \mu^*(E \setminus A),$$

hence we conclude that A is [Carathéodory measurable](#) with respect to μ^* if $\mu^*(A) = 0$. ■

2. **Claim:** If A is [μ-subnull](#), then $A \in \mathcal{A}$.

Proof. Let \mathcal{A} denotes the [Carathéodory σ-algebra](#), and $\mu := \mu^*|_{\mathcal{A}}$. We want to show if A is [μ-subnull](#), then $A \in \mathcal{A}$.

Firstly, if A is [μ-subnull](#), then there exists a $B \in \mathcal{A}$ such that $A \subset B$ and $\mu(B) = 0$. But since from the [monotonicity](#) of μ^* , we further have

$$0 = \mu(B) = \mu^*(B) \geq \mu^*(A),$$

hence $\mu^*(A) = 0$.

From the first claim, we immediately see that A is [Carathéodory measurable](#) with respect to μ^* , which implies A is in [Carathéodory σ-algebra](#), hence $A \in \mathcal{A}$. ■

We see that the second claim directly proves that (X, \mathcal{A}, μ) is a [complete measure space](#). ■

Lemma A.1. The function F defined in [this example](#) is a [distribution function](#)

Proof. We define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n \end{cases}$$

where $\{r_1, r_2, \dots\} = \mathbb{Q}$, and

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} = \sum_{n; r_n \leq x} \frac{1}{2^n}$$

is both increasing and right-continuous.

- Increasing. Consider $x < y$. We see that

$$F(x) = \sum_{n; r_n \leq x} \frac{1}{2^n} \leq \sum_{n; r_n \leq y} \frac{1}{2^n} = F(y)$$

clearly.^{[26](#)}

- Right-continuous. We want to show $F(x^+) = F(x)$. Let $x^+(\epsilon) := x + \epsilon$ with $\epsilon > 0$, we'll show that

$$\lim_{\epsilon \rightarrow 0} F(x^+(\epsilon)) = \lim_{\epsilon \rightarrow 0} F(x + \epsilon) = F(x).$$

Firstly, we have

$$F(x^+(\epsilon)) - F(x) = \sum_{n; r_n \leq x+\epsilon} \frac{1}{2^n} - \sum_{n; r_n \leq x} \frac{1}{2^n} = \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n},$$

and we want to show

$$\lim_{\epsilon \rightarrow 0} F(x^+(\epsilon)) - F(x) = \lim_{\epsilon \rightarrow 0} \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} = 0.$$

Before we show how we choose ϵ ,^{[28](#)} we see that

$$\sum_{n=k}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

²⁶This is trivial since we're always going to sum more strictly positive terms in $F(y)$ than in $F(x)$.

²⁷The strict is crucial to show the result, since if $x = r_k$ for some fixed k , then we can't make the summation arbitrarily small.

²⁸To be precise, how ϵ depends on r_n .

Now, we observe that

$$\sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} \leq \sum_{n=\arg \min_k x < r_k \leq x+\epsilon}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

With this observation, it should be fairly easy to see that we can choose ϵ based on how small we want to make 2^{1-k} be,²⁹ and we indeed see that

$$\lim_{k \rightarrow \infty} 2^{1-k} = 0,$$

which implies that F is right-continuous by squeeze theorem. ■

Lemma A.2. The function F defined in [this example](#) satisfies

- $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$

given in [this example](#).

Proof. We prove them one by one. And notice that F is indeed a distribution function as we proved in [Lemma A.1](#).

1. To show $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$, we first note that $\{r\} = \bigcap_{a-1 \leq x < r} (x, r]$.

Then, we see that

$$\mu_F(\{r\}) = \mu_F \left(\bigcap_{a-1 \leq x < a} (x, r] \right),$$

where each $(x, r] \in \mathcal{A}$ and $(x, r] \supset (y, r]$ whenever $r-1 \leq x \leq y < r$. Notice that we implicitly assign the order of the index by the order of x . Then, we see that $\mu_F(r-1, r] < \infty$.³⁰ Then, from continuity from above, we see that

$$\mu_F(\{r\}) = \lim_{i \rightarrow \infty} \mu_F((x_i, r]),$$

where we again implicitly assign an order to x as the usual order on \mathbb{R} by given index i . It's then clear that as $i \rightarrow \infty$, $x_i \rightarrow r$. From the definition of F , we see that

$$F((x_i, r]) = F(r) - F(x_i) = \sum_{n; r_n \leq r} \frac{1}{2^n} - \sum_{n; r_n \leq x_i} \frac{1}{2^n} = \sum_{n; x_i < r_n \leq r} \frac{1}{2^n}.$$

It's then clear that since $r \in \mathbb{Q}$, there exists an i' such that $r_{i'} = r$. Then, we immediately see that no matter how close $x_i \rightarrow r$, this sum is at least

$$\frac{1}{2^{i'}}$$

for a fixed i' . Hence, we conclude that $\mu_F(\{r\}) > 0$ for every $r \in \mathbb{Q}$.

²⁹We're referring to $\epsilon - \delta$ proof approach.

³⁰This will be $\mu(A_1)$ in the condition of continuity from above. Furthermore, since \mathbb{Q} is countable, hence $F(x) < \infty$ is promised.

2. Now, we show $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$. Firstly, we claim that

$$\mu_F(\mathbb{Q}) = 1$$

and

$$\mu_F(\mathbb{R}) = 1$$

as well. Since $\mu_F(\mathbb{Q}) = 1$ is clear, we note that the second one essentially follows from the fact that we can write

$$\mathbb{R} = \lim_{N \rightarrow \infty} \bigcup_{i=1}^N (a - i, a + i]$$

for any $a \in \mathbb{R}$, say 0. From continuity from below, we have

$$\mu_F\left(\bigcup_{i=1}^{\infty} (-i, +i]\right) = \lim_{n \rightarrow \infty} \mu_F((-n, n]) = \sum_{n; r_n \in \mathbb{Q}} \frac{1}{2^n} = 1.$$

Given the above, from countable additivity of μ_F , we have

$$\mu_F(\mathbb{R} \setminus \mathbb{Q}) + \underbrace{\mu_F(\mathbb{Q})}_1 = \underbrace{\mu_F(\mathbb{R})}_1 \implies \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$$

as we desired. ■

Lemma A.3 (Cantor set has measure 0). Let C denotes the [middle thirds Cantor set](#), then the [Lebesgue measure](#) of C is 0. i.e.,

$$m(C) = 0.$$

Proof. Since we're removing $\frac{1}{3}$ of the whole interval at each n , we see that the measure of those removing parts, denoted by r , is

$$m(r) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 1.$$

Then, by [countable additivity](#) of m , we see that

$$m(C) = m([0, 1]) - m(r) = 1 - 1 = 0. \quad \blacksquare$$

A.2 Integration

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