MATH681
Mathematical Logic

Pingbang Hu

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Abstract

This is a graduate-level mathematical logic course taught by Matthew Harrison-Trainor, aiming to obtain insights into all other branches of mathematics, such as algebraic geometry, analysis, etc. Specifically, we will cover model theory beyond the basic foundational ideas of logic.

While there are no required textbooks, some books do cover part of the material in the class. For example, Marker's *Model Theory: An Introduction* [Mar02], Hodges's *A Shorter Model Theory* [HH97], and Hinman's *Fundamentals of Mathematical Logic* [Hin05].

This course is taken in Winter 2023, and the date on the covering page is the last updated time.

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Chapter 1

Language, Logic, and Structures

Lecture 1: Introduction to Mathematical Logic

The goal of mathematical logic is to obtain insights into other areas of mathematics – algebra, analysis, combinatorics, and so on, by formalizing the **process** of mathematics. 5 Jan. 14:30

Remark. More concretely, there are different branches:

(a) Model Theory: Study subsets of an object defined by a formula (i.e., first-order logic).

(b) Computability Theory / Recursion Theory: Formalizing what it means to have an algorithm and studying relative computability.

(c) Set Theory: Study the structure of the mathematical universe.

(d) Proof Theory: Study the syntactic nature of proofs.

In this class, we study model theory in nature; specifically, we will cover

- basic definitions of logic:
  - What is a formula?
  - What does it mean for a formula to be true?
  - What is a proof?
- Soundness & completeness theorems:
  - Anything provable is true.
  - Anything true is provable.
- Compactness theorem:
  - Non-standard objects exist.
- Using compactness theorem for applications:
  - Chevalley's theorem.

The main theme of this course will be *syntax* v.s. *semantics*:

Syntax	v.s.	Semantics
proofs		truth
form of a formula		mathematical structures
number and type of quantifiers		isomorphisms, embeddings

1.1 Syntax and Semantics

1.1.1 Languages and Structures

Let's start with the fundamental object, language.

Definition 1.1.1 (Language). A *language* $\mathcal{L}$ consists of:

- a set $\mathcal{F}$ of function symbols f with arities n_f;
- a set $\mathcal{R}$ of relation symbols R with arities n_R;
- a set $\mathcal{C}$ of constant symbols c.

A language is also sometimes called a *signature*, in which case we use σ rather than $\mathcal{L}$.

Note. A constant is the same as a 0-ary function.

Remark. Any or all sets in Definition 1.1.1 might be empty.

Example (Graph). The language of graphs, $\mathcal{L}_{\text{graph}} = \{E\}$ where E is a binary (2-ary) relation symbol.

Example (Ring). The language of rings, $\mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot, -\}$, where $0, 1$ are constants, $+, \cdot$ are binary functions, and $-$ is a unary function.

Example (Ordered ring). The language of ordered rings, $\mathcal{L}_{\text{ord}} = \mathcal{L}_{\text{ring}} \cup \{\leq\}$ where $\leq$ is the binary relation for an ordered ring.

Then, given a language, we can now interpret it in the following way.

Definition 1.1.2 (Structure). Given a language $\mathcal{L}$, an *$\mathcal{L}$-structure* $\mathcal{M}$ consists of:

- a non-empty set M called the *universe*, *domain*, or *underlying set* of $\mathcal{M}$;
- for each function symbol $f \in \mathcal{F}$, a function $f^{\mathcal{M}} \colon M^{n_f} \to M$;
- for each relation symbol $R \in \mathcal{R}$, a relation $R^{\mathcal{M}} \subseteq M^{n_R}$;
- for each constant symbol $c \in \mathcal{C}$, an element $c^{\mathcal{M}} \in M$.

Notation (Interpretation). The *interpretation* of symbols f, R, c in $\mathcal{M}$ is $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$, respectively.

Basically, a structure gives meaning to the symbols from the language, and we often write

$$\mathcal{M} = (M, f^{\mathcal{M}}, \ldots, R^{\mathcal{M}}, \ldots, c^{\mathcal{M}}, \ldots) = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} \colon f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C}).$$

Notation. We usually use $\mathcal{M}, \mathcal{N}, \ldots, \mathcal{A}, \mathcal{B}, \ldots$ to refer to structures, and $M, N, \ldots, A, B, \ldots$ for the domains.[a]

[a] Some people use $|\mathcal{M}|$ for the domain of $\mathcal{M}$.

It's time to look at some examples.

Example. The rationals $\mathbb{Q}$ and integers $\mathbb{Z}$ are both $\mathcal{L}_{\text{ring}}$-structures.

Proof. Clearly, the domain is the set of rationals, and naively, we let $+^{\mathbb{Q}} = +$ in $\mathbb{Q}$, $0^{\mathbb{Q}} = 0$ in

$\mathbb{Q}$, $1^{\mathbb{Q}} = 1$ in $\mathbb{Q}$, etc. In this way, $\mathbb{Q} = (\mathbb{Q}, 0, 1, +, \cdot, -)$ is an $\mathcal{L}_{ring}$-structure. Similarly, $\mathbb{Z} = (\mathbb{Z}, 0, 1, +, \cdot, -)$ is as well. ⊛

While the language we have seen are all intuitively correct with their name, e.g., $\mathcal{L}_{\text{ring}}$, $\mathcal{L}_{\text{ord}}$, and $\mathcal{L}_{\text{graph}}$, they are really just the high-level abstraction of the objects in the subscript.

Example. Nothing forces an $\mathcal{L}_{\text{ring}}$-structure to be a ring.

Proof. Since an $\mathcal{L}_{\text{ring}}$-structure is just any structure with two binary functions, a unary function, and two constants interpreting the symbols of the language; hence we can define an $\mathcal{L}_{\text{ring}}$-structure $\mathcal{M}$ as

- $\mathcal{M} = \{0, 5, 11\}$;
- $0^{\mathcal{M}} = 5$;
- $1^{\mathcal{M}} = 11$;
- $+^{\mathcal{M}}$ is the constant function 0;
- $\cdot^{\mathcal{M}}$ is the function 5;
- $-^{\mathcal{M}}$ is the identity.

This is clearly not a ring since it fails nearly every axiom of a ring. ⊛

Note. Later, we will talk about theories that let us restrict to structures we want.

1.1.2 Embeddings and Isomorphisms

We can now consider the relation between structures.

Definition 1.1.3 (Embedding)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. A map $\eta\colon \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-*embedding* if it is one-to-one and preserves the interpretation of all symbols of $\mathcal{L}$:

(a) for each function symbol $f \in \mathcal{F}$ of arity n_f, and $a_1, \ldots, a_{n_f} \in M$,

$$\eta(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \ldots, \eta(a_{n_f}));$$

(b) for each relation symbol $R \in \mathcal{R}$ of arity n_R, and $a_1, \ldots, a_{n_R} \in M$,

$$(a_1, \ldots, a_{n_R}) \in R^{\mathcal{M}} \Leftrightarrow (\eta(a_1), \ldots, \eta(a_{n_R})) \in R^{\mathcal{N}};$$

(c) for each constant symbol $c \in \mathcal{C}$, $c^{\mathcal{M}} = c^{\mathcal{N}}$.

From the definition, an $\mathcal{L}$-embedding is an injection, and naturally, we have the following.

Definition 1.1.4 (Isomorphism)**.** An $\mathcal{L}$-*isomorphism* is a bijective $\mathcal{L}$-embedding.

Definition 1.1.5 (Automorphism)**.** An $\mathcal{L}$-*automorphism* of $\mathcal{M}$ is an $\mathcal{L}$-isomorphism from $\mathcal{M}$ to $\mathcal{M}$.

Definition. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Suppose $M \subseteq N$ and the inclusion map $\iota\colon \mathcal{M} \hookrightarrow \mathcal{N}$ is an $\mathcal{L}$-embedding.

Definition 1.1.6 (Substructure)**.** $\mathcal{M}$ is a *substructure* of $\mathcal{N}$.

Definition 1.1.7 (Extension)**.** $\mathcal{N}$ is an *extension* of $\mathcal{M}$.

Example. Ring embeddings are $\mathcal{L}_{\text{ring}}$-embeddings.

This generalizes the notions of embedding and isomorphism for many mathematical structures.

Remark. Asking that η be injective is the same as (b) in Definition 1.1.3 for the relation $=$ since

$$a = b \in \mathcal{M} \Leftrightarrow \eta(a) = \eta(b) \in \mathcal{N}.$$

The notion of substructure is language sensitive. For groups, there are two possible languages:

(a) $\mathcal{L}_1 = \{e, \cdot\}$;

(b) $\mathcal{L}_2 = \left\{e, \cdot, ^{-1}\right\}$, i.e., with the unary inverse operation.

While both seem valid at the first glance, we should use the second one.

To see why, if we use $\mathcal{L}_2$, the substructure of a group is the same thing as a subgroup. But if we use $\mathcal{L}_1$, then $(\mathbb{N}, +, 0)$ is a substructure of $(\mathbb{Z}, +, 0)$, while $\mathbb{N}$ is not a group for sure.[1]

Similarly, we include $-$ in $\mathcal{L}_{\text{ring}}$ for a similar reason as in the previous example.

Example. An $\mathcal{L}_{\text{ring}}$-substructure of a field will be a subring, not a subfield. If we want subfields, use $\mathcal{L}_{\text{ring}} \cup \left\{^{-1}\right\}$.[a]

[a] We can set $0^{-1} = 0$, but never use this.

Lecture 2: Formulas and First-Order Logic

We start by asking that given a function symbol f of arity n, could we replace f with an $(n+1)$-ary R relation to represent its graph? 10 Jan. 14:30

Example. Let $\mathcal{L}$ be a language with only relation symbols. Let $\mathcal{A}$ be an $\mathcal{L}$-structure. For any $B \subseteq A$, there is a substructure $\mathcal{B}$ of $\mathcal{A}$ with domain B.

Proof. For each relation symbol R, leting $R^{\mathcal{B}} = R^{\mathcal{A}} \cap B^{n_R}$ will make $\mathcal{B}$ a substructure of $\mathcal{A}$. ⊛

The above is not true for function symbols though.

Example. If $G = (\mathbb{Z}, 0, +)$, then $\mathbb{N}$ is not the domain of a subgroup. So if we took $\mathcal{L} = \left\{0, +, ^{-1}\right\}$, where 0 is the unary relation, $+$ is the ternary relation, and $^{-1}$ is the binary relation, an $\mathcal{L}$-substructure of a group might not be a subgroup.

1.1.3 Terms

Intuitive, an $\mathcal{L}$-formula is an expression built using the symbols in a language $\mathcal{L}$, $=$, the logical connectives $\wedge, \vee, \neg$, and variable symbols $v_1, v_2, \ldots, x, y, z$, and also quantifiers $\exists$ and $\forall$.

Definition 1.1.8 (Term). Given a language $\mathcal{L}$, the set of $\mathcal{L}$-*terms* are defined inductively by:

(a) each constant symbol is a *term*;

(b) each variable symbol $v_1, \ldots$ is a *term*;

(c) if f is a function symbol, and $t_1, \ldots, t_{n_f}$ are terms, then $f(t_1, \ldots, t_{n_f})$ is a *term*.

If $\mathcal{M}$ is an $\mathcal{L}$-structure, and t is a term involving only variables among $v_1, \ldots, v_n$, then t has an interpretation $t^{\mathcal{M}} \colon M^n \to M$ as a function as follows. On input $a_1, \ldots, a_n \in M$,

(a) if t is a constant c, $t^{\mathcal{M}}(a_1, \ldots, a_n) = c^{\mathcal{M}}$.

(b) if t is a variable v_i, $t^{\mathcal{M}}(a_1, \ldots, a_n) = v_i$;

[1] Simply observe that both $(\mathbb{N}, 0, +), (\mathbb{Z}, 0, +)$ are $\mathcal{L}_1$-structures.

(c) if t is $f(s_1, \dots, s_k)$, then $t^{\mathcal{M}}(a_1, \dots, a_n) = f^{\mathcal{M}}\big(s_1^{\mathcal{M}}(a_1, \dots, a_n), \dots, s_k^{\mathcal{M}}(a_1, \dots, a_n)\big)$.

Intuition. We are basically substituting for variables and evaluating the expression.

Example. In $(\mathbb{R}, 0, 1, +, \cdot, -)$, a term is essentially just a polynomial with integer coefficients, assuming we interpret them in a ring. Technically, a term looks like

$$\cdot(+(1,1), +(x,y)),$$

but we will write terms the natural way, i.e.,

$$(1+1)(x+y).$$

Also, we will use $\underline{n}$ or n to represent the term $\underline{n} = \underbrace{1+1+\ldots+1}_{n \text{ times}}$. So we could write the above term as $2 \cdot (x+y)$.

1.1.4 Formulas

Definition 1.1.9 (Formula)**.** The set of $\mathcal{L}$-*formulas* is defined inductively:

(a) If s, t are terms, then $s = t$ is a *formula*.

(b) If R is a relation symbol of arity n_R and $s_1, \dots, s_{n_R}$ are terms, then $R(s_1, \dots, s_{n_R})$ is a *formula*.

(c) If f is a formula, then $\neg f$ is a *formula*.

(d) If φ and ψ are formulas, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are *formulas*.

(e) If φ is a formula and v_i are variables, then $\exists v_i\ \varphi$ and $\forall v_i\ \varphi$ are *formulas*.

Notation (Atomic formula)**.** Definition 1.1.9 (a) and (b) are called *atomic formulas*.

Notation (Quantifier-free formula)**.** Definition 1.1.9 (a), (b), (c), and (d) are called *quantifier-free formulas*.

This logic is called *first-order logic* (FO logic), since the quantifiers range over elements of the structures, but not over, e.g., subsets.

Example. We can say that an element x of a ring has a square root by $\exists y\ y^2 = x$.

Example. A group is torsion of order 2 can be said by $\forall x\ x \cdot x = e$.

Example. We can write down all the field/group/... axioms as formulas.

Notice that for the first example, the formula $\exists y\ y^2 = x$ only has meaning if we assign what x is. In this case, we say that y is *bound* by $\exists y$. But this is local:

Example. Consider

$$y = 1 \wedge \exists y\ y^2 = x,$$

while the first appearance of y is free, the second appearance of y is bound by (in the scope of) $\exists y$.

While our definitions work perfectly fine with the above example, but sometimes we don't want this to happen. In such a case, we simply replace the bound instances of y with a new variable z. This idea of variables being free or bound is defined formally as follows.

Definition 1.1.10 (Free variable)**.** The *free variables* $\mathrm{FV}(\varphi)$ of a formula φ are defined inductively:

(a) $\mathrm{FV}(s = t)$ is the set of variables showing up in s or t.

(b) $\mathrm{FV}(R(s_1, \ldots, s_{n_R}))$ is the set of variables showing up in $s_1, \ldots, s_{n_R}$.

(c) $\mathrm{FV}(\neg\varphi) = \mathrm{FV}(\varphi)$.

(d) $\mathrm{FV}(\varphi \wedge \psi) = \mathrm{FV}(\varphi \vee \psi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$.

(e) $\mathrm{FV}(\exists x\ \varphi) = \mathrm{FV}(\forall x\ \varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$.

Example. $\mathrm{FV}(\exists y\ y^2 = x) = \{x\}$.

Example. $\mathrm{FV}(\forall x\ x \cdot x = e) = \varnothing$.

Definition 1.1.11 (Sentence)**.** A formula φ is called a *sentence* if it has no free variables.

Notation. If φ is a formula with free variables among $x_1, \ldots, x_n$ we often write $\varphi(x_1, \ldots, x_n)$.

Remark. So given $\varphi(x_1, \ldots, x_n)$, we know that φ has no other free variables than $x_1, \ldots, x_n$.

Example. It's valid to write $\varphi(x, y, z) \coloneqq x = y$.

1.1.5 Truths

Finally, we define the notion of truth.

Definition 1.1.12 (Truth)**.** Given an $\mathcal{L}$-structure $\mathcal{M}$, let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula and let $a_1, \ldots, a_n \in M$. Then we say φ is *true* of $\overline{a}$ in $\mathcal{M}$,[a] denoted as $\mathcal{M} \models \varphi(\overline{a})$, as follows:

(a) If φ is $s = t$, then $\mathcal{M} \models \varphi(\overline{a})$ if $s^{\mathcal{M}}(\overline{a}) = t^{\mathcal{M}}(\overline{a})$.

(b) If φ is $R(t_1, \ldots, t_{n_R})$, then $\mathcal{M} \models \varphi(\overline{a})$ if $\left(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_{n_R}^{\mathcal{M}}(\overline{a})\right) \in R^{\mathcal{M}}$.

(c) If φ is $\neg\psi$, then $\mathcal{M} \models \varphi(\overline{a})$ if $\mathcal{M} \not\models \psi(\overline{a})$.

(d) If φ is $\psi_1 \wedge \psi_2$, then $\mathcal{M} \models \varphi(\overline{a})$ if $\mathcal{M} \models \psi_1(\overline{a})$ and $\mathcal{M} \models \psi_2(\overline{a})$.

(e) If φ is $\psi_1 \vee \psi_2$, then $\mathcal{M} \models \varphi(\overline{a})$ if $\mathcal{M} \models \psi_1(\overline{a})$ or $\mathcal{M} \models \psi_2(\overline{a})$.

(f) If φ is $\exists y\ \psi(\overline{x}, y)$, then $\mathcal{M} \models \varphi(\overline{a})$ if there's $b \in M$ such that $\mathcal{M} \models \psi(\overline{a}, b)$.

(g) If φ is $\forall y\ \psi(\overline{x}, y)$, then $\mathcal{M} \models \varphi(\overline{a})$ if for all $b \in M$ such that $\mathcal{M} \models \psi(\overline{a}, b)$.

[a] Or $\mathcal{M}$ satisfies $\varphi(\overline{a})$.

Remark. Every formula is true, or its negation is.

Lecture 3: Logical Consequence and Equivalence

Notation (Material implication)**.** The *material implication* $\varphi \to \psi$ between two formulas φ, ψ is an abbreviation of $\neg\varphi \vee \psi$. 12 Jan. 14:30

Notation. We use $\varphi \leftrightarrow \psi$ as an abbreviation of $((\varphi \to \psi) \land (\psi \to \varphi))$.

Essentially, $\to$ and $\leftrightarrow$ is different from $\Rightarrow$ and $\Leftrightarrow$, where the former are only shown in formula. Now, consider the language of graphs $\mathcal{L}_{\text{graph}} = \{E\}$, let's see some examples.

Example. An undirected graph can be written as

$$\forall x \forall y\ (xEy \to yEx).$$

Example. A vertex has at least three neighbors can be written as

$$\varphi(x) := \exists u \exists v \exists w\ (xEu \land xEv \land xEw \land u \neq v \land v \neq w \land u \neq w)$$

in non-reflexive graphs.

Example. For a vertex has exactly three neighbors,

$$\psi(x) := \exists u \exists v \exists w \forall y\ \big(xEu \land xEv \land xEw \land u \neq v \land v \neq w \land u \neq w \land (y = u \lor y = v \lor y = w \lor \neg yEx)\big).$$

Problem. Can we say that x has an even number of neighbors?

Answer. We can't. Some things are not expressible in FO logic. ⊛

Example. For a vertex x has a path of length 4 to y,

$$\Theta(x, y) := \exists u \exists v \exists w\ (xEu \land uEv \land vEw \land wEy).$$

We can also express that there is a path of length at most 4.

Problem. Can we say that there is a path from x to y?

Answer. We still can't! Not in FO logic (using compactness theorem). ⊛

Remark. When we prove results by induction on formulas, we only need to prove for $\neg, \land, \exists$, instead of for both $\land, \lor$, and both $\exists$ and $\forall$.

Proof. Since we can view $\varphi \lor \psi$ as an abbreviation for $\neg(\neg\varphi \land \neg\psi)$ and $\forall x\ \varphi$ as an abbreviation for $\neg(\exists x\ \neg\varphi)$. ⊛

Remark (Sheffer stroke)**.** In fact, we can get $\land, \lor, \neg$ from one logical connective, e.g., the *sheffer stroke* $\uparrow$, which is defined as

$$\varphi \uparrow \psi := \neg(\varphi \land \psi),$$

and we can use $\uparrow$ to define $\neg, \lor, \land$.

Notation. Let Φ be a (possibly infinite) set of sentences, we write $\mathcal{M} \models \Phi$ if $\mathcal{M} \models \varphi$ for all $\varphi \in \Phi$.

Definition 1.1.13 (Logical consequence)**.** Let Φ be a set of sentences, and φ be a sentence. We say that φ is a *logical consequence* of Φ, written $\Phi \models \varphi$, if $\mathcal{M} \models \varphi$ whenever $\mathcal{M} \models \Phi$.

If $\Phi = \varnothing$ is the empty set, Definition 1.1.13 is written as $\models \varphi$, i.e., φ is true in all $\mathcal{L}$-structures.[2]

[2]Recall that we always have a language $\mathcal{L}$ implicitly.

Definition 1.1.14 (Equivalent). Given two formulas φ, ψ, $\varphi(\overline{x})$ and $\psi(\overline{x})$ are *equivalent* if

$$\models \forall \overline{x}\ \big(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})\big).$$

Problem. Two sentences φ and ψ are equivalent if and only if $\varphi \models \psi$ and $\psi \models \varphi$.

DIY

As previously seen. $\mathcal{A}$ is a substructure of $\mathcal{B}$, or $\mathcal{A} \subseteq \mathcal{B}$, means that $A \subseteq B$ and $\mathrm{id}\colon A \hookrightarrow B$ is an $\mathcal{L}$-embedding.

Proposition 1.1.1. Suppose that $\mathcal{A}$ is a substructure of $\mathcal{B}$, and $\varphi(\overline{x})$ is a quantifier-free formula. Let $\overline{a} \in \mathcal{A}$,[a] then $\mathcal{A} \models \varphi(\overline{a})$ if and only if $\mathcal{B} \models \varphi(\overline{a})$.

[a]Formally, we need to write $\mathcal{A}$ to be the Cartesian product with a fixed length.

Proof. We start with terms by proving that if t is a term and $\overline{b} \in \mathcal{A}$, then $t^{\mathcal{A}}(\overline{b}) = t^{\mathcal{B}}(\overline{B})$. The proof is induction on terms.

(a) If t is a constant symbol c, then $t^{\mathcal{A}}(\overline{b}) = c^{\mathcal{A}} = c^{\mathcal{B}} = t^{\mathcal{B}}(\overline{b})$.

(b) If t is a variable x_i, then $t^{\mathcal{A}}(\overline{b}) = b_i = t^{\mathcal{B}}(\overline{b})$.

(c) If t is a function symbol $f(s_1, \dots, s_n)$ where s_i are terms, then $t^{\mathcal{A}}(\overline{b}) = f^{\mathcal{A}}\big(s_1^{\mathcal{A}}(\overline{b}), \dots, s_n^{\mathcal{A}}(\overline{b})\big)$. By the induction hypothesis, $s_i^{\mathcal{A}}(\overline{b}) = s_i^{\mathcal{B}}(\overline{b}) \in \mathcal{A}$, and hence

$$t^{\mathcal{B}}(\overline{b}) = f^{\mathcal{B}}\big(s_1^{\mathcal{B}}(\overline{b}), \dots, s_n^{\mathcal{B}}(\overline{b})\big) = f^{\mathcal{A}}\big(s_1^{\mathcal{A}}(\overline{b}), \dots, s_n^{\mathcal{A}}(\overline{b})\big) = t^{\mathcal{A}}(\mathcal{B}),$$

i.e., $f^{\mathcal{B}} \restriction_{\mathcal{A}} = f^{\mathcal{A}}$, so $t^{\mathcal{A}}(\overline{b}) = t^{\mathcal{B}}(\overline{b})$.

Now we turn to formulas, and prove that for φ quantifier-free, then $\mathcal{A} \models \varphi(\overline{a}) \Leftrightarrow \mathcal{B} \models \varphi(\overline{a})$ for $\overline{a} \in \mathcal{A}$. The proof is, again, induction on formulas.[a]

(a) If φ is $s = t$, then $s^{\mathcal{A}}(\overline{a}) = s^{\mathcal{B}}(\overline{a})$ and $t^{\mathcal{A}}(\overline{a}) = t^{\mathcal{B}}(\overline{a})$, so

$$\mathcal{A} \models \varphi(\overline{a}) \Leftrightarrow s^{\mathcal{A}}(\overline{a}) = t^{\mathcal{A}}(\overline{a}) \Leftrightarrow s^{\mathcal{B}}(\overline{a}) = t^{\mathcal{B}}(\overline{a}) \Leftrightarrow \mathcal{B} \models \varphi(\overline{a}).$$

(b) If φ is $R(s_1, \dots, s_n)$, then

$$A \models \varphi(\overline{a}) \Leftrightarrow \big(s_1^{\mathcal{A}}(\overline{a}), \dots, s_n^{\mathcal{A}}(\overline{a})\big) \in R^{\mathcal{A}} \Leftrightarrow \big(s_1^{\mathcal{B}}(\overline{a}), \dots, s_n^{\mathcal{B}}(\overline{a})\big) \in R^{\mathcal{B}} \Leftrightarrow \mathcal{B} \models \varphi(\overline{a}).$$

(c) If φ is $\neg\psi$,

$$\mathcal{A} \models \varphi(\overline{a}) \Leftrightarrow \mathcal{A} \not\models \psi(\overline{a}) \Leftrightarrow \mathcal{B} \not\models \psi(\overline{a}) \Leftrightarrow \mathcal{B} \models \varphi(\overline{a}),$$

where we use the induction hypothesis in the second $\Leftrightarrow$.

(d) If φ is $\psi_1 \vee \psi_2$,

$$\mathcal{A} \models \varphi(\overline{a}) \Leftrightarrow \mathcal{A} \models \psi_1(\overline{a}) \text{ or } \mathcal{A} \models \psi_2(\overline{a}) \Leftrightarrow \mathcal{B} \models \psi_1(\overline{a}) \text{ or } \mathcal{B} \models \psi_2(\overline{a}) \Leftrightarrow \mathcal{B} \models \varphi(\overline{a}),$$

where we use the induction hypothesis in the second $\Leftrightarrow$.

■

[a]Recall that we only need to show one of $\vee$ or $\wedge$, and here we pick $\vee$ and treat $\wedge$ as an abbreviation.

As previously seen (Characteristic). Given a field K, the *characteristic* p of K is the number of 1 you need to add 1 in order to get 0, i.e., $\underbrace{1 + 1 + \dots + 1}_{p} = 0$.

Example. Let L be a subfield of K, for each $p > 0$, $\varphi_p \coloneqq \underbrace{1+1+\ldots+1}_{p} = 0$, which says the characteristic p. φ_p is quantifier-free, so

$$L \models \varphi_p \Leftrightarrow K \models \varphi_p.$$

Example. Consider $\mathbb{Z} = (\mathbb{Z}, 0, 1, +, -, \cdot)$, and let $\varphi(x) \coloneqq \neg\exists y\ y + y = x$. We see that $\mathbb{Z} \models \varphi(1)$ but $\mathbb{Q} \models \neg\varphi(1)$.

Proposition 1.1.2. Suppose that $\mathcal{A}$ is a substructure of $\mathcal{B}$, and $\varphi(\overline{x}, y_1, \ldots, y_n)$ is a quantifier-free formula. Let $\overline{a} \in \mathcal{A}$, then

(a) if $\mathcal{A} \models \exists y_1 \ldots \exists y_n\ \varphi(\overline{a}, y_1, \ldots, y_n)$, then $\mathcal{B} \models \exists y_1 \ldots \exists y_n\ \varphi(\overline{a}, y_1, \ldots, y_n)$;

(b) if $\mathcal{B} \models \forall y_1 \ldots \forall y_n\ \varphi(\overline{a}, y_1, \ldots, y_n)$, then $\mathcal{A} \models \forall y_1 \ldots \forall y_n\ \varphi(\overline{a}, y_1, \ldots, y_n)$.

Proof. Suppose that $\mathcal{A} \models \exists y_1 \ldots \exists y_n\ \varphi(\overline{a}, y_1, \ldots, y_n)$, so there are $b_1, \ldots, b_n \in \mathcal{A}$ such that $\mathcal{A} \models \varphi(\overline{a}, b_1, \ldots, b_n)$. Since φ is quantifier-free, so $\mathcal{B} \models \varphi(\overline{a}, b_1, \ldots, b_n)$ from Proposition 1.1.1, and hence $\mathcal{B} \models \exists y_1 \ldots \exists y_n\ \varphi(\overline{a}, y_1, \ldots, y_n)$.

On the other hand, it's easy to see that (b) is implied by (a). ∎

Notation. In Proposition 1.1.2, formulas as in (a) are called *existential ($\exists_1$ or $\exists$) formulas*; and formulas as in (b) are called *universal ($\forall_1$ or $\forall$) formulas*.

Example. Recall $\mathcal{L}_1 = \{e, \cdot\}$, $\mathcal{L}_2 = \left\{e, \cdot, ^{-1}\right\}$.

- Associativity: $\forall x \forall y \forall z\ (xy)z = x(yz)$.
- Identity: $\forall x\ ex = xe$.

These are $\forall$-formulas in either language.

- Inverses in $\mathcal{L}_1$: $\forall x \exists y\ xy = yx = e$, which is **not** an $\forall$-formula.
- Inverses in $\mathcal{L}_2$: $\forall x\ xx^{-1} = x^{-1}x = e$, which is an $\forall$-formula.

Hence, group axioms in $\mathcal{L}_1$ are not universal, but in $\mathcal{L}_2$ they are.

The above discrepancy is the reason why $\mathcal{L}_2$ is better than $\mathcal{L}_1$, i.e., $\mathcal{L}_1$-substructure might not be a group.

Problem. Show that $\forall x \exists y\ xy = yx = e$ in the above example is not equivalent to an $\forall$-formula.

Lecture 4: Theories and Axioms

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Example. Let $\mathcal{L}_1 = \{E\}$, where E is a binary relation representing edge relation; and $\mathcal{L}_2 = \{V, E, I\}$, where V, E are unary relations and I is a binary relation representing incidence such that $I(v, e)$ for $v \in V$, $e \in E$ means that v is a vertex on edge e. Then,

- Let G be a graph, viewed as an $\mathcal{L}_1$-structure. A substructure of G is an induced subgraph $H \subseteq G$ such that any edge in G between two vertices of H is in H.
- If we view G as an $\mathcal{L}_2$-substructure, a substructure is a subgraph H such that H has some vertices and edges from G.[a]

[a]But there might be edges in H with no vertices, which can be fixed by having two functions $I_1(e) = v$, $I_2(e) = w$ when $e \colon v \to w$.

The difference is that for $\mathcal{L}_1$, having an edge is quantifier-free, while in $\mathcal{L}_2$ is existential. To elaborate a bit further, for $\mathcal{L}_2$, vEw is quantifier-free, while in $\mathcal{L}_2$,

$$\exists\ (v \in V \land w \in V \land e \in E \land I(v,e) \land I(w,e))$$

is not quantifier-free.

1.2 Theories

Let's start by the notion of theory.

Definition 1.2.1 (Theory). An $\mathcal{L}$-*theory* is a set of $\mathcal{L}$-sentences.

Definition 1.2.2 (Model). $\mathcal{M}$ is a *model* of a theory T, written as $\mathcal{M} \models T$, if $\mathcal{M} \models \varphi$ for all $\varphi \in T$.

Note. Not every theory has a model, e.g., $\{\exists x\ x \neq x\}$.

The above note motivates the following.

Definition 1.2.3 (Satisfiable). A theory is *satisfiable* if it has a model.

Definition 1.2.4 (Elementary class). A class $\mathcal{K}$ of $\mathcal{L}$-structures $\mathcal{M}$ is called an *elementary class* if there is an $\mathcal{L}$-theory T such that

$$\mathcal{K} = \{\mathcal{M} \mid \mathcal{M} \models T\}\,.$$

One way to get an elementary class is to take an $\mathcal{L}$-structure $\mathcal{M}$ and take the full theory.

Definition 1.2.5 (Full theory). The *full theory* $\mathrm{Th}(\mathcal{M})$ of an $\mathcal{L}$-structure $\mathcal{M}$ is defined as $\mathrm{Th}(\mathcal{M}) = \{\varphi \mid \mathcal{M} \models \varphi\}$.

From the definition, $\mathcal{M} \models \mathrm{Th}(\mathcal{M})$, and $\mathrm{Th}(\mathcal{M})$ characterizes the structures satisfying the same sentences as $\mathcal{M}$.

Definition 1.2.6 (Complete). A theory T is *complete* if for any sentence φ, either $\varphi \in T$ or $\neg\varphi \in T$.

Remark. $\mathrm{Th}(\mathcal{M})$ is complete.

Definition 1.2.7 (Elementarily equivalent). $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*$\mathcal{M} \equiv \mathcal{N}$ if for all sentences φ,

$$\mathcal{M} \models \varphi \Leftrightarrow \mathcal{N} \models \varphi.$$

Remark. There are $\mathcal{N} \models \mathrm{Th}(\mathbb{N})$, but $\mathcal{N}$ is not isomorphic to $\mathbb{N}$. $\mathcal{N}$ is called a *non-standard model of arithmetic*, and $\mathcal{N}$ might have *infinite element* larger than all of $\mathcal{M}$. Here, $\mathbb{N} = (\mathbb{N}, 0, 1, +, \cdot, -)$

Example. $\mathbb{Z} \oplus \mathbb{Z} \not\equiv \mathbb{Z}$ as groups.

The other way to define a theory is to write down axioms.

Example (Infinite set). Let $\mathcal{L} = \varnothing$, and let T consist of

$$\varphi_n \coloneqq \exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j.$$

Example (Linear order). Let $\mathcal{L} = \{\leq\}$, and let T consist of the axioms of linear orders, e.g.,

$$\forall x \forall y \; (x \leq y \land y \leq x \to x = y).$$

There are other interesting theories of linear orders, e.g., dense ones.

Example (Dense linear order). Consider

$$\forall x \forall y \; (x < y \to \exists z \; x < z < y),$$

where we use $a < b$ as shorthand of saying $a \leq b \land a \neq b$.

Example (Group). In $\mathcal{L}_{\text{group}} = \{e, \cdot, ^{-1}\}$, let T be the group axioms.

Other theories of groups include Abelson group, divisible, etc.

Definition 1.2.8 (Finitely axiomatizable). A theory is *finitely axiomatizable* if it has a finite set of axioms.

Given a theory, consider $T^{\models} = \{\varphi \mid T \models \varphi\}$,[3] so $\mathcal{M} \models T$ if and only if $\mathcal{M} \models T^{\models}$. Often we think of T and $T^{\models}$ as the same. A theory T is finitely axiomatizable if there is a finite Φ such that $T^{\models} = \Phi^{\models}$.

1.2.1 Elementary Embeddings

Let's now consider the following notion.

Definition 1.2.9 (Elementary embedding). Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures, and $f \colon \mathcal{M} \to \mathcal{N}$ an $\mathcal{L}$-embedding. Then f is an *elementary embedding* if for any formula $\varphi(\overline{x})$ and $\overline{a} \in M$,

$$\mathcal{M} \models \varphi(\overline{a}) \Leftrightarrow \mathcal{N} \models \varphi(f(\overline{a})).$$

Definition 1.2.10 (Elementary substructure). If $f \colon \mathcal{M} \hookrightarrow \mathcal{N}$ is a elementary embedding where $\mathcal{M}$ is a substructure of $\mathcal{N}$, then $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$.

Example. As groups, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is not elementary. In fact, $\mathbb{Z} \not\equiv \mathbb{Q}$. Wheres, if $f \colon \mathcal{M} \hookrightarrow \mathcal{N}$ is an elementary embedding, $\mathcal{M} \equiv \mathcal{N}$.[a]

[a] And also much more is true.

Proposition 1.2.1. Every isomorphism is an elementary embedding.

Proof. Let $f \colon \mathcal{M} \to \mathcal{N}$ be an isomorphism. We will argue by induction on formulas φ, that for all $\overline{a} \in M$,

$$\mathcal{M} \models \varphi(\overline{a}) \Leftrightarrow \mathcal{N} \models \varphi(f(\overline{a})).$$

Firstly, observe that all cases except quantifiers are the same as Proposition 1.1.1. For quantifiers, suppose that $\varphi(\overline{x})$ is $\exists y \; \psi(\overline{x}, y)$ and $\mathcal{M} \models \varphi(\overline{a})$. This means that there is $b \in M$ such that $\mathcal{M} \models \psi(\overline{a}, b)$. By the induction hypothesis, $\mathcal{N} \models \psi(f(\overline{a}, f(b)))$, so $\mathcal{N} \models \varphi(f(\overline{a}))$.

Now suppose $\mathcal{N} \models \varphi(f(\overline{a}))$, then there is $c \in N$ such that $\mathcal{N} \models \psi(f(\overline{a}), c)$. Since f is an isomorphism, so there is a $b \in M$ such that $f(b) = c$. By the induction hypothesis, $\mathcal{M} \models \psi(\overline{a}, b)$, so $\mathcal{M} \models \varphi(\overline{a})$. ■

Corollary 1.2.1. If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.

[3] Recall Definition 1.1.13.

1.2.2 Definable Sets

Consider the following.

Definition 1.2.11 (Definable)**.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure, then $X \subseteq M^n$ is *definable* if there is a formula $\varphi(x_1, \ldots, x_n, \overline{y})$ and $\overline{b} \in M$ such that

$$X = \left\{\overline{a} \in M^n \mid \mathcal{M} \models \varphi(\overline{a}, \overline{b})\right\}.$$

Notation (Define)**.** We say that $\varphi(\overline{x}, \overline{b})$ *defines* X over $\overline{b}$, written as $X = \varphi(\mathcal{M}, \overline{b})$.

Notation (Parameter)**.** The tuple $\overline{b}$ is called the *parameters* when X is definable over $\overline{b}$.

Remark. Sometimes X is definable without parameters, or definable over $\varnothing$.

Example. Take $\mathbb{R} = (\mathbb{R}, 0, 1, +, \cdot, -)$ in $\mathcal{L}_{\text{ring}}$, then

$$\leq = \{(a, b) \colon a \leq b\}$$

is definable.

Example. Let $\mathbb{Z} = (\mathbb{Z}, +, -, \cdot, 0, 1)$, then $\mathbb{N}$ is $\varnothing$-definable in $\mathbb{Z}$ by[a]

$$\mathbb{N} = \left\{z \in \mathbb{Z} \colon \exists u, v, x, y\ u^2 + v^2 + x^2 + y^2 = z\right\}.$$

[a] From the Langrange's four-square theorem, which says that every natural number is the sum of four squares.

Example. $\mathbb{Z}$ is $\varnothing$-definable in $\mathbb{Q} = (\mathbb{Q}, +, -, \cdot, 0, 1)$. This is a result of Julia Robinson [Rob49], and the formulation is very complicated.

Problem. How does one show that a set is not definable? For example, $\mathbb{R}$ is not definable in $\mathbb{C} = (\mathbb{C}, 0, 1, +, \cdot, -)$.

Lecture 5: Hilbert-Style Deductive System

We start by asking whether $\mathbb{R}$ is definable in $\mathbb{C} = (\mathbb{C}, 0, 1, +, \cdot, -)$? 19 Jan. 14:30

Proposition 1.2.2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $X \subseteq M^n$ be a set which is definable over $\overline{a}$. Then any automorphism of $\mathcal{M}$ that fixes $\overline{a}$ pointwise[a] fixes X setwise.[b]

[a] If $\overline{a} = (a_1, \ldots, a_m)$, then $f(a_i) = a_i$.
[b] If $b \in X$, then $f(b) \in X$.

Proof. Let f be an automorphism of $\mathcal{M}$ fixing $\overline{a}$ pointwise, and $X = \left\{\overline{b} \in M^n \colon \mathcal{M} \models \varphi(\overline{b}, \overline{a})\right\}$. Fix $\overline{b}$, and suppose $\overline{b} \in X$, so $\mathcal{M} \models \varphi(\overline{b}, \overline{a})$. Because f is an elementary embedding from Proposition 1.2.1,

$$\mathcal{M} \models \varphi(f(\overline{b}), f(\overline{a})) \Rightarrow \mathcal{M} \models \varphi(f(\overline{b}), \overline{a}),$$

hence $f(\overline{b}) \in X$. Similarly, if $\overline{b} \notin X$, $\mathcal{M} \models \neg\varphi(\overline{b}, \overline{a}) \Rightarrow \mathcal{M} \models \neg\varphi(f(\overline{b}, \overline{a}))$, so $f(\overline{b}) \notin X$. ■

Remark. If X is $\varnothing$-definable, it is fixed setwise by any automorphism.

Example. $\mathbb{N}$ is fixed setwise by any automorphism of the ring $\mathbb{Z}$. In fact, the only automorphism of $\mathbb{Z}$ is the identity.

Example. $\mathbb{N}$ is not $\varnothing$-definable in $\mathbb{Z} = (\mathbb{Z}, 0, +)$.

Proof. Consider an automorphism $f(x) = -x$ of the group $\mathbb{Z}$, which does not fix $\mathbb{N}$ setwise. ⊛

Problem. Is $\mathbb{N}$ definable in $\mathbb{Z} = (\mathbb{Z}, 0, +)$ over some parameters $\overline{a}$?

Answer. For example, if $\overline{a} = (1)$, then f does not fix 1. In fact, any automorphism fixing 1 also fixes all of $\mathbb{Z}$, but $\mathbb{N}$ is not definable in $(\mathbb{Z}, 0, +)$. To prove this we need compactness. ⊛

As previously seen. Given a field F, then $F(a) \cong F(b)$ if a and b have the same minimal polynomial over F or if both do not satisfy any polynomial over F.

Example. $\mathbb{Q}(\pi) \cong \mathbb{Q}(e)$ because π and e are both transcendental.

We now return to the big question: is $\mathbb{R}$ definable in $\mathbb{C} = (\mathbb{C}, 0, 1, +, \cdot, -)$? If $f\colon \mathbb{Q}(a) \to \mathbb{Q}(b)$ such that $a \mapsto b$, then there is an automorphism $\hat{f}\colon \mathbb{C} \to \mathbb{C}$ such that $a \mapsto b$, i.e., $\hat{f}$ extends f. In other words, we need to find such an f with $a \in \mathbb{R}$ and $b \notin \mathbb{R}$.

Example. $a = \pi$, $b = i\pi$ are both transcendental.

Example. a is a real $\sqrt[4]{2}$, b is a complex $\sqrt[4]{2}$.

The above two examples show that $\mathbb{R}$ is not $\varnothing$-definable in $\mathbb{C}$. In fact, $\mathbb{R}$ is not definable over any $\overline{a}$ because there are elements of $\mathbb{R}$ and $\mathbb{C} \setminus \mathbb{R}$ transcendental over any $\overline{a}$.

Intuition. There are so many a, b such that given any $\overline{a}$, we can still find a pair that works.

1.3 Completeness and Compactness

In this section, we're going to formalize proofs.

1.3.1 Proofs

There are all sorts of different proof systems, and the one we use is the so-called Hilbert-style deductive system. Before that, we first see some common notions.

Notation (Schema)**.** A *schema* is written in symbols for formulas, variables, etc.

Example. $\varphi \to (\psi \to \varphi)$ is a schema, i.e., an infinite set with all possible choices of φ and ψ.

Specifically, every logical axiom is written in schema, meaning that any instance of a symbol for a formula, e.g., φ, can be replaced by any formula.

Definition 1.3.1 (Generalization)**.** A formula φ is a *generalization* of a formula ψ if φ is $\forall x_1 \ldots \forall x_n\ \psi$ where $x_1, \ldots, x_n$ are variables.

Notation (Hypothesis)**.** *Hypotheses* are formulas that we may assume in a proof.

Definition 1.3.2 (Proof)**.** A *proof* is a sequence of formulas $\{\varphi_i\}_{i=1}^n$ such that φ_n is the conclusion, and each formula is either an axiom or is obtained from the previous formulas by a rule of inference.

Moreover, for a proof based on a set of hypotheses Γ, then in addition to a logical axiom, we can assert a formula $\varphi \in \Gamma$. If we prove ψ using Γ as hypotheses, we write $\Gamma \vdash \psi$.

Definition 1.3.3 (Valid)**.** If we prove ψ without hypotheses, we write $\vdash \psi$ and say ψ is *valid*.

Definition 1.3.4 (Logical axioms)**.** The *logical axioms* are the following formulas written in schema, as well as all of their generalizations:

Definition 1.3.5 (Propositional axioms)**.** The *propositional axioms* are

(A1) $\varphi \to (\psi \to \varphi)$.

(A2) $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$.

(A3) $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$.

(A4) $\forall x\ \varphi(x, \dots) \to \varphi(t, \dots)$ where t is any term.

(A5) $[\forall x\ (\varphi \to \psi)] \to [(\forall x\ \varphi) \to (\forall x \psi)]$.

(A6) $\varphi \to \forall x\ \varphi$, where x is not free in φ.

Definition 1.3.6 (Axioms for equality)**.** The *axioms for equality* is

(A7) for any terms $t, u, v, \dots$, function symbols f, and relation symbols R,

- (a) $t = t$.
- (b) $t = u \to u = t$.
- (c) $(t = u \land u = v) \to (t = v)$.
- (d) $(u_1 = t_1 \land \dots \land u_{n_f} = t_{n_f}) \to f(u_1, \dots, u_{n_f}) = f(t_1, \dots, t_{n_f})$.
- (e) $(u_1 = t_1 \land \dots \land u_{n_R} = t_{n_R}) \to (R(u_1, \dots, u_{n_R}) \leftrightarrow R(t_1, \dots, t_{n_R}))$.

Definition 1.3.7 (Rule of inference)**.** From φ and $\varphi \to \psi$, deduces ψ.[a]

[a] This is called modus ponens.

These formulas might have free variables.

Example. A proof from calculus of a limit, e.g., $\forall\epsilon\exists\delta\dots$. And we start by stating

- let $\epsilon > 0$,
- choose $\delta = \epsilon$,

  $\vdots$
- $|f(x) - f(y)| < \epsilon$.

We should interpret free variables as anything.

As previously seen (Propositional logic)**.** $(p \land q) \lor (r \land \neg q)$.

Remark. We can check whether the propositional axioms are true with a truth table.

Definition 1.3.8 (Propositional tautology). A *propositional tautology* is a boolean combination $\vee, \wedge, \neg$ of formulas $\varphi_1, \ldots, \varphi_n$ which is true via a truth table assigning true or false to each of $\varphi_1, \ldots, \varphi_n$.

So instead of using propositional axioms, we could instead allow as logical axioms any propositional tautology. To prove completeness, we will need 5 propositional tautologies. We will prove some of these, but take others on faith.

Remark. Propositional axioms are enough to prove all propositional tautologies.

Notation. We write $\Gamma \vdash_{\mathcal{L}} \varphi$ if there is a proof of φ from Γ in the language $\mathcal{L}$.

Note. Passing to a larger language will not let you prove more, so we can just write $\vdash$.

Lecture 6: Soundness Theorem

To see why propositional axioms are enough to prove all propositional tautologies, we see one example. 24 Jan. 14:30

Problem. Prove $\varphi \to \varphi$ using propositional axioms.

Answer. We see that

1. $\varphi \to ((\psi \to \varphi) \to \varphi)$ from (A1), where ψ is any formula (possibly $\psi = \varphi$).
2. $[\varphi \to ((\psi \to \varphi) \to \varphi)] \to [(\varphi \to (\psi \to \varphi)) \to (\varphi \to \varphi)]$ from (A2).
3. $(\varphi \to (\psi \to \varphi)) \to (\varphi \to \varphi)$ from (MP) and the two above.
4. $\varphi \to (\psi \to \varphi)$ from (A1).
5. $\varphi \to \varphi$ from (MP) and the two above.

⊛

In general, we can prove

(a) $\varphi \to \varphi$;

(b) $\varphi \to \neg\neg\varphi$;

(c) $\neg\neg\varphi \to \varphi$;

(d) $(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)$;

(e) $\varphi \to (\psi \to (\varphi \to \psi))$,

and so on.

Note. As we said, we may replace propositional axioms by every propositional tautologies.

Now, to see how (A6) is useful, consider the following.

Theorem 1.3.1. If $\Gamma \vdash \varphi$, and x does not occur freely in Γ, then $\Gamma \vdash \forall x\ \varphi$.

Proof. Fix Γ and x, we use *induction on proofs*. Consider the set $\{\varphi \mid \Gamma \vdash \forall x\ \varphi\}$, we will show that this set contains all the logical axioms, formulas from Γ, and is closed under MP. Thus, if $\Gamma \vdash \theta$, then $\theta \in \{\varphi \mid \Gamma \vdash \forall x\ \varphi\}$.

(a) If φ is a logical axiom, so is its generalization $\forall x\ \varphi$, so $\Gamma \vdash \forall x\ \varphi$.

(b) If $\varphi \in \Gamma$, then x is not free in φ, then from (A6), $\varphi \to \forall x\ \varphi$. Since $\Gamma \vdash \varphi$, $\Gamma \vdash \varphi \to \forall x\ \varphi$, by (MP), $\Gamma \vdash \forall x\ \varphi$.

(c) Suppose $\Gamma \vdash \forall x\ \varphi$ and $\Gamma \vdash \forall x\ (\varphi \to \psi)$, we want to show that $\Gamma \vdash \forall x\ \psi$.

- By (A5), $\forall x\ (\varphi \to \psi) \to (\forall x\ \varphi \to \forall x\ \psi)$, Γ proves this.
- By (MP), $\Gamma \vdash \forall x\ \varphi \to \forall x \psi$.
- By (MP) again, $\Gamma \vdash \forall x\ \psi$.

∎

Corollary 1.3.1. If $\vdash \varphi$, then $\vdash \forall x\ \varphi$. So the generalization of anything valid is also valid.

We now ask a critical question: is our proof system a good one?

1.3.2 Soundness

The idea is that if an $\mathcal{L}$-sentence φ is provable, then it is true in all $\mathcal{L}$-structures, i.e., every thing we prove should be true, in other words, we can't prove wrong things. Let's start with our first glance on soundness.

Lemma 1.3.1 (Soundness)**.** If Γ is a set of $\mathcal{L}$-sentences and φ is a sentence, and $\Gamma \vdash_{\mathcal{L}} \varphi$, then $\Gamma \models \varphi$.

Proof. Suppose that $\Gamma \vdash \varphi$, let $\psi_1, \psi_2, \ldots, \psi_n = \varphi$ be such a proof.[a] Let $\overline{x} = (x_1, \ldots, x_m)$ be the free variable that appears in the ψ_i. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\mathcal{M} \models \Gamma$, we want to show that $\mathcal{M} \models \varphi$. We will show by induction on i that for all $\overline{a} \in M^m$, $\mathcal{M} \models \psi_i(\overline{a})$.

For ψ_i, we have three cases:

(a) If $\psi_i \in \Gamma$, then $\mathcal{M} \models \Gamma$ so $\mathcal{M} \models \psi_i$.

(b) If ψ_i is a (generalization of) a logical axiom, then we can check that $\mathcal{M} \models \psi_i(\overline{a})$. For example, if ψ_i is (A1), $\theta \to (\gamma \to \theta)$, it's easy to check that

$$\mathcal{M} \models \theta(\overline{a}) \to (\gamma(\overline{a}) \to \theta(\overline{a})).$$

(c) If there are $j, k < i$ such that ψ_k is $\psi_j \to \psi_i$, from inductive hypothesis, for all $\overline{a}$, $\mathcal{M} \models \psi_j(\overline{a}), \mathcal{M} \models \psi_k(\overline{a})$, then $\mathcal{M} \models \psi_j(\overline{a}) \to \psi_i(\overline{a})$. Checking our definition of truth, we get $\mathcal{M} \models \psi_i(\overline{a})$.

∎

[a] Some ψ_i might be formulas, but φ should be a sentence.

There are some remarks to make.

Remark. If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.

Remark. If $\Delta \subseteq \Gamma$, and $\Delta \vdash \varphi$, then $\Gamma \vdash \varphi$.

Remark. If $\Gamma \vdash_{\mathcal{L}} \varphi$, and $\mathcal{L}^+ \supseteq \mathcal{L}$, then $\Gamma \vdash_{\mathcal{L}^+} \varphi$.

Remark. If $\Gamma \vdash \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$.

We can prove the following.

Theorem 1.3.2 (Deduction theorem). For any set of formulas Γ, formulas θ and ψ,

$$\Gamma \cup \{\theta\} \vdash \psi \Leftrightarrow \Gamma \vdash \theta \to \psi.$$

Proof. The backward direction is easier. Suppose $\Gamma \vdash \theta \to \psi$, then $\Gamma \cup \{\theta\} \vdash \psi$ since we can have a proof like:

1. θ

$\vdots$ (the proof of $\Gamma \vdash \theta \to \psi$)

n. $\theta \to \psi$

$n+1$. ψ.

Now, suppose that $\Gamma \cup \{\theta\} \vdash \psi$, then there is a proof $\psi_1, \ldots, \psi_n = \psi$ of ψ from $\Gamma \cup \{\theta\}$. We argue inductively that $\Gamma \vdash \theta \to \psi_i$. Suppose we know that for $j < i$, prove it for i. Divide into cases:

(a) $\psi_i \in \Gamma$ or it is a logical axiom. By (A1), $\psi_i \to (\theta \to \psi_j)$, so $\Gamma \vdash \psi_j$.

(b) $\psi_i = \theta$. Then $\Gamma \vdash \theta \to \theta$ by (A1) and (A2) from here.

(c) ψ_i follows from ψ_j, $\psi_k = \psi_j \to \psi_i$, using (MP) with $j, k < i$.

- From the induction hypothesis, $\Gamma \vdash \theta \to \psi_j$ and $\Gamma \vdash \theta \to (\psi_j \to \psi_i)$.
- By (A2), $\Gamma \vdash [\theta \to (\psi_j \to \psi_i)] \to [(\theta \to \psi_j) \to (\theta \to \psi_i)]$.
- By (MP), $\Gamma \vdash (\theta \to \psi_j) \to (\theta \to \psi_i)$.
- By (MP), $\Gamma \vdash \theta \to \psi_i$.

■

Lecture 7: Soundness, Completeness, and Compactness

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Proposition 1.3.1 (Contraposition). If $\Gamma \cup \{\varphi\} \vdash \neg\psi$, then $\Gamma \cup \{\psi\} \vdash \neg\varphi$.

Proof. Suppose $\Gamma \cup \{\varphi\} \vdash \neg\psi$, by the deduction theorem says that

$$\Gamma \vdash \varphi \to \neg\psi.$$

From (A1), (A2), and (A3), we can prove $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$. By (MP), $\Gamma \vdash \psi \to \neg\varphi$. By the deduction theorem, $\Gamma \cup \{\psi\} \vdash \neg\varphi$. ■

Definition 1.3.9 (Consistent). A theory T is *consistent* if for all φ, it is not the case that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Definition 1.3.10 (Inconsistent). If a theory T is not consistent, then it's *inconsistent*.

Remark. We could make the same definition for a set of formulas.

Proposition 1.3.2 (Proof by contradiction). If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \vdash \neg\varphi$.

Proof. There is ψ such that $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\varphi\} \vdash \psi$, so $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \varphi \to \neg\psi$. Using (A1), (A2), and (A3), we can prove that

$$(\varphi \to \psi) \to \big((\varphi \to \neg\psi) \to \neg\varphi\big).$$

By (MP), $\Gamma \vdash (\varphi \to \neg\psi) \to \neg\varphi$, and by (MP) again, we have $\Gamma \vdash \neg\varphi$. ■

Proposition 1.3.3. If a theory T is consistent, and φ is a sentence, then either $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is consistent.

Proof. If they were both inconsistent, $T \vdash \neg\varphi$ and $T \vdash \neg\neg\varphi$, so T would be inconsistent, but T is consistent$\lightning$ ■

Note. The above is also true for formula.

Proposition 1.3.4. If a theory T is inconsistent, then $T \cup \{\varphi\}$ is inconsistent for all φ. Hence, $T \vdash \varphi$ for all φ.

Definition 1.3.11 (Maximal)**.** A theory T is called *maximal* if it is consistent and for all sentences φ, either $\varphi \in T$ or $\neg\varphi \in T$.

In particular, if $T \vdash \varphi$, then $\varphi \in T$.

Theorem 1.3.3 (Zorn's lemma)**.** Let $(P, \leq)$ be a partially ordered set. If every non-empty chain in P has an upper bound, then P has a maximal element.

Theorem 1.3.4. Any consistent theory T can be extended to a maximal consistent theory $T' \supseteq T$.

Proof. We first consider the case that T is countable by considering $\mathcal{L}$ is countable since if $\mathcal{L}$ is countable, then there are only countable many formulas since there are only countable many formulas of each length.

Claim. The result holds for $\mathcal{L}$ being countable.

Proof. Firstly, list out all sentences $\varphi_1, \varphi_2, \ldots$, start with $T_0 = T$. Given T_i consistent, one of $T_i \cup \{\varphi_i\}$ or $T_i \cup \{\neg\varphi_i\}$ is consistent. Let T_{i+1} be one of these that is consistent. Let $T^* = \bigcup_i T_i$, we now see that T^* is consistent.

Suppose not, then $T^* \vdash \theta$ and $T^* \vdash \neg\theta$. In this case, there is some T_i such that $T_i \vdash \theta$ and $T_i \vdash \neg\theta$ because proofs are finite. But T_i is consistent, so this cannot happen, hence T^* is maximal. ⊛

Claim. The result holds for arbitrary $\mathcal{L}$.

Proof. For arbitrary $\mathcal{L}$, let $(P, \leq)$ be the set of consistent theories extending T_i ordered by inclusion. Let C be a non-empty chain, and let $T^* = \bigcup_{T' \in C} T' \supseteq T$. We see that T^* is consistent because if $t^* \vdash \theta$ and $T^* \vdash \neg\theta$, there are finitely many formulas used in those proofs, from, say, $T_1, \ldots, T_n \in C$.

Because C is a chain, by reordering, we may assume that $T_1 \subseteq \ldots \subseteq T_n$. So $T_n \vdash \theta$ and $T_n \vdash \neg\theta$, contradicting the consistency of T_n, so T^* is consistent, i.e., $T^* \in P$, and T^* is an upper bound on C. By Zorn's lemma, $(P, \leq)$ has a maximal lemma $T^* \supseteq T$, consistent. If T^* is not maximal, then there is φ such that $\varphi \notin T^*$, $\neg\varphi \notin T^*$. But one of $T^* \cup \{\varphi\}$ or $T^* \cup \{\neg\varphi\}$ is consistent, hence in P, contradicting the fact that T^* is maximal. ⊛

■

Remark. We can do that same proof for any $\mathcal{L}$ using transfinite recursion for the uncountable case.

Motivated by Lemma 1.3.1, we have the following.

Theorem 1.3.5 (Soundness)**.** Let T be a theory and φ be a sentence.

(a) If $T \vdash \varphi$, then $T \models \varphi$.

(b) If T is satisfiable, then it is consistent.

Proof. (a) is exactly Lemma 1.3.1. For (b), let $\mathcal{M} \models T$, suppose that T was inconsistent, then $T \vdash \varphi$ and $T \vdash \neg\varphi$ for some φ. By (a), $T \models \varphi$ and $T \models \neg\varphi$, so $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \neg\varphi$. But $\mathcal{M} \models \neg\varphi$ means $\mathcal{M} \not\models \varphi$, so this is a contradiction, hence T is consistent. ■

1.3.3 Completeness

If φ is true in all $\mathcal{L}$-structures, then it is provable.

Theorem 1.3.6 (Completeness)**.** Let T be a theory and φ be a sentence.

(a) If $T \models \varphi$, then $T \vdash \varphi$.

(b) If T is consistent, then it is satisfiable.

Proof. We leave (b) for later, and prove that (b) implies (a). Suppose that $T \models \varphi$, so $T \cup \{\neg\varphi\}$ is unsatisfiable. By (b), $T \cup \{\neg\varphi\}$ is inconsistent. By proof by contradiction, $T \vdash \varphi$. ■

1.3.4 Compactness

Theorem 1.3.7 (Compactness)**.** Let T be a theory and φ be a sentence.

(a) If $T \models \varphi$, then there is a finite $T_0 \subseteq T$ such that $T_0 \models \varphi$.

(b) T is satisfiable if and only if every finite subset of T is satisfiable.

Proof. Consider:

(a*) If $T \vdash \varphi$, then there is a finite $T_0 \subseteq T$ such that $T_0 \vdash \varphi$.

(b*) If T is consistent if and only if every finite subset of T is consistent.

We see that (a*) and (b*) are true because proofs are finite, and soundness and completeness translate directly between (a) and (a*) (and (b) and (b*)). ■

Let's see some examples using compactness.

Example. Let $\mathcal{L} = \{0, 1, +, \cdot, -, <\}$, and $\mathcal{L}^* = \mathcal{L} \cup \{c\}$, where c is a new constant symbol. Let

$$T = \mathrm{Th}_{\mathcal{L}}(\mathbb{N}) \cup \{c > \underline{n} \mid n \in \mathbb{N}\},$$

then T is finitely satisfiable.

Proof. Given $T_0 \subseteq T$ finite, $T_0 \subseteq \mathrm{Th}_{\mathcal{L}}(\mathbb{N}) \cup \{c > \underline{n}, \ldots, c > \underline{n_\ell}\}$, and may assume they are equal and show that T_0 is satisfiable. Let $\mathcal{N}$ be the $\mathcal{L} \cup \{c\}$ -structure which is the expansion of the $\mathcal{L}$-structure $\mathbb{N}$, with

$$c^{\mathcal{N}} = 1 + \max(n_1, \ldots, n_\ell),$$

then $\mathcal{N} \models T_0$, and T_0 is satisfiable. By compactness, T is satisfiable, say $\mathcal{A} \models T$. Then $\mathcal{A} \equiv \mathbb{N}$ and $\mathcal{A}$ contains an element $c^{\mathcal{A}}$ bigger than 1, $1+1$, $1+1+1$, ..., but $\mathcal{A} \not\cong \mathbb{N}$, so $\mathcal{A}$ is a non-standard model of arithmetic. ⊛

Lecture 8

We first proveTheorem 1.3.6 (b), but before that we need an additional definition and a technical lemma. 31 Jan. 14:30

Definition 1.3.12 (Henkin constant). An $\mathcal{L}^*$-theory T^* has *He kin constants* if for each formula $\varphi(x)$ with one free variable, there is a constant symbol $c \in \mathcal{L}^*$ such that

$$\big(\exists x\ \varphi(x)\big) \to \varphi(c) \text{ is in } T^*.$$

We see that the above is equivalent to

$$\big(\neg\forall x\ \varphi(x)\big) \to \neg\varphi(x) \text{ is in } T^*,$$

and we will use this version ($\forall$) and view $\exists$ being a shorthand for $\neg\forall\neg$; also, we will use $\to$ and $\neg$ as primitive, and $\wedge, \vee$ are shorthand.

Lemma 1.3.2. If $\Gamma \vdash \varphi(x)$, and c does not occur in Γ or in $\varphi(x)$, then there is a variable y, not appearing in $\varphi(x)$, such that $\Gamma \vdash \forall y\ \varphi(y)$. Moreover, there is a proof of $\forall y\ \varphi(y)$ in which c does not appear.

Proof. Let $\alpha_1(x), \ldots, \alpha_n(x) = \varphi(c)$ be a proof of $\varphi(c)$ from Γ. Let y be a variable not appearing in this proof. We claim that $\alpha_1(y), \ldots, \alpha_n(y) = \varphi(y)$ is still a valid proof of $\varphi(y)$. There are three cases to consider (for each $i = 1, \ldots, n$):

(a) If $\alpha_i(c)$ is in Γ, then c does not actually occur in $\alpha_i(c)$ because it does not appear in Γ. So $\alpha_i(y)$ is the same as $\alpha_i(c)$.

(b) If $\alpha_i(c)$ is a logical axiom, then $\alpha_i(y)$ is a logical axiom as well. For most of these it is easy to check, but for (A6), i.e., $\varphi \to \forall x\ \varphi$ if x is not free in φ, there is a little more. But y did not appear in $\alpha_i(c)$, so $y \neq x$, and substituting y for c will not stop x from being free.

(c) If $\alpha_i(c)$ follows by (MP) from $\alpha_j(c)$ and $\alpha_k(c) = \alpha_j(c) \to \alpha_i(c)$ for $j, k < i$, then $\alpha_i(y)$ follows by (MP) from $\alpha_j(y)$ and $\alpha_k(y) = \alpha_j(y) \to \alpha_i(y)$.

So $\Gamma \vdash \varphi(y)$ and the proof does not involve c. If y does not appear in Γ, then $\Gamma \vdash \forall y\ \varphi(y)$.[a] In general, let $\Phi \subseteq \Gamma$ be the subset of Γ that was used in the proof, so y does not appear in Φ. $\Phi \vdash \varphi(y)$, so $\Phi \vdash Aay\ \varphi(y)$, and $\Gamma \vdash \forall y\ \varphi(y)$.

■

[a] And the proof does not involve c.

So Lemma 1.3.2 implies that we have $\Gamma \vdash \varphi(y)$ and the proof does not involve c.

Corollary 1.3.2. If $\Gamma \vdash \varphi(c)$, and c does not occur in Γ or in $\varphi(x)$. Then $\Gamma \vdash \forall x\ \varphi(x)$, and there is a proof not involving c.[a]

[a] Here, x is any variable that does not appear in $\varphi(c)$.

Proof. We know that for some y, $\Gamma \vdash \forall y\ \varphi(y)$, (A4) says $\forall y\ \varphi(y) \to \varphi(x)$. So $\forall y\ \varphi(y) \vdash \varphi(x)$ since x does not appear in $\forall y \varphi(y)$, $\forall y\ \varphi(y) \vdash \forall x\ \varphi(x)$. ■

Note. x might appear in Γ.

Theorem 1.3.8. Let T be a consistent $\mathcal{L}$-theory. There is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a consistent $\mathcal{L}^*$-theory such that T^* has Henkin constants. We ca choose $\mathcal{L}^*$ such that $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.

Proof. Let $\mathcal{L}_0 = \mathcal{L}$ and $T_0 = T$. Let $\mathcal{L}_1$ be the expansion of $\mathcal{L}_0$ by adding a new constant symbol c_ℓ for each $\mathcal{L}_0$-formula ℓ. First, we show that T_0 is still a consistent $\mathcal{L}_1$-theory.

Remark. Technically, $\vdash$ is really $\vdash_{\mathcal{L}}$. This is a key step for seeing that it does not matter.

If not, there is a proof from T_0 of a contradiction. This proof uses only finitely many of the new constants c_ℓ. By Corollary 1.3.2, we can replace these constants one-by-one by new variables, e.g., if the original contradiction was $\varphi(c_1, \ldots, c_n)$ and $\neg\varphi(c_1, \ldots, c_n)$, then T_0 proves $\forall x_1, \ldots, \forall x_n\ \varphi(x_1, \ldots, x_n)$ and $\forall x_1, \ldots, \forall x_n\ \neg\varphi(x_1, \ldots, x_n)$. Moreover, these proofs take place in $\mathcal{L}_0$ By (A4), $T_0 \vdash_{\mathcal{L}_0} \varphi(x_1, \ldots, x_n)$, and $T_0 \vdash_{\mathcal{L}_0} \neg\varphi(x_1, \ldots, x_n)$, which is a contradiction. So T_0 is a consistent $\mathcal{L}_1$-theory. ■

Now, we can prove Theorem 1.3.6 (b) to complete the proof of Theorem 1.3.6.

Proof of Theorem 1.3.6 (b). Let T be a consistent theory in a language $\mathcal{L}$. We now proceed in steps.

1. Expand $\mathcal{L}$ to $\mathcal{L} \supseteq \mathcal{L}$ with new constant symbols, and then expand T to an $\mathcal{L}^*$-theory T^* with the following property.

   If φ is of the form $\neg\forall x\ \psi(x)$, then let

   $$\theta_\varphi := (\neg\forall x\ \psi(x)) \to \neg\psi(c_\ell)$$

   $((\exists\neg\psi(x)) \to \neg\psi(c_\ell))$.[a] Let $\theta = \{\theta_\ell \mid \ell \text{ on } \mathcal{L}_0\text{-formula}\}$.

   Claim. $T_1 = T_0 \cup \theta$ is consistent.

   [a]If φ is not in this form, let $\theta_\varphi = \forall x\ (x = x)$.

   Proof. Note that T_1 *has Hekin constants* for $\mathcal{L}_0$. If T_1 is inconsistent, there are $\varphi_n, \ldots, \varphi_{m+1}$ such that $T_0 \cup \left\{\theta_\ell, \ldots, \theta_{\ell_{m'}}, \theta_{\ell_m}\right\}$ is inconsistent. Taking m to be as small as possible, $T_0 \cup \{\theta_{\varphi_1}, \ldots, \theta_{\varphi_m}\}$.

   > **Note.** This makes sense as T_0 is consistent.

   So

   $$T_0 \cup \{\theta_{\ell_1}, \ldots, \theta_{\ell_m}\} \vdash \neg\theta_{\varphi_{m+1}},$$

   and φ_{m+1} is of the form $\neg\forall x\ \psi(x)$, and $\theta_{\ell_{m+1}}$ is $\left(\neg\forall x\ \psi(x)\right) \to \neg\psi(c_\ell)$. By (A1), (A2), (A3),

   $$T_0 \cup \{\theta_{\ell_1}, \ldots, \theta_{\ell_m}\} \vdash \neg\forall x\ \psi(x)$$

   and

   $$T_0 \cup \{\theta_{\ell_1}, \ldots, \theta_{\ell_m}\} \vdash \psi(c_\ell).$$

   Since c_ℓ does not appear in $T_0 \cup \{\theta_{\ell_1}, \ldots, \theta_{\ell_m}\}$, so

   $$T_0 \cup \{\theta_{\ell_1}, \ldots, \theta_{\ell_m}\} \vdash \forall x\ \psi(x).$$

   So $T_0 \cup \{\theta_{\ell_1}, \ldots, \theta_{\ell_m}\}$ is inconsistent, a contradiction, so T_1 is consistent. ⊛

   Given T_i and $\mathcal{L}_i$, define a T_{i+1} and $\mathcal{L}_{i+1}$ in this way. Each T_i is consistent, then, $T^* = \bigcup T_i$ is an $\mathcal{L}^* = \bigcup \mathcal{L}_i$-theory. T^* is consistent as a nested union of consistent theories, and T^* has Henkin constants because every $\mathcal{L}^*$-formula φ is an $\mathcal{L}_i$-formula for some i, and $\theta_\ell \in T_{i+1} \subseteq T^*$.

2. Extend T^* to a maximal theory T^{**}[b]

3. Turn T^{**} into a model. The elements of the model are constant symbols from $\mathcal{L}^*$, modulo the equivalence relation $c \sim d$ if $c = d$ is in T^{**}, i.e., $T^{**} \vdash c = d$.

■

[b]Which still has Henkin constants.

Appendix

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