MATH681
Mathematical Logic

Pingbang Hu

January 5, 2023

Abstract

This is a graduate-level mathematical logic course taught by Matthew Harrison-Trainor, aiming to obtain insights into all other branches of mathematics, such as algebraic geometry, analysis, etc. Specifically, we will cover model theory beyond the basic foundational ideas of logic.

While there are no required textbooks, some books do cover part of the material in the class. For example, Marker's *Model Theory: An Introduction* [Mar02], Hodges's *A Shorter Model Theory* [HH97], and Hinman's *Fundamentals of Mathematical Logic* [Hin05].

This course is taken in Fall 2022, and the date on the covering page is the last updated time.

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Chapter 1

Language, Logic, and Structures

Lecture 1: Introduction to Mathematical Logic

The goal of mathematical logic is to obtain insights into other areas of mathematics – algebra, analysis, combinatorics, and so on, by formalizing the **process** of mathematics. More concretely, there are different branches:

(a) Model Theory: Study subsets of an object defined by a formula (i.e., first-order logic).

(b) Computability Theory / Recursion Theory: Formalizing what it means to have an algorithm and studying relative computability.

(c) Set Theory: Study the structure of the mathematical universe.

(d) Proof Theory: Study the syntactic nature of proofs.

In this class, we study model-theory in nature; specifically, we will cover

- basic definitions of logic:
    - What is a formula?
    - What does it mean for a formula to be true?
    - What is a proof?

- Soundness & completeness theorems:
    - Anything provable is true.
    - Anything true is provable.

- Compactness theorem:
    - Non-standard objects exist.

- Using compactness theorem for applications:
    - Chevalley's theorem

The main theme of this course will be *syntax* v.s. *semantics*:

Syntax	v.s.	Semantics
proofs		truth
form of a formula		mathematical structures
number and type of quantifiers		isomorphisms, embeddings

1.1 Syntax and Semantics

1.1.1 Languages and Structures

Let's start by the very basic object, language.

Definition 1.1.1 (Language). A *language* $\mathcal{L}$ consists of:

- a set $\mathcal{F}$ of function symbols f with arities n_f;
- a set $\mathcal{R}$ of relation symbols R with arities n_R;
- a set $\mathcal{C}$ of constant symbols c.

A language is also sometimes called a *signature*, in which case we use σ rather than $\mathcal{L}$.

Note. A constant is the same as a 0-ary function.

Remark. Any or all sets in Definition 1.1.1 might be empty.

Example (Graph). The language of graphs, $\mathcal{L}_{\mathrm{graph}} = \{\mathcal{E}\}$ where $\mathcal{E}$ is a binary (2-ary) relation symbol.

Example (Ring). The language of rings, $\mathcal{L}_{\mathrm{ring}} = \{0, 1, +, \cdot, -\}$, where $0, 1$ are constants, $+, \cdot$ are binary functions, and $-$ is a unary function.

Example (Ordered ring). The language of ordered rings, $\mathcal{L}_{\mathrm{ord}} = \mathcal{L}_{\mathrm{ring}} \cup \{\leq\}$ where $\leq$ is the binary relation for an ordered ring.

Then, given a language, we can now interpret it in the following way.

Definition 1.1.2 (Structure). Given a language $\mathcal{L}$, an $\mathcal{L}$-*structure* $\mathcal{M}$ consists of:

- a non-empty set M called the *universe*, *domain*, or *underlying set* of $\mathcal{M}$;
- for each function symbol $f \in \mathcal{F}$, a function $f^{\mathcal{M}} : M^{n_f} \to M$;
- for each relation symbol $R \in \mathcal{R}$, a relation $R^{\mathcal{M}} \subseteq M^{n_R}$;
- for each constant symbol $c \in \mathcal{C}$, an element $c^{\mathcal{M}} \in M$.

Note (Interpretation). We call $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ the *interpretation in* $\mathcal{M}$ of symbols f, R, c, respectively.

Basically, a structure gives meaning to the symbols from the language, and we often write

$$\mathcal{M} = (M, f^{\mathcal{M}}, \ldots, R^{\mathcal{M}}, \ldots, c^{\mathcal{M}}, \ldots) = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C}).$$

Notation. We usually use $\mathcal{M}, \mathcal{N}, \ldots, \mathcal{A}, \mathcal{B}, \ldots$ to refer to structures, and $M, N, \ldots, A, B, \ldots$ for the domains.[a]

[a]Some people use $|\mathcal{M}|$ for the domain of $\mathcal{M}$.

It's time to look at some examples.

Example. The rationals $\mathbb{Q}$ and integers $\mathbb{Z}$ are both $\mathcal{L}_{\mathrm{ring}}$-structures.

Proof. Clearly, the domain is the set of rationals, and naively, we let $+^{\mathbb{Q}} = +$ in $\mathbb{Q}$, $0^{\mathbb{Q}} = 0$ in $\mathbb{Q}$, $1^{\mathbb{Q}} = 1$ in $\mathbb{Q}$, etc. In this way, $\mathbb{Q} = (\mathbb{Q}, 0, 1, +, \cdot, -)$ is an $\mathcal{L}_{ring}$-structure. Similarly, $\mathbb{Z} =$

$(\mathbb{Z}, 0, 1, +, \cdot, -)$ is as well. ⊛

While the language we have seen are all intuitively correct with their name, i.e., $\mathcal{L}_{\text{ring}}$, $\mathcal{L}_{\text{ord}}$, and $\mathcal{L}_{\text{graph}}$, they are really just the high-level abstraction of the objects in the subscript.

Example. Nothing forces an $\mathcal{L}_{\text{ring}}$-structure to be a ring.

Proof. Since an $\mathcal{L}_{\text{ring}}$-structure is just any structure with two binary functions, a unary function, and two constants interpreting the symbols of the language; hence we can define an $\mathcal{L}_{\text{ring}}$-structure $\mathcal{M}$ as

- $\mathcal{M} = \{0, 5, 11\}$;

- $0^{\mathcal{M}} = 5$;

- $1^{\mathcal{M}} = 11$;

- $+^{\mathcal{M}}$ is the constant function 0;

- $\cdot^{\mathcal{M}}$ is the function 5;

- $-^{\mathcal{M}}$ is the identity.

This is clearly not a ring since it fails nearly every axiom of a ring. ⊛

Note. Later, we will talk about theories that let us restrict to structures we want.

1.1.2 Embeddings and Isomorphisms

We can now consider the relation between structures.

Definition 1.1.3 (Embedding). Given a language $\mathcal{L}$ and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. A map $\eta \colon \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-*embedding* if it is one-to-one and preserves the interpretation of all symbols of $\mathcal{L}$:

(a) for each $f \in \mathcal{F}$ of arity n_f, and $a_1, \ldots, a_{n_f} \in \mathcal{M}$,

$$\eta\big(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})\big) = f^{\mathcal{N}}\big(\eta(a_1), \ldots, \eta_{a_{n_f}}\big);$$

(b) for each relation $R \in \mathcal{R}$ of arity n_R, and $a_1, \ldots, a_{n_R} \in \mathcal{M}$,

$$(a_1, \ldots, a_{n_R}) \in R^{\mathcal{M}} \Leftrightarrow (\eta(a_1), \ldots, \eta(a_{n_R})) \in R^{\mathcal{N}};$$

(c) for each constant $c \in \mathcal{C}$, $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

From the definition, an $\mathcal{L}$-embedding is an injection, and naturally, we have the following.

Definition 1.1.4 (Isomorphism). An $\mathcal{L}$-*isomorphism* is a bijective $\mathcal{L}$-embedding.

Definition. Given a language $\mathcal{L}$ and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Suppose $M \subseteq N$ and the inclusion map $\iota \colon M \hookrightarrow N$ is an $\mathcal{L}$-embedding.

Definition 1.1.5 (Substructure). $\mathcal{M}$ is a *substructure* of $\mathcal{N}$.

Definition 1.1.6 (Extension). $\mathcal{N}$ is an *extension* of $\mathcal{M}$.

Example. Ring embeddings are $\mathcal{L}_{\text{ring}}$-embeddings.

This generalizes the notions of embedding and isomorphism for many kinds of mathematical structures.

> **Remark.** Asking that η be injective is the same as (b) in Definition 1.1.3 for the relation $=$ of equality since
> $$a = b \in \mathcal{M} \Leftrightarrow \eta(a) = \eta(b) \in \mathcal{N}.$$

However, the notion of substructure is language sensitive. For groups, there are two possible languages:

(a) $\mathcal{L}_1 = \{e, \cdot\}$;

(b) $\mathcal{L}_2 = \left\{e, \cdot, ^{-1}\right\}$, i.e., with the unary inverse operation.

While both of them seem fine at the first glance, but actually we should use the second one.

> **Remark.** Using $\mathcal{L}_2$, the substructure of a group is the same things as a subgroup. But if we use $\mathcal{L}_1$, then $(\mathbb{N}, +, 0)$ is a substructure of $(\mathbb{Z}, +, 0)$, while $\mathbb{N}$ is not a group for sure.
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> **Proof.** Simply observe that both $(\mathbb{N}, 0, +), (\mathbb{Z}, 0, +)$ are $\mathcal{L}_1$-structures. ⍟

Similarly, we include $-$ in $\mathcal{L}_{\mathrm{ring}}$ for a similar reason as in the previous example.

> **Example.** An $\mathcal{L}_{\mathrm{ring}}$-substructure of a field will be a subring, not a subfield. If we want subfields, use $\mathcal{L}_{\mathrm{ring}} \cup \left\{^{-1}\right\}$.[a]
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> [a] Set $0^{-1} = 0$, but never use this.

Appendix

Bibliography

[HH97]   W. Hodges and S.M.S.W. Hodges. *A Shorter Model Theory.* Cambridge University Press, 1997. ISBN: 9780521587136. URL: https://books.google.com/books?id=S6QYeuo4p1EC.

[Hin05] P.G. Hinman. *Fundamentals of Mathematical Logic.* Taylor & Francis, 2005. ISBN: 9781568812625. URL: https://books.google.com/books?id=xA6D8o72qAgC.

[Mar02] D. Marker. *Model Theory : An Introduction.* Graduate Texts in Mathematics. Springer New York, 2002. ISBN: 9780387987606. URL: https://books.google.com/books?id=gkvogoiEnuYC.