MATH592
Introduction to Algebraic Topology

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January 5, 2022

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Lecture 1: Homotopies of Maps

05 Jan. 10:00

1 Homotopy

Definition 1. Let X, Y be topological spaces. Let f, $g\colon X \to Y$ continuous maps. Then a *homotopy* from f to g is a 1−parameter family of maps that continuously deforms f to g, i.e., it's a continuous function $F\colon X \times I \to Y$, where $I = [0, 1]$, such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x).$$

We often write $F_t(x)$ for $F(x, t)$.

If a homotopoy exists between f and g, we say they are *homotopic* and write

$$f \simeq g.$$

If f is homotopic to a constant map, we call it *nullhomotopic*.

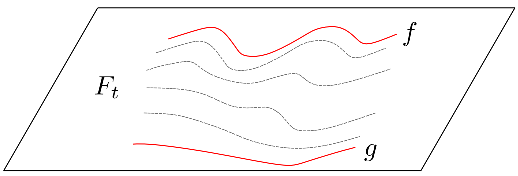

Figure 1: The continuous deforming from f to g described by F_t

Remark. Later, we'll not state that a map is continuous explicitly, since it's almost always assumed in this context.

Example. We first see some examples.

1. Any two maps (continuous) with specification

$$f,\ g\colon X \to \mathbb{R}^n$$

are homotopic by considering

$$F_t(x) = (1-t)f(x) + tg(x).$$

We call it *the straight line homotopy.*

2. Let S^1 denotes the unit circle in $\mathbb{R}^2$, and D^2 denotes the unit disk in $\mathbb{R}^2$. Then the inclusion $f\colon S^1 \hookrightarrow D^2$ is nullhomotopic by considering

$$F_t(x) = (1-t)f(x)(+t\cdot 0).$$

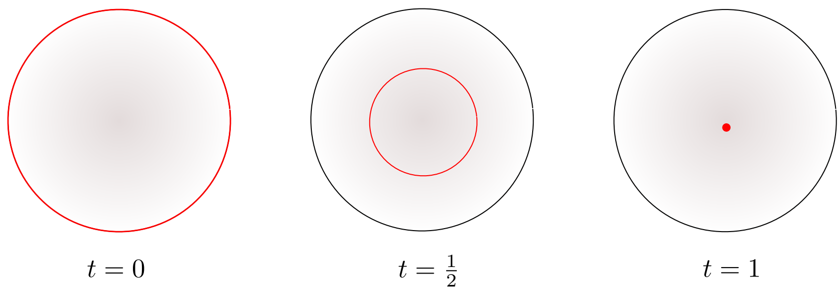

Figure 2: The illustration of $F_t(x)$

We see that there is a homotopy from $f(x)$ to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a nullhomotopy.

3. The maps

$$\begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not** homotopy.

Remark. It will essentially **flip** the orientation, hence we can't deform one to another continuously.

Exercise. We first see some exercises.

1. A subset $S \subseteq \mathbb{R}^n$ is star-shaped if

$$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from x_0 to x lies in S.

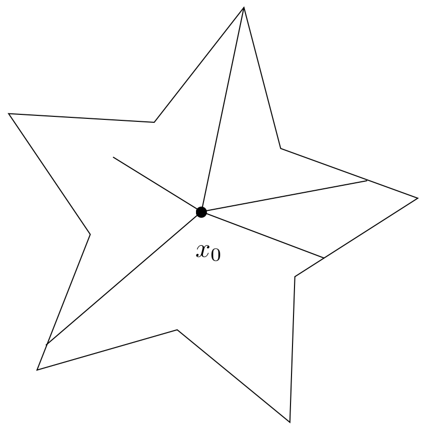

Figure 3: Star-shaped illustration

Show that $\mathrm{id}\colon S \to S$ is nullhomotopic.

Answer. Consider
$$F_t(x) \coloneqq (1-t)x + tx_0,$$
which essentially just concentrates all points x to x_0.

2. Suppose
$$X \underset{f_0}{\overset{f_1}{\rightarrow}} Y \underset{g_0}{\overset{g_1}{\rightarrow}} Z$$
where
$$f_0 \underset{F_t}{\backsimeq} f_1, \quad g_0 \underset{G_t}{\backsimeq} g_1.$$
Show
$$g_0 \circ f_0 \backsimeq g_1 \circ f_1.$$

Answer. Consider $I \times X \to Z$. Then
$$\begin{array}{ccccc} X \times I & \to & Y \times I & \to & Z \\ (x,t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

Remark. Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** homotopic?

Answer.