

# MATH592

## Introduction to Algebraic Topology

Pingbang Hu

February 6, 2022

### Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

## Contents

<b>1</b>	<b>Foundation of Algebraic Topology</b>	<b>1</b>
1.1	Homotopy	1
1.2	Homotopy Equivalence	5
1.3	CW Complexes	9
1.4	Operations on CW Complexes	13
1.4.1	Products	13
1.4.2	Wedge Sum	14
1.4.3	Quotients	14
1.5	Category Theory	15
1.5.1	Functor	17
1.6	Free Groups	19
1.6.1	Constructing the Free Groups $F_S$	21
<b>2</b>	<b>The Fundamental Group</b>	<b>22</b>
2.1	Definition	22
2.2	Calculations with $\pi_1(S^n)$	27
2.3	Group Presentation	35
2.4	Presentations for $\pi_1$ of CW Complexes	37

## Lecture 1: Homotopies of Maps

05 Jan. 10:00

### 1 Foundation of Algebraic Topology

#### 1.1 Homotopy

**Definition 1.1 (Homotopy, homotopic, nullhomotopic).** Let  $X, Y$  be topological spaces. Let  $f, g: X \rightarrow Y$  continuous maps. Then a *homotopy* from  $f$  to  $g$  is a 1-parameter family of maps that continuously deforms  $f$  to  $g$ , i.e., it's a continuous function  $F: X \times I \rightarrow Y$ , where  $I = [0, 1]$ , such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x).$$

We often write  $F_t(x)$  for  $F(x, t)$ .

If a homotopy exists between  $f$  and  $g$ , we say they are *homotopic* and write

$$f \simeq g.$$

If  $f$  is homotopic to a constant map, we call it *nullhomotopic*.

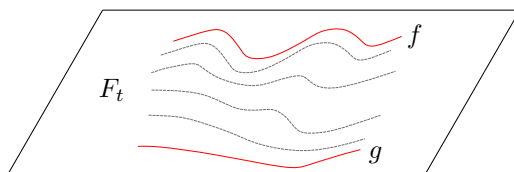


Figure 1: The continuous deforming from  $f$  to  $g$  described by  $F_t$

**Remark.** Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

**Example.** We first see some examples.

1. Any two maps (continuous) with specification

$$f, g: X \rightarrow \mathbb{R}^n$$

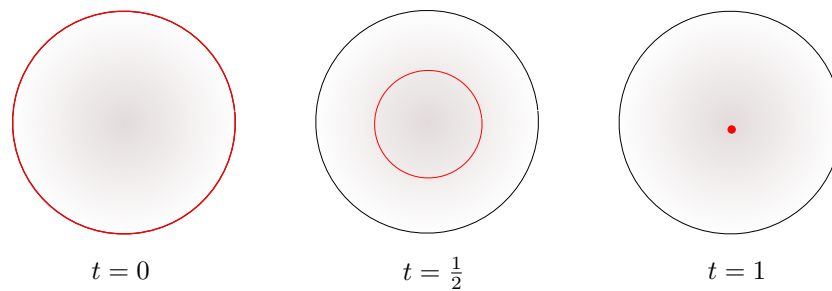
are *homotopic* by considering

$$F_t(x) = (1 - t)f(x) + tg(x).$$

We call it *the straight line homotopy*.

2. Let  $S^1$  denotes the unit circle in  $\mathbb{R}^2$ , and  $D^2$  denotes the unit disk in  $\mathbb{R}^2$ . Then the inclusion  $f: S^1 \hookrightarrow D^2$  is *nullhomotopic* by considering

$$F_t(x) = (1 - t)f(x) + (t \cdot 0).$$


 Figure 2: The illustration of  $F_t(x)$ 

We see that there is a **homotopy** from  $f(x)$  to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a **nullhomotopy**.

3. The maps

$$\begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not** **homotopy**.

**Remark.** It will essentially **flip** the orientation, hence we can't deform one to another continuously.

**Exercise.** We first see some exercises.

1. A subset  $S \subseteq \mathbb{R}^n$  is star-shaped if

$$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from  $x_0$  to  $x$  lies in  $S$ .

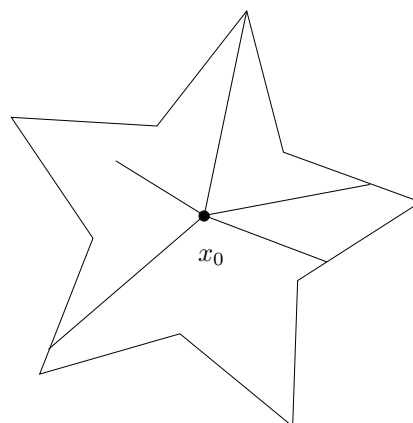


Figure 3: Star-shaped illustration

Show that  $\text{id}: S \rightarrow S$  is nullhomotopic.

**Answer.** Consider

$$F_t(x) := (1-t)x + tx_0,$$

which essentially just concentrates all points  $x$  to  $x_0$ .

2. Suppose

$$X \xrightarrow[f_0]{f_1} Y \xrightarrow[g_0]{g_1} Z$$

where

$$f_0 \simeq_{F_t} f_1, \quad g_0 \simeq_{G_t} g_1.$$

Show

$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

**Answer.** Consider  $I \times X \rightarrow Z$ . Then

$$\begin{array}{ccccc} X \times I & \rightarrow & Y \times I & \rightarrow & Z \\ (x, t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

**Remark.** Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** homotopic?

**Answer.**

## Lecture 2: Homotopy Equivalence

07 Jan. 10:00

**As previously seen.** Two maps  $f, g: X \rightarrow Y$  is homotopy if there exists a map

$$F_t(x): X \times I \rightarrow Y$$

with the properties

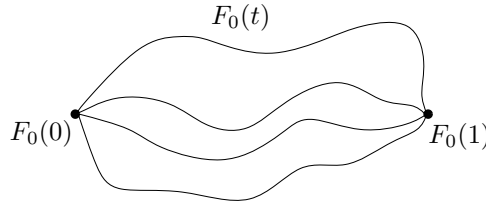
1. Continuous
2.  $F_0(x) = f(x)$
3.  $F_1(x) = g(x)$

**Remark.** The continuity of  $F_t$  is an even stronger condition for the continuity of  $F_t$  for a fixed  $t$ .

We now introduce another concept.

**Definition 1.2 (Homotopy relative).** Given two spaces  $X, Y$ , and let  $B \subseteq X$ . Then a homotopy  $F_t(x): X \rightarrow Y$  is called *homotopy relative  $B$*  (denotes  $\text{rel}B$ ) if  $F_t(b)$  is independent of  $t$  for all  $b$ .

**Example.** Let  $X = [0, 1]$  and  $B = \{0, 1\}$ . Then the [homotopy](#) of paths from  $[0, 1] \rightarrow X$  is  $\text{rel}\{0, 1\}$ .



## 1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

**Definition 1.3 (Homotopy equivalence, homotopy inverse).** A map  $f: X \rightarrow Y$  is a *homotopy equivalence* if  $\exists g: Y \rightarrow X$  such that

$$f \circ g \simeq \text{id}_Y, \quad g \circ f \simeq \text{id}_X.$$

We say that  $X, Y$  are *homotopy equivalent*, and  $g$  is called *homotopy inverse* of  $f$ .

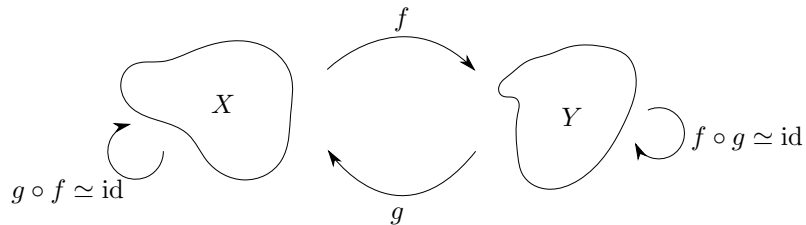


Figure 4: [Homotopy Equivalence](#)

If  $X, Y$  are [homotopy equivalent](#), then we say that they have the same *homotopy type*.

**Notation.** We denote a closed  $n$ -disk as  $D^n$ .

**Example.**  $D^n$  is [homotopy equivalent](#) to a point.



We see that  $f \circ g = \text{id}_*$  and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is [homotopic](#) to  $\text{id}_{D^n}$  by [straight line homotopy](#)  $F_t(x) = tx$ .

**Note.** We say that a space is *contractible* if  $H$  is [homotopy equivalent](#) to a point.

Before doing exercises, we introduce two new concepts.

**Definition 1.4 (Retraction, retract).** Given  $B \subseteq X$ , a *retraction* from  $X$  to  $B$  is a map  $f: X \rightarrow X$  (or  $X \rightarrow B$ ) such that  $\forall b \in B$   $f(b) = b$ , namely  $r|_B = \text{id}_B$ . Or one can see this from

$$\begin{array}{ccc} B & \xrightarrow{i} & X \\ & \searrow r \circ i & \nearrow r \\ & & B \end{array}$$

where  $r$  is a retraction if and only if  $r \circ i = \text{id}_B$ , where  $i$  is an inclusion identity. If  $r$  exists,  $B$  is a *retract* of  $X$ .

**Definition 1.5 (Deformation retraction).** Given  $X$  and  $B \subseteq X$ , a *(strong) deformation retraction*  $F_t: X \rightarrow X$  onto  $B$  is a [homotopy](#)  $\text{rel} B$  from the  $\text{id}_X$  to a [retraction](#) from  $X$  to  $B$ . i.e.,

$$\begin{aligned} F_0(x) &= x & \forall x \in X \\ F_1(x) &\in B & \forall x \in X \\ F_t(b) &= b & \forall t \forall b \in B. \end{aligned}$$

**Exercise.** We now see some problems.

1. Let  $X \simeq Y$ . Show  $X$  is path-connected if and only if  $Y$  is.

**Answer.** Suppose  $X$  is path-connected. Then we see that given two points  $x_1$  and  $x_2$  in  $X$ , there exists a path  $\gamma(t)$  with

$$\gamma: [0, 1] \rightarrow X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Since  $X \simeq Y$ , then there exists a pair of  $f$  and  $g$  such that  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  with

$$f \circ g \underset{F}{\simeq} \text{id}_Y, \quad g \circ f \underset{G}{\simeq} \text{id}_X.$$

(Notice the abuse of notation)

For any two  $y_1$  and  $y_2 \in Y$ , we want to construct a path  $\gamma'(t)$  such that

$$\gamma': [0, 1] \rightarrow Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

Firstly, we let  $g(y_1) =: x_1$  and  $g(y_2) =: x_2$ . From the argument above, we know there exists such a  $\gamma$  starting at  $x_1 = g(y_1)$  ending at  $x_2 = g(y_2)$ . Now, consider  $f(\gamma(t)) = (f \circ \gamma)(t)$  such that

$$f \circ \gamma: I \rightarrow Y, \quad f \circ \gamma(0) = y'_1, \quad f \circ \gamma(1) = y'_2,$$

we immediately see that  $y'_1$  and  $y'_2$  is path connected. Now, we claim that  $y_1$  and  $y'_1$  are path connected in  $Y$ , hence so are  $y_2$  and  $y'_2$ . To see this, note that

$$f \circ g \underset{F}{\simeq} \text{id}_Y,$$

which means that there exists  $F: Y \times I \rightarrow Y$  such that

$$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y'_1 \\ F(y_1, 1) = \text{id}_Y(y_1) = y_1. \end{cases}$$

Since  $F$  is continuous in  $I$ , we see that there must exist a path connects  $y_1$  and  $y'_1$ . The same argument applies to  $y_2$  and  $y'_2$ . Now, we see that the path

$$y_1 \rightarrow y'_1 \rightarrow y'_2 \rightarrow y_2$$

is a path in  $Y$  for any two  $y_1$  and  $y_2$ , which shows  $Y$  is path-connected.



Figure 5: Demonstration of the proof

**Challenge:** One can further show that the connectedness is also preserved by any [homotopy equivalence](#).

2. Show that if there exists [deformation retraction](#) from  $X$  to  $B \subseteq X$ , then  $X \simeq B$ .

### Lecture 3: Deformation Retraction

10 Jan. 10:00

**As previously seen.** A [deformation retraction](#) is a [homotopy](#) of maps  $\text{rel} B$   $X \rightarrow X$  from  $\text{id}_X$  to a [retraction](#) from  $X$  to  $B$ . Then  $B$  is a [deformation retract](#).

**Example.** We can also show

1.  $S^1$  is a [deformation retraction](#) of  $D^2 \setminus \{0\}$ . Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$

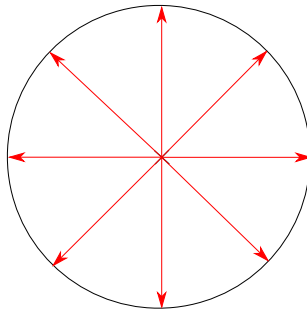


Figure 6: The [deformation retraction](#) of  $D^2 \setminus \{0\}$  is just to *enlarge* that hold and push all the interior of  $D^2$  to the boundary, which is  $S^1$

2.  $\mathbb{R}^n$  [deformation retracts](#) to 0. Indeed, since

$$F_t(x) = (1-t)x.$$

This implies that  $\mathbb{R}^n \simeq *$ , hence we see that

- dimension
- compactness
- etc.

are not [homotopy](#) invariants.

3.  $S^1$  is a [deformation retract](#) of a cylinder and a Möbius band.

For a cylinder, consider  $X \times I \rightarrow X$ . Define [homotopy](#) on a closed rectangle, then verify it induces map on quotient.

For a Möbius band, we define a [homotopy](#) on a closed rectangle, then verify that it respect the equivalence relation.



Finally, we use the universal property of quotient topology to argue that we get a [homotopy](#) on Möbius band.

**Upshot:** Möbius band  $\simeq S^1 \simeq$  cylinder, hence the orientability is not [homotopy](#) invariant.

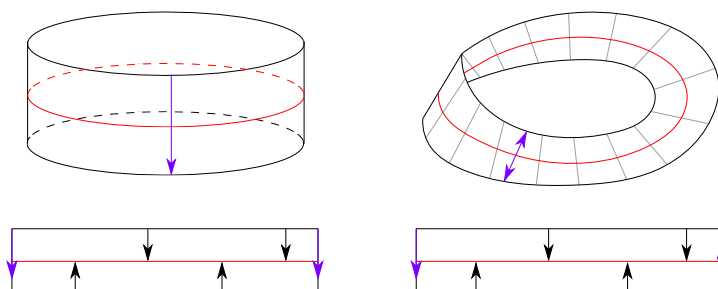


Figure 7: The [deformation retraction](#) for Cylinder and Möbius band

## Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

As previously seen. We saw that

- [homotopy equivalence](#)
- [homotopy](#) invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

### 1.3 CW Complexes

**Example.** Let's start with a few examples.

1. Constructing spheres:

- $S^1$  (up to homeomorphism<sup>1</sup>)

<sup>1</sup>This is just the term for isomorphism in topology.

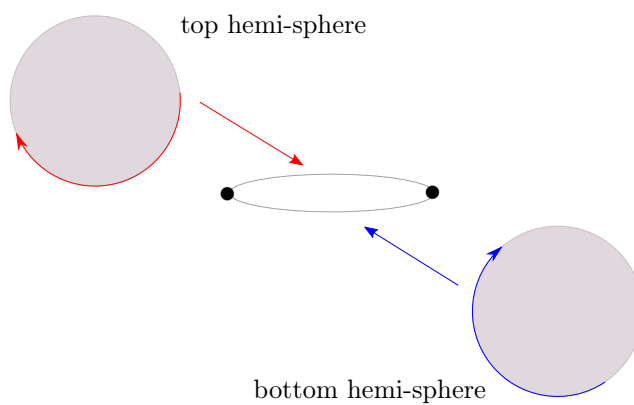


- $S^2$ 
  - glue boundary of 2-disk to a point
  - glue 2 disks onto a circle



Figure 8: **Left:** Glue a 2-disk to a point along its boundary. **Right:** Glue 2 disks to  $S^1$ .

The gluing instruction to construct  $S^2$  in the right-hand side can be demonstrated as follows.



- $T = S^1 \times S^1$



view as gluing instructions

vertex + 2 edges + 2-disks.

Specifically, we have




---

Formally, we have the following definition.

**Notation.** Let  $D^n$  denotes a closed  $n$ -disk (or  $n$ -ball)

$$D^n \simeq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

And let  $S^n$  denotes an  $n$ -sphere

$$S^n \simeq \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

Lastly, we call a point as a  $0$ -cell, and the interior of  $D^n$   $\text{int}(D^n)$  for  $n \geq 1$  as a  $n$ -cell.

**Definition 1.6 (CW Complex).** A *CW Complex* is a topological space constructed inductively as

1.  $X^0$  (the 0-skeleton) is a set of discrete points.
2. We inductively construct the n-skeleton  $X^n$  from  $X^{n-1}$  by attaching  $n$ -cells  $e_\alpha^n$ , where  $\alpha$  is the index.

The gluing instructions glued by an attaching map is that  $\forall \alpha, \exists$  continuous map  $\varphi_\alpha$

$$\varphi_\alpha: \partial D_\alpha^n \rightarrow X^{n-1},$$

then

$$X^n = \left( X^{n-1} \coprod_\alpha D_\alpha^n \right) / x \sim \varphi_\alpha(x)$$

with identification  $x \sim \varphi_\alpha(x)$  for all  $x \in \partial D_\alpha^n$  with quotient topology.

- 3.

$$X = \bigcup_{n=0} X^n,$$

and let  $\bar{w}$  denotes weak topology. Then

$$u \subseteq X \text{ is open} \iff \forall n \ u \cap X^n \text{ is open}.$$

If all cells have dimension less than  $N$  and a  $\exists N$ -cell, then  $X = X^N$  and we call it  $N$ -dim CW complex.

**Remark.** We write  $X^{(n)}$  for  $n$ -skeleton if we need to distinguish from the Cartesian product.

**Example.** Let's look at some examples.

1. 0-dim **CW complex** is a discrete space.
2. 1-dim **CW complex** is a graph.
3. A CW complex  $X$  is finite if it has finitely many cells.

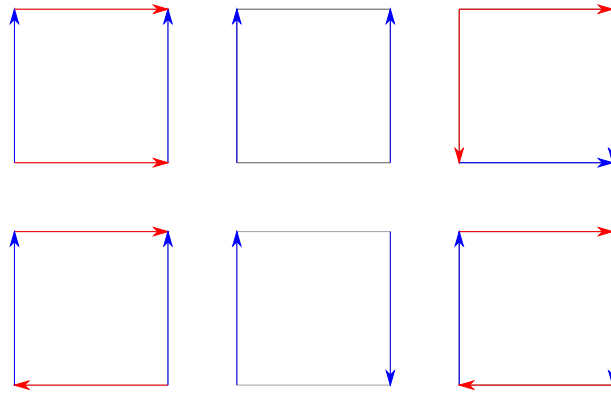
**Definition 1.7 (CW subcomplex).** A *CW subcomplex*  $A \subseteq X$  is a closed subset equal to a union of cells

$$e_\alpha^n = \text{int}(D_\alpha^n).$$

**Remark.** This inherits a **CW complex** structure.

**Exercise.** Given the following gluing instruction:

Check the images of attaching maps.



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere,  $\mathbb{R}P$ .

**Answer.** We see that

1. Torus
2. Cylinder
3. 2-sphere
4. Klein bottle
5. Möbius band
6.  $\mathbb{R}P$

**Notation.** We call the real projection space as  $\mathbb{R}P$ , and we also have so-called complex projection space, denote as  $\mathbb{C}P$ .

## Lecture 5: Operation on Spaces

14 Jan. 10:00

### 1.4 Operations on CW Complexes

#### 1.4.1 Products

We can consider the product of two CW complexes given by a [CW complex](#) structure. Namely, given  $X$  and  $Y$  two [CW complexes](#), we can take two cells  $e_\alpha^n$  from  $X$  and  $e_\beta^m$  from  $Y$  and form the product space  $e_\alpha^n \times e_\beta^m$ , which is homeomorphic to an  $n + m$ -cell. We then take these products as the cells for  $X \times Y$ .

Specifically, given  $X, Y$  are [CW complexes](#), then  $X \times Y$  has a cell structure

$$\{e_\alpha^m \times e_\alpha^n : e_\alpha^m \text{ is a } m\text{-cell on } X, e_\alpha^n \text{ is an } n\text{-cell on } Y\}.$$

**Remark.** The product topology may not agree with the weak topology on the  $X \times Y$ . However, they do agree if  $X$  or  $Y$  is locally compact or if  $X$  and  $Y$  both have at most countably many cells.

**Note.** Notice that if the product is wild enough, then the product topology may not agree with the weak topology.

### 1.4.2 Wedge Sum

Given  $X, Y$  are CW complexes, and  $x_0 \in X^0, y_0 \in Y^0$  (only points). Then we define

$$X \vee Y = X \amalg Y$$

with quotient topology.

**Remark.**  $X \vee Y$  is a CW complex.

### 1.4.3 Quotients

Let  $X$  be a CW complex, and  $A \subseteq X$  subcomplex (closed union of cells), then

$$X / A$$

is a quotient space collapse  $A$  to one point and inherits a CW complex structure.

**Remark.**  $X / A$  is a CW complex.

0-skeleton

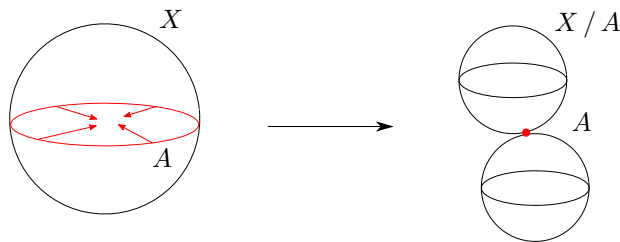
$$(X^0 - A^0) \amalg *$$

where  $*$  is a point for  $A$ . Each cell of  $X - A$  is attached to  $(X / A)^n$  by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n / A^n$$

**Example.** Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a wedge of two spheres.



2. We can take the torus and squish down a ring around the hole.



Figure 9: We see that  $X / A$  is [homotopy equivalent](#) to a 2-sphere wedged with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other points, forming a circle.

## Lecture 6: A Foray into Category Theory

19 Jan. 10:00

### 1.5 Category Theory

We start with a definition.

**Definition 1.8 (Category, object, morphism).** A *category*  $\mathcal{C}$  is 3 pieces of data

- A class of *objects*  $\text{Ob}(\mathcal{C})$
- $\forall X, Y \in \text{Ob}(\mathcal{C})$  a class of *morphisms* or arrows,  $\text{Hom}_{\mathcal{C}}(X, Y)$ .
- $\forall X, Y, Z \in \text{Ob}(\mathcal{C})$ , there exists a composition law

$$\begin{aligned} \text{Hom}(X, Y) \times \text{Hom}(Y, Z) &\rightarrow \text{Hom}(X, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

and 2 axioms

- Associativity.  $(f \circ g) \circ h = f \circ (g \circ h)$  for all [morphisms](#)  $f, g, h$  where composites are defined.
- Identity.  $\forall X \in \text{Ob}(\mathcal{C}) \exists \text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$  such that

$$f \circ \text{id}_X = f, \quad \text{id}_X \circ g = g$$

for all  $f, g$  where this makes sense.

Let's see some examples.

**Example.** We introduce some common [category](#).

$\mathcal{C}$	$\text{Ob}(\mathcal{C})$	$\text{Mor}(\mathcal{C})$
$\underline{\text{set}}$	Sets $X$	All maps of sets
$\underline{\text{fset}}$	Finite sets	All maps
$\underline{\text{Gp}}$	Groups	Group Homomorphisms
$\underline{\text{Ab}}$	Abelian groups	Group Homomorphisms
$\underline{k\text{-vect}}$	Vector spaces over $k$	$k$ -linear maps
$\underline{\text{Rng}}$	Rings	Ring Homomorphisms
$\underline{\text{Top}}$	Topological spaces	Continuous maps
$\underline{\text{Haus}}$	Hausdorff Spaces	Continuous maps
$\underline{\text{hTop}}$	Topological spaces	Homotopy classes of continuous maps
$\underline{\text{Top}^*}$	Based topological spaces <sup>2</sup>	Based maps <sup>3</sup>

**Remark.** Any **diagram** plus composition law.

$$\text{id}_A \hookrightarrow A \longrightarrow B \hookleftarrow \text{id}_B .$$

**Definition 1.9 (monic, epic).** A **morphism**  $f: M \rightarrow N$  is *monic* if

$$\forall g_1, g_2 \quad f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$A \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} M \xrightarrow{f} N$$

Dually,  $f$  is *epic* if

$$\forall g_1, g_2 \quad g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$M \xrightarrow{f} N \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} B$$

**Lemma 1.1.** In  $\underline{\text{set}}, \underline{\text{Ab}}, \underline{\text{Top}}, \underline{\text{Gp}}$ , a map is **monic** if and only if  $f$  is injective, and **epic** if and only if  $f$  is surjective.

*Proof.* In  $\underline{\text{set}}$ , we prove that  $f$  is **monic** if and only if  $f$  is injective. Suppose  $f \circ g_1 = f \circ g_2$  and  $f$  is injective, then for any  $a$ ,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence  $g_1 = g_2$ .

<sup>2</sup>Topological spaces with a distinguished base point  $x_0 \in X$

<sup>3</sup>Continuous maps that presence base point  $f: (x, x_0) \rightarrow (y, y_0)$  such that

$$f: X \rightarrow Y, \quad f(x_0) = y_0$$

is continuous.



Now we prove another direction, with contrapositive. Namely, we assume that  $f$  is not injective and show that  $f$  is not **monic**. Suppose  $f(a) = f(b)$  and  $a \neq b$ , we want to show such  $g_i$  exists. This is easy by considering

$$g_1: * \mapsto a, \quad g_2: * \mapsto b.$$

■

### 1.5.1 Functor

After introducing the **category**, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

**Definition 1.10 (Functor).** Given  $\mathcal{C}, \mathcal{D}$  be two **categories**. A (covariant) *functor*

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is

1. a map on **objects**

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of **morphisms**

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(X) \rightarrow F(Y)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(f) \circ F(g)$

## Lecture 7: Functors

21 Jan. 10:00

**As previously seen.** Assume that we initially have a commutative diagram in  $\mathcal{C}$  as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & Z \end{array}$$

After applying  $F$ , we'll have

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ & \searrow F(g \circ f) = F(g) \circ F(f) & \downarrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in  $\mathcal{D}$ .

We can also have a so-called contravariant **functor**.

**Definition 1.11 (Contravariant functor).** Given  $\mathcal{C}, \mathcal{D}$  be two categories. A *contravariant functor*

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is

1. a map on objects

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(Y), F(X)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(Y) \rightarrow F(X)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a *contravariant functor*  $F$ , the diagram becomes

$$\begin{array}{ccc} F(X) & \xleftarrow{F(f)} & F(Y) \\ & \nwarrow F(g \circ f) = F(f) \circ F(g) & \uparrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in  $\mathcal{D}$ .

**Example.** Let see some examples.

1. Identity functor.

$$I: \mathcal{C} \rightarrow \mathcal{C}.$$

2. Forgetful functor.

•

$$\begin{aligned} F: \underline{\text{Gp}} &\rightarrow \underline{\text{set}} \\ G &\mapsto G^4 \\ [f: G \rightarrow H] &\mapsto [f: G \rightarrow H] \end{aligned}$$

•

$$\begin{aligned} F: \underline{\text{Top}} &\rightarrow \underline{\text{set}} \\ X &\mapsto X^5 \\ [f: X \rightarrow Y] &\mapsto [f: X \rightarrow Y] \end{aligned}$$

<sup>4</sup> $G$  is now just the underlying set of the group  $G$ .

## 3. Free functor.

$$\begin{aligned} \underline{\text{set}} &\rightarrow \underline{k\text{-vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

i.e., vector space with basis  $s$

$$[f: A \rightarrow B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

## 4.

$$\begin{aligned} \underline{k\text{-vect}} &\rightarrow \underline{k\text{-vect}} \\ V &\mapsto V^* = \text{Hom}_k(V, k) \end{aligned}$$

If we are working on a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functor.

## 1.

$$\begin{aligned} \underline{\text{Top}}^* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \Pi_1(X, x_0) \end{aligned}$$

where  $\Pi_1$  is so-called *fundamental group*.

## 2.

$$\begin{aligned} \underline{\text{Top}} &\rightarrow \underline{\text{Ab}} \\ X &\mapsto \text{Hp}(X) \end{aligned}$$

where Hp is so-called  $p^{\text{th}}$  *homology*.

Let see the formal definition.

## 1.6 Free Groups

We start with a definition.

**Definition 1.12 (Free group).** Given a set  $S$ , the *free group* is a group  $F_S$  on  $S$  with a map  $S \rightarrow F_S$  satisfying the universal property.

If  $G$  is any group,  $f: S \rightarrow G$  is any map of sets,  $f$  extends uniquely to group homomorphism  $\bar{f}: F_S \rightarrow G$ .

$$\begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \bar{f}: \text{gp hom} \\ & & G \end{array}$$

<sup>5</sup> $X$  is now just the underlying set of the topological space  $X$ .

**Note.** This defines a *natural bijection*

$$\mathrm{Hom}_{\mathrm{set}}(S, \mathcal{U}(G)) \cong \mathrm{Hom}_{\mathrm{Grp}}(F_S, G),$$

where  $\mathcal{U}(G)$  is the **forgetful functor** from the **category** of groups to the **category** of sets. This is the statement that the **free functor** and the forgetful functor are **adjoint**; specifically that the **free functor** is the left **adjoint** (appears on the left in the Hom above).

**Definition 1.13 (Adjoint functor).** A **free** and **forgetful** functors are *adjoints*.

**Remark.** Whenever we state a universal property for an **object** (plus a map), an **object** (plus a map) may or may not exist. If such **object** exists, then it defines the **object uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the **object** (plus a map).

**Lemma 1.2.** Universal property defines  $F_S$  (plus a map  $S \rightarrow F(S)$ ) uniquely up to unique isomorphism.

*Proof.* Fix  $S$ . Suppose

$$S \rightarrow F_S, \quad S \rightarrow \tilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{array}{ccc} S & \longrightarrow & \tilde{F}_S \\ & \searrow f & \downarrow \exists! \varphi \\ & & F_S \end{array} \quad \begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \psi \\ & & \tilde{F}_S \end{array}$$

We'll show  $\varphi$  and  $\psi$  are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \mathrm{id}_{F_S} & \searrow f \\ S & & \\ f \searrow & \downarrow & \nearrow \\ & F_S & \end{array} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \mathrm{id}_{\tilde{F}_S} & \searrow f \\ S & & \\ f \searrow & \downarrow & \nearrow \\ & \tilde{F}_S & \end{array}$$

Hence, we see that

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \psi & \searrow f \\ S & \longrightarrow & \tilde{F}_S \\ f \searrow & \downarrow \varphi & \nearrow \\ & F_S & \end{array} \quad \varphi \circ \psi = \mathrm{id}_{F_S} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \varphi & \searrow f \\ S & \longrightarrow & F_S \\ f \searrow & \downarrow \psi & \nearrow \\ & \tilde{F}_S & \end{array} \quad \psi \circ \varphi = \mathrm{id}_{\tilde{F}_S}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \text{id}_{F_S}, \quad \psi \circ \varphi = \text{id}_{\tilde{F}_S},$$

so  $\varphi$  and  $\psi$  are inverses (thus group isomorphism). ■

## Lecture 8: The Fundamental Group $\pi_1$

24 Jan. 10:00

**Example.** In category Ab **free** Abelian group on a set  $S$  is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on  $S$** .

### 1.6.1 Constructing the Free Groups $F_S$

**Proposition 1.1.** The **free group** defined by the universal property exists.

*Proof.* We'll just give a construction below. First, we see the definition.

**Definition 1.14.** Fix a set  $S$ , and we define a *word* as a finite sequence (possibly  $\emptyset$ ) in the formal symbols

$$\{s, s^{-1} \mid s \in S\}.$$

Then we see that elements in  $F_S$  are equivalence classes of words with the equivalence relation being

- delete  $ss^{-1}$  or  $s^{-1}s$ . i.e.,

$$vs^{-1}sw \sim vw$$

$$vss^{-1}w \sim vw$$

for every **word**  $v, w, s \in S$ ,

with the group operation being concatenation. ■

**Example.** Given words  $ab^{-1}, bba$ , their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

**Exercise.** There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that  $F_S$  satisfies the universal property with respect to the map

$$S \rightarrow F_S, \quad s \mapsto s.$$

## 2 The Fundamental Group

### 2.1 Definition

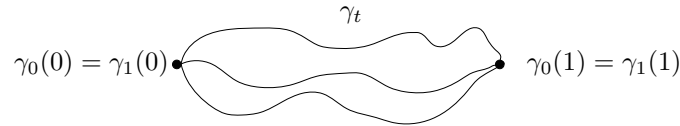
We start with the definition.

**Definition 2.1 (Path).** A *path* in a space  $X$  is a continuous map

$$\gamma: I \rightarrow X$$

where  $I = [0, 1]$ .

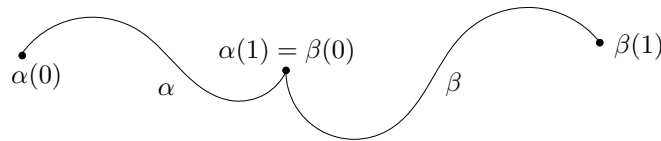
**Definition 2.2 (Homotopy path).** A *homotopy of paths*  $\gamma_0, \gamma_1$  is a *homotopy* from  $\gamma_0$  to  $\gamma_1$  rel  $\{0, 1\}$ .



**Example.** Fix  $x_1, x_0 \in X$ , then  $\exists$  *homotopy of paths* is an equivalence relation on *paths* from  $x_0$  to  $x_1$  (i.e.,  $\gamma$  with  $\gamma(0) = x_0, \gamma(1) = x_1$ ).

**Definition 2.3 (Path composition).** For *paths*  $\alpha, \beta$  in  $X$  with  $\alpha(1) = \beta(0)$ , the *composition*<sup>a</sup>  $\alpha \cdot \beta$  is

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



<sup>a</sup>Also named *product*, *concatenation*.

**Remark.** By the pasting lemma, this is continuous, hence  $\alpha \cdot \beta$  is actually a path from  $\alpha(0)$  to  $\beta(1)$ .

**Definition 2.4 (Reparameterization).** Let  $\gamma: I \rightarrow X$  be a path, then a *reparameterization* of  $\gamma$  is a path

$$\gamma': I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where  $\varphi$  is continuous and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

**Exercise.** A path  $\gamma$  is homotopic rel $\{0, 1\}$  to all of its reparameterizations.

*Proof.* We show that  $\gamma$  and  $\gamma \circ \phi$  are homotopic rel $\{0, 1\}$  by showing that there exists a continuous  $F_t$  such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since  $\phi$  is continuous, so we define

$$F_t(x) = (1-t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all  $x, t \in I$ .

Now, we check that  $F_t$  really gives us a homotopic rel $\{0, 1\}$ . We have

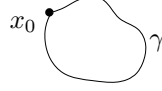
$$\begin{aligned} F_t(0) &= (1-t)\gamma(0) + t \cdot \gamma \circ \phi(0) = (1-t)\gamma(0) + t \cdot \underbrace{\gamma(\phi(0))}_0 = \gamma(0), \\ F_t(1) &= (1-t)\gamma(1) + t \cdot \gamma \circ \phi(1) = (1-t)\gamma(1) + t \cdot \underbrace{\gamma(\phi(1))}_1 = \gamma(1), \end{aligned}$$

which shows that 0 and 1 are independent of  $t$ , hence  $\gamma$  and  $\gamma \circ \phi$  are homotopic rel $\{0, 1\}$ . ■

**Exercise.** Fix  $x_0, x_1 \in X$ . Then Homotopy of paths (relative  $\{0, 1\}$ ) is an equivalence relation on paths from  $x_0$  to  $x_1$ .

**Definition 2.5 (Fundamental Group).** Let  $X$  denotes the space and let  $x_0 \in X$  be the base point. The *fundamental group of  $X$  based at  $x_0$* , denoted by  $\pi_1(X, x_0)$ , is a group such that

- Elements: **Homotopy** classes  $\text{rel}\{0, 1\}$  of **paths**  $[\gamma]$  where  $\gamma$  is a **loop** with  $\gamma(0) = \gamma(1) = x_0$ <sup>a</sup>

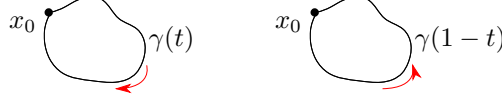


- Operation: **Composition of paths**.
- Identity: Constant loop  $\gamma$  based at  $x_0$  such that

$$\gamma: I \rightarrow X, \quad t \mapsto x_0$$

- Inverses: The inverse  $[\gamma]^{-1}$  of  $[\gamma]$  is represented by the loop  $\bar{\gamma}$  such that

$$\bar{\gamma}(t) = \gamma(1 - t).$$



<sup>a</sup>We say  $\gamma$  is **based** at  $x_0$ .

*Proof.* We prove that

- Associativity:  $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$ . We break this down into

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t - 2), & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t - 3), & \text{if } t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t - 1), & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



Then, we define  $\phi: I \rightarrow I$  such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \frac{1}{2}\right], & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ t + \frac{1}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right], & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{t+1}{2} \in \left[\frac{3}{4}, 1\right], & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and  $\phi(t)$  is continuous and satisfied  $\phi(0) = 0$  and  $\phi(1) = 1$ , which implies that the associativity holds.

- Identity: We want to show that  $[\gamma \cdot c] = [\gamma]$ . Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, consider  $\phi: I \rightarrow I$  such that

$$\phi(t) = \begin{cases} 2t, & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ 1, & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and  $\phi(t)$  is continuous and satisfied  $\phi(0) = 0$  and  $\phi(1) = 1$ .

- Inverses: We want to show that  $\gamma \cdot \bar{\gamma} \simeq c$ , where  $\bar{\gamma}(t) = \gamma(1-t)$ . Firstly, we have

$$(\gamma \cdot \bar{\gamma})(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2t), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider  $F_t$  given by

$$F_t(x) = \begin{cases} \gamma(2xt), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2xt), & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

If  $t = 0$ , we have

$$F_0(x) = \begin{cases} \gamma(0), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0$$

for all  $x \in I$ , namely  $F_0 = c$ , while when  $t = 1$ , we have

$$F_1(x) = \begin{cases} \gamma(2x), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2x), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \bar{\gamma})(x),$$

and we see that  $F_t$  is continuous since at  $x = \frac{1}{2}$ , we have

$$\gamma(2x) = \gamma(1) = \bar{\gamma}(0) = \bar{\gamma}(1-2x),$$

hence we see that  $F_t$  is the **homotopy** between  $\gamma \cdot \bar{\gamma}$  and  $c$ .

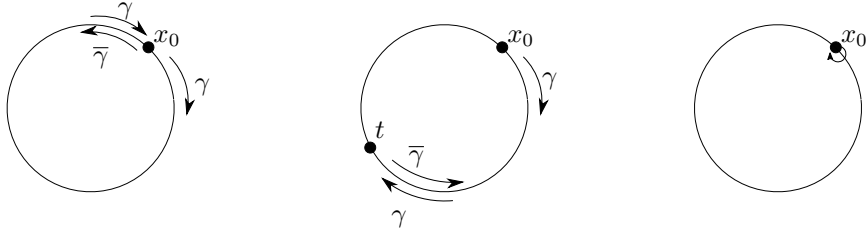


Figure 10: Illustration of  $F_t$ . Intuitively, the path  $\gamma \cdot \bar{\gamma}$  is  $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\bar{\gamma}} x_0$ . But now,  $F_t$  is  $x_0 \xrightarrow{\gamma} t \xrightarrow{\bar{\gamma}} x_0$ . We can think of this **homotopy** is *pulling back* the turning point along the original path.

■

**Theorem 2.1.** If  $X$  is **path**-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

**Remark.** We often write  $\pi_1(X)$  up to isomorphism.

*Proof.* To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if  $[h \cdot \gamma \cdot \bar{h}] = [h \cdot \gamma' \cdot \bar{h}]$ , then since we know that  $h^{-1} = \bar{h}$ , hence in the **fundamental group**  $\pi_1(X, x_0)$ , we see that

$$\bar{h} \cdot h \cdot \gamma \cdot \bar{h} \cdot h = \bar{h} \cdot h \cdot \gamma' \cdot \bar{h} \cdot h. \implies \gamma = \gamma'$$

as we desired.

- onto. We see that for every  $\alpha \in \pi_1(X, x_0)$ , there exists a  $\gamma \in \pi_1(X, x_0)$  such that

$$\gamma = \bar{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)^6$$

since  $h \cdot \gamma \cdot \bar{h} = \alpha$ .

<sup>6</sup>Notice that this is indeed the case, one can verify this by the fact that  $h: x_0 \rightarrow x_1$  and  $\bar{h}: x_1 \rightarrow x_0$ .

We then see that the **fundamental group** of  $X$  does not depend on the choice of basepoint, only on the choice of the **path** component of the basepoint. If  $X$  is **path**-connected, it now makes sense to refer to *the fundamental group* of  $X$  and write  $\pi_1(X)$  for the abstract group (up to isomorphism). ■

**Exercise.** Composition of paths is well-defined on **homotopy** classes  $\text{rel}\{0, 1\}$ .

**Exercise.** If  $X$  is a contractible space, then  $X$  is **path**-connected and  $\pi_1(X)$  is trivial.

## Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

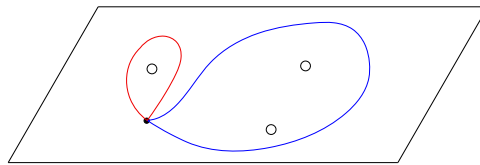


Figure 11: **Fundamental Group** is basically a *hole detector*!

### 2.2 Calculations with $\pi_1(S^n)$

Let's start with a simple theorem.

**Theorem 2.2.**  $\pi_1(S^1) \cong \mathbb{Z}$ , and this identification is given by the paths

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))].$$

**Remark.** Intuitively, this winds around  $S^1$   $n$  times. The key to this proof was to understand  $S^1$  via the covering space  $\mathbb{R} \rightarrow S^1$ . We will talk about covering spaces more later.

*Proof.*



HW

**Theorem 2.3.** Given  $(X, x_0)$  and  $(Y, y_0)$ , then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\left[ \begin{array}{l} r: I \rightarrow X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{array} \right] \mapsto (r_X, r_Y),$$

where  $\gamma$  is continuous  $\iff f_X, f_Y$  are continuous.

*Proof.* Let  $Z \xrightarrow{f} X \times Y$  with  $z \mapsto (f_X(z), f_Y(z))$ . Then we have

$$\text{continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply above to

- **Paths**  $I \rightarrow X \times Y$ .
- **Homotopies of paths**  $I \times I \rightarrow X \times Y$ .

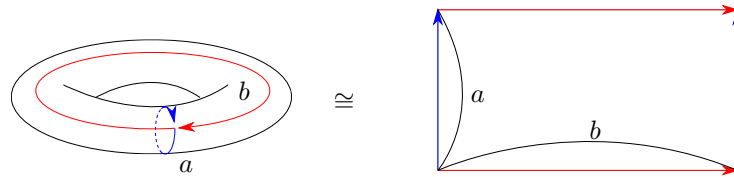
■

**Corollary 2.1.** The torus  $T \cong S^1 \times S^1$  has **fundamental group**  $\pi_1(T) \cong \mathbb{Z}^2$ . Additionally, for a  $k$ -torus  $\underbrace{S^1 \times S^1 \times \dots \times S^1}_{k \text{ times}} = (S^1)^k$ , the **fundamental group** is then  $\mathbb{Z}^k$ , i.e.

$$\pi_1((S^1)^k) \cong \mathbb{Z}^k.$$

*Proof.* Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

**Remark.** One way to think of the  $k$ -torus is as a  $k$ -dimensional cube with opposite  $(k-1)$ -dimensional faces identified by translation.

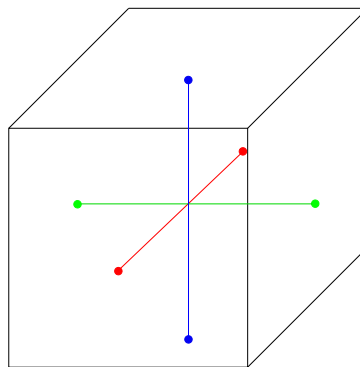


Figure 12: 3-torus with cube identified with parallel sides.

**Example.** We now see some examples.

1.  $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2.  $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong 0 \times \mathbb{Z} = \mathbb{Z}$  since

$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R},$$

which means that the generators are just loops around the hold intuitively.

**Theorem 2.4.**  $\pi_1$  is a **functor** such that

$$\begin{aligned} \pi_1 : \underline{\text{Top}}_* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0). \end{aligned}$$

A map  $f: X \rightarrow Y$  taking base point  $x_0$  to  $y_0$  induces a map

$$\begin{aligned} f_* : \pi_1(X, x_0) &\rightarrow \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

i.e.,

$$[f: X \rightarrow Y] \mapsto [f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))].$$

**Notation.** We usually write  $f_*$  if it's a **covariant functor**, while writing  $f^*$  if it's a **contravariant functor**.

*Proof.* We need to check

- well-defined on **path homotopy** classes.
- $f_*$  is a group homomorphism.

$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \frac{1}{2}\right] \\ f(\beta(1-2s)), & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

- $(\text{id}_{(X, x_0)})_* = \text{id}_{\pi_1(X, x_0)}$
- $(f_* \circ g_*) = (f \circ g)_*$

$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \implies f_*(\gamma_*(\gamma)).$$

DIY

■

## Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

The goal is to compute  $\pi_1(X)$  where  $X = A \cup B$  using the data

$$\pi_1(A), \pi_1(B), \pi_1(A \cap B).$$

We first introduce a definition.

**Definition 2.6 (Free product with amalgamation).** Given some collections of groups  $\{G_\alpha\}_\alpha$ , the *free product*, denoted by  $*_\alpha G_\alpha$  is a group such that

- Elements: **Words** in  $\{g: g \in G_\alpha \text{ for any } \alpha\}$  modulo by the equivalence relation generated by

$$wg_i g_j v \sim w(g_i g_j) v$$

when both  $g_i, g_j \in G_\alpha$ . Also, for the identity element  $\text{id} = e_\alpha \in G_\alpha$  for any  $\alpha$  such that

$$we_\alpha v \sim wv.$$

- Operation: Concatenation of **words**.

Furthermore, if two groups  $G_\alpha$  and  $G_\beta$  have a common subgroup  $S_{\{\alpha, \beta\}}$ <sup>a</sup>, given two inclusion maps<sup>b</sup>  $i_{\alpha\beta}: S_{\{\alpha, \beta\}} \rightarrow G_\alpha$  and  $i_{\beta\alpha}: S_{\{\alpha, \beta\}} \rightarrow G_\beta$ , the *free product with amalgamation*  ${}_{\alpha} *_{S} G_\alpha$  is defined as  $*_\alpha G_\alpha$  modulo the normal subgroup generated by

$$\{i_{\alpha\beta}(s_{\{\alpha, \beta\}})i_{\beta\alpha}(s_{\{\alpha, \beta\}})^{-1} \mid s_{\{\alpha, \beta\}} \in S_{\{\alpha, \beta\}}\},$$

Namely<sup>c</sup>,

$${}_{\alpha} *_{S} G_\alpha = {}_\alpha G_\alpha / \langle i_{\alpha\beta}(s_{\{\alpha, \beta\}})i_{\beta\alpha}(s_{\{\alpha, \beta\}})^{-1} \rangle$$

and satisfies the universal property

$$\begin{array}{ccc} S & \xrightarrow{i_{\alpha\beta}} & G_\alpha \\ i_{\beta\alpha} \downarrow & & \downarrow \\ G_\beta & \longrightarrow & G_\alpha *_{S} G_\beta \\ & \searrow & \downarrow \text{ } \exists! \\ & & X \end{array}$$

<sup>a</sup>In general, we don't need  $S_{\{\alpha, \beta\}}$  to be a subgroup.

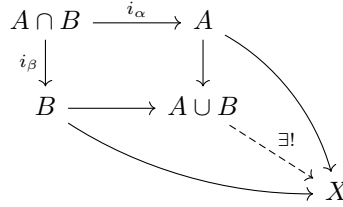
<sup>b</sup>We don't actually need  $i_{\alpha\beta}, i_{\beta\alpha}$  to be inclusive as well.

<sup>c</sup>i.e.,  $i_{\alpha\beta}(s)$  and  $i_{\beta\alpha}(s)$  will be identified in the quotient.

**Remark.** We see that

- We can then write out **words** such as  $g_\alpha \cdot s \cdot g_\beta$  for  $s \in S$ , and view  $s$  as an element of  $G_\alpha$  or  $G_\beta$ . In fact, we can do this construction even when  $i_\alpha$  and  $i_\beta$  are not injective, though this means we are not working with a subgroup.

- Aside, in Top, the same universal property defines union



for  $A, B$  are open subsets and the inclusion of intersection.

**Theorem 2.5 (Seifert-Van Kampen Theorem).** Given  $(X, x_0)$  such that  $X = \bigcup_{\alpha} A_{\alpha}$  with

- $A_{\alpha}$  are open and **path**-connected and  $\forall \alpha \ x_0 \in A_{\alpha}$
- $A_{\alpha} \cap A_{\beta}$  is **path**-connected for all  $\alpha, \beta$ .

Then there exists a surjective group homomorphism

$$*_\alpha: \pi_1(A_{\alpha}, x_0) \rightarrow \pi_1(X, x_0).$$

If we additionally have  $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$  where they are all **path**-connected for every  $\alpha, \beta, \gamma$ , then

$$\pi_1(X, x_0) \cong *_\alpha \pi_1(A_{\alpha} \cap A_{\beta}, x_0) \pi_1(A_{\alpha}, x_0)$$

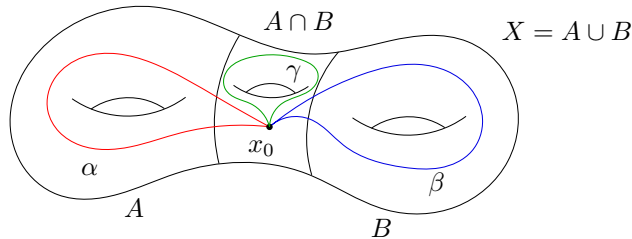
associated to all maps  $\pi_{\alpha}(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha}), \pi_1(A_{\beta})$  induced by inclusions of spaces. i.e.,  $\pi_1(X, x_0)$  is a quotient of the **free product**  $*_{\alpha} \pi_1(A_{\alpha})$  where we have

$$(i_{\alpha\beta})_*: \pi_1(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha} + A_{\beta})$$

which is induced by the inclusion  $i_{\alpha\beta}: A_{\alpha} \cap A_{\beta} \rightarrow A_{\alpha}$ . We then take the quotient by the normal subgroup generated by

$$\{(i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_{\alpha} \cap A_{\beta})\}.$$

**Example.** We first see a great visualization of the [Theorem 2.5](#).



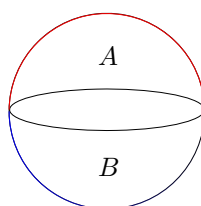
Intuitively we see the [fundamental group](#) of  $X$ , which is built by gluing  $A$  and  $B$  along their intersection. As the [fundamental group](#) of  $A$  and  $B$  glued along the [fundamental group](#) of their intersection. In essence,  $\pi_1(X, x_0)$  is the quotient of  $\pi_1(A) * \pi_1(B)$  by relations to impose the condition that loops like  $\gamma$  lying in  $A \cap B$  can be viewed as elements of either  $\pi_1(A)$  or  $\pi_1(B)$ .

## Lecture 11: Group Presentations

31 Jan. 10:00

**Example.** We now see some applications of [Theorem 2.5](#).

1. We can use [Seifert Van Kampen Theorem](#) to compute the [fundamental group](#) of  $S^2$ . We see that



We see that  $\pi_1(S^2)$  must be a quotient of  $\pi_1(A) * \pi_1(B)$ , but since  $A, B \simeq D^2$ , we know that  $\pi_1(A)$  and  $\pi_1(B)$  are both zero groups, thus  $\pi_1(A) * \pi_1(B)$  is the zero group, and  $\pi_1(S^2)$  is also the zero group.

**Remark.** Note that the inclusion of  $A \cap B \rightarrow A$  induces the zero map  $\pi_1(A \cap B) \rightarrow \pi_1(A)$ , which cannot be an injection. In fact, we know that  $\pi_1(A \cap B) \cong \mathbb{Z}$  since  $A \cap B \simeq S^1$ .

2. In the case of torus, consider the following.

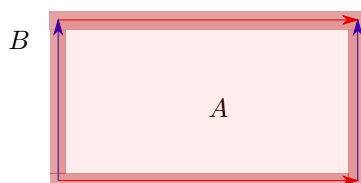


Figure 13:  $A$  is the interior, while  $B$  is the neighborhood of the boundary.

Now note that  $A \simeq D^2$  and  $B \simeq S^1 \vee S^1$ , and since it's a thickening of the two loops around the torus in both ways, this suggests the question of how do we find  $\pi_1(B)$ ? We grab a bit of knowledge from [Seifert Van Kampen Theorem](#) before we continue.

**Exercise.** Suppose we have [path](#)-connected spaces  $(X_\alpha, x_\alpha)$ , and we take their wedge sum  $\bigvee_\alpha X_\alpha$  by identifying the points  $x_\alpha$  to a single point  $x$ .

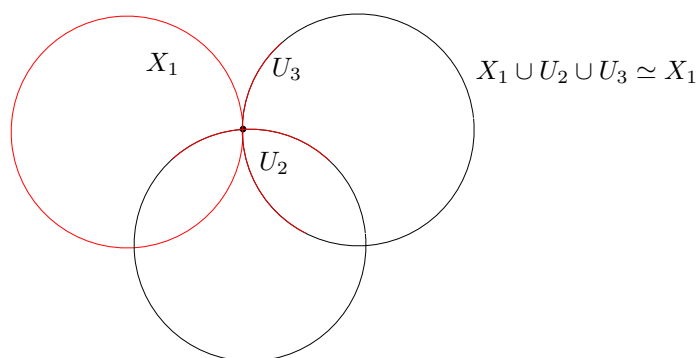


We also suppose a mild condition for all  $\alpha$ , the point  $x_\alpha$  is a [deformation retract](#) of some neighborhood of  $x_\alpha$ .

For example, this doesn't work if we choose the *bad point* on the Hawaiian earring. Then we can use [Seifert Van Kampen Theorem](#) to show that

$$\pi_1 \left( \bigvee_{\alpha} X_{\alpha}, x \right) \cong {}_{\alpha} \pi_1 (X_{\alpha}, x_{\alpha}).$$

*Proof.* If we denote



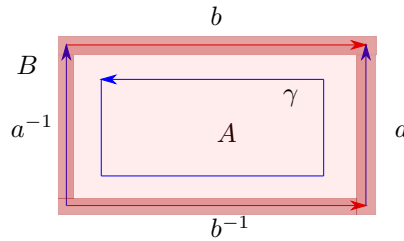
as  $C_n$ , then  $\pi_1(C_n) \cong F_n$ . Then we apply [Theorem 2.5](#) to  $A_{\alpha} = X_{\alpha} \cup_{\beta} U_{\beta}$ . Specifically, take  $A_{\alpha} = X_{\alpha} \cup_{\beta} U_{\beta} \simeq X_{\alpha}$ , where  $U_{\beta}$  is a neighborhood of  $x_{\beta}$  which [deformation retracts](#) to  $x_{\beta}$ . This makes  $A_{\alpha}$  open as desired. ■

**Corollary 2.2.** The wedge sum of circles  $\pi_1(\bigvee_{\alpha \in A} S^1) = {}_{\alpha} \pi_1 \mathbb{Z}$  is a [free group](#) on  $A$ . In particular, when  $A$  is finite, the [fundamental group](#) of a bouquet of circles is the [free group](#) on  $|A|$ .

Returning to the [example of torus](#), we see that

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$

Further, we know that  $\pi_1(A \cap B) \rightarrow \pi_1(A)$  is the zero map. We need to understand  $\pi_1(A \cap B) \rightarrow \pi_1(B)$ . To do so we need to understand how we're able to identify  $\pi_1(S^1 \vee S^1)$  with  $F_2$  and how we identify  $\pi_1(S^1)$  with  $\mathbb{Z}$ . We update our [Figure 13](#) to talk about this.



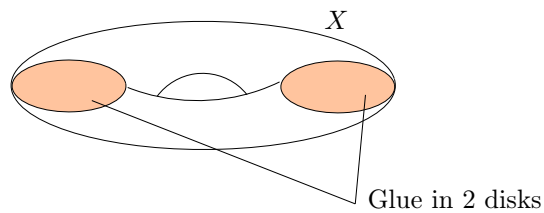
From this, we have

$$\pi_1(A \cap B) \rightarrow \pi_1(B) \cong F_{a,b}, \quad \gamma \mapsto aba^{-1}b^{-1}.$$

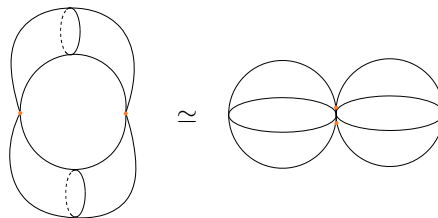
By [Seifert Van Kampen Theorem](#), we identify the image of \$\gamma\$ in \$\pi\_1(B)[aba^{-1}b^{-1}]\$ with its image in \$\pi\_1(A)\$, which is just trivial. Therefore, we have

$$\pi_1(T^2) = F_{a,b} / \langle aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2.$$

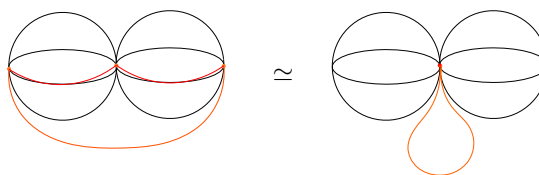
- Let's see the last example which illustrate the power of [Seifert Van Kampen Theorem](#). Start with a torus, and we glue in two disks into the hollow inside.



We'll call this space \$X\$, and our goal is to find \$\pi\_1(X)\$. We can place a [CW complex](#) structure on this space so that each disk is a [subcomplex](#). Then, we take quotient of each disk to a point without changing the [homotopy type](#), hence \$X\$ is [homotopy](#) to



By the same property, we can expand one of those points into an interval, and then contract the red [path](#) as follows.



This is exactly  $S^2 \vee S^2 \vee S^1$ . With [Seifert Van Kampen Theorem](#), we have

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}.$$

**Exercise.** Consider  $\mathbb{R}^2 \setminus \{x_1, \dots, x_n\}$ , that is the plane punctured at  $n$  points. Then  $X \simeq \bigvee_n S^1$ , so then

$$\pi_1(X) \simeq F_n.$$

One way to do this is to convince yourself that you can do a [deformation retract](#) the plane onto the following [wedge](#).

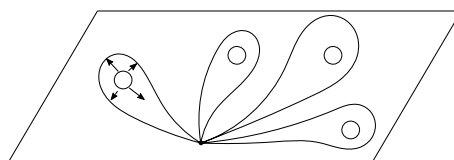


Figure 14: [Deformation retract](#)  $X$  onto wedge.

## 2.3 Group Presentation

In order to go further, we introduce the concept of *group presentation*.

**Definition 2.7 (Group presentation).** A *presentation*  $\langle S \mid R \rangle$  of a group  $G$  is

- $S$ : set of *generators*
- $R$ : set of *relators* ([words](#) in a generator and inverses)

such that

$$G \cong F_S / \langle R \rangle,$$

where  $\langle R \rangle$  is a subgroup normally generated by the elements of  $R$ .

Notice that  $\langle S \mid R \rangle$  is [finite](#) if  $S, R$  are, and  $G$  is *finitely presented* if there exists a *finite presentation*.

**Note.** One way to think about whether  $G$  is [finitely presented](#) is that if  $r$  is a [word](#) in  $R$  then  $r = 1$ , where 1 is the identity of  $G$ .

**Example.** We see that

1.  $F_2 = \langle a, b \mid \rangle$
2.  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$
3.  $\mathbb{Z} / 3\mathbb{Z} = \langle a \mid a^3 \rangle$
4.  $S_3 = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$

**Theorem 2.6.** Any group  $G$  has a [presentation](#).

*Proof.* We first choose a generating set  $S$  for  $G$ . Notice that we can even choose  $S = G$  directly. From the universal property of free group, we see that there exists a surjective map  $\varphi: F_S \rightarrow G, s \mapsto s$ . Now, let  $R$  be the generating set for  $\ker(\varphi)$ , by the first isomorphism theorem<sup>7</sup>,  $G \cong F_S / \ker \varphi$ . In fact, we have  $G = \langle S \mid R \rangle$ . ■

---

**Remark.** The advantages are that given  $G = \langle S \mid R \rangle$ , it's now easy to define a homomorphism  $\psi: G \rightarrow H$  given a map  $\varphi: S \rightarrow H$ ,  $\psi$  extends to a group homomorphism  $G \rightarrow H$  if and only if  $\psi$  vanishes on  $R$ , i.e.,  $\phi(r) = 0$  for all  $r \in R$ . We see an example to illustrate this.

**Example.** If we have  $G = \langle a, b \mid aba \rangle$ , a map  $\varphi: \{a, b\} \rightarrow H$  gives a group homomorphism if and only if

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H.$$

This essentially uses the universal property of quotients.

---

**Remark.** It's sometimes easy to calculate  $G^{\text{Ab}}$

$$G^{\text{Ab}} = \langle S \mid R, \text{commutators in } S \rangle.$$

**Example.** Suppose all relations in  $R$  are commutators, so  $R \subseteq [G, G]$ . Then,

$$G^{\text{Ab}} = (F_S)^{\text{Ab}} = \bigoplus_S \mathbb{Z}.$$

**Remark.** The disadvantages are that, the computationally **very difficult**.

---

**Example.** Given  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ , let

$$\psi: \{a, b\} \rightarrow H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1_H \in H.$$

Namely, this is a [presentation](#) of the trivial group, but this is entirely unclear.

---

<sup>7</sup>[https://en.wikipedia.org/wiki/Isomorphism\\_theorems](https://en.wikipedia.org/wiki/Isomorphism_theorems)

## Lecture 12: Presentations for $\pi_1$ of CW Complexes

2 Feb. 10:00

Let's first see an exercise.

**Exercise.** Consider  $G_1 = \langle S_1 \mid R_1 \rangle$  and  $G_2 = \langle S_2 \mid R_2 \rangle$ . Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\} \rangle$
- $G_1 *_H G_2$  where  $f_1: H \rightarrow G_1$  and  $f_2: H \rightarrow G_2$ . Then we have

$$G_1 *_H G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{f_1(h)f_2(h)^{-1} \mid h \in H\} \rangle.$$

### 2.4 Presentations for $\pi_1$ of CW Complexes

For  $X$  a **CW complex**, we have

1. A 1-dimensional **CW complex** has free  $\pi_1$  (call its generators as  $a_1, \dots, a_n$ ).
2. Gluing a 2-disk by its boundary along a word  $w$  in the generators *kills*  $w$  in  $\pi_1$ . We then get a **presentation** for  $\pi_1(X^2)$  given by

$$\langle a_1, \dots, a_n \mid w \text{ for each 2-cell in } X_2 \rangle.$$

3. Gluing in any higher dimensional cells along their boundary will not change  $\pi_1$ . That is, in a **CW complex**, we have  $\pi_1(X) = \pi_1(X^2)$ .

**Remark.** We can write the above more precise.

1. Find free generators  $\{a_i\}_{i \in I}$  for  $\pi_1(X^1)$ .
2. For each 2-disk  $D_\alpha^2$ , write attaching map as word  $w_\alpha$  in  $a_i$ . i.e.,

$$\pi_1(X^2) = \langle a_i \mid w_\alpha \rangle.$$

3.  $\pi_1(X) = \pi_1(X^2)$ .

**Example.** Given  $G = \mathbb{Z}/n\mathbb{Z} = \langle a, a^n \rangle$ , then we take a loop and then wind a 2-disk around the loop  $a$  for  $n$  times.

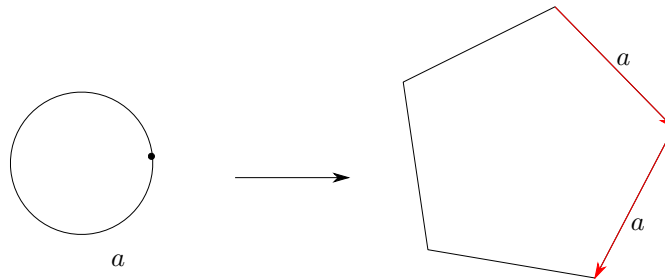


Figure 15: For  $G = \mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$ , we wind the boundary around  $a$  for  $n$  times.

We then see that given a group  $G$  with [presentation](#)  $\langle S \mid R \rangle$ , one can construct a 2-dimensional [CW complex](#) with  $\pi_1 = G$  by

- Set  $X^1 = \bigvee_{s \in S} S^1$
- For each relation  $r \in R$ , glue in a 2-disk along loops specified by the [word](#)  $r$ .

Every group is then  $\pi_1$  of some space.

**Theorem 2.7.** If  $X$  is a [CW complex](#) and  $\iota_1: X^1 \hookrightarrow X$  and  $\iota: X^2 \hookrightarrow X$ , then  $(\iota_1)_*$  surjects onto  $\pi_1$  and  $(\iota_2)_*$  is an isomorphism on  $\pi_1$ .

*Proof.*

HW

**Definition 2.8.** (Graph, subgraph, tree, maximal tree) We import some topological definitions of graph theoretic concepts.

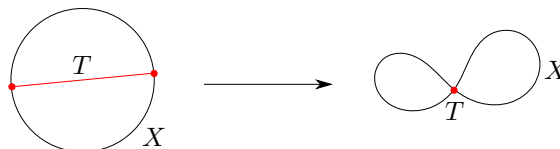
- A *graph* is a 1-dimensional [CW complex](#).
- A *subgraph* is a [subcomplex](#).
- A *tree* is a contractible [graph](#).
- A [tree](#) in [graph](#)  $X$  (necessarily a [subgraph](#)) is *maximal* or *spanning* if it contains all the vertices.

**Theorem 2.8.** Every connected [graph](#) has a [maximal tree](#). Every [tree](#) is contained in a [maximal tree](#).

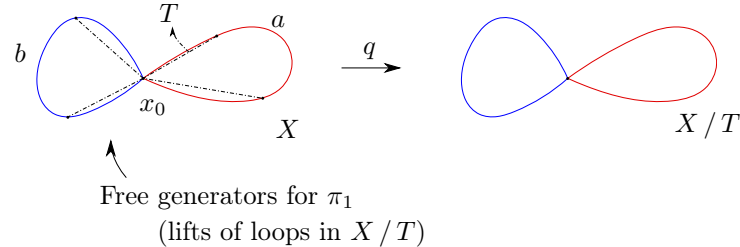
**Corollary 2.3.** Suppose  $X$  is a connected [graph](#) with basepoint  $x_0$ . Then  $\pi_1(X, x_0)$  is a [free group](#).

Furthermore, we can give a [presentation](#) for  $\pi_1(X, x_0)$  by finding a [spanning tree](#)  $T$  in  $X$ . The generators of  $\pi_1$  will be indexed by cells  $e_\alpha \in X - T$ , and  $e_\alpha$  will correspond to a loop that passes through  $T$ , traverses  $e_\alpha$  once, then returns to the basepoint  $x_0$  through  $T$ .

*Proof.* The idea is simple.  $X$  is [homotopy equivalent](#) to  $X/T$  via previous work on the homework,  $T$  contains all the vertices, so the quotient has a single vertex. Thus, it is a [wedge](#) of circles, and each  $e_\alpha$  projects to a loop in  $X/T$ .



The current plan is to calculate the **fundamental group** of **CW complexes**. For now, we need to see that the **fundamental group** of a 1-skeleton (a graph) can be found by taking a **maximal tree**, and then quotienting out the space by the **tree** to get a **wedge** of circles.



We now prove that the **maximal trees** exist. Recall that  $X$  is a quotient of

$$X^0 \coprod_{\alpha} I_{\alpha}.$$

Each subset  $U$  is open if and only if it intersects each edge  $\bar{e}_{\alpha}$  in an open subset. A map  $X \rightarrow Y$  if and only if its restriction to each edge  $\bar{e}_{\alpha}$  is continuous. Now, take  $X_0$  to be a **subgraph**. Our goal is to construct a **subgraph**  $Y$  with

- $X_0 \subset Y \subset X$
- $Y$  **deformation retracts** to  $X_0$
- $Y$  contains all vertices of  $X$ .

So if we take  $X_0$  to be a vertex, then  $Y$  is our **tree** and we're done!

Our strategy now is to build a sequence  $X_0 \subset X_1 \subset \dots$  and correspondingly,  $Y_0 \subset Y_1 \subset \dots$ . We start with  $X_0$  and inductively define

$$X_i := X_{i-1} \cup \text{all edges } \bar{e}_{\alpha} \text{ with one or both vertices in } X_{i-1}.$$

We then see that  $X = \bigcup_i X_i$ .<sup>8</sup> Now, let  $Y_0 = X_0$ . By induction, we'll assume that  $Y_i$  is a **subgraph** of  $X_i$  such that

- $Y_i$  contains all vertices of  $X_i$ .
- $Y_i$  **deformation retracts** to  $Y_{i-1}$ .

We can then construct  $Y_{i+1}$  by taking  $Y_i$  and adding to it one edge to adjoin every vertex of  $X_{i+1}$ , namely

$$Y_{i+1} := Y_i \cup \text{one edge to adjoin every vertex of } X_i$$

We then see that  $Y_{i+1}$  **deformation retracts** to  $Y_i$  by just smashing down each edge. Now, we can show that  $Y$  **deformation retracts** to  $Y_0 = X_0$  by performing the **deformation retraction** from  $Y_i$  to  $Y_{i-1}$  during the time interval  $[1/2^i, 1/2^{i-1}]$ . ■

<sup>8</sup>[HPM02] do this by arguing the union on the right is both open and closed.

<sup>9</sup>This is possible if we assume Axiom of Choice.

---

**Example.** Let

- $S^n$ : decompose into 2 open disks
- $A_1$ : neighborhood of top hemisphere
- $A_2$ : neighborhood of lower hemisphere

We see that  $A_1 \cap A_2 \simeq S^{n-1}$ , where we need  $n \geq 2$  to let  $S^{n-1}$  be connected. We then have

$$\pi_1(S^n) \cong 0 \underset{\pi_1(A_1 \cap A_2)}{*} 0 = 0.$$

On the other hand, if  $n \geq 3$ , then we see that

$$S^n = D^n \cup * \Big/ \sim.$$

Since 2-skeleton is a point, thus  $\pi_1(S^n) = 0$ .

## Lecture 13

4 Feb. 10:00



---

## Appendix

---

## References

- [HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: <https://books.google.com/books?id=BjKs86kosqC>.