

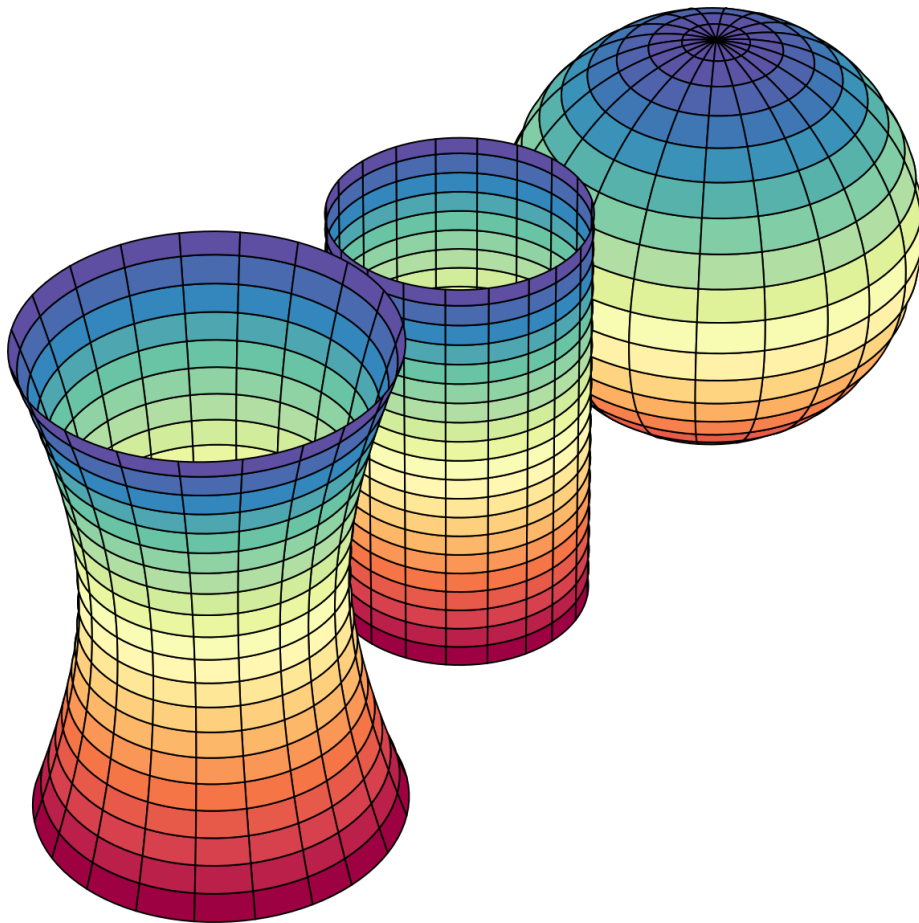
MATH635  
Riemannian Geometry

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## Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by [Lydia Bieri](#). Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [[FC13](#)] as our reference; while not required, but highly recommended have on.



This course is taken in Winter 2023, and the date on the covering page is the last updated time.

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# Chapter 1

## Smooth Manifolds

### Lecture 1: A Foray to Smooth Manifolds

#### 1.1 Topological Manifolds

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Let's start with a common definition.

**Definition 1.1.1 (Topological manifold).** A *topological manifold*  $\mathcal{M}$  of dimension  $n$  is a (topological) Hausdorff space such that each point  $p \in \mathcal{M}$  has a neighborhood  $U$  homeomorphic via  $\varphi: U \rightarrow U'$  to an open subset  $U' \subseteq \mathbb{R}^n$ .

**Definition 1.1.2 (Local coordinate map).** For every  $p \in \mathcal{M}$ , the corresponding homeomorphism  $\varphi$  is called the *local coordinate map*.

**Definition 1.1.3 (Local coordinate).** The pull-back  $(x^1, \dots, x^n)$  of the *local coordinate map*  $\varphi$  from  $\mathbb{R}^n$  is called the *local coordinates* on  $U$ , given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

**Definition 1.1.4 (Coordinate chart).** The pair  $(U, \varphi)$  is called a (*coordinate*) *chart* on  $\mathcal{M}$ .

In other words, a *topological manifold* can be thought of as a space such that it looks like  $\mathbb{R}^n$  locally.



**Definition 1.1.5 (Atlas).** An *atlas*  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$  for a *manifold*  $\mathcal{M}$  is a collection of *charts* such that  $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$  are an open covering of  $\mathcal{M}$ , i.e.,  $\mathcal{M} = \bigcup_\alpha U_\alpha$ .

In other words, for all  $p \in \mathcal{M}$ , there exists a neighborhood  $U \subseteq \mathcal{M}$  and homeomorphism  $h: U \rightarrow U' \subseteq \mathbb{R}^n$  open.

**Definition 1.1.6 (Locally finite).** An *atlas* is said to be *locally finite* if each point  $p \in \mathcal{M}$  is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as  $\mathbb{R}^n$ , there's no clear way to make sense of differentiability of a *manifold*. But thankfully, we now have an explicit relation to the ambient space  $\mathbb{R}^n$  via  $\varphi_\alpha$ . To formalize, let  $\mathcal{A}$  be an *atlas* for a *manifold*  $\mathcal{M}$ , and assume that  $(U_1, \varphi_1), (U_2, \varphi_2)$  are 2 elements

of  $\mathcal{A}$ . Then clearly, the map  $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$  is a homeomorphism between 2 open sets of Euclidean spaces since both  $\varphi_1$  and  $\varphi_2$  are homeomorphism. Due to this map's importance, it has its own name.

**Definition 1.1.7 (Coordinate transition).** The map  $\varphi_2 \circ \varphi_1^{-1}$  is called the *coordinate transition* of  $\mathcal{A}$  for the pair of charts  $(U_1, \varphi_1), (U_2, \varphi_2)$ .



## 1.2 Differentiable Manifolds

Notice that the *coordinate transitions* are from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ ; hence differentiability makes sense now, which induces the following.

**Definition 1.2.1 (Differentiable atlas).** The atlas  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$  is *differentiable* if all *transitions* are differentiable.

**Remark.** Here, the differentiability depends on the content. Sometimes, we may want it to be  $C^\infty$ , and sometimes may be  $C^k$  for some finite  $k$ . On the other hand, smooth always refers to  $C^\infty$ . We'll use them interchangeably if it's clear which case we're referring to.

**Definition 1.2.2 (Equivalence atlas).** Two atlases  $\mathcal{U}, \mathcal{V}$  of a manifold are equivalent if for every  $(U, \varphi) \in \mathcal{U}, (V, \psi) \in \mathcal{V}$ ,

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

**Notation (Smoothly compatible).** Two charts  $(U, \varphi)$  and  $(V, \psi)$  are *smoothly compatible* if either  $U \cap V = \emptyset$  or  $\psi \circ \varphi^{-1}$  is a diffeomorphism.

This suggests the following.

**Definition 1.2.3 (Smooth structure).** A *smooth structure* on  $\mathcal{M}$  is an equivalence class  $\mathcal{U}$  of *coordinate atlas* with the property that all *transition functions* are diffeomorphisms.

**Remark.** We can also use the *maximal differentiable atlas* to be our differentiable structure.

**Definition 1.2.4 (Smooth manifold).** A *smooth manifold* is a manifold  $\mathcal{M}$  with a *smooth structure*.

In this way, we can do calculus on *smooth manifolds*! Furthermore, it now makes sense to say that a function  $f : \mathcal{M} \rightarrow \mathbb{R}$  is differentiable (or  $C^\infty$ ) by considering differentiability of  $f \circ \varphi^{-1}$  around  $p$ .

**Notation.** The collection of smooth functions on [smooth manifold](#)  $\mathcal{M}$  is denoted by  $C^\infty(\mathcal{M}, \mathbb{R})$ , or  $C^k(\mathcal{M}, \mathbb{R})$ .

**Remark.** The class  $C^\infty(\mathcal{M}, \mathbb{R})$  consists of functions with property is well-defined.

**Proof.** Let  $\mathcal{A}$  be any given [atlas](#) from [equivalence class](#) that defines the [smooth structure](#), and as we have shown, if  $(U, \varphi) \in \mathcal{A}$ , then  $f \circ \varphi^{-1}$  is smooth on  $\mathbb{R}^n$ . This requirement defines the same set of smooth functions no matter the choice of representative [atlas](#) by the nature of [Definition 1.2.2](#) requirement that defines the equivalent [manifolds](#).  $\circledast$

### 1.2.1 Orientation

Another essential property of a [manifold](#) is its orientability.

**Definition.** Consider an [atlas](#)  $\mathcal{A}$  for a [differentiable manifold](#)  $\mathcal{M}$ .

**Definition 1.2.5 (Oriented).**  $\mathcal{A}$  is *oriented* if all [transitions](#) have positive functional determinant.

**Definition 1.2.6 (Orientable).**  $\mathcal{M}$  is *orientable* if  $\mathcal{A}$  is an [oriented atlas](#).

Motivated by the above definitions, we see that we can actually use an [atlas](#) to define an [orientation](#).

**Definition 1.2.7 (Orientation).** Let  $\mathcal{M}$  be an [orientable manifold](#). Then a [oriented differentiable structure](#) is called an *orientation* of  $\mathcal{M}$ .

If  $\mathcal{M}$  possesses an [orientation](#), we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with [Definition 1.2.5](#).

**Remark.** Two [differentiable structures](#) obeying [Definition 1.2.5](#) determine the same [orientation](#) if the union again satisfying [Definition 1.2.5](#).

**Remark.** If  $\mathcal{M}$  is [orientable](#) and connected, then there exists exactly 2 distinct [orientations](#) on  $\mathcal{M}$ .

Now, we can see some examples of [smooth manifolds](#).

**Example (Sphere).** The sphere  $S^n \subseteq \mathbb{R}^{n+1}$  given by

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Consider  $U_i^+ = \{x \in S^n \mid x_i > 0\}$ ,  $U_i^- = \{x \in S^n \mid x_i < 0\}$  for  $i = 1, \dots, n+1$ , and  $h_i^\pm: U_i^\pm \rightarrow \mathbb{R}^n$  such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1}).$$

Note that the minimum [charts](#) needed to cover  $S^n$  is 2.

**Example.** Let  $\mathcal{M} = U \subseteq \mathbb{R}^n$ , then  $\{(U, \varphi)\}$  is a [smooth structure](#) with  $\varphi = \text{id}$ .

**Example.** Open sets of  $C^\infty$ -[manifolds](#) are  $C^\infty$ -[manifolds](#).

**Example (General linear group).**  $\text{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ , open.

**Example (Real projective space).**  $\mathbb{R}P^n = S^n / \sim$  where  $x \sim -x$  with  $\pi: S^n \rightarrow \mathbb{R}P^n$ ,  $x \mapsto [x]$ .

**Proof.**  $\pi$  is a homeomorphism on each  $U_i^+$  for  $i = 1, \dots, n+1$ , with

$$\{(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1}), i = 1, \dots, n+1\}$$

is a  $C^\infty$ -atlas for  $\mathbb{R}P^n$ . \*

**Note.** Observe that  $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ .

## Lecture 2: Maps Between Smooth Manifolds

### 1.2.2 Smooth Maps

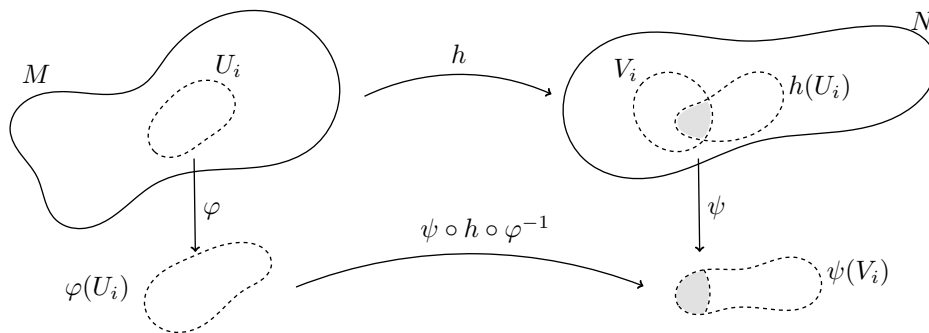
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We can now consider the maps between manifolds, specifically, the smooth manifolds.

**Definition 1.2.8 (Smooth function).** Let  $M, N$  be two smooth manifolds, and let  $\mathcal{U}$  be locally finite atlas from the equivalence class that gives the smooth structure on  $M$ , and let  $\mathcal{V}$  be the corresponding for  $N$ . A map  $h: M \rightarrow N$  is said to be *smooth* if each map in the collection

$$\{\psi \circ h \circ \varphi^{-1} : h(U) \cap V \neq \emptyset\},$$

where  $(U, \varphi) \in \mathcal{U}$ ,  $(V, \psi) \in \mathcal{V}$  is  $C^\infty$ -differentiable as a map from one Euclidean space to another.



**Remark.** Equivalence relation guarantees that Definition 1.2.8 depends only on the smooth structure of  $M, N$ , but not on the chosen representative coordinate atlas.

**Definition.** Consider two smooth manifolds  $M, N$  and a smooth homeomorphism  $h: M \rightarrow N$  with smooth inverse.

**Definition 1.2.9 (Diffeomorphic).** The two manifolds  $M, N$  are said to be *diffeomorphic*.

**Definition 1.2.10 (Diffeomorphism).** The map  $h$  is said to be a *diffeomorphism*.

Let  $M_1, M_2$  be two smooth manifolds, and let  $\varphi: M_1 \rightarrow M_2$  be a diffeomorphism. Then the following hold.

- $M_1$  is orientable if and only if  $M_2$  is orientable.
- If in addition,  $M_1$  and  $M_2$  are both connected and oriented, then  $\varphi$  induces an orientation on  $M_2$  that may or may not coincide with the initial orientation of  $M_2$ .

If the induced orientation coincides, then we say  $\varphi$  preserves the orientation, otherwise  $\varphi$  reverses the orientation.

Check

### 1.2.3 Grassmannian Manifold

Before proceeding, let's consider an interesting [smooth manifold](#).

**Definition 1.2.11** (Grassmannian manifold). Given  $m, n \in \mathbb{N}$ , the so-called *Grassmannian manifold*  $G(n, m)$  is the set of all  $n$ -dimensional subspaces of  $\mathbb{R}^{n+m}$ .

**Note.**  $G(1, m)$  is just  $\mathbb{R}P^m$ , and  $G(0, m)$ ,  $G(n, 0)$  are one-point sets.

As we will soon see,  $G(n, m)$  has the [smooth structure](#) of an  $mn$ -dimensional [manifold](#).

**Intuition.** We obtain the [structure](#) by exhibiting an [atlas](#) whose [transitions](#) are [diffeomorphisms](#).

Firstly, we give  $G(n, m)$  a suitable topology, i.e., the metric topology. Let  $\Pi \in G(n, m)$ , and let  $\mathcal{L}(\Pi, \Pi^\perp)$  denote the  $mn$ -dimensional space of linear maps from  $\Pi$  to  $\Pi^\perp$ . Define the map

$$\varphi_\Pi: \mathcal{L}(\Pi, \Pi^\perp) \rightarrow G(n, m), \quad \varphi_\Pi(\alpha) = (\mathbb{1}_\Pi \oplus \alpha)(\Pi)$$

where  $\mathbb{1}_\Pi \oplus \alpha$  is regarded as a map  $\Pi \rightarrow \Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$ .<sup>1</sup> Clearly,  $\varphi_\Pi$  is injective, and thus,  $(\mathcal{L}(\Pi, \Pi^\perp), \varphi_\Pi)$  is an  $mn$ -dimensional [chart](#) of  $G(n, m)$ .

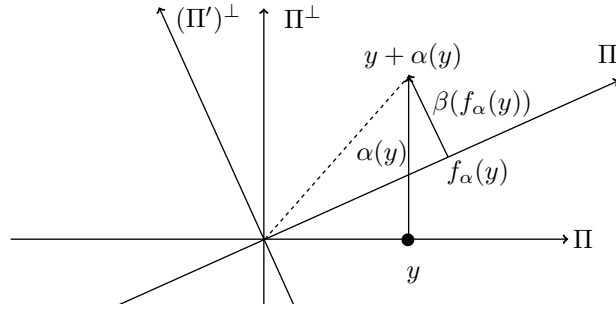
**Remark.** The images  $\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$  cover  $G(n, m)$ .

**Example.**  $\Pi = \varphi_\Pi(0) \in \varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$ .

We can now prove that these [charts](#) are mutually [compatible](#). Let  $\Pi, \Pi' \in G(n, m)$ , and let  $P, P'$  be orthogonal projections from  $\mathbb{R}^{n+m}$  onto  $\Pi, \Pi'$  respectively. Firstly,

$$F = \varphi_{\Pi'}^{-1} \varphi_\Pi: \varphi_\Pi^{-1}(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))) \rightarrow \varphi_{\Pi'}^{-1}(\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp)))$$

is smooth.



Consider  $\alpha \in \mathcal{L}(\Pi, \Pi^\perp)$ , and  $\beta \in \mathcal{L}(\Pi', (\Pi')^\perp)$ , then for  $\alpha, \beta$ , the equality  $F(\alpha) = \beta$  means that  $\varphi_\Pi(\alpha) = \varphi_{\Pi'}(\beta)$ . Let  $f_\alpha: \Pi \rightarrow \Pi'$  be defined by

$$f_\alpha = P' \circ (\mathbb{1}_\Pi \oplus \alpha).$$

We need to check

- (a)  $f_\alpha$  is invertible, and
- (b)  $\forall y \in \Pi, y + \alpha(y) = f_\alpha(y) + \beta(f_\alpha(y))$ .

**Note.** The condition that  $\det f_\alpha \neq 0$  gives an exact description of the subset

$$\varphi_{\Pi^{-1}}(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp)))$$

<sup>1</sup>In other words,  $\varphi_\Pi(\alpha)$  is the graph of  $\alpha$  in  $\Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$ .



of  $\mathcal{L}(\Pi, \Pi^\perp)$ , which is therefore open.

For  $\beta$ , it is  $(\mathbb{1}_{\Pi'} \oplus \beta) \circ f_\alpha = \mathbb{1}_\Pi \oplus \alpha$ , and hence

$$\beta = F(\alpha) = (\mathbb{1}_\Pi \oplus \alpha) \circ f_\alpha^{-1} - \mathbb{1}_{\Pi'}.$$

It follows by the construction that the image of  $\beta$  is contained in  $(\Pi')^\perp$ .

**Remark.** We obtain an infinite atlas for  $G(n, m)$  with charts labeled by  $\Pi \in G(n, m)$ . But it's suffices to consider only  $\binom{n+m}{n}$  charts corresponding to subspaces  $\Pi$  spanned with  $n$  coordinate axes.

### 1.3 Manifolds with Boundaries

We first introduce two notions.

**Definition 1.3.1** (Closed manifold). A manifold is *closed* if it is compact and without boundary.

**Definition 1.3.2** (Open manifold). A manifold is *open* if it has only non-compact components without boundary.

**Lemma 1.3.1.** If  $M$  can be covered by two coordinate neighborhoods  $V_1, V_2$  such that  $V_1 \cap V_2$  is connected, then  $M$  is *orientable*.

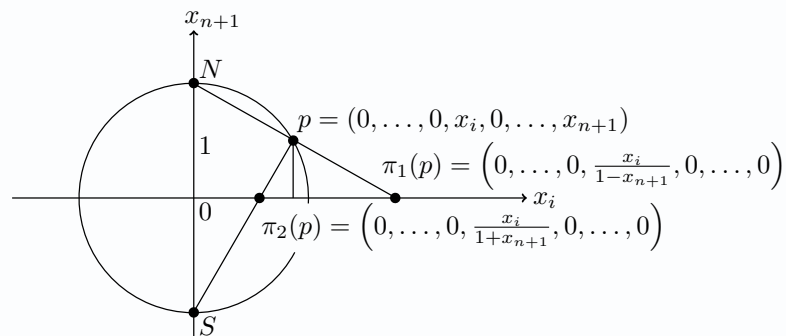
**Proof.** The determinant of the differential of the coordinate change  $\neq 0$ , so it does not change sign in  $V_1 \cap V_2$ . If it's negative at a single point, it's enough to change the sign of one of the coordinates to make it positive at that point, hence on  $V_1 \cap V_2$ . ■

**Example.** Let  $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$  is *orientable*.

**Proof.** Let  $N = (0, \dots, 0, 1)$  and  $S = (0, \dots, 0, -1)$ , consider given  $p = (0, \dots, 0, x_i, 0, \dots, x_{n+1})$ , then  $\pi_1: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  given by

$$\pi_1(p) = \left(0, \dots, 0, \frac{x_i}{1 - x_{n+1}}, 0, \dots, 0\right)$$

to be the stereographic projection from the north pole  $N$ .



More generally, it takes  $p(x_1, \dots, x_{n+1}) \in S^n - \{N\}$  into the intersection at the hyperplane  $x_{n+1} = 0$  with the line passing through  $p$  and  $N$ . In this way, we have

$$\pi_1(x_1, \dots, x_n) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right),$$

hence  $\pi_1: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  is differentiable, and is injective. Similarly,  $\pi_2: S^n \setminus \{S\} \rightarrow \mathbb{R}^n$  for  $S$  can also be defined and everything holds similarly. We see that these two parametrizations  $(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})$  cover  $S^n$ . The change of coordinate is given by

$$y_j = \frac{x_j}{1 - x_{n+1}} \leftrightarrow y'_j = \frac{x_j}{1 + x_{n+1}}, \quad (y_1, \dots, y_n) \in \mathbb{R}^n, \quad j = 1, \dots, n,$$

where

$$y'_j = \frac{y_j}{\sum_{i=1}^n y_i^2}.$$

This implies that  $\{(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})\}$  is a **differentiable structure** for  $S^n$ . Now, consider  $\pi_1^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^n) = S^n \setminus \{N \cup S\}$ , which is connected, and hence  $S^n$  is **orientable**, and the above **structure** gives an **orientation** of  $S^n$ . ⊗

## Lecture 3: Complex Manifolds, Tangent Spaces and Bundles

Let's look at two more examples about **orientation**.

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**Example.** Let  $A: S^n \rightarrow S^n$  be the antipodal map given by  $A(p) = -p$  for  $p \in \mathbb{R}^{n+1}$ . It's easy to see that  $A$  is differentiable with  $A^2 = \mathbb{1}$ . Furthermore,  $A$  is **diffeomorphism** of  $S^n \subseteq \mathbb{R}^{n+1}$ . We see that

- if  $n$  is even,  $A$  reverses the **orientation**;
- if  $n$  is odd,  $A$  preserves the **orientation**.

**Example.**  $G(k, n)$  is **orientable** if and only if  $n$  is even or  $n = 1$ .

### 1.4 Complex Manifolds

Here we introduce the notion of **complex manifold**.

**Definition 1.4.1** (Complex manifold). A *complex manifold*  $\mathcal{M}$  of complex dimension  $d$  ( $\dim_{\mathbb{C}} \mathcal{M} = d$ ) is a **differentiable manifold** of (real) dimension  $2d$  ( $\dim_{\mathbb{R}} \mathcal{M} = 2d$ ) whose **charts** take values in open subsets of  $\mathbb{C}^d$  with holomorphic **chart transitions**.

**As previously seen.** The **chart transitions**  $z_\beta \circ z_\alpha^{-1}: z_\alpha(U_\alpha \cap U_\beta) \rightarrow z_\beta(U_\alpha \cap U_\beta)$  is holomorphic if  $\partial z_\beta^j / \partial \bar{z}_\alpha^k = 0$  for all  $j, k$  where

$$\frac{\partial}{\partial \bar{z}^k} = \frac{1}{2} \left( \frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k} \right).$$

**Remark.** **Complex Grassmannians**  $G_{\mathbb{C}}(k, n)$  are all **orientable**. More generally, **complex manifolds** are always **orientable** because holomorphic maps always have positive functional determinant.

### 1.5 Partition of Unity

We state, without proof, of an important lemma about the **partition of unity**.

**Definition 1.5.1** (Partition of unity). Let  $\mathcal{M}$  be a **differentiable manifold**, and let  $(U_\alpha)_{\alpha \in \mathcal{A}}$  be an open covering of  $\mathcal{M}$ . Then a *partition of unity* is a **locally finite** refinement  $(V_\beta)_{\beta \in \mathcal{B}}$  of  $(U_\alpha)$  and  $C^\infty$ -functions  $\varphi_\beta: \mathcal{M} \rightarrow \mathbb{R}$  with

- (a)  $\text{supp}(\varphi_\beta) \subseteq V_\beta$  for all  $\beta \in \mathcal{B}$ ;

(b)  $0 \leq \varphi_\beta(x) \leq 1$  for all  $x \in \mathcal{M}$ ,  $\beta \in \mathcal{B}$ ;

(c)  $\sum_{\beta \in \mathcal{B}} \varphi_\beta = 1$  for all  $x \in \mathcal{M}$ .<sup>a</sup>

<sup>a</sup>There are only finitely many non-vanishing summands of each point, since only finitely many  $\varphi_\beta$  are non-zero of any given point as the covering  $(V_\beta)$  is [locally finite](#).

**Lemma 1.5.1** (Partition of unity). Let  $\mathcal{M}$  be a [differentiable manifold](#), and let  $(U_\alpha)_{\alpha \in \mathcal{A}}$  be an open covering of  $\mathcal{M}$ . Then there exists a [partition of unity](#) subordinate to  $(U_\alpha)$ ,

## 1.6 Tangent Spaces and Cotangent Spaces

### 1.6.1 Tangent Spaces in Euclidean Spaces

To discuss the concept of calculus between [manifolds](#) formally, we start with our discussion in Euclidean spaces, where we naturally have the coordinates for every point.

**Definition.** Let  $\mathcal{M}$  be a Euclidean [manifold](#) of dimension  $d$ ,  $x = (x^1, \dots, x^d)$  be Euclidean coordinates of  $\mathbb{R}^d$ , and  $x_0 \in \Omega \subseteq \mathbb{R}^d$  where  $\Omega$  is open.

**Definition 1.6.1** (Tangent space of Euclidean space). The *tangent space*  $T_{x_0}\Omega$  of  $\Omega$  at  $x_0$  is the vector space  $\{x_0\} \times E^a$  spanned by the basis  $(\partial/\partial x^1, \dots, \partial/\partial x^d)$ .

<sup>a</sup> $E$  is a  $d$ -dimensional Euclidean space.

**Definition 1.6.2** (Tangent vector of Euclidean space). The elements in the [tangent space of Euclidean spaces](#) is called *tangent vectors*.

Before proceeding, we introduce a shorthand notation.

**Notation** ([Einstein notation](#)). The *Einstein notation* abbreviates the summation  $\sum_i v^i x_i$  as  $v^i x_i$ , where we implicitly sum over the upper and lower index.

**Definition 1.6.3** (Differential of Euclidean space). If  $\Omega \subseteq \mathbb{R}^d$ ,  $\Omega' \subseteq \mathbb{R}^d$  are open, and  $f: \Omega \rightarrow \Omega'$  is differentiable, then the *differential*  $df(x_0)$  for  $x_0 \in \Omega$  is the induced linear map between [tangent spaces](#)

$$df(x_0): T_{x_0}\Omega \rightarrow T_{f(x_0)}\Omega', \quad v = v^i \frac{\partial}{\partial x^i} \mapsto v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}.$$

**Definition 1.6.4** (Tangent bundle of Euclidean space). The *tangent bundle* is defined as  $T\Omega := \bigsqcup_{x \in \Omega} T_x\Omega \cong \Omega \times E \cong \Omega \times \mathbb{R}^d$ , which is an open subset of  $\mathbb{R}^d \times \mathbb{R}^d$ .

**Note** ([Total space](#)).  $T\Omega$  is also called the *total space*.

**Remark.** Given a [tangent bundle](#)  $T\Omega$ , we define  $\pi$  to be the projection  $\pi: T\Omega \rightarrow \Omega$  given by  $\pi(x, v) = x$ . This makes  $T\Omega$  naturally a [differentiable manifold](#).

With the notion of [tangent bundle](#), given  $f: \Omega \rightarrow \Omega'$ , we can also define  $df: T\Omega \rightarrow T\Omega'$  as

$$\left(x, v^i \frac{\partial}{\partial x^i}\right) \mapsto \left(f(x), v^i \frac{\partial f^j}{\partial x^i}(x) \frac{\partial}{\partial f^j}\right).$$

**Notation.** We often write  $df(x)(v)$  instead of  $df(x, v)$  to coincide with the notation of [differential](#).

In particular, for  $v = v^i \partial / \partial x^i$ , we have

$$df(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R},$$

and we write  $v(f)(x)$  for  $df(x)(v)$ .

### 1.6.2 Tangent Spaces in Manifolds

We now try to formally define the [tangent space](#) on a [smooth manifold](#). A natural idea is the following.

**Intuition.** Let  $\mathcal{M}^d$  be a [differentiable manifold](#) with a [chart](#)  $x: U \rightarrow \Omega \subseteq \mathbb{R}^d$  and  $p \in U \subseteq \mathcal{M}$  where  $U$  is open. The *tangent space*  $T_p\mathcal{M}$  of  $\mathcal{M}$  at  $p$  should be represented in the [chart](#)  $x$  by  $T_{x(p)}x(U)$ .

To see that the above are well-defined, i.e.,  $T_p\mathcal{M}$  are independent of the choice of [charts](#), let  $x': U' \rightarrow \mathbb{R}^d$  to be another [chart](#) with  $p \in U' \subseteq \mathcal{M}$  where  $U'$  is also open. Denote  $\Omega := x(U)$ , and  $\Omega' := x'(U')$ , then the transition map

$$x' \circ x^{-1}: x(U \cap U') \rightarrow x'(U \cap U')$$

induces a vector space isomorphism

$$L := d(x' \circ x^{-1})(x(p)): T_{x(p)}\Omega \rightarrow T_{x'(p)}\Omega',$$

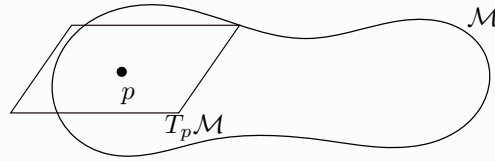
such that  $v \in T_{x(p)}\Omega$  and  $L(v) \in T_{x'(p)}\Omega'$  represent the same [tangent vector](#) in  $T_p\mathcal{M}$ .

**Remark.** A [tangent vector](#) in  $T_p\mathcal{M}$  is given by the family of the [coordinate representations](#).

Now, we want to define the similar notion of [differential of Euclidean spaces](#). Let consider a simple case first, where we let  $f: \mathcal{M} \rightarrow \mathbb{R}$  to be a differentiable function, and assume that the [tangent vector](#)  $w \in T_p\mathcal{M}$  is represented by  $v \in T_{x(p)}x(U)$ .

**Intuition.** We want to define  $df(p)$  as a linear map from  $T_p\mathcal{M} \rightarrow \mathbb{R}$ . In [chart](#)  $x$ , let  $w \in T_p\mathcal{M}$  be given as  $v = v^i \partial / \partial x^i \in T_{x(p)}x(U)$ . Say that  $df(p)(w)$  in this chart represented by

$$d(f \circ x^{-1})(x(p))(v).$$



**Remark.**  $T_p\mathcal{M}$  is a vector space of dimension  $d$  isomorphic to  $\mathbb{R}^d$ , where the isomorphism depends on choice of [chart](#).

**Intuition.** Pull functions on  $\mathcal{M}$  back by a [chart](#) to an open subset of  $\mathbb{R}^d$ , differentiate there.

In order to obtain a [tangent space](#) which does not depend on [charts](#), we need to have transformation behavior under change of [charts](#). Let  $F: \mathcal{M}^d \rightarrow \mathcal{N}^c$  be a differentiable map where  $\mathcal{M}, \mathcal{N}$  are [smooth manifolds](#). Then we want to represent  $dF$  in [local charts](#)  $x: U \subseteq \mathcal{M} \rightarrow \mathbb{R}^d, y: V \subseteq \mathcal{N} \rightarrow \mathbb{R}^c$  by  $d(y \circ F \circ x^{-1})$ . The [local coordinates](#) on  $U$  is given by  $(x^1, \dots, x^d)$ , and on  $V$  is  $(F^1, \dots, F^c)$  such that

$$F(x) = (F^1(x^1, \dots, x^d), \dots, F^c(x^1, \dots, x^d)).$$

Then,  $dF$  induces a linear map  $dF: T_p\mathcal{M} \rightarrow T_{F(x)}\mathcal{N}$  which in our [coordinate representation](#) is given by the matrix

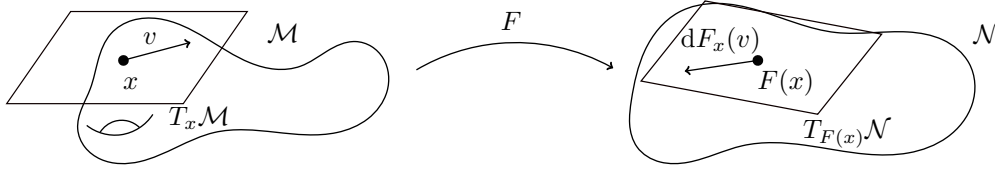
$$\left( \frac{\partial F^\alpha}{\partial x^i} \right)_{\substack{\alpha=1, \dots, c \\ i=1, \dots, d}},$$

and a change of **charts** is then just the base change at **tangent spaces**: if

$$\begin{aligned} (x^1, \dots, x^d) &\mapsto (\xi^1, \dots, \xi^d) \\ (F^1, \dots, F^c) &\mapsto (\phi^1, \dots, \phi^c) \end{aligned}$$

are **coordinate changes**, then  $dF$  represented in the new **coordinates** is given by

$$\left( \frac{\partial \phi^\beta}{\partial \xi^j} \right) = \left( \frac{\partial \phi^\beta}{\partial F^\alpha} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial x^i}{\partial \xi^j} \right).$$



## Lecture 4: Tangent Bundles, Vector Fields, and Submanifolds

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**Definition.** Let  $\mathcal{M}^d$  be a **differentiable manifold** with a **chart**  $x: U \rightarrow \Omega \subseteq \mathbb{R}^d$  and  $p \in U \subseteq \mathcal{M}$  where  $U$  is open. On  $\{(x, v) \mid v \in T_{x(p)}\Omega\}$ , we define an equivalence relation by  $(x, v) \sim (y, w)$  if and only if  $w = d(y \circ x^{-1})v$ .

**Definition 1.6.5 (Tangent space).** The space of equivalence classes is called the *tangent space*  $T_p \mathcal{M}$  at point  $p$  to  $\mathcal{M}$ .

**Definition 1.6.6 (Tangent vector).** The elements in the **tangent space** is called *tangent vectors*.

**Remark.**  $T_p \mathcal{M}$  naturally carries the structure of a vector space.

Now,  $T\mathcal{M}$  is defined as

$$T\mathcal{M} := \coprod_{p \in \mathcal{M}} T_p \mathcal{M}.$$

Recall the projection  $\pi: T\mathcal{M} \rightarrow \mathcal{M}$  with  $\pi(w) = p$  for  $w \in T_p \mathcal{M}$ . Then we can define the following.

**Definition 1.6.7 (Derivation).** If  $x: U \rightarrow \mathbb{R}^d$  be a **chart** for  $\mathcal{M}$ , and let  $TU = \coprod_{p \in U} T_p U$ . Then we define the *derivation*  $dx: TU \rightarrow Tx(U) := \coprod_{p \in x(U)} T_p \mathbb{R}^d$  by  $w \mapsto dx(\pi(w))(w) \in T_{x(\pi(w))}x(U)$ .

The transition maps

$$dx' \circ (dx)^{-1} = d(x' \circ x^{-1})$$

are differentiable.  $\pi$  is local represented by  $x \circ \pi \circ dx^{-1}$  maps  $(x_0, v) \in Tx(U)$  to  $x_0$ .

**Definition 1.6.8 (Tangent bundle).** The triple  $(T\mathcal{M}, \pi, \mathcal{M})$  is called the *tangent bundle* of  $\mathcal{M}$ .

**Definition 1.6.9 (Total space).**  $T\mathcal{M}$  is called the *total space* of the **tangent bundle**.

### 1.6.3 Cotangent Spaces

Another important objects is the **cotangent spaces**.

**Definition.** Let  $\mathcal{M}^d$  be a **differentiable manifold**, and  $T_p \mathcal{M}$  be the **tangent space** at  $p$  to  $\mathcal{M}$ .

**Definition 1.6.10 (Cotangent space).** The *cotangent space*  $T_p^*\mathcal{M}$  to  $\mathcal{M}$  is the dual of  $T_p\mathcal{M}$ , i.e.,  $T_p^*\mathcal{M} = (T_p\mathcal{M})^*$ .

**Definition 1.6.11 (Cotangent vector).** The elements in the *cotangent space* is called *cotangent vectors*.

**Remark.**  $T_p^*\mathcal{M}$  is the space of 1-forms on  $T_p\mathcal{M}$ .

**Notation** (Covariant vector). The *cotangent vectors* are also called *covariant vectors*.

**Notation** (Contravariant vector). The *tangent vectors* are also called *contravariant vectors*.

## 1.7 Vector Fields and Brackets

### 1.7.1 Vector Fields

We now introduce the notion of *vector field*.

**Definition 1.7.1 (Vector field).** A *vector field*  $X$  on a *differentiable manifold*  $\mathcal{M}$  is a correspondence associating to each point  $p \in \mathcal{M}$  a vector  $X(p) \in T_p\mathcal{M}$ , i.e.,  $X: \mathcal{M} \rightarrow T\mathcal{M}$ .

**Definition 1.7.2 (Section).** A *section* of the *tangent bundle* is a differentiable map  $s: \mathcal{M} \rightarrow T\mathcal{M}$  such that  $\pi \circ s = \text{id}_{\mathcal{M}}$ .

**Remark.** Naturally, we say that the *field*  $X$  is differentiable if the map  $X$  is differentiable.

Considering a *local chart*  $x: U \subseteq \mathbb{R}^n \rightarrow \mathcal{M}$ , we can write

$$X(p) = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i},$$

where  $a_i: U \rightarrow \mathbb{R}$  are functions on  $U$  for  $i = 1, \dots, n$ , and  $\{\partial/\partial x_i\}_i$  is the basis associated to  $x$ .

**Remark.**  $X$  is differentiable if and only if  $a_i$  are differentiable for some (and, therefore, for any)  $x$ .

It's convenient to think of a *vector field* as a mapping  $X: \mathcal{D} \rightarrow \mathcal{F}$  from the set  $\mathcal{D}$  of differentiable functions on  $\mathcal{M}$  to the set  $\mathcal{F}$  of the functions on  $\mathcal{M}$ , defined by

$$(Xf)(p) = \sum_{i=1}^n a_i(p) \frac{\partial f}{\partial x_i}(p),$$

where  $f$  is implicitly denoting the expression of  $f$  in the *chart*  $x$ .

**Intuition.** This idea of a vector as a directional derivative is precisely what was used to define the notion of *tangent vector*.

**Remark.**  $Xf$  does not depend on the choice of  $x$ .

**Remark.**  $X$  is differentiable if and only if  $X: \mathcal{D} \rightarrow \mathcal{D}$ , i.e.,  $Xf \in \mathcal{D}$  for all  $f \in \mathcal{D}$ .

Observe that if  $\varphi: \mathcal{M} \rightarrow \mathcal{M}$  is a *diffeomorphism*,  $v \in T_p\mathcal{M}$  and  $f$  differentiable function in a neighborhood of  $\varphi(p)$ , we have

$$(d\varphi(v)f)\varphi(p) = v(f \circ \varphi)(p)$$

since by letting  $\alpha: (-\epsilon, \epsilon) \rightarrow \mathcal{M}$  be a differentiable **curve** with  $\alpha'(0) = v$ ,  $\alpha(0) = p$ ,<sup>2</sup> then

$$(\mathrm{d}\varphi(v)f)\varphi(p) = \left. \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi \circ \alpha) \right|_{t=0} = v(f \circ \varphi)(p).$$

### 1.7.2 Brackets

By viewing  $X$  as an operator on  $\mathcal{D}$ , we can consider the iterates of  $X$ , i.e, given differentiable **fields**  $X$  and  $Y$  and  $f: M \rightarrow \mathbb{R}$  being a differentiable function, consider  $X(Yf)$  and  $Y(Xf)$ .

**Note.** In general,  $X(Yf)$  (and hence  $Y(Xf)$ ) is not a **field**.

**Proof.** It involves derivatives of order higher than one. ⊛

But we have the following.

**Lemma 1.7.1.** Let  $X, Y$  be differentiable **vector fields** on a **smooth manifold**  $\mathcal{M}$ . Then there exists a unique **vector field**  $Z$  such that for all  $f \in \mathcal{D}$ ,  $Zf = (XY - YX)f$ .

**Proof.** See do Carmo [FC13, Chapter 0, Lemma 5.2]. ■

This  $Z$  is called the **bracket**.

**Definition 1.7.3 (Bracket).** Given two differentiable **vector fields**  $X, Y$  on a **smooth manifold**  $\mathcal{M}$ , the *bracket* of  $X$  and  $Y$  is defined by

$$[X, Y] := XY - YX.$$

Clearly,  $[X, Y]$  is differentiable.

**Proposition 1.7.1.** If  $X, Y$  and  $Z$  are differentiable **vector fields** on  $\mathcal{M}$ ,  $a, b \in \mathbb{R}$ ,  $f, g$  are differentiable functions, then we have the following.

- (a)  $[X, Y] = -[Y, X]$  (*anti-commutativity*),
- (b)  $[aX + bY, Z] = a[X, Z] + b[Y, Z]$  (*linearity*),
- (c)  $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$  (*Jacobi identity*),
- (d)  $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$ .

**Proof.** See do Carmo [FC13, Chapter 0, Proposition 5.3]. ■

## 1.8 Submanifolds, Immersions, and Embeddings

We now study the relation between **manifolds**.

**Definition 1.8.1 (Immersion).** Let  $\mathcal{M}^m, \mathcal{N}^n$  be **smooth manifolds**. A differentiable mapping  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$  is an *immersion* if

$$\mathrm{d}\varphi_p: T_p\mathcal{M} \rightarrow T_{\varphi(p)}\mathcal{N}$$

is injective for every  $p \in \mathcal{M}$ .

**Definition 1.8.2 (Embedding).** An **immersion**  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$  is an *embedding* if it is also a homeomorphism onto  $\varphi(\mathcal{M}) \subseteq \mathcal{N}$ , with  $\varphi(\mathcal{M})$  having the subspace topology induced from  $\mathcal{N}$ .

<sup>2</sup>This is the way do Carmo [FC13] used to define **tangent vectors**.

**Definition 1.8.3 (Submanifold).** If the inclusion  $\iota: \mathcal{M} \hookrightarrow \mathcal{N}$  between two manifolds is an embedding, then  $\mathcal{M}$  is a submanifold of  $\mathcal{N}$ .

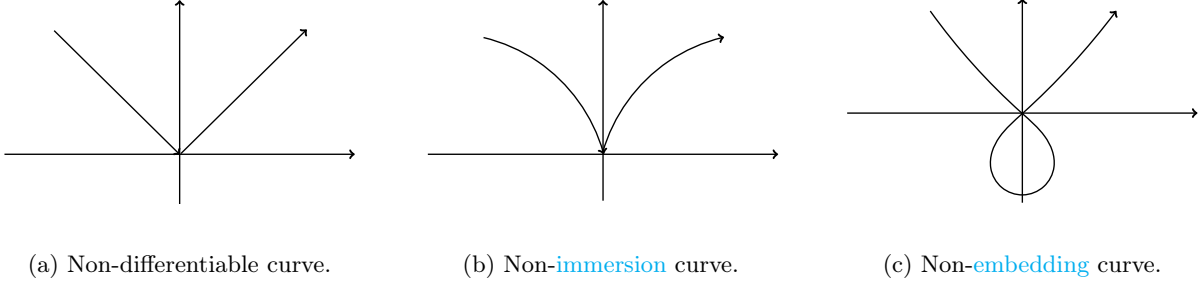


Figure 1.1: Three simple examples

**Lemma 1.8.1.** Let  $f: \mathcal{M}^m \rightarrow \mathcal{N}^n$  be an immersion and  $x \in \mathcal{M}$ .<sup>a</sup> Then there exists a neighborhood  $U$  of  $x$  and a chart  $(V, y)$  on  $\mathcal{N}$  with  $f(x) \in V$  such that  $f|_U$  is a differentiable embedding and  $y^{m+1}(p) = \dots = y^n(p) = 0$  for all  $p \in f(U \cap V)$ .

<sup>a</sup>Hence,  $n \geq m$ .

**Proof.** In the local coordinates  $(z^1, \dots, z^n)$  on  $\mathcal{N}$ , and  $(x^1, \dots, x^m)$  on  $\mathcal{M}$ , without loss of generality,<sup>a</sup> let

$$\left( \frac{\partial z^\alpha(f(x))}{\partial x^i} \right)_{i, \alpha=1, \dots, m}$$

be non-singular. Consider

$$F(z, x) := (z^1 - f^1(x), \dots, z^n - f^n(x)),$$

which has maximal rank in  $x^1, \dots, x^m, z^{m+1}, \dots, z^n$ . By the implicit function theorem, locally, there exists a map  $\varphi: U \rightarrow \mathbb{R}^n$  such that

$$(z^1, \dots, z^m) \mapsto (\varphi^1(z^1, \dots, z^m), \dots, \varphi^n(z^1, \dots, z^m)) = x$$

such that  $F(z, x) = 0$ , i.e.,

$$\varphi^i(z^1, \dots, z^m) = \begin{cases} x^i, & \text{if } i = 1, \dots, m; \\ z^i, & \text{if } i = m+1, \dots, n, \end{cases}$$

for which

$$\left( \frac{\partial \varphi^i}{\partial z^\alpha} \right)_{\alpha, i=1, \dots, m}$$

has maximal rank. Now, we choose a new coordinate

$$(y^1, \dots, y^n) = (\varphi^1(z^1, \dots, z^m), \dots, \varphi^m(z^1, \dots, z^m), \\ z^{m+1} - \varphi^{m+1}(z^1, \dots, z^m), \dots, z^n - \varphi^n(z^1, \dots, z^m)).$$

Then, we have  $z = f(x) \Leftrightarrow F(z, x) = 0$ , i.e.,  $(y^1, \dots, y^n) = (x^1, \dots, x^n, 0, \dots, 0)$ , proving the result. ■

<sup>a</sup>Since  $df(x)$  is injective.

**Lemma 1.8.2.** Let  $f: \mathcal{M}^m \rightarrow \mathcal{N}^n$  be a differentiable map such that  $m \geq n$  with  $p \in \mathcal{N}$ . Let  $df(x)$  has rank  $n$  for all  $x \in \mathcal{M}$  with  $f(x) = p$ . Then  $f^{-1}(p)$  is the union of differentiable submanifolds of  $\mathcal{M}$  of dimension  $m - n$ .



---

**Remark.** Let  $\mathcal{N}^n$  be a smooth manifold, and let  $1 \leq m \leq n$ . Then an arbitrary subset  $\mathcal{M} \subseteq \mathcal{N}$  has the structure of differentiable submanifold of  $\mathcal{N}$  of dimension  $m$  if and only if for all  $p \in \mathcal{M}$ , there exists a smooth chart  $(U, \varphi)$  of  $\mathcal{N}$  such that  $p \in U$ ,  $\varphi(p) = 0$ ,  $\varphi(U)$  is open, and

$$\varphi(U \cap \mathcal{M}) = (-\epsilon, +\epsilon)^n \times \{0\}^{n-m},$$

where  $(-\epsilon, +\epsilon)^n$  is the cube. Noticeably, the  $C^\infty$ -manifold structure of  $\mathcal{M}$  is uniquely determined.

**Remark.** Let  $\mathcal{M} \subseteq \mathcal{N}$  be a differentiable submanifold of  $\mathcal{N}$ , and let  $\iota: \mathcal{M} \hookrightarrow \mathcal{N}$  be the inclusion. Then, for  $p \in \mathcal{M}$ ,  $T_p\mathcal{M}$  can be considered as subspace of  $T_p\mathcal{N}$ , namely as the image of  $d\iota(T_p\mathcal{M})$ .

**Lemma 1.8.3.** Let  $f: \mathcal{M}^m \rightarrow \mathcal{N}^n$  be a differentiable map such that  $m \geq n$  with  $p \in \mathcal{N}$ . Let  $df(x)$  has rank  $n$  for all  $x \in \mathcal{M}$  with  $f(x) = p$ . For the submanifold  $X = f^{-1}(p)$  and for  $q \in X$ , it is true that

$$T_qX = \ker df(q) \subseteq T_q\mathcal{M}.$$

# Chapter 2

## Riemannian Manifolds

### Lecture 5: Riemannian Manifolds

In this chapter, we start our discussion on [Riemannian manifolds](#).

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#### 2.1 Riemannian Metrics

We start by defining the [Riemannian metric](#).

**Definition 2.1.1** (Riemannian metric). A *Riemannian metric*  $g$  on a [differentiable manifold](#)  $\mathcal{M}$  is given by a scalar product  $I$  on each  $T_p\mathcal{M}$  which depends smoothly on the base point  $p$ .

**Definition 2.1.2** (Riemannian manifold). A *Riemannian manifold*  $(\mathcal{M}, g)$  is a [smooth manifold](#)  $\mathcal{M}$  equipped with a [Riemannian metric](#)  $g$ .

Let  $x = (x^1, \dots, x^d)$  be the [local coordinates](#). In these, a [metric](#) is represented by a positive definite symmetric matrix

$$(g_{ij}(x))_{i,j=1,\dots,d},$$

i.e.,  $g_{ij} = g_{ji}$ , and  $g_{ij}\xi^i\xi^j > 0$  for all  $\xi = (\xi^1, \dots, \xi^d) \neq 0$  with coefficients smoothly depending on  $x$ .

##### 2.1.1 Transformation Behavior

We now see that the smoothness does not depend on [coordinates](#), i.e., the smooth dependence on the base point (as required in [Definition 2.1.1](#)) can be represented in the [local coordinates](#). Given 2 [tangent vectors](#)  $v, w \in T_p\mathcal{M}$  with [coordinate representations](#)  $(v^1, \dots, v^d), (w^1, \dots, w^d)$  given by  $x$  such that  $v = v^i \frac{\partial}{\partial x^i}$  and  $w = w^i \frac{\partial}{\partial x^i}$ , their product is

$$\langle v, w \rangle := g_{ij}(x(p))v^i w^j.$$

In particular,

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = g_{ij}.$$

**Remark.** The length of  $v$  is given as  $\|v\| := \langle v, v \rangle^{1/2}$ .

Let  $y = f(x)$  define different [local coordinates](#). In these,  $v, w$  are given as

$$(\tilde{v}^1, \dots, \tilde{v}^d), (\tilde{w}^1, \dots, \tilde{w}^d)$$

with  $\tilde{v}^j = v^i \frac{\partial f^j}{\partial x^i}$  and  $\tilde{w}^j = w^i \frac{\partial f^j}{\partial x^i}$ . Denote the [metric](#) in new [coordinates](#)  $y$  by  $h_{k\ell}(y)$ , then we have

$$h_{k\ell}(f(x))\tilde{v}^k \tilde{w}^\ell = \langle v, w \rangle = g_{ij}(x)v^i w^j.$$

Plug everything in, we have

$$h_{k\ell}(f(x)) \frac{\partial f^k}{\partial x^i} \frac{\partial f^\ell}{\partial x^j} v^i w^j = g_{ij}(x) v^i w^j.$$

We see that this holds for any **tangent vectors**  $v, w$ , therefore,

$$h_{k\ell}(f(x)) \frac{\partial f^k}{\partial x^i} \frac{\partial f^\ell}{\partial x^j} = g_{ij}(x),$$

which is the transformation behavior under **coordinates changes**.

**Remark.** This shows that the smoothness does not depend on the choice of coordinates!

**Example.** Consider the Euclidean space  $\Omega$ , then given  $v, w \in T_p\Omega$ , we have

$$\langle v, w \rangle = \delta_{ij} v^i w^j = v^i w_i.$$

**Theorem 2.1.1.** Every **differentiable manifold** can be equipped with a **Riemannian metric**.

**Proof.** From **Lemma 1.5.1**, there exists a differentiable **partition of unity**  $\{f_\alpha\}$  of  $\mathcal{M}$  subordinate to a covering  $\{V_\alpha\}$  of  $\mathcal{M}$ . Consider the induced **metric**  $\langle \cdot, \cdot \rangle^\alpha$  of the system of **local coordinates** on each  $V_\alpha$ . Then, for every  $p \in M$ , a **Riemannian metric**  $\langle \cdot, \cdot \rangle_p$  can be defined naturally as

$$\langle u, v \rangle_p = \sum_{\alpha} f_{\alpha}(p) \langle u, v \rangle_p^{\alpha}$$

for all  $u, v \in T_p M$ . Given the fact that  $\{f_\alpha\}$  is the **partition of unity**, we know that

- (a)  $f_\alpha \geq 0$ , and  $f_\alpha = 0$  on  $\overline{V_\alpha}^c$ ,
- (b)  $\sum_{\alpha} f_{\alpha}(p) = 1$  for all  $p$  on  $M$ ,

it's then immediate that the defined is indeed a **Riemannian metric**. ■

## 2.1.2 Isometry

After introducing any type of mathematical structure, we must introduce a notion of when two objects are the same.

**Definition 2.1.3 (Isometry).** A **diffeomorphism**  $h: \mathcal{M} \rightarrow \mathcal{N}$  is an **isometry** between two **Riemannian manifolds** if it preserves the **Riemannian metric**, i.e., for  $p \in \mathcal{M}$ ,  $v, w \in T_p \mathcal{M}$ ,

$$\langle v, w \rangle_{\mathcal{M}} = \langle dh(v), dh(w) \rangle_{\mathcal{N}}.$$

**Definition 2.1.4 (Local isometry).** A **diffeomorphism**  $h: \mathcal{M} \rightarrow \mathcal{N}$  is a **local isometry** between two **Riemannian manifolds** if for every  $p \in \mathcal{M}$ , there exists a neighborhood  $U$  such that  $h|_U: U \rightarrow h(U): \mathcal{M} \rightarrow \mathcal{N}$  is an **isometry** and  $h(U) \subseteq \mathcal{N}$  is open.

It's common to say that a **Riemannian manifold**  $\mathcal{M}$  is **locally isometric** to a **Riemannian manifold**  $\mathcal{N}$  if for every  $p \in \mathcal{M}$ , there exists a neighborhood  $U$  of  $p$  in  $\mathcal{M}$  and a **local isometry**  $f: U \rightarrow f(U) \subseteq \mathcal{N}$ .

Let's first look at an almost trivial example.

**Example (Euclidean space).** Let  $\mathcal{M} = \mathbb{R}^n$  with  $\partial/\partial x_i$  identified with  $e_i = (0, \dots, 1, \dots, 0)$ . The metric is given by

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

$\mathbb{R}^n$  is called *Euclidean space of dimension  $n$*  and the Riemannian geometry of this space is metric Euclidean geometry.

**Example** (Lie group). See [Appendix A](#) for reference.

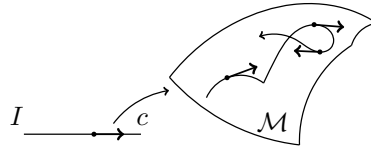
## 2.2 Curves, Lengths, and Energies

### 2.2.1 Curves

We are now going to show how a [Riemannian metric](#) can be used to calculate the [length](#) of a [curve](#).

**Definition 2.2.1** (Curve). A differentiable mapping  $c: I \rightarrow \mathcal{M}$  of an open interval  $I \subseteq \mathbb{R}$  into a [differentiable manifold](#)  $\mathcal{M}$  is called a (parametrized) *curve*.

**Note.** A parametrized curve can admit self-intersections as well as corners.



**Definition 2.2.2** (Vector field along a curve). We say that a *vector field along a curve*  $c: I \rightarrow \mathcal{M}$  is a differentiable mapping that associates to every  $t \in I$  a [tangent vector](#)  $V(t) \in T_{c(t)}\mathcal{M}$ .

To say  $V$  is differentiable means that for any differentiable function  $f$  on  $\mathcal{M}$ , the function  $t \mapsto V(t)f$  is a differentiable function on  $I$ .

**Example** (Velocity field). The [vector field](#)  $dc(d/dt)$ , denoted by  $dc/dt$ , is called the *velocity field* or *tangent vector field*, of course.

**Remark.** A [vector field along  \$c\$](#)  cannot necessarily be extended to a [vector field](#) on an open set of  $\mathcal{M}$ .

**Notation** (Segment). The restriction of a [curve](#)  $c$  to a closed interval  $[a, b] \subseteq I$  is called a *segment*.

### 2.2.2 Lengths and Energies

We're interested in the following two quantities.

**Definition.** Let  $\gamma: [a, b] \rightarrow \mathcal{M}$  be a [curve](#) on a [Riemannian manifold](#)  $(\mathcal{M}, g)$ .

**Definition 2.2.3** (Length). The *length* of  $\gamma$  is defined as

$$L(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt.$$

**Definition 2.2.4** (Energy). The *energy* of  $\gamma$  is defined as

$$E(\gamma) := \frac{1}{2} \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt.$$

We now want to compute  $L(\gamma)$ ,  $E(\gamma)$  in [local coordinates](#). Let the [local coordinates](#) be

$$(x^1(\gamma(t)), \dots, x^d(\gamma(t))),$$

we write

$$\dot{x}^i(t) = \frac{d}{dt}(x^i(\gamma(t))).$$

Then, we have

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)} dt, \quad E(\gamma) = \frac{1}{2} \int_a^b g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t) dt.$$

**Definition 2.2.5 (Distance).** Given a Riemannian manifold  $(\mathcal{M}, g)$ , the *distance* between 2 points  $p, q \in \mathcal{M}$  is defined as

$$d(p, q) := \inf \{L(\gamma) \mid \gamma: [a, b] \rightarrow \mathcal{M} \text{ piecewise curve with } \gamma(a) = p, \gamma(b) = q\}.$$

**Note.** Any 2 points  $p, q \in \mathcal{M}$  can be connected by a piecewise curve, hence  $d(p, q)$  always exists.

**Corollary 2.2.1.** The topology of  $\mathcal{M}$  induced by the distance function  $d$  coincides with the original manifold topology of  $\mathcal{M}$ .

**Lemma 2.2.1.** If  $\gamma: [a, b] \rightarrow \mathcal{M}$  is a curve, and  $\psi: [\alpha, \beta] \rightarrow [a, b]$  is a change of parameter, then  $L(\gamma \circ \psi) = L(\gamma)$ .

**Proof.** This can be proved by computation, and the take-away is that the length functional is invariant under parameter changes. ■

# Chapter 3

## Geodesics

This is the first focus on the study of Riemannian geometry, i.e., the [geodesics](#). The up-shot is that a [geodesic](#) minimizes the [arc length](#) for points *sufficiently close* (in a sense to be made precise); in addition, if a [curve](#) minimizes [arc length](#) between any two of its points, it is a [geodesic](#).

### 3.1 Euler-Lagrange Equations

Let's first fix some common notations.

**Notation.**  $(g^{ij})_{i,j=1,\dots,d} = (g_{ij})_{i,j=1,\dots,d}^{-1}$ .

**Note.**  $g^{i\ell}g_{\ell j} = \delta_j^i$ .

**Notation.**  $g_{j\ell,k} := \frac{\partial}{\partial x^k} g_{j\ell}$ .

And the following is particularly important.

**Notation** (Christoffel symbol). The *Christoffel symbol* is defined as

$$\Gamma_{jk}^i := \frac{1}{2} g^{i\ell} (g_{j\ell,k} + g_{k\ell,j} - g_{jk,\ell})$$

for all  $i$ .

Recall the definition of [energy](#), and recall that we want to find a [curve](#) which minimizes the [length](#) between sufficiently close two points. It turns out that instead of working with [length](#) directly, we should work with [energy](#) instead.

**Proposition 3.1.1.** The [Euler-Lagrange equations](#) for the [energy](#)  $E$  are

$$\ddot{x}^i(t) + \Gamma_{jk}^i(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0 \tag{3.1}$$

for  $i = 1, \dots, d$ .

**Proof.** The [Euler-Lagrange equations](#) of a functional

$$I(x) = \int_a^b f(t, x(t), \dot{x}(t)) dt$$

are

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0$$

for  $i = 1, \dots, d$ . Just by plugging in, we obtain for  $E$ , we have

$$\frac{d}{dt} (g_{ik}(x(t))\dot{x}^k(t) + g_{ji}(x(t))\dot{x}^j(t)) - g_{jk,i}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$$

for  $i = 1, \dots, d$ . Hence,

$$g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j + g_{ik,\ell}\dot{x}^\ell\dot{x}^k + g_{ji,\ell}\dot{x}^\ell\dot{x}^j - g_{jk,i}\dot{x}^\ell\dot{x}^j = 0$$

Rename some indices and use  $g_{ij} = g_{ji}$ , we have that

$$2g_{\ell m}\ddot{x}^m + (g_{k\ell,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0$$

for  $\ell = 1, \dots, d$ . Hence, we have

$$g^{i\ell}g_{\ell m}\ddot{x}^m + \frac{1}{2}g^{i\ell}(g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0$$

for  $i = 1, \dots, d$ . Finally, observe that

$$g^{i\ell}g_{\ell m} = \delta_{im} \Rightarrow g^{i\ell}g_{\ell m}\ddot{x}^m = \ddot{x}^i,$$

hence the claim follows. ■

Finally, we define the **geodesics** as the solution of [Equation 3.1](#).

**Definition 3.1.1 (Geodesic).** A **curve**  $\gamma: [a, b] \rightarrow \mathcal{M}$  that obeys [Equation 3.1](#) is called a *geodesic*.

In other words, from [Proposition 3.1.1](#), we naturally define **geodesic** by the solution of [Equation 3.1](#) since it finds the critical points of **energy**.

### 3.1.1 Action Functional

Consider the following.

**Definition 3.1.2 (Action).** Let  $\mathcal{L}$  be the Lagrangian, then let

$$I[w(\cdot)] := \int_0^t \mathcal{L}(\dot{w}(s), w(s)) ds$$

defined for functions  $w(\cdot) = (w^1(\cdot), \dots, w^n(\cdot))$  of the admissible class

$$\mathcal{A} = \{w(\cdot) \in C^2([0, t]; \mathbb{R}^n) \mid w(0) = y, w(t) = x\}.$$

From the calculus of variation, we can find a **curve**  $x(\cdot) \in \mathcal{A}$  such that

$$I[x(\cdot)] = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)].$$

**Theorem 3.1.1 (Euler-Lagrangian equations).**  $x(\cdot)$  from  $I[x(\cdot)] = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)]$  solves the system of Euler-Lagrangian equations

$$\frac{d}{ds} (D_{\dot{x}}\mathcal{L}(\dot{x}(s), x(s)) + D_x\mathcal{L}(\dot{x}(s), x(s))) = 0$$

for  $0 \leq s \leq t$ .

## Lecture 6: Geodesic and the Exponential Map

Now, we draw some relations between **length** and **energy** and see why starting from **energy** makes sense. 24 Jan. 14:30

**Proposition 3.1.2.** For all **curves**  $\gamma: [a, b] \rightarrow \mathcal{M}$ ,

$$\mathcal{L}(\gamma)^2 \leq 2(b-a)E(\gamma)$$

with equality if and only if  $\|\mathrm{d}\gamma/\mathrm{d}t\|$  is a constant.

**Proof.** From **Hölder's inequality**,

$$\int_a^b \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t} \right\| \mathrm{d}t \leq (b-a)^{1/2} \left( \int_a^b \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t} \right\|^2 \mathrm{d}t \right)^{1/2}$$

with equality if and only if  $\|\mathrm{d}\gamma/\mathrm{d}t\|$  is a constant. ■

**Example.** Let

$$\mathcal{L}(q, x) = \frac{1}{2}m|q|^2 - V(x)$$

with  $m > 0$ ,  $q = \dot{x}$ , the Euler-Lagrangian equations is given by

$$m\ddot{x}(s) = F(x(s))$$

for  $F := -DV$ .

**As previously seen.** Regular curves can be parametrized by **arc length** with unit speed  $\|\mathrm{d}\gamma/\mathrm{d}t\| = \|\dot{\gamma}\| \equiv 1$ .

**Lemma 3.1.1.** Each **geodesic** is parametrized proportionally to the **arc length**.<sup>a</sup>

<sup>a</sup>This means that we have constant speed, i.e.,  $\|\dot{\gamma}\|$  is a constant.

**Proof.** For a solution of  $\ddot{x}^i(t) + \Gamma_{jk}^i(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$ ,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \dot{x}, \dot{x} \rangle = \frac{\mathrm{d}}{\mathrm{d}t} (g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)) = 0.$$

Do the computation!

Our goal now is to minimize the **length** within class of regular **smooth curves**.

**As previously seen.** The **length** and the **energy** functionals are invariants under parameter changes.

This means that it's enough to look at **curves** parametrized by **arc length**.

**Theorem 3.1.2.** Let  $\mathcal{M}$  be a **Riemannian manifold**,  $p \in \mathcal{M}$  and  $v \in T_p\mathcal{M}$ . Then there exists an  $\epsilon > 0$  and a unique **geodesic** such that  $c: [0, \epsilon] \rightarrow \mathcal{M}$  with  $c(0) = p$  and  $\dot{c}(0) = v$ . In addition,  $c$  smoothly depend on  $p, v$ .

**Proof.** Since **Equation 3.1** is a system of second order ODE, by **Picard-Lindelöf theorem**, we have local existence and uniqueness of the solution with prescribed initial values and derivative such that the solution depends smoothly on  $p, v$ . ■

If  $x(t)$  is the solution of **Equation 3.1**, then  $x(\lambda t)$  is also a solution for any constant  $\lambda \in \mathbb{R}$ . Denote **geodesic** from **Theorem 3.1.2** by  $c_v$ , then

$$c_v(t) = c_{\lambda v}(t/\lambda)$$

for  $\lambda > 0$ ,  $t \in [0, \epsilon]$ , and hence  $c_{\lambda v}$  defined on  $[0, \epsilon/\lambda]$ .

**Remark.** Since  $c_v$  depends smoothly on  $v$ , the set  $\{v \in T_p\mathcal{M} \mid \|v\| = 1\}$  is compact, hence there exists  $\epsilon_0 > 0$  such that for  $\|v\| = 1$ ,  $c_v$  defined at least on  $[0, \epsilon_0]$ , implying that for all  $w \in T_p\mathcal{M}$



with  $\|w\| \leq \epsilon_0$ ,  $c_w$  is defined at least on  $[0, 1]$ .

## 3.2 Exponential Maps

The above discussion permits us to introduce the concept of the [exponential map](#) in the following manner.

**Definition 3.2.1** (Exponential map). Let  $(\mathcal{M}, g)$  be a [Riemannian manifold](#),  $p \in \mathcal{M}$ , and  $V_p := \{v \in T_p\mathcal{M} \mid c_v \text{ defined on } [0, 1]\}$ . The *exponential map of  $\mathcal{M}$  at  $p$* ,  $\exp_p: V_p \rightarrow \mathcal{M}$ , is defined as  $v \mapsto c_v(1)$ .

Clearly,  $\exp$  is differentiable, and we shall utilize the restriction of  $\exp$  to an open subset of the [tangent space](#)  $T_q\mathcal{M}$ , i.e., we define

$$\exp_p: B(0, \epsilon) \subseteq T_p\mathcal{M} \rightarrow \mathcal{M},$$

where  $B(0, \epsilon)$  is an open ball with center at the origin 0 of  $T_p\mathcal{M}$  of radius  $\epsilon$ . It's easy to see that  $\exp_p$  is differentiable and that  $\exp_p(0) = p$ .

**Intuition.** Geometrically,  $\exp_p(v)$  is a point of  $\mathcal{M}$  obtained by going out the [length](#) equal to  $|v|$ , starting from  $p$ , along a [geodesic](#) which passes through  $p$  with velocity equal to  $v/|v|$ .

**Proposition 3.2.1.** The [exponential map](#)  $\exp_p$  maps a neighborhood of  $0 \in T_p\mathcal{M}$  [diffeomorphically](#) onto a neighborhood of  $p \in \mathcal{M}$ .

**Proof.** We see that

$$d(\exp_p)_0(v) = \left. \frac{d}{dt} \exp_p(tv) \right|_{t=0} = \left. \frac{d}{dt} c_{tv}(1) \right|_{t=0} = \left. \frac{d}{dt} c_v(t) \right|_{t=0} = v,$$

i.e.,  $d(\exp_p)_0$  is the identity of  $T_p\mathcal{M}$ . By the inverse function theorem,  $\exp_p$  is a local [diffeomorphism](#) on a neighborhood of 0. ■

Consider  $\exp_p: B(0, \epsilon) \subseteq T_p\mathcal{M} \rightarrow \mathcal{M}$ , maps [diffeomorphically](#) onto its image, we can then introduce the coordinates around  $p$ . Let  $(e_1, \dots, e_n)$  be the orthonormal basis of  $T_p\mathcal{M}$ , and  $(x_1, \dots, x_n)$  be the associated [local coordinates](#). Given  $p \in \mathcal{M}^n$ ,  $0 \in \mathbb{R}^n$ , we have

$$g_{ij}(p) = \delta_{ij}, \quad \Gamma_{ij}^k(p) = 0, \quad g_{ij,k} = 0$$

for all  $i, j, k$ .

**Definition 3.2.2** (Normal coordinate).

**Note.** The first derivative vanishes, so locally, the [manifold](#) looks Euclidean.

**Theorem 3.2.1.** For all  $p \in \mathcal{M}$ , there exists  $\rho > 0$  such that the Riemannian polar coordinates may be introduced on  $B(p, \rho) = \{q \in \mathcal{M} \mid d(p, q) \leq \rho\}$ . For any such  $\rho$  and  $q \in \partial B(p, \rho)$ , there exists a unique [geodesic](#) of shortest length ( $= \rho$ ) from  $p$  to  $q$ . And in the polar coordinates, this [geodesic](#) is given by the straight line  $x(t) = (t, \varphi_0)$ ,  $0 \leq t \leq \rho$ , with  $q$  represented by coordinates  $(\rho, \varphi_0)$ ,  $\varphi_0 \in S^{d-1}$ .

**Proof.** Take an arbitrary curve from  $p$  to  $q$ , namely  $c(t) = (r(t), \varphi(t))$ ,  $0 \leq t \leq T$ , which does not have to be entirely contained in  $B(p, \rho)$ . Let  $t_0$  be defined as

$$t_0 := \inf \{t \leq T \mid d(x(t), p) \geq \rho\}.$$

Then  $t_0 \leq T$  such that  $c|_{[0, t_0]}$  lies entirely in  $B(p, \rho)$ . We want to show that

- (a)  $L(c|_{[0, t_0]}) \geq \rho$ , and

(b)  $L(c|_{[0,t_0]}) = \rho$  only for a straight line in the polar coordinates,

where

$$L(c|_{[0,t_0]}) := \int_0^{t_0} \sqrt{g_{ij}(c(t))\dot{c}^i\dot{c}^j} dt.$$

Observe that  $g_{r\varphi} = 0$ , with  $g_{\varphi\varphi}$  being positive definite, hence

$$L(c|_{[0,t_0]}) \geq \int_0^{t_0} \sqrt{g_{rr}(c(t))\dot{r}^2} dt = \int_0^{t_0} |\dot{r}| dt \geq \int_0^{t_0} \dot{r} dt = r(t_0) = \rho,$$

where we know that  $g_{rr} \equiv 1$ . ■

**Remark (Compact manifold).** For compact manifold, from [Theorem 3.2.1](#), we can prove that Riemannian polar coordinates can be introduced. Also, there exists  $\rho_0 > 0$  such that for any 2 points  $p, q \in \mathcal{M}$  with  $d(p, q) \leq \rho_0$  can be connected by minimizing [geodesic](#).

## Lecture 7: Hopf-Rinow Theorem

### 3.3 Hopf-Rinow Theorem

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We have shown the following in the homework.

**Theorem 3.3.1.** Let  $(\mathcal{M}, g)$  be a compact [Riemannian manifold](#).

- (a) Any 2 points  $p, q \in \mathcal{M}$  can be connected by a minimizing [geodesic](#).
- (b) For all  $p \in \mathcal{M}$ , the [exponential map](#)  $\exp_p$  is defined on all of  $T_p\mathcal{M}$  and any [geodesic](#) may be extended indefinitely in each direction.

We now want to generalize it. However, this is not true in the most general setting, and we need one more requirement.

**Definition 3.3.1 (Geodesically complete).** A [Riemannian manifold](#)  $(\mathcal{M}, g)$  is *geodesically complete* if for all  $p \in \mathcal{M}$ ,  $\exp_p$  is defined on all of  $T_p\mathcal{M}$ , if any [geodesic](#)  $c(t)$  with  $c(0) = p$  can be extended for all  $t \in \mathbb{R}$ .

Finally, we have the following.

**Theorem 3.3.2 (Hopf-Rinow theorem).** Let  $(\mathcal{M}, g)$  be a compact [Riemannian manifold](#), then the following statements are equivalent.

- (a)  $\mathcal{M}$  is complete as a metric space.<sup>a</sup>
- (b) The closed and bounded subsets of  $\mathcal{M}$  are compact.
- (c) There exists  $p \in \mathcal{M}$  such that  $\exp_p$  is defined on all  $T_p\mathcal{M}$ .
- (d)  $\mathcal{M}$  is [geodesically complete](#).

Furthermore, (d) (and hence (a), (b), and (c)) implies

- (e) for two points  $p, q \in \mathcal{M}$  can be joined by a minimizing [geodesic](#), i.e., [geodesic](#) of the shortest [distance](#)  $d(p, q)$ .

<sup>a</sup>Hence, equivalently, complete as a topological space w.r.t. the underlying topology.

**Proof.** We start by proving (d) implies (e). Let  $\mathcal{M}$  be [geodesically complete](#), and let  $r := d(p, q)$ , and let  $\rho$  be as in the corollary from handout for HW1. Let  $p_0 \in \partial B(p, \rho)$  be a point where the continuous functional  $d(q, \cdot)$  attains its minimum on the compact set  $\partial B(p, \rho)$ . Then, for some

$$V \in T_p \mathcal{M},$$

$$p_0 = \exp_p \rho V.$$

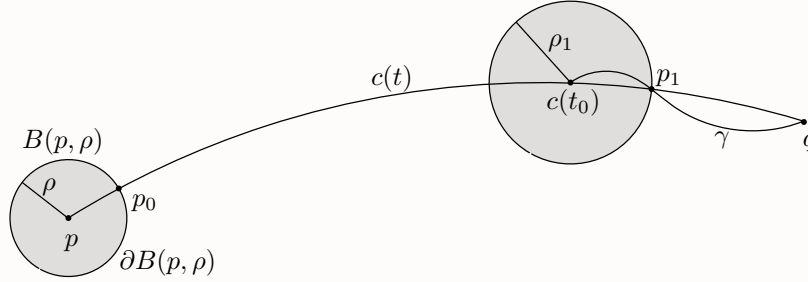
Consider the **geodesic**  $c(t) = \exp_p tV$ , by showing

$$c(r) = q,$$

$c|_{[0,r]}$  will be the shortest **geodesic** from  $p$  to  $q$ . We start by defining

$$I := \{t \in [0, r] \mid d(c(t), q) = r - t\},$$

and referring to the following diagram to guide us.



Now, we want to show that  $I = [0, r]$ , which will follow from showing that  $I$  is open.

**Note.**  $I$  is not empty since by definition it contains 0 and  $r$ . Further,  $I$  is closed by continuity.

Let  $t_0 \in I$ , and let  $\rho_1 > 0$  be the radius as in the corollary, without loss of generality,  $\rho_1 < r - t_0$ . Let  $p_1 \in \partial B(c(t_0), \rho_1)$  be the point where the continuous functional  $d(q, \cdot)$  attains its minimum on the compact set  $\partial B(c(t_0), \rho_1)$ . By the triangle inequality,

$$d(p, q) \leq d(p, p_1) + d(p_1, q).$$

For every curve  $\gamma$  from  $c(t_0)$  to  $q$ , there exists  $\gamma(t) \in \partial B(c(t_0), \rho_1)$ , hence

$$L(\gamma) \geq \underbrace{d(c(t_0), \gamma(t))}_{\rho_1} + d(\gamma(t), q) = \rho_1 + d(p_1, q),$$

implying  $d(q, c(t_0)) \geq \rho_1 + d(p_1, q)$ . But from the triangle inequality, we actually have

$$d(q, c(t_0)) = \rho_1 + d(p_1, q) \Leftrightarrow d(p_1, q) = \underbrace{d(q, c(t_0))}_{r - t_0} - \rho_1,$$

hence  $d(p_1, p) \geq r - (r - t_0 - \rho_1) = t_0 + \rho_1$ , i.e., this is a minimizing curve!

On the other hand, there exists a curve from  $p$  to  $p_1$  of length  $t_1 + \rho_1$  since it's composed by the portion from  $p$  to  $c(t_0)$  along  $c(t)$  and the portion being the **geodesic** from  $c(t_0)$  to  $p_1$  of length  $\rho_1$ . Then, by the theorem we have proved in the HW1#5, this curve is a **geodesic** curve. Finally, from the uniqueness of **geodesic** with the given extra data, this **geodesic** coincides with  $c$ . Hence,

$$p_1 = c(t_0 + \rho_1),$$

with  $d(p_1, q) = r - t_0 - \rho_1$ ,

$$d(c(t_0 + \rho_1), q) = d(p_1, q) = r - t_0 - \rho_1 = r - (t_0 + \rho_1),$$

thus  $t_0 + \rho_1 \in I$ , hence  $I$  is open, i.e.,  $I = [0, r]$ , so  $c(r) = q$  follows.

## Lecture 8: Injectivity Radius and Vector Bundles

In the proof we did last time, the last step can be shown via [FC13, Corollary 3.9].

**Proof of Hopf-Rinow theorem (Continued).** We see that (d) implies (e), hence we only need to

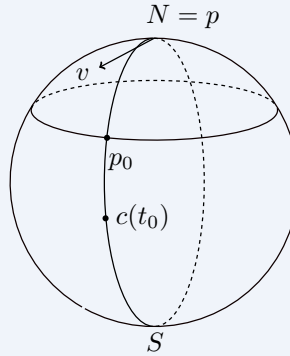
show that (a), (b), (c), and (d) are equivalent.

- (d)  $\Rightarrow$  (c) is trivial.
- (c)  $\Rightarrow$  (b): Let  $K \subseteq \mathcal{M}$  be closed and bounded. As  $K$  bounded,  $K \subseteq B(p, r)$  for some  $r > 0$ . Then any point in  $B(p, r)$  can be joined with  $p$  by **geodesic** of length  $\leq r$ , and  $B(p, r)$  is the image of the compact ball in  $T_p\mathcal{M}$  of radius  $r$  under continuous map  $\exp_p$ , hence  $B(p, r)$  is compact. As  $K$  closed and  $K \subseteq B(p, r)$ ,  $K$  is compact.
- (b)  $\Rightarrow$  (a): Let  $(p_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$  be a Cauchy sequence, so it's bounded, and by (b), its closure is compact. It contains a convergent subsequence, so it converges, i.e.,  $\mathcal{M}$  is **complete**.
- (a)  $\Rightarrow$  (d): Let  $c$  be a **geodesic** in  $\mathcal{M}$ , parametrized by arc length defined on a maximal interval  $I$ . Since  $I$  is non-empty, and we can show that  $I$  is both open and closed.

Exercise

It's worth mentioning that we do have uniqueness after choosing  $p_0$ , in other words, after choosing  $p_0$ , everything is fixed, so the non-uniqueness really comes from the initial choice of  $p_0$ .

**Example.** Consider  $S^2$ , after fixing  $p_0$ ,  $c(t_0)$  is extended uniquely.



### 3.4 Injectivity Radius

Consider the following.

**Definition 3.4.1** (Injectivity radius). Let  $\mathcal{M}$  be a **Riemannian manifold**, and  $p \in \mathcal{M}$ . The *injectivity radius*  $i(p)$  of  $p$  is

$$i(p) := \sup \{ \rho > 0 \mid \exp_p \text{ defined on } B(0, \rho) \subseteq T_p\mathcal{M} \text{ and injective} \}.$$

Similarly, the *injectivity radius*  $i(\mathcal{M})$  of  $\mathcal{M}$  is defined as  $i(\mathcal{M}) := \inf_{p \in \mathcal{M}} i(p)$ .

**Example** (Sphere).  $i(S^n) = \pi$ .

**Example** (Torus).  $i(T^n) = 1/2$ .

Any manifold carries a **complete Riemannian metric**.

If  $(\mathcal{M}, g_1)$  is not **complete**, we can find  $g_2$  such that  $(\mathcal{M}, g_2)$  is **complete**.

**Example** (Hyperbolic half-plane). The half-plane  $P = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$  equipped with metric induced by the Euclidean metric on  $\mathbb{R}^2$ , which is not **complete**.

However, it becomes **complete** when equipped with the following metric

$$\frac{1}{y^2}(dx^2 + dy^2).$$

In fact,  $P$  with the above metric is called the *hyperbolic half-plane*  $H^2$ , and we can extend it to  $H^n$ . Another question we may ask is the following.

**Problem.** Is the converse of **Hopf-Rinow theorem** true? I.e., can we show that **(e)** implies **(d)**?

**Answer.** No! Any 2 points in the open half-sphere can be joined by a unique minimal **geodesic**, but this manifold is not **geodesically complete**.  $\otimes$

**Example.** The **injectivity radius** of  $H^n$  is  $\infty$ .

**Remark.** Given a compact  $\mathcal{M}$ , the **injectivity radius** is always  $> 0$  by continuity argument.

Now, given a **complete** but not compact  $\mathcal{M}$ , the **injectivity radius** can be 0.

**Example.** Take the quotient of the Poincaré half-plane by the translations

$$(x, y) \mapsto (x + n, y), \quad n \in \mathbb{Z}.$$

We then obtain a **complete Riemannian manifold**  $\mathcal{M}$  with  $i(\mathcal{M}) = 0$ .

**Note.** Finding lower bounds for  $i(\mathcal{M})$  introduces curvature estimates.

# Chapter 4

## Affine and Riemannian Connections

### 4.1 Vector Bundles and Tensor Fields

#### 4.1.1 Vector Bundles

We first see one definition.

**Definition 4.1.1** (Vector bundle). A (differentiable) *vector bundle* of rank  $n$  is the tuple  $(E, \pi, \mathcal{M})$  consists of *base space*  $\mathcal{M}$ , *total space*  $E$ , and *bundle projection*  $\pi: E \rightarrow \mathcal{M}$  such that each *fiber*  $E_x := \pi^{-1}(x)$  of  $x \in \mathcal{M}$  carries a structure of an  $n$ -dimensional (real) vector space, and *local triviality* condition holds.

**Definition 4.1.2** (Base space). The *differentiable manifold*  $\mathcal{M}$  is called the *base space*.

**Definition 4.1.3** (Total space). The *differentiable manifold*  $E$  is called the *total space*.

**Definition 4.1.4** (Bundle projection). The (differentiable) continuous surjection  $\pi: E \rightarrow \mathcal{M}$  is called the *bundle projection*.

**Definition 4.1.5** (Local trivialization). For all  $x \in \mathcal{M}$ , the *local trivialization*  $(U, \varphi)$  consists a neighborhood  $U$  and *diffeomorphism*  $\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  such that for all  $y \in U$ ,

$$\varphi_y := \varphi|_{E_y} : E_y \rightarrow \{y\} \times \mathbb{R}^n$$

is a vector space isomorphism.

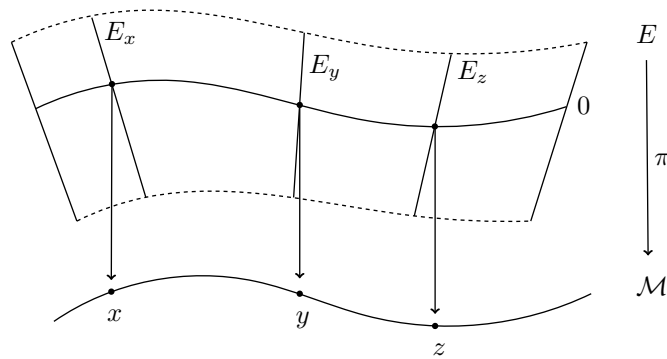


Figure 4.1: An illustration of *vector bundle*  $(E, \pi, \mathcal{M})$ .

**Notation** (Fiber). Given  $f: X \rightarrow Y$ , the *fiber* of  $y \in Y$  under  $f$  is the preimage of a  $\{y\}$ , i.e.,  $f^{-1}(\{y\})$ .

**Definition 4.1.6** (Tivial). A **vector bundle** is *trivial* if it's isomorphic to  $\mathcal{M} \times \mathbb{R}^n$ .<sup>a</sup>

<sup>a</sup> $n$  is the rank of the **vector bundle**.

**Intuition.** The **local trivialization** shows that *locally* the map  $\pi$  looks like the **projection** of  $U \times \mathbb{R}^n$  on  $U$ .

**Notation** (Bundle chart). The pair  $(\varphi, U)$  is also called the *bundle chart* in **local trivialization**.

**Remark.** From **Definition 4.1.1**, **vector bundle** is locally, but not necessarily globally a product of **base space** and the **fiber**.

**Intuition.** We may look at a **vector bundle** as a family of vector spaces, all isomorphic to a fixed  $\mathbb{R}^n$ , “parametrized” (**locally trivially**) by a **manifold**.

Finally, we note that given  $\pi: E \rightarrow \mathcal{M}$ , we have all information for defining a **vector bundle**, hence we sometimes just say something like *let  $\pi$  be the **vector bundle***.

## Lecture 9: Tensors and Connections

### 4.1.2 Contravariant and Covariant Tensors

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Let's introduce more tools in order to develop our theory.

**Definition 4.1.7** (Tensor field). Let  $V$  be a vector space of dimension  $m < \infty$ , and the dual space  $V^*$ .<sup>a</sup> Then the  *$r$ -times contravariant and  $s$ -times covariant tensors over  $V$  tensor field*, denoted as  $T_s^r(V)$ , is the vector field defined as

$$T_s^r(V) = \{A: \underbrace{V^* \times \dots \times V^*}_r \times \underbrace{V \times \dots \times V}_s \rightarrow \mathbb{R}\} = \underbrace{V \otimes \dots \otimes V}_r \otimes \underbrace{V^* \otimes \dots \otimes V^*}_s.$$

<sup>a</sup>I.e.,  $V^* := \{\lambda: V \rightarrow \mathbb{R} \mid \lambda \text{ linear}\}$ .

**Definition.** Let  $\Lambda^s(V^*) := \{A \in T_s^0(V) \mid A \text{ skew-symmetric}\}$ , where  $s \in \mathbb{N}$ . Let  $\mathcal{M}^n$  be a **manifold**, and  $\pi: E \rightarrow \mathcal{M}$  a  **$C^\infty$ -vector bundle**.

- (a)  $\Gamma(E) := \{s \in C^\infty(\mathcal{M}, E) \mid \pi \circ s = \text{id}_{\mathcal{M}}\}$ .
- (b)  $\Gamma(TM) := \{\text{vector fields on } \mathcal{M}\}$ .
- (c)  $\Gamma(\Lambda_s \mathcal{M}) := \{s\text{-forms on } \mathcal{M}\}$  with  $\Lambda_s \mathcal{M} = \Lambda^s \left( \bigcup_{p \in \mathcal{M}} T_p^* \mathcal{M} \right)$ .
- (d)  $\Gamma(T_s^r \mathcal{M}) := \{(r, s)\text{-tensor fields on } \mathcal{M}\}$  with  $T_s^r \mathcal{M}$  is the **section** of  $TM \otimes \dots \otimes TM \otimes T^* \mathcal{M} \otimes \dots \otimes T^* \mathcal{M}$ .

**Example.** A **Riemannian metric**  $g$  on  $\mathcal{M}$  is a **(0, 2)-tensor field**, i.e.,  $g \in \Gamma(T_2^0(\mathcal{M}))$  for all  $p \in \mathcal{M}$ .

**Proof.** Since  $g_p: T_p \mathcal{M} \times T_p \mathcal{M} \rightarrow \mathbb{R}$ .

⊛

## 4.2 Metrics, Connections and Curvatures

### 4.2.1 Metrics

We now discuss some other metrics on a [manifold](#).

**Definition 4.2.1** (Pseudo-Riemannian metric). A *pseudo-Riemannian metric* on a [differentiable manifold](#)  $\mathcal{M}$  is a [tensor field](#)  $g \in T_2^0(\mathcal{M})$  with

- (a)  $g(X, Y) = g(Y, X)$  for all  $X, Y \in T\mathcal{M}$ .
- (b) For all  $p \in \mathcal{M}$ ,  $g_p$  is non-degenerate bilinear form on  $T_p\mathcal{M}$ , i.e.,  $g_p(X, Y) = 0$  for all  $X, Y \in T_p\mathcal{M}$  if and only if  $Y = 0$ .

**Definition 4.2.2** (Lorentzian metric). A *Lorentzian metric*  $g$  is a continuous assignment of a non-degenerate<sup>a</sup> quadratic form  $g_p$  of index 1<sup>b</sup> in  $T_p\mathcal{M}$  for all  $p \in \mathcal{M}$ .

<sup>a</sup> $g_p(X, Y) = 0$  for all  $Y \in T_p\mathcal{M}$  implies  $X = 0$ .

<sup>b</sup>It means that the maximal dimension of a subspace of  $T_p\mathcal{M}$  on which  $g_p$  is negative definite is 1.

An equivalent definition is the following.

**Definition 4.2.3** (Lorentzian). A quadratic form  $g_p$  in  $T_p\mathcal{M}$  is *Lorentzian* if there exists a vector  $V \in T_p\mathcal{M}$  such that  $g_p(V, V) < 0$  while setting  $\Sigma_V = \{X \mid g_p(X, V) = 0\}$  such that  $g_p|_{\Sigma_V}$ <sup>a</sup> is positive definite.

<sup>a</sup>The  $g_p$ -orthogonal complement of  $V$ .

### 4.2.2 Connections

We now can talk about the notion of *connections*.

**Definition 4.2.4** (Linear connection). A *linear connection* (covariant derivative)  $\nabla$  (or  $D$ ) on  $T\mathcal{M}$  is a bilinear map

$$\nabla: \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M}),$$

and we write  $\nabla(X, Y) = \nabla_X Y$  with

- (a)  $\nabla_f X Y = f \nabla_X Y$ ;
- (b)  $\nabla_X f Y = X(f)Y + f \nabla_X Y$  for all [vector fields](#)  $X, Y \in \Gamma(T\mathcal{M})$ ,  $f \in C^\infty(\mathcal{M})$ .

**Definition 4.2.5** (Torsion tensor). Let  $\nabla$  be a [linear connection](#). The map  $T: \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M})$  defined as

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

is the *torsion tensor* of  $\nabla$ .

**Definition.** Let  $\nabla$  be a [linear connection](#).

**Definition 4.2.6** (Torsion-free).  $\nabla$  is *torsion-free* if  $T = 0$ .

**Definition 4.2.7** (Riemannian). If  $g$  is a [Riemannian metric](#) on  $\mathcal{M}$ , then  $\nabla$  is *Riemannian* (metric) if

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

for all  $X, Y, Z \in \Gamma(T\mathcal{M})$ .



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**Proposition 4.2.1** (Koszul formula). On each Riemannian manifold  $(\mathcal{M}, g)$ , there exists a unique Riemannian, torsion-free connection  $\nabla$  on  $T\mathcal{M}$  determined by the Koszul formula

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} (X \langle Y, Z \rangle - Z \langle X, Y \rangle + Y \langle Z, X \rangle - \langle X, [Y, Z] \rangle + \langle Z, [X, Y] \rangle + \langle Y, [Z, X] \rangle). \quad (4.1)$$

**Proof.** Firstly, we prove that for each Riemannian and torsion-free connection satisfies Equation 4.1. Then it will imply uniqueness. As for existence, we verify that the unique  $\mathbb{R}$ -bilinear map

$$\nabla: \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M})$$

given by Equation 4.1 has the desired properties, i.e., 2 product rules from connection, torsion-free, and being metric. ■

**Definition 4.2.8** (Levi-Civita connection). The *Levi-Civita connection* is the unique linear connection  $\nabla$  defined by the Koszul formula.

**Definition 4.2.9** (Riemannian curvature tensor). Let  $\nabla$  be the Levi-Civita connection on  $T\mathcal{M}$ . Then the *Riemannian curvature tensor*  $R: \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M})$  of  $\nabla$  is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

## Lecture 10

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# Appendix

# Appendix A

## Lie Groups and Lie Algebra

### A.1 Lie Groups

**Lie groups** are an important topic to study for Riemannian geometry, hence we now introduce it now.

**Definition A.1.1** (Lie group). A *Lie group* is a group  $G$  with a **differentiable structure** such that the mapping  $G \times G \rightarrow G$  given by  $(x, y) \rightarrow xy^{-1}$ ,  $x, y \in G$ , is differentiable.

**Definition** (Transformation). Let  $G$  be a **Lie group**.

**Definition A.1.2** (Left transformation). The *translations from the left*  $L_x: G \rightarrow G$  is defined as  $L_x(y) = xy$ .

**Definition A.1.3** (Right transformation). The *translations from the right*  $R_x: G \rightarrow G$  is defined as  $R_x(y) = yx$ .

**Remark.** Both  $L_x$  and  $R_x$  are **diffeomorphisms**.

In the following discussion, let  $G$  be a **Lie group**. Turns out that  $G$  admits some nice properties on **left invariant vector fields**.

**Definition** (Invariant of Riemannian metric). Let  $g$  be a **Riemannian metric** on  $G$ .

**Definition A.1.4** (Left invariant).  $g$  is *left invariant* if

$$\langle u, v \rangle_y = \langle d(L_x)_y u, d(L_x)_y v \rangle_{L_x(y)}$$

for all  $x, y \in G$ ,  $u, v \in T_y G$ , i.e.,  $L_x$  is an **isometry**.

**Definition A.1.5** (Right invariant).  $g$  is *right invariant* if

$$\langle u, v \rangle_y = \langle d(R_x)_y u, d(R_x)_y v \rangle_{R_x(y)}$$

for all  $x, y \in G$ ,  $u, v \in T_y G$ , i.e.,  $R_x$  is an **isometry**.

**Definition A.1.6** (Bi-invariant).  $g$  is *bi-invariant* if it's both **right** and **left invariant**.

**Definition** (Invariant of vector field). Let  $X$  be a **vector field** on  $G$ .

**Definition A.1.7** (Left invariant).  $X$  is *left invariant* if  $dL_x X = X$  for all  $x \in G$ .

**Definition A.1.8** (Right invariant).  $X$  is *right invariant* if  $dR_x X = X$  for all  $x \in G$ .

**Definition A.1.9** (Bi-invariant).  $X$  is *bi-invariant* if it's both [right](#) and [left invariant](#).

As we mentioned, the [left invariant vector fields](#) are completely determined by their values at a single point of  $G$ , which allows us to introduce an additional structure on the [tangent space](#) to the neutral element  $e \in G$  in the following manner.

To each [vector](#)  $X_e \in T_e G$ , we associate the [left invariant](#)  $X$  defined by

$$X_a := dL_a X_e, \quad a \in G.$$

## A.2 Lie Algebras

Let  $X, Y$  be [left invariant vector fields](#) on  $G$ . Since for each  $x \in G$  and for any differentiable function  $f$  on  $G$ ,

$$dL_x[X, Y]f = [X, Y](f \circ L_x) = X(dL_x Y)f - Y(dL_x X)f = (XY - YX)f = [X, Y]f,$$

i.e.,  $[X, Y]$  is again a [left invariant vector field](#) if  $X, Y$  are. Now, if  $X_e, Y_e \in T_e G$ , we put  $[X_e, Y_e] = [X, Y]_e$ .

**Definition A.2.1** (Lie algebra). The *Lie algebra* of  $G$ , denoted by  $\mathfrak{g}$ , is the vector space  $T_e G$  with the [bracket](#)  $[\cdot, \cdot]$ .

**Note.** The elements in the [Lie algebra](#)  $\mathfrak{g}$  will be thought of either as [vectors](#) in  $T_e G$  or as [left invariant vector fields](#) on  $G$ .

To introduce a [left invariant metric](#) on  $g$ , take any arbitrary inner product  $\langle \cdot, \cdot \rangle_e$  on  $\mathfrak{g}$  and define

$$\langle u, v \rangle_x := \langle (dL_{x^{-1}})_x(u), (dL_{x^{-1}})_x(v) \rangle_e \quad (\text{A.1})$$

for  $x \in G$ ,  $u, v \in T_x G$ . Since  $L_x$  depends differentiably on  $x$ , this is actually a [Riemannian metric](#), which is clearly [left invariant](#).

**Remark.** We can also construct a [right invariant metric](#) on  $G$ , and if  $G$  is compact,  $G$  possesses a [bi-invariant metric](#).

One important characterization for  $G$  having a [bi-invariant metric](#) is that the inner product that the [metric](#) determines on  $\mathfrak{g}$  satisfies the following relation.

**Proposition A.2.1.** If  $G$  has a [bi-invariant metric](#), then for any  $U, V, X \in \mathfrak{g}$ , the inner product that the [metric](#) determines on  $\mathfrak{g}$  satisfies

$$\langle [U, X], V \rangle = -\langle U, [V, X] \rangle.$$

**Proof.** See do Carmo [FC13, Page 40, 41]. ■

The important point about this relation is that it characterizes the [bi-invariant metrics](#) of  $G$  in the following sense.

**Remark.** If a positive bilinear form  $\langle \cdot, \cdot \rangle_e$  defined on  $\mathfrak{g}$  satisfies this relation, then the [Riemannian metrics](#) defined on  $G$  by [Equation A.1](#) is [bi-invariant](#).

# Bibliography

- [FC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013. ISBN: 9780817634902. URL: <https://books.google.com/books?id=ct91XCWkWEUC>.