

MATH635  
Riemannian Geometry

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## Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by [Lydia Bieri](#). Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [[FC13](#)] as our reference; while not required, but highly recommended have on. Apart from this, I also found [[Sch15](#)] very useful.



This course is taken in Winter 2023, and the date on the covering page is the last updated time.

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# Chapter 1

## Smooth Manifolds

### Lecture 1: A Foray to Smooth Manifolds

#### 1.1 Topological Manifolds

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Let's start with a common definition.

**Definition 1.1.1 (Topological manifold).** A *topological manifold*  $\mathcal{M}$  of dimension  $n$  is a (topological) Hausdorff space such that each point  $p \in \mathcal{M}$  has a neighborhood  $U$  homeomorphic via  $\varphi: U \rightarrow U'$  to an open subset  $U' \subseteq \mathbb{R}^n$ .

**Definition 1.1.2 (Local coordinate map).** For every  $p \in \mathcal{M}$ , the corresponding homeomorphism  $\varphi$  is called the *local coordinate map*.

**Definition 1.1.3 (Local coordinate).** The pull-back  $(x^1, \dots, x^n)$  of the *local coordinate map*  $\varphi$  from  $\mathbb{R}^n$  is called the *local coordinates* on  $U$ , given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

**Definition 1.1.4 (Coordinate chart).** The pair  $(U, \varphi)$  is called a (*coordinate*) *chart* on  $\mathcal{M}$ .

In other words, a *topological manifold* can be thought of as a space such that it looks like  $\mathbb{R}^n$  locally.



**Definition 1.1.5 (Atlas).** An *atlas*  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$  for a *manifold*  $\mathcal{M}$  is a collection of *charts* such that  $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$  are an open covering of  $\mathcal{M}$ , i.e.,  $\mathcal{M} = \bigcup_\alpha U_\alpha$ .

In other words, for all  $p \in \mathcal{M}$ , there exists a neighborhood  $U \subseteq \mathcal{M}$  and homeomorphism  $h: U \rightarrow U' \subseteq \mathbb{R}^n$  open.

**Definition 1.1.6 (Locally finite).** An *atlas* is said to be *locally finite* if each point  $p \in \mathcal{M}$  is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as  $\mathbb{R}^n$ , there's no clear way to make sense of differentiability of a *manifold*. But thankfully, we now have an explicit relation to the ambient space  $\mathbb{R}^n$  via  $\varphi_\alpha$ . To formalize, let  $\mathcal{A}$  be an *atlas* for a *manifold*  $\mathcal{M}$ , and assume that  $(U_1, \varphi_1), (U_2, \varphi_2)$  are 2 elements

of  $\mathcal{A}$ . Then clearly, the map  $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$  is a homeomorphism between 2 open sets of Euclidean spaces since both  $\varphi_1$  and  $\varphi_2$  are homeomorphism. Due to this map's importance, it has its own name.

**Definition 1.1.7 (Coordinate transition).** The map  $\varphi_2 \circ \varphi_1^{-1}$  is called the *coordinate transition* of  $\mathcal{A}$  for the pair of charts  $(U_1, \varphi_1), (U_2, \varphi_2)$ .



## 1.2 Differentiable Manifolds

Notice that the *coordinate transitions* are from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ ; hence differentiability makes sense now, which induces the following.

**Definition 1.2.1 (Differentiable atlas).** The atlas  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$  is *differentiable* if all *transitions* are differentiable.

**Remark.** Here, the differentiability depends on the content. Sometimes, we may want it to be  $C^\infty$ , and sometimes may be  $C^k$  for some finite  $k$ . On the other hand, smooth always refers to  $C^\infty$ . We'll use them interchangeably if it's clear which case we're referring to.

**Definition 1.2.2 (Equivalence atlas).** Two atlases  $\mathcal{U}, \mathcal{V}$  of a manifold are equivalent if for every  $(U, \varphi) \in \mathcal{U}, (V, \psi) \in \mathcal{V}$ ,

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

**Notation (Smoothly compatible).** Two charts  $(U, \varphi)$  and  $(V, \psi)$  are *smoothly compatible* if either  $U \cap V = \emptyset$  or  $\psi \circ \varphi^{-1}$  is a diffeomorphism.

This suggests the following.

**Definition 1.2.3 (Smooth structure).** A *smooth structure* on  $\mathcal{M}$  is an equivalence class  $\mathcal{U}$  of *coordinate atlas* with the property that all *transition functions* are diffeomorphisms.

**Remark.** We can also use the *maximal differentiable atlas* to be our differentiable structure.

**Definition 1.2.4 (Smooth manifold).** A *smooth manifold* is a manifold  $\mathcal{M}$  with a *smooth structure*.

In this way, we can do calculus on *smooth manifolds*! Furthermore, it now makes sense to say that a function  $f : \mathcal{M} \rightarrow \mathbb{R}$  is differentiable (or  $C^\infty$ ) by considering differentiability of  $f \circ \varphi^{-1}$  around  $p$ .

**Notation.** The collection of smooth functions on [smooth manifold](#)  $\mathcal{M}$  is denoted by  $C^\infty(\mathcal{M}, \mathbb{R})$ , or  $C^k(\mathcal{M}, \mathbb{R})$ .

**Remark.** The class  $C^\infty(\mathcal{M}, \mathbb{R})$  consists of functions with property is well-defined.

**Proof.** Let  $\mathcal{A}$  be any given [atlas](#) from [equivalence class](#) that defines the [smooth structure](#), and as we have shown, if  $(U, \varphi) \in \mathcal{A}$ , then  $f \circ \varphi^{-1}$  is smooth on  $\mathbb{R}^n$ . This requirement defines the same set of smooth functions no matter the choice of representative [atlas](#) by the nature of [Definition 1.2.2](#) requirement that defines the equivalent [manifolds](#).  $\circledast$

### 1.2.1 Orientation

Another essential property of a [manifold](#) is its orientability.

**Definition.** Consider an [atlas](#)  $\mathcal{A}$  for a [differentiable manifold](#)  $\mathcal{M}$ .

**Definition 1.2.5 (Oriented).**  $\mathcal{A}$  is *oriented* if all [transitions](#) have positive functional determinant.

**Definition 1.2.6 (Orientable).**  $\mathcal{M}$  is *orientable* if  $\mathcal{A}$  is an [oriented atlas](#).

Motivated by the above definitions, we see that we can actually use an [atlas](#) to define an [orientation](#).

**Definition 1.2.7 (Orientation).** Let  $\mathcal{M}$  be an [orientable manifold](#). Then a [oriented differentiable structure](#) is called an *orientation* of  $\mathcal{M}$ .

If  $\mathcal{M}$  possesses an [orientation](#), we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with [Definition 1.2.5](#).

**Remark.** Two [differentiable structures](#) obeying [Definition 1.2.5](#) determine the same [orientation](#) if the union again satisfying [Definition 1.2.5](#).

**Remark.** If  $\mathcal{M}$  is [orientable](#) and connected, then there exists exactly 2 distinct [orientations](#) on  $\mathcal{M}$ .

Now, we can see some examples of [smooth manifolds](#).

**Example (Sphere).** The sphere  $S^n \subseteq \mathbb{R}^{n+1}$  given by

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Consider  $U_i^+ = \{x \in S^n \mid x_i > 0\}$ ,  $U_i^- = \{x \in S^n \mid x_i < 0\}$  for  $i = 1, \dots, n+1$ , and  $h_i^\pm: U_i^\pm \rightarrow \mathbb{R}^n$  such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1}).$$

Note that the minimum [charts](#) needed to cover  $S^n$  is 2.

**Example.** Let  $\mathcal{M} = U \subseteq \mathbb{R}^n$ , then  $\{(U, \varphi)\}$  is a [smooth structure](#) with  $\varphi = \text{id}$ .

**Example.** Open sets of  $C^\infty$ -[manifolds](#) are  $C^\infty$ -[manifolds](#).

**Example (General linear group).**  $\text{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ , open.

**Example (Real projective space).**  $\mathbb{R}P^n = S^n / \sim$  where  $x \sim -x$  with  $\pi: S^n \rightarrow \mathbb{R}P^n$ ,  $x \mapsto [x]$ .

**Proof.**  $\pi$  is a homeomorphism on each  $U_i^+$  for  $i = 1, \dots, n+1$ , with

$$\{(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1}), i = 1, \dots, n+1\}$$

is a  $C^\infty$ -atlas for  $\mathbb{R}P^n$ . \*

**Note.** Observe that  $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ .

## Lecture 2: Maps Between Smooth Manifolds

### 1.2.2 Smooth Maps

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We can now consider the maps between manifolds, specifically, the smooth manifolds.

**Definition 1.2.8 (Smooth function).** Let  $M, N$  be two smooth manifolds, and let  $\mathcal{U}$  be locally finite atlas from the equivalence class that gives the smooth structure on  $M$ , and let  $\mathcal{V}$  be the corresponding for  $N$ . A map  $h: M \rightarrow N$  is said to be *smooth* if each map in the collection

$$\{\psi \circ h \circ \varphi^{-1} : h(U) \cap V \neq \emptyset\},$$

where  $(U, \varphi) \in \mathcal{U}$ ,  $(V, \psi) \in \mathcal{V}$  is  $C^\infty$ -differentiable as a map from one Euclidean space to another.



**Remark.** Equivalence relation guarantees that Definition 1.2.8 depends only on the smooth structure of  $M, N$ , but not on the chosen representative coordinate atlas.

**Definition.** Consider two smooth manifolds  $M, N$  and a smooth homeomorphism  $h: M \rightarrow N$  with smooth inverse.

**Definition 1.2.9 (Diffeomorphic).** The two manifolds  $M, N$  are said to be *diffeomorphic*.

**Definition 1.2.10 (Diffeomorphism).** The map  $h$  is said to be a *diffeomorphism*.

Let  $M_1, M_2$  be two smooth manifolds, and let  $\varphi: M_1 \rightarrow M_2$  be a diffeomorphism. Then

- (a)  $M_1$  is orientable if and only if  $M_2$  is orientable.
- (b) If in addition,  $M_1$  and  $M_2$  are both connected and oriented, then  $\varphi$  induces an orientation on  $M_2$  that may or may not coincide with the initial orientation of  $M_2$ .

If the induced orientation coincides, then we say  $\varphi$  preserves the orientation, otherwise  $\varphi$  reverses the orientation.

### 1.2.3 Grassmannian Manifold

Before proceeding, let's consider an interesting [smooth manifold](#).

**Definition 1.2.11 (Grassmannian manifold).** Given  $m, n \in \mathbb{N}$ , the so-called *Grassmannian manifold*  $G(n, m)$  is the set of all  $n$ -dimensional subspaces of  $\mathbb{R}^{n+m}$ .

**Note.**  $G(1, m)$  is just  $\mathbb{R}P^m$ , and  $G(0, m)$ ,  $G(n, 0)$  are one-point sets.

As we will soon see,  $G(n, m)$  has the [smooth structure](#) of an  $mn$ -dimensional [manifold](#).

**Intuition.** We obtain the [structure](#) by exhibiting an [atlas](#) whose [transitions](#) are [diffeomorphisms](#).

Firstly, we give  $G(n, m)$  a suitable topology, i.e., the metric topology. Let  $\Pi \in G(n, m)$ , and let  $\mathcal{L}(\Pi, \Pi^\perp)$  denote the  $mn$ -dimensional space of linear maps from  $\Pi$  to  $\Pi^\perp$ . Define the map

$$\varphi_\Pi: \mathcal{L}(\Pi, \Pi^\perp) \rightarrow G(n, m), \quad \varphi_\Pi(\alpha) = (\mathbb{1}_\Pi \oplus \alpha)(\Pi)$$

where  $\mathbb{1}_\Pi \oplus \alpha$  is regarded as a map  $\Pi \rightarrow \Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$ .<sup>1</sup> Clearly,  $\varphi_\Pi$  is injective, and thus,  $(\mathcal{L}(\Pi, \Pi^\perp), \varphi_\Pi)$  is an  $mn$ -dimensional [chart](#) of  $G(n, m)$ .

**Remark.** The images  $\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$  cover  $G(n, m)$ .

**Example.**  $\Pi = \varphi_\Pi(0) \in \varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$ .

We can now prove that these [charts](#) are mutually [compatible](#). Let  $\Pi, \Pi' \in G(n, m)$ , and let  $P, P'$  be orthogonal projections from  $\mathbb{R}^{n+m}$  onto  $\Pi, \Pi'$  respectively. Firstly,

$$F = \varphi_{\Pi'}^{-1} \varphi_\Pi: \varphi_\Pi^{-1}(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))) \rightarrow \varphi_{\Pi'}^{-1}(\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp)))$$

is smooth.



Consider  $\alpha \in \mathcal{L}(\Pi, \Pi^\perp)$ , and  $\beta \in \mathcal{L}(\Pi', (\Pi')^\perp)$ , then for  $\alpha, \beta$ , the equality  $F(\alpha) = \beta$  means that  $\varphi_\Pi(\alpha) = \varphi_{\Pi'}(\beta)$ . Let  $f_\alpha: \Pi \rightarrow \Pi'$  be defined by

$$f_\alpha = P' \circ (\mathbb{1}_\Pi \oplus \alpha).$$

We need to check

- (a)  $f_\alpha$  is invertible, and
- (b)  $\forall y \in \Pi, y + \alpha(y) = f_\alpha(y) + \beta(f_\alpha(y))$ .

**Note.** The condition that  $\det f_\alpha \neq 0$  gives an exact description of the subset  $\varphi_{\Pi'}^{-1}(\varphi_\Pi(\mathcal{L}(\Pi', (\Pi')^\perp)))$  of  $\mathcal{L}(\Pi, \Pi^\perp)$ , which is therefore open.

For  $\beta$ , it is  $(\mathbb{1}_{\Pi'} \oplus \beta) \circ f_\alpha = \mathbb{1}_\Pi \oplus \alpha$ , and hence

$$\beta = F(\alpha) = (\mathbb{1}_\Pi \oplus \alpha) \circ f_\alpha^{-1} - \mathbb{1}_{\Pi'}.$$

It follows by the construction that the image of  $\beta$  is contained in  $(\Pi')^\perp$ .

<sup>1</sup>In other words,  $\varphi_\Pi(\alpha)$  is the graph of  $\alpha$  in  $\Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$ .



**Remark.** We obtain an infinite atlas for  $G(n, m)$  with charts labeled by  $\Pi \in G(n, m)$ . But it's suffices to consider only  $\binom{n+m}{n}$  charts corresponding to subspaces  $\Pi$  spanned with  $n$  coordinate axes.

We now introduce two notions.

**Definition 1.2.12** (Closed manifold). A manifold is *closed* if it is compact and without boundary.

**Definition 1.2.13** (Open manifold). A manifold is *open* if it has only non-compact components without boundary.

**Lemma 1.2.1.** If  $M$  can be covered by two coordinate neighborhoods  $V_1, V_2$  such that  $V_1 \cap V_2$  is connected, then  $M$  is orientable.

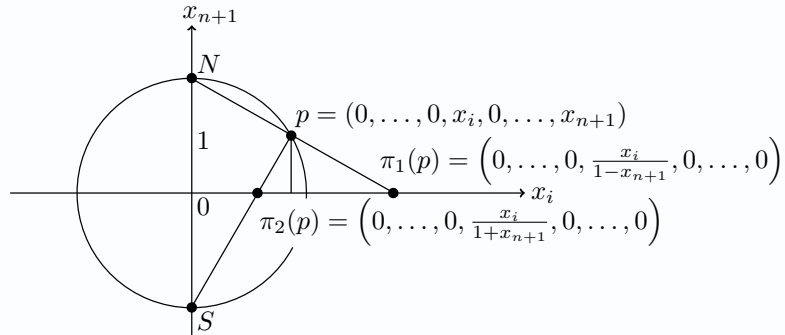
**Proof.** The determinant of the differential of the coordinate change  $\neq 0$ , so it does not change sign in  $V_1 \cap V_2$ . If it's negative at a single point, it's enough to change the sign of one of the coordinates to make it positive at that point, hence on  $V_1 \cap V_2$ . ■

**Example.** Let  $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$  is orientable.

**Proof.** Let  $N = (0, \dots, 0, 1)$  and  $S = (0, \dots, 0, -1)$ , consider given  $p = (0, \dots, 0, x_i, 0, \dots, x_{n+1})$ , then  $\pi_1: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  given by

$$\pi_1(p) = \left(0, \dots, 0, \frac{x_i}{1 - x_{n+1}}, 0, \dots, 0\right)$$

to be the stereographic projection from the north pole  $N$ .



More generally, it takes  $p(x_1, \dots, x_{n+1}) \in S^n - \{N\}$  into the intersection at the hyperplane  $x_{n+1} = 0$  with the line passing through  $p$  and  $N$ . In this way, we have

$$\pi_1(x_1, \dots, x_n) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right),$$

hence  $\pi_1: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  is differentiable, and is injective. Similarly,  $\pi_2: S^n \setminus \{S\} \rightarrow \mathbb{R}^n$  for  $S$  can also be defined and everything holds similarly. We see that these two parametrizations  $(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})$  cover  $S^n$ . The change of coordinate is given by

$$y_j = \frac{x_j}{1 - x_{n+1}} \leftrightarrow y'_j = \frac{x_j}{1 + x_{n+1}}, \quad (y_1, \dots, y_n) \in \mathbb{R}^n, \quad j = 1, \dots, n,$$

where

$$y'_j = \frac{y_j}{\sum_{i=1}^n y_i^2}.$$

This implies that  $\{(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})\}$  is a **differentiable structure** for  $S^n$ . Now, consider  $\pi_1^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^n) = S^n \setminus \{N \cup S\}$ , which is connected, and hence  $S^n$  is **orientable**, and the above **structure** gives an **orientation** of  $S^n$ .  $\circledast$

## Lecture 3: Complex Manifolds, Tangent Spaces and Bundles

Let's look at two more examples about **orientation**.

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**Example.** Let  $A: S^n \rightarrow S^n$  be the antipodal map given by  $A(p) = -p$  for  $p \in \mathbb{R}^{n+1}$ . It's easy to see that  $A$  is differentiable with  $A^2 = 1$ . Furthermore,  $A$  is **diffeomorphism** of  $S^n \subseteq \mathbb{R}^{n+1}$ . We see that

- if  $n$  is even,  $A$  reverses the **orientation**;
- if  $n$  is odd,  $A$  preserves the **orientation**.

**Example.**  $G(k, n)$  is **orientable** if and only if  $n$  is even or  $n = 1$ .

Finally, we introduce the notion of **complex manifolds**.

**Definition 1.2.14 (Complex manifold).** A *complex manifold*  $\mathcal{M}$  of complex dimension  $d$  ( $\dim_{\mathbb{C}} \mathcal{M} = d$ ) is a **differentiable manifold** of (real) dimension  $2d$  ( $\dim_{\mathbb{R}} \mathcal{M} = 2d$ ) whose **charts** take values in open subsets of  $\mathbb{C}^d$  with holomorphic **chart transitions**.

**As previously seen.** The **chart transitions**  $z_{\beta} \circ z_{\alpha}^{-1}: z_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow z_{\beta}(U_{\alpha} \cap U_{\beta})$  is holomorphic if  $\partial z_{\beta}^j / \partial z_{\alpha}^k = 0$  for all  $j, k$  where

$$\frac{\partial}{\partial z^k} = \frac{1}{2} \left( \frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k} \right).$$

**Remark.** **Complex Grassmannians**  $G_{\mathbb{C}}(k, n)$  are all **orientable**. More generally, **complex manifolds** are always **orientable** because holomorphic maps always have positive functional determinant.

### 1.3 Partition of Unity

We state, without proof, of an important lemma about the **partition of unity**.

**Definition 1.3.1 (Partition of unity).** Let  $\mathcal{M}$  be a **differentiable manifold**, and let  $(U_{\alpha})_{\alpha \in \mathcal{A}}$  be an open covering of  $\mathcal{M}$ . Then a *partition of unity* is a **locally finite** refinement  $(V_{\beta})_{\beta \in \mathcal{B}}$  of  $(U_{\alpha})$  and  $C^{\infty}$ -functions  $\varphi_{\beta}: \mathcal{M} \rightarrow \mathbb{R}$  with

- $\text{supp}(\varphi_{\beta}) \subseteq V_{\beta}$  for all  $\beta \in \mathcal{B}$ ;
- $0 \leq \varphi_{\beta}(x) \leq 1$  for all  $x \in \mathcal{M}$ ,  $\beta \in \mathcal{B}$ ;
- $\sum_{\beta \in \mathcal{B}} \varphi_{\beta} = 1$  for all  $x \in \mathcal{M}$ .<sup>a</sup>

<sup>a</sup>There are only finitely many non-vanishing summands of each point, since only finitely many  $\varphi_{\beta}$  are non-zero of any given point as the covering  $(V_{\beta})$  is **locally finite**.

**Lemma 1.3.1 (Partition of unity).** Let  $\mathcal{M}$  be a **differentiable manifold**, and let  $(U_{\alpha})_{\alpha \in \mathcal{A}}$  be an open covering of  $\mathcal{M}$ . Then there exists a **partition of unity** subordinate to  $(U_{\alpha})$ ,

## 1.4 Tangent and Cotangent Spaces

### 1.4.1 Tangent Spaces in Euclidean Spaces

To discuss the concept of calculus between [manifolds](#) formally, we start with our discussion in Euclidean spaces, where we naturally have the coordinates for every point.

**Definition.** Let  $\mathcal{M}$  be a Euclidean [manifold](#) of dimension  $d$ ,  $x = (x^1, \dots, x^d)$  be Euclidean coordinates of  $\mathbb{R}^d$ , and  $x_0 \in \Omega \subseteq \mathbb{R}^d$  where  $\Omega$  is open.

**Definition 1.4.1** (Tangent space of Euclidean space). The *tangent space*  $T_{x_0}\Omega$  of  $\Omega$  at  $x_0$  is the vector space  $\{x_0\} \times E^a$  spanned by the basis  $(\partial/\partial x^1, \dots, \partial/\partial x^d)$ .

<sup>a</sup> $E$  is a  $d$ -dimensional Euclidean space.

**Definition 1.4.2** (Tangent vector of Euclidean space). The elements in the [tangent space of Euclidean space](#) is called *tangent vectors*.

Before proceeding, we introduce a shorthand notation.

**Notation** ([Einstein notation](#)). The *Einstein notation* abbreviates the summation  $\sum_i v^i x_i$  as  $v^i x_i$ , where we implicitly sum over the upper and lower index.

**Definition 1.4.3** (Differential of Euclidean space). If  $\Omega \subseteq \mathbb{R}^d$ ,  $\Omega' \subseteq \mathbb{R}^d$  are open, and  $f: \Omega \rightarrow \Omega'$  is differentiable, then the *differential*  $df(x_0)$  for  $x_0 \in \Omega$  is the induced linear map between [tangent spaces](#)

$$df(x_0): T_{x_0}\Omega \rightarrow T_{f(x_0)}\Omega', \quad v = v^i \frac{\partial}{\partial x^i} \mapsto v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}.$$

**Definition 1.4.4** (Tangent bundle of Euclidean space). The *tangent bundle* is defined as  $T\Omega := \bigsqcup_{x \in \Omega} T_x\Omega \cong \Omega \times E \cong \Omega \times \mathbb{R}^d$ , which is an open subset of  $\mathbb{R}^d \times \mathbb{R}^d$ .

**Note** ([Total space](#)).  $T\Omega$  is also called the *total space*.

**Remark.** Given a [tangent bundle](#)  $T\Omega$ , we define  $\pi$  to be the projection  $\pi: T\Omega \rightarrow \Omega$  given by  $\pi(x, v) = x$ . This makes  $T\Omega$  naturally a [differentiable manifold](#).

With the notion of [tangent bundle](#), given  $f: \Omega \rightarrow \Omega'$ , we can also define  $df: T\Omega \rightarrow T\Omega'$  as

$$\left(x, v^i \frac{\partial}{\partial x^i}\right) \mapsto \left(f(x), v^i \frac{\partial f^j}{\partial x^i}(x) \frac{\partial}{\partial f^j}\right).$$

**Notation.** We often write  $df(x)(v)$  instead of  $df(x, v)$  to coincide with the notation of [differential](#).

In particular, for  $v = v^i \partial/\partial x^i$ , we have

$$df(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R},$$

and we write  $v(f)(x)$  for  $df(x)(v)$ .

### 1.4.2 Tangent Spaces in Manifolds

We now try to formally define the [tangent space](#) on a [smooth manifold](#). A natural idea is the following.

**Intuition.** Let  $\mathcal{M}^d$  be a differentiable manifold with a chart  $x: U \rightarrow \Omega \subseteq \mathbb{R}^d$  and  $p \in U \subseteq \mathcal{M}$  where  $U$  is open. The tangent space  $T_p\mathcal{M}$  of  $\mathcal{M}$  at  $p$  should be represented in the chart  $x$  by  $T_{x(p)}x(U)$ .

To see that the above are well-defined, i.e.,  $T_p\mathcal{M}$  are independent of the choice of charts, let  $x': U' \rightarrow \mathbb{R}^d$  to be another chart with  $p \in U' \subseteq \mathcal{M}$  where  $U'$  is also open. Denote  $\Omega := x(U)$ , and  $\Omega' := x'(U')$ , then the transition map

$$x' \circ x^{-1}: x(U \cap U') \rightarrow x'(U \cap U')$$

induces a vector space isomorphism

$$L := d(x' \circ x^{-1})(x(p)): T_{x(p)}\Omega \rightarrow T_{x'(p)}\Omega',$$

such that  $v \in T_{x(p)}\Omega$  and  $L(v) \in T_{x'(p)}\Omega'$  represent the same tangent vector in  $T_p\mathcal{M}$ .

**Remark.** A tangent vector in  $T_p\mathcal{M}$  is given by the family of the coordinate representations.

Now, we want to define the similar notion of differential of Euclidean spaces. Let consider a simple case first, where we let  $f: \mathcal{M} \rightarrow \mathbb{R}$  to be a differentiable function, and assume that the tangent vector  $w \in T_p\mathcal{M}$  is represented by  $v \in T_{x(p)}x(U)$ .

**Intuition.** We want to define  $df(p)$  as a linear map from  $T_p\mathcal{M} \rightarrow \mathbb{R}$ . In chart  $x$ , let  $w \in T_p\mathcal{M}$  be given as  $v = v^i \partial/\partial x^i \in T_{x(p)}x(U)$ . Say that  $df(p)(w)$  in this chart represented by

$$d(f \circ x^{-1})(x(p))(v).$$



**Remark.**  $T_p\mathcal{M}$  is a vector space of dimension  $d$  isomorphic to  $\mathbb{R}^d$ , where the isomorphism depends on choice of chart.

**Intuition.** Pull functions on  $\mathcal{M}$  back by a chart to an open subset of  $\mathbb{R}^d$ , differentiate there.

In order to obtain a tangent space which does not depend on charts, we need to have transformation behavior under change of charts. Let  $F: \mathcal{M}^d \rightarrow \mathcal{N}^c$  be a differentiable map where  $\mathcal{M}, \mathcal{N}$  are smooth manifolds. Then we want to represent  $dF$  in local charts  $x: U \subseteq \mathcal{M} \rightarrow \mathbb{R}^d, y: V \subseteq \mathcal{N} \rightarrow \mathbb{R}^c$  by  $d(y \circ F \circ x^{-1})$ . The local coordinates on  $U$  is given by  $(x^1, \dots, x^d)$ , and on  $V$  is  $(F^1, \dots, F^c)$  such that

$$F(x) = (F^1(x^1, \dots, x^d), \dots, F^c(x^1, \dots, x^d)).$$

Then,  $dF$  induces a linear map  $dF: T_p\mathcal{M} \rightarrow T_{F(x)}\mathcal{N}$  which in our coordinate representation is given by the matrix

$$\left( \frac{\partial F^\alpha}{\partial x^i} \right)_{\substack{\alpha=1, \dots, c \\ i=1, \dots, d}},$$

and a change of charts is then just the base change at tangent spaces: if

$$\begin{aligned} (x^1, \dots, x^d) &\mapsto (\xi^1, \dots, \xi^d) \\ (F^1, \dots, F^c) &\mapsto (\phi^1, \dots, \phi^c) \end{aligned}$$

are coordinate changes, then  $dF$  represented in the new coordinates is given by

$$\left( \frac{\partial \phi^\beta}{\partial \xi^j} \right) = \left( \frac{\partial \phi^\beta}{\partial F^\alpha} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial x^i}{\partial \xi^j} \right).$$



## Lecture 4: Tangent Bundles, Vector Fields, and Submanifolds

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**Definition.** Let  $\mathcal{M}^d$  be a **differentiable manifold** with a **chart**  $x: U \rightarrow \Omega \subseteq \mathbb{R}^d$  and  $p \in U \subseteq \mathcal{M}$  where  $U$  is open. On  $\{(x, v) \mid v \in T_{x(p)}\Omega\}$ , we define an equivalence relation by  $(x, v) \sim (y, w)$  if and only if  $w = d(y \circ x^{-1})v$ .

**Definition 1.4.5 (Tangent space).** The space of equivalence classes is called the *tangent space*  $T_p \mathcal{M}$  at point  $p$  to  $\mathcal{M}$ .

**Definition 1.4.6 (Tangent vector).** The elements in the **tangent space** is called *tangent vectors*.

**Remark.**  $T_p \mathcal{M}$  naturally carries the structure of a vector space.

Now,  $T\mathcal{M}$  is defined as

$$T\mathcal{M} := \coprod_{p \in \mathcal{M}} T_p \mathcal{M}.$$

Recall the projection  $\pi: T\mathcal{M} \rightarrow \mathcal{M}$  with  $\pi(V) = p$  for  $V \in T_p \mathcal{M}$ . Then we can define the following.

**Definition 1.4.7 (Derivation).** If  $x: U \rightarrow \mathbb{R}^d$  be a **chart** for  $\mathcal{M}$ , and let  $TU = \coprod_{p \in U} T_p U$ . Then we define the *derivation*  $dx: TU \rightarrow T_x(U) := \coprod_{p \in x(U)} T_p \mathcal{M}$  by  $w \mapsto dx(\pi(w))(w) \in T_{x(\pi(w))}x(U)$ .

The transition maps

$$dx' \circ (dx)^{-1} = d(x' \circ x^{-1})$$

are differentiable.  $\pi$  is local represented by  $x \circ \pi \circ dx^{-1}$  maps  $(x_0, v) \in T_x(U)$  to  $x_0$ .

**Definition 1.4.8 (Tangent bundle).** The triple  $(T\mathcal{M}, \pi, \mathcal{M})$  is called the *tangent bundle* of  $\mathcal{M}$ .

**Definition 1.4.9 (Total space).**  $T\mathcal{M}$  is called the *total space* of the **tangent bundle**.

We can choose the courses (the initial) topology for **total space**  $T\mathcal{M}$  such that  $\pi$  is continuous. Furthermore, we can construct a  **$C^\infty$ -atlas**  $\mathcal{A}_{T\mathcal{M}}$  on  $T\mathcal{M}$  from the  **$C^\infty$ -atlas**  $\mathcal{A}$  of  $\mathcal{M}$ . Specifically, consider  $\mathcal{A}_{T\mathcal{M}} := \{(TU, \xi_x) \mid (U, x) \in \mathcal{A}\}$  where  $\xi_x: TU \rightarrow \mathbb{R}^{2 \cdot d}$  such that

$$x \mapsto ((x^1 \circ \pi)(x), \dots, (x^d \circ \pi)(x), (dx^1)_{\pi(x)}(X), \dots, (dx^d)_{\pi(x)}(X)).$$

**Intuition.** We know that  $X = X_x^i (\partial/\partial x^i)_{\pi(x)}$ , and we might tempt to write  $X^i$  as the last  $d$  components. But we write it in the above way is because

$$(dx^j)_{\pi(x)}(X) = (dx^j)_{\pi(x)} \left( X_x^i \left( \frac{\partial}{\partial x^i} \right)_{\pi(x)} \right) = X_x^i \delta_i^j = X_x^j.$$

**Note.** We can check that  $\xi_x^{-1}$  exists, and it's also smooth, hence  $T\mathcal{M}$  has a natural topology and a  **$C^\infty$ -atlas** making it a  $2 \dim \mathcal{M}$ -dimensional **smooth manifold**.

### 1.4.3 Cotangent Spaces

Another important objects is the [cotangent spaces](#).

**Definition.** Let  $\mathcal{M}^d$  be a [differentiable manifold](#), and  $T_p\mathcal{M}$  be the [tangent space](#) at  $p$  to  $\mathcal{M}$ .

**Definition 1.4.10** (Cotangent space). The *cotangent space*  $T_p^*\mathcal{M}$  to  $\mathcal{M}$  is the dual of  $T_p\mathcal{M}$ , i.e.,  $T_p^*\mathcal{M} = (T_p\mathcal{M})^*$ .

**Definition 1.4.11** (Cotangent vector). The elements in the [cotangent space](#) is called *cotangent vectors*.

**Remark.**  $T_p^*\mathcal{M}$  is the space of 1-forms on  $T_p\mathcal{M}$ .

**Notation** (Covariant vector). The [cotangent vectors](#) are also called *covariant vectors*.

**Notation** (Contravariant vector). The [tangent vectors](#) are also called *contravariant vectors*.

Similarly, we can define the projection  $\pi: T^*\mathcal{M} \rightarrow \mathcal{M}$  with  $\pi(\omega) = p$  for  $\omega \in T_p^*\mathcal{M}$ , and we have the following.

**Definition 1.4.12** (Cotangent bundle). The triple  $(T^*\mathcal{M}, \pi, \mathcal{M})$  is called the *cotangent bundle* of  $\mathcal{M}$ .

## 1.5 Vector Fields and Brackets

### 1.5.1 Vector Fields

We now introduce the notion of [vector field](#).

**Definition 1.5.1** (Vector field). A (*tangent*) *vector field*  $X$  on a [differentiable manifold](#)  $\mathcal{M}$  is a correspondence associating to each point  $p \in \mathcal{M}$  a vector  $X(p) \in T_p\mathcal{M}$ , i.e.,  $X: \mathcal{M} \rightarrow T\mathcal{M}$ .

**Remark.** Naturally, we say that the [field](#)  $X$  is differentiable if the map  $X$  is differentiable.

Considering a [local chart](#)  $x: U \subseteq \mathbb{R}^n \rightarrow \mathcal{M}$ , we can write

$$X(p) = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i},$$

where  $a_i: U \rightarrow \mathbb{R}$  are functions on  $U$  for  $i = 1, \dots, n$ , and  $\{\partial/\partial x_i\}_i$  is the basis associated to  $x$ .

**Remark.**  $X$  is differentiable if and only if  $a_i$  are differentiable for some (and, therefore, for any)  $x$ .

It's convenient to think of a [vector field](#) as a mapping  $X: \mathcal{D} \rightarrow \mathcal{F}$  from the set  $\mathcal{D}$  of differentiable functions on  $\mathcal{M}$  to the set  $\mathcal{F}$  of the functions on  $\mathcal{M}$ , defined by

$$(Xf)(p) = \sum_{i=1}^n a_i(p) \frac{\partial f}{\partial x_i}(p),$$

where  $f$  is implicitly denoting the expression of  $f$  in the [chart](#)  $x$ .

**Intuition.** This idea of a vector as a directional derivative is precisely what was used to define the notion of [tangent vector](#).

**Remark.**  $Xf$  does not depend on the choice of  $x$ .

**Remark.**  $X$  is differentiable if and only if  $X: \mathcal{D} \rightarrow \mathcal{D}$ , i.e.,  $Xf \in \mathcal{D}$  for all  $f \in \mathcal{D}$ .

Observe that if  $\varphi: \mathcal{M} \rightarrow \mathcal{M}$  is a **diffeomorphism**,  $v \in T_p\mathcal{M}$  and  $f$  differentiable function in a neighborhood of  $\varphi(p)$ , we have

$$(d\varphi(v)f)\varphi(p) = v(f \circ \varphi)(p)$$

since by letting  $\alpha: (-\epsilon, \epsilon) \rightarrow \mathcal{M}$  be a differentiable **curve** with  $\alpha'(0) = v$ ,  $\alpha(0) = p$ ,<sup>2</sup> then

$$(d\varphi(v)f)\varphi(p) = \left. \frac{d}{dt}(f \circ \varphi \circ \alpha) \right|_{t=0} = v(f \circ \varphi)(p).$$

### 1.5.2 Brackets

By viewing  $X$  as an operator on  $\mathcal{D}$ , we can consider the iterates of  $X$ , i.e, given differentiable **fields**  $X$  and  $Y$  and  $f: \mathcal{M} \rightarrow \mathbb{R}$  being a differentiable function, consider  $X(Yf)$  and  $Y(Xf)$ .

**Note.** In general,  $X(Yf)$  (and hence  $Y(Xf)$ ) is not a **field**.

**Proof.** It involves derivatives of order higher than one. ⊗

But we have the following.

**Lemma 1.5.1.** Let  $X, Y$  be differentiable **vector fields** on a **smooth manifold**  $\mathcal{M}$ . Then there exists a unique **vector field**  $Z$  such that for all  $f \in \mathcal{D}$ ,  $Zf = (XY - YX)f$ .

**Proof.** See do Carmo [FC13, Chapter 0, Lemma 5.2]. ■

This  $Z$  is called the **bracket**.

**Definition 1.5.2 (Bracket).** Given two differentiable **vector fields**  $X, Y$  on a **smooth manifold**  $\mathcal{M}$ , the **bracket** of  $X$  and  $Y$  is defined by

$$[X, Y] := XY - YX.$$

Clearly,  $[X, Y]$  is differentiable.

**Proposition 1.5.1.** If  $X, Y$  and  $Z$  are differentiable **vector fields** on  $\mathcal{M}$ ,  $a, b \in \mathbb{R}$ ,  $f, g$  are differentiable functions, then we have the following.

- (a)  $[X, Y] = -[Y, X]$  (*anti-commutativity*),
- (b)  $[aX + bY, Z] = a[X, Z] + b[Y, Z]$  (*linearity*),
- (c)  $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$  (*Jacobi identity*),
- (d)  $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$ .

**Proof.** See do Carmo [FC13, Chapter 0, Proposition 5.3]. ■

**Example.**  $[\partial/\partial x^i, \partial/\partial x^j] = 0$  for  $i = j$ .

## 1.6 Submanifolds, Immersions, and Embeddings

We now study the relation between **manifolds**.

<sup>2</sup>This is the way do Carmo [FC13] used to define **tangent vectors**.

**Definition 1.6.1 (Immersion).** Let  $\mathcal{M}^m, \mathcal{N}^n$  be smooth manifolds. A differentiable mapping  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$  is an *immersion* if

$$d\varphi_p: T_p\mathcal{M} \rightarrow T_{\varphi(p)}\mathcal{N}$$

is injective for every  $p \in \mathcal{M}$ .

**Definition 1.6.2 (Embedding).** An immersion  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$  is an *embedding* if it is also a homeomorphism onto  $\varphi(\mathcal{M}) \subseteq \mathcal{N}$ , with  $\varphi(\mathcal{M})$  having the subspace topology induced from  $\mathcal{N}$ .

**Definition 1.6.3 (Submanifold).** If the inclusion  $\iota: \mathcal{M} \hookrightarrow \mathcal{N}$  between two manifolds is an embedding, then  $\mathcal{M}$  is a *submanifold* of  $\mathcal{N}$ .



(a) Non-differentiable curve.

(b) Non-immersion curve.

(c) Non-embedding curve.

Figure 1.1: Three simple examples

**Lemma 1.6.1.** Let  $f: \mathcal{M}^m \rightarrow \mathcal{N}^n$  to be an immersion and  $x \in \mathcal{M}$ .<sup>a</sup> Then there exists a neighborhood  $U$  of  $x$  and a chart  $(V, y)$  on  $\mathcal{N}$  with  $f(x) \in V$  such that  $f|_U$  is a differentiable embedding and  $y^{m+1}(p) = \dots = y^n(p) = 0$  for all  $p \in f(U \cap V)$ .

<sup>a</sup>Hence,  $n \geq m$ .

**Proof.** In the local coordinates  $(z^1, \dots, z^n)$  on  $\mathcal{N}$ , and  $(x^1, \dots, x^m)$  on  $\mathcal{M}$ , without loss of generality,<sup>a</sup> let

$$\left( \frac{\partial z^\alpha(f(x))}{\partial x^i} \right)_{i, \alpha=1, \dots, m}$$

be non-singular. Consider

$$F(z, x) := (z^1 - f^1(x), \dots, z^n - f^n(x)),$$

which has maximal rank in  $x^1, \dots, x^m, z^{m+1}, \dots, z^n$ . By the implicit function theorem, locally, there exists a map  $\varphi: U \rightarrow \mathbb{R}^n$  such that

$$(z^1, \dots, z^m) \mapsto (\varphi^1(z^1, \dots, z^m), \dots, \varphi^n(z^1, \dots, z^m)) = x$$

such that  $F(z, x) = 0$ , i.e.,

$$\varphi^i(z^1, \dots, z^m) = \begin{cases} x^i, & \text{if } i = 1, \dots, m; \\ z^i, & \text{if } i = m+1, \dots, n, \end{cases}$$

for which

$$\left( \frac{\partial \varphi^i}{\partial z^\alpha} \right)_{\alpha, i=1, \dots, m}$$

has maximal rank. Now, we choose a new coordinate

$$(y^1, \dots, y^n) = (\varphi^1(z^1, \dots, z^m), \dots, \varphi^m(z^1, \dots, z^m), \\ z^{m+1} - \varphi^{m+1}(z^1, \dots, z^m), \dots, z^n - \varphi^n(z^1, \dots, z^m)).$$



Then, we have  $z = f(x) \Leftrightarrow F(z, x) = 0$ , i.e.,  $(y^1, \dots, y^n) = (x^1, \dots, x^n, 0, \dots, 0)$ , proving the result.  $\blacksquare$

<sup>a</sup>Since  $df(x)$  is injective.

**Lemma 1.6.2.** Let  $f: \mathcal{M}^m \rightarrow \mathcal{N}^n$  be a differentiable map such that  $m \geq n$  with  $p \in \mathcal{N}$ . Let  $df(x)$  has rank  $n$  for all  $x \in \mathcal{M}$  with  $f(x) = p$ . Then  $f^{-1}(p)$  is the union of differentiable submanifolds of  $\mathcal{M}$  of dimension  $m - n$ .

**Remark.** Let  $\mathcal{N}^n$  be a smooth manifold, and let  $1 \leq m \leq n$ . Then an arbitrary subset  $\mathcal{M} \subseteq \mathcal{N}$  has the structure of differentiable submanifold of  $\mathcal{N}$  of dimension  $m$  if and only if for all  $p \in \mathcal{M}$ , there exists a smooth chart  $(U, \varphi)$  of  $\mathcal{N}$  such that  $p \in U$ ,  $\varphi(p) = 0$ ,  $\varphi(U)$  is open, and

$$\varphi(U \cap \mathcal{M}) = (-\epsilon, +\epsilon)^n \times \{0\}^{n-m},$$

where  $(-\epsilon, +\epsilon)^n$  is the cube. Noticeably, the  $C^\infty$ -manifold structure of  $\mathcal{M}$  is uniquely determined.

**Remark.** Let  $\mathcal{M} \subseteq \mathcal{N}$  be a differentiable submanifold of  $\mathcal{N}$ , and let  $\iota: \mathcal{M} \hookrightarrow \mathcal{N}$  be the inclusion. Then, for  $p \in \mathcal{M}$ ,  $T_p\mathcal{M}$  can be considered as subspace of  $T_p\mathcal{N}$ , namely as the image of  $d\iota(T_p\mathcal{M})$ .

**Lemma 1.6.3.** Let  $f: \mathcal{M}^m \rightarrow \mathcal{N}^n$  be a differentiable map such that  $m \geq n$  with  $p \in \mathcal{N}$ . Let  $df(x)$  has rank  $n$  for all  $x \in \mathcal{M}$  with  $f(x) = p$ . For the submanifold  $X = f^{-1}(p)$  and for  $q \in X$ , it is true that

$$T_qX = \ker df(q) \subseteq T_q\mathcal{M}.$$

# Chapter 2

## Riemannian Manifolds

### Lecture 5: Riemannian Manifolds

In this chapter, we start our discussion on [Riemannian manifolds](#).

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#### 2.1 Riemannian Metrics

We start by defining the [Riemannian metric](#).

**Definition 2.1.1** (Riemannian metric). A *Riemannian metric*  $g$  on a [differentiable manifold](#)  $\mathcal{M}$  is given by a scalar product  $I$  on each  $T_p\mathcal{M}$  which depends smoothly on the base point  $p$ .

**Definition 2.1.2** (Riemannian manifold). A *Riemannian manifold*  $(\mathcal{M}, g)$  is a [smooth manifold](#)  $\mathcal{M}$  equipped with a [Riemannian metric](#)  $g$ .

Let  $x = (x^1, \dots, x^d)$  be the [local coordinates](#). In these, a [metric](#) is represented by a positive definite symmetric matrix  $(g_{ij}(x))_{i,j=1,\dots,d}$ , i.e.,  $g_{ij} = g_{ji}$ , and  $g_{ij}\xi^i\xi^j > 0$  for all  $\xi = (\xi^1, \dots, \xi^d) \neq 0$  with coefficients smoothly depending on  $x$ .

##### 2.1.1 Transformation Behavior

We now see that the smoothness does not depend on [coordinates](#), i.e., the smooth dependence on the base point (as required in [Definition 2.1.1](#)) can be represented in the [local coordinates](#). Given 2 [tangent vectors](#)  $v, w \in T_p\mathcal{M}$  with [coordinate representations](#)  $(v^1, \dots, v^d), (w^1, \dots, w^d)$  given by  $x$  such that  $v = v^i \frac{\partial}{\partial x^i}$  and  $w = w^j \frac{\partial}{\partial x^j}$ , their product is

$$\langle v, w \rangle := g_{ij}(x(p))v^i w^j.$$

In particular,

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = g_{ij}.$$

**Remark.** The length of  $v$  is given as  $\|v\| := \langle v, v \rangle^{1/2}$ .

Let  $y = f(x)$  define different [local coordinates](#). In these,  $v, w$  are given as

$$(\tilde{v}^1, \dots, \tilde{v}^d), (\tilde{w}^1, \dots, \tilde{w}^d)$$

with  $\tilde{v}^j = v^i \frac{\partial f^j}{\partial x^i}$  and  $\tilde{w}^j = w^i \frac{\partial f^j}{\partial x^i}$ . Denote the [metric](#) in new [coordinates](#)  $y$  by  $h_{k\ell}(y)$ , then we have

$$h_{k\ell}(f(x))\tilde{v}^k \tilde{w}^\ell = \langle v, w \rangle = g_{ij}(x)v^i w^j.$$

Plug everything in, we have

$$h_{k\ell}(f(x)) \frac{\partial f^k}{\partial x^i} \frac{\partial f^\ell}{\partial x^j} v^i w^j = g_{ij}(x)v^i w^j.$$

We see that this holds for any **tangent vectors**  $v, w$ , therefore,

$$h_{k\ell}(f(x)) \frac{\partial f^k}{\partial x^i} \frac{\partial f^\ell}{\partial x^j} = g_{ij}(x),$$

which is the transformation behavior under **coordinates changes**.

**Remark.** This shows that the smoothness does not depend on the choice of coordinates!

**Example.** Consider the Euclidean space  $\Omega$ , then given  $v, w \in T_p\Omega$ , we have

$$\langle v, w \rangle = \delta_{ij} v^i w^j = v^i w_i.$$

**Theorem 2.1.1.** Every **differentiable manifold** can be equipped with a **Riemannian metric**.

**Proof.** From **Lemma 1.3.1**, there exists a differentiable **partition of unity**  $\{f_\alpha\}$  of  $\mathcal{M}$  subordinate to a covering  $\{V_\alpha\}$  of  $\mathcal{M}$ . Consider the induced **metric**  $\langle \cdot, \cdot \rangle^\alpha$  of the system of **local coordinates** on each  $V_\alpha$ . Then, for every  $p \in M$ , a **Riemannian metric**  $\langle \cdot, \cdot \rangle_p$  can be defined naturally as

$$\langle u, v \rangle_p = \sum_\alpha f_\alpha(p) \langle u, v \rangle_p^\alpha$$

for all  $u, v \in T_p M$ . Given the fact that  $\{f_\alpha\}$  is the **partition of unity**, we know that

- (a)  $f_\alpha \geq 0$ , and  $f_\alpha = 0$  on  $\overline{V_\alpha}^c$ ,
- (b)  $\sum_\alpha f_\alpha(p) = 1$  for all  $p$  on  $M$ ,

it's then immediate that the defined is indeed a **Riemannian metric**. ■

### 2.1.2 Isometry

After introducing any type of mathematical structure, we must introduce a notion of when two objects are the same, hence we now characterize  $g$ .

**Definition 2.1.3 (Isometry).** A **diffeomorphism**  $h: \mathcal{M} \rightarrow \mathcal{N}$  is an *isometry* between two **Riemannian manifolds** if it preserves the **Riemannian metric**, i.e., for  $p \in \mathcal{M}$ ,  $v, w \in T_p \mathcal{M}$ ,

$$\langle v, w \rangle_{\mathcal{M}} = \langle dh(v), dh(w) \rangle_{\mathcal{N}}.$$

**Definition 2.1.4 (Local isometry).** A **diffeomorphism**  $h: \mathcal{M} \rightarrow \mathcal{N}$  is a *local isometry* between two **Riemannian manifolds** if for every  $p \in \mathcal{M}$ , there exists a neighborhood  $U$  such that  $h|_U: U \rightarrow h(U): \mathcal{M} \rightarrow \mathcal{N}$  is an **isometry** and  $h(U) \subseteq \mathcal{N}$  is open.

It's common to say that a **Riemannian manifold**  $\mathcal{M}$  is **locally isometric** to a **Riemannian manifold**  $\mathcal{N}$  if for every  $p \in \mathcal{M}$ , there exists a neighborhood  $U$  of  $p$  in  $\mathcal{M}$  and a **local isometry**  $f: U \rightarrow f(U) \subseteq \mathcal{N}$ .

**Example (Euclidean space).** The *Euclidean space of dimension  $n$*   $\mathcal{M} = \mathbb{R}^n$  with  $\partial/\partial x_i$  identified with  $e_i = (0, \dots, 1, \dots, 0)$  is with the metric

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

The Riemannian geometry of this space is metric Euclidean geometry.

**Example (Lie group).** See **Appendix A.4** for reference.

## 2.2 Geodesics

This is the first focus on the study of Riemannian geometry, i.e., the [geodesics](#). The up-shot is that a [geodesic](#) minimizes the [arc length](#) for points *sufficiently close* (in a sense to be made precise); in addition, if a [curve](#) minimizes [arc length](#) between any two of its points, it is a [geodesic](#).

### 2.2.1 Vector Fields along Curves

We are now going to show how a [Riemannian metric](#) can be used to calculate the [length](#) of a [curve](#).

**Definition 2.2.1 (Curve).** A (parametrized) *curve* is a differentiable mapping  $c: I \subseteq \mathbb{R} \rightarrow \mathcal{M}$  to a [smooth manifold](#)  $\mathcal{M}$ .

**Note.** A parametrized [curve](#) can admit self-intersections as well as corners.



**Definition 2.2.2 (Vector field along a curve).** A (smooth) *vector field*  $X$  along a curve  $c: I \subseteq \mathbb{R} \rightarrow \mathcal{M}$  on a [smooth manifold](#)  $\mathcal{M}$  is defined as  $X: I \rightarrow T\mathcal{M}$  such that  $X(t) \in T_{c(t)}\mathcal{M}$  for all  $t \in I$ .

**Notation.** The set of smooth [vector fields along](#)  $c$  is denoted as  $\chi_c(\mathcal{M})$ .

**Note.** To say  $V$  is differentiable means that for any differentiable function  $f$  on  $\mathcal{M}$ , the function  $t \mapsto V(t)f$  is a differentiable function on  $I$ .

**Example (Velocity field).** The [vector field along](#)  $c$   $dc/dt := dc(d/dt)$  is called the *velocity field* or *tangent vector field*.

**Remark.** A [vector field along](#)  $c$  can't necessarily be extended to a [vector field](#) on an open set of  $\mathcal{M}$ .

**Notation (Segment).** The restriction of a [curve](#)  $c$  to a closed interval  $[a, b] \subseteq I$  is called a *segment*.

### 2.2.2 Lengths and Energies

We're interested in the following two quantities.

**Definition.** Let  $\gamma: [a, b] \rightarrow \mathcal{M}$  be a [curve](#) on a [Riemannian manifold](#)  $(\mathcal{M}, g)$ .

**Definition 2.2.3 (Length).** The *length* of  $\gamma$  is defined as

$$L(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt.$$

**Definition 2.2.4 (Energy).** The *energy* of  $\gamma$  is defined as

$$E(\gamma) := \frac{1}{2} \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt.$$

We now want to compute  $L(\gamma)$ ,  $E(\gamma)$  in **local coordinates**. Let the **local coordinates** be

$$(x^1(\gamma(t)), \dots, x^d(\gamma(t))),$$

we write

$$\dot{x}^i(t) = \frac{d}{dt}(x^i(\gamma(t))).$$

Then, we have

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t))) \dot{x}^i(t) \dot{x}^j(t)} dt, \quad E(\gamma) = \frac{1}{2} \int_a^b g_{ij}(x(\gamma(t))) \dot{x}^i(t) \dot{x}^j(t) dt.$$

**Definition 2.2.5 (Distance).** Given a **Riemannian manifold**  $(\mathcal{M}, g)$ , the *distance* between 2 points  $p, q \in \mathcal{M}$  is defined as

$$d(p, q) := \inf \{L(\gamma) \mid \gamma: [a, b] \rightarrow \mathcal{M} \text{ piecewise curve with } \gamma(a) = p, \gamma(b) = q\}.$$

**Note.** Any 2 points  $p, q \in \mathcal{M}$  can be connected by a piecewise **curve**, hence  $d(p, q)$  always exists.

**Corollary 2.2.1.** The topology of  $\mathcal{M}$  induced by the **distance function**  $d$  coincides with the original manifold topology of  $\mathcal{M}$ .

**Lemma 2.2.1.** If  $\gamma: [a, b] \rightarrow \mathcal{M}$  is a **curve**, and  $\psi: [\alpha, \beta] \rightarrow [a, b]$  is a reparametrization, then  $L(\gamma \circ \psi) = L(\gamma)$ .

**Proof.** This can be proved by computation, and the take-away is that the **length functional** is invariant under parameter changes. ■

### 2.2.3 Geodesic Equations as Euler-Lagrange Equations

We want to find a **curve** which minimizes the **length** between sufficiently close two points. It turns out that instead of working with **length** directly, we should work with **energy** instead.

**Notation.** Let's first fix some common notations.

(a)  $(g^{ij})_{i,j=1,\dots,d} = (g_{ij})_{i,j=1,\dots,d}^{-1}$ .<sup>a</sup>

(b)  $g_{j\ell,k} := \frac{\partial}{\partial x^k} g_{j\ell}$ .

<sup>a</sup>Technically,  $g^{-1}$  is not an inverse:  $g$  is a **(0, 2)-tensor field**, while  $g^{-1}$  is a **(2, 0)-tensor field**.

**Note.** In the above notations, we have  $g^{i\ell} g_{\ell j} = \delta_j^i$ .

And the following is particularly important.

**Notation (Christoffel symbol).** The *Christoffel symbol* is defined for all  $i$  as

$$\Gamma_{jk}^i := \frac{1}{2} g^{i\ell} (g_{j\ell,k} + g_{k\ell,j} - g_{j\ell,k}).$$

**Remark.** The notion of  $\Gamma$  is a bit cryptic at first, and we will come back to this after. Now, just treat it as a calculation tool.

The up-shot is that the **Euler-Lagrange equations** for the **energy**  $E$  has a nice form, and the solution of which has exactly the properties we want, hence we define it as **geodesics**.

**Proposition 2.2.1.** The **Euler-Lagrange equations** for the **energy**  $E$  are

$$\ddot{x}^i(t) + \Gamma_{jk}^i(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0 \text{ for } i = 1, \dots, d. \quad (2.1)$$

**Proof.** The **Euler-Lagrange equations** of a functional<sup>a</sup>

$$I(x) = \int_a^b f(t, x(t), \dot{x}(t)) dt$$

are

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0$$

for  $i = 1, \dots, d$ . Just by plugging in, we obtain for  $E$ , we have

$$\frac{d}{dt} (g_{ik}(x(t))\dot{x}^k(t) + g_{ji}(x(t))\dot{x}^j(t)) - g_{jk,i}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$$

for  $i = 1, \dots, d$ . Hence,

$$g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j + g_{ik,\ell}\dot{x}^\ell\dot{x}^k + g_{ji,\ell}\dot{x}^\ell\dot{x}^j - g_{jk,i}\dot{x}^\ell\dot{x}^j = 0$$

Rename some indices and use  $g_{ij} = g_{ji}$ , we have that

$$2g_{\ell m}\ddot{x}^m + (g_{k\ell,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0$$

for  $\ell = 1, \dots, d$ . Hence, we have

$$g^{i\ell}g_{\ell m}\ddot{x}^m + \frac{1}{2}g^{i\ell}(g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0$$

for  $i = 1, \dots, d$ . Finally, observe that  $g^{i\ell}g_{\ell m} = \delta_{im}$ , i.e.,  $g^{i\ell}g_{\ell m}\ddot{x}^m = \ddot{x}^i$ , hence the claim follows. ■

<sup>a</sup>The Lagrangian is  $\mathcal{L} = \frac{1}{2}g_{jk}\dot{x}^j\dot{x}^k$ .

Finally, we define the **geodesics** as the solution of **Equation 2.1**.

**Definition 2.2.6 (Geodesic).** A **curve**  $\gamma: [a, b] \rightarrow \mathcal{M}$  that obeys **Equation 2.1** is called a *geodesic*.

**Intuition.** **Geodesic** is the critical points of **energy**.<sup>a</sup>

<sup>a</sup>In fact, we can also start from **length** and get the same thing, which might be more natural.

## 2.2.4 Solving the Geodesic Equations

We now discuss why **geodesic** is well-defined, i.e., we want to show that **Equation 2.1** has a unique solution. We solve this via the variational principal, and we first define the **action functional**.

**Definition 2.2.7 (Action functional).** Let  $\mathcal{L}$  be the Lagrangian, then the *action functional*

$$I[w(\cdot)] := \int_0^t \mathcal{L}(\dot{w}(s), w(s)) ds$$

is defined for functions  $w(\cdot) = (w^1(\cdot), \dots, w^n(\cdot))$  of the admissible class

$$\mathcal{A} = \{w(\cdot) \in C^2([0, t]; \mathbb{R}^n) \mid w(0) = y, w(t) = x\}.$$

**Example.** Both **length** and **energy** are **action functionals**.

From the calculus of variation, we can find a **curve**  $x(\cdot) \in \mathcal{A}$  such that  $I[x(\cdot)] = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)]$ .

**Theorem 2.2.1** (Euler-Lagrangian equations). The solution  $x(\cdot)$  from  $I[x(\cdot)] = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)]$  solves the system of **Euler-Lagrangian equations**

$$\frac{d}{ds} (D_{\dot{x}} \mathcal{L}(\dot{x}(s), x(s)) + D_x \mathcal{L}(\dot{x}(s), x(s))) = 0$$

for  $0 \leq s \leq t$ .

## Lecture 6: Geodesics and the Exponential Map

Now, we draw some relations between **length** and **energy** and see why starting from **energy** makes sense. 24 Jan. 13:00

**Proposition 2.2.2.** For all **curves**  $\gamma: [a, b] \rightarrow \mathcal{M}$ ,

$$\mathcal{L}(\gamma)^2 \leq 2(b-a)E(\gamma)$$

with equality if and only if  $\|d\gamma/dt\|$  is a constant.

**Proof.** From **Hölder's inequality**,

$$\int_a^b \left\| \frac{d\gamma}{dt} \right\| dt \leq (b-a)^{1/2} \left( \int_a^b \left\| \frac{d\gamma}{dt} \right\|^2 dt \right)^{1/2}$$

with equality if and only if  $\|d\gamma/dt\|$  is a constant. ■

**Intuition.** We see that if we can minimize  $E$ , we also minimize  $L$  since both are positive!

**Example.** Let

$$\mathcal{L}(q, x) = \frac{1}{2}m|q|^2 - V(x)$$

with  $m > 0, q = \dot{x}$ , the Euler-Lagrangian equations is given by  $m\ddot{x}(s) = F(x(s))$  for  $F := -DV$ .

Since regular curves can be parametrized by **arc length** with unit speed  $\|d\gamma/dt\| = \|\dot{\gamma}\| \equiv 1$ , the following is natural.

**Lemma 2.2.2.** Each **geodesic** is parametrized proportionally to the **arc length**, i.e.,  $\|\dot{\gamma}\|$  is a constant.

**Proof.** For a solution  $x(t)$  of  $\ddot{x}^i(t) + \Gamma_{jk}^i(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$  (i.e., the **geodesic**), we have

$$\frac{d}{dt} \langle \dot{x}, \dot{x} \rangle = \frac{d}{dt} (g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)) = 0.$$

■

Since the **length** and the **energy** functionals are invariants under parameter changes, it's enough to look at **curves** parametrized by **arc length**.

**Theorem 2.2.2.** Let  $\mathcal{M}$  be a **Riemannian manifold**,  $p \in \mathcal{M}$  and  $v \in T_p\mathcal{M}$ . Then there exists an  $\epsilon > 0$  and a unique **geodesic** such that  $c: [0, \epsilon] \rightarrow \mathcal{M}$  with  $c(0) = p$  and  $\dot{c}(0) = v$ . In addition,  $c$  smoothly depend on  $p, v$ .

**Proof.** Since **Equation 2.1** is a system of second order ODE, by **Picard-Lindelöf theorem**, we have local existence and uniqueness of the solution with prescribed initial values and derivative such that the solution depends smoothly on  $p, v$ . ■

If  $x(t)$  is the solution of **Equation 2.1**, then  $x(\lambda t)$  is also a solution for any constant  $\lambda \in \mathbb{R}$ . Denote **geodesic** from **Theorem 2.2.2** by  $c_v$ , then

$$c_v(t) = c_{\lambda v}(t/\lambda)$$

for  $\lambda > 0$ ,  $t \in [0, \epsilon]$ , and hence  $c_{\lambda v}$  defined on  $[0, \epsilon/\lambda]$ .

**Remark.** Since  $c_v$  depends smoothly on  $v$ , the set  $\{v \in T_p\mathcal{M} \mid \|v\| = 1\}$  is compact, hence there exists  $\epsilon_0 > 0$  such that for  $\|v\| = 1$ ,  $c_v$  defined at least on  $[0, \epsilon_0]$ , implying that for all  $w \in T_p\mathcal{M}$  with  $\|w\| \leq \epsilon_0$ ,  $c_w$  is defined at least on  $[0, 1]$ .

### 2.2.5 Exponential Maps and Normal Coordinates

The above discussion permits us to introduce the concept of the [exponential map](#) in the following manner.

**Definition 2.2.8 (Exponential map).** Let  $(\mathcal{M}, g)$  be a [Riemannian manifold](#),  $p \in \mathcal{M}$ , and  $V_p := \{v \in T_p\mathcal{M} \mid c_v \text{ defined on } [0, 1]\}$ . The *exponential map* of  $\mathcal{M}$  at  $p$ ,  $\exp_p: V_p \rightarrow \mathcal{M}$ , is defined as  $v \mapsto c_v(1)$ .

Clearly,  $\exp$  is differentiable, and we shall utilize the restriction of  $\exp$  to an open subset of the [tangent space](#)  $T_q\mathcal{M}$ , i.e., we define

$$\exp_p: B(0, \epsilon) \subseteq T_p\mathcal{M} \rightarrow \mathcal{M},$$

where  $B(0, \epsilon)$  is an open ball with center at the origin 0 of  $T_p\mathcal{M}$  of radius  $\epsilon$ . It's easy to see that  $\exp_p$  is differentiable and that  $\exp_p(0) = p$ .

**Intuition.** Geometrically,  $\exp_p(v)$  is a point of  $\mathcal{M}$  obtained by going out the [length](#) equal to  $|v|$ , starting from  $p$ , along a [geodesic](#) which passes through  $p$  with velocity equal to  $v/|v|$ .

**Proposition 2.2.3.** The [exponential map](#)  $\exp_p$  maps a neighborhood of  $0 \in T_p\mathcal{M}$  [diffeomorphically](#) onto a neighborhood of  $p \in \mathcal{M}$ .

**Proof.** We see that

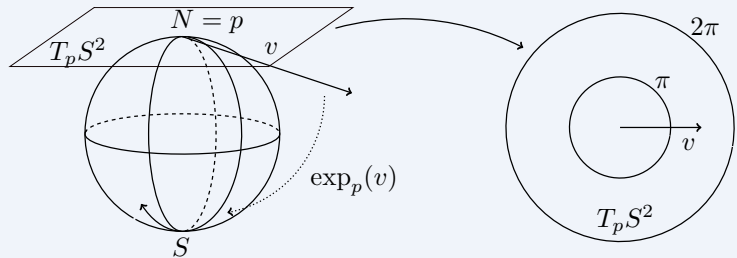
$$d(\exp_p)_0(v) = \left. \frac{d}{dt} \exp_p(tv) \right|_{t=0} = \left. \frac{d}{dt} c_{tv}(1) \right|_{t=0} = \left. \frac{d}{dt} c_v(t) \right|_{t=0} = v,$$

i.e.,  $d(\exp_p)_0$  is the identity of  $T_q\mathcal{M}$ . By the inverse function theorem,  $\exp_p$  is a local [diffeomorphism](#) on a neighborhood of 0. ■

**Example.** Let  $\mathcal{M} = \mathbb{R}^n$ , then the [exponential map](#) is the identity.<sup>a</sup>

<sup>a</sup>With the usual identification of  $T_p\mathbb{R}^n$  at  $p$  with  $\mathbb{R}^n$ .

**Example.** Let  $\mathcal{M} = S^2$ .



Now we know that  $\exp_p: B(0, \epsilon) \subseteq T_p\mathcal{M} \rightarrow \mathcal{M}$  maps [diffeomorphically](#) onto its image, we then define the following.

**Definition 2.2.9 (Normal coordinate).** Given an [exponential map](#)  $\exp_p: B(0, \epsilon) \rightarrow \mathcal{M}$ , let  $(e_1, \dots, e_n)$  be the orthonormal basis of  $T_p\mathcal{M}$ . Then the associated [local coordinates](#) are the *normal coordinates*.



In this case, given  $p \in \mathcal{M}^n$ ,  $0 \in \mathbb{R}^n$ , for all  $i, j, k$ ,<sup>1</sup>

$$g_{ij}(p) = \delta_{ij}, \quad \Gamma_{ij}^k(p) = 0, \quad g_{ij,k} = 0.$$

**Intuition.** The first derivative vanishes, so locally, the manifold looks Euclidean.

**Note.** [FC13] introduces everything above using  $T\mathcal{M}$  instead of  $T_p\mathcal{M}$ .

## 2.3 Hopf-Rinow Theorem

With all the tools we have developed, we now want to characterize the minimizing property of geodesics.

### 2.3.1 Riemannian Polar Coordinates

A particular useful tool is the Riemannian polar coordinates, which is introduced as follows.

**Theorem 2.3.1.** For all  $p \in \mathcal{M}$ , there exists  $\rho > 0$  such that the Riemannian polar coordinates may be introduced on  $B(p, \rho) = \{q \in \mathcal{M} \mid d(p, q) \leq \rho\}$ . For any such  $\rho$  and  $q \in \partial B(p, \rho)$ , there exists a unique geodesic of shortest length ( $= \rho$ ) from  $p$  to  $q$ . In the polar coordinates, this geodesic is given by the straight line  $x(t) = (t, \varphi_0)$ ,  $0 \leq t \leq \rho$ , with  $q$  represented by coordinates  $(\rho, \varphi_0)$ ,  $\varphi_0 \in S^{d-1}$ .

**Proof.** Take an arbitrary curve from  $p$  to  $q$ , namely  $c(t) = (r(t), \varphi(t))$ ,  $0 \leq t \leq T$ , which does not have to be entirely contained in  $B(p, \rho)$ . Let  $t_0$  be defined as

$$t_0 := \inf \{t \leq T \mid d(x(t), p) \geq \rho\}.$$

Then  $t_0 \leq T$  such that  $c|_{[0, t_0]}$  lies entirely in  $B(p, \rho)$ . We want to show that

- (a)  $L(c|_{[0, t_0]}) \geq \rho$ , and
- (b)  $L(c|_{[0, t_0]}) = \rho$  only for a straight line in the polar coordinates,

where

$$L(c|_{[0, t_0]}) := \int_0^{t_0} \sqrt{g_{ij}(c(t)) \dot{c}^i \dot{c}^j} dt.$$

Observe that  $g_{r\varphi} = 0$ , with  $g_{\varphi\varphi}$  being positive definite and  $g_{rr} \equiv 1$ , we have

$$L(c|_{[0, t_0]}) \geq \int_0^{t_0} \sqrt{g_{rr}(c(t)) \dot{r}^2} dt = \int_0^{t_0} |\dot{r}| dt \geq \int_0^{t_0} \dot{r} dt = r(t_0) = \rho.$$

■

**Corollary 2.3.1.** Let  $\mathcal{M}$  be a compact Riemannian manifold. Then there exists  $\rho_0 > 0$  such that

- (a) for any  $p \in \mathcal{M}$ , Riemannian polar coordinates may be introduced on  $B(p, \rho_0)$ ;
- (b) for any  $p, q \in \mathcal{M}$  with  $d(p, q) \leq \rho_0$ , they can be connected by precisely one geodesic or shortest length which depends continuously on  $p$  and  $q$ .

## Lecture 7: Hopf-Rinow Theorem

### 2.3.2 Hopf-Rinow Theorem

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By using Corollary 2.3.1, we have shown the following in the homework.

<sup>1</sup>Note that this only holds at  $p$ . We will come back to this when we formally introduce the linear connection.

**Lemma 2.3.1.** Let  $(\mathcal{M}, g)$  be a compact Riemannian manifold. Then for all  $p \in \mathcal{M}$ , the exponential map  $\exp_p$  is defined on all of  $T_p\mathcal{M}$  and any geodesic may be extended indefinitely in each direction.

Then we use Lemma 2.3.1 to show the following.

**Theorem 2.3.2.** Let  $(\mathcal{M}, g)$  be a compact Riemannian manifold.

- (a) For any 2 points  $p, q \in \mathcal{M}$ , there exists a geodesic in every homotopy class of curves from  $p$  to  $q$ . Moreover, we can choose a shortest curve as the geodesic in the homotopy class.
- (b) Every homotopy class of closed curves in  $\mathcal{M}$  contains a curve that is shortest and geodesic.

We now want to generalize it. However, this is not true in the most general setting, and we need one more requirement.

**Definition 2.3.1** (Geodesically complete). A Riemannian manifold  $(\mathcal{M}, g)$  is *geodesically complete* if for all  $p \in \mathcal{M}$ ,  $\exp_p$  is defined on all of  $T_p\mathcal{M}$ .

In other words, a Riemannian manifold  $\mathcal{M}$  is *geodesically complete* if any geodesic  $c(t)$  with  $c(0) = p$  can be extended for all  $t \in \mathbb{R}$ . Then, we have the following.

**Theorem 2.3.3** (Hopf-Rinow theorem). Let  $(\mathcal{M}, g)$  be a Riemannian manifold, then the following statements are equivalent.

- (a)  $\mathcal{M}$  is complete as a metric space.<sup>a</sup>
- (b) The closed and bounded subsets of  $\mathcal{M}$  are compact.
- (c) There exists  $p \in \mathcal{M}$  such that  $\exp_p$  is defined on all  $T_p\mathcal{M}$ .
- (d)  $\mathcal{M}$  is geodesically complete.

Furthermore, (d) (and hence (a), (b), and (c)) implies

- (e) for two points  $p, q \in \mathcal{M}$  can be joined by a minimizing geodesic, i.e., geodesic of the shortest distance  $d(p, q)$ .

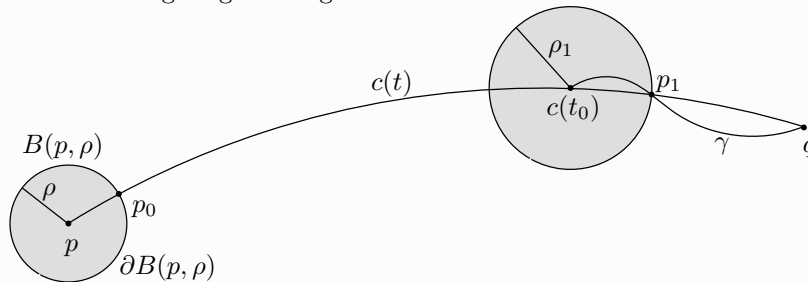
<sup>a</sup>Hence, equivalently, complete as a topological space w.r.t. the underlying topology.

**Proof.** We start by proving (d) implies (e). Let  $\mathcal{M}$  be geodesically complete, and let  $r := d(p, q)$ , and let  $\rho$  be as in Corollary 2.3.1. Let  $p_0 \in \partial B(p, \rho)$  be a point where the continuous functional  $d(q, \cdot)$  attains its minimum on the compact set  $\partial B(p, \rho)$ . Then, for some  $V \in T_{p_0}\mathcal{M}$ ,  $p_0 = \exp_{p_0} \rho V$ .

Consider the geodesic  $c(t) = \exp_{p_0} tV$ , by showing  $c(r) = q$ ,  $c|_{[0, r]}$  will be the shortest geodesic from  $p$  to  $q$ . We start by defining

$$I := \{t \in [0, r] \mid d(c(t), q) = r - t\},$$

and referring to the following diagram to guide us.



Now, we want to show that  $I = [0, r]$ , which will follow from showing that  $I$  is open.

**Note.**  $I$  is not empty since by definition it contains 0 and  $r$ . Further,  $I$  is closed by continuity.

Let  $t_0 \in I$ , and let  $\rho_1 > 0$  be the radius as in the corollary, without loss of generality,  $\rho_1 < r - t_0$ . Let  $p_1 \in \partial B(c(t_0), \rho_1)$  be the point where the continuous functional  $d(q, \cdot)$  attains its minimum on the compact set  $\partial B(c(t_0), \rho_1)$ . By the triangle inequality,

$$d(p, q) \leq d(p, p_1) + d(p_1, q).$$

For every curve  $\gamma$  from  $c(t_0)$  to  $q$ , there exists  $\gamma(t) \in \partial B(c(t_0), \rho_1)$ , hence

$$L(\gamma) \geq d(c(t_0), \gamma(t)) + d(\gamma(t), q) = \rho_1 + d(p_1, q) \Rightarrow d(q, c(t_0)) \geq \rho_1 + d(p_1, q)$$

where we use  $d(c(t_0), \gamma(t)) = \rho_1$ . But from the triangle inequality, we actually have

$$d(q, c(t_0)) = \rho_1 + d(p_1, q) \Leftrightarrow d(p_1, q) = \underbrace{d(q, c(t_0))}_{r-t_0} - \rho_1,$$

hence  $d(p_1, p) \geq r - (r - t_0 - \rho_1) = t_0 + \rho_1$ , i.e., this is a minimizing curve!

On the other hand, there exists a curve from  $p$  to  $p_1$  of length  $t_1 + \rho_1$  since it's composed by the portion from  $p$  to  $c(t_0)$  along  $c(t)$  and the portion being the **geodesic** from  $c(t_0)$  to  $p_1$  of length  $\rho_1$ . Then, by [Theorem 2.3.2](#), this curve is a **geodesic** curve. Finally, from the uniqueness of **geodesic** with the given extra data, this **geodesic** coincides with  $c$ . Hence,  $p_1 = c(t_0 + \rho_1)$ . With  $d(p_1, q) = r - t_0 - \rho_1$ ,

$$d(c(t_0 + \rho_1), q) = d(p_1, q) = r - t_0 - \rho_1 = r - (t_0 + \rho_1),$$

so  $t_0 + \rho_1 \in I$ , implying that  $I$  is open, i.e.,  $I = [0, r]$ , so  $c(r) = q$  follows.<sup>a</sup>

<sup>a</sup>For a detailed proof, see [\[FC13, Corollary 3.9\]](#).

## Lecture 8: Injectivity Radius and Vector Bundles

Let's finish the proof of [Hopf-Rinow theorem](#).

31 Jan. 13:00

**Proof of Theorem 2.3.3 (Continued).** We see that (d) implies (e), hence we only need to show that (a), (b), (c), and (d) are equivalent.

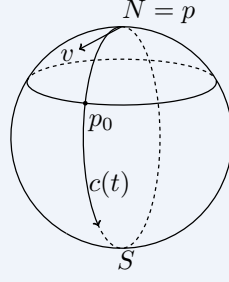
- (d)  $\Rightarrow$  (c): It is trivial.
- (c)  $\Rightarrow$  (b): Let  $K \subseteq \mathcal{M}$  be closed and bounded. As  $K$  bounded,  $K \subseteq B(p, r)$  for some  $r > 0$ . Then any point in  $B(p, r)$  can be joined with  $p$  by **geodesic** of length  $\leq r$ , and  $B(p, r)$  is the image of the compact ball in  $T_p \mathcal{M}$  of radius  $r$  under continuous map  $\exp_p$ , hence  $B(p, r)$  is compact. As  $K$  closed and  $K \subseteq B(p, r)$ ,  $K$  is compact.
- (b)  $\Rightarrow$  (a): Let  $(p_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$  be a Cauchy sequence, so it's bounded, and by (b), its closure is compact. It contains a convergent subsequence, so it converges, i.e.,  $\mathcal{M}$  is **complete**.
- (a)  $\Rightarrow$  (d): Let  $c$  be a **geodesic** in  $\mathcal{M}$ , parametrized by arc length defined on a maximal interval  $I$ . Since  $I$  is non-empty, and we can show that  $I$  is both open and closed.

■

**Remark.** It's worth mentioning that we do have uniqueness after choosing  $p_0$ ,<sup>a</sup> so the non-uniqueness really comes from the initial choice of  $p_0$ .

<sup>a</sup>In other words, after choosing  $p_0$ , everything is fixed.

**Example.** Consider  $S^2$ , after fixing  $p_0$ ,  $c(t_0)$  is extended uniquely.



### 2.3.3 Injectivity Radius

One might wonder, though we have [Lemma 2.3.1](#), how far can  $\exp_p$  extend while maintaining injectivity?

**Definition 2.3.2** (Injectivity radius). Let  $\mathcal{M}$  be a [Riemannian manifold](#), and  $p \in \mathcal{M}$ . The *injectivity radius*  $i(p)$  of  $p$  is

$$i(p) := \sup \{ \rho > 0 \mid \exp_p \text{ defined on } B(0, \rho) \subseteq T_p \mathcal{M} \text{ and injective} \}.$$

Similarly, the *injectivity radius*  $i(\mathcal{M})$  of  $\mathcal{M}$  is defined as  $i(\mathcal{M}) := \inf_{p \in \mathcal{M}} i(p)$ .

**Example** (Sphere).  $i(S^n) = \pi$ .

**Example** (Torus).  $i(T^n) = 1/2$ .

**Remark.** Any [manifold](#) carries a [complete Riemannian metric](#). If  $(\mathcal{M}, g_1)$  is not [complete](#), we can find  $g_2$  such that  $(\mathcal{M}, g_2)$  is [complete](#).

**Example** (Hyperbolic half-plane). The half-plane  $P = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$  equipped with metric induced by the Euclidean metric on  $\mathbb{R}^2$ , which is not [complete](#).

However, it becomes [complete](#) when equipped with the following metric

$$\frac{1}{y^2} (dx^2 + dy^2).$$

In fact,  $P$  with the above metric is called the *hyperbolic half-plane*  $H^2$ , and we can extend it to  $H^n$ .

Another question we may ask is the following.

**Problem.** Is the converse of [Hopf-Rinow theorem](#) true? I.e., can we show that (e) implies (d)?

**Answer.** No! Any 2 points in the open half-sphere can be joined by a unique minimal [geodesic](#), but this manifold is not [geodesically complete](#). ⊗

**Example.** The [injectivity radius](#) of  $H^n$  is  $\infty$ .

**Remark.** Given a compact  $\mathcal{M}$ , the [injectivity radius](#) is always  $> 0$  by continuity argument.

Now, given a [complete](#) but not compact  $\mathcal{M}$ , the [injectivity radius](#) can be 0.

**Example.** Take the quotient of the Poincaré half-plane by the translations

$$(x, y) \mapsto (x + n, y), \quad n \in \mathbb{Z}.$$

We then obtain a [complete Riemannian manifold](#)  $\mathcal{M}$  with  $i(\mathcal{M}) = 0$ .

**Note.** Finding lower bounds for  $i(\mathcal{M})$  introduces curvature estimates.

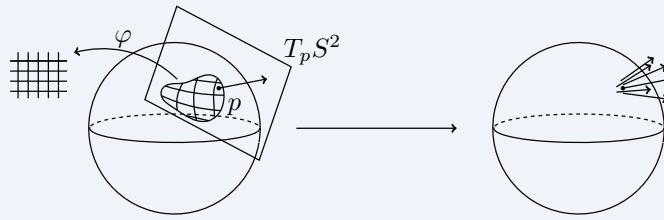
## 2.4 Bundles and Fields

We now introduce the theory of **bundles**, which allows us to introduce **vector fields**, which is a more general notion of **tensor fields**. Noticeably, nearly every structure we can put on a **Riemannian manifold** will be in the form of **tensor fields**, which is why we care about them.

As a motivating example, recall the **tangent bundle**<sup>2</sup>  $(T\mathcal{M}, \pi, \mathcal{M})$ , which captures the idea of “for every  $p \in \mathcal{M}$ , we associate a space  $T_p\mathcal{M}$ ”.

**Intuition.** This helps us construct **tangent vector fields** since a **tangent vector field**  $X$  of  $\mathcal{M}$  is defined by associating  $p$  to a **tangent vector**  $X(p)$  in the associated **tangent space**  $T_p\mathcal{M}$ .

**Example.** The **tangent vector field** assigns every  $p \in S^2$  a “point” in the associated **tangent space**.



### 2.4.1 Bundles

The above example of **tangent bundle** generalizes quite easily for defining a general **bundle**.

**Definition 2.4.1 (Bundle).** A *bundle* is a tuple  $(E, \pi, \mathcal{M})$  consists of the **total space**  $E$ , the **base space**  $\mathcal{M}$ , and the **bundle projection**  $\pi: E \rightarrow \mathcal{M}$ .

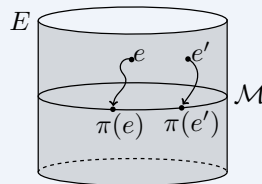
**Definition 2.4.2 (Total space).** The **differentiable manifold**  $E$  is called the *total space*.

**Definition 2.4.3 (Base space).** The **differentiable manifold**  $\mathcal{M}$  is called the *base space*.

**Definition 2.4.4 (Bundle projection).** The (differentiable) continuous surjection  $\pi: E \rightarrow \mathcal{M}$  is called the *bundle projection*.

**Example.** A **tangent bundle**  $(T\mathcal{M}, \pi, \mathcal{M})$  is a **bundle**.

**Example.** Let  $E$  be a cylinder,  $\mathcal{M}$  be a circle.



As we can see, the number of possible  $\pi$  is enormous, as long as it's surjective and smooth.

<sup>2</sup>Where just use the name “**bundle**” and don't know what it is. We'll see now!

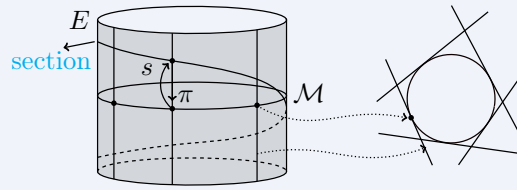
**Notation.** Sometimes, we will just denote a **bundle** as  $E \xrightarrow{\pi} \mathcal{M}$ , or even more compactly, just  $\pi$  since it captures all the data.

**Definition 2.4.5 (Fiber).** Given a **bundle**  $(E, \pi, \mathcal{M})$ , the *fiber* over  $p \in \mathcal{M}$  under  $\pi$  is the preimage of a  $\{p\}$ , i.e.,  $\pi^{-1}(\{p\})$ .

**Definition 2.4.6 (Section).** A *section* of a **bundle**  $(E, \pi, \mathcal{M})$  is a differentiable map  $s: \mathcal{M} \rightarrow E$  such that  $\pi \circ s = \text{id}_{\mathcal{M}}$ .

**Remark.** We see that a **section**  $s$  encodes lots of information of a **bundle**, since  $s$  includes  $E, \mathcal{M}$ , and the condition deal with  $\pi$ .

**Example.** Again let  $E$  be a cylinder,  $\mathcal{M}$  be a circle. This time, we choose  $\pi$  to be the trivial one.



We see that in this way, this **bundle** really captures all the **tangent spaces** structure of a circle!

## 2.4.2 Vector Bundles

Then, we're interested in the so-called **vector bundle**.

**Definition 2.4.7 (Vector bundle).** A (differentiable) *vector bundle* of rank  $n$  is a **bundle**  $(E, \pi, \mathcal{M})$  such that each **fiber**  $E_x := \pi^{-1}(x)$  of  $x \in \mathcal{M}$  carries a structure of an  $n$ -dimensional (real) vector space, and **local triviality** condition holds.

**Definition 2.4.8 (Local trivialization).** For all  $x \in \mathcal{M}$ , the *local trivialization*  $(U, \varphi)$  consists a neighborhood  $U$  and **diffeomorphism**  $\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  such that for all  $y \in U$ ,

$$\varphi_y := \varphi|_{E_y} : E_y \rightarrow \{y\} \times \mathbb{R}^n$$

is a vector space isomorphism.

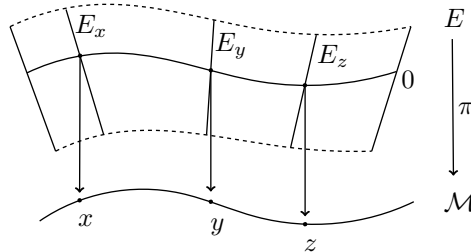


Figure 2.1: An illustration of **vector bundle**  $(E, \pi, \mathcal{M})$ .

**Definition 2.4.9 (Trivial).** A **vector bundle** is *trivial* if it's isomorphic to  $\mathcal{M} \times \mathbb{R}^n$ .<sup>a</sup>

<sup>a</sup> $n$  is the rank of the **vector bundle**.

**Intuition.** Local trivialization shows that *locally*  $\pi$  looks like the **projection** of  $U \times \mathbb{R}^n$  on  $U$ .

**Definition 2.4.10** (Bundle chart). The pair  $(\varphi, U)$  is the *bundle chart* in **local trivialization**.

**Remark.** From **Definition 2.4.7**, **vector bundle** is locally, but not necessarily globally a product of **base space** and the **fiber**.

**Intuition.** We may look at a **vector bundle** as a family of vector spaces, all isomorphic to a fixed  $\mathbb{R}^n$ , “parametrized” (**locally trivially**) by a **manifold**.

### 2.4.3 Vector Fields

We can now introduce the notion of **vector fields** in terms of **section**.

**Definition 2.4.11** (Vector field). A (smooth) *vector field*  $X$  is a smooth **section** of a **bundle**.

**Note.** We see that a smooth **tangent vector field** is indeed a smooth **vector field** with the **bundle** being the **tangent bundle**.

**Notation.** Since we will nearly always be talking about **tangent vector fields**, we will abuse the notation a bit and just simply call it **vector fields**. But always keep in mind that more broadly, a **vector field** should be a **section** of a **bundle**, not always  $T\mathcal{M}$ .

## Lecture 9: Tensors and Connections

### 2.4.4 Tensor Fields

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We can introduce the notion of “**tensor fields**” in a brute-force way<sup>3</sup> by first introducing **tensors**.

**Definition 2.4.12** (Tensor). Let  $V$  be a vector space of dimension  $m < \infty$ , and the dual space  $V^*$ . Then the vector space of the  *$r$ -times contravariant and  $s$ -times covariant tensors over  $V$* , denoted as  $T_s^r(V)$ , is the **vector field** defined as

$$T_s^r(V) = \{T: \underbrace{V^* \times \cdots \times V^*}_r \times \underbrace{V \times \cdots \times V}_s \rightarrow \mathbb{R}\} = \underbrace{V \otimes \cdots \otimes V}_r \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_s.$$

**Notation.** Let  $\mathcal{M}^n$  be a **smooth manifold** and  $\pi: E \rightarrow \mathcal{M}$  a **smooth vector bundle**, then the set of **sections** is denoted as

$$\Gamma(E) := \{s \in C^\infty(\mathcal{M}, E) \mid \pi \circ s = \text{id}_{\mathcal{M}}\}.$$

**Example.** Consider the **vector bundle**  $(T\mathcal{M}, \pi, \mathcal{M})$ , then  $\Gamma(T\mathcal{M}) := \{\text{vector fields on } \mathcal{M}\}$ .

**Notation.** For  $s \in \mathbb{N}$ , let  $\Lambda^s(V^*) := \{A \in T_s^0(V) \mid A \text{ skew-symmetric}\}$ .

**Example.**  $\Gamma(\Lambda_s \mathcal{M}) := \{s\text{-forms on } \mathcal{M}\}$  with  $\Lambda_s \mathcal{M} = \Lambda^s \left( \bigcup_{p \in \mathcal{M}} T_p^* \mathcal{M} \right)$ .

Then, we have the following.

<sup>3</sup>See **Appendix A.3.2** for another view point.

**Definition 2.4.13 (Tensor field).** The  $(r, s)$ -tensor fields on  $\mathcal{M}$  is defined as elements in  $\Gamma(T_s^r \mathcal{M})$  with  $T_s^r \mathcal{M} = \bigcup_{p \in \mathcal{M}} T_s^r(T_p \mathcal{M})$ .

**Example.** A Riemannian metric  $g$  on  $\mathcal{M}$  is a  $(0, 2)$ -tensor field, i.e.,  $g \in \Gamma(T_2^0(\mathcal{M}))$  for all  $p \in \mathcal{M}$ .

**Proof.** Since  $g_p: T_p \mathcal{M} \times T_p \mathcal{M} \rightarrow \mathbb{R}$ , so by regarding  $p$  as the argument of the map  $g$ ,  $g: \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ .  $\ast$

**Note.** It's in fact unnecessary to have such a general Definition 2.4.13 on a Riemannian manifold.

**Proof.** Since given a Riemannian metric  $g$ , it associates to each  $X \in \Gamma(T\mathcal{M})$  a unique  $\omega \in \Gamma(T^*\mathcal{M})$  given by

$$\omega(Y) = g(X, Y)$$

for all  $X, Y \in \Gamma(T\mathcal{M})$ .  $\ast$

## 2.5 Other Metrics

Finally, we discuss some other metrics we may let a manifold equipped with.

**Definition 2.5.1 (Pseudo-Riemannian metric).** A pseudo-Riemannian metric on a differentiable manifold  $\mathcal{M}$  is a  $(0, 2)$ -tensor field  $g \in \Gamma(T_2^0(\mathcal{M}))$  for all  $p \in \mathcal{M}$  with

- (a)  $g(X, Y) = g(Y, X)$  for all  $X, Y \in T\mathcal{M}$ ;
- (b) for all  $p \in \mathcal{M}$ ,  $g_p$  is non-degenerate bilinear form on  $T_p \mathcal{M}$ , i.e.,  $g_p(X, Y) = 0$  for all  $X, Y \in T_p \mathcal{M}$  if and only if  $Y = 0$ .

**Note.** A pseudo Riemannian metric is actually a Riemannian metric if it's positive definite at every  $p \in \mathcal{M}$ .

**Definition 2.5.2 (Lorentzian metric).** A Lorentzian metric  $g$  is a continuous assignment of a non-degenerate<sup>a</sup> quadratic form  $g_p$  of index 1<sup>b</sup> in  $T_p \mathcal{M}$  for all  $p \in \mathcal{M}$ .

<sup>a</sup> $g_p(X, Y) = 0$  for all  $Y \in T_p \mathcal{M}$  implies  $X = 0$ .

<sup>b</sup>It means that the maximal dimension of a subspace of  $T_p \mathcal{M}$  on which  $g_p$  is negative definite is 1.

An equivalent definition is the following.

**Definition 2.5.3 (Lorentzian).** A quadratic form  $g_p$  in  $T_p \mathcal{M}$  is Lorentzian if there exists a vector  $V \in T_p \mathcal{M}$  such that  $g_p(V, V) < 0$  while setting  $\Sigma_V = \{X \mid g_p(X, V) = 0\}$  such that  $g_p|_{\Sigma_V}$ <sup>a</sup> is positive definite.

<sup>a</sup>The  $g_p$ -orthogonal complement of  $V$ .

**Example (Minkowski space).** The Minkowski space on  $\mathbb{R}^4$  is the prototypical example from physics (flat spacetime). Namely, the metric is given by the quadratic form

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with the coordinates being  $(t, x, y, z)$ .



## Chapter 3

# Connections and Curvatures

So far, we saw that a [vector field](#)  $X$  can be used to provide a directional derivative since it gives us a [tangent vector](#) at each point smoothly. Now, we will introduce a new symbol  $\nabla$  where we let

$$\nabla_X f := Xf$$

for  $f \in C^\infty(\mathcal{M})$ .

**Problem.** Does this notation overkill? We already know that  $Xf = (df)(X)$ !

**Answer.** No! While  $\nabla, X: C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ , while  $df: \Gamma(T\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ , we can generalize  $\nabla_X$  to act from [vector fields](#) to [vector fields](#)! The insight is that if  $X$  can be extended naturally (without providing any extra structures), then we certainly won't bother introducing a new symbol. However, as you might guess, to let  $\nabla$  doing this, we do need to provide extra structures, and  $\nabla$  stands exactly for these extra structures!  $\otimes$

In some sense, this new notions  $\nabla$  allows us to “connect” [tangent spaces](#), which allows us to make sense of “curvatures” and other geometric property of a [Riemannian manifold](#).

### 3.1 Levi-Civita Connections

We start by talking about [linear connections](#), and then realize that after specifying a [Riemannian metric](#)  $g$ , with an additional (technical) assumption, a unique [linear connection](#), defined as [Levi-Civita connections](#), exists for any [Riemannian manifold](#). In other words, specifying  $g$  is the same as specifying the “shape of the space.” We'll make sense of all these on the way.

#### 3.1.1 Affine Connections

We first formulate a *wish list* of properties which the  $\nabla_X$  should have. Any remaining freedom in choosing  $\nabla$  will need to be provided as additional structures beyond the structures on  $\mathcal{M}$  we already have.

**Definition 3.1.1** (Linear connection). A *linear connection* (or *affine connection*) on a [smooth manifold](#)  $\mathcal{M}$  is a bilinear map

$$\nabla: \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M}),$$

which is denoted by  $\nabla(X, Y) = \nabla_X Y$  and which satisfies

- (a)  $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ ;
- (b)  $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$ ;
- (c)  $\nabla_X fY = f\nabla_X Y + X(f)Y$ ;

for all [vector fields](#)  $X, Y, Z \in \Gamma(T\mathcal{M})$  and  $f, g \in C^\infty(\mathcal{M})$ .

**Remark.** Definition 3.1.1 (c) shows that this is actually a local notion as we will see.

**Note.** There's a similar notation called **covariant derivative**, denoted by  $D$ , satisfies similar properties as a **linear connection**. Hence, we often write  $D$  and  $\nabla$  interchangeably.<sup>a</sup>

<sup>a</sup> $\nabla$  is more general than  $D$ ; however, we treat them as the same as suggested by Proposition 3.4.1.

Now, one might be wondering that, after fixing these rules we want, how much freedom is left? To see this, let's first do some calculations...

### 3.1.2 Connection Coefficients

Choose a **system of coordinates**  $(x_1, \dots, x_n)$  at  $p \in \mathcal{M}$ , we can write  $X = X^i \frac{\partial}{\partial x_i}$ ,  $Y = Y^j \frac{\partial}{\partial x_j}$ , then

$$\nabla_X Y = \nabla_{X^i \frac{\partial}{\partial x_i}} \left( Y^j \frac{\partial}{\partial x_j} \right) = X^i Y^j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + X^i \frac{\partial}{\partial x_i} (Y^j) \frac{\partial}{\partial x_j}.$$

Now, we see that  $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$  is another **vector field**, hence can again write

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} =: \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

in terms of the basis with a new set of coefficients  $\Gamma$ .

**Notation** (Connection coefficient). The coefficients  $\Gamma_{ij}^k$  is called the *connection coefficients*.<sup>a</sup>

<sup>a</sup>It's tempting to say that the **connection coefficients** are the same as **Christoffel symbols** since we're using the same symbols. Indeed, they are! For a deeper understanding, see Appendix A.1.

**Note.** It's clear that  $\Gamma_{ij}^k$  are differentiable and **charts**-dependent and hence  $\nabla$  is local.

Finally, we have

$$\nabla_X Y = \left( X^i Y^j \Gamma_{ij}^k + X(Y^k) \right) \frac{\partial}{\partial x_k} \Rightarrow (\nabla_X Y)^k = X(Y^k) + \Gamma_{ij}^k X^i Y^j,$$

meaning that we have  $(\dim \mathcal{M})^3$  many  $\Gamma$ 's (freedom) when choosing  $\Gamma_{ij}^k$  with Definition 3.1.1.<sup>1</sup>

**Remark.** One might ask what about other **tensor fields**? Fortunately, the same set of  $\Gamma$ 's fix the action of  $\nabla$  on any **tensor fields**.

**Proof.** The key observation is that if we define  $\nabla_{\frac{\partial}{\partial x^j}} (dx^i) =: \Sigma_{jk}^i dx^k$ , then

$$\nabla_{\frac{\partial}{\partial x^j}} \left( dx^i \left( \frac{\partial}{\partial x^k} \right) \right) = \begin{cases} \frac{\partial}{\partial x^j} (\delta_k^i) = 0; \\ \left( \nabla_{\frac{\partial}{\partial x^j}} dx^i \right) \frac{\partial}{\partial x^k} + dx^i \left( \underbrace{\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}}_{\Gamma_{jk}^\ell \frac{\partial}{\partial x^\ell}} \right), \end{cases}$$

leading to

$$\left( \nabla_{\frac{\partial}{\partial x^j}} dx^i \right) \frac{\partial}{\partial x^k} = -dx^i \left( \Gamma_{jk}^\ell \frac{\partial}{\partial x^\ell} \right) \Rightarrow \left( \nabla_{\frac{\partial}{\partial x^j}} dx^i \right)_k = -\Gamma_{jk}^i$$

since  $dx^i \frac{\partial}{\partial x^\ell} = \delta_\ell^i$ . \*

In summary, we have

$$\begin{cases} (\nabla_X Y)^k = X(Y^k) + \Gamma_{ij}^k X^i Y^j, & \text{if } Y \text{ is a vector field;} \\ (\nabla_X \omega)_k = X(\omega_k) - \Gamma_{ik}^j X^i \omega_j, & \text{if } \omega \text{ is a co-vector field.} \end{cases}$$

<sup>1</sup>This is for a particular domain  $U$ .

### 3.1.3 Levi-Civita Connections

The basic insight is that, after choosing a particular [connection](#),<sup>2</sup> the space is basically fixed: i.e., the *shape*, or “curvature”, of the space is determined by the choice of  $\nabla$ ! We now formalize this idea. A particularly natural notion related to “curvature” is the [torsion](#), defined as follows.

**Definition 3.1.2 (Torsion).** The *torsion*  $T$  of a [linear connection](#)  $\nabla$  is the [\(1,2\)-tensor field](#)

$$T(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X, Y]).$$

**Notation.** We usually write this as  $T(X, Y)$  by neglecting  $\omega$ .

**Remark.**  $T$  is actually  $C^\infty$ -linear in each entry,<sup>a</sup> hence a [tensor field](#).

<sup>a</sup>See [Appendix A.3.2](#).

**Proof.** Since  $T(f \cdot \omega, X, Y) = f \cdot \omega(\dots) = fT(\omega, X, Y)$  and  $T(\omega + \psi, X, Y) = \dots = T(\omega, X, Y) + T(\psi, X, Y)$ , and also

$$\begin{aligned} T(\omega, fX, Y) &= \omega(\nabla_{fX} Y - \nabla_Y(fX) - [fX, Y]) \\ &= \omega(f\nabla_X Y - (Yf)X - f\nabla_Y X - f[X, Y] + (Yf)X) = f \cdot T(\omega, X, Y) \end{aligned}$$

since

$$([fX, Y])g = f \cdot X(Yg) - Y(f \cdot Xg) = f \cdot X(Yg) - (Yf)(Xg) - f \cdot Y(Xg) = (f \cdot [X, Y] - (Yf)X)g.$$

Finally, we claim that the additivity at  $X$  holds, with  $T(\omega, X, Y) = -T(\omega, Y, X)$ , we’re done.  $\circledast$

**Intuition.** [Definition 3.1.2](#) makes sense (in such a form) since this will make  $T$  actually a [tensor field](#). For example, without the [Lie bracket](#) term, we don’t have the linearity at  $X$  (hence  $Y$ ).

**Definition 3.1.3 (Torsion-free).** A [linear connection](#)  $\nabla$  is *torsion-free* if  $T = 0$ .

**Notation** (symmetric). A [torsion-free](#)  $\nabla$  is sometimes said to be *symmetric*.

In a [chart](#),

$$T_{jk}^i := T\left(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \Gamma_{jk}^i - \Gamma_{kj}^i = 2\Gamma_{[jk]}^i,$$

hence if  $T = 0$ , we can interchange the lower two indexes of  $\Gamma_{ij}^k$ , i.e.,  $\Gamma_{ij}^k = \Gamma_{ji}^k$ .

**Definition 3.1.4 (Riemannian).** Let  $\nabla$  be a [linear connection](#) and  $g$  be a [Riemannian metric](#) on  $\mathcal{M}$ . Then  $\nabla$  is *Riemannian* (or *metric*) if for all  $X, Y, Z \in \Gamma(T\mathcal{M})$ ,<sup>a</sup>

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

<sup>a</sup>We view  $g(X, Y) \in C^\infty(\mathcal{M})$  as suggested by [Appendix A.3.2](#).

**Notation** (Compatible). A [Riemannian](#)  $\nabla$  is sometimes said to be *compatible*.

**Remark.** Equivalently, [Definition 3.1.4](#) can be formulated as  $\nabla g = 0$ .

We are now able to state the fundamental theorem of this section.

<sup>2</sup>Remember that we have freedom to choose  $\Gamma$ ’s.

**Theorem 3.1.1 (Levi-Civita).** On each Riemannian manifold  $(\mathcal{M}, g)$ , there exists a unique Riemannian, torsion-free connection  $\nabla$  on  $T\mathcal{M}$  determined by the Koszul formula

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} (X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle). \quad (3.1)$$

**Proof sketch.** Firstly, we show that every Riemannian and torsion-free connection satisfies Koszul formula, which implies uniqueness. For existence, we verify that the unique map  $\nabla: \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M})$  given by Koszul formula is Riemannian and torsion-free.<sup>a</sup> ■

<sup>a</sup>For a detail proof, see [FC13, §2 Theorem 3.6].

**Note.** I rearrange the Koszul formula to make it easier to memorize.

Finally, we define the following.

**Definition 3.1.5 (Levi-Civita connection).** The Levi-Civita connection is the unique linear connection  $\nabla$  defined by the Koszul formula.

**Remark.** This means, given a Riemannian metric  $g$ , with the condition of torsion-free, the shape of the space is also fixed since there's a unique linear connection  $\nabla$  such that  $T = \nabla g = 0$ .

## Lecture 10: Curvatures and Flow of Vector Fields

### 3.2 Riemannian Curvatures

7 Feb. 13:00

Given all these definitions, we can now introduce the notion of “curvatures.” Consider the following.

**Definition 3.2.1 (Riemannian curvature).** The Riemannian curvature  $R$  of a Levi-Civita connection  $\nabla$  is the  $(1, 3)$ -tensor field<sup>a</sup>

$$R(\omega, Z, X, Y) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z).$$

<sup>a</sup> $R$  is indeed  $C^\infty$ -linear in each entry,<sup>b</sup> although we omit the proof here.

**Notation.** We usually write this as  $R(X, Y)Z$  by emphasizing  $Z$  and neglecting  $\omega$ .

**Note.** In do Carmo [FC13], the corresponding definition of  $R$  differs from Definition 3.2.1 by a sign.

**Example (Euclidean space).** If  $\mathcal{M} = \mathbb{R}^n$  (with the “flat”  $\nabla$ ),  $R(X, Y)Z = 0$ ,  $\forall X, Y, Z \in \Gamma(T\mathbb{R}^n)$ .

**Proof.** Since given  $Z = (z_1, \dots, z_n)$  with the components from natural coordinates of  $\mathbb{R}^n$ ,  $\nabla_X Z = (X z_1, \dots, X z_n)$ , then  $\nabla_Y \nabla_X Z = (Y X z_1, \dots, Y X z_n)$ , hence  $R(X, Y)Z = 0$ . ⊛

**Intuition.**  $R(X, Y)Z$  is trying to measure how much  $\mathcal{M}$  deviates from being Euclidean.

Another way to look at this is that, consider a system of coordinates  $\{x_i\}$  around  $p \in \mathcal{M}$ . Since  $[\partial/\partial x_i, \partial/\partial x_j] = 0$ ,<sup>3</sup> we have

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \frac{\partial}{\partial x_k} = \left(\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}}\right) \frac{\partial}{\partial x_k}.$$

<sup>3</sup>For second derivative, we can exchange the order due to smoothness.

**Intuition.**  $R(X, Y)Z$  is trying to measure the non-commutativity of the [covariant derivative](#).

Consider expressing things in a [chart](#)  $(U, x)$  at  $p \in \mathcal{M}$ . Let  $\partial/\partial x_i = X_i$ , then

$$R(X_i, X_j)X_k =: R_{ijk}^\ell X_\ell$$

as how we define [connection coefficients](#), i.e.,  $R_{ijk}^\ell$  are components of  $R$  in  $(U, x)$ .<sup>4</sup> If  $X = u^i X_i, Y = v^j X_j, Z = w^k X_k$ , from the linearity of  $R$ ,

$$R(X, Y)Z = R_{ijk}^\ell u^i v^j w^k X_\ell.$$

Then the above computation formally can be written as follows.

**Remark** (Algebraic significant of Riemannian curvature). Since

$$(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) = R(\cdot, Z, X, Y) + \nabla_{[X, Y]} Z,$$

by letting  $\nabla_i := \nabla_{\frac{\partial}{\partial x^i}}, \nabla_j := \nabla_{\frac{\partial}{\partial x^j}}$ , in a [chart](#)  $(U, x)$ , we have

$$(\nabla_i \nabla_j Z)^k - (\nabla_j \nabla_i Z)^k = R_{lij}^k Z^\ell + \underbrace{\nabla_{[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]} Z}_{=0} = R_{lij}^k,$$

i.e., the components of  $R$  contains all the information of how  $\nabla_i$  and  $\nabla_j$  fail to commute.

Finally, it's worth-noting that

$$\langle R(X_i, X_j)X_k, X_\ell \rangle = R_{ijk}^s g_{s\ell} = R_{ijk\ell}.$$

### 3.2.1 Identities

There are many important identities related to  $R$ , and we should see some of them.

**Proposition 3.2.1** (First Bianchi identity). Given the [Riemannian curvature tensor](#)  $R$ , for all [vector fields](#)  $X, Y, Z$ ,

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0;$$

or equivalently,  $R_{klij} + R_{kijl} + R_{kjli} = 0$ .

**Proof.** See do Carmo [FC13, Proposition 2.4] (and also homework 2). ■

**Proposition 3.2.2** (Second Bianchi identity). Given the [Riemannian curvature tensor](#)  $R$ ,

$$\frac{\partial}{\partial x^h} R_{klij} + \frac{\partial}{\partial x^k} R_{lhij} + \frac{\partial}{\partial x^\ell} R_{hkij} = 0;$$

or equivalently,  $\nabla_{[\alpha} R_{\beta\gamma]\delta\epsilon} := \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0$ .<sup>a</sup>

<sup>a</sup>This notation is a bit cryptic: see [Ricci calculus](#).

**Proof.** See homework 2. ■

**Proposition 3.2.3.** Given the [Riemannian curvature tensor](#)  $R$ ,

- (a)  $R(X, Y)Z - R(Y, X)Z$ , i.e.,  $R_{klij} = -R_{lkij}$ ;
- (b)  $\langle R(X, Y)Z, W \rangle = -\langle R(X, Y)W, Z \rangle$ , i.e.,  $R_{klij} = -R_{lkij}$ ;
- (c)  $\langle R(X, Y)Z, W \rangle = -\langle R(Y, X)Z, W \rangle$ , i.e.,  $R_{klij} = -R_{lkij}$ ;

<sup>4</sup>do Carmo [FC13, Page 93] shows that  $R_{ijk}^\ell = \Gamma_{ik}^p \Gamma_{jp}^\ell - \Gamma_{jk}^p \Gamma_{ip}^\ell + \Gamma_{ik,j}^\ell - \Gamma_{jk,i}^\ell$  (note the sign difference).

(d)  $\langle R(X, Y)Z, W \rangle = -\langle R(Z, W)X, Y \rangle$ , i.e.,  $R_{k\ell ij} = R_{ij\ell k}$ .

**Proof.** See do Carmo [FC13, Proposition 2.5] (and also homework 2). ■

### 3.2.2 Other Curvatures

There are other notions of curvature, but they all depend on the [Riemannian curvature](#), and appearing to be some sorts of “average” of  $R$ .

**Definition 3.2.2** (Riemannian-Christoffel curvature). The *Riemannian-Christoffel curvature* is defined by

$$R_{k\ell ij} := g_{km} R_{\ell ij}^m = \left\langle R \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k} \right\rangle.$$

**Definition 3.2.3** (Ricci curvature). The *Ricci curvature* is defined by  $R_{ab} = g^{cm} R_{camb} = R_{amb}^m$ .

**Definition 3.2.4** (Ricci scalar curvature). The *(Ricci) scalar curvature* is defined by  $R = g^{ab} R_{ab}$ .

**Note.** For a more formal treatment, see do Carmo [FC13, §4.4].<sup>a</sup>

<sup>a</sup>Notice that the order in do Carmo [FC13] is a bit different: it introduces [sectional curvature](#) first.

## 3.3 Flows of Vector Fields

Let  $\mathcal{M}$  be a [smooth manifold](#), and  $X$  a [vector field](#) on  $\mathcal{M}$ . Then  $X$  defines 1<sup>st</sup> order differential equations<sup>5</sup>

$$\dot{c} = X(c).$$

And this ODE has a solution, as guaranteed by [Proposition 3.3.1](#).

**Proposition 3.3.1.** For all  $p \in \mathcal{M}^d$ , there exists an open interval  $I = I_p \subseteq \mathbb{R}$  with  $0 \in I_p$  such that a [smooth curve](#)  $c: I_p \rightarrow \mathcal{M}$  solves

$$\begin{cases} \frac{dc(t)}{dt} = X(c(t)), & t \in I; \\ c(0) = p. \end{cases}$$

Further, the solution depends smoothly on the initial data (i.e.,  $p$ ).<sup>a</sup>

<sup>a</sup>This directly follows from ODE theory.

**Proof.** For all  $p \in \mathcal{M}$ , we want to find an open interval  $I = I_p$  around  $0 \in \mathbb{R}$  and a solution of the following ODE for  $c: I \rightarrow \mathcal{M}$ :

$$\begin{cases} \frac{dc(t)}{dt} = X(c(t)), & t \in I; \\ c(0) = p. \end{cases}$$

We can check in [local coordinates](#) that this is a system of ODE. In such [coordinates](#), let  $c(t)$  be given by  $c(t) = (c^1(t), c^2(t), \dots, c^d(t))$ . Let  $X =: X^i \partial / \partial x^i$ , then the above system becomes

$$\frac{dc^i(t)}{dt} = X^i(c(t)), \quad i = 1, \dots, d.$$

From the [Picard-Lindelöf theorem](#), with the initial data  $c(0) = p$ , there is a unique solution. ■

<sup>5</sup>If  $\dim \mathcal{M} > 1$ , it is a system of 1<sup>st</sup>-order differential equations.

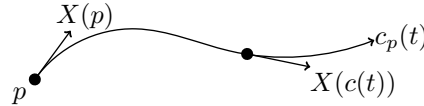
**Proposition 3.3.2.** For all  $p \in \mathcal{M}$ , there exists an open neighborhood  $U$  of  $p$  and an open interval  $I_p$  with  $0 \in I_p$  such that for all  $q \in U$ , the **curve**  $c_q$  with

$$\dot{c}_q(t) = X(c_q(t)), \quad c_q(0) = q$$

is defined on  $I$  and the map  $c: I \times U \rightarrow \mathcal{M}$ ,  $(t, q) \mapsto c_q(t)$  is smooth.

**Proposition 3.3.2** suggests the following definition.

**Definition 3.3.1 (Local flow).** The map  $c_q(t): I \times U \rightarrow \mathcal{M}$ ,  $(t, q) \mapsto c_q(t)$  from **Proposition 3.3.2** is called the *local flow* of the **vector field**  $X$ .



**Definition 3.3.2 (Integral curve).** The **local flow**  $c_q(t)$  is called the *integral curve* of  $X$  through  $q$ .

### 3.3.1 Local 1-Parameter Groups

Now, given a **local flow**  $c_q(t)$  of a **vector field**  $X$ , by fixing  $t$ , we can vary  $q$  and see the following.

**Theorem 3.3.1.** Let  $\varphi_t(q) := c_q(t)$  such that  $\varphi_t \circ \varphi_s(q) = \varphi_{t+s}(q)$  for  $s, t, (t+s) \in I_q$ . If  $\varphi_t$  is defined on  $U \subseteq \mathcal{M}$ , it maps  $U$  **diffeomorphically** onto its image.

We see that  $\varphi_t$  defines a family of **diffeomorphism** around  $p$ , which gives the following.

**Definition 3.3.3 (Local 1-parameter group).** A family  $(\varphi_t)_{t \in I}$  of **diffeomorphism** from  $\mathcal{M}$  to  $\mathcal{M}$  satisfying **Theorem 3.3.1** is called a *local 1-parameter group* of **diffeomorphisms**.

In general, a **local 1-parameter group** needs not be extendible to a group because the maximum interval  $I = I_q$  in **Definition 3.3.3** need not be all of  $\mathbb{R}$ .

**Example.** Let  $\mathcal{M} = \mathbb{R}$ ,  $X(t) = \tau^2 d/d\tau$ . Then the solution of  $\dot{c}(t) = c^2(t)$  is not defined over all  $\mathbb{R}$ .

To get the whole group structure, consider the following.

**Theorem 3.3.2.** Let  $X$  be a **vector field** on a **smooth manifold**  $\mathcal{M}$  with a compact support. Then the corresponding **local flow** is defined for every  $q \in \mathcal{M}$  and  $t \in \mathbb{R}$ , and the **local 1-parameter group** becomes a group of **diffeomorphisms**.

**Proof.** By using  $\text{supp}(X) \subseteq K$ ,  $K$  compact, we can cover  $K$  by a finite covering, then using **Proposition 3.3.2**, we're done. ■

This leads to the following.

**Corollary 3.3.1.** On a compact **differentiable manifold**  $\mathcal{M}$ , any **vector field** generates a **local 1-parameter group**.

## Lecture 11: Geodesic & Cogeodesic Flows and Parallel Transport

### 3.3.2 Geodesic and Cogeodesic Flows

9 Feb. 13:00

A particularly interesting **flow** is the **cogeodesic-flow**, which can be constructed as follows. Let's first transform **Equation 2.1** (which is a second order ODE) into a first order system on the **cotangent bundle**

$T^*\mathcal{M}$ , and locally trivialize  $T^*\mathcal{M}$  by chart  $T^*\mathcal{M}|_U \cong U \times \mathbb{R}^d$  with coordinates  $(x^1, \dots, x^d, p_1, \dots, p_d)$ . Now, set

$$H(x, p) = \frac{1}{2} g^{ij}(x) p_i p_j, \quad (3.2)$$

**Theorem 3.3.3.** Equation 2.1 is equivalent to the system on  $T^*\mathcal{M}$ :

$$\begin{cases} \dot{x}^i = \frac{\partial H}{\partial p_i} g^{ij}(x) p_j; \\ \dot{p}_i = -\frac{\partial H}{\partial x^i} = -\frac{1}{2} g_{,i}^{jk}(x) p_j p_k. \end{cases} \quad (3.3)$$

**Proof.** This is just computation (recall that  $g^{ik} g_{kj} = \delta_j^i$ ). ■

**Definition 3.3.4** (Cogeodesic flow). The *cogeodesic flow* is the local flow determined by Equation 3.3.

**Definition 3.3.5** (Geodesic flow). The *geodesic flow* on  $T\mathcal{M}$  is obtained from the *cogeodesic flow* by the first equation in Equation 3.3.

Thus, the *geodesic* is the projection of the *integral curve* of the *geodesic flow* onto  $\mathcal{M}$ .

**Remark** (Hamiltonian flow). The *cogeodesic flow* is a *Hamiltonian flow* for the Hamiltonian  $H$ .

**Proof.** By Equation 3.3, along the *integral curves*,

$$\frac{dH}{dt} = H_{x^i} \dot{x}^i + H_{p_i} \dot{p}^i = -\dot{p}_i \dot{x}^i + \dot{x}^i \dot{p}_i = 0.$$

Observe that the *cogeodesic flow* maps the set  $E_\lambda := \{(x, p) \in T^*\mathcal{M} \mid H(x, p) = \lambda\}$  onto itself for all  $\lambda \geq 0$ . \*

**Remark.** If  $\mathcal{M}$  is compact, then all  $E_\lambda$  are compact, then all *geodesic flows* are defined on all  $E_\lambda$  for all  $\lambda$ .

**Remark.**  $\mathcal{M} = \bigcup_{\lambda \geq 0} P E_\lambda$  for  $P$  being the projection.

### 3.4 Parallelism

An important concept related to curvatures is “parallelism,” which needs a formal introduction of *covariant derivatives*.<sup>6</sup> As a motivating example, the following is an equivalent definition of *geodesic*.

**Example** (Autoparallel). The *geodesic*  $c$  satisfies  $\nabla_{\dot{c}} \dot{c} = 0$ . This is called *autoparallel*.

**Proof.** In the *local coordinates*, we have  $\dot{c} = \dot{c}^i \partial / \partial x^i$ , and note that

$$\nabla_{\dot{c}} \dot{c} = \dot{c}^i \nabla_{\frac{\partial}{\partial x^i}} \dot{c}^j \frac{\partial}{\partial x^j} = \dot{c}^i \dot{c}^j \Gamma_{ij}^k \frac{\partial}{\partial x^k} + \dot{c}^k \frac{\partial}{\partial x^k} = (\dot{c}^k + \Gamma_{ij}^k \dot{c}^i \dot{c}^j) \frac{\partial}{\partial x^k} = 0 \quad (3.4)$$

since a *geodesic* is the solution of Equation 2.1. \*

**Intuition.** A *geodesic* is a *curve* with “zero acceleration”.<sup>a</sup>

<sup>a</sup>Recall why defining  $\nabla$  allows us to talk about acceleration.

To understand what  $\nabla_{\dot{c}} \dot{c}$  is doing beyond just calculation, we need to understand *parallel transports*.

<sup>6</sup>Although we say we’re going to treat them the same as  $\nabla$ .



### 3.4.1 Covariant Derivatives

**As previously seen.** The set of smooth **vector fields along  $c$**  is denoted as  $\mathcal{X}_c(\mathcal{M})$ .

We can now finally define **covariant derivative** formally.

**Definition 3.4.1 (Covariant derivative).** The *covariant derivative* of  $V$  along  $c$  is the **vector field**  $DV/dt$  in **Proposition 3.4.1**.

**As previously seen.** Let  $X = X^i \frac{\partial}{\partial x_i}$ ,  $V = V^k \frac{\partial}{\partial x_k}$ , and let  $D$  be the **Levi-Civita connection**. Then

$$D_V X = D_V \left( X^i \frac{\partial}{\partial x_i} \right) = V(X^i) \frac{\partial}{\partial x_i} + \underbrace{X^i D_V \frac{\partial}{\partial x_i}}_{V^k D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}} = V(X^i) \frac{\partial}{\partial x_i} + V^k X^i \Gamma_{ki}^j \frac{\partial}{\partial x_j}.$$

**Proposition 3.4.1 (Covariant derivative).** Let  $(\mathcal{M}, g)$  be a **Riemannian manifold**,  $D$  the canonical (**Levi-Civita**) **connection**, and  $c$  a **smooth curve** in  $\mathcal{M}$ . Then there exists a unique operator  $D/dt$  defined as the vector space of **vector fields along  $c$**  satisfying

- (i) (a)  $\frac{D}{dt}(fY)(t) = f'(t)Y(t) + f(t)\frac{D}{dt}Y(t)$  for all  $f \in C^\infty(I)$  and  $Y \in \mathcal{X}_c(\mathcal{M})$ ;
- (b)  $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$  for all  $V, W \in \mathcal{X}_c(\mathcal{M})$ ;
- (ii) if there exists a neighborhood of in  $I$  such that  $Y$  is the restriction to  $c$  of a **vector field**  $X$  defined on a neighborhood of  $c(t_0)$  in  $\mathcal{M}$ , then  $\frac{D}{dt}Y(t_0) = (D_{c(t_0)}X)_{c(t_0)}$ .

**Proof.** Consider defining such an operator  $D/dt$  as

$$\frac{D}{dt} \left( Y^i(t) \frac{\partial}{\partial x_i} \right) = \frac{dY^i}{dt} \frac{\partial}{\partial x_i} + \dot{c} Y^i \Gamma_{ji}^k(c(t)) \frac{\partial}{\partial x_k},$$

where  $\dot{c} = \dot{c}^k \frac{\partial}{\partial x_k}$ . This shows (i) (a) and (b) hold. Next, to show (ii), let  $x$  be a smooth **vector field** in  $\mathcal{M}$ . Then the induced **vector field along  $c$**  is given by  $Y(t) = X_{c(t)}$ , i.e., in terms of the coordinate basis, we have

$$Y(t) = Y^i(t) \frac{\partial}{\partial x_i}, \quad X_x = X^i(x) \frac{\partial}{\partial x_i}, \quad Y^i(t) = X^i(c(t)).$$

Then,

$$\begin{aligned} D_i X &= D_i \left( X^i \frac{\partial}{\partial x_i} \right) = \dot{c}(X^i) \frac{\partial}{\partial x_i} + X^i D_i \frac{\partial}{\partial x_i} = X^i \underbrace{\dot{c}^k D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}}_{\Gamma_{ki}^\ell \frac{\partial}{\partial x_\ell}} \\ &= \partial_t(X^i \circ c) \frac{\partial}{\partial x_i} + \dot{c}^k X^i \Gamma_{ki}^\ell \frac{\partial}{\partial x_\ell} = \partial_t(X^i \circ c) \frac{\partial}{\partial x_i} + \dot{c}^k Y^i \Gamma_{ki}^\ell \frac{\partial}{\partial x_\ell} = \frac{D}{dt} Y. \end{aligned}$$

■

**Problem 3.4.1.** Why not just define  $DY/dt$  by (ii)?

**Answer.** A **vector field  $Y$  along a curve** may not always be extended to a neighborhood of  $c$  in  $\mathcal{M}$ . But, in **local coordinates**,

$$Y(t) = \sum_{i=1}^n Y^i(t) \left( \frac{\partial}{\partial x^i} \right)_{c(t)},$$

which shows that a **vector field along  $c$**  is always a linear combination of **vector fields along  $c$**  that can be extended. ⊛

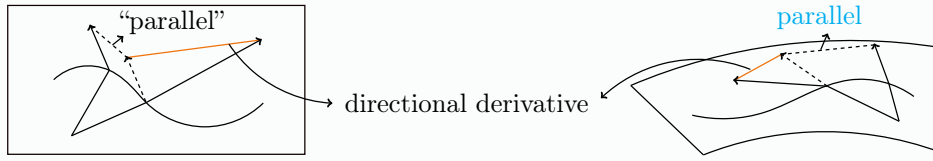
**Remark.** Proposition 3.4.1 shows that the choice of a linear connection on  $\mathcal{M}$  leads to a bona fide (satisfying (a) and (b)) derivative of vector fields along curves. The notion of “connection” furnishes, therefore, a manner of differentiating vectors along curves.

### 3.4.2 Parallel Transports

Finally, we introduce the notion of parallel.

**Definition 3.4.2 (Parallel).** A vector field  $X$  on  $\mathcal{M}$  along a curve  $c$  is *parallel* (or *parallelly transported*) along  $c$  if  $DX/dt = 0$  for all  $t \in I$ .

**Intuition.** In the (flat) Euclidean space, we know what is “parallel,” and hence we can define the directional derivative. But now the logic is reversed: we first define what is parallel in a curved space, and then we can make sense of directional derivative in a curved space!

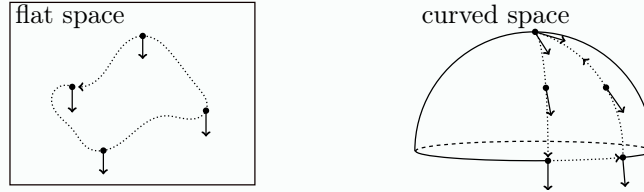


Given the definition of a parallel vector fields along curves, we can talk about parallel transport.

**Definition 3.4.3 (Parallel transport).** The *parallel transport* from  $c(0)$  to  $c(t)$  along the curve  $c$  in a Riemannian manifold  $(\mathcal{M}, g)$  is the linear map  $P_t: T_{c(0)}\mathcal{M} \rightarrow T_{c(t)}\mathcal{M}$  associating  $v \in T_{c(0)}\mathcal{M}$  with  $X_v(t) \in T_{c(t)}\mathcal{M}$  with  $X_v$  being the parallel vector field along  $c$  such that  $X_v(0) = v$ .

It's clear that how we can extend Definition 3.4.3 for a piece-wise smooth curve.

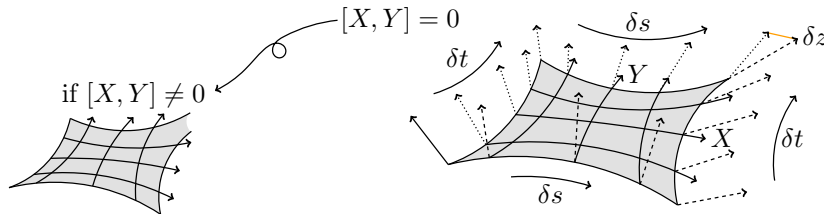
**Intuition.** When the space is flat, keeping the “arrow” (which defines a vector field) in one direction and moving around won't produce any changes, while when the space is curved, it will.



We make a surprising remark on the relation between Riemannian curvature and parallel transport.

**Remark (Geometric significant of Riemannian curvature).** The idea is that for a manifold with torsion free  $\nabla$ , if we parallel transporting along two paths on an infinitesimal patch (which induces  $X, Y$ ) such that  $[X, Y] = 0$ , we can detect curvature in terms of  $\delta z$ , where<sup>a</sup>

$$(\delta z)^i = \dots = R^i_{jkl} X^k Y^l Z^j \cdot \delta s \delta t + O(\delta s^2 \delta t, \delta s \delta t^2).$$



We will come back to this later.

<sup>a</sup>This is a deep theorem! In the ..., we use  $T \equiv 0$ .

**Proposition 3.4.2.** The [parallel transport](#) exists, uniquely.

**Proof.** do Carmo [FC13, Proposition 2.6] ■

**Proposition 3.4.3.** Let  $(\mathcal{M}, g)$  be a [Riemannian manifold](#). The [parallel transport](#) defines for all  $t$  an [isometry](#) from  $T_{c(0)}\mathcal{M}$  onto  $T_{c(t)}\mathcal{M}$ ; more generally, if  $X, Y$  are [vector fields along  \$c\$](#) , then

$$\frac{d}{dt}g(x(t), y(t)) = g\left(\frac{DX(t)}{dt}, Y(t)\right) + g\left(X(t), \frac{DY(t)}{dt}\right).$$

**Proof.** See do Carmo [FC13, Proposition 3.2] ■

### 3.4.3 Autoparallel Curves

Now we can formally introduce the notion of [autoparallel](#).

**Definition 3.4.4 (Autoparallel).** Let  $\nabla$  be a [connection](#) on  $T\mathcal{M}$  of a [differentiable manifold](#)  $\mathcal{M}$ . A [curve](#)  $c: I \rightarrow \mathcal{M}$  is called *autoparallel* (or *geodesic*) w.r.t.  $\nabla$  if

$$\nabla_{\dot{c}}\dot{c} = 0.$$

**Intuition.** An [autoparallel curve](#) is the *straightest line* (hence [geodesic](#)) in the space w.r.t.  $\nabla$ !

**Remark (Physical interpretation).** One can start from introducing  $\nabla$ , considering  $\nabla_{\dot{c}}\dot{c} := 0$  (which is just [Equation 2.1](#)), and realize that we don't need to consider gravity as a force, rather a “curvature of spacetime,” in order to make sense of Newton's first law, i.e., mass without forces will undergo a [autoparallel curve](#).

**Example (Euclidean plane).** Let  $U = \mathbb{R}^2$ ,  $x = \text{id}_{\mathbb{R}^2}$ ,  $\Gamma_{jk}^i = 0$ , then  $\ddot{c}^k = 0$  in [Equation 3.4](#). Hence,

$$c^k(t) = a^k t + b^k \text{ for } a^k, b^k \in \mathbb{R}^d.$$

**Example (Round sphere).** The [geodesics](#) on a “round sphere” are the great circles.

**Proof.** Consider a “unit round sphere”  $\mathcal{M} = S^2$  with spherical coordinates  $x(p) = (r, \theta, \varphi)$  such that  $r = 1$ ,  $\theta \in (0, \pi)$ , and  $\varphi \in [0, 2\pi]$ . The “roundness” is given by  $\nabla_{\text{round}}$  where we specify (at one point)

$$\Gamma_{22}^1 := -\sin \theta \cos \theta, \quad \Gamma_{21}^2 = \Gamma_{12}^2 := \cot \theta,$$

where we let  $x^1(p) = \theta(p)$ ,  $x^2(p) = \varphi(p)$ . The [autoparallel equation](#) tells us

$$\begin{cases} \ddot{\theta} + \Gamma_{22}^1 \dot{\varphi} \dot{\varphi} = 0; \\ \ddot{\varphi} + 2\Gamma_{12}^2 \dot{\theta} \dot{\varphi} = 0; \end{cases} \Leftrightarrow \begin{cases} \ddot{\theta} - \sin(\theta) \cos(\theta) \dot{\varphi} \dot{\varphi} = 0; \\ \ddot{\varphi} + 2 \cot(\theta) \dot{\theta} \dot{\varphi} = 0. \end{cases}$$

Then, we see that  $\theta(t) = \pi/2$ ,  $\varphi(t) = \omega t + \varphi_0$  is a solution.<sup>a</sup> Hence, we conclude that if we run at a constant speed around the great circle of  $S^2$ , it'll be [autoparallel](#), hence a [geodesic](#). \*<sup>⊛</sup>

<sup>a</sup>Note that  $\theta(t) = \pi/2$ ,  $\varphi(t) = \omega t^2 + \varphi_0$  is not a solution.

Similarly, given any  $\nabla$  on a space, we can find the straightest [curve](#) on which.

## Lecture 12: Tangent and Cotangent Bundles

### 3.5 More on Tangent and Cotangent Bundles

Let  $f: \mathcal{M} \rightarrow \mathcal{N}$  be a differentiable map between two [differentiable manifolds](#), until now, we have only talked about how to transform [tangent vectors](#) or 1-form via  $f$ . Implicitly, these are just [pullback](#) ( $f^*$ ) and [pushforward](#) ( $f_*$ ), as we now define formally.

**Definition.** Let  $f: \mathcal{M} \rightarrow \mathcal{N}$  be a smooth map between two [smooth manifolds](#) and  $p \in \mathcal{M}$ .

**Definition 3.5.1 (Pushforward).** The *pushforward* is the linear map  $f_* := df_p: T_p\mathcal{M} \rightarrow T_{f(p)}\mathcal{N}$ .

**Definition 3.5.2 (Pullback).** The *pullback* is the linear map  $f^*: T_{f(p)}^*\mathcal{N} \rightarrow T_p^*\mathcal{M}$  where

$$(f^*\omega)(X) = \omega(f_*X)$$

for  $\omega \in T_{f(p)}^*\mathcal{N}$  and  $X \in T_p\mathcal{M}$ .

In all, the following diagram commutes:

$$\begin{array}{ccc} T_p^*\mathcal{M} & \xleftarrow{f^*} & T_p^*\mathcal{N} \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N} \end{array} \quad \begin{array}{ccc} T_p\mathcal{M} & \xrightarrow{f_*} & T_{f(p)}\mathcal{N} \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N} \end{array}$$

#### 3.5.1 Pullbacks and Pushforwards on Bundles

Now, consider a [vector bundle](#)  $(E, \pi, \mathcal{N})$  over  $\mathcal{N}$ , we want to use  $f$  to “pull back” the [vector bundle](#), i.e., construct a [vector bundle](#), denote as  $f^*E$ , for which the [fiber](#) over  $x \in \mathcal{M}$  is  $E_{f(x)}$ .

**Definition 3.5.3 (Pullback bundle).** The *pullback bundle*  $f^*E$  is the [vector bundle](#) over  $\mathcal{M}$  with the [bundle charts](#)  $(\varphi \circ f, f^{-1}(U))$  if  $(\varphi, U)$  is the [bundle charts](#) of  $E$ .

Similarly, we can “push forward” a [vector bundle](#)  $(E, \pi, \mathcal{M})$  over  $\mathcal{M}$  via  $f$  in the same fashion.

**Definition 3.5.4 (Pushforward bundle).** The *pushforward bundle*  $f_*E$  is the [vector bundle](#) over  $\mathcal{N}$  with the [bundle charts](#)  $(\varphi \circ f^{-1}, f(U))$  if  $(\varphi, U)$  is the [bundle charts](#) of  $E$ .

**Note.** In [Definition 3.5.4](#), it only makes sense if  $\mathcal{M} \hookrightarrow \mathcal{N}$ .

**Definition 3.5.5 (Bundle homomorphism).** Consider 2 [vector bundles](#)  $(E_1, \pi_1, \mathcal{M}), (E_2, \pi_2, \mathcal{M})$  over  $\mathcal{M}$ , and let the differentiable map  $f: E_1 \rightarrow E_2$  be [fiber](#) preserving, i.e.,  $\pi_2 \circ f = \pi_1$ . If the [fiber](#) maps  $f_x: E_{1,x} \rightarrow E_{2,x}$  is linear,<sup>a</sup> then  $f$  is called a *bundle homomorphism*.

<sup>a</sup>I.e., vector homomorphisms.

**Definition 3.5.6 (Subbundle).** Let  $(E, \pi, \mathcal{M})$  of rank  $n$  be a [vector bundle](#). Let  $E^1 \subseteq E$ , and assume that for all  $x \in \mathcal{M}$ , there exists a [bundle chart](#)  $(\varphi, U)$  for  $x \in U$  and

$$\varphi(\pi^{-1}(U) \cap E^1) = U \times \mathbb{R}^m \subseteq U \times \mathbb{R}^n$$

for  $m \leq n$ . Then the *subbundle* of  $E$  of rank  $m$  is the [vector bundle](#)  $(E^1, \pi|_{E^1}, \mathcal{M})$ .

**Example.** Consider  $f: \mathcal{M} \hookrightarrow \mathcal{N}$  where  $g_{\mathcal{N}}$  is a [metric](#) on  $\mathcal{N}$ . Then,  $g_{\mathcal{N}}$  induces a [metric](#)  $g_{\mathcal{M}}$  on  $\mathcal{M}$

by  $f$  since we can define

$$g_{\mathcal{M}}(X, Y) := g_{\mathcal{N}}(f_*(X), f_*(Y)).$$

### 3.5.2 Pullbacks and Pushforwards of Vector Fields

Now, we consider to “pull back” or “push forward” a **vector field**, i.e., a **section** of a **bundle**.

**Definition 3.5.7 (Pushforward).** Let  $\psi: \mathcal{M} \rightarrow \mathcal{N}$  be a **diffeomorphism** between **smooth manifolds**, and let  $X$  be a **vector field** on  $\mathcal{M}$ . Then the *pushforward vector field*  $Y = \psi_*X = d\psi X$  on  $\mathcal{N}$  is

$$Y(p) = d\psi(X(\psi^{-1}(p))).$$

**Definition 3.5.8 (Pullback).** Let  $\psi: \mathcal{M} \rightarrow \mathcal{N}$  be a **diffeomorphism** between **smooth manifolds**, and let  $Y$  be a **vector field** on  $\mathcal{N}$ . Then the *pullback vector field*  $X = \psi^*Y$  on  $\mathcal{M}$  is just  $X(p) = Y_{\psi(p)}$ .

**Note.** We let  $\psi$  be a **diffeomorphism** just for convenient.

**Lemma 3.5.1.** For every differentiable function  $f: \mathcal{N} \rightarrow \mathbb{R}$ ,  $(\psi_*X)(f)(p) = X(f \circ \psi)(\psi^{-1}p)$ .

**Lemma 3.5.2.** Let  $X$  be a **vector field** on  $\mathcal{M}$  and  $\psi: \mathcal{M} \rightarrow \mathcal{N}$  be a **diffeomorphism**. If the **local 1-parameter group**  $(\varphi_t)_{t \in I}$  generated by  $X$ , then the **local 1-parameter group** generated by  $\psi_*X$  is  $\psi \circ \varphi_t \circ \psi^{-1}$ .

### 3.5.3 Induced Bundle Metrics

Let  $(\mathcal{M}, g)$  be a **Riemannian manifold**, then  $g$  induces the *bundle metrics* on all **vector bundles** over  $\mathcal{M}$ : for  $T^*\mathcal{M}$ , it is given by

$$g(\omega, \eta) := g^{ij}\omega_i\eta_j$$

for  $\omega = \omega_i dx^i, \eta = \eta_i dx^i$ . Hence, we can talk about the identification between  $T\mathcal{M}$  and  $T^*\mathcal{M}$  through  $g$ :

$$\begin{array}{c} V = V^i \frac{\partial}{\partial x^i} \in T\mathcal{M} \\ \Updownarrow \\ \omega = \omega_j dx^j \in T^*\mathcal{M} \end{array}$$

with  $\omega_j = g_{ij}V^i$  (or  $V^i = g^{ij}\omega_j$ ) such that

- (a)  $g(X, Y) = g_{ij}X^iY^j$  for  $X, Y \in T\mathcal{M}$ ;
- (b)  $g(\omega, \eta) = g^{ij}\omega_i\eta_j$  for  $\omega, \eta \in T^*\mathcal{M}$ .

Thus, for  $V \in T_x\mathcal{M}$ , there corresponds a 1-form  $\omega \in T_x^*\mathcal{M}$  via the metric  $\omega(Y) := g(V, Y)$  for all  $Y$ , and we further have  $\|\omega\| = \|V\|$ .

We can also consider the **coordinate** transformation behavior. Let  $(e_i)_{i=1, \dots, d}$  be a basis of  $T_x\mathcal{M}$  and  $(\omega^j)_{j=1, \dots, d}$  the dual basis of  $T_x^*\mathcal{M}$ , i.e.,  $\omega^j(e_i) = \delta_i^j$ . Given  $V = V^i e_i \in T_x\mathcal{M}$ ,  $\eta = \eta_j \omega^j \in T_x^*\mathcal{M}$ , we then have  $\eta(V) = \eta_i V^i$ . Now, consider bases  $(e_i), (\omega^j)$  in the **local coordinates**, i.e.,  $e_i = \partial/\partial x^i$  and  $\omega^j = dx^j$ . Let  $f$  be a **local coordinates** change, then  $V$  and  $\eta$  transformed as

$$f_*(V) := V^i \frac{\partial f^\alpha}{\partial x^i} \frac{\partial}{\partial f^\alpha}, \quad f^*(\eta) := \eta_j \frac{\partial x^j}{\partial f^\beta} df^\beta$$

correspondingly, and we see that

$$f^*(\eta)(f_*(V)) = \eta_j \frac{\partial x^j}{\partial f^\alpha} V^i \frac{\partial f^\alpha}{\partial x^i} = \eta_i V^i = \eta(V).$$

**Intuition.** The above means that

- the **tangent vectors** transform with the functional matrix of **coordinates** change;
- the **cotangent vectors** transform with the transposed inverse of the above matrix.

To compute the **coordinates** change  $y \mapsto x(y)$  for  $\omega = \omega_i dx^i$ ,  $\eta = \eta_i dx^i$  with  $\langle \omega, \eta \rangle = g^{ij} \omega_i \eta_j$ , we have

$$\omega_i dx^i = \omega_i \frac{\partial x^i}{\partial y^\alpha} dy^\alpha =: \tilde{\omega}_\alpha dy^\alpha.$$

**As previously seen.**  $g^{ij}$  is transformed as

$$h^{\alpha\beta} = g^{ij} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}.$$

Then, we see that  $h^{\alpha\beta} \tilde{\omega}_\alpha \tilde{\eta}_\beta = g^{ij} \omega_i \eta_j$  and  $\|\omega(x)\| = \sup \{\omega(x)(V) \mid V \in T_x \mathcal{M}, \|v\| = 1\}$ .

**Remark.** If we consider  $T\mathcal{M} \otimes T\mathcal{M}$ , then metric is

$$\langle V \otimes Y, \xi \otimes \eta \rangle = g_{ij} V^i Y^j g_{kl} \xi^k \eta^l.$$

**As previously seen** (Lie derivative). Consider a **vector field**  $X$  with a **local 1-parameter group**  $(\psi_t)_{t \in I}$  and a **tensor field**  $S$  on  $\mathcal{M}$ . The **Lie derivative** of  $S$  in the direction of  $X$  is defined as

$$\mathcal{L}_X S := \left. \frac{d}{dt} (\psi_t^* S) \right|_{t=0}.$$

## Lecture 13: Sectional Curvatures and Space Forms

Let  $X = X^i \partial / \partial x^i$  be a **vector field**. Then consider  $(\psi_t)_* X(\psi_t(x))$  to get a **curve**  $X_t$  in  $T_x \mathcal{M}$  for  $t \in I$ . 16 Feb. 13:00  
By differentiate that curve, i.e.,

$$(\psi_t)_* \frac{\partial}{\partial x^i} (\psi_t(x)) = \frac{\partial \psi_t^k}{\partial x^i} \frac{\partial}{\partial x^k}.$$

**Note.** For  $\varphi: \mathcal{M} \rightarrow \mathcal{N} := \mathcal{M}$  and  $X$  and  $\varphi(x)$  are in the same **coordinate neighborhood**,

$$\varphi_* \frac{\partial}{\partial x^i} = \frac{\partial \varphi^k}{\partial x^i} \frac{\partial}{\partial \varphi^k}$$

since  $\frac{\partial}{\partial \varphi^k} = \frac{\partial}{\partial x^k}$ .

On the other hand, let  $\omega = \omega_i dx^i$  be a 1-form, then we have

$$(\psi_t^*)(\omega)(x) = \omega_i(\psi_t(x)) \frac{\partial \psi_t^i}{\partial x^k} dx^k,$$

which is a **curve** in  $T_x^* \mathcal{M}$ .

**Note.** For  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ ,<sup>a</sup> with for the 1-form  $\omega = \omega_i dx^i$  on  $\mathcal{N}$ ,

$$\varphi^* \omega = \omega_i(\varphi(x)) \frac{\partial z^i}{\partial x^k} dx^k.$$

<sup>a</sup> $\varphi$  need not be a **diffeomorphism**.

Let  $\varphi: \mathcal{M} \rightarrow \mathcal{N}$  be a **diffeomorphism**,  $Y$  be a **vector field** on  $\mathcal{N}$ . Then, set

$$\varphi^* Y := (\varphi^{-1})_* Y,$$

and for other **contravariant tensors**,  $\varphi^*$  can be defined in an analogous way.

**Example.** For a vector field  $X$  and a local 1-parameter group  $(\psi_t)_{t \in I}$ , it is  $(\psi_t^* X) = (\psi_t)_* X$ .

### 3.6 Sectional Curvatures

Consider the following.

**Definition 3.6.1** (Sectional curvature). The *sectional curvature* of the plane spanned by the (linearly independent) tangent vectors  $X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i} \in T_x \mathcal{M}$  of a Riemannian manifold  $(\mathcal{M}, g)$  is

$$K(X \wedge Y) := \frac{g(R(X, Y)Y, X)}{|X \wedge Y|^2}$$

where  $|X \wedge Y|^2 = g(X, X)g(Y, Y) - g(X, Y)^2$ .

**Intuition.** Given a vector space  $V$  and  $x, y \in V$ ,  $|x \wedge y| := \sqrt{|x|^2 + |y|^2 - \langle x, y \rangle^2}$  represents the area of the two-dimensional parallelogram spanned by  $x, y$ .

**Remark.** Sectional curvature determines the whole Riemannian curvature.

**Proof.** Given  $g(R(X, Y)Z, W)$ , we can express this entirely by  $K$ . See do Carmo [FC13, Lemma 3.3]. \*

**Remark** (Gauss curvature). For  $\dim \mathcal{M} = 2$ ,  $R_{ijkl} = K(g_{ik}g_{jl} - g_{ij}g_{kl})$  since  $T_x \mathcal{M}$  contains only one plane, i.e.,  $T_x \mathcal{M}$  itself. In this case,  $K$  is called the *Gauss curvature*.

In particular, the space form considers the space with constant sectional curvature.

**Definition 3.6.2** (Space form). A Riemannian manifold  $(\mathcal{M}, g)$  is a *space form* if  $K(X \wedge Y)$  is a constant for all linearly independent tangent vectors  $X, Y \in T_p \mathcal{M}$  for all  $p \in \mathcal{M}$ .

**Definition 3.6.3** (Spherical). A space form is called *spherical* if  $K > 0$ .

**Definition 3.6.4** (Flat). A space form is called *flat* if  $K = 0$ .

**Definition 3.6.5** (Hyperbolic). A space form is called *hyperbolic* if  $K < 0$ .

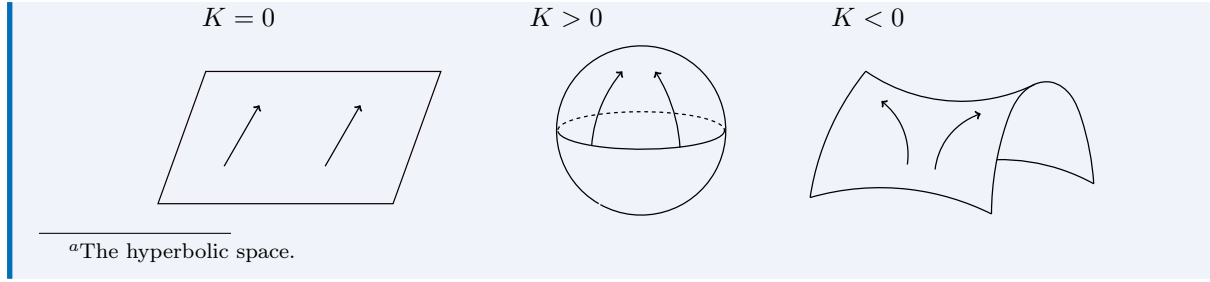
Generalize Definition 3.6.2 a bit, we have the so-called Einstein manifolds.

**Definition 3.6.6** (Einstein manifold). A Riemannian manifold  $(\mathcal{M}, g)$  is called an *Einstein manifold* if  $R_{ik} = cg_{ik}$  for a constant  $c$ .<sup>a</sup>

<sup>a</sup>Which does not depend on the choice of local coordinates.

**Remark.** Every space form is an Einstein manifold.

**Example.**  $\mathbb{R}^n$  is flat,  $S^n$  is spherical, and  $\mathbb{H}^n$  is hyperbolic. And all are Einstein manifolds.



**Definition 3.6.7 (Flat).** A connection  $\nabla$  on  $T\mathcal{M}$  is *flat* if each point in  $\mathcal{M}$  has a neighborhood  $U$  with *local coordinates* for which all the coordinate *vector fields*  $\partial/\partial x^i$  are *parallel*, i.e.,  $\nabla \partial/\partial x^i = 0$ .

**Theorem 3.6.1.** A connection  $\nabla$  on  $T\mathcal{M}$  is *flat* if and only if its *curvature* and *torsion* vanish identically.

**Proof.** *Flat connection* implies  $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ , hence all  $\Gamma_{ij}^k = 0$ , so  $T, R$  vanish. Conversely, find the *local coordinates* such that  $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$  for all  $i, j$  and use *Frobenius theorem*. ■

**Example.** The following are *flat manifolds* with their usual shape, i.e., *connections*.

- $\mathbb{R}^n$ .
- Products of *flat manifolds*.
- Torus  $T^2$ .
- Every 1-dimensional *Riemannian manifold*.
- Tori.

**Theorem 3.6.2 (Schur theorem).** Let  $(\mathcal{M}, g)$  be a *Riemannian manifold* with  $\dim \mathcal{M} \geq 3$ .

- (a) If the *sectional curvature* of  $\mathcal{M}$  is constant at each point, i.e.,  $K(X \wedge Y) = f(x)$  for  $X, Y \in T_x \mathcal{M}$ , then  $f(x)$  is a constant on  $\mathcal{M}$ , hence  $\mathcal{M}$  is a *space form*.
- (b) If the *Ricci curvature* is a constant at each point, i.e.,  $R_{ik} = c(x)g_{ik}$ , then  $c(x)$  is a constant, hence  $\mathcal{M}$  is an *Einstein manifold*.

**Remark.** *Schur theorem* says that the isotropy<sup>a</sup> of a *Riemannian manifold* implies the homogeneity.<sup>b</sup> Hence, a point-wise property implies a global one!

<sup>a</sup>I.e., the property that at each point, all directions are geometrically indistinguishable.

<sup>b</sup>I.e., all points are geometrically indistinguishable.



# Chapter 4

## Isometric Immersions

### 4.1 Covering Maps

**Definition 4.1.1** (Covering map). Let  $\mathcal{M}, \widetilde{\mathcal{M}}$  be 2 manifolds. a map  $p: \widetilde{\mathcal{M}} \rightarrow \mathcal{M}$  is a *covering map* if

- (a)  $p$  is smooth and surjective;
- (b) for all  $m \in \mathcal{M}$ , there exists a neighborhood  $U$  at  $m$  in  $\mathcal{M}$  with  $p^{-1}(U) = \coprod_{i \in I} U_i$  with  $p: U_i \rightarrow U$  being a [diffeomorphism](#) and  $U_i$  are disjoint open subsets of  $\widetilde{\mathcal{M}}$ .

**Notation** (Covering space).  $\widetilde{\mathcal{M}}$  in [Definition 4.1.1](#) is called the *covering space*.

**Notation** (Universal covering space). A [covering space](#) is *universal* if it's simply connected.

**Definition 4.1.2** (Riemannian covering map). Let  $(\mathcal{M}, g), (\mathcal{N}, h)$  be [Riemannian manifolds](#). A map  $p: \mathcal{N} \rightarrow \mathcal{M}$  is a *Riemannian covering map* if  $p$  is a smooth [covering map](#) and is a [local isometry](#).

**Proposition 4.1.1.** Let  $p: \mathcal{N} \rightarrow \mathcal{M}$  be a smooth [covering map](#). For every [Riemannian metric](#)  $g$  on  $\mathcal{M}$ , there exists a unique [Riemannian metric](#)  $h$  on  $\mathcal{N}$  such that  $p$  is a [Riemannian covering map](#).

**Note.** The converse of [Proposition 4.1.1](#) is generally not true.

**Example.** Every space [covers](#) itself trivially.

**Example.**  $\mathbb{R}$  is the [universal covering space](#) of  $S^1$ .

**Example.**  $U(n)$  has [universal covers](#)  $U(n) \times \mathbb{R}$ .

**Example.**  $S^n$  is a double [cover](#) for  $\mathbb{R}P^n$  and is [universal](#) for  $n > 1$ .

### Lecture 14: The Second Fundamental Form

**Proposition 4.1.2.** Let  $(\mathcal{N}, h)$  be a [Riemannian manifold](#) and  $G$  be a [free](#) and [proper](#) group of [isometries](#) of  $(\mathcal{N}, h)$ , then there exists a unique [Riemannian metric](#)  $g$  on the quotient manifold  $\mathcal{M} = \mathcal{N} / G$  such that the connected projection  $p: \mathcal{N} \rightarrow \mathcal{M}$  is a [Riemannian covering map](#).

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**Proof.** Let  $n, n' \in \mathcal{N}$  such that  $n, n' \in p^{-1}(m)$  for  $m \in \mathcal{M}$ . Hence, there exists an **isometry**  $f \in G$  such that  $f(n) = n'$ . Also,  $p \circ f = p$ , and  $p$  is a local **diffeomorphism**, so we can define a scalar product  $g_m$  on  $T_m\mathcal{M}$ : for all  $u, v \in T_m\mathcal{M}$ ,

$$g_m(u, v) = h_n((T_n p)^{-1}u, (T_n p)^{-1}v)$$

for  $n \in p^{-1}(m)$ . This does not depend on the choice of  $n \in p^{-1}(m)$  since  $(T_n p)^{-1} = T_n f \circ (T_{n'} p)^{-1}$  and  $T_n f$  is an **isometry** of the Euclidean vector spaces  $T_n\mathcal{N}$  and  $T_{n'}\mathcal{N}$ . It can be shown that  $g$  is smooth. Thus, we have constructed a **metric**  $g$  on  $\mathcal{M}$  such that  $p$  is a **Riemannian covering map**, which is unique. ■

**Definition 4.1.3** (Totally geodesic). A **submanifold**  $\mathcal{M}$  of  $(\tilde{M}, \tilde{g})$  is called *totally geodesic* if for all  $m \in \mathcal{M}$  and  $v \in T_m\mathcal{M}$ , the **geodesic**  $c$  of  $(\tilde{M}, \tilde{g})$  with  $c(0) = m$  and  $c'(0) = v$  is contained fully in  $\mathcal{M}$ .

**Proposition 4.1.3.** Let  $p: (\mathcal{N}, h) \rightarrow (\mathcal{M}, g)$  be a **Riemannian covering map**. The **geodesic** of  $(\mathcal{M}, g)$  are the projections of the **geodesic** in  $(\mathcal{N}, h)$ ; and the **geodesic** of  $(\mathcal{N}, h)$  are the liftings of those in  $(\mathcal{M}, g)$ .

**Proof.** Since  $p$  is a **local isometry**, if  $\gamma$  is a **geodesic** of  $\mathcal{N}$ , then  $c = p \circ \gamma$  is also a **geodesic** of  $\mathcal{M}$ . From the **uniqueness theorem** for **geodesics** shows that these are indeed the only **geodesics** on  $\mathcal{M}$ . Conversely, if  $p \circ \gamma$  is a **geodesic** in  $\mathcal{M}$ , then  $\gamma$  is a **geodesic** in  $\mathcal{N}$ . ■

**Example.** In Euclidean spaces, the **totally geodesic submanifolds** are affine linear subspaces and their open subsets.

**Example.** Each closed **geodesic** in **Riemannian manifolds** defines a 1-dimensional compact **totally geodesic submanifold**.

**Example.** The **totally geodesic submanifolds** of  $S^n \subseteq \mathbb{R}^{n+1}$  are the intersections of  $S^n$  with linear subspaces of  $\mathbb{R}^{n+1}$ .

**Example.** In general, **Riemannian manifolds** do not have any **totally geodesic submanifolds** of dimension  $> 1$ .

**Note.** We will see that  $\mathcal{M}$  is **totally geodesic** in  $\tilde{M}$  if and only if all the  $2^{nd}$ -fundamental forms vanish identically.

## 4.2 The Second Fundamental Form

Let  $\mathcal{M}^m \subseteq \mathcal{N}^n$  be two **Riemannian manifolds**, and we know that a **metric** on  $\mathcal{N}$  induces a **metric** on  $\mathcal{M}$  naturally. Now, we want to see that given the **Levi-Civita connection**  $\nabla^{\mathcal{N}}$  of  $\mathcal{N}$ , how to get  $\nabla^{\mathcal{M}}$  of  $\mathcal{M}$ .

This is given by the central object  $(\nabla_X^{\mathcal{N}} Y)^\top$  we will study in this chapter, where  $\top: T_x\mathcal{N} \rightarrow T_x\mathcal{M}$  for  $x \in \mathcal{M}$  is the orthogonal projection. We see the following.

**Theorem 4.2.1.** For  $X, Y \in \Gamma(T\mathcal{M})$ ,  $\nabla_X^{\mathcal{M}} Y = (\nabla_X^{\mathcal{N}} Y)^\top$ .

**Proof.** Firstly, we have to make sure that the right-hand side is defined. This can be done by extending **vector fields**  $X, Y$  locally to a neighborhood of  $\mathcal{M}$  in  $\mathcal{N}$ . Do this in the **local coordinates** around  $x \in \mathcal{M}$  locally mapping  $\mathcal{M}$  to  $\mathbb{R}^m \subseteq \mathbb{R}^n$ .

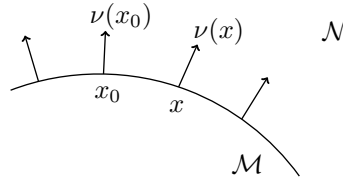
Specifically, the extension of  $X = \xi^i(x)\partial/\partial x^i$  is

$$\tilde{X}(x^1, \dots, x^n) = \sum_{i=1}^m \xi^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i}.$$

Then  $\langle \tilde{X}, \tilde{Y} \rangle(x) = \langle X, Y \rangle(x)$  and  $[\tilde{X}, \tilde{Y}](x) = [X, Y](x)$ . From [Levi-Civita theorem](#), the [Koszul formula](#) holds for both  $\mathcal{N}$  and  $\mathcal{M}$ . Finally, we see that

- $(\nabla_X^{\mathcal{N}} Y)^\top$  does not depend on the chosen extensions: follows from the fact that the representation of  $\nabla^{\mathcal{N}}$  is done by  $\Gamma$ ;
- $(\nabla_X^{\mathcal{N}} Y)^\top$  defines a [torsion-free connection](#) on  $\mathcal{M}$ : as  $\nabla_X^{\mathcal{N}} Y - \nabla_Y^{\mathcal{N}} X - [X, Y]$  vanishes, also the tangential part to  $\mathcal{M}$  has to vanish.

Let  $\nu(x)$  be a [vector field](#) in a neighborhood of  $x_0 \in \mathcal{M} \subseteq \mathcal{N}$  that is orthogonal to  $\mathcal{M}$ , i.e.,  $\langle \nu(x), X \rangle = 0$  for all  $X \in T_x \mathcal{M}$ . Also, let  $T_x \mathcal{M}^\perp$  be the orthogonal complement of  $T_x \mathcal{M}$  in  $T_x \mathcal{N}$ , and  $T\mathcal{M}^\perp$  with [fiber](#)  $T_x \mathcal{M}^\perp$  of  $x \in \mathcal{M}$ .



**Notation** (Normal bundle).  $T\mathcal{M}^\perp$  is the *normal bundle* of  $\mathcal{M}$  in  $\mathcal{N}$ .

We see that  $\langle \nu(x), X \rangle = 0$  for all  $X \in T_x \mathcal{M}$  means  $\nu(x) \in T_x \mathcal{M}^\perp$ .

**Lemma 4.2.1.**  $(\nabla_X^{\mathcal{N}} \nu)^\top(x)$  only depends on  $\nu(x)$ .

**Proof.** This follows directly from

$$(\nabla_X^{\mathcal{N}} f\nu)^\top(x) = (X(f)(x)\nu(x))^\top + f(x)(\nabla_X^{\mathcal{N}} \nu)^\top(x) = f(x)(\nabla_X^{\mathcal{N}} \nu)^\top(x)$$

for  $f$  smooth, since  $(X(f)(x)\nu(x))^\top = 0$ . ■

**Definition 4.2.1** (Second fundamental tensor). The *second fundamental tensor*  $S: T_x \mathcal{M} \times T_x \mathcal{M}^\perp \rightarrow T_x \mathcal{M}$  of  $\mathcal{M}$  at point  $x \in \mathcal{M}$  is defined by

$$S(X, \nu) = (\nabla_X^{\mathcal{N}} \nu)^\top.$$

**Lemma 4.2.2.** For  $X, Y \in T_x \mathcal{M}$ ,  $\ell_\nu(X, Y) := \langle S(X, \nu), Y \rangle$  is symmetric in  $X, Y$ .

**Proof.** Since

$$\ell_\nu(X, Y) = \langle (\nabla_X^{\mathcal{N}} \nu)^\top, Y \rangle = \langle \nabla_X^{\mathcal{N}} \nu, Y \rangle = -\langle \nu, \nabla_X^{\mathcal{N}} Y \rangle$$

as  $\nabla^{\mathcal{N}}$  is [metric](#) and  $\langle \nu, Y \rangle = 0$ . Now, since  $\nabla^{\mathcal{N}}$  is [torsion-free](#), we further have

$$\ell_\nu(X, Y) = -\langle \nu, \nabla_Y^{\mathcal{N}} X + [X, Y] \rangle = -\langle \nu, \nabla_Y^{\mathcal{N}} X \rangle - \langle \nu, [X, Y] \rangle = -\langle \nu, \nabla_Y^{\mathcal{N}} X \rangle$$

as  $\nu \in T_x \mathcal{M}^\perp$ ,  $[X, Y] \in T_x \mathcal{M}$ , so  $\langle \nu, [X, Y] \rangle = 0$ . Finally, since again,  $\nabla^{\mathcal{N}}$  is [metric](#),

$$\ell_\nu(X, Y) = \langle \nabla_Y^{\mathcal{N}} \nu, X \rangle = \langle (\nabla_Y^{\mathcal{N}} \nu)^\top, X \rangle = \ell_\nu(Y, X).$$

**Definition 4.2.2** (Second fundamental form). The *second fundamental form*  $\ell_\nu(\cdot, \cdot)$  of  $\mathcal{M}$  in  $\mathcal{N}$  is defined as  $\ell_\nu(X, Y) := \langle S(X, \nu), Y \rangle$ .

Now, fix a **normal field**  $\nu$ , and let  $S_\nu(X) := S(X, \nu)$ , then

$$S_\nu: T_x\mathcal{M} \rightarrow T_x\mathcal{M}$$

is self-adjoint w.r.t. the **metric**  $\langle \cdot, \cdot \rangle$  by **Lemma 4.2.2**.

**Definition.** Assume that  $\langle \nu, \nu \rangle \equiv 1$ , i.e.,  $\nu$  is the unit **normal field**, then  $S_\nu$  has  $m$  real eigenvalues.

**Definition 4.2.3** (Principal curvature). The eigenvalues are called *principal curvatures* of  $\mathcal{M}$  in direction  $\nu$ .

**Definition 4.2.4** (Principal curvature vector). The corresponding eigenvectors are called *principal curvature vectors* of  $\mathcal{M}$  in direction  $\nu$ .

**Definition 4.2.5** (Mean curvature). The *mean curvature* of  $\mathcal{M}$  in direction  $\nu$  is defined by

$$H_\nu := \frac{1}{m} \text{Tr } S_\nu.$$

**Definition 4.2.6** (Gauss-Kronecker curvature). The *Gauss-Kronecker curvature* of  $\mathcal{M}$  in direction  $\nu$  is defined by

$$K_\nu := \det S_\nu.$$

## Lecture 15: Immersions and the Second Fundamental Form

Given a 1-form  $\omega$ , and **vector fields**  $X, Y$ , we have

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$$X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y).$$

For arbitrary **tensors**  $S, T$ , we similarly have

$$\nabla_X(S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T.$$

If  $S$  is a  $p$ -times covariant tensor, and  $Y_1, \dots, Y_p$  **vector fields**,

$$(\nabla_X S)(Y_1, \dots, Y_p) = X(S(Y_1, \dots, Y_p)) - \sum_{i=1}^p S(Y_1, \dots, Y_{i-1}, \nabla_X Y_i, Y_{i+1}, \dots, Y_p).$$

For  $T$  a **(p, q)-tensor field**,

$$\begin{aligned} (\nabla_Y T)(\alpha_1, \dots, \alpha_q, X_1, \dots, X_p) &= Y(T(\alpha_1, \dots, \alpha_q, X_1, \dots, X_p)) \\ &\quad - \sum_{i=1}^q T(\alpha_1, \dots, \nabla_Y \alpha_i, \dots, \alpha_q, X_1, \dots, X_p) \\ &\quad - \sum_{i=1}^p T(\alpha_1, \dots, \alpha_q, X_1, \dots, \nabla_Y X_i, \dots, X_p). \end{aligned}$$

If  $S = g_{ij} dx^i \otimes dx^j$ , then  $\nabla_X g = 0$  for all **vector fields**  $X$ .

Also,

$$\begin{aligned} (\mathcal{L}_X S)(Y_1, \dots, Y_p) &= X(S(Y_1, \dots, Y_p)) - \sum_{i=1}^p S(Y_1, \dots, [X, Y_i], \dots, Y_p) \\ &= (\nabla_X S)(Y_1, \dots, Y_p) + \sum_{i=1}^p S(Y_i, \dots, \nabla_{Y_i} X, \dots, Y_p) \end{aligned}$$

since  $\nabla$  is **torsion-free**, we have  $\nabla_X Y_i - \nabla_{Y_i} X = [X, Y_i]$ .

**Definition 4.2.7** (Killing field). Consider a Riemannian manifold  $(\mathcal{M}, g)$ , and  $g = g_{ij}dx^i \otimes dx^j$ . Then a vector field  $X$  such that

$$\mathcal{L}_X g = 0$$

is called a *killing field* (or *infinitesimal isometry*).

**Lemma 4.2.3.** A vector field  $X$  on  $(\mathcal{M}, g)$  is a killing field if and only if the local 1-parameter group generated by  $X$  consisted of local isometries.

**Lemma 4.2.4.** The killing fields of a Riemannian manifold constitute a Lie algebra.

Let  $\dim \mathcal{N} = m + 1$ ,  $\dim \mathcal{M} = m$ , then for all  $x \in \mathcal{M}$ , there are exactly 2 normal vectors  $\nu \in T_x \mathcal{M}^\perp$  with  $\langle \nu, \nu \rangle \equiv 1$ , i.e.,  $\nabla_X^\mathcal{N} \nu$  always tangential to  $\mathcal{M}$ .

**Remark.**  $\nabla_X^\mathcal{N} \nu$  measures the “tilting velocity” with which  $\nu$  is tilted relative to a fixed parallel vector field in  $\mathcal{N}$ , when on  $\mathcal{M}$  in direction  $X$ .

**Theorem 4.2.2.** Given  $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$  such that  $\mathcal{M}$  is totally geodesic in  $\widetilde{\mathcal{M}}$  if and only if all 2<sup>nd</sup> fundamental form of  $\mathcal{M}$  vanish identically.

**Proof.** Let  $c: I \rightarrow \mathcal{M}$  be a geodesic in  $\mathcal{M}$ , i.e.,  $\nabla_{\dot{c}}^\mathcal{M} \dot{c} = 0$ . By Theorem 4.2.1, we have that

$$\nabla_{\dot{c}}^\mathcal{M} \dot{c} = (\nabla_{\dot{c}}^{\widetilde{\mathcal{M}}} \dot{c})^\top = 0,$$

i.e.,  $c$  is a geodesic in  $\widetilde{\mathcal{M}}$  if and only if  $(\nabla_{\dot{c}}^{\widetilde{\mathcal{M}}} \dot{c})^\top = 0$ , i.e.,

$$\langle \nabla_{\dot{c}}^{\widetilde{\mathcal{M}}} \dot{c}, \nu \rangle = 0$$

for all  $\nu \in T\mathcal{M}^\perp$ . Notice that  $\langle \dot{c}, \nu \rangle = 0$  and  $\dot{c} \langle \dot{c}, \nu \rangle = \langle \nabla_{\dot{c}}^{\widetilde{\mathcal{M}}} \dot{c}, \nu \rangle + \langle \dot{c}, \nabla_{\dot{c}}^{\widetilde{\mathcal{M}}} \nu \rangle = 0$ , we have

$$0 = \langle \nabla_{\dot{c}}^{\widetilde{\mathcal{M}}} \dot{c}, \nu \rangle = \langle \dot{c}, \nabla_{\dot{c}}^{\widetilde{\mathcal{M}}} \nu \rangle = -\ell_\nu(\dot{c}, \dot{c}).$$

■

**Note.** Theorem 4.2.2 holds for Lorentzian manifolds  $(\widetilde{\mathcal{M}}, \widetilde{g})$ .

**Example.** The initial value problem for Einstein equations. Given a  $(\widetilde{\mathcal{M}}^4, \widetilde{g})$  a Lorentzian manifold satisfying Einstein equations.  $(\mathcal{M}^3, g)$  non-degenerate Riemannian manifold. If the 2<sup>nd</sup> fundamental form of  $\mathcal{M}^3$  in  $\widetilde{\mathcal{M}}^4$  vanishes identically, then  $\mathcal{M}^3$  is totally geodesic. This is a special case and not in general.

**Notation.** Greek indices  $(\alpha, \beta, \dots)$  occurring twice are summed over from 1 to  $k$  for  $X, Y, Z, W \in T_x \mathcal{M}$ .

**Theorem 4.2.3** (Gauss equations). Let  $\mathcal{N}$  be a Riemannian manifold with  $\dim \mathcal{N} = n$ , and let  $\mathcal{M} \subseteq \mathcal{N}$  be a submanifold with  $\dim \mathcal{M} = m$ . Let  $k = n - m$ , and  $x \in \mathcal{M}$ ,  $\nu_1, \dots, \nu_k$  be an orthonormal basis of  $(T_x \mathcal{M})^\perp$ ,  $S_\alpha := {}^2_{\nu_\alpha}$ ,  $\ell_\alpha := \ell_{\nu_\alpha}$ ,  $\alpha = 1, \dots, k$ . Then,

$$R^\mathcal{M}(X, Y)Z - (R^\mathcal{N}(X, Y)Z)^\top = \ell_\alpha(Y, Z)S_\alpha(X) - \ell_\alpha(X, Z)S_\alpha(Y).$$

Thus, we also have

$$\langle R^\mathcal{M}(X, Y)Z, W \rangle - \langle R^\mathcal{N}(X, Y)Z, W \rangle = \ell_\alpha(Y, Z)\ell_\alpha(X, W) - \ell_\alpha(X, Z)\ell_\alpha(Y, W).$$

**Proof.** We can extend  $X, Y, Z, W$ , and  $\nu, \dots, \nu_k$  to **vector fields** in  $T\mathcal{M}$  and  $T\mathcal{M}^\perp$ , respectively. Let  $\nu_\alpha$  be orthonormal, then

$$\nabla_Y^{\mathcal{N}} Z = (\nabla_Y^{\mathcal{N}} Z)^\top = (\nabla_X^{\mathcal{N}} Z)^\perp = \nabla_Y^{\mathcal{M}} Z + \langle \nu_\alpha, \nabla_Y^{\mathcal{N}} Z \rangle \nu_\alpha$$

as  $\nu_\alpha$  form orthonormal basis of  $T\mathcal{M}^\perp$ . Hence,

$$\nabla_X^{\mathcal{N}} \nabla_Y^{\mathcal{N}} Z = \nabla_X^{\mathcal{N}} \nabla_Y^{\mathcal{M}} Z + X(\langle \nu_\alpha, \nabla_Y^{\mathcal{N}} Z \rangle) \nu_\alpha + \langle \nu_\alpha, \nabla_Y^{\mathcal{N}} Z \rangle \nabla_X^{\mathcal{N}} \nu_\alpha.$$

Then,

$$(\nabla_X^{\mathcal{N}} \nabla_Y^{\mathcal{N}} Z)^\top = \nabla_X^{\mathcal{M}} \nabla_Y^{\mathcal{M}} Z + \underbrace{\langle \nu_\alpha, \nabla_Y^{\mathcal{N}} Z \rangle}_{-\ell_\alpha(Y, Z)} \underbrace{(\nabla_X^{\mathcal{N}} \nu_\alpha)^\top}_{S_\alpha(X)} = \nabla_X^{\mathcal{M}} \nabla_Y^{\mathcal{M}} Z - \ell_\alpha(Y, Z) S_\alpha(X).$$

Analogously, we have

$$(\nabla_Y^{\mathcal{N}} \nabla_X^{\mathcal{N}} Z)^\top = \nabla_Y^{\mathcal{M}} \nabla_X^{\mathcal{M}} Z - \ell_\alpha(X, Z) S_\alpha(Y),$$

and also, we have

$$(\nabla_{[X, Y]}^{\mathcal{N}} Z)^\top = \nabla_{[X, Y]}^{\mathcal{M}} Z.$$

By collecting terms, we have

$$\begin{aligned} & (\nabla_X^{\mathcal{N}} \nabla_Y^{\mathcal{N}} Z)^\top - (\nabla_Y^{\mathcal{N}} \nabla_X^{\mathcal{N}} Z)^\top - (\nabla_{[X, Y]}^{\mathcal{N}} Z)^\top \\ &= \nabla_X^{\mathcal{M}} \nabla_Y^{\mathcal{M}} Z - \nabla_Y^{\mathcal{M}} \nabla_X^{\mathcal{M}} Z - \nabla_{[X, Y]}^{\mathcal{M}} Z - \ell_\alpha(Y, Z) S_\alpha(X) + \ell_\alpha(X, Z) S_\alpha(Y), \end{aligned}$$

equivalently,

$$R^{\mathcal{M}}(X, Y)Z - (R^{\mathcal{N}}(X, Y)Z)^\top = \ell_\alpha(Y, Z) S_\alpha(X) - \ell_\alpha(X, Z) S_\alpha(Y).$$

■

**Theorem 4.2.3** tells us that for a surface  $\mathcal{M}$  in  $\mathbb{R}^3$ , the **Gauss-Kronecker curvature** coincides with the **Riemannian curvature** of  $\mathcal{M}$ , which is independent of the **embedding**. Therefore, **Gauss-Kronecker curvature** does not depend on **embeddings** of  $\mathcal{M}$  into  $\mathbb{R}^3$ .

**Remark (Codazzi equations).** Let  $\mathcal{M}^m \subseteq \mathcal{N}^{m+1}$  where  $N$  is unit normal on  $\mathcal{M}$

$$\langle R(X, Y)e_j, N \rangle = (\nabla_X^{\mathcal{M}} \ell)(Y, e_j) - (\nabla_Y^{\mathcal{M}} \ell)(X, e_j) = X^i Y^j \nabla_k^{\mathcal{M}} \ell_{ij} - Y^j X^i \nabla_k^{\mathcal{M}} \ell_{ij},$$

i.e.,  $\langle R(X, Y)Z, N \rangle = (\nabla_X^{\mathcal{M}} \ell)(Y, Z) - (\nabla_Y^{\mathcal{M}} \ell)(X, Z)$ .

# Chapter 5

## Jacobi Fields

### Lecture 16: Jacobi Field

#### 5.1 Jacobi Fields

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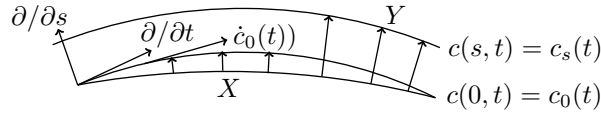
**Definition 5.1.1** (Jacobi field). Let  $\mathcal{M}$  be a  $d$ -dimensional Riemannian manifold. Let  $c: I \rightarrow \mathcal{M}$  be a geodesic. A vector field  $X$  along  $c$  is called a *Jacobi field* if

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} X + R(X, \dot{c})\dot{c} = 0. \quad (5.1)$$

**Notation.** We write  $\dot{X} := \nabla_{\frac{d}{dt}} X$  and  $\ddot{X} := \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} X$ . So the Jacobi equation writes as

$$\ddot{X} + R(X, \dot{c})\dot{c} = 0.$$

Then, we can label  $c$  as follows



Notice that we might either fix the endpoints or left them open. Formally, we have the following.

**Definition 5.1.2** (Geodesic variation). Let  $\mathcal{M}$  be a Riemannian manifold. A variation of curves  $c: (-\epsilon, \epsilon) \times I \rightarrow \mathcal{M}$  is called the *geodesic variation* if for all  $s \in (-\epsilon, \epsilon)$ , the curve  $t \mapsto c_s(t) := c(s, t)$  is a geodesic.

In general, for a smooth curve  $c: [a, b] \rightarrow \mathcal{M}$  and  $\epsilon > 0$ , a *geodesic variation* of  $c$  is a differentiable map

$$F: \underbrace{[a, b]}_t \times \underbrace{(-\epsilon, \epsilon)}_s \rightarrow \mathcal{M}$$

such that  $F(t, 0) = c(t)$  for  $t \in [a, b]$ .

**Definition 5.1.3** (Proper variation). A *proper variation* is a *geodesic variation* where the endpoints are fixed, i.e.,

$$F(a, s) = c(a), F(b, s) = c(b)$$

for all  $s \in (-\epsilon, \epsilon)$ .

**Notation.** We set  $c_s(t) = c(t, s) = F(t, s)$ , and

- $\dot{c}(t, s) := \frac{\partial}{\partial t} c(t, s)$ , i.e.,  $dF(\partial/\partial t)c(t, s)$ , and

- $c'(t, s) = \partial/\partial s c(t, s)$ , i.e.,  $dF(\partial/\partial s)c(t, s)$ .

Let  $\mathcal{M}$  be a Riemannian manifold of dimension  $d$ , and  $\mathcal{H}$  be a differentiable manifold.<sup>1</sup> Let  $f: \mathcal{H} \rightarrow \mathcal{M}$ , smooth.<sup>2</sup> What is the tangent space of  $f(\mathcal{H})$  of point  $p \in f(\mathcal{H})$ ?

**Example.** Let  $p = f(x) = f(y)$  for  $x \neq y$ . For  $f$  being an immersion, we may restrict  $f$  to a sufficiently small neighborhood  $U, V$  at  $x, y$ , respectively, such that  $f(U), f(V)$  have well-defined tangent spaces at  $p$ . Then, in a double point of  $f(\mathcal{H})$ , the tangent space can be specified by specifying the preimage ( $x$  or  $y$ ), i.e., consider  $f^*(T\mathcal{M})$ , the fiber over  $x \in \mathcal{H}$  is  $T_{f(x)}\mathcal{M}$  introduce connection  $f^*(\nabla)$  on  $f^*(T\mathcal{M})$ . Let  $X \in T_x\mathcal{H}$ ,  $Y$  a section of  $f^*(T\mathcal{M})$ . Set

$$(f^*\nabla)_x Y := \nabla_{d(x)} Y,$$

where  $f^*(T\mathcal{M}_x)$  is identified with  $T_{f(x)}\mathcal{M}$  with  $\nabla$  for  $f^*\nabla$ . All this means that a section of  $f^*(T\mathcal{M})$  is a vector field along  $f$ .

### 5.1.1 First Variations

Recall the following.

**As previously seen.** The energy is defined as

$$E(s) := \frac{1}{2} \int_a^b \left\langle \frac{\partial c(t, s)}{\partial t}, \frac{\partial c(t, s)}{\partial t} \right\rangle dt,$$

and the length is defined as

$$L(s) := \int_a^b \left\langle \frac{\partial c(t, s)}{\partial t}, \frac{\partial c(t, s)}{\partial t} \right\rangle^{1/2} dt.$$

Now, we consider

- the first variations  $E'(0)$  and  $L'(0)$ , i.e., the first derivatives;
- for  $c = c_0$  geodesic, compute the second variations  $E''(0)$  and  $L''(0)$ , i.e., the second derivatives.

**Lemma 5.1.1.** If  $L(s)$ ,  $E(s)$  are differentiable w.r.t.  $s$ , then

$$L'(0) = \int_a^b \left( \frac{\frac{\partial}{\partial t} \langle c', \dot{c} \rangle}{\langle \dot{c}, \dot{c} \rangle^{1/2}} - \frac{\langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c} \rangle}{\langle \dot{c}, \dot{c} \rangle^{1/2}} \right) dt,$$

and

$$E'(0) = \langle c'(b, 0), \dot{c}(b, 0) \rangle - \langle c'(a, 0), \dot{c}(a, 0) \rangle - \int_a^b \left\langle \frac{\partial c}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial t}(t, s) \right\rangle dt.$$

**Proof.** We have already proved this in different notations. ■

**Note.** If  $c = c_0$  is parametrized proportionally to the arc-length, i.e.,  $\|\dot{c}(t, 0)\|$  is a constant. Then  $L'(0)$  becomes

$$L'(0) = \frac{1}{\langle \dot{c}, \dot{c} \rangle^{1/2}} \left( \langle c', \dot{c} \rangle \Big|_{t=a, s=0}^{t=b, s=0} - \int_a^b \langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c} \rangle dt \right).$$

If we consider the fixed endpoints case, we observe that  $E$  and  $L$  are stationary if and only if

$$\nabla_{\frac{\partial}{\partial t}} \dot{c}(t, 0) = 0,$$

i.e., when  $c$  is a geodesic.

<sup>1</sup>Often times,  $H$  is an interval  $I \subseteq \mathbb{R}$  or a square  $I \times I \subseteq \mathbb{R}^2$ .

<sup>2</sup> $f$  may not be injective.



### 5.1.2 Second Variations

Now, let  $c = c_0$  be a [geodesic](#). Then we compute the second derivatives w.r.t.  $s$  of  $E$  and  $L$  at  $s = 0$ .

**Theorem 5.1.1.** Let  $c: [a, b] \rightarrow \mathcal{M}$  be a [geodesic](#). Then

$$E''(0) = \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} c'(t, 0), \nabla_{\frac{\partial}{\partial t}} c'(t, 0) \right\rangle dt - \int_a^b \left\langle R(\dot{c}, c') c', \dot{c} \right\rangle dt \Big|_{s=0} + \left\langle \nabla_{\frac{\partial}{\partial s}} c', \dot{c} \right\rangle \Big|_{t=a, s=0}^{t=b, s=0}.$$

By letting  $c'^\perp := c' - \left\langle \frac{\dot{c}}{\|\dot{c}\|}, c' \right\rangle \frac{\dot{c}}{\|\dot{c}\|}$ ,<sup>a</sup> we have

$$L''(0) = \frac{1}{\|\dot{c}\|} \left( \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} c'^\perp, \nabla_{\frac{\partial}{\partial t}} c'^\perp \right\rangle dt - \int_a^b \left\langle R(\dot{c}, c'^\perp) c'^\perp, \dot{c} \right\rangle dt + \left\langle \nabla_{\frac{\partial}{\partial s}} c', \dot{c} \right\rangle \Big|_{t=a, s=0}^{t=b, s=0} \right).$$

<sup>a</sup>I.e., the component of  $c'$  orthogonal to  $\dot{c}$ .

**Remark.** By keeping the endpoints fixed, if the [sectional curvature](#) of  $\mathcal{M}$  is non-positive, then the [Riemannian curvature](#) in  $E''(0)$  and  $L''(0)$  are non-negative. This implies  $E''(0) > 0$ , then  $E(c_s) > E(c_0)$  for small  $|s|$ .

**Corollary 5.1.1.** On a manifold with non-positive [sectional curvature](#), the [geodesics](#) with fixed endpoints are always locally minimizing.

### 5.1.3 Index Form

Let  $X$  be a [vector field along  \$c\$](#)  where  $c$  is a [geodesic](#). Then, there exists a [geodesic variation](#)

$$c: [a, b] \times (-\epsilon, \epsilon) \rightarrow \mathcal{M}$$

of  $c(t)$  with

$$\frac{\partial c}{\partial s} \Big|_{s=0} = X.$$

Put

$$I(X, X) := \int_a^b \left( \left\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} X \right\rangle - \left\langle R(\dot{c}, X) X, \dot{c} \right\rangle \right) dt,$$

i.e.,  $I(X, X) = \frac{d^2}{ds^2} E(0)$  if  $X(a) = X(b) = 0$ . Also, put

$$I(X, Y) := \int_a^b \left( \left\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} Y \right\rangle - \left\langle R(\dot{c}, X) Y, \dot{c} \right\rangle \right) dt.$$

We see that  $I(X, Y)$  is a bilinear symmetric in  $X, Y$  where  $Y := \frac{\partial c}{\partial x}$

**Definition 5.1.4** (Index form).  $I$  defined above is called the *index form* of [geodesic  \$c\$](#) .

As previously seen. Recall the [Jacobi equation](#), i.e.,

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} X + R(X, \dot{c}) \dot{c} = 0.$$

**Proposition 5.1.1** (Jacobi field). A [vector field  \$X\$  along a geodesic  \$c: \[a, b\] \rightarrow \mathcal{M}\$](#)  is a [Jacobi-field](#) if and only if the [index form](#) of  $c$  satisfies  $I(X, Y) = 0$  for all [vector fields  \$Y\$  along  \$c\$](#)  with  $Y(a) = Y(b) = 0$ .

**Proof.** Observe that

$$\begin{aligned} I(X, Y) &:= \int_a^b \left( \left\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} Y \right\rangle - \langle R(\dot{c}, X)Y, \dot{c} \rangle \right) dt \\ &= \int_a^b \left( \left\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} Y \right\rangle - \langle R(X, \dot{c})\dot{c}, Y \rangle \right) dt = \int_a^b \left( \left\langle -\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} X, Y \right\rangle - \langle R(X, \dot{c})\dot{c}, Y \rangle \right) dt, \end{aligned}$$

where the second inequality follows from the fact that  $\nabla$  is [Riemannian](#), and  $Y(a) = 0 = Y(b)$ . We see that the right-hand side of the above vanishes for every  $Y$  if and only if

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} X + R(X, \dot{c})\dot{c} = 0,$$

which is just the [Jacobi equation](#), so the result follows.  $\blacksquare$

**Remark.** [Proposition 5.1.1](#) is really where the [Jacobi equation](#) comes from.

## Lecture 17: Jacobi Fields and General Relativity

**Lemma 5.1.2.** A [vector field along a geodesic](#)  $c: [a, b] \rightarrow \mathcal{M}$  is a [Jacobi field](#) if and only if it is a critical point of  $I(X, X)$  w.r.t. all variations with fixed endpoints, i.e.,

$$\left. \frac{d}{ds} I(X + sY, X + sY) \right|_{s=0} = 0$$

for every [vector field along  \$c\$](#)  with  $Y(a) = 0 = Y(b)$ .

**Proof.** We just use the proof of [Proposition 5.1.1](#).  $\blacksquare$

This tells us that the [Jacobi equation](#) is the [Euler-Lagrange equations](#) for  $I(X) := I(X, X)$ .

### 5.1.4 Existence and Uniqueness of Jacobi Fields

Given the initial data, how can we characterize the [Jacobi equation](#) on a [Riemannian manifold](#)  $(\mathcal{M}, g)$  with  $\dim \mathcal{M} = d$ ? Firstly, we know that the [Jacobi equation](#) is a system of  $d$  linear  $2^{nd}$ -order ODE.

**Theorem 5.1.2.** Let  $c: [a, b] \rightarrow \mathcal{M}$  be a [geodesic](#). For all  $v, w \in T_{c(a)}\mathcal{M}$ , there exists a unique [Jacobi field](#)  $X$  along  $c$  with  $X(a) = v$ ,  $\dot{X}(a) = w$ .

**Proof.** Let  $\{v_i\}_{i=1}^d$  be an orthonormal basis of  $T_{c(a)}\mathcal{M}$ . Let  $\{X_i\}_{i=1}^d$  be [parallel vector field along](#) with  $X_i(a) = v_i$  for  $i = 1, \dots, d$ . Then for all  $t \in [a, b]$ ,  $X_1(t), \dots, X_d(t)$  is an orthonormal basis of  $T_{c(t)}\mathcal{M}$ . Choose arbitrary [vector field  \$X\$  along  \$c\$](#)  as  $X = \xi^i X_i$ , i.e.,  $\xi^i(t) = \langle X(t), X_i(t) \rangle$ . As [vector field  \$X\_i\$](#)  are [parallel](#), we have

$$\nabla_{\frac{d}{dt}} X = \frac{d\xi^i}{dt} X_i + \underbrace{\xi_i \nabla_{\frac{d}{dt}} X_i}_0 = \frac{d\xi^i}{dt} X_i,$$

hence

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} X = \frac{d^2 \xi^i}{dt^2} X_i.$$

To write the [Jacobi equation](#) in these coordinates, we first write the [curvature](#) as

$$R(X, \dot{c})\dot{c} = \xi^i \rho_i^k X_k.$$

**Notation** (Rotation). Let  $\rho_i^k := \langle R(X_i, \dot{c})\dot{c}, X_k \rangle$ , i.e.,  $R(X_i, \dot{c})\dot{c} = \rho_i^k X_k$ .

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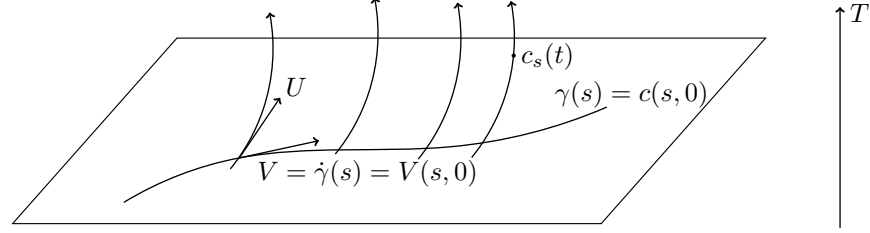
Then, the **Jacobi equation** becomes

$$\left( \frac{d^2 \xi^k}{dt^2} + \xi^i \rho_i^k \right) X_k = 0 \Rightarrow \frac{d^2 \xi^k(t)}{dt^2} + \xi^i(t) \rho_i^k(t) = 0, \quad k = 1, \dots, d$$

since  $\{X_i\}$  is a orthonormal basis. Then, by the linear algebra and ODE theory, we have existence and uniqueness. ■

## 5.2 Application of General Relativity

Consider the universe as a  $(\mathcal{M}^4, g)$  a **Lorentzian manifold**,



Here, we have  $[\partial/\partial s, \partial/\partial t] = [U, V] = 0$ . Hence, the **Jacobi equation** is now

$$\nabla_U^2 V + R(V, U, U) = 0.$$

For given  $U$ , the right-hand side defines of each  $p \in \mathcal{M}$  a linear map

$$N \mapsto R(N, U)U$$

for  $N$  unit normal of subspace of  $T_p \mathcal{M}$  perpendicular to  $U$ . This is often called the *field force operator*. Hence, locally,

- the gravitational field  $g$ , the “fields strengths”  $\Gamma$  can be transformed away;
- variation of gravitational fields strengths can be described by **Riemannian curvature tensor**, hence cannot be transformed away.

All these imply that the **Jacobi equation** with **Riemannian curvature tensor** can describe the relative accelerations (or field forces) of nearby **geodesics**.

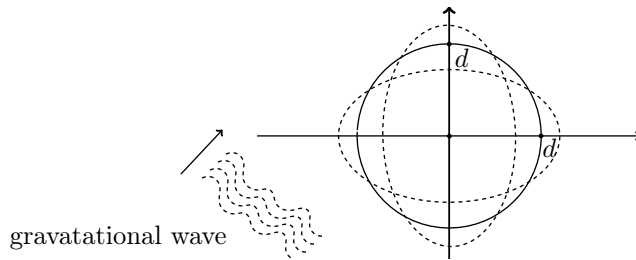


Figure 5.1: LIGO 2015,  $\frac{\Delta \lambda}{\lambda} \approx 10^{-21}$ .

**Example ( $\mathbb{R}^n$ ).** The **Jacobi field** in  $\mathbb{R}^n$ . Since the **geodesics** are straight lines, the **Jacobi field**  $X$  along straight line  $c$  with  $X(a) = v, \dot{X}(a) = w$ . Let  $V(t), W(t)$  be **parallel vector fields** along  $c$  with  $V(a) = v, W(a) = w$ , by linearizing, we have

$$X(t) = V(t) + (t - a)W(t).$$

**Example** ( $S^n \subseteq \mathbb{R}^{n+1}$ ). Let  $c: [0, T] \rightarrow S^n$  be a **geodesic** with  $\|\dot{c}\| = 1$ , and  $v, w \in T_{c(0)}S^n$ ,  $V, W$  **parallel vector fields along  $c$**  with  $V(0) = v, W(0) = w$ . Also, assume that  $\langle v, \dot{c}(0) \rangle = 0 = \langle w, \dot{c}(0) \rangle$ , then the **Jacobi field**  $X$  is

$$X(t) = V(t) \cos t + W(t) \sin t.$$

**Proof.** We see that

$$\dot{X}(t) = -V(t) \sin t + W(t) \cos t,$$

and

$$\ddot{X}(t) = -V(t) \cos t - W(t) \sin t.$$

By using the **Riemannian curvature** on  $S^n$ , we have

$$R(X, \dot{c})\dot{c} = \underbrace{\langle \dot{c}, \dot{c} \rangle}_1 X - \underbrace{\langle X, \dot{c} \rangle}_0 \dot{c} = X.$$

Then  $\ddot{X} + R(X, \dot{c})\dot{c} = 0$ . \*

**Remark.** We can also consider  $S_\rho^n \subseteq \mathbb{R}^{n+1}$  with  $\|\dot{c}\| = 1$  and play the above game, i.e., by letting

$$X(t) = V(t) \cos \frac{t}{\rho} + W(t) \sin \frac{t}{\rho}.$$

### 5.3 Jacobi Fields and Geodesics

Consider a **Jacobi field** transversal along  $c$ , then we can split the **Jacobi field** into

- tangential component: do not depend on geometry of  $\mathcal{M}$ , hence no information about  $\mathcal{M}$ ;
- normal component: very useful!

Specifically, consider  $X = X^\top + X^\perp$ , we have the following.

**Lemma 5.3.1.** Let  $c: [a, b] \rightarrow \mathcal{M}$  be a **geodesic**, and  $\lambda, \mu \in \mathbb{R}$ . Then, the **Jacobi field**  $X$  along  $c$  with  $X(a) = \lambda \dot{c}(a)$ ,  $\dot{X}(a) = \mu \dot{c}(a)$  is given by

$$X(t) = (\lambda + (t - a)\mu)\dot{c}(t).$$

## Lecture 18: Jacobi Fields and Geodesics

**Theorem 5.3.1.** Consider a **geodesic**  $c: [0, 1] \rightarrow \mathcal{M}$ ,  $t \mapsto c(t)$ , and the variation  $c: [0, 1] \times (-\epsilon, \epsilon) \rightarrow \mathcal{M}$  of  $c$  for all curves  $c(\cdot, s) =: c_s(\cdot)$  are **geodesics**. Then  $X(t) := \frac{\partial}{\partial s} c(t, s)|_{s=0}$  is a **Jacobi field** along  $c(t) = c_0(t)$ .

Conversely, every **Jacobi field** along  $c(t)$  can be obtained in this way, i.e., by variation of **geodesics**.

**Proof.** The forward direction is straightforward: since  $c(t, s)$  for a fixed  $s$  is a **geodesic**, hence

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} c(t, s) = 0$$

for all  $s$ , hence  $\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} c(t, s) = 0$ . Then,

$$\begin{aligned} & \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} c(t, s) + \left( -\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} + \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \right) \frac{\partial}{\partial t} c(t, s) \\ &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} c(t, s) + R \left( \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \right) \frac{\partial c}{\partial t} \\ &= 0, \end{aligned}$$

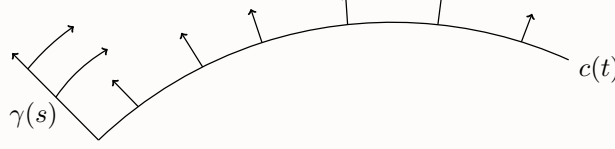
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where we use the fact that  $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$ , so  $\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} = \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}$ . Plugging in the definition of  $X$ , we have

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} X + R\left(X, \frac{\partial c}{\partial t}\right) \frac{\partial c}{\partial t} = 0,$$

i.e.,  $X$  is a **Jacobi field**.

The converse direction is left as a homework. As a hint, consider the following:



Then, let

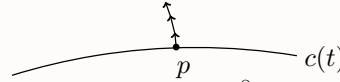
$$c(t, s) = \exp_{\gamma(s)}(t(\dot{c}(0) + s \cdot V))$$

for some  $V$ . Once we have this, we just let  $X(t) = \frac{\partial}{\partial s} c(t, s)|_{s=0}$ . ■

**Intuition.** The **Jacobi equation** can be viewed as the linearization of the **geodesic equation**.

**Corollary 5.3.1.** Every **killing field**  $X$  on  $\mathcal{M}$  is a **Jacobi field** along any **geodesic** in  $\mathcal{M}$ .

**Proof idea.** Since we have a **killing field**  $X$ , we use it to construct  $\Phi_s: \mathcal{M} \rightarrow \mathcal{M}$ , which is an **isometry** since  $X$  is a **killing field**.

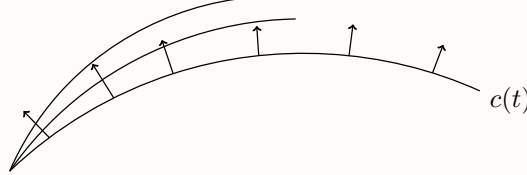


The idea is to consider  $c(t, s) = \Phi_s \circ c(t)$ , and let  $X = \frac{\partial}{\partial s} c(t, s)$ . By **Theorem 5.3.1**, we're done. ■

**Corollary 5.3.2.** Let  $c: [0, T] \rightarrow \mathcal{M}$  be a **geodesic** with  $p = c(0)$ , i.e.,  $c(t) = \exp_p(t\dot{c}(0))$ . For  $W \in T_p \mathcal{M}$ , the **Jacobi field**  $x$  along  $c$  with  $X(0) = 0$ ,  $\dot{X}(0) = W$  then is given as

$$X(t) = D(\exp_p)|_{(t\dot{c}(0))}(tW).$$

**Proof.** This is a direct consequence of **Theorem 5.3.1**, since now  $X(0) = 0$ , we don't need to worry about constructing  $\gamma(s)$ , i.e., we have the following:



Now, we consider  $c(t, s) = \exp_p(t(\dot{c}(0) + s \cdot W))$ , hence

$$\frac{\partial}{\partial s} c(t, s) = \frac{\partial}{\partial s} \exp_p(t\dot{c} + s \cdot W) \Big|_{s=0}.$$

This means that if we want to have  $D(\exp_p)|_V(W)$ , and construct a **Jacobi field**  $W$  such that  $X(0) = 0$ ,  $\dot{X}(0) = W$ . ■

### 5.3.1 Conjugate Points

**Definition 5.3.1 (Conjugate point).** Let  $c: I \rightarrow \mathcal{M}$  be a **geodesic**. For  $t_0, t_1 \in I$  with  $t_0 \neq t_1$ ,  $c(t_0)$  and  $c(t_1)$  are called *conjugate* along  $c$  if there exists a **Jacobi field**  $X(t)$  along  $c$  which does not vanish identically and satisfies  $X(t_0) = 0 = X(t_1)$ .

**Note.** We see that  $\langle X(t), \dot{c}(t) \rangle = 0$  for all  $t$ .

**Proof.** Since  $\nabla_{\partial t}\langle X(t), \dot{c}(t) \rangle = \langle \dot{X}, \dot{c} \rangle$ , so

$$\nabla_{\partial t}\nabla_{\partial t}\langle X(t), \dot{c}(t) \rangle = \langle \ddot{X}, \dot{c} \rangle = -\langle R(X, \dot{c})\dot{c}, \dot{c} \rangle = 0.$$

This is a linear function, and if two endpoints are both 0, everything is 0.  $\circledast$

**Note.** If  $t_0, t_1 \in I$ ,  $t_0 \neq t_1$  are not **conjugate** along  $c$ , then for  $V \in T_{c(t_0)}\mathcal{M}, W \in T_{c(t_1)}\mathcal{M}$ , there exists a unique **Jacobi field**  $Y(t)$  along  $c$  such that  $Y(t_0) = V, Y(t_1) = W$ .

**Proof.** Let  $\mathcal{J}_c$  be the **Jacobi fields** along  $c$ , then construct the linear map

$$A: \mathcal{J}_c \rightarrow T_{c(t_0)}\mathcal{M} \times T_{c(t_1)}\mathcal{M}, \quad Y \mapsto (Y(t_0), Y(t_1)).$$

Since  $\mathcal{J}_c$  is a vector space with  $\dim \mathcal{J}_c = 2n$ , and the target space is also with dimension  $2n$ , it suffices to show that  $\ker A = \{0\}$ . This is true because  $t_0 \neq t_1$  are not **conjugate**.  $\circledast$

**Example.** Any antipodal points of  $S^n$  are **conjugate points**.

**Example.** Consider  $\mathbb{R}^n$  with flat **metric** doesn't have **conjugate points**.

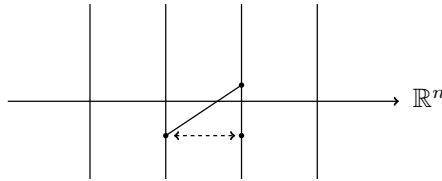
**Example.** **Riemannian manifolds** with non-positive **sectional curvature** has no **conjugate points**.

### 5.3.2 Length-Minimizing Geodesics

**Theorem 5.3.2.** Let  $c: [a, b] \rightarrow \mathcal{M}$  be a **geodesic**.

- (a) If there does not exist a point **conjugate** to  $c(a)$  along  $c(t)$ , then there exists  $\epsilon > 0$  such that for all piecewise **smooth curve**  $g: [a, b] \rightarrow \mathcal{M}$  with  $g(a) = c(a), g(b) = c(b)$  and  $d(g(t), c(t)) < \epsilon$  for all  $t \in [a, b]$ , we have  $L(c) \leq L(g)$ , and the equality holds when if and only if  $g$  is a reparametrization of  $c$ .
- (b) If there is  $\tau \in (a, b)$  such that  $c(a)$  and  $c(\tau)$  are **conjugate points** along  $c$ , then there exists a proper variation  $c(t, s): [a, b] \times (-\epsilon, \epsilon) \rightarrow \mathcal{M}$  such that  $L(c_s) < L(c)$  for  $s \in (-\epsilon, \epsilon) \setminus \{0\}$ .

**Remark.** Consider the cylinder, where we identify every line of  $\mathbb{R}^n$  below.



There are two **geodesics**, but one is strictly longer.

**Corollary 5.3.3.** Let  $p \in \mathcal{M}$  and  $V \in T_p\mathcal{M}$  is contained in the domain of definition of  $\exp_p$ . Let  $c(t) = \exp_p(tV)$ , and  $\gamma: [0, 1] \rightarrow T_p\mathcal{M}$  be a piecewise **smooth curve** contained in the domain of  $\exp_p$  with  $\gamma(0) = 0, \gamma(1) = V$ . Then

$$L\left(\exp_p(tV)|_{t \in [0, 1]}\right) \leq L(\exp_p \circ \gamma(t))$$

and the equality holds if and only if  $\gamma$  differs from the curve  $tV$ ,  $t \in [0, 1]$  only by reparametrization.

**Proof hint.** We directly estimate

$$L(\exp \circ \gamma) = \int_0^1 \left| \frac{d}{dt} \exp \circ \gamma \right| dt = \int_0^1 |D \exp \circ \gamma| dt.$$

■

## Lecture 19: Length-Minimizing Geodesics and Conjugacy

Now, let's prove [Theorem 5.3.2](#).

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**Proof of Theorem 5.3.2.** We prove them one by one.

- (a) First steps: last lecture. Now, cover  $\{tV \mid 0 \leq t \leq 1\}$  by finitely many such neighborhoods  $\Omega_i$ ,  $i = 1, \dots, k$ , and let  $U_i = \exp_p \Omega_i$ . Assume that  $tV \in \Omega_i$ , for  $t_{i-1} \leq t \leq t_i$  (with  $t_0 = 0, t_k = 1$ ). Let  $\epsilon > 0$  sufficiently small. Then, for all [curve](#)  $g: [0, 1] \rightarrow \mathcal{M}$  satisfying the assumption,  $g([t_{i-1}, t_i]) \subseteq U_i$ .

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**Claim.** For all  $g$  satisfying  $g([t_{i-1}, t_i]) \subseteq U_i$ , there exists a [curve](#)  $\gamma \subseteq T_p \mathcal{M}$  such that

$$\exp_p \gamma = g$$

with  $\gamma(0) = 0, \gamma(1)gV$ .

**Proof.** Put  $\gamma(t) = \left( \exp_p|_{\Omega_i} \right)^{-1} (g(t))$  for  $t_{i-1} \leq t \leq t_i$ , so  $\gamma$  satisfies [Corollary 5.3.3](#).  $\otimes$

- (b) Without loss of generality, let  $a = 0, b = 1$ . Let  $X$  be a non-trivial [Jacobi field](#) along  $c$  with  $X(0) = 0 = X(\tau)$ . We have  $\dot{X}(\tau) \neq 0$ , as otherwise  $X \equiv 0$  by the uniqueness. Let  $Z(t)$  be an arbitrary [vector field](#)  $X$  along  $c$  with  $Z(0) = 0 = Z(1)$ ,  $Z(\tau) = -\dot{X}(\tau)$ . Let  $\eta > 0$ , set

$$Y_\eta(t) = \begin{cases} Y_\eta^1(t) = X(t) + \eta Z(t), & \text{if } 0 \leq t \leq \tau; \\ Y_\eta^2(t) = \eta Z(t), & \text{if } \tau \leq t \leq 1, \end{cases}$$

and we let  $Z^1 := Z|_{[0, \tau]}, Z^2 := Z|_{[\tau, 1]}$ . Now, since

$$I(Y_\eta^1, Y_\eta^1) = \left\langle \dot{X}(\tau), 2\eta Z(\tau) \right\rangle + \eta^2 I(Z^1, Z^1) - 2\eta \|\dot{X}(\tau)\|^2 + \eta^2 I(Z^1, Z^1),$$

with

$$I(Y_\eta^2, Y_\eta^2) = \eta^2 I(Z^2, Z^2),$$

with

$$I(Y_\eta, Y_\eta) = I(Y_\eta^1, Y_\eta^1) + I(Y_\eta^2, Y_\eta^2) = -2\eta \|\dot{X}(\tau)\|^2 + \eta^2 I(Z, Z)$$

for sufficiently small  $\eta > 0$ . Now, consider the variation

$$c(t, s) := \exp_{c(t)} s Y_\eta(t),$$

we have  $L'(0) = 0^a$  and

$$L''(0) = I(Y_\eta, Y_\eta) < 0.$$

By the Taylor theorem, this is a minimum, i.e.,  $L(c_s) < L(c)$ .

■

<sup>a</sup>Note that  $L(s) := L(c_s)$ ,  $L(0) = L(c)$ .

**Definition 5.3.2 (Order).** The *order (or multiplicity) of conjugacy* is the dimension of space of [Jacobi fields](#) vanishing at two [conjugate points](#).

Given  $\dim \mathcal{M} = n$ , by the existence and uniqueness theorem for [Jacobi fields](#), we see the following examples.

**Example.** There is an  $n$ -dimensional space of [Jacobi fields](#) vanishing at  $p \in \mathcal{M}$ .

**Example.** There is at most  $(n - 1)$ -dimensional space of [Jacobi fields](#) vanishing at  $p, q \in \mathcal{M}$ , as tangential [Jacobi fields](#) vanishes at most at one point.

We now characterize the [conjugate points](#): they are precisely the images of singularities of the [exponential map](#).

**Proposition 5.3.1.** Let  $p \in \mathcal{M}$ ,  $V \in T_p \mathcal{M}$ ,  $q = \exp V$ . Then  $\exp_p$  is a local [diffeomorphism](#) in a neighborhood of  $V$  if and only if  $q$  not [conjugates](#) to  $p$  along [geodesic](#)  $\gamma(t) = \exp_p tV$ ,  $t \in [0, 1]$ .

Let  $c: [a, b] \rightarrow \mathcal{M}$  be a [curve](#), and let  $\nu_c$  be the space of [vector field  \$X\$  along  \$c\$](#) , i.e.,  $\nu_c = \Gamma(c^*T\mathcal{M})$ . Let  $\dot{\nu}_c$  be the space of [vector field  \$X\$  along  \$c\$](#)  with  $V(a) = V(b) = 0$ .

**Lemma 5.3.2.** Let  $c: [a, b] \rightarrow \mathcal{M}$  be a [geodesic](#). Then there is no pair of [conjugate points](#) along  $c$  if and only if the [index form](#)  $I$  of  $c$  is strictly positive definite on  $\dot{\nu}_c$ .

**Proof.** Assume that  $c$  has no [conjugate points](#), then [Theorem 5.3.2](#) implies that  $I(X, X) \geq 0$  for all  $X \in \dot{\nu}_c$  because otherwise  $c(t, s) := \exp_{c(t)} sX(t)$  would be locally length-decreasing.

If  $I(Y, Y) = 0$  for some  $Y \in \dot{\nu}_c$ , then by  $I(X, X) \geq 0$ , we have

$$0 \leq I(Y - \lambda Z, Y - \lambda Z) = 0 - 2\lambda I(Y, Z) + \lambda^2 I(Z, Z)$$

for all  $Z \in \dot{\nu}_c$  and  $\lambda \in \mathbb{R}$ . We see that this inequality hold only if  $I(Y, Z) = 0$  for all  $Z \in \dot{\nu}_c$ , which implies  $Y$  is a [Jacobi field](#) from [Proposition 5.1.1](#). As no [conjugate points](#) along  $c$ ,  $Y = 0$ , i.e.,  $I$  is strictly positive definite.

## Lecture 20

**As previously seen.**  $\nabla_c$  is the space of [vector fields along  \$c\$](#) , and  $\dot{\nu}_c$  is the space of [vector fields along  \$c\$](#)  with  $V(a) = V(b) = 0$ .

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Let's resume our proof about [Lemma 5.3.2](#).

**Proof of Lemma 5.3.2 (Continue).** For the backward direction, assume that for  $t_0, t_1 \in [a, b]$  (without loss of generality, let  $t_0 < t_1$ ). Suppose for a contradiction, let  $c(t_0), c(t_1)$  be two [conjugate points](#) along  $c$ , then there exists a non-trivial [Jacobi field](#)  $x$  along  $c$  such that  $X(t_0) = 0 = X(t_1)$ . Set

$$Y(t) = \begin{cases} 0, & \text{if } a \leq t \leq t_0; \\ X(t), & \text{if } t_0 \leq t \leq t_1; \\ 0, & \text{if } t_1 \leq t \leq b; \end{cases} \Rightarrow J(Y, Y) = 0,$$

hence  $I$  is not positive definite. ■

### 5.3.3 sobolev Spaces

Now, on  $\dot{\nu}_c$ , we introduce the norm

$$\|X\| := \left( \int_a^b (\langle \dot{X}, \dot{X} \rangle + \langle X, X \rangle) dt \right)^{1/2},$$

and denote  $\dot{H}_c$  the completion of  $\dot{\nu}_c$  w.r.t.  $\|\cdot\|$ .



**Definition 5.3.3** (Schwartz space). A *Schwartz space*  $\mathcal{S}(\mathbb{R}^d)$  is defined as

$$\mathcal{S}(\mathbb{R}^d) := \left\{ u \in C^\infty(\mathbb{R}^d) \mid \forall \alpha, \beta \in \mathbb{N}^d \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta u(x)| < \infty \right\}.$$

**Definition 5.3.4** (Tempered distribution). A *tempered distribution* is a continuous linear functional  $f$  on  $\mathcal{S}(\mathbb{R}^d)$ , i.e.,  $f: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$ .

**Notation.** The space of **tempered distributions** is denoted as  $\mathcal{S}'(\mathbb{R}^d)$ .

**Definition 5.3.5** (Locally integrable). Let  $\Omega \subseteq \mathbb{R}^n$  be open, and let  $f: \Omega \rightarrow \mathbb{C}$  be Lebesgue measurable. Then

$$L^1_{\text{loc}}(\Omega) := \{f: \Omega \rightarrow \mathbb{C} \text{ measurable} \mid f|_K \in L^1(K) \forall \text{ compact } K \subseteq \Omega\}$$

is the *locally integrable* (or locally summable) space.

**Definition 5.3.6** (Weak derivative). Let  $U \subseteq \mathbb{R}^n$  be open, and  $u, v \in L^1_{\text{loc}}(U)$ . Let  $\alpha$  be a multi-index. Then  $v$  is the *weak derivative* of  $u$ , denoted as  $D^\alpha_u = v$  provided

$$\int_U u \cdot D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx$$

for all test functions  $\phi \in C_c^\infty(U)$ .

**Remark.** If the **weak derivative** exists, then it's unique up to a set of measure zero.

**Definition 5.3.7** (Sobolev space). Fix  $1 \leq p \leq \infty$ , and let  $k$  be a non-negative integer. The *Sobolev space*  $W^{k,p}(U)$  consists of all **locally integrable** functions  $u: U \rightarrow \mathbb{R}$  for all  $\alpha$  with  $|\alpha| \leq k$  such that  $D^\alpha u$  exists in the **weak** sense and belongs to  $L^p(U)$ .

**Remark.** If  $p = 2$ ,  $H^k(U) := W^{k,2}(U)$  for  $k = 0, 1, \dots$  is a Hilbert space.

**Example.**  $H^0(U) = L^2(U)$ .

**Definition 5.3.8.** If  $u \in W^{k,p}(U)$ , define its norm as

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left( \sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p \, dx \right)^{1/p}, & \text{if } 1 \leq p < \infty; \\ \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_U |D^\alpha u|, & \text{if } p = \infty. \end{cases}$$

**Notation.** The closure of  $C_c^\infty(U)$  in  $W^{k,p}(U)$  be  $W_0^{k,p}(U)$ .

Thus,  $u \in W_0^{k,p}(U)$  if and only if there exists functions  $u_n \in C_c^\infty(U)$  such that  $u_n \rightarrow u$  in  $W^{k,p}(U)$ .

**Remark.** Lastly, the upshot is that  $u \in W_0^{k,p}(U)$  if  $u \in W^{k,p}(U)$  such that “ $D^\alpha u = 0$  on  $\partial U$ ” for all  $|\alpha| \leq k - 1$ , more precisely, use traces.

## 5.4 Cut Locus

Let  $\{V_i\}$  be an orthonormal basis of [parallel vector fields](#). Now, write  $X = \xi^i V_i$ , so  $\dot{X}_i = \dot{\xi}^i V_i$ , hence

$$\|X\| = \left( \int_a^b \left( \dot{\xi}^i \dot{\xi}^j + \xi^i \xi^j \right) dt \right)^{1/2}.$$

Then,  $\dot{H}_c^1$  can be identified with [sobolev space](#)  $\dot{H}^{1,2}(I, \mathbb{R}^d)$ . Consider  $I$  of  $c$  as quadratic form on  $\dot{H}_c^1$ , i.e.,  $I: \dot{H}_c^1 \times \dot{H}_c^1 \rightarrow \mathbb{R}$ .

**Definition 5.4.1 (Index).** The *index* of  $c$ ,  $\text{Ind}(c)$ , is the dimension of the largest subspace of  $\dot{H}_c^1$ , on which  $I$  is negative definite.

**Definition 5.4.2 (Extended index).** The *extended index* of  $c$ ,  $\text{Ind}_0(c)$ , is the dimension of the largest subspace of  $\dot{H}_c^1$ , on which  $I$  is negative semi-definite.

**Definition 5.4.3 (Nullity).** The *nullity* is defined as  $N(c) := \text{Ind}_0 - \text{Ind}(c)$ .

For  $t \in (a, b]$ , let  $\mathcal{J}_c^t$  be the space of [Jacobi fields](#)  $x$  along  $c$  with  $X(a) = 0 = X(t)$ .

**Lemma 5.4.1.**  $\dim \mathcal{J}_c^b = N(c)$ .

Let  $(\mathcal{M}^n, g)$  be a [complete Riemannian manifold](#). Let  $p \in \mathcal{M}$ , and denote  $d := d(p, \cdot)$ . For all  $v \in S^{n-1}$ , we can find a [geodesic](#)  $c_v(t) = \exp_t(tv)$ . Let  $R(v) := \sup \left\{ T \mid c_v|_{[0, T]} \text{ minimizing} \right\}$ . If  $t < R(v)$ , then  $d(p, c_v(t)) = t$ .

**Example.** If  $R(v) = \infty$ ,  $c_v$  minimizing.

**Definition 5.4.4 (Cut locus).** The *cut locus* of  $p$  as

$$C(p) := \{c_v(R(v)) \mid v \in S^{n-1} \text{ such that } R(v) < \infty\}.$$

**Definition 5.4.5 (Cut point).** Consider [geodesic](#)  $c$  with

$$d(\underbrace{c(0), c(t)}_p) = t$$

on  $[0, t_0]$  where  $t_0$  is the last point this holds. Then we say  $c(t_0)$  as a *cut point*.

# Chapter 6

## Morse Theory

### Lecture 21: Morse Index Theorem

#### 6.1 Morse Index Theorem

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**As previously seen.** Fix  $p \in \mathcal{M}$ , let  $q \in C(p)$  a **cut point**. Then there exists a **geodesic**  $c$  such that

1.  $c$  minimizing up to and including  $q$ , and
2.  $c$  uniquely minimizing up to but not including  $q$ .

Thus,  $c$  is not minimizing after that point.

##### 6.1.1 Conjugate Locus

Let's first see the following under the above setting.

**Proposition 6.1.1.** At each **cut point**  $q \in C(p)$ ,  $q$  is either a **conjugate point** or there exists two minimizing **geodesics** connecting  $p, q$ .

**Proof.** See [FC13]. ■

**As previously seen.** We have the **index form**  $I(X, Y)$ , **index**  $\text{Ind}(c)$ , and also  $\text{Ind}_0$ .

For  $t \in (a, b]$ , let  $\mathcal{J}_c^t$  be the space of **Jacobi field** along  $c$  such that  $X(a) = 0 = X(t)$ .

**Lemma 6.1.1.**  $\text{Ind}(c)$  and  $N(c)$  are always finite.

**Proof.** See [FC13] (by contradiction). ■

**Remark (Reformulation).** Equivalently, let  $c: [0, T] \rightarrow \mathcal{M}$  be a **geodesic**. Then the **index**  $\text{Ind}(c)$  on the space  $\mathcal{V}_c$  is finite and equals the number of points  $c(t)$  **conjugate** to  $c(0)$  for  $t \in (0, T)$ , counted with multiplicities.

**Intuition.** The **Morse index theorem** gives information about **conjugate points** via **index form**.

**Definition 6.1.1 (Conjugate locus).** Let  $(\mathcal{M}, g)$  be a **Riemannian manifold**. The set of (first) **conjugate points** of point  $p \in \mathcal{M}$  for all **geodesics** starting at  $p$  is called the *conjugate locus* of  $p$ .

**Proposition 6.1.2.** Let  $(\mathcal{M}, g)$  be a **complete Riemannian manifold**. Let  $c: [0, \infty) \rightarrow \mathcal{M}$  be a

normalized **geodesic** with  $c(0) = p$ . Assume that  $c(t_0)$  is the **cut point** at  $p = c(0)$  along  $c$ . Then, either  $c(t_0)$  is the first **conjugate point** of  $c(0)$  along  $c$  or there exists another **geodesic**  $\sigma \neq c$  from  $p$  to  $c(t_0)$  such that  $\ell(\sigma) = \ell(c)$ .

Conversely, if either the above are true, then there exists  $t_0 \in (0, t_0]$  such that  $c(t_1)$  is the **cut point** of  $p$  along  $c$ .

**Proof.** See [FC13]. ■

### 6.1.2 Morse Index Theorem

Now, we prove the following.

**Theorem 6.1.1 (Morse index theorem).** Let  $c: [a, b] \rightarrow \mathcal{M}$  be a **geodesic**. Then, there are at most finitely many points **conjugate** to  $c(a)$  along  $c$ , and

$$\text{Ind}(c) = \sum_{t \in (a, b)} \dim \mathcal{J}_c^t, \quad \text{Ind}_0(c) = \sum_{t \in (a, b]} \dim \mathcal{J}_c^t.$$

**Proof.** For all  $t_i \in (a, b]$ , for which  $c(t_i)$  **conjugate** to  $c(a)$ , there exists a **Jacobi field**  $X_i$  along with  $X_i(a) = 0 = X_i(t_i)$ . Set

$$Y_i(t) := \begin{cases} X_i(t), & \text{if } a \leq t \leq t_i; \\ 0, & \text{otherwise,} \end{cases}$$

we have that  $Y_i(t)$  are linearly independent such that  $I(Y_i, Y_i) = 0$  for all  $i$ . This implies that the number of **conjugate points** is at most  $\text{Ind}_0(c)$ , which is finite from **Lemma 6.1.1**.

For  $\tau \in (a, b]$ , set

$$\varphi(\tau) := \text{Ind}\left(c|_{[a, \tau]}\right), \quad \varphi_0(\tau) := \text{Ind}_0\left(c|_{[a, \tau]}\right).$$

**Claim.**  $\varphi(\tau)$  is left-continuous.

**Proof.** For  $\tau \in (a, b]$ , let  $I_\tau$  be the **index form** of  $c|_{[a, \tau]}$ , and let  $X$  be a **vector field** along  $c|_{[a, \tau]}$  satisfy  $I_\tau(X, X) < 0$  and  $\|X\| = 1$ .

Let  $\tilde{X}$  be **vector field** defined by  $\tilde{X}(t) := X(\tau t / \sigma)$  on  $[a, \sigma]$ . Then,

$$\int_0^\sigma \langle \dot{\tilde{X}}(t), \dot{\tilde{X}}(t) \rangle dt = \int_0^\sigma \left(\frac{\tau}{\sigma}\right)^2 \langle \dot{X}(\tau t / \sigma), \dot{X}(\tau t / \sigma) \rangle dt = \frac{\tau}{\sigma} \int_0^\tau \langle \dot{X}(s), \dot{X}(s) \rangle ds,$$

implying

$$\int_0^\sigma \langle \dot{\tilde{X}}(t), \dot{\tilde{X}}(t) \rangle dt \rightarrow \int_0^\tau \langle \dot{X}(t), \dot{X}(t) \rangle dt$$

for  $\sigma \rightarrow \tau$ . Also, we have  $\|X\| = 1$ , and  $X$  is continuous,<sup>a</sup> we see that  $\tilde{X}$  converges point-wise to  $X$  as  $\sigma \rightarrow \tau$ , hence

$$\int_0^\sigma \langle R(\dot{c}, \tilde{X})\tilde{X}, \dot{c} \rangle dt \rightarrow \int_0^\tau \langle R(\dot{c}, X)X, \dot{c} \rangle dt$$

as  $\sigma \rightarrow \tau$ , hence  $I_\sigma(\tilde{X}, \tilde{X}) \rightarrow I_\tau(X, X)$  as  $\sigma \rightarrow \tau$ . Notice that the above also implies  $I_\sigma(\tilde{X}, \tilde{X}) < 0$  if  $\sigma$  is sufficiently close to  $\tau$ .

Finally, for all orthonormal basis of a space on which  $I_\tau$  is negative definite, we may also find a basis of some space on which  $I_\sigma$  is negative definite if  $\sigma$  is sufficiently close to  $\tau$ . As  $\varphi$  is monotonically increasing, we have left-continuity. ⊗

<sup>a</sup>This is from something called Sobolev theorem.

**Claim.**  $\varphi_0(\tau)$  is right-continuous.

**Proof.** Let  $(\tau_n)_{n \in \mathbb{N}} \subseteq (a, b]$  converge to  $\tau \in (a, b]$  for all  $n \in \mathbb{N}$ , let  $X_n$  be a **vector field along**  $c|_{[0, \tau_n]}$  with  $\|X\| = 1$  and  $I_{\tau_n}(X_n, X_n) \leq 0$ . After selecting a subsequence,  $X_n$  converges weakly in Sobolev space  $H^{1,2}$  topology to some **vector field  $X$  along**  $c|_{[a, \tau]}$ . Then, we just check every ingredient of **index form** (see [FC13]).  $\otimes$

Finally, let  $a < t_1 < t_2 < \dots < t_k \leq b$  be the points  $c(t_i)$  **conjugate** to  $c(a)$ . Then,  $\varphi_0(t) - \varphi(t) = 0$  for  $t_i \in (a, b]$ . Then,

$$\sum_{t \in (a, b]} \dim \mathcal{J}_c^t = \sum_{t \in (a, b]} (\varphi_0(t) - \varphi(t)) = \sum_{i=1}^k (\varphi_0(t_i) - \varphi(t_i)).$$

Since  $\varphi$  is left-continuous, and  $\varphi_0$  is right-continuous, hence we have

$$\varphi_0(t_i) = \varphi(t_{i+1})$$

for  $i = 1, \dots, k-1$ , we finally have

$$\sum_{i=1}^k (\varphi_0(t_i) - \varphi(t_i)) = \varphi_0(t_k) - \varphi(t_1).$$

From  $\varphi$  being left-continuous again,  $\varphi(t_1) = 0$ . Finally, again, from the continuity properties of  $\varphi, \varphi_0$ , they can “jump” only at the points  $\tau$  where  $\varphi_0(\tau) \neq \varphi(\tau)$ , i.e., at the **conjugate points**. In particular,  $\varphi_0$  is constant on  $[t_k, b]$  hence,  $\varphi_0(t_k) = \varphi_0(b)$ , i.e.,

$$\varphi_0(b) = \sum_{t \in (a, b]} \dim \mathcal{J}_c^t.$$

■

**Intuition.** The “jump” only happens at **conjugate points**.

## Lecture 22: Bonnet-Mayers Theorem and Morse Functions

### 6.1.3 Bonnet-Mayers Theorem

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**Definition 6.1.2 (Diameter).** The *diameter* of a **manifold**  $\mathcal{M}$  is defined as

$$\text{diam}(\mathcal{M}) := \sup_{p, q \in \mathcal{M}} d(p, q).$$

**Theorem 6.1.2 (Bonnet-Mayers theorem).** Let  $(\mathcal{M}^n, g)$  be a **complete<sup>a</sup> Riemannian manifold** and **Ricci curvature**  $\geq \lambda > 0$ , i.e.,

$$\text{Ric}(X, X) \geq \lambda \langle X, X \rangle$$

for all  $X \in T\mathcal{M}$ . Then the diameter of  $\mathcal{M}$  is less than  $\pi \sqrt{(n-1)/\lambda}$ . In particular,  $\mathcal{M}$  is compact and has finite fundamental group  $\pi_1(\mathcal{M})$ .

<sup>a</sup>I.e., closed and any two points can be joined by a minimizing **geodesic**.

**Proof.** For all  $\rho < \text{diam}(\mathcal{M})$ , there exists  $p, q \in \mathcal{M}$  with  $d(p, q) = \rho$ . As  $\mathcal{M}$  **complete**, there exists a shortest **geodesic** arc  $c: [0, \rho] \rightarrow \mathcal{M}$  with  $c(0) = p$  and  $c(\rho) = q$ . Now, let  $\{e_i\}_{i=1}^n$  be an orthonormal basis of  $T_p\mathcal{M}$ , such that  $e_1 = \dot{c}(0)$ . Now, consider a parallel orthonormal basis  $\{\dot{c}(t), X_1(t), \dots, X_n(t)\}$  along  $c$ . Furthermore, consider  $Y_i(t) := (\sin \pi t / \rho) X_i(t)$  with  $i = 2, \dots, n$ .

Then,

$$I(Y_i, Y_i) = \int_0^\rho -\langle \ddot{Y}_i, Y_i \rangle - \langle R(Y_i, \dot{c})\dot{c}, Y_i \rangle dt = \int_0^\rho \sin^2 \frac{\pi t}{\rho} \left( \frac{\pi^2}{\rho^2} - \langle R(X_i, \dot{c})\dot{c}, X_i \rangle \right) dt.$$

Since  $c$  is the shortest [curve](#) connecting  $p, q$ , it follows that there are no [conjugate points](#) between  $p, q$ . Hence,  $I(Y_i, Y_i) \geq 0$  for all  $i$ , so

$$0 \leq \sum_{i=2}^n I(Y_i, Y_i) = \int_0^\rho \sin^2 \frac{\pi t}{\rho} \left( \frac{\pi^2}{\rho^2} (n-1) - R(\dot{c}, \dot{c}) \right) dt \leq \left( \frac{\pi^2}{\rho^2} (n-1) - \lambda \right) \int_0^\rho \sin^2 \frac{\pi t}{\rho} dx,$$

implying

$$0 \leq \frac{1}{2}\rho \left( \frac{\pi^2(n-1)}{\rho^2} - \lambda \right) \Rightarrow \rho^2 \leq \frac{\pi^2(n-1)}{\lambda} \Rightarrow \rho \leq \pi \sqrt{\frac{n-1}{\lambda}}.$$

Since this is true for all  $\rho < \text{diam}(\mathcal{M})$ , hence we see that  $\text{diam}(\mathcal{M}) \leq \pi \sqrt{(n-1)/\lambda}$ .

Furthermore, the universal cover of  $\mathcal{M}$  satisfies the same assumptions as [Ricci curvature](#), by computation, we have finite  $\pi_1(\mathcal{M})$ . ■

**Remark.** We choose  $Y_i(t) = \sin(\pi t/\rho)X_i(t)$  is just because it satisfies the needed condition, and makes the computation works out nicely.

**Intuition.** [Bonnet-Mayers theorem](#) says that if  $\mathcal{M}$  has [Ricci curvature](#) not less than the one of  $S_r^n$ , then  $\text{diam}(\mathcal{M})$  is at most the one of  $S_r^n$ .

Consider the hyperbolic space  $\mathbb{H}^n$  in  $\mathbb{R}^{n+1}$  where we define

$$\langle x, x \rangle := -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2$$

for  $x = (x^0, \dots, x^1)$ . Then

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -1, x^0 > 0\}.$$

Also, consider the half-space of  $\mathbb{R}^n$  such that

$$H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

with metric on  $\mathbb{H}^n$ , we have

$$g_{ij}(x_1, \dots, x_n) = \frac{\delta_{ij}}{x_n^2}.$$

Then, we see that we have a constant sectional curvature of  $-1$ .

## 6.2 Flow Homology

We now turn our focus to critical points of functions.

### 6.2.1 Morse Functions

Let  $(\mathcal{M}, g)$  be a [complete Riemannian manifold](#). Let  $f: \mathcal{M} \rightarrow \mathbb{R}$  be a smooth<sup>1</sup> function. Then

$$df(x) = 0$$

means that  $x$  is a critical point of  $f$ .

**Definition 6.2.1 (Non-degenerate).** A critical point  $a$  of  $f$  is *non-degenerate* if the Hessian of  $f$  is non-singular at  $a$ .

<sup>1</sup>It's typically enough to ask for  $f \in C^3(\mathcal{M}, \mathbb{R})$ .

**Definition 6.2.2 (Morse index).** The *index*  $\mu(p)$  of **non-degenerate** critical point  $p$  of  $f$  is the dimension of the largest subspace of  $T_p\mathcal{M}$  on which the Hessian is negative definite.

**Intuition.** That is, the number of directions in which  $f$  decreases.

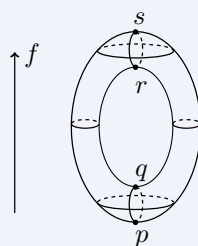
**Note.** The **degeneracy** and **index** are independent of coordinate choice.

Now, we define the critical set of  $f$  as

$$C(f) := \{x \in \mathcal{M} \mid df(x) = 0\}.$$

**Definition 6.2.3 (Morse function).** A *Morse function*  $f$  is a function as introduced such that all critical points are **non-degenerate**.

**Example.** Consider  $f$  is the height function, which is a **Morse function** such that the **index** of  $s$  is 2,  $r$  is 1,  $q$  is 1, and  $p$  is 0 by looking at the decreasing directions.



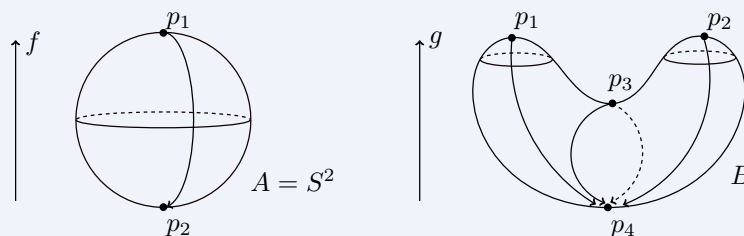
Now, define  $M^a = f^{-1}(-\infty, a]$ , then we see that

- (a) Pass  $p$ :  $M^a$  for  $0 < a < f(q)$  is a disk, which is homotopy equivalent to a point, i.e., 0-cell.
- (b) Pass  $q$ :  $M^a$  for  $f(q) < a < f(r)$  is a cylinder, where we attach a 1-cell.
- (c) Pass  $r$ :  $M^a$  for  $f(r) < a < f(s)$  is a torus with disk removed.
- (d) Pass  $s$ :  $M^a$  for  $a > f(s)$  is a torus.

## Lecture 23: Morse Theory and Flow Homology

Throughout this lecture, let  $\mathcal{M}$  be a compact **Riemannian manifold**, also, we'll keep mentioning the following example. 30 Mar. 13:00

**Example.** Consider the following.



Then,

- $f$ :  $\mu_f(p_1) = 2$ ,  $\mu_f(p_2) = 0$ .
- $g$ :  $\mu_g(p_1) = \mu_g(p_2) = 2$ ,  $\mu_g(p_3) = 1$ , and  $\mu_g(p_4) = 0$ .

Additionally, we can study some “invariants” about spaces, such as the Euler characteristic<sup>2</sup> of  $\mathcal{M}$ , which is “defined” as

$$\chi(\mathcal{M}) = \sum_{p: \text{critical point of } f} (-1)^{\mu(p)} \mu(p).$$

**Example.**  $\chi(A) = \chi(S^2) = 2$ , and  $\chi(B) = 4 - 1 = 3$ .

To make  $\chi$  formal, we need to consider the homology as we did in algebraic topology, i.e., we need to define the “boundary map” and “complexes”.

**Intuition.** Intuitively, our complexes should be a vector space built on top of critical points; on the other hand, for two critical points  $p, q$  such that  $\mu(p) - \mu(q) = 1$ , we want to count the trajectories from  $p$  to  $q$  modulo 2, i.e., the boundary map  $\partial$  should be somehow defined as

$$\partial p := \sum_{\substack{p \text{ critical point of } f \\ \mu(q) = \mu(p) - 1}} (\#\{\text{flow lines from } p \text{ to } q\} \bmod 2) \cdot q.$$

### 6.2.2 Morse Complex

We study so-called **Morse complex**, and define the so-called *flow homology*. We start by defining the so-called **negative gradient flow**.

**Definition 6.2.4 (Negative gradient flow).** The *negative gradient flow* of  $f$  on  $\mathcal{M}$  is defined as the solution  $\phi: \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M}$  of

$$\begin{cases} \frac{\partial}{\partial t} \phi(x, t) = -\text{grad}(f(\phi(x, t))); \\ \phi(x, 0) = x \end{cases} \quad \text{for } x \in \mathcal{M}.$$

**Note.** We will simply call **negative gradient flow** a *flow* for simplicity.

**Remark.** From the **Picard-Lindelöf theorem**, local existence of the **flow** is guaranteed. Moreover, if we have “very good” conditions, such a **flow** may even exist globally.

More generally, the Euler characteristic is an example of a **flow**  $\phi: \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M}$  such that

$$\begin{cases} \frac{\partial}{\partial t} \phi(x, t) = -V(f(\phi(x, t))); \\ \phi(x, 0) = x \end{cases}$$

with some **vector field**  $V$  on  $\mathcal{M}$ .

**Note (Autonomous).** We have  $V(\phi(x, t))$ , not  $V(\phi(x, t), t)$ , i.e., it doesn’t explicitly depend on  $t$ .

**Remark.** The **flow** satisfies group property, i.e.,  $\phi(x, t_1 + t_2) = \phi(\phi(x, t_1), t_2)$  for all  $t_1, t_2 \in \mathbb{R}$ .

- Moreover, for all  $x \in \mathcal{M}$ , the **flow line** or orbit  $\gamma_x := \{\phi(x, t) \mid t \in \mathbb{R}\}$  through point  $x$  is flow-invariant, i.e., for all  $y \in \gamma_x$ ,  $t \in \mathbb{R}$ , we have  $\phi(y, t) \in \gamma_x$ .
- Finally, for all  $t \in \mathbb{R}$ ,  $\phi(\cdot, t): \mathcal{M} \rightarrow \mathcal{M}$  is a **diffeomorphism** of  $\mathcal{M}$  onto its image.

Naturally, we can consider the following two kinds of points of  $\mathcal{M}$ .

<sup>2</sup>We will make it formal.



**Definition 6.2.5** (Stable manifold). The *stable manifold* at  $x_0$  of the flow  $\phi$  are defined as

$$W^s(x_0) := \left\{ y \in \mathcal{M} \mid \lim_{t \rightarrow \infty} \phi(y, t) = x_0 \right\}.$$

**Definition 6.2.6** (Unstable manifold). The *unstable manifold* at  $x_0$  of the flow  $\phi$  are defined as

$$W^u(x_0) := \left\{ y \in \mathcal{M} \mid \lim_{t \rightarrow -\infty} \phi(y, t) = x_0 \right\}.$$

That is to say, we should focus on the dimension of  $W^u(p)$ .

**Intuition.** For  $t \rightarrow \pm\infty$ , each flow line  $x(t)$  defined as  $x: \mathbb{R} \rightarrow \mathcal{M}$  with  $\dot{x}(t) = -\text{grad } f(x(t))$  for all  $t \in \mathbb{R}$  converges to critical point, i.e.,

$$p = x(-\infty), \quad p = x(+\infty) \Rightarrow W^u(p) \text{ the all flow lines } x(t) \text{ with } x(-\infty) = p.$$

**As previously seen.** For a Morse function  $f$ , the set of critical points  $C(f) := \{x \in \mathcal{M} \mid df(x) = 0\}$

**Notation.** Denote the set of critical points of  $f$  of index  $k$  as  $\text{Crit}_k(f)$ .

**Definition 6.2.7** (Morse complex). Define the vector space over  $\mathbb{Z}/2\mathbb{Z}$  as

$$C_k(f, \mathbb{Z}_2) = C_k(f) := \left\{ \sum_{a \in \text{Crit}_k(f)} m_a a \mid m_a \in \mathbb{Z}/2\mathbb{Z} \right\}.$$

**Definition 6.2.8** (Boundary operator). The *boundary operator*  $\partial_k: C_k(f) \rightarrow C_{k-1}(f)$  by specifying its behavior on the basis elements. Given a critical point  $a \in C_k(f)$ ,  $\partial_k$  sends  $a$  to a linear combination of points in  $\text{Crit}_{k-1}(f)$  defined as

$$\partial_k(a) = \sum_{b \in \text{Crit}_{k-1}(f)} m(a, b) b$$

with  $m(a, b) \in \mathbb{Z}/2\mathbb{Z}$  being the number mod 2 of trajectories from  $a$  to  $b$ .<sup>a</sup>

<sup>a</sup>We can check that  $\partial \circ \partial = 0$ .

### 6.2.3 Flow Homology

With all the notions we have established, we have the following naturally.

**Definition 6.2.9** (Flow homology group). The *flow homology group* is defined as

$$H_k(\mathcal{M}, f, \mathbb{Z}_2) := \ker \partial \text{ on } C_k(f) / \text{Im } \partial \text{ from } C_{k+1}(f).$$

**Remark.** The image of  $\partial$  from  $C_{k+1}(f, \mathbb{Z}_2)$  is always contained in the kernel of  $\partial$  on  $C_k(f, \mathbb{Z}_2)$ .

**Definition 6.2.10** (Betti number). The *Betti number* is defined as  $b_k := \dim_{\mathbb{Z}_2} H_k(\mathcal{M}, f, \mathbb{Z}_2)$ .

**Definition 6.2.11 (Euler characteristic).** The *Euler characteristic* of  $\mathcal{M}$  is defined as

$$\chi(\mathcal{M}) = \sum_i (-1)^i b^i.$$

Let's now calculate all these on our examples  $A = S^2$  and  $B$ .

**Example.** Revisit the example for  $f$ , we have

- $C_2(f) = \mathbb{Z} / 2\mathbb{Z}[p_1];$
- $C_0(f) = \mathbb{Z} / 2\mathbb{Z}[p_2];$
- $C_1(f) = 0.$

Also, we see that the chain complexes are

- $C_2 = \mathbb{Z}_2[p_1];$
- $C_0 = \mathbb{Z}_2[p_2];$
- $C_1 = 0;$

For kernels,

- $\ker \partial_2 = \{p_1\};$
- $\ker \partial_0 = \{p_2\};$
- and since  $\partial_1$  is trivial, so all images are trivial.

Finally, we calculate the homology groups as

- $H_2(A, f, \mathbb{Z}_2) = \mathbb{Z}_2;$
- $H_1(A, f, \mathbb{Z}_2) = 0;$
- $H_0(A, f, \mathbb{Z}_2) = \mathbb{Z}_2.$

**Example.** Revisit the example for  $g$ , we have

- $C_2(g) = \mathbb{Z} / 2\mathbb{Z}[p_1];$
- $C_1(g) = \mathbb{Z} / 2\mathbb{Z}[p_3];$
- $C_0(g) = \mathbb{Z} / 2\mathbb{Z}[p_4];$
- $C_k(g) = 0$  for  $k \geq 3.$

Also, we see that the chain complexes are

- $\partial_2(p_1) = p_3 = \partial_2(p_2),$  and  $\partial_2(p_1 + p_2) = 2p_3 = 0;$
- $\partial_1(p_3) = 2p_4 = 0.$

So the kernels are

- $\ker \partial_1 = \text{Im } \partial_2 = \mathbb{Z}_2[p_3];$
- $\ker \partial_2 = \mathbb{Z}_2[p_1 + p_2];$
- $\ker \partial_0 = \mathbb{Z}_2[p_4],$

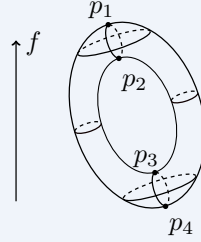
and all other images and kernels are trivial. Finally, we calculate the homology groups as

- $H_2(B, g, \mathbb{Z}_2) = \ker \partial_2 = \mathbb{Z}_2$ ;
- $H_1(B, g, \mathbb{Z}_2) = \ker \partial_1 / \text{Im } \partial_2 = 0$ ;
- $H_0(B, g, \mathbb{Z}_2) = \mathbb{Z}_2$ .

## Lecture 24: Introduction to Rauch Comparison Theorems

**Example (Tilted torus).** Consider the tilted torus

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We see that

- $C_2 = \mathbb{Z}_2[p_1]$ ;
- $C_1 = \mathbb{Z}_2[p_2] \oplus \mathbb{Z}_2[p_3]$ ;
- $C_0 = \mathbb{Z}_2[p_1]$ .

Moreover, the chain complex is

$$0 \xrightarrow{\partial_3} \mathbb{Z}[p_1] \xrightarrow{\partial_2} \mathbb{Z}_2[p_2] + \mathbb{Z}_2[p_3] \xrightarrow{\partial_1} \mathbb{Z}_2[p_4] \xrightarrow{\partial_0} 0$$

### 6.3 Rauch Comparison Theorems

In this section, our goal is to compare [Riemannian manifolds](#)  $(\mathcal{M}, g)$  with other [Riemannian manifolds](#) of constant [curvatures](#) model spaces, e.g.,  $S^n$ ,  $\mathbb{R}^n$ , and  $\mathbb{H}^n$ .

**Notation (Model space).** The set of *model spaces* is denoted as  $\mathcal{M}_m \in \{S^n, \mathbb{R}^n, \mathbb{H}^n\}$ .

Now, let  $c(t)$  be a [geodesic](#) with  $\|\dot{c}\| = 1$ ,  $v \in T_{c(0)}\mathcal{M}$ . Furthermore, let  $\mathcal{J}(t)$  be the [Jacobi field](#) along  $c(t)$  with  $\mathcal{J}(0) = 0$  and  $\dot{\mathcal{J}}(0) = v$  given by

$$\begin{cases} (\sin t)v, & \text{for } S^n; \\ tv, & \text{for } \mathbb{R}^n; \\ (\sinh t)v, & \text{for } \mathbb{H}^n. \end{cases}$$

Now, consider  $(\mathcal{M}, g)$  such that  $\lambda \leq \kappa \leq \mu$  with  $\lambda \leq 0$  and  $\mu \geq 0$ .

**Notation.** For  $\rho \in \mathbb{R}$ ,

$$c_\rho(t) = \begin{cases} \cos(\sqrt{\rho}t), & \text{if } \rho > 0; \\ 1, & \text{if } \rho = 0; \\ \cosh(\sqrt{-\rho}t), & \text{if } \rho < 0, \end{cases}$$

and also,

$$s_\rho(t) = \begin{cases} \frac{1}{\sqrt{\rho}} \sin(\sqrt{\rho}t), & \text{if } \rho > 0; \\ t, & \text{if } \rho = 0; \\ \frac{1}{\sqrt{-\rho}} \sinh(\sqrt{-\rho}t), & \text{if } \rho < 0, \end{cases}$$

These are solutions of **Jacobi equations** for constant **sectional curvature**  $\rho$ , i.e.,

$$\ddot{f}(t) + \rho f(t) = 0$$

with corresponding initial values  $f(0) = 0$ ,  $\dot{f}(0) = 1$ , respectively,  $f(0) = 1$ ,  $\dot{f}(0) = 0$ .

**Theorem 6.3.1.** Assume  $\kappa \leq \mu$  and  $\|\dot{c}\| \equiv 1$ . Assume either  $\mu \geq 0$  or  $\mathcal{J}^{\text{tan}} \equiv 0$ . Let  $f_\mu := |\mathcal{J}(0)|c_\mu + |\mathcal{J}'(0)s_\mu$  solve

$$\ddot{f} + \mu f = 0$$

with  $f(0) = |\mathcal{J}(0)|$  and  $\dot{f}(0) = |\mathcal{J}'(0)|$ . If  $f_\mu(t) > 0$  for  $0 < t < \tau$ , then the following holds.

- (a)  $\langle \mathcal{J}, \dot{\mathcal{J}} \rangle f_\mu \geq \langle \mathcal{J}, \mathcal{J} \rangle \dot{f}_\mu$  on  $[0, \tau]$ .
- (b)  $1 \leq \frac{|\mathcal{J}(t_0)|}{f_\mu(t_1)} \leq \frac{|\mathcal{J}(t_2)|}{f_\mu(t_2)}$  if  $0 < t_1 \leq t_2 < \tau$ .
- (c)  $|\mathcal{J}(0)|c_\mu(t) + |\mathcal{J}'(0)s_\mu(t) \leq |\mathcal{J}(t)|$  for  $0 \leq t \leq \tau$ .

**Proof.** Firstly, we have that

$$|\mathcal{J}'| = \frac{\langle \mathcal{J}, \dot{\mathcal{J}} \rangle}{|\mathcal{J}|}, \quad |\mathcal{J}''| = \frac{\langle \dot{\mathcal{J}}, \dot{\mathcal{J}} \rangle}{|\mathcal{J}|} + \frac{\langle \mathcal{J}, \ddot{\mathcal{J}} \rangle}{|\mathcal{J}|} - \frac{\langle \mathcal{J}, \dot{\mathcal{J}} \rangle^2}{|\mathcal{J}|^3},$$

so

$$|\mathcal{J}''| + \mu|\mathcal{J}| = \frac{1}{|\mathcal{J}|} (-\langle R(\mathcal{J}, \dot{c})\dot{c}, \mathcal{J} \rangle + \mu\langle \mathcal{J}, \mathcal{J} \rangle) + \frac{1}{|\mathcal{J}|^3} (|\dot{\mathcal{J}}|^2|\mathcal{J}|^2 - \langle \mathcal{J}, \dot{\mathcal{J}} \rangle^2) \geq 0$$

since  $\kappa \leq \mu$  for  $0 < t < \tau$ , provided  $\mathcal{J}$  has no zeros on  $(0, \tau)$ . Moreover,

$$(|\mathcal{J}'|f_\mu - |\mathcal{J}|\dot{f}_\mu)' = |\mathcal{J}''|f_\mu - |\mathcal{J}|\ddot{f}_\mu \geq 0$$

since  $\ddot{f}_\mu + \mu f_\mu = 0$  for  $f_\mu(t) \geq 0$ . Also, we have  $|\mathcal{J}|(0) = f_\mu(0)$ ,  $|\mathcal{J}'|(0) = \dot{f}_\mu(0)$ , implying

$$|\mathcal{J}'|f_\mu - |\mathcal{J}|\dot{f}_\mu \geq 0,$$

which proves the first claim.

Furthermore,

$$\left( \frac{|\mathcal{J}|}{f_\mu} \right)' = \frac{1}{f_\mu^2} (|\mathcal{J}'|f_\mu - |\mathcal{J}|\dot{f}_\mu) \geq 0,$$

then since first zero of  $\mathcal{J}$  cannot occur before the first zero of  $f_\mu$ , the second claim is proved. The last claim follows directly from this.  $\blacksquare$

**Remark.**  $f_\mu(t) > 0$  for  $0 < t < \tau$  is necessary.

**Proof.** Take  $S^{nn(\mu-\epsilon)}$  with  $\mathcal{J}(0) = 0$ . We see that  $f_\mu(t)$  has a zero at  $t = \pi/\sqrt{\mu}$  and  $\mathcal{J}(t)$  has one at  $t = \pi/\sqrt{\mu-\epsilon}$ . For small  $\epsilon > 0$  and any  $t$ , only slightly longer than  $\pi/\sqrt{\mu-\epsilon}$ , we have

$$\frac{|\mathcal{J}(t)|}{f(t)} < 1.$$

⊛

**Corollary 6.3.1.** Suppose that  $\kappa \leq \mu$ ,  $c_\mu \geq 0$  on  $(0, \tau)$ , and  $\mu \geq 0$  or  $\mathcal{J}^{\tan} \equiv 0$ . Let  $\|\dot{c}\| \equiv 1$ ,  $\mathcal{J}(0) = 0$ ,  $|R| < \Lambda$  with  $R$  being the [curvature tensor](#). Then,

$$|\mathcal{J}(t) - t\dot{\mathcal{J}}(t)| \leq |\mathcal{J}(\tau)| \frac{1}{2} \Lambda t^2.$$

**Theorem 6.3.2.** Assume that  $\lambda \leq \kappa \leq \mu$ , and either  $\lambda \leq 0$  or  $\mathcal{J}^{\tan} \equiv 0$ ,  $\|\dot{c}\| \equiv 1$ , and  $\mathcal{J}(0)$ ,  $\dot{\mathcal{J}}(0)$  be linearly dependent. Finally, assume  $s_{(\lambda+\mu)/2} > 0$  on  $(0, \tau)$ . Then, for  $0 \leq t \leq \tau$ ,

$$|\mathcal{J}(t)| \leq |\mathcal{J}(0)|c_\lambda(t) + |\mathcal{J}'(0)|s_\lambda(t).$$

**Proof idea.** Let  $\rho \in \mathbb{R}$ ,  $\eta := \max(\mu - \rho, \rho - \lambda)$ . Let  $A$  be a [vector field along  \$c\$](#)  with  $\ddot{A} + \rho A = 0$ ,  $A(0) = \mathcal{J}(0)$ , and  $\dot{A}(0) = \dot{\mathcal{J}}(0)$ .

Let  $a: I \rightarrow \mathbb{R}$  being a solution of  $\ddot{a} + (\rho - \eta)a = \eta|A|$ ,  $a(0) = \dot{a}(0) = 0$ , and  $b: I \rightarrow \mathbb{R}$  solving  $\ddot{b} + \rho b = \eta|\mathcal{J}|$ ,  $b(0) = \dot{b}(0) = 0$ . ■

## Lecture 25: Rauch Comparison Theorems

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**Theorem 6.3.3 (Rauch comparison theorem).** Let  $(\mathcal{M}^m, g)$ ,  $(\overline{\mathcal{M}}^m, \overline{g})$  be [Riemannian manifolds](#) and  $\gamma: [0, a] \rightarrow \mathcal{M}$ ,  $\overline{\gamma}: [0, a] \rightarrow \overline{\mathcal{M}}$  be normalized [geodesics](#) with  $\gamma(0) = p$ ,  $\overline{\gamma}(0) = \overline{p}$ . Let  $X, \overline{X}$  be [Jacobi fields](#) along  $\gamma, \overline{\gamma}$ , respectively such that  $X(0) = \overline{X}(0) = 0$ ,  $|\nabla_{\dot{\gamma}(0)} X| = |\nabla_{\dot{\overline{\gamma}}(0)} \overline{X}|$ , and  $\langle \dot{\gamma}(0), \nabla_{\dot{\gamma}(0)} X \rangle = \langle \dot{\overline{\gamma}}(0), \nabla_{\dot{\overline{\gamma}}(0)} \overline{X} \rangle$ .

Furthermore, assume that

- (a)  $\gamma$  has no [conjugate points](#) on  $[0, a]$ ;
- (b) [sectional curvatures](#)  $K, \overline{K}$  of  $\mathcal{M}, \overline{\mathcal{M}}$  satisfy  $\overline{K} \leq K$  for all 2-planes containing  $\dot{\gamma}, \dot{\overline{\gamma}}$ .

Then,  $\overline{\gamma}$  has no [conjugate points](#) on  $[0, a]$ , and for all  $t \in [0, a]$ ,

$$|X(t)| \leq |\overline{X}(t)|.$$

**Proof idea.** To prove this, we first see a lemma.

**Lemma 6.3.1.** Let  $(\mathcal{M}^m, g)$ ,  $(\overline{\mathcal{M}}^m, \overline{g})$  be [Riemannian manifolds](#) and  $\gamma: [0, a] \rightarrow \mathcal{M}$ ,  $\overline{\gamma}: [0, a] \rightarrow \overline{\mathcal{M}}$  be normalized [geodesics](#) with  $\gamma(0) = p$ ,  $\overline{\gamma}(0) = \overline{p}$ . Let  $X, \overline{X}$  be [Jacobi fields](#) along  $\gamma, \overline{\gamma}$ , respectively such that  $X(0) = \overline{X}(0) = 0$ . Furthermore, assume that

- (a)  $\gamma$  has no [conjugate points](#) on  $[0, a]$ ;
- (b) [sectional curvatures](#)  $K, \overline{K}$  of  $\mathcal{M}, \overline{\mathcal{M}}$  satisfy  $\overline{K} \leq K$  for all 2-planes containing  $\dot{\gamma}, \dot{\overline{\gamma}}$ .

Finally, assume that  $|X(a)| = |\overline{X}(a)|$ . Then,  $I(X, X) \leq I(\overline{X}, \overline{X})$ .

**Proof idea.** We first choose an orthonormal frame in  $(\mathcal{M}, g)$  and  $(\overline{\mathcal{M}}, \overline{g})$  with  $e_1 = \dot{\gamma}$  and  $\overline{e}_1 = \dot{\overline{\gamma}}$ , and  $e_2 = X(a)/|X(a)| \neq 0$ , etc. Consider  $X(t) = X^i(t)e_i(t)$  and the same for  $\overline{X}$ . Then, the second variation of the [energy](#) shows  $I(X, X) \leq I(\overline{X}, \overline{X})$ . ■

Now, consider normal components of  $X, \overline{X}$  only, and we can show that

$$\lim_{t \rightarrow 0} \frac{|X(t)|^2}{|X(t)|^2} =: \lim_{t \rightarrow 0} \frac{\overline{u}(t)}{u(t)} = 1,$$

thus to prove  $|X| \leq |\overline{X}|$ , it's enough to show that

$$\frac{d}{dt} \frac{|X(t)|^2}{|X(t)|^2} \geq 0,$$

equivalently,  $\dot{\bar{u}} - \bar{u}\dot{u} \geq 0$ . Then, since  $\gamma$  has no [conjugate points](#), we have  $u(t) > 0$ . Let  $c \in [0, a]$  be the greatest number such that  $\bar{u}(t) > 0$  on  $(0, c)$ . Then, for all  $b \in (0, c)$ , define

$$X_b(t) = \frac{X(t)}{|X(b)|}, \quad \bar{X}_b(t) = \frac{\bar{X}(t)}{|\bar{X}(b)|}.$$

From [Lemma 6.3.1](#) to  $I(X_b, X_b), I(\bar{X}_b, \bar{X}_b)$ , then we're done.  $\blacksquare$

**Corollary 6.3.2.** Let  $(\mathcal{M}, g)$  be a [complete](#) and simply-connected [Riemannian manifold](#) with non-positive [sectional curvature](#), and  $\triangle ABC$  is a [geodesic](#) triangle in  $\mathcal{M}$ , then

- (a)  $|AB|^2 + |AC|^2 - 2|AB||AC|\cos \angle A \leq |BC|^2$ ;
- (b)  $\angle A + \angle B + \angle C \leq \pi$ .

**Corollary 6.3.3.** Suppose that [sectional curvature](#) of  $(\mathcal{M}, g)$  satisfies

$$0 < C_1 \leq K \leq C_2$$

for some constants  $C_1, C_2$ . Let  $\gamma$  be any [geodesic](#) in  $\mathcal{M}$ . Then, the distance  $d$  between any two [conjugate points](#) of  $\gamma$  satisfies

$$\frac{\pi}{\sqrt{C_2}} \leq d \leq \frac{\pi}{\sqrt{C_1}}.$$

**Corollary 6.3.4.** Let  $(\mathcal{M}, g)$  be compact [Riemannian manifold](#) where the [sectional curvature](#)  $K$  satisfies  $K \leq C$  for some constant  $C$ . Then, either the [injectivity radius](#)

$$i(\mathcal{M}, g) \geq \pi/\sqrt{C},$$

or there exists a closed [geodesic](#)  $\gamma$  in  $\mathcal{M}$  whose [length](#) is minimal among all closed [geodesics](#) such that

$$i(\mathcal{M}, g) \geq \frac{1}{2}L(\gamma).$$

## 6.4 Sphere Theorems

Consider the following.

**Theorem 6.4.1** (Sphere theorem). Let  $\mathcal{M}^n$  be a compact and simply-connected [Riemannian manifold](#) with [sectional curvature](#)  $K$  such that

$$0 < hK_{\max} < K \leq K_{\max}.$$

Then if  $h = 1/4$ , then  $\mathcal{M}$  is homeomorphic to a sphere  $S^n$ .

**Notation** (Pinching number).  $h$  in the [sphere theorem](#) is called the *pinching number* of  $\mathcal{M}$ .

**Remark.** Another version of the [sphere theorem](#) is to assume  $0 < h < K \leq 1$  by scaling.

To prove this, Borger, Klingenberg used [Rauch comparison theorem](#) with [Morse index theorem](#) in the 1960s.

### 6.4.1 Gauss-Bonnet Theorem and Theorem by Hamilton

To understand the [sphere theorem](#), we should consider  $n = 2, 3$ . In this case, it suffices to assume  $h \geq 0$ , i.e., for a compact and simply-connected [Riemannian manifold](#)  $\mathcal{M}^n$  with  $n = 2, 3$  such that it has positive

sectional curvature, then  $\mathcal{M}^n$  is homeomorphic to  $S^n$ .

**Note.** For  $n = 2$ , it follows from the Gauss-Bonnet theorem, and for  $n = 3$ , it follows from a theorem by R. Hamilton.

**Theorem 6.4.2** (Gauss-Bonnet theorem). Let  $\mathcal{M}$  be a compact connected 2-dimensional Riemannian manifold  $\mathcal{M}$  with curvature  $K$ . Then, its characteristic is given by

$$\chi(\mathcal{M}) = \frac{1}{2\pi} \int_{\mathcal{M}} K \, d\mu_{\mathcal{M}}.$$

The Gauss-Bonnet theorem generalizes the so-called Gauss-Bonnet formula.

**Note** (Gauss-Bonnet formula). Let  $\gamma$  be a curved polygon on an oriented Riemannian 2-manifold  $(\mathcal{M}, g)$  such that  $\gamma$  is positive oriented as the boundary of an open set  $\Omega$  with compact closure.

Then,

$$\int_{\Omega} K \, dA + \int_{\gamma} k_N \, ds + \sum_i \epsilon_i = 2\pi,$$

where  $k_N(t) = \langle D_t \dot{\gamma}(t), N(t) \rangle$ ,<sup>a</sup> and  $\epsilon_i$  are the exterior angles.

<sup>a</sup> $N(t)$  is the normal vector field.

To understand all these, we need the following concept.

**Definition 6.4.1** (Smooth triangulation). For  $\mathcal{M}$  smooth, compact 2-manifold, a *smooth triangulation* of  $\mathcal{M}$  is a finite collection of curved triangles such that

- the union of the closed regions  $\overline{\Omega}_i$  bounded by the triangles is actually  $\mathcal{M}$ ;
- the intersection of any pair (if not empty) is either a single vertex of each or a single edge of each.

**Theorem 6.4.3** (Radó [Rad25]). Every compact topological 2-manifold has a triangulation.

Let  $\mathcal{M}$  be a triangulated 2-manifold. Then, the Euler characteristic is

$$\chi(\mathcal{M}) = N_v - N_e + N_f.$$

This implies that

$$\int_{\mathcal{M}} K \, dA = 2\pi\chi(\mathcal{M}).$$

Then, we can start proving Gauss-Bonnet theorem.

**Proof of Theorem 6.4.2.** Let  $\{\Omega_i\}_{i=1}^{N_f}$  denote the faces of triangulation, and for all  $i$ , let  $\{\gamma_{ij} \mid j = 1, 2, 3\}$  be the edges of  $\Omega_i$  and  $\{\theta_{ij} \mid j = 1, 2, 3\}$  be its interior angles.

As each exterior angle is  $\pi$  minus the interior angle, by applying the Gauss-Bonnet formula to each triangle and sum over  $i$ , we have

$$\begin{aligned} & \sum_{i=1}^{N_f} \int_{\Omega_i} K \, dA + \sum_{i=1}^{N_f} \sum_{j=1}^3 \int_{\gamma_{ij}} k_N \, ds + \sum_{i=1}^{N_f} \sum_{j=1}^3 (\pi - \theta_{ij}) = \sum_{i=1}^{N_f} 2\pi \\ & \Leftrightarrow \int_{\mathcal{M}} K \, d\mu_{\mathcal{M}} + 0 + \left( 3\pi N_f - \sum_{i=1}^{N_f} \sum_{j=1}^3 \theta_{ij} \right) = 2\pi N_f \end{aligned}$$

where the second term vanishes since each edge appears twice but with opposite sign, and ■

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**Theorem 6.4.4** (Hamilton). Let  $\mathcal{M}$  be a compact and simply-connected 3-dimensional Riemannian manifold  $\mathcal{M}$  with strictly positive Ricci curvature. Then,  $\mathcal{M}$  is diffeomorphic to  $S^3$ .



# Appendix

# Appendix A

## Additional Notes

### A.1 Christoffel Symbols

In this section, we dive deep into the notion of the [Christoffel symbols](#)  $\Gamma$  in various ways.

In particular, we will see that  $\Gamma$  are really just the corrections to an ordinary derivative on a “curved” manifold w.r.t. the [Levi-Civita connection](#), i.e., in the context of [torsion free](#) and [Riemannian connection](#)  $\nabla$ , we have also defined the so-called [connection coefficients](#), and we use the same notation  $\Gamma$ , and indeed they’re the same.

See [this](#)

#### A.1.1 Geometric Interpretation

#### A.1.2 Metric Interpretation

#### A.1.3 A Visual Guide

### A.2 Tensor Calculus

### A.3 Algebra

This chapter will collect some notion about algebras which you might not be familiar with.

#### A.3.1 Modules

**Definition A.3.1 (Left module).** Suppose  $R$  is a ring with 1. A *left  $R$ -module*  $M$  consists of an Abelian group  $(M, +)$  and an operation  $\cdot : R \times M \rightarrow M$  such that for all  $r, s \in R$  and  $x, y \in M$ ,

- (a)  $r \cdot (x + y) = r \cdot x + r \cdot y$ ;
- (b)  $(r + s) \cdot x = r \cdot x + s \cdot x$ ;
- (c)  $(rs) \cdot x = r \cdot (s \cdot x)$ ;
- (d)  $1 \cdot x = x$ .

**Note.** A *right  $R$ -module*  $M$  can also be defined similarly by consider  $\cdot : M \times R \rightarrow M$ .

**Definition A.3.2 (Module).** If  $R$  is commutative, then the [left and right  \$R\$ -module](#)  $M$  are the same, and we call  $M$  a *module*.

**Intuition.** We’re basically relaxing the notion of  $\mathbb{F}$ -vector field, but this time, the field  $\mathbb{F}$  is replaced by a ring  $R$ .

**Remark.** The most noticeable difference between a [module](#) and a vector field is that a [module](#) usually don't have a basis.

### A.3.2 The $C^\infty(\mathcal{M})$ -Module Viewpoint of Tensor Fields

The reason why we introduce the notion of [module](#) is because of the following: we can understand [tensor-field](#) better in the following way. Firstly, let's introduce the so-called [tensor bundles](#).

**Definition A.3.3** (Tensor bundle). A *tensor bundle* is a [fiber bundle](#) where the [fiber](#) is the product of any number of [tangent spaces](#) and/or [cotangent spaces](#).

So in a [tensor bundle](#), the [fiber](#) is a vector space and the [tensor bundle](#) is a special kind of [vector bundle](#).<sup>1</sup> Then, recall how we introduce [Definition 1.5.1](#):

**As previously seen.** A  $(r, s)$ -[tensor field](#)  $T$  is just a [section](#) of a [tensor bundle](#).

But there's actually a deeper explanation: observe that  $\Gamma(TM) = \{X : \text{vector fields on } \mathcal{M}\}$  is actually a  $C^\infty(\mathcal{M})$ -[module](#):

**Claim.**  $\Gamma(TM)$  carries a natural  $C^\infty(\mathcal{M})$ -[module](#) structure.

**Proof.** Firstly, observe that  $C^\infty(\mathcal{M}) = ((C^\infty(\mathcal{M}), +, \cdot))$  is not a field but a ring.<sup>a</sup> Then, naturally, the  $C^\infty(\mathcal{M})$ -[module](#)  $(\Gamma(TM), \oplus, \odot)$  where

- $\oplus: (X \oplus \tilde{X})(f) := (Xf) + \tilde{X}(f);$
- $\odot: (g \odot X)(f) := g \cdot X(f),$

for  $X, \tilde{X} \in \Gamma(TM)$ ,  $g, f \in C^\infty(\mathcal{M})$ . ⊗

<sup>a</sup>Since given  $f \in C^\infty(\mathcal{M})$ , we might not have  $f^{-1}$ .

**Notation.** Notice that given a [vector field](#)  $X: \mathcal{M} \rightarrow TM$  with  $p \mapsto X(p)$ , we let

$$Xf: \mathcal{M} \rightarrow \mathbb{R}, \quad p \mapsto X(p)f.$$

This makes sense since we can't always do things globally, e.g., [Hairy ball theorem](#). Specifically, we can't choose a basis  $X_1, \dots, X_d \in \Gamma(TM)$  for our [vector field](#) globally as we already know. Similarly, we can define  $\Gamma(T^*\mathcal{M})$ , i.e., the set of "convector field"<sup>2</sup> is again a  $C^\infty(\mathcal{M})$ -[module](#).

**Example.** Given  $\omega \in \Gamma(T^*\mathcal{M})$  and  $X \in \Gamma(TM)$ ,  $\omega$  acts on  $X$  to yield smooth functions by point-wise evaluation, i.e., we define

$$(\omega(X))(p) := \omega(p)(X(p)).$$

Then, the action of  $\omega$  on  $X$  is a  $C^\infty(\mathcal{M})$ -linear map since

$$(\omega(fX))(p) = f(p)\omega(p)(X(p)) = (f\omega)(p)(X(p)) = (f\omega(X))(p)$$

for  $f \in C^\infty(\mathcal{M})$ . This suggests that we should not regard  $\omega$  just as a [section](#) of  $T^*\mathcal{M}$ , but also a linear mapping of  $X \in \Gamma(TM)$  into  $C^\infty(\mathcal{M})$ .

Then, in this view point, we have the following.

**Definition A.3.4** (Tensor field\*). A  $(r, s)$ -*tensor field*  $T$  on a [smooth manifold](#)  $\mathcal{M}$  is a  $C^\infty(\mathcal{M})$

<sup>1</sup>There are [vector bundles](#) which are not [tensor bundles](#).

<sup>2</sup>We won't define it formally, but it's defined similarly.

multilinear map

$$T: \underbrace{\Gamma(T^*\mathcal{M}) \times \cdots \times \Gamma(T^*\mathcal{M})}_r \times \underbrace{\Gamma(T\mathcal{M}) \times \cdots \times \Gamma(T\mathcal{M})}_s \rightarrow C^\infty(\mathcal{M}).$$

Comparing to Definition 2.4.13, this definition is more general!

**Example.** The linear connection  $\nabla (X, Y) \mapsto \nabla_X Y$  does not define a tensor field.

**Proof.** Since  $\nabla$  is only  $\mathbb{R}$ -linear in  $Y$ . \*

## A.4 Lie Groups and Lie Algebra

### A.4.1 Lie Groups

Lie groups are an important topic to study for Riemannian geometry, hence we now introduce it.

**Definition A.4.1 (Lie group).** A Lie group is a group  $G$  with a differentiable structure such that the mapping  $G \times G \rightarrow G$  given by  $(x, y) \mapsto xy^{-1}$ ,  $x, y \in G$ , is differentiable.

**Definition (Transformation).** Let  $G$  be a Lie group.

**Definition A.4.2 (Left transformation).** The translations from the left  $L_x: G \rightarrow G$  is defined as  $L_x(y) = xy$ .

**Definition A.4.3 (Right transformation).** The translations from the right  $R_x: G \rightarrow G$  is defined as  $R_x(y) = yx$ .

**Remark.** Both  $L_x$  and  $R_x$  are diffeomorphisms.

In the following discussion, let  $G$  be a Lie group. Turns out that  $G$  admits some nice properties on left invariant vector fields.

**Definition (Invariant of Riemannian metric).** Let  $g$  be a Riemannian metric on  $G$ .

**Definition A.4.4 (Left invariant).**  $g$  is left invariant if

$$\langle u, v \rangle_y = \langle d(L_x)_y u, d(L_x)_y v \rangle_{L_x(y)}$$

for all  $x, y \in G$ ,  $u, v \in T_y G$ , i.e.,  $L_x$  is an isometry.

**Definition A.4.5 (Right invariant).**  $g$  is right invariant if

$$\langle u, v \rangle_y = \langle d(R_x)_y u, d(R_x)_y v \rangle_{R_x(y)}$$

for all  $x, y \in G$ ,  $u, v \in T_y G$ , i.e.,  $R_x$  is an isometry.

**Definition A.4.6 (Bi-invariant).**  $g$  is bi-invariant if it's both right and left invariant.

**Definition (Invariant of vector field).** Let  $X$  be a vector field on  $G$ .

**Definition A.4.7** (Left invariant).  $X$  is *left invariant* if  $dL_x X = X$  for all  $x \in G$ .

**Definition A.4.8** (Right invariant).  $X$  is *right invariant* if  $dR_x X = X$  for all  $x \in G$ .

**Definition A.4.9** (Bi-invariant).  $X$  is *bi-invariant* if it's both *right* and *left invariant*.

As we mentioned, the *left invariant vector fields* are completely determined by their values at a single point of  $G$ , which allows us to introduce an additional structure on the *tangent space* to the neutral element  $e \in G$  in the following manner.

To each *vector*  $X_e \in T_e G$ , we associate the *left invariant*  $X$  defined by

$$X_a := dL_a X_e, \quad a \in G.$$

## A.4.2 Lie Algebras

Let  $X, Y$  be *left invariant vector fields* on  $G$ . Since for each  $x \in G$  and for any differentiable function  $f$  on  $G$ ,

$$dL_x[X, Y]f = [X, Y](f \circ L_x) = X(dL_x Y)f - Y(dL_x X)f = (XY - YX)f = [X, Y]f,$$

i.e.,  $[X, Y]$  is again a *left invariant vector field* if  $X, Y$  are. Now, if  $X_e, Y_e \in T_e G$ , we put  $[X_e, Y_e] = [X, Y]_e$ .

**Definition A.4.10** (Lie algebra). Given a *Lie group*  $G$ , the *Lie algebra*  $\mathfrak{g}$  is the vector space  $T_e G$  with the *bracket*  $[\cdot, \cdot]$ .

**Note.** The elements in the *Lie algebra*  $\mathfrak{g}$  will be thought of either as *vectors* in  $T_e G$  or as *left invariant vector fields* on  $G$ .

To introduce a *left invariant metric* on  $\mathfrak{g}$ , take any arbitrary inner product  $\langle \cdot, \cdot \rangle_e$  on  $\mathfrak{g}$  and define

$$\langle u, v \rangle_x := \langle (dL_{x^{-1}})_x(u), (dL_{x^{-1}})_x(v) \rangle_e \quad (\text{A.1})$$

for  $x \in G$ ,  $u, v \in T_x G$ . Since  $L_x$  depends differentiably on  $x$ , this is actually a *Riemannian metric*, which is clearly *left invariant*.

**Remark.** We can also construct a *right invariant metric* on  $G$ , and if  $G$  is compact,  $G$  possesses a *bi-invariant metric*.

One important characterization for  $G$  having a *bi-invariant metric* is that the inner product that the *metric* determines on  $\mathfrak{g}$  satisfies the following relation.

**Proposition A.4.1.** If  $G$  has a *bi-invariant metric*, then for any  $U, V, X \in \mathfrak{g}$ , the inner product that the *metric* determines on  $\mathfrak{g}$  satisfies

$$\langle [U, X], V \rangle = -\langle U, [V, X] \rangle.$$

**Proof.** See do Carmo [FC13, Page 40, 41]. ■

The important point about this relation is that it characterizes the *bi-invariant metrics* of  $G$  in the following sense.

**Remark.** If a positive bilinear form  $\langle \cdot, \cdot \rangle_e$  defined on  $\mathfrak{g}$  satisfies this relation, then the *Riemannian metrics* defined on  $G$  by Equation A.1 is *bi-invariant*.

## A.4.3 Lie Subalgebra

Consider  $(h_t^X)$  be a *local 1-parameter group* for a *vector field*  $X$ , and let  $\Gamma(TM)$  still denotes the set of all *vector fields*, but now view it as just an  $\mathbb{R}$ -vector space. Then, we revise Definition A.4.10 as follows.

**Definition A.4.11** (Lie algebra\*). Let  $\mathcal{M}$  be a smooth manifold, the  $(\Gamma(TM), [\cdot, \cdot])$  is the Lie algebra.

This induces the following.

**Definition A.4.12** (Lie subalgebra). Let  $X_1, \dots, X_n$  be  $n$  vector fields on  $\mathcal{M}$  such that for all  $i, j$ ,

$$[X_i, X_j] = C_{ij}^k X_k$$

for  $C_{ij}^k \in \mathbb{R}$ . Then,  $L := (\text{span}_{\mathbb{R}}(\{X_1, \dots, X_n\}), [\cdot, \cdot])$  is called a Lie subalgebra.

**Notation** (Structure constant).  $C_{ij}^k$  in Definition A.4.12 are called structure constants.

**Example.** On  $S^2$ , given  $[X_1, X_2] = X_3$ ,  $[X_2, X_3] = X_1$ ,  $[X_3, X_1] = X_2$ , we have

$$(\text{span}_{\mathbb{R}}(\{X_1, X_2, X_3\}), [\cdot, \cdot]) = \mathfrak{so}(3).$$

**Definition A.4.13** (Symmetry). A finite-dimensional Lie subalgebra  $(L, [\cdot, \cdot])$  is said to be a symmetry of a metric tensor field  $g$  if for every  $X \in L$  and  $t \in \mathbb{R}$ ,

$$g((h_t^X)_*(A), (h_t^X)_*(B)) = g(A, B).$$

This means that  $(h_t^X)_*$  defines an isometry.

**Note.** Or equivalently,  $(h_t^X)^*g = g$  where for  $\varphi: \mathcal{M} \rightarrow \mathcal{M}$ ,

$$(\varphi^*g)(X, Y) := g(\varphi_*(X), \varphi_*(Y)).$$

#### A.4.4 Lie Derivatives

Observe that for all  $X \in L$  with the corresponding local 1-parameter group  $(h_t^X)$ , if

$$\mathcal{L}_X := \lim_{t \rightarrow 0} \frac{(h_t^X)^*g - g}{t} = 0,$$

then  $L$  is a symmetry of  $g$ .

**Definition A.4.14** (Lie derivative). The Lie derivative  $\mathcal{L}$  on a smooth manifold  $\mathcal{M}$  sends a pair of a vector field  $X$  and a  $(p, q)$ -tensor field to a  $(p, q)$ -tensor field such that

- (a)  $\mathcal{L}_X f = Xf$ ;
- (b)  $\mathcal{L}_X Y = [X, Y]$ ;
- (c)  $\mathcal{L}_X (T + S) = \mathcal{L}_X T + \mathcal{L}_X S$ ;
- (d)  $\mathcal{L}_X (T(\omega, Y)) = (\mathcal{L}_X T)(\omega, Y) + T(\mathcal{L}_X \omega, Y) + T(\omega, \mathcal{L}_X Y)$ , similarly for any other valence of  $T$ ;
- (e)  $\mathcal{L}_{X+Y} T = \mathcal{L}_X T + \mathcal{L}_Y T$ .

**Remark.**  $\nabla_X$  is  $C^\infty(\mathcal{M})$ -linear in the lower slot, while  $\mathcal{L}_X$  is not.

**Intuition.** Study neighboring fibers using a local 1-parameter group of diffeomorphisms  $(\psi_t)_{t \in I}$ .

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