MATH635
Riemannian Geometry

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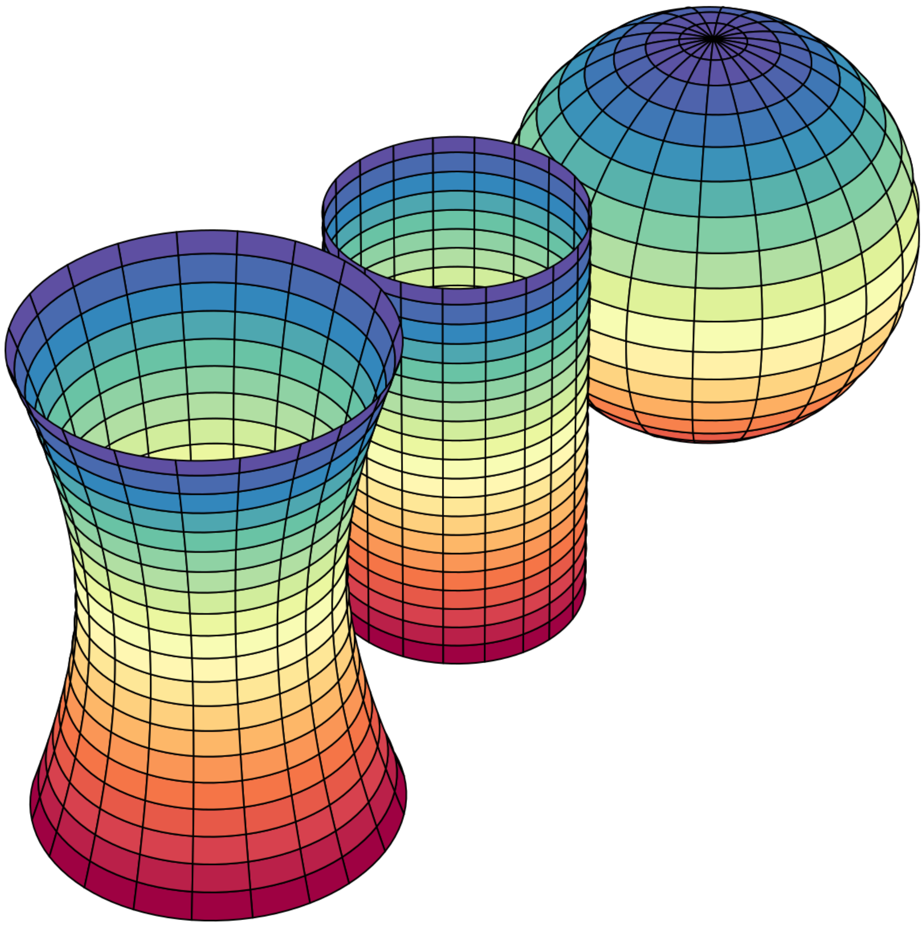
Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by Lydia Bieri. Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [FC13] as our reference; while not required, but highly recommended have on.



This course is taken in Winter 2023, and the date on the covering page is the last updated time.

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Chapter 1

Smooth Manifolds

Lecture 1: A Foray to Smooth Manifolds

1.1 Topological Manifolds

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Let's start with a common definition.

Definition 1.1.1 (Topological manifold). A *topological manifold* $\mathcal{M}$ of dimension n is a (topological) Hausdorff space such that each point $p \in \mathcal{M}$ has a neighborhood U homeomorphic via $\varphi\colon U \to U'$ to an open subset $U' \subseteq \mathbb{R}^n$.

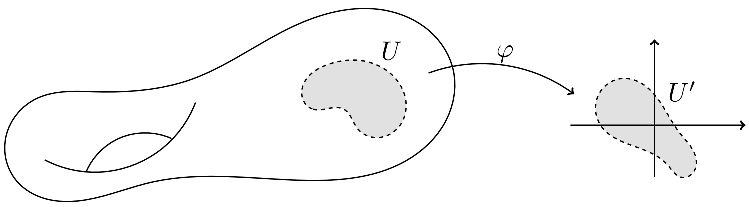
Definition 1.1.2 (Local coordinate map). For every $p \in \mathcal{M}$, the corresponding homeomorphism φ is called the *local coordinate map*.

Definition 1.1.3 (Local coordinate). The pull-back $(x^1, \dots, x^n)$ of the local coordinate map φ from $\mathbb{R}^n$ is called the *local coordinates* on U, given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

Definition 1.1.4 (Coordinate chart). The pair (U, φ) is called a *(coordinate) chart* on M.

In other words, a topological manifold can be thought of as a space such that it looks like $\mathbb{R}^n$ locally.



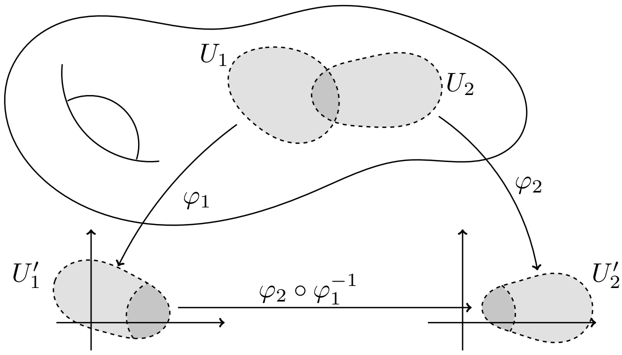
Definition 1.1.5 (Atlas). An *atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ for a manifold $\mathcal{M}$ is a collection of charts such that $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$ are an open covering of $\mathcal{M}$, i.e., $\mathcal{M} = \bigcup_\alpha U_\alpha$.

In other words, for all $p \in \mathcal{M}$, there exists a neighborhood $U \subseteq \mathcal{M}$ and homeomorphism $h\colon U \to U' \subseteq \mathbb{R}^n$ open.

Definition 1.1.6 (Locally finite). An atlas is said to be *locally finite* if each point $p \in \mathcal{M}$ is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as $\mathbb{R}^n$, there's no clear way to make sense of differentiability of a manifold. But thankfully, we now have an explicit relation to the ambient space $\mathbb{R}^n$ via φ_α. To formalize, let $\mathcal{A}$ be an atlas for a manifold $\mathcal{M}$, and assume that $(U_1, \varphi_1), (U_2, \varphi_2)$ are 2 elements of $\mathcal{A}$. Then clearly, the map $\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a homeomorphism between 2 open sets of Euclidean spaces since both φ_1 and φ_2 are homeomorphism. Due to this map's importance, it has its own name.

Definition 1.1.7 (Coordinate transition)**.** The map $\varphi_2 \circ \varphi_1^{-1}$ is called the *coordinate transition* of $\mathcal{A}$ for the pair of charts $(U_1, \varphi_1), (U_2, \varphi_2)$.



1.2 Differentiable Manifolds

Notice that the coordinate transitions are from $\mathbb{R}^n$ to $\mathbb{R}^n$; hence differentiability makes sense now, which induces the following.

Definition 1.2.1 (Differentiable atlas)**.** The atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is *differentiable* if all transitions are differentiable.

Remark. Here, the differentiability depends on the content. Sometimes, we may want it to be C^∞, and sometimes may be C^k for some finite k. On the other hand, smooth always refers to C^∞. We'll use them interchangeably if it's clear which case we're referring to.

Definition 1.2.2 (Equivalence atlas)**.** Two atlases $\mathcal{U}, \mathcal{V}$ of a manifold are equivalent if for every $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$,

$$\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

Notation (Smoothly compatible)**.** Two charts (U, φ) and (V, ψ) are *smoothly compatible* if either $U \cap V = \varnothing$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

This suggests the following.

Definition 1.2.3 (Smooth structure)**.** A *smooth structure* on $\mathcal{M}$ is an equivalence class $\mathcal{U}$ of coordinate atlas with the property that all transition functions are diffeomorphisms.

Remark. We can also use the *maximal* differentiable atlas to be our differentiable structure.

Definition 1.2.4 (Smooth manifold)**.** A *smooth manifold* is a manifold $\mathcal{M}$ with a smooth structure.

In this way, we can do calculus on smooth manifolds! Furthermore, it now makes sense to say that a function $f \colon \mathcal{M} \to \mathbb{R}$ is differentiable (or C^∞) by considering differentiability of $f \circ \varphi^{-1}$ around p.

Notation. The collection of smooth functions on smooth manifold $\mathcal{M}$ is denoted by $C^\infty(\mathcal{M}, \mathbb{R})$, or $C^k(\mathcal{M}, \mathbb{R})$.

Remark. The class $C^\infty(\mathcal{M}, \mathbb{R})$ consists of functions with property is well-defined.

Proof. Let $\mathcal{A}$ be any given atlas from equivalence class that defines the smooth structure, and as we have shown, if $(U, \varphi) \in \mathcal{A}$, then $f \circ \varphi^{-1}$ is smooth on $\mathbb{R}^n$. This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.2.2 requirement that defines the equivalent manifolds. ⊛

1.2.1 Orientation

Another essential property of a manifold is its orientability.

Definition. Consider an atlas $\mathcal{A}$ for a differentiable manifold $\mathcal{M}$.

Definition 1.2.5 (Oriented). $\mathcal{A}$ is *oriented* if all transitions have positive functional determinant.

Definition 1.2.6 (Orientable). $\mathcal{M}$ is *orientable* if $\mathcal{A}$ is an oriented atlas.

Motivated by the above definitions, we see that we can actually use an atlas to define an orientation.

Definition 1.2.7 (Orientation). Let $\mathcal{M}$ be an orientable manifold. Then a oriented differentiable structure is called an *orientation* of $\mathcal{M}$.

If $\mathcal{M}$ possesses an orientation, we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with Definition 1.2.5.

Remark. Two differentiable structures obeying Definition 1.2.5 determine the same orientation if the union again satisfying Definition 1.2.5.

Remark. If $\mathcal{M}$ is orientable and connected, then there exists exactly 2 distinct orientations on $\mathcal{M}$.

Now, we can see some examples of smooth manifolds.

Example (Sphere). The sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by

$$S^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\right\}.$$

Consider $U_i^+ = \{x \in S^n \mid x_i > 0\}$, $U_i^- = \{x \in S^n \mid x_i < 0\}$ for $i = 1, \ldots, n+1$, and $h_i^\pm \colon U_i^\pm \to \mathbb{R}^n$ such that

$$h_i^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, .., x_{n+1}).$$

Note that the minimum charts needed to cover S^n is 2.

Example. Let $\mathcal{M} = U \subseteq \mathbb{R}^n$, then $\{(U, \varphi)\}$ is a smooth structure with $\varphi = \mathbb{1}$.

Example. Open sets of C^∞-manifolds are C^∞-manifolds.

Example (General linear group). $\mathrm{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, open.

Example (Real projective space). $\mathbb{R}P^n = S^n / \sim$ where $x \sim -x$ with $\pi \colon S^n \to \mathbb{R}P^n$, $x \mapsto [x]$.

Proof. π is a homeomorphism on each U_i^+ for $i = 1, \ldots, n+1$, with

$$\left\{ \left(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1} \right), i = 1, \ldots, n+1 \right\}$$

is a C^∞-atlas for $\mathbb{R}P^n$. ⊛

Note. Observe that $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} \ / \ \sim$.

Lecture 2: Maps Between Smooth Manifolds

1.2.2 Smooth Maps

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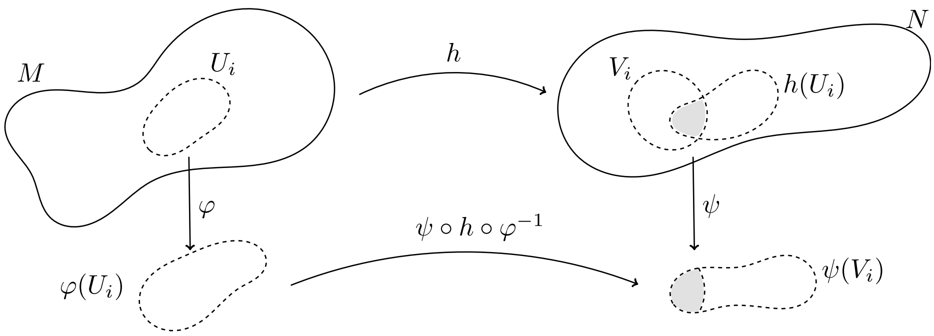
We can now consider the maps between manifolds, specifically, the smooth manifolds.

Definition 1.2.8 (Smooth function). Let M, N be two smooth manifolds, and let $\mathcal{U}$ be locally finite atlas from the equivalence class that gives the smooth structure on M, and let $\mathcal{V}$ be the corresponding for N. A map $h\colon M \to N$ is said to be *smooth* if each map in the collection

$$\left\{ \psi \circ h \circ \varphi^{-1} \colon h(U) \cap V \neq \varnothing \right\},$$

where $(U, \varphi) \in \mathcal{U}$, $(V, \psi) \in \mathcal{V}$ is C^∞-differentiable as a map from one Euclidean space to another.



Remark. Equivalence relation guarantees that Definition 1.2.8 depends only on the smooth structure of M, N, but not on the chosen representative coordinate atlas.

Definition. Consider two smooth manifolds M, N and a smooth homeomorphism $h\colon M \to N$ with smooth inverse.

Definition 1.2.9 (Diffeomorphic). The two manifolds M, N are said to be *diffeomorphic*.

Definition 1.2.10 (Diffeomorphism). The map h is said to be a *diffeomorphism*.

Let M_1, M_2 be two smooth manifolds, and let $\varphi\colon M_1 \to M_2$ be a diffeomorphism. Then the following hold. Check

(a) M_1 is orientable if and only if M_2 is orientable.

(b) If in addition, M_1 and M_2 are both connected and oriented, then φ induces an orientation on M_2 that may or may not coincide with the initial orientation of M_2.

If the induced orientation coincides, then we say φ preserves the orientation, otherwise φ reverses the orientation.

1.2.3 Grassmannian Manifold

Before proceeding, let's consider an interesting smooth manifold.

Definition 1.2.11 (Grassmannian manifold)**.** Given $m, n \in \mathbb{N}$, the so-called *Grassmannian manifold* $G(n, m)$ is the set of all n-dimensional subspaces of $\mathbb{R}^{n+m}$.

Note. $G(1, m)$ is just $\mathbb{R}P^m$, and $G(0, m)$, $G(n, 0)$ are one-point sets.

As we will soon see, $G(n, m)$ has the smooth structure of an mn-dimensional manifold.

Intuition. We obtain the structure by exhibiting an atlas whose transitions are diffeomorphisms.

Firstly, we give $G(n, m)$ a suitable topology, i.e., the metric topology. Let $\Pi \in G(n, m)$, and let $\mathcal{L}(\Pi, \Pi^\perp)$ denote the mn-dimensional space of linear maps from Π to $\Pi^\perp$. Define the map

$$\varphi_\Pi \colon \mathcal{L}(\Pi, \Pi^\perp) \to G(n, m), \qquad \varphi_\Pi(\alpha) = (\mathbb{1}_\Pi \oplus \alpha)(\Pi)$$

where $\mathbb{1}_\Pi \oplus \alpha$ is regarded as a map $\Pi \to \Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.[1] Clearly, φ_Π is injective, and thus, $(\mathcal{L}(\Pi, \Pi^\perp), \varphi_\Pi)$ is an mn-dimensional chart of $G(n, m)$.

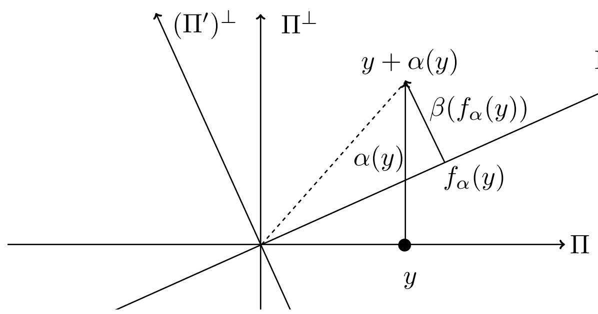
Remark. The images $\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$ cover $G(n, m)$.

Example. $\Pi = \varphi_\Pi(0) \in \varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$.

We can now prove that these charts are mutually compatible. Let $\Pi, \Pi' \in G(n, m)$, and let P, P' be orthogonal projections from $\mathbb{R}^{n+m}$ onto Π, Π' respectively. Firstly,

$$F = \varphi_{\Pi'}^{-1}\varphi_\Pi \colon \varphi_\Pi^{-1}\left(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))\right) \to \varphi_{\Pi'}^{-1}\left(\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))\right)$$

is smooth.



Consider $\alpha \in \mathcal{L}(\Pi, \Pi^\perp)$, and $\beta \in \mathcal{L}(\Pi', (\Pi')^\perp)$, then for α, β, the equality $F(\alpha) = \beta$ means that $\varphi_\Pi(\alpha) = \varphi_{\Pi'}(\beta)$. Let $f_\alpha \colon \Pi \to \Pi'$ be defined by

$$f_\alpha = P' \circ (\mathbb{1}_\Pi \oplus \alpha).$$

We need to check

(a) f_α is invertible, and

(b) $\forall y \in \Pi$, $y + \alpha(y) = f_\alpha(y) + \beta(f_\alpha(y))$.

Note. The condition that $\det f_\alpha \neq 0$ gives an exact description of the subset

$$\varphi_{\Pi^{-1}}\left(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))\right)$$

[1] In other words, $\varphi_\Pi(\alpha)$ is the graph of α in $\Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$.

of $\mathcal{L}(\Pi, \Pi^\perp)$, which is therefore open.

For β, it is $\left(\mathbb{1}_{\Pi'} \oplus \beta\right) \circ f_\alpha = \mathbb{1}_\Pi \oplus \alpha$, and hence

$$\beta = F(\alpha) = (\mathbb{1}_\Pi \oplus \alpha) \circ f_\alpha^{-1} - \mathbb{1}_{\Pi'}.$$

It follows by the construction that the image of β is contained in $(\Pi')^\perp$.

Remark. We obtain an infinite atlas for $G(n, m)$ with charts labeled by $\Pi \in G(n, m)$. But it's suffices to consider only $\binom{n+m}{n}$ charts corresponding to subspaces Π spanned with n coordinate axes.

We now introduce two notions.

Definition 1.2.12 (Closed manifold)**.** A manifold is *closed* if it is compact and without boundary.

Definition 1.2.13 (Open manifold)**.** A manifold is *open* if it has only non-compact components without boundary.

Lemma 1.2.1. If M can be covered by two coordinate neighborhoods V_1, V_2 such that $V_1 \cap V_2$ is connected, then M is orientable.

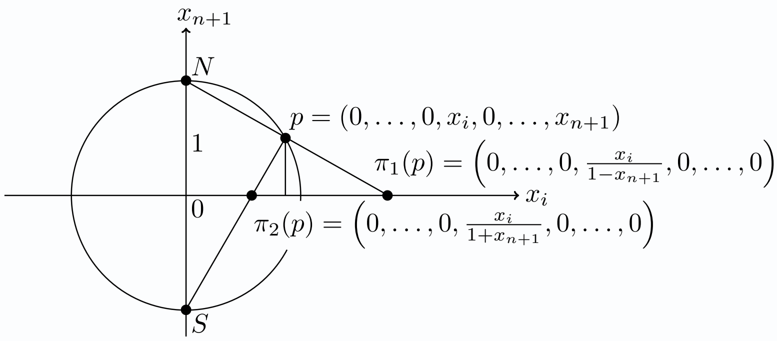
Proof. The determinant of the differential of the coordinate change $\neq 0$, so it does not change sign in $V_1 \cap V_2$. If it's negative at a single point, it's enough to change the sign of one of the coordinates to make it positive at that point, hence on $V_1 \cap V_2$. ■

Example. Let $S^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\right\} \subseteq \mathbb{R}^{n+1}$ is orientable.

Proof. Let $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$, consider given $p = (0, \dots, 0, x_i, 0, \dots, x_{n+1})$, then $\pi_1 \colon S^n \setminus \{N\} \to \mathbb{R}^n$ given by

$$\pi_1(p) = \left(0, \dots, 0, \frac{x_i}{1 - x_{n+1}}, 0, \dots, 0\right)$$

to be the stereographic projection from the north pole N.



More generally, it takes $p(x_1, \dots, x_{n+1}) \in S^n - \{N\}$ into the intersection at the hyperplane $x_{n+1} = 0$ with the line passing through p ad N. In this way, we have

$$\pi_1(x_1, \dots, x_n) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right),$$

hence $\pi_1 \colon S^n \setminus \{N\} \to \mathbb{R}^n$ is differentiable, and is injective. Similarly, $\pi_2 \colon S^n \setminus \{S\} \to \mathbb{R}^n$ for S can also be defined and everything holds similarly. We see that these two parametrizations

$(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})$ cover S^n. The change of coordinate is given by

$$y_j = \frac{x_j}{1 - x_{n+1}} \leftrightarrow y_j' = \frac{x_j}{1 + x_{n+1}},\ (y_1, \ldots, y_n) \in \mathbb{R}^n,\ j = 1, \ldots, n,$$

where

$$y_j' = \frac{y_j}{\sum_{i=1}^n y_i^2}.$$

This implies that $\left\{(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})\right\}$ is a differentiable structure for S^n. Now, consider $\pi_1^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^n) = S^n \setminus \{N \cup S\}$, which is connected, and hence S^n is orientable, and the above structure gives an orientation of S^n. ⊛

Lecture 3: Complex Manifolds, Tangent Spaces and Bundles

Let's look at two more examples about orientation. 12 Jan. 14:30

Example. Let $A\colon S^n \to S^n$ be the antipodal map given by $A(p) = -p$ for $p \in \mathbb{R}^{n+1}$. It's easy to see that A is differentiable with $A^2 = \mathbb{1}$. Furthermore, A is diffeomorphism of $S^n \subseteq \mathbb{R}^{n+1}$. We see that

- if n is even, A reverses the orientation;
- if n is odd, A preserves the orientation.

Example. $G(k, n)$ is orientable if and only if n is even or $n = 1$.

Finally, we introduce the notion of complex manifolds.

Definition 1.2.14 (Complex manifold). A *complex manifold* $\mathcal{M}$ of complex dimension d ($\dim_{\mathbb{C}} \mathcal{M} = d$) is a differentiable manifold of (real) dimension $2d$ ($\dim_{\mathbb{R}} \mathcal{M} = 2d$) whose charts take values in open subsets of $\mathbb{C}^d$ with holomorphic chart transitions.

As previously seen. The chart transitions $z_\beta \circ z_\alpha^{-1}\colon z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ is holomorphic if $\partial z_\beta^j / \partial \overline{z_\alpha^k} = 0$ for all j, k where

$$\frac{\partial}{\partial \overline{z^k}} = \frac{1}{2}\left(\frac{\partial}{\partial \overline{x^k}} + i \frac{\partial}{\partial \overline{y^k}}\right).$$

Remark. Complex Grassmannians $G_{\mathbb{C}}(k, n)$ are all orientable. More generally, complex manifolds are always orientable because holomorphic maps always have positive functional determinant.

1.3 Partition of Unity

We state, without proof, of an important lemma about the partition of unity.

Definition 1.3.1 (Partition of unity). Let $\mathcal{M}$ be a differentiable manifold, and let $(U_\alpha)_{\alpha \in \mathcal{A}}$ be an open covering of $\mathcal{M}$. Then a *partition of unity* is a locally finite refinement $(V_\beta)_{\beta \in \mathcal{B}}$ of (U_α) and C^∞-functions $\varphi_\beta\colon \mathcal{M} \to \mathbb{R}$ with

(a) $\operatorname{supp}(\varphi_\beta) \subseteq V_\beta$ for all $\beta \in \mathcal{B}$;

(b) $0 \le \varphi_\beta(x) \le 1$ for all $x \in \mathcal{M}$, $\beta \in \mathcal{B}$;

(c) $\sum_{\beta \in \mathcal{B}} \varphi_\beta = 1$ for all $x \in \mathcal{M}$.[a]

[a] There are only finitely many non-vanishing summands of each point, since only finitely many φ_β are non-zero of any given point as the covering (V_β) is locally finite.

Lemma 1.3.1 (Partition of unity)**.** Let $\mathcal{M}$ be a differentiable manifold, and let $(U_\alpha)_{\alpha\in\mathcal{A}}$ be an open covering of $\mathcal{M}$. Then there exists a partition of unity subordinate to (U_α),

1.4 Tangent and Cotangent Spaces

1.4.1 Tangent Spaces in Euclidean Spaces

To discuss the concept of calculus between manifolds formally, we start with our discussion in Euclidean spaces, where we naturally have the coordinates for every point.

Definition. Let $\mathcal{M}$ be a Euclidean manifold of dimension d, $x = (x^1, \dots, x^d)$ be Euclidean coordinates of $\mathbb{R}^d$, and $x_0 \in \Omega \subseteq \mathbb{R}^d$ where Ω is open.

Definition 1.4.1 (Tangent space of Euclidean space)**.** The *tangent space* $T_{x_0}\Omega$ *of* Ω *at* x_0 is the vector space $\{x_0\} \times E$[a] spanned by the basis $(\partial/\partial x^1, \dots, \partial/\partial x^d)$.

[a] E is a d-dimensional Euclidean space.

Definition 1.4.2 (Tangent vector of Euclidean space)**.** The elements in the tangent space of Euclidean spaces is called *tangent vectors*.

Before proceeding, we introduce a shorthand notation.

Notation (Einstein notation)**.** The *Einstein notation* abbreviates the summation $\sum_i v^i x_i$ as $v^i x_i$, where we implicitly sum over the upper and lower index.

Definition 1.4.3 (Differential of Euclidean space)**.** If $\Omega \subseteq \mathbb{R}^d$, $\Omega' \subseteq \mathbb{R}^d$ are open, and $f\colon \Omega \to \Omega'$ is differentiable, then the *differential* $\mathrm{d}f(x_0)$ for $x_0 \in \Omega$ is the induced linear map between tangent spaces

$$\mathrm{d}f(x_0)\colon T_{x_0}\Omega \to T_{f(x_0)}\Omega', \quad v = v^i \frac{\partial}{\partial x^i} \mapsto v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}.$$

Definition 1.4.4 (Tangent bundle of Euclidean space)**.** The *tangent bundle* is defined as $T\Omega \coloneqq \bigsqcup_{x\in\Omega} T_x\Omega \cong \Omega \times E \cong \Omega \times \mathbb{R}^d$, which is an open subset of $\mathbb{R}^d \times \mathbb{R}^d$.

Note (Total space)**.** $T\Omega$ is also called the *total space*.

Remark. Given a tangent bundle $T\Omega$, we define π to be the projection $\pi\colon T\Omega \to \Omega$ given by $\pi(x, v) = x$. This makes $T\Omega$ naturally a differentiable manifold.

With the notion of tangent bundle, given $f\colon \Omega \to \Omega'$, we can also define $\mathrm{d}f\colon T\Omega \to T\Omega'$ as

$$\left(x, v^i \frac{\partial}{\partial x^i}\right) \mapsto \left(f(x), v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}\right).$$

Notation. We often write $\mathrm{d}f(x)(v)$ instead of $\mathrm{d}f(x, v)$ to coincide with the notation of differential.

In particular, for $v = v^i \partial/\partial x^i$, we have

$$\mathrm{d}f(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R},$$

and we write $v(f)(x)$ for $\mathrm{d}f(x)(v)$.

1.4.2 Tangent Spaces in Manifolds

We now try to formally define the tangent space on a smooth manifold. A natural idea is the following.

> **Intuition.** Let $\mathcal{M}^d$ be a differentiable manifold with a chart $x\colon U \to \Omega \subseteq \mathbb{R}^d$ and $p \in U \subseteq \mathcal{M}$ where U is open. The *tangent space* $T_p\mathcal{M}$ of $\mathcal{M}$ at p *should be* represented in the chart x by $T_{x(p)}x(U)$.

To see that the above are well-defined, i.e., $T_p\mathcal{M}$ are independent of the choice of charts, let $x'\colon U' \to \mathbb{R}^d$ to be another chart with $p \in U' \subseteq \mathcal{M}$ where U' is also open. Denote $\Omega \coloneqq x(U)$, and $\Omega' \coloneqq x'(U')$, then the transition map

$$x' \circ x^{-1}\colon x(U \cap U') \to x'(U \cap U')$$

induces a vector space isomorphism

$$L \coloneqq \mathrm{d}(x' \circ x^{-1})(x(p))\colon T_{x(p)}\Omega \to T_{x'(p)}\Omega',$$

such that $v \in T_{x(p)}\Omega$ and $L(v) \in T_{x'(p)}\Omega'$ represent the same tangent vector in $T_p\mathcal{M}$.

> **Remark.** A tangent vector in $T_p\mathcal{M}$ is given by the family of the coordinate representations.

Now, we want to define the similar notion of differential of Euclidean spaces. Let consider a simple case first, where we let $f\colon \mathcal{M} \to \mathbb{R}$ to be a differentiable function, and assume that the tangent vector $w \in T_p\mathcal{M}$ is represented by $v \in T_{x(p)}x(U)$.

> **Intuition.** We want to define $\mathrm{d}f(p)$ as a linear map from $T_p\mathcal{M} \to \mathbb{R}$. In chart x, let $w \in T_p\mathcal{M}$ be given as $v = v^i\partial/\partial x^i \in T_{x(p)}x(U)$. Say that $\mathrm{d}f(p)(w)$ in this chart represented by
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> $$\mathrm{d}(f \circ x^{-1})(x(p))(v).$$
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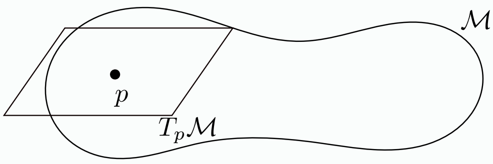
> 
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> **Remark.** $T_p\mathcal{M}$ is a vector space of dimension d isomorphic to $\mathbb{R}^d$, where the isomorphism depends on choice of chart.

> **Intuition.** Pull functions on $\mathcal{M}$ back by a chart to an open subset of $\mathbb{R}^d$, differentiate there.

In order to obtain a tangent space which does not depend on charts, we need to have transformation behavior under change of charts. Let $F\colon \mathcal{M}^d \to \mathcal{N}^c$ be a differentiable map where $\mathcal{M}, \mathcal{N}$ are smooth manifolds. Then we want to represent $\mathrm{d}F$ in local charts $x\colon U \subseteq \mathcal{M} \to \mathbb{R}^d, y\colon V \subseteq \mathcal{N} \to \mathbb{R}^c$ by $\mathrm{d}(y \circ F \circ x^{-1})$. The local coordinates on U is given by $(x^1, \dots, x^d)$, and on V is $(F^1, \dots, F^c)$ such that

$$F(x) = \left(F^1(x^1, \dots, x^d), \dots, F^c(x^1, \dots, x^d)\right).$$

Then, $\mathrm{d}F$ induces a linear map $\mathrm{d}F\colon T_p\mathcal{M} \to T_{F(x)}\mathcal{N}$ which in our coordinate representation is given by the matrix

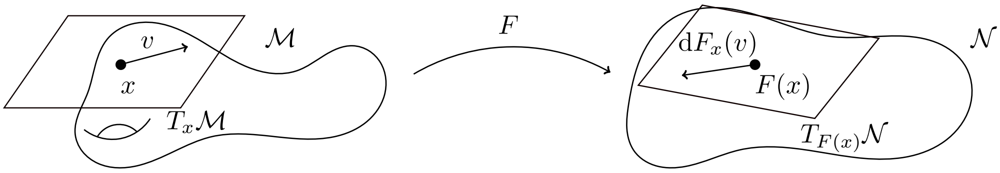
$$\left(\frac{\partial F^\alpha}{\partial x^i}\right)_{\substack{\alpha=1,\dots,c\\ i=1,\dots,d}},$$

and a change of charts is then just the base change at tangent spaces: if

$$\begin{aligned}(x^1, \dots, x^d) &\mapsto (\xi^1, \dots, \xi^d)\\ (F^1, \dots, F^c) &\mapsto (\phi^1, \dots, \phi^c)\end{aligned}$$

are coordinate changes, then $\mathrm{d}F$ represented in the new coordinates is given by

$$\left(\frac{\partial \phi^\beta}{\partial \xi^j}\right) = \left(\frac{\partial \phi^\beta}{\partial F^\alpha}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial x^i}{\partial \xi^j}\right).$$



Lecture 4: Tangent Bundles, Vector Fields, and Submanifolds

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Definition. Let $\mathcal{M}^d$ be a differentiable manifold with a chart $x\colon U \to \Omega \subseteq \mathbb{R}^d$ and $p \in U \subseteq \mathcal{M}$ where U is open. On $\left\{(x, v) \mid v \in T_{x(p)}\Omega\right\}$, we define an equivalence relation by $(x, v) \sim (y, w)$ if and only if $w = \mathrm{d}(y \circ x^{-1})v$.

Definition 1.4.5 (Tangent space)**.** The space of equivalence classes is called the *tangent space* $T_p\mathcal{M}$ at point p to $\mathcal{M}$.

Definition 1.4.6 (Tangent vector)**.** The elements in the tangent space is called *tangent vectors*.

Remark. $T_p\mathcal{M}$ naturally caries the structure of a vector space.

Now, $T\mathcal{M}$ is defined as

$$T\mathcal{M} \coloneqq \coprod_{p\in\mathcal{M}} T_p\mathcal{M}.$$

Recall the projection $\pi\colon T\mathcal{M} \to \mathcal{M}$ with $\pi(w) = p$ for $w \in T_p\mathcal{M}$. Then we can define the following.

Definition 1.4.7 (Derivation)**.** If $x\colon U \to \mathbb{R}^d$ be a chart for $\mathcal{M}$, and let $TU = \coprod_{p\in U} T_pU$. Then we define the *derivation* $\mathrm{d}x\colon TU \to Tx(U) \coloneqq \coprod_{p\in x(U)} T_p\mathcal{M}$ by $w \mapsto \mathrm{d}x(\pi(w))(w) \in T_{x(\pi(w))}x(U)$.

The transition maps

$$\mathrm{d}x' \circ (\mathrm{d}x)^{-1} = \mathrm{d}(x' \circ x^{-1})$$

are differentiable. π is local represented by $x \circ \pi \circ \mathrm{d}x^{-1}$ maps $(x_0, v) \in Tx(U)$ to x_0.

Definition 1.4.8 (Tangent bundle)**.** The triple $(T\mathcal{M}, \pi, \mathcal{M})$ is called the *tangent bundle* of $\mathcal{M}$.

Definition 1.4.9 (Total space)**.** $T\mathcal{M}$ is called the *total space* of the tangent bundle.

We can choose the courses (the initial) topology for total space $T\mathcal{M}$ such that π is continuous. Furthermore, we can construct a C^∞-atlas $\mathcal{A}_{T\mathcal{M}}$ on $T\mathcal{M}$ from the C^∞-atlas $\mathcal{A}$ of $\mathcal{M}$. Specifically, consider $\mathcal{A}_{T\mathcal{M}} \coloneqq \{(TU, \xi_x) \mid (U, x) \in \mathcal{A}\}$ where $\xi_x\colon TU \to \mathbb{R}^{2\cdot d}$ such that

$$x \mapsto \left((x^1 \circ \pi)(x), \ldots, (x^d \circ \pi)(x), (\mathrm{d}x^1)_{\pi(x)}(X), \ldots, (\mathrm{d}x^d)_{\pi(x)}(X)\right).$$

Intuition. We know that $X = X^i_x(\partial/\partial x^i)_{\pi(x)}$, and we might tempt to write X^i as the last d components. But we write it in the above way is because

$$(\mathrm{d}x^j)_{\pi(x)}(X) = (\mathrm{d}x^j)_{\pi(x)}\left(X^i_x\left(\frac{\partial}{\partial x^i}\right)_{\pi(x)}\right) = X^i_x\delta^j_i = X^j_x.$$

Note. We can check that ξ_x^{-1} exists, and it's also smooth, hence $T\mathcal{M}$ has a natural topology and a C^∞-atlas making it a $2\dim\mathcal{M}$-dimensional smooth manifold.

1.4.3 Cotangent Spaces

Another important objects is the cotangent spaces.

Definition. Let $\mathcal{M}^d$ be a differentiable manifold, and $T_p\mathcal{M}$ be the tangent space at p to $\mathcal{M}$.

Definition 1.4.10 (Cotangent space)**.** The *cotangent space* $T_p^*\mathcal{M}$ to $\mathcal{M}$ is the dual of $T_p\mathcal{M}$, i.e., $T_p^*\mathcal{M} = (T_p\mathcal{M})^*$.

Definition 1.4.11 (Cotangent vector)**.** The elements in the cotangent space is called *cotangent vectors.*

Remark. $T_p^*\mathcal{M}$ is the space of 1-forms on $T_p\mathcal{M}$.

Notation (Covariant vector)**.** The cotangent vectors are also called *covariant vectors.*

Notation (Contravariant vector)**.** The tangent vectors are also called *contravariant vectors.*

1.5 Vector Fields and Brackets

1.5.1 Vector Fields

We now introduce the notion of vector field.

Definition 1.5.1 (Vector field)**.** A *(tangent) vector field* X on a differentiable manifold $\mathcal{M}$ is a correspondence associating to each point $p \in \mathcal{M}$ a vector $X(p) \in T_p\mathcal{M}$, i.e., $X\colon \mathcal{M} \to T\mathcal{M}$.

Remark. Naturally, we say that the field X is differentiable if the map X is differentiable.

Considering a local chart $x\colon U \subseteq \mathbb{R}^n \to \mathcal{M}$, we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i},$$

where $a_i\colon U \to \mathbb{R}$ are functions on U for $i = 1, \ldots, n$, and $\{\partial/\partial x_i\}_i$ is the basis associated to x.

Remark. X is differentiable if and only if a_i are differentiable for some (and, therefore, for any) x.

It's convenient to think of a vector field as a mapping $X\colon \mathcal{D} \to \mathcal{F}$ from the set $\mathcal{D}$ of differentiable functions on $\mathcal{M}$ to the set $\mathcal{F}$ of the functions on $\mathcal{M}$, defined by

$$(Xf)(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p),$$

where f is implicitly denoting the expression of f in the chart x.

Intuition. This idea of a vector as a directional derivative is precisely what was used to define the notion of tangent vector.

Remark. Xf does not depend on the choice of x.

Remark. X is differentiable if and only if $X\colon \mathcal{D} \to \mathcal{D}$, i.e., $Xf \in \mathcal{D}$ for all $f \in \mathcal{D}$.

Observe that if $\varphi\colon \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, $v \in T_p\mathcal{M}$ and f differentiable function in a neighborhood of $\varphi(p)$, we have

$$(\mathrm{d}\varphi(v)f)\varphi(p) = v(f \circ \varphi)(p)$$

since by letting $\alpha\colon (-\epsilon, \epsilon) \to \mathcal{M}$ be a differentiable curve with $\alpha'(0) = v$, $\alpha(0) = p$,[2] then

$$(\mathrm{d}\varphi(v)f)\varphi(p) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi \circ \alpha)\Big|_{t=0} = v(f \circ \varphi)(p).$$

1.5.2 Brackets

By viewing X as an operator on $\mathcal{D}$, we can consider the iterates of X, i.e, given differentiable fields X and Y and $f\colon M \to \mathbb{R}$ being a differentiable function, consider $X(Yf)$ and $Y(Xf)$.

Note. In general, $X(Yf)$ (and hence $Y(Xf)$) is not a field.

Proof. It involves derivatives of order higher than one. ⊛

But we have the following.

Lemma 1.5.1. Let X, Y be differentiable vector fields on a smooth manifold $\mathcal{M}$. Then there exists a unique vector field Z such that for all $f \in \mathcal{D}$, $Zf = (XY - YX)f$.

Proof. See do Carmo [FC13, Chapter 0, Lemma 5.2]. ■

This Z is called the bracket.

Definition 1.5.2 (Bracket)**.** Given two differentiable vector fields X, Y on a smooth manifold $\mathcal{M}$, the *bracket* of X and Y is defined by

$$[X, Y] \coloneqq XY - YX.$$

Clearly, $[X, Y]$ is differentiable.

Proposition 1.5.1. If X, Y and Z are differentiable vector fields on $\mathcal{M}$, $a, b \in \mathbb{R}$, f, g are differentiable functions, then we have the following.

(a) $[X, Y] = -[Y, X]$ (*anti-commutativity*),

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (*linearity*),

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (*Jacobi identity*),

(d) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

Proof. See do Cargo [FC13, Chapter 0, Proposition 5.3]. ■

Example. $[\partial/\partial x^i, \partial/\partial x^j] = 0$ for $i = j$.

1.6 Submanifolds, Immersions, and Embeddings

We now study the relation between manifolds.

Definition 1.6.1 (Immersion)**.** Let $\mathcal{M}^m, \mathcal{N}^n$ be smooth manifolds. A differentiable mapping $\varphi\colon \mathcal{M} \to \mathcal{N}$ is an *immersion* if

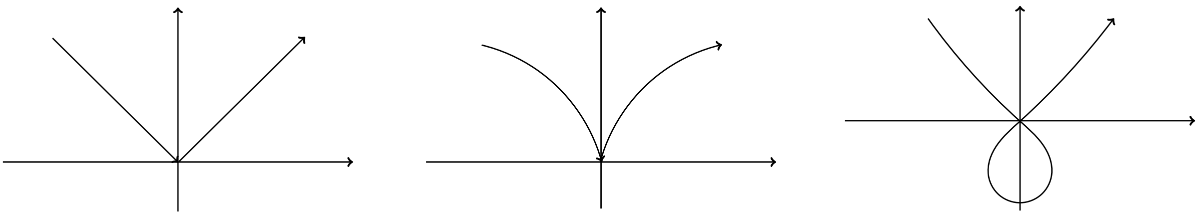
$$\mathrm{d}\varphi_p\colon T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$$

[2] This is the way do Carmo [FC13] used to define tangent vectors.

is injective for every $p \in \mathcal{M}$.

Definition 1.6.2 (Embedding)**.** An immersion $\varphi\colon \mathcal{M} \to \mathcal{N}$ is an *embedding* if it is also a homeomorphism onto $\varphi(\mathcal{M}) \subseteq \mathcal{N}$, with $\varphi(\mathcal{M})$ having the subspace topology induced from $\mathcal{N}$.

Definition 1.6.3 (Submanifold)**.** If the inclusion $\iota\colon \mathcal{M} \hookrightarrow \mathcal{N}$ between two manifolds is an embedding, then $\mathcal{M}$ is a *submanifold* of $\mathcal{N}$.



(a) Non-differentiable curve. (b) Non-immersion curve. (c) Non-embedding curve.

Figure 1.1: Three simple examples

Lemma 1.6.1. Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ to be an immersion and $x \in \mathcal{M}$.[a] Then there exists a neighborhood U of x and a chart (V, y) on $\mathcal{N}$ with $f(x) \in V$ such that $f|_U$ is a differentiable embedding and $y^{m+1}(p) = \ldots = y^n(p) = 0$ for all $p \in f(U \cap V)$.

[a]Hence, $n \geq m$.

Proof. In the local coordinates $(z^1, \ldots, z^n)$ on $\mathcal{N}$, and $(x^1, \ldots, x^m)$ on $\mathcal{M}$, without loss of generality,[a] let

$$\left(\frac{\partial z^\alpha(f(x))}{\partial x^i}\right)_{i,\alpha=1,\ldots,m}$$

be non-singular. Consider

$$F(z, x) \coloneqq \left(z^1 - f^1(x), \ldots, z^n - f^n(x)\right),$$

which has maximal rank in $x^1, \ldots, x^m, z^{m+1}, \ldots, z^n$. By the implicit function theorem, locally, there exists a map $\varphi\colon U \to \mathbb{R}^n$ such that

$$(z^1, \ldots, z^m) \mapsto (\varphi^1(z^1, \ldots, z^m), \ldots, \varphi^n(z^1, \ldots, z^m)) = x$$

such that $F(z, x) = 0$, i.e.,

$$\varphi^i(z^1, \ldots, z^m) = \begin{cases} x^i, & \text{if } i = 1, \ldots, m; \\ z^i, & \text{if } i = m+1, \ldots, n, \end{cases}$$

for which

$$\left(\frac{\partial \varphi^i}{\partial z^\alpha}\right)_{\alpha,i=1,\ldots,m}$$

has maximal rank. Now, we choose a new coordinate

$$\begin{aligned}(y^1, \ldots, y^n) = \big(&\varphi^1(z^1, \ldots, z^m), \ldots, \varphi^m(z^1, \ldots, z^m), \\ &z^{m+1} - \varphi^{m+1}(z^1, \ldots, z^m), \ldots, z^n - \varphi^n(z^1, \ldots, z^m)\big).\end{aligned}$$

Then, we have $z = f(x) \Leftrightarrow F(z, x) = 0$, i.e., $(y^1, \ldots, y^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$, proving the result. ∎

[a]Since $\mathrm{d}f(x)$ is injective.

Lemma 1.6.2. Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ be a differentiable map such that $m \geq n$ with $p \in \mathcal{N}$. Let $\mathrm{d}f(x)$ has rank n for all $x \in \mathcal{M}$ with $f(x) = p$. Then $f^{-1}(p)$ is the union of differentiable submanifolds of $\mathcal{M}$ of dimension $m - n$.

Remark. Let $\mathcal{N}^n$ be a smooth manifold, and let $1 \leq m \leq n$. Then an arbitrary subset $\mathcal{M} \subseteq \mathcal{N}$ has the structure of differentiable submanifold of $\mathcal{N}$ of dimension m if and only if for all $p \in \mathcal{M}$, there exists a smooth chart (U, φ) of $\mathcal{N}$ such that $p \in U$, $\varphi(p) = 0$, $\varphi(U)$ is open, and

$$\varphi(U \cap \mathcal{M}) = (-\epsilon, +\epsilon)^n \times \{0\}^{n-m},$$

where $(-\epsilon, +\epsilon)^n$ is the cube. Noticeably, the C^∞-manifold structure of $\mathcal{M}$ is uniquely determined.

Remark. Let $\mathcal{M} \subseteq \mathcal{N}$ be a differentiable submanifold of $\mathcal{N}$, and let $\iota\colon \mathcal{M} \hookrightarrow \mathcal{N}$ be the inclusion. Then, for $p \in \mathcal{M}$, $T_p\mathcal{M}$ can be considered as subspace of $T_p\mathcal{N}$, namely as the image of $\mathrm{d}\iota(T_p\mathcal{M})$.

Lemma 1.6.3. Let $f\colon \mathcal{M}^m \to \mathcal{N}^n$ be a differentiable map such that $m \geq n$ with $p \in \mathcal{N}$. Let $\mathrm{d}f(x)$ has rank n for all $x \in \mathcal{M}$ with $f(x) = p$. For the submanifold $X = f^{-1}(p)$ and for $q \in X$, it is true that

$$T_qX = \ker \mathrm{d}f(q) \subseteq T_q\mathcal{M}.$$

Chapter 2

Riemannian Manifolds

Lecture 5: Riemannian Manifolds

In this chapter, we start our discussion on Riemannian manifolds. 19 Jan. 14:30

2.1 Riemannian Metrics

We start by defining the Riemannian metric.

Definition 2.1.1 (Riemannian metric)**.** A *Riemannian metric* g on a differentiable manifold $\mathcal{M}$ is given by a scalar product I on each $T_p\mathcal{M}$ which depends smoothly on the base point p.

Definition 2.1.2 (Riemannian manifold)**.** A *Riemannian manifold* $(\mathcal{M}, g)$ is a smooth manifold $\mathcal{M}$ equipped with a Riemannian metric g.

Let $x = (x^1, \dots, x^d)$ be the local coordinates. In these, a metric is represented by a positive definite symmetric matrix

$$\big(g_{ij}(x)\big)_{i,j=1,\dots,d},$$

i.e., $g_{ij} = g_{ji}$, and $g_{ij}\xi^i\xi^j > 0$ for all $\xi = (\xi^1, \dots, \xi^d) \neq 0$ with coefficients smoothly depending on x.

2.1.1 Transformation Behavior

We now see that the smoothness does not depend on coordinates, i.e., the smooth dependence on the base point (as required in Definition 2.1.1) can be represented in the local coordinates. Given 2 tangent vectors $v, w \in T_p\mathcal{M}$ with coordinate representations $(v^1, \dots, v^d), (w^1, \dots, w^d)$ given by x such that $v = v^i\frac{\partial}{\partial x^i}$ and $w = w^i\frac{\partial}{\partial x^i}$, their product is

$$\langle v, w\rangle \coloneqq g_{ij}(x(p))v^iw^j.$$

In particular,

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle = g_{ij}.$$

Remark. The length of v is given as $\lVert v\rVert \coloneqq \langle v, v\rangle^{1/2}$.

Let $y = f(x)$ define different local coordinates. In these, v, w are given as

$$(\widetilde{v}^1, \dots, \widetilde{v}^d), (\widetilde{w}^1, \dots, \widetilde{w}^d)$$

with $\widetilde{v}^j = v^i\frac{\partial f^j}{\partial x^i}$ and $\widetilde{w}^j = w^i\frac{\partial f^j}{\partial x^i}$. Denote the metric in new coordinates y by $h_{k\ell}(y)$, then we have

$$h_{k\ell}(f(x))\widetilde{v}^k\widetilde{w}^\ell = \langle v, w\rangle = g_{ij}(x)v^iw^j.$$

Plug everything in, we have

$$h_{k\ell}(f(x))\frac{\partial f^k}{\partial x^i}\frac{\partial f^\ell}{\partial x^j}v^i w^j = g_{ij}(x)v^i w^j.$$

We see that this holds for any tangent vectors v, w, therefore,

$$h_{k\ell}(f(x))\frac{\partial f^k}{\partial x^i}\frac{\partial f^\ell}{\partial x^j} = g_{ij}(x),$$

which is the transformation behavior under coordinates changes.

Remark. This shows that the smoothness does not depend on the choice of coordinates!

Example. Consider the Euclidean space Ω, then given $v, w \in T_p\Omega$, we have

$$\langle v, w\rangle = \delta_{ij}v^i w^j = v^i w_i.$$

Theorem 2.1.1. Every differentiable manifold can be equipped with a Riemannian metric.

Proof. From Lemma 1.3.1, there exists a differentiable partition of unity $\{f_\alpha\}$ of $\mathcal{M}$ subordinate to a covering $\{V_\alpha\}$ of $\mathcal{M}$. Consider the induced metric $\langle\cdot,\cdot\rangle^\alpha$ of the system of local coordinates on each V_α. Then, for every $p \in M$, a Riemannian metric $\langle\cdot,\cdot\rangle_p$ can be defined naturally as

$$\langle u, v\rangle_p = \sum_\alpha f_\alpha(p)\,\langle u, v\rangle_p^\alpha$$

for all $u, v \in T_pM$. Given the fact that $\{f_\alpha\}$ is the partition of unity, we know that

(a) $f_\alpha \geq 0$, and $f_\alpha = 0$ on $\overline{V}_\alpha^c$,

(b) $\sum_\alpha f_\alpha(p) = 1$ for all p on M,

it's then immediate that the defined is indeed a Riemannian metric. ■

2.1.2 Isometry

After introducing any type of mathematical structure, we must introduce a notion of when two objects are the same.

Definition 2.1.3 (Isometry)**.** A diffeomorphism $h\colon \mathcal{M} \to \mathcal{N}$ is an *isometry* between two Riemannian manifolds if it preserves the Riemannian metric, i.e., for $p \in \mathcal{M}$, $v, w \in T_p\mathcal{M}$,

$$\langle v, w\rangle_{\mathcal{M}} = \langle \mathrm{d}h(v), \mathrm{d}h(w)\rangle_{\mathcal{N}}.$$

Definition 2.1.4 (Local isometry)**.** A diffeomorphism $h\colon \mathcal{M} \to \mathcal{N}$ is a *local isometry* between two Riemannian manifolds if for every $p \in \mathcal{M}$, there exists a neighborhood U such that $h|_U : U \to h(U)\colon \mathcal{M} \to \mathcal{N}$ is an isometry and $h(U) \subseteq \mathcal{N}$ is open.

If's common to say that a Riemannian manifold $\mathcal{M}$ is locally isometric to a Riemannian manifold $\mathcal{N}$ if for every $p \in \mathcal{M}$, there exists a neighborhood U of p in $\mathcal{M}$ and a local isometry $f\colon U \to f(U) \subseteq \mathcal{N}$.

Let's first look at an almost trivial example.

Example (Euclidean space)**.** Let $\mathcal{M} = \mathbb{R}^n$ with $\partial/\partial x_i$ identified with $e_i = (0, \ldots, 1, \ldots, 0)$. The metric is given by

$$\langle e_i, e_j\rangle = \delta_{ij}.$$

$\mathbb{R}^n$ is called *Euclidean space of dimension n* and the Riemannian geometry of this space is metric Euclidean geometry.

Example (Lie group)**.** See Appendix A for reference.

2.2 Geodesics

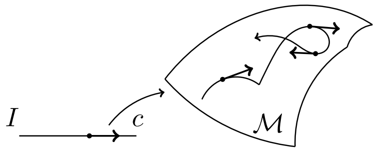
This is the first focus on the study of Riemannian geometry, i.e., the geodesics. The up-shot is that a geodesic minimizes the arc length for points *sufficiently close* (in a sense to be made precise); in addition, if a curve minimizes arc length between any two of its points, it is a geodesic.

2.2.1 Curves

We are now going to show how a Riemannian metric can be used to calculate the length of a curve.

Definition 2.2.1 (Curve)**.** A differentiable mapping $c\colon I \to \mathcal{M}$ of an open interval $I \subseteq \mathbb{R}$ into a differentiable manifold $\mathcal{M}$ is called a (parametrized) *curve*.

Note. A parametrized curve can admit self-intersections as well as corners.



Definition 2.2.2 (Vector field along a curve)**.** We say that a *vector field along a curve* $c\colon I \to \mathcal{M}$ is a differentiable mapping that associates to every $t \in I$ a tangent vector $V(t) \in T_{c(t)}\mathcal{M}$.

To say V is differentiable means that for any differentiable function f on $\mathcal{M}$, the function $t \mapsto V(t)f$ is a differentiable function on I.

Example (Velocity field)**.** The vector field $\mathrm{d}c(\mathrm{d}/\mathrm{d}t)$, denoted by $\mathrm{d}c/\mathrm{d}t$, is called the *velocity field* or *tangent vector field*, of course.

Remark. A vector field along c cannot necessarily be extended to a vector field on an open set of $\mathcal{M}$.

Notation (Segment)**.** The restriction of a curve c to a closed interval $[a, b] \subseteq I$ is called a *segment*.

2.2.2 Lengths and Energies

We're interested in the following two quantities.

Definition. Let $\gamma\colon [a, b] \to \mathcal{M}$ be a curve on a Riemannian manifold $(\mathcal{M}, g)$.

Definition 2.2.3 (Length)**.** The *length* of γ is defined as

$$L(\gamma) \coloneqq \int_a^b \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \right\| \,\mathrm{d}t.$$

Definition 2.2.4 (Energy)**.** The *energy* of γ is defined as

$$E(\gamma) \coloneqq \frac{1}{2} \int_a^b \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \right\|^2 \,\mathrm{d}t.$$

We now want to compute $L(\gamma)$, $E(\gamma)$ in local coordinates. Let the local coordinates be

$$(x^1(\gamma(t)), \ldots, x^d(\gamma(t))),$$

we write

$$\dot{x}^i(t) = \frac{\mathrm{d}}{\mathrm{d}t}(x^i(\gamma(t))).$$

Then, we have

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)}\,\mathrm{d}t, \quad E(\gamma) = \frac{1}{2}\int_a^b g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)\,\mathrm{d}t.$$

Definition 2.2.5 (Distance)**.** Given a Riemannian manifold $(\mathcal{M}, g)$, the *distance* between 2 points $p, q \in \mathcal{M}$ is defined as

$$d(p,q) \coloneqq \inf \left\{ L(\gamma) \mid \gamma\colon [a,b] \to \mathcal{M} \text{ piecewise curve with } \gamma(a) = p, \gamma(b) = q \right\}.$$

Note. Any 2 points $p, q \in \mathcal{M}$ can be connected by a piecewise curve, hence $d(p,q)$ always exists.

Corollary 2.2.1. The topology of $\mathcal{M}$ induced by the distance function d coincides with the original manifold topology of $\mathcal{M}$.

Lemma 2.2.1. If $\gamma\colon [a,b] \to \mathcal{M}$ is a curve, and $\psi\colon [\alpha,\beta] \to [a,b]$ is a change of parameter, then $L(\gamma \circ \psi) = L(\gamma)$.

Proof. This can be proved by computation, and the take-away is that the length functional is invariant under parameter changes. ■

2.2.3 Euler-Lagrange Equations

We want to find a curve which minimizes the length between sufficiently close two points. It turns out that instead of working with length directly, we should work with energy instead.

Notation. Let's first fix some common notations.

(a) $\left(g^{ij}\right)_{i,j=1,\ldots,d} = (g_{ij})^{-1}_{i,j=1,\ldots,d}$.

(b) $g_{j\ell,k} \coloneqq \frac{\partial}{\partial x^k} g_{j\ell}$.

Note. In the above notations, we have $g^{i\ell}g_{\ell j} = \delta^i_j$.

And the following is particularly important.

Notation (Christoffel symbol)**.** The *Christoffel symbol* is defined for all i as

$$\Gamma^i_{jk} \coloneqq \frac{1}{2} g^{i\ell} \left(g_{j\ell,k} + g_{k\ell,j} - g_{jk,\ell}\right).$$

The up-shot is that the Euler-Lagrange equations for the energy E has a nice form, and the solution of which has exactly the properties we want, hence we define it as geodesics.

Proposition 2.2.1. The Euler-Lagrange equations for the energy E are

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0 \text{ for } i = 1, \ldots, d. \tag{2.1}$$

Proof. The Euler-Lagrange equations of a functional

$$I(x) = \int_a^b f(t, x(t), \dot{x}(t))\,\mathrm{d}t$$

are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0$$

for $i = 1, \ldots, d$. Just by plugging in, we obtain for E, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ik}(x(t))\dot{x}^k(t) + g_{ji}(x(t))\dot{x}^j(t)\right) - g_{jk,i}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$$

for $i = 1, \ldots, d$. Hence,

$$g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j + g_{ik,\ell}\dot{x}^\ell\dot{x}^k + g_{ji,\ell}\dot{x}^\ell\dot{x}^j - g_{jk,i}\dot{x}^\ell\dot{x}^j = 0$$

Rename some indices and use $g_{ij} = g_{ji}$, we have that

$$2g_{\ell m}\ddot{x}^m + (g_{k\ell,j} + g_{j\ell,k} - g_{jk,\ell})\,\dot{x}^j\dot{x}^k = 0$$

for $\ell = 1, \ldots, d$. Hence, we have

$$g^{i\ell}g_{\ell m}\ddot{x}^m + \frac{1}{2}g^{i\ell}\,(g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\,\dot{x}^j\dot{x}^k = 0$$

for $i = 1, \ldots, d$. Finally, observe that $g^{i\ell}g_{\ell m} = \delta_{im}$, i.e., $g^{i\ell}g_{\ell m}\ddot{x}^m = \ddot{x}^i$, hence the claim follows. ■

Finally, we define the geodesics as the solution of Equation 2.1.

Definition 2.2.6 (Geodesic). A curve $\gamma\colon [a, b] \to \mathcal{M}$ that obeys Equation 2.1 is called a *geodesic*.

In other words, from Proposition 2.2.1, we naturally define geodesic by the solution of Equation 2.1, which is the critical points of energy.

2.2.4 Action Functional

Consider the following.

Definition 2.2.7 (Action). Let $\mathcal{L}$ be the Lagrangian, then let

$$I[w(\cdot)] \coloneqq \int_0^t \mathcal{L}(\dot{w}(s), w(s))\,\mathrm{d}s$$

defined for functions $w(\cdot) = (w^1(\cdot), \ldots w^n(\cdot))$ of the admissible class

$$\mathcal{A} = \left\{ w(\cdot) \in C^2([0, t]; \mathbb{R}^n) \mid w(0) = y, w(t) = x \right\}.$$

From the calculus of variation, we can find a curve $x(\cdot) \in \mathcal{A}$ such that

$$I[x(\cdot)] = \min_{w(\cdot)\in\mathcal{A}} I[w(\cdot)].$$

Theorem 2.2.1 (Euler-Lagrangian equations). $x(\cdot)$ from $I[x(\cdot)] = \min_{w(\cdot)\in\mathcal{A}} I[w(\cdot)]$ solves the system of Euler-Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(D_{\dot{x}}\mathcal{L}(\dot{x}(s), x(s)) + D_x\mathcal{L}(\dot{x}(s), x(s))\right) = 0$$

for $0 \leq s \leq t$.

Lecture 6: Geodesic and the Exponential Map

Now, we draw some relations between length and energy and see why starting from energy makes sense. 24 Jan. 14:30

Proposition 2.2.2. For all curves $\gamma\colon [a,b] \to \mathcal{M}$,

$$\mathcal{L}(\gamma)^2 \leq 2(b-a)E(\gamma)$$

with equality if and only if $\|\mathrm{d}\gamma/\mathrm{d}t\|$ is a constant.

Proof. From Hölder's inequality,

$$\int_a^b \left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\| \,\mathrm{d}t \leq (b-a)^{1/2}\left(\int_a^b \left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\|^2 \,\mathrm{d}t\right)^{1/2}$$

with equality if and only if $\|\mathrm{d}\gamma/\mathrm{d}t\|$ is a constant. ∎

Example. Let

$$\mathcal{L}(q,x) = \frac{1}{2}m|q|^2 - V(x)$$

with $m>0, q=\dot{x}$, the Euler-Lagrangian equations is given by

$$m\ddot{x}(s) = F(x(s))$$

for $F := -\mathrm{D}V$.

As previously seen. Regular curves can be parametrized by arc length with unit speed $\|\mathrm{d}\gamma/\mathrm{d}t\| = \|\dot{\gamma}\| \equiv 1$.

Lemma 2.2.2. Each geodesic is parametrized proportionally to the arc length.[a]

[a] This means that we have constant speed, i.e., $\|\dot{\gamma}\|$ is a constant.

Proof. For a solution of $\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$,

Do the computation!

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{x},\dot{x}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)\right) = 0.$$

∎

Our goal now is to minimize the length within class of regular smooth curves.

As previously seen. The length and the energy functionals are invariants under parameter changes.

This means that it's enough to look at curves parametrized by arc length.

Theorem 2.2.2. Let $\mathcal{M}$ be a Riemannian manifold, $p\in\mathcal{M}$ and $v\in T_p\mathcal{M}$. Then there exists an $\epsilon>0$ and a unique geodesic such that $c\colon [0,\epsilon]\to\mathcal{M}$ with $c(0)=p$ and $\dot{c}(0)=v$. In addition, c smoothly depend on p,v.

Proof. Since Equation 2.1 is a system of second order ODE, by Picard-Lindelöf theorem, we have local existence and uniqueness of the solution with prescribed initial values and derivative such that the solution depends smoothly on p,v. ∎

If $x(t)$ is the solution of Equation 2.1, then $x(\lambda t)$ is also a solution for any constant $\lambda\in\mathbb{R}$. Denote geodesic from Theorem 2.2.2 by c_v, then

$$c_v(t) = c_{\lambda v}(t/\lambda)$$

for $\lambda>0$, $t\in[0,\epsilon]$, and hence $c_{\lambda v}$ defined on $[0,\epsilon/\lambda]$.

Remark. Since c_v depends smoothly on v, the set $\{v \in T_p\mathcal{M} \mid \|v\| = 1\}$ is compact, hence there exists $\epsilon_0 > 0$ such that for $\|v\| = 1$, c_v defined at least on $[0, \epsilon_0]$, implying that for all $w \in T_p\mathcal{M}$ with $\|w\| \leq \epsilon_0$, c_w is defined at least on $[0, 1]$.

2.3 Exponential Maps

The above discussion permits us to introduce the concept of the exponential map in the following manner.

Definition 2.3.1 (Exponential map). Let $(\mathcal{M}, g)$ be a Riemannian manifold, $p \in \mathcal{M}$, and $V_p := \{v \in T_p\mathcal{M} \mid c_v \text{ defined on } [0,1]\}$. The *exponential map of $\mathcal{M}$ at p*, $\exp_p \colon V_p \to \mathcal{M}$, is defined as $v \mapsto c_v(1)$.

Clearly, exp is differentiable, and we shall utilize the restriction of exp to an open subset of the tangent space $T_q\mathcal{M}$, i.e., we define

$$\exp_p \colon B(0, \epsilon) \subseteq T_p\mathcal{M} \to \mathcal{M},$$

where $B(0, \epsilon)$ is an open ball with center at the origin 0 of $T_p\mathcal{M}$ of radius ϵ. It's easy to see that $\exp_p$ is differentiable and that $\exp_p(0) = q$.

Intuition. Geometrically, $\exp_p(v)$ is a point of $\mathcal{M}$ obtained by going out the length equal to $|v|$, starting from p, along a geodesic which passes through p with velocity equal to $v/|v|$.

Proposition 2.3.1. The exponential map $\exp_p$ maps a neighborhood of $0 \in T_p\mathcal{M}$ diffeomorphically onto a neighborhood of $p \in \mathcal{M}$.

Proof. We see that

$$\mathrm{d}(\exp_p)_0(v) = \frac{\mathrm{d}}{\mathrm{d}t} \exp_p(tv)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} c_{tv}(1)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} c_v(t)\bigg|_{t=0} = v,$$

i.e., $\mathrm{d}(\exp_p)_0$ is the identity of $T_q\mathcal{M}$. By the inverse function theorem, $\exp_p$ is a local diffeomorphism on a neighborhood of 0. ∎

Consider $\exp_p \colon B(0, \epsilon) \subseteq T_p\mathcal{M} \to \mathcal{M}$, maps diffeomorphically onto its image, we can then introduce the coordinates around m. Let $(e_1, \ldots, e_n)$ be the orthonormal basis of $T_m\mathcal{M}$, and $(x_1, \ldots, x_n)$ be the associated local coordinates. Given $p \in \mathcal{M}^n$, $0 \in \mathbb{R}^n$, we have

$$g_{ij}(p) = \delta_{ij}, \quad \Gamma^k_{ij}(p) = 0, \quad g_{ij,k} = 0$$

for all i, j, k.

Definition 2.3.2 (Normal coordinate).

Note. The first derivative vanishes, so locally, the manifold looks Euclidean.

Theorem 2.3.1. For all $p \in \mathcal{M}$, there exists $\rho > 0$ such that the Riemannian polar coordinates may be introduced on $B(p, \rho) = \{q \in \mathcal{M} \mid d(p, q) \leq \rho\}$. For any such ρ and $q \in \partial B(p, \rho)$, there exists a unique geodesic of shortest length $(= \rho)$ from p to q. And in the polar coordinates, this geodesic is given by the straight line $x(t) = (t, \varphi_0)$, $0 \leq t \leq \rho$, with q represented by coordinates (ρ, φ_0), $\varphi_0 \in S^{d-1}$.

Proof. Take an arbitrary curve from p to q, namely $c(t) = \big(r(t), \varphi(t)\big)$, $0 \leq t \leq T$, which does not have to be entirely be contained in $B(p, \rho)$. Let t_0 be defined as

$$t_0 := \inf \{t \leq T \mid d(x(t), p) \geq \rho\}.$$

Then $t_0 \le T$ such that $c|_{[0,t_0]}$ lies entirely in $B(p,\rho)$. We want to show that

(a) $L\left(c|_{[0,t_0]}\right) \ge \rho$, and

(b) $L\left(c|_{[0,t_0]}\right) = \rho$ only for a straight line in the polar coordinates,

where

$$L\left(c|_{[0,t_0]}\right) := \int_0^{t_0} \sqrt{g_{ij}(c(t))\dot{c}^i\dot{c}^j}\,\mathrm{d}t.$$

Observe that $g_{r\varphi} = 0$, with $g_{\varphi\varphi}$ being positive definite, hence

$$L\left(c|_{[0,t_0]}\right) \ge \int_0^{t_0} \sqrt{g_{rr}(c(t))\dot{r}\dot{r}}\,\mathrm{d}t = \int_0^{t_0} |\dot{r}|\,\mathrm{d}t \ge \int_0^{t_0} \dot{r}\,\mathrm{d}t = r(t_0) = \rho,$$

where we know that $g_{rr} \equiv 1$. ■

Remark (Compact manifold)**.** For compact manifold, from Theorem 2.3.1, we can prove that Riemannian polar coordinates can be introduced. Also, there exists $\rho_0 > 0$ such that for any 2 points $p, q \in \mathcal{M}$ with $d(p,q) \le \rho_0$ can be connected by minimizing geodesic.

Lecture 7: Hopf-Rinow Theorem

2.4 Hopf-Rinow Theorem

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We have shown the following in the homework.

Theorem 2.4.1. Let $(\mathcal{M}, g)$ be a compact Riemannian manifold.

(a) Any 2 points $p, q \in \mathcal{M}$ can be connected by a minimizing geodesic.

(b) For all $p \in \mathcal{M}$, the exponential map $\exp_p$ is defined on all of $T_p\mathcal{M}$ and any geodesic may be extended indefinitely in each direction.

We now want to generalize it. However, this is not true in the most general setting, and we need one more requirement.

Definition 2.4.1 (Geodesically complete)**.** A Riemannian manifold $(\mathcal{M}, g)$ is *geodesically complete* if for all $p \in \mathcal{M}$, $\exp_p$ is defined on all of $T_p\mathcal{M}$, if any geodesic $c(t)$ with $c(0) = p$ can be extended for all $t \in \mathbb{R}$.

Finally, we have the following.

Theorem 2.4.2 (Hopf-Rinow theorem)**.** Let $(\mathcal{M}, g)$ be a compact Riemannian manifold, then the following statements are equivalent.

(a) $\mathcal{M}$ is complete as a metric space.[a]

(b) The closed and bounded subsets of $\mathcal{M}$ are compact.

(c) There exists $p \in \mathcal{M}$ such that $\exp_p$ is defined on all $T_p\mathcal{M}$.

(d) $\mathcal{M}$ is geodesically complete.

Furthermore, (d) (and hence (a), (b), and (c)) implies

(e) for two points $p, q \in \mathcal{M}$ can be joined by a minimizing geodesic, i.e., geodesic of the shortest distance $d(p,q)$.

[a] Hence, equivalently, complete as a topological space w.r.t. the underlying topology.

Proof. We start by proving (d) implies (e). Let $\mathcal{M}$ be geodesically complete, and let $r := d(p,q)$, and let ρ be as in the corollary from handout for HW1. Let $p_0 \in \partial B(p,\rho)$ be a point where the continuous functional $d(q,\cdot)$ attains its minimum on the compact set $\partial B(p,\rho)$. Then, for some $V \in T_p\mathcal{M}$,

$$p_0 = \exp_p \rho V.$$

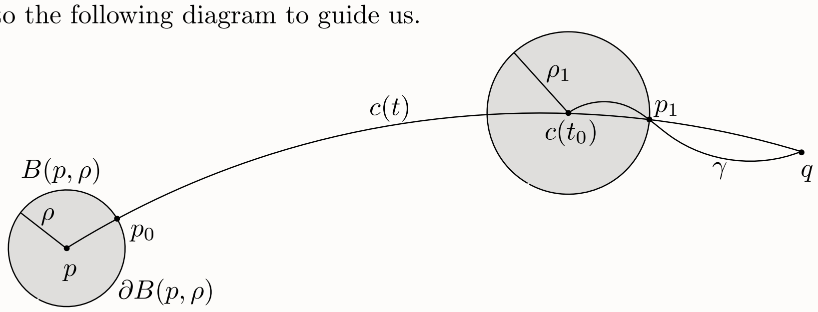
Consider the geodesic $c(t) = \exp_p tV$, by showing

$$c(r) = q,$$

$c|_{[0,r]}$ will be the shortest geodesic from p to q. We start by defining

$$I := \{t \in [0,r] \mid d(c(t),q) = r - t\},$$

and referring to the following diagram to guide us.



Now, we want to show that $I = [0,r]$, which will follow from showing that I is open.

> **Note.** I is not empty since by definition it contains 0 and r. Further, I is closed by continuity.

Let $t_0 \in I$, and let $\rho_1 > 0$ be the radius as in the corollary, without loss of generality, $\rho_1 < r - t_0$. Let $p_1 \in \partial B(c(t_0),\rho_1)$ be the point where the continuous functional $d(q,\cdot)$ attains its minimum on the compact set $\partial B(c(t_0),\rho_1)$. By the triangle inequality,

$$d(p,q) \leq d(p,p_1) + d(p_1,q).$$

For every curve γ from $c(t_0)$ to q, there exists $\gamma(t) \in \partial B(c(t_0),\rho_1)$, hence

$$L(\gamma) \geq \underbrace{d(c(t_0),\gamma(t))}_{\rho_1} + d(\gamma(t),q) = \rho_1 + d(p_1,q),$$

implying $d(q,c(t_0)) \geq \rho_1 + d(p_1,q)$. But from the triangle inequality, we actually have

$$d(q,c(t_0)) = \rho_1 + d(p_1,q) \Leftrightarrow d(p_1,q) = \underbrace{d(q,c(t_0))}_{r-t_0} - \rho_1,$$

hence $d(p_1,p) \geq r - (r - t_0 - \rho_1) = t_0 + \rho_1$, i.e., this is a minimizing curve!

On the other hand, there exists a curve from p to p_1 of length $t_1 + \rho_1$ since it's composed by the portion from p to $c(t_0)$ along $c(t)$ and the portion being the geodesic from $c(t_0)$ to p_1 of length ρ_1. Then, by the theorem we have proved in the HW1#5, this curve is a geodesic curve. Finally, from the uniqueness of geodesic with the given extra data, this geodesic coincides with c. Hence,

$$p_1 = c(t_0 + \rho_1),$$

with $d(p_1,q) = r - t_0 - \rho_1$,

$$d(c(t_0+\rho_1),q) = d(p_1,q) = r - t_0 - \rho = r - (t_0 + \rho_1),$$

thus $t_0 + \rho_1 \in I$, hence I is open, i.e., $I = [0,r]$, so $c(r) = q$ follows.

Lecture 8: Injectivity Radius and Vector Bundles

In the proof we did last time, the last step can be shown via [FC13, Corollary 3.9]. 31 Jan. 14:30

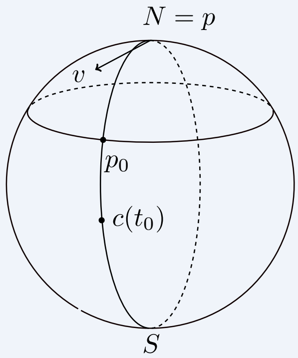
Proof of Hopf-Rinow theorem (Continued). We see that (d) implies (e), hence we only need to show that (a), (b), (c), and (d) are equivalent.

- (d) $\Rightarrow$ (c) is trivial.
- (c) $\Rightarrow$ (b): Let $K \subseteq \mathcal{M}$ be closed and bounded. As K bounded, $K \subseteq B(p, r)$ for some $r > 0$. Then any point in $B(p, r)$ can be joined with p by geodesic of length $\leq r$, and $B(p, r)$ is the image of the compact ball in $T_p\mathcal{M}$ of radius r under continuous map $\exp_p$, hence $B(p, r)$ is compact. As K closed and $K \subseteq B(p, r)$, K is compact.
- (b) $\Rightarrow$ (a): Let $(p_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be a Cauchy sequence, so it's bounded, and by (b), its closure is compact. It contains a convergent subsequence, so it converges, i.e., $\mathcal{M}$ is complete.
- (a) $\Rightarrow$ (d): Let c be a geodesic in $\mathcal{M}$, parametrized by arc length defined on a maximal interval I. Since I s non-empty, and we can show that I is both open and closed. Exercise

■

It's worth mentioning that we do have uniqueness after choosing p_0, in other words, after choosing p_0, everything is fixed, so the non-uniqueness really comes from the initial choose of p_0.

Example. Consider S^2, after fixing p_0, $c(t_0)$ is extended uniquely.



2.5 Injectivity Radius

Consider the following.

Definition 2.5.1 (Injectivity radius). Let $\mathcal{M}$ be a Riemannian manifold, and $p \in \mathcal{M}$. The *injectivity radius* $i(p)$ of p is

$$i(p) \coloneqq \sup \left\{ \rho > 0 \mid \exp_p \text{ defined on } B(0, \rho) \subseteq T_p\mathcal{M} \text{ and injective} \right\}.$$

Similarly, the *injectivity radius* $i(\mathcal{M})$ of $\mathcal{M}$ is defined as $i(\mathcal{M}) \coloneqq \inf_{p\in\mathcal{M}} i(p)$.

Example (Sphere). $i(S^n) = \pi$.

Example (Torus). $i(T^n) = 1/2$.

Any manifold carries a complete Riemannian metric.
If $(\mathcal{M}, g_1)$ is not complete, we can find g_2 such that $(\mathcal{M}, g_2)$ is complete.

Example (Hyperbolic half-plane). The half-plane $P = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with metric induced by the Euclidean metric on $\mathbb{R}^2$, which is not complete.

However, it becomes complete when equipped with the following metric

$$\frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2).$$

In fact, P with the above metric is called the *hyperbolic half-plane* H^2, and we can extend it to H^n.

Another question we may ask is the following.

Problem. Is the converse of Hopf-Rinow theorem true? I.e., can we show that (e) implies (d)?

Answer. No! Any 2 points in the open half-sphere can be joint by a unique minimal geodesic, but this manifold is not geodesically complete. ⊛

Example. The injectivity radius of H^n is ∞.

Remark. Given a compact $\mathcal{M}$, the injectivity radius is always > 0 by continuity argument.

Now, given a complete but not compact $\mathcal{M}$, the injectivity radius can be 0.

Example. Take the quotient of the Poincaré half-plane by the translations

$$(x, y) \mapsto (x + n, y), \quad n \in \mathbb{Z}.$$

We then obtain a complete Riemannian manifold $\mathcal{M}$ with $i(\mathcal{M}) = 0$.

Note. Finding lower bounds for $i(\mathcal{M})$ introduces curvature estimates.

2.6 Bundles and Fields

Let's first introduce the theory of bundles, which allows us to introduce the notion of vector fields, which is a more general notion of tensor fields. And noticeably, nearly every structure we can put on a Riemannian manifold will be in the form of tensor fields.

Example. Given a tangent vector field X of a smooth manifold $\mathcal{M}$ is where we simply associate $X(p)$ to a tangent vector:

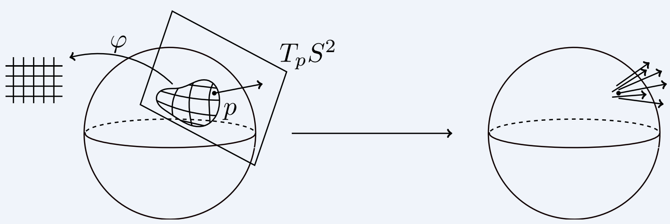


Figure 2.1: Given $\mathcal{M} = S^2$, a vector field assigns every point a "point" in the associated "space." In this case, a tangent vector field associates every p a vector in the corresponding tangent space.

Recall the tangent bundle $(T\mathcal{M}, \pi, \mathcal{M})$, where we only take the name "bundle" for granted and don't know why it is: however, we should see that it helps us construct the vector field, since it captures the idea of "for every point p, we have an associated space $T_p\mathcal{M}$," which is exactly what we need here. This idea generalizes quite easily.

2.6.1 Bundles

We start by introducing the notion of bundles.

Definition 2.6.1 (Bundle). A *bundle* is a tuple $(E, \pi, \mathcal{M})$ consists of the total space E, the base space $\mathcal{M}$, and the bundle projection $\pi\colon E \to \mathcal{M}$.

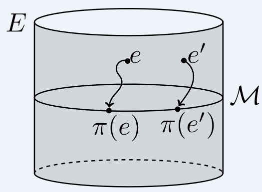
> **Definition 2.6.2** (Total space). The differentiable manifold E is called the *total space*.

> **Definition 2.6.3** (Base space). The differentiable manifold $\mathcal{M}$ is called the *base space*.

> **Definition 2.6.4** (Bundle projection). The (differentiable) continuous surjection $\pi\colon E \to \mathcal{M}$ is called the *bundle projection*.

Note. We see that a tangent bundle $(T\mathcal{M}, \pi, \mathcal{M})$ is actually a bundle.

Example. Let E be a cylinder, $\mathcal{M}$ be a circle.



As we can see, the number of possible π is enormous, as long as it's surjective and smooth.

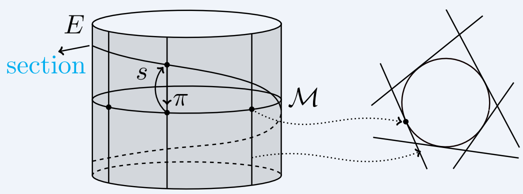
Notation. Sometimes, we will just denote a bundle as $E \overset{\pi}{\to} \mathcal{M}$, or even more compactly, just π since it captures all the data.

Definition 2.6.5 (Fiber). Given a bundle $(E, \pi, \mathcal{M})$, the *fiber* over $p \in \mathcal{M}$ under π is the preimage of a $\{p\}$, i.e., $\pi^{-1}(\{p\})$.

Definition 2.6.6 (Section). A *section* of a bundle $(E, \pi, \mathcal{M})$ is a differentiable map $s\colon \mathcal{M} \to E$ such that $\pi \circ s = \mathrm{id}_{\mathcal{M}}$.

Remark. We see that a section s encodes lots of information of a bundle, since s includes $E, \mathcal{M}$, and the condition deal with π.

Example. Again let E be a cylinder, $\mathcal{M}$ be a circle. This time, we choose π to be the trivial one.



We see that in this way, this bundle really captures all the tangent spaces structure of a circle!

2.6.2 Vector Bundles

Then, we're interested in the so-called vector bundle.

Definition 2.6.7 (Vector bundle)**.** A (differentiable) *vector bundle* of rank n is a bundle $(E, \pi, \mathcal{M})$ such that each fiber $E_x := \pi^{-1}(x)$ of $x \in \mathcal{M}$ carries a structure of an n-dimensional (real) vector space, and local triviality condition holds.

> **Definition 2.6.8** (Local trivialization)**.** For all $x \in \mathcal{M}$, the *local trivialization* (U, φ) consists a neighborhood U and diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that for all $y \in U$,
>
> $$\varphi_y := \varphi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^n$$
>
> is a vector space isomorphism.

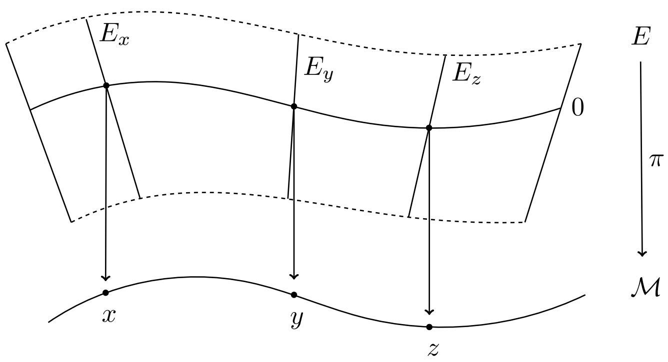


Figure 2.2: An illustration of vector bundle $(E, \pi, \mathcal{M})$.

Definition 2.6.9 (Trivial)**.** A vector bundle is *trivial* if it's isomorphic to $\mathcal{M} \times \mathbb{R}^n$.[a]

[a] n is the rank of the vector bundle.

Intuition. The local trivialization shows that *locally* the map π looks like the projection of $U \times \mathbb{R}^n$ on U.

Definition 2.6.10 (Bundle chart)**.** The pair (φ, U) is also called the *bundle chart* in local trivialization.

Remark. From Definition 2.6.7, vector bundle is locally, but not necessarily globally a product of base space and the fiber.

Intuition. We may look at a vector bundle as a family of vector spaces, all isomorphic to a fixed $\mathbb{R}^n$, "parametrized" (locally trivially) by a manifold.

Lecture 9: Tensors and Connections

2.6.3 Vector Fields

2 Feb. 14:30

We can now introduce the notion of vector fields in terms of section.

Definition 2.6.11 (Vector field). A (smooth) *vector field* X is a smooth section of a bundle.

Note. We see that a smooth tangent vector field is indeed a smooth vector field with the bundle being the tangent bundle.

Notation. Since we will nearly always be talking about tangent vector fields, we will abuse the notation a bit and just simply call it vector fields. But always keep in mind that more broadly, a vector field should be a section of a bundle, not always $T\mathcal{M}$.

2.6.4 Tensor Fields

We can introduce the notion of "tensor fields" in a brute-fore way.[1] To do this, given a vector space V, we first introduce tensors.

Definition 2.6.12 (Tensor). Let V be a vector space of dimension $m < \infty$, and the dual space V^*.[a] Then the vector space of the *r-times contravariant and s-times covariant tensors over V*, denoted as $T^r_s(V)$, is the vector field defined as

$$T^r_s(V) = \{T\colon \underbrace{V^* \times \ldots \times V^*}_{r} \times \underbrace{V \times \ldots \times V}_{s} \to \mathbb{R}\} = \underbrace{V \otimes \ldots \otimes V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s}.$$

[a] I.e., $V^* \coloneqq \{\lambda\colon V \to \mathbb{R} \mid \lambda \text{ linear}\}$.

Notation. Let $\mathcal{M}^n$ be a smooth manifold, and $\pi\colon E \to \mathcal{M}$ a smooth vector bundle, then

$$\Gamma(E) \coloneqq \{s \in C^\infty(\mathcal{M}, E) \mid \pi \circ s = \mathrm{id}_{\mathcal{M}}\},$$

i.e., the set of sections.

Example. Consider the vector bundle $(T\mathcal{M}, \pi, \mathcal{M})$, then $\Gamma(T\mathcal{M}) \coloneqq \{$vector fields on $\mathcal{M}\}$.

Example. $\Gamma(\Lambda_s\mathcal{M}) \coloneqq \{s\text{-forms on } \mathcal{M}\}$ with $\Lambda_s\mathcal{M} = \Lambda^s\left(\bigcup_{p\in\mathcal{M}} T^*_p\mathcal{M}\right)$.[a]

[a] Here, $\Lambda^s(V^*) \coloneqq \{A \in T^0_s(V) \mid A \text{ skew-symmetric}\}$, where $s \in \mathbb{N}$.

Then, we have the following.

Definition 2.6.13 (Tensor field). The *(r, s)-tensor fields* on $\mathcal{M}$ is defined as elements in $\Gamma(T^r_s\mathcal{M})$ with $T^r_s\mathcal{M} = \bigcup_{p\in\mathcal{M}} T^r_s(T_p\mathcal{M})$.

Example. A Riemannian metric g on $\mathcal{M}$ is a $(0, 2)$-tensor field, i.e., $g \in \Gamma(T^0_2(\mathcal{M}))$ for all $p \in \mathcal{M}$.

Proof. Since $g_p\colon T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$. ⊛

2.7 Other Metrics

Finally, we discuss some other metrics we may let a manifold equipped with.

Definition 2.7.1 (Pseudo-Riemannian metric). A *pseudo-Riemannian metric* on a differentiable manifold $\mathcal{M}$ is a $(0, 2)$-tensor field $g \in \Gamma(T^0_2(\mathcal{M}))$ for all $p \in \mathcal{M}$ with

[1] See Appendix B.1.1 for another view point.

(a) $g(X, Y) = g(Y, X)$ for all $X, Y \in T\mathcal{M}$;

(b) for all $p \in \mathcal{M}$, g_p is non-degenerate bilinear form on $T_p\mathcal{M}$, i.e., $g_p(X, Y) = 0$ for all $X, Y \in T_p\mathcal{M}$ if and only if $Y = 0$.

Note. A pseudo Riemannian metric is actually a Riemannian metric if it's positive definite at every $p \in \mathcal{M}$.

Definition 2.7.2 (Lorentzian metric)**.** A *Lorentzian metric* g is a continuous assignment of a non-degenerate[a] quadratic form g_p of index 1[b] in $T_p\mathcal{M}$ for all $p \in \mathcal{M}$.

[a] $g_p(X, Y) = 0$ for all $Y \in T_p\mathcal{M}$ implies $X = 0$.
[b] It means that the maximal dimension of a subspace of $T_p\mathcal{M}$ on which g_p is negative definite is 1.

An equivalent definition is the following.

Definition 2.7.3 (Lorentzian)**.** A quadratic form g_p in $T_p\mathcal{M}$ is *Lorentzian* if there exists a vector $V \in T_p\mathcal{M}$ such that $g_p(V, V) < 0$ while setting $\Sigma_V = \{X \mid g_p(X, V) = 0\}$ such that $g_p|_{\Sigma_V}$[a] is positive definite.

[a] The g_p-orthogonal complement of V.

Example (Minkowski space)**.** The Minkowski space on $\mathbb{R}^4$ is the prototypical example from physics (flat spacetime). Namely, the metric is given by the quadratic form

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with the coordinates being (t, x, y, z).

Chapter 3

Affine and Riemannian Connections

So far, we saw that a vector field X can be used to provide a directional derivative since it gives us a tangent vector at each point smoothly. Now, we will introduce a new symbol ∇ where we let

$$\nabla_X f \coloneqq Xf$$

for $f \in C^\infty(\mathcal{M})$.

Problem. Does this notation overkill? We already know that $Xf = (\mathrm{d}f)(X)$!

Answer. No! While $\nabla, X \colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, while $\mathrm{d}f \colon \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$, we can generalize ∇_X to act from vector fields to vector fields! The insight is that if X can be extended naturally (without providing any extra structures), then we certainly won't bother introducing a new symbol. However, as you might guess, to let ∇ doing this, we do need to provide extra structures, and ∇ stands exactly for these extra structures! ⊛

3.1 Affine Connections

We first formulate a *wish list* of properties which the ∇_X should have. Any remaining freedom in choosing ∇ will need to be provided as additional structures beyond the structures on $\mathcal{M}$ we already have.

Definition 3.1.1 (Linear connection)**.** A *linear connection* (*affine connection*) on a smooth manifold $\mathcal{M}$ is a bilinear map

$$\nabla \colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M}),$$

which is denoted by $\nabla(X, Y) = \nabla_X Y$ and which satisfies

(a) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$;

(b) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$;

(c) $\nabla_X fY = f\nabla_X Y + X(f)Y$;

for all vector fields $X, Y, Z \in \Gamma(T\mathcal{M})$ and $f, g \in C^\infty(\mathcal{M})$.

Note (Covariant derivative)**.** There's a similar notation called *covariant derivative*, denoted by D, satisfies similar properties as a linear connection. Hence, we often write D and ∇ interchangeably.[a]

[a]Linear connection is more general than a covariant derivative; however, we treat them as the same.

Remark (Γ and ∇)**.**

Definition 3.1.2 (Torsion tensor). Let ∇ be a linear connection. The *torsion tensor* of ∇ is the map $T\colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ defined as

$$T(X, Y) \coloneqq \nabla_X Y - \nabla_Y X - [X, Y].$$

Definition. Let ∇ be a linear connection.

Definition 3.1.3 (Torsion-free). ∇ is *torsion-free* if $T = 0$.

Definition 3.1.4 (Riemannian). If g is a Riemannian metric on $\mathcal{M}$, then ∇ is *Riemannian* (metric) if

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

for all $X, Y, Z \in \Gamma(T\mathcal{M})$.

Proposition 3.1.1 (Koszul formula). On each Riemannian manifold $(\mathcal{M}, g)$, there exists a unique Riemannian, torsion-free connection ∇ on $T\mathcal{M}$ determined by the *Koszul formula*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X\langle Y, Z\rangle - Z\langle X, Y\rangle + Y\langle Z, X\rangle - \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle\right). \quad (3.1)$$

Proof. Firstly, we prove that for each Riemannian and torsion-free connection satisfies Equation 3.1, then it will imply uniqueness. As for existence, we verify that the unique $\mathbb{R}$-bilinear map

$$\nabla\colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$$

given by Equation 3.1 has the desired properties, i.e., 2 product rules from connection, torsion-free, and being metric. ∎

Definition 3.1.5 (Levi-Civita connection). The *Levi-Civita connection* is the unique linear connection ∇ defined by the Koszul formula.

Definition 3.1.6 (Riemannian curvature tensor). Let ∇ be the Levi-Civita connection on $T\mathcal{M}$. Then the *Riemannian curvature tensor* $R\colon \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ of ∇ is defined by

$$R(X, Y)Z \coloneqq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Lecture 10: Flow of Vector Fields

Consider plugging in basis vectors $\partial/\partial x^i$ into Definition 3.1.6, with $[\partial/\partial x^i, \partial/\partial x^k] = 0$, we have 7 Feb. 14:30

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^\ell} = R^k_{\ell ij}\frac{\partial}{\partial x^k}$$

where

$$R_{k\ell ij} \coloneqq g_{km} R^m_{\ell ij} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k}\right\rangle.$$

Remark. We have $\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle$ and $R_{k\ell ij} = -R_{\ell kij}$, etc.

Remark. $R^\ell_{ijk} = \Gamma^p_{ik}\Gamma^\ell_{jp} - \Gamma^p_{jk}\Gamma^\ell_{ip} + \Gamma^\ell_{ik,j} - \Gamma^\ell_{jk,i}$.

Definition 3.1.7 (Ricci curvature tensor)**.** The *Ricci curvature tensor* is defined by $R_{ab} = g^{cd} R_{cadb}$.

Definition 3.1.8 (Scalar curvature tensor)**.** The *Ricci curvature tensor* is defined by $R = g^{ab} R_{ab}$.

Proposition 3.1.2 (Bianchi identity)**.**

$$\nabla_\alpha R_{\beta\gamma\delta\epsilon} \coloneqq \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\alpha R_{\beta\gamma\delta\epsilon} = 0$$

The following is an equivalent definition of geodesic:

$$\nabla_{\dot{c}} \dot{c} = \theta.$$

Notation. $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij} rk \frac{\partial}{\partial x^k}$.

Definition 3.1.9 (Autoparallel)**.** Let ∇ be a connection on $T\mathcal{M}$ of a differentiable manifold $\mathcal{M}$. A curve $c\colon I \to \mathcal{M}$ is called *autoparallel* or *geodesic* w.r.t. ∇ if

$$\nabla_{\dot{c}} \dot{c} = 0.$$

In the local coordinates, we have $\dot{c} = \dot{c}^i \partial / \partial x^i$.

Note that

$$\nabla_{\dot{c}} \dot{c} = \dot{c}^i \nabla_{\frac{\partial}{\partial x^i}} \dot{c}^j \frac{\partial}{\partial x^j} = \dot{c}^i \dot{c}^j \Gamma_{ij} rk \frac{\partial}{\partial x^k} + \ddot{c}^k \frac{\partial}{\partial x^k} = \left(\ddot{c}^k + \Gamma_{ij}^k \dot{c}^i \dot{c}^j \right) \frac{\partial}{\partial x^k} = 0.$$

3.2 Flow of Vector Fields

Let $\mathcal{M}$ be a differentiable manifold. Let x be a vector field on $\mathcal{M}$ (i.e., a smooth section of the tangent bundle $T\mathcal{M}$). Then X defined a 1^{st} order differential equations, i.e., if $\dim \mathcal{M} > 1$, it is a system of 1^{st} differential equations

$$\dot{c} = X(c).$$

Proposition 3.2.1. For all $p \in \mathcal{M}$, there exists an open interval $I = I_p \subseteq \mathbb{R}$ with $0 \in I_p$ such that a smooth curve $c\colon I_p \to \mathcal{M}$ with

$$\begin{cases} \frac{\mathrm{d}c(t)}{\mathrm{d}t} = X(c(t)), & t \in I; \\ c(0) = 0. \end{cases}$$

Further, the solution depends smoothly on the initial data (i.e., p).[a]

[a] This directly follows from ODE theory.

Proof. For all $p \in \mathcal{M}$, we want to find an open interval $I = I_p$ around $0 \in \mathbb{R}$ and a solution of the following ODE for $c\colon I \to \mathcal{M}$:

$$\begin{cases} \frac{\mathrm{d}c(t)}{\mathrm{d}t} = X(c(t)), & t \in I; \\ c(0) = 0. \end{cases}$$

We can check in local coordinates that this is a system of ODE. In such coordinates, let $c(t)$ be given by

$$c(t) = \left(c^1(t), c^2(t), \ldots, c^d(t) \right)$$

for $d = \dim \mathcal{M}$. Let X be $X^i \partial / \partial x^i$, then the above system becomes

$$\frac{\mathrm{d}c^i(t)}{\mathrm{d}t} = X^i\big(c(t)\big), \quad i = 1, \ldots, d.$$

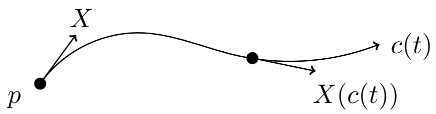
From the ODE theory, specifically Picard-Lindelöf theorem, with the initial data $c(0) = p$, there is a unique solution. ∎

Proposition 3.2.2. For all $p \in \mathcal{M}$, there exists an open neighborhood U of p and an open interval I_p with $0 \in I_p$ such that for all $q \in U$, the curve c_q with

$$\dot{c}_q(t) = X(c_q(t)), \quad c_q(0) = q$$

is defined on I. The map[a] $(t, q) \mapsto c_q(t)$ is smooth.

[a] $I \times U \to \mathcal{M}$.



Definition 3.2.1 (Local flow). The map $c_q(t) \colon I \times U \to \mathcal{M}$, $(t, q) \mapsto c_q(t)$ from Proposition 3.2.2 is called the *local flow* of the vector field X.

Definition 3.2.2 (Integral curve). The curve c_q is called the *integral curve* of X through q.

Now, fixing t, we can vary q and see the following.

Theorem 3.2.1. Let $\varphi_t(q) \coloneqq c_q(t)$, $\varphi_t \circ \varphi_s(q) = \varphi_{t+s}(q)$ for $s, t, (t+s) \in I_q$ and if φ_t define don $U \subseteq \mathcal{M}$, it maps U diffeomorphically onto its image.

Proof. ∎ Exercise

Definition 3.2.3 (Local 1-parameter group). A family $(\varphi_t)_{t \in I}$ of diffeomorphism from $\mathcal{M}$ to $\mathcal{M}$ satisfying Theorem 3.2.1 is called a *local 1-parameter group* of diffeomorphisms.

Remark. In general, a local 1-parameter group needs not be extendible to a group because the maximum interval of definition $I = I_q$ in Definition 3.2.3 need not be all of $\mathbb{R}$.

Example. Let $\mathcal{M} = \mathbb{R}$, $X(t) = \tau^2 \mathrm{d}/\mathrm{d}\tau$. Then the solution of $\dot{c}(t) = c^2(t)$ as an ODE is not defined over all $\mathbb{R}$.

Now, we want the whole group structure.

Theorem 3.2.2. Let X be a vector field on $\mathcal{M}$ with a compact support. Then the corresponding local flow is defined for every $q \in \mathcal{M}$ and $t \in \mathbb{R}$, and the local 1-parameter group becomes a group of diffeomorphisms.

Proof. By using $\operatorname{supp}(X) \subseteq K$, K compact, we can cover K by a finite covering, then using Proposition 3.2.2, we're done. ∎

Corollary 3.2.1. On a compact differentiable manifold $\mathcal{M}$, any vector field generates a local 1-parameter group.

Lecture 11: Parallel Transport

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Let's first transform Equation 2.1 into 1^{st} order system on the cotangent bundle $T^*\mathcal{M}$, and locally trivialize $T^*\mathcal{M}$ by chart $T^*\mathcal{M}|_U \cong U \times \mathbb{R}^d$ with coordinates $(x^1, \ldots, x^d, p_1, \ldots, p_d)$. Set

$$H(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j, \tag{3.2}$$

recall $g^{ik} g_{kj} = \delta^i_j$.

Theorem 3.2.3. Equation 2.1 is equivalent to the system on $T^*\mathcal{M}$:

$$\begin{cases} \dot{x}^i = \dfrac{\partial H}{\partial p_i} g^{ij}(x) p_j; \\ \dot{p}_i = -\dfrac{\partial H}{\partial x^i} = -\dfrac{1}{2} g^{jk}_{,i}(x) p_j p_k. \end{cases} \tag{3.3}$$

Remark (Cogeodesic flow)**.** The flow determined by Equation 3.3 is called the *cogeodesic flow*.

Note. The geodesic flow on $T\mathcal{M}$ is obtained from the cogeodesic flow by the first equation in Equation 3.3.

Remark (Hamiltonian flow)**.** The cogeodesic flow is a *Hamiltonian flow* for the Hamiltonian H.

Proof. By Equation 3.3, along the integral curves,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = H_{x^i} \dot{x}^i + H_{p_i} \dot{p}^i = -\dot{p}_i x \dot{x}^i + \dot{x}^i \dot{p}_i = 0.$$

Observe that the cogeodesic flow maps the set

$$E_\lambda \coloneqq \{(x,p) \in T^*\mathcal{M} \mid H(x,p) = \lambda\}$$

onto itself for all $\lambda \geq 0$. ⊛

Remark. If $\mathcal{M}$ is compact, then all E_λ are compact, then all geodesic flow define don all of E_λ for all λ. Also, $\mathcal{M} = \bigcup_{\lambda \geq 0} P E_\lambda$ for P being the projection.

Definition 3.2.4 (Vector field along a curve)**.** A vector field along a curve $c \colon I \subseteq \mathbb{R} \to \mathcal{M}$ is $X \colon I \to T\mathcal{M}$ such that $X(t) \in T_{c(t)}\mathcal{M}$ for all $t \in I$.

Notation. Denote the set of smooth vector field along c by $\chi_c(\mathcal{M})$.

Theorem 3.2.4. Let $(\mathcal{M}, g)$ be a Riemannian manifold, D is the canonical (Levi-Civita) connection and c a curve in $\mathcal{M}$. Then there exists a unique operator $\mathrm{D}/\mathrm{d}t$ defined as the vector space of vector field along c satisfying

(i) (a) $\frac{\mathrm{D}}{\mathrm{d}t}(fY)(t) = f'(t)Y(t) + f(t)\frac{\mathrm{D}}{\mathrm{d}t}Y(t)$ for all real function f on I.
    (b) $\frac{\mathrm{D}}{\mathrm{d}t}(V + W) = \frac{\mathrm{D}V}{\mathrm{d}t} + \frac{\mathrm{D}W}{\mathrm{d}t}$.

(ii) If there exists a neighborhood of in I such that Y is the restriction to c of a vector field X defined on a neighborhood of $c(t_0)$ in $\mathcal{M}$, then $\frac{\mathrm{D}}{\mathrm{d}t}Y(t_0) = (\mathrm{D}_{c(t_0)}X)_{c(t_0)}$.

Problem 3.2.1. Why not just define $\mathrm{D}Y/\mathrm{d}t$ by (ii)?

Answer. A vector field Y along a curve may not always be extended to a neighborhood of c in $\mathcal{M}$. But, in local coordinates,

$$Y(t)=\sum_{i=1}^{n}Y^i(t)\left(\frac{\partial}{\partial x^i}\right)_{c(t)},$$

which shows that a vector field along c always a linear combination of vector fields along c that can be extended. ⊛

As previously seen. Let $X=X^i\partial_i$, $V=V^k\partial_k$, and let D be the Levi-Civita connection. Then

$$\mathrm{D}_V X=\mathrm{D}_V(X^i\partial_i)=V(X^i)\partial_i+X^i\underbrace{\mathrm{D}_V\partial_i}_{V^kD_{\partial_k}\partial_i}=V(X^i)\partial_i+V^kX^i\Gamma^j_{ki}\partial_j.$$

Proof. The covariant derivative along c is defined by

$$\frac{\mathrm{D}}{\mathrm{d}t}(Y^i(t)\partial_i)=\frac{\mathrm{d}V^i}{\mathrm{d}t}\partial_i+\dot{c}Y^i\Gamma^k_{ji}(c(t))\partial_k,$$

where $\dot{c}=\dot{c}^k\partial_k$. This shows (i) (a) and (b) hold. Next, to show (ii), let x be a smooth vector field in $\mathcal{M}$. Then the induced vector field along c is given by $Y(t)=X_{c(t)}$, i.e., in terms of the coordinate basis, we have

$$Y(t)=Y^i(t)\partial_i,\quad X_x=X^i(x)\partial_i,\quad Y^i(t)=X^i(c(t)).$$

Then,

$$\begin{aligned}\mathrm{D}_iX=\mathrm{D}_i(X^i\partial_i)=\dot{c}(Xri)\partial_i+X^i\mathrm{D}_i\partial_i=X^i\dot{c}^k\underbrace{\mathrm{D}_{\partial_k}\partial_i}_{\Gamma^\ell_{ki}\partial_\ell}\\=\partial_t(X^i\circ c)\partial_i+\dot{c}^kX^i\Gamma^\ell_{ki}\partial_\ell=\partial_t(X^i\circ c)\partial_i+\dot{c}^kY^i\Gamma^\ell_{ki}\partial_\ell=\frac{\mathrm{D}}{\mathrm{d}t}Y.\end{aligned}$$

■

Definition 3.2.5 (Parallel). A vector field along c is called *parallel* if $\mathrm{D}X/\mathrm{d}t=0$.

Definition 3.2.6 (Parallel transport). The *parallel transport* from $c(0)$ to $c(t)$ along the curve c in $(\mathcal{M},g)$ is the linear map $P_i\colon T_{c(0)}\mathcal{M}\to T_{c(t)}\mathcal{M}$ associating to $v\in T_{c(0)}\mathcal{M}$ the vector X_v with X_v being the parallel vector field along c such that $X_v(0)=v$.

Proposition 3.2.3. The parallel transport defines for all t an isometry from $T_{c(0)}\mathcal{M}$ onto $T_{c(t)}\mathcal{M}$. More generally, if X,Y vector fields along c, then

$$\frac{\mathrm{d}}{\mathrm{d}t}g(x(t),y(t))=g\left(\frac{\mathrm{D}X(t)}{\mathrm{d}t},Y(t)\right)+g\left(X(t),\frac{\mathrm{D}Y(t)}{\mathrm{d}t}\right).$$

Lecture 12: Tangent and Cotangent Bundles

3.3 More on Tangent and Cotangent Bundles

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3.3.1 Preliminaries

Let $f\colon\mathcal{M}\to\mathcal{N}$ be a differentiable map between two differentiable manifolds, and let $(E,\pi,\mathcal{N})$ be a vector bundle over $\mathcal{N}$. We want to pull back the bundle via f, i.e., construct a bundle $f^*\in E$for which the fiber over $x\in\mathcal{M}$ is $E_{f(x)}$:

Definition 3.3.1 (Pullback)**.** The *pullback* bundle f^*E is the bundle over $\mathcal{M}$ with the bundle charts $\varphi \circ f, f^{-1}(U)$ with (φ, U) being the bundle charts of E.

Definition 3.3.2 (Bundle homomorphism)**.** Consider 2 vector bundles $(E_1, \pi_1, \mathcal{M}), (E_2, \pi_2, \mathcal{M})$ over $\mathcal{M}$, and let the differentiable map $f\colon E_1 \to E_2$ be fiber preserving, i.e., $\pi_2 \circ f = \pi_1$. Let the fiber maps $f_x\colon E_{1,x} \to E_{2,x}$ be linear.[a] Then f is called a *bundle homomorphism.*

[a] I.e., vector homomorphisms.

Definition 3.3.3 (Subbundle)**.** Let $(E, \pi, \mathcal{M})$ of rank n be a vector bundle. Let $E^1 \subseteq E$, and assume that for all $x \in \mathcal{M}$, there exists a bundle chart (φ, U) for $x \in U$ and $\varphi(\pi^{-1}(U) \cap E^1) = U \times \mathbb{R}^m \subseteq U \times \mathbb{R}^n$ for $m \leq n$. The so constructed vector bundle $(E^1, \pi|_{E^1}, \mathcal{M})$ is called the *subbundle* of E of rank m.

3.3.2 Identifications

Let $\mathcal{M}$ be a differentiable manifold.

As previously seen. The elements of $T\mathcal{M}$ is (x, V) with $x \in \mathcal{M}$, V the tangent vector to $\mathcal{M}$ at x. Also, the tangent bundle is the section of $T\mathcal{M}$.

Also, the elements of $T * 8\mathcal{M}$ are called cotangent vector, i.e., linear functionals $\alpha \in T_x^*\mathcal{M}$ such that $\alpha\colon T_x\mathcal{M} \to \mathbb{R}$. Then, the section of $T^*\mathcal{M}$ are 1-forms, i.e., $\alpha\colon T\mathcal{M} \to \mathbb{R}$, $\alpha_x\colon\ \alpha|_{T_x\mathcal{M}} \to \mathbb{R}$.

A Riemannian metric induces bundle metrics on all tensor bundles over $\mathcal{M}$. Metric of $T^*\mathcal{M}$ given by

$$\langle \omega, \eta \rangle = g(\omega, \eta) = g^{ij}\omega_i\eta_i$$

where $\omega = \omega_i \mathrm{d}x^i$, $\eta = \eta_i \mathrm{d}x^i$. The identification between $T\mathcal{M}$ such that $T^*\mathcal{M}$ through Riemannian metric is given by

$$V = V^i \frac{\partial}{\partial x^i} \in T\mathcal{M} \text{ corresponds to } \omega = \omega_j \mathrm{d}x^j \in T^*\mathcal{M}$$

with $\omega_j = g_{ij}V^i$ or $V^i = g^{ij}\omega_j$ with

(a) $g(X, Y) = g_{ij}X^iY^j$ for $X, Y \in T\mathcal{M}$.

(b) $g(\omega, \eta) = g^{ij}\omega_i\eta_j$ for $\omega, \eta \in T^*\mathcal{M}$.

Then, for $V \in T_x\mathcal{M}$, there corresponds a 1-form $\omega \in T_x^*\mathcal{M}$ via the metric $\omega(Y) \coloneqq g(V, Y)$ for all Y, and $\|\omega\| = \|V\|$. Let $(e_i)_{i=1,\dots,d}$ be a basis of $T_x\mathcal{M}$ and $(\omega^j)_{j=1,\dots,d}$ the dual basis of $T_x^*\mathcal{M}$, i.e., $w^j(e_i) = \delta_i^j$. Let $V = V^ie_i \in T_x\mathcal{M}$, $\eta = \eta_j\omega^j \in T_x^*\mathcal{M}$, then $\eta(V) = \eta_iV^i$.

Consider basis $(e_i), (\omega^j)$ in the local coordinates: $e_i = \partial/\partial x^i$ and $\omega^j = \mathrm{d}x^j$. Let f be a local coordinates change. Then V transform as

$$f_*(V) \coloneqq V^i \frac{\partial f^\alpha}{\partial x^i} \frac{\partial}{\partial f^\alpha};$$

and η transforms as

$$f^*(\eta) \coloneqq \eta_j \frac{\partial x^j}{\partial f^\beta} \mathrm{d}f^\beta.$$

Hence,

$$f^*(\eta)(f_*(V)) = \eta_j \frac{\partial x^j}{\partial f^\alpha} V^i \frac{\partial f^\alpha}{\partial x^i} = \eta_i V^i = \eta(V).$$

Intuition. The above means that

- the tangent vectors transform with the functional matrix of coordinates change;

- the cotangent vectors transform with the transposed inverse of the above matrix.

To compute the coordinates change $y \mapsto x(y)$ for $\omega = \omega_i \mathrm{d}x^i$, $\eta = \eta_i \mathrm{d}x^i$ with $\langle \omega, \eta \rangle = g^{ij}\omega_i\eta_j$, we have

$$\omega_i \mathrm{d}x^i = \omega_i \frac{\partial x^i}{\partial y^\alpha} \mathrm{d}y^\alpha =: \widetilde{w}_\alpha \mathrm{d}y^\alpha,$$

and g^{ij} is transformed as

$$h^{\alpha\beta} = g^{ij} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}$$

and $h^{\alpha\beta}\widetilde{w}_\alpha\widetilde{\eta}_\beta = g^{ij}\omega_i\eta_j$ and

$$\|\omega(x)\| = \sup\left\{\omega(x)(V) \mid V \in T_x\mathcal{M}, \|v\| = 1\right\}.$$

Consider $T\mathcal{M} \otimes T\mathcal{M}$, the metric is

$$\langle V \otimes Y, \xi \otimes \eta \rangle = g_{ij} V^i Y^i g_{k\ell} \omega^k \eta^\ell.$$

Definition 3.3.4 (Lie derivative)**.** Consider a vector field X with a local 1-parameter group $(\psi_t)_{t\in I}$ of local diffeomorphism, and a tensor field S on $\mathcal{M}$. The *Lie derivative* of S in the direction of X is defined as

$$L_x S := \frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^* S)\bigg|_{t=0}.$$

Definition 3.3.5 (Pushforward)**.** Let $\psi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism between two differentiable manifolds. Let X be a vector field on $\mathcal{M}$. Then define a vector field $Y = \psi_* X$ on $\mathcal{N}$ be by $Y(p) = \mathrm{d}\psi(X(\psi^{-1}(p)))$.

Lemma 3.3.1. The following holds.

1. For every differentiable function $f\colon \mathcal{N} \to \mathbb{R}$, then $(\psi_* X)(f)(p) = X(f \circ \psi)(\psi^{-1} p)$.
2. Let X be a vector field on $\mathcal{M}$, $\psi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. If the local 1-parameter group generated by X given by φ_t, then the local group generated by $\psi_* X$ is $\psi \circ \varphi_t \circ \psi^{-1}$.

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Let $X = X^i \partial/\partial x^i$ be a vector field. Then 16 Feb. 14:30

$$(\psi_t)_* \frac{\partial}{\partial x^i}(\psi_t(x)) = \frac{\partial \psi_t^k}{\partial x^i} \frac{\partial}{\partial x^k}.$$

For $\mathcal{M} = \mathcal{N}$, X and $\varphi(x)$ in the same coordinate neighborhood, we have

$$\frac{\partial}{\partial \varphi^k} = \frac{\partial}{\partial x^k}.$$

On the other hand, let $\omega g \omega_i \mathrm{d}x^i$ be a 1-form, then we have

$$(\psi_t^*)(\omega)(x) = \omega(\psi_t(x)) \frac{\partial \psi_t^i}{\partial x^k} \mathrm{d}x^k,$$

which is the curve in $T_x^*\mathcal{M}$. For $\varphi\colon \mathcal{M} \to \mathcal{N}$, with the 1-form $\omega = \omega_i \mathrm{d}x^i$ on $\mathcal{N}$,

$$\varphi^* \omega = \omega_i(\varphi(x)) \frac{\partial z^i}{\partial x^k} \mathrm{d}x^k.$$

Let $\varphi\colon \mathcal{M} \to \mathcal{N}$ be a diffeomorphism, Y be a vector field on $\mathcal{N}$. Set

$$\varphi^* Y := (\varphi^{-1})_* Y.$$

Remark. φ^* can be defined in an analogous way in terms of contravariant tensors.

In particular, let X be a vector field, and a local 1-parameter group $(\psi_t)_{t\in I}$,

$$(\psi_t^* X) = (\psi_t)_* X.$$

Let $x \in \mathcal{M}$, $X \in T_x\mathcal{M}$, and $\mathrm{d}f_x \colon T_x\mathcal{M} \to T_{f(x)}\mathcal{N}$. $\mathrm{d}f_x X$ or $(f_*)_x X$ is called the pushforward of x by f.

Let $\omega \in T^*\mathcal{N}$, a 1-form on $\mathcal{N}$. Then

$$(f^*\omega)_x X = \omega_{f(x)}(\mathrm{d}f_x X)$$

is called pullback of ω by f.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f"] & \mathcal{N} \\ T\mathcal{M} \arrow[u, "\pi_{\mathcal{M}}"] \arrow[r, "\mathrm{d}f"] & T\mathcal{N} \arrow[u, "\pi_{\mathcal{N}}"'] \end{tikzcd}$$

Definition 3.3.6 (Sectional curvature). The *sectional curvature* of the plane spanned by the (linearly independent) tangent vectors $X = X^i\partial/\partial x^i, Y = Y^i\partial/\partial x^i \in T_x\mathcal{M}$ of the Riemannian manifold $\mathcal{M}$ is

$$K(X \wedge Y) = \frac{g(R(X,Y)Y, X)}{|X \wedge Y|^2}$$

where $|X \wedge Y|^2 = g(X,X)g(Y,Y) - g(X,Y)^2$.

Remark. Sectional curvature determines the whole curvature tensor.

Proof. Given $g(R(X,Y)Z,W)$, we can express this entirely by K. ⊛

Remark (Gauss curvature). For $\dim \mathcal{M} = 2$,

$$R_{ijk\ell} = K(g_{ik}g_{j\ell} - g_{ij}g_{k\ell})$$

where here, K is called the *Gauss curvature*. Since $T_x\mathcal{M}$ contains only one plane, i.e., $T_x\mathcal{M}$ itself.

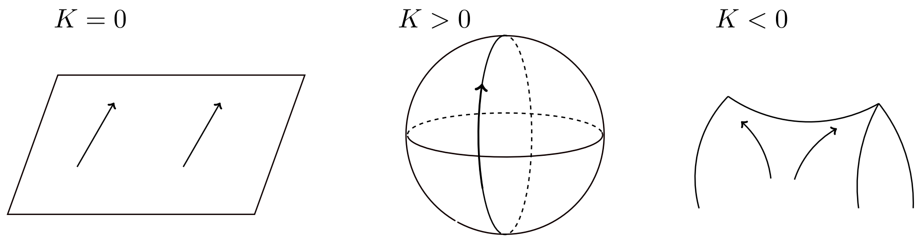
Definition 3.3.7 (Space form). Let $(\mathcal{M}, g)$ be a Riemannian manifold. $\mathcal{M}$ is called a *space of constant sectional curvature* or a *space form* if $K(X \wedge Y)$ is a constant for all linearly independent tangent vectors $X, Y \in T_x\mathcal{M}$ and for all $x \in \mathcal{M}$.

Definition 3.3.8 (Spherical). A space form is called *spherical* if $K > 0$.

Definition 3.3.9 (Flat). A space form is called *flat* if $K = 0$.

Definition 3.3.10 (Hyperbolic). A space form is called *hyperbolic* if $K > 0$.

Definition 3.3.11 (Einstein manifold). $\mathcal{M}$ is called an *Einstein manifold* if $R_{ik} = cg_{ik}$ for c being a constant which does not depend on the choice of local coordinates.



Remark. Every manifold with constant sectional curvature is an Einstein manifold.

Definition 3.3.12 (Flat)**.** A connection ∇ on $T\mathcal{M}$ is called *flat* if each point in $\mathcal{M}$ has a neighborhood U with local coordinates for which all the coordinate vector fields $\partial/\partial x^i$ are parallel, i.e., $\nabla\partial/\partial x^i = 0$.

Theorem 3.3.1. A connection ∇ on $T\mathcal{M}$ is flat if and only if its curvature and torsion vanish identically.

Proof. Flat connection implies $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$, hence all $\Gamma^k_{ij} = 0$, so T, R vanish. Conversely, find the local coordinates such that $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ for all i, j. Then, by computation with Frobenius theorem, we're done. ■

Theorem 3.3.2 (Schur theorem)**.** Let $(\mathcal{M}, g)$ be a Riemannian manifold with $\dim \mathcal{M} \geq 3$.

(a) If the sectional curvature of $\mathcal{M}$ is constant at each point, i.e., $K(X \wedge Y) = f(x)$ for $X, Y \in T_x\mathcal{M}$. Then $f(x)$ is a constant on $\mathcal{M}$, so $\mathcal{M}$ is a space form.

(b) If the Ricci curvature is a constant at each point, i.e., $R_{ik} = c(x)g_{ik}$, then $c(x)$ is a constant and $\mathcal{M}$ is an Einstein manifold.

Remark. Schur theorem says that the isotropy of a Riemannian manifold, i.e., the property that at each point, all directions are geometrically indistinguishable, implies the homogeneity, i.e., all points are geometrically indistinguishable.

Note. A point-wise property implies a global one!

3.3.3 Covering Maps

Definition 3.3.13 (Covering map)**.** Let $\mathcal{M}, \widetilde{\mathcal{M}}$ be 2 manifolds.a map $p\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ is a *covering map* if

(a) p is smooth and surjective, and

(b) for all $m \in \mathcal{M}$, there exists a neighborhood U at m in $\mathcal{M}$ with $p^{-1}(U) = \coprod_{i \in I} U_i$ with $p\colon U_i \to U$ being a diffeomorphism and U_i are disjoint open subsets of $\widetilde{\mathcal{M}}$.

Notation (Covering space)**.** $\widetilde{\mathcal{M}}$ in Definition 3.3.13 is called the *covering space*.

Notation (Universal covering space)**.** A covering space is *universal* if it's simply connected.

Definition 3.3.14 (Riemannian covering map)**.** Let $(\mathcal{M}, g), (\mathcal{N}, h)$ be 2 Riemannian manifolds. A map $p\colon \mathcal{N} \to \mathcal{M}$ is a *Riemannian covering map* if p is a smooth covering map and is a local isometry.

Proposition 3.3.1. Let $p \colon \mathcal{N} \to \mathcal{M}$ be a smooth covering map. For every Riemannian metric g on $\mathcal{M}$, there exists a unique Riemannian metric h on $\mathcal{N}$ such that p is a Riemannian covering map.

Note. The converse of the above is generally not true.

Example. Every space covers itself trivially.

Example. $\mathbb{R}$ is the universal covering space of S^1.

Example. $U(n)$ has universal covers $U(n) \times \mathbb{R}$.

Example. S^n is a double cover for $P_n(\mathbb{R})$ and is universal for $n > 1$.

Proposition 3.3.2. Let $(\mathcal{N}, h)$ be a Riemannian manifold. Let G be a free and proper group of isometries of $(\mathcal{N}, h)$. Then there exists a unique Riemannian manifold $(\mathcal{M}, g)$ on the quotient manifold $\mathcal{M} = \mathcal{N} / G$ such that the connected projection $p \colon \mathcal{N} \to \mathcal{M}$ is a Riemannian covering map.

Appendix

Appendix A

Lie Groups and Lie Algebra

A.1 Lie Groups

Lie groups are an important topic to study for Riemannian geometry, hence we now introduce it now.

Definition A.1.1 (Lie group). A *Lie group* is a group G with a differentiable structure such that the mapping $G \times G \to G$ given by $(x, y) \to xy^{-1}$, $x, y \in G$, is differentiable.

Definition (Transformation). Let G be a Lie group.

Definition A.1.2 (Left transformation). The *translations from the left* $L_x \colon G \to G$ is defined as $L_x(y) = xy$.

Definition A.1.3 (Right transformation). The *translations from the right* $R_x \colon G \to G$ is defined as $R_x(y) = yx$.

Remark. Both L_x and R_x are diffeomorphisms.

In the following discussion, let G be a Lie group. Turns out that G admits some nice properties on left invariant vector fields.

Definition (Invariant of Riemannian metric). Let g be a Riemannian metric on G.

Definition A.1.4 (Left invariant). g is *left invariant* if

$$\langle u, v \rangle_y = \langle \mathrm{d}(L_x)_y u, \mathrm{d}(L_x)_y v \rangle_{L_x(y)}$$

for all $x, y \in G$, $u, v \in T_y G$, i.e., L_x is an isometry.

Definition A.1.5 (Right invariant). g is *right invariant* if

$$\langle u, v \rangle_y = \langle \mathrm{d}(R_x)_y u, \mathrm{d}(R_x)_y v \rangle_{R_x(y)}$$

for all $x, y \in G$, $u, v \in T_y G$, i.e., R_x is an isometry.

Definition A.1.6 (Bi-invariant). g is *bi-invariant* if it's both right and left invariant.

Definition (Invariant of vector field). Let X be a vector field on G.

Definition A.1.7 (Left invariant). X is *left invariant* if $dL_x X = X$ for all $x \in G$.

Definition A.1.8 (Right invariant). X is *right invariant* if $dR_x X = X$ for all $x \in G$.

Definition A.1.9 (Bi-invariant). X is *bi-invariant* if it's both right and left invariant.

As we mentioned, the left invariant vector fields are completely determined by their values at a single point of G, which allows us to introduce an additional structure on the tangent space to the neutral element $e \in G$ in the following manner.

To each vector $X_e \in T_e G$, we associate the left invariant X defined by

$$X_a := dL_a X_e, \quad a \in G.$$

A.2 Lie Algebras

Let X, Y be left invariant vector fields on G. Since for each $x \in G$ and for any differentiable function f on G,

$$dL_x[X,Y]f = [X,Y](f \circ L_x) = X(dL_x Y)f - Y(dL_x X)f = (XY - YX)f = [X,Y]f,$$

i.e., $[X, Y]$ is again a left invariant vector field if X, Y are. Now, if $X_e, Y_e \in T_e G$, we put $[X_e, Y_e] = [X, Y]_e$.

Definition A.2.1 (Lie algebra). The *Lie algebra* of G, denoted by $\mathfrak{g}$, is the vector space $T_e G$ with the bracket $[\cdot, \cdot]$.

Note. The elements in the Lie algebra $\mathfrak{g}$ will be thought of either as vectors in $T_e G$ or as left invariant vector fields on G.

To introduce a left invariant metric on g, take any arbitrary inner product $\langle \cdot, \cdot \rangle_e$ on $\mathfrak{g}$ and define

$$\langle u, v \rangle_x := \langle (dL_{x^{-1}})_x(u), (dL_{x^{-1}})_x(v) \rangle_e \tag{A.1}$$

for $x \in G$, $u, v \in T_x G$. Since L_x depends differentiably on x, this is actually a Riemannian metric, which is clearly left invariant.

Remark. We can also construct a right invariant metric on G, and if G is compact, G possesses a bi-invariant metric.

One important characterization for G having a bi-invariant metric is that the inner product that the metric determines on $\mathfrak{g}$ satisfies the following relation.

Proposition A.2.1. If G has a bi-invariant metric, then for any $U, V, X \in \mathfrak{g}$, the inner product that the metric determines on $\mathfrak{g}$ satisfies

$$\langle [U, X], V \rangle = - \langle U, [V, X] \rangle .$$

Proof. See do Carmo [FC13, Page 40, 41]. ■

The important point about this relation is that it characterizes the bi-invariant metrics of G in the following sense.

Remark. If a positive bilinear form $\langle \cdot, \cdot \rangle_e$ defined on $\mathfrak{g}$ satisfies this relation, then the Riemannian metrics defined on G by Equation A.1 is bi-invariant.

Appendix B

Algebra

This chapter will collect some notion about algebras which you might not be familiar with.

B.1 Modules

Definition B.1.1 (Left module)**.** Suppose R is a ring with 1. A *left R-module M* consists of an Abelian group $(M, +)$ and n operation $\cdot \colon R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$,

(a) $r \cdot (x + y) = r \cdot x + r \cdot y$;

(b) $(r + s) \cdot x = r \cdot x + s \cdot x$;

(c) $(rs) \cdot x = r \cdot (s \cdot x)$;

(d) $1 \cdot x = x$.

Note. A *right R-module M* can also be defined similarly by consider $\cdot \colon M \times R \to M$.

Definition B.1.2 (Module)**.** If R is commutative, then the left and right R-module M are the same, and we call M a *module*.

Intuition. We're basically relaxing the notion of $\mathbb{F}$-vector field, but this time, the field $\mathbb{F}$ is replaced by a ring R.

Remark. The most noticeable difference between a module and a vector field is that a module usually don't have a basis.

B.1.1 The $C^\infty(\mathcal{M})$-Module Viewpoint of Tensor Fields

The reason why we introduce the notion of module is because of the following: we can understand tensor-field better in the following way. Firstly, let's introduce the so-called tensor bundles.

Definition B.1.3 (Tensor bundle)**.** A *tensor bundle* is a fiber bundle where the fiber is the product of any number of tangent spaces and/or cotangent spaces.

So in a tensor bundle, the fiber is a vector space and the tensor bundle is a special kind of vector bundle.[1] Then, we have the following notion similar to the way we define vector fields.

Definition B.1.4 (Tensor field)**.** A *(r, s)-tensor field T* is a section of a tensor bundle.

To elaborate this idea, observe that $\Gamma(T\mathcal{M}) = \{X \colon \text{vector fields on } \mathcal{M}\}$ is actually a $C^\infty(\mathcal{M})$-module:

[1] There are vector bundles which are not tensor bundles.

Claim. $\Gamma(T\mathcal{M})$ carries a natural $C^\infty(\mathcal{M})$-module structure.

Proof. Firstly, observe that $C^\infty(\mathcal{M}) = \big((C^\infty(\mathcal{M}), +, \cdot)\big)$ is not a field but a ring.[a] Then, naturally, the $C^\infty(\mathcal{M})$-module $(\Gamma(T\mathcal{M}), \oplus, \odot)$ where

- $\oplus$: $(X \oplus \widetilde{X})(f) \coloneqq (Xf) + \widetilde{X}(f)$;
- $\odot$: $(g \odot X)(f) \coloneqq g \cdot X(f)$,

for $X, \widetilde{X} \in \Gamma(T\mathcal{M})$, $g, f \in C^\infty(\mathcal{M})$. ⊛

[a] Since given $f \in C^\infty(\mathcal{M})$, we might not have f^{-1}.

Notation. Notice that given a vector field $X\colon \mathcal{M} \to T\mathcal{M}$ with $p \mapsto X(p)$, we let

$$Xf\colon \mathcal{M} \to \mathbb{R}, \quad p \mapsto X(p)f.$$

Remark. This makes sense since we can't always do things globally, e.g., Hairy ball theorem. More precisely, we can't choose a basis $X_1, \dots, X_d \in \Gamma(T\mathcal{M})$ for our vector field globally as we already know.

Then, similarly, we can define $\Gamma(T^*\mathcal{M})$, i.e., the set of "convector field,"[2] which is again a $C^\infty(\mathcal{M})$-module.

Example. Given $f \in C^\infty(\mathcal{M})$, let $\mathrm{d}f\colon \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$ with $X \mapsto \mathrm{d}f(X) \coloneqq Xf$. We see that $\mathrm{d}f$ is a $(0,1)$-tensor field since $\mathrm{d}f$ is linear.

Then, in this view point,[3] tensor field T is a $C^\infty(\mathcal{M})$ multilinear map

$$T\colon \underbrace{\Gamma(T^*\mathcal{M}) \times \ldots \times \Gamma(T^*\mathcal{M})}_{r} \times \underbrace{\Gamma(T\mathcal{M}) \times \ldots \Gamma(T\mathcal{M})}_{s} \to C^\infty(\mathcal{M}).$$

Remark. Comparing to Definition 2.6.13, they're essentially the same.

[2] We won't define it formally, but it's defined similarly.
[3] This is a more abstract way (but often useful) to characterize a tensor field.

Bibliography

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