

MATH602  
Real Analysis II

Pingbang Hu

September 6, 2022

### **Abstract**

This is a graduate level functional analysis taught by [Joseph Conlon](#). The prerequisites include linear algebra, complex analysis and also [real analysis](#). We'll use Peter Lax[[Lax02](#)] and Reed-Simon[[RS80](#)] as textbooks.

This course is taken in Fall 2022, and the date on the covering page is the last updated time.

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Linear Space . . . . .	2
1.2	Quotient Space . . . . .	2
1.3	Normed Spaces . . . . .	3
<b>2</b>	<b>Banach Spaces</b>	<b>7</b>
2.1	Completion of Normed Space to Banach Space . . . . .	8
<b>A</b>	<b>Additional Proofs</b>	<b>10</b>

# Chapter 1

## Introduction

### Lecture 1: Introduction

We first briefly review different kinds of vector spaces.

30 Aug. 14:30

#### 1.1 Linear Space

**Definition 1.1.1** (Linear vector space). A set with operations of addition and multiplication (by a scalar) is called a *linear vector space*.

**Example.** Denote the multiplicative scalar by  $\lambda$ , then

- $\lambda \in \mathbb{R} \Rightarrow$  real vector space.
- $\lambda \in \mathbb{C} \Rightarrow$  complex vector space

**Lemma 1.1.1.** Given  $E$  a linear vector space, if  $v, w \in E$ ,  $\lambda, \mu \in \mathbb{R}$  (or  $\mathbb{C}$ ), then  $\lambda v + \mu w \in E$ .

we also have usual rules of associativity and commutativity.

**Example.**  $\mathbb{R}^n$  a  $n$  dimensional linear vector space,  $\mathbb{C}^n$  a  $n$  dimensional complex linear vector space.

We concentrate on  $\infty$  dimensional linear vector space.

**Example.** Let  $K$  is a compact Hausdorff space, then

$$E = \{f: K \rightarrow \mathbb{R} \mid f(\cdot) \text{ is continuous}\}.$$

We then see that  $E$  is an  $\infty$  dimensional real linear vector space.

#### 1.2 Quotient Space

Observe that a linear vector space can have many subspaces. Say  $E$  is a linear vector space, and  $E_1 \subset E$  where  $E_1$  is a proper subspace, i.e.,  $E_1 \neq E$ .

**Definition 1.2.1** (Quotient Space). The *quotient space*  $E/E_1$  is the set of equivalence classes of vectors in  $E$  where equivalence is given by  $x \sim y$  if  $x - y \in E_1$ . Additionally, denote  $[x]$  as the equivalence class of  $x \in E$ , i.e.,  $[x] = x + E_1$ .

Note that  $E/E_1$  is a linear vector space since if  $x_1 + x_2 \in E$ ,  $[x_1] + [x_2] = [x_1 + x_2]$ , and also,  $\lambda[x] = [\lambda x]$  for  $\lambda \in \mathbb{R}$  or  $\mathbb{C}$ , i.e.,  $v, w \in E/E_1$ ,  $\lambda, \mu \in \mathbb{R}$  or  $\mathbb{C}$  implies  $\lambda v + \mu w \in E$ .

**Definition 1.2.2 (Codimension).** If  $E / E_1$  has finite dimension, then the dimension of  $E / E_1$  is called the *codimension* of  $E_1$  in  $E$ .

**Example.** There exists the case that  $\dim(E) = \infty$ ,  $\dim(E_1) < \infty$  where  $\dim(E / E_1) < \infty$ .

**Proof.** Let  $E = \{f: K \rightarrow \mathbb{R} \mid f(\cdot) \text{ continuous}\}$ , and  $E_1 = \{f \in E: f(k_1) = 0\}$  where  $k_1 \in K$  is fixed. We see that the dimension of  $E / E_1$  is exactly 1 since  $E / E_1$  is the set of constant functions.  $\circledast$

**Theorem 1.2.1.** If  $E$  is finite dimensional, then  $\text{codim}(E_1) + \dim(E_1) = \dim(E)$

**Definition 1.2.3 (Linear operator).** A map  $T: E \rightarrow F$  between 2 linear spaces is a *linear operator* if it preserves the properties of addition and multiplication by a scalar, i.e.,  $T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$  for  $v, w \in E$  and  $\lambda, \mu \in \mathbb{R}$  or  $\mathbb{C}$ .

**Definition.** Given a linear operator  $T: E \rightarrow F$  we have the following.

**Definition 1.2.4 (Kernel).** The *kernel* of  $T$  is the subspace  $\ker(T) = \{x \in E \mid Tx = 0\}$ .

**Definition 1.2.5 (Image).** The *image* of  $T$  is the subspace  $\text{Im}(T) = \{Tx \in F \mid x \in E\}$ .

## 1.3 Normed Spaces

We review some basic notions.

**Definition 1.3.1 (Norm).** Let  $E$  be a linear vector space. A *norm*  $\|\cdot\|: E \rightarrow \mathbb{R}$  on  $E$  is a function from  $E$  to  $\mathbb{R}$  with the properties:

- (a)  $\|x\| \geq 0$  and  $\|x\| = 0 \Leftrightarrow x = 0$ .
- (b)  $\|\lambda x\| = |\lambda| \|x\|$ ,  $\lambda \in \mathbb{R}$  or  $\mathbb{C}$ .
- (c)  $\|x + y\| \leq \|x\| + \|y\|$ .

**Notation (Dilation).** We say that the second condition is the *dilation* property.

**Definition 1.3.2 (Normed vector space).** A linear vector space  $E$  equipped with a norm  $\|\cdot\|$  is called a *normed vector space*.

**Remark (Induced metric space).** A normed vector space  $E$  induces a *metric space* with metric  $d(x, y) = \|x - y\|$ , where the metric has properties

- (a)  $d(x, y) \geq 0$ . Also,  $d(x, x) = 0$  and  $d(x, y)$  implies  $x = y$ .
- (b)  $d(x, y) = d(y, x)$ .
- (c)  $d(x, z) \leq d(x, y) + d(y, z)$ .

**Example (Bounded sequences  $\ell_\infty$ ).** Let  $\ell_\infty$  be the space of bounded sequences  $x = (x_1, x_2, \dots)$  with  $x_i \in \mathbb{R}$  for  $i = 1, 2, \dots$ . Then we define  $\|x\| = \|x\|_\infty = \sup_{i \geq 1} |x_i|$ .

**Example (Absolutely summable sequences  $\ell_1$ ).** Let  $\ell_1$  be the space of absolutely summable sequences  $x = (x_1, x_2, \dots)$  and  $\sum_{i=1}^{\infty} |x_i| < \infty$ . Then we define  $\|x\| = \|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$ .

**Example (Continuous functions  $C(k)$ ).** The space  $C(k)$  of continuous functions  $f: K \rightarrow \mathbb{R}$  where  $K$  is compact Hausdorff. Then we define  $\|f\| = \|f\|_{\infty} = \sup_{x \in K} |f(x)|$ .

### 1.3.1 Geometry of Normed Spaces

**Definition 1.3.3 (Ball).** A (closed) *ball* centered at a point  $x_0 \in E$  with radius  $r > 0$  is the set  $B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\}$ .

**Definition 1.3.4 (Sphere).** The *sphere* centered at  $x_0$  with radius  $r > 0$  is the set  $S(x_0, r) = \{x \in E \mid \|x - x_0\| = r\}$ .

**Remark.** We see that  $S(x_0, r)$  is the **boundary** of  $B(x_0, r)$ , i.e.,  $S(x_0, r) = \partial B(x_0, r)$ .

**Note (Nonequivalency in infinite dimensional spaces).** We know that in finite dimensional, all **norms** are equivalent, which is not true for infinite dimensional vector spaces.

This has something to do with the geometry of **balls**.

Explicitly, **balls** can have different geometries depending on the properties of the **norms**. We see that an  $\|\cdot\|_{\infty}$  can have multiple supporting hyperplane at the corner, while for an  $\|\cdot\|_2$  can have only one at each point.

Also, unit **balls** for  $\|\cdot\|_1$  is also a **square**, where we have

$$B(0, 1) = \{x = (x_1, x_2, \dots) \mid -1 < y_{\epsilon} < 1 \forall \epsilon\}$$

such that  $y_{\epsilon} = \sum_{i=1}^{\infty} \epsilon_i x_i$ ,  $\epsilon_i = \pm 1$  and  $\epsilon = (\epsilon_1, \epsilon_2, \dots)$ .

We see that different **norms** give different geometry, but they have important common features, most notably, convexity properties.

**Definition 1.3.5 (Convex set).** Given  $E$  a **linear vector space**, a set  $K \subset E$  is *convex* if  $x, y \in K$  and  $0 \leq \lambda \leq 1$ , we have  $\lambda x + (1 - \lambda)y \in K$ .

**Definition 1.3.6 (Convex function).** Given  $E$  a **linear vector space**, a function  $f: E \rightarrow \mathbb{R}$  is called *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for  $x, y \in E$ ,  $0 \leq \lambda \leq 1$ .

**Remark.** If  $f: E \rightarrow \mathbb{R}$  is a **convex function**, then for any  $M \in \mathbb{R}$  the set  $\{x \in E \mid f(x) \leq M\}$  is **convex**.

The upshot is that **norms** are **convex**, and the unit **balls** are **convex** as well.

## Lecture 2: Banach Spaces and Completion

Let's first see a proposition.

01 Sep. 14:30

**Proposition 1.3.1.** Let  $\{E, \|\cdot\|\}$  be a **normed linear space**. Then the norm is **convex** and continuous.

**Proof.** Let  $f: E \rightarrow \mathbb{R}$  be  $f(x) = \|x\|$ . Then  $f(x) - f(y) = \|x\| - \|y\| \leq \|x - y\|$ , which implies  $|f(x) - f(y)| \leq \|x - y\|$  for  $x, y \in E$ , i.e.,  $f$  is Lipschitz continuous. For **convexity**, let  $0 < \lambda < 1$ ,

we have

$$f(\lambda x + (1 - \lambda)y) = \|\lambda x + (1 - \lambda)y\| \leq \|\lambda x\| + \|(1 - \lambda)y\| = \lambda \|x\| + (1 - \lambda) \|y\| = \lambda f(x) + (1 - \lambda)f(y).$$

■

**Note.** Note that  $f(\cdot)$  is continuous implies the closed ball

$$B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\} = \{x \in E \mid f(x - x_0) \leq r\}$$

is closed in topology of  $E$ . Also,  $f(\cdot)$  is **convex** implies  $B(x_0, r)$  is **convex**.

**Remark.** If  $f: E \rightarrow \mathbb{R}$  is **convex**, then the sets  $\{x \in E \mid f(x) \leq M\}$  is also **convex**. However, it's possible to have non-**convex functions**  $f$  such that all sets  $\{x \in E \mid f(x) \leq M\}$  are **convex**.

**Example.** Take  $f(x) = |x|^p$  for  $x \in \mathbb{R}$  and  $p > 0$ . We see that  $f$  is **convex** if  $p > 1$ , and non-**convex** if  $p < 1$ . The sets  $\{x \in \mathbb{R} \mid f(x) \leq M\}$  all **convex** since it's independent of  $p$ .

**Lemma 1.3.1.** Suppose  $x \mapsto \|x\|$  satisfies

- (a)  $\|x\| \geq 0$  and  $\|x\| = 0 \Leftrightarrow x = 0$ .
- (b)  $\|\lambda x\| = |\lambda| \|x\|$ ,  $\lambda \in \mathbb{R}$  or  $\mathbb{C}$ .
- (c) The unit ball  $B(0, 1)$  is **convex**.

Then  $f(x) = \|x\|$  satisfies the triangle inequality  $\|x + y\| \leq \|x\| + \|y\|$ .

**Proof.** We see that if the third condition is true, then for  $u, v \in B(0, 1)$  and  $0 < \lambda < 1$ , we have  $\lambda u + (1 - \lambda)v \in B(0, 1)$ . Let  $x, y \in E$ , and

$$\lambda = \frac{\|x\|}{\|x\| + \|y\|} \Rightarrow 1 - \lambda = \frac{\|y\|}{\|x\| + \|y\|}.$$

By letting  $u = x / \|x\|$ ,  $v = y / \|y\|$  we see that

$$\lambda u + (1 - \lambda)v = \frac{\|x\|}{\|x\| + \|y\|} \frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|} \frac{y}{\|y\|} \in B(0, 1) \Rightarrow \left\| \frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|} \right\| \leq 1.$$

From the second condition, it follows that  $\|x + y\| \leq \|x\| + \|y\|$ , which is the triangle inequality. ■

**Remark.** If  $x \mapsto \|x\|$  satisfies the first two condition and is a **convex**, then it satisfies the triangle inequality.

**Proof.** Since  $\frac{1}{2} \|x + y\| = \left\| \frac{x}{2} + \frac{y}{2} \right\| \leq \frac{1}{2} \|x\| + \frac{1}{2} \|y\|$ . ⊗

Now, given a **quotient space**  $E / E_1$ , the question is can we try to define a **norm**?

**Problem 1.3.1.** On  $E / E_1$ , is  $\|[x]\| := \inf_{y \in E_1} \|x + y\|$  a **norm**?

**Answer.** We see that if  $x \in \overline{E_1} \setminus E_1$ , then  $\|[x]\| = 0$  but  $[x] \neq 0 \in E / E_1$ . ⊗

**Note.** Notice the difference from finite dimensional situation. All finite dimensional spaces  $E_1$  are closed but not in general if  $E_1$  has  $\infty$  dimensions.

**Example.** Let  $\ell_1(\mathbb{R})$  be the sequence of  $x_n$  for  $n \geq 1$  in  $\mathbb{R}$  such that  $\sum_{i=1}^{\infty} |x_i| \leq \infty$ . Define

$$\|x\|_1 := \sum_{i=1}^{\infty} |x_i|,$$

and let  $E_1$  be all sequences with finite number of the  $x_n$  are nonzero. We see that  $\overline{E_1} = \ell_1(\mathbb{R})$  is infinite dimensional.

**Proposition 1.3.2.** Let  $\{E, \|\cdot\|\}$  be a **normed space** and  $E_1 \subseteq E$ ,  $E_1$  is closed. Then

$$\|\cdot\| : E/E_1 \rightarrow \mathbb{R}, \quad \|[x]\| = \inf_{y \in E_1} \|x + y\|$$

is a **norm** on  $E/E_1$ .

**Proof.** If  $\|[x]\| = 0$ , then  $\inf_{y \in E_1} \|x - y\| = 0$ , which implies  $x \in E_1$  since  $E_1$  is closed, so  $[x] = 0$ . Also, since

$$\|\lambda[x]\| = \inf_{y \in E_1} \|\lambda x + y\| = \inf_{z \in E_1} \|\lambda x + \lambda z\| = |\lambda| \inf_{z \in E_1} \|x + z\| = |\lambda| \|[x]\|,$$

the dilation property is satisfied. Finally, for triangle inequality, we have

$$\|[x] + [y]\| = \inf_{x_1, y_1 \in E_1} \|x + y + x_1 + y_1\| \leq \inf_{x_1 \in E_1} \|x + x_1\| + \inf_{y_1 \in E_1} \|y + y_1\| = \|[x]\| + \|[y]\|.$$

■

**Remark.** This shows that the only obstacle for this kind of **norm** being an actual **norm** is the closeness of  $E_1$ .



## Chapter 2

# Banach Spaces

**Definition 2.0.1** (Banach space). A **linear normed space** is a *Banach space* if it's complete, i.e., every Cauchy sequence converges.

**Note.** If  $x_n \in E$ ,  $n \geq 1$  is a sequence with property such that  $\lim_{m \rightarrow \infty} \sup_{n \geq m} \|x_n - x_m\| > 0$ , then  $\exists x_\infty \in E$  such that  $\lim_{n \rightarrow \infty} \|x_n - x_m\| = 0$ .

**Example.** The spaces  $\ell_1$ ,  $\ell_\infty$  and  $C(K)$  are **Banach spaces**.

We want to give a different criterion for showing  $\{E, \|\cdot\|\}$  is **Banach**. Let  $E$  be a **linear normed space** and  $\{x_\ell \mid \ell \geq 1\}$  a sequence in  $E$ .

**Definition 2.0.2** (Absolutely summable). A sequence is *absolutely summable* if  $\sum_{i=1}^{\infty} \|x_i\| < \infty$ .

**Theorem 2.0.1** (Criterion for completeness). A **normed space**  $\{E, \|\cdot\|\}$  is a **Banach space** if and only if every series in  $E$  converges.

**Proof.** We need to prove two directions.

( $\Rightarrow$ ) Suppose  $E$  is a **Banach space** and  $\{x_k \mid k \geq 1\}$  an **absolutely summable** series. Set  $s_n = \sum_{k=1}^n x_k$ ,  $n \geq 1$ , we want to show  $s_n$  is Cauchy, and if this is the case, completeness of  $E$  implies  $\exists s_\infty$  and  $\lim_{n \rightarrow \infty} \|s_n - s_\infty\| = 0$ . Let  $n > m$ , we see that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \leq \sum_{k=m+1}^{\infty} \|x_k\|.$$

Observe that  $\lim_{m \rightarrow \infty} \sum_{k=m+1}^{\infty} \|x_k\| = 0$ , we see that the sequence  $\{s_n\}$  is Cauchy.

( $\Leftarrow$ ) Conversely, suppose  $E$  is **not** complete. Then there exists a Cauchy sequence  $\{x_n \mid n \geq 1\}$  which does not converge. Furthermore, no subsequence of  $\{x_n \mid n \geq 1\}$  converges.<sup>a</sup> We now construct an **absolutely summable** series which does not converge.

Define  $n(1) \geq 1$  such that  $\|x_n - x_{n(1)}\| \leq \frac{1}{2}$  if  $n \geq n(1)$ , similarly, let  $n(2) > n(1)$  be such that  $\|x_n - x_{n(2)}\| \leq \frac{1}{2^2}$  if  $n \geq n(2)$ . In all, we have  $n(1) < n(2) < n(3) < \dots$  such that  $\|x_n - x_{n(k)}\| \leq \frac{1}{2^k}$  if  $n \geq n(k)$ . Define  $w_j := x_{n(j+1)} - x_{n(j)}$  for  $j = 1, 2, \dots$ . We see that

$$x_{n(m)} = x_{n(1)} + \sum_{j=1}^m w_j$$

---

for  $m = 1, 2, \dots$ , and  $\{x_{n(m)}\}$  does not converge, hence so does the series  $\sum_{j=1}^{\infty} w_j$ . However,  $\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$ , which implies  $\{w_j\}$  is **absolutely summable**. ■

---

<sup>a</sup>Otherwise, the whole sequence converges by the fact that it's Cauchy.

## 2.1 Completion of Normed Space to Banach Space

**Theorem 2.1.1.** Suppose  $E$  is a **normed space**. Then there exists a **Banach space**  $\hat{E}$  called a completion of  $E$  with the following properties:

- (a) There exists a linear map  $i: E \rightarrow \hat{E}$  such that  $\|ix\| = \|x\|$ .<sup>a</sup>
- (b)  $\text{Im}(i)$  is dense in  $\hat{E}$ , and  $\hat{E}$  is the smallest **Banach space** containing image of  $E$ .

---

<sup>a</sup>This is called an *isometric embedding* of  $E$  into  $\hat{E}$ .

## Lecture 3

07 Sep. 14:30

# Appendix

## Appendix A

### Additional Proofs

# Bibliography

- [Lax02] P.D. Lax. *Functional Analysis*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2002. ISBN: 9780471556046. URL: <https://books.google.com/books?id=18VqDwAAQBAJ>.
- [RS80] M. Reed and B. Simon. *I: Functional Analysis*. Methods of Modern Mathematical Physics. Elsevier Science, 1980. ISBN: 9780125850506. URL: <https://books.google.com/books?id=hInvAAAAMAAJ>.