MATH592
Introduction to Algebraic Topology

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Contents

Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

As previously seen. We saw that

- homotopy equivalence
- homotopy invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

0.1 CW Complexes

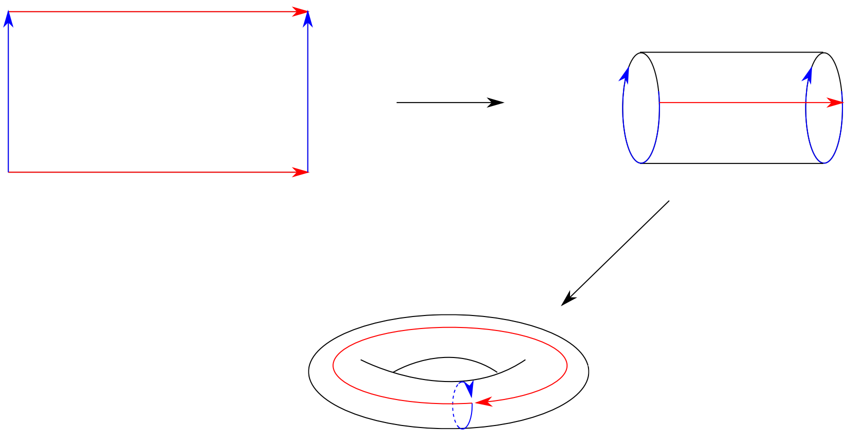
Example. Let's start with a few examples.

1. Constructing spheres:
   - S^1 (up to homeomorphism)

- S^2
  - glue boundary of 2-disk to a point
  - glue 2 disks onto a circle

Figure 1: title

- $T = S^1 \times S^1$



view as gluing instructions

$$\text{vertex } + 2 \text{ edges } + 2\text{-disks.}$$

Formally, we have the following definition.

Notation. Let D^n denotes a closed n-disk (or n-ball)

$$D^n \simeq \{x \in \mathbb{R}^n \colon \ \|x\| \leq 1\} .$$

And let S^n denotes an n-sphere

$$S^n \simeq \left\{x \in \mathbb{R}^{n+1} \colon \ \|x\| = 1\right\} .$$

Lastly, we call a point as a *0-cell*, and the interior of D^n $\mathrm{int}(D^n)$ for $n \geq 1$ as a *n-cell*.

Definition 0.1 (CW Complex). A *CW Complex* is a topological space constructed inductively as

1. X^0 (the 0-skeleton) is a set of discrete points.
2. We inductively construct the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α, where α is the index.

   The gluing instructions glued by an attaching map is that $\forall \alpha$, $\exists$ continuous map φ_α
   $$\varphi_\alpha \colon \partial D^n_\alpha \to X^{n-1},$$
   then
   $$X^n = \left(X^{n-1} \coprod_\alpha D^n_\alpha\right) \Big/ x \sim \varphi_\alpha(x)$$
   with identification $x \sim \varphi_\alpha(x)$ for all $x \in \partial D^n_\alpha$ with quotient topology.
3. $$X = \bigcup_{n=0} X^n,$$
   and let $\overline{w}$ denotes weak topology. Then
   $$u \subseteq X \text{ is open } \iff \forall n \; u \cap X^n \text{ is open } .$$
   If all cells have dimension less than N and a $\exists N$-cell, then $X = X^N$ and we call it N-dim CW complex.

Remark. We write $X^{(n)}$ for n-skeleton if we need to distinguish from the Cartesian product.

Example. Let's look at some examples.

1. 0-dim CW complex is a discrete space.
2. 1-dim CW complex is a graph.
3. A CW complex X is finite if it has finitely many cells.

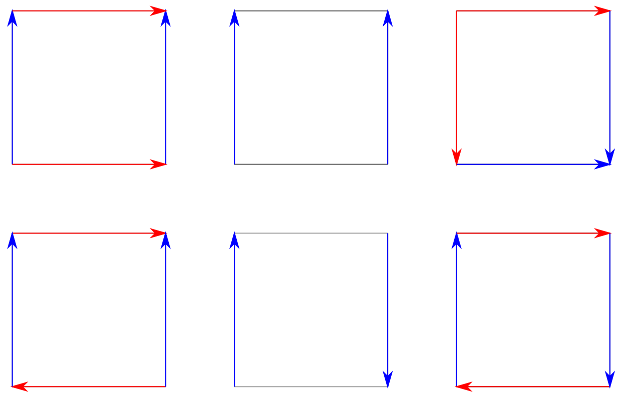
Definition 0.2 (CW subcomplex). A *CW subcomplex* $A \subseteq X$ is a closed subset equal to a union of cells

$$e^n_\alpha = \operatorname{int}(D^n_\alpha) .$$

Remark. This inherits a CW complex structure.

Check the images of attaching maps.

Exercise. Given the following gluing instruction:



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere, $\mathbb{R}P$.

Answer. We see that

1. Torus	2. Cylinder	3. 2-sphere
4. Klein bottle	5. Möbius band	6. $\mathbb{R}P$

Notation. We call the real projection space as $\mathbb{R}P$, and we also have so-called complex projection space, denote as $\mathbb{C}P$.

Lecture 5: Operation on Spaces

14 Jan. 10:00

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0.2 Operations on CW Complexes

0.2.1 Products

Given X, Y are CW complexes, then $X \times Y$ has a cell structure

$$\{e^m_\alpha \times e^n_\alpha : e^m_\alpha \text{ is a } m\text{-cell on } X, e^n_\alpha \text{ is a } n\text{-cell on } Y\} .$$

...

Remark. The product topology may not agree with the weak topology on the $X \times Y$. However, they do agree if X or Y is locally compact <u>or</u> if X and Y both have at most countably many cells.

Note. Notice that if the product is wild enough, then the product topology may not agree with the weak topology.

0.2.2 Wedge Sum

Given X, Y are CW complexes, and $x_0 \in X^0$, $y_0 \in Y^0$ (only points). Then we define

$$X \vee Y = X \coprod Y$$

with quotient topology.

Remark. $X \vee Y$ is a CW complex.

0.2.3 Quotients

Let X be a CW complex, and $A \subseteq X$ subcomplex (closed union of cells), then

$$X \big/ A$$

is a quotient space collapse A to one point.

Remark. $X \,/\, A$ is a CW complex.

0-skeleton

$$(X^0 - A^0) \coprod *$$

where $*$ is a point for A. Each cell of $X - A$ is attached to $(X \,/\, A)^n$ by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n \,/\, A^n$$

Lecture 6: A Foray into Category Theory

19 Jan. 10:00

1 Category Theory

We start with a definition.

Definition 1.1 (Category). A category $\mathcal{C}$ is 3 pieces of data

- A class of objects $\mathrm{Ob}(\mathcal{C})$
- $\forall(x, y) \in \mathrm{Ob}(\mathcal{C})$ a class of morphisms or arrows, $\mathrm{Hom}_{\mathcal{C}}(x, y)$.
- $\forall(x, y, z)$, there exists a composition law

$$\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z) \text{ such that } (f, g) \mapsto g \circ f,$$

and 2 axioms

- Associativity. $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms f, g, h where composites are defined.
- Identity. $\forall x \in \mathrm{Ob}(\mathcal{C})$ $\exists \mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$ such that

$$f \circ \mathrm{id}_x = f, \quad \mathrm{id}_x \circ g = g$$

  for all f, g where this makes sense.

Let's see some examples.

Example. We introduce some common category.

$\mathcal{C}$	$\mathrm{Ob}(\mathcal{C})$	$\mathrm{Mor}(\mathcal{C})$
set	sets X	all maps of sets
fset	finite sets	all maps
Gp	groups	group homos
Ab	Abelian groups	group homs
k-vect (Fix k)	vector spaces over k	k-linear maps
Rng	rings	rings maps
Top	Topological spaces	continuous maps
hTop	Topological spaces	homotopy classes of maps
Top_*	based topological spaces[1]	based maps[2]

Remark. Any **diagram** plus composition law.

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[r] & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd} .$$

[1] space + choice of distinguished base point $x_0 \in X$

[2] continuous maps that presence base point $f\colon (x, x_0) \to (y, y_0)$ such that

$$f\colon X \to Y, \quad f(x_0) = y_0$$

continuous.

Definition 1.2 (monic, epic). A morphism $f\colon M \to N$ is *monic* if

$$\forall g_1, g_2 \; f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$\begin{tikzcd} A \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

Dually, f is *epic* if

$$\forall g_1, g_2 \; g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & B \end{tikzcd}$$

Lemma 1.1. In set, Ab, Top, Gp, a map is monic if and only if f is injective, and epic if and only if f is surjective.

Proof. In set, we prove that f is monic if and only if f is injective. Suppose $f \circ g_1 = f \circ g_2$, then for any a,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence $g_1 = g_2$.

Now we prove another direction, with contrapositive. Let f be <u>not</u> injective, suppose $f(a) = f(b)$ and $a \neq b$. Then take

$$g_1 : * \mapsto a, \quad g_2 : * \mapsto b.$$

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1.1 Functor

Again, we start with a definition.

Definition 1.3 (Functor). Given $\mathcal{C}, \mathcal{D}$ be two categories. A (covariant) *functor*

$$F : \mathcal{C} \to \mathcal{D}$$

is

1. a map on objects
$$F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}).$$

2. maps of morphisms
$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(x, y) &\to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y)) \\ [f : X \to Y] &\mapsto [F(f) : F(X) \to F(Y)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(g) \circ F(g)$