

MATH681
Mathematical Logic

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Abstract

This is a graduate-level mathematical logic course taught by [Matthew Harrison-Trainor](#), aiming to obtain insights into all other branches of mathematics, such as algebraic geometry, analysis, etc. Specifically, we will cover model theory beyond the basic foundational ideas of logic.

While there are no required textbooks, some books do cover part of the material in the class. For example, Marker's *Model Theory: An Introduction* [[Mar02](#)], Hodges's *A Shorter Model Theory* [[HH97](#)], and Hinman's *Fundamentals of Mathematical Logic* [[Hin05](#)].



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Chapter 1

Language, Logic, and Structures

Lecture 1: Introduction to Mathematical Logic

The goal of mathematical logic is to obtain insights into other areas of mathematics – algebra, analysis, combinatorics, and so on, by formalizing the **process** of mathematics. 5 Jan. 14:30

Remark. More concretely, there are different branches:

- (a) Model Theory: Study subsets of an object defined by a **formula** (i.e., first-order logic).
- (b) Computability Theory / Recursion Theory: Formalizing what it means to have an algorithm and studying relative computability.
- (c) Set Theory: Study the structure of the mathematical universe.
- (d) Proof Theory: Study the syntactic nature of **proofs**.

In this class, we study model theory in nature; specifically, we will cover

- basic definitions of logic:
 - What is a **formula**?
 - What does it mean for a **formula** to be **true**?
 - What is a **proof**?
- **Soundness** & completeness theorems:
 - Anything **provable** is **true**.
 - Anything **true** is **provable**.
- Compactness theorem:
 - Non-standard objects exist.
- Using compactness theorem for applications:
 - **Chevalley's theorem**.

The main theme of this course will be *syntax* v.s. *semantics*:

Syntax	v.s.	Semantics
proofs		truth
form of a formula		mathematical structures
number and type of quantifiers		isomorphisms, embeddings

1.1 Syntax and Semantics

1.1.1 Languages and Structures

Let's start with the fundamental object, [language](#).

Definition 1.1.1 (Language). A *language* \mathcal{L} consists of:

- a set \mathcal{F} of function symbols f with arities n_f ;
- a set \mathcal{R} of relation symbols R with arities n_R ;
- a set \mathcal{C} of constant symbols c .

A [language](#) is also sometimes called a *signature*, in which case we use σ rather than \mathcal{L} .

Note. A constant is the same as a 0-ary function.

Remark. Any or all sets in [Definition 1.1.1](#) might be empty.

Example (Graph). The [language](#) of graphs, $\mathcal{L}_{\text{graph}} = \{E\}$ where E is a binary (2-ary) relation symbol.

Example (Ring). The [language](#) of rings, $\mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot, -\}$, where $0, 1$ are constants, $+, \cdot$ are binary functions, and $-$ is a unary function.

Example (Ordered ring). The [language](#) of ordered rings, $\mathcal{L}_{\text{ord}} = \mathcal{L}_{\text{ring}} \cup \{\leq\}$ where \leq is the binary relation for an ordered ring.

Then, given a [language](#), we can now interpret it in the following way.

Definition 1.1.2 (Structure). Given a [language](#) \mathcal{L} , an \mathcal{L} -*structure* \mathcal{M} consists of:

- a non-empty set M called the *universe*, *domain*, or *underlying set* of \mathcal{M} ;
- for each function symbol $f \in \mathcal{F}$, a function $f^{\mathcal{M}}: M^{n_f} \rightarrow M$;
- for each relation symbol $R \in \mathcal{R}$, a relation $R^{\mathcal{M}} \subseteq M^{n_R}$;
- for each constant symbol $c \in \mathcal{C}$, an element $c^{\mathcal{M}} \in M$.

Notation (Interpretation). The *interpretation* of symbols f, R, c in \mathcal{M} is $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$, respectively.

Basically, a [structure](#) gives meaning to the symbols from the [language](#), and we often write

$$\mathcal{M} = (M, f^{\mathcal{M}}, \dots, R^{\mathcal{M}}, \dots, c^{\mathcal{M}}, \dots) = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}: f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C}).$$

Notation. We usually use $\mathcal{M}, \mathcal{N}, \dots, \mathcal{A}, \mathcal{B}, \dots$ to refer to [structures](#), and M, N, \dots, A, B, \dots for the domains.^a

^aSome people use $|\mathcal{M}|$ for the domain of \mathcal{M} .

It's time to look at some examples.

Example. The rationals \mathbb{Q} and integers \mathbb{Z} are both $\mathcal{L}_{\text{ring}}$ -structures.

Proof. Clearly, the domain is the set of rationals, and naively, we let $+^{\mathbb{Q}} = +$ in \mathbb{Q} , $0^{\mathbb{Q}} = 0$ in

\mathbb{Q} , $1^{\mathbb{Q}} = 1$ in \mathbb{Q} , etc. In this way, $\mathbb{Q} = (\mathbb{Q}, 0, 1, +, \cdot, -)$ is an $\mathcal{L}_{\text{ring}}$ -structure. Similarly, $\mathbb{Z} = (\mathbb{Z}, 0, 1, +, \cdot, -)$ is as well. \circledast

While the language we have seen are all intuitively correct with their name, e.g., $\mathcal{L}_{\text{ring}}$, \mathcal{L}_{ord} , and $\mathcal{L}_{\text{graph}}$, they are really just the high-level abstraction of the objects in the subscript.

Example. Nothing forces an $\mathcal{L}_{\text{ring}}$ -structure to be a ring.

Proof. Since an $\mathcal{L}_{\text{ring}}$ -structure is just any structure with two binary functions, a unary function, and two constants interpreting the symbols of the language; hence we can define an $\mathcal{L}_{\text{ring}}$ -structure \mathcal{M} as

- $\mathcal{M} = \{0, 5, 11\}$;
- $0^{\mathcal{M}} = 5$;
- $1^{\mathcal{M}} = 11$;
- $+^{\mathcal{M}}$ is the constant function 0;
- $\cdot^{\mathcal{M}}$ is the function 5;
- $-^{\mathcal{M}}$ is the identity.

This is clearly not a ring since it fails nearly every axiom of a ring. \circledast

Note. Later, we will talk about theories that let us restrict to structures we want.

1.1.2 Embeddings and Isomorphisms

We can now consider the relation between structures.

Definition 1.1.3 (Embedding). Let \mathcal{M} and \mathcal{N} be \mathcal{L} -structures. A map $\eta: \mathcal{M} \rightarrow \mathcal{N}$ is an \mathcal{L} -embedding if it is one-to-one and preserves the interpretation of all symbols of \mathcal{L} :

- (a) for each function symbol $f \in \mathcal{F}$ of arity n_f , and $a_1, \dots, a_{n_f} \in M$,

$$\eta(f^{\mathcal{M}}(a_1, \dots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \dots, \eta(a_{n_f}));$$

- (b) for each relation symbol $R \in \mathcal{R}$ of arity n_R , and $a_1, \dots, a_{n_R} \in M$,

$$(a_1, \dots, a_{n_R}) \in R^{\mathcal{M}} \Leftrightarrow (\eta(a_1), \dots, \eta(a_{n_R})) \in R^{\mathcal{N}};$$

- (c) for each constant symbol $c \in \mathcal{C}$, $c^{\mathcal{M}} = c^{\mathcal{N}}$.

From the definition, an \mathcal{L} -embedding is an injection, and naturally, we have the following.

Definition 1.1.4 (Isomorphism). An \mathcal{L} -isomorphism is a bijective \mathcal{L} -embedding.

Definition 1.1.5 (Automorphism). An \mathcal{L} -automorphism of \mathcal{M} is an \mathcal{L} -isomorphism from \mathcal{M} to \mathcal{M} .

Definition. Let \mathcal{M} and \mathcal{N} be \mathcal{L} -structures. Suppose $M \subseteq N$ and the inclusion map $\iota: \mathcal{M} \hookrightarrow \mathcal{N}$ is an \mathcal{L} -embedding.

Definition 1.1.6 (Substructure). \mathcal{M} is a substructure of \mathcal{N} .

Definition 1.1.7 (Extension). \mathcal{N} is an extension of \mathcal{M} .

Example. Ring embeddings are $\mathcal{L}_{\text{ring}}$ -embeddings.

This generalizes the notions of embedding and isomorphism for many mathematical structures.

Remark. Asking that η be injective is the same as (b) in Definition 1.1.3 for the relation $=$ since

$$a = b \in \mathcal{M} \Leftrightarrow \eta(a) = \eta(b) \in \mathcal{N}.$$

The notion of substructure is language sensitive. For groups, there are two possible languages:

- (a) $\mathcal{L}_1 = \{e, \cdot\}$;
- (b) $\mathcal{L}_2 = \{e, \cdot, {}^{-1}\}$, i.e., with the unary inverse operation.

While both seem valid at the first glance, we should use the second one.

To see why, if we use \mathcal{L}_2 , the substructure of a group is the same thing as a subgroup. But if we use \mathcal{L}_1 , then $(\mathbb{N}, +, 0)$ is a substructure of $(\mathbb{Z}, +, 0)$, while \mathbb{N} is not a group for sure.¹

Similarly, we include $-$ in $\mathcal{L}_{\text{ring}}$ for a similar reason as in the previous example.

Example. An $\mathcal{L}_{\text{ring}}$ -substructure of a field will be a subring, not a subfield. If we want subfields, use $\mathcal{L}_{\text{ring}} \cup \{{}^{-1}\}$.^a

^aWe can set $0^{-1} = 0$, but never use this.

Lecture 2: Formulas and First-Order Logic

We start by asking that given a function symbol f of arity n , could we replace f with an $(n+1)$ -ary R relation to represent its graph? 10 Jan. 14:30

Example. Let \mathcal{L} be a language with only relation symbols. Let \mathcal{A} be an \mathcal{L} -structure. For any $B \subseteq A$, there is a substructure \mathcal{B} of \mathcal{A} with domain B .

Proof. For each relation symbol R , letting $R^{\mathcal{B}} = R^{\mathcal{A}} \cap B^{n_R}$ will make \mathcal{B} a substructure of \mathcal{A} . \circledast

The above is not true for function symbols though.

Example. If $G = (\mathbb{Z}, 0, +)$, then \mathbb{N} is not the domain of a subgroup. So if we took $\mathcal{L} = \{0, +, {}^{-1}\}$, where 0 is the unary relation, $+$ is the ternary relation, and ${}^{-1}$ is the binary relation, an \mathcal{L} -substructure of a group might not be a subgroup.

1.1.3 Terms

Intuitive, an \mathcal{L} -formula is an expression built using the symbols in a language \mathcal{L} , $=$, the logical connectives \wedge, \vee, \neg , and variable symbols v_1, v_2, \dots, x, y, z , and also quantifiers \exists and \forall .

Definition 1.1.8 (Term). Given a language \mathcal{L} , the set of \mathcal{L} -terms are defined inductively by:

- (a) each constant symbol is a term;
- (b) each variable symbol v_1, \dots is a term;
- (c) if f is a function symbol, and t_1, \dots, t_{n_f} are terms, then $f(t_1, \dots, t_{n_f})$ is a term.

If \mathcal{M} is an \mathcal{L} -structure, and t is a term involving only variables among v_1, \dots, v_n , then t has an interpretation $t^{\mathcal{M}}: M^n \rightarrow M$ as a function as follows. On input $a_1, \dots, a_n \in M$,

- (a) if t is a constant c , $t^{\mathcal{M}}(a_1, \dots, a_n) = c^{\mathcal{M}}$.
- (b) if t is a variable v_i , $t^{\mathcal{M}}(a_1, \dots, a_n) = v_i$;

¹Simply observe that both $(\mathbb{N}, 0, +), (\mathbb{Z}, 0, +)$ are \mathcal{L}_1 -structures.

(c) if t is $f(s_1, \dots, s_k)$, then $t^{\mathcal{M}}(a_1, \dots, a_n) = f^{\mathcal{M}}(s_1^{\mathcal{M}}(a_1, \dots, a_n), \dots, s_k^{\mathcal{M}}(a_1, \dots, a_n))$.

Intuition. We are basically substituting for variables and evaluating the expression.

Example. In $(\mathbb{R}, 0, 1, +, \cdot, -)$, a **term** is essentially just a polynomial with integer coefficients, assuming we interpret them in a ring. Technically, a **term** looks like

$$\cdot(+ (1, 1), + (x, y)),$$

but we will write **terms** the natural way, i.e.,

$$(1 + 1)(x + y).$$

Also, we will use \underline{n} or n to represent the **term** $\underline{n} = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}}$. So we could write the above **term** as $2 \cdot (x + y)$.

1.1.4 Formulas

Definition 1.1.9 (Formula). The set of \mathcal{L} -formulas is defined inductively:

- (a) If s, t are **terms**, then $s = t$ is a *formula*.
- (b) If R is a relation symbol of arity n_R and s_1, \dots, s_{n_R} are **terms**, then $R(s_1, \dots, s_{n_R})$ is a *formula*.
- (c) If f is a **formula**, then $\neg f$ is a *formula*.
- (d) If φ and ψ are **formulas**, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are *formulas*.
- (e) If φ is a **formula** and v_i are variables, then $\exists v_i \varphi$ and $\forall v_i \varphi$ are *formulas*.

Notation (Atomic formula). **Definition 1.1.9 (a)** and **(b)** are called *atomic formulas*.

Notation (Quantifier-free formula). **Definition 1.1.9 (a)**, **(b)**, **(c)**, and **(d)** are called *quantifier-free formulas*.

This logic is called *first-order logic* (FO logic), since the quantifiers range over elements of the **structures**, but not over, e.g., subsets.

Example. We can say that an element x of a ring has a square root by $\exists y \, y^2 = x$.

Example. A group is torsion of order 2 can be said by $\forall x \, x \cdot x = e$.

Example. We can write down all the field/group/... axioms as **formulas**.

Notice that for the first example, the **formula** $\exists y \, y^2 = x$ only has meaning if we assign what x is. In this case, we say that y is *bound* by $\exists y$. But this is local:

Example. Consider

$$y = 1 \wedge \exists y \, y^2 = x,$$

while the first appearance of y is free, the second appearance of y is bound by (in the scope of) $\exists y$.

While our definitions work perfectly fine with the above example, but sometimes we don't want this to happen. In such a case, we simply replace the bound instances of y with a new variable z . This idea of variables being free or bound is defined formally as follows.

Definition 1.1.10 (Free variable). The *free variables* $\text{FV}(\varphi)$ of a **formula** φ are defined inductively:

- (a) $\text{FV}(s = t)$ is the set of variables showing up in s or t .
- (b) $\text{FV}(R(s_1, \dots, s_{n_R}))$ is the set of variables showing up in s_1, \dots, s_{n_R} .
- (c) $\text{FV}(\neg\varphi) = \text{FV}(\varphi)$.
- (d) $\text{FV}(\varphi \wedge \psi) = \text{FV}(\varphi \vee \psi) = \text{FV}(\varphi) \cup \text{FV}(\psi)$.
- (e) $\text{FV}(\exists x \varphi) = \text{FV}(\forall x \varphi) = \text{FV}(\varphi) \setminus \{x\}$.

Example. $\text{FV}(\exists y y^2 = x) = \{x\}$.

Example. $\text{FV}(\forall x x \cdot x = e) = \emptyset$.

Definition 1.1.11 (Sentence). A **formula** φ is called a *sentence* if it has no **free variables**.

Notation. If φ is a **formula** with **free variables** among x_1, \dots, x_n we often write $\varphi(x_1, \dots, x_n)$.

Remark. So given $\varphi(x_1, \dots, x_n)$, we know that φ has no other **free variables** than x_1, \dots, x_n .

Example. It's valid to write $\varphi(x, y, z) := x = y$.

1.1.5 Truths

Finally, we define the notion of **truth**.

Definition 1.1.12 (Truth). Given an \mathcal{L} -**structure** \mathcal{M} , let $\varphi(x_1, \dots, x_n)$ be an \mathcal{L} -**formula** and let $a_1, \dots, a_n \in M$. Then we say φ is *true* of \bar{a} in \mathcal{M} ,^a denoted as $\mathcal{M} \models \varphi(\bar{a})$, as follows:

- (a) If φ is $s = t$, then $\mathcal{M} \models \varphi(\bar{a})$ if $s^{\mathcal{M}}(\bar{a}) = t^{\mathcal{M}}(\bar{a})$.
- (b) If φ is $R(t_1, \dots, t_{n_R})$, then $\mathcal{M} \models \varphi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{n_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$.
- (c) If φ is $\neg\psi$, then $\mathcal{M} \models \varphi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$.
- (d) If φ is $\psi_1 \wedge \psi_2$, then $\mathcal{M} \models \varphi(\bar{a})$ if $\mathcal{M} \models \psi_1(\bar{a})$ and $\mathcal{M} \models \psi_2(\bar{a})$.
- (e) If φ is $\psi_1 \vee \psi_2$, then $\mathcal{M} \models \varphi(\bar{a})$ if $\mathcal{M} \models \psi_1(\bar{a})$ or $\mathcal{M} \models \psi_2(\bar{a})$.
- (f) If φ is $\exists y \psi(\bar{x}, y)$, then $\mathcal{M} \models \varphi(\bar{a})$ if there's $b \in M$ such that $\mathcal{M} \models \psi(\bar{a}, b)$.
- (g) If φ is $\forall y \psi(\bar{x}, y)$, then $\mathcal{M} \models \varphi(\bar{a})$ if for all $b \in M$ such that $\mathcal{M} \models \psi(\bar{a}, b)$.

^aOr \mathcal{M} satisfies $\varphi(\bar{a})$.

Remark. Every **formula** is **true**, or its negation is.

Lecture 3: Logical Consequence and Equivalence

Notation (Material implication). The *material implication* $\varphi \rightarrow \psi$ between two **formulas** φ, ψ is an abbreviation of $\neg\varphi \vee \psi$.

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Notation. We use $\varphi \leftrightarrow \psi$ as an abbreviation of $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.

Essentially, \rightarrow and \leftrightarrow is different from \Rightarrow and \Leftrightarrow , where the former are only shown in [formula](#). Now, consider the [language of graphs](#) $\mathcal{L}_{\text{graph}} = \{E\}$, let's see some examples.

Example. An undirected graph can be written as

$$\forall x \forall y (xEy \rightarrow yEx).$$

Example. A vertex has at least three neighbors can be written as

$$\varphi(x) := \exists u \exists v \exists w (xEu \wedge xEv \wedge xEw \wedge u \neq v \wedge v \neq w \wedge u \neq w)$$

in non-reflexive graphs.

Example. For a vertex has exactly three neighbors,

$$\psi(x) := \exists u \exists v \exists w \forall y (xEu \wedge xEv \wedge xEw \wedge u \neq v \wedge v \neq w \wedge u \neq w \wedge (y = u \vee y = v \vee y = w \vee \neg yEx)).$$

Problem. Can we say that x has an even number of neighbors?

Answer. We can't. Some things are not expressible in FO logic. ⊛

Example. For a vertex x has a path of length 4 to y ,

$$\Theta(x, y) := \exists u \exists v \exists w (xEu \wedge uEv \wedge vEw \wedge wEy).$$

We can also express that there is a path of length at most 4.

Problem. Can we say that there is a path from x to y ?

Answer. We still can't! Not in FO logic (using [compactness theorem](#)). ⊛

Remark. When we prove results by induction on [formulas](#), we only need to prove for \neg, \wedge, \exists , instead of for both \wedge, \vee , and both \exists and \forall .

Proof. Since we can view $\varphi \vee \psi$ as an abbreviation for $\neg(\neg\varphi \wedge \neg\psi)$ and $\forall x \varphi$ as an abbreviation for $\neg(\exists x \neg\varphi)$. ⊛

Remark (Sheffer stroke). In fact, we can get \wedge, \vee, \neg from one logical connective, e.g., the *sheffer stroke* \uparrow , which is defined as

$$\varphi \uparrow \psi := \neg(\varphi \wedge \psi),$$

and we can use \uparrow to define \neg, \vee, \wedge .

Notation. Let Φ be a (possibly infinite) set of [sentences](#), we write $\mathcal{M} \models \Phi$ if $\mathcal{M} \models \varphi$ for all $\varphi \in \Phi$.

Definition 1.1.13 (Logical consequence). Let Φ be a set of [sentences](#), and φ be a [sentence](#). We say that φ is a *logical consequence* of Φ , written $\Phi \models \varphi$, if $\mathcal{M} \models \varphi$ whenever $\mathcal{M} \models \Phi$.

If $\Phi = \emptyset$ is the empty set, [Definition 1.1.13](#) is written as $\models \varphi$, i.e., φ is [true](#) in all \mathcal{L} -structures.²

²Recall that we always have a [language](#) \mathcal{L} implicitly.

Definition 1.1.14 (Equivalent). Given two formulas φ, ψ , $\varphi(\bar{x})$ and $\psi(\bar{x})$ are *equivalent* if

$$\models \forall \bar{x} (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Problem. Two sentences φ and ψ are *equivalent* if and only if $\varphi \models \psi$ and $\psi \models \varphi$.

DIY

As previously seen. \mathcal{A} is a *substructure* of \mathcal{B} , or $\mathcal{A} \subseteq \mathcal{B}$, means that $A \subseteq B$ and $\text{id}: A \hookrightarrow B$ is an \mathcal{L} -embedding.

Proposition 1.1.1. Suppose that \mathcal{A} is a *substructure* of \mathcal{B} , and $\varphi(\bar{x})$ is a *quantifier-free formula*. Let $\bar{a} \in \mathcal{A}$,^a then $\mathcal{A} \models \varphi(\bar{a})$ if and only if $\mathcal{B} \models \varphi(\bar{a})$.

^aFormally, we need to write \mathcal{A} to be the Cartesian product with a fixed length.

Proof. We start with *terms* by proving that if t is a *term* and $\bar{b} \in \mathcal{A}$, then $t^{\mathcal{A}}(\bar{b}) = t^{\mathcal{B}}(\bar{b})$. The proof is induction on *terms*.

- (a) If t is a constant symbol c , then $t^{\mathcal{A}}(\bar{b}) = c^{\mathcal{A}} = c^{\mathcal{B}} = t^{\mathcal{B}}(\bar{b})$.
- (b) If t is a variable x_i , then $t^{\mathcal{A}}(\bar{b}) = b_i = t^{\mathcal{B}}(\bar{b})$.
- (c) If t is a function symbol $f(s_1, \dots, s_n)$ where s_i are *terms*, then $t^{\mathcal{A}}(\bar{b}) = f^{\mathcal{A}}(s_1^{\mathcal{A}}(\bar{b}), \dots, s_n^{\mathcal{A}}(\bar{b}))$.
By the induction hypothesis, $s_i^{\mathcal{A}}(\bar{b}) = s_i^{\mathcal{B}}(\bar{b}) \in \mathcal{A}$, and hence

$$t^{\mathcal{B}}(\bar{b}) = f^{\mathcal{B}}(s_1^{\mathcal{B}}(\bar{b}), \dots, s_n^{\mathcal{B}}(\bar{b})) = f^{\mathcal{A}}(s_1^{\mathcal{A}}(\bar{b}), \dots, s_n^{\mathcal{A}}(\bar{b})) = t^{\mathcal{A}}(\bar{b}),$$

i.e., $f^{\mathcal{B}} \upharpoonright_{\mathcal{A}} = f^{\mathcal{A}}$, so $t^{\mathcal{A}}(\bar{b}) = t^{\mathcal{B}}(\bar{b})$.

Now we turn to *formulas*, and prove that for φ *quantifier-free*, then $\mathcal{A} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{B} \models \varphi(\bar{a})$ for $\bar{a} \in \mathcal{A}$. The proof is, again, induction on *formulas*.^a

- (a) If φ is $s = t$, then $s^{\mathcal{A}}(\bar{a}) = s^{\mathcal{B}}(\bar{a})$ and $t^{\mathcal{A}}(\bar{a}) = t^{\mathcal{B}}(\bar{a})$, so

$$\mathcal{A} \models \varphi(\bar{a}) \Leftrightarrow s^{\mathcal{A}}(\bar{a}) = t^{\mathcal{A}}(\bar{a}) \Leftrightarrow s^{\mathcal{B}}(\bar{a}) = t^{\mathcal{B}}(\bar{a}) \Leftrightarrow \mathcal{B} \models \varphi(\bar{a}).$$

- (b) If φ is $R(s_1, \dots, s_n)$, then

$$\mathcal{A} \models \varphi(\bar{a}) \Leftrightarrow (s_1^{\mathcal{A}}(\bar{a}), \dots, s_n^{\mathcal{A}}(\bar{a})) \in R^{\mathcal{A}} \Leftrightarrow (s_1^{\mathcal{B}}(\bar{a}), \dots, s_n^{\mathcal{B}}(\bar{a})) \in R^{\mathcal{B}} \Leftrightarrow \mathcal{B} \models \varphi(\bar{a}).$$

- (c) If φ is $\neg\psi$,

$$\mathcal{A} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{A} \not\models \psi(\bar{a}) \Leftrightarrow \mathcal{B} \not\models \psi(\bar{a}) \Leftrightarrow \mathcal{B} \models \varphi(\bar{a}),$$

where we use the induction hypothesis in the second \Leftrightarrow .

- (d) If φ is $\psi_1 \vee \psi_2$,

$$\mathcal{A} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{A} \models \psi_1(\bar{a}) \text{ or } \mathcal{A} \models \psi_2(\bar{a}) \Leftrightarrow \mathcal{B} \models \psi_1(\bar{a}) \text{ or } \mathcal{B} \models \psi_2(\bar{a}) \Leftrightarrow \mathcal{B} \models \varphi(\bar{a}),$$

where we use the induction hypothesis in the second \Leftrightarrow .

■

^aRecall that we only need to show one of \vee or \wedge , and here we pick \vee and treat \wedge as an abbreviation.

As previously seen (Characteristic). Given a field K , the *characteristic* p of K is the number of 1 you need to add 1 in order to get 0, i.e., $\underbrace{1 + 1 + \dots + 1}_p = 0$.

Example. Let L be a subfield of K , for each $p > 0$, $\varphi_p := \underbrace{1 + 1 + \dots + 1}_p = 0$, which says the characteristic p . φ_p is **quantifier-free**, so

$$L \models \varphi_p \Leftrightarrow K \models \varphi_p.$$

Example. Consider $\mathbb{Z} = (\mathbb{Z}, 0, 1, +, -, \cdot)$, and let $\varphi(x) := \neg \exists y \ y + y = x$. We see that $\mathbb{Z} \models \varphi(1)$ but $\mathbb{Q} \models \neg \varphi(1)$.

Proposition 1.1.2. Suppose that \mathcal{A} is a **substructure** of \mathcal{B} , and $\varphi(\bar{x}, y_1, \dots, y_n)$ is a **quantifier-free formula**. Let $\bar{a} \in \mathcal{A}$, then

- (a) if $\mathcal{A} \models \exists y_1 \dots \exists y_n \ \varphi(\bar{a}, y_1, \dots, y_n)$, then $\mathcal{B} \models \exists y_1 \dots \exists y_n \ \varphi(\bar{a}, y_1, \dots, y_n)$;
- (b) if $\mathcal{B} \models \forall y_1 \dots \forall y_n \ \varphi(\bar{a}, y_1, \dots, y_n)$, then $\mathcal{A} \models \forall y_1 \dots \forall y_n \ \varphi(\bar{a}, y_1, \dots, y_n)$.

Proof. Suppose that $\mathcal{A} \models \exists y_1 \dots \exists y_n \ \varphi(\bar{a}, y_1, \dots, y_n)$, so there are $b_1, \dots, b_n \in \mathcal{A}$ such that $\mathcal{A} \models \varphi(\bar{a}, b_1, \dots, b_n)$. Since φ is **quantifier-free**, so $\mathcal{B} \models \varphi(\bar{a}, b_1, \dots, b_n)$ from **Proposition 1.1.1**, and hence $\mathcal{B} \models \exists y_1 \dots \exists y_n \ \varphi(\bar{a}, y_1, \dots, y_n)$.

On the other hand, it's easy to see that (b) is implied by (a). ■

Notation. In **Proposition 1.1.2**, formulas as in (a) are called *existential* (\exists_1 or \exists) *formulas*; and formulas as in (b) are called *universal* (\forall_1 or \forall) *formulas*.

Example. Recall $\mathcal{L}_1 = \{e, \cdot\}$, $\mathcal{L}_2 = \{e, \cdot, {}^{-1}\}$.

- Associativity: $\forall x \forall y \forall z \ (xy)z = x(yz)$.
- Identity: $\forall x \ ex = xe$.

These are \forall -formulas in either language.

- Inverses in \mathcal{L}_1 : $\forall x \exists y \ xy = yx = e$, which is **not** an \forall -formula.
- Inverses in \mathcal{L}_2 : $\forall x \ xx^{-1} = x^{-1}x = e$, which is an \forall -formula.

Hence, group axioms in \mathcal{L}_1 are not universal, but in \mathcal{L}_2 they are.

The above discrepancy is the reason why \mathcal{L}_2 is better than \mathcal{L}_1 , i.e., \mathcal{L}_1 -substructure might not be a group.

Problem. Show that $\forall x \exists y \ xy = yx = e$ in the above example is not **equivalent** to an \forall -formula.

Lecture 4: Theories and Axioms

Example. Let $\mathcal{L}_1 = \{E\}$, where E is a binary relation representing edge relation; and $\mathcal{L}_2 = \{V, E, I\}$, where V, E are unary relations and I is a binary relation representing incidence such that $I(v, e)$ for $v \in V, e \in E$ means that v is a vertex on edge e . Then,

- Let G be a graph, viewed as an \mathcal{L}_1 -structure. A **substructure** of G is an induced subgraph $H \subseteq G$ such that any edge in G between two vertices of H is in H .
- If we view G as an \mathcal{L}_2 -substructure, a **substructure** is a subgraph H such that H has some vertices and edges from G .^a

^aBut there might be edges in H with no vertices, which can be fixed by having two functions $I_1(e) = v, I_2(e) = w$ when $e: v \rightarrow w$.

The difference is that for \mathcal{L}_1 , having an edge is **quantifier-free**, while in \mathcal{L}_2 is existential. To elaborate a bit further, for \mathcal{L}_2 , vEw is **quantifier-free**, while in \mathcal{L}_2 ,

$$\exists (v \in V \wedge w \in V \wedge e \in E \wedge I(v, e) \wedge I(w, e))$$

is not **quantifier-free**.

1.2 Theories

Let's start by the notion of **theory**.

Definition 1.2.1 (Theory). An \mathcal{L} -theory is a set of \mathcal{L} -sentences.

Definition 1.2.2 (Model). \mathcal{M} is a *model* of a **theory** T , written as $\mathcal{M} \models T$, if $\mathcal{M} \models \varphi$ for all $\varphi \in T$.

Note. Not every **theory** has a **model**, e.g., $\{\exists x x \neq x\}$.

The above note motivates the following.

Definition 1.2.3 (Satisfiable). A **theory** is *satisfiable* if it has a **model**.

Definition 1.2.4 (Elementary class). A class \mathcal{K} of \mathcal{L} -structures \mathcal{M} is called an *elementary class* if there is an \mathcal{L} -theory T such that

$$\mathcal{K} = \{\mathcal{M} \mid \mathcal{M} \models T\}.$$

One way to get an **elementary class** is to take an \mathcal{L} -structure \mathcal{M} and take the **full theory**.

Definition 1.2.5 (Full theory). The *full theory* $\text{Th}(\mathcal{M})$ of an \mathcal{L} -structure \mathcal{M} is defined as $\text{Th}(\mathcal{M}) = \{\varphi \mid \mathcal{M} \models \varphi\}$.

From the definition, $\mathcal{M} \models \text{Th}(\mathcal{M})$, and $\text{Th}(\mathcal{M})$ characterizes the **structures** satisfying the same **sentences** as \mathcal{M} .

Definition 1.2.6 (Complete). A **theory** T is *complete* if for any **sentence** φ , either $\varphi \in T$ or $\neg\varphi \in T$.

Remark. $\text{Th}(\mathcal{M})$ is **complete**.

Definition 1.2.7 (Elementarily equivalent). \mathcal{M} and \mathcal{N} are *elementarily equivalent* $\mathcal{M} \equiv \mathcal{N}$ if for all **sentences** φ ,

$$\mathcal{M} \models \varphi \Leftrightarrow \mathcal{N} \models \varphi.$$

Remark. There are $\mathcal{N} \models \text{Th}(\mathbb{N})$, but \mathcal{N} is not isomorphic to \mathbb{N} . \mathcal{N} is called a *non-standard model of arithmetic*, and \mathcal{N} might have *infinite element* larger than all of \mathbb{N} . Here, $\mathbb{N} = (\mathbb{N}, 0, 1, +, \cdot, -)$

Example. $\mathbb{Z} \oplus \mathbb{Z} \not\equiv \mathbb{Z}$ as groups.

The other way to define a **theory** is to write down axioms.

Example (Infinite set). Let $\mathcal{L} = \emptyset$, and let T consist of

$$\varphi_n := \exists x_1 \dots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j.$$

Example (Linear order). Let $\mathcal{L} = \{\leq\}$, and let T consist of the axioms of linear orders, e.g.,

$$\forall x \forall y (x \leq y \wedge y \leq x \rightarrow x = y).$$

There are other interesting theories of linear orders, e.g., dense ones.

Example (Dense linear order). Consider

$$\forall x \forall y (x < y \rightarrow \exists z x < z < y),$$

where we use $a < b$ as shorthand of saying $a \leq b \wedge a \neq b$.

Example (Group). In $\mathcal{L}_{\text{group}} = \{e, \cdot, ^{-1}\}$, let T be the group axioms.

Other theories of groups include Abelson group, divisible, etc.

Definition 1.2.8 (Finitely axiomatizable). A theory is *finitely axiomatizable* if it has a finite set of axioms.

Given a theory, consider $T^{\models} = \{\varphi \mid T \models \varphi\}$,³ so $\mathcal{M} \models T$ if and only if $\mathcal{M} \models T^{\models}$. Often we think of T and T^{\models} as the same. A theory T is *finitely axiomatizable* if there is a finite Φ such that $T^{\models} = \Phi^{\models}$.

1.2.1 Elementary Embeddings

Let's now consider the following notion.

Definition 1.2.9 (Elementary embedding). Let \mathcal{M} and \mathcal{N} be \mathcal{L} -structures, and $f: \mathcal{M} \rightarrow \mathcal{N}$ an \mathcal{L} -embedding. Then f is an *elementary embedding* if for any formula $\varphi(\bar{x})$ and $\bar{a} \in M$,

$$\mathcal{M} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{N} \models \varphi(f(\bar{a})).$$

Definition 1.2.10 (Elementary substructure). If $f: \mathcal{M} \hookrightarrow \mathcal{N}$ is a *elementary embedding* where \mathcal{M} is a *substructure* of \mathcal{N} , then \mathcal{M} is an *elementary substructure* of \mathcal{N} .

Example. As groups, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is not *elementary*. In fact, $\mathbb{Z} \not\equiv \mathbb{Q}$. Whereas, if $f: \mathcal{M} \hookrightarrow \mathcal{N}$ is an *elementary embedding*, $\mathcal{M} \equiv \mathcal{N}$.^a

^aAnd also much more is true.

Proposition 1.2.1. Every *isomorphism* is an *elementary embedding*.

Proof. Let $f: \mathcal{M} \rightarrow \mathcal{N}$ be an *isomorphism*. We will argue by induction on formulas φ , that for all $\bar{a} \in M$,

$$\mathcal{M} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{N} \models \varphi(f(\bar{a})).$$

Firstly, observe that all cases except quantifiers are the same as [Proposition 1.1.1](#). For quantifiers, suppose that $\varphi(\bar{x})$ is $\exists y \psi(\bar{x}, y)$ and $\mathcal{M} \models \varphi(\bar{a})$. This means that there is $b \in M$ such that $\mathcal{M} \models \psi(\bar{a}, b)$. By the induction hypothesis, $\mathcal{N} \models \psi(f(\bar{a}), f(b))$, so $\mathcal{N} \models \varphi(f(\bar{a}))$.

Now suppose $\mathcal{N} \models \varphi(f(\bar{a}))$, then there is $c \in N$ such that $\mathcal{N} \models \psi(f(\bar{a}), c)$. Since f is an *isomorphism*, so there is a $b \in M$ such that $f(b) = c$. By the induction hypothesis, $\mathcal{M} \models \psi(\bar{a}, b)$, so $\mathcal{M} \models \varphi(\bar{a})$. ■

Corollary 1.2.1. If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.

³Recall [Definition 1.1.13](#).

1.2.2 Definable Sets

Consider the following.

Definition 1.2.11 (Definable). Let \mathcal{M} be an \mathcal{L} -structure, then $X \subseteq M^n$ is *definable* if there is a formula $\varphi(x_1, \dots, x_n, \bar{y})$ and $\bar{b} \in M$ such that

$$X = \{\bar{a} \in M^n \mid \mathcal{M} \models \varphi(\bar{a}, \bar{b})\}.$$

Notation (Define). We say that $\varphi(\bar{x}, \bar{b})$ *defines* X over \bar{b} , written as $X = \varphi(\mathcal{M}, \bar{b})$.

Notation (Parameter). The tuple \bar{b} is called the *parameters* when X is *definable* over \bar{b} .

Remark. Sometimes X is *definable* without *parameters*, or *definable* over \emptyset .

Example. Take $\mathbb{R} = (\mathbb{R}, 0, 1, +, \cdot, -)$ in $\mathcal{L}_{\text{ring}}$, then

$$\leq = \{(a, b) : a \leq b\}$$

is *definable*.

Example. Let $\mathbb{Z} = (\mathbb{Z}, +, -, \cdot, 0, 1)$, then \mathbb{N} is \emptyset -*definable* in \mathbb{Z} by^a

$$\mathbb{N} = \{z \in \mathbb{Z} : \exists u, v, x, y \ u^2 + v^2 + x^2 + y^2 = z\}.$$

^aFrom the *Langrange's four-square theorem*, which says that every natural number is the sum of four squares.

Example. \mathbb{Z} is \emptyset -*definable* in $\mathbb{Q} = (\mathbb{Q}, +, -, \cdot, 0, 1)$. This is a result of Julia Robinson [Rob49], and the formulation is very complicated.

Problem. How does one show that a set is not *definable*? For example, \mathbb{R} is not *definable* in $\mathbb{C} = (\mathbb{C}, 0, 1, +, \cdot, -)$.

Lecture 5: Hilbert-Style Deductive System

We start by asking whether \mathbb{R} is *definable* in $\mathbb{C} = (\mathbb{C}, 0, 1, +, \cdot, -)$?

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Proposition 1.2.2. Let \mathcal{M} be an \mathcal{L} -structure, and let $X \subseteq M^n$ be a set which is *definable* over \bar{a} . Then any *automorphism* of \mathcal{M} that fixes \bar{a} pointwise^a fixes X setwise.^b

^aIf $\bar{a} = (a_1, \dots, a_m)$, then $f(a_i) = a_i$.

^bIf $b \in X$, then $f(b) \in X$.

Proof. Let f be an *automorphism* of \mathcal{M} fixing \bar{a} pointwise, and $X = \{\bar{b} \in M^n : \mathcal{M} \models \varphi(\bar{b}, \bar{a})\}$. Fix \bar{b} , and suppose $\bar{b} \in X$, so $\mathcal{M} \models \varphi(\bar{b}, \bar{a})$. Because f is an *elementary embedding* from Proposition 1.2.1,

$$\mathcal{M} \models \varphi(f(\bar{b}), f(\bar{a})) \Rightarrow \mathcal{M} \models \varphi(f(\bar{b}), \bar{a}),$$

hence $f(\bar{b}) \in X$. Similarly, if $\bar{b} \notin X$, $\mathcal{M} \models \neg\varphi(\bar{b}, \bar{a}) \Rightarrow \mathcal{M} \models \neg\varphi(f(\bar{b}), \bar{a})$, so $f(\bar{b}) \notin X$. ■

Remark. If X is \emptyset -*definable*, it is fixed setwise by any *automorphism*.

Example. \mathbb{N} is fixed setwise by any **automorphism** of the ring \mathbb{Z} . In fact, the only **automorphism** of \mathbb{Z} is the identity.

Example. \mathbb{N} is not **\emptyset -definable** in $\mathbb{Z} = (\mathbb{Z}, 0, +)$.

Proof. Consider an **automorphism** $f(x) = -x$ of the group \mathbb{Z} , which does not fix \mathbb{N} setwise. \circledast

Problem. Is \mathbb{N} **definable** in $\mathbb{Z} = (\mathbb{Z}, 0, +)$ over some parameters \bar{a} ?

Answer. For example, if $\bar{a} = (1)$, then f does not fix 1. In fact, any **automorphism** fixing 1 also fixes all of \mathbb{Z} , but \mathbb{N} is not **definable** in $(\mathbb{Z}, 0, +)$. To prove this we need compactness. \circledast

As previously seen. Given a field F , then $F(a) \cong F(b)$ if a and b have the same minimal polynomial over F or if both do not satisfy any polynomial over F .

Example. $\mathbb{Q}(\pi) \cong \mathbb{Q}(e)$ because π and e are both transcendental.

We now return to the big question: is \mathbb{R} **definable** in $\mathbb{C} = (\mathbb{C}, 0, 1, +, \cdot, -)$? If $f: \mathbb{Q}(a) \rightarrow \mathbb{Q}(b)$ such that $a \mapsto b$, then there is an **automorphism** $\hat{f}: \mathbb{C} \rightarrow \mathbb{C}$ such that $a \mapsto b$, i.e., \hat{f} extends f . In other words, we need to find such an f with $a \in \mathbb{R}$ and $b \notin \mathbb{R}$.

Example. $a = \pi$, $b = i\pi$ are both transcendental.

Example. a is a real $\sqrt[4]{2}$, b is a complex $\sqrt[4]{2}$.

The above two examples show that \mathbb{R} is not **\emptyset -definable** in \mathbb{C} . In fact, \mathbb{R} is not **definable** over any \bar{a} because there are elements of \mathbb{R} and $\mathbb{C} \setminus \mathbb{R}$ transcendental over any \bar{a} .

Intuition. There are so many a, b such that given any \bar{a} , we can still find a pair that works.

1.3 Completeness and Compactness

In this section, we're going to formalize **proofs**.

1.3.1 Proofs

There are all sorts of different proof systems, and the one we use is the so-called Hilbert-style deductive system. Before that, we first see some common notions.

Notation (Schema). A *schema* is written in symbols for **formulas**, variables, etc.

Example. $\varphi \rightarrow (\psi \rightarrow \varphi)$ is a **schema**, i.e., an infinite set with all possible choices of φ and ψ .

Specifically, every **logical axiom** is written in **schema**, meaning that any instance of a symbol for a **formula**, e.g., φ , can be replaced by any **formula**.

Definition 1.3.1 (Generalization). A **formula** φ is a *generalization* of a **formula** ψ if φ is $\forall x_1 \dots \forall x_n \psi$ where x_1, \dots, x_n are variables.

Notation (Hypothesis). *Hypotheses* are **formulas** that we may assume in a **proof**.

Definition 1.3.2 (Proof). A *proof* is a sequence of formulas $\{\varphi_i\}_{i=1}^n$ such that φ_n is the conclusion, and each formula is either an axiom or is obtained from the previous formulas by a rule of inference.

Moreover, for a proof based on a set of hypotheses Γ , then in addition to a logical axiom, we can assert a formula $\varphi \in \Gamma$. If we prove ψ using Γ as hypotheses, we write $\Gamma \vdash \psi$.

Definition 1.3.3 (Valid). If we prove ψ without hypotheses, we write $\vdash \psi$ and say ψ is *valid*.

Definition 1.3.4 (Logical axioms). The *logical axioms* are the following formulas written in schema, as well as all of their generalizations:

Definition 1.3.5 (Propositional axioms). The *propositional axioms* are

- (A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$.
- (A2) $(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta))$.
- (A3) $(\neg\varphi \rightarrow \neg\psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \varphi)$.
- (A4) $\forall x \varphi(x, \dots) \rightarrow \varphi(t, \dots)$ where t is any term.
- (A5) $[\forall x (\varphi \rightarrow \psi)] \rightarrow [(\forall x \varphi) \rightarrow (\forall x \psi)]$.
- (A6) $\varphi \rightarrow \forall x \varphi$, where x is not free in φ .

Definition 1.3.6 (Axioms for equality). The *axioms for equality* is

- (A7) for any terms t, u, v, \dots , function symbols f , and relation symbols R ,
 - (a) $t = t$.
 - (b) $t = u \rightarrow u = t$.
 - (c) $(t = u \wedge u = v) \rightarrow (t = v)$.
 - (d) $(u_1 = t_1 \wedge \dots \wedge u_{n_f} = t_{n_f}) \rightarrow f(u_1, \dots, u_{n_f}) = f(t_1, \dots, t_{n_f})$.
 - (e) $(u_1 = t_1 \wedge \dots \wedge u_{n_R} = t_{n_R}) \rightarrow (R(u_1, \dots, u_{n_R}) \leftrightarrow R(t_1, \dots, t_{n_R}))$.

Definition 1.3.7 (Rule of inference). From φ and $\varphi \rightarrow \psi$, deduces ψ .^a

^aThis is called **modus ponens**.

These formulas might have free variables.

Example. A proof from calculus of a limit, e.g., $\forall \epsilon \exists \delta \dots$. And we start by stating

- let $\epsilon > 0$,
- choose $\delta = \epsilon$,
- \vdots
- $|f(x) - f(y)| < \epsilon$.

We should interpret free variables as anything.

As previously seen (Propositional logic). $(p \wedge q) \vee (r \wedge \neg q)$.

Remark. We can check whether the **propositional axioms** are **true** with a truth table.

Definition 1.3.8 (Propositional tautology). A *propositional tautology* is a boolean combination \vee, \wedge, \neg of **formulas** $\varphi_1, \dots, \varphi_n$ which is **true** via a truth table assigning true or false to each of $\varphi_1, \dots, \varphi_n$.

So instead of using **propositional axioms**, we could instead allow as **logical axioms** any **propositional tautology**. To prove completeness, we will need 5 **propositional tautologies**. We will **prove** some of these, but take others on faith.

Remark. **Propositional axioms** are enough to **prove** all **propositional tautologies**.

Notation. We write $\Gamma \vdash_{\mathcal{L}} \varphi$ if there is a **proof** of φ from Γ in the **language** \mathcal{L} .

Note. Passing to a larger **language** will not let you **prove** more, so we can just write \vdash .

Lecture 6: Soundness Theorem

To see why **propositional axioms** are enough to **prove** all **propositional tautologies**, we see one example.

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Problem. **Prove** $\varphi \rightarrow \varphi$ using **propositional axioms**.

Answer. We see that

1. $\varphi \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$ from (A1), where ψ is any **formula** (possibly $\psi = \varphi$).
2. $[\varphi \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)] \rightarrow [(\varphi \rightarrow (\psi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)]$ from (A2).
3. $(\varphi \rightarrow (\psi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)$ from (MP) and the two above.
4. $\varphi \rightarrow (\psi \rightarrow \varphi)$ from (A1).
5. $\varphi \rightarrow \varphi$ from (MP) and the two above.

⊛

In general, we can **prove**

- (a) $\varphi \rightarrow \varphi$;
- (b) $\varphi \rightarrow \neg\neg\varphi$;
- (c) $\neg\neg\varphi \rightarrow \varphi$;
- (d) $(\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \psi)$;
- (e) $\varphi \rightarrow (\psi \rightarrow (\varphi \rightarrow \psi))$,

and so on.

Note. As we said, we may replace **propositional axioms** by every **propositional tautologies**.

Now, to see how (A6) is useful, consider the following.

Theorem 1.3.1. If $\Gamma \vdash \varphi$, and x does not occur **freely** in Γ , then $\Gamma \vdash \forall x \varphi$.

Proof. Fix Γ and x , we use *induction on proofs*. Consider the set $\{\varphi \mid \Gamma \vdash \forall x \varphi\}$, we will show that this set contains all the **logical axioms**, **formulas** from Γ , and is closed under **MP**. Thus, if $\Gamma \vdash \theta$, then $\theta \in \{\varphi \mid \Gamma \vdash \forall x \varphi\}$.

- (a) If φ is a **logical axiom**, so is its **generalization** $\forall x \varphi$, so $\Gamma \vdash \forall x \varphi$.
- (b) If $\varphi \in \Gamma$, then x is not **free** in φ , then from (A6), $\varphi \rightarrow \forall x \varphi$. Since $\Gamma \vdash \varphi$, $\Gamma \vdash \varphi \rightarrow \forall x \varphi$, by (MP), $\Gamma \vdash \forall x \varphi$.
- (c) Suppose $\Gamma \vdash \forall x \varphi$ and $\Gamma \vdash \forall x (\varphi \rightarrow \psi)$, we want to show that $\Gamma \vdash \forall x \psi$.
 - By (A5), $\forall x (\varphi \rightarrow \psi) \rightarrow (\forall x \varphi \rightarrow \forall x \psi)$, Γ **proves** this.
 - By (MP), $\Gamma \vdash \forall x \varphi \rightarrow \forall x \psi$.
 - By (MP) again, $\Gamma \vdash \forall x \psi$.

■

Corollary 1.3.1. If $\vdash \varphi$, then $\vdash \forall x \varphi$. So the **generalization** of anything **valid** is also **valid**.

We now ask a critical question: is our **proof** system a good one?

1.3.2 Soundness

The idea is that if an **\mathcal{L} -sentence** φ is **provable**, then it is **true** in all **\mathcal{L} -structures**, i.e., every thing we **prove** should be **true**, in other words, we can't **prove** wrong things. Let's start with our first glance on **soundness**.

Lemma 1.3.1 (Soundness). If Γ is a set of **\mathcal{L} -sentences** and φ is a **sentence**, and $\Gamma \vdash_{\mathcal{L}} \varphi$, then $\Gamma \models \varphi$.

Proof. Suppose that $\Gamma \vdash \varphi$, let $\psi_1, \psi_2, \dots, \psi_n = \varphi$ be such a **proof**.^a Let $\bar{x} = (x_1, \dots, x_m)$ be the **free variable** that appears in the ψ_i . Let \mathcal{M} be an **\mathcal{L} -structure**, $\mathcal{M} \models \Gamma$, we want to show that $\mathcal{M} \models \varphi$. We will show by induction on i that for all $\bar{a} \in M^m$, $\mathcal{M} \models \psi_i(\bar{a})$.

For ψ_i , we have three cases:

- (a) If $\psi_i \in \Gamma$, then $\mathcal{M} \models \Gamma$ so $\mathcal{M} \models \psi_i$.
- (b) If ψ_i is a (**generalization** of) a **logical axiom**, then we can check that $\mathcal{M} \models \psi_i(\bar{a})$. For example, if ψ_i is (A1), $\theta \rightarrow (\gamma \rightarrow \theta)$, it's easy to check that

$$\mathcal{M} \models \theta(\bar{a}) \rightarrow (\gamma(\bar{a}) \rightarrow \theta(\bar{a})).$$

- (c) If there are $j, k < i$ such that ψ_k is $\psi_j \rightarrow \psi_i$, from inductive hypothesis, for all \bar{a} , $\mathcal{M} \models \psi_j(\bar{a}), \mathcal{M} \models \psi_k(\bar{a})$, then $\mathcal{M} \models \psi_j(\bar{a}) \rightarrow \psi_i(\bar{a})$. Checking our definition of **truth**, we get $\mathcal{M} \models \psi_i(\bar{a})$.

■

^aSome ψ_i might be **formulas**, but φ should be a **sentence**.

There are some remarks to make.

Remark. If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.

Remark. If $\Delta \subseteq \Gamma$, and $\Delta \vdash \varphi$, then $\Gamma \vdash \varphi$.

Remark. If $\Gamma \vdash_{\mathcal{L}} \varphi$, and $\mathcal{L}^+ \supseteq \mathcal{L}$, then $\Gamma \vdash_{\mathcal{L}^+} \varphi$.

Remark. If $\Gamma \vdash \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$.

We can prove the following.

Theorem 1.3.2 (Deduction theorem). For any set of formulas Γ , formulas θ and ψ ,

$$\Gamma \cup \{\theta\} \vdash \psi \Leftrightarrow \Gamma \vdash \theta \rightarrow \psi.$$

Proof. The backward direction is easier. Suppose $\Gamma \vdash \theta \rightarrow \psi$, then $\Gamma \cup \{\theta\} \vdash \psi$ since we can have a proof like:

1. θ
- \vdots (the proof of $\Gamma \vdash \theta \rightarrow \psi$)
- n . $\theta \rightarrow \psi$
- $n + 1$. ψ .

Now, suppose that $\Gamma \cup \{\theta\} \vdash \psi$, then there is a proof $\psi_1, \dots, \psi_n = \psi$ of ψ from $\Gamma \cup \{\theta\}$. We argue inductively that $\Gamma \vdash \theta \rightarrow \psi_i$. Suppose we know that for $j < i$, prove it for i . Divide into cases:

- (a) $\psi_i \in \Gamma$ or it is a logical axiom. By (A1), $\psi_i \rightarrow (\theta \rightarrow \psi_j)$, so $\Gamma \vdash \psi_j$.
- (b) $\psi_i = \theta$. Then $\Gamma \vdash \theta \rightarrow \theta$ by (A1) and (A2) from here.
- (c) ψ_i follows from $\psi_j, \psi_k = \psi_j \rightarrow \psi_i$, using (MP) with $j, k < i$.
 - From the induction hypothesis, $\Gamma \vdash \theta \rightarrow \psi_j$ and $\Gamma \vdash \theta \rightarrow (\psi_j \rightarrow \psi_i)$.
 - By (A2), $\Gamma \vdash [\theta \rightarrow (\psi_j \rightarrow \psi_i)] \rightarrow [(\theta \rightarrow \psi_j) \rightarrow (\theta \rightarrow \psi_i)]$.
 - By (MP), $\Gamma \vdash (\theta \rightarrow \psi_j) \rightarrow (\theta \rightarrow \psi_i)$.
 - By (MP), $\Gamma \vdash \theta \rightarrow \psi_i$.

■

Lecture 7: Soundness, Completeness, and Compactness

Proposition 1.3.1 (Contraposition). If $\Gamma \cup \{\varphi\} \vdash \neg\psi$, then $\Gamma \cup \{\psi\} \vdash \neg\varphi$.

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Proof. Suppose $\Gamma \cup \{\varphi\} \vdash \neg\psi$, by the deduction theorem says that

$$\Gamma \vdash \varphi \rightarrow \neg\psi.$$

From (A1), (A2), and (A3), we can prove $(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$. By (MP), $\Gamma \vdash \psi \rightarrow \neg\varphi$. By the deduction theorem, $\Gamma \cup \{\psi\} \vdash \neg\varphi$. ■

Definition 1.3.9 (Consistent). A theory T is *consistent* if for all φ , it is not the case that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Definition 1.3.10 (Inconsistent). If a theory T is not consistent, then it's *inconsistent*.

Remark. We could make the same definition for a set of formulas.

Proposition 1.3.2 (Proof by contradiction). If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \vdash \neg\varphi$.

Proof. There is ψ such that $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\varphi\} \vdash \neg\psi$, so $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \varphi \rightarrow \neg\psi$. Using (A1), (A2), and (A3), we can prove that

$$(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi).$$

By (MP), $\Gamma \vdash (\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi$, and by (MP) again, we have $\Gamma \vdash \neg\varphi$. ■

Proposition 1.3.3. If a theory T is consistent, and φ is a sentence, then either $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is consistent.

Proof. If they were both inconsistent, $T \vdash \neg\varphi$ and $T \vdash \neg\neg\varphi$, so T would be inconsistent, but T is consistent. ■

Note. The above is also true for formula.

Proposition 1.3.4. If a theory T is inconsistent, then $T \cup \{\varphi\}$ is inconsistent for all φ . Hence, $T \vdash \varphi$ for all φ .

Definition 1.3.11 (Maximal). A theory T is called *maximal* if it is consistent and for all sentences φ , either $\varphi \in T$ or $\neg\varphi \in T$.

In particular, if $T \vdash \varphi$, then $\varphi \in T$.

Theorem 1.3.3 (Zorn's lemma). Let (P, \leq) be a partially ordered set. If every non-empty chain in P has an upper bound, then P has a maximal element.

Theorem 1.3.4. Any consistent theory T can be extended to a maximal consistent theory $T' \supseteq T$.

Proof. We first consider the case that T is countable by considering \mathcal{L} is countable since if \mathcal{L} is countable, then there are only countable many formulas since there are only countable many formulas of each length.

Claim. The result holds for \mathcal{L} being countable.

Proof. Firstly, list out all sentences $\varphi_1, \varphi_2, \dots$, start with $T_0 = T$. Given T_i consistent, one of $T_i \cup \{\varphi_i\}$ or $T_i \cup \{\neg\varphi_i\}$ is consistent. Let T_{i+1} be one of these that is consistent. Let $T^* = \bigcup_i T_i$, we now see that T^* is consistent.

Suppose not, then $T^* \vdash \theta$ and $T^* \vdash \neg\theta$. In this case, there is some T_i such that $T_i \vdash \theta$ and $T_i \vdash \neg\theta$ because proofs are finite. But T_i is consistent, so this cannot happen, hence T^* is maximal. ⊛

Claim. The result holds for arbitrary \mathcal{L} .

Proof. For arbitrary \mathcal{L} , let (P, \leq) be the set of consistent theories extending T ordered by inclusion. Let C be a non-empty chain, and let $T^* = \bigcup_{T' \in C} T' \supseteq T$. We see that T^* is consistent because if $T^* \vdash \theta$ and $T^* \vdash \neg\theta$, there are finitely many formulas used in those proofs, from, say, $T_1, \dots, T_n \in C$.

Because C is a chain, by reordering, we may assume that $T_1 \subseteq \dots \subseteq T_n$. So $T_n \vdash \theta$ and $T_n \vdash \neg\theta$, contradicting the consistency of T_n , so T^* is consistent, i.e., $T^* \in P$, and T^* is an upper bound on C . By Zorn's lemma, (P, \leq) has a maximal lemma $T^* \supseteq T$, consistent. If T^* is not maximal, then there is φ such that $\varphi \notin T^*$, $\neg\varphi \notin T^*$. But one of $T^* \cup \{\varphi\}$ or $T^* \cup \{\neg\varphi\}$ is consistent, hence in P , contradicting the fact that T^* is maximal. ⊛

Remark. We can do that same proof for any \mathcal{L} using transfinite recursion for the uncountable case.

Motivated by Lemma 1.3.1, we have the following.

Theorem 1.3.5 (Soundness). Let T be a [theory](#) and φ be a [sentence](#).

- (a) If $T \vdash \varphi$, then $T \models \varphi$.
- (b) If T is [satisfiable](#), then it is [consistent](#).

Proof. (a) is exactly [Lemma 1.3.1](#). For (b), let $\mathcal{M} \models T$, suppose that T was [inconsistent](#), then $T \vdash \varphi$ and $T \vdash \neg\varphi$ for some φ . By (a), $T \models \varphi$ and $T \models \neg\varphi$, so $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \neg\varphi$. But $\mathcal{M} \models \neg\varphi$ means $\mathcal{M} \not\models \varphi$, so this is a contradiction, hence T is [consistent](#). ■

1.3.3 Completeness

If φ is [true](#) in all \mathcal{L} -structures, then it is [provable](#).

Theorem 1.3.6 (Completeness). Let T be a [theory](#) and φ be a [sentence](#).

- (a) If $T \models \varphi$, then $T \vdash \varphi$.
- (b) If T is [consistent](#), then it is [satisfiable](#).

Proof. We leave (b) for later, and prove that (b) implies (a). Suppose that $T \models \varphi$, so $T \cup \{\neg\varphi\}$ is [unsatisfiable](#). By (b), $T \cup \{\neg\varphi\}$ is [inconsistent](#). By [proof by contradiction](#), $T \vdash \varphi$. ■

1.3.4 Compactness

Theorem 1.3.7 (Compactness). Let T be a [theory](#) and φ be a [sentence](#).

- (a) If $T \models \varphi$, then there is a finite $T_0 \subseteq T$ such that $T_0 \models \varphi$.
- (b) T is [satisfiable](#) if and only if every finite subset of T is [satisfiable](#).

Proof. Consider:

- (a*) If $T \vdash \varphi$, then there is a finite $T_0 \subseteq T$ such that $T_0 \vdash \varphi$.
- (b*) If T is [consistent](#) if and only if every finite subset of T is [consistent](#).

We see that (a*) and (b*) are true because [proofs](#) are finite, and [soundness](#) and [completeness](#) translate directly between (a) and (a*) (and (b) and (b*)). ■

Let's see some examples using [compactness](#).

Example. Let $\mathcal{L} = \{0, 1, +, \cdot, -, <\}$, and $\mathcal{L}^* = \mathcal{L} \cup \{c\}$, where c is a new constant symbol. Let

$$T = \text{Th}_{\mathcal{L}}(\mathbb{N}) \cup \{c > \underline{n} \mid n \in \mathbb{N}\},$$

then T is finitely [satisfiable](#).

Proof. Given $T_0 \subseteq T$ finite, $T_0 \subseteq \text{Th}_{\mathcal{L}}(\mathbb{N}) \cup \{c > \underline{n}, \dots, c > \underline{n}_\ell\}$, and may assume they are equal and show that T_0 is [satisfiable](#). Let \mathcal{N} be the $\mathcal{L} \cup \{c\}$ -structure which is the expansion of the \mathcal{L} -structure \mathbb{N} , with

$$c^{\mathcal{N}} = 1 + \max(n_1, \dots, n_\ell),$$

then $\mathcal{N} \models T_0$, and T_0 is [satisfiable](#). By [compactness](#), T is [satisfiable](#), say $\mathcal{A} \models T$. Then $\mathcal{A} \equiv \mathbb{N}$ and \mathcal{A} contains an element $c^{\mathcal{A}}$ bigger than $1, 1+1, 1+1+1, \dots$, but $\mathcal{A} \not\equiv \mathbb{N}$, so \mathcal{A} is a non-standard model of arithmetic. ⊛

Appendix

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