MATH602
Real Analysis II

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Abstract

This is a graduate level functional analysis taught by Joseph Conlon. The prerequisites include linear algebra, complex analysis and also real analysis. We'll use Peter Lax[Lax02] and Reed-Simon[RS80] as textbooks.

This course is taken in Fall 2022, and the date on the covering page is the last updated time.

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Chapter 1

Introduction

Lecture 1: Introduction

We first briefly review different kinds of vector spaces. 30 Aug. 14:30

1.1 Linear Space

Definition 1.1.1 (Linear vector space). A set with operations of addition and multiplication (by a scalar) is called a *linear vector space*.

Example. Denote the multiplicative scalar by λ, then

- $\lambda \in \mathbb{R} \Rightarrow$ real vector space.
- $\lambda \in \mathbb{C} \Rightarrow$ complex vector space

Lemma 1.1.1. Given E a linear vector space, if $v, w \in E$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), then $\lambda v + \mu w \in E$.

we also have usual rules of associativity and commutativity.

Example. $\mathbb{R}^n$ a n dimensional linear vector space, $\mathbb{C}^n$ a n dimensional complex linear vector space.

We concentrate on ∞ dimensional linear vector space.

Example. Let K is a compact Hausdorff space, then

$$E = \{f \colon K \to \mathbb{R} \mid f(\cdot) \text{ is continuous}\}.$$

We then see that E is an ∞ dimensional **real** linear vector space.

1.2 Quotient Space

Observe that a linear vector space can have many subspaces. Say E is a linear vector space, and $E_1 \subset E$ where E_1 is a proper subspace, i.e., $E_1 \neq E$.

Definition 1.2.1 (Quotient Space). The *quotient space* E / E_1 is the set of equivalence classes of vectors in E where equivalence is given by $x \sim y$ if $x - y \in E_1$. Additionally, denote $[x]$ as the equivalence class of $x \in E$, i.e., $[x] = x + E_1$.

Note that E / E_1 is a linear vector space since if $x_1 + x_2 \in E$, $[x_1] + [x_2] = [x_1 + x_2]$, and also, $\lambda[x] = [\lambda x]$ for $\lambda \in \mathbb{R}$ or $\mathbb{C}$, i.e., $v, w \in E / E_1$, $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$ implies $\lambda v + \mu w \in E$.

Definition 1.2.2 (Codimension)**.** If $E \,/\, E_1$ has finite dimension, then the dimension of $E \,/\, E_1$ is called the *cdimension* of E_1 in E.

Example. There exists the case that $\dim(E) = \infty$, $\dim(E_1) < \infty$ where $\dim(E \,/\, E_1) < \infty$.

Proof. Let $E = \{f\colon K \to \mathbb{R} \mid f(\cdot) \text{ continuous}\}$, and $E_1 = \{f \in E\colon f(k_1) = 0\}$ where $k_1 \in K$ is fixed. We see that the dimension of $E \,/\, E_1$ is exactly 1 since $E \,/\, E_1$ is the set of constant functions. ⊛

Theorem 1.2.1. If E is finite dimensional, then $\operatorname{codim}(E_1) + \dim(E_1) = \dim(E)$

Definition 1.2.3 (Linear operator)**.** A map $T\colon E \to F$ between 2 linear spaces is a *linear operator* if it preserves the properties of addition and multiplication by a scalar, i.e., $T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$ for $v, w \in E$ and $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$.

Definition. Given a inear operator $T\colon E \to F$ we have the following.

Definition 1.2.4 (Kernel)**.** The *kernel* of T is the subspace $\ker(T) = \{x \in E \mid Tx = 0\}$.

Definition 1.2.5 (Image)**.** The *image* of T is the subspace $\operatorname{Im}(T) = \{Tx \in F \mid x \in E\}$.

1.3 Normed Spaces

We review some basic notions.

Definition 1.3.1 (Norm)**.** Let E be a linear vector space. A *norm* $\|\cdot\| : E \to \mathbb{R}$ on E is a function from E to $\mathbb{R}$ with the properties:

(a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Notation (Dilation)**.** We say that the second condition is the *dilation* property.

Definition 1.3.2 (Normed vector space)**.** A linear vector space E equipped with a norm $\|\cdot\|$ is called a *normed vector space*.

Remark (Induced metric space)**.** A normed vector space E induces a *metric space* with metric $d(x, y) = \|x - y\|$, where the metric has properties

(a) $d(x, y) \geq 0$. Also, $d(x, x) = 0$ and $d(x, y)$ implies $x = y$.

(b) $d(x, y) = d(y, x)$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

Example (Bounded sequences ℓ_∞)**.** Let ℓ_∞ be the space of bounded sequences $x = (x_1, x_2, \ldots)$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots$. Then we define $\|x\| = \|x\|_\infty = \sup_{i \geq 1} |x_i|$.

Example (Absolutely summable sequences ℓ_1). Let ℓ_1 be the space of absolutely summable sequences $x = (x_1, x_2, \ldots)$ and $\sum_{i=1}^{\infty} |x_i| < \infty$. Then we define $\|x\| = \|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$.

Example (Continuous functions $C(k)$). The space $C(k)$ of continuous functions $f \colon K \to \mathbb{R}$ where K is compact Hausdorff. Then we define $\|f\| = \|f\|_\infty = \sup_{x \in K} |f(x)|$.

1.3.1 Geometry of Normed Spaces

Definition 1.3.3 (Ball). A (closed) *ball* centered at a point $x_0 \in E$ with radius $r > 0$ is the set $B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\}$.

Definition 1.3.4 (Sphere). The *sphere* centered at x_0 with radius $r > 0$ is the set $S(x_0, r) = \{x \in E \mid \|x - x_0\| = r\}$.

Remark. We see that $S(x_0, r)$ is the **boundary** of $B(x_0, r)$, i.e., $S(x_0, r) = \partial B(x_0, r)$.

Note (Nonequivalency in infinite dimensional spaces). We know that in finite dimensional, all norms are equivalent, which is not true for infinite dimensional vector spaces.

This has something to do with the geometry of balls.

Explicitly, balls can have different geometries depending on the properties of the norms. We see that an $\|\cdot\|_\infty$ can have multiple supporting hyperplane at the corner, while for an $\|\cdot\|_2$ can have only one at each point.

Also, unit balls for $\|\cdot\|_1$ is also a **square**, where we have

$$B(0,1) = \{x = (x_1, x_2, \ldots) \mid -1 < y_\epsilon < 1 \forall \epsilon\}$$

such that $y_\epsilon = \sum_{i=1}^{\infty} \epsilon_i x_i$, $\epsilon_i = \pm 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots)$.

We see that different norms give different geometry, but they have important common features, most notably, convexity properties.

Definition 1.3.5 (Convex set). Given E a linear vector space, a set $K \subset E$ is *convex* if $x, y \in K$ and $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda) y \in K$.

Definition 1.3.6 (Convex function). Given E a linear vector space, a function $f \colon E \to \mathbb{R}$ is called *convex* if

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for $x, y \in E$, $0 \leq \lambda \leq 1$.

Remark. If $f \colon E \to \mathbb{R}$ is a convex function, then for any $M \in \mathbb{R}$ the set $\{x \in E \mid f(x) \leq M\}$ is convex.

The upshot is that norms are convex, and the unit balls are convex as well.

Lecture 2: Banach Spaces and Completion

Let's first see a proposition. 01 Sep. 14:30

Proposition 1.3.1. Let $\{E, \|\cdot\|\}$ be a normed linear space. Then the norm is convex and continuous.

Proof. Let $f \colon E \to \mathbb{R}$ be $f(x) = \|x\|$. Then $f(x) - f(y) = \|x\| - \|y\| \leq \|x - y\|$, which implies $|f(x) - f(y)| \leq \|x - y\|$ for $x, y \in E$, i.e., f is Lipschitz continuous. For convexity, let $0 < \lambda < 1$, we have

$$f(\lambda x+(1-\lambda)y) = \|\lambda x + (1-\lambda)y\| \leq \|\lambda x\|+\|(1-\lambda)y\| = \lambda\|x\|+(1-\lambda)\|y\| = \lambda f(x)+(1-\lambda)f(y).$$

■

Note. Note that $f(\cdot)$ is continuous implies the closed ball

$$B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\} = \{x \in E \mid f(x - x_0) \leq r\}$$

is closed in topology of E. Also, $f(\cdot)$ is convex implies $B(x_0, r)$ is convex.

Remark. If $f\colon E \to \mathbb{R}$ is convex, then the sets $\{x \in E \mid f(x) \leq M\}$ is also convex. However, it's possible to have non-convex functions f such that all sets $\{x \in E \mid f(x) \leq M\}$ are convex.

Example. Take $f(x) = |x|^p$ for $x \in \mathbb{R}$ and $p > 0$. We see that f is convex if $p > 1$, and non-convex if $p < 1$. The sets $\{x \in \mathbb{R} \mid f(x) \leq M\}$ all convex since it's independent of p.

Lemma 1.3.1. Suppose $x \mapsto \|x\|$ satisfies

(a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda|\,\|x\|$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) The unit ball $B(0,1)$ is convex.

Then $f(x) = \|x\|$ satisfies the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$.

Proof. We see that if the third condition is true, the for $u, v \in B(0,1)$ and $0 < \lambda < 1$, we have $\lambda u + (1-\lambda)v \in B(0,1)$. Let $x, y \in E$, and

$$\lambda = \frac{\|x\|}{\|x\| + \|y\|} \Rightarrow 1 - \lambda = \frac{\|y\|}{\|x\| + \|y\|}.$$

By letting $u = x/\|x\|$, $v = y/\|y\|$ we see that

$$\lambda u + (1-\lambda)v = \frac{\|x\|}{\|x\| + \|y\|}\frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|}\frac{y}{\|y\|} \in B(0,1) \Rightarrow \left\|\frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|}\right\| \leq 1.$$

From the second condition, it follows that $\|x + y\| \leq \|x\| + \|y\|$, which is the triangle inequality. ■

Remark. If $x \mapsto \|x\|$ satisfies the first two condition and is a convex, then it satisfies the triangle inequality.

Proof. Since $\frac{1}{2}\|x + y\| = \left\|\frac{x}{2} + \frac{y}{2}\right\| \leq \frac{1}{2}\|x\| + \frac{1}{2}\|y\|$. ⊛

Now, given a quotient space $E \,/\, E_1$, the question is can we try to define a norm?

Problem 1.3.1. On $E \,/\, E_1$, is $\|[x]\| \coloneqq \inf_{y \in E_1}\|x + y\|$ a norm?

Answer. We see that if $x \in \overline{E}_1 \setminus E_1$, then $\|[x]\| = 0$ but $[x] \neq 0 \in E \,/\, E_1$. ⊛

Note. Notice the difference from finite dimensional situation. All finite dimensional spaces E_1 are closed but not in general if E_1 has ∞ dimensions.

Example. Let $\ell_1(\mathbb{R})$ be the sequence of x_n for $n \geq 1$ in $\mathbb{R}$ such that $\sum_{i=1}^{\infty} |x_i| \leq \infty$. Define

$$\|x\|_1 \coloneqq \sum_{i=1}^{\infty} |x_i| \,,$$

and let E_1 be all sequences with finite number of the x_n are nonzero. We see that $\overline{E}_1 = \ell_1(\mathbb{R})$ is infinite dimensional.

Proposition 1.3.2. Let $\{E, \|\cdot\|\}$ be a normed space and $E_1 \subseteq E$, E_1 is closed. Then

$$\|\cdot\| : {}^{E}\big/_{E_1} \to \mathbb{R}, \quad \|[x]\| = \inf_{y \in E_1} \|x + y\|$$

is a norm on $E \,/\, E_1$.

Proof. If $\|[x]\| = 0$, then $\inf_{y \in E_1} \|x - y\| = 0$, which implies $x \in E_1$ since E_1 is closed, so $[x] = 0$. Also, since

$$\|\lambda[x]\| = \inf_{y \in E_1} \|\lambda x + y\| = \inf_{z \in E_1} \|\lambda x + \lambda z\| = |\lambda| \inf_{z \in E_1} \|x + z\| = |\lambda| \, \|[x]\| \,,$$

the dilation property is satisfied. Finally, for triangle inequality, we have

$$\|[x] + [y]\| = \inf_{x_1, y_1 \in E} \|x + y + x_1 + y_1\| \leq \inf_{x_1 \in E_1} \|x + x_1\| + \inf_{y_1 \in E_1} \|y + y_1\| = \|[x]\| + \|[y]\| \,.$$

■

Remark. This shows that the only obstacle for this kind of norm being an actual norm is the closeness of E_1.

Chapter 2

Banach Spaces

2.1 Introduction

Definition 2.1.1 (Banach space)**.** A linear normed space is a *Banach space* if it's complete, i.e., every Cauchy sequence converges.

Note. If $x_n \in E$, $n \geq 1$ is a sequence with property such that $\lim_{m\to\infty} \sup_{n\geq m} \|x_n - x_m\| > 0$, then $\exists x_\infty \in E$ such that $\lim_{n\to\infty} \|x_n - x_m\| = 0$.

Example. The spaces ℓ_1, ℓ_∞ and $C(K)$ are Banach spaces.

We want to give a different criterion for showing $\{E, \|\cdot\|\}$ is Banach. Let E be a linear normed space and $\{x_\ell \mid \ell \geq 1\}$ a sequence in E.

Definition 2.1.2 (Absolutely summable)**.** A sequence is *absolutely summable* if $\sum_{i=1}^\infty \|x_i\| < \infty$.

Theorem 2.1.1 (Criterion for completeness)**.** A normed space $(E, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in E converges.

Proof. We need to prove two directions.

($\Rightarrow$) Suppose E is a Banach space and $\{x_k \mid x \geq 1\}$ an absolutely summable series. Set $s_n = \sum_{k=1}^n x_k$, $n \geq 1$, we want to show s_n is Cauchy, and if this is the case, completeness of E implies $\exists s_\infty$ and $\lim_{n\to\infty} \|s_n - s_\infty\| = 0$. Let $n > m$, we see that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \leq \sum_{k=m+1}^\infty \|x_k\|.$$

Observe that $\lim_{m\to\infty} \sum_{k=m+1}^\infty \|x_k\| = 0$, we see that the sequence $\{s_n\}$ is Cauchy.

($\Leftarrow$) Conversely, suppose E is **not** complete. Then there exists a Cauchy sequence $\{x_n \mid n \geq 1\}$ which does not converge. Furthermore, no subsequence of $\{x_n \mid n \geq 1\}$ converges.[a] We now construct an absolutely summable series which does not converge.

Define $n(1) \geq 1$ such that $\left\|x_n - x_{n(1)}\right\| \leq \frac{1}{2}$ if $n \geq n(1)$, similarly, let $n(2) > n(1)$ be such that $\left\|x_n - x_{n(2)}\right\| \leq \frac{1}{2^2}$ if $n > n(2)$. In all, we have $n(1) < n(2) < n(3) < \dots$ such that $\left\|x_n - x_{n(k)}\right\| \leq \frac{1}{2^k}$

if $n > n(k)$. Define $w_j \coloneqq x_{n(j+1)} - x_{n(j)}$ for $j = 1, 2, \ldots$. We see that

$$x_{n(m)} = x_{n(1)} + \sum_{j=1}^{m} w_j$$

for $m = 1, 2, \ldots$, and $\{x_{n(m)}\}$ does not converge, hence so does the series $\sum_{j=1}^{\infty} w_j$. However, $\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$, which implies $\{w_j\}$ is absolutely summable. ■

[a]Otherwise, the whole sequence converges by the fact that it's Cauchy.

2.2 Completion of Normed Space to Banach Space

Theorem 2.2.1. Suppose E is a normed space. Then there exists a Banach space $\hat{E}$ called a completion of E with the following properties:

(a) There exists a linear map $i\colon E \to \hat{E}$ such that $\|ix\| = \|x\|$.[a]

(b) $\operatorname{Im}(i)$ is dense in $\hat{E}$, and $\hat{E}$ is the smallest Banach space containing image of E.

[a]This is called an *isometric embedding* of E into $\hat{E}$.

Lecture 3: Banach, Inner Product Spaces

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Example (Banach spaces)**.** We already showed spaces ℓ_1 and ℓ_∞ are Banach spaces.

We now want to generalize to ℓ_p with $1 < p < \infty$. For $x = \{x_n, n \geq 1\} \in \ell_p$ and if $\sum_{n=1}^{\infty} |x_n|^p < \infty$, for $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$, we want to show that $x \to \|x\|_p$ satisfies properties of a norm. The first two properties of a norm is easy check. As for triangle inequality, we have the following.

Lemma 2.2.1 (Minkowski inequality)**.** Let $1 \leq p < \infty$, for $x, y \in \ell_p$,

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p .$$

Proof. Recall that from Lemma 1.3.1, we only need to show that $B(0,1)$ is convex, where

$$B(0,1) = \left\{ x = \{x_n \colon n \geq 1\} \mid f(x) = \sum_{n=1}^{\infty} |x_n|^p \leq 1 \right\}.$$

But $f(x)$ is convex since $x \mapsto |x|^p$, $x \in \mathbb{R}$ is convex if $p \geq 1$, we're done. Hence, $\|x+y\|_p \leq \|x\|_p + \|y\|_p$, i.e.,

$$\left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p} .$$

■

Lemma 2.2.2 (Hölder's inequality)**.** Let $1 < p < \infty$, for $x \in \ell_p$, $y \in \ell_q$, we have

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q$$

where $1/p + 1/q = 1$.

Proof. Note first that we can assume without loss of generality, $\sum_{j=1}^{\infty} |x_j|^p = \sum_{j=1}^{\infty} |y_j|^q = 1$.

Then, result follows from the Young's inequality,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

for $x, y > 0$, $x, y \in \mathbb{R}$.

Remark (Legendre transform and the inequality). Young's inequality is a special case of the inequality

$$xy \leq f(x) + \mathcal{L}f(y)$$

where $\mathcal{L}f(\cdot)$ is the *Legendre transform* of $f(\cdot)$, i.e., $\mathcal{L}f(y) = \sup_x [xy - f(x)]$.

If f is convex, then the function $xy \mapsto xy - f(x)$ is concave so has unique maximum. And $\mathcal{L}f(\cdot)$ always convex even if $f(\cdot)$ is not. In particular, if $f(x) = x^p/p$, then $\mathcal{L}f(y) = y^q/q$.

∎

Note. Minkowski inequality is usually proved via the Hölder's inequality. To show this, since

$$\sum_{j=1}^{\infty} |x_j + y_j|^p \leq \sum_{j=1}^{\infty} |x_j| \, |x_j + y_j|^{p-1} + \sum_{j=1}^{\infty} |y_j| \, |x_j + y_j|^{p-1}.$$

Then Holder inequality implies

$$\sum_{j=1}^{\infty} |x_j| \, |x_j y_j|^{p-1} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |x_j + y_j|^{(p-1)q} \right)^{1/q},$$

and we're done.[a]

[a] Note that $(p-1)q = p$.

Remark. The above argument applies to more general spaces of p integrable functions. Let (Ω, Σ, μ) be a measure space and $L_p(\Omega, \Sigma, \mu)$ where all Σ measure functions $f\colon \Omega \to \mathbb{R}$ (or $\mathbb{C}$) such that $\int_\Omega |f|^p \,\mathrm{d}\mu < \infty$. Then, $L_p(\Omega, \Sigma, \mu)$ is a normed space with norm

$$\|f\|_p = \left(\int_\Omega |f|^p \,\mathrm{d}\mu \right)^{1/p}.$$

It's more tricky to show that L^p is a Banach space, but it's indeed still the case.

Theorem 2.2.2. The p-integrable space $L_p(\Omega, \Sigma, \mu)$ is a Banach space.

Proof. Let $\{f_n : n \geq 1\}$ be an absolutely summable sequence in L^p. Then the norm satisfies

$$\left\| \sum_{k=1}^{N} f_k \right\|_p \overset{!}{\leq} \sum_{k=1}^{N} \|f_k\|_p \leq C.$$

Hence, $\int_\Omega \left| \sum_{k=1}^N f_k \right|^p \,\mathrm{d}\mu \leq C^p$.

- Assume all f_k are non-negative. From monotone convergence theorem, we have

$$\lim_{N \to \infty} \int_\Omega \left(\sum_{k=1}^{N} f_k \right)^p \mathrm{d}\mu = \int_\Omega \left(\sum_{k=1}^{\infty} f_k \right)^p \mathrm{d}\mu \leq C^p.$$

  Hence, $g = \sum_{k=1}^{\infty} f_k \in L_p$. We now want to show that $\sum_{k=1}^{N} f_k \to g$ in L_p. Set $r_n =$

$\sum_{k=n+1}^{\infty} f_k$ where r_n is a decreasing sequence where $r_n \to 0$ a.e. and also

$$\int_{\Omega} r_1^p \,\mathrm{d}\mu < \infty.$$

This means that $\lim\limits_{n\to\infty} \|r_n\|_p = 0$ by dominate convergence theorem.

- For arbitrary $f_k \colon \Omega \to \mathbb{R}$, write $f_k = f_k^+ + f_k^-$ where $f_k^+ = \sup(f_k, 0)$ and $f_k^- = \inf(f_k, 0)$. The sequence $\left\{f_k^+ \colon k \geq 1\right\}$ are absolutely summable, and we just proceed as before. Similarly, if $f_k \colon \Omega \to \mathbb{C}$.

$\blacksquare$

Chapter 3

Hilbert Spaces

3.1 Inner Product Spaces

Definition 3.1.1 (Inner product)**.** Let E be a linear space over $\mathbb{C}$. An *inner product* $\langle\cdot,\cdot\rangle : E\times E \to \mathbb{C}$ is a function which has the following properties:

(a) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

(b) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for $a, b \in \mathbb{C}$.

(c) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

Remark (Real inner product)**.** We can also define inner products of spaces over $\mathbb{R}$ with no extra conjugation in the last property.

Definition 3.1.2 (Inner product space)**.** An *inner product space* is a linear space E with an inner product $\langle\cdot,\cdot\rangle : E\times E \to \mathbb{C}$.

Definition 3.1.3 (Orthogonal)**.** Given a linear space E, $x, y \in E$ are *orthogonal* if $\langle x, y\rangle = 0$, denote as $x \perp y$.

Theorem 3.1.1 (Cauchy-Schwarz inequality)**.** Let $x, y \in E$ and an inner product $\langle\cdot,\cdot\rangle$, then

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}} .$$

Proof. Define $Q(t)$ by $Q(t) = \langle x + ty, x + ty\rangle = \langle y, y\rangle t^2 + 2t \operatorname{Re}\langle x, y\rangle + \langle x, x\rangle$ if $t \in \mathbb{R}$. Then we see that $Q(t) \geq 0$ with $t \in \mathbb{R}$ and the equation $Q(t) = 0$ has no real roots, implying $(\operatorname{Re}\langle x, y\rangle)^2 \leq \langle x, x\rangle \langle y, y\rangle$. Finally, the result follows by choosing $\theta \in \mathbb{R}$ such that

$$\langle x, y\rangle = \operatorname{Re}\left\langle xe^{i\theta}, y\right\rangle .$$

■

Corollary 3.1.1. The function $x \mapsto \|x\| \coloneqq \langle x, x\rangle^{\frac{1}{2}}$ is a norm on E.

Proof. The triangle inequality is a consequence of Theorem 3.1.1 such that

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \overset{!}{\leq} \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = \langle\|x\| + \|y\|\rangle^2 .$$

■

Example. The space ℓ_2 of square summable sequences $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$,

$$\langle x, y \rangle \coloneqq \sum_{j=1}^{\infty} x_j \overline{y}_j.$$

Example. The space $L_2(\Omega, \Sigma, \mu)$ of $f, g \in L_2(\Omega, \Sigma, \mu)$,

$$\langle f, g \rangle = \int_{\Omega} f(x) \overline{g}(x) \, \mathrm{d}\mu(x).$$

Example. The space of $m \times n$ matrices $A = (a_{ij})$, $1 \leq i \leq m, 1 \leq j \leq n$. Then

$$\langle A, B \rangle = \operatorname{Tr} AB^*,$$

where B^* is the Hermitian adjoint of B, i.e., for $B = (b_{ij})$, then $B^* = (b^*_{ij})$ for $b^*_{ij} = \overline{b}_{ji}$.

Remark (Hilbert-Schmidt norm)**.** Specifically, the norm corresponding to this inner product is

$$\|A\|_{\mathrm{HS}} \coloneqq \sum_{i,j}^{\infty} \left(|a_{ij}|^2 \right)^{1/2},$$

which is known as the *Hilbert-Schmidt* norm.

For an inner product space, the inner product can be expressed in terms of the norm. This is because both parallelogram law and polarization identity hold.

Lemma 3.1.1 (Parallelogram law)**.** Given E an inner product space, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Lemma 3.1.2 (Polarization identity)**.** Given E an inner product space, we have

$$\langle x, y \rangle = \frac{1}{4} \left\{ \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right\}$$

Lecture 4: Orthogonality and Projection

As previously seen. Recall the parallelogram law and polarization identity. The proof is just to expand the right-hand side in terms of inner product.

08 Sep. 14:30

Check it!

Remark. Polarization identity shows that the function $x \mapsto \|x\|^2$ determines the inner product.

3.2 Hilbert Spaces

Definition 3.2.1 (Hilbert space)**.** A complete inner product space is called a *Hilbert space*.

Example. We have seen that ℓ_2 and $L^2(\Omega, \Sigma, \mu)$ are complete, hence are Hilbert space.

We'll soon see that the key notion in Hilbert space theory is orthogonality.

Definition 3.2.2 (Orthogonal complement). Let $A \subseteq \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space. Then the *orthogonal complement* $A^\perp$ of A is

$$A^\perp := \{x \in \mathcal{H} \colon \langle x, y\rangle = 0 \text{ for } y \in A\}.$$

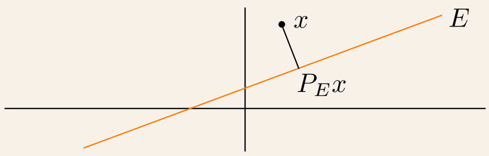
Remark. $A^\perp$ is also a Hilbert space.

Proof. Since $A^\perp$ is closed linear subspace of $\mathcal{H}$, where the closure follows from the continuity of the function $x \mapsto \langle x, y\rangle$ for $x \in \mathcal{H}$ by looking at the inverse image of $\{0\}$. ⊛

Theorem 3.2.1 (Orthogonality principle). Assume $E \subseteq \mathcal{H}$ is a closed linear subspace of the Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$. Then we have the following.

(a) Then there exists a unique closest point $y = P_E x \in E$ to x, i.e., $\|x - P_E x\| = \inf_{y' \in E} \|x - y'\|$.

(b) The point $y = P_E x \in E$ is the unique vector such that $x - y \in E^\perp$.



Proof. Note that the function $y' \mapsto \|x - y'\|$ for $y' \in E$ is convex. We expect a minimizer y'.

> **Note.** To show this exists, we typically need
>
> 1. Compactness properties
> 2. Non-degeneracy properties for uniqueness

(a) Here by using parallelogram law, we don't need compactness. Let $y_n \in E$ for $n = 1, 2, \ldots$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} \|x - y_n\| = \inf_{y' \in E} \|x - y'\| = d.$$

From parallelogram law, we have

$$\|y_n - y_m\|^2 + 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2.$$

As $n, m \to \infty$, the right-hand side goes to $4d^2$. But since $\frac{1}{2}(y_n + y_m) \in E$, we have $\left\|x - \frac{1}{2}(y_n - y_m)\right\| \geq d$, so

$$\lim_{m\to\infty} \sup_{m \geq n} \|y_n - y_m\|^2 = 0,$$

which further implies $\{y_n\}$ is a Cauchy sequence. As $\mathcal{H}$ is complete, we see that $y_n \to y_\infty \in E$, with $\|x - y_\infty\| = d$.

Now, with the fact that E is closed, we set $y_\infty = P_E x$ where y_∞ is unique since if $\|x - y_\infty\| = \|x - y'_\infty\| = d$, again by the parallelogram law where we now plug in y_∞ and y'_∞ instead of y_n and y_m as above, we see that $\|y_\infty - y'_\infty\| = 0$. In all, $P_E x \in E$ is uniquely defined.

(b) We now shoe $P_E x$ is the unique vector $y \in E$ such that $x - y \perp E$, i.e., $x - y \in E^\perp$. Let $y' \in E$ and let $Q(t)$ be the quadratic

$$Q(t) := \langle x - P_E x + ty', x - P_E x + ty'\rangle = \|x - P_E x + ty'\|^2.$$

Since $t \mapsto Q(t)$ has a **strict** minimum at $t = 0$, which implies $Q'(0) = 0$, i.e., $\operatorname{Re}(x - P_E x, y') = 0$ for all $y' \in E$, which further implies $\langle x - P_E x, y' \rangle = 0$ for all $y' \in E$. This shows that $x - P_E x \in E^\perp$. Finally, we need to show $P_E x \in E$ is the unique vector such $x - P_E x \in E^\perp$. This can be seen from $Q(t) = \|x - P_E x\|^2 + t^2 \|y'\|^2$ for any $y' \in E$.

■

Remark. Theorem 3.2.1 shows that the minimizer for the function $y' \mapsto \|x - y'\|$ for $y' \in E$ is characterized by the orthogonality condition, i.e., $x - y \perp E$ for some $y \in E$.

Definition 3.2.3 (Orthogonal projection)**.** Let $\mathcal{H}$ be a Hilbert space ad let $E \subseteq \mathcal{H}$ be a closed subspace. The *orthogonal projection operator* $P_E \colon \mathcal{H} \to E$ is given by $x \mapsto P_E x$ where $P_E x$ is defined uniquely via $x - P_E x \in E^\perp$.

Definition 3.2.4 (Bounded linear map)**.** Given a mapping $A \colon \mathcal{B} \to \mathcal{B}$ on a Banach space $\mathcal{B}$, we say it's a *bounded linear map* if it's bounded and linear.

Definition 3.2.5 (Linear map)**.** The operator A is *linear* if for $x, y \in \mathcal{B}$, $a, b \in \mathbb{C}$,

$$A(ax + by) = aA(x) + bB(y).$$

Definition 3.2.6 (Bounded map)**.** The operator A is *bounded* if

$$\|A\| \coloneqq \sup_{\|x\|=1} \|Ax\| < \infty.$$

Remark. Note that $\|Ax\| \leq \|A\| \|x\|$ for $x \in \mathcal{B}$.

We see that $P_E x$ is a bounded linear operator $P_E \colon \mathcal{H} \to E$ with the properties $P_E^2 g P_E$ and $\|x\|^2 = \|P_E x\|^2 + \|(I - P_E)x\|^2$ since $(I - P_E)x \perp P_E x$. The latter property shows that

$$\|P_E\| \leq 1, \quad \|(I - P_E)\| \leq 1,$$

and fact, $\|P_E\| = \|I - P_E\| = 1$. Also, $I - P_E$ is also an orthogonal projection onto $E^\perp$.

3.2.1 Orthogonal Systems

We first give the definition.

Definition 3.2.7 (Orthogonal system)**.** A sequence $\{x_k \colon k \geq 1\}$ of non-zero vectors in a Hilbert space $\mathcal{H}$ is *orthogonal* if $\langle x_k, x_\ell \rangle = 0$ for all $\ell \neq k$.

Definition 3.2.8 (Orthonormal system)**.** A orthogonal system is called an *orthonormal system* if in addition, we have $\|x_k\| = 1$ for $k = 1, 2, \ldots$.

Remark (Equivalence definition of orthonormal system)**.** $\{x_k \colon k \geq 1\}$ is orthonormal if $\langle x_k, x_\ell \rangle = \delta_{k,\ell}$ where δ is the Kronecker delta.

We now see some examples.

Example. $x_k = (0, 0, \ldots, \delta, 0, \ldots, 0) \in \ell_2$ for $k = 1, 2, \ldots$ is orthonormal sequence in ℓ_2.

Example (Fourier basis). For $L_2([-\pi,\pi])$,

$$e_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$$

for $k \in \mathbb{R}$ is orthonormal. In addition, this is the Fourier basis associated with the Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \frac{1}{\sqrt{2\pi}} e^{ikt}$$

where

$$a_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \,\mathrm{d}t.$$

Remark. We can further generalize Fourier series to any Hilbert space by letting $\{x_k : k \geq 1\}$ be an orthonormal set in $\mathcal{H}$. For $n = 1, 2, \ldots$, we define $S_n \colon \mathcal{H} \to E_n$ such that

$$S_n(x) = \sum_{k=1}^{n} \langle x, x_k \rangle \, x_k$$

for $x \in \mathcal{H}$ where $E_n = \operatorname{span}\{x_1, \ldots, x_n\}$. We see that S_n is a linear operator and $S_n = P_{E_n}$ is the orthogonal projection onto E_n since $\langle x - S_n(x), x_k \rangle = 0$ for $k = 1, \ldots, n$ and $S_n(x) \in E_n$, $x - S_n(x) \perp E_n$.

Remark (Bessel's inequality). Additionally,

$$\left\| S_n(x) \right\|^2 = \sum_{k=1}^{n} \left| \langle x, x_k \rangle \right|^2,$$

with $S_n = P_{E_n}$ and $\left\| P_{E_n} x \right\|^2 \leq \left\| x \right\|^2$, we have

$$\sum_{k=1}^{n} \left| \langle x, x_k \rangle \right|^2 \leq \left\| x \right\|^2$$

for $x \in \mathcal{H}$. This is the so-called *Bessel's inequality*.

Theorem 3.2.2. Let $\{x_k : k \geq 1\}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$. Then the corresponding Fourier expansion $S_n(x) = \sum_{k=1}^{n} \langle x, x_k \rangle \, x_k$ converges, i.e., $\lim\limits_{n\to\infty} S_n(x) = S_\infty(x)$ exists for $x \in \mathcal{H}$. Furthermore, $S_n = P_{E_n}$ for every n where E_n is the space spanned by $\{x_i\}_{i=1}^{n}$.[a]

[a]This includes $n = \infty$, where E_∞ is the **closure** of the space spanned by $\{x_i\}_i$.

Proof. We show that the sequence $S_n(x)$ for $n = 1, 2, \ldots$ is Cauchy. This is because

$$\left\| S_n(x) - S_m(x) \right\|^2 = \sum_{k=m+1}^{n} \left| \langle x, x_k \rangle \right|^2,$$

and Bessel's inequality implies $\sum_{k=1}^{\infty} \left| \langle x, x_k \rangle \right|^2 \leq \left\| x \right\|^2$. Hence, for any $\epsilon > 0$, there exists $m(\epsilon)$ such that

$$\sum_{k=m(\epsilon)+1}^{\infty} \left| \langle x, x_k \rangle \right|^2 < \epsilon,$$

which implies $\left\| S_n(x) - S_m(x) \right\|^2 < \epsilon$ if $n > m(\epsilon)$, hence $\{S_n(x) : n \geq 1\}$ is Cauchy, and $\lim\limits_{n\to\infty} S_n(x) = S_\infty(x) \in \mathcal{H}$. Also, $S_\infty = P_{E_\infty}$ where E_∞ is the closure of the linear space generated by the sequence $\{x_k : k \geq 1\}$. ∎

Remark. Note that the closeness of E_∞ makes sense since the self-dual of a set's orthogonal complement is itself if it's closed in the first place.

Lecture 5: Abstract Fourier Series

Let's start with a definition. 13 Sep. 14:30

Definition 3.2.9 (Complete system). A system of vector $\{x_k : k \geq 1\}$ in Hilbert space $\mathcal{H}$ is *complete* if the space spanned by $\{x_k : k \geq 1\}$ is **dense** in $\mathcal{H}$.

Example (Fourier inversion formula). If an orthogonal set $\{x_k : k \geq 1\}$ is complete, then $E_\infty = \mathcal{H}$, $P_{E_\infty} = I$. This implies

$$x = \sum_{k=1}^{\infty} \langle x, x_k \rangle \, x_k$$

for $x \in \mathcal{H}$. This is Fourier inversion formula.

Remark (Parseval's identity). We have $\|x\|^2 = \|P_{E_n} x\|^2 + \|(I - P_{E_n})\|^2$. By letting $n \to \infty$, we have

$$\|x\|^2 = \lim_{n\to\infty} \|P_{E_n} x\|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 = \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 .$$

Definition 3.2.10 (Separable). A metric space is *separable* if it contains a countable dense subset.

Remark (Banach space). For Banach space, separability follows from finding a countable set of vectors $\{x_k : k \geq 1\}$ such that the span of $\{x_k : k \geq 1\}$ is dense in E.

3.3 Gram-Schmidt Orthogonalization

Suppose $x_1, x_2, \ldots \in \mathcal{H}$ is a set of vectors and $E_n = \operatorname{span}(\{x_1, \ldots, x_n\})$. Then we can find an orthonormal set $\{y_k \in \mathcal{H} : k \geq 1\}$ such that $E_n = \operatorname{span}(\{y_1, y_2, \ldots, y_{m(n)}\})$ where $m(n) \leq n$.

Firstly, set $y_1 = x_1 / \|x_1\|$, and

$$y_n = \frac{(I - P_{E_{n-1}}) x_n}{\left\|(I - P_{E_{n-1}}) x_n\right\|}$$

if $x_n \notin E_{n-1}$, i.e., E_{n-1} is properly contained in E_n.

Remark. Proving completeness of a set of vectors $\{x_k : k \geq 1\}$ in $\mathcal{H}$ can be **non**-trivial.

Example (Haar basis). We consider the *Haar basis* for $L^2([0,1])$. Let $h\colon (0,1) \to \mathbb{R}$ where

$$h(t) = \begin{cases} 1, & \text{if } 0 < t < 1/2; \\ -1, & \text{if } 1/2 < t < 1. \end{cases}$$

Extend $h(\cdot)$ by zero outside $(0,1)$, we get $h\colon \mathbb{R} \to \mathbb{R}$, $h(t) = 0$ if $t \notin (0,1)$, otherwise it's the same as above. The function $t \mapsto h(2^k t)$ has support in interval $0 < t < 2^{-k}$. Move the support to interval $\ell 2^{-k} < t < (\ell+1) 2^{-k}$ by translation. Set

$$h_{k,\ell}(t) = h(2^k t - \ell), \quad \ell = 0, 1, \ldots, 2^k - 1.$$

The constant function plus functions $h_{k,\ell}$, $k = 0, 1, 2, \ldots$, $0 \leq \ell \leq 2^k - 1$ are a complete orthogonal set for $\mathcal{H} = L^2([0,1])$.

Proof. The span of the Haar functions includes characteristics functions χ_F for all dyadic intervals $[2^{-k}\ell, 2^{-k}(\ell+1)]$ for $\ell = 0, 1, \ldots, 2^{k-1}$, $k = 0, 1, \ldots$. If the set is **not** complete, then there exists $f \in L^2([0,1])$ such that

$$\int_F f \, \mathrm{d}t = 0$$

for all dyadic intervals F. Since we can approximate any measurable set $E \subseteq (0,1)$ by a union of dyadic intervals.

> **Intuition.** an easy way to see this is to consider
>
> $$\left\{ F \in \mathcal{B} \colon : \int_F f \, \mathrm{d}t = 0 \right\},$$
>
> which is the Borel subalgebra of $\mathcal{B}$, which indeed is a Borel algebra on $(0,1)$. Then observe that dyadic intervals generate all open intervals.

Hence, we see that $\int_F f \, \mathrm{d}t = 0$ for all measurable $F \subseteq (0,1)$. Let $F = \{t \in (0,1) \colon f(t) > 0\}$, if $m(F) > 0$, then

$$\int_F f \, \mathrm{d}t > 0.$$

Hence, a contradiction, so $m(F) = 0$. ⊛

Example (Fourier basis)**.** Consider the Fourier basis $e_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$ for $k \in \mathbb{Z}$, $-\pi < t < \pi$. This is complete in $L^2([-\pi, \pi])$.

Proof. We use Stone-Weierstrass theorem and apply it to Fourier basis. All $e_k(\cdot)$ are in $C[-\pi, \pi]$, i.e., continuous functions $f \colon [-\pi, \pi] \to \mathbb{C}$. We know that $C([-\pi, \pi])$ is a Banach space with supremum norm $\|f\| \coloneqq \sup_{t \in [-\pi, \pi]} |f(t)|$. Stone-Weierstrass theorem implies density of the space spanned by $e_k(\cdot)$, $k \in \mathbb{Z}$ in $C([-\pi, \pi])$, hence the completeness in $L^2([-\pi, \pi])$ follows from the density of continuous functions in $L^2([-\pi, \pi])$. ⊛

3.4 Bounded Linear Functional

Definition. Let E be a linear space over $\mathbb{R}$ or $\mathbb{C}$.

> **Definition 3.4.1** (Linear functional)**.** A *linear functional* on E is a linear operator $f \colon E \to \mathbb{R}$ of $\mathbb{C}$ such that
>
> $$f(ax + by) = af(x) + bf(y)$$
>
> for $x, y \in E$, $a, b \in \mathbb{R}$ or $\mathbb{C}$.

> **Definition 3.4.2** (Bounded linear functional)**.** We say a linear functional $f(\cdot)$ is a *bounded linear functional* if
>
> $$\|f\| \coloneqq \sup_{\|x\|=1} |f(x)| < \infty$$
>
> by dilation and additive.

Remark. The boundedness of $f(\cdot)$ implies $|f(x-y)| \leq \|f\| \, \|x-y\|$ for $x, y \in E$. Hence, $f(\cdot)$ is continuous and in fact Lipschitz continuous.

Remark. Conversely, if a linear functional is continuous then it is bounded.

Proof. Suppose $f(\cdot)$ is not bounded, then there exists a sequence $x_n \in E$ such that $|f(x_n)| \geq n \|x_n\|$ for $n = 1, 2, \ldots$. By linearity,

$$\left| f\left(\frac{x_n}{n\|x_n\|}\right) \right| \geq 1, \quad n = 1, 2, \ldots.$$

But we know $\lim_{n\to\infty} \frac{x_n}{n\|x_n\|} = 0$ and $f(0) = 0$, hence $f(\cdot)$ is not continuous at 0. ⊛

Definition 3.4.3 (Dual space)**.** Let E be a normed space. The space of all bounded linear functionals $f(\cdot)$ on E is known as the *dual space* E^* of E.

Remark. The dual space is also a normed space with norm $\|f\| \coloneqq \sup_{\|x\|=1} |f(x)|$, which is in fact a Banach space. And it is a Banach space even if the original E is not.

Definition 3.4.4 (Hyperplane)**.** Let E be a linear space and $H \subseteq E$ is a subspace. Say H is a *hyperplane* if $\operatorname{codim}(H) = 1$, i.e., $\dim(E \,/\, H) = 1$.

The goal is to make an equivalence between bounded linear functionals on E and *closed* hyperplanes in E.

Problem 3.4.1. Does there exist a **non**-closed hyperplane?

Answer. We know that this is not the case in finite dimension. And this question is analogous to asking *does there exist a subset $F \subseteq \mathbb{R}$ which is **not** Lebesgue measurable?* The answer to this is yes in both cases. However, construction uses axiom of choice. ⊛

Proposition 3.4.1. Let E be a linear space.

(a) For every linear functional on E, $\ker(f)$ is a hyperplane in E. If E is a Banach space, and $f(\cdot)$ is bounded, then $\ker(f) = H$ is closed.

(b) If $f, g \neq 0$ are linear functionals on E such that $\ker(f) = \ker(g)$, then $f = ag$ for some $a \neq 0$.

(c) For every hyperplane $H \subseteq E$, there exists a linear functional $f \neq 0$ on E such that $\ker(f) = H$. If E is a Banach space, and $\ker(f) = H$ is closed, then $f(\cdot)$ is bounded.

Lecture 6: Abstract Fourier Series

Let's first see the proof of Proposition 3.4.1. 15 Sep. 14:30

Proof of Proposition 3.4.1. We prove them in order.

(a) Let $x, y \notin \ker(f)$, so $f(x), f(y) \neq 0$. Hence, there exists a scalar $\lambda \neq 0$ such that $f(x) = \lambda f(y)$, i.e., $x - \lambda y \in \ker(f)$. Hence, if $[x], [y] \in E \,/\, \ker(f)$, $[x] = \lambda[y]$, implying $\dim E \,/\, \ker(f) = 1$. Now, if f is bounded, then f is continuous, so $\ker(f) = f^{-1}(\{0\})$ is closed.

(b) Consider the induced functionals $\widetilde{f}, \widetilde{g} \colon E \,/\, H \to \mathbb{R}$ or $\mathbb{C}$ where $H = \ker(f) = \ker(g)$. This implies

$$\dim\left({}^{E}\!/_{H}\right) = 1 \Rightarrow \widetilde{f} = a\widetilde{g} \text{ for some } a \neq 0 \Rightarrow f = ag.$$

(c) Assume $\dim(E \,/\, H) = 1$, so $E \,/\, H = \{a[x_0] \colon a \in \mathbb{C} \text{ (or } \mathbb{R})\}$ for some $x_0 \in E$. Then, for any $x \in E$, then $[x] = a(x)[x_0]$ for some $a(x) \in \mathbb{C}$ or $\mathbb{R}$. Define $f(x) \coloneqq a(x)$, we see that f is linear and $\ker(f) = H$. Now, if E is a Banach space and H is closed with $\dim(E \,/\, H) = 1$. Recall that $E \,/\, H$ is also a Banach space with norm $\|[x]\| = \inf_{y\in H} \|x + y\|$ for $x \in E$.[a]

Let $\widetilde{f}$ be a linear functional on E / H. Then $\dim(E / H)$ is finite, $\widetilde{f}$ is continuous, implying $\left|\widetilde{f}([x])\right| \leq A \|[x]\|$ for all $x \in E$. Finally, we define $f(x) = \widetilde{f}([x])$ for $x \in E$, then $\ker(f) = H$ and $|f(x)| \leq A \|[x]\| \leq A \|x\|$.

■

[a] We see now why we need the closure: otherwise we'll get a non-zero function with norm 0.

3.5 Riesz Representation Theorem

Let's first see the statement.

Theorem 3.5.1 (Riesz Representation Theorem). Let $\mathcal{H}$ be a Hilbert space. Then we have the following.

(a) For every $y \in \mathcal{H}$, then function $f(x) = \langle x, y \rangle$ for $x \in \mathcal{H}$ is a bounded linear functional on $\mathcal{H}$.

(b) If $f\colon \mathcal{H} \to \mathbb{C}$ or $\mathbb{R}$ is a bounded linear functional on $\mathcal{H}$, then there exists $y \in \mathcal{H}$ such that $f(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$. Hence, the dual $\mathcal{H}^*$ of $\mathcal{H}$ is isometric to $\mathcal{H}$.

Proof. We prove this in order.

(a) This follows form Cauchy-Schwarz inequality such that

$$|f(x)| = |\langle x, y \rangle| \leq \|x\| \|y\|,$$

i.e., $\|f\| = \|y\|$ by setting $x = y / \|y\|$.

> **Note.** Note that there exists x_f such that $\|x_f\| = 1$ since $\|f\| = \sup_{\|x\|=1} |f(x)| = f(x_f)$, i.e., the supremum is achieved, although we're working on an infinite dimensional space. This property does not always hold for bounded linear functionals on Banach space. But this holds for Hilbert space.

(b) Let $f\colon \mathcal{H} \to \mathbb{C}$ or $\mathbb{R}$ be a bounded linear functional on $\mathcal{H}$. Let $H = \ker(f)$, which is closed from Proposition 3.4.1. Let $H^{\perp}$ be the orthogonal complement of H, i.e., $\mathcal{H} = H \oplus H^{\perp}$. Then $\dim(\mathcal{H} / H) = 1 \Rightarrow \dim(H^{\perp}) = 1$. Choose $y \in H^{\perp}$ such that $f(y) = \langle y, y \rangle$ which can be done since $f(y') = \lambda y'$ for some $\lambda \in \mathbb{C}$ or $\mathbb{R}$. This implies $f(x) = \langle x, y \rangle$ for $x \in \mathcal{H}$.

■

We can use Riesz representation theorem to give a proof of Radon-Nikodym theorem.

Theorem 3.5.2 (Radon-Nikodym theorem). Let μ, ν be two finite measures such that $v \ll \mu$, i.e., ν is absolutely continuous w.r.t, μ.[a] Then there exists $g \geq 0$ such that g is μ integrable and

$$\nu(A) = \int_A g \,\mathrm{d}\mu$$

for A measurable.

[a] This means $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

Proof. Consider the linear functional $F\colon L^2(\mu) \to \mathbb{R}$ or $\mathbb{C}$ such that

$$F(f) = \int_\Omega f \,\mathrm{d}\mu.$$

Then we have $\|F(f)\| \leq \|f\|_2 \sqrt{\mu(\Omega)}$. We see that F is also a bounded linear functional on $L^2(\mu+\nu)$, hence by Theorem 3.5.1, there exists $h \in L^2(\mu+\nu)$ such that

$$F(f) = \int_\Omega fh \,\mathrm{d}(\mu+\nu)$$

for $f \in L^2(\mu+\nu)$, i.e.,

$$\int_\Omega f \,\mathrm{d}\mu = \int_\Omega fh \,\mathrm{d}\mu + \int_\Omega fh \,\mathrm{d}\nu \tag{3.1}$$

if $f \in L^2(\mu+\nu)$. This further implies

$$\int_\Omega fh \,\mathrm{d}\nu = \int_\Omega f[1-h] \,\mathrm{d}\mu \tag{3.2}$$

for $f \in L^2(\mu+\nu)$. Let $A \subseteq \Omega$ where A is measurable, and we set $f = \frac{1}{h}\chi_A$, then $\nu(A) = \int_A g \,\mathrm{d}\mu$, $g = \frac{1-h}{h} \Rightarrow g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. But we still need to show that this is well-defined, i.e., there are no zeros of h.

Claim. This h satisfies $0 < h \leq 1$ a.e. i.e., outside a set of measure $\mu+\nu$ zero.

Proof. We first note that $\mu(A) = 0 \Leftrightarrow \mu(A) + \nu(A) = 0$. Let $A = \{h \leq 0\}$, $f = \mathbb{1}_A$ be the characteristic function on A. Then Equation 3.1 implies

$$\int_A h(\,\mathrm{d}\mu + \,\mathrm{d}\nu) \leq 0 \Rightarrow \mu(A) = 0 \Rightarrow h > 0 \ \mu \text{ a.e.}$$

But since g is a positive function, so we also need $h \leq 1$. Again, set $B = \{h > 1\}$, $f = \mathbb{1}_B$. Then Equation 3.1 implies

$$\mu(B) = \int_B h\,(\,\mathrm{d}\mu + \,\mathrm{d}\nu) > \mu(B)$$

unless $\mu(B) = 0$. ⊛

We see that $0 < h \leq 1$ μ a.e. This implies Equation 3.2 holds for all $f \geq 0$, $f \in L^2(\mu+\nu)$. Now, by using monotone convergence theorem, we conclude that Equation 3.2 holds for all $f \geq 0$,[a] hence we just plug in $f = \frac{1}{h}\chi_A$ and get the result. ■

[a]Both sides could be ∞.

Notation (Radon-Nikodym derivative). g in Theorem 3.5.2 is referred to as the *Radon-Nikodym derivative* where $g := \mathrm{d}\nu/\mathrm{d}\mu$.

Note (Uniqueness). The uniqueness of Radon-Nikodym derivatives can be shown via

$$\int_A g \,\mathrm{d}\mu = 0$$

for all μ-measurable A, i.e., $g = 0$ μ a.e.

Remark. We see that given a Hilbert space $\mathcal{H}$, Riesz representation theorem identifies the dual space $\mathcal{H}^*$, which can be used to show Radon-Nikodym theorem. Consider spaces $L^p(\Omega,\mu)$ for $1 \leq p < \infty$, Radon-Nikodym theorem implies that the dual of $L^p(\Omega,\mu)$ is $L^q(\Omega,\mu)$ where $1/p + 1/q = 1$, and if $p = 1$, $q = \infty$.

Proof. The easy part is that $g \in L^q$ induces a bounded linear functional on L^p by setting

$$F(f) = \int_\Omega fg \,\mathrm{d}\mu.$$

By Hölder's inequality, $|F(f)| \leq \|f\|_p \|g\|_q$. We then see that $\|F\| = \|g\|_q$ by choosing $f = |g|^{q-1} \overline{g} / |g|$ since

$$F(f) = \int_\Omega |g|^q \,\mathrm{d}\mu = \|g\|_q^q,$$

and from $1/p + 1/q = 1 \Rightarrow q - 1 = q/p$, we have

$$\|f\|_p^p = \int |f|^p \,\mathrm{d}\mu = \int_\Omega |g|^q \,\mathrm{d}\mu = \|g\|_q^q \Rightarrow \|f\|_p = \|g\|_q^{q/p} = \|g\|_q^{q-1}.$$

This implies

$$F(f) = \int_\Omega |g|^q \,\mathrm{d}\mu \Rightarrow \|g\|_q^q = \|g\|_q \|f\|_p.$$

Note. We see that $\sup_{\|f\|_p = 1} |F(f)|$ is attained for these functionals c.

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Appendix

Appendix A

Additional Proofs

Bibliography

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