EECS544
Analysis of Societal Networks

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Abstract

We'll use Easley, D. and Kleinberg, J.[EK10] as our main reference. We focus on mathematically rigorous approach to analyze social network and also game theory.

This course is taken in Fall 2021, and the date on the covering page is the last updated time.

Contents

Chapter 1

Graph

Lecture 1: Introduction to Graph Theory

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1.1 Basic Definition

Let's start with some of the most fundamental definitions.

Definition 1.1.1 (Vertex set)**.** A set is called a *vertex set* if we view its elements as vertices.[a]

[a] Sometimes *nodes*.

Definition 1.1.2 (Graph)**.** There are two kinds of *graph* we're interested in, undirected graph and directed graph.

Definition 1.1.3 (Undirected graph)**.** An *undirected graph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set.

Definition 1.1.4 (Edge set)**.** An *edge set* $\mathcal{E}$ of an undirected graph $\mathcal{G}$ is a set of paired vertices, which we call edges.[a]

[a] Sometimes *links*.

Definition 1.1.5 (Directed grpah)**.** A *directed graph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the directed edge set.

Definition 1.1.6 (Directed edge set)**.** A *directed edge set* $\mathcal{E}$ of a directed graph $\mathcal{G}$ is a set of ordered-paired vertices, which we call directed edges.[a]

[a] Sometimes *directed links*.

Example (Undirected and directed graph)**.** Let $A = \{a, b, c, d, e\}$ be a vertex set, then an example of edge sets can be

$$B \coloneqq \{\{a, b\}, \{b, e\}, \{c, d\}\},$$

and

$$B' \coloneqq \{(a, b), (b, e), (c, d)\}$$

is an example of directed edge set. Together, (A, B) is a valid undirected graph, while (A, B') is a directed graph.

Notation. We some time will write an edge between a and b as (a, b) instead of $\{a, b\}$ and a directed edge from a to b as $a \to b$ if the content is clear.

Note. While the definition of a vertex set is purely abstract, we often just deal with concrete graph where the structure of the graph is clear (e.g., in social network, every people are vertices, and relationships between people are edges).

Definition. Given a directed graph $\mathcal{G}$, we can define the followings.

Definition 1.1.7 (In-degree). For a node $v \in \mathcal{V}$, the *in-degree* of v in a directed graph $\mathcal{G}$ is defined as the number of directed edges pointing in to v.

Definition 1.1.8 (Out-degree). For a node $v \in \mathcal{V}$, the *out-degree* of v in a directed graph $\mathcal{G}$ is defined as the number of directed edges pointing out from v.

Moreover, if we generalize above to an undirected graph we have the following.

Definition 1.1.9 (Degree). Given an undirected graph, for a node $v \in \mathcal{V}$, the *degree* of v in an undirected graph is defined as the number of edges link with v.

Definition 1.1.10 (Subgraph). A *subgraph*[a] $\mathcal{G}'$ is defined by a pair $(\mathcal{V}', \mathcal{E}')$ such that $\mathcal{V}' \subseteq \mathcal{V}$, $\mathcal{E}' \subseteq \mathcal{E}$.

[a] Sometimes *component*.

Note. Notice that the edges in $\mathcal{E}'$ need to be well-defined. Namely, for every $e = (a, b) \in \mathcal{E}'$, a, b need to also be in $\mathcal{V}'$.

Finally, we introduce the following definition.

Definition 1.1.11 (Simple graph). A *simple graph* is a graph without loops and multiple edges.[a]

[a] i.e., no edges like $\{a, a\}$ and there is only one edge between any two nodes.

Lecture 2: Graph Theory

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Definition 1.1.12 (Path). Given a list of vertices indexed in order as $v_1, \ldots, v_n$ where $v_i \in \mathcal{V}$,

(a) A *path* P connecting v_i in an undirected graph is a subgraph $P = (V, E)$, where $V = \{v_i\}_{i=1}^{n}$, and $E = \{\{v_i, v_{i+1}\}\}_{i=1}^{n-1}$.

(b) A *path* P connecting v_i in a directed graph is a subgraph $P = (V, E)$, where $V = \{v_i\}_{i=1}^{n}$, and $E = \{(v_i, v_{i+1})\}_{i=1}^{n-1}$.

With the definitions we introduced, we can now try to characterize a graph.

Definition. We can then define connectivity of a graph as follows.

Definition 1.1.13 (Connected). An undirected graph is *connected* if for every two nodes, there exists at least one path connect them together.

Definition 1.1.14 (Strongly connected). A directed graph is *strongly connected* if for every two nodes, there exists at least one path connect them together.

Notation (Connected component). The above definitions can be generalized to a subgraph as well. In this case, we say that this subgraph is connected, or more often we'll say this subgraph is a *connected component*. Same for strongly connected.

Definition 1.1.15 (Giant component). A subgraph is called a *giant component* of a graph if it is a connected component and with a significant number of nodes of the original graph.

1.2 Matrices Associated with Graphs

1.2.1 Adjacency Matrix

Definition 1.2.1 (Adjacency matrix). For an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the *adjacency matrix of an undirected graph* is a matrix A such that

$$A_{ij} = \begin{cases} 1, & \text{if } \{i,j\} \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases}$$

For a directed graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, the *adjacency matrix of a directed graph* is a matrix A' such that

$$A'_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{E}' \\ 0, & \text{otherwise.} \end{cases}$$

Remark. The adjacency matrix of an undirected graph must be symmetric by definition.

Proof. Since if $\{i,j\} \in \mathcal{E}$, we know that $\{j,i\} \in \mathcal{E}$ as well since $\{j,i\} = \{i,j\}$, so $A_{ij} = A_{ji} = 1$.
On the other hand, if $\{i,j\} \notin \mathcal{E}$, then $\{j,i\} \notin \mathcal{E}$, hence in this case $A_{ij} = A_{ji} = 0$ clearly. ⊛

Chapter 2

Network

Lecture 3: Network Property

A network is essentially a graph with some interesting properties. We start with connectivity. 8 Sep. 12:30

2.1 Measuring Connectivity

Intuition. If there are more paths in the graph between different parts of the graph, then this graph is *more connected*.

2.1.1 Measuring the Difference between Two Paths

Definition 2.1.1 (Edge independent)**.** Two paths are said to be *edge independent* (EI) if they do not share a common edge.

Definition 2.1.2 (Vertex independent)**.** To paths are said to be *vertex independent* (VI) if they do not share a common vertex except the starting vertex and the ending vertex.

2.1.2 Cut Set

Before we formulate the definition of cut set, we'll need to define so-called induced subgraph since they're highly related.

Definition 2.1.3 (Induced subgraph)**.** Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the *induced subgraph* with respect to

- $\mathcal{V}' \subseteq \mathcal{V}$ is defined as $\mathcal{G}[\mathcal{V}'] \coloneqq (\mathcal{V}', \hat{\mathcal{E}})$, where $\hat{\mathcal{E}} \coloneqq \{e \in \mathcal{E} \colon e = (i, j), i, j \in \mathcal{V}'\}$. [a]
- $\mathcal{E}' \subseteq \mathcal{E}$ is defined as $\mathcal{G}[\mathcal{E}'] \coloneqq (\mathcal{V}, \mathcal{E}')$.

[a] This is for directed graph, but the case for undirected graph is similar, just change (i, j) to $\{i, j\}$.

Intuition. This is just a convenient notion for graph subtraction which is essentially just the subtraction between the edge set and the vertex set. Notice that the case of edge-subtraction is trivial, while node-subtraction needs to be well-defined, e.g, it's nonsense to say (i, j) is still in the induced subgraph if one end of the edge is not in the node set anymore.

Definition 2.1.4 (Edge cut set)**.** Given $u, v \in \mathcal{V}$, a set of edges $\mathcal{E}' \subseteq \mathcal{E}$ is called an *edge cut set* if u, v are in different connected components in the subgraph induced from $\mathcal{E}'$.

Definition 2.1.5 (Vertex cut set)**.** Given $u, v \in \mathcal{V}$, a set of vertices $\mathcal{V}' \subseteq \mathcal{V}$ is called a *vertex cut set* if u, v are in different connected components in the subgraph induced from $\mathcal{V}'$.

Note. It's worth noting that in the case of vertex cut set, if u or v is not in $\mathcal{V}'$, then we still say u and v are not in the same connected component (even if neither u nor v is in $\mathcal{V}'$).

Remark. Note that the above definitions are for undirected graph since we only consider connected but not strongly connected. But it's obvious that one can generalize the notion to directed graph.

Theorem 2.1.1 (Mengur's theorem)**.** If for any pair of nodes (i, j) we have $|\mathrm{ECS}(i, j)| > n$ where $\mathrm{ECS}(i, j)$ denotes the collection of edge cut set respect to (i, j), then for any pair of nodes (i, j),

$$\#\text{ edge independent paths between } i \text{ and } j > n.$$

Proof. A nice proof can be found here. ■

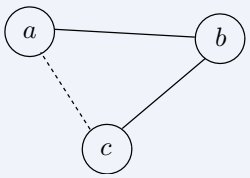
Lecture 4: Network Properties

We now try to further characterize the connectedness of a network. We first give a common notion. 13 Sep. 12:30

Definition 2.1.6 (Complete graph)**.** A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *complete* if $\mathcal{E}$ contains every possible edges.

Notation. We often denote a complete graph with n nodes as K_n.

Example (Triadic closure)**.** The *triadic closure* says that if a network has this property, then given $a, b, c \in \mathcal{V}$, if $\{a, b\}, \{b, c\} \in \mathcal{E}$, it's very likely that $\{a, c\} \in \mathcal{E}$ as well.



This motivates us that the number of triangles maybe can be used to quantify the connectedness of a network.

Definition 2.1.7 (Clustering coefficient)**.** For each node i, define the *clustering coefficient*

$$c_i = \frac{\#\text{ of triangles in the graph that include node } i}{d_i(d_i - 1)/2},$$

where d_i is the degree[a] of i.

[a] We assume our network is an undirected for simplicity.

Remark. We have, for any node i, $c_i \in [0, 1]$. If $\forall i\ c_i = 1$, then this graph is complete. If $\forall i\ c_i = 0$, then this graph is a tree.

2.1.3 Bridges and Local Bridges

Definition 2.1.8 (Bridge)**.** A *bridge* is the only edge that connects together two subgraphs that are themselves connected.

Remark. Deleting a bridge may change the distance between two nodes from some finite value to ∞. (disconnects those two nodes)

Definition 2.1.9 (Local bridge)**.** An edge (i, j) is said to be a *local bridge* if i, j have no neighbors in common.

Note. Equivalently, a local bridge is an edge that is not included in any triangles.

Remark. If (i, j) is a local bridge, then we have

$$d_{\mathcal{G}}(i, j) \geq 3$$

after removing this local bridge. Furthermore, a bridge must also be a local bridge by definition.

Definition 2.1.10 (Span of a local bridge)**.** The *span of a local bridge* b is defined as $d_{\mathcal{G}[\mathcal{E}\setminus\{b\}]}$.

2.2 Triadic closure and Overlapping Triangles

Intuition ([EK10] pp.62)**.** If two people in a social network have a friend in common, then there is an increased likelihood that they will become friends themselves at some point in the future.

Definition 2.2.1 (Strong triadic closure)**.** If a is a node and there are two nodes b, c such that (a, b) and (a, c) are strong ties, then (b, c) will form an edge. If for any nodes a, b, c in this graph $\mathcal{G}$ satisfy this property, then we say this graph has *strong triadic closure property* (STC).

Definition 2.2.2 (Embeddedness)**.** The *embeddedness* of an edge is defined as the number of common neighbors shared by the end points.

Chapter 3

Partition a Network

Lecture 5: Partition Networks Hierarchically

We want to partition a network in order to study its structure. We do this in the following ways 15 Sep. 12:30

1. Remove bridges and local bridge
2. Assign importance for each node

Problem. In what order?

Answer. Assign importance to local bridge. ⊛

This idea is the backbone of the Girvan-Newman method/algorithm.

3.1 Girvan-Newman method

Intuition. A brief summary of this algorithm is

1. Rank all local bridges.
2. Remove the highest ranked one, which yields a partition.
3. Re-calculate ranking.
4. Repeat from 2. until no edges remain.

Note. At the end all nodes become isolated.

Problem. How to rank order local bridges?

Intuition. An edge is more important if more shortest paths use it. Notice that a *normalization* is needed: Between pairs of nodes, there should be lots of variation in the number of the shortest paths and we want to balance these out.

3.1.1 Betweenness

Definition 3.1.1 (Betweenness). Imagine that there is a unit flow of water between each (distinct) pair of nodes. If the flow has k different ways to go to,[a] we divide this unit flow into $1/k$. We then equally divide this flow amongst the shortest paths coming out of the starting node.

[a] Actually not only depends on where can the flow go, but also where were that node added into BFS in the first place

With the definition of betweenness, the Girvan-Newman method becomes

Algorithm 1: Girvan-Newman Algorithm

Data: A network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

for $e \in \mathcal{E}$ **do**
  $\text{between}_e \leftarrow 0$ // Initialize betweenness variable for every edge

while $\mathcal{E} \neq \varnothing$ **do**
  Betweenness($\mathcal{G}$, $\{\text{between}_e\}_{e\in\mathcal{E}}$) // Calculate betweenness of all remaining edges
  $\text{between}_{\text{highest}} \leftarrow \max_{e\in\mathcal{E}} \text{between}_e$
  $\mathcal{E}_{\text{highest}} \leftarrow \{e \in \mathcal{E} : \text{between}_e = \text{between}_{\text{highest}}\}$
  $\mathcal{E} \leftarrow \mathcal{E} \setminus \mathcal{E}_{\text{highest}}$
  $\mathcal{G} \leftarrow \mathcal{G}[\mathcal{E}]$
return

Remark. Run time analysis: $O(\ln n)$.

Problem. But how do we calculate betweenness in reality?

3.1.2 Calculate Betweenness

Intuition. Use breadth-first search (BFS) in a slight different manner.

Remark. The key is that any sub-path to destination is also a shortest path.

With BFS exploration (i: starting point, m is the destination), we have

$$\#\text{ the shortest path from } i \text{ to } m = \#\text{ the shortest path from } i \text{ to } j + \#\text{ the shortest path from } i \text{ to } k$$

while j, k are at same level $n-1$, and m at level n, with the fact that there are edges between j, k to m. This can be done by adding a *counter* when running BFS.

3.1.3 Counter

From the above equation, we will have

$$c_v^{n+1} = \sum_{v^n} c_{v^n}^n$$

where the subscript indicates the node belongs to the counter, and c_v^{n+1} maintains the number of shortest path to v, which is just $n+1$.

The unit flow flows from nodes back from the path it is added into the BFS algorithm needs to keep its own 1, and then distribute.

Remark. An algorithm for calculating the importance of each local bridge, which turns out to be betweenness. And this is accomplished by BFS with a counter.

Lecture 6: Homophily

3.2 Shortest Path Algorithm

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We want to develop some algorithms similar to BFS to find the distance between nodes. We first make the notion of distance between nodes rigorous.

Definition 3.2.1 (Shortest path). Given a graph, a *shortest path* between two nodes u, v is a path whose end nodes are u, u while the sum of the weights[a] of its constituent edges is minimized.

[a] We only discuss unweighted graph so far, where we treat the weight as 1 for all edges.

Definition 3.2.2 (Distance between nodes). We usually let the length of the shortest path between two nodes in a graph $\mathcal{G}$ be the *distance between these two nodes*, denotes as

$$d_{\mathcal{G}}(\cdot, \cdot),$$

which takes two input for two corresponding nodes.

Something like

$$d_{\mathcal{G}}(i, j) = 1 + \min_{k \in N_{\mathcal{G}}(j)} d_{\mathcal{G}}(i, k).$$

3.2.1 Dynamic programming

There are many approaches to calculate the distance between nodes, one way is *Djikstra's algorithm*.

3.3 Homophily

Links in a (social) network are formed between people that are *similar*. This is an important feature that leads to the formation of communities.

3.3.1 Define *Like* and *Opposite*

Consider a network with two types of individuals:

Example (Middle school). Consider a school of boys and girls. Suppose the fraction of boy is p and the fraction of girls is $q = 1 - p$. We have

$$\frac{\binom{np}{2}}{\binom{n}{2}} \cong p^2$$

and also

$$1 - p^2 - q^2 = 2pq.$$

If the fraction of cross connection $\ll 2pq$, homophily is considered likely. Otherwise, if the fraction $\gg 2pq$, inverse-homophily is likely to occur.

We see that some social implications arise from this

1. schelling model
2. parameters
   (a) satisfaction threshold
   (b) population of each partition
   (c) vacancy
3. a grid of X and O (denoting partition of agents).

- Satisfied Agent: A satisfied agent is one that is surrounded by at least t percent of agents that are like itself.
- Dynamics: When an agent is not satisfied, it can be moved to any vacant location in the grid.

Chapter 4

Diagonalization

Lecture 7: Spectral Theorem

We now want to study a very useful theorem called spectral theorem. Before this, we need to study *eigenvalue*. 22 Sep. 12:30

4.1 Matrices with Real Eigenvalues

Need to rely on some structural properties.

4.1.1 Symmetric matrices

Definition 4.1.1 (Symmetric matrix). A *symmetric matrix* is a square matrix such that

$$A = A^\top.$$

Theorem 4.1.1 (Spectral Theorem). If A is a symmetric matrix, then all eigenvalues are real. Furthermore, the left and right eigenvectors are the same, and they construct an orthonormal basis.

Proof. We verify the second part of the theorem. If we have A is symmetric, then suppose $\vec{x}$ is a right eigenvector, we have

$$A\vec{x} = \lambda\vec{x} \Rightarrow (A\vec{x})^\top = \lambda\vec{x}^\top \Rightarrow \vec{x}^\top A^\top = \lambda\vec{x}^\top \Rightarrow \vec{x}^\top A = \lambda\vec{x}^\top,$$

where we see that $\vec{x}$ is just left eigenvector, hence the left and right eigenvectors are the same. ■

Lecture 8: Diagonalization

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As previously seen (The diagonalization). We see that

1. Theorem 4.1.1: For an $n \times n$ symmetric matrix $A = A^\top$, the characteristic equation is

$$\det(\lambda I - A) = 0.$$

2. Denote n right eigenvectors by $u_1, \ldots, u_n$ where u_i corresponds to eigenvalue λ_i such that

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

Define U as

$$U = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{pmatrix},$$

we have

(a) u_i are linear independent, and they form a basis in $\mathbb{R}^n$.

(b) $u_i^\top u_j (= \langle u_i, u_j \rangle) = 0$ for $i \neq j$.

(c) $u_i^\top u_i = \sum_{j=1}^n u_{ij}^2 = 1$ ($length = 1$).

3. U is invertible. We see that $U^{-1} = U^\top$ since

$$U^\top U = \begin{pmatrix} — & u_1^\top & — \\ — & u_2^\top & — \\ & \vdots & \\ — & v_n^\top & — \end{pmatrix} \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{pmatrix} = I.$$

4. $A = UDU^\top$ where

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

From $U^\top = U^{-1}$, we have $AU = (UDU^\top)U = UD$ since

$$AU = A \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Au_1 & \cdots & Au_n \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ \lambda_1 u_1 & \cdots & \lambda_n u_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = UD$$

4.2 Diagonalization

Example.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & a & 2 \end{pmatrix}.$$

The determinant is

$$\det(\lambda I - A) = \det\left(\begin{pmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 2 & 1 \\ 0 & 2 & \lambda - 2 \end{pmatrix}\right)$$

$$= (\lambda - 1)((\lambda - 2)^2 - a) = (\lambda - 1)(\lambda^2 - 4\lambda + 4 - a).$$

Since $\triangle = 16 - 4(4 - a) = 4a$, $a = -1$, $\triangle = -4 < 0 \Rightarrow$ complex roots. Hence,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -1 & 2 \end{pmatrix}$$

has complex roots.

Definition 4.2.1 (Laplician). For an undirected graph, the *Laplician* is defined as $L := D - A$, where A is the adjacent matrix, D is diagonal degree matrix.

Remark. Since

$$L^\top = (D - A)^\top = D^\top - A^\top = D - A = L,$$

we see that L is symmetric, hence its eigenvalues are real by Theorem 4.1.1. Many incidence matrices $B_{|\mathcal{V}|\times|\mathcal{E}|}$ which encode the relation between edges and nodes satisfies

$$L = BB^\top.$$

We'll show that L has non-negative eigenvalues later.

4.3 Positive Definite and Positive Semi-Definite

Definition 4.3.1 (Positive definite). A is *positive definite* (PD) if and only if

$$\forall \vec{x} \neq \vec{0}, \quad \vec{x}^\top A\vec{x} > 0.$$

Definition 4.3.2 (Positive semi-definite). A is *positive semi-definite* (PSD) if and only if

$$\forall \vec{x} \neq \vec{0}, \quad \vec{x}^\top A\vec{x} \geq 0.$$

Lemma 4.3.1. If λ is a real eigenvalue, then $\lambda \geq 0 \Leftrightarrow A$ is positive semi-definite; also, $\lambda > 0 \Leftrightarrow A$ is positive-definite.

Proof. Suppose λ is a real eigenvalue, let $\vec{u}$ be the eigenvector such that $A\vec{u} = \lambda\vec{u}$, and

$$\vec{u}^\top A\vec{u} = \vec{u}^\top(\lambda\vec{u}) = \lambda \underbrace{\vec{u}^\top\vec{u}}_{>0} \geq 0,$$

which implies A is positive semi-definite. Similarly, if $\lambda > 0 \Rightarrow \vec{u}^\top\vec{u} > 0$, A is positive definite. Lastly, the backward direction is obvious. ■

Lemma 4.3.2. A is positive semi-definite and symmetric $\Leftrightarrow \exists \widetilde{B}$ such that $A = \widetilde{B}\widetilde{B}^\top$.

Proof. We see that if $A = \widetilde{B}\widetilde{B}^\top$, it's clearly symmetric and

$$x^\top Ax = x^\top \widetilde{B}\widetilde{B}^\top x = (\widetilde{B}x)^\top \widetilde{B}x = y^\top y \geq 0,$$

hence A is positive semi-definite.

For another direction, from Theorem 4.1.1, since A is symmetric, we know that $A = UDU^\top$ where all entries of D is ≥ 0 by Lemma 4.3.1. We see that we can decompose D into $D = CC^\top$ where

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix} =: CC^\top,$$

then by defining $\widetilde{B} := UC$, we see that

$$A = UDU^\top = UCC^\top U^\top = (UC)(UC)^\top = \widetilde{B}\widetilde{B}^\top$$

as desired. ■

Remark (Laplacian). Given a Laplacian $L = BB^\top$, from Lemma 4.3.2, L is positive semi-definite, which further implies

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

But $\lambda_1 = 0$ since we have $L = D - A$ where $\mathrm{diag}(d_1, \ldots, d_n)$ being the diagonal degree matrix, and

furthermore,

$$L\vec{1} = (D - A)\vec{1} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} - A\vec{1} = \vec{0}$$

since $d_i = \sum_{j=1}^{n} A_{ij}$ from the definition of A, so

$$L\vec{1} = 0\vec{1}.$$

Hence, 0 is an eigenvalue with eigenvector $\vec{1}$, so $\lambda_1 = 0$.

We now study the multiplicity of an eigenvalue of L. We first introduce the following definition.

Definition 4.3.3 (Multiplicity). We say the eigenvalue λ^* of A has *multiplicity* k if

$$\det(\lambda I - A) = (\lambda - \lambda^*)^k g(\lambda)$$

where $g(\cdot)$ is a polynomial in λ with $g(\lambda^*) \neq 0$.

If the network has two connected components, we can then relabel the nodes and obtain a concatenation of two adjacency matrices that doesn't intersect

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

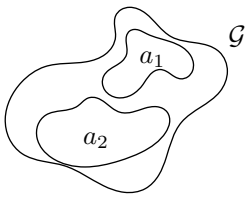


Figure 4.1: Two connected components graph.

Then we see that we can also represent the Laplician in block matrix form as

$$L = \begin{bmatrix} D_1 - A_1 & 0 \\ 0 & D_2 - A_2 \end{bmatrix} =: \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}.$$

We then see that the characteristic equation of L is[1]

$$\det(\lambda I - L) = \det(\lambda I_1 - L_2)\det(\lambda I_2 - L_2)$$

Then, as we just remarked, both L_1 and L_2 has an eigenvalue being 0, which implies that 0 is an eigenvalue of L with multiplicity 2.

We see that multiplicity of eigenvalue of Laplician can help us find communities. This suggests the following definition.

Definition 4.3.4 (Fiedler eigenvalue). The *Fiedler eigenvalue* of a graph $\mathcal{G}$ is the second-smallest eigenvalue (neglect multiplicity) of the Laplacian of $\mathcal{G}$.

We see that by first doing the eigen-decomposition on Laplacian and find all eigenvalues that are close to zero, which can be used to find communities, and nearly all community detection algorithms rely on this fact.

[1] Detailed derivation is omitted. It's essentially followed from the block structure.

As previously seen. Turning back to Givran-Newman algorithm, though it works at social network, but what about other tasks like internet?

Problem. Hyperlinks (directed): How important is the edge?

Answer. Internet search seems to rank nodes, not edges. (the website itself is the point?) ⊛

An immediate question arises.

Problem. How does one rank nodes?

Answer. Network centrality. ⊛

Example (Network centrality)**.** There are many examples of methods to quantify network centrality, e.g.,

- Degree. Ranks nodes by edges.
- Eigen-centrality. Adjacency matrix, finds an eigenvector and use its entries.

We now study this in depth.

Chapter 5

HITS Algorithm

HITS is a hyperlinks-induced topic search algorithm proposed by John Kleinberg, whose backbones are the idea of **hubs** and **authorities**.

- Hubs: nodes that *aggregate*.
- Authorities: nodes that are *important*.

The key idea is that

1. If many websites (nodes) point to it, then this website is important (authority property).
2. If a node points to many authority nodes, then it is important as well.

Then by iteratively identifying hubs and authorities, we can rank the nodes in the network. We now describe in detailed how this algorithm works.

5.1 HITS Algorithm

The pseudocode of the algorithm is as follows.

Algorithm 2: HITS Algorithm

Data: A network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
Result: hub, auth

for $v \in \mathcal{V}$ **do** // Initialize
  $\mathsf{auth}(v) \leftarrow 1$
  $\mathsf{hub}(v) \leftarrow 1$

while *not converge* **do**
  for $v \in \mathcal{V}$ **do** // auth update
    $\mathsf{auth}(v) \leftarrow \sum_{u \in \mathcal{V}:\, u \to v \in \mathcal{E}} \mathsf{hub}(u)$
  for $v \in \mathcal{V}$ **do** // hub update
    $\mathsf{hub}(v) \leftarrow \sum_{u \in \mathcal{V}:\, v \to u \in \mathcal{E}} \mathsf{auth}(u)$
  for $v \in \mathcal{V}$ **do** // auth normalize
    $\mathsf{auth}(v) \leftarrow \mathsf{auth}(v) / \sum_{u \in \mathcal{V}} \mathsf{auth}(u)$
  for $v \in \mathcal{V}$ **do** // hub normalize
    $\mathsf{hub}(v) \leftarrow \mathsf{hub}(v) / \sum_{u \in \mathcal{V}} \mathsf{hub}(u)$
return hub, auth

Note. There are two facts are worth noting.

1. $\texttt{hub}(v)$ and $\texttt{auth}(v)$ will converge as long as initial score are positive.

2. Final scores will be independent of the initial scores.

Lecture 9: Analysis of HITS

5.1.1 Analysis of HITS

29 Sep. 12:30

- Authority update:

$$\mathtt{a}_{\mathtt{k}}(i) = \sum_{j\in\mathcal{V}:\ j\to i\in\mathcal{E}} \mathtt{h}_{\mathtt{k}-1}(j) = \sum_{j\in\mathcal{V}:\ A_{ji}=1} \mathtt{h}_{\mathtt{k}-1}(j) = \sum_{j\in\mathcal{V}} A_{ji}\mathtt{h}_{\mathtt{k}-1}(j) = \left(A^\top \mathtt{h}_{\mathtt{k}-1}\right)_i .$$

  We see that

$$\mathtt{a}_{\mathtt{k}} = A^\top \mathtt{h}_{\mathtt{k}-1}.$$

- Hub update:

$$\mathtt{h}_{\mathtt{k}}(j) = \sum_{j\in\mathcal{V}:\ i\to j\in\mathcal{E}} \mathtt{a}_{\mathtt{k}}(j) = \sum_{j\in\mathcal{V}} \mathtt{a}_{\mathtt{k}}(j) = \sum_{j\in\mathcal{V}} A_{ij}\mathtt{a}_{\mathtt{k}}(j) = \left(A\mathtt{a}_{\mathtt{k}}\right)_i .$$

  We see that

$$\mathtt{h}_{\mathtt{k}} = A\mathtt{a}_{\mathtt{k}}.$$

  Furthermore,

$$\mathtt{h}_{\mathtt{k}} = A\mathtt{a}_{\mathtt{k}} \Rightarrow \mathtt{h}_{\mathtt{k}} = A\mathtt{a}_{\mathtt{k}} = A(A^\top \mathtt{h}_{\mathtt{k}-1}) = AA^\top \mathtt{h}_{\mathtt{k}-1}.$$

If we ignore the normalization for now, then

$$\mathtt{h}_{\mathtt{k}} = AA^\top \mathtt{h}_{\mathtt{k}-1} = (AA^\top)AA^\top h_{k-2} = \cdots = (AA^\top)^k \mathtt{h}_0.$$

Hence, we need to study $AA^\top$. Firstly, from Lemma 4.3.2, we see that $AA^\top$ is positive semi-definite and symmetric. Furthermore, from Theorem 4.1.1, since $AA^\top$ is symmetric, all eigenvalues are real. Then from Lemma 4.3.1, we see that all eigenvalue of $AA^\top$ are positive, i.e.,

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{|\mathcal{V}|}.$$

From the Theorem 4.1.1, let the eigenvectors of $AA^\top$ be $u_1, \ldots, u_{|\mathcal{V}|}$, then

$$\mathtt{h}_0 = \sum_{i=1}^{|\mathcal{V}|} \alpha_i u_i$$

where α_i are unique since $\{u_i\}$ form a basis, and

$$\alpha_i = u_i^\top \mathtt{h}_0 = \sum_{j=1}^{|\mathcal{V}|} \alpha_j u_i^\top u_j.$$

Moreover, we have

$$\mathtt{h}_1 = (AA^\top)\mathtt{h}_0 = \sum_{i=1}^{|\mathcal{V}|} \alpha_i \lambda_i u_i;$$

$$\vdots$$

$$\mathtt{h}_{\mathtt{k}} = (AA^\top)^k \mathtt{h}_0 = \sum_{i=1}^{|\mathcal{V}|} \alpha_i \lambda_i^k u_i.$$

With the above formula, we now consider

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \underbrace{\lambda_{|\mathcal{V}|}}_{\text{largest}} .$$

Case i. Unique largest eigenvalue: $\lambda_{|\mathcal{V}|} > \lambda_{|\mathcal{V}|-1} \geq \cdots \geq \lambda_1 \geq 0$, then the term λ^k will dominate, namely

$$\mathtt{h_k} = \sum_{i=1}^{|\mathcal{V}|} \alpha_i \lambda_i^k u_i.$$

Dividing both sides by $\lambda_{|\mathcal{V}|}^k$, we have

$$\frac{\mathtt{h_k}}{\lambda_{|\mathcal{V}|}^k} = \sum_{i=1}^{|\mathcal{V}|-1} \alpha_i \left(\frac{\lambda_i}{\lambda_{|\mathcal{V}|}}\right)^k u_i + \alpha_{|\mathcal{V}|} u_{|\mathcal{V}|}.$$

Then since $\lambda_i/\lambda_{|\mathcal{V}|} < 1$ for $1 \leq i \leq |\mathcal{V}| - 1$, hence when $k \to \infty$, all terms in the sum will converge to 0, leaving us with

$$\lim_{k\to\infty} \frac{\mathtt{h_k}}{\lambda_{|\mathcal{V}|}^k} = \alpha_{|\mathcal{V}|} u_{|\mathcal{V}|}.$$

Since $u_{|\mathcal{V}|}$ is a non-zero vector, we consider the following two cases:

(a) All terms are non-negative with one positive. Firstly, suppose $\mathtt{h_0} = x > 0$ such that $u_{|\mathcal{V}|}^\top x = \alpha_{|\mathcal{V}|} > 0$. Then we have

$$\lim_{h\to\infty} \frac{\left(AA^\top\right)^k x}{\lambda_{|\mathcal{V}|}^k} = \alpha_{|\mathcal{V}|} u_{|\mathcal{V}|}.$$

We see that the left-hand side is greater or equal to 0, and stays in positive for every $k \Rightarrow$ the limit is also positive. Now, since $\alpha_{|\mathcal{V}|} > 0$, we can conclude that $u_{|\mathcal{V}|} \geq 0$. In other words, $\mathtt{h_0}$ can be any vector with all coefficients positive and then $\alpha_{|\mathcal{V}|} = u_{|\mathcal{V}|}^\top \mathtt{h_0} > 0$.

(b) At least one term is positive. $\mathtt{h_0} = x > 0$ such that $u_{|\mathcal{V}|}^\top x > 0$.

> **Remark.** To rank nodes, all we need is the vector $u_{|\mathcal{V}|}$. And the limiting value of $u_{|\mathcal{V}|}$ is called the **hub score**.

Case ii. Non-unique largest eigenvalues: For discussion, we assume that the largest eigenvalue has algebraic multiplicity of 2. Namely

$$\lambda_{|\mathcal{V}|} = \lambda_{|\mathcal{V}|-1} \geq \cdots \geq \lambda_1 \geq 0.$$

Again, from the same discussion, we see that

$$\mathtt{h_0} = \sum_{i=1}^{|\mathcal{V}|} \alpha_i u_i.$$

Furthermore, we see that

$$\frac{\left(AA^\top\right)^k h_0}{\lambda_{|\mathcal{V}|}^k} = \sum_{i=1}^{|\mathcal{V}|-2} \alpha_i \left(\frac{\lambda_i}{\lambda_{|\mathcal{V}|}}\right)^k u_i + \alpha_{|\mathcal{V}|-1} u_{|\mathcal{V}|-1} + \alpha_{|\mathcal{V}|} u_{|\mathcal{V}|}.$$

Since $\lambda_i/\lambda_{|\mathcal{V}|} < 1$ for $1 \leq i \leq V - 2$, hence when $k \to \infty$, all terms in the sum will converge to 0, leaving us with

$$\lim_{k\to\infty} \frac{(AA^\top)^k \mathtt{h_0}}{\lambda_V^k} = \alpha_{|\mathcal{V}|-1} u_{|\mathcal{V}|-1} + \alpha_{|\mathcal{V}|} u_{|\mathcal{V}|}.$$

> **Remark.** It still converges but it may not be $u_{|\mathcal{V}|}$.

Chapter 6

Page Rank

Lecture 10: Page Rank

6.1 Page Rank Algorithm

4 Oct. 12:30

> **As previously seen** (Voting procedure). The hubs vote authorities, while authorities vote for hubs in HITS algorithm.

But this is not always the case. Rankings that can be collaborated as the following new algorithm will show, which will determine the importance of a node we called *page rank*.

Algorithm 3: Basic Page Rank Algorithm

Data: A network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, iteration L
Result: pageRank

```
for v ∈ 𝒱 do                                                          // Initialize
    pageRank(v) ← 1/|𝒱|

for i = 1, ..., L do
    for v ∈ 𝒱 do                                                      // Send
        neighborNum ← |{v → u ∈ ℰ}|
        if neighborNum ≠ 0 then          // divides pageRank(v) to its (out)neighbor
            for v → u ∈ ℰ do
                pageRankNew(u) ← pageRankNew(u) + pageRank(v)/neighborNum
        else                                                          // no outgoing links
            pageRankNew(v) ← pageRankNew(v) + pageRank(v)

    for v ∈ 𝒱 do                                                      // Receive
        pageRank(v) ← pageRankNew(v)                    // Update the page rank
        pageRankNew(v) ← 0          // Clean up for the next iteration's updating

return pageRank
```

> **Remark** (Algorithm explanation). The following is the detailed explanation of the basic page rank algorithm.
>
> (a) Initialize: For all nodes in $\mathcal{V}$, set page rank to be $1/|\mathcal{V}|$.
>
> (b) Send: Basic page rank updates: each node divides of its current page rank equally across its outgoing links and sends the value to the node at the end. If there are no outgoing links, then the node retains its page rank.

(c) Receive: Each node updates its page rank to be the sum of the values it receives on its incoming links. If a node has retained its page rank, then it adds it to this value.

Note (Conservation of the page rank). Page rank is never lost over gained in the network, it is just passed around. This means no normalization is required, the page ranks always sum to 1, with all terms non-negative.

But then there are some problems we care about.

Problem. Will it converge?

Answer. It will converge in most cases. ⊛

Example. Consider a cycle graph, with different initial page rank. It will not converge! All values just cycling around.

Problem. To what if it does?

Answer. Eigenvector of some eigenvalues of some matrix associated with the network. ⊛

Problem. Is the final state always interesting?

Answer. It may not always be interesting. ⊛

Our goal is to have

1. convergence for all starting vectors;
2. the final vector to be relevant to the graph.

6.2 Analysis on Page Rank

Let's denote the page rank of node i as r_i throughout the following discussion.

(a) Initialize: Denote r^{old} as the column vector of page ranks.

(b) Send:

- If $d_i^{\text{out}} > 0$: then $\frac{r_i^{\text{old}}}{d_i^{\text{out}}}$ is sent to neighbor at the end of the outgoing links.
- If $d_i^{\text{out}} = 0$: then r_i^{old} sent to itself.

(c) Receive: Collect all the page rank values. Therefore,

$$r_i^{\text{new}} = \sum_{j:j\to i} \frac{r_j^{\text{old}}}{d_j^{\text{out}}} + \begin{cases} r_i^{\text{old}} & , \text{ if } d_i^{\text{out}} = 0; \\ 0 & , \text{ if } d_i^{\text{out}} > 0. \end{cases}$$

We can indeed write it in a more compact form as follows. We have $r^{\text{new}} = N^\top r^{\text{old}}$, where

$$\text{if } i \neq j,\ (N^\top)_{ij} = \begin{cases} \frac{1}{d_j^{\text{out}}} & , \text{ if } A_{ji} = 1; \\ 0 & , \text{ if } A_{ji} = 0, \end{cases} = \begin{cases} \frac{A_{ji}}{d_j^{\text{out}}} & , \text{ if } d_j^{\text{out}} > 0; \\ A_{ji} & , \text{ otherwise.} \end{cases}$$

$$\text{if } i = j,\ (N^\top)_{ij} = \begin{cases} 1 & , \text{ if } d_i^{\text{out}} = 0; \\ 0 & , \text{ if } d_i^{\text{out}} > 0, \end{cases} = \begin{cases} 1 & , \text{ if } d_i^{\text{out}} = 0; \\ A_{ii} & , \text{ if } d_i^{\text{out}} > 0. \end{cases}$$

Hence,

$$N_{ij} = \begin{cases} \frac{A_{ij}}{d_i^{\text{out}}} & , \text{ if } d_i^{\text{out}} > 0; \\ A_{ij} & , \text{ if } d_i^{\text{out}} = 0 \wedge i \neq j; \\ 1 & , \text{ if } d_i^{\text{out}} = 0 \wedge i = j. \end{cases}$$

6.2.1 Eigenvalues of N

After L steps, we have

$$r(L) = (N^\top)^L r(0).$$

To study what happens as $L \to \infty$, we suppose there is a convergence vector r such that

$$r = (N^\top) r.$$

We see that r must be an eigenvector for $N^\top$ for eigenvalue 1. Notice that u should be *non-negative* and sums to 1 over entries. Now we have

$$(N^\top) r = r \Rightarrow r^\top N = r^\top,$$

r is a left eigenvector of N with eigenvalue 1.

Note. N has an eigenvalue 1.

Proof. From what we have derived for N_{ij}, we have

$$\sum_{j \in \mathcal{V}} N_{ij} = \begin{cases} \displaystyle\sum_{j \in \mathcal{V}} \frac{A_{ij}}{d_i^{\text{out}}} = \frac{\displaystyle\sum_{j \in \mathcal{V}} A_{ij}}{d_i^{\text{out}}} = \frac{d_i^{\text{out}}}{d_i^{\text{out}}} = 1 & \text{, if } d_i^{\text{out}} > 0; \\ \underbrace{1}_{j=i} + \underbrace{\displaystyle\sum_{j \neq i} A_{ij}}_{= d_i^{\text{out}} = 0} = 1 + 0 = 1 & \text{, if } d_i^{\text{out}} = 0. \end{cases}$$

Hence, we conclude that N is a matrix such that the row sums are always 1, so

$$N\vec{1} = \vec{1},$$

which shows that 1 is an eigenvalue of N, with $\vec{1}$ being a right eigenvector! ⊛

We can ask about the left eigenvector — it exists. Corresponding to $\vec{1}$.

Problem. What do we know about the entries?

Answer. We have

1. row sums on 1
2. non-negative entries
3. row stochastic matrix (associated with Markov chains)

⊛

Remark. Each row vector is a probability mass function (PMF) on $\{1, 2, \ldots, |\mathcal{V}|\}$.

Chapter 7

Scaled Page Rank

7.1 Perron-Frobenius Theorem

7.1.1 Irreducible

In the following lectures, we will study so-called Perron-Frobenius theorem, which is a theorem about non-negative matrices. Before this, we need to study irreducible matrix.

Definition 7.1.1 (Irreducible)**.** A non-negative matrix $A_{n\times n}$ is *irreducible* if for all $i, j = 1, \dots, n$, there exists $k(i,j) \in \mathbb{Z}$ depends on i, j such that $k(i,j) > 0$ and

$$(A^k)_{ij} > 0.$$

Remark. We can express the condition to be irreducible as there exists a $k > 0$ such that

$$\underbrace{A_{im_1}A_{m_1m_2}\dots A_{m_{k-1}j}}_{k \text{ terms}} > 0.$$

Proof. We have

$$\begin{aligned}
(A^2)_{ij} &= \sum_k A_{ik}A_{kj} \\
(A^3)_{ij} &= (AA^2)_{ij} = \sum_k A_{ik}(A^2)_{kj} = \sum_k \sum_l A_{ik}A_{kl}A_{lj} \\
&\vdots \\
(A^n)_{ij} &= \sum_{k_1}\sum_{k_2}\dots\sum_{k_n} A_{ik_1}A_{k_1k_2}A_{k_2k_3}\dots A_{k_nj}
\end{aligned}$$

A is non-negative so $A_{ij}^{n+1} > 0 \Rightarrow$ at least one of the sums > 0, by re-indexing we have the result. ⊛

We see that the above equivalent condition is still not easy to work with, hence we introduce the following definition.

Definition 7.1.2 (Adjacency matrix of a matrix)**.** Given a matrix A, we define the *adjacency matrix of* A as $\widetilde{A}$ such that

$$\widetilde{A}_{ij} = \begin{cases} 1, & \text{if } A_{ij} > 0 \\ 0, & \text{if } A_{ij} = 0 \end{cases}.$$

Note. $\widetilde{A}$ is an adjacency matrix of some directed graph with n nodes.

We then see that

$$A_{im_1}A_{m_1m_2}\dots A_{m_{k-1}j} > 0 \Leftrightarrow \widetilde{A}_{im_1} = \widetilde{A}_{m_1m_2} = \dots = \widetilde{A}_{m_{k-1}j} = 1.$$

This implies the irreducibility of $A \Leftrightarrow$ irreducibility of $\widetilde{A} \Leftrightarrow \mathcal{G}$ corresponding to $\widetilde{A}$ is strongly connected, i.e., $\forall i, j$ we can find a path on $\mathcal{G}$.

Remark. This graph $\mathcal{G}$ can be non-simple graph.

We see that to check for a non-negative matrix A, whether $(A^{n+1})_{ij} > 0$ or not, we first find the graph $\mathcal{G}$ corresponding to $\widetilde{A}$. If $(A^{n+1})_{ij}$ is positive $\Leftrightarrow$ there is a path in $\mathcal{G}$ from i to j.

Lecture 11: Scaled Page Rank

6 Oct. 12:30

As previously seen. Basic page rank algorithm: If the page rank converges, then we must have

$$r^* = (N^\top)r^* \Leftrightarrow (r^*)^\top N = (r^*)^\top,$$

where r^* is a left eigenvector of N for eigenvalue 1.

Remark. Row stochastic property $\Rightarrow$ 1 is eigenvalue.

7.1.2 Periodicity

Definition 7.1.3 (Period). Given $A_{n\times n}$ and $i \in \{1, \dots, n\}$, find all $m > 0$ such that

$$(A^m)_{ii} > 0.$$

And the greatest common divisor of them is called the *period of* i.

Definition 7.1.4 (Aperiodic). Given $A_{n\times n}$ and $i \in \{1, \dots, n\}$, if the period is 1 for node i, then i is said to be *aperiodic*. And A is said to be aperiodic all i are *aperiodic*.

Remark. We see that

- A sufficient condition for aperiodic is $A_{ii} > 0$ since it implies the period is 1 for i.
- If A is a positive matrix[a], then we know that

$$\begin{aligned} A_{ij} > 0 \ \forall i, j &\Rightarrow \text{irreducible} \\ A_{ii} > 0 \ \forall i \quad &\Rightarrow \text{aperiodic}. \end{aligned}$$

  Notice that this is only a simple sufficient condition for a matrix to be irreducible and aperiodic.

[a] All entries are positive

7.1.3 Perron-Frobenius Theorem

We now start to study our main theorem, the Perron-Frobenius theorem. Before that, we introduce a useful definition.

Definition 7.1.5 (Spectral radius). The eigenvalue of A with the largest absolute value is called the *spectral radius* or the *Perron-Frobenius eigenvalue* of A, denoted as $\rho(A)$.

Theorem 7.1.1 (Perron-Frobenius theorem). Let A be an irreducible non-negative matrix, then the spectral radius is positive and has multiplicity 1.

Both the right eigenvector $(\vec{v}, A\vec{v} = \rho(A)\vec{v})$ and the left eigenvector $(\vec{w}, \vec{w}^\top A = \rho(A)\vec{w}^\top)$ can be taken to be positive with

$$\vec{w}^\top \vec{v} = 1.$$

Furthermore, we have

(a) The following holds

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \frac{A^k}{(\rho(A))^k} = \vec{v}\vec{w}^\top_{(n\times n)}.$$

(b) The following holds

- Row sum bound:
$$\min_i \sum_j A_{ij} \le \rho(A) \le \max_i \sum_j A_{ij}.$$
- Column sum bound:
$$\min_j \sum_i A_{ij} \le \rho(A) \le \max_j \sum_i A_{ij}.$$

(c) If A is also aperiodic, then

$$\lim_{m\to\infty} \frac{A^m}{(\rho(A))^m} = \vec{v}\vec{w}^\top.$$

Proof. We omit the proof, but it's worth checking out. See here for a short proof. ■

As previously seen. For the basic page rank algorithm:

1. N is non-negative and row stochastic, so we know $\rho(N) \le 1$. But since we already showed 1 is one of the eigenvalues of $N \Rightarrow \rho(N) = 1$.
2. $N \to \widetilde{A}$: Adjacency matrix of original graph plus self-loops for any nodes with no outgoing links.

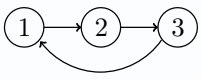
We can now say something about N.

Remark. We don't certainly have that N is irreducible.

Proof. Since $\rho(N) = 1$ but the multiplicity can be > 1 or eigenvectors may not all be positive. From Theorem 7.1.1, N cannot be irreducible in both cases. ⊛

Remark. We don't certainly have that N is aperiodic.

Proof. Consider the following network:



For this network, we have

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and for higher power it just repeats. We see that the period for all three nodes are all 3, which means that N is not aperiodic. ⊛

This is problematic, since for basic page rank algorithm, we have

$$r(k) = (N^\top)^k r(0),$$

hence we are interested in the case that $(N^\top)^k$ converges.

7.2 Scaled Page Rank

Scaled page rank is a modification of page rank that gives a matrix $\widetilde{N}(s)$ that is always irreducible and aperiodic with $s \in (0,1)$. We first give the algorithm to calculate it.

Algorithm 4: Scaled Page Rank Algorithm

Data: A network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, scale factor $s \in (0,1)$, iteration L
Result: scaledPageRank

for $v \in \mathcal{V}$ **do** // Initialize
  scaledPageRank$(v) \leftarrow 1/|\mathcal{V}|$

for $i = 1, \ldots, L$ **do**
  for $v \in \mathcal{V}$ **do** // Send
    neighborNum $\leftarrow |\{v \to u \in \mathcal{E}\}|$
    if neighborNum $\neq 0$ **then** // divides scaledPageRank(v) to its (out)neighbor
      for $v \to u \in \mathcal{E}$ **do**
        scaledPageRankNew$(u) \leftarrow$ scaledPageRankNew(u) + scaledPageRank(v)/neighborNum
    else // no outgoing links
      scaledPageRankNew$(v) \leftarrow$ scaledPageRankNew(v) + scaledPageRank(v)
  for $v \in \mathcal{V}$ **do** // Receive
    scaledPageRank$(v) \leftarrow$ scaledPageRankNew(v) // Update the page rank
    scaledPageRankNew$(v) \leftarrow 0$ // Clean up for the next iteration's updating
  for $v \in \mathcal{V}$ **do** // Scale
    scaledPageRank$(v) \leftarrow s \times$ scaledPageRankNew(v) // Scaled down by s
    scaledPageRank$(v) \leftarrow \frac{1-s}{|\mathcal{V}|}$ // Re-distribute page rank

return scaledPageRank

Remark (Algorithm explanation). The following is the detailed explanation of the scaled page rank algorithm.

(a) Initialization: same as basic page rank algorithm.

(b) Send: same as basic page rank algorithm.

(c) Receive: same as basic page rank algorithm.

(d) Scale: Scale down all page ranks by s (multiply by s) and add $(1-s)/|\mathcal{V}|$ to the page ranks of all nodes.

7.3 Analysis on Scaled Page Rank

Again, let's denote the scaled page rank of node i as r_i throughout the following discussion.

$$r_i^{\text{new}} = s\left(\sum_{j:\, A_{ji}=1} \frac{u_j^{\text{old}}}{d_j^{\text{old}}} + \begin{cases} u_i^{\text{old}} & , \text{ if } d_i^{\text{out}} = 0; \\ 0 & , \text{ otherwise;} \end{cases}\right) + \frac{1-s}{|\mathcal{V}|}.$$

Then we have

$$r^{\text{new}} = s(N^\top r^{\text{old}}) + \frac{1-s}{|\mathcal{V}|}\vec{1} = s(N^\top u^{\text{old}}) + \frac{1-s}{|\mathcal{V}|}\vec{1}\underbrace{(\vec{1}^\top r^{\text{old}})}_{\text{add to 1}} = \underbrace{\left(sN + \frac{1-s}{|\mathcal{V}|}\vec{1}\vec{1}^\top\right)^\top}_{\widetilde{N}(s)} r^{\text{old}}.$$

Finally, we have the rule of updating scaled page rank being $r^{\text{new}} = (\widetilde{N}(s))^\top r^{\text{old}}$, hence

$$r(k) = \left(\widetilde{N}(s)^\top\right)^k r(0).$$

> **Note.** We see that $\widetilde{N}(s)$ is
>
> - non-negative
> - row stochastic
> - strictly positive with all entries at least $(1-s)/|\mathcal{V}|$

Firstly, since $\widetilde{N}(s)$ is non-negative and is row-stochastic, we see that $\widetilde{N}(s)\vec{1} = \vec{1}$, hence $\rho(\widetilde{N}(s)) = 1$.

Moreover, since $\widetilde{N}(s)$ is strictly positive, from the remark, we have that $\widetilde{N}(s)$ is both irreducible and aperiodic. With the non-negativity, we can then apply Theorem 7.1.1 (c) to get

$$\lim_{m\to\infty} \frac{(\widetilde{N}(s))^m}{(\rho(\widetilde{N}(s)))^m} = \lim_{m\to\infty} (\widetilde{N}(s))^m = \vec{v}\vec{w}^\top,$$

where $\vec{v}$ is the right eigenvector of $\widetilde{N}(s)$ and $\vec{w}$ is the left eigenvector of $\widetilde{N}(s)$ with $\vec{w}^\top\vec{v} = 1$.

And since $\widetilde{N}(s)$ is row stochastic, we have $\vec{v} = \vec{1}$, hence $\vec{w}^\top\vec{1} = 1$, which means the sum of all entries is 1 and $\vec{w}$ has all positive entries.

If $(\vec{v}, \vec{w})$ are right and left eigenvectors of $\widetilde{N}(s) \Leftrightarrow (\vec{w}, \vec{v})$ are the right and left eigenvectors of $\widetilde{N}(s)^\top$.

> **Remark.** Scaled page rank converges to r^* where
>
> $$r^* = \widetilde{N}(s)^\top r^*,$$
>
> which is the right eigenvector of $\widetilde{N}(s)^\top$. But it's the same as the left eigenvector of $\widetilde{N}(s)$, which is just $\vec{w}(s)$.

> **Note.** In practice, the scale factor is usually set as $s \cong 0.8 \sim 0.85$.

Lecture 12: Page Rank, Markov Chain and Randomness

11 Oct. 12:30

Recall that we want to measure the centrality of a network. We did this by introducing Theorem 4.1.1 and Theorem 7.1.1.

We first look at following two problems.

> **Problem** (Eigenvector centrality). For A be an adjacency matrix, which is a non-negative matrix with $r(0) = \vec{1}$ and
>
> $$r(k) = Ar(k-1), \qquad \forall k = 1\dots$$
>
> Then if it is irreducible and aperiodic, then
>
> $$\lim_{k\to\infty} \frac{r(k)}{(\rho(A))^k} = v(w^\top r(0))$$
>
> where $\rho(A)$ is the unique spectral radius, v is the right eigenvector of A and w is the left eigenvector of A, and $w^\top v = 1$ for $\rho(A)$.

We then have

$$\lim_{k\to\infty} \frac{A^k}{(\rho(A))^k} = vw^\top.$$

Remark. Notice that

- v would be the eigenvector centrality measure.
- $r^* = Ar^*$ need not hold because 1 need not be an eigenvalue.
- Since
$$(r(k))_i = \sum_j A_{ij} r_j(k-1) = \sum_{j\,:\, i\to j} r_j(k-1),$$
so we see that nodes that either point to many nodes, or important nodes or both, will have higher value.

Problem (Katz centrality). Again, consider $r(k) = Ar(k-1)$, but this time we modify the equation to be

$$r(k) = \alpha A r(k-1) + \beta\vec{1}$$

where α is aiming to dampen the degree, and β is aiming to add a constants weight for each node to increase the fairness. Similarly, any solution will satisfy

$$r^* = \alpha A r^* + \beta\vec{1} \Leftrightarrow (I - \alpha A) r^* = \beta\vec{1}.$$

Definition 7.3.1 (Katz centrality). The solution r^* of $r^* = \alpha A r^* + \beta\vec{1}$ is called the *Katz centrality* if it exits.

Now, if $I - \alpha A$ is invertible, then

$$r^* = \beta(I - \alpha A)^{-1}\vec{1}.$$

Noting that

$$\det(\lambda I - A) = 0 \Leftrightarrow \det(I - \lambda A) = 0,$$

which implies α should not be the inverse of an eigenvalue of A. Also, we need that r^* to be positive.

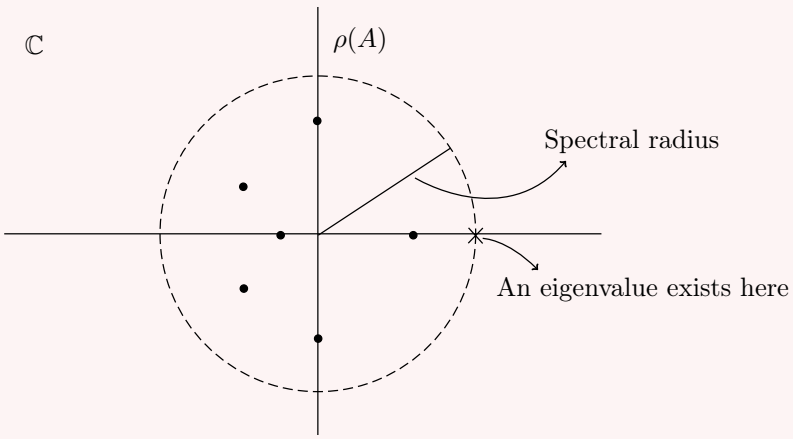


Figure 7.1: Katz Centrality

Then for all $1/\alpha > \rho(A) \Leftrightarrow \alpha\rho(A) < 1 \Leftrightarrow \rho(\alpha A) < 1$, α can't be the inverse of any eigenvalues of A since we see that αA is non-negative with spectral radius < 1.

Now, if αA is non-negative with spectral radius strictly less than one, then we have

$$(I - \alpha A)^{-1} = I + (\alpha A) + (\alpha A)^2 + \dots$$

just like the geometry series of $\frac{1}{1-k}$ for $|k| < 1$. We then have

$$r^* = \beta \left(\sum_{i=0}^{\infty} (\alpha A)^i \right) \vec{1}.$$

Remark. Since

$$\rho(A) \leq \text{ the maximum of row sum of } A = \text{ the highest out-degree},$$

we see that

$$\underline{\alpha \times \text{the highest out-degree} < 1}$$

is a sufficient condition for this to hold.

Note. Katz centrality is commonly used.

Chapter 8

Scaled Page Rank via Markov Chain

Problem. What's the motivation?

Answer. $\widetilde{N}(s)$ is formed in a way such that

$$\widetilde{r}(k) = \widetilde{N}^{\top}(s)\widetilde{r}(k-1)$$

with $\widetilde{r}(0) = \frac{1}{V}\vec{1}$, where $\widetilde{r}(k)$ is the vector form of the scaled page rank.

Further,

$$\widetilde{r}(k) = \left(\widetilde{N}^{\top}(s)\right)^{k} r(0).$$

This is so-called *Power iteration* or *Power Method*. This provides a uniform decrease in the relative error of the entries.

Now, let v be the left eigenvalues, then we have

$$\max_{i\in\mathcal{V}} |(r(k))_i - v_i| \leq c_0 s^k$$

where s is the scaled PageRank factor. Then, by using this, if we set the error tolerance as ϵ(RHS $\leq \epsilon$), we have

$$\#\text{iteration} = \frac{\log(\epsilon)}{\log(s)} \times \text{nnz}(N),$$

where $\text{nnz}(N)$ is the number of non-zero elements in N.

> **Remark.** A sparse graph like Internet will typically have
>
> $$\text{nnz}(N) \sim \Theta(n^2),$$
>
> where n is the number of the nodes.

⊛

8.1 Probability

We first look at some definition.

Definition 8.1.1. We let

- Σ being our sample space.
- $\mathcal{F}$ being the σ-algebra (field). This is essentially just a collection of events subsets of the sample space.

- $\mathbb{P}$ is the probability.

Example. Skip... Enough Ve401...

8.1.1 Properties and Notions

1. Mutually disjoint sets. Let A and $B \in \mathcal{F}$ and $A \cap B = \varnothing$. Then we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

This can be easily generalized to any finite collection of mutually disjoint subsets

$$A_1, A_2, \ldots, A_n$$

such that

$$A_i \cup A_j = \begin{cases} A_i, & \text{if } i = j \\ \varnothing, & \text{if } i \neq j. \end{cases}$$

We then have

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

Remark. Axioms of probability says that this extends to countably infinity collections. Namely, consider mutually disjoint sets

$$A_1, A_2, \ldots$$

Then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

2. Let Ω be the sample space, then we have

$$\mathbb{P}(\Omega) = 1.$$

This implies

$$\mathbb{P}(\varnothing) = 0.$$

Further, if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ and $A \cap A^c = \varnothing$. We then see

$$\mathbb{P}(\Omega) = 1 = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c) \Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

Notice that $\varnothing = \Omega^c$.

3. Mutually independent sets. We say that A, B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Also, we have similar definition for countably infinite collections of sets. We call a countably infinite collection mutually independent if *any finite sub-collection* is independent. Namely, for any index set $K \subsetneq \mathbb{N}$,

$$\mathbb{P}\left(\bigcap_{k\in K} A_k\right) = \prod_{k\in K} \mathbb{P}(A_k).$$

8.1.2 Random Variable

A random variable is essentially just a mappings(functions) from the sample space to real number. Typically, we let

$$X\colon \Omega \to \mathbb{R} \text{ with measurability.}$$

And we denote the Borel $\sigma-$algebra as $\mathcal{B}(\mathbb{R})$. In particular, we have

$$(-\infty, x] \in \mathcal{B}(\mathbb{R}),$$

or more generally,

$$\{\omega\colon X(\omega) \leq x\} \in \mathcal{F} \quad \forall x \in (-\infty, +\infty).$$

Above allows us to define the cumulative density function(CDF) of x. In particular,

$$\mathbb{P}_X\left((-\infty, x)\right) = \mathbb{P}\left(\underbrace{\{\omega\colon X(\omega) \leq\}}_{\mathcal{F}}\right).$$

Same as probability, we can define **mutual independence** of a collection of random variables.

Definition 8.1.2 (Independent). Let X and Y being two random variables. We say X, Y are *independent* if

$$\mathbb{P}\left(\{x \in A\} \cap \{y \in B\}\right) = \mathbb{P}\left(\{x \in A\}\right)\mathbb{P}\left(\{y \in B\}\right) \quad \forall A, B \text{ in } \mathcal{B}(\mathbb{R}).$$

Remark. Notice that we are abusing the notation here. In the definition, x is sampled from X and y is sampled from Y without explicitly mentioning. In reality, we have

$$\{x \in A\} \cup \{y \in B\} = \{\omega \in \Omega\colon X(\omega) \in A \wedge Y(\omega) \in B\},$$

we see that we need

$$\{\omega \in \Omega\colon X(\omega) \in A\} \in \mathcal{F} \text{ and } \{\omega \in \Omega\colon Y(\omega) \in B\} \in \mathcal{F}.$$

Another important property is that $\mathcal{F}$ is closed under (countably many) intersections.

We can also view this from the probability density function $f(x)$. We have

$$\mathbb{E}\left[f(x)\right] \coloneqq \begin{cases} \sum_{i=1}^{n} f(x_i)\mathbb{P}\left(x = x_i\right) \\ \int_{-\infty}^{\infty} f(x)\,\mathrm{d}\underbrace{F_x(x)}_{\text{CDF}} \end{cases}$$

where $f_x(x)$ is probability density function.

The in this case, we say X and Y are independent if for any f and g,

$$\mathbb{E}\left[f(x)g(y)\right] = \mathbb{E}\left[f(x)\right]\mathbb{E}\left[g(y)\right].$$

Example. Let $f(x) = x^2$ and $g(y) = y^3$. Then

$$\mathbb{E}\left[x^2y^3\right] = \mathbb{E}\left[x^2\right]\mathbb{E}\left[y^3\right].$$

Again, we have similar definition for countably infinite collection of random variables being independent. Let random variables being

$$X_1, X_2, \ldots$$

associated with the probability density function

$$f_1, f_2, \ldots$$

Then, we say this collection of random variables is independent if for any finite sub-collection is independent. Namely, for any index set $K \subsetneq \mathbb{N}$,

$$\mathbb{E}\left[\prod_{k\in K} f_k(x_k)\right] = \prod_{k\in K} \mathbb{E}\left[f_k(x_k)\right].$$

8.2 Stochastic Process(Random Process)

This is essentially a collection of random variable induced by a discrete (time) index

$$X_1, X_2, X_3, \ldots$$

or

$$\{X_i\}_{i=1}^{\infty}.$$

Intuition (Stochastic process). A stochastic process is essentially just a collection of (mutually) independent and identical (i.i.d) distinct random variables.

Lecture 13: Markov Chain

20 Oct. 12:30

Example. Some example for stochastic process.

1. Let $X_i \sim$ Beronulli(p), where $X_i \in \{0,1\}$ and
$$\mathbb{P}\left(X_i = 1\right) = p.$$
Intuition. Biased coin toss.
2. Let $X_i \sim$ Uniform$(\{1,2,3,4,5,6\})$, where $X_i \in \{1,2,3,4,5,6\}$ and
$$\mathbb{P}\left(X_i = j\right) = \frac{1}{6}.$$
Intuition. Random tosses of dice.
3. Let $X_i \sim$ Uniform$([0,1])$, where $X_i \in [0,1]$.
4. Let $X_i \sim \exp(\lambda)$, then
$$\mathbb{P}\left(X_i > u\right) = e^{-\lambda u} \qquad \forall u \geq 0.$$
5. Let $X_i \sim \mathcal{N}(\mu, \sigma^2)$. This is so-called normal distribution with *Gaussian mean* μ and variance σ^2.

8.2.1 i.i.d. Sequence

Remark. Notice that our discussion will focus on $\mathbb{E}_{X_i}\left[<\right] + \infty$.

Note. The core property of i.i.d. sequence is **strong law of large numbers**(SLLN). Namely,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i \to \mathbb{E}_X\left[.\right]$$

In general, we let $Y_i = \mathbb{1}_{\{X_i \in B\}}$(indicator function), where

$$\mathbb{1}_{\{X_i \in B\}}(\omega) = \begin{cases} 0, & \text{if } X_i(\omega) \in B \\ 1, & \text{otherwise,} \end{cases}$$

hence $y_i \sim \text{Bernoulli}(p = \mathbb{P}(X_i \in B))$.

Note. Markov chains will not be i.i.d. sequence of random variables since there will be dependence. But still, the SLLN will hold under conditions.

8.2.2 Conditional Property

As previously seen. We first note that if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, then A and B are independent.

Definition 8.2.1 (Conditional probability). Let $\mathbb{P}_(B)) > 0$, then we can define so-called *conditional probability* between event A and B as

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} =: \mathbb{P}(A \mid B).$$

Definition 8.2.2 (Conditional expectation). Similarly, we can have so-called *conditional expectation* defined as

$$\mathbb{E}[X \mid X \in A] = \mathbb{E}[X \mid A].$$

Example. We look at an example for discrete random variable X. Let $X \in \Omega$, then

$$\begin{aligned} \mathbb{E}[X \mid X \in A] &= \sum_{\omega \in \Omega} \omega \mathbb{P}(X = \omega \mid X \in A) \\ &= \sum_{\omega \in \Omega} \omega \frac{\mathbb{P}(\{X = \omega\} \cap \{x \in A\})}{\mathbb{P}(X \in A)} \\ &= \sum_{\omega \in \Omega} \omega \frac{\mathbb{P}(X = \omega)}{\mathbb{P}(X \in A)} \\ &= \frac{\sum_{\omega \in \Omega} \omega \mathbb{P}(X = \omega)}{\mathbb{P}(X \in A)} \end{aligned}$$

We also have so-called conditional independence.

Definition 8.2.3 (Conditionally independent). Let A, B, C be events. Then

$$\mathbb{P}(X \in A, X \in B, X \in C) = \mathbb{P}(X \in A \cap B \cap C).$$

Then if A, B, C are independent, then we further have

$$\mathbb{P}(X \in A \cap B \cap C) = \mathbb{P}(X \in A)\mathbb{P}(X \in B)\mathbb{P}(X \in C)$$

as we have seen. If they are dependent, given $\mathbb{P}(X \in C) > 0$, then

$$\mathbb{P}(X \in A, X \in B, X \in C) = \mathbb{P}(X \in A, X \in B \mid X \in C)\mathbb{P}(X \in C).$$

Now, if

$$\mathbb{P}(X \in A, X \in B \mid X \in C) = \mathbb{P}(X \in A \mid X \in C)\mathbb{P}(X \in B \mid X \in C),$$

then A and B are *conditionally independent* given C.

8.3 Markov Chain

Markov Chains are dependent processes but with a specific form of conditional independence.

Consider random variables sequence

$$X_1, X_2, X_3, \dots$$

with discrete time index $1, 2, 3, \dots$. Noting that they lie on a discrete space, like number, labels, etc.

Note (Conditional Independence properties). Knowing the present, the past and the future are *conditionally independent*.

Say we are at time $n > 1$, then

$$X_n = a_n$$

where a_n is taking values in a discrete set. Then

$$\begin{aligned}&\mathbb{P}\left(X_1 = a_1, X_2 = a_2, \dots X_{n-1} = a_{n-1}, X_{n+1} = a_{n+1}, \dots, X_{n+m} = a_{n+m} \mid X_n = a_n\right)\\ =&\mathbb{P}\left(X_1 = a_1, X_2 = a_2, \dots X_{n-1} = a_{n-1} \mid X_n = a_n\right)\\ &\cdot \mathbb{P}\left(X_{n+1} = a_{n+1}, \dots, X_{n+m} = a_{n+m} \mid X_n = a_n\right).\end{aligned}$$

Intuition. Knowing where we are, the progress in the future is conditionally independent of how we got to the present.

8.4 Time Homogeneous Markov Chain

We need to know two things:

- Initial distribution. $X_0 \sim \mu$(or $X_1 \sim \mu$)
- One-step transition matrix.

$$P_{ij} \coloneqq \mathbb{P}\left(X_{n+1} = j \mid X_n = i\right) \qquad \forall i, j.$$

Note. Note that

1. Notice that this is not a function of n.
2. The condition here is just for an example from the lecture.

Review the example in lecture

Then we see that the total probability of $\mathbb{P}\left(X_n = i\right)$ is

$$\begin{aligned}\mathbb{P}\left(X_n = i\right) &= \sum_j \mathbb{P}\left(X_n = i, X_0 = j\right)\\ &= \sum_j \mathbb{P}\left(X_0 = j\right)\mathbb{P}\left(X_n = i \mid X_0 = j\right)\\ &= \sum_j \mu_j (P^n)_{ji}\\ &= (\mu^\top P^n)_i\end{aligned}$$

where the first step is from the definition of *total probability*. Recall that

$$P_{ij} = \mathbb{P}\left(X_{n+1} = j \mid X_n = i\right),$$

hence

$$\begin{aligned}\sum_{j\in\Omega} P_{ij} &= \sum_{j\in\Omega} \mathbb{P}\left(X_{n+1}=j \mid X_n=i\right)\\ &= \mathbb{P}\left(X_{n+1}\in\Omega \mid X_n=i\right)\\ &= \frac{\mathbb{P}\left(X_{n+1}\in\Omega, X_n=i\right)}{\mathbb{P}\left(X_n=i\right)} = \frac{\mathbb{P}\left(X_n=i\right)}{\mathbb{P}\left(X_n=i\right)} = 1.\end{aligned}$$

We see that P is row stochastic and $P_{i\cdot}$ is a probability distribution on Ω. So $\rho(P)=1 \Rightarrow \exists\pi$(distribution) such that

$$\pi^\top P = \pi^\top.$$

We see that the left eigenvector is a distribution. We have P is irreducible and Ω is finite, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{1}_{\{X_m=i\}} = \pi_i$$

almost surely.

Refinement with aperiodic P,

$$\lim_{n\to\infty} \mathbb{P}\left(X_n=i\right) = \pi_i$$

8.5 Connection with Basic Page Rank

As previously seen. The $r(n)$ is updated by the equation

$$r(n+1) = N^\top r(n) \Leftrightarrow r^\top(n+1) = r^\top(n)N.$$

Further, recall that N is a row stochastic matrix, so compare to Markov Chain,

$$\mathbb{P}\left(X_n=\cdot\right)^\top = \mu^\top P^n = \mu^\top P^{n-1}P = \mathbb{P}\left(X_{n-1}=\cdot\right)^\top P$$

where $\mu \sim \mathbb{P}\left(X_0=\cdot\right)$.

We see that the above two equations are the same! With

$$r(0) \coloneqq \begin{pmatrix} 1/V \\ \vdots \\ 1/V \end{pmatrix}.$$

Recall that N_{ij} is defined as

$$N_{ij} = \begin{cases} \dfrac{1}{d_i^{out}}, & \text{if } A_{ij}\geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

We pick every outgoing neighbors uniformly at random, but no one else.

8.6 Random Walker

Intuition. At location i, regardless of how the random walker came to i, pick the next destination uniformly at random from outgoing neighbors.

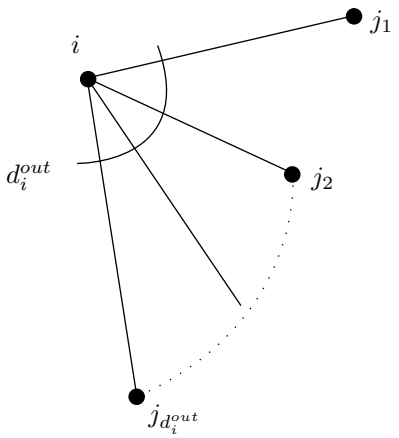


Figure 8.1: Random Walker

Note. If $d_i^{out} = 0$, then the random walker get stuck.

Lecture 14: Random Walker and Monte Carlo Estimator

8.7 Personalized Page Rank

25 Oct. 12:30

As previously seen. Recall that we want to associate Page Rank with random walker.

- Basic PageRank: Random walker on the graph.
- Scaled PageRank: Random walker with reset to a uniformly chosen node.

This is a good view point for the *population as a whole* user surfing the webs.

Across population reset is to a randomly chosen webpage. We can rank webpages across the population.

Problem. Can we be more specific?

Answer. With the help of reset, one can make this happen:

- Starting point of random walk
  - Random Walk
- Reset

where we restart/reset to a specific node $v^* \in \mathcal{V}$.

Remark. Personalizing the restart distribution is what allows this to happen.

⊛

Definition 8.7.1. *Personalized PageRank* is denoted by

$$\pi_{v^*}(s).$$

Collect $\pi_{v^*}(s)$ for all $v^* \in \mathcal{V}$, then if we reset uniformly,

$$\pi(s) = \frac{1}{V} \sum_{v^* \in \mathcal{V}} \pi_{v^*}(s).$$

Remark. Average of all the personalized PageRank, one gets the scaled PageRank. We see this as follows. Similar average with respect to any specific distribution μ distributed on $\mathcal{V}$, then

$$\pi_\mu(s) = \sum_{v^* \in \mathcal{V}} \mu(v^*) \pi_{v^*}(s).$$

Compare to the Scaled PageRank, we see that μ is uniformly distributed for Scaled PageRank.

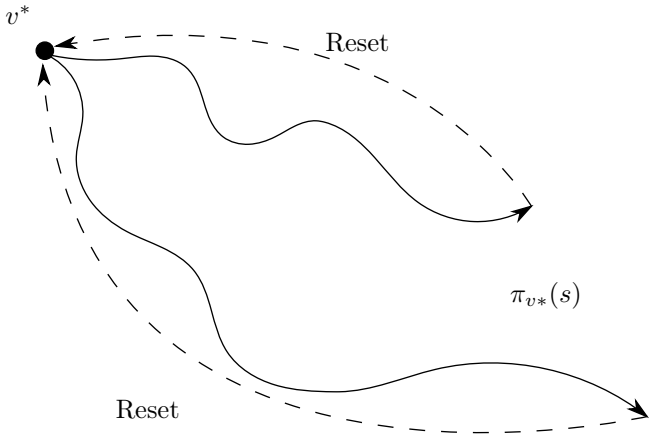


Figure 8.2: Random Walk for personalized PageRank

8.8 Monte Carlo Algorithm

Note. This is a simulation-based algorithm.

The basic algorithm is

0. Start at a uniformly random node.
1. First toss a biased coin such that

$$\mathbb{P}(\text{heads}) = s, \qquad \mathbb{P}(\text{tails}) = 1 - s.$$

   - If Heads, then perform random walk if possible(namely when $d_i^{out} = 0$).
   - If Tails, then reset to a uniformly chosen node.
2. **GOTO 1.**(Repeat T times).

We can describe this algorithm as Markov Chain. Let

$$\{X_n\}_{n\geq 0}$$

where X_n is the location of the random walker at time n. We see that

$$\begin{aligned} X_0 &\sim \text{uniform}(\{1,2,\ldots,V\}) \\ X_0 &= \begin{cases} \text{uniform}(\{1,2,\ldots,V\}) & \text{with probability } 1-s \\ \text{NextHop}(x_0) & \text{with probability } s \end{cases}. \\ \text{NextHop}(i) &= \begin{cases} \text{uniform}(\{N_i^{out}\}), & \text{if } d_i^{out} > 0 \\ i, & \text{if } d_i^{out} = 0 \end{cases} \end{aligned}$$

This is irreducible and aperiodic. And by the SLLN property, we now analyze this algorithm by Markov Chain. Let run Monte Carlo Algorithm for T long.

8.8.1 1^{st} Estimator for $\pi_V(s)$

The estimation of π is then

$$\hat{\pi}_V^\top(s) = \frac{1}{T}\sum_{n=0}^{T-1} \mathbb{1}_{\{X_n=v\}} \qquad \forall v \in \mathcal{V}.$$

With $T \to \infty$, we have

$$\lim_{T\to\infty} \frac{1}{T}\sum_{n=0}^{T-1} \mathbb{1}_{\{X_n=v\}} = \pi_V(s)$$

by SLLN. But this algorithm has two main issues.

1. Not clear how long we should run this for. Namely, we need to be able to say something about

   $$\mathbb{E}\left[\left|\hat{\pi}_V^\top(s) - \pi_V(s)\right|^2\right]$$

   in T to ensure the speed of convergence.

2. The estimator is a biased estimator. Recall that an estimator is unbiased if

   $$\mathrm{E}\left[\hat{\pi}_V^\top(s)\right] = \pi_V(s);$$

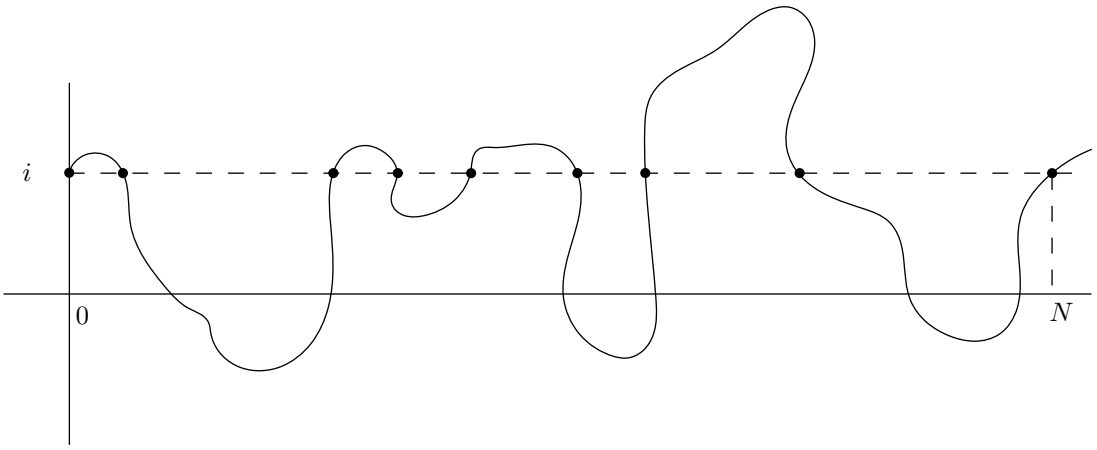
   and if it is biased,

   $$\hat{\pi}_V^\top(s) - \pi_V(s) \neq 0$$

   for this particular estimator $\hat{\pi}_V^\top(s)$. We see that $\hat{\pi}_V^\top(s)$ is indeed biased by

   $$\begin{aligned} \mathrm{E}\left[\hat{\pi}_V^\top(s)\right] &= \frac{1}{T}\sum_{n=0}^{T-1} \mathbb{E}\left[\mathbb{1}_{\{X_n=v\}}\right] \\ &= \frac{1}{T}\sum_{n=0}^{T-1} \mathbb{P}\left(X_n = v\right) \\ &= \frac{1}{T}\sum_{n=0}^{T-1} (\mu_0 P^n)_v \underset{T\to\infty}{\to} \pi_V(s), \end{aligned}$$

   where P is the *one step trransform matrix*. We see that this only converges when $T \to \infty$ by Perron-Frobenius Theorem(**??**) since aperiodicity.



$$\hat{\pi}_i(s) = 9/N$$

Figure 8.3: First Monte Carlo Estimator

8.8.2 2^{nd} Estimator for $\pi_V(s)$

Recall that

$$\widetilde{N}(s) = sN + (1-s)\frac{1}{N}\vec{1}\vec{1}^\top.$$

Notice that $\pi(s)$ is the left eigenvector of $\widetilde{N}(s)$ with eigenvalue 1. Write

$$\begin{aligned}\pi^\top(s)\widetilde{N}(s) &= \pi^\top(s) \\ &= \pi^\top(s)sN + (1-s)\underbrace{\pi^\top(s)\vec{1}}_{=1}\frac{1}{V}\vec{1}^\top \\ &= \pi^\top(s)sN + (1-s)\frac{1}{V}\vec{1}^\top,\end{aligned}$$

where $\pi^\top(s)\vec{1} = 1$ because $\pi(s)$ is a probability vector, which sums to 1. Then

$$\pi^\top(s)(I - sN) = (1-s)\frac{1}{V}\vec{1}^\top.$$

> **As previously seen.**
>
> $$(I-A)^{-1} = I + A + A^2 + \dots.$$

Hence,

$$\begin{aligned}\pi^\top(s) &= (1-s)\frac{1}{V}\vec{1}^\top(I-sN)^{-1} \\ &= (1-s)\frac{1}{V}\vec{1}^\top\left(\sum_{i=0}^{\infty} s^i N^i\right) \\ &= \sum_{i=0}^{\infty} s^i(1-s)\frac{1}{V}\vec{1}^\top N^i.\end{aligned}$$

Notice that $N^0 \coloneqq I$.

We now analysis this expression.

$$\pi^\top(s) = \sum_{i=0}^{\infty}\underbrace{s^i(1-s)}_{\tau}\frac{1}{V}\vec{1}^\top N^i,$$

where

- τ
  - Tossing a series of independent biased coins until first tail occurs.

$$\begin{aligned}\mathbb{P}(\tau = 0) &= 1 - s \\ \mathbb{P}(\tau = 1) &= s(1-s) \\ &\vdots \\ \mathbb{P}(\tau = i) &= s^i(1-s).\end{aligned}$$

  - τ is so-called *geometric random variable.*

  We see that reset happens at time N, hence we look at **time $N-1$**.

- $\frac{1}{V}\vec{1}^\top N^i$
  - $u = \frac{1}{V}\vec{1}^\top$ is the uniform distribution on the nodes of the graph.
  - N^i represents random walk repeated i times.

  In all, this term represents that we start at a uniformly random node and take i steps as per random walk. Namely,

Distribution of locations of the random walk after i steps.

Now, the algorithm becomes

0. Toss a geometric random variable τ such that

$$\mathbb{P}(\tau = i) = s^i(1-s).$$

1. Pick a random node of the graph and run random walk τ times to find where we end up.
2. **GOTO 1.**(Repeat K times)

Denote those K random variables τ as

$$\tau_1, \tau_2, \ldots, \tau_K$$

and the associated X_τ as

$$X_{\tau_1}, X_{\tau_2}, \ldots, X_{\tau_K}.$$

We see that

$$\hat{\pi}_V^K(s) = \frac{1}{K}\sum_{k=1}^{K} \mathbb{1}_{\{X_{\tau_k}=v\}}.$$

This is an unbiased estimator since

$$\mathrm{E}\left[\mathbb{1}_{\{X_{\tau_k}=v\}}\right] = \mathbb{P}(x_{\tau_k} = v) = \pi_v(s).$$

Also, the variance can be derived as

$$\mathrm{Var}\left[\hat{\pi}_V^K(s)\right] = \frac{\pi_v(s)(1-\pi_v(s))}{K} \leq \frac{1}{4K}.$$

Remark. Since $n(1-n) \leq 1/4$ for every $n \in [0,1]$ with equality if $x = 1/2$.

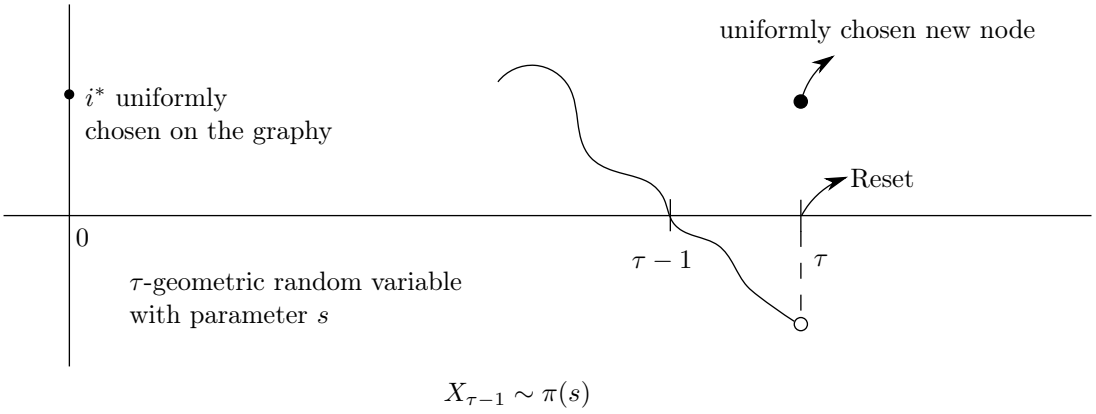


Figure 8.4: Second Monte Carlo Estimator

Note. We note that the empirical mean is unbiased while the empirical variance is biased. Let $X_1, X_2, \ldots, X_N$ are i.i.d. with mean μ and variance σ^2.

- We see that $\hat{\mu}_N$ is the empirical mean which is defined as

$$\hat{\mu}_N \coloneqq \frac{1}{N}\sum_{i=1}^{N} X_i$$

with

$$\mathbb{E}\left[\hat{\mu}_N\right] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[X_i\right] = \frac{1}{N}\sum_{i=1}^{N}\mu = \mu,$$

which implies this is an unbiased estimator.

- Denote $\hat{\mathrm{Var}}(N)$ as the empirical variance estimator. It's defined as

$$\hat{\mathrm{Var}}(N) \coloneqq \frac{1}{N}\sum_{i=1}^{N}(X_i - \hat{\mu}_N)^2.$$

This is indeed a biased estimator since

$$\mathbb{E}\left[\hat{\mathrm{Var}}(N)\right] = \left(\frac{N-1}{N}\right)\sigma^2,$$

which is slightly less than the true variance. We then have an unbiased estimator defined as

$$\overline{\mathrm{Var}}(N) \coloneqq \frac{1}{N-1}\sum_{i=1}^{N}(X_{i-\hat{\mu}_N})^2$$

with

$$\mathbb{E}\left[\overline{\mathrm{Var}}(N)\right] = \sigma^2.$$

8.8.3 3^{rd} Estimator for $\pi_V(s)$

Recall that

$$\pi(s) = \frac{1-s}{V}\vec{1}^\top (I - sN)^{-1}.$$

Define

$$Z \coloneqq (I - sN)^{-1} = \sum_{i=0}^{\infty} s^i N^i$$

with $s^0 \coloneqq 1$ and $N^0 \coloneqq I$. Z is a non-negative matrix since $Z_{ij} \geq 0$. Let

$$\sigma_{ij} \coloneqq (1-s)Z_{ij}, \qquad (\Sigma)_{ij} \coloneqq \sigma_{ij}$$

where $\sigma_{i\cdot}$ is the row of Σ corresponding to node i. We then modify the algorithm as

0. PageRank corresponding to reset distribution where we always reset to node i.
1. Start at node i.
   - Find geometric length τ^i.
   - Run random walk until τ^i steps to find the node that we end up with.
2. **GOTO 1.**(Repeat M times for every i)

If we only take one walk, namely $M = 1$, then

$$\pi_V(s) = \frac{1}{V}\sum_{i\in\mathcal{V}} \sigma_{iv}(s).$$

Now, for every node i in the network, we run M times with

$$\tau_1^i, \tau_2^i, \ldots, \tau_M^i$$

and

$$X_{\tau_1}^i, X_{\tau_2}^i, \ldots, X_{\tau_M}^i.$$

We count the number of arrivals in node v and find the average. More explicitly, we have

$$\hat{\pi}_V(s) = \frac{1}{V}\sum_{i=1}^{V}\frac{1}{M}\sum_{m=1}^{M}\mathbb{1}_{\{X_{\tau_m^i}^i = v\}} = \frac{1}{VM}\sum_{i=1}^{V}\sum_{m=1}^{M}\mathbb{1}_{\{X_{\tau_m^i}^i = v\}}.$$

This is another unbiased estimator for the same reason given in the second estimator, but we see that this converges faster and with smaller variance. Specifically, we have

$$\mathrm{Var}\left[\hat{\pi}_V(s)\right] = \frac{1}{VM}\left(\pi_j - \frac{\sum_{i=1}^{V}\sigma_{ij}^2}{V}\right).$$

We see that this is a better estimator.

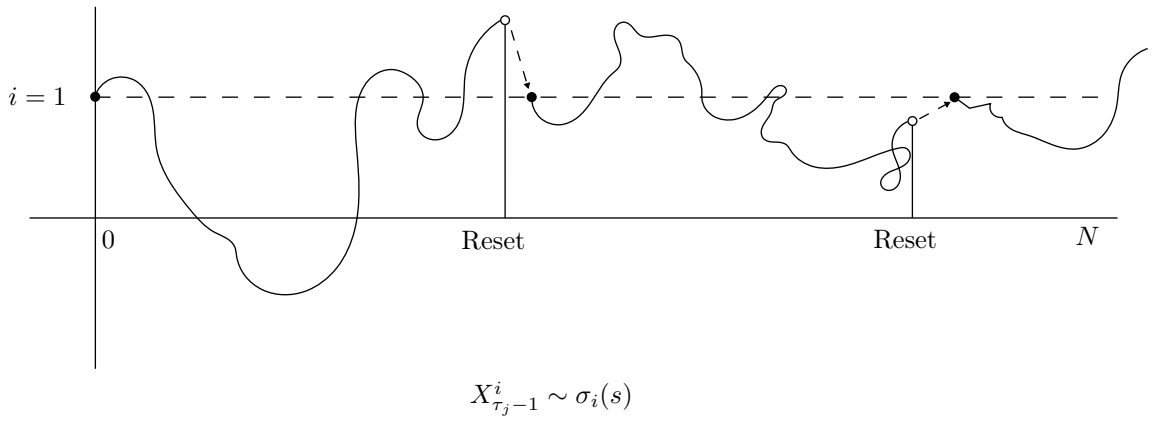


Figure 8.5: Third Monte Carlo Estimator

Lecture 15: Random Graph

27 Oct. 12:30

As previously seen. Main idea is that to construct a Markov Chain on nodes of the graph. At none i, the random walker toss a biased coin and determine where to go.

8.8.4 4^{th} Estimator for $\pi_V(s)$

Although the 3^{rd} estimator for $\pi_V(s)$ is better than the other two, but it still has its flaws. The main thing is, it throws away so much data, which means it will converge slower. We introduce a final estimator, which make the full use of the data given.

We first introduce Renewal Theorem.[1]

Theorem 8.8.1. Renewal Theorem.

$$\frac{\mathbb{E}\left[\#\text{ of visits to node } k \text{ in this reset}\right]}{\underbrace{\mathbb{E}\left[\text{Duration of reset}\right]}_{=\frac{1}{1-s}}} = \pi_k(s).$$

Noting that each reset duration is independent of the other T rest duration starting at uniformly chosen node. Then we have

$$\hat{\pi}_k(s) = \frac{\frac{1}{T}\sum_{i=1}^{\top} N_t(k)}{1/1-s} = \frac{1-s}{T}\sum_{t=1}^{\top} N_t(k)$$

where $N_t(k)$ is the number of time we visited k in reset duration.

Remark. We see that

- This is also a non-biased estimator.
- $N_t(k)$ is i.i.d.

[1] https://en.wikipedia.org/wiki/Renewal_theory

- This estimator does not throw out those useful data, hence the estimator computed by this algorithm converges faster.

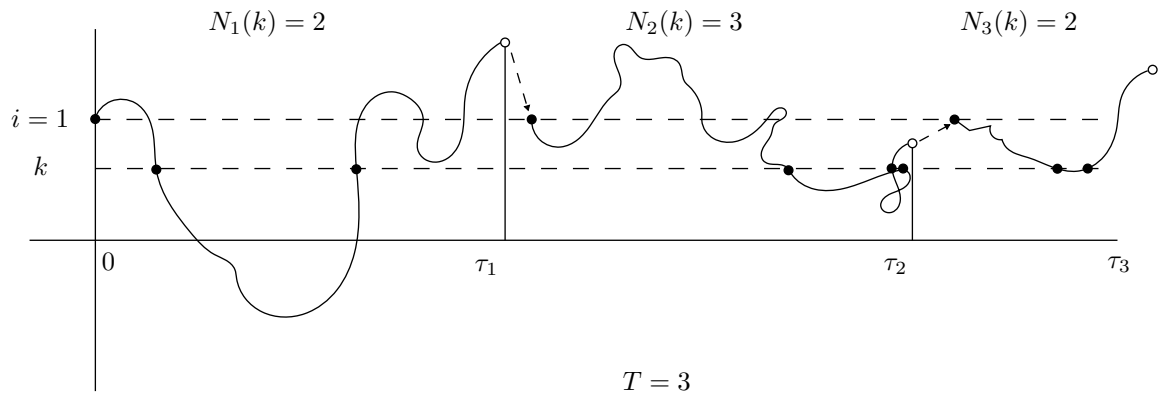


Figure 8.6: Forth Monte Carlo Estimator

Chapter 9

Random Graph

We are interested in some functions depend on the number of nodes n in the graph such that

$$\lim_{n\to\infty} f(n),$$

namely we are interested in so-called *large graph property*.

9.1 Order Relationship between Functions

We first introduce some common notations for asymptomatic comparison.

Definition 9.1.1. $f(n)$ is said to be $O(g(n))$ if there is a positive k such that

$$f(n) \leq k \cdot g(n)$$

for all large enough n.

Definition 9.1.2. $f(n)$ is said to be $\Omega(g(n))$ if there is a positive k such that

$$f(n) \geq k \cdot g(n)$$

for all large enough n.

Definition 9.1.3. $f(n)$ is said to be $\Theta(g(n))$ if there are positive $k_1 \leq k_2$ such that

$$k_1 \cdot g(n) \leq f(n) \leq k_2 \cdot g(n)$$

for all large enough n.

Definition 9.1.4. $f(n)$ is said to be $o(g(n))$ if for every positive ϵ, there exists a N large enough such that

$$f(n) \geq \epsilon \cdot g(n) \ \forall n \geq N.$$

Definition 9.1.5. $f(n)$ is said to be $\omega(g(n))$ if for every positive ϵ, there exists a N large enough such that

$$f(n) \leq \epsilon \cdot g(n) \ \forall n \geq N.$$

Definition 9.1.6. $f(n)$ is said to be of the same order as $g(n)$ asymptotically if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

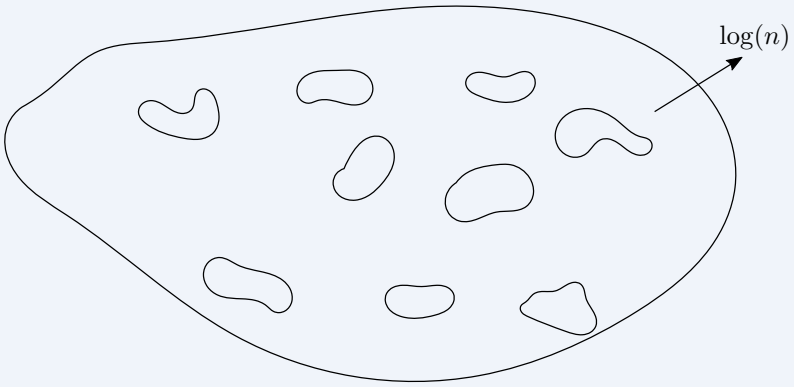
We can write $f(n) \sim g(n)$ for short in this case.

Remark. If $f(n)$ or $g(n)$ are not non-negative, we simply use $|f(n)|$ and $|g(n)|$ and apply the definition.

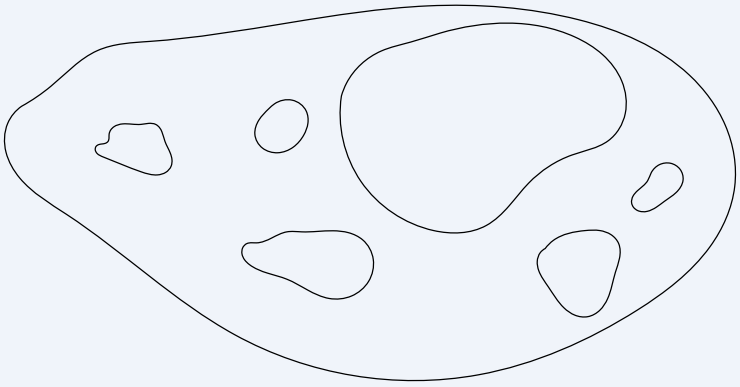
We consider the size of the largest connected component, and see how does this relation behave as $n \to \infty$.

Example. We see some example for a function $f(n)$ which represent the size of a graph, where n is the number of nodes in the graph.

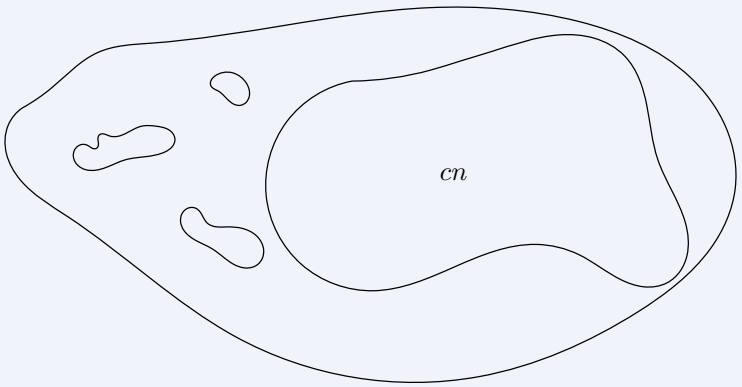
- $f(n) = \log n$: Islands with $\frac{n}{\log(n)}$ of them. When $n \to \infty$, $\frac{n}{\log(n)} \to 0$.



- $f(n) = n^c$ with $c < 1$: We see that $\frac{n^c}{n} \to 0$ for $c < 1$ as $n \to \infty$. Better than island but not that great.



- $f(n) = cn$: There is a giant component roughly c percent of nodes in the graph. We see that as $n \to \infty$, $\frac{cn}{n} \to c$.



Lecture 16: Random Graph

9.2 Erdős-Rényi Random Graphs Family

1 Nov. 12:30

We introduce two main models for generating random graphs.

- Erdős-Rényi. The random graph is given by $G(n, M)$. A graph is chosen uniformly at random from the collection of **all graphs with n nodes and M edges**.
- Edward Gilbert. The random graph is given by $G(n, p)$. The graph is constructed by connecting nodes randomly, with each edge is included in the graph with probability p **independently**.

Remark. We compare these two models.

- The number of expected edges in $G(n, p)$ is $\binom{n}{2}p$. Hence, by SLLN, nearly all graphs in $G(n, p)$ will have $\binom{n}{2}p$ edges. Then if $pn^2 \to \infty$ and $M = \binom{n}{2}p$, then as n grows, the behavior of $G(n, p)$ should be similar with $G(n, M)$.
- The most commonly used model is $G(n, p)$ since the independent property of edges simplifies lots of analysis.

We first focus on Erdős-Rényi random graphs family.

- Uniform undirected case. A is symmetric. Construct the adjacency matrix A such that

$$A = \begin{pmatrix} 0 & & & \\ & 0 & & U \\ & & \ddots & \\ & L & & 0 \\ & & & & 0 \end{pmatrix},$$

where we need to specify the upper triangular part U, and the lower triangular part L can then be obtained by symmetry.

Let V denotes the nodes $\{1, 2, \ldots, V\}$ and let $p \in (0, 1)$, we have the following algorithm to generate a random graph:

1. For node $i = 1, 2, \ldots, V-1$,
   (a) For $j = i+1, \ldots, V$
      i. Generate an independent Bernoulli(p) random variable, and set A_{ij} to be that random variable with
      $$\mathbb{P}(A_{ij} = 1) = p, \qquad \mathbb{P}(A_{ij} = 0) = 1 - p.$$

Soon, we will study some properties as $V \to \infty$.

- Non-uniform undirected graph. With $P = P^\top$, where P is the probability matrix with diagonal entries being 0. Then, we have the following algorithm to generate a random graph:

1. For node $i = 1, 2, \ldots, V-1$,
   (a) For $j = i+1, \ldots, V$
      i. Generate a Bernoulli(p) random variable. Set A_{ij} to be that random variable with
      $$\mathbb{P}(A_{ij} = 1) = p, \qquad \mathbb{P}(A_{ij} = 0) = 1 - p.$$
      Namely,
      $$A_{ij} = \text{Bernoulli}(P_{ij}).$$

- Uniform directed graph. Again, with $p \in (0, 1)$, and
$$\underset{i,j}{\forall}\, i \neq j \quad A_{ij} = \text{Bernoulli}(p) \text{ chosen independently.}$$
And because of independence, $A_{ij} \neq A_{ji}$ can occur and
$$\mathbb{P}(A_{ij} = A_{ji} = 1) = \mathbb{P}(A_{ij} = 1)\,\mathbb{P}(A_{ji} = 1) = p^2.$$
Compared to undirected case, we see that
$$\mathbb{P}(A_{ij} = A_{ji} = 1) = p$$
in an undirected graph.

- Non-uniform directed graph. Denote P as the probability matrix with diagonal entries being 0. Then
$$\underset{i,j}{\forall}\, i \neq j \quad A_{ij} = \text{Bernoulli}(P_{ij}) \text{ chosen independently.}$$

> **Remark.** To summarize, we have
> - Uniform case. With $p \in (0, 1)$ such that
>   * $p = 0$ - V has isolated nodes.
>   * $p = 1$ - Complete graph.
> - Non-uniform case. Some values can be 1 or 0, but not all.

- Particular case of non-uniform and directed graph. This is called Stochastic Block Model(SBM) with the probability matrix being:
$$P = \begin{matrix} \\ V_1 \\ V_2 \end{matrix} \begin{matrix} \begin{matrix} V_1 & V_2 \end{matrix} \\ \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{matrix}$$

with $P_{12} = P_{21}^\top$, $P_{11} = P_{11}^\top$ and $P_{22} = P_{22}^\top$. In particular,

$$P_{11} = p_{11} \begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}_{V_1 \times V_1} \quad \text{uniform within the block}$$

$$P_{22} = p_{22} \begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}_{V_2 \times V_2}$$

$$P_{12} = p_{12} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}_{V_1 \times V_2}$$

$$P_{21} = p_{21} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}_{V_2 \times V_1}$$

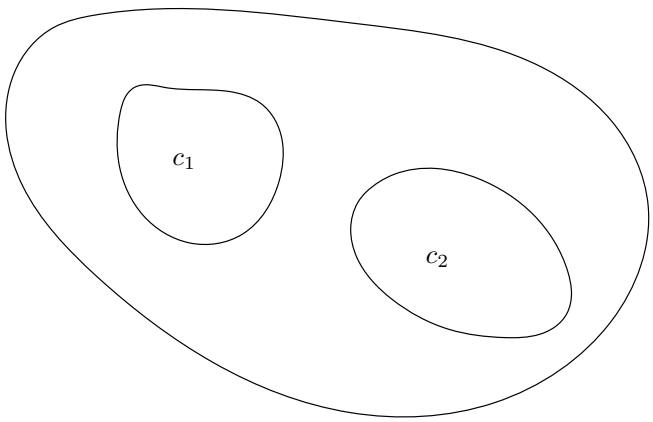
where in the typical set up, we will have $p_{11}, p_{22} \gg p_{12}$. This creates a community structure such that

- p_{11} governs connectivity in community 1.
- p_{22} governs connectivity in community 2.
- p_{12} governs the *cross* connectivity structure.

Remark. Bipartite graph generalization, which happens when

$$p_{12} \gg p_{11}, p_{22}.$$

This generalizes to multiple communities by an appropriate block structure.



$p_{12} = 0$
$p_{12}, p_{22} > 0$

Also, this is related to the community detection algorithm. We look at the graph Laplacian

$$L = D - A$$

and find spectral decomposition and find the smallest eigenvalues and eigenvectors to study community structure.

Let's look at the properties from follows the above discussion. In particular, we consider the uniform directed case with $p \in (0, 1)$.

1. Each undirected edge is chosen with probability p. For node i, we have
   - The degree is associated with Binomial$(V-1, p)$ such that
   $$\mathbb{P}(\deg = k) = \binom{V-1}{k} p^k (1-p)^{V-1-k}.$$
   - The mean degree is just $(V-1)p$.
   - The total number of edges is associated with Binomial$(\frac{V(V-1)}{2}, p)$.
   - The mean number of edges is $\frac{V(V-1)}{2}p$.
2. Extremely sparse regime. With $p = \min(\frac{c}{V}, 1)$ for some $c > 0$. Now, for node i, we have
   - The degree is associated with Binomial$(V-1, \frac{c}{V}) \to$ Poisson(c) as $V \to \infty$.
   - The mean degree is
   $$c\frac{V-1}{V} \to c \qquad \text{as } V \to \infty.$$

   Note. Recall the Poisson distribution.
   $$\mathbb{P}(\text{Poisson}(c) = k) = e^{-c}\frac{c^k}{k!}, \qquad k = 0, 1, \dots$$
   We can have isolated nodes with
   $$\mathbb{P}(\deg = 0) = e^{-c} > 0.$$
   As $V \to \infty$, e^{-c} fraction of nodes will be isolated since
   $$\mathbb{P}(\text{isolated nodes}) = (1-p)^{V-1} = (1 - \frac{c}{v})^{V-1} \to e^{-c},$$
   where
   $$\lim_{n\to\infty}(1 - \frac{1}{x})^x = e^{-1}.$$
   - The total number of edges is associated with Binomial$(\frac{V(V-1)}{2}, \frac{c}{V})$.
   - The mean number of edges is $\frac{V(V-1)}{2}\frac{c}{V} = \frac{c(V-1)}{2} \sim \Theta(\frac{c}{2}V)$, which is linear in V.

   We can further divide this cases into following sub-cases:

   (a) Sub-critical regime with $0 < c < 1$. Essentially "isolated nodes." The largest connected component will have size $O(\log V)$ as $V \to \infty$. And the number of connected components will be
   $$\Omega(\frac{V}{\log V}) \qquad \text{as } V \to \infty.$$

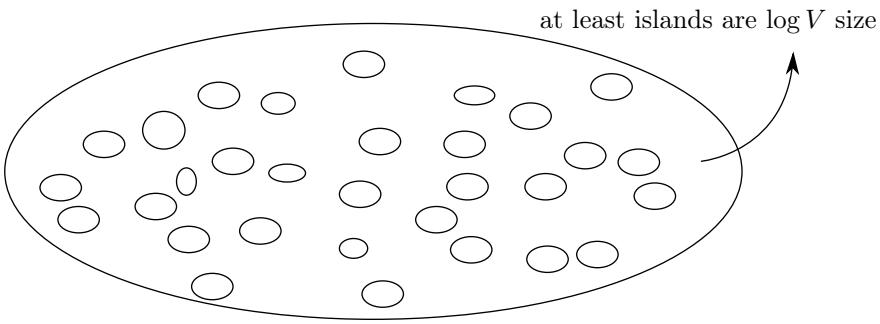


Figure 9.1: Sub-critical Regime.

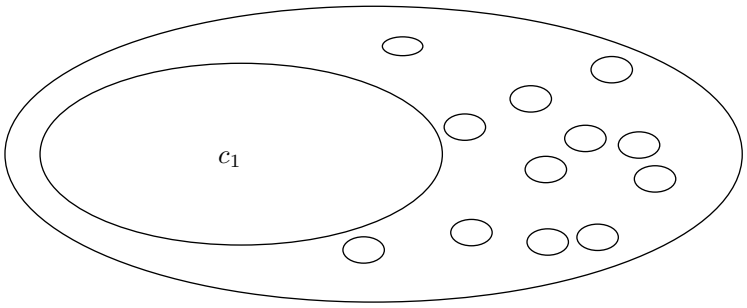
(b) Super-critical regime with $c > 1$. This essentially leads to a (Single) Giant component. There will exist a c_1 associated with the largest component such that

$$|c_1| \sim f(c_1)V, \qquad f \in (0,1)$$

where f is an increasing function of c with $f(0) = 0$ and $\lim_{c\to\infty} f(c) = 1$. And since $e^{-c} > 0$, so isolated nodes exist.
Further, the second-largest component will have

$$|c_2| \sim O(\log V),$$

while all other components are very small.



$e^{-c} > 0$, so isolated nodes exist

Figure 9.2: Super-critical Regime.

(c) Critical case such that $c = 1$. We see that

$$|c_1| \sim \Theta(V^{\frac{2}{3}}), \qquad |c_2|\, O(\log V).$$

Remark (Branching processes). These analyses are using so-called *Branching Processes*.[a]

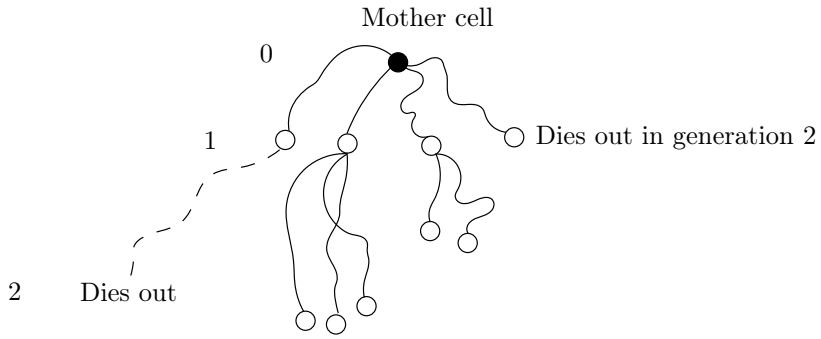


Figure 9.3: Branching processes.

Let c be the mean number of children. Then we have

- $c < 1$: dies out fast
- $c > 1$: can keep growing
- $c = 1$: critical - dies out but goes much further compare to the first case

[a] https://en.wikipedia.org/wiki/Branching_process

3. Sparse regime(Connectivity regime) with $p = \frac{c \log V + a}{V}$ where $a \in \mathbb{R}$ and $c > 0$. More precisely,

$$p \sim \frac{c \log V + a}{V} \text{ as } V \to \infty.$$

Asymptotically, as $V \to \infty$, node degrees go to infinity(not necessarily for all nodes). Now, for node i, we have

- The mean degree is

$$(V-1)p = \frac{V-1}{V}(c \log V + a) \sim c \log V + a = \Theta(\log V).$$

- The number of edges is

$$\frac{V(V-1)}{2} p = (V-1)(c \log V + a) = \Theta(V \log V).$$

Now consider following regimes.

(a) $0 < c < 1$: As $V \to \infty$, there will always be isolated nodes. There will be one giant component with almost all nodes.

(b) $c > 1$: As $V \to \infty$, the graph is **always connected**.

(c) $c = 1$:

- With probability $e^{-e^{-a}}$, there are isolated nodes.
- With probability $1 - e^{-e^{-a}}$, the graph is connected.

> **Remark.** For these graphs, isolated nodes are the ones stopping, preventing connectivity. Moment you eliminate isolated nodes, connectivity emerges.

4. Extremely dense and connected graphs with $p \sim \Omega(\frac{\log V}{V})$.

> **Remark.**
> - Generally, we say a graph is sparse when the number of edges is linear with respect to V, namely
> $$E \sim \Theta(V).$$
> And we say a graph is dense when the number of edges is quadratic with respect to V, namely
> $$E \sim \Theta(V^2),$$
> note that this case is not so common.
> - We can also see this by considering the number of triangles. In the connectivity regime, the number of edges is
> $$\Theta(V \log V),$$
> which is small relative to what's observed in real0world graphs.

Lecture 17: Random Graph

9.3 Real-World Graphs

3 Nov. 12:30

In this section, we compare the random graph models with the real-world graphs. In practice, a real-world graph will have the following properties:

1. The number of edges is linear in V.
2. Giant component usually entice graph(connectivity)
3. Lots of triangles

Compare to the real-world graphs, the Erdős-Rényi random graphs family usually have the following properties, for each different settings:

1. Very sparse
   - edges is linear in V
   - giant component
   - isolated nodes
   - no triangles as $V \to \infty$
2. Sparse
   - edges is $\sim \Theta(V \log V)$
   - most nodes in the giant component
   - can have connectivity
   - Many triangles
3. Dense
   - edges is $\sim \Omega(V^{1+\epsilon})$ with $\epsilon > 0$
   - fully connected
   - lots of triangles

9.4 R-MAT Graphs

Besides Erdős-Rényi random graph family, we can use different approaches to generate a random graph. We now introduce so-called *R-MAT Graphs*.

9.4.1 Recursion Adjacency Matrix Method

We assume that the number of nodes is

$$V = 2^k$$

for some k. Now, given four numbers $a, b, c, d \geq 0$ with $a + b + c + d = 1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

which is a distribution on $\{1, 2, 3, 4\}$. Further, we denote

- E: Target number of edges to be placed on a directed graph. Furthermore, we add entries in the adjacency matrix one edge of a time.
- A: $2k \times 2k$ matrix.

$$\begin{array}{c} \\ 2^{k-1} \\ 2^{k-1} \end{array} \begin{array}{c} \begin{array}{cc} 2^{k-1} & 2^{k-1} \end{array} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2^k \times 2^k} \end{array}$$

Then the algorithm to generate a R-MAT graph just works like the following.

1. For $e = 1 : E$
   (a) $l = k : -1 : 2$
   - Choose a region of $A_{2^l \times 2^l}$ with size $2^{l-1} \times 2^{l-1}$:

$$\begin{array}{c} \\ 2^{l-1} \\ 2^{l-1} \end{array} \begin{array}{c} \begin{array}{cc} 2^{l-1} & 2^{l-1} \end{array} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2^l \times 2^l} \end{array}$$

   with the probability $1 \sim a$, $2 \sim b$, $3 \sim c$, $4 \sim c$.
   - Determine the region for the next step. If i, then we simply pick the region i and **GOTO (a).**

   (b) Place edge in the chosen region if entry is 0, else we ignore. **GOTO 1.**
2. Clean of. Delete any entries on the diagonal and produce the adjacency matrix A to make it a simple directed graph.

Remark. • We implement $1 \sim a, 2 \sim b, 3 \sim c, 4 \sim d$ by letting $r =$ `rand(1)`(uniform in $[0, 1]$) such that if

- $r \leq a$, then 1
- $a < r \leq a + b$, then 2
- $a + b < r \leq a + b + c$, then 3
- $a + b + c < r \leq a + b + c + d = 1$, then 4

Example.

$$M = \left(\begin{array}{cc|cc} & & & \\ & & \cdot & \\ \hline & & & \\ & & & \end{array} \right),$$

where we choose $k = 3$ with the sequence

$$2, 1, 3.$$

- Clean-up stage. In this step, we delete any entries on the diagonal and produce the adjacency matrix A for a simple directed graph.

Example. Let A be

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

After cleaning up, we have

$$A' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

9.5 Preferential Attachment Graph(Directed)

Yet, we have another algorithm to produce a random graph. Let the (undirected) degree distribution on V being

$$d_1, d_2, \ldots, d_V$$

with $\sum_{i=1}^{V} d_i$ being even. Then by considering a histogram, for $k = \{0, 1, \ldots, V-1\}$, we have

$$P_k^{\alpha} = \frac{1}{V} \sum_{i=1}^{V} \mathbb{1}_{\{d_i = k\}} = \frac{\#\text{ of nodes with degree } k}{V}.$$

Remark. We can do some comparison between what we have already seen.

1. Erdős-Rényi Random Graph: Poisson degree distribution with $p = \frac{c}{V}$. Then the tail probability(CCDF) is
$$\mathbb{P}(\deg \geq k) \cong ce^{-\alpha k},$$
which decreases exponentially.
2. Real-World graphs. Tail probability is
$$\mathbb{P}(\deg \geq k) = \frac{c}{k^{\alpha - 1}},$$
which is polynomial decrease if $\alpha \in (2, 3)$. More specifically,
   - $\alpha > 2$:
     - mean degree exists, such that
     $$\text{mean degree } \cong \sum_{k=1}^{\infty} k \frac{\widetilde{c}}{k^{\alpha}} = \sum_{k=1}^{\infty} \frac{\widetilde{c}}{k^{\alpha - 1}}.$$
     We see that $\alpha - 1 > 1$ is needed for above to converge to a finite number.
     - mean squared of degree exists, such that
     $$\text{mean squared of degree } \cong \sum_{k=1}^{\infty} k^2 \frac{\widetilde{c}}{k^{\alpha}} = \sum_{k=1}^{\infty} \frac{\widetilde{c}}{k^{\alpha - 2}}.$$
   - $\alpha < 3 \Rightarrow \alpha - 2 < 1$: Above will just diverge to $+\infty$.
3. In practice, the variance of degrees is exponentially high.

We make the following assumptions for constructing a preferential attachment graph.

- Nodes arrive in sequence $1, 2, \ldots, V$

- Each node has an out-degree. Out going edges that need to be paired with nodes earlier in sequence such that
$$d_i^{out} \leq i - 1.$$

Now, with $p \in (0, 1)$, we do the following to generate a preferential attachment graph.

1. for each edge in d_i^{out}(d_i^{out} time)
   - Toss a biased coin with probability p for Heads($q \coloneqq 1 - p$ for Tails).
     - If Heads, then choose one of unpaired nodes at random and pair.
     - If Tails, then choose one of the unpaired nodes in **proportion** to their incoming edges.
3. **GOTO 1.**

Remark. Node 1 always has out-degree 0.

Now, lets analysis the situation such that if $d_i^{out} = 1$ for all $i = 2, 3, \ldots$ with $t = 2, 3, \ldots, V$. Further, we define
$$X_i(t) \coloneqq \#\text{ of incoming edges for node } i \text{ after node } t \text{ is added to the graph.}$$
We then see
$$\sum_{i=1}^{V} X_i(t) = t - 1 = \sum_{i=1}^{t-1} X_i(t)$$
since $X_i(t) = 0$ for all i such that $t \leq i \leq V$. Look at the change of $X_i(t)$ with respect to $X_i(t-1)$, we further have
$$X_j(t) = \begin{cases} X_j(t-1) = 0, & \text{if } t \leq j \leq V \\ X_j(t-1), & \text{w.p. } p\dfrac{t-2}{t-1} + q\dfrac{t-1-X_j(t-1)}{t-1} \\ X_j(t-1) + 1 & \text{w.p. } p\dfrac{1}{t-1} + q\dfrac{X_j(t-1)}{t-1}. \end{cases}$$

Remark. This is a vector valued random process.

In order to simply the analysis, we take a look at the expectation. We have
$$x_j(t) = \mathbb{E}_{X_j(t)}[]\, x_j(t) = \begin{cases} x_j(t-1) = 0, & \text{if } t \leq j \leq V \\ x_j(t-1) + p\dfrac{1}{t-1} + q\dfrac{x_j(t-1)}{t-1}, & \text{otherwise.} \end{cases}$$
Then
$$\underbrace{x_j(t) - x_j(t-1)}_{\text{like a derivative}} = \begin{cases} 0, & \text{if } t \leq j \leq V \\ \dfrac{p}{t-1} + q\dfrac{x_j(t-1)}{t-1}, & \text{otherwise.} \end{cases}$$
Let $V \to \infty$ and $\widetilde{x}_i(t) = \frac{x_i(t)}{V} \cong \frac{X_i(t)}{V}$ will be well-approximated by a differential equation
$$\frac{\mathrm{d}\widetilde{x}_j(t)}{\mathrm{d}t} = \frac{p}{t} + q\frac{\widetilde{x}_j(t)}{t} = \frac{p + q\widetilde{x}_j(t)}{t}.$$
Rearranging, we have
$$\frac{1}{p + q\widetilde{x}_j(t)} \frac{\mathrm{d}\widetilde{x}_j(t)}{\mathrm{d}t} = \frac{1}{t}.$$
Then since
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln(p + q\widetilde{x}_j(t))\right) = \frac{1}{p + q\widetilde{x}_j(t)} \cdot q\frac{\mathrm{d}\widetilde{x}_j(t)}{\mathrm{d}t},$$
hence we have
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln(p + q\widetilde{x}_j(t))\right) = \frac{q}{t},$$

In all, we have

$$\ln(p + q\widetilde{x}_j(t)) = q\ln(t) + qc.$$

Take exponential on both sides,

$$p + q\widetilde{x}_j(t) = t^q e^{cq}.$$

Solving for $\widetilde{x}_j$, we see that

$$\widetilde{x}_j(t) = \frac{t^q e^{cq} - p}{q}.$$

We now need the initial value. In this case, we have

$$x_j(t) = 0, \forall\ t \leq j \leq V \Rightarrow \widetilde{x}_j(j) = 0.$$

This further implies

$$\widetilde{x}_j(j) = 0 \Leftrightarrow j^q e^{cq} = p \Rightarrow e^{cq} = \frac{p}{j^q} \Rightarrow \widetilde{x}_j(t) = \frac{p}{q}\left(\left(\frac{t}{j}\right)^q - 1\right) \forall\ t \geq j.$$

From this, we see that the fraction of nodes with degree greater than k is

$$\frac{1}{\left(1 + \frac{q}{p}\frac{k}{V}\right)^{\frac{1}{q}}} \cong \frac{c}{k^\alpha}$$

where $\alpha = \frac{1}{q} = \frac{1}{1-p}$.

Note. This is what we called **power law tail**.

Remark. This process is biasing towards incoming degrees, which is caused from the second step.

Remark. A more formal setup is the following. Let

$$[X_j(t)]_{j=1,\ldots,V}$$

be a Markov Chain. Considering two aggregations:

1. For $i = 0, 1, \ldots$, we determine the fraction of nodes with in-degree at least i at time t. Then

$$y_i(t) = \frac{1}{V}\sum_{k=1}^{V}\sum_{j=1}^{V} \mathbb{1}_{\{X_j(t)=k\}} = \frac{1}{V}\sum_{j=1}^{V} \mathbb{1}_{\{X_j(t)\geq i\}}.$$

2. The for $s \in [0, 1]$, define $t = \lfloor sV \rfloor$. Now, for $i = 0, 1, \ldots$, set $\widetilde{y}_t^V(s) = y_i(\lfloor sV \rfloor)$. Then the formal comparison of $\widetilde{y}_i^V(s)$ can be made with an appropriate O.D.E. solution

$$\{y_i(s)\}_{i=0,1,\ldots}$$

for $s \in [0, 1]$. This is the formal mean-field analysis that is not in the scope of the course. Hence, we give a flavour of the results using a hand-wavy argument.

Chapter 10

Game Theory

Lecture 18: Game Theory

The goal is to analyze situations with rational agents who want to maximize their won rewards under a game with a particular structure. 8 Nov. 12:30

10.1 Game Structure

We restrict a game with the following structure.

1. Players/Agents: **finite** or infinite
2. Strategies/Actions: Each player has a set of strategies/actions to choose from - **finite** or infinite, homogeneous or inhomogeneous
3. Payoff/Utility/Rewards: Return for each agent based on the action or strategy of all agents(including self).

Besides the game structure, we also have some assumption for the player in the game.

- Common knowledge: All agents know all players' strategy set and payoffs. Basically means each player knows everything about the structure of the game.
- Rationality: All agents are fully rational and self utility maximizer. As a result of common knowledge, everyone knows that all agents are rational.

Note. By above two assumptions, players succeed in selecting optimal strategies, and our goal is to understand these optimal strategies.

10.2 Normal Form, One-Shot Games

For simplicity, we assume that the number of players and the size of the actions set per player and also the payoffs are all finite.

Definition 10.2.1. We call a game a *One-Shot Game* if players will simultaneously and independently choose their actions, and they do so only once in this game.

We note that contrarily, there are games called *dynamic game*.

Definition 10.2.2 (Dynamic game)**.** We call a game a *dynamic game* if actions can be played sequentially over time for players in the game.

Example. We first see some examples for one-shot games describing in *normal form*[a].

1. **Prisoner's Dilemma**. Assume that two people are taken for investigation of a crime. They're interrogated simultaneously and in different rooms so they can't communicate. Each of them is offered the choice to confess or not.

   - Two players.
   - Action Set: {confess(C), not confess(NC)}.
   - Payoffs: There are four possibilities:
     - If you confess and partner doesn't, then you are released and partner gets 10 years in jail.
     - If you don't confess and partner confess, then you'll get 10 years and partner released.
     - If both confess, both get 4 years.
     - If both do not confess, both get charges on minor crime, which will let them get 1 year.

   We can then define so-called *payoff matrix*,

		Player 2	
		NC	C
Player 1	NC	$(-1,-1)$	$(-10,0)$
	C	$(0,-10)$	$(-4,-4)$

   Problem. How will you react?

   [a] https://en.wikipedia.org/wiki/Normal-form_game

   Answer. We first see what action should suspect 1 plays based on rational analysis.

   Suspect 1's reasoning

   - If suspect 2 confess $\rightarrow$ since $-4 > -10 \Rightarrow$ best action is to confess.
   - If suspect 2 don't confess $\rightarrow$ since $0 > -1 \Rightarrow$ best action is to confess.

   Since this problem is symmetric, namely the strategy for both suspects are the same, so both of them will confess, leading to an equilibrium. ⊛

   Remark. We see that

   - No hidden payoffs other than what we have in payoff matrix.
   - Both not confessing is better as a pair(if we sum the payoffs). We see that utilize sum is not maximized, which means this action is not *efficient*.
   - Confessing is the best option of the other suspect's choice. In this case, confessing is a *dominant strategy*. Where *dominant strategy* is the best response to every other strategy of the other player.

2. **Golden ball** example. Two players have two options, one is split, and another is steal. And there are x amounts of money in total. If both players choose split, then they can both get half of the money; if one chooses split and another chooses steal, the player chooses steal can get all the money while another player get nothing. If both choose steal, then they both get nothing.

   - Two players.
   - Action Set: {split, steal}.
   - Payoff: Assume the total rewards is x.

(a) (split, split): $\frac{x}{2}$ for both.
(b) (split, steal): player who plays steal gets x, steal gets 0
(c) (steal, steal): 0 for both.

The payoff matrix for Golden Ball is

		Player 2	
		split	steal
Player 1	split	$(x/2, x/2)$	$(0, x)$
	steal	$(x, 0)$	$(0, 0)$

Player 1's reasoning

- If player 2 chooses to split $\rightarrow$ since $x > \frac{x}{2}$, the best action is to steal.
- If player 2 chooses to steal $\rightarrow$ since $0 = 0$, the best action is *undetermined*.

We see that stealing is a *weakly dominant strategy*.

> **Remark.** Comparison between strictly and weakly dominant strategy:
>
> - Strictly dominant strategy: payoff is strictly higher in all cases.
> - Weakly dominant strategy: Strategy is always in the best action set. In some cases, some other strategies will give you the same payoff.

3. Consider again the prisoner's dilemma, but with different payoff matrix.

		Player 2	
		NC	C
Player 1	NC	$(-1, -1)$	$(-3, -2)$
	C	$(-2, -3)$	$(-4, -4)$

With the same analysis, we see that the equilibrium is not confessing, namely (NC, NC) is a dominant strategy equibrilium, furthermore, it's a strictly dominant strategy.

4. **Two firms** game. There are two firms competing with each other. They both have options to produce lower-priced products or upscale product. And we simply assume that their goal is to maximize the market share. We further assume that in the whole market, there are 60% of population will only buy low-priced products, and other 40% of population will only buy upscale products.

   - Two firms.
   - Action Set: $\{\text{low-priced}, \text{upscales}\}$
   - Payoff:

		firm 2	
		Lp	Uc
firm 1	Lp	$(0.48, 0.12)$	$(0.6, 0.4)$
	Uc	$(0.4, 0.6)$	$(0.32, 0.08)$

Firm 1's reasoning

- Lp is strictly dominant strategy

Firm 2's reasoning

- If firm 1 choose Lp $\to$ Uc
- If firm 1 choose Uc $\to$ Lp

We see that with common knowledge and rationality assumptions, firm 2 will assume that firm 1 will play its dominant strategy. As a result, firm 2 will play Uc. Hence, (Lp, Uc) is the equibrilium point, and it's also efficient.

5. Two firms game, but with three actions. The payoff matrix now becomes

		firm 2		
		A	B	C
firm 1	A	$(4,4)$	$(0,2)$	$(0,2)$
	B	$(0,0)$	$(1,1)$	$(0,2)$
	C	$(0,0)$	$(0,2)$	$(1,1)$

Best Response

(a) For firm 1:
- If firm 2 play $A \to$ pick A
- If firm 2 play $B \to$ pick B
- If firm 2 play $C \to$ pick C

(b) For firm 2:
- If firm 1 play $A \to$ pick A
- If firm 1 play $B \to$ pick C
- If firm 1 play $C \to$ pick B

We see that there are no dominant strategy. But (A, A) is the equilibrium point, which we will call it as *Nash equilibrium*.

Note. One may notice that we are using normal form to describe one-shot games. Contrarily, we will use so-called *Extensive form* to describe a dynamic game.

10.3 Nash Equilibrium

Definition 10.3.1. Nash Equilibrium.

- A set of action when each player doesn't have an incitement to **unilaterally deviate** is called a *Nash Equilibrium*.
- Comparison between pure or mixed strategy:
  - Pure Strategy: Pick exactly one strategy deterministically.
  - Mixed Strategy: Play strategy with some probability.

We then define a game structure mathematically in the following way.

Definition 10.3.2. Denote the set of player $\mathcal{I}$, with the number of the player being $I := |\mathcal{I}|$. Assume that each player has a finite set $\mathcal{S}_i$ of actions to choose from with size $|\mathcal{S}_i|$. Then, we can define a *strategy vector*, which denotes the action chosen by all players such that

$$s := (s_1, \ldots, s_I)$$

with dimension being I for every player i. We also define the vector of opponents' strategy s_{-i} such

that

$$s_{-i} \coloneqq (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$$

with dimension being $I - 1$.

Finally, the utility function for player i is

$$u_i \coloneqq \prod_{j=1}^{I} \mathcal{S}_j \to \mathbb{R},$$

where $u_i(s) = u_i(s_1, \ldots, s_I) = u_i(s_i, s_{-i})$.

Definition 10.3.3. Given the strategies of all other players, the subset of s_i that maximize the payoff of player i is called the *Best response correspondence*. Let

$$\mathrm{BR}_i(s_{-i})$$

denotes the *best response* for player i when the other player is s_{-i}. Note that

$$\mathrm{BR}_i(s_{-i}) \subseteq \mathcal{S}_i$$

such that

$$\mathrm{BR}_i(s_{-i}) = \arg\max_{s_i \in \mathcal{S}_i} u_i(s_i, s_{-i}).$$

Lecture 19: Nash Equilibrium

10 Nov. 12:30

As previously seen. We assume that all players have

- Common knowledge
- Rationality

Note. We also have something called *bounded rationality*.

- Some finite levels
- Not fully rational

As previously seen. Nash equilibrium.

- $\mathcal{I}$: the set of players.
- I: $|\mathcal{I}|$, the number of players.
- $\mathcal{S}_i$: player i choose strategy/action from $\mathcal{S}_i$ such that

  $$s_i \in \mathcal{S}_i$$

  with

  $$S_i = |\mathcal{S}_i|.$$

- s: $(s_1, s_2, \ldots, s_I)$, strategy chosen by all players
- Player $i \in \mathcal{I}$, then $-i \coloneqq \mathcal{I} \setminus \{i\}$: Everyone else but i.

Example.

$$\begin{aligned} i = 1: &\quad -1 = \{2, 3, \ldots, I\}, \\ i = 2: &\quad -2 = \{1, 3, \ldots, I\}, \\ &\vdots \end{aligned}$$

since

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I).$$

- Best-response correspondence(set): For player i, give the strategies of $-i$ (s_{-i} is given). Find the subset of strategies in $\mathcal{S}_i$ that give player i the highest payoff/utility.

  Intuition. $\text{BR}_i(s_{-i})$ can be though as follows: For player i, given every other players' strategies they will play, find the best response. And since there may be several strategies can achieve the optimal result, so it's should be a set.

- Utilities/Payoffs: $\forall i \in \mathcal{I}$,

  $$u_i \colon \prod_{j=1}^{I} \mathcal{S}_j \to \mathbb{R}.$$

  For every vector $s = (s_1, s_2, \ldots, s_I) \in \prod_{j \in \mathcal{I}} \mathcal{S}_j$, $u_i(s)$ is a real number.

Remark. We see that s_i, s_{-i} form the entire vector of strategies every player plays, so $u_i(s_i, s_{-i})$ is **well-defined**. Moreover,

$$\max_{s_i \in \mathcal{S}_i} u_i(s_i, s_{-i})$$

finds all strategies that attain the maximum rewards.

As previously seen. We now view the **Golden Balls** game with what we just defined. The payoff matrix is

		Player 2	
		split	steal
Player 1	split	$(x/2, x/2)$	$(0, x)$
	steal	$(x, 0)$	$(0, 0)$

For player 1, given

1. $s_{-1} = \text{steal}$:
   - $u_1(\text{split}, \underbrace{\text{steal}}_{s_{-1}}) = 0$
   - $u_1(\text{steal}, \underbrace{\text{steal}}_{s_{-1}}) = 0$

   So

   $$\arg\max_{s_1 \in \mathcal{S}_1} u_1(s_1, \text{steal}) = \{\text{split}, \text{steal}\},$$

   which is our best response correspondence.

2. $s_{-1} = \text{split}$:
   - $u_1(\text{split}, \underbrace{\text{split}}_{s_{-1}}) = \frac{x}{2}$

- $u_1(\text{steal}, \underbrace{\text{split}}_{s_{-1}}) = x$

So

$$\arg\max_{s_1 \in \mathcal{S}_1} u_1(s_1, \text{split}) = \{\text{steal}\},$$

which is our best response correspondence, and moreover, since it's unique and strictly greater than any other choices, so we call it *strictly best response*.

Definition 10.3.4. $\overline{s}_i$ is called a *strict best response* to s_{-i} if

$$u_i(\overline{s}_i, s_{-i}) > u_i(s_i, s_{-i})$$

for every $s_i \in \mathcal{S}_i \setminus \{\overline{s}_i\}$.

Remark. Obviously, a strict best response will be unique.

Definition 10.3.5. s_i is *strictly dominant* if it is the strict best response for all s_{-i}.

Definition 10.3.6. s_i is *weakly dominant* if it is one of the best response for all s_{-i}.

Remark. We see that

- In the Golden balls example, where steal is a weakly dominant strategy.
- Rationality plus common knowledge implies that if a player with dominant strategies, then the player will play them.

As previously seen. Nash equilibrium. Let $s^* = (s_1^*, s_2^*, \ldots, s_I^*)$ be the **strategy profile**. Then we call it as a *pure strategy* Nash equilibrium(NE) if

$$s_i^* \in \text{BR}(s_{-i}^*)$$

for all $i \in \mathcal{I}$.

Note. We see that

- Every agent is playing a BR to the others.
- Utility(Return/Payoff/Reward) is the incentive for agents. Setting everyone else at s_{-i}^*, there is no incentive for player i to deviate(not necessary to change action). This implies there are no *unilateral dominants* are possible.

$$\forall_{i \in \mathcal{I}} \forall_{s_i \in \mathcal{S}_i} u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

As previously seen. There are two different types of strategies:

- Pure strategy: Fixing an action for every player.
- Mixed strategy: A randomized action is played by every player.

Remark. We see that

1. Pure strategy Nash Equilibrium needs not exist for every game.
2. Even if they exist, they need not be unique.
3. If there are multiple equilibrium, then which one gets played is a tough question and usually involves external quantities(outside information).

10.3.1 Pure Strategy Nash Equilibrium

We first introduce the concept of *Coordination Games*. Coordination games are games such that players get a higher payoff by working together(taking the same action).

Example. We first see some coordination game examples.

1. **Shaking hands**. There are two men want to shake their hands. They can either position their hand up(U) or down(D).

		player 2	
		U	D
player 1	U	$(1,1)$	$(0,0)$
	D	$(0,0)$	$(1,1)$

   We see that (U,U) and (D,D) are pure-strategy Nash Equilibrium.

2. **Battle of the sexes**. A man and a woman are going to a date. The man prefer to see football(F) while the woman prefer to go to theater(T). If their opinion are not equal, then they can't go to anywhere, so their fulfillment(payoff) will both be 0.

		woman	
		T	F
man	T	$(1,2)$	$(0,0)$
	F	$(0,0)$	$(2,1)$

   We see that (T,T) and (F,F) are pure-strategy Nash Equilibrium.

3. **Stay Hunt**.

		player 1	
		S	H/R
player 2	S	$(4,4)$	$(0,3)$
	H/R	$(3,0)$	$(3,3)$

   We see that (S,S) and (H,H) are pure-strategy Nash Equilibrium.

We also have so-called *Anti-Coordination Games*. Such games mean that one player prefers to coordinate, and the other does not.

Example. Again, we first see some examples.

1. **Rock, paper, scissors**. This is an example that a game without pure Nash Equilibrium. The payoff matrix is

		player 2		
		Rock	Paper	Scissors
player 1	Rock	$(0,0)$	$(-1,\underline{+1})$	$(\underline{+1},-1)$
	Paper	$(\underline{+1},-1)$	$(0,0)$	$(-1,\underline{+1})$
	Scissors	$(-1,\underline{+1})$	$(\underline{+1},-1)$	$(0,0)$

And it's clear that no matter what we fixed a strategy for one player first, another player will have incentive to deviate right after another player changes its strategy.

2. **Hawk-Dove** game. Two neighboring countries are likely going to fight. If one choose Hawk, then that country will fight; otherwise if choosing Dove, then that country will not going to fight and will not going to fight back either.

		Country 2	
		H	D
Country 1	H	$(0,0)$	$(5,1)$
	D	$(1,5)$	$(3,3)$

Problem. Is (D,D) a Nash Equilibrium?

Answer. False. Since if player 2 is playing D. We see that $u_1(H,D)=5$, $u_1(D,D)=3$. Then player 1 has a deviation to get a higher payoff. ⊛

Problem. Is (H,D) a Nash Equilibrium?

Answer. True. Since

- Player 1: H is the best response
- Player 2: Player 1 is H. Then $u_2(H,H)=0$, and $u_2(H,D)=1$.

We see that neither player 1 nor player 2 has an incentive to deviate. So (D,H)(also (H,D)) is a Nash Equilibrium. ⊛

3. **Matching Pennies**. This is a simple example for *attack-defense* game. Suppose there are two people each hold a penny and simultaneously choose whether to show heads(H) or tails(T) on their penny. Player 1 loses his penny to Player 2 if they match; Player 1 wins Player 2's penny if they don't match.

		Player 2	
		H	T
Player 1	H	$(-1,1)$	$(1,-1)$
	T	$(1,-1)$	$(-1,1)$

We see that there are no pure strategy Nash Equilibrium. So we have no idea what strategy each player will play.

Note. An *attack-defense* game can be simply described as follows. There are two players, one is attacker and another is defender. Attacker has two actions, they are attack with strategy A or B, respectively. Correspondingly, defender also has two actions to choose from, namely defend against A or defend against B. Clearly, if defender correctly defend attacker's attack in terms of strategy attack chooses, then defender gets a higher payoff, otherwise the attacker gets a higher payoff.

> **Remark.** This is so-called a *Zero-Sum Game*. One player's gain is exactly another player's loss.

> **Remark.** From matching pennies games example, we see that there are no Nash Equilibrium. This is due to the fact that all player will play their strategy in a deterministic way, if the best response exists. And if not, then we have no idea what they will play in this case, which is hard to analyze. Now if we allow some randomness, then by John Nash, he states that with such set up, the Nash Equilibrium always exists.

10.3.2 Mixed Strategy Nash Equilibrium

Follows the intuition of matching pennies game, we now develop the concept so-called *mixed strategy*. Essentially, we are just randomizing one's strategies such that the randomization is done independently. This will bring the game structure with much more interesting properties.

> **As previously seen.** The reason why we call it *mixed strategies* is initially, we only have distinct strategy to choose from, but now, with probability, we have some kind of **mixed** strategy between choices. In matching pennies game example, we have strategies which mixes between H and T.

For player i, denote $\overrightarrow{P}_i$ as a probability distribution on $\mathcal{S}_i$(PMF) such that

$$\{s_{i1}, s_{i2}, s_{i3}, s_{i4}\} = \mathcal{S}_i,$$

and $S_i = 4$. More specifically, $\overrightarrow{P}_i$ looks like

$$(P_{i1}, P_{i2}, P_{i3}, P_{i4})$$

with all entries non-negative and their sum is 1. There are lots of example for such P_i. For example,

- $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$: uniformly mixed.
- $(1, 0, 0, 0)$: pure strategy.

We can then define *probability simplex* $\Delta(\mathcal{S}_i)$, which is equal to

$$\left\{ (P_{i1}, P_{i2}, \ldots, P_{iS_i}) : P_{ij} \geq 0 \wedge \sum_{j=1}^{S_i} P_{ij} = 1 \quad \forall j = 1, \ldots, S_i \right\}.$$

- 2 actions.

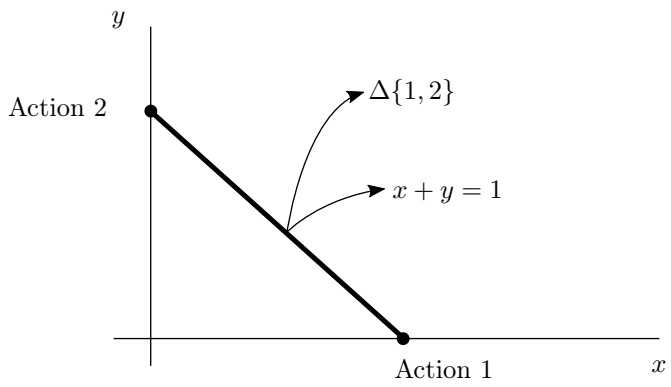


Figure 10.1: Probability Simplex with 2 Actions.

- 3 actions.

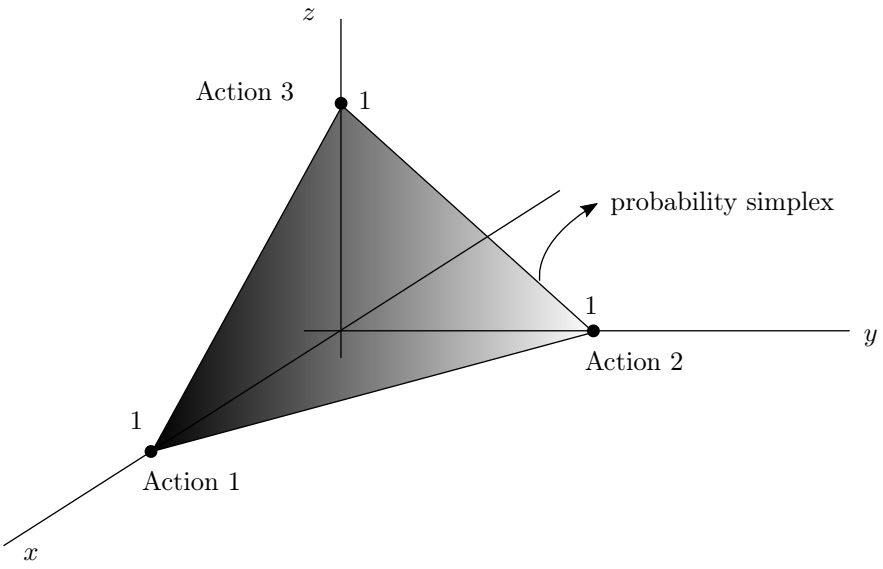


Figure 10.2: Probability Simplex with 3 Actions.

Intuition. This is just a set of probability distributions over $\mathcal{S}_i$, but with a good geometry representation.

Players' actions are enhanced to picking probability distribution(adding uncertainty to your choice).

Problem. How to evaluate payoff?

Answer. Let

$$u_i(\overrightarrow{P}_i, \overrightarrow{P}_{-i}) = \mathbb{E}_{\overrightarrow{P}_1 \times \overrightarrow{P}_2 \times \ldots \times \overrightarrow{P}_I} \left[u_i \left(\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_I \right) \right],$$

such that $\overline{S}_i$ is a random variable that takes values in $\mathcal{S}_i$ with distribution $\overrightarrow{P}_i$, and they are mutually independent. This is further equal to

$$\sum_{j_1 \in \mathcal{S}_1} \sum_{j_2 \in \mathcal{S}_2} \cdots \sum_{j_I \in \mathcal{S}_I} P_{1j_1} \times P_{2j_2} \times \ldots \times P_{Ij_I} \times u_i \left(\overline{S}_{1j_1}, \overline{S}_{2j_2}, \ldots, \overline{S}_{Ij_I} \right).$$

⊛

Example. Recall the last game we consider, namely the **Matching Pennies** game.

		Player 2	
		H	T
Player 1	H	$(1,-1)$	$(-1,1)$
	T	$(-1,1)$	$(1,-1)$

with $P_1 = (\frac{3}{4}, \frac{1}{4})$ and $P_2 = (\frac{1}{3}, \frac{2}{3})$. Then

$$\begin{aligned} u_1(P_1, P_2) &= \frac{3}{4} \times \frac{1}{3} u_1(H,H) + \frac{3}{4} \times \frac{2}{3} u_1(H,T) + \frac{1}{3} \times \frac{1}{3} u_1(T,H) + \frac{1}{4} \times \frac{2}{3} u_1(T,T) \\ &= \frac{3}{12} - \frac{6}{12} - \frac{1}{12} + \frac{2}{12} \\ &= -\frac{1}{6}. \end{aligned}$$

Also, $u_2(P_1, P_2)$ is $\frac{1}{6}$ from Zero-Sum setting.

We can now combine the probability set up with the definition of the best response, which gives us

$$\mathrm{BR}_i(\overrightarrow{P}_{-i}) = \left\{ \overrightarrow{P}_i \in \Delta(\mathcal{S}_i) \colon u_i(\overrightarrow{P}_i, \overrightarrow{P}_{-i}) = \max_{\hat{P}_i \in \Delta(\mathcal{S}_i)} u_i(\hat{P}_i, \overrightarrow{P}_{-i}) \right\}.$$

Also, for Nash Equilibrium, let P^* be the same probability distribution for all $i \in \mathcal{I}$, then

$$P_i^* \in \mathrm{BR}_i(P_{-i}^*) \Leftrightarrow \forall_{P_i \in \Delta(\mathcal{S}_i)} u_i(P_i^*, P_{-i}^*) \geq u_i(P_i, P_{-i}^*).$$

Theorem 10.3.1. Nash's Existence Theorem. Every finite game(finite action) has at least one mixed strategy Nash Equilibrium.

Remark. This is a theorem given by Nash by using fixed-point theorem.

Lecture 20: Mixed strategy

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As previously seen. Recall the game matching pennies. Assume that Player 2 plays $(\frac{1}{2}, \frac{1}{2})$. Now, if player 1 plays

- $(1,0)$: Utility for player 1 is $u_1(H, P_2^*) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot -1 = 0$.
- $(0,1)$: Utility for player 1 is $u_1(T, P_2^*) = \frac{1}{2} \cdot -1 + \frac{1}{2} \cdot 1 = 0$.

Hence, the utility of player 1 is just

$$u_1(P_1, P_2^*) = P_1(H) u_1(H, P_2^*) + P_1(T) u_1(T, P_2^*) = 0.$$

Therefore, any mixed strategy is the best response, so we just use $(\frac{1}{2}, \frac{1}{2})$ for player 1, and the same is true for player 2. In all, we have

$$P_1^* := \left(\frac{1}{2}, \frac{1}{2}\right), \quad P_2^* := \left(\frac{1}{2}, \frac{1}{2}\right)$$

as the Nash Equilibrium.

Problem. What if player 2 was laying $(\frac{2}{3}, \frac{1}{3})$?

Answer. The best response for player 1 is to always play H. ⊛

work this out

Problem. The mixed strategies are everywhere in real-life. But how can we understand mixed strategies?

Answer. We see that

- Actions are distributions or beliefs.
- In real life, people do randomize-serving in tennis, bluffing in cards game, openings in chess, etc.
- Variations in the population and lack of knowledge of how the other person will play.

⊛

Example. Again, we first see an example.

- **Football**. Let player 1 be offense-side, and player 2 be defense-side. They both have two options, one is run(or to defense against run), denotes by R, another is pass(or defense against pass), denotes by P.

		Player 2	
		R	P
Player 1	R	$(0, 0)$	$(5, -5)$
	P	$(10, -10)$	$(0, 0)$

  we then see that there is a Nash Equilibrium such that player 1 plays $(\frac{2}{3}, \frac{1}{3})$, while player 2 plays $(\frac{1}{3}, \frac{2}{3})$.

10.3.3 Principle of Indifference

For player i, let $P_i \in \Delta(\mathcal{S}_i)$. Now, let *support* of P_i to be the actions with strictly positive probability, which is just

$$\{s_i \in \mathcal{S}_i \colon P_i(s_i) > 0\} \subseteq \mathcal{S}_i.$$

Example. For example, let $\{a, b, c\} = \mathcal{S}_i$. Then

- $P_i = (1, 0, 0)$: the support is $\{a\}$
- $P_i = (\frac{1}{6}, \frac{5}{6} - \frac{1}{10^6}, \frac{1}{10^6})$: the support is $\{a, b, c\}$
- $P_i = (0, \frac{1}{2}, \frac{1}{2})$: the support is $\{b, c\}$

Note. In the second case, we call this a *fully mixed*.

Theorem 10.3.2. Principle of Indifference. For a Nash Equilibrium P^*, for every player i, if

$$\forall s_i^j, s_i^k \in \text{support}(P_i^*),$$

we have

$$u_i(s_i^j, P_{-i}^*) = u_i(s_i^k, P_{-i}^*) = u_i(P_i^*, P_{-i}^*).$$

Intuition. In other words, given the strategies of the opponents, player i is *indifferent* between the strategies in the support(P_i^*) because they yield the same payoff.

Proof. Let $s_i^j, s_i^k \in \text{support}(P_i^*)$. Then

$$P_i^*(s_i^j) > 0 \wedge P_i^*(s_i^k) > 0.$$

Now, denotes γ as

$$\gamma = \min\left(P_i^*(s_i^j), P_i^*(s_i^k)\right) > 0,$$

and let $\epsilon \coloneqq [-\gamma, \gamma]$. We can then construct a new mixed strategy $\widetilde{P}_i^\epsilon$ for player i as follows. For s_i^j and s_i^k

$$\begin{aligned}\widetilde{P}_i^\epsilon(s_i^j) &= P_i^*(s_i^j) - \epsilon \\ \widetilde{P}_i^\epsilon(s_i^k) &= P_i^*(s_i^k) + \epsilon,\end{aligned}$$

with

$$\underset{s_i \in \mathcal{S}_i \setminus \{s_i^j, s_i^k\}}{\forall} \widetilde{P}_i^\epsilon(s_i) = P_i^*(s_i),$$

which means that the other actions remain the same. Since P^* is a Nash Equilibrium, so we have

$$u_i\left(P_i^*, P_i^*\right) = \max_{P_i \in \Delta(\mathcal{S}_i)} u_i(P_i, P_{-i}^*) \geq u_i(\widetilde{P}_i^\epsilon, P_{-i}^*).$$

Furthermore, the left-hand side is

$$\begin{aligned}u_i\left(P_i^*, P_{-i}^*\right) = P_i^*(s_i^j)u_i(s_i^j, P_{-i}^*) + P_i^*(s_i^k)u_i(s_i^k, P_{-i}^*) \\ + \sum_{s_i \in \mathcal{S}_i \setminus \{s_i^j, s_i^*\}} P_i^*(s_i)u_i(s_i, P_{-i}^*),\end{aligned}$$

and the right-hand side is

$$\begin{aligned}u_i\left(\widetilde{P}_i^\epsilon, P_{-i}^*\right) = \widetilde{P}_i^\epsilon(s_i^j)u_i(s_i^j, P_{-i}^*) + \widetilde{P}_i^\epsilon(s_i^k)u_i(s_i^k, P_{-i}^*) \\ + \sum_{s_i \in \mathcal{S}_i \setminus \{s_i^j, s_i^*\}} P_i^*(s_i)u_i(s_i, P_{-i}^*).\end{aligned}$$

Then we see that the inequality is

$$P_i^*(s_i^j)u_i(s_i^j, P_{-i}^*) + P_i^*(s_i^k)u_i(s_i^k, P_{-i}^*) \geq \widetilde{P}_i^\epsilon(s_i^j)u_i(s_i^j, P_{-i}^*) + \widetilde{P}_i^\epsilon(s_i^k)u_i(s_i^k, P_{-i}^*)$$

since the summation part is the same. Rearranging, we have

$$\epsilon u_i(s_i^j, P_{-i}^*) \geq \epsilon u_i(s_i^k, P_{-i}^*).$$

Since $\epsilon \in [-\gamma, \gamma]$, we further have

$$u_i(s_i^j, P_{-i}^*) = u_i(s_i^k, P_{-i}^*) = u.$$

Hence,

$$u_i(P_i^*, P_i^*) = \sum_{s_i \in \text{support}(P_i^*)} P_i^*(s_i) \underbrace{u_i(s_i, P_{-i}^*)}_{u} = u.$$

The last equality follows since

$$\sum_{s_i \in \text{support}(P_i^*)} P_i^*(s_i) = 1.$$

■

Now, we show a general method to find all Nash Equilibrium in a 2 players and 2 actions game. Let the payoff matrix be like

		Player 2	
		L	R
Player 1	U	$(a_{U,L}, b_{U,L})$	$(a_{U,R}, b_{U,R})$
	D	$(a_{D,L}, b_{D,L})$	$(a_{D,R}, b_{D,R})$

Let

- Player 1 plays strategy $(p, 1-p)$ with $p \in [0,1]$
- Player 2 plays strategy $(q, 1-q)$ with $q \in [0,1]$

We then find all the best-response of each player in a parametric form. Says player 1 with $p^*(q)$, while player 2 with $q^*(p)$, where they can be a set. We see that

$$p^*(q) = \arg\max_{\widetilde{p} \in [0,1]} u_1\left((\widetilde{p}, 1-\widetilde{p}), (q, 1-q)\right).$$

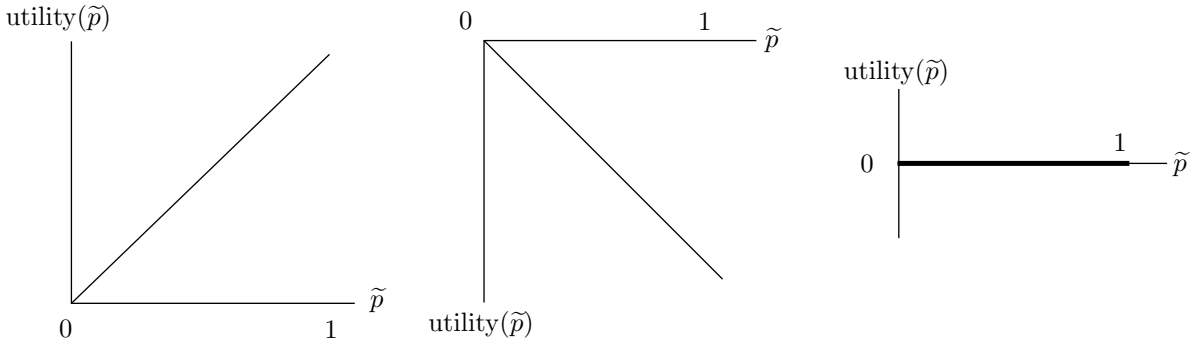
Then

$$\begin{aligned}
&u_1\left((\widetilde{p}, 1-\widetilde{p}), (q, 1-q)\right) \\
=&\widetilde{p}\left(q \cdot a_{U,L} + (1-q)a_{U,R}\right) + (1-\widetilde{p})\left(q \cdot a_{D,L} + (1-q)a_{D,R}\right) \\
=&\widetilde{p}\left([q \cdot a_{U,L} + (1-q)a_{U,R}] - [q \cdot a_{D,L} + (1-q)a_{D,R}]\right) + (q \cdot a_{D,L} + (1-q)a_{D,R})
\end{aligned}$$

Now, denotes C as

$$C \coloneqq [q \cdot a_{U,L} + (1-q)a_{U,R}] - [q \cdot a_{D,L} + (1-q)a_{D,R}],$$

there are three cases for C:



$C > 0$ $C < 0$ $C = 0$

We see that

- $C > 0$: maximizes in $\widetilde{p} = 1$

- $C < 0$: maximizes in $\widetilde{p} = 0$
- $C = 0$: maximizes in $\widetilde{p} = [0, 1]$

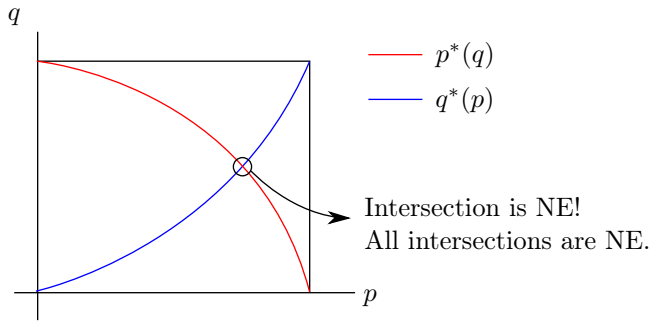
Then we denote

$$\arg\max_{\widetilde{p}\in[0,1]} u_1\left((\widetilde{p}, 1-\widetilde{p}), (q, 1-q)\right) =: p^* := \begin{cases} 1, & \text{if } C > 0 \\ 0, & \text{if } C < 0 \\ [0,1], & \text{if } C = 0 (\text{Indifference}). \end{cases}$$

Use the similar calculations to find $q^*(p)$ using the best response. We conclude that

$$(\overline{p}, \overline{q}) \text{ is a Nash Equilibrium} \Leftrightarrow \overline{p} \in p^*(\overline{q}) \wedge \overline{q} \in q^*(\overline{p}),$$

namely they are the best response of each other.



- We come back to the matching pennies games. We see that

$$p^*(q) = \begin{cases} 1, & \text{if } q > \frac{1}{2} \\ [0,1], & \text{if } q = \frac{1}{2} \\ 0, & \text{if } q < \frac{1}{2} \end{cases}, \quad q^*(p) = \begin{cases} 1, & \text{if } p > \frac{1}{2} \\ [0,1], & \text{if } p = \frac{1}{2} \\ 0, & \text{if } p < \frac{1}{2}. \end{cases}$$

After plotting, we see that $(\frac{1}{2}, \frac{1}{2})$ is the only intersection, so it's the only Nash Equilibrium.

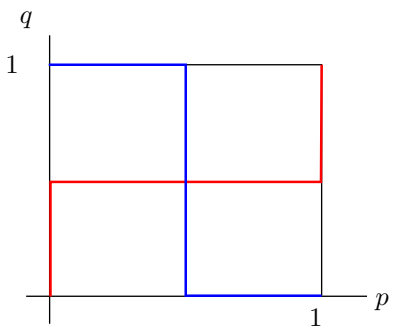


Figure 10.3: The diagram for matching pennies game

- We also look at the football game example. We see that for $p^*(q)$, we have

– R: utility is
$$q(0) + (1-q)5 = 5 - 5q.$$
– P: utility is
$$q(10) + (1-q)0 = 10q.$$

Hence, the difference is $5 - 5q - 10q = 5 - 15q$, hence

$$5 - 15q \Rightarrow \begin{cases} \text{Difference } > 0, p^*(q) = 1, & 5 > 15q \Rightarrow q < \frac{1}{3} \\ \text{Difference } < 0, p^*(q) = 0, & 5 < 15q \Rightarrow q > \frac{1}{3} \\ \text{Difference } = 0, p^*(q) = [0,1], & 5 = 15q \Rightarrow q = \frac{1}{3}. \end{cases}$$

As for $q^*(p)$, we have

– R: utility is
$$p(0) + (1-p)(-10) = -10 + 10p.$$
– P: utility is
$$p(-5) + (1-p)0 = -5q.$$

Hence, the difference is $-10 + 10p - (-5p) = -10 + 15p$, hence

$$-10 + 15p \Rightarrow \begin{cases} \text{Difference } > 0, q^*(p) = 1, & 5 > 15q \Rightarrow p > \frac{2}{3} \\ \text{Difference } < 0, q^*(p) = 0, & 5 < 15q \Rightarrow p < \frac{2}{3} \\ \text{Difference } = 0, q^*(p) = [0,1], & 5 = 15q \Rightarrow p = \frac{2}{3}. \end{cases}$$

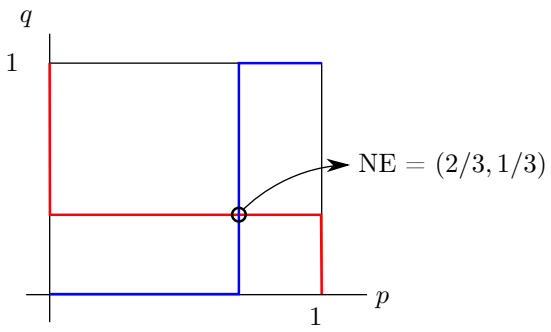


Figure 10.4: The diagram for football game.

- We now look at a new game. Consider the **Market entering** game. Let the payoff matrix being

		Player 2	
		E	D
Player 1	E	$(-1,-1)$	$(3,0)$
	D	$(0,3)$	$(0,0)$

We immediately see that (E,D) and (D,E) are pure Nash Equilibrium. And since we always have odd number of Nash Equilibrium, hence there are one more mixed-strategy Nash Equilibrium. By using the principle of indifference, we see that

$$\begin{cases} u_1(E,\overline{q}) & = \overline{q}(-1) + (1-\overline{q})3 = 3 - 4\overline{q} \\ u_1(D,\overline{q}) & = \overline{q}(0) + (1-\overline{q})0 = 0, \end{cases}$$

so we have

$$u_1(E, \overline{q}) = u_1(D, \overline{q})$$

by the principle of indifference. This implies $\overline{p} = \frac{3}{4}$.

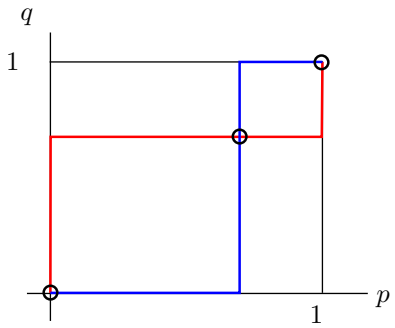


Figure 10.5: The diagram for market entering Game. There are 3 N.E.

Lecture 21: Bayesian Game

17 Nov. 12:30

As previously seen. We look back some games and introduce another measure of Nash Equilibrium.

- Coordination game. The payoff matrix is defined as

		Player 2	
		S	H
Player 1	S	$(5,5)$	$(0,3)$
	H	$(3,0)$	$(3,3)$

  As we can see, there are two Nash Equilibrium (S, S) and (H, H) with payoffs $(5, 5)$ and $(3, 3)$ respectively. We can further consider so-called **Social welfare**, then in order to maximize the sum of payoffs, one will choose to play (S, S) with social welfare as 10 rather than 6.

- **Prisoner's Dilemma**.

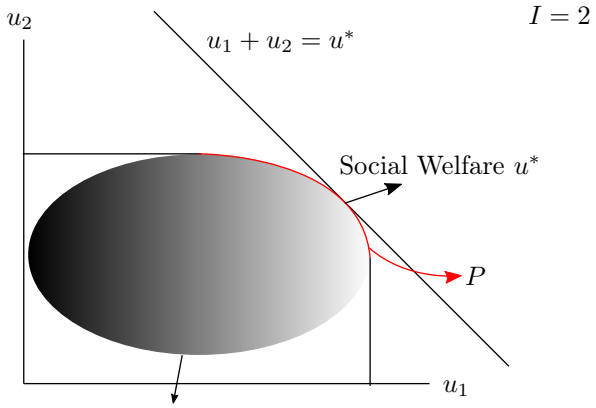
		Player 2	
		NC	C
Player 1	NC	$(-1,-1)$	$(-10,0)$
	C	$(0,-10)$	$(-4,-4)$

  We see that (C, C) is the only Nash Equilibrium, but $(\mathrm{NC}, \mathrm{NC})$ is the social welfare maximization, which is not in the Nash Equilibrium.

Definition 10.3.7. A Nash Equilibrium reaches *Social Welfare* if it's the maximizer of the **sum of the payoffs** among all other states.

10.3.4 Operating Points

Let $i \in \mathcal{I}$ be the set of players, and P_i be the corresponding distribution of strategy on player i. Then we see the so-called *operating points*.



feasible region: All possible payoff vectors

Let P denotes the set of operating points. We see that we cannot increase the payoff of one agent without decreasing the payoff of at least one other agent. From here, we have the following criteria for *Pareto Optimal*.

Definition 10.3.8. A solution is called *Pareto Optimal* if no individual or preference criterion can be better off without making at least one individual or preference criterion worse off or without any loss thereof.

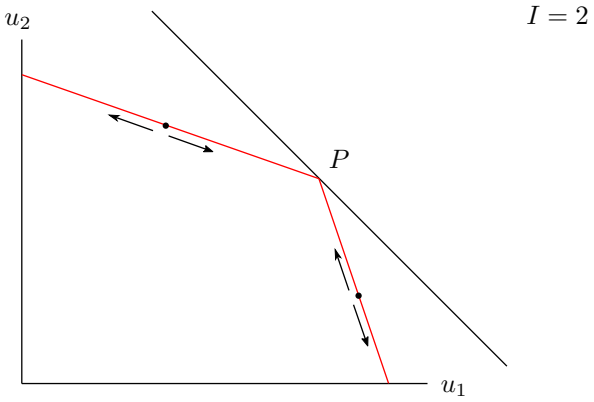


Figure 10.6: Parteo Optimality

Note. For either social welfare maximizing or Pareto operating point,

- Nash Equilibrium need not achieve either goal.
- Social welfare maximizing points are always Pareto optimal.

Remark. We can (re)designing games such that some nice objective is additive, then social welfare maximization can be achieved.

- The goal is that any Nash Equilibrium should be social welfare maximizing.

Until this point, we only study the games are complete and symmetric information - Every agent knows actions, utility function of everyone(common knowledge assumption). Now, we can look at some more complicated case such that we have an Incomplete and/or asymmetric information setting.

10.4 Bayesian Game

This is a framework for asymmetric/incomplete games such as auctions. And since we need to use *Bayesian Theorem* to analysis such games, so it gets its name.

An important difference is that in Bayesian games, nation is also a player. Further, randomness decides the state of everyone and people may know their state but not of the others.

Note. Randomness is different from mixed strategies. This happens **at the beginning**.

10.4.1 Basic setup

Let $\mathcal{I}$ denotes the set of players, and $I = |\mathcal{I}|$. Let each player i has a type $\tau_i \in T_i$ chosen with distribution π_i such that

$$\mathbb{P}(\tau_i = t_i) = \pi_i(t_i).$$

And also, we have the set of actions $\mathcal{S}_i$, and the set of feasible action

$$c_i(t_i) \in \mathcal{S}_i.$$

Furthermore, the utility of player i is

$$u_i(t_1, s_1, t_2, s_2, \ldots, t_I, s_I),$$

where we need types and action of all players to determine this. Also, we have

$$\vec{t} \coloneqq (t_1, \ldots, t_I), \quad \vec{s} \coloneqq (s_1, \ldots, s_I),$$

with the common notation for $-i$, namely

$$\vec{t}_{-i} \coloneqq (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_I), \quad \vec{s}_{-i} \coloneqq (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I).$$

We see that player i has to choose a strategy $\sigma_i \colon T_i \to \mathcal{S}_i$ such that

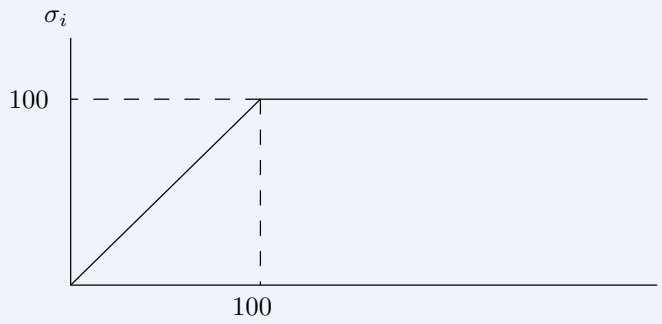
$$\forall t_i \in T_i, \text{ choose } \sigma_i(t_i) \in c_i(t_i) \subseteq \mathcal{S}_i.$$

Intuition. We are simply picking functions as strategies.

Example. Let $T_i = \mathbb{R}_+ = [0, +\infty)$ be non-negative real number. Let

$$\sigma_i(t_i) = \min(100, t_i),$$

and $\mathcal{S}_i = \mathbb{R}_+$.



Then we can have so-called **Independent types model**. Let τ_i, $i = 1, \ldots, I$ be the set of mutually independent variables, then

$$\mathbb{P}\left(\tau_1 = t_1, \ldots, \tau_I = t_I\right) = \prod_{i=1}^{I} \pi_i(t_i).$$

We see that we can express all games we have previously seen like follows. Let $T_i = \{i\}$, and $c_i(i) = \mathcal{S}_i$ with type being $\pi_i(i) = 1$ such that

$$u_i(1, s_1, 2, s_2, \ldots, I, s_I) = u_i(s_1, s_2, \ldots, s_I).$$

Now we define the notions of incomplete information.

- Ex-Ante: No player has information about anyone's types but only the distribution information where strategy is chosen. Since we don't know the exact utility, hence we simply take the expected value for the payoff based on the strategy functions:

  $$u_i(\sigma_1, \ldots, \sigma_I) = \mathbb{E}_{\tau_1, \ldots, \tau_I}\left[u_i(\tau_1, \sigma_1(\tau_1), \ldots, \tau_I, \sigma_I(\tau_I))\right].$$

  Then the Nash Equilibrium $(\sigma_1^*, \ldots, \sigma_I^*)$ such that

  $$\underset{i \in \mathcal{I}}{\forall} \underset{\sigma_i}{\forall} u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*).$$

  This type of information mechanism can be summarized as follows.

  1. Given player $i, \ldots, I$.
  2. $\sigma_1, \ldots, \sigma_I$ are chosen: Each player picks strategy σ_i.
  3. $\tau_1, \ldots \tau_I$ are realized: Nation chooses types for each player.
  4. $\underset{i}{\forall}\ s_i = \sigma_i(\tau_i)$ is chosen: Actions produced, and the game is realized.
  5. $u_i(\tau_1, s_1, \ldots, \tau_I, s_I)$, namely the payoff is received.

- Interim: Players get to know their types since Nation play first, and they can reason based on that. So we see that player i knows τ_i but not τ_{-i}. Then $u_i(t_i, \sigma_i(t_i), \sigma_{-i})$ can be described as

  $$\mathbb{E}_{\tau_{-i}}\Big[u_i(\underline{t_i, \sigma_i(t_i)}, \tau_1, \sigma_1(\tau_1), \ldots, \underline{\tau_{i-1}, \sigma_{i-1}(\tau_{i-1}),} \\ \underline{\tau_{i+1}, \sigma_{i+1}(\tau_{i+1})}, \ldots, \tau_I, \sigma_I(\tau_I))\Big].$$

  This type of information mechanism can be summarized as follows.

  1. Given player $i, \ldots, I$.
  2. $\tau_1, \ldots \tau_I$ are realized: Nation chooses types for each player.
  3. $\sigma_1, \ldots, \sigma_I$ are chosen: Each player picks strategy σ_i.
  4. $\underset{i}{\forall}\ s_i = \sigma_i(\tau_i)$ is chosen: Actions produced, and the game is realized.

5. $u_i(\tau_1, s_1, \ldots, \tau_I, s_I)$, namely the payoff is received.

- Ex-Post: Every player gets to know everyone's type.

$$u_i(t_1, \sigma_1(t_1), \ldots, t_I, \sigma_I(t_I)).$$

We will focus on **Interim** setting. We first see the example of auctions, which model this well.

Think about the best response. Suppose σ_{-i} is fixed. Then for all $s_i \in \mathcal{S}_i$, $u_i(t_i, s_i, \sigma_{-i})$ is defined.

$$\arg\max_{s_i \in \mathcal{S}_i} u_i(t_i, s_i, \sigma_{-i})$$

for every t_i, $\mathrm{BR}_i(\sigma_{-i}) = \overline{\sigma}_i(\sigma_{-i})$. Then we call that

$$(\sigma_1^*, \ldots, \sigma_I^*)$$

is a (Bayes interim) Nash Equilibrium if

$$\underset{i \in \mathcal{I}}{\forall} \sigma_i^* \in \overline{\sigma}_i(\sigma_{-i}).$$

We now use **auctions** to see how it applies.

Chapter 11

Auctions

We now introduce a specific kind of game, auction. Rather than the simple set up we have seen, the auction is more complex and interesting. We'll see how can we extend what we have discussed in Game Theory into this set up. Firstly, we see some examples.

Example. There are mainly four types of auction:

1. First-price auction(Sealed bid): Each player bids an amount for the item (how much the player is willing to pay), and the sellers selects the highest bidder and charges him/her the price that was bid by him/her and give the item. Others pay nothing, receive nothing.
2. Dutch auction(Descending price auction): Seller announces a price(high price). If no players raise hands, seller drops price until the first time some player ha hand up. Item sold to the player at that price.

   Ties breaking:

   - Random order.
   - Choose some fixed order(specified earlier).

Remark. The first two auctions basically implement the same result.

Lecture 22: Auctions

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Example. There are more types of auction.

3. Second-price auction(Sealed bid):
   (a) Sort the bids from the highest
   (b) Find the highest bidder
   (c) Give the good to this bidder but charge him/her the second-highest bid.

   Ties breaking:

   - Random order.
   - Choose some fixed order(specified earlier).

Example. For example, we have

	A	B	C	D
b_i	10	2	3	7

After sorted:

	A	D	C	B
b_i	10	7	3	2

We then sell the good to A with price 7.

4. English auction(Ascending price auction):
   (a) Start from price 0
   (b) Keeping increasing price until only one left
   (c) Sell the good to the person remaining and at the price at which everyone else drops out.

Example. For example, we start with 0, and everyone wants the good. Then we slightly increase the price, until:

(a) Price is 2 or $(2+\epsilon)$: A, D and C want the good, B drops out.
(b) Price is 3 or $(3+\epsilon)$: A and D want the good, C drops out too.
(c) Price is 7 or $(7+\epsilon)$: A want the good, D drops out too.

Then A gets the good with price being 7.

11.1 Order Statistics

For all kinds of auctions we mentioned, we see that to analyze auctions, it'll involve some kind of order of bidding. In the content of auctions, we often refer this as *Order Statistics*.

Let $x_1, x_2, \ldots, x_n$ be i.i.d. and non-negative, continuous valued random variables. Let $F_X(\cdot)$ be the CDF, and $f_x(\cdot)$ be the probability density function.

Example. Consider

1. $X = \text{uniform}([0,1])$:

$$F_X(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ \min(x,1) & \text{if } x > 1 \end{cases}$$

with

$$f_X(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{if } x > 1. \end{cases}$$

2. $\text{Exp}(\lambda)$:

$$F_X(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

with

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

The idea is simple, we first sort $x_1, x_2, \ldots, x_n$ in decreasing order and denotes

$$y_1, y_2, \ldots, y_n$$

such that

$$\begin{aligned} y_1 &= \max\{x_1, \ldots, x_n\} \\ y_2 &= \max\{x_1, \ldots, x_n\} \setminus \{y_1\} \\ y_3 &= \max\{x_1, \ldots, x_n\} \setminus \{y_2, y_3\} \\ &\vdots \end{aligned}$$

Then we have

$$\max\{x_1, \ldots, x_n\} = y_1 \geq y_2 \geq \ldots \geq y_n = \min\{x_1, \ldots, x_n\}.$$

Now, we want to find out $\mathbb{E}[y_1]$, which is just

$$\mathbb{E}[y_1] = \int_0^\infty y f_{y_1}(y)\,\mathrm{d}y$$

where f_{y_1} is the probability density function of y_1. Since $f_{y_1}(y) = \frac{\mathrm{d}}{\mathrm{d}y}(F_{y_1}(y))$, where

$$\begin{aligned} F_{y_1}(y) &= \mathbb{P}(y_1 \leq y) \\ &= \mathbb{P}(\max\{x_1, \ldots, x_n\} \leq y) \\ &= \mathbb{P}(x_1 \leq y, x_2 \leq y, \ldots, x_n \leq y) \\ &= \prod_{i=1}^n \mathbb{P}(x_i \leq y) \\ &= \prod_{i=1}^n F_{x_i}(y) \\ &= (F_x(y))^n, \end{aligned}$$

where we have used the properties like independence and identically distributed(i.i.d.) assumptions.

Further, since we now know

$$f_{y_1}(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left((F_x(y))^n\right) = n(F_x(y))^{n-1}\frac{\mathrm{d}}{\mathrm{d}y}F_x(y) = n(F_x(y))^{n-1} f_x(y),$$

we have

$$\mathbb{E}_{y_1}[=] \int_0^\infty n y (F_x(y))^{n-1} f_x(y)\,\mathrm{d}y.$$

Example. We now revisit the examples we just show.

1. Let $F_x(\cdot) \sim \text{uniform}([0,1])$. Then

$$\mathbb{E}_{y_1}[=] \int_0^1 n y y^{n-1}\,\mathrm{d}y = n\int_0^1 y^n\,\mathrm{d}y = \left.\frac{n y^{n+1}}{n+1}\right|_0^1 = \frac{n}{n+1}.$$

   Hence, $\mathbb{E}_{x_1}[=] \frac{1}{2}$.

2. Exercise - Try exponential.

Remark. For the uniform case, the expected value of the k^{th} element is exactly

$$\frac{k}{n+1}$$

as one can check.

11.2 Game-Form for Auction

We relate auctions with game as we discussed before. Given the types for player $i \in \mathcal{I}$ being i.i.d. such that

$$t_1, t_2, \ldots, t_I,$$

and also for the valuations such that

$$v_1, v_2, \ldots, v_I.$$

Note. We see that

- For simplicity, without specifying, we assume they are distributed in

$$\text{uniform}([0,1]).$$

- The valuation v_i is taken as the satisfaction in dollar amount from buying the good.

Then, the utility function is determined based on specific forms of auction. We now see some explicit examples.

11.3 Second-price Auction

We first analyze second-price auction, since it's easier to analyze. Recall the Interim Setting, namely each agent know its valuation but only the distribution of the valuation of others. Now, let

$$\sigma_i \colon [0,1] \to [0,1] \qquad v_i \mapsto \sigma_i(v_i)$$

where σ_i is just the strategy of bidding. Then the utility is

$$\mathbb{E}_{-i}\left[u_i(\underbrace{v_i, \sigma_i(v_i)}_{\text{known for } i}, \underbrace{v_{-i}, \sigma_{-i}(v_{-i})}_{\text{random}})\right].$$

The goal is to choose $\sigma_i(\cdot)$ such that expected utility is maximized for each v_i assuming σ_{-i} is fixed. Then,

$$\begin{aligned} & u_i(v_i, \sigma_i(v_i), v_{-i}, \sigma_{-i}(v_{-i})) \\ = & \left(v_i - \max_{j \in I \setminus \{i\}} \sigma_j(v_j)\right) \mathbb{1}_{\{i \text{ wins the auction}\}} \\ = & \left(v_i - \max_{j \in -i} \sigma_j(v_j)\right) \mathbb{1}_{\{\sigma_i(v_i) \geq \sigma_j(v_j) \forall j \in -i\}}. \end{aligned}$$

We now aim to maximize

$$\mathbb{E}_{-i}\left[u_i(v_i, \sigma_i(v_i), v_{-i}, \sigma_{-i}(v_{-i}))\right].$$

11.3.1 Reason of Response

We need to reason how agents will bid strategically.

- Case 1: If $b_i = \sigma_i(v_i) > v_i$.
  - (a) $b_i < \widetilde{P}_i = \max_{j \in -i} \sigma_j(v_j) = b_j$, then we see that

$$\max(b_i, \widetilde{P}_i) = \widetilde{P}_i,$$

    so i never wins the auction.
    Continue to do so even if he reduces bid to v_i.
  - (b) $b_i = \widetilde{P}_i$, then i can win the auction.
    Since $v_i - \widetilde{P}_i < 0$, and the second-highest bid is $\widetilde{P}_i$. So He better to bid v_i and lost the auction.

(c) $b_i > \widetilde{P}_i$, then i definitely wins the auction.

- $v_i < \widetilde{P}_i$: utility is $v_i - \widetilde{P}_i < 0$, not a viable option.
- $v_i \geq \widetilde{P}_i$: utility is $v_i - \widetilde{P}_i \geq 0$, but same result would also hold if b_i is exactly v_i.

- Case 2: If $b_i = \sigma_i(v_i) < v_i$.

(a) $\widetilde{P}_i \leq b_i < v_i$. Agent i wins with utility being $v_i - \widetilde{P}_i \geq 0$ and no lose in increasing bid to v_i.

(b) $b_i < v_i \leq \widetilde{P}_i$. Don't win the auction. No lose in increasing bid to v_i.

(c) $b_i \leq \widetilde{P}_i \leq v_i$. Lose the auction. However, bidding v_i the agent could have won the auction and gotten non-negative utility.

> **Remark.** Bidding v_i is a (weakly) dominant strategy for agent i. This further implies that for second-price auction, agents bid their true value. This is essentially *Incentive Compatibility*, or *Truth-telling desirable*.

> **Remark.** We see that 2^{nd} price auction is truth-telling in dominant strategies. Hence, irrespective of how the other players bid, truthfully bidding is optimal, namely
>
> $$\sigma_i(v_i) = v_i.$$

11.3.2 Revenue of the Auctioneer

Given $v_1, v_2, \ldots, v_I$, defined $\widetilde{v}_i$ as before, namely

$$\max\{v_1, \ldots, v_I\} = \widetilde{v}_I \geq \widetilde{v}_{I-1} \geq \ldots \geq \widetilde{v}_1 = \min\{v_1, \ldots, v_I\}.$$

And since it's a 2^{nd} auction, so the revenue is $\widetilde{v}_{I-1}$.

> **Note.** The expected revenue of $\widetilde{v}_{I-1}$ is
>
> $$\mathbb{E}_{\widetilde{v}_{I-1}}[=] \frac{I-1}{I+1}$$
>
> if we assume uniform valuation.

Hence, we see that the winner gets $\widetilde{v}_I - \widetilde{v}_{I-1}$, and the expected utility is

$$\mathbb{E}_{\widetilde{v}_I}[-]\,\mathbb{E}_{\widetilde{v}_{I-1}}[.]$$

In the uniform case, the above further equals to

$$\frac{I}{I+1} - \frac{I-1}{I+1} = \frac{1}{I+1}.$$

With winner and auctioneer together, their utility is

$$\widetilde{v}_I - \widetilde{v}_{I-1} + \widetilde{v}_{I-1} = \widetilde{v}_I.$$

> **Remark.** We see that the good always goes to the highest valuation.

Sums of utilities of all players in auctioneer is Social welfare - max value possible of $\widetilde{v}_I$.

> **Remark.** 2^{nd} price auction is social-welfare maximizing.

We see that if agent i bids v_i and wins, then he gets

$$v_i = \max_{j \in -i} v_j \geq 0,$$

and if lose then get 0. Hence, the expected utility is always non-negative if you attend the auction, while staying out will get you 0, hence we see that it's better to participate in the auction. This phenomenon is called *voluntary participation*(Individual rationality, IR).

And since the probability of winning is $\frac{1}{I}$, hence the expected utility is

$$\frac{1}{I} \times \frac{1}{I+1} = \frac{1}{I(I+1)}$$

for every agent.

> **Note.** Even if we are using Ex Post setting, it's still optimal to bid truthfully in a 2^{nd} auction.

11.4 First-price Auction

We'll see that the first price auction is not as "simple". There are no dominant strategy equilibrium, only so-called Bayes-Nash Equilibrium in interim.

Consider $I = 2$, and v_i choose $b_i = \sigma_i(v_i)$. The utility is

$$u_i(v_i, b_i, v_{-i}, b_{-i}) = (v_i - \max_{j \in \mathcal{I}} b_i) \mathbb{1}_{\{i \text{ is the winner}\}} = (v_i - b_i) \mathbb{1}_{\{b_i \geq b_{-i}\}}$$

If $I > 2$, then the utility is now

$$(v_i - b_i) \mathbb{1}_{\{\sigma_i(v_i) \geq \sigma_j(v_j) \forall j \in -i\}} = (v_i - b_i) \prod_{j \in -i} \mathbb{1}_{\{\sigma_i(v_i) \geq \sigma_j(v_j)\}}$$

Then, since we're in the interim set-up, the expected value of the utility is

$$\begin{aligned}
&\mathbb{E}_{-i} \left[u_i(v_i, \sigma_i(v_i), v_{-i}, \sigma_{-i}(v_{-i})) \right] \\
=&\mathbb{E}_{-i} \left[(v_i - \sigma_i(v_i)) \prod_{j \in -i} \mathbb{1}_{\{\sigma_i(v_i) \geq \sigma_j(v_j)\}} \right] \\
=&(v_i - \sigma_i(v_i)) \prod_{j \in -i} \mathbb{P} \left(\sigma_j(v_j) \leq \sigma_i(v_i) \right),
\end{aligned}$$

where the last equality comes from independence.

> **Note.** There are two facts to note.
>
> - Fact 1: Identical valuation implies $\sigma_i \equiv \sigma$ for all $i \in \mathcal{I}$. Same bidding function so it's symmetric.
> - Fact 2: $\sigma(\cdot)$ is monotonically increasing. Then
>
> $$\mathbb{P} \left(\sigma_j(v_j) \leq \sigma_i(v_i) \right) = \mathbb{P} \left(\sigma(v_i) \leq \sigma(v_i) \right) = \mathbb{P} \left(v_j \leq v_i \right) = F(v_i) = v_i,$$
>
> where F is the CDF, and the last equality comes from the fact that we assume the valuation is uniform($[0, 1]$).
>
> This is a fairly reasonable assumption, since it'll rule out the situations like if two buyers' true value are different, but they somehow decide to bid the same value.

With the second fact of the note above, the expected utility is

$$(v_i - \sigma(v_i)) F^{I-1}(v_i).$$

If we assume that this is uniform valuation, then above further equals to

$$(v_i - \sigma(v_i)) v_i^{I-1}.$$

Lecture 23: Bayesian Nash Equilibrium

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As previously seen. The 1^{st} price auction.

Now we look at the expected utility of agent i over other agents' valuation and bids. From the monotonically increasing property of $\sigma(\cdot)$, the expected utility of i is

$$\begin{aligned}
&\mathbb{E}_{-i}\left[u_i(v_i, \sigma_i(v_i), v_{-i}, \sigma_{-i}(v_{-i}))\right] \\
=&\mathbb{E}_{-i}\left[(v_i - \sigma_i(v_i)) \prod_{j\in -i} \mathbb{1}_{\{\sigma_i(v_i)\geq \sigma_j(v_j)\}}\right] \\
=&(v_i - \sigma_i(v_i)) \prod_{j\in -i} \mathbb{P}\left(\sigma_j(v_j) \leq \sigma_i(v_i)\right) \\
=&(v_i - \sigma_i(v_i)) \prod_{j\in -i} \mathbb{P}\left(v_j \leq v_i\right) \\
=&(v_i - \sigma_i(v_i))F^{I-1}(v_i) \\
=&(v_i - \sigma_i(v_i))v_i^{I-1}.
\end{aligned}$$

Now the question is how should an agent bid? Namely, how to find $\sigma_i(\cdot)$?

11.4.1 Reason of Response

To reason about the bidding strategy, we first see an important principle.

Theorem 11.4.1. Revelation Principle. Let σ be the equilibrium bidding function. Then any deviation can be decomposed into two steps:

1. $v_i \to v$: Map value to same *false* value.
2. Apply σ to the *false* value, so that the bid is $\sigma(v)$.

Intuition. This principle simply tells us that we don't need to perturb σ itself to analyze incentive of deviation, we can simply perturb v to achieve the same result.

Remark. The *Equilibrium bidding* is

- Step 1: $v_i \to v_i$: identical mapping.
- Step 2: The same, so bid is $\sigma(v_i)$.

Assume a deviation of $v_i \to v$ and the bid is $\sigma(v)$. Then the utility becomes

$$\begin{aligned}
u_i &= (v_i - \sigma(v))\mathbb{P}\left(\text{Winning with bid } \sigma(v)\right) \\
&= (v_i - \sigma(v)) \prod_{j\in -i} \mathbb{P}\left(\sigma(v_j) \leq \sigma(v)\right) \\
&= (v_i - \sigma(v))F^{I-1}(v) \\
&= (v_i - \sigma(v))v^{I-1} =: f_\sigma(v; v_i).
\end{aligned}$$

Then the best response calculation for i is as follows. Given a σ, maximize utility by properly choosing v for each v_i. We simply differentiate the function and set the value be zero

$$\frac{\mathrm{d}}{\mathrm{d}v} f_\sigma(v; v_i) = v_i(I-1)v^{I-2} - (I-1)v^{I-2}\sigma(v) - \sigma^1(v)v^{I-1} := 0$$

for the **first order condition**. It's equivalent to solve the following O.D.E.

$$v_i(I-1)v^{I-2} = \frac{\mathrm{d}}{\mathrm{d}v}(\sigma(v)v^{I-1}).$$

We see that $\sigma(v) \leq v_i$ is needed as you never bid higher than your valuation in a first-price auction.

For the equilibrium bidding $v_i \to v_i$, the maximizer should be v_i itself. For the equilibrium bidding function σ^*,

$$\frac{\mathrm{d}}{\mathrm{d}v} f_{\sigma^*}(v; v_i)\Big|_{v=v_i} = 0.$$

Expand it out, we have

$$\frac{\mathrm{d}}{\mathrm{d}v} f_{\sigma^*}(v; v_i) = v_i(I-1)v^{I-2} - \left((I-1)v^{I-2}\sigma^*(v) + \frac{\mathrm{d}\sigma^*(v)}{\mathrm{d}v} v^{I-1}\right).$$

By setting $v = v_i$, we should get zero!

$$(I-1)v_i^{I-1} = (I-1)v_i^{I-2}\sigma^*(v_i) + \frac{\mathrm{d}\sigma^*(v)}{\mathrm{d}v}\Big|_{v=v_i} {v_i}^{I-1}.$$

This equation has to hold for every v_i in $[0, 1]$.

Now, for a $\overline{v}$, $\sigma^*(\overline{v})$ should be

$$(I-1)v_i^{I-1} = \frac{\mathrm{d}}{\mathrm{d}v_i}\left(v_i^{I-1}\sigma^*(v_i)\right).$$

Integrate on both sides, we have

$$\int_0^{\overline{v}} (I-1)v_i^{I-1}\,\mathrm{d}v_i = v_i^{I-1}\sigma^*(v_i)\Big|_0^{\overline{v}}.$$

Solve for the left-hand side, we have

$$\frac{I-1}{I}\overline{v}^I = \overline{v}^{I-1}\sigma^*(\overline{v}).$$

Hence,

$$\sigma^*(\overline{v}) = \frac{I-1}{I}\overline{v} = (1 - \frac{1}{I})\overline{v},$$

which means that agent bids $\frac{I-1}{I}$ times the value who estimate!

> **Note.** We see that
>
> - Agents will shade their bid, hence it's not truthful.
> - The good goes to the agent with the highest valuation, hence it's a social-welfare maximizer.

> **Remark.** If an agent whose valuation is zero, then from the function, we see that he would bid zero, hence the utility is zero as well.

11.4.2 Revenue of the Auctioneer

The expected revenue for the auctioneer is equal to

$$\frac{I-1}{I}\mathbb{E}\left[\widetilde{v}_I\right] = \frac{I-1}{I}\frac{I}{I+1} = \frac{I-1}{I+1}.$$

> **Remark.** Recall that the expected revenue for 2^{nd} price auction for the auctioneer is $\mathbb{E}\left[\widetilde{v}_{I-2}\right] = \frac{I-1}{I+1}$. We see that both auctions give the same expected revenue.

> **Note.** If two auctions implement the same outcome in Bayesian Nash Equilibrium, and agent with value 0 bid 0, then the expected revenue will be the same. This is so-called *Revenue Equivalence*. And this equivalence relation holds for more general kinds of auctions, namely under some conditions of auctions, the revenue for auctioneer is always the same. We'll come back to this relation soon.

11.5 Mechanism Design of Auctions

After seeing two kinds of auctions, we now try to give a general framework of auctions and try to design a mechanism of such games. Assume that the auctioneer doesn't know values of agent. How the auction will be conducted?

- For every $b_1, \dots, b_I$:
  1. Who wins the auction
  2. What that person pays

Example. We first see an example.

- **All-pay auction**. For bids $b_1, \dots, b_I$,
  - The highest bidder wins.
  - Everyone pays their bid.

For a general auction, the bids are just messages. Auctioneer is specifying two things.

- Allocation rule: $(x_1, \dots, x_I)$, where x_i is the amount of good goes to agent i.
- Payment rule: $(p_1, \dots, p_I)$, where p_i to be paid by agent i.

Definition 11.5.1. In the specification, if the messages are in the same space as values, then we call such a mechanism as a *direct mechanism*.

As previously seen. From Revelation Principle, *direct mechanisms* are sufficient. Since bids are messages, which can be anything, hence it suffices to consider messages to be in the same space as the values.

We assume that the message space is the same as value space, namely we are designing a direct mechanism. For every message vector, specifies the allocation rule and the payment rule. Furthermore, we let

$$\text{Utility} = \text{Value} - \text{Payment},$$

which turns out to be so-called Quasilinear utility[1].

Note. There are things that a mechanism designer may want.

- Incentive Compatible(IC)
- Individual rationality(IR)

Also, a mechanism designer may have a goal in mind.

- Social welfare maximization(Efficiency)
- Revenue maximization
- Budget-Balance: Payments should sum to 0.

Remark. All can't be met simultaneously.

[1] https://en.wikipedia.org/wiki/Quasilinear_utility

Assume every agent i values $v_1, \dots, v_I$, and bids $b_1, \dots, b_I$ respectively. With the specification being

$$\begin{aligned} \text{Allocations} &: x_1, \dots, x_I \\ \text{Payments} &: p_1, \dots, p_I, \end{aligned}$$

where all are functions of b_i. Then the utility is

$$u_i(v_i, n_i(\vec{v})) - p_i(\vec{v}).$$

With interim set-up, we average across all other agents.

Now, we assume that $x_i(\vec{v}) \in \{0, 1\}$ and will allow multiple agents to get

$$x_i(\vec{v}) = 1.$$

If $x_i(\vec{v}) = 1$, then utility is v_i, else 0. We then see that

$$u_i(v_i, x_i(\vec{v})) = v_i x_i(\vec{v}) - p_i(\vec{v}).$$

Then the expected value is

$$\begin{aligned} \mathbb{E}_{-i}\left[u_i(v_i, x_i(\vec{v}))\right] &= v_i \mathbb{E}_{-i}\left[x_i(\vec{v})\right] - \mathbb{E}_{-i}\left[p_i(\vec{v})\right] \\ &= v_i \mathbb{E}_{-i}\left[x_i(v_i, v_{-i})\right] - \mathbb{E}_{-i}\left[p_i(v_i, v_{-i})\right] \\ &= v_i \overline{x}_i(v_i) - \overline{p}_i(v_i). \end{aligned}$$

Now we see a theorem.

Theorem 11.5.1. Bayesian Nash Equilibrium characterization. When the types are drawn from a continuous joint distribution $\vec{F}$ with independent types, strategies

$$\vec{\sigma}^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_I^*) \qquad (b_i = \sigma_i^*(v_i))$$

are in a Bayesian Nash Equilibrium if and only if

- Monotonicity. $\overline{x}_i(v_i)$ is monotonically non-decreasing.
- Payment identity.
$$\overline{p}_i(v_i) = v_i \overline{x}_i(v_i) - \int_0^{v_i} \overline{x}_i(z)\,\mathrm{d}z + \overline{p}_i(0),$$
where often $\overline{p}_i(0) = 0$.

If the strategy profile is onto, the converse is true as well.

Lecture 24: Matching Market

11.6 Bayesian Nash Equilibrium

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As previously seen. For a valuation $\vec{v}$, the bid $\sigma^*(\vec{v})$ is in the same space as $\vec{v}$ by revelation principle. With allocation being specified as

$$\vec{x}_i(v_i)$$

which is monotonically increasing, and the payment being specified as

$$\overline{p}_i(v_i) = v_i \overline{x}_i(v_i) - \int_0^{v_i} \overline{x}_i(z)\,\mathrm{d}z + \overline{p}_i(0),$$

we now try to analyze a general auction.

11.6.1 Revenue Equivalence

We start with the expected revenue. The expected revenue is

$$\mathbb{E}_v\left[\sum_i \overline{p}_i(v_i)\right] = \mathbb{E}_v\left[F(\vec{v})\right] + \sum_{i=1}^{I} \overline{p}_i(v).$$

1. If two mechanisms have the same allocation rule in equilibrium, then
   - $\overline{x}_i(v_i)$ will be the same
   - $\overline{p}_i(v_i)$ will be the same except for the $\overline{p}_i(0)$ term.
2. If agent with valuation zero, pay zero, then
   $$\overline{p}_i(0) = 0$$
   in both mechanisms. This implies that the revenue will be the same.

> **Note.** Similar characterization holds for dominant strategy equilibrium but those ex-post.
>
> - Allocation rule property
> - Payment rule property

Assume the independent and identical valuation setting, together with the valuation being uniform($[0,1]$). Then the 2^{nd} price auction and 1^{st} price auction are *revenue equivalent*.

By symmetry, the payment rules will be identical in each auction, hence the revenue is

$$I \times \text{payment rule for each agent.}$$

- For the second price auction(SP),
  $$\overline{p}_1^{\text{SP}}(v_1) = \mathbb{E}\left[\left(\max_{j\in -1} v_j\right)\mathbb{1}_{\{\max_{j\in -1} v_j \le v_1\}}\right].$$
  Denote $\widetilde{p}_1 = \max_{j\in -1} v_j$, then
  $$F_{\widetilde{p}_i^{\text{SP}}}(x) = \mathbb{P}\left(\widetilde{p}_1 \le x\right) = \mathbb{P}\left(v_2 \le x, v_3 \le x, \ldots, v_I \le x\right) = x^{I-1},$$
  where $x \in [0,1]$. Then,
  $$f_{\widetilde{p}^{\text{SP}}}(x) = (I-1)x^{I-2}.$$
  We see that
  $$\begin{aligned}
  \widetilde{p}_1^{\text{SP}} &= \mathbb{E}\left[\widetilde{p}_1 \mathbb{1}_{\{\widetilde{p}_1 \le v_1\}}\right] \\
  &= \int_0^{v_1} x f_{\widetilde{p}_1}(x)\,\mathrm{d}x \\
  &= (I-1)\int_0^{v_1} x \cdot x^{I-2}\,\mathrm{d}x \\
  &= (I-1)\int_0^{v_1} x^{I-1}\,\mathrm{d}x \\
  &= \frac{I-1}{I} v_1^I.
  \end{aligned}$$
- For the first price auction(FP),
  $$\overline{p}_1^{\text{FP}}(v_1) = \sigma^{\text{FP}}(v_1)\mathbb{P}\left(\text{Agent 1 has the highest value } v_1\right) = \sigma^{\text{FP}}(v_1)v_1^{I-1},$$
  where every other agent's value is less or equal to v_1.
  From Revenue equivalence,
  $$\widetilde{p}^{\text{SP}}(v_1) = \widetilde{p}^{\text{FP}}(v_1),$$
  which implies
  $$\frac{I-1}{I}v_1^I = \sigma^{\text{FP}}(v_1)v_1^{I-1} \Rightarrow \sigma^{\text{FP}} = \frac{I-1}{I}v_1,$$
  as we discussed before.

Chapter 12

Matching Market

Beyond the current setup of auctions, we can see a more general setup. Specifically, we introduce so-called **Matching Market**. The comparison is as follows.

- One good to many buyers: Auctions.
- Many goods to many buyers: Matching Market.

Example. We first see an example of matching market in terms of a general form of auction, namely **Ad-Auctions**. Specifically, consider a particular keyword searching result, the ads related to which may show up in the following order:

- AD1 with r_1
- AD2 with r_2
- AD3 with r_3
- AD4 with r_4
- $\vdots$

where r_i is the *click-through rate* for ad location i. We see that the location of ads matter.

Note. We see that

- Click-through rate is location dependent. Person needs to click on ad and then buy the product the ad is about.
- Often, $r_i < r_j$ for $i > j$.

We see that there is a *bipartite structure*. On the one side of the graph contains buyers, and the other side contains sellers(Ad positions). Now, assuming for each buyer's good, the cost is b_i. Then the expected revenue to advertiser i is

$$b_i r_j$$

if one purchase the location j. Now, let $v_{ij} \coloneqq b_i r_i$, where i denotes the buyers, and j denotes the sellers. Then we have the so-called *Valuation matrix* V such that

$$V \coloneqq \vec{r}^{\top} \vec{b},$$

which is also called the **rank one valuation matrix**.

If $V_{ij} \geq 0$ and beyond rank-one imply good allocation here.

We allow the sellers to set prices to goods.

Remark. There are different types of prices.

- Posted. Item comes with price.
- Anonymous. The price is the same for any buyers.

Definition 12.0.1. The market is called *balance* if the number of sellers and buyers are the same.

Without loss of generality, we assume that the market is **balanced** by letting

$$V \in \mathbb{R}^{N\times N},$$

which implies there are a same number(N) of buyers and sellers. This can be achieved by adding dummy buyers or sellers.

- Dummy Buyers: Value every good as 0.
- Dummy Sellers: Valued as 0 for every buyer.

Consider each seller/item coming with a posted and anonymous price, namely item j has price $p_j \geq 0$.

Problem. How does a buyer i react to this? Assuming i knows her/his valuation, and each buyer can only purchase one good.

Answer. Suppose the valuation for buyer i is

$$V_{i1}, V_{i2}, \ldots, V_{iN},$$

and the prices for each good is

$$p_1, p_2, \ldots, p_N.$$

Then it's clear that buyer i will

- only consider item whose valuation exceeds the price
- further, focus only on goods whose payoff is maximum(valuation $-$ price)

Mathematically,

$$\arg\max_{j \text{ s.t. } V_{ij}\geq p_j} V_{ij} - p_j.$$

For convenience, we define a function $[\,\cdot\,]_+$ such that

$$[x]_+ := \max(x, 0) = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

The utility of buyer i at price vector $\vec{p}$ is

$$u_i = \left[\max_{j\in\{1,\ldots,N\}} V_{ij} - p_j\right]_+ = \max_{j\in\{1,\ldots,N\}} [V_{ij} - p_j]_+ .$$

We then defined so-called *preferred sellers list* $s(i)$ for buyer i, which contains goods that buyer i is interested in.

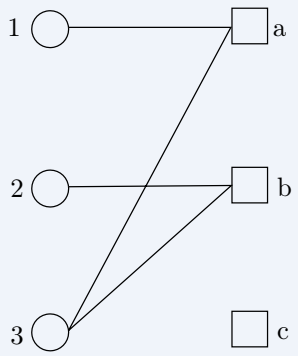
$$s(i) = \begin{cases} \varnothing, & \text{if } \forall j\ V_{ij} < p_j \\ \arg\max_j \left(V_{ij} - p_j\right), & \text{otherwise.} \end{cases}$$

Remark. If $s(i) \neq \varnothing$, then there is at last one seller that buyer i is interested in.

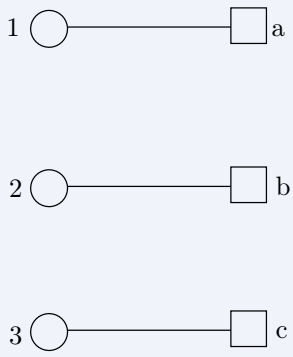
We then use the preferred sellers lists to construct an undirected bipartite graph between buyers and sellers. ⊛

Example. We see some examples.

1. Let $N = 3$, and
$$V \coloneqq \begin{pmatrix} 5 & 3 & 4 \\ 3 & 8 & 3 \\ 5 & 4 & 7 \end{pmatrix}, \quad p \coloneqq \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}.$$
Then, we have
   (a) Buyer 1. $V_{11} - p_1 = 1$, $V_{12} - p_2 = 0$, $V_{13} - p_3 = -3$. Hence,
   $$s(1) = \{1\}.$$
   (b) Buyer 2. $V_{21} - p_1 = -1$, $V_{22} - p_2 = 5$, $V_{23} - p_3 = -4$. Hence,
   $$s(2) = \{2\}.$$
   (c) Buyer 3. $V_{31} - p_1 = 1$, $V_{32} - p_2 = 1$, $V_{33} - p_3 = 0$. Hence,
   $$s(3) = \{1, 2\}.$$
   We see that c is not desired by any buyers at current price.



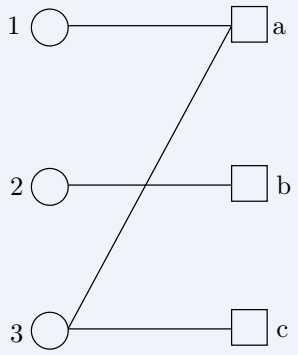
2. Consider changing p into
$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
Then we see that satisfaction is possible.



3. Consider changing p into

$$p = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}.$$

Then we see that satisfaction is possible as well.



Remark. We see that the satisfaction match is not always possible.

Now, the task is that for any given price, find the preferred sellers bipartite graph and perform matching. Furthermore, we try to maximize the matching in terms of size. In other words, we are trying to satisfy as many buyers as possible.

Note. We call a matching is *perfect* if the size of the maximum matching is N.

12.1 Perfect Matching

To discuss perfect matching, we first defined some notations. Let

$$S \subseteq \{1, \dots, N\}$$

be the subset of sellers, and

$$B \subseteq \{1, \dots, N\}$$

be the subset of buyers. We further denote $N(S)$ be the union of all buyers that like goods in S, and $N(B)$ be the union of preferred sellers of all the buyers in B.

If $|S| > |N(S)|$, then we call S is a *constricted set*. Similarly, if $|B| > |N(B)|$ then B is constricted.

Theorem 12.1.1. Kőnig-Hall Maximize theorem. A bipartite graph has a perfect matching if and only if there are no constricted sets.

Note. This is a necessary and sufficient conditions for a perfect matching to exist given a bipartite graph.

Remark. We see that

- With naive approach, we need to check every non-empty subset of sellers. Hence, there are $2^N - 1$ checks.
- Polynomial time checks exists.
- We can also find a perfect matching in polynomial time.
- Prices for which a perfect matching exists are called *market clearing prices*.

Lecture 25: Perfect Matching

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Problem. We now want to solve the following questions.

1. Given a bipartite graph, can we check if there is a perfect matching?
   Answer. Yes, by Hopcroft-Karp algorithm, which is an *asymmetric BFS algorithm.* ⊛
2. Market-clearing prices. Do they exist, can we find them?
   Answer. Yes. We can find it in polynomial time by Hungarian algorithm. ⊛
3. Social welfare maximizing. Connection of market clearing prices.
4. VCG principle. To obtain specific market clearing prices.

12.2 Hopcroft–Karp Algorithm

To answer the first question, we need to use Kőnig-Hall Maximize theorem.

Intuition. It's essentially an asymmetric BFS with labels.

We first define buyer nodes such that

$$\text{Buyer nodes} \in \{\text{free}, \text{matched}, \text{exhausted}\}$$

and seller nodes such that

$$\text{Seller nodes} \in \{\text{free}, \text{matched}\},$$

and also edges such that

$$\text{Edges} \in \{\text{free}, \text{matched}\}.$$

Then the algorithm can be described as follows.

0. All nodes and edges are marked as free.
1. Initial step. Pick some matching and go through free buyers and match to free sellers they are connected to. After match labels of buyers, sellers and edge to matched. If any buyers is not able to be matched, let it be free for now.
2. If after this initial step, there are no free buyers, then we're done and perfect match is obtained.
3. Otherwise, there are some free buyers left.
   - Pick one of these free buyers, connect it to all **matched** sellers that this free buyers has edges with(all edges here are free)
   - For each **matched** sellers, go to the other side using the **matched** edges, we'll end up at matched buyers.
   - For each **matched** buyers, use free edges to connect to the other side, which is seller.
   - Return to the buyers side from any **matched** sellers using their **matched** edges.
4. Repeat until no exploration can be done.
   - Matched buyer has no free edges.
   - All new sellers are free.

Remark. We see that

- At termination, if there is a path from a **free** buyer to a **free** seller, then it has to have alternating labels. We call such a path an *augmenting path*, since we can augment the total size of the matching by 1 by altering this path.
- Specifically, if an augmenting path exists, then flip all the labels of the edges, and label the free buyers and sellers to matched. This will increase the size of the matching by 1.
- If for a free buyer node, no augmenting paths are found, then label it as exhausted. Repeat through all free buyers nodes, until none remain.
- This is a polynomial time algorithm, since we will only explore every node once.

We then state a fundamental theorem about the **only** constraint when constructing a perfect matching, without proving it

Theorem 12.2.1. If no exhausted buyer nodes, then we have a perfect matching. Otherwise, if there are exhausted buyer nodes, then we can find constricted sets.

Note. In the first scenario, we will obtain it using matched edges. Also, in the second scenario, we will only obtain a maximal matching using the matched edges.

Remark. This theorem essentially tells us that the only obstacle one will meet when constructing a perfect matching is the existence of constricted set.

12.3 Market Clearing Prices

We now come back to market clearing prices problem.

Problem. Given valuation matrix V and price vector p, is it always the case that a market clearing price exists?

Answer. Yes, and we can find it in polynomial time.(Hungarian algorithm do it in $O(n^3)$, which is strongly polynomial time) ⊛

Without loss of generality. Assume that $V_1 > V_2 > V_3 \geq \ldots \geq V_N$. Then we easily see that if

- $p = (V_1, 0, \ldots, 0)$, then a perfect matching exists.

  Note. This is essentially a first price auction.

- $p = (V_2, 0, \ldots, 0)$, then a perfect matching exists as well.

  Note. This is essentially a second price auction.

  Furthermore, if

  - $p_1 > V_1$, then no perfect matching exists.
  - $p_1 < V_2$, then no perfect matching exists as well.

We claim that market clearing prices always exists, and will follow so-called *Lattice structure*. Says if p^1 and p^2 are both market clearing prices, then

$$p^1 \vee p^2 := \text{element-wise max}$$
$$p^1 \wedge p^2 := \text{element-wise min}$$

are also market clearing prices, where

- Max market clearing price is by taking the maximum in all components.
- Min market clearing price is by taking the minimum in all components.

Remark. There is a nice property for market clearing prices: It's always social-welfare maximizing, respect to the utility of all buyers and the sum of prices(seller made).

We can actually formulate the above question as a linear programming. Define $x_{ij} \in [0, 1]$ as the part of good j in assigned to buyer i, namely we allow fractional assignment. Then we model the above question as

$$\begin{aligned} V^* := \max \quad & \sum_{i,j} V_{ij} x_{ij} \\ & \sum_{j} x_{ij} = 1 \ \forall i \\ & \sum_{i} x_{ij} = 1 \ \forall j \\ & x_{ij} \in [0, 1]. \end{aligned}$$

Intuition. We see that

- The first constraint says that buyer want one good.
- The second constraint says that seller has one good for type j.

If we further restrict the fractional assignment property, then we have the integer programming

version of above, namely

$$\begin{aligned} V^* \coloneqq \max \quad & \sum_{i,j} V_{ij} x_{ij} \\ & \sum_{j} x_{ij} = 1 \ \forall i \\ & \sum_{i} x_{ij} = 1 \ \forall j \\ & x_{ij} \in \{0,1\}. \end{aligned}$$

Remark. We see that

- Perfect matching is the only possible solution.
- Solving the LP leads to an integer solution.
- The market clearing price induced by perfect matching will achieve V^* in the IP, which is social welfare maximizing.

12.4 Demange-Gale-Sotomayor(DGS) Algorithm

Given an English auction for multiple goods, with valuations and prices being all in integers. Then by running the following DGS algorithm, we'll have a set of market clearing prices.

1. Set initial price to $p = (0, 0, \ldots, 0)$
2. At p fixed the preferred sellers bipartite graph.
3. Check if there is a perfect matching.(Can be done by Hopcroft-Karp Algorithm)
   - If yes, then end.
   - Otherwise, determine a constricted set of buyers B, namely find a B such that
   $$|N(B)| < |B|.$$
   Pick any non-empty subset S of $N(B)$ and increase the price of the goods in S all by 1.
4. **GOTO 2**.

Intuition. If there is more demand and less supply, then we increase the price.

Remark. This algorithm will terminate in finite-time, further, it's actually a polynomial time algorithm.

If one carefully enough for choosing S, he will always terminate in the minimum market clearing prices.

12.5 VCG Principle

Consider a 2^{nd} price auction. There is only 1 item to be sold with N buyers with valuation $V_1, V_2, \ldots, V_N$. We further let

$$V_1 > V_2 > V_3 \geq \ldots V_N.$$

In this case, we see that agent 1 gets the good with price V_2. We denote the utility as $\Delta \coloneqq V_1 - V_2 > 0$.

Problem. What's the utility of other players?

Answer. 0. ⊛

We now conduct a thought experiment. If we remove agent 1 and replace him with a dummy agent, says $\tilde{1}$, then we see that agent 2 will get the good with price V_3. Further, we see that the sum of the values obtained by agents other than $\tilde{1}$ is V_2, since now agent 2 gets the good and his valuation is V_2.

We see that by the appearance of agent 1, other agents collectively lost V_2 values. We call such collectively lost values as *Externality* agent 1 imposes.

With this view point to view a second price auction, we see the following principle.

Theorem 12.5.1. VCG Principle. Charge a price to an agent as the externality agent imposes on others.

Example. We see that for

- Agent 1 should be charged with price V_2
- Agent 2 should be charged with:

  In the original auction, the value of all other agents are V_1. Now, replace agent 2 by a dummy agent $\tilde{2}$, then the value of all agents other than $\tilde{2}$ is V_1. Hence, the difference is

  $$\Delta_2 = V_1 - V_1 = 0,$$

  namely by VCG principle, we charge agent 2 with price 0.

To apply VCG principle in a matching market problem mathematically, we first denote the total set of sellers as S, and the total set of buyers as B. Recall the definition of V^*, we further specify it by adding the superscript and subscript as

$$V_B^{*,S}$$

for considering S and B in the programming. Then, for $i \in B$, denote $j^*(i)$ as the matched seller respect to i in a perfect matching that maximize social welfare. Now, consider the *reduced problem*, namely calculate

$$V_{B-i}^{*,S-j^*(i)},$$

which stands for the value of everyone else when agent i is not present.

Lastly, we consider the *alternate problem*, where we need to calculate the value that everyone else gets if i is not present, namely we calculate

$$V_{\widetilde{B}}^{*,S}$$

where $\widetilde{B} \coloneqq B - i + \varnothing$, where $\varnothing$ stands for a dummy agent $\widetilde{i}$.

Combining the reduced problem and the alternate problem, together with the VCG principle, we see that the auctioneer should charge buyer i

$$p_i^{\text{VCG}} = V_{\widetilde{B}}^{*,S} - V_{B-i}^{*,S-j^*(i)}.$$

We now simply let

$$p_{j^*(i)} = p_i^{\text{VCG}},$$

namely we make the price as posted price.

Remark. We finally note that

- It's market clearing price, furthermore, it's actually the minimum market clearing price.
- There is a conceptual difference between the VCG approach and the DGS algorithm approach. One is posted, one is not. We see that in VCG approach, we determine the prices for each

agent by their valuation, while this is not the case in DGS algorithm approach.

Appendix

This note is completed in LaTeXwith Inkscape, in case of anyone is interested, please check out this blog[1] together with my configuration[2].

[1] https://castel.dev/
[2] https://github.com/sleepymalc/VSCode-LaTeX-Inkscape

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