MATH597
Analysis II

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\stackrel{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition we're using to deduce that particular equality.

Notice that there are some proofs is **intended** left as assignments, and for completeness, I put them in Appendix A, use it in your **own risks**! You'll lose the chance to practice and really understand the materials.

Additionally, we'll use [FF99] as our main text, while using [Tao13] and [Axl19] as supplementary references.

Contents

Lecture 1: σ-algebra

05 Jan. 11:00

1 Measure

Example. Before we start, we first see some examples.

1. Let $X = \{a, b, c\}$. Then
$$\mathcal{P}(X) := \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$
which is the *power set* of X. We see that
$$\#X = n \implies \#\mathcal{P}(X) = 2^n$$
for $n < \infty$.

2. If $n = \infty$, say $X = \mathbb{N}$, then
$$\mathcal{P}(\mathbb{N})$$
is an uncountable set while $\mathbb{N}$ is a countable set. We can see this as follows. Consider
$$\phi\colon \mathcal{P}(\mathbb{N}) \to [0, 1], \quad A \mapsto 0.a_1a_2a_3\ldots \text{(base 2)},$$
where
$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$
and for example, A can be $A = \{2, 3, 6, \ldots\} \subseteq \mathbb{N}$. Note that ϕ is surjective, hence we have
$$\#\mathcal{P}(\mathbb{N}) \geq \#[0, 1].$$
But since $[0, 1]$ is uncountable, so is $\mathcal{P}(\mathbb{N})$.

We like to *measure* the *size* of subsets of X. Hence, we are intriguing to define a map μ such that
$$\mu\colon \mathcal{P}(X) \to [0, \infty].$$

Example. We first see some examples.

1. Let $X = \{0, 1, 2\}$. Then we want to define $\mu\colon \mathcal{P}(X) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$. Then we have
     - $\mu(\{0, 1\}) = 2$
     - $\mu(\{0\}) = 1$
   - $\mu(A) = \sum\limits_{i\in A} 2^i$. Then we have
     - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$
2. Let $X = \{0\} \cup \mathbb{N}$. Then we want to define $\mu\colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$. Then we have
     - $\mu(\{2, 3, 4, 5, \ldots\}) = \infty = \mu(\{\text{even numbers}\})$
   - $\mu(A) = e^{-1} \sum\limits_{i\in A} \frac{1}{i!}$. Then we have
     - $\mu(\{0, 2, 4, 6, \ldots\}) = e^{-1}\left(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right)$
   - $\mu(A) = \sum\limits_{i\in A} a_i$
3. Let $X = \mathbb{R}$. Then we want to define $\mu\colon \mathcal{P}(\mathbb{R}) \to [0, \infty]$, we can have
   - $\mu(A) = \#A$
   - $\mu((a, b)) = b - a$.

     Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

     Answer. No!
   - $\mu((a, b)) = e^b - e^a$.

     Problem. Can we extend this map to all of $\mathcal{P}(\mathbb{R})$?

     Answer. No!

We immediately see the problems. To extend our native measure method into $\mathbb{R}$ is hard and will cause something counter-intuitive![1] Hence, rather than define measurement on *all* subsets in the power set of X, we only focus on *some* subsets. In other words, we want to define

$$\mu\colon \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \to [0, \infty] \,.$$

1.1 σ-algebras

We start from the definition of the most fundamental element in measure theory.

[1] https://en.wikipedia.org/wiki/Banach-Tarski_paradox

Definition 1.1 (σ-algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra on* X if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\varnothing}_{\in \mathcal{A}}$ and $\mathcal{A}$ is closed under complement.
- $\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c$, namely $\mathcal{A}$ is closed under countable intersections.
- $A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \varnothing \cup \varnothing \cup \ldots$, hence $\mathcal{A}$ is closed under finite unions and intersections.

An immediate definition can be given. We now define so-called *Borel set*.

Definition 1.2 (Borel set). Given a topological space X, a *Borel set* is any set in X that can be formed from open sets through the operations of countable union, countable intersection and relative complement.

Lecture 2: Measure

07 Jan. 11:00

Example. Again, we first see some examples.

1. Let $\mathcal{A} = \mathcal{P}(X)$, which is the power σ-algebra.
2. Let $\mathcal{A} = \{\varnothing, X\}$, which is a trivial σ-algebra.
3. Let $B \subset X$, $B \neq \varnothing$, $B \neq X$. Then we see that $\mathcal{A} = \{\varnothing, B, B^c, X\}$ is a σ-algebra.

Lemma 1.1. Let $\mathcal{A}_\alpha$, $\alpha \in I$, be a family of σ-algebra on X. Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is a σ-algebra on X.

Remark. Notice that I may be an uncountable intersection.

Proof. A simple proof can be made as follows. Firstly, $\varnothing \in \mathcal{A}_\alpha$ for every α clearly. Moreover, closure under complement and countable unions for every $\mathcal{A}_\alpha$ implies the same must be true for $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$. Hence, $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ-algebra. ■

The above allows us to give the following definition.

Definition 1.3 (Generation of σ-algebra). Given $\mathcal{E} \subset \mathcal{P}(X)$, where $\mathcal{E}$ is not necessarily a σ-algebra. Let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X containing $\mathcal{E}$, then we call $\langle \mathcal{E} \rangle$ the *σ-algebra generated by $\mathcal{E}$*.

Remark. Clearly, $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$, and it is unique. To check the uniqueness, we suppose there are two different $\langle \mathcal{E} \rangle_1$ and $\langle \mathcal{E} \rangle_2$ generated from $\mathcal{E}$. It's easy to show

$$\langle \mathcal{E} \rangle_1 \subseteq \langle \mathcal{E} \rangle_2 ,$$

and by symmetry, they are equal.

Example. We see that $\{\varnothing, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle$.

Lemma 1.2. We have

1. Given $\mathcal{A}$ a σ-algebra, $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \mathcal{A}$
2. $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$

Proof. We'll see that after proving the first claim, the second follows smoothly.

1. The first claim is trivial, since we know that $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$, then if $\mathcal{E} \subset \mathcal{A}$, we clearly have $\langle \mathcal{E} \rangle \subset \mathcal{A}$ by the definition.
2. The second claim is also easy. From the first claim and the definition, we have
$$\mathcal{E} \subset \mathcal{F} \subset \langle \mathcal{F} \rangle \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle .$$

■

At this point, we haven't put any specific structure on X. Now we try to describe those spaces with good structure, which will give the space some nice properties.

Definition 1.4 (Borel σ-algebra). For a topological space X, the *Borel σ-algebra on X*, denotes as $\mathcal{B}(X)$, is the σ-algebra generated by the collection of all open sets in X.

Example. We see that $\mathcal{B}(\mathbb{R})$ contains

- $\mathcal{E}_1 = \{(a, b) \mid a < b; a, b \in \mathbb{R}\}$.
- $\mathcal{E}_2 = \{[a, b] \mid a < b; a, b \in \mathbb{R}\}$ since $[a, b] = \bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.
- $\mathcal{E}_3 = ((a, b] \mid a < b; a, b \in \mathbb{R})$ since $(a, b] = \bigcap\limits_{n=1}^{\infty} (a, b + \frac{1}{n})$.
- $\mathcal{E}_4 = ([a, b) \mid a < b; a, b \in \mathbb{R})$ since $[a, b) = \bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b)$.

- $\mathcal{E}_5 = ((a, \infty) \mid a \in \mathbb{R})$ since $(a, \infty) = \bigcup\limits_{n=1}^{\infty} (a, a+n)$.
- $\mathcal{E}_6 = ([a, \infty) \mid a \in \mathbb{R})$ since $[a, \infty) = \bigcup\limits_{n=1}^{\infty} [a, a+n)$.
- $\mathcal{E}_7 = ((-\infty, b) \mid b \in \mathbb{R})$ since $(-\infty, b) = \bigcup\limits_{n=1}^{\infty} (b-n, b)$.
- $\mathcal{E}_8 = ((-\infty, b] \mid a \in \mathbb{R})$ since $(-\infty, b] = \bigcup\limits_{n=1}^{\infty} (b-n, b]$.

Proposition 1.1. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \ldots, 8$.

Proof. Firstly, we see that $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \implies \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$ by Lemma 1.2. Secondly, by definition, $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$ where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R}\}.$$

It's enough to show $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$ since if so, $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$, and clearly $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$, then we will have $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$. Let $O \subset \mathbb{R}$ be an open set, i.e., $O \in \mathcal{E}$. We claim that every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.[2]

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

where I_j open interval with the form of $(a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$.

For example, $\mathcal{E}_1$ is trivially true, and

$$(a, b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{[a + \frac{1}{n}, b - \frac{1}{n}]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for $\mathcal{E}_2$ and

$$(a, \infty) = \bigcup_{k=1}^{\infty} (a, a+k)$$

shows the case for $\mathcal{E}_5$. It's now straightforward to check open intervals are in $\langle \mathcal{E}_i \rangle$ for every i. ■

Now, to put a structure on a space, we define the following.

Definition 1.5 (Measurable space, $\mathcal{A}$-measurable set). A *measurable space* or *Borel space* is a tuple of a set X and a σ-algebra $\mathcal{A}$ on X, denoted by $(X, \mathcal{A})$.

Furthermore, for every $E \in \mathcal{A}$ is called an $\mathcal{A}-$*measurable set*.

[2] https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals

1.2 Measures

With the definition of measurable space, we now can refine our measure function μ as follows.

Definition 1.6 (Measure, Measure space). Given a measurable space on $(X, \mathcal{A})$, a *measure* is a function μ such that

$$\mu\colon \mathcal{A} \to [0, \infty]$$

with

1. $\mu(\varnothing) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_1, A_2, \ldots \in \mathcal{A}$ are **disjoint**. We call this *Countable additivity*.

We denote $(X, \mathcal{A}, \mu)$ a *measure space*.

Notation. We denote $[0, \infty] \coloneqq [0, \infty) \cup \{\infty\}$.

Remark. The motivation of why we only want *countable additivity* but not uncountable additivity can be seen by the following example. We'll consider the most intuitive measure on $\mathbb{R}, \mathcal{B}(\mathbb{R})$.

Since we have

$$(0, 1] = (1/2, 1] \cup (1/4, 1/2] \cup (1/8, 1/4] \cup \ldots$$

and also

$$(0, 1] = \bigcup_{x \in (0,1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2}, 1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4}, \frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8}, \frac{1}{4}])} + \ldots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0,1]} 0$$

since $\mu(x) = 0$ for $x \in \mathbb{R}$, which is clearly not what we want.

Example. We see some examples.

1. For any $(X, \mathcal{A})$, we let $\mu(A) \coloneqq \# A$. This is called *counting measure*.
2. Let $x_0 \in X$. For any $(X, \mathcal{A})$, the *Dirac measure at x_0* is

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A. \end{cases}$$

3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$,
$$\mu(A) = \sum_{i \in A} a_i,$$
where $a_1, a_2, \ldots \in [0, \infty)$.

Lecture 3: Construct a Measure

10 Jan. 11:00

Note. If $A, B \in \mathcal{A}$ and $A \subset B$, then
$$\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

Theorem 1.1. Given $(X, \mathcal{A}, \mu)$ be a measure space.

1. (monotonicity)
$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$
2. (countable subadditivity)
$$A_1, A_2, \ldots \in \mathcal{A} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$
3. (continuity from below/ monotone convergence theorem (MCT) for sets)
$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \ldots \end{cases} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$
4. (continuity from above)
$$\begin{cases} A_1, A_2, \ldots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \ldots \\ \mu(A_1) < \infty \end{cases} \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

Proof. We prove this theorem one by one.

1. Since $A \subset B$, hence we have
$$\mu(B) = \mu\Big(\underbrace{\underline{(B \setminus A)} \cup \underline{A}}_{\text{disjoint}}\Big) \overset{!}{=} \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A).$$
2. This should be trivial from countable additivity with the fact that $\mu(A) \geq 0$ for all A. DIY!
3. Let $B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$, then
$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

are a disjoint union and $B_i \in \mathcal{A}$, hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i).$$

With $\mu\left(\bigcup\limits_{i=1}^{n} B_i\right) = \mu(A_n)$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu(A_n).$$

4. Let $E_i = A_1 \setminus A_i \implies E_i \in \mathcal{A}$, $E_1 \subset E_2 \subset \ldots$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right),$$

which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left(\bigcup_{i=1}^{\infty} E_i\right) \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$$

since $\mu\left(\bigcup\limits_{i=1}^{\infty} E_i\right) \leq \mu(A_1) < \infty$. Then from continuity from below, we further have

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right).$$

From monotonicity, we see that $\mu(A_n) \leq \mu(A_1) < \infty$, hence we can split the limit and further get

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1) - \mu(A_1) + \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(A_n).$$

■

Example. Given $(\mathbb{N}, \mathcal{P}(\mathbb{N})$, counting measure). Then we see

- $A_n = \{n, n+1, n+2, \ldots\} \implies \mu(A_n) = \infty$
- $A_1 \supset A_2 \supset A_3 \supset \ldots$
- $\bigcap\limits_{i=1}^{\infty} A_i = \varnothing \implies \mu\left(\bigcap\limits_{i=1}^{\infty} A_i\right) = 0$

Remark. We see that in this case, since $\mu(A_1) \not< \infty$, hence continuity from above doesn't hold.

We now try to characterize some properties of a measure space.

Definition 1.7 (μ-null, μ-subnull, Complete measure space). Given $(X, \mathcal{A}, \mu)$,

- $A \subset X$ is a *μ-null set* if $A \in \mathcal{A}$ and $\mu(A) = 0$.
- $A \subset X$ is a *μ-subnull set* if $\exists \mu$-null set B such that $A \subset B$.
- $(X, \mathcal{A}, \mu)$ is a *complete* measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Note. We see that for a μ-subnull set, it's not necessary $\mathcal{A}$-measurable.

There are some useful terminologies we'll use later relating to μ-null.

Definition 1.8 (Almost everywhere). Given $(X, \mathcal{A}, \mu)$, a statement $P(x)$, $x \in X$ holds *μ-almost everywhere (a.e.)* if the set

$$\{x \in X : P(x) \text{ does not hold}\}$$

is μ-null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

Definition 1.9 (Finite measure). , Given $(X, \mathcal{A}, \mu)$

- μ is a *finite measure* if $\mu(X) < \infty$.
- μ is a *σ-finite measure* if $X = \bigcup\limits_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.

Exercise. Every measure space can be **completed**. Namely, we can always find a bigger σ-algebra to complete the space.

1.3 Outer Measures

We start by giving a definition.

Definition 1.10 (Outer measure). An *outer measure* on X is a map

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that

- $\mu^*(\varnothing) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$
- (countable subadditivity) $\mu^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) \leq \sum\limits_{i=1}^{\infty} \mu^*(A_i)$ for every $A_i \subset X$.

Example. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \colon \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

is an outer measure due to the Proposition 1.2 we're going to show.

Remark. We see that an outer measure need not be a measure. Check Definition 1.6.

Proposition 1.2. Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\varnothing, X \in \mathcal{E}$. Let

$$\rho \colon \mathcal{E} \to [0, \infty]$$

such that $\rho(\varnothing) = 0$. Then

$$\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

is an outer measure on X.

Note. Recall the Tonelli's Theorem[3] for series:

If $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Specifically, in [Tao13] Theorem 0.0.2.

Lecture 4: Carathéodory extension Theorem

12 Jan. 11:00

As previously seen. We now prove the Proposition 1.2.

Proof. We need to prove

- μ^* is well-defined. i.e., inf is taken over a non-empty set. This is trivial since $X \in \mathcal{E}$ and $X \supset A$ for any $A \in \mathcal{E}$.
- $\mu^*(\varnothing) = 0$. Since $\varnothing \in \mathcal{E}$ and

$$\mu^*(\varnothing) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \colon \underset{i \in \mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \varnothing \right\} = 0$$

since $\rho(\varnothing) = 0$ for all i and further, by Squeeze Theorem[4], we see that $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \rho(\varnothing) = 0$.

[3] https://en.wikipedia.org/wiki/Fubini%27s_theorem

[4] https://en.wikipedia.org/wiki/Squeeze_theorem

- $A \subset B \implies \mu^*(A) \leq \mu^*(B)$. We simply show this by contradiction. Suppose $A \subset B$ and $\mu^*(A) > \mu^*(B)$, then by definition of μ^*, we have

$$\begin{aligned}\mu^*(A) &= \inf\left\{\sum_{i=1}^{\infty}\rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A\right\}\\ &> \inf\left\{\sum_{i=1}^{\infty}\rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B\right\} = \mu^*(B).\end{aligned}$$

Now, let $B =: (B \setminus A) \cup A$, then we have

$$\begin{aligned}\mu^*(A) &= \inf\left\{\sum_{i=1}^{\infty}\rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A\right\}\\ &> \inf\left\{\sum_{i=1}^{\infty}\rho(E_i) \colon \underset{i\in\mathbb{N}}{\forall} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B\setminus A)\cup A\right\} = \mu^*(B).\end{aligned}$$

Now, since $B\setminus A \supseteq \varnothing$, then this inequality can't hold, hence a contradiction $\lightning$.

- Countable subadditivity. Let $A_1, A_2, \ldots \in X$. If one of $\mu^*(A_n) = \infty$, then result holds. So we may assume $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Now, fix any $\epsilon > 0$, we will show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\mu^*(A_n) + \epsilon.$$

For each $n \in \mathbb{N}$, $\exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ such that

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n$$

and

$$\mu^*(A_n) + \frac{\epsilon}{2^n}{}^{5} \geq \sum_{k=1}^{\infty}\rho(E_{n,k}).$$

Then we see that

$$\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{k,n},$$

which implies

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{(n,k)\in\mathbb{N}^2}\rho\left(E_{k,n}\right) \overset{!}{=} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\rho(E_{k,n}) \leq \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right)$$

from the inequality just derived. Now, since the last term is just

$$\sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) = \sum_{n=1}^{\infty}\mu^*(A_n) + \epsilon,$$

[5] This is an important trick!!

hence we finally have

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon$$

for arbitrarily small fixed $\epsilon > 0$, hence the subadditivity is proved.

■

Definition 1.11 (Carathéodory measurable). Let μ^* be an outer measure on X. We say $A \subset X$ is *Carathéodory measurable (C-measurable) with respect to* μ^* if

$$\forall E \subset X,\ \mu^*(E) = \mu^*\left(E \cap A\right) + \mu^*\left(E \setminus A\right).$$

Lemma 1.3. Let μ^* be an outer measure on X. Suppose $B_1, \ldots, B_N$ are disjoint C-measurable sets. Then,

$$\forall E \subset X,\ \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) = \sum_{i=1}^{N} \mu^*\left(E \cap B_i\right).$$

Proof. Since we have

$$\begin{aligned}
\mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) &= \mu^*\left(E' \cap B_1\right) + \mu^*\left(E' \setminus B_1\right)^{6} \\
&= \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i \cap B_1\right)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{N} B_i\right) \cap B_1^c\right) \\
&= \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_{i=2}^{N} B_i\right)\right)
\end{aligned}$$

where the equality comes from the fact that B_1 is C-measurable and disjoint from B_i, $i \neq 1$. Then, we simply iterate this argument and have the result. ■

Remark. This implies that if we restrict an outer measure on a C-measurable set, then it becomes finite additive.

Theorem 1.2 (Carathéodory extension Theorem). Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets (with respect to μ^*). Then,

1. $\mathcal{A}$ is a σ-algebra on X.
2. $\mu = \mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.
3. $(X, \mathcal{A}, \mu)$ is a complete measure space.

[6] Here, $E' := E \cap \left(\bigcup_{i=1}^{N} B_i\right)$ for the simplicity of notation.

Proof. We divide the proof in several steps.

1. We show $\mathcal{A}$ is a σ-algebra by showing
   (a) $\varnothing \in \mathcal{A}$. To show this, we simply check that $\varnothing$ is C-measurable. We see that
   $$\underset{E\subset X}{\forall}\ \mu^*(E) = \mu^*(E\cap\varnothing) + \mu^*(E\setminus\varnothing) = \mu^*(E),$$
   which just shows $\varnothing \in \mathcal{A}$.
   (b) $\mathcal{A}$ closed under complements. This is equivalent to say that if A is C-measurable, so is A^c. We see that if A is C-measurable, then for every $E \subset X$,
   $$\mu^*(E) = \mu^*(E\cap A) + \mu^*(E\setminus A).$$
   Observing that $E\cap A = E\setminus A^c$ and $E\setminus A = E\cap A^c$, hence
   $$\mu^*(E) = \mu^*(E\setminus A^c) + \mu^*(E\cap A^c).$$
   We immediately see that above implies $A^c \in \mathcal{A}$.
   (c) $\mathcal{A}$ closed under countable unions.

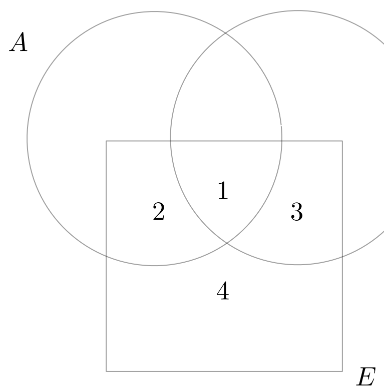
   Note. To show $\mathcal{A}$ closed under countable unions, we show that $\mathcal{A}$ is closed under:

   finite unions $\overset{\text{then}}{\Longrightarrow}$ countable <u>disjoint</u> unions $\overset{\text{then}}{\Longrightarrow}$ countable unions.

   - We show $\mathcal{A}$ is closed under finite unions.

     Claim. $A, B \in \mathcal{A} \implies A\cup B \in \mathcal{A}$.

     Fix $E \subset X$ arbitrary. We need to show that
     $$\mu^*(E) = \mu^*(E\cap(A\cup B)) + \mu^*(E\setminus(A\cup B)),$$
     i.e.,
     $$\mu^*(1\cup 2\cup 3\cup 4) = \mu^*(1\cup 2\cup 3) + \mu^*(4)$$
     given $A, B \in \mathcal{A}$.



- Since A is C-measurable,
  * $\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$
  * $\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$
- Since B is C-measurable,
  * $\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$

Hence, we have

$$\begin{aligned}\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4)\\ &= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4)\\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).\end{aligned}$$

- We show $\mathcal{A}$ is closed under countable disjoint unions.

  Let $A_1, A_2, \ldots \in \mathcal{A}$ and disjoint. Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

  $$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^{\infty} A_i\right),$$

  hence we only need to show another way around.

  Fix $N \in \mathbb{N}$, we have $\bigcup\limits_{n=1}^{N} A_n \in \mathcal{A}$ since N is finite, and

  $$\begin{aligned}\mu^*(E) &= \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} A_n\right)\right) + \mu^*\left(E \setminus \left(\bigcup_{n=1}^{N} A_n\right)\right)\\ &\geq \underbrace{\sum_{n=1}^{N} \mu^*(E \cap A_n)}_{\overset{!}{=}\mu^*\left(E\cap\left(\bigcup_{n=1}^{N} A_n\right)\right)} + \underbrace{\mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right)}_{\leq \mu^*\left(E\setminus\left(\bigcup_{n=1}^{N} A_n\right)\right)}.\end{aligned}$$

  Now, take $N \to \infty$ then we are done.

- We show $\mathcal{A}$ is closed under countable unions. DIY

The proof will be continued...

Lecture 5: Hahn-Kolmogorov Theorem

14 Jan. 11:00

Firstly, we see a stronger version of Lemma 1.3 we have seen before.

Lemma 1.4. Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots$ are disjoint C-measurable sets. Then,

$$\forall E \subset X,\ \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \sum_{i=1}^{\infty} \mu^*\left(E \cap B_i\right).$$

Proof.

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_i) \geq \mu^*\left(E \cap \bigcup_{n=1}^{\infty} B_n\right) \geq \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} B_n\right)\right) \overset{!}{=} \sum_{n=1}^{N} \mu^*\left(E \cap B_n\right).$$

Now, we just take $N \to \infty$ (or note that $N \in \mathbb{N}$ is arbitrary, we then get the result according to Squeeze Theorem[7]). ■

Let's continue the proof of Theorem 1.2.

2. Since from Definition 1.6, we need to show
   - $\mu(\varnothing) = 0$. This means that we need to show $\mu^*|_{\mathcal{A}}(\varnothing) = 0$. Since $\varnothing \in \mathcal{A}$ and μ^* is an outer measure, hence from the property of outer measure, it clearly holds.
   - Countable additivity of μ^* on $\mathcal{A}$ follows from the Lemma 1.4 with $E = X$
3. The proof is given in Theorem A.1.

■

1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets $\mathcal{E}$ and any ρ such that it assigns measure on $\mathcal{E}$, then it induces an outer measure by Proposition 1.2, finally complete the outer measure by Theorem 1.2.

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.2}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to σ-algebra we are working on till now.

Definition 1.12 (Algebra). Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called an *algebra on* X if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under **finite** unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ for $n < \infty$.

Remark. The only difference between an algebra and a σ-algebra is whether they closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an outer measure.

[7] https://en.wikipedia.org/wiki/Squeeze_theorem

Definition 1.13 (Pre-measure). Let $\mathcal{A}_0$ be an algebra on X. We say

$$\mu_0 \colon \mathcal{A}_0 \to [0, \infty]$$

is a *pre-measure* if

1. $\mu_0(\varnothing) = 0$
2. (finite additivity) $\mu_0\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu_0(A_i)$ if $A_1, \dots, A_n \in \mathcal{A}_0$ are disjoint.
3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}_0$, disjoint, then

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Lemma 1.5. (1) + (3) $\implies$ (2) in Definition 1.13.

Proof. It's easy to see that since μ_0 is monotone. ∎

Theorem 1.3 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in Proposition 1.2. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^*, then $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$. i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

Proof. We prove this theorem in two parts.

- We first show $\mathcal{A} \supset \mathcal{A}_0$. Let $A \in \mathcal{A}_0$, we want to show $A \in \mathcal{A}$, i.e., A is C-measurable, i.e.,

$$\forall E \subset X \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We first fix an $E \subset X$. From countable subadditivity of μ^*, we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Hence, we only need to show another direction. If $\mu^*(E) = \infty$, then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ clearly. So, assume $\mu^*(E) < \infty$.

Fix $\epsilon > 0$. By the Proposition 1.2 of μ^*, $\exists B_1, B_2, \dots \in \mathcal{A}_0$, $\bigcup_{n=1}^{\infty} B_n \supset E$ such that

$$\mu^*(E) + \epsilon \overset{!}{\geq} \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left(\mu_0(\underbrace{B_n \cap A}_{\in \mathcal{A}_0}) + \mu_0(\underbrace{B_n \cap A^c}_{\in \mathcal{A}_0}) \right)$$

by the finite additivity of μ_0. Note that

$$\begin{cases} \bigcup_{n=1}^{\infty}(B_n \cap A) \supset E \cap A \\ \bigcup_{n=1}^{\infty}(B_n \cap A^c) \subset E \cap A^c \end{cases} \implies \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

since

$$\mu^*(E \cap A) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n \cap A)$$

and

$$\mu^*(E \cap A^c) \leq \mu^*\left(\bigcup_{n=1}^{\infty}(B_n \cap A^c)\right) \leq \sum_{n=1}^{\infty}\mu^*(B_n \cap A^c).$$

We then see that for any $\epsilon > 0$, the inequality

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

holds, hence so does

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

which implies $\mathcal{A} \supset \mathcal{A}_0$.

The proof will be continued...

Lecture 6: Hahn-Kolmogorov Theorem and Extension.

18 Jan. 11:00

Let's continue the proof of Theorem 1.3.

- Let $A \in \mathcal{A}_0$, we want to show that

$$\mu(A) = \mu_0(A).$$

  - Firstly, let

$$B_i = \begin{cases} A, & \text{if } i = 1 \\ \varnothing, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,$$

    hence $\bigcup_{i=1}^{\infty} B_i = A$, then we see that

$$\mu^*(A) \leq \sum_{i=1}^{\infty}\mu_0(B_i) = \mu_0(A)$$

    from the definition of μ^* and countable additivity within the algebra of μ_0.

- Secondly, let $B_i \in \mathcal{A}_0$, $\bigcup_{i=1}^{\infty} B_i \supset A$ be arbitrary. Let $C_1 = A \cap B_1 \in \mathcal{A}_0$, $C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right) \in \mathcal{A}_0$ for $i \geq 2$ since the operations are finite. Then we see

  $$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

  are disjoint countable unions, by countable additivity within the algebra, we therefore have

  $$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i) \implies \mu_0(A) \leq \mu^*(A)$$

  by taking the infimum from the definition of μ^*.

Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every $A \in \mathcal{A}_0$, which implies

$$\mu(A) = \mu_0(A)$$

for every $A \in \mathcal{A}_0$ from Theorem 1.2, where we extend μ^* to μ respect to $\mathcal{A}_0$.

■

Definition 1.14 (HK extension). $(\mathcal{A}, \mu)$ obtained from Theorem 1.3 is the *Hahn-Kolmogorov extensions* of $(\mathcal{A}_0, \mu_0)$.

We can show the uniqueness of HK extension.

Theorem 1.4 (Uniqueness of HK extension). Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$. Let $(\mathcal{A}, \mu)$ be the HK extension of $(\mathcal{A}_0, \mu_0)$. Let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$. Then if μ_0 is σ-finite, $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$.

Note. Notice that $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$ since they both extend $\mathcal{A}_0$.

Proof. Let $A \in \mathcal{A} \cap \mathcal{A}'$, we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that $\mu^*(A) \geq \mu'(A)$ by choosing the arbitrary cover of A and using the definition of μ^*.

Secondly, we will show that $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) < \infty$, and fix $\epsilon > 0$. Then there exists $B_i \in \mathcal{A}_0$ with $B := \bigcup_{i=1}^{\infty} B_i \supset A$ such that

$$\mu(A) + \epsilon = \mu^*(A) + \epsilon \overset{!}{\geq} \sum_{i=1}^{\infty} \mu_0(B_i) \overset{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

  This implies that
  $$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$
  where the first equality comes from $A \subset B$ and $\mu(A) < \infty$. On the other hand,

$$\mu(B) = \lim_{N\to\infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) \overset{8}{=} \lim_{N\to\infty} \mu'\left(\bigcup_{i=1}^{N} B_i\right) = \mu'(B),$$

  hence,

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \overset{9}{\leq} \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon$$

  for arbitrary ϵ, so we conclude $\mu(A) \leq \mu'(A)$.

- Assume $\mu(A) = \infty$. Since μ_0 is σ-finite, so we know $X = \bigcup_{n=1}^{\infty} X_n$ for some $X_n \in \mathcal{A}_0$ such that
  $$\mu_0(X_n) < \infty.$$
  Replacing X_n by $X_1 \cup \ldots \cup X_n \in \mathcal{A}_0$, we may assume that
  $$X_1 \subset X_2 \subset \ldots.$$
  Then,
  $$\forall_{n\in\mathbb{N}}\ \mu(A \cap X_n) < \infty \overset{!}{\implies} \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$
  From the continuity of measure, we then have
  $$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) \leq \lim_{n\to\infty} \mu'(A \cap X_n) = \mu'(A).$$

■

[8] $\mu = \mu'$ on $\mathcal{A}_0$.
[9] From the first part.

Corollary 1.1. Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then

$$\exists! \text{ measure } \mu \text{ on } \langle \mathcal{A}_0 \rangle \text{ that extends } \mathcal{A}_0.$$

Furthermore,

- The completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.
- $$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \underset{i \in \mathbb{N}}{\forall} \bigcup_{i=1}^{\infty} B_i \supset A \right\}$$

  for all $A \in \overline{\langle \mathcal{A}_0 \rangle}$.

Lecture 7: Borel Measures

21 Jan. 11:00

1.5 Borel Measures on $\mathbb{R}$

We first introduce so-called *distribution function.*

Definition 1.15 (Distribution function). An increasing[a] function

$$F \colon \mathbb{R} \to \mathbb{R}$$

and right-continuous. F is then a *distribution function.*

[a]Here, increasing means $F(x) \leq F(y)$ for $x < y$.

Example. Here are some examples of right-continuous functions.

1. $F(x) = x$.
2. $F(x) = e^x$.
3. Define
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$
4. Let $\mathbb{Q} \coloneqq \{r_1, r_2, \ldots\}$. Define
$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$
   and
$$F(x) \coloneqq \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$
   Then F is a distribution function (hence right-continuous). This is shown in Lemma A.1.

Note. If F is increasing, and

$$F(\infty) := \lim_{x \nearrow \infty} F(x), \quad F(-\infty) := \lim_{x \searrow \infty} F(x)$$

exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.[10]

Now, we can define a *Borel measure* on $(X, \mathcal{B}(\mathbb{R}))$.

Definition 1.16 (Borel mesaure). A *Borel measure* is any measure μ defined on the σ-algebra of Borel sets.

Definition 1.17 (Locally finite). Let X be a Hausdorff topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Note. Some authors will require a Borel measure equipped with the locally finite property. But formally, this is not so common.

Lemma 1.6. Let μ be a locally finite Borel measure on $\mathbb{R}$, then

$$F_\mu(x) = \begin{cases} \mu\left((0, x]\right), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu\left((x, 0]\right), & \text{if } x < 0 \end{cases}$$

is a distribution function.

Proof. To show F_μ is increasing, consider $x < y$ such that

$$F_\mu(x) \leq F_\mu(y)$$

by considering

- $x > 0$: Then $F_\mu(x) = \mu((0, x])$ and

$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$

- $x = 0$: Then $F_\mu(x) = 0$ and

$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$

  since $y > 0$.

- $x < 0$: Follows the same argument with $x > 0$.

[10] There are distributions [FF99] Ch9., but these are different from distribution functions.

Now, we need to show F_μ is right-continuous. Firstly, assume that $x \geq 0$, then we see that

$$F_\mu(x) = \mu((0, x]) = \mu((0, x^+])$$

from the fact that a measure is right-continuous.[11] Now, if $x \leq 0$, the same argument follows since multiplying -1 will not change the fact that a measure is continuous. ■

Definition 1.18 (Half intervals). We call

$$\varnothing, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

half-intervals.

Lemma 1.7. Let $\mathcal{H}$ be the collection of finite disjoint unions of half-intervals. Then, $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. We see that

- $\varnothing \in \mathcal{H}$. Clearly.
- To show $\mathcal{H}$ is closed under complements, we have
  - $\varnothing^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$.
  - $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$.[12]
  - $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$.
  - $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$.
  - $(-\infty, \infty)^c = \varnothing \in \mathcal{H}$.
- $\mathcal{H}$ is closed under finite unions, clearly.

■

[11] Actually, a measure is always continuous.
[12] Since it's a two disjoint union of half intervals.

Proposition 1.3 (Distribution function defines a pre-measure). Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function. For a half interval I, define

$$\ell(I) \coloneqq \ell_F(I) = \begin{cases} 0, & \text{if } I = \varnothing; \\ F(b) - F(a), & \text{if } I = (a, b]; \\ F(\infty) - F(a), & \text{if } I = (a, \infty]; \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b]; \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 \coloneqq \mu_{0,F}$ as

$$\mu_{0,F}\colon \mathcal{H} \to [0, \infty]$$

by

$$\mu_0(A) = \sum_{k=1}^{N} \ell(I_k) \text{ if } A = \bigcup_{k=1}^{N} I_k,$$

where A is a finite disjoint union of half intervals $I_1, \ldots, I_N$. Then, μ_0 is a pre-measure on $\mathcal{H}$.

Proof. We see that

1. μ_0 is well-defined.
2. $\mu_0(\varnothing) = 0$.
3. μ_0 is finite additive.
4. μ_0 is countable additive within $\mathcal{H}$.

   Suppose $A \in \mathcal{H}$ where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.[13]

   Focus on the case $I = (a, b]$. Let

   $$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

   which is a disjoint union. Then we only need to check

   $$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

   - Since $(a, b] \supset \bigcup_{n=1}^{N} (a_n, b_n]$ for any fixed $N \in \mathbb{N}$, hence

     $$\forall_{N \in \mathbb{N}}\ F(b) - F(a) \geq \sum_{n=1}^{N} (F(b_n) - F(a_n)).$$

[13] Since $\mathcal{H}$ is only a collection of *finite* disjoint half intervals, hence after considering $A = I$, we can apply the same argument iteratively and stop in finite steps. Formally, we can consider $H \in \mathcal{H}$, $H = \bigcup_{i=1}^{\infty} A^i$, where A^i being a half interval. Then by the above argument, we have $A^i = I^i$ and so on.

By letting $N \to \infty$, we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right).$$

- Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

$$F(a') - F(a) < \epsilon.$$

For each $n \in \mathbb{N}$, $\exists b_n' > b_n$ such that

$$F(b_n') - F(b_n) < \frac{\epsilon}{2^n}.$$

Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b_n'),$$

hence

$$\underset{N\in\mathbb{N}}{\exists}\ [a', b] \subset \bigcup_{n=1}^{N} (a_n, b_n'),$$ [14]

which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^{N} F(b_n') - F(a_n).$$

Finally, we see that

$$\begin{aligned} F(b) - F(a) &\leq F(b) - F(a') + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n') - F(a_n)\right) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\epsilon}{2^n}\right) + \epsilon \\ &= \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right) + 2\epsilon \end{aligned}$$

for any fixed $\epsilon > 0$, hence

$$F(b) - F(a) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

Combine these two inequalities, we have

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

as we desired.

[14] This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.

$\blacksquare$

Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8: Lebesgue-Stieltjes Measure on $\mathbb{R}$

24 Jan. 11:00

To classify all measures, we now see this last theorem to complete the task.

Theorem 1.5 (Locally finite Borel measures on $\mathbb{R}$). We have

1. $F\colon \mathbb{R} \to \mathbb{R}$ a distribution function, then there exists a **unique** *locally finite Borel measure* μ_F on $\mathbb{R}$ satisfying

$$\mu_F((a, b]) = F(b) - F(a)$$

for every $a < b$.

2. Suppose $F, G\colon \mathbb{R} \to \mathbb{R}$ are distribution functions. Then,

$$\mu_F = \mu_G$$

on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. $\blacksquare$

HW.

Remark. Theorem 1.5 simply states that given a distribution function, if we restrict our attention on locally finite measures on $\mathbb{R}$ following our usual convention, then it defines the measure on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

1.6 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that

$$F \text{ distribution function} \overset{!}{\implies} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 1.19 (Lebesgue-Stieltjes measure). Given a distribution function F, we say μ_F on $\mathcal{A}_{\mu_F}$ is called the *Lebesgue-Stieltjes measure* corresponding to F.

Definition 1.20 (Lebesgue measure, Lebesgue σ-algebra). From Definition 1.19, if $F(x) = x$, then the induced $(\mathcal{A}_{\mu_F}, \mu_F)$ is denoted as $(\mathcal{L}, m)$, where $\mathcal{L}$ is called *Lebesgue σ-algebra*, and m is called *Lebesgue measure*.

Remark. Recall that $\mathcal{L}$ is induced by Theorem 1.2, namely given m, for all $A \subset \mathbb{R}$, we have

$$\mathcal{L} := \left\{ A \subset \mathbb{R} \mid \underset{E \subset \mathbb{R}}{\forall} m(A) = m(A \cap E) + m(A \setminus E) \right\}$$

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).^{15}$$

Example. We first see some examples.

1. $\mu_F((a, b]) = F(b) - F(a)$. Then
    - $\mu_F(\{a\}) = F(a) - F(a^-)$
    - $\mu_F([a, b]) = F(b) - F(a^-)$
    - $\mu_F((a, b)) = F(b^-) - F(a)$
2. We define
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$
Then
    - $\mu_F(\{0\}) = 1$
    - $\mu_F(\mathbb{R}) = 1$
    - $\mu_F(\mathbb{R} \setminus \{0\}) = 0$. This is easy to see since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, hence
$$\begin{aligned} \mu_F(\mathbb{R} \setminus \{0\}) &= \mu_F((-\infty, 0) \cup (0, \infty)) \\ &= \underbrace{\mu_F((-\infty, 0))}_{0-0^{16}} + \underbrace{\mu_F((0, \infty))}_{1-1^{17}} = 0. \end{aligned}$$

    We call that μ_F is the *Dirac measure* at 0.
3. Denote $\mathbb{Q} = \{r_1, r_2, \ldots\}$, and we define
$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r. \end{cases}$$
Then
    - $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
    - $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$.

[15] Some text will use $x-$ and $x+$ instead of x^- and x^+, respectively.
[16] It follows from $F(0^-) - F(-\infty) = 0 - 0 = 0$.
[17] It follows from $F(\infty) - F(0) = 1 - 1 = 0$.

This is shown in Lemma A.2.

4. If F is continuous at a, then $\mu_F(\{a\}) = 0$.
5. $F(x) = x$, then recall that we denote $\mu_F \coloneqq m$, and we have
   - $m((a, b]) = m((a, b)) = m([a, b]) = b - a$.
6. $F(x) = e^x$
   - $\mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$.

Remark. We see that the first two examples are *discrete measures*.

Example (Middle thirds Cantor set). Let $C \coloneqq \bigcap\limits_{n=1}^{\infty} K_n$, where we have

$$\begin{aligned}
K_0 &\coloneqq [0, 1] \\
K_1 &\coloneqq K_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \\
K_2 &\coloneqq K_1 \setminus \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \\
&\vdots \\
K_n &\coloneqq K_{n-1} \setminus \bigcup_{k=1}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3^{k+2}}{3^{n+1}}\right).
\end{aligned}$$

We see that C is uncountable and with $m(C) = 0$. And observe that $x \in C$ if and only if $x = \sum\limits_{n=1}^{\infty} \frac{a_n}{3}$ for some $a_n \in \{0, 2\}$. Hence, we can instead formulate K_n by

$$K_n = \bigcup_{\substack{a_i \in \{0,2\} \\ 1 \leq i \leq n}} \left[\sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n}\right].$$

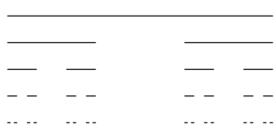


Figure 1: The top line corresponds to K_0, and then K_1, etc.

The proof of $m(C) = 0$ is given in Lemma A.3.

1.6.1 Cantor Function

Consider F as follows. We define a function F to be 0 to the left of 0, and 1 to the right of 1. Then, define F to be $\frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$, $\frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9}\right)$ and so on. This is so-called *Cantor Function*. We can show F is continuous and increasing, which makes F a distribution function. Also, we see that the measure this F induced is called *Cantor measure*.

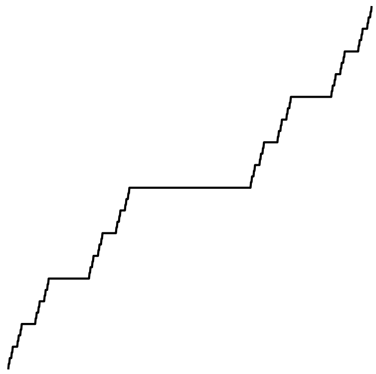


Figure 2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

Cantor Measure μ_F		Lebesgue Measure m
$\mu_F(\mathbb{R} \setminus C) = 0$		$m(\mathbb{R} \setminus C) = \infty > 0$
$\mu_F(C) = 1$	$\Longleftrightarrow$	$m(C) = 0$
$\mu_F(\{a\}) = 0$		$m(\{a\}) = 0$

Remark. μ_F and m are said to be **singular** to each other.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 1.8. Let μ be Lebesgue-Stieltjes measure on $\mathbb{R}$. Then we have

$$\begin{aligned}\mu(A) &\overset{!}{=} \inf\left\{\sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supset A\right\} \\ &= \inf\left\{\sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A\right\}\end{aligned}$$

for every $A \in \mathcal{A}_\mu$

Proof. The second equality follows from the continuity of the measure. ∎

Remark. This is similar to

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - 1/n], \quad (a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n].$$

Lecture 9: Properties of Lebesgue-Stieltjes measure

26 Jan. 11:00

As previously seen. Let $X \subset [0, \infty]$. Recall that

- Finite supremum.

$$\alpha = \sup X < \infty \iff \begin{cases} \underset{x \in X}{\forall} \ \alpha \geq x \\ \underset{\epsilon > 0}{\forall} \ \underset{x \in X}{\exists} \ x + \epsilon \geq \alpha. \end{cases}$$

- Infinite supremum.

$$\alpha = \sup X = \infty \iff \underset{L > 0}{\forall} \ \underset{x \in X}{\exists} \ x \geq L.$$

This should be useful latter on.

Theorem 1.6 (Regularity). Let μ be Lebesgue-Stieltjes measure. Then, for all $A \in \mathcal{A}_\mu$,

1. (outer regularity) $\mu(A) = \inf\{\mu(O) \mid O \supset A, O \text{ is open}\}$
2. (inner regularity) $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$

Proof. We check them separately.

1. DIY
2. Let $s \coloneqq \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$, then by monotonicity, we have $\mu(A) \geq s$. To show the other direction, we consider

   - A is a bounded set.

     Then $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, $\overline{A}$ is also bounded $\implies$ $\mu(\overline{A}) < \infty$. Fix $\epsilon > 0$, then by outer regularity, there exists an open $O \supset \overline{A} \setminus A$, and $\mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \epsilon$. Let $K \coloneqq \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$, we show that

     $$\mu(K) \geq \mu(A) - \epsilon.$$

     DIY

   - A is an unbounded set with $\mu(A) < \infty$.

     Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n = A \cap [-n, n]$ where $A_1 \subset A_2 \subset \ldots$, then

     $$\lim_{n \to \infty} \mu(A_n) = \mu(A) < \infty.$$

   - A is an unbounded set with $\mu(A) = \infty$.

     We can show that

     $$\lim_{n \to \infty} \mu(A_n) = \mu(A) = \infty.$$

     Fix $L > 0$, then $\exists N$ such that $\mu(A_N) \geq L$.

■

Definition 1.21 (G_δ-set, F_σ-set). Let X be a topological space. Then

- A G_δ*-set* is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open.
- A F_σ*-set* is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.7. Let μ be a Lebesgue-Stieltjes measure. Then *TFAE*[a]:

1. $A \in \mathcal{A}_\mu$
2. $A = G \setminus M$, G is a G_δ-set, M is a μ-null set.
3. $A = F \setminus N$, F is a F_σ-set, N is a μ-null set.

[a] *TFAE*: The following are equivalent.

Proof. We see that (2.) $\implies$ (1.) and (3.) $\implies$ (1.) are clear.

- (1.) $\implies$ (3.)
  - Assume $\mu(A) < \infty$. From the inner regularity, we have

$$\forall n \in \mathbb{N}\ \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

  Let $F = \bigcup_{n=1}^{\infty} K_n$, then $N = A \setminus F$ is μ-null. Check!

  - Assume $\mu(A) = \infty$. Let $A = \bigcup_{k \in \mathbb{Z}} A_k$, $A_k = A \cap (k, k+1]$. From what we have just shown above,

$$\forall k \in \mathbb{Z}\ A_k = F_k \cup N_k,\ A = \underbrace{\left(\bigcup_k F_k\right)}_{F_\sigma\text{-set}} \cup \underbrace{\left(\bigcup_k N_k\right)}_{\mu\text{-null}}.$$

- (1.) $\implies$ (2.)

  We see that

$$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

∎

Proposition 1.4. Let μ be a Lebesgue-Stieltjes measure, and $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then we have

$$\forall \epsilon > 0\ \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that $\mu(A \triangle I) \leq \epsilon$.

Proof. Using outer regularity and the fact that every open set is $\bigcup_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. ∎ DIY

We now see some properties of Lebesgue measure.

Theorem 1.8. Let $A \in \mathcal{L}$, then we have $A+s \in \mathcal{L}$, $rA \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. i.e.,

$$m(A+s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

Proof. ∎ DIY

Example. We now see some examples.

1. Let $\mathbb{Q} =: \{r_i\}_{i=1}^{\infty}$ which is dense in $\mathbb{R}$. Let $\epsilon > 0$, and

$$O = \bigcup_{i=1}^{\infty} \left(r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i}\right).$$

We see that O is open and dense[18] in $\mathbb{R}$. But we see

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$

Furthermore, $\partial O = \overline{O} \setminus O$, $m(\partial O) = \infty$

2. There exists uncountable set A with $m(A) = 0$.
3. There exists A with $m(A) > 0$ but A contains no non-empty open intervals.
4. There exists $A \notin \mathcal{L}$. e.g. Vitali set.[19]
5. There exists $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$.

Lecture 10: Integration

26 Jan. 11:00

2 Integration

2.1 Measurable Function

We start with a definition.

Definition 2.1 (Measurable function). Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$ are measurable spaces. Then we say $f \colon X \to Y$ is $(\mathcal{A}, \mathcal{B})$*-measurable* if

$$\forall_{B \in \mathcal{B}} \ f^{-1}(B) \in \mathcal{A}.$$

Remark. If $\mathcal{A}$ and $\mathcal{B}$ are given, we'll sometimes say f is measurable if it'll not cause any confusions.

[18] https://en.wikipedia.org/wiki/Dense_set
[19] https://en.wikipedia.org/wiki/Vitali_set

Lemma 2.1. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, and suppose $\mathcal{B} = \langle \mathcal{E} \rangle$ for some $\mathcal{E} \subset Y$. Then,

$$f\colon X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \underset{E \in \mathcal{E}}{\forall} f^{-1}(E) \in \mathcal{A}.$$

Proof. We see that the *only if* part ($\implies$) is clear. On the other direction, we consider the following. Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$, then

- $\mathcal{E} \subset \mathcal{D}$ by assumption
- $\mathcal{D}$ is a σ-algebra

Check!

hence, we see that $\langle \mathcal{E} \rangle = \mathcal{B} \subset \mathcal{D}$ from Lemma 1.2. The result then follows from the definition of $(\mathcal{A}, \mathcal{B})$-measurable. ■

Note. Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup_\alpha E_\alpha\right) = \bigcup_\alpha f^{-1}(E_\alpha)$

Definition 2.2 ($\mathcal{A}$-measurable). Let $(X, \mathcal{A})$ be a measurable space. Then,

$$\left.\begin{array}{l} f\colon X \to \mathbb{R} \\ f\colon X \to \overline{\mathbb{R}} \\ f\colon X \to \mathbb{C} \end{array}\right\} \text{ is } \mathcal{A}\textit{-measurable} \text{ if } \begin{cases} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f\colon X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{cases}$$

Notation. Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$
- $\mathcal{B}(\overline{\mathbb{R}}) = \left\{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\right\}$.
- $\Re f$ is the real part of f, while $\Im f$ is the imaginary part of f.

Example. We see that

- $\mathcal{A} = \mathcal{P}(X) \implies$ Every function is $\mathcal{A}$-measurable.
- $\mathcal{A} = \{\varnothing, X\} \implies$ The only $\mathcal{A}$-measurable functions are constant functions.

Definition 2.3 (Lebesgue measurable). A *Lebesgue measurable function* f is a measurable function

$$f\colon (\mathbb{R}, \mathcal{L}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})).$$

Lemma 2.2. Given $f\colon X \to \mathbb{R}$, *TFAE.*

1. f is $\mathcal{A}$-measurable
2. $\forall a \in \mathbb{R},\ f^{-1}((a, \infty)) \in \mathcal{A}$
3. $\forall a \in \mathbb{R},\ f^{-1}([a, \infty)) \in \mathcal{A}$
4. $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a)) \in \mathcal{A}$
5. $\forall a \in \mathbb{R},\ f^{-1}((-\infty, a]) \in \mathcal{A}$

Proof. The result follows from Lemma 2.1 we just saw. ■

Remark (Operations preserve $\mathcal{A}$-measurability). Given $f, g\colon X \to \mathbb{R}$ and is $\mathcal{A}$-measurable, then

1. $\phi\colon \mathbb{R} \to \mathbb{R}$, $\mathcal{A}$-measurable[20], then
$$\phi \circ f\colon X \to \mathbb{R}$$
is $\mathcal{A}$-measurable.
2. $-f,\ 3f,\ f^2,\ |f|$ are all $\mathcal{A}$-measurable, and $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) \neq 0, \forall x \in X$.
3. $f + g$ is $\mathcal{A}$-measurable. We see this from
$$(f+g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty)) \right)$$
with Lemma 2.2.
4. $f \cdot g$ is $\mathcal{A}$-measurable. We see this from
$$f(x)g(x) = \frac{1}{2} \left((f(x) + g(x))^2 - f(x)^2 - g(x)^2 \right).$$
5. We see that
$$(f \vee g)(x) \coloneqq \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) \coloneqq \min\{f(x), g(x)\}$$
are $\mathcal{A}$-measurable.
6. Let $f_n\colon X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable. Then
$$\sup_{n \in \mathbb{N}} f_n,\ \inf_{n \in \mathbb{N}} f_n,\ \limsup_{n \to \infty} f_n,\ \liminf_{n \to \infty} f_n$$
are $\mathcal{A}$-measurable.

   Proof. Consider $\sup_{n \in \mathbb{N}} f_n =: g$, then
$$g^{-1}((a, \infty]) = \bigcup_{n \in \mathbb{N}} f_n^{-1}((a, \infty])$$

for $\sup_n f_n(x) = g(x) > a$. A similar argument can prove the case of $\inf_{n\in\mathbb{N}} f_n$. check

And notice that $\limsup_{n\to\infty} f_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} f_n$, then the similar argument also proves this case. ■

7. If $\lim_{n\to\infty} f_n(x)$ converges for every $x \in X$, then f is $\mathcal{A}$-measurable.
8. If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous

   $\implies$ f is Borel measurable

   $\implies$ f is Lebesgue measurable

   since the preimage of an open set of a continuous function is open, then we consider $f^{-1}((a,\infty))$.

Definition 2.4 (Support). The *support* of function $f\colon X \to \overline{\mathbb{R}}$ is

$$\operatorname{supp} f := \{x \in X \mid f(x) \neq 0\}.$$

Definition 2.5 (Positive and Negative part). For $f\colon X \to \overline{\mathbb{R}}$, let $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$,[a] where we call f^+ the *positive part* of f while f^- the *negative part* of f.

[a] i.e., $f^+(x) = \max\{f(x), 0\}$, $\quad f^-(x) = \max\{-f(x), 0\}$

Remark. If $\operatorname{supp} f^+ \cap \operatorname{supp} f^- = \varnothing$ and $f(x) = f^+(x) - f^-(x)$, then

$$f \text{ is } \mathcal{A}\text{-measurable} \iff f^+, f^- \text{ are } \mathcal{A}\text{-measurable}.$$

Definition 2.6 (Characteristic (Indicator) function). For $E \subset X$, the *characteristic (indicator) function* of E is

$$\mathcal{X}_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

Remark. We see that $\mathbb{1}_E$ is $\mathcal{A}$-measurable $\iff E \in \mathcal{A}$.

Definition 2.7 (Simple function). Let $(X, \mathcal{A})$ be a measurable space. Then a *simple function* $\phi\colon X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

Remark. We see that if

$$\phi(X) = \{c_1, \ldots, c_N\},$$

[20] In this case, we also call it *Borel measurable*.

then

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \implies \phi = \sum_{i=1}^{N} \underbrace{c_i}_{\neq \pm\infty} \mathbb{1}_{\underbrace{E_i}_{\in \mathcal{A}}}.$$

Lecture 11: Integration of nonnegative functions

31 Jan. 11:00

As previously seen. For a simple function ϕ, c_i can actually be in $\mathbb{C}$.

Theorem 2.1. Given a measurable space $(X, \mathcal{A})$ and let $f\colon X \to [0, \infty]$, the followings are equivalent.

1. f is a $\mathcal{A}$-measurable function.
2. There exists simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ such that
$$\forall_{x \in X} \lim_{n\to\infty} \phi_n(x) = f(x)$$
i.e., f is a pointwise upward limit of simple functions.

Proof. We'll prove both directions.

- It's clear that (2.) $\implies$ (1.) from the fact that $f(x) = \sup_n \phi_n(x)$ and the remark.
- We want to show that (1.) $\implies$ (2.). Assume f is $\mathcal{A}$-measurable, and fix $n \in \mathbb{N}$.

  Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. Also, for $0 \leq k \leq 2^{2n} - 1$, $E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}$.

  Then, define ϕ_n be

  $$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n},$$

  we have

  - $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ for every $x \in X$
  - $\forall x \in X \setminus F_n$, we have $0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$

  Furthermore, we see that

  $$F_1 \supset F_2 \supset \ldots, \qquad \bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\}),$$

  then

  - $x \in f^{-1}([0, \infty]) = X \setminus \bigcap_{n=1}^{\infty} F_n \implies \lim_{n\to\infty} \phi_n(x) = f(x)$
  - $x \in f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} F_n \implies f_n(x) \geq 2^n \implies \lim_{n\to\infty} \phi_n(x) = \infty = f(x)$

■

Corollary 2.1. If f is bounded on a set $A \subset \mathbb{R}$, i.e., $\exists L > 0$ such that

$$\forall_{x \in A} \ |f(x)| \leq L,$$

then there exists a sequence of simple functions $\{\phi_n\}$ such that $\phi_n \to f$ uniformly on A.

Proof. ■ DIY

Corollary 2.2. If $f \colon X \to \mathbb{C}$ is a measurable function if and only if there exists simple functions $\phi_n \colon X \to \mathbb{C}$ such that

$$0 \leq |\phi_1(x)| \leq |\phi_2(x)| \leq \ldots \leq |f(x)|$$

with

$$\forall_{x \in X} \lim_{n \to \infty} \phi_n(x) = f(x).$$

Proof. ■ DIY

2.2 Integration of Nonnegative Functions

We start with our first definition about integral.

Definition 2.8 (Integration of nonnegative funciton). Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\phi \colon X \to [0, \infty]$ such that

$$\phi = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}$$

be a simple function. Define

$$\int \phi = \int \phi \,\mathrm{d}\mu = \int_X \phi \,\mathrm{d}\mu = \sum_{i=1}^{N} c_i \mu(E_i).$$

Furthermore, for $A \in \mathcal{A}$,

$$\int_A \phi = \int_A \phi \,\mathrm{d}\mu = \int \phi \mathbb{1}_A \,\mathrm{d}\mu.$$

Note. Note that

- In the expression $\sum\limits_{i=1}^{N} c_i \mu(E_i)$, we're using the convention $0 \cdot \infty = 0$.

- The function $\phi\mathbb{1}_A$ is also a simple function since both ϕ and $\mathbb{1}_A$ are simple function.

Proposition 2.1. Suppose we have $\phi, \psi \geq 0$ be two simple functions. Then,

- Definition 2.8 is well-defined.
- $\int c\phi = c\int\phi$ for $c \in [0, \infty)$.
- $\int \phi + \psi = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x)$ for all $x \implies \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi \,\mathrm{d}\mu$ is a measure on $(X, \mathcal{A})$.

Proof. ■ DIY

Definition 2.9 (Generatlization of Integration of nonnegative function). Given $(X, \mathcal{A}, \mu)$ with $f\colon X \to [0, \infty]$ be $\mathcal{A}$-measurable. Define

$$\int f = \int f \,\mathrm{d}\mu = \sup\left\{\int \phi \colon 0 \leq \phi \leq f \text{ such that } \phi \text{ is simple}\right\}.$$

Note. Note that

- If f is a simple function, the Definition 2.8 and Definition 2.9 of $\int f$ are the same.
- $\int cf = c\int f$ for $c \in [0, \infty)$.
- If $f \geq g \geq 0 \implies \int f \geq \int g$.
- But $\int f + g = \int f + \int g$ is not trivial.

Theorem 2.2 (Monotone Convergence Theorem (MCT)). Given $(X, \mathcal{A}, \mu)$ be a measure space. Then if

- $f_n\colon X \to [0, \infty]$ be $\mathcal{A}$-measurable for every $n \in \mathbb{N}$;
- $0 \leq f_1(x) \leq f_2(x) \leq \ldots$ for every $x \in X$;
- $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$,

we have

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. Note that if $\lim\limits_{n\to\infty} \int f_n$ exists, then it's equal to $\sup_n \int f_n$.

Then

- $f_n \leq f \implies \int f_n \leq \int f \implies \lim\limits_{n\to\infty} \int f_n \leq \int f$.

- Fix a simple function $0 \le \phi \le f$, then it's enough to show $\lim_{n\to\infty} \int f_n \ge \int \phi$.

  We first fix $\alpha = (0,1)$, then it's also enough to show

  $$\lim_{n\to\infty} \int f_n \ge \alpha \int \phi.$$

  Let $A_n \coloneqq \{x \in X \mid f_n(x) \ge \alpha\phi(x)\}$, then since f_n is measurable,

  - $A_n \in \mathcal{A}$
  - $A_1 \subset A_2 \subset A_3 \subset \dots$
  - $\bigcup_{n=1}^{\infty} A_n = X$ Check!

  We then have

  $$\int f_n \ge \int f_n \mathbb{1}_{A_n} \ge \int \alpha\phi\mathbb{1}_{A_n} = \alpha \int_{A_n} \phi = \alpha\nu(A_n)$$

  where $\nu(A) = \int_A \phi$ is a measure. This implies

  $$\lim_{n\to\infty} \int f_n \ge \alpha \lim_{n\to\infty} \nu(A_n) \overset{21}{=} \alpha\nu(X) = \alpha \int \phi.$$

∎

Corollary 2.3 (Linearity of nonnegative integral). Let $f, g \ge 0$ be measurable, then

$$\int f + g = \int f + \int g.$$

Proof. There exists simple functions ϕ_n and ψ_n such that

- $0 \le \phi_1 \le \phi_2 \le \dots$ and $\phi_n \to f$ pointwise
- $0 \le \psi_1 \le \psi_2 \le \dots$ and $\psi_n \to g$ pointwise

Then,

$$\int (f+g) \overset{!}{=} \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \int \phi_n + \int \psi_n \overset{!}{=} \int f + \int g.$$

∎

Lecture 12: Fatou's Lemma

2 Feb. 11:00

We start with a useful corollary.

[21] This follows from the continuity of measure from below

Corollary 2.4 (Tonelli's theorem for nonnegative series and integrals). Given $g_n \geq 0$ for every $n \in \mathbb{N}$ and let g_n be measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

Remark. Recall that we have seen two series case before. We'll later see two integrals cases.

Proof. Let $f_N := \sum_{n=1}^{N} g_n$ such that $\lim_{N\to\infty} f_N \sum_{n=1}^{\infty} g_n =: f$, then since $g_n \geq 0$, we have $0 \leq f_1 \leq f_2 \leq \ldots$ with

$$\lim_{N\to\infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

By Theorem 2.2, we have

$$\lim_{N\to\infty} \int \underbrace{\sum_{n=1}^{N} g_n}_{f_N} = \int \underbrace{\sum_{n=1}^{\infty} g_n}_{f}.$$

Now, since the terms in the limit on the left-hand side is just a finite sum, by Corollary 2.3, we have

$$\underbrace{\lim_{N\to\infty} \sum_{n=1}^{N}}_{\sum_{n=1}^{\infty}} \int g_n = \lim_{N\to\infty} \int \sum_{n=1}^{N} g_n = \int \sum_{n=1}^{\infty} g_n,$$

hence

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

∎

Theorem 2.3 (Fatou's Lemma). Suppose $f_n \geq 0$ and measurable, then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Remark. Recall that

$$\liminf_{n\to\infty} f_n := \lim_{k\to\infty} \inf_{n\geq k} f_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} f_n$$

and

$$\exists \lim_{n\to\infty} a_n \iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Proof. Let $g_k = \inf_{n\geq k} f_n$, then g_k is measurable and $0 \leq g_1 \leq g_2 \leq \ldots$. Now, from Theorem 2.2, we have

$$\int \lim_{k\to\infty} g_k = \lim_{k\to\infty} \int g_k.$$

Notice that the left-hand side is just $\int \liminf\limits_{n\to\infty} f_n$, while the right-hand side is just $\lim\limits_{k\to\infty} \int \inf\limits_{n\geq k} f_n$, i.e.,

$$\int \liminf_{n\to\infty} f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n.$$

We see that we want to take the inf outside the integral on the right-hand side. Observe that

$$\forall_{m\geq k} \inf_{n\geq k} f_n \leq f_m \implies \forall_{m\geq k} \int \inf_{n\geq k} f \leq \int f_m \implies \int \inf_{n\geq k} f_n \leq \inf_{m\geq k} \int f_m.$$

Then, we have

$$\int \liminf_{n\to\infty} f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n \leq \lim_{k\to\infty} \inf_{m\geq k} \int f_m = \liminf_{m\to\infty} \int f_m.$$

■

Example. Given $(\mathbb{R}, \mathcal{L}, m)$.

1. **Escape to horizontal infinity**. Let $f_n := \mathbb{1}_{(n,n+1)}$. We immediately see that
   - $f_n \to 0$ pointwise
   - $\int f_n = 1$ for every n
   - $\int f = 0$

   From Theorem 2.3, we have a strict inequality

   $$0 = \int \liminf_{n\to\infty} f_n, \liminf_{n\to\infty} \int f_n = 1.$$

2. **Escape to width infinity**. Let $f_n := \frac{1}{n}\mathbb{1}_{(0,n)}$.
3. **Escape to vertical infinity**. Let $f_n := n\mathbb{1}_{(0,\frac{1}{n})}$.

Lemma 2.3 (Markov's inequiality). Let $f \geq 0$ be measurable. Then

$$\forall_{c\in(0,\infty)} \mu\left(\{x \mid f(x) \geq c\}\right) \leq \frac{1}{c}\int f.$$

Proof. Denote $\{x \mid f(x) \geq c\} =: E$, then

$$f(x) \geq c\mathbb{1}_E(x) \implies \int f \geq c\int \mathbb{1}_E = c \cdot \mu(E).$$

■

Remark. Notice that $E = f^{-1}([c, \infty])$, hence E is measurable.

Proposition 2.2. Let $f \geq 0$ be measurable. Then,

$$\int f = 0 \iff f = 0 \text{ a.e.}.$$

i.e.,

$$\int f \, \mathrm{d}\mu = 0 \iff \mu(A) = 0$$

where $A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$.

Proof. Firstly, assume that $f = \phi$ is a simple function. We may write

$$\phi = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}$$

where E_i are disjoint and $c_i \in (0, \infty)$. Then,

$$\begin{aligned} \int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0 \\ \iff \mu(E_1) = \ldots = \mu(E_N) = 0 \\ \iff \mu(A) = 0,\ A = \bigcup_{i=1}^{N} E_i. \end{aligned}$$

Now, assume that f is a general function where $f \geq 0$ is the only constraint.

1. Assume $\mu(A) = 0$ (i.e., $f = 0$ a.e.). Let $0 \leq \phi \leq f$, where ϕ is simple. Then
$$\forall_{x \in A^c}\ \phi(x) = 0$$
since $f(x) = 0$, $\forall x \in A^c$. This implies that $\phi = 0$ a.e. since $\mu(A) = 0$, so $\int \phi = 0$. We then have
$$\int f = 0$$
from Definition 2.9.
2. Assume $\int f = 0$. Let $A_n = f^{-1}\left(\left[\frac{1}{n}, \infty\right]\right)$. Then we see that
   - $A_1 \subset A_2 \subset \ldots$
   - $\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0, \infty)) = A.$

   We then have
$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \overset{!}{\leq} n \int f = 0,$$
which further implies
$$\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0$$

from the continuity of measure from below.

■

Corollary 2.5. If $f, g \geq 0$ are both measurable and $f = g$ a.e., then

$$\int f = \int g.$$

Proof. Let $A = \{x \mid f(x) \neq g(x)\}$[22]. Then by assumption, $\mu(A) = 0$, hence

$$f\mathbb{1}_A = 0 \text{ a.e.}, \quad g\mathbb{1}_A = 0 \text{ a.e.}.$$

This further implies that

$$\begin{aligned}\int f &= \int f(\mathbb{1}_A + \mathbb{1}_{A^c}) \\ &\overset{!}{=} \int f\mathbb{1}_A + \int f\mathbb{1}_{A^c} \\ &= \int f\mathbb{1}_{A^c} = \int g\mathbb{1}_{A^c} \\ &= \int g\mathbb{1}_{A^c} + \int g\mathbb{1}_A = \int g.\end{aligned}$$

■

Corollary 2.6. Let $f_n \geq 0$ be measurable. Then

1. $$\left.\begin{aligned} &0 \leq f_1 \leq f_2 \leq \ldots \leq f \text{ a.e.} \\ &\lim_{n\to\infty} f_n = f \text{ a.e.}\end{aligned}\right\} \implies \lim_{n\to\infty} \int f_n = \int f.$$
2. $\lim_{n\to\infty} f_n = f$ a.e. $\implies \int f \leq \liminf_{n\to\infty} \int f_n$.

Proof. ■ DIY

Remark. Almost all the theorems we've proved can be replaced by theorems dealing with almost everywhere condition.

Lecture 13: Integration of Complex Functions

4 Feb. 11:00

2.3 Integration of Complex Functions

As usual, we start with a definition.

[22] A is measurable indeed.

Definition 2.10 (Integrable). Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f\colon X \to \overline{\mathbb{R}}$ and $g\colon X \to \mathbb{C}$ be measurable.[a]

Then f, g are called *integrable* if $\int |f| < \infty$ and $\int |g| < \infty$, and we define

$$\int f = \int f^+ - \int f^-, \qquad \int g = \int \Re g + i \int \Im g.$$

Furthermore, for $f\colon X \to \overline{\mathbb{R}}$, we define

$$\int f = \begin{cases} \infty, & \text{if } \int f^+ = \infty, \int f^- < \infty; \\ -\infty, & \text{if } \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

[a] Recall that for a complex-valued function like g, this means that both $\Re g$ and $\Im g$ are measurable.

We now see a lemma.

Lemma 2.4. Let $f, g\colon X \to \overline{\mathbb{R}}$ or $\mathbb{C}$ integrable. Assume that $f(x) + g(x)$ is well-defined for all $x \in X$.[a]

Then we have

1. $f + g$, cf for all $c \in \mathbb{C}$ are integrable.
2. $\int f + g = \int f + \int g$. This is not trivial since $(f + g)^+ \neq f^+ + g^+$.
3. $\left|\int f\right| \leq \int |f|$.

[a] That is, we never see $\infty + (-\infty)$ or $(-\infty) + \infty$.

Proof. Check [FF99] page 53. ∎

Lemma 2.5. Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be an integrable function on X. Then

1. f is finite a.e.. i.e., $\{x \in X \mid |f(x)| = \infty\}$ is a null set.
2. The set $\{x \in X \mid f(x) \neq 0\}$ is σ-finite.

Proof. ∎

HW 5 Q8 by Lemma 2.3

Proposition 2.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, then

1. If h is integrable on X, then

$$\underset{E\in\mathcal{A}}{\forall} \int_E h = 0 \iff \int |h| = 0 \iff h = 0 \text{ a.e.}$$

2. If f, g are integrable on X, then

$$\underset{E\in\mathcal{A}}{\forall} \int_E f = \int_E g \iff f = g \text{ a.e.}$$

Proof. We prove this one by one.

1. We see that the second equivalence is done in Proposition 2.2, hence we prove the first equivalence only. Since we have

$$\int |h| = 0 \implies \left|\int_E h\right| \leq \int_E |h| \leq \int |h| = 0,$$

which shows one implication. Now assume that $\int_E h = 0$ for all $E \in \mathcal{A}$, then we can write h as

$$h = u + iv = (u^+ - u^-) + i(v^+ - v^-).$$

Let $B := \{x \in X \mid u^+(x) > 0\}$, then by assumption, we have

$$0 = \int_B h = \Re \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+,$$

hence $u^+ = 0$ almost everywhere. Similarly, we have u^-, v^+, v^- are all zero almost everywhere. This gives us that h is zero almost everywhere as desired.

2. DIY

$\blacksquare$

Theorem 2.4 (Dominated Convergence Theorem). Let $(X, \mathcal{A}, \mu)$ be a measure space, and

- Let f_n be integrable on X.
- $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere.
- There is a $g\colon X \to [0, \infty]$ such that g is integrable and

$$\underset{n\in\mathbb{N}}{\forall} |f_n(x)| \leq g(x) \text{ a.e.}$$

Then we have

$$\lim_{n\to\infty} \int f_n = \int f = \int \lim_{n\to\infty} f_n.$$

Proof. Let F be the countable union of null set on which the three conditions may fail. Then we see that after modifying the definition of f_n, f and g on F, we may assume that all three conditions hold everywhere since modifying on a null set does not change the integral.

We now consider the $\overline{\mathbb{R}}$-valued case only. Note that the second and the third conditions imply that f is integrable since $|f| \leq g(x)$. We then see that $g + f_n \geq 0$ and $g - f_n \geq 0$ because $-g \leq f_n \leq g$. From Theorem 2.3, we have

Check $\mathbb{C}$-valued case

$$\int g + f \leq \liminf_{n\to\infty} \int g + f_n, \quad \int g - f \leq \liminf_{n\to\infty} \int g - f_n.$$

From the linearity of integral, we have

$$\int g + \int f \leq \int g + \liminf_{n\to\infty} \int f_n, \quad \int g - \int f \leq \int g - \liminf_{n\to\infty} \int f_n.$$

Now, since $\int g < \infty$, we can cancel it, which gives

$$\int f \leq \liminf_{n\to\infty} \int f_n, \quad -\int f \leq \liminf_{n\to\infty} \int -f_n = -\limsup_{n\to\infty} \int f_n,$$

which implies

$$\int f \leq \liminf_{n\to\infty} f_n \leq \limsup_{n\to\infty} \int f_n \leq \int f.$$

This shows that the limit exists, and the desired result indeed holds. ∎

Corollary 2.7 (Tonelli's theorem for series and integrals). Suppose f_n are integrable functions such that

$$\sum_{n=1}^{\infty} \int |f_n| < \infty,$$

then we have

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

Proof. Take $G(x)$ to be

$$G(x) \coloneqq \sum_{n=1}^{\infty} |f_n(x)|,$$

then we see

$$G(x) \geq |F_N(x)|$$

where

$$F_N(x) \coloneqq \sum_{n=1}^{N} f_n(x).$$

By Corollary 2.4, we have

$$\int G(x) = \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Lastly, from Theorem 2.4, the result follows. ∎

Remark. Compare to Corollary 2.4, we see that we further generalize the result!

Lecture 14: L^1 Space

7 Feb. 11:00

2.4 L^1 Space

We now introduce another space called L^p *spaces*, which are function spaces defined using a natural generalization of the p-norm for finite-dimensional vector spaces. We sometimes call it Lebesgue spaces also.

Before we start, we need to define a *norm*.

Definition 2.11 (Seminorm). Let V be a vector space over filed $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* on V is

$$\|\cdot\| : V \to [0, \infty)$$

such that

- $\|cv\| = |c| \|v\|$ for every $v \in V$ and every scalar c.
- $\|v + w\| \leq \|v\| + \|w\|$ for every $v, w \in V$.

Definition 2.12 (Norm). A *norm* is a seminorm with

- $\|v\| = 0 \iff v = 0$.

Lemma 2.6. A normed vector space is a metric space with metric

$$\rho(v, w) = \|v - w\|.$$

Proof. ∎ DIY

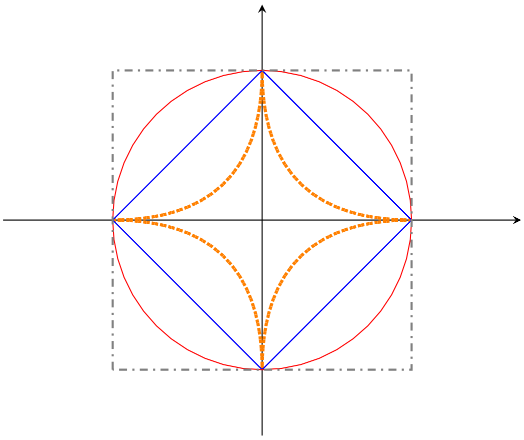
Example (p-norm). $V = \mathbb{R}^d$ with

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}, & \text{if } p \in [0, \infty); \\ \max_{1 \leq i \leq d} |x_i|, & \text{if } p = \infty \end{cases}$$

is a normed vector space. The unit ball

$$\{x \in \mathbb{R}^d \mid \|x\|_p \leq 1\}$$

for different p has the following figures.



Remark. All $\|\cdot\|_p$ norms induce the same topology. i.e., if U is open in p-norm, it is open in p'-norm as well.

Note. Recall that we say f is integrable means

$$\int |f| < \infty,$$

and if $f = g$ a.e., then

$$\int f = \int g$$

Definition 2.13 (L^1 **Space**)**.** Given $(X, \mathcal{A}, \mu)$,

$$f \in L^1(X, \mathcal{A}, \mu) \left(= L^1(X, \mu) = L^1(X) = L^1(\mu)\right)$$

means that f is an integrable function on X.

Lemma 2.7. $L^1(X, \mathcal{A}, \mu)$ is a vector space with seminorm

$$\|f\|_1 = \int |f|.$$

Proof. ∎

Check this is indeed a seminorm.

Definition 2.14 (L^1 **Space with equivalence class**)**.** Define $f \sim g$ if $f = g$ a.e., then

$$L^1(X, \mathcal{A}, \mu) \big/ \sim \; = L^1(X, \mathcal{A}, \mu),$$

i.e., we simply denote the collection of equivalence classes by itself.[a]

[a] By some abusing of notation of L^1.

Remark. We have

- With Definition 2.14, $L^1(X, \mathcal{A}, \mu)$ is a normed vector space.
- We say that the L^1-*metric* $\rho(f, g)$ is simply

$$\rho(f, g) = \int |f - g| .$$

2.4.1 Dense Subsets of L^1

Note. Recall the definition of a *dense set*[23].

Definition 2.15 (Step function). A *step function* on $\mathbb{R}$ is

$$\psi = \sum_{i=1}^{N} c_i \mathbb{1}_{I_i},$$

where I_i is an interval.

Theorem 2.5. We have the following.

1. {integrable simple functions} is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1-metric).
2. $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$, where μ is a Lebesgue-Stieltjes-measure. Then {integrable simple functions} is dense in $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu)$.
3. $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

Notation. We denote the collection of continuous functions with compact support by $C_c(\mathbb{R})$.

Proof. We prove this one by one.

1. Since there exists simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$, where $\phi_n \to f$ pointwise. Then by Theorem 2.4, we have

$$\lim_{n\to\infty} \int \underbrace{|f_n - f|}_{\leq |\phi_n| + |f| \leq 2|f|} = 0$$

   where $2|f|$ is in L^1.

2. Let $\mathbb{1}_E$ approximate by $\sum_{i=1}^{\infty} c_i \mathbb{1}_{I_i}$. From Theorem 1.6 for Lebesgue-Stieltjes-measure,

$$\forall \epsilon' > 0 \; \exists I = \bigcup_{i=1}^{N} I_i \text{ such that } \mu(E \triangle I) \leq \epsilon'.$$

[23] https://en.wikipedia.org/wiki/Dense_set

3. To approximate $\mathbb{1}_{(a,b)}$, we simply consider function $g \in C_c(\mathbb{R})$ such that

$$\int \left|\mathbb{1}_{(a,b)} - g\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

■

Lecture 15: Riemann Integral

9 Feb. 11:00

2.5 Riemann Integrability

We are now working in $(\mathbb{R}, \mathcal{L}, m)$. Let's first revisit the definition of Riemann Integral. Let P be a partition of $[a, b]$ as

$$P = \{a = t_0 < t_1 < \ldots < t_k = b\}.$$

Then the *lower Riemann sum* of f using P is equal to L_P, which is defined as

$$L_P = \sum_{i=1}^{K} \left(\inf_{[t_{i-1}, t_i]} f\right)(t_i - t_{i-1}),$$

and the *upper Riemann sum* of f using P is equal to U_P, which is defined as

$$U_P = \sum_{i=1}^{K} \left(\sup_{[t_{i-1}, t_i]} f\right)(t_i - t_{i-1}).$$

Then we call

- *Lower Riemann integral* of $f = \underline{I} = \sup_P L_P$
- *Upper Riemann integral* of $f = \overline{I} = \inf_P U_P$

Definition 2.16 (Riemann (Darboux) integrable). A bounded function $f\colon [a, b] \to \mathbb{R}$ is called *Riemann (Darboux) integrable* if

$$\underline{I} = \overline{I}$$

If so, then $\underline{I} = \overline{I} = \int_a^b f(x)\,\mathrm{d}x$.

Note. We see that

- If $P \subset P'$, then
$$L_P \leq L_{P'}, \quad U_{P'} \leq U_P.$$
- Recall that continuous functions on $[a, b]$ are Reimann integrable on $[a, b]$.

Theorem 2.6. Let $f\colon [a,b]\to\mathbb{R}$ be a bounded function. Then

1. If f is Reimann integrable, then f is Lebesgue measurable, thus Lebesgue integrable. Further,

$$\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathrm{d}m.$$

2. If f is Reimann integrable $\iff$ f is continuous Lebesgue a.e.[a]

[a]Here, we mean that the set where f is discontinuous is a null set under Lebesgue measure.

Proof. There exists $P_1\subset P_2\subset\dots$ such that $L_{P_n}\nearrow \underline{I}$ and $U_{P_n}\searrow \overline{I}$.[24] Now, define simple (step) functions

- $\phi_n = \sum_{i=1}^K \left(\inf_{[t_{i-1},t_i]} f\right)\mathbb{1}_{(t_{i-1},t_i]}$
- $\psi_n = \sum_{i=1}^K \left(\sup_{[t_{i-1},t_i]} f\right)\mathbb{1}_{(t_{i-1},t_i]}$

if $P_n=\{a=t_0<t_1<\dots<t_K\}$. Let $\phi\coloneqq\sup_n\phi_n$ and $\psi\coloneqq\inf_n\psi_n$. We then see that ϕ,ψ are Lebesgue (Borel) measurable function.

Note. Note that

- $\exists M>0$ such that $\forall_{n\in\mathbb{N}}\ \lvert\phi_n\rvert,\lvert\psi_n\rvert\leq M\mathbb{1}_{[a,b]}$
- $\int\phi_n\,\mathrm{d}m = L_{P_n}$, $\int\psi_n\,\mathrm{d}m = U_{P_n}$

By Theorem 2.4 and the fact that $M\mathbb{1}_{[a,b]}\in L^1(\mathbb{R},\mathcal{L},m)$, we have

$$\underline{I} = \lim_{n\to\infty}\int\phi_n\,\mathrm{d}m = \int\phi\,\mathrm{d}m,\quad \overline{I} = \int\psi\,\mathrm{d}m.$$

Thus,

$$\begin{aligned}
f\text{ is Riemann integrable} &\iff \int\phi = \int\psi\\
&\iff \int(\psi-\phi)=0\\
&\iff \psi=\phi\text{ Lebesgue a.e.}
\end{aligned}$$

■

2.6 Modes of Convergence

As we should already see, there are different *modes* of convergence. Let's formalize them.

[24]Here, we took refinements of P_n if needed.

Definition 2.17 (Pointwise, Uniformly convergence). Let

$$f_n, f\colon X \to \mathbb{C},$$

and $S \subset X$. Then we say

- $f_n \to f$ *pointwise* on S if

$$\underset{x \in S}{\forall}\ \underset{\epsilon > 0}{\forall}\ \underset{N \in \mathbb{N}}{\exists}\ \underset{n \geq N}{\forall}\ |f_n(x) - f(x)| < \epsilon.$$

- $f_n \to f$ *uniformly* on S if

$$\underset{\epsilon > 0}{\forall}\ \underset{N \in \mathbb{N}}{\exists}\ \underset{x \in S}{\forall}\ \underset{n \geq N}{\forall}\ |f_n(x) - f(x)| < \epsilon.$$

Remark. We see that we can replace $\forall \epsilon > 0$ by $\forall k \in \mathbb{N}$ with ϵ changing to $\frac{1}{k}$.

Lemma 2.8. Let $B_{n,k}$ be

$$B_{n,k} \coloneqq \left\{ x \in X \mid |f_n(x) - f(x)| < \frac{1}{k} \right\}.$$

Then

1. $f_n \to f$ pointwise on S if and only if

$$S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$

2. $f_n \to f$ uniformly on S if and only if $\exists N_1, N_2, \ldots \in \mathbb{N}$ such that

$$S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Proof. This essentially follows from Definition 2.17. ∎

Definition 2.18 (Converges a.e., Converges in L^1). Let $(X, \mathcal{A}, \mu)$ be a measure space. Assuming that f_n, f are measurable function, then

1. $f_n \to f$ *almost everywhere* means

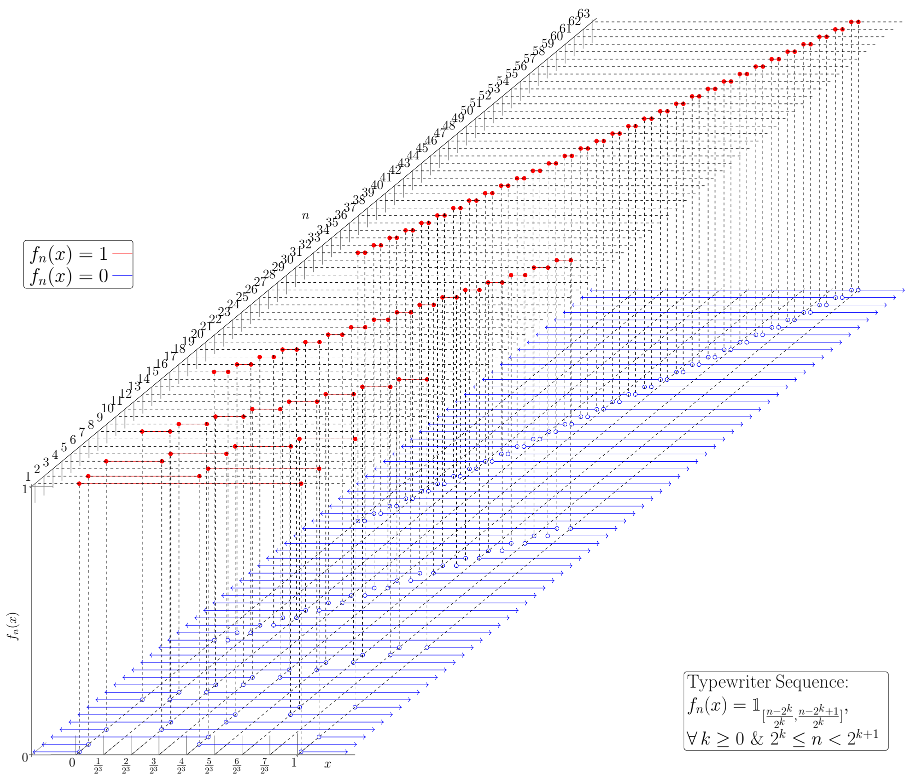
$$\exists \text{ null set } E \text{ such that } f_n \to f \text{ pointwise on } E^c.$$

2. $f_n \to f$ *in L^1* means

$$\lim_{n \to \infty} \|f_n - f\| = 0.$$

Example. Given $(\mathbb{R}, \mathcal{L}, m)$ and let $f = 0$. We see the followings.

1. $f_n = \mathbb{1}_{(n,n+1)}$
2. $f_n = \frac{1}{n}\mathbb{1}_{(0,n)}$
3. $f_n = n\mathbb{1}_{(0,\frac{1}{n})}$
4. **Typewriter functions**.



Lecture 16: Modes of Convergence

11 Feb. 11:00

Let's start with a proposition.

Proposition 2.4 (Fast L^1 convergence leads to a.e. convergence). Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n, f are all measurable functions on X. Then

$$\sum_{n=1}^{\infty} \|f_n - f\|_1 < \infty \implies f_n \to f \quad \text{a.e.}$$

Proof. Let

$$E := \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c = \{x \in X \mid f_n(x) \nrightarrow f(x)\}.$$

By Lemma 2.3, we see that

$$\underset{k}{\forall}\ \underset{N}{\forall}\ \mu\left(B_{n,k}^c\right) \leq k \int |f_n - f| \implies \underset{k}{\forall}\ \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) \leq \sum_{n=N}^{\infty} k \left\|f_n - f\right\|_1 \to 0$$

as $N \to \infty$. Now, by continuity of measure from above,

$$\underset{k}{\forall}\ \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \mu(E) = 0$$

since $f_n \to f$ pointwise on E^c. ■

> **Corollary 2.8.** Given $\{f_n\}_n$ such that $f_n \to f$ in L^1, there exists a subsequence $\{f_{n_j}\}_{n_j}$ where $f_{n_j} \to f$ a.e.

Proof. Since

$$\underset{j\in\mathbb{N}}{\forall}\ \underset{n_j\in\mathbb{N}}{\forall}\ \left\|f_{n_j} - f\right\|_1 \leq \frac{1}{j^2}.$$

Then,

$$\sum_{j=1}^{\infty} \left\|f_{n_j} - f\right\|_1 < \infty.$$

From Proposition 2.4, we have the desired result. ■

> **Definition 2.19 (Converge in measure).** Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. Then $f_n \to f$ *in measure* means
>
> $$\underset{\epsilon>0}{\forall}\ \lim_{n\to\infty} \mu\left(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}\right) = 0.$$

Example. Let $f_n = n\mathbb{1}_{(0,\frac{1}{n})}$ and $f = 0$. We see that

$$\forall \epsilon > 0\ \ \{x \in X \mid |f_n(x) - f(x)| > \epsilon\} = \left(0, \frac{1}{n}\right),$$

$f_n \to 0$ in measure. (Recall that $f_n \not\to 0$ in L^1)

Remark. We see that

$$\begin{tikzcd}
 & f_n \to f \text{ in } L^1 \arrow[r, Rightarrow, shift left, "\text{Check}"] \arrow[dd, Leftrightarrow, "/" marking] & f_n \to f \text{ in measure} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[dd, Rightarrow, "\text{Read}"] \arrow[ddl, Leftrightarrow, "/" marking] \\
f_n \to f \text{ fast in } L^1 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & f_n \to f \text{ a.e.} & \exists f_{n_j} \to f \text{ a.e.}
\end{tikzcd}$$

Definition 2.20 (Uniformly a.e., Almost uniformly). Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$.

1. $f_n \to f$ *uniformly almost everywhere* means $\exists$null set F such that $f_n \to f$ uniformly on F^c.
2. $f_n \to f$ *almost uniformly* means $\forall \epsilon > 0$ $\exists F \in \mathcal{A}$ such that $\mu(F) < \epsilon$, $f_n \to f$ uniformly on F^c.

Lemma 2.9. We have

$$f_n \to f \text{ uniformly on } S \iff \exists N_1, N_2, \ldots \in \mathbb{N} \ S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Theorem 2.7 (Egorov's Theorem). Let f_n, f be measurable functions on $(X, \mathcal{A}, \mu)$. Suppose $\mu(X) < \infty$, then

$$f_n \to f \text{ a.e.} \iff f_n \to f \text{ almost uniformly}.$$

Proof. We prove two directions.

- $\Longleftarrow$ DIY
- $\implies$ Fix $\epsilon > 0$. We see that

$$\begin{aligned} f_n \to f \text{ a.e.} &\implies \mu\left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \\ &\implies \forall_k \, \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0. \end{aligned}$$

From continuity of measure from above and $\mu(X) < \infty$, we further have

$$\forall_k \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \forall_k \exists_{N_k \in \mathbb{N}} \mu\left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c\right) < \frac{\epsilon}{2^k}.$$

Now, let

$$F := \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c,$$

we see that $\mu(F) < \epsilon$, hence $f_n \to f$ uniformly.

∎

3 Product Measure

3.1 Product σ-algebra

Before we start, we see the setup.

- Product space.

$$X = \prod_{\alpha \in I} X_\alpha$$

where $x = (x_\alpha)_{\alpha \in I} \in X$.

- Coordinate map.

$$\pi_\alpha \colon X \to X_\alpha.$$

Now we see the formal definition.

Definition 3.1 (Product σ-algebra). Let $(X_\alpha, \mathcal{A}_\alpha)$ be a measurable space for all $\alpha \in I$. Then a *product σ-algebra* on $X = \prod_{\alpha \in I} X_\alpha$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha \in I} \pi_\alpha^{-1}(\mathcal{A}_\alpha) \right\rangle,$$

where

$$\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \left\{ \pi_\alpha^{-1}(E) \mid E \in \mathcal{A}_\alpha \right\}.$$

Notation. We denote $I = \{1, \ldots, d\} \implies X = \prod_{i=1}^{d} X_i, x = (x_1, \ldots, x_d)$. Also,

$$\bigotimes_{i=1}^{d} \mathcal{A}_i = \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_d.$$

Lemma 3.1. If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \forall_i \, E_i \in \mathcal{A}_i \right\} \right\rangle.$$

Proof. If $E_i \in \mathcal{A}_i$, then $\pi_i^{-1}(E_i) = \prod_{j=1}^{\infty} E_j$, where $E_j = X$ for $j \neq i$. On the other hand, since

$$\prod_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} \pi_i^{-1}(E_i),$$

from Lemma 1.2, the result follows. ∎

Lecture 17: Product Measure

14 Feb. 11:00

We now see a lemma.

Lemma 3.2. Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle$ for every $\alpha \in I$. Then

1. $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$
2. $\bigotimes_\alpha \mathcal{A}_\alpha = \left\langle \bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$
3. If I is countable, then

$$\bigotimes_{i=1}^{\infty} \mathcal{A}_i = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid \forall_i \ E_i \in \mathcal{E}_i \right\} \right\rangle$$

Proof. We prove this one by one.

1. Note that for $f\colon Y \to Z$, and $\mathcal{B}$ be a σ-algebra on Z, then $f^{-1}(\mathcal{B})$ is also a σ-algebra.[25] Hence, π_α^{-1} is a σ-algebra on X, i.e.,

$$\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \overset{!}{\implies} \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha).$$

To show the other direction, let $\mathcal{M}$ being

$$\mathcal{M} = \left\{ B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle \right\}.$$

We now check

- $\mathcal{M}$ is a σ-algebra. Check (Easy)!
- $\mathcal{E}_\alpha \subset \mathcal{M}$. This is true by definition of $\mathcal{M}$.

Thus, $\langle \mathcal{E}_\alpha \rangle = \mathcal{A}_\alpha \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha$, $E \in \mathcal{M}$, implying

$$\pi_\alpha^{-1}(E) \in \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle,$$

i.e., $\mathcal{A}_\alpha \subset \left\langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$.

2. DIY
3. DIY

∎

[25] Since $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(B)^c = f^{-1}(B^c)$, and $\bigcup_n f^{-1}(B_n) = f^{-1}(\bigcup_n B_n)$.

Theorem 3.1. Suppose $X_1, \ldots, X_d$ are metric spaces. Let $X = \prod_{i=1}^{d} X_i$ with product metric defined as

$$\rho(x, y) = \sum_{i=1}^{d} \rho_i(x_i, y_i).$$

Then,

1. $\bigotimes_{i=1}^{d} \mathcal{B}(X_i) \subset \mathcal{B}(X)$
2. If in addition, each X_i has a countable dense subset,

$$\bigoplus_{i=1}^{d} \mathcal{B}(X_i) = \mathcal{B}(X).$$

Proof. ■ DIY

Remark. We see that

- $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$
- let $f = u + iv \colon X \to \mathbb{C}$, and $\mathcal{A}$ be a σ-algebra on X. Then

$$\forall_{E \in \mathcal{B}(\mathbb{R})} \; u^{-1}(E), v^{-1}(E) \in \mathcal{A} \iff f^{-1}(F) \in \mathcal{A}, \forall\, F \in \mathcal{B}(\mathbb{C})$$

  with $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Definition 3.2 (x-section, y-section). Let X, Y be two sets. Then

1. For $E \subset X \times Y$,

$$E_x = \{y \in Y \mid (x, y) \in E\}, \quad E^y = \{x \in X \mid (x, y) \in E\}.$$

2. For $f \colon X \times Y \to \mathbb{C}$, define

$$f_x \colon Y \to \mathbb{C}, \quad f^y \colon X \to \mathbb{C}$$

   by

$$f_x(y) = f(x, y) = f^y(x).$$

Example. We see that

$$(\mathbb{1}_E)_x = \mathbb{1}_{E_x}$$

and

$$(\mathbb{1}_E)^y = \mathbb{1}_{E^y}.$$

Proposition 3.1. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, then

1. If $E \in \mathcal{A} \otimes \mathcal{B}$, then
$$\forall_{x \in X} \forall_{y \in Y} E_x \in \mathcal{B}, E^y \in \mathcal{A}.$$
2. If $f\colon X \times Y \to \mathbb{C}$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then
$$\forall_{x \in X} \forall_{y \in Y} f_x \text{ is } \mathcal{B}\text{-measurable}, f^y \text{ is } \mathcal{A}\text{-measurable}.$$

Proof. We prove this one by one.

1. Let $\mathcal{F} := \left\{ E \subset X \times Y \mid \forall_{x \in X} \forall_{y \in Y} E_x \in \mathcal{B}, E^y \in \mathcal{A} \right\}$, then
   - $\mathcal{F}$ is a σ-algebra.
     - $\varnothing \in \mathcal{F}$.
     - $(E^c)_x = E_x^c$.
     - $\left(\bigcup_{j=1}^{\infty} E_j \right)_x = \bigcup_{j=1}^{\infty} (E_j)_x$.

     And the same is true for y.
   - Let $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$, which is again easy to show from definition.

   Hence, we see that $\langle R_0 \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{F}$.
2. Since
$$(f_x)^{-1}(B) = (f^{-1}(B))_x$$
   and
$$(f^y)^{-1}(B) = (f^{-1}(B))^y,$$
   the result follows from 1.

■

3.2 Product Measures

We start with the definition.

Definition 3.3 (Rectangle). Given two measurable spaces, a *(measurable) rectangle* is $R = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, we let
$$\mathcal{R}_0 := \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$
and
$$\mathcal{R} := \left\{ \bigcup_{i=1}^{N} R_i \mid N \in \mathbb{N}, R_1, \ldots, R_N \text{ disjoint rectangles} \right\}.$$

Note. Whenever we're talking about rectangle, they're always measurable.

Lemma 3.3. $\mathcal{R}$ is an algebra, and

$$\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}.$$

Proof. Simply observe that

$$(A \times B)^c = (A^c \times Y) \cup (A \times B)$$

■ DIY

Lecture 18: Monotone Class

16 Feb. 11:00

Let's start with a theorem.

Theorem 3.2. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces. Then

1. There is a measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

for every $A \in \mathcal{A}, B \in \mathcal{B}$.

2. If μ, ν are σ-finite , then $\mu \times \nu$ is unique.

Proof. We prove this one by one.

1. Define $\mu\colon \mathcal{R} \to [0, \infty]$ by $\mu(A \times B) = \mu(A)\nu(B)$, and extending linearly, we have

$$\pi(A \times B) = \mu(A)\nu(B),$$

hence

$$\pi\left(\coprod_{i=1}^{N} A_i \times B_i\right) = \sum_{i=1}^{n} \pi(A_i \times B_i).$$

We claim that π is a pre-measure. To show this, it's enough to check that $\pi(A \times B) = \sum\limits_{n=1}^{\infty} \pi(A_n \times B_n)$ if $A \times B = \coprod\limits_{n} A_n \times B_n$. Since $A_n \times B_n$ are disjoint, so

$$\mathbb{1}_{A \times B}(x, y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n \times B_n}(x, y).$$

Thus,

$$\mathbb{1}_A(x)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y).$$

Integrating with respect to x, and applying Proposition 1.3, we have

$$\int_X \mathbb{1}_A(x)\mathbb{1}_B(y)\,\mathrm{d}\mu(x) = \sum_{n=1}^{\infty} \int_x \mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y)\,\mathrm{d}\mu(x),$$

which implies

$$\mu(A)\mathbb{1}_B(y) = \sum_{n=1}^{\infty} \mu(A_n)\mathbb{1}_{B_n}(y)$$

for every y. We can then integrate again with respect to y and apply Proposition 1.3, we have

$$\int_Y \mu(A)\mathbb{1}_B(y)\,\mathrm{d}\nu(y) = \sum_{n=1}^{\infty} \int_Y \mu(A_n)\mathbb{1}_{B_n}(y)\,\mathrm{d}\nu(y),$$

which gives us

$$\mu(A)\nu(B) = \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n).$$

Hence, we see that μ is indeed a pre-measure, so Theorem 1.3 gives $\mu \times \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on $\mathcal{R}$.

2. If μ, ν are σ-finite, then π is σ-finite on $\mathcal{R}$, then Theorem 1.4 applies. Moreover, we have that

$$(\mu \times \nu)(E) = \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \mid E \subset \bigcup_{i=1}^{\infty} A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\}.$$

■

3.3 Monotone Class Lemma

Let's start with a definition.

Definition 3.4 (Monotone Class). If X is a set, and $C \subset \mathcal{P}(X)$, we say that C is a *monotone class* on X if

- C is closed under countable increasing unions.
- C is closed under countable decreasing intersections.

Example. We see that

1. Every σ-algebra is a monotone class.
2. If C_α are (arbitrarily many) monotone classes on a set X, then $\bigcap_\alpha C_\alpha$ is a monotone class. Furthermore, if $\mathcal{E} \subset \mathcal{P}(X)$, there is a unique smallest monotone class containing $\mathcal{E}$, denoted by $\langle \mathcal{E} \rangle$, which follows the same idea as in Definition 1.3.

Theorem 3.3 (Monotone Class Lemma). Suppose $\mathcal{A}_0$ is an algebra on X. Then $\langle \mathcal{A}_0 \rangle$[a] is the monotone class generated by $\mathcal{A}_0$.

[a] $\langle \mathcal{A}_0 \rangle$ is the σ-algebra generated by $\mathcal{A}_0$ by Definition 1.3.

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$ and let $\mathcal{C}$ be the monotone class generated by $\mathcal{A}_0$. Since $\mathcal{A}$ is a σ-algebra, it's a monotone class. Note that it contains $\mathcal{A}_0$, hence $\mathcal{A} \supset \mathcal{C}$.

To show $\mathcal{C} \supset \mathcal{A}$, it's enough to show that $\mathcal{C}$ is a σ-algebra. We check that

1. $\varnothing \in \mathcal{A}_0 \subseteq \mathcal{C}$.
2. Let $\mathcal{C}' = \{E \subset X \mid E^c \in \mathcal{C}\}$.
   - $\mathcal{C}'$ is a monotone class.
   - $\mathcal{A}_0 \subset \mathcal{C}'$ because if $E \in \mathcal{A}_0$, then $E^c \in \mathcal{A}_0$, so $E^c \in \mathcal{C}$, thus $E \in \mathcal{C}'$.

   We see that $\mathcal{C}' \subset \mathcal{C}'$, so $\mathcal{C}$ is closed under complements.
3. For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.
   - $\mathcal{D}(E) \subset \mathcal{C}$.
   - $\mathcal{D}(E)$ is a monotone class.
   - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. We see this by picking $F \in \mathcal{A}_0$, then $E \cup F \in \mathcal{A}_0 \supset \mathcal{C}$.

   Hence, $C = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.
4. Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\}$. That is $\mathcal{D} = \{E \in \mathcal{C} \mid E \cup F, \forall F \in \mathcal{C}\}$. Then we have
   - $A_0 \subset \mathcal{D}$ by 3.
   - $\mathcal{D}$ is a monotone class.
   - $\mathcal{D} \subset \mathcal{C}$ by definition.

   Thus, $\mathcal{D} = \mathcal{C}$, so if $E, F \in \mathcal{C}$, then $E \cup F \in \mathcal{C}$. This implies that $\mathcal{C}$ is closed under finite unions.
5. Now to show that $\mathcal{C}$ is closed under countable unions, let $E_1, E_2, \ldots \in \mathcal{C}$. We may then define
$$F_N = \bigcup_{n=1}^{N} E_n \in \mathcal{C}.$$
   Then we see that $F_1 \subset F_2 \subset \ldots$, hence $\bigcup_N F_N \in \mathcal{C}$. But this simply implies
$$\bigcup_N F_N = \bigcup_n E_n,$$
   so we're done.

■

Lecture 19: Fubini-Tonelli's Theorem

18 Feb. 11:00

As previously seen. If $E \in A \otimes B \implies E_x \in \mathcal{B}, E^y \in \mathcal{A}\ \forall x \in X, \forall y \in Y$. Note that the reverse is not true.

3.4 Fubini-Tonelli Theorem

We start with a theorem.

Theorem 3.4 (Tonelli's theorem for characteristic functions). Given $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure space. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$, then

1. $\alpha(x) \coloneqq \nu(E_x)\colon X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.
2. $\beta(x) \coloneqq \mu(E^y)\colon Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.
3. $(\mu \times \nu)(E) = \int_X \nu(E_x)\,\mathrm{d}\mu(x) = \int_Y \mu(E^y)\,\mathrm{d}\nu(y)$.

Proof. We prove this one by one.

1. Assume that μ, ν are finite measure. Let

$$C \coloneqq \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{ Conditions 1., 2., 3., hold}\}.$$

It's enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset C$. We further observe that from the Theorem 3.3 and the fact that $\mathcal{R}$ is an algebra, it's also enough to show that

- $\mathcal{R} \subset C$.
- C is a monotone class.

From condition 1.,

$$\alpha(x) = \nu\left((A \times B)_x\right) = \begin{cases} \nu(B), & \text{if } x \in A; \\ 0, & \text{if } x \notin A \end{cases} = \nu(B)\mathbb{1}_A.$$

And from condition 2.,

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

and

$$\int_X \nu((A \times B)_x)\,\mathrm{d}\mu(x) = \nu(B)\mu(A).$$

Let $E_n \in C$, $E_1 \subset E_2 \subset \ldots$. We need to show $E = \bigcup\limits_{n=1}^{\infty} E_n \in C$. We now see that

$$E_x = \bigcup_{n=1}^{\infty} (E_n)_x, (E_1)_x \subset (E_2)_x \subset \ldots$$

$$\implies \alpha(x) = \nu(E_n)_x \overset{!}{=} \lim_{n\to\infty} \nu((E_n)_x)\ \forall x \in X.$$

This implies that 1. is proved.

For 3., we see that

$$\begin{aligned}(\mu \times \nu)(E) &\overset{!}{=} \lim_{n\to\infty} (\mu \times \nu)(E_n) \\ &= \lim_{n\to\infty} \int_X \nu((E_n)_x)\,\mathrm{d}\mu(x) \\ &\overset{!}{=} \int_X \nu(E_x)\,\mathrm{d}\mu(x).\end{aligned}$$

Now let $F_n \in C$, $F_1 \supset F_2 \supset \ldots$. We need to show that $F = \bigcap_{n=1}^{\infty} F_n \in C$. Instead of using Theorem 2.2, we now want to use Theorem 2.4, which is applicable since $\mu(X), \nu(Y) < \infty$ by assumption. Then assume that μ, ν are σ-finite, then

$$X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n), \begin{cases} X_1 \subset X_2 \subset \ldots, & \mu(X_k) < \infty \\ Y_1 \subset Y_2 \subset \ldots, & \nu(Y_k) < \infty. \end{cases}$$

DIY

■

Theorem 3.5 (Fubini-Tonelli's Theorem). Given two σ-finite measure space $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$, we have the following two versions.

(Tonelli) If $f \colon X \times Y \to [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then

1. $g(x) := \int_Y f(x, y) \,\mathrm{d}\nu(y)$, $X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.
2. $h(x) := \int_X f(x, y) \,\mathrm{d}\mu(x)$, $Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.
3. We have

$$\int_{X \times Y} f \,\mathrm{d}(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \,\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x, y) \,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y).$$

(Fubini) If $f \in L^1(X \times Y, \mu \times \nu)$, then

1. $f_x \in L^1(Y, \nu)$ for μ-a.e. x, and $g(x) \in L^1(X, \mu)$ defined μ-a.e.
2. $f^y \in L^1(X, \mu)$ for ν-a.e. y, and $h(x) \in L^1(Y, \nu)$ defined μ-a.e.
3. The iterated integral formulas hold. Namely, we have

$$\int_{X \times Y} f \,\mathrm{d}(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \,\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x, y) \,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y).$$

Proof. Read [FF99]. ■

Lecture 20: Lebesgue Measure on $\mathbb{R}^d$

21 Feb. 11:00

3.5 Lebesgue Measure on $\mathbb{R}^d$

Example. We first see some examples.

1. $(\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m)$ is not complete.
   - Let $A \in \mathcal{L}$, $A \neq \varnothing$, $m(A) = 0$.
   - Let $B \subset [0, 1]$, $B \notin \mathcal{L}$ (Vital set for example).
   - Let $E = A \times B$, $F = A \times [0, 1]$.

We see that $E \subset F$, $F \in \mathcal{L} \otimes \mathcal{L}$, $(m \times m)(F) = m(A)m([0,1]) = 0$, i.e., F is a null set. But E is **not** $\mathcal{L} \otimes \mathcal{L}$-measurable-function since otherwise, its sections are all measurable.

Definition 3.5. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the *completion* of

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \ldots \times m),$$

which is same as the *completion* of

$$(\mathbb{R}^d, \mathcal{L} \otimes \ldots \otimes \mathcal{L}, m \times \ldots m).$$

Remark. We see that

$$\mathcal{L}^d \supsetneq \mathcal{L} \otimes \ldots \otimes \mathcal{L} = \left\langle \left\{ \prod_{i=1}^{d} E_i \mid E_i \in \mathcal{L} \right\} \right\rangle.$$

Definition 3.6 (General rectangle). A *rectangle* in $\mathbb{R}^d$ is $R = \prod_{i=1}^{d} E_i$ where $E_i \in \mathcal{B}(\mathbb{R})$.

Definition 3.7. We let

$$m^d(E) := \inf \left\{ \sum_{k=1}^{\infty} m^d(R_k) \mid E \subset \bigcup_{k=1}^{\infty} R_k, R_k \text{ is rectangles} \right\}.$$

Theorem 3.6. Let $E \subset \mathcal{L}^d$. Then

1. $m^d(E) = \inf \left\{ m^d(0) \mid \text{open } O \supset E \right\} = \sup \left\{ m^d(K) \mid \text{compact } K \subset E \right\}$.
2. $E = A_1 \cup N_1 = A_2 \setminus N_2$, where A_1 is F_σ, A_2 is G_δ , and N_i are null.
3. If $m^d(E) < \infty$, $\forall \epsilon > 0$, $\exists R_1, \ldots, R_m$ rectangles whose sides are intervals such that

$$m^d \left(E \triangle \left(\bigcup_{i=1}^{m} R_i \right) \right) < \epsilon.$$

Proof. Similar to $d = 1$ case. ∎

Theorem 3.7. Integrable step functions and $C_c(\mathbb{R}^d)$, the collection of continuous functions, are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$

Proof. See [FF99]. ∎

Theorem 3.8. Lebesgue measure in $\mathbb{R}^d$ is translation-invariant.

Proof. See [FF99]. ∎

Theorem 3.9 (Effect of linear transformation on Lebesgue measure). If $T \in \mathrm{GL}(\mathbb{R}^d)$, $e \in \mathcal{L}^d$, then $T(E)$ is measurable and

$$m(T(E)) = |\det T| \cdot m(E).$$

Proof. See [FF99]. ∎

4 Differentiation on Euclidean Space

As previously seen. Given $f\colon [a,b] \to \mathbb{R}$, there are two versions of **fundamental theorem of calculus**:

1. $$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a).$$

2. $$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x),$$

   which follows from

   $$\lim_{r\to 0^+} \frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t = f(x) = \lim_{r\to 0^+} \frac{1}{r}\int_{x-r}^{x} f(t)\,\mathrm{d}t.$$

Remark. We see that

$$\lim_{r\to 0^+} \frac{1}{r}\int_x^{x+r} (f(t) - f(x))\,\mathrm{d}t = 0 = \lim_{r\to 0^+} \frac{1}{r}\int_{x-r}^{x} (f(t) - f(x))\,\mathrm{d}t,$$

where we have

$$f(x) = \frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t.$$

This generalized to $f\colon \mathbb{R}^d \to \mathbb{R}$, namely

$$\lim_{r\to 0^+} \frac{1}{\mathrm{vol}\,(B(x,r))}\int_{B(x,r)} (f(t) - f(x))\,\underbrace{\mathrm{d}t}_{\mathbb{R}^d} \stackrel{?}{=} 0.$$

4.1 Hardy-Littlewood Maximal Function

We first see our notation.

Notation. Given a(n) (open) ball in $\mathbb{R}^d$, $B = B(a,r)$, denote $cB = B(a,cr)$ for $c > 0$.

Lemma 4.1 (Vitali-type covering lemma). Let $B_1, \ldots, B_k$ be a finite collection of open balls in $\mathbb{R}^d$. Then there exists a sub-collection $B_1', \ldots, B_m'$ of disjoint open balls such that

$$\bigcup_{i=1}^{m} (3B_j') \supset \bigcup_{i=1}^{k} B_i.$$

Proof. Greedy Algorithm. ■

Lecture 21: Hardy-Littlewood Maximal Function and Inequality

25 Feb. 11:00

Notation. We let

$$\int_E f \,\mathrm{d}m = \int_E f(x) \,\mathrm{d}x.$$

The problem we're working on is

$$\frac{1}{m(B(w,r))} \int_{B(x,r)} f(y) \,\mathrm{d}y \overset{r\to 0}{\underset{?}{\longrightarrow}} f(x).$$

Definition 4.1 (Locally integrable). Given $f\colon \mathbb{R}^d \to \mathbb{C}$ be Lebesgue measurable function. Then we say f is *locally integrable* if for every compact $K \subset \mathbb{R}^d$,

$$\int_K |f| \,\mathrm{d}m < \infty.$$

We write $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Definition 4.2 (Hardy-Littlewood maximal function). Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, the *Hardy-Littlewood maximal function* for f is defined as

$$\mathrm{H}f(x) \coloneqq \sup \{\mathrm{A}_r(x) \mid r > 0\},$$

where

$$\mathrm{A}_r(x) \coloneqq \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| \,\mathrm{d}y.$$

Note. We note that $\mathrm{A}_r(\cdot)$ means *averaging function*.

Lemma 4.2. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then

1. $\mathrm{A}_r(x)$ is jointly continuous for $(x, r) \in \mathbb{R}^d \times (0, \infty)$.
2. $\mathrm{Hf}(x)$ is Borel measurable.

Proof. We outline the proof.

1. Let $(x, r) \to (x^*, r^*) \implies \mathrm{A}_r(x) \to \mathrm{A}_{r^*}(x^*)$. Let (x_n, r_n) be any sequence which converges to x^*, r^*, then we consider $\lim\limits_{n\to\infty} \mathrm{A}_{r_n}(x_n)$ and we can calculate
$$\int \underbrace{|f(y)|\, \mathbb{1}_{B(x_n,r_n)}(y)}_{:=h_n(y)},$$
then we apply Theorem 2.4 to h_n.
2. Observe that
$$(\mathrm{H}f)^{-1}\underbrace{((a,\infty))}_{\text{open}} = \bigcup_{r>0} \mathrm{A}_r^{-1}\left((a,\infty)\right)$$
is open, since $\mathrm{A}_r^{-1}\left((a,\infty)\right)$ is open from the 1. Note that the equality comes from the fact that $\mathrm{H}f = \sup_r \mathrm{A}_r$.

■

Theorem 4.1 (Hardy-Littlewood maximal inequality). There exists $C_d > 0$ such that for every $f \in L^1(\mathbb{R}^d)$,
$$\underset{\alpha>0}{\forall}\; m\left(\left\{x \in \mathbb{R}^d \mid \mathrm{H}f(x) > \alpha\right\}\right) \leq \frac{C_d}{\alpha} \int |f(x)|\, \mathrm{d}x.$$

Proof. We first fix $f \in L^1$ and $\alpha > 0$. We define
$$E := \{x \mid \mathrm{H}f(x) > \alpha\},$$
which is a Borel measurable set by Lemma 4.2. Then
$$x \in E \implies \underset{r_x>0}{\exists}\; \mathrm{A}_{r_x}(x) > \alpha \implies m(B(x, r_x)) < \frac{1}{\alpha}\int_{B(x,r_x)} |f(y)|\, \mathrm{d}y.$$
From inner regularity, we have
$$m(E) = \sup\{m(K) \mid \text{compact } K \subset E\}.$$
Let $K \subset E$ be compact, then
$$K \subset \bigcup_{x\in K} B(x, r_x) \overset{K \text{ compact}}{\implies} K \subset \bigcup_{i=1}^{N} B_i \overset{!}{\implies} K \subset \bigcup_{i=1}^{m} \{3B_j'\}.$$
From here, we further have
$$m(K) \leq \sum_{i=1}^{m} m(3B_j') = 3^d \sum_{j=1}^{m} m(B_j') \leq \frac{3^d}{\alpha}\sum_{j=1}^{m}\int_{B_j'} |f(y)|\, \mathrm{d}y.$$
Now, since $B_i', \ldots, B_m'$ are disjoint, hence we finally have
$$m(K) \leq \frac{3^d}{\alpha}\int_{\mathbb{R}^d} |f(y)|\, \mathrm{d}y.$$

■

Lecture 22: Lebesgue Differentiation Theorem

07 Mar. 11:00

We should compare the Hardy-Littlewood maximal inequality to Markov's inequality. Namely, there exists $C_d > 0$ (can take 3^d) such that for all $f \in L^1(\mathbb{R}^d)$, $\alpha > 0$, we have

$$\begin{cases} m(\{x \mid \mathrm{H}f(x) > \alpha\}) \leq \dfrac{C_d}{\alpha} \displaystyle\int \lvert f \rvert\,; \\ m(\{x \mid \lvert f(x) \rvert > \alpha\}) \leq \dfrac{1}{\alpha} \displaystyle\int \lvert f \rvert\,. \end{cases}$$

4.2 Lebesgue Differentiation Theorem

We start with a theorem!

Theorem 4.2 (Lebesgue Differentiation Theorem). Let $f \in L^1$, then

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} \lvert f(y) - f(x) \rvert \,\mathrm{d}y = 0$$

for a.e. x.

Proof. The result holds for $f \in C_c(\mathbb{R}^d)$, namely for those continuous functions with **compact** support. This is because for any $\epsilon > 0$, if r is small and $\lvert f(y) - f(x) \rvert < \epsilon$, then

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} \lvert f(y) - f(x) \rvert \,\mathrm{d}y < \epsilon.$$

Now, let $f \in L^1(\mathbb{R}^d)$ and fix $\epsilon > 0$. By density, there exists $g \in C_c(\mathbb{R}^d)$ with $\lVert f - g \rVert_1 < \epsilon$. We then have

$$\begin{aligned} \int_{B(x,r)} \lvert f(y) - f(x) \rvert \,\mathrm{d}y \leq \int_{B(x,r)} & \lvert f(y) - g(y) \rvert \,\mathrm{d}y \\ & + \int_{B(x,r)} \lvert g(y) - g(x) \rvert \,\mathrm{d}y \\ & \quad + \int_{B(x,r)} \lvert g(x) - f(x) \rvert \,\mathrm{d}y. \end{aligned}$$

Divide all of these by $m(B(x,r))$, and take $\limsup_{r \to \infty}$, we need to understand the error terms, namely

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} \lvert f(x) - g(x) \rvert \,\mathrm{d}y = \lvert g(x) - f(x) \rvert$$

and

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} \lvert f(y) - g(y) \rvert \,\mathrm{d}y \leq (\mathrm{H}(f-g))(x).$$

We define

$$Q(x) \coloneqq \limsup_{r \to \infty} \frac{1}{m(B(x,r))} \int_{B(x,r)} \lvert f(y) - f(x) \rvert \,\mathrm{d}y.$$

We want to show $m(\{x \in X \mid Q(x) > 0\}) = 0$. Let $E_\alpha = \{x \in X \mid Q(x) > \alpha\}$. It is enough to show $m(E_\alpha) = 0$ for all $\alpha > 0$ because $\{x \in X \mid Q(x) > 0\} = \bigcup_n E_{\frac{1}{n}}$. We know by the above that

$$Q(x) \leq (\mathrm{H}(f-g))(x) + 0 + |g(x) - f(x)|\,.$$

Therefore,

$$E_\alpha \subset \{x \in X \mid (\mathrm{H}(f-g))(x) > \alpha/2\} \cup \{x \in X \mid |g(x) - f(x)| > \alpha/2\}.$$

By the Hardy-Littlewood maximal inequality and Markov's inequality, we have

$$\begin{cases} m(\{x \mid (\mathrm{H}(f-g))(x) > \alpha/2\}) \leq \dfrac{2C_d}{\alpha} \displaystyle\int |f-g|\,; \\ m(\{x \mid |g(x) - f(x)| > \alpha/2\}) \leq \dfrac{2}{\alpha} \displaystyle\int |f-g|\,. \end{cases}$$

Thus,

$$0 \leq m(E_\alpha) \leq \frac{2C_d}{\alpha} \|f-g\|_1 + \frac{2}{\alpha} \|f-g\|_1 \leq \frac{2(C_d+1)}{\alpha}\epsilon.$$

Taking $\epsilon \to 0$, $m(E_\alpha)$ does not depend on ϵ and g, hence $m(E_\alpha) = 0$. ∎

Corollary 4.1. Theorem 4.2 also holds for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Proof. Using the fact that m^d is σ-finite, and apply Theorem 4.2. Specifically, partition $\mathbb{R}^d$ into countably many compact sets K_i and apply Theorem 4.2 to $f \mathbb{1}_{K_i}$ for all i. ∎

Corollary 4.2. For $f \in L^1_{\mathrm{loc}}$, we have

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\,\mathrm{d}y = f(x)$$

for a.e. x.

Proof. . Use that DIY

$$f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x)\,\mathrm{d}y$$

and the triangle inequality. ∎

Definition 4.3 (Lebesgue point). Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, the point $x \in \mathbb{R}^d$ is called a *Lebesgue point of f* if

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\,\mathrm{d}y = 0.$$

Remark. Corollary 4.1 tells us that almost all points in $\mathbb{R}^d$ in $\mathbb{R}^d$ are Lebesgue points for f.

Definition 4.4 (Shrink nicely). We say that $\{E_r\}_{r>0}$ *shrinks nicely* to x as $r \to 0$ if $E_r \subset B(x,r)$ and

$$\underset{c>0}{\exists}\ c \cdot m(B(x,r)) \leq m(E_r).$$

Corollary 4.3. Suppose E_r shrink nicely to 0, and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, and x is a Lebesgue point. Then

$$\begin{cases} \lim\limits_{r\to 0} \dfrac{1}{m(E_r)} \displaystyle\int_{E_r+x} |f(y) - f(x)| \,\mathrm{d}y = 0; \\ \lim\limits_{r\to 0} \dfrac{1}{m(E_r)} \displaystyle\int_{E_r+x} |f(y)| \,\mathrm{d}y = f(x). \end{cases}$$

Corollary 4.4. If $f \in L^1_{\text{loc}}(\mathbb{R})$, then $F(x) = \int_0^x f(y)\,\mathrm{d}y$ is differentiable and $F'(x) = f(x)$ almost everywhere.

Lecture 23: Metric, normed and L^p Spaces

09 Mar. 11:00

5 Normed Vector Space

5.1 Metric Spaces and Normed Spaces

We have seen the definition of a norm before, now we formally introduce the concept of *metric*.

Definition 5.1 (Metric). Let Y be a set, a function $\rho\colon Y \times Y \to [0,\infty)$ is a *metric* on Y if

- $\rho(x,y) = \rho(y,x)$ for all $x,y \in Y$.
- $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x,y,z \in Y$.
- $\rho(x,y) = 0$ if and only if $x = y$.

Note. The followings make sense in a metric space.

1. Open/closed balls.
2. Open/closed sets.
3. Convergence sequences ($x_n \to x$ with respect to ρ if and only if $\lim\limits_{n\to\infty} \rho(x_n,x) = 0$).
4. Continuous functions.

Example. We have the following metric spaces.

1. $\mathbb{Q}$ with $\rho(x,y) = |x-y|$.

2. $\mathbb{R}$ with $\rho(x,y) = |x-y|$.
3. $\mathbb{R}_+$ with $\rho(x,y) = |\ln(y/x)|$.
4. $\mathbb{R}^d$ with
$$\rho_p(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p}$$
and
$$\rho_\infty(x,y) = \max_{1\leq i\leq d} |x_i - y_i|.$$
These all give the same open sets, hence they are topologically equivalent.
5. $C([0,1])$ with
$$\rho_p(f,g) = \left(\int_0^1 |f-g|^p\right)^{1/p}$$
and
$$\rho_\infty(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|.$$
6. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Let Y be the set of measurable functions on X, then
$$\rho(f,g) = \int \min\left\{|f(x) - g(x)|, 1\right\} \,\mathrm{d}\mu(x)$$
is a metric and $f_n \to f$ in ρ if and only if $f_n \to f$ in measure.

Let V be a vector space over scalar field $K = \mathbb{R}$ or $K = \mathbb{C}$.

As previously seen (Metric induced by a norm). Recall the definition of seminorm and norm. We see that a norm induces a metric
$$\rho(v,w) := \|v - w\|,$$
and we have
$$v_n \to v \iff \lim_{n\to\infty} \|v_n - v\| = 0.$$

Example. We first see some common examples of normed vector space.

1. $L^1(X, \mathcal{A}, \mu)$ with $\|f\|_1 := \int |f| \,\mathrm{d}\mu$.
2. $C([0,1])$ with $\|f\|_1 := \int_0^1 |f(x)| \,\mathrm{d}x$, $\|f\|_\infty := \max_{0\leq x\leq 1} |f(x)|$.
3. For $\mathbb{R}^d$ and $0 < p < \infty$, we have
$$\|x\|_p := \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}, \qquad \|x\|_\infty := \max_{1\leq i\leq d} |x_i|.$$

5.2 L^p Space

It turns out that we can generalize L^1 into L^p.

Definition 5.2 (L^p space). Given a measure space $(X, \mathcal{A}, \mu)$ and a measurable function f and p such that $0 < p < \infty$, we define a seminorm $\|\cdot\|_p$ such that

$$\|f\|_p := \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p},$$

which induces the so-called L^p *space* $L^p(X, \mathcal{A}, \mu)$, where

$$L^p(X, \mathcal{A}, \mu) := \left\{ f \mid \|f\|_p < \infty \right\}.$$

Remark. Note that $\|\cdot\|_p$ is only a seminorm. But if we identity functions which are equal almost everywhere, then it's indeed a norm.

Example. $(\mathbb{R}, \mathcal{L}, m)$ has $f(x) = x^{-\alpha}\mathbb{1}_{(1,\infty)}(x) \in L^p$ if and only if $\alpha p > 1$. In contrast, $g(X) = x^{-\beta}\mathbb{1}_{(0,1)}(x) \in L^p$ if and only if $\beta p < 1$.

Similar to Definition 5.2, we have the following.

Definition 5.3 (ℓ^p space). If $(X, \mathcal{P}(X), \nu)$ is equipped with the counting measure, then we say it's an ℓ^p *space* such that

$$\ell^p(X) := L^p(X, \mathcal{P}(X), \nu).$$

Remark. We are interested in $\ell^p(\mathbb{N})$ in particular. We have

$$\ell^p := \ell^p(\mathbb{N}) = \left\{ a = (a_1, a_2, \dots) \mid \|a\|_p = \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p} < \infty \right\}.$$

Lemma 5.1. $L^p(X, \mathcal{A}, \nu)$ is a vector space for all $p \in (0, \infty)$.

Proof. We verify the following.

- $c \cdot f \in L^p(X, \mathcal{A}, \mu)$ for $c \in \mathbb{R}$. Indeed, since

$$\|cf\|_p = \left(\int |cf|^p \,\mathrm{d}\mu\right)^{1/p} = |c| \, \|f\|_p < \infty \iff \|f\|_p < \infty,$$

  which implies $c \cdot f \in L^p(X, \mathcal{A}, \mu)$.

- $f + g \in L^p(X, \mathcal{A}, \mu)$. Indeed, since for any real numbers α, β, we have

$$(\alpha + \beta)^p \le (2 \cdot \max\{|\alpha|, |\beta|\})^p = 2^p \cdot \max\{|\alpha|^p, |\beta|^p\} \le 2^p \left(|\alpha|^p + |\beta|^p\right),$$

which implies that for $f, g \in L^p(X, \mathcal{A}, \mu)$, we have

$$\|f+g\|_p < \infty \iff \|f+g\|_p^p = \int |f+g|^p \,\mathrm{d}\mu \leq 2^p \int (|f|^p + |g|^p) < \infty.$$

This further implies

$$\|f+g\|_p < \infty \iff \|f\|_p, \|g\|_p < \infty,$$

which is what we want.

■

We see that in the above derivation, it doesn't give us the triangle inequality, namely

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p,$$

hence we need some new results.

Theorem 5.1 (Hölder's inequality). Let $1 < p < \infty$, and let $q := p/(p-1)$ so that $1/p + 1/p = 1$. Then we have

$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. We prove this in steps.

1. Note that
$$t \leq \frac{t^p}{p} + 1 - \frac{1}{p} = \frac{t^p}{p} + \frac{1}{q}$$
for all $t \geq 0$. Hence, by taking $F(t) := t - t^p/p$ and $t \geq 0$, we see that the maximum of F implies the above inequality.

2. Young's Inequality.[26] We have
$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$
for $\alpha, \beta > 0$. This follows by taking $t := \alpha/\beta^{q-1}$ in the first inequality we obtained.

3. Without loss of generality, we can assume that $0 < \|f\|_p, \|g\|_q < \infty$. Now, consider $F(x) = f(x)/\|f\|_p$, $G(x) = g(x)/\|g\|_q$. We know that $\|F\|_p = 1 = \|G\|_q$. Then by Young's Inequality, we have
$$\int |F(x)G(x)| \,\mathrm{d}\mu \leq \int \frac{|F(x)|^p}{p} + \frac{|G(x)|^q}{q} \implies \frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1,$$
which implies our desired result.

■

[26] https://en.wikipedia.org/wiki/Young's_inequality_for_products

Example. For $p = q = 2$, $X = \{1, \dots, d\}$ with μ being the counting measure, then for any $x, y \in \mathbb{R}^d$, we have

$$\sum_{i=1}^{d} |x_i y_i| \leq \sqrt{\sum_{i=1}^{d} x_i^2} \sqrt{\sum_{i=1}^{d} y_i^2}$$

We now see how we can obtain the desired triangle inequality.

Theorem 5.2 (Minkowski's Inequality). Let $1 \leq p < \infty$, then for $f, g \in L^p$,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p .$$

Proof. For $p = 1$, it's easy since it's just triangle inequality. Now, we assume that $1 < p < \infty$, and we may assume also that $\|f + g\| \neq 0$ without loss of generality. Then

$$\begin{aligned}
\int |f(x) + g(x)|^p &\leq \int |f(x) + g(x)|^{p-1} \left(|f(x)| + |g(x)|\right) \\
&\leq \left(\int |f + g|^{(p-1)q}\right)^{1/q} \left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right] \\
&\leq \left(\int |f + g|^p\right)^{1/q} \left(\|f\|_p + \|g\|_p\right).
\end{aligned}$$

We then see that

$$\underbrace{\left(|f(x) + g(x)|^p\right)^{1-1/q}}_{\left(|f(x)+g(x)|^p\right)^{1/p}} \leq \|f\|_p + \|g\|_p ,$$

which is just $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. ■

Lecture 24: Embedding L^p Space

11 Mar. 11:00

Definition 5.4 (Essential supremum). For a measurable function f on $(X, \mathcal{A}, \mu)$, we define

$$\begin{aligned}
S &:= \{\alpha \geq 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\} \\
&= \{\alpha \geq 0 \mid |f(x)| \leq \alpha \text{ a.e.}\} .
\end{aligned}$$

Then, we say that the *essential supremum of* f, denoted as $\|f\|_\infty$, is defined as

$$\|f\|_\infty := \begin{cases} \inf S, & \text{if } S \neq \varnothing; \\ \infty, & \text{if } S = \varnothing. \end{cases}$$

Definition 5.5 (L^∞ space). Let $L^\infty(X, \mathcal{A}, \mu)$ be

$$L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\} .$$

Definition 5.6 (ℓ^∞ space). We let ℓ^∞ be defined as

$$\ell^\infty = L^\infty(\mathcal{N}, \mathcal{P}(\mathcal{N}), \nu),$$

where ν is the counting measure.

Example. Consider $(\mathbb{R}, \mathcal{L}, m)$. Then

$$f(x) = \frac{1}{x}\mathbb{1}_{(0,\infty)}(x) \notin L^\infty;$$
$$g(x) = x\mathbb{1}_{\mathbb{Q}}(x) + \frac{1}{1+x^2} \in L^\infty.$$

If f is continuous on $(\mathbb{R}, \mathcal{L}, m)$, then $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. For $a \in \ell^\infty$, we have $\|a\|_\infty = \sup_{i\in\mathbb{N}} |a_i|$, and sequences in ℓ^∞ are exactly the bounded sequences.

Lemma 5.2. We have the following.

1. Suppose $f \in L^\infty(X, \mathcal{A}, \mu)$. Then,

$$\begin{cases} \mu(\{x \mid |f(x)| > \alpha\}) = 0, & \text{if } \alpha \geq \|f\|_\infty ; \\ \mu(\{x \mid |f(x)| > \alpha\}) > 0, & \text{if } \alpha < \|f\|_\infty . \end{cases}$$

2. $|f(x)| \leq \|f\|_\infty$ almost everywhere.
3. $f \in L^\infty$ if and only if there exists a bounded measurable function g such that $f = g$ almost everywhere.

Proof. . ■ DIY

Theorem 5.3. We have the following.

1. $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.
2. $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
3. $f_n \to f$ in L^∞ if and only if $f_n \to f$ uniformly almost everywhere.

Remark. The motivation for 1. is that

$$\frac{1}{1} + \frac{1}{\infty} = 1,$$

and we want to have the similar result as in Theorem 5.1.

Proof. . We'll do one implication in 3. Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$. Let $A = \bigcup_n A_n$, we see that $\mu(A) = 0$ as well. DIY

For $x \in A^c$ and for every n, we have

$$|f_n(x) - f(x)| \leq \|f_n - f\|_\infty .$$

Given $\epsilon > 0$, there is an N so that

$$\|f_n - f\| < \epsilon$$

for all $n \geq N$. But then for all $x \in A^c$, $|f_n(x) - f(x)| < \epsilon$ as well. ■

Proposition 5.1. We have the following.

1. For $1 \leq p < \infty$, the collection of simple functions with finite measure support is dense in $L^p(X, \mathcal{A}, \mu)$.
2. For $1 \leq p < \infty$, the collection of step functions with finite measure support is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$, so is $C_c(\mathbb{R})$.
3. For $p = \infty$, the collection of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.

Remark. Note that $C_c(\mathbb{R})$ is **not** dense in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

Proof. . ■ DIY

5.3 Embedding Properties of L^p Spaces

Definition 5.7 (Equivalent norm). Two norms $\|\cdot\|, \|\cdot\|'$ on V are *equivalent* if there exists $c_1, c_2 > 0$, such that

$$c_1 \|v\| \leq \|v\|' \leq c_2 \|v\|$$

for all $v \in V$.

Note. We see that

1. These norms gives the same topological properties (open sets, closed sets, convergence, etc.).
2. Definition 5.7 is an equivalence relation on norms.

Example. For $\mathbb{R}^d$ we have the norms $\|\cdot\|_p$ for $1 \leq p \leq \infty$. All of these are equivalent. We see that for $1 \leq p < \infty$,

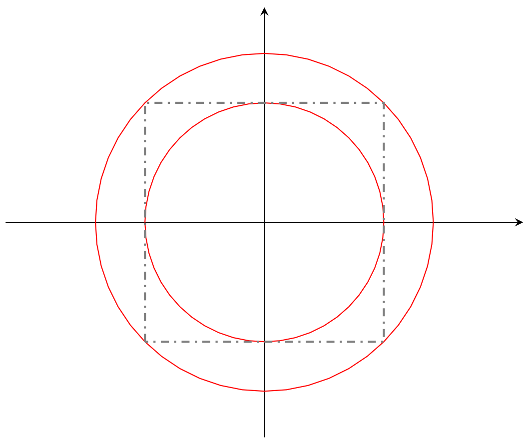
$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p} \leq (d \|x\|_\infty^p)^{1/p} = d^{1/p} \|x\|_\infty .$$

Also,

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p} \geq (\|x\|_\infty^p)^{1/p} = \|x\|_\infty .$$

Thus, $\|\cdot\|_p$ is equivalent to $\|\cdot\|_\infty$ for every $1 \leq p < \infty$, and transitivity gives that they are all equivalent.

Another way of thinking of this, by assuming $v \neq 0$, and scaling by some t, we may assume v lies on the unit circle in one of the norms. Then we are squeezing a unit circle in $\|\cdot\|'$ between two circles of radius c_1, c_2 in $\|\cdot\|$. In picture, we have to show that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent, we have



since the circles in $\|\cdot\|_\infty$ are squares.

Example. Fro $1 \leq p, q \leq \infty$, we have $L^p(\mathbb{R}, m)$-norm and $L^q(\mathbb{R}, m)$-norm are not equivalent, even worse, we have that

$$L^p(\mathbb{R}, m) \nsubseteq L^1(\mathbb{R}, m), \quad L^p(\mathbb{R}, m) \nsupseteq L^1(\mathbb{R}, m).$$

Lecture 25: Banach Spaces

14 Mar. 11:00

Proposition 5.2. Suppose $\mu(X) < \infty$, then for every $0 < p < q \leq \infty$, $L^q \subseteq L^p$.

Proof. Suppose $q < \infty$, then

$$\int |f|^p \leq \left(\int (|f|^p)^{q/p} \right)^{p/q} \left(\int 1^{q/(q-p)} \right)^{1-p/q} = \left(\int |f|^q \right)^{p/q} \mu(x)^{1-p/q} < \infty$$

where we split $\int |f|^p$ into $\int |f|^p \cdot 1$. From Hölder's inequality with $q/p > 1$, we have

$$\|f\|_p \leq \|f\|_q \mu(X)^{1/p-1/q} < \infty.$$

The case that $q = \infty$ is left as an exercise. ∎ DIY

Proposition 5.3. If $0 < p < q \leq \infty$, then $\ell^p \subseteq \ell^q$.

Proof. When $q = \infty$, we have

$$\|a\|_\infty^p = \left(\sup_i |a_i| \right)^p = \sup_i |a_i|^p \leq \sum_{i=1}^\infty |a_i|^p.$$

Thus $\|a\|_\infty \leq \|a\|_p$.

When $q < \infty$, we see that

$$\sum_{i=1}^{\infty} |a_i|^q = \sum_{i=1}^{\infty} |a_i|^p \cdot |a_i|^{q-p} \leq \|a\|_{\infty}^{q-p} \sum_{i=1}^{\infty} |a_i|^p \leq \|a\|_p^{q-p} \cdot \|a\|_p^p = \|a\|_p^q .$$

Therefore,

$$\|a\|_q \leq \|a\|_p .$$

■

Proposition 5.4. For all $0 < p < q < r \leq \infty$, $L^p \cap L^r \subseteq L^q$.

Proof. ■ DIY

5.4 Banach Spaces

Let's start with a definition.

Definition 5.8 (Cauchy sequence). Let Y, ρ be a metric space. We call x_n a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\rho(x_n, x_m) < \epsilon$.

Note. Convergent sequence are Cauchy.

Definition 5.9 (Complete). A metric space (Y, ρ) is called *complete* if every Cauchy sequence in Y converges.

Example. We first see some examples.

1. We see that $\mathbb{Q}$ with $\rho(x, y) = |x - y|$ is **not** complete, but $\mathbb{R}$ with the same metric is complete.
2. $C([0, 1])$ with $\rho(f, g) = \|f - g\|_{\infty}$ is complete, but with $\rho(f, g) = \int |f - g|$ is not.

Definition 5.10 (Banach space). A *Banach space* is a complete normed vector space.

Remark. Namely, a vector space equipped with a norm whose metric induced by the norm is complete.

Theorem 5.4. Let $(V, \|\cdot\|)$ be a normed space. Then,

$$V \text{ is complete} \iff \text{every absolutely convergent series is convergent.}$$

i.e., if $\sum_{i=1}^{\infty} \|v_i\| < \infty$, then $\left\{ \sum_{i=1}^{N} v_i \right\}_{N \in \mathbb{N}}$ converges to some $s \in V$.

Before we prove Theorem 5.4, we first see one of the result based on this theorem.[27]

Theorem 5.5 (Riesz-Fischer theorem). For every $1 \leq p \leq \infty$, we have $L^p(X, \mathcal{A}, \mu)$ is complete, hence a Banach space.

Proof. We prove this in steps.

1. We first prove this for $1 \leq p < \infty$. Suppose $f_n \in L^p$ and $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

   We need to show that there is an $F \in L^p$ such that $\left\| \sum_{n=1}^{N} f_n - F \right\|_p \to 0$ as $N \to \infty$. i.e., we need to show

   (a) $\sum_{n=1}^{\infty} f_n(x)$ is convergent a.e. In fact, we can show $\int \sum_{n=1}^{\infty} |f_n(x)| < \infty$.

   Let $G(x) = \sum_{n=1}^{\infty} |f_n(x)| = \sup_N \sum_{n=1}^{N} |f_n(x)|$, $G\colon X \to [0, \infty]$. Also, let $G_N(x) = \sum_{n=1}^{N} |f_n(x)|$. Then, we have

   $$0 \leq G_1 \leq G_2 \leq \ldots \leq G,$$

   and $G_N \to G$. Furthermore,

   $$0 \leq G_1^p \leq G_2^p \leq \ldots \leq G^p,$$

   and $G_N^p \to G^p$. From monotone convergence theorem,

   $$\int G^p = \lim_{N\to\infty} \int G_N^p.$$

   From Minkowski inequality, we further have

   $$\|G_N\|_p \leq \sum_{n=1}^{N} \|f_n\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p \coloneqq B < \infty.$$

   Thus,

   $$\int G(x)^p = \lim_{N\to\infty} \int G_N^p = \lim_{N\to\infty} \|G_N\|_p^p \leq B^p < \infty.$$

   We see that G is finite a.e. as desired. This implies that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ a.e., so $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. Now, we simply let

   $$F(x) = \begin{cases} \sum_{n=1}^{\infty} f_n(x), & \text{if it converges;} \\ 0, & \text{otherwise.} \end{cases}$$

[27] The proof can be found in here.

(b) $F \in L^p$, where $F(x) \coloneqq \sum_{n=1}^{\infty} f_n(x)$ a.e. and say is zero elsewhere.

This is clear since

$$|F(x)| \leq G(x) \implies \int |F|^p \leq \int G^p < \infty,$$

hence $F \in L^p$.

(c) $\left\| \sum_{n=1}^{N} f_n - F \right\|_p \to 0$ as $N \to \infty$.

We now see that

$$\left| \sum_{n=1}^{N} f_n(x) - F(x) \right|^p \leq \left(\sum_{n=1}^{\infty} |f_n(x)| + |F(x)| \right)^p \leq (2G(x))^p.$$

Since $2G \in L^p$, so $2G^p \in L^1$. Thus, by dominated convergence theorem, we have

$$\lim_{N\to\infty} \int \left| \sum_{n=1}^{N} f_n(x) - F(x) \right|^p \,\mathrm{d}x = 0.$$

This implies

$$\left\| \sum_{n=1}^{N} f_n - F \right\|_p \to 0$$

as $N \to \infty$.

2. The case that $1 \leq p \leq \infty$ is left as an exercise. DIY

■

Lecture 26: Bounded Linear Transformations

16 Mar. 11:00

We now prove Theorem 5.4, completing the proof of Theorem 5.5 since the latter relies on this result.

Proof. We prove it by proving two directions.

($\implies$) Suppose V is complete, and fix an absolutely convergent series $\sum_n v_n$. Define $s_N = \sum_{n=1}^{N} v_n$. It suffices to show the partial sums are a Cauchy Sequence.

Fix $\epsilon > 0$, then because $\sum_{n=1}^{\infty} \|v_n\| < \infty$, there is a $K \in \mathbb{N}$ so that

$$\sum_{n=K}^{\infty} \|v_n\| < \epsilon.$$

Now let $M > N > K$, we see that

$$\|s_M - s_N\| = \left\| \sum_{n=N+1}^{M} v_n \right\| \leq \sum_{n=N+1}^{M} \|v_n\| \leq \sum_{n=N}^{\infty} \|v_n\| < \epsilon,$$

so this is Cauchy.

($\Longleftarrow$) Now suppose $v_n, n \in \mathbb{N}$ is a Cauchy sequence. For all $j \in \mathbb{N}$, there exists an $N_j \in \mathbb{N}$ such that

$$\|v_n - v_m\| < \frac{1}{2^j}$$

for all $n, m \geq N_j$. Without loss of generality, we may assume $N_1 < N_2 < \ldots$.

Let $w_1 = v_{N_1}$, $w_j = v_{N_j} - v_{N_j - 1}$ for $j \geq 2$. Therefore,

$$\sum_{j=1}^{\infty} \|w_j\| \leq \|v_{N_1}\| + \sum_{j=2}^{\infty} \frac{1}{2^{j-1}} < \infty.$$

Thus, $\sum_{j=1}^{k} w_j \to s \in V$ as $k \to \infty$. But by telescoping, we have

$$v_{N_k} = \sum_{j=1}^{k} w_j \to s.$$

Now we claim that since v_n is Cauchy, so $v_n \to s$.

Explicitly, take $\epsilon > 0$, and let k be large enough so that $\|v_{N_k} - s\| < \epsilon$ and $1/2^k < \epsilon$. Then if $n > N_k$ then

$$\|v_n - s\| \leq \|v_n - v_{N_k}\| + \|v_{N_k} - s\| < \epsilon + \epsilon = 2\epsilon.$$

Thus, $v_n \to s$. ■

5.5 Bounded Linear Transformations

Definition 5.11 (Bounded linear transformation). Given two normed vector spaces $(V, \|\cdot\|)$, $(W, \|\cdot\|')$, a linear map $T\colon V \to W$ is called a *bounded map* if there exists $c \geq 0$ such that

$$\|Tv\|' \leq c\,\|v\|$$

for all $v \in V$.

Proposition 5.5. Suppose $T\colon (V, \|\cdot\|) \to (W, \|\cdot\|')$ is a linear map. Then the followings are equivalent.

1. T is continuous.
2. T is continuous at 0.
3. T is a bounded map.

Proof. 1. $\Longrightarrow$ 2. is clear. For 2. $\Longrightarrow$ 3., take $\epsilon = 1$, then there exists a $\delta > 0$ such that $\|Tu\|' < 1$ if $\|u\| < \delta$.

Now take an arbitrary $\|v\| \in V$, $v \neq 0$. Let $u = \frac{\delta}{2\|v\|} v$. Then $\|u\| < \delta$. Therefore,

$$\|Tu\|' < 1 \implies \frac{\delta}{2\,\|v\|}\,\|Tv\|' < 1 \implies \|Tv\|' < \frac{2}{\delta}\,\|v\|\,.$$

Then $2/\delta$ is our constant.

For 3. $\implies$ 1., fix $v_0 \in V$. Then for some constant c

$$\|Tv - Tv_0\|' = \|T(v - v_0)\|' \leq c\,\|v - v_0\|\,.$$

Thus, T is continuous, as when $v \to v_0$ the right-hand side goes to zero, and so $Tv \to Tv_0$. ■

Example. Let's see some examples.

1. We can look at

$$\begin{aligned} T\colon \ell^1 &\to \ell^1 \\ (a_1, a_2, \ldots) &\mapsto (a_2, a_3, \ldots). \end{aligned}$$

   Then clearly $\|Ta\|_1 \leq \|a\|_1$, so T is a bounded linear transformation.

2. We can also look at $S\colon (C([-1,1]), \|\cdot\|_1) \to \mathbb{C}$, where $Sf = f(0)$. S is not a bounded linear transformation, because we can make

$$\begin{cases} \|Sf\| & = |f(0)| = n \\ \|f\|_1 & = 1 \end{cases}$$

   for every $n \in \mathbb{N}$ (take f's graph to be a skinny triangle shooting up to n at 0).

3. But $U\colon (C([-1,1]), \|\cdot\|_\infty) \to \mathbb{C}$ defined by $Uf = f(0)$ is a bounded linear transformation, because $|f(0)| \leq \|f\|_\infty$.

4. Let A be an $n \times m$ matrix. Then $T\colon \mathbb{R}^m \to \mathbb{R}^n$ defined by $v \mapsto Av$ is a bounded linear transformation.

   Explicitly this is

$$(Tv)_i = (Av)_i = \sum_{j=1}^{m} A_{ij} v_j.$$

5. Let $K(x, y)$ be a continuous function on $[0,1] \times [0,1]$. We'll define

$$T\colon (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$$

   by

$$(Tf)(x) = \int_0^1 K(x,y) f(y)\,\mathrm{d}y.$$

   This is an analogue of matrix multiplication (K is like a continuous matrix). This is a bounded linear transformation.

6. Let us look at $T\colon L^1(\mathbb{R}) \to (C(\mathbb{R}), \|\cdot\|_\infty)$ defined by

$$(Tf)(t) = \int_{-\infty}^{\infty} e^{-itx} f(x)\,\mathrm{d}x$$

   that is the Fourier transform of f.

7. $T : (C^\infty[0,1], \|\cdot\|_\infty) \to (C^\infty[0,1], \|\cdot\|_\infty)$. Define
$$(Tf)(x) = f'(x).$$
This is not a bounded linear transformation. In contrast, S, defined on the same spaces
$$(Sf)(x) = \int_0^x f(t)\,\mathrm{d}t$$
is bounded.

Definition 5.12 (Operator norm). Let $L(V, W)$ be defined as a vector space such that

$$L(V, W) \coloneqq \{T\colon V \to W \mid T \text{ is a bounded linear transformation}\}.$$

Then for $T \in L(V, W)$, the *operator norm* of T is

$$\begin{aligned}\|T\| &\coloneqq \inf\{c \geq 0 \mid \|Tv\|'' \leq c\,\|v\|' \text{ for all } v \in V\}\\ &= \sup\left\{\frac{\|Tv\|''}{\|v\|'} \mid v \neq 0, v \in V\right\}\\ &= \sup\left\{\|Tv\|'' \mid \|v\|' = 1, v \in V\right\}.\end{aligned}$$

Lemma 5.3. We have that

1. The three definitions of $\|T\|$ above are all equal.
2. $(L(V, W), \|\cdot\|)$ is indeed a normed space.

Proof. ■ DIY

Lecture 27: Dual Space

18 Mar. 11:00

As previously seen. From Definition 5.12, we have that
$$\|Tv\|'' \leq \|T\|\,\|v\|'.$$

Remark. Notice that this Definition 5.12 is only for bounded linear transformation.

Theorem 5.6. If W is complete, then $L(V, W)$ is complete.

Proof. Suppose T_n is a Cauchy sequence in $L(V, W)$. Fix $v \in V$ and let $w_n \coloneqq T_n v \in W$, we then have

$$\|w_n - w_m\| = \|T_n v - T_m v\| = \|(T_n - T_m)v\| \leq \underbrace{\|T_n - T_m\|}_{\to 0}\underbrace{\|v\|}_{\text{fixed value}}.$$

Thus, w_n is Cauchy, so it converges since W is complete. We call its unique limit Tv. This makes $T\colon V \to W$ into a function. We must show it is a bounded linear transformation and that $\|T_n - T\| \to 0$. ■ DIY

5.6 Dual of L^p Spaces

Example. Let $w \in \mathbb{R}^d$, and denote the inner product between w and $v \in \mathbb{R}^d$ by

$$v \cdot w := \langle v, w \rangle .$$

Then we can consider

$$\max\{v \cdot w \mid \|v\|_2 = 1\} = \|w\|_2 .$$

If $w \in \mathbb{C}^d$, this is similar since

$$\max\{|v \cdot w| \mid \|v\|_2 = 1\} = \|w\|_2 .$$

These maximums are achieved by $v = \frac{\overline{w}}{\|w\|}$ if $w \neq 0$.

Proposition 5.6. Let $1/p + 1/q = 1$ with $1 \leq q < \infty$. For every $g \in L^q$,

$$\|g\|_q = \sup\left\{\left|\int fg\right| \mid \|f\|_p = 1\right\}.$$

Suppose μ is σ-finite. Then the result also holds for $q = \infty$, $p = 1$.

As previously seen. For $\alpha \in \mathbb{C}$, $\mathrm{sgn}(\alpha) := e^{i\theta}$ where $\alpha = |\alpha| e^{i\theta}$.

Proof. By Hölder's inequality, we know that

$$\left|\int fg\right| \leq \int |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q = \|g\|_q .$$

Thus, the supremum is less or equal to $\|g\|_q$.

1. Let
$$f(x) = \frac{|g(x)|^{q-1} \cdot \overline{\mathrm{sgn}(g(x))}}{\|g\|_q^{q-1}}$$
Then $\int |f|^p = 1$, and $\int fg = \|g\|_q$. Check
2. The case that μ is σ-finite and $q = \infty, p = 1$ can be shown. DIY

∎

Remark. One could use the above to prove Minkowski's inequality (as it only uses Hölder's inequality).

Definition 5.13 (Dual space). For a normed space $(V, \|\cdot\|)$, its *dual space* is $V^* = L(V, \mathbb{R})$ or $V^* = L(V, \mathbb{C})$.

Remark. Namely, the dual space of V contains bounded linear transformations with codomain being the scalar field.

Definition 5.14 (Linear functional). Given a normed space $(V, \|\cdot\|)$, $\ell \in V^*$ is called a *linear functional* on V. i.e.,

- $\ell \colon V \to \mathbb{R}$ (or $\mathbb{C}$).
- ℓ is linear.
- There exists a $c \geq 0$ such that $|\ell(v)| = c\,\|v\|$.

Note. V^* is always a Banach space (even if V is not complete).

Corollary 5.1. We have the followings.

1. Let $1/p + 1/q = 1, 1 \leq q < \infty$. For $g \in L^q$ define $\ell_g \in L^p \to \mathbb{C}$ by
$$\ell_g(f) = \int fg.$$
Then $\ell_g \in (L^p)^*$. Furthermore, $\|\ell_g\| = \|g\|_q$.
2. If μ is σ-finite then this also holds for $q = \infty, p = 1$.

Proof. ℓ_g is clearly linear in f because the integral is linear. Then Proposition 5.6 gives in both 1. and 2. that
$$\|g\|_q = \sup\{|\ell_g(f)| \mid \|g\|_p = 1\} = \|\ell_g\|$$
and so ℓ_g is a bounded linear transformation with the desired properties. ∎

Theorem 5.7. We have the followings.

1. Let $1/p + 1/q = 1$, $1 \leq q < \infty$. The map $T \colon L^q \to (L^p)^*$ given by $Tg = \ell_g$ is an isometric[a] linear isomorphism.

   This means that

   - T is a bounded linear transformation.
   - T is bijective.
   - T is norm-preserving.
2. If μ is σ-finite then this also holds for $q = \infty, p = 1$.

[a]A map T is called isometric if for a given g, $\|Tg\| = \|g\|$.

Appendix

A Additional Proofs

A.1 Measure

This section gives all additional proofs in Section 1.

Theorem A.1 (Theorem 1.2 3.). Under the setup of Theorem 1.2, $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. We see this in two parts.

1. **Claim:** If $A \subset X$ satisfies $\mu^*(A) = 0$, then A is Carathéodory measurable with respect to μ^*.

   Proof. If $A \subset X$ and $\mu^*(A) = 0$, where μ^* is an outer measure on X, we'll show that A is Carathéodory measurable with respect to μ^*.

   Equivalently, we want to show that for any $E \subset X$,

   $$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

   Firstly, noting that $(E \cap A) \subset A$, and by monotonicity of μ^*, we see that

   $$\mu^*(E \cap A) \leq \mu^*(A) = 0,$$

   and since $\mu^* \geq 0$, hence $\mu^*(E \cap A) = 0$. Now, we only need to show that

   $$\mu^*(E) = \mu^*(E \setminus A).$$

   Since $E \setminus A = E \cap A^c$, and hence we have $E \cap A^c \subset E$, so

   $$\mu^*(E) \geq \mu^*(E \setminus A).$$

   To show another direction, we note that

   $$\mu^*(E) \leq \mu^*(E \cup A) = \mu^*((E \setminus A) \cup A) \leq \mu^*(E \setminus A),$$

   hence we conclude that A is Carathéodory measurable with respect to μ^* if $\mu^*(A) = 0$. ■

2. **Claim:** If A is μ-subnull, then $A \in \mathcal{A}$.

   Proof. Let $\mathcal{A}$ denotes the Carathéodory σ-algebra, and $\mu := \mu^*|_{\mathcal{A}}$. We want to show if A is μ-subnull, then $A \in \mathcal{A}$.

   Firstly, if A is μ-subnull, then there exists a $B \in \mathcal{A}$ such that $A \subset B$ and $\mu(B) = 0$. But since from the monotonicity of μ^*, we further have

   $$0 = \mu(B) = \mu^*(B) \geq \mu^*(A),$$

   hence $\mu^*(A) = 0$.

   From the first claim, we immediately see that A is Carathéodory measurable with respect to μ^*, which implies A is in Carathéodory σ-algebra, hence $A \in \mathcal{A}$. ■

We see that the second claim directly proves that $(X, \mathcal{A}, \mu)$ is a complete measure space. ■

Lemma A.1. The function F defined in this example is a distribution function

Proof. We define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n \end{cases}$$

where $\{r_1, r_2, \ldots\} = \mathbb{Q}$, and

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} = \sum_{n; r_n \leq x} \frac{1}{2^n}$$

is both increasing and right-continuous.

- Increasing. Consider $x < y$. We see that

$$F(x) = \sum_{n; r_n \leq x} \frac{1}{2^n} \leq \sum_{n; r_n \leq y} \frac{1}{2^n} = F(y)$$

 clearly.[28]

- Right-continuous. We want to show $F(x^+) = F(x)$. Let $x^+(\epsilon) := x + \epsilon$ with $\epsilon > 0$, we'll show that

$$\lim_{\epsilon \to 0} F(x^+(\epsilon)) = \lim_{\epsilon \to 0} F(x + \epsilon) = F(x).$$

 Firstly, we have

$$F(x^+(\epsilon)) - F(x) = \sum_{n; r_n \leq x+\epsilon} \frac{1}{2^n} - \sum_{n; r_n \leq x} \frac{1}{2^n} = \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n},$$

 [29]

 and we want to show

$$\lim_{\epsilon \to 0} F(x^+(\epsilon)) - F(x) = \lim_{\epsilon \to 0} \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} = 0.$$

 Before we show how we choose ϵ,[30] we see that

$$\sum_{n=k}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

[28] This is trivial since we're always going to sum more strictly positive terms in $F(y)$ than in $F(x)$.

[29] The strict is crucial to show the result, since if $x = r_k$ for some fixed k, then we can't make the summation arbitrarily small.

[30] To be precise, how ϵ depends on r_n.

Now, we observe that

$$\sum_{n;x<r_n\leq x+\epsilon}\frac{1}{2^n}\leq\sum_{n=\arg\min_k x<r_k\leq x+\epsilon}^{\infty}\frac{1}{2^n}=2^{1-k}.$$

With this observation, it should be fairly easy to see that we can choose ϵ based on how small we want to make 2^{1-k} be,[31] and we indeed see that

$$\lim_{k\to\infty}2^{1-k}=0,$$

which implies that F is right-continuous by squeeze theorem.

■

Lemma A.2. The function F defined in this example satisfies

- $\mu_F(\{r_i\})>0$ for all $r_i\in\mathbb{Q}$.
- $\mu_F(\mathbb{R}\setminus\mathbb{Q})=0$

given in this example.

Proof. We prove them one by one. And notice that F is indeed a distribution function as we proved in Lemma A.1.

1. To show $\mu_F(\{r\})>0$ for every $r\in\mathbb{Q}$, we first note that $\{r\}=\bigcap_{a-1\leq x<r}(x,r]$. Then, we see that

$$\mu_F(\{r\})=\mu_F\left(\bigcap_{a-1\leq x<a}(x,r]\right),$$

where each $(x,r]\in\mathcal{A}$ and $(x,r]\supset(y,r]$ whenever $r-1\leq x\leq y<r$. Notice that we implicitly assign the order of the index by the order of x. Then, we see that $\mu_F(r-1,r]<\infty$.[32] Then, from continuity from above, we see that

$$\mu_F(\{r\})=\lim_{i\to\infty}\mu_F((x_i,r]),$$

where we again implicitly assign an order to x as the usual order on $\mathbb{R}$ by given index i. It's then clear that as $i\to\infty$, $x_i\to r$. From the definition of F, we see that

$$F((x_i,r])=F(r)-F(x_i)=\sum_{n;r_n\leq r}\frac{1}{2^n}-\sum_{n;r_n\leq x_i}\frac{1}{2^n}=\sum_{n;x_i<r_n\leq r}\frac{1}{2^n}.$$

It's then clear that since $r\in\mathbb{Q}$, there exists an i' such that $r_{i'}=r$. Then, we immediately see that no matter how close $x_i\to r$, this sum is at least

$$\frac{1}{2^{i'}}$$

for a fixed i'. Hence, we conclude that $\mu_F(\{r\})>0$ for every $r\in\mathbb{Q}$.

[31] We're referring to $\epsilon-\delta$ proof approach.
[32] This will be $\mu(A_1)$ in the condition of continuity from above. Furthermore, since $\mathbb{Q}$ is countable, hence $F(x)<\infty$ is promised.

2. Now, we show $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$. Firstly, we claim that

$$\mu_F(\mathbb{Q}) = 1$$

and

$$\mu_F(\mathbb{R}) = 1$$

as well. Since $\mu_F(\mathbb{Q}) = 1$ is clear, we note that the second one essentially follows from the fact that we can write

$$\mathbb{R} = \lim_{N\to\infty} \bigcup_{i=1}^{N} (a - i, a + i]$$

for any $a \in \mathbb{R}$, say 0. From continuity from below, we have

$$\mu_F\left(\bigcup_{i=1}^{\infty} (-i, +i]\right) = \lim_{n\to\infty} \mu_F((-n, n]) = \sum_{n;r_n\in\mathbb{Q}} \frac{1}{2^n} = 1.$$

Given the above, from countable additivity of μ_F, we have

$$\mu_F(\mathbb{R} \setminus \mathbb{Q}) + \underbrace{\mu_F(\mathbb{Q})}_{1} = \underbrace{\mu_F(\mathbb{R})}_{1} \implies \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$$

as we desired.

■

Lemma A.3 (Cantor set has measure 0). Let C denotes the middle thirds Cantor set, then the Lebesgue measure of C is 0. i.e.,

$$m(C) = 0.$$

Proof. Since we're removing $\frac{1}{3}$ of the whole interval at each n, we see that the measure of those removing parts, denoted by r, is

$$m(r) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 1.$$

Then, by countable additivity of m, we see that

$$m(C) = m([0, 1]) - m(r) = 1 - 1 = 0.$$

■

A.2 Integration

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