

EECS598-001

# Approximation Algorithms & Hardness of Approximation

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### Abstract

This is an advanced graduate-level algorithm course taught in University of Michigan by [Euiwoong Lee](#). Topics include both approximation algorithms like covering, clustering, network design, and constraint satisfaction problems (the first half), and also the hardness of approximation algorithms (the second half).

The first half of the course is classical and well-studied, and we'll use Williamson and Shmoys[[WS11](#)], Vazirani[[Vaz02](#)] as our reference. The second half of the course is still developing, and we'll look into papers by Barak and Steurer[[BS14](#)], O'donnell[[ODo21](#)], etc.

This course is taken in Fall 2022, and the date on the covering page is the last updated time.

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# Chapter 1

## Introduction

### Lecture 1: Overview, Set Cover

#### 1.1 Computational Problem

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We're interested in the following optimization problem: Given a problem with an input, we want to either maximize or minimize some objectives. This suggests the following definition.

**Definition 1.1.1 (Computational problem).** A *computational problem*  $P$  is a function from input  $I$  to  $(X, f)$ , where  $X$  is the feasible set of  $I$  and  $f$  is the objective function.

We see that by replacing  $f$  with  $-f$ , we can unify the notion and only consider either minimization or maximization, but we will not bother to do this.

**Example ( $s$ - $t$  shortest path).** The  $s$ - $t$  shortest path problem  $P$  can be formalized as follows. Given input  $I$ , it defines

- Input: Graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  and two vertices  $s, t \in \mathcal{V}$ .
- Feasible set:  $X = \{\text{set of all (simple) paths } s \text{ to } t\}$ .
- Objective function:  $f: X \rightarrow \mathbb{R}$  where  $f(x) = \text{length}(x)$  (# of edges of  $x$ ).

The output of  $P$  should be some  $x \in X$  (i.e., some valid  $s$ - $t$  paths) such that it minimizes  $f(x)$ .

We see that the [computational problem](#) we focus on is an optimization problem, and more specifically, we're interested in [combinatorial optimization](#).

**Definition 1.1.2 (Combinatorial optimization).** A *combinatorial optimization* problem is a [problem](#) where the feasible set  $X$  is a finite set.

**Example ( $s$ - $t$  shortest path).** The  $s$ - $t$  shortest path problem is an [combinatorial optimization](#) problem since given a graph  $\mathcal{G}$  with  $n = |\mathcal{V}|$ ,  $m = |\mathcal{E}|$ , there are at most  $n!$  different paths, i.e.,  $|X| \leq n! < \infty$ .

**Note.** We'll also look into some continuous optimization problem, where  $X$  is now infinite (or even uncountable). For example, find  $x \in \mathbb{R}$  that minimizes  $f(x) = x^2 + 2x + 1$ . In this case,  $X = \mathbb{R}$  which is uncountable (hence infinite).

#### 1.2 Efficient Algorithms

Given a [problem](#)  $P$ , we want to solve it fast with [algorithms](#). Before we characterize the speed of an [algorithm](#), we should first define what exactly an algorithm is.

**Definition 1.2.1 (Algorithm).** Given a **problem**  $P$  and input  $I$  (which defines  $X$  and  $f$ ), an *algorithm*  $A$  outputs solution  $y = A(I)$  such that  $y \in X$  and  $y = \arg \max_{x \in X} f(x)$  or  $\arg \min_{x \in X} f(x)$ , depending on  $I$ .

**Definition 1.2.2 (Efficient).** We say that an **algorithm**  $A$  is *efficient* if it runs in **polynomial time**.

**Remark (Runtime parametrization).** The *runtime* of an **algorithm**  $A$  should be parametrized by the size of input  $I$ . Formally, given input  $I$  represented in  $s$  bits, runtime of  $A$  on  $I$  should be  $\text{poly}(s)$  for  $A$  to be **efficient**.

**Note.** In most cases, there are 1 or 2 parameters that essentially define the size of input.

**Example (Graph).** A natural representation of a graph with  $n$  vertices and  $m$  edges are

- (a) Adjacency matrix:  $n^2$  numbers.
- (b) Adjacency list:  $O(m + n)$  numbers.

**Example (Set system).** A **set system** with  $n$  elements and  $m$  sets has a natural representation which uses  $O(nm)$  numbers.

**Example.** If an input  $I$  can be represented by  $s$  bits, then the runtime of an **algorithm** can be  $O(s \log s)$ ,  $O(s^2)$ , or  $O(s^{100})$ , which are considered as **efficient**. On the other hand, something like  $2^s$  or  $s!$  are not.

Hence, our goal is to get  $\text{poly}((n, m))$ -time **algorithm**!

## 1.3 Approximation Algorithms

We first note that many interesting **combinatorial optimization problems** are NP-hard, hence it's impossible to find optimum in polynomial time unless P is NP. This suggests one problem: *How well can we do in polynomial time?*

In normal cases, we may assume that objective function value is always positive, i.e.,  $f: X \rightarrow \mathbb{R}^+ \cup \{0\}$ . Then, we have the following definition which characterizes the *slackness*.

**Definition 1.3.1 (Approximation algorithm).** Given an **algorithm**  $A$ , we say  $A$  is an  $\alpha$ -*approximation algorithm* for a **problem**  $P$  if for every input  $I$  of  $P$ ,

- Min:  $f(A(I)) \leq \alpha \cdot \text{OPT}(I)$  for  $\alpha \geq 1$
- Max:  $f(A(I)) \geq \alpha \cdot \text{OPT}(I)$  for  $\alpha \leq 1$

where we define  $\text{OPT}(I)$  as  $\max_{x \in X} f(x)$  for maximization,  $\min_{x \in X} f(x)$  if minimization.

We see that  $\alpha$  characterizes the slackness allowed for our **algorithm**  $A$ . Now, we're ready to look at some interesting **problems**. Broadly, there are around 10 classes of them which are actively studied:

- We'll see: **Covering**, Clustering, Network design, Constraint satisfaction.
- We'll not see: graph cuts, Packing, Scheduling, String, etc.

The above list is growing! For example, applications of continuous optimization in **combinatorial optimization** is getting attention recently. Also, there are around 8 techniques developed, e.g., greedy, local search, LP rounding, SDP rounding, primal-dual, cuts and metrics, etc.

## 1.4 Hardness

For most problems we saw, we can even say that getting an  $\alpha$ -approximation is NP-hard for some  $\alpha > 1$ . This bound is sometimes tight, but not always, and we'll focus on this part in the second half of this course.

# Chapter 2

## Covering

Before we jump into any problem formulations, we define a fundamental object in combinatorial optimization, the [set system](#).

**Definition 2.0.1 (Set system).** Given a ground set  $\Omega$  (often called *universe*), the *set system* is an order pair  $(\Omega, \mathcal{S})$  where  $\mathcal{S}$  is a collection of subsets of  $\Omega$ .

**Note.** For a [set system](#)  $(\Omega, \mathcal{S})$ , we often let  $m := |\mathcal{S}|$  and  $n := |\Omega|$ .

**Definition 2.0.2 (Degree).** Given a [set system](#)  $(\Omega, \mathcal{S})$ , the *degree* of  $x \in \Omega$ ,  $\deg(x)$ , is defined as

$$\deg(x) := |\{S \in \mathcal{S} \mid x \in S\}|.$$

**Remark (Bipartite representation).** Naturally, for a [set system](#), we have a bipartite representation.

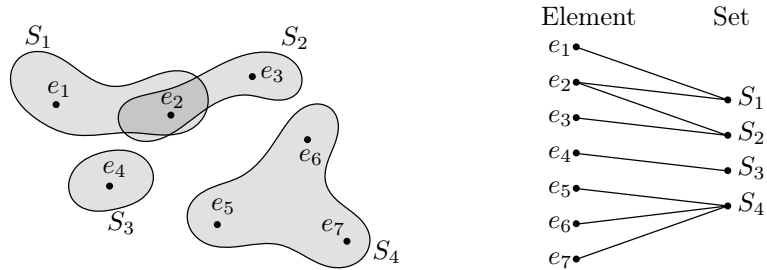


Figure 2.1: Bipartite representation of a [set system](#).

Denote  $d := \max_{e \in \Omega} \deg(e) \leq m$  and  $k := \max_{i \in [m]} |S_i| \leq n$ , which is just the maximum vertex degree on two sides of the bipartite graph representation of this [set system](#).

Finally, we have the following.

**Definition 2.0.3 (Covering).** A *covering*  $\mathcal{S}' \subseteq \mathcal{S}$  of  $(\Omega, \mathcal{S})$  is a (sub)collection of subsets such that  $\bigcup_{S \in \mathcal{S}'} S = \Omega$ .

### 2.1 Set Cover

Let's first consider the classical problem called [set cover](#).

**Problem 2.1.1 (Set cover).** Given a finite set system  $(U, \mathcal{S})$  where  $\mathcal{S} := \{S_i \subseteq U\}_{i=1}^m$  along with a weight function  $w: \mathcal{S} \rightarrow \mathbb{R}^+$ , find a covering  $\mathcal{S}'$  while minimizing  $\sum_{S \in \mathcal{S}'} w(S)$ .

Assuming there always exists at least one covering, we can in fact get two types of non-comparable approximation ratio in terms of  $k$  and  $d$ . Specifically, we get  $\log k$  and  $d$ -approximation ratio via either greedy, LP rounding or dual-methods.

### 2.1.1 Greedy

We first see the algorithm when  $w(S) = 1$  for all  $S \in \mathcal{S}$ .

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**Algorithm 1:** Greedy I – set cover

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**Data:** A set system  $(U, \mathcal{S})$   
**Result:** A covering  $\mathcal{S}'$

```

1  $\mathcal{S}' \leftarrow \emptyset, i \leftarrow 0$ 
2 while  $U \neq \emptyset$  do                                     //  $O(n)$ 
3   Choose  $S_i$  with maximum  $|U \cap S_i|$                      //  $O(mn)$ 
4    $\mathcal{S}' \leftarrow \mathcal{S}' \cup \{S_i\}$ 
5    $U \leftarrow U \setminus S_i$ 
6   for  $e \in U \cap S_i$  do
7      $y_e \leftarrow 1/|U \cap S_i|$                              // Average costs
8    $i \leftarrow i + 1$ 
9 return  $\mathcal{S}'$ 

```

---

**Remark.** It's clear that Algorithm 1 is a polynomial time algorithm, also, the output  $\mathcal{S}'$  is always a valid covering.

**Theorem 2.1.1.** Algorithm 1 is an  $H_k$ -approximation<sup>a</sup> algorithm.

<sup>a</sup> $H_k$  is the so-called *harmonic number*, which is defined as  $\sum_{i=1}^k 1/i \leq \ln k + 1$ .

**Proof.** Denote the OPT as  $\mathcal{S}^* := \{S_1^*, \dots, S_\ell^*\}$ , and first note that the average cost  $y_e$  essentially maintains  $\sum_{e \in U} y_e = |\mathcal{S}'|$ , hence we just need to bound  $y_e$  w.r.t.  $\mathcal{S}^*$ . To do this, for any  $S^* \in \mathcal{S}^*$ , say  $S_1^* = \{e_1, \dots, e_k\}$  where we number  $e_i$  in terms of the order of which being deleted, i.e.,  $e_1$  is deleted first from  $U$  (line 5), etc.

**Note.**  $S_1^*$  can have less than  $k$  element, but in that case similar argument will follow. Also, if some elements are deleted at the same time, we just order them arbitrarily.

Then, we have the following claim.

**Claim.** For all  $e_i$ ,  $y_{e_i} \leq \frac{1}{k-i+1}$ .

**Proof.** Consider the iteration when  $e_i$  was picked by  $\mathcal{S}'$ , i.e.,  $|U \cap \mathcal{S}'| \geq |U \cap S_1^*| \geq k - i + 1$ , then by definition (line 5) we have  $y_{e_i} = \frac{1}{|U \cap \mathcal{S}'|} \leq \frac{1}{|U \cap S_1^*|} \leq \frac{1}{k-i+1}$ . ⊛

We immediately see that whenever the optimal solution pays 1 (for choosing  $S_1^*$  for instance), Algorithm 1 pays at most  $H_k$  since  $\sum_{e_i \in S_1^*} y_{e_i} \leq \sum_{i=1}^k \frac{1}{k-i+1} = H_k$ , or more formally,

$$|\mathcal{S}'| = \sum_{e \in U} y_e \leq \sum_{S_i^* \in \mathcal{S}^*} \underbrace{\sum_{e \in S_i^*} y_e}_{\leq H_k} \leq \ell \cdot H_k = H_k \cdot |\text{OPT}|,$$

which finishes the proof. ■

In all, observe that  $H_k \leq \ln k + 1$ , we see that Algorithm 1 is a  $(\ln k)$ -approximation algorithm.



## Lecture 2: Linear Programming with Set Covers

### 2.1.2 Linear Programming Rounding

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To get a  $d$ -approximation algorithm, instead of seeing the greedy algorithm, we first see the LP<sup>1</sup> dual method, which turns out to be exactly the same as the greedy algorithm.

**As previously seen.** Both linear programming and convex programming can be solved in polynomial time.

Notice that it's more natural to define **set cover** in terms of ILP (integer LP). Define our integer variables  $\{x_i\}_{i \in [n]}$  such that

$$x_i = \begin{cases} 1, & \text{if } S_i \in \mathcal{S}'; \\ 0, & \text{otherwise.} \end{cases}$$

In this way, we have the following ILP formulation for **set cover** as

$$\begin{aligned} \min \quad & \sum_i w_i \cdot x_i \\ & \sum_{S_i \ni e} x_i \geq 1 & \forall e \in U \\ \text{(IP)} \quad & x_i \in \{0, 1\} & \forall i. \end{aligned}$$

But we know that this is an NP-hard problem, so we relax it to be

$$\begin{aligned} \min \quad & \sum_i w_i \cdot x_i \\ & \sum_{S_i \ni e} x_i \geq 1 & \forall e \in U \\ \text{(LP)} \quad & x_i \geq 0 & \forall i. \end{aligned}$$

Write it in a more compact form, we have

$$\begin{aligned} \min \quad & \langle w, x \rangle \\ & Ax \geq \mathbf{1} \\ & x \geq 0 \end{aligned}$$

where  $A \in \mathbb{R}^{n \times m}$  such that

$$A_{ij} = \begin{cases} 1, & \text{if } e_i \in S_j; \\ 0, & \text{otherwise.} \end{cases}$$

**Note.** Note when we do relaxation, we want  $x \in \text{fes}(\text{IP}) \Rightarrow x \in \text{fea}(\text{LP})$ , i.e.,  $\text{fes}(\text{LP}) \supseteq \text{fes}(\text{IP})$ . Note that in this case, for a minimization problem, we have

$$f(x) = \text{OPT}_{\text{LP}} \leq \text{OPT}_{\text{IP}}.$$

In this case, we see that the most natural way to get an integer solution from the fractional solution obtained from the relaxed LP is to **round**  $x$  to integral solution. This leads to the following **algorithm**.

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**Algorithm 2:** LP Rounding – **set cover**

---

**Data:** A **set system**  $(U, \mathcal{S})$

**Result:** A **covering**  $\mathcal{S}'$

```

1  $x \leftarrow \text{solve}(\text{LP})$  // Solve the relaxed (LP)
2  $\mathcal{S}' \leftarrow \{S_i : x_i \geq 1/d\}$ 
3 return  $\mathcal{S}'$ 
```

---

We now prove the correctness and **Algorithm 2**'s approximation ratio.

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<sup>1</sup>See **MATH561** for a complete reference.

**Lemma 2.1.1.**  $\mathcal{S}'$  is a covering.

**Proof.** Fix  $e \in U$ , let  $S_1, \dots, S_d$  be the sets containing  $e$ . We see that

$$\sum_{i=1}^d x_i \geq 1 \Rightarrow \exists j \in [d] \text{ s.t. } x_j \geq \frac{1}{d} \Rightarrow S_j \in \mathcal{S}'.$$

■

**Theorem 2.1.2.** Algorithm 2 is  $d$ -approximation algorithm.

**Proof.** By comparing  $w(\mathcal{S}')$  and  $\text{OPT}_{\text{LP}} = \sum_{i=1}^m x_i w_i$ , we see that

$$\text{OPT} \leq \sum_{S_i \in \mathcal{S}'} w_i \leq d \sum_{S_i \in \mathcal{S}'} w_i x_i \leq d \cdot \text{OPT}_{\text{LP}} \leq d \cdot \text{OPT},$$

which implies  $\text{OPT} / d \leq \text{OPT}_{\text{LP}} \leq \text{OPT}$ .

**Note.** Note that  $\text{OPT}$  is assumed to be  $\text{OPT}_{\text{IP}}$ , i.e., the optimum of the original IP formulation of Problem 2.1.1.

■

**Definition 2.1.1** (Integrality gap). The *integrality gap* as

$$\sup_{\text{input } I} \frac{\text{OPT}(I)}{\text{OPT}_{\text{LP}}(I)}.$$

**Remark.** We see that the integrality gap of Algorithm 2 is  $d$  from Theorem 2.1.2.

### 2.1.3 Randomized Linear Programming Rounding

And indeed, we can use a more natural way to do the rounding, i.e., respect to the  $x_i$  value.

**Intuition.** If  $x_i$  is close to 1, it's reasonable to include it, vice versa.

We see that algorithm first.

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**Algorithm 3:** Randomized LP Rounding – set cover

---

**Data:** A set system  $(U, \mathcal{S})$

**Result:** A (possible) covering  $\mathcal{S}'$

```

1  $x \leftarrow \text{solve}(\text{LP})$  // Solve the relaxed (LP)
2  $\mathcal{S} \leftarrow \emptyset$ 
3 for  $i = 1, \dots, m$  do
4    $\lfloor$  add  $S_i$  to  $\mathcal{S}'$  w.p.  $x_i$  // independently
5 return  $\mathcal{S}'$ 
```

---

Now, the question is, how is this  $\mathcal{S}'$ 's quality? In other words, fix  $e \in U$ , what's  $\Pr(e \text{ is covered})$ ?

**Lemma 2.1.2.**  $\Pr(e \text{ is covered}) \geq 1 - 1/e \approx 0.63$ .

**Proof.** We bound  $\Pr(\overline{e \text{ is covered}})$  instead. Say  $S_1, \dots, S_d$  are the sets containing  $e$ , then we see that

$$\Pr(\overline{e \text{ is covered}}) = \prod_{i=1}^d (1 - x_i) \leq \prod_{i=1}^d e^{-x_i} = e^{-\overbrace{(x_1 + \dots + x_d)}^{\geq 1}} \leq e^{-1}.$$

**Note.** For every  $x$ , we have  $1 + x \leq e^x$ , and this approximation is close when  $|x|$  is small. ■

A standard way to boost the correctness of a randomized algorithm is to run it multiple time, which leads to the following.

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**Algorithm 4:** Multi-time Randomized LP Rounding – [set cover](#)

---

**Data:** A [set system](#)  $(U, \mathcal{S})$ ,  $\alpha$

**Result:** A (possible) [covering](#)  $\mathcal{S}'$

```

1  $x \leftarrow \text{solve}(\text{LP})$  // Solve the relaxed (LP)
2  $\mathcal{S} \leftarrow \emptyset$ 
3 for  $t = 1, \dots, \alpha$  do // independently
4   for  $i = 1, \dots, m$  do
5      $\lfloor$  add  $S_i$  to  $\mathcal{S}'$  w.p.  $x_i$  // independently
6 return  $\mathcal{S}'$ 
```

---

**Lemma 2.1.3.** With  $\alpha = 2 \ln n$ ,  $\mathcal{S}'$  returned from [Algorithm 4](#) is a [covering](#) w.p. at least  $1 - \frac{1}{n}$ .

**Proof.** We have  $\Pr(e \text{ is not covered}) \leq e^{-\alpha}$  from independence of each run. Let  $\alpha = 2 \ln n$ , then  $\Pr(e \text{ is not covered}) \leq e^{-\alpha} = 1/n^2$ . By union bound,

$$\Pr(\text{some elements is not covered}) \leq \sum_{e \in U} \Pr(e \text{ not covered}) \leq n \cdot \frac{1}{n^2} = \frac{1}{n}.$$

This implies w.p. at least  $1 - 1/n$ ,  $\mathcal{S}'$  is a [covering](#). ■

In other words, with  $\alpha = 2 \ln n$ , [Algorithm 4](#) is correct with probability at least  $1 - 1/n$ .

**Lemma 2.1.4.** With  $\alpha = 2 \ln n$ ,  $\mathcal{S}'$  returned from [Algorithm 4](#) has an approximation ratio  $4 \ln n$  w.p. at least  $\frac{1}{2}$ .<sup>a</sup>

<sup>a</sup>Note that  $\mathcal{S}'$  is not necessary a [covering](#).

**Proof.** Since  $\mathbb{E}[w(\mathcal{S}')] \leq \alpha \sum_i w_i x_i = \alpha \text{OPT}_{\text{LP}}$ , we have  $\Pr(w(\mathcal{S}') \geq 2 \cdot \alpha \text{OPT}_{\text{LP}}) \leq 1/2$  from Markov inequality. Now, we see that w.p.  $\geq 1/2$ , we have  $w(\mathcal{S}') \leq 2 \cdot 2 \ln n \cdot \text{OPT}_{\text{LP}} \leq 4 \ln n \text{OPT}$ . ■

**Theorem 2.1.3.** By running [Algorithm 4](#) many times, we get a  $(4 \ln n)$ -approximation algorithm with high probability.<sup>a</sup>

<sup>a</sup>Note that we still need to choose  $\mathcal{S}'$ .

**Proof.** Together with [Lemma 2.1.3](#) and [Lemma 2.1.4](#) and using the union bound, the probability of  $\mathcal{S}'$  not being a [covering](#) or with weight higher than  $4 \ln n \text{OPT}$  is at most  $\frac{1}{n} + \frac{1}{2}$ , which is less than 1. Hence, by running [Algorithm 4](#) many times (independently), the failing possibility is exponential small. ■

**Note.** With [Theorem 2.1.3](#), we still need to find a valid [covering](#) with the lowest cost, where a valid [covering](#) with low enough weight is guaranteed to exist with high probability. Note that this is still a polynomial time algorithm since we know that checking  $\mathcal{S}'$  is a [covering](#) is just linear.

**Remark.** Indeed, with some smarter algorithm modified from [Algorithm 4](#), we can get a  $H_k$  approximation ratio.

# Appendix

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