MATH602
Real Analysis II

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August 30, 2022

Abstract

Additionally, we'll use .
This course is taken in Fall 2022, and the date on the covering page is the last updated time.

Contents

Chapter 1

Introduction

Lecture 1: Introduction

We first briefly review different kinds of vector space. 05 Jan. 11:00

1.1 Linear Vector Space

Definition 1.1.1 (Linear vector space)**.** A set with operations of addition and multiplication (by a scalar) is called a *linear vector space*.

Example. Denote the multiplicative scalar by λ, then

- $\lambda \in \mathbb{R} \Rightarrow$ real vector space.
- $\lambda \in \mathbb{C} \Rightarrow$ complex vector space

Lemma 1.1.1. Given E a linaer vector space, if $v, w \in E$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), then $\lambda v + \mu w \in E$.

we also have usual rules of associativity and commutativity.

Example. $\mathbb{R}^n$ a n dimensional linear vector space, $\mathbb{C}^n$ a n dimensional complex vector space.

We concentrate on ∞ dimensional vector spaces.

Example. Let K is a compact Hausdorff space, then

$$E = \{ f \colon K \to \mathbb{R} \mid f(\cdot) \text{ is continuous} \} .$$

We then see that E is an ∞ dimensional **real** vector space.

1.2 Quotient Space

Observe that a vector space can have many subspaces. Say E is a vector space, and $E_1 \subset E$ where E_1 is a proper subspace, i.e., $E_1 \neq E$.

Definition 1.2.1 (Quotient Space)**.** The *quotient space* E / E_1 is the set of equivalence classes of vectors in E where equivalence is given by $x \sim y$ if $x - y \in E_1$. Additionally, denote $[x]$ as the equivalence class of $x \in E$, i.e., $[x] = x + E_1$.

Note that E / E_1 is a linear vector space since if $x_1 + x_2 \in E$, $[x_1] + [x_2] = [x_1 + x_2]$, and also, $\lambda[x] = [\lambda x]$ for $\lambda \in \mathbb{R}$ or $\mathbb{C}$, i.e., $v, w \in E / E_1$, $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$ implies $\lambda v + \mu w \in E$.

Definition 1.2.2 (Codimension). If $E \,/\, E_1$ has finite dimension, then the dimension of $E \,/\, E_1$ is called the *cdimension* of E_1 in E.

Example. Let $E = \{f : K \to \mathbb{R} \mid f(\cdot) \text{ continuous}\}$, and $E_1 = \{f(\cdot) \in E : f(k_1) = 0\}$ where $k_1 \in K$ is fixed. We see that the dimension of $E \,/\, E_1$ is exactly 1 since $E \,/\, E_1$ is the set of constant functions.

Theorem 1.2.1. If E is finite dimensional, then

$$\operatorname{codim}(E_1) + \dim(E_1) = \dim(E).$$

Definition 1.2.3 (Linear operator). A map $T : E \to F$ between 2 linear spaces is a *linear operator* if it preserves the properties of addition and multiplication by a scalar, i.e., $T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$ for $v, w \in E$ and $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$.

Definition 1.2.4 (Kernel). The *kernel* of T is the subspace $\ker(T) = \{x \in E \mid Tx = 0\}$.

Definition 1.2.5 (Image). The *image* of T is the subspace $\operatorname{Im}(T) = \{Tx \in F \mid x \in E\}$.

1.3 Normed Spaces

Definition 1.3.1 (Norm). Let E be a linear vector space. A *norm* $\|\cdot\|$ on E is a function from E to $\mathbb{R}$ with the properties:

(a) $\|x\| \geq 0$, $x \in E$ and $\|x\| = 0 \Rightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda| \, \|x\|$, $x \in E$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in E$.

Definition 1.3.2 (Normed space). A linear space E equipped with a norm $\|\cdot\|$ is called a *normed space*.

Remark. This makes e a metric space with metric $d(x, y) = \|x - y\|$, where a matrix has properties

(a) $d(x, y) \geq 0$. Also, $d(x, x) = 0$ and $d(x, y)$ implies $x = y$.

(b) $d(x, y) = d(y, x)$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

Example (ℓ_∞). Let ℓ_∞ be the space of bounded sequences $x = (x_1, x_2, \ldots)$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots$. Define $\|x\| = \|x\|_\infty = \sup_{i \geq 1} |x_i|$.

Example (ℓ_1). Let ℓ_1 be the space of absolutely summable sequences $x = (x_1, x_2, \ldots)$ and $\sum_{i=1}^{\infty} |x_i| < \infty$. Then we define $\|x\| = \|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$.

Example ($C(k)$). The space $C(k)$ of continuous functions $f : K \to \mathbb{R}$ where K is compact Hausdorff. Then we define $\|f\| = \|f\|_\infty = \sup_{x \in K} |f(x)|$.

1.4 Geometry of Normed Spaces

Definition 1.4.1 (Ball). A (closed) *ball* centered at a point $x_0 \in E$ with radius $r > 0$ is the set $B(x_0, r) = \{x \in E \mid \|x - x_0\| \le r\}$.

Definition 1.4.2 (Sphere). The *sphere* centered at x_0 with radius $r > 0$ is the set $S(x_0, r) = \{x \in E \mid \|x - x_0\| = r\}$.

Remark. We see that $S(x_0, r)$ is the **boundary** of $B(x_0, r)$, i.e., $S(x_0, r) = \partial B(x_0, r)$.

We know that in finite dimensional, all norms are equivalent, which is not true for infinite dimensional vector spaces. This has something to do with the geometry of balls. Explicitly, balls can have different geometries depending on the properties of the norm. We see that an $\|\cdot\|_\infty$ can have multiple supporting hyperplane at the corner, while for an $\|\cdot\|_2$ can have only one at each point.

Also, unit ball for $\|\cdot\|_1$ is also a **square**, where we have

$$B(0, 1) = \{x = (x_1, x_2, \dots) \mid -1 < y_\epsilon < 1 \forall \epsilon\}$$

such that $y_\epsilon = \sum_{i=1}^{\infty} \epsilon_i x_i$, $\epsilon_i = \pm 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \dots)$.

We see that different norms give different geometry, but they have important common features, most notably, convexity properties.

Definition 1.4.3 (Convex set). Given E a linear space, a set $K \subset E$ is *convex* if $x, y \in K$ and $0 \le \lambda \le 1$, we have $\lambda x e (1 - \lambda) y \in K$.

Definition 1.4.4 (Convex function). Given E a linear space, a function $f \colon E \to \mathbb{R}$ is called *convex* if

$$f(\lambda x + (1 - \lambda) y) \le \lambda f(x) + (1 - \lambda) f(y)$$

for $x, y \in E$, $0 \le \lambda \le 1$.

Remark. If $f \colon E \to \mathbb{R}$ is a convex function, then for any $M \in \mathbb{R}$ the set $\{x \in E \mid f(x) \le M\}$ is convex.

The upshot is that norms are convex, and the unit balls are convex as well.

Appendix

Appendix A

Additional Proofs