MATH597
Analysis II

Pingbang Hu

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Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over $=$, like $\overset{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition.

Additionally, we'll use [FF99] as our main text, while using [Tao13] and [Axl19] as supplementary references.

Contents

Lecture 7: Borel Measures

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0.1 Borel Measures on $\mathbb{R}$

We first introduce so-called *distribution function*.

Definition 0.1 (Distribution function). An increasing[a] function

$$F\colon \mathbb{R} \to \mathbb{R}$$

and right-continuous. F is then a *distribution function*.

[a]Here, increasing means $F(x) \leq F(y)$ for $x < y$.

Example. Here are some examples of right-continuous functions.

1. $F(x) = x$.
2. $F(x) = e^x$.

3. Define
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

4. Let $\mathbb{Q} \coloneqq \{r_1, r_2, \ldots\}$. Define
$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$
and
$$F(x) \coloneqq \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$
Then F is a distribution function (hence right-continuous).

Note. If F is increasing, and
$$F(\infty) \coloneqq \lim_{x \nearrow \infty} F(x), \quad F(-\infty) \coloneqq \lim_{x \searrow \infty} F(x)$$
exist in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$, $F(-\infty) = 0$.[1]

Definition 0.2 (Locally finite). Let X be a topological space, μ on $(X, \mathcal{B}(X))$ is called *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$.

Lemma 0.1. Let μ be a locally finite Borel measure on $\mathbb{R}$, then
$$F_\mu(x) = \begin{cases} \mu\left((0, x]\right), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu\left((x, 0]\right), & \text{if } x < 0 \end{cases}$$
is a distribution function.

Proof. To show F_μ is increasing, consider $x < y$ such that
$$F_\mu(x) \leq F_\mu(y)$$
by considering

- $x > 0$: Then $F_\mu(x) = \mu((0, x])$ and
$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$
- $x = 0$: Then $F_\mu(x) = 0$ and
$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$
since $y > 0$.

[1] There are distributions [FF99] Ch9., but these are different from distribution functions.

- $x < 0$: Follows the same argument with $x > 0$.

Now, we need to show F_μ is right-continuous. ■

DIY, use continuity of measure

Definition 0.3 (Half intervals). We call

$$\varnothing, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

half-intervals.

Lemma 0.2. Let $\mathcal{H}$ be the collection of finite disjoint unions of half-intervals. Then, $\mathcal{H}$ is an algebra on $\mathbb{R}$.

Proof. We see that

- $\varnothing \in \mathcal{H}$. Clearly.
- To show $\mathcal{H}$ is closed under complements, we have
  - $\varnothing^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$.
  - $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$.[2]
  - $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$.
  - $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$.
  - $(-\infty, \infty)^c = \varnothing \in \mathcal{H}$.
- $\mathcal{H}$ is closed under finite unions, clearly.

■

[2]Since it's a two disjoint union of half intervals.

Proposition 0.1 (Distribution function defines a pre-measure). Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function. For a half-interval I, define

$$\ell(I) \coloneqq \ell_F(I) = \begin{cases} 0, & \text{if } I = \varnothing \\ F(b) - F(a), & \text{if } I = (a,b] \\ F(\infty) - F(a), & \text{if } I = (a,\infty] \\ F(b) - F(-\infty), & \text{if } I = (-\infty,b] \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty,\infty). \end{cases}$$

Define $\mu_0 \coloneqq \mu_{0,F}$ as

$$\mu_{0,F}\colon \mathcal{H} \to [0,\infty]$$

by

$$\mu_0(A) = \sum_{k=1}^{N} \ell(I_k) \text{ if } A = \bigcup_{k=1}^{N} I_k,$$

where A is a finite disjoint union of half-intervals $I_1, \ldots, I_N$. Then, μ_0 is a pre-measure on $\mathcal{H}$.

Proof. We see that

1. μ_0 is well-defined.
2. $\mu_0(\varnothing) = 0$.
3. μ_0 is finite additive.
4. μ_0 is countable additive within $\mathcal{H}$.

   Suppose $A \in \mathcal{H}$ where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable disjoint union. It is enough to consider the case that $A = I$, $A_k = I_k$ are all half-intervals.[3] Focus on the case $I = (a,b]$. Let

   $$(a,b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

   which is a disjoint union. Then we only need to check

   $$F(b) - F(a) = \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right).$$

   - Since $(a,b] \supset \bigcup_{n=1}^{N} (a_n, b_n]$ for any fixed $N \in \mathbb{N}$, hence

     $$\underset{N\in\mathbb{N}}{\forall}\ F(b) - F(a) \geq \sum_{n=1}^{N} \left(F(b_n) - F(a_n)\right).$$

[3] why?

By letting $N \to \infty$, we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) \right).$$

- Fix $\epsilon > 0$. Since F is right-continuous, $\exists a' > a$ such that

$$F(a') - F(a) < \epsilon.$$

For each $n \in \mathbb{N}$, $\exists b_n' > b_n$ such that

$$F(b_n') - F(b_n) < \frac{\epsilon}{2^n}.$$

Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b_n'),$$

hence

$$\underset{N\in\mathbb{N}}{\exists} \; [a', b] \subset \bigcup_{n=1}^{N} (a_n, b_n'),$$ [4]

which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^{N} F(b_n') - F(a_n).$$

Finally, we see that

$$\begin{aligned} F(b) - F(a) &\leq F(b) - F(a') + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n') - F(a_n) \right) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\epsilon}{2^n} \right) + \epsilon \\ &= \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) \right) + 2\epsilon \end{aligned}$$

for any fixed $\epsilon > 0$, hence

$$F(b) - F(a) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

Combine these two inequalities, we have

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

as we desired.

[4] This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.

$\blacksquare$

Remark. It's again the $\frac{\epsilon}{2^n}$ trick we saw before!

Lecture 8: Lebesgue-Stieltjes Measure on $\mathbb{R}$

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To classify all measures, we now see this last theorem to complete the task.

Theorem 0.1 (Locally finite Borel measures on $\mathbb{R}$). We have

1. $F\colon \mathbb{R} \to \mathbb{R}$ a distribution function, then there exists a **unique** locally finite Borel measure μ_F on $\mathbb{R}$ satisfying

$$\mu_F((a,b]) = F(b) - F(a)$$

for every $a < b$.

2. Suppose $F, G\colon \mathbb{R} \to \mathbb{R}$ are distribution functions. Then,

$$\mu_F = \mu_G$$

on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. $\blacksquare$ HW.

Remark. Theorem 0.1 simply states that given a distribution function, if we restrict our attention on locally finite measures on $\mathbb{R}$ following our usual convention, then it defines the measure on $\mathcal{B}(\mathbb{R})$ uniquely up to a *constant shift*.

0.2 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that

$$F \text{ distribution function} \overset{!}{\implies} \mu_F \text{ on Carathéodory } \sigma\text{-algebra } \mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R}).$$

Furthermore, we actually have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

Definition 0.4 (Lebesgue-Stieltjes measure). Given a distribution function F, we define

- μ_F on $\mathcal{A}_{\mu_F}$ is called the *Lebesgue-Stieltjes measure* corresponding to F.
- Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{L}, m)$, where $\mathcal{L}$ is called *Lebesgue σ-algebra*, and m is called *Lebesgue measure*.

Note. We see that since F is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).$$[5]

Example. We first see some examples.

1. $\mu_F((a,b]) = F(b) - F(a)$. Then
   - $\mu_F(\{a\}) = F(a) - F(a^-)$
   - $\mu_F([a,b]) = F(b) - F(a^-)$
   - $\mu_F((a,b)) = F(b^-) - F(a)$
2. We define
   $$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$
   Then
   - $\mu_F(\{0\}) = 1$
   - $\mu_F(\mathbb{R}) = 1$
   - $\mu_F(\mathbb{R} \setminus \{0\}) = 0$.

   We call that μ_F is the *Dirac measure* at 0.
3. Denote $\mathbb{Q} = \{r_1, r_2, \ldots\}$, and we define
   $$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r. \end{cases}$$
   Then HW
   - $\mu_F(\{r_i\}) > 0$ for all $r_i \in \mathbb{Q}$.
   - $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$
4. If F is continuous at a, then $\mu_F(\{a\}) = 0$.
5. $F(x) = x$
   - $m((a,b]) = m((a,b)) = m([a,b]) = b - a$.
6. $F(x) = e^x$
   - $\mu_F((a,b]) = \mu_F((a,b)) = e^b - e^a$.

Remark. We see that the first two examples are *discrete measures*.

Example (Middle thirds Cantor set). Let $C := \bigcap_{n=1}^{\infty} K_n$.

[5]Some text will use $x-$ and $x+$ instead of x^- and x^+, respectively.

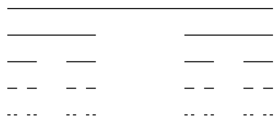

Figure 1: The top line corresponds to K_1, and then K_2, etc.

Since C is uncountable set, hence $m(C) = 0$. And notice that

$$x \in C \iff x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \; a_n \in \{0, 2\}.$$

0.2.1 Cantor Function

Consider F as follows.

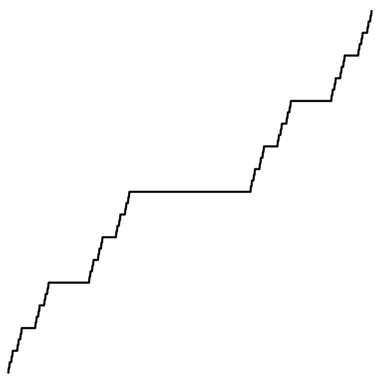

Figure 2: Cantor Function (Devil's Staircase).

We see that F is *continuous* and increasing. Furthermore,

$$\begin{array}{lcl} \mu_F(\mathbb{R} \setminus C) = 0 & & m(\mathbb{R} \setminus C) = \infty > 0 \\ \mu_F(C) = 1 & \iff & m(C) = 0 \\ \mu_F(\{a\}) = 0 & & m(\{a\}) = 0 \end{array}$$

Remark. μ_F and m are said to be **singular** to each other.

0.3 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

Lemma 0.3. Let μ be Lebesgue-Stieltjes measure on $\mathbb{R}$. Then we have

$$\begin{aligned}\mu(A) &\overset{!}{=} \inf\left\{\sum_{i=1}^{\infty}\mu((a_i,b_i]) \mid \bigcup_{i=1}^{\infty}(a_i,b_i] \supset A\right\}\\ &= \inf\left\{\sum_{i=1}^{\infty}\mu((a_i,b_i)) \mid \bigcup_{i=1}^{\infty}(a_i,b_i) \supset A\right\}\end{aligned}$$

for every $A \in \mathcal{A}_\mu$

Proof. The second equality follows from the continuity of the measure. ■

Lecture 9

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Appendix

References

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