MATH602
Real Analysis II

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Abstract

This is a graduate level functional analysis taught by Joseph Conlon. The prerequisites include linear algebra, complex analysis and also real analysis. We'll use Peter Lax[Lax02] and Reed-Simon[RS80] as textbooks.

This course is taken in Fall 2022, and the date on the covering page is the last updated time.

Contents

1 Introduction **2**
1.1 Linear Space . . . 2
1.2 Quotient Space . . . 2
1.3 Normed Spaces . . . 3

2 Banach Spaces **7**
2.1 Introduction . . . 7
2.2 Completion of Normed Space to Banach Space . . . 8

3 Inner Product Spaces **11**
3.1 Introduction . . . 11

A Additional Proofs **14**

Chapter 1

Introduction

Lecture 1: Introduction

We first briefly review different kinds of vector spaces. 30 Aug. 14:30

1.1 Linear Space

Definition 1.1.1 (Linear vector space)**.** A set with operations of addition and multiplication (by a scalar) is called a *linear vector space*.

Example. Denote the multiplicative scalar by λ, then

- $\lambda \in \mathbb{R} \Rightarrow$ real vector space.
- $\lambda \in \mathbb{C} \Rightarrow$ complex vector space

Lemma 1.1.1. Given E a linear vector space, if $v, w \in E$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), then $\lambda v + \mu w \in E$.

we also have usual rules of associativity and commutativity.

Example. $\mathbb{R}^n$ a n dimensional linear vector space, $\mathbb{C}^n$ a n dimensional complex linear vector space.

We concentrate on ∞ dimensional linear vector space.

Example. Let K is a compact Hausdorff space, then

$$E = \{f : K \to \mathbb{R} \mid f(\cdot) \text{ is continuous}\}.$$

We then see that E is an ∞ dimensional **real** linear vector space.

1.2 Quotient Space

Observe that a linear vector space can have many subspaces. Say E is a linear vector space, and $E_1 \subset E$ where E_1 is a proper subspace, i.e., $E_1 \neq E$.

Definition 1.2.1 (Quotient Space)**.** The *quotient space* E / E_1 is the set of equivalence classes of vectors in E where equivalence is given by $x \sim y$ if $x - y \in E_1$. Additionally, denote $[x]$ as the equivalence class of $x \in E$, i.e., $[x] = x + E_1$.

Note that E / E_1 is a linear vector space since if $x_1 + x_2 \in E$, $[x_1] + [x_2] = [x_1 + x_2]$, and also, $\lambda[x] = [\lambda x]$ for $\lambda \in \mathbb{R}$ or $\mathbb{C}$, i.e., $v, w \in E / E_1$, $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$ implies $\lambda v + \mu w \in E$.

Definition 1.2.2 (Codimension)**.** If $E \,/\, E_1$ has finite dimension, then the dimension of $E \,/\, E_1$ is called the *cdimension* of E_1 in E.

Example. There exists the case that $\dim(E) = \infty$, $\dim(E_1) < \infty$ where $\dim(E \,/\, E_1) < \infty$.

Proof. Let $E = \{f \colon K \to \mathbb{R} \mid f(\cdot) \text{ continuous}\}$, and $E_1 = \{f \in E \colon f(k_1) = 0\}$ where $k_1 \in K$ is fixed. We see that the dimension of $E \,/\, E_1$ is exactly 1 since $E \,/\, E_1$ is the set of constant functions. ⊛

Theorem 1.2.1. If E is finite dimensional, then $\operatorname{codim}(E_1) + \dim(E_1) = \dim(E)$

Definition 1.2.3 (Linear operator)**.** A map $T \colon E \to F$ between 2 linear spaces is a *linear operator* if it preserves the properties of addition and multiplication by a scalar, i.e., $T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$ for $v, w \in E$ and $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$.

Definition. Given a inear operator $T \colon E \to F$ we have the following.

Definition 1.2.4 (Kernel)**.** The *kernel* of T is the subspace $\ker(T) = \{x \in E \mid Tx = 0\}$.

Definition 1.2.5 (Image)**.** The *image* of T is the subspace $\operatorname{Im}(T) = \{Tx \in F \mid x \in E\}$.

1.3 Normed Spaces

We review some basic notions.

Definition 1.3.1 (Norm)**.** Let E be a linear vector space. A *norm* $\|\cdot\| : E \to \mathbb{R}$ on E is a function from E to $\mathbb{R}$ with the properties:

(a) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Notation (Dilation)**.** We say that the second condition is the *dilation* property.

Definition 1.3.2 (Normed vector space)**.** A linear vector space E equipped with a norm $\|\cdot\|$ is called a *normed vector space*.

Remark (Induced metric space)**.** A normed vector space E induces a *metric space* with metric $d(x, y) = \|x - y\|$, where the metric has properties

(a) $d(x, y) \geq 0$. Also, $d(x, x) = 0$ and $d(x, y)$ implies $x = y$.

(b) $d(x, y) = d(y, x)$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

Example (Bounded sequences ℓ_∞)**.** Let ℓ_∞ be the space of bounded sequences $x = (x_1, x_2, \ldots)$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots$. Then we define $\|x\| = \|x\|_\infty = \sup_{i \geq 1} |x_i|$.

Example (Absolutely summable sequences ℓ_1)**.** Let ℓ_1 be the space of absolutely summable sequences $x = (x_1, x_2, \ldots)$ and $\sum_{i=1}^{\infty} |x_i| < \infty$. Then we define $\|x\| = \|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$.

Example (Continuous functions $C(k)$)**.** The space $C(k)$ of continuous functions $f: K \to \mathbb{R}$ where K is compact Hausdorff. Then we define $\|f\| = \|f\|_{\infty} = \sup_{x\in K} |f(x)|$.

1.3.1 Geometry of Normed Spaces

Definition 1.3.3 (Ball)**.** A (closed) *ball* centered at a point $x_0 \in E$ with radius $r > 0$ is the set $B(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\}$.

Definition 1.3.4 (Sphere)**.** The *sphere* centered at x_0 with radius $r > 0$ is the set $S(x_0, r) = \{x \in E \mid \|x - x_0\| = r\}$.

Remark. We see that $S(x_0, r)$ is the **boundary** of $B(x_0, r)$, i.e., $S(x_0, r) = \partial B(x_0, r)$.

Note (Nonequivalency in infinite dimensional spaces)**.** We know that in finite dimensional, all norms are equivalent, which is not true for infinite dimensional vector spaces.

This has something to do with the geometry of balls.

Explicitly, balls can have different geometries depending on the properties of the norms. We see that an $\|\cdot\|_{\infty}$ can have multiple supporting hyperplane at the corner, while for an $\|\cdot\|_2$ can have only one at each point.

Also, unit balls for $\|\cdot\|_1$ is also a **square**, where we have

$$B(0, 1) = \{x = (x_1, x_2, \ldots) \mid -1 < y_\epsilon < 1 \forall \epsilon\}$$

such that $y_\epsilon = \sum_{i=1}^{\infty} \epsilon_i x_i$, $\epsilon_i = \pm 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots)$.

We see that different norms give different geometry, but they have important common features, most notably, convexity properties.

Definition 1.3.5 (Convex set)**.** Given E a linear vector space, a set $K \subset E$ is *convex* if $x, y \in K$ and $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda) y \in K$.

Definition 1.3.6 (Convex function)**.** Given E a linear vector space, a function $f: E \to \mathbb{R}$ is called *convex* if

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for $x, y \in E$, $0 \leq \lambda \leq 1$.

Remark. If $f: E \to \mathbb{R}$ is a convex function, then for any $M \in \mathbb{R}$ the set $\{x \in E \mid f(x) \leq M\}$ is convex.

The upshot is that norms are convex, and the unit balls are convex as well.

Lecture 2: Banach Spaces and Completion

Let's first see a proposition. 01 Sep. 14:30

Proposition 1.3.1. Let $\{E, \|\cdot\|\}$ be a normed linear space. Then the norm is convex and continuous.

Proof. Let $f: E \to \mathbb{R}$ be $f(x) = \|x\|$. Then $f(x) - f(y) = \|x\| - \|y\| \leq \|x - y\|$, which implies $|f(x) - f(y)| \leq \|x - y\|$ for $x, y \in E$, i.e., f is Lipschitz continuous. For convexity, let $0 < \lambda < 1$, we have

$$f(\lambda x+(1-\lambda)y) = \|\lambda x + (1-\lambda)y\| \le \|\lambda x\|+\|(1-\lambda)y\| = \lambda \|x\|+(1-\lambda)\|y\| = \lambda f(x)+(1-\lambda)f(y).$$

■

Note. Note that $f(\cdot)$ is continuous implies the closed ball

$$B(x_0,r) = \{x \in E \mid \|x-x_0\| \le r\} = \{x \in E \mid f(x-x_0) \le r\}$$

is closed in topology of E. Also, $f(\cdot)$ is convex implies $B(x_0,r)$ is convex.

Remark. If $f\colon E \to \mathbb{R}$ is convex, then the sets $\{x \in E \mid f(x) \le M\}$ is also convex. However, it's possible to have non-convex functions f such that all sets $\{x \in E \mid f(x) \le M\}$ are convex.

Example. Take $f(x) = |x|^p$ for $x \in \mathbb{R}$ and $p > 0$. We see that f is convex if $p > 1$, and non-convex if $p < 1$. The sets $\{x \in \mathbb{R} \mid f(x) \le M\}$ all convex since it's independent of p.

Lemma 1.3.1. Suppose $x \mapsto \|x\|$ satisfies

(a) $\|x\| \ge 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

(b) $\|\lambda x\| = |\lambda|\,\|x\|$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(c) The unit ball $B(0,1)$ is convex.

Then $f(x) = \|x\|$ satisfies the triangle inequality $\|x+y\| \le \|x\| + \|y\|$.

Proof. We see that if the third condition is true, the for $u, v \in B(0,1)$ and $0 < \lambda < 1$, we have $\lambda u + (1-\lambda)v \in B(0,1)$. Let $x, y \in E$, and

$$\lambda = \frac{\|x\|}{\|x\|+\|y\|} \Rightarrow 1-\lambda = \frac{\|y\|}{\|x\|+\|y\|}.$$

By letting $u = x/\|x\|$, $v = y/\|y\|$ we see that

$$\lambda u + (1-\lambda)v = \frac{\|x\|}{\|x\|+\|y\|}\frac{x}{\|x\|} + \frac{\|y\|}{\|x\|+\|y\|}\frac{y}{\|y\|} \in B(0,1) \Rightarrow \left\| \frac{x}{\|x\|+\|y\|} + \frac{y}{\|x\|+\|y\|} \right\| \le 1.$$

From the second condition, it follows that $\|x+y\| \le \|x\| + \|y\|$, which is the triangle inequality. ■

Remark. If $x \mapsto \|x\|$ satisfies the first two condition and is a convex, then it satisfies the triangle inequality.

Proof. Since $\frac{1}{2}\|x+y\| = \left\|\frac{x}{2}+\frac{y}{2}\right\| \le \frac{1}{2}\|x\| + \frac{1}{2}\|y\|$. ⊛

Now, given a quotient space $E\,/\,E_1$, the question is can we try to define a norm?

Problem 1.3.1. On $E\,/\,E_1$, is $\|[x]\| := \inf_{y\in E_1}\|x+y\|$ a norm?

Answer. We see that if $x \in \overline{E}_1 \setminus E_1$, then $\|[x]\| = 0$ but $[x] \ne 0 \in E\,/\,E_1$. ⊛

Note. Notice the difference from finite dimensional situation. All finite dimensional spaces E_1 are closed but not in general if E_1 has ∞ dimensions.

Example. Let $\ell_1(\mathbb{R})$ be the sequence of x_n for $n \geq 1$ in $\mathbb{R}$ such that $\sum_{i=1}^{\infty} |x_i| \leq \infty$. Define

$$\|x\|_1 \coloneqq \sum_{i=1}^{\infty} |x_i| \, ,$$

and let E_1 be all sequences with finite number of the x_n are nonzero. We see that $\overline{E}_1 = \ell_1(\mathbb{R})$ is infinite dimensional.

Proposition 1.3.2. Let $\{E, \|\cdot\|\}$ be a normed space and $E_1 \subseteq E$, E_1 is closed. Then

$$\|\cdot\| : {}^{E}\!\big/_{E_1} \to \mathbb{R}, \quad \|[x]\| = \inf_{y \in E_1} \|x + y\|$$

is a norm on $E \,/\, E_1$.

Proof. If $\|[x]\| = 0$, then $\inf_{y \in E_1} \|x - y\| = 0$, which implies $x \in E_1$ since E_1 is closed, so $[x] = 0$. Also, since

$$\|\lambda[x]\| = \inf_{y \in E_1} \|\lambda x + y\| = \inf_{z \in E_1} \|\lambda x + \lambda z\| = |\lambda| \inf_{z \in E_1} \|x + z\| = |\lambda| \, \|[x]\| \, ,$$

the dilation property is satisfied. Finally, for triangle inequality, we have

$$\|[x] + [y]\| = \inf_{x_1, y_1 \in E} \|x + y + x_1 + y_1\| \leq \inf_{x_1 \in E_1} \|x + x_1\| + \inf_{y_1 \in E_1} \|y + y_1\| = \|[x]\| + \|[y]\| \, .$$

■

Remark. This shows that the only obstacle for this kind of norm being an actual norm is the closeness of E_1.

Chapter 2

Banach Spaces

2.1 Introduction

Definition 2.1.1 (Banach space)**.** A linear normed space is a *Banach space* if it's complete, i.e., every Cauchy sequence converges.

Note. If $x_n \in E$, $n \geq 1$ is a sequence with property such that $\lim\limits_{m\to\infty} \sup\limits_{n\geq m} \|x_n - x_m\| > 0$, then $\exists x_\infty \in E$ such that $\lim\limits_{n\to\infty} \|x_n - x_m\| = 0$.

Example. The spaces ℓ_1, ℓ_∞ and $C(K)$ are Banach spaces.

We want to give a different criterion for showing $\{E, \|\cdot\|\}$ is Banach. Let E be a linear normed space and $\{x_\ell \mid \ell \geq 1\}$ a sequence in E.

Definition 2.1.2 (Absolutely summable)**.** A sequence is *absolutely summable* if $\sum_{i=1}^\infty \|x_i\| < \infty$.

Theorem 2.1.1 (Criterion for completeness)**.** A normed space $(E, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in E converges.

Proof. We need to prove two directions.

($\Rightarrow$) Suppose E is a Banach space and $\{x_k \mid x \geq 1\}$ an absolutely summable series. Set $s_n = \sum_{k=1}^n x_k$, $n \geq 1$, we want to show s_n is Cauchy, and if this is the case, completeness of E implies $\exists s_\infty$ and $\lim\limits_{n\to\infty} \|s_n - s_\infty\| = 0$. Let $n > m$, we see that

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \leq \sum_{k=m+1}^\infty \|x_k\|.$$

Observe that $\lim\limits_{m\to\infty} \sum\limits_{k=m+1}^\infty \|x_k\| = 0$, we see that the sequence $\{s_n\}$ is Cauchy.

($\Leftarrow$) Conversely, suppose E is **not** complete. Then there exists a Cauchy sequence $\{x_n \mid n \geq 1\}$ which does not converge. Furthermore, no subsequence of $\{x_n \mid n \geq 1\}$ converges.[a] We now construct an absolutely summable series which does not converge.

Define $n(1) \geq 1$ such that $\left\|x_n - x_{n(1)}\right\| \leq \frac{1}{2}$ if $n \geq n(1)$, similarly, let $n(2) > n(1)$ be such that $\left\|x_n - x_{n(2)}\right\| \leq \frac{1}{2^2}$ if $n > n(2)$. In all, we have $n(1) < n(2) < n(3) < \ldots$ such that $\left\|x_n - x_{n(k)}\right\| \leq \frac{1}{2^k}$

if $n > n(k)$. Define $w_j \coloneqq x_{n(j+1)} - x_{n(j)}$ for $j = 1, 2, \ldots$. We see that

$$x_{n(m)} = x_{n(1)} + \sum_{j=1}^{m} w_j$$

for $m = 1, 2, \ldots$, and $\{x_{n(m)}\}$ does not converge, hence so does the series $\sum_{j=1}^{\infty} w_j$. However, $\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$, which implies $\{w_j\}$ is absolutely summable. ■

[a]Otherwise, the whole sequence converges by the fact that it's Cauchy.

2.2 Completion of Normed Space to Banach Space

Theorem 2.2.1. Suppose E is a normed space. Then there exists a Banach space $\hat{E}$ called a completion of E with the following properties:

(a) There exists a linear map $i\colon E \to \hat{E}$ such that $\|ix\| = \|x\|$.[a]

(b) $\operatorname{Im}(i)$ is dense in $\hat{E}$, and $\hat{E}$ is the smallest Banach space containing image of E.

[a]This is called an *isometric embedding* of E into $\hat{E}$.

Lecture 3: Banach, Inner Product Spaces

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Example (Banach spaces)**.** We already showed spaces ℓ_1 and ℓ_∞ are Banach spaces.

We now want to generalize to ℓ_p with $1 < p < \infty$. For $x = \{x_n, n \geq 1\} \in \ell_p$ and if $\sum_{n=1}^{\infty} |x_n|^p < \infty$, for $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$, we want to show that $x \to \|x\|_p$ satisfies properties of a norm. The first two properties of a norm is easy check. As for triangle inequality, we have the following.

Lemma 2.2.1 (Minkowski inequality)**.** Let $1 \leq p < \infty$, for $x, y \in \ell_p$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p .$$

Proof. Recall that from Lemma 1.3.1, we only need to show that $B(0, 1)$ is convex, where

$$B(0,1) = \left\{ x = \{x_n \colon n \geq 1\} \mid f(x) = \sum_{n=1}^{\infty} |x_n|^p \leq 1 \right\}.$$

But $f(x)$ is convex since $x \mapsto |x|^p$, $x \in \mathbb{R}$ is convex if $p \geq 1$, we're done. Hence, $\|x + y\|_p \leq \|x\|_p + \|y\|_p$, i.e.,

$$\left(\sum_{j=1}^{\infty} |x_j + y_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |y_j|^p \right)^{1/p} .$$

■

Lemma 2.2.2 (Hölder's inequality)**.** Let $1 < p < \infty$, for $x \in \ell_p$, $y \in \ell_q$, we have

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q$$

where $1/p + 1/q = 1$.

Proof. Note first that we can assume without loss of generality, $\sum_{j=1}^{\infty} |x_j|^p = \sum_{j=1}^{\infty} |y_j|^q = 1$.

Then, result follows from the Young's inequality,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

for $x, y > 0$, $x, y \in \mathbb{R}$.

Remark (Legendre transform and the inequality)**.** Young's inequality is a special case of the inequality

$$xy \leq f(x) + \mathcal{L}f(y)$$

where $\mathcal{L}f(\cdot)$ is the *Legendre transform* of $f(\cdot)$, i.e., $\mathcal{L}f(y) = \sup_x [xy - f(x)]$.

If f is convex, then the function $xy \mapsto xy - f(x)$ is concave so has unique maximum. And $\mathcal{L}f(\cdot)$ always convex even if $f(\cdot)$ is not. In particular, if $f(x) = x^p/p$, then $\mathcal{L}f(y) = y^q/q$.

■

Note. Minkowski inequality is usually proved via the Hölder's inequality. To show this, since

$$\sum_{j=1}^{\infty} |x_j + y_j|^p \leq \sum_{j=1}^{\infty} |x_j| \, |x_j + y_j|^{p-1} + \sum_{j=1}^{\infty} |y_j| \, |x_j + y_j|^{p-1}.$$

Then Holder inequality implies

$$\sum_{j=1}^{\infty} |x_j| \, |x_j y_j|^{p-1} \leq \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |x_j + y_j|^{(p-1)q} \right)^{1/q},$$

and we're done.[a]

[a] Note that $(p-1)q = p$.

Remark. The above argument applies to more general spaces of p integrable functions. Let (Ω, Σ, μ) be a measure space and $L_p(\Omega, \Sigma, \mu)$ where all Σ measure functions $f\colon \Omega \to \mathbb{R}$ (or $\mathbb{C}$) such that $\int_\Omega |f|^p \, \mathrm{d}\mu < \infty$. Then, $L_p(\Omega, \Sigma, \mu)$ is a normed space with norm

$$\|f\|_p = \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{1/p}.$$

It's more tricky to show that L^p is a Banach space, but it's indeed still the case.

Theorem 2.2.2. The p-integrable space $L_p(\Omega, \Sigma, \mu)$ is a Banach space.

Proof. Let $\{f_n \colon n \geq 1\}$ be an absolutely summable sequence in L^p. Then the norm satisfies

$$\left\| \sum_{k=1}^{N} f_k \right\|_p \overset{!}{\leq} \sum_{k=1}^{N} \|f_k\|_p \leq C.$$

Hence, $\int_\Omega \left| \sum_{k=1}^N f_k \right|^p \mathrm{d}\mu \leq C^p$.

- Assume all f_k are non-negative. From monotone convergence theorem, we have

$$\lim_{N \to \infty} \int_\Omega \left(\sum_{k=1}^{N} f_k \right)^p \mathrm{d}\mu = \int_\Omega \left(\sum_{k=1}^{\infty} f_k \right)^p \mathrm{d}\mu \leq C^p.$$

Hence, $g = \sum_{k=1}^{\infty} f_k \in L_p$. We now want to show that $\sum_{k=1}^{N} f_k \to g$ in L_p. Set $r_n =$

$\sum_{k=n+1}^{\infty} f_k$ where r_n is a decreasing sequence where $r_n \to 0$ a.e. and also

$$\int_{\Omega} r_1^p \,\mathrm{d}\mu < \infty.$$

This means that $\lim_{n\to\infty} \|r_n\|_p = 0$ by dominate convergence theorem.

- For arbitrary $f_k \colon \Omega \to \mathbb{R}$, write $f_k = f_k^+ + f_k^-$ where $f_k^+ = \sup(f_k, 0)$ and $f_k^- = \inf(f_k, 0)$. The sequence $\left\{f_k^+ \colon k \geq 1\right\}$ are absolutely summable, and we just proceed as before. Similarly, if $f_k \colon \Omega \to \mathbb{C}$.

■

Chapter 3

Inner Product Spaces

3.1 Introduction

Definition 3.1.1 (Inner product)**.** Let E be a linear space over $\mathbb{C}$. An *inner product* $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$ is a function which has the following properties:

(a) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

(b) $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$ for $a, b \in \mathbb{C}$.

(c) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

Remark (Real inner product)**.** We can also define inner products of spaces over $\mathbb{R}$ with no extra conjugation in the last property.

Definition 3.1.2 (Inner product space)**.** An *inner product space* is a linear space E with an inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$.

Definition 3.1.3 (Orthogonal)**.** Given a linear space E, $x, y \in E$ are *orthogonal* if $\langle x, y \rangle = 0$, denote as $x \perp y$.

Theorem 3.1.1 (Cauchy-Schwarz inequality)**.** Let $x, y \in E$ and an inner product $\langle \cdot, \cdot \rangle$, then

$$|\langle x, y \rangle| \leq \langle x, y \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}.$$

Proof. Define $Q(t)$ by $Q(t) = \langle x + ty, x + ty \rangle = \langle y, y \rangle t^2 + 2t \operatorname{Re} \langle x, y \rangle + \langle x, x \rangle$ if $t \in \mathbb{R}$. Then we see that $Q(t) \geq 0$ with $t \in \mathbb{R}$ and the equation $Q(t) = 0$ has no real roots, implying $(\operatorname{Re} \langle x, y \rangle)^2 \leq \langle x, x \rangle \langle y, y \rangle$. Finally, the result follows by choosing $\theta \in \mathbb{R}$ such that

$$\langle x, y \rangle = \operatorname{Re} \left\langle x e^{i\theta}, y \right\rangle.$$

■

Corollary 3.1.1. The function $x \mapsto \|x\| := \langle x, x \rangle^{\frac{1}{2}}$ is a norm on E.

Proof. The triangle inequality is a consequence of Theorem 3.1.1 such that

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2 \overset{!}{\leq} \|x\|^2 + 2 \|x\| \|y\| + \|y\|^2 = \langle \|x\| + \|y\| \rangle^2.$$

■

Example. The space ℓ_2 of square summable sequences $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$,

$$\langle x, y \rangle \coloneqq \sum_{j=1}^{\infty} x_j \overline{y}_j.$$

Example. The space $L_2(\Omega, \Sigma, \mu)$ of $f, g \in L_2(\Omega, \Sigma, \mu)$,

$$\langle f, g \rangle = \int_{\Omega} f(x) \overline{g}(x) \, \mathrm{d}\mu(x).$$

Example. The space of $m \times n$ matrices $A = (a_{ij})$, $1 \leq i \leq m, 1 \leq j \leq n$. Then

$$\langle A, B \rangle = \operatorname{Tr} AB^*,$$

where B^* is the Hermitian adjoint of B, i.e., for $B = (b_{ij})$, then $B^* = (b^*_{ij})$ for $b^*_{ij} = \overline{b}_{ji}$.

Remark (Hilbert-Schmidt norm)**.** Specifically, the norm corresponding to this inner product is

$$\|A\|_{\mathrm{HS}} \coloneqq \sum_{i,j}^{\infty} \left(|a_{ij}|^2 \right)^{1/2},$$

which is known as the *Hilbert-Schmidt* norm.

For an inner product space, the inner product can be expressed in terms of the norm. This is because both parallelogram law and polarization identity hold.

Lemma 3.1.1 (Parallelogram law)**.** Given E an inner product space, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Lemma 3.1.2 (Polarization identity)**.** Given E an inner product space, we have

$$\langle x, y \rangle = \frac{1}{4} \left\{ \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right\}$$

Appendix

Appendix A

Additional Proofs

Bibliography

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