

# MATH597

## Analysis II

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### Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over  $=$ , like  $\stackrel{!}{=}$ , then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition.

Notice that there are some proofs is **intended** left as assignments, and for completeness, I put them in [Appendix A](#), use it in your **own risks**! You'll lose the chance to practice and really understand the materials.

Additionally, we'll use [\[FF99\]](#) as our main text, while using [\[Tao13\]](#) and [\[Ax19\]](#) as supplementary references.

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Lecture 1:  $\sigma$ -algebra

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## 1 Measure

**Example.** Before we start, we first see some examples.

1. Let  $X = \{a, b, c\}$ . Then

$$\mathcal{P}(X) := \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which is the *power set* of  $X$ . We see that

$$\#X = n \implies \#\mathcal{P}(X) = 2^n$$

for  $n < \infty$ .

2. If  $n = \infty$ , say  $X = \mathbb{N}$ , then

$$\mathcal{P}(\mathbb{N})$$

is an uncountable set while  $\mathbb{N}$  is a countable set. We can see this as follows. Consider

$$\phi: \mathcal{P}(\mathbb{N}) \rightarrow [0, 1], \quad A \mapsto 0.a_1a_2a_3 \dots \text{ (base 2),}$$

where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

and for example,  $A$  can be  $A = \{2, 3, 6, \dots\} \subseteq \mathbb{N}$ . Note that  $\phi$  is surjective, hence we have

$$\#\mathcal{P}(\mathbb{N}) \geq \#[0, 1].$$

But since  $[0, 1]$  is uncountable, so is  $\mathcal{P}(\mathbb{N})$ .

We like to *measure* the *size* of subsets of  $X$ . Hence, we are intriguing to define a map  $\mu$  such that

$$\mu: \mathcal{P}(X) \rightarrow [0, \infty].$$

**Example.** We first see some examples.

1. Let  $X = \{0, 1, 2\}$ . Then we want to define  $\mu: \mathcal{P}(X) \rightarrow [0, \infty]$ , we can have

- $\mu(A) = \#A$ . Then we have
  - $\mu(\{0, 1\}) = 2$
  - $\mu(\{0\}) = 1$
- $\mu(A) = \sum_{i \in A} 2^i$ . Then we have
  - $\mu(\{0, 1\}) = 2^0 + 2^1 = 3$

2. Let  $X = \{0\} \cup \mathbb{N}$ . Then we want to define  $\mu: \mathcal{P}(\mathbb{N}) \rightarrow [0, \infty]$ , we can have

- $\mu(A) = \#A$ . Then we have
  - $\mu(\{2, 3, 4, 5, \dots\}) = \infty = \mu(\{\text{even numbers}\})$
- $\mu(A) = e^{-1} \sum_{i \in A} \frac{1}{i!}$ . Then we have

$$- \mu(\{0, 2, 4, 6, \dots\}) = e^{-1} \left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right)$$

$$\bullet \mu(A) = \sum_{i \in A} a_i$$

3. Let  $X = \mathbb{R}$ . Then we want to define  $\mu: \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$ , we can have

- $\bullet \mu(A) = \#A$
- $\bullet \mu((a, b)) = b - a.$

**Problem.** Can we extend this map to all of  $\mathcal{P}(\mathbb{R})$ ?

**Answer.** No!

$$\bullet \mu((a, b)) = e^b - e^a.$$

**Problem.** Can we extend this map to all of  $\mathcal{P}(\mathbb{R})$ ?

**Answer.** No!

We immediately see the problems. To extend our native measure method into  $\mathbb{R}$  is hard and will cause something counter-intuitive!<sup>1</sup> Hence, rather than define measurement on *all* subsets in the power set of  $X$ , we only focus on *some* subsets. In other words, we want to define

$$\mu: \mathcal{P}(\mathbb{R}) \supset \mathcal{A} \rightarrow [0, \infty].$$

## 1.1 $\sigma$ -algebras

We start from the definition of the most fundamental element in measure theory.

**Definition 1.1 ( $\sigma$ -algebra).** Let  $X$  be a set. A collection  $\mathcal{A}$  of subsets of  $X$ , i.e.,  $\mathcal{A} \subset \mathcal{P}(X)$  is called a  $\sigma$ -algebra on  $X$  if

- $\bullet \emptyset \in \mathcal{A}.$
- $\bullet \mathcal{A}$  is closed under complements. i.e., if  $A \in \mathcal{A}$ ,  $A^c = X \setminus A \in \mathcal{A}.$
- $\bullet \mathcal{A}$  is closed under countable unions. i.e., if  $A_i \in \mathcal{A}$ , then  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}.$

**Remark.** There are some easy properties we can immediately derive.

- $\bullet X \in \mathcal{A}$  from  $X = X \setminus \underbrace{\emptyset}_{\in \mathcal{A}}$  and  $\mathcal{A}$  is closed under complement.
- $\bullet \bigcap_{i=1}^{\infty} A_i = \left( \bigcup_{i=1}^{\infty} A_i^c \right)^c$ , namely  $\mathcal{A}$  is closed under countable intersections.
- $\bullet A_1 \cup A_2 \cup \dots \cup A_n = A_1 \cup A_2 \cup \dots \cup A_n \cup \emptyset \cup \emptyset \cup \dots$ , hence  $\mathcal{A}$  is closed under finite unions and intersections.

An immediate definition can be given. We now define so-called *Borel set*.

<sup>1</sup>[https://en.wikipedia.org/wiki/Banach-Tarski\\_paradox](https://en.wikipedia.org/wiki/Banach-Tarski_paradox)

**Definition 1.2 (Borel set).** Given a topological space  $X$ , a *Borel set* is any set in  $X$  that can be formed from open sets through the operations of countable union, countable intersection and relative complement.

## Lecture 2: Measure

07 Jan. 11:00

**Example.** Again, we first see some examples.

1. Let  $\mathcal{A} = \mathcal{P}(X)$ , which is the power  $\sigma$ -algebra.
2. Let  $\mathcal{A} = \{\emptyset, X\}$ , which is a trivial  $\sigma$ -algebra.
3. Let  $B \subset X$ ,  $B \neq \emptyset$ ,  $B \neq X$ . Then we see that  $\mathcal{A} = \{\emptyset, B, B^c, X\}$  is a  $\sigma$ -algebra.

**Lemma 1.1.** Let  $\mathcal{A}_\alpha$ ,  $\alpha \in I$ , be a family of  $\sigma$ -algebra on  $X$ . Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is a  $\sigma$ -algebra on  $X$ .

**Remark.** Notice that  $I$  may be an uncountable intersection.

*Proof.* A simple proof can be made as follows. Firstly,  $\emptyset \in \mathcal{A}_\alpha$  for every  $\alpha$  clearly. Moreover, closure under complement and countable unions for every  $\mathcal{A}_\alpha$  implies the same must be true for  $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ . Hence,  $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$  is a  $\sigma$ -algebra. ■

The above allows us to give the following definition.

**Definition 1.3 (Generation of  $\sigma$ -algebra).** Given  $\mathcal{E} \subset \mathcal{P}(X)$ , where  $\mathcal{E}$  is not necessarily a  $\sigma$ -algebra. Let  $\langle \mathcal{E} \rangle$  be the intersection of all  $\sigma$ -algebras on  $X$  containing  $\mathcal{E}$ , then we call  $\langle \mathcal{E} \rangle$  the  $\sigma$ -algebra generated by  $\mathcal{E}$ .

**Remark.** Clearly,  $\langle \mathcal{E} \rangle$  is the smallest  $\sigma$ -algebra containing  $\mathcal{E}$ , and it is unique. To check the uniqueness, we suppose there are two different  $\langle \mathcal{E} \rangle_1$  and  $\langle \mathcal{E} \rangle_2$  generated from  $\mathcal{E}$ . It's easy to show

$$\langle \mathcal{E} \rangle_1 \subseteq \langle \mathcal{E} \rangle_2,$$

and by symmetry, they are equal.

**Example.** We see that  $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle$ .

**Lemma 1.2.** We have

1. Given  $\mathcal{A}$  a  $\sigma$ -algebra,  $\mathcal{E} \subset \mathcal{A} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \mathcal{A}$
2.  $\mathcal{E} \subset \mathcal{F} \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$

*Proof.* We'll see that after proving the first claim, the second follows smoothly.

1. The first claim is trivial, since we know that  $\langle \mathcal{E} \rangle$  is the smallest  $\sigma$ -algebra containing  $\mathcal{E}$ , then if  $\mathcal{E} \subset \mathcal{A}$ , we clearly have  $\langle \mathcal{E} \rangle \subset \mathcal{A}$  by the definition.
2. The second claim is also easy. From the first claim and the definition, we have

$$\mathcal{E} \subset \mathcal{F} \subset \langle \mathcal{F} \rangle \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle.$$

■

At this point, we haven't put any specific structure on  $X$ . Now we try to describe those spaces with good structure, which will give the space some nice properties.

**Definition 1.4 (Borel  $\sigma$ -algebra).** For a topological space  $X$ , the *Borel  $\sigma$ -algebra on  $X$* , denoted as  $\mathcal{B}(X)$ , is the  $\sigma$ -algebra generated by the collection of all open sets in  $X$ .

**Example.** We see that  $\mathcal{B}(\mathbb{R})$  contains

- $\mathcal{E}_1 = \{(a, b) \mid a < b; a, b \in \mathbb{R}\}$ .
- $\mathcal{E}_2 = \{[a, b] \mid a < b; a, b \in \mathbb{R}\}$  since  $[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$ .
- $\mathcal{E}_3 = \{(a, b] \mid a < b; a, b \in \mathbb{R}\}$  since  $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$ .
- $\mathcal{E}_4 = \{[a, b) \mid a < b; a, b \in \mathbb{R}\}$  since  $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b)$ .
- $\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}$  since  $(a, \infty) = \bigcup_{n=1}^{\infty} (a, a + n)$ .
- $\mathcal{E}_6 = \{[a, \infty) \mid a \in \mathbb{R}\}$  since  $[a, \infty) = \bigcup_{n=1}^{\infty} [a, a + n)$ .
- $\mathcal{E}_7 = \{(-\infty, b) \mid b \in \mathbb{R}\}$  since  $(-\infty, b) = \bigcup_{n=1}^{\infty} (b - n, b)$ .
- $\mathcal{E}_8 = \{(-\infty, b] \mid b \in \mathbb{R}\}$  since  $(-\infty, b] = \bigcup_{n=1}^{\infty} (b - n, b]$ .

**Proposition 1.1.**  $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$  for each  $i = 1, \dots, 8$ .

*Proof.* Firstly, we see that  $\mathcal{E}_i \subset \mathcal{B}(\mathbb{R}) \implies \langle \mathcal{E}_i \rangle \subset \mathcal{B}(\mathbb{R})$  by Lemma 1.2. Secondly, by definition,  $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E} \rangle$  where

$$\mathcal{E} = \{O \subseteq \mathbb{R} \mid O \text{ is open in } \mathbb{R}\}.$$

It's enough to show  $\mathcal{E} \subset \langle \mathcal{E}_i \rangle$  since if so,  $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{E}_i \rangle$ , and clearly  $\langle \mathcal{E} \rangle \supseteq \langle \mathcal{E}_i \rangle = \mathcal{B}(\mathbb{R})$ , then we will have  $\langle \mathcal{E} \rangle = \langle \mathcal{E}_i \rangle$ . Let  $O \subset \mathbb{R}$  be an open set, i.e.,  $O \in \mathcal{E}$ . We claim that every open set in  $\mathbb{R}$  is a countable union of disjoint open intervals.<sup>2</sup>

Thus,

$$O = \bigcup_{j=1}^{\infty} I_j,$$

where  $I_j$  open interval with the form of  $(a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$ .

For example,  $\mathcal{E}_1$  is trivially true, and

$$(a, b) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{\left[a + \frac{1}{n}, b - \frac{1}{n}\right]}_{\in \mathcal{E}_2}}_{\in \langle \mathcal{E}_2 \rangle}$$

shows the case for  $\mathcal{E}_2$  and

$$(a, \infty) = \bigcup_{k=1}^{\infty} (a, a + k)$$

shows the case for  $\mathcal{E}_5$ . It's now straightforward to check open intervals are in  $\langle \mathcal{E}_i \rangle$  for every  $i$ . ■

Now, to put a structure on a space, we define the following.

**Definition 1.5 (Measurable space).**  $(X, \mathcal{A})$  is called a *measurable space*, and  $E \in \mathcal{A}$  is called an  *$\mathcal{A}$ -measurable set*.

## 1.2 Measures

With the definition of measurable space, we now can refine our measure function  $\mu$  as follows.

**Definition 1.6 (Measure).** Given a measurable space on  $(X, \mathcal{A})$ , a *measure* is a function  $\mu$  such that

$$\mu: \mathcal{A} \rightarrow [0, \infty]$$

with

1.  $\mu(\emptyset) = 0$
2.  $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$  if  $A_1, A_2, \dots \in \mathcal{A}$  are **disjoint**. We call this *Countable additivity*.

We denote  $(X, \mathcal{A}, \mu)$  a *measure space*.

<sup>2</sup><https://math.stackexchange.com/questions/318299/any-open-subset-of-bbb-r-is-a-countable-union-of-disjoint-open-intervals>

**Notation.** We denote  $[0, \infty] := [0, \infty) \cup \{\infty\}$ .

**Remark.** The motivation of why we only want *countable additivity* but not uncountable additivity can be seen by the following example. We'll consider the most intuitive measure on  $\mathbb{R}, \mathcal{B}(\mathbb{R})$ .

Since we have

$$(0, 1] = \left(\frac{1}{2}, 1\right] \cup \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left(\frac{1}{8}, \frac{1}{4}\right] \cup \dots$$

and also

$$(0, 1] = \bigcup_{x \in (0, 1]} \{x\}.$$

Specifically, in the first case, we are claiming that

$$1 = \underbrace{\frac{1}{2}}_{\mu((\frac{1}{2}, 1])} + \underbrace{\frac{1}{4}}_{\mu((\frac{1}{4}, \frac{1}{2}])} + \underbrace{\frac{1}{8}}_{\mu((\frac{1}{8}, \frac{1}{4}])} + \dots;$$

while in the second case, we are claiming that

$$1 = \sum_{x \in (0, 1]} 0$$

since  $\mu(x) = 0$  for  $x \in \mathbb{R}$ , which is clearly not what we want.

**Example.** We see some examples.

1. For any  $(X, \mathcal{A})$ , we let  $\mu(A) := \#A$ . This is called *counting measure*.
2. Let  $x_0 \in X$ . For any  $(X, \mathcal{A})$ , the *Dirac measure at  $x_0$*  is

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A; \\ 0, & \text{if } x_0 \notin A. \end{cases}$$

3. For  $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ ,

$$\mu(A) = \sum_{i \in A} a_i,$$

where  $a_1, a_2, \dots \in [0, \infty)$ .

## Lecture 3: Construct a Measure

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**Note.** If  $A, B \in \mathcal{A}$  and  $A \subset B$ , then

$$\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A) \text{ if } \mu(A) < \infty.$$

**Theorem 1.1.** Given  $(X, \mathcal{A}, \mu)$  be a measure space.

1. (monotonicity)  $A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B)$ .
2. (countable subadditivity)  $A_1, A_2, \dots \in \mathcal{A} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$
3. (continuity from below/ monotone convergence theorem (MCT) for sets)

$$\begin{cases} A_1, A_2, \dots \in \mathcal{A} \\ A_1 \subset A_2 \subset A_3 \subset \dots \end{cases} \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

4. (continuity from above)

$$\begin{cases} A_1, A_2, \dots \in \mathcal{A} \\ A_1 \supset A_2 \supset A_3 \supset \dots \\ \mu(A_1) < \infty \end{cases} \implies \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

*Proof.* We prove this theorem one by one.

1. Since  $A \subset B$ , hence we have

$$\mu(B) = \mu\left(\underbrace{(B \setminus A) \cup A}_{\text{disjoint}}\right) \stackrel{!}{=} \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A).$$

2. This should be trivial from [countable additivity](#) with the fact that  $\mu(A) \geq 0$  for all  $A$ .

DIY!

3. Let  $B_1 = A_1, B_i = A_i \setminus A_{i-1}$  for  $i \geq 2$ , then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

are a disjoint union and  $B_i \in \mathcal{A}$ , hence we see that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i).$$

With  $\mu\left(\bigcup_{i=1}^n B_i\right) = \mu(A_n)$ , we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n B_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

4. Let  $E_i = A_1 \setminus A_i \implies E_i \in \mathcal{A}, E_1 \subset E_2 \subset \dots$ . We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right),$$



which implies

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \left( \bigcup_{i=1}^{\infty} E_i \right) \implies \mu \left( \bigcap_{i=1}^{\infty} A_i \right) = \mu(A_1) - \mu \left( \bigcup_{i=1}^{\infty} E_i \right)$$

since  $\mu \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \mu(A_1) < \infty$ . Then from [continuity from below](#), we further have

$$\mu \left( \bigcap_{i=1}^{\infty} A_i \right) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(E_n) = \mu(A_1) - \lim_{n \rightarrow \infty} (\mu(A_1) - \mu(A_n)).$$

From monotonicity, we see that  $\mu(A_n) \leq \mu(A_1) < \infty$ , hence we can split the limit and further get

$$\mu \left( \bigcap_{i=1}^{\infty} A_i \right) = \mu(A_1) - \mu(A_1) + \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \mu(A_n).$$

■

**Example.** Given  $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting measure})$ . Then we see

- $A_n = \{n, n+1, n+2, \dots\} \implies \mu(A_n) = \infty$
- $A_1 \supset A_2 \supset A_3 \supset \dots$
- $\bigcap_{i=1}^{\infty} A_i = \emptyset \implies \mu \left( \bigcap_{i=1}^{\infty} A_i \right) = 0$

**Remark.** We see that in this case, since  $\mu(A_1) \not< \infty$ , hence [continuity from above](#) doesn't hold.

We now try to characterize some properties of a measure space.

**Definition 1.7 ( $\mu$ -null,  $\mu$ -subnull, Complete measure space).** Given  $(X, \mathcal{A}, \mu)$

- $A \subset X$  is a  $\mu$ -null set if  $A \in \mathcal{A}$  and  $\mu(A) = 0$ .
- $A \subset X$  is a  $\mu$ -subnull set if  $\exists \mu$ -null set  $B$  such that  $A \subset B$ . Note that  $A$  is not necessarily  $\mathcal{A}$ -measurable.
- $(X, \mathcal{A}, \mu)$  is a *complete* measure space if every  $\mu$ -subnull set is  $\mathcal{A}$ -measurable.

There are some useful terminologies we'll use later relating to  $\mu$ -null.

**Definition 1.8 (Almost everywhere).** Given  $(X, \mathcal{A}, \mu)$ , a statement  $P(x)$ ,  $x \in X$  holds  $\mu$ -almost everywhere (a.e.) if the set

$$\{x \in X : P(x) \text{ does not hold}\}$$

is  $\mu$ -null.

It's always pleasurable working with finite rather than infinite, hence we give the following definition.

**Definition 1.9 (finite measure).** Given  $(X, \mathcal{A}, \mu)$

- $\mu$  is a *finite measure* if  $\mu(X) < \infty$ .
- $\mu$  is a  *$\sigma$ -finite measure* if  $X = \bigcup_{n=1}^{\infty} X_n$ ,  $X_n \in \mathcal{A}$ ,  $\mu(X_n) < \infty$ .

**Exercise.** Every measure space can be **completed**. Namely, we can always find a bigger  **$\sigma$ -algebra** to **complete** the space.

### 1.3 Outer Measures

We start by giving a definition.

**Definition 1.10 (Outer measure).** An *outer measure* on  $X$  is a map

$$\mu^*: \mathcal{P}(X) \rightarrow [0, \infty]$$

such that

- $\mu^*(\emptyset) = 0$
- (monotonicity)  $\mu^*(A) \leq \mu^*(B)$  if  $A \subset B$
- (countable subadditivity)  $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$  for every  $A_i \subset X$ .

**Example.** For  $A \subset \mathbb{R}$ ,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\}$$

is an outer measure due to the [Proposition 1.2](#) we're going to show.

**Remark.** We see that an outer measure need not be a measure. Check the [Definition 1.6](#) for a measure function.

**Proposition 1.2.** Let  $\mathcal{E} \subset \mathcal{P}(X)$  such that  $\emptyset, X \in \mathcal{E}$ . Let

$$\rho: \mathcal{E} \rightarrow [0, \infty]$$

such that  $\rho(\emptyset) = 0$ . Then

$$\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall_{i \in \mathbb{N}} E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\}$$

is an outer measure on  $X$ .

**Note.** Recall the Tonelli's Theorem<sup>3</sup> for series:

If  $a_{ij} \in [0, \infty]$ ,  $\forall i, j \in \mathbb{N}$ , then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Specifically, in [Tao13] Theorem 0.0.2.

## Lecture 4: Carathéodory extension Theorem

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**As previously seen.** We now prove the Proposition 1.2.

*Proof.* We need to prove

- $\mu^*$  is well-defined. i.e.,  $\inf$  is taken over a non-empty set. This is trivial since  $X \in \mathcal{E}$  and  $X \supset A$  for any  $A \in \mathcal{E}$ .
- $\mu^*(\emptyset) = 0$ . Since  $\emptyset \in \mathcal{E}$  and

$$\mu^*(\emptyset) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset \emptyset \right\} = 0$$

since  $\rho(\emptyset) = 0$  for all  $i$  and further, by Squeeze Theorem<sup>4</sup>, we see that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(\emptyset) = 0.$$

- $A \subset B \implies \mu^*(A) \leq \mu^*(B)$ . We simply show this by contradiction. Suppose  $A \subset B$  and  $\mu^*(A) > \mu^*(B)$ , then by definition of  $\mu^*$ , we have

$$\begin{aligned} \mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\} \\ &> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset B \right\} = \mu^*(B). \end{aligned}$$

Now, let  $B = (B \setminus A) \cup A$ , then we have

$$\begin{aligned} \mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset A \right\} \\ &> \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : \forall E_i \in \mathcal{E}, \bigcup_{i=1}^{\infty} E_i \supset (B \setminus A) \cup A \right\} = \mu^*(B). \end{aligned}$$

Now, since  $B \setminus A \supseteq \emptyset$ , then this inequality can't hold, hence a contradiction ⚡.

<sup>3</sup>[https://en.wikipedia.org/wiki/Fubini%27s\\_theorem](https://en.wikipedia.org/wiki/Fubini%27s_theorem)

<sup>4</sup>[https://en.wikipedia.org/wiki/Squeeze\\_theorem](https://en.wikipedia.org/wiki/Squeeze_theorem)

- Countable subadditivity. Let  $A_1, A_2, \dots \in \mathcal{X}$ . If one of  $\mu^*(A_n) = \infty$ , then result holds. So we may assume  $\mu^*(A_n) < \infty$  for all  $n \in \mathbb{N}$ . Now, fix any  $\epsilon > 0$ , we will show that

$$\mu^* \left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon.$$

For each  $n \in \mathbb{N}$ ,  $\exists E_{n,1}, E_{n,2}, \dots \in \mathcal{E}$  such that

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n$$

and

$$\mu^*(A_n) + \frac{\epsilon}{2^n} \geq \sum_{k=1}^{\infty} \rho(E_{n,k}).$$

Then we see that

$$\bigcup_{k=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{k,n} = \bigcup_{(n,k) \in \mathbb{N}^2} E_{k,n},$$

which implies

$$\mu^* \left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{(n,k) \in \mathbb{N}^2} \rho(E_{k,n}) \stackrel{!}{=} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{k,n}) \leq \sum_{n=1}^{\infty} \left( \mu^*(A_n) + \frac{\epsilon}{2^n} \right)$$

from the inequality just derived. Now, since the last term is just

$$\sum_{n=1}^{\infty} \left( \mu^*(A_n) + \frac{\epsilon}{2^n} \right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon,$$

hence we finally have

$$\mu^* \left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) + \epsilon$$

for arbitrarily small fixed  $\epsilon > 0$ , hence the subadditivity is proved. ■

**Definition 1.11 (Carathéodory measurable).** Let  $\mu^*$  be an outer measure on  $X$ . We say  $A \subset X$  is *Carathéodory measurable* (*C-measurable*) with respect to  $\mu^*$  if

$$\forall E \subset X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

<sup>5</sup>This is an important trick!!

**Lemma 1.3.** Let  $\mu^*$  be an **outer measure** on  $X$ . Suppose  $B_1, \dots, B_N$  are disjoint C-measurable sets. Then,

$$\forall E \subset X, \mu^* \left( E \cap \left( \bigcup_{i=1}^N B_i \right) \right) = \sum_{i=1}^N \mu^* (E \cap B_i).$$

*Proof.* Since we have

$$\begin{aligned} \mu^* \left( E \cap \left( \bigcup_{i=1}^N B_i \right) \right) &= \mu^* (E' \cap B_1) + \mu^* (E' \setminus B_1)^6 \\ &= \mu^* \left( E \cap \left( \bigcup_{i=1}^N B_i \cap B_1 \right) \right) + \mu^* \left( E \cap \left( \bigcup_{i=1}^N B_i \right) \cap B_1^c \right) \\ &= \mu^* (E \cap B_1) + \mu^* \left( E \cap \left( \bigcup_{i=2}^N B_i \right) \right) \end{aligned}$$

where the equality comes from the fact that  $B_1$  is **C-measurable** and disjoint from  $B_i, i \neq 1$ . Then, we simply iterate this argument and have the result. ■

**Remark.** This implies that if we restrict an **outer measure** on a **C-measurable** set, then it becomes finite additive.

**Theorem 1.2 (Carathéodory extension Theorem).** Let  $\mu^*$  be an **outer measure** on  $X$ . Let  $\mathcal{A}$  be the collection of **C-measurable** sets (with respect to  $\mu^*$ ). Then,

1.  $\mathcal{A}$  is a  **$\sigma$ -algebra** on  $X$ .
2.  $\mu = \mu^*|_{\mathcal{A}}$  is a **measure** on  $(X, \mathcal{A})$ .
3.  $(X, \mathcal{A}, \mu)$  is a **complete measure space**.

*Proof.* We divide the proof in several steps.

1. We show  $\mathcal{A}$  is a  **$\sigma$ -algebra** by showing

- (a)  $\emptyset \in \mathcal{A}$ . To show this, we simply check that  $\emptyset$  is **C-measurable**. We see that

$$\forall_{E \subset X} \mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E \setminus \emptyset) = \mu^*(E),$$

which just shows  $\emptyset \in \mathcal{A}$ .

- (b)  $\mathcal{A}$  closed under complements. This is equivalent to say that if  $A$  is **C-measurable**, so is  $A^c$ . We see that if  $A$  is **C-measurable**, then for every  $E \subset X$ ,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

---

<sup>6</sup>Here,  $E' := E \cap \left( \bigcup_{i=1}^N B_i \right)$  for the simplicity of notation.

Observing that  $E \cap A = E \setminus A^c$  and  $E \setminus A = E \cap A^c$ , hence

$$\mu^*(E) = \mu^*(E \setminus A^c) + \mu^*(E \cap A^c).$$

We immediately see that above implies  $A^c \in \mathcal{A}$ .

(c)  $\mathcal{A}$  closed under countable unions.

**Note.** To show  $\mathcal{A}$  closed under countable unions, we show that  $\mathcal{A}$  is closed under:

finite unions  $\xRightarrow{\text{then}}$  countable disjoint unions  $\xRightarrow{\text{then}}$  countable unions.

- We show  $\mathcal{A}$  is closed under finite unions.

**Claim.**  $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$ .

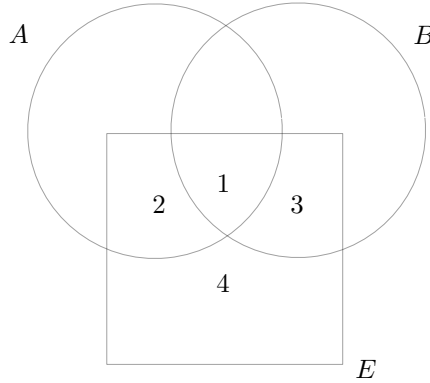
Fix  $E \subset X$  arbitrary. We need to show that

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)),$$

i.e.,

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

given  $A, B \in \mathcal{A}$ .



- Since  $A$  is **C-measurable**,

$$* \mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$$

$$* \mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$$

- Since  $B$  is **C-measurable**,

$$* \mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$$

Hence, we have

$$\begin{aligned} \mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4). \end{aligned}$$

- We show  $\mathcal{A}$  is closed under countable disjoint unions.

Let  $A_1, A_2, \dots \in \mathcal{A}$  and disjoint. Fix  $E \subset X$  arbitrary. Since  $\mu^*$  is countably subadditive,

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^{\infty} A_i\right),$$

hence we only need to show another way around.

Fix  $N \in \mathbb{N}$ , we have  $\bigcup_{n=1}^N A_n \in \mathcal{A}$  since  $N$  is finite, and

$$\begin{aligned} \mu^*(E) &= \mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right) + \mu^*\left(E \setminus \left(\bigcup_{n=1}^N A_n\right)\right) \\ &\geq \underbrace{\sum_{n=1}^N \mu^*(E \cap A_n)}_{\stackrel{!}{=} \mu^*\left(E \cap \left(\bigcup_{n=1}^N A_n\right)\right)} + \underbrace{\mu^*\left(E \setminus \bigcup_{n=1}^{\infty} A_n\right)}_{\leq \mu^*\left(E \setminus \left(\bigcup_{n=1}^N A_n\right)\right)}. \end{aligned}$$

Now, take  $N \rightarrow \infty$  then we are done.

- We show  $\mathcal{A}$  is closed under countable unions.

DIY

The proof will be *continued*...

## Lecture 5: Hahn-Kolmogorov Theorem

14 Jan. 11:00

Firstly, we see a stronger version of [Lemma 1.3](#) we have seen before.

**Lemma 1.4.** Let  $\mu^*$  be an [outer measure](#) on  $X$ . Suppose  $B_1, B_2, \dots$  are disjoint [C-measurable](#) sets. Then,

$$\forall E \subset X, \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i).$$

*Proof.*

$$\sum_{n=1}^{\infty} \mu^*(E \cap B_i) \geq \mu^*\left(E \cap \bigcup_{n=1}^{\infty} B_n\right) \geq \mu^*\left(E \cap \left(\bigcup_{n=1}^N B_n\right)\right) \stackrel{!}{=} \sum_{n=1}^N \mu^*(E \cap B_n).$$

Now, we just take  $N \rightarrow \infty$  (or note that  $N \in \mathbb{N}$  is arbitrary, we then get the result according to Squeeze Theorem<sup>7</sup>). ■

Let's continue the proof of [Theorem 1.2](#).

2. Since from [Definition 1.6](#), we need to show

<sup>7</sup>[https://en.wikipedia.org/wiki/Squeeze\\_theorem](https://en.wikipedia.org/wiki/Squeeze_theorem)

- $\mu(\emptyset) = 0$ . This means that we need to show  $\mu^*|_{\mathcal{A}}(\emptyset) = 0$ . Since  $\emptyset \in \mathcal{A}$  and  $\mu^*$  is an outer measure, hence from the [property](#) of outer measure, it clearly holds.
- [Countable additivity](#) of  $\mu^*$  on  $\mathcal{A}$  follows from the [Lemma 1.4](#) with  $E = X$

3. The proof is given in [Theorem A.1](#). ■

## 1.4 Hahn-Kolmogorov Theorem

We see that we can start with any collection of open sets  $\mathcal{E}$  and any  $\rho$  such that it assigns measure on  $\mathcal{E}$ , then induces an outer measure by [Proposition 1.2](#), finally complete the outer measure by [Theorem 1.2](#).

Specifically, we have

$$(\mathcal{E}, \rho) \xrightarrow{\text{Proposition 1.2}} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{Theorem 1.2}} (\mathcal{A}, \mu)$$

To introduce this concept, we see that we can start with a more general definition compared to [σ-algebra](#) we are working on till now.

**Definition 1.12 (Algebra).** Let  $X$  be a set. A collection  $\mathcal{A}$  of subsets of  $X$ , i.e.,  $\mathcal{A} \subset \mathcal{P}(X)$  is called an *algebra on  $X$*  if

- $\emptyset \in \mathcal{A}$ .
- $\mathcal{A}$  is closed under complements. i.e., if  $A \in \mathcal{A}$ ,  $A^c = X \setminus A \in \mathcal{A}$ .
- $\mathcal{A}$  is closed under **finite** unions. i.e., if  $A_i \in \mathcal{A}$ , then  $\bigcup_{i=1}^n A_i \in \mathcal{A}$  for  $n < \infty$ .

**Remark.** The only difference between an [algebra](#) and a [σ-algebra](#) is whether they closed under **countable** unions in the definition.

Now, we can look at a more general setup compared to an [outer measure](#).



**Definition 1.13 (Pre-measure).** Let  $\mathcal{A}_0$  be an algebra on  $X$ . We say

$$\mu_0: \mathcal{A}_0 \rightarrow [0, \infty]$$

is a *pre-measure* if

1.  $\mu_0(\emptyset) = 0$
2. (finite additivity)  $\mu_0\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu_0(A_i)$  if  $A_1, \dots, A_n \in \mathcal{A}_0$  are disjoint.
3. (countable additivity within the algebra) If  $A \in \mathcal{A}_0$  and  $A = \bigcup_{n=1}^{\infty} A_n$ ,  $A_n \in \mathcal{A}_0$ , disjoint, then

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

**Lemma 1.5.** (1) + (3)  $\implies$  (2) in Definition 1.13.

*Proof.* It's easy to see that since  $\mu_0$  is monotone. ■

**Theorem 1.3 (Hahn-Kolmogorov Theorem).** Let  $\mu_0$  be a pre-measure on algebra  $\mathcal{A}_0$  on  $X$ . Let  $\mu^*$  be the outer measure induced by  $(\mathcal{A}_0, \mu_0)$  in Proposition 1.2. Let  $\mathcal{A}$  and  $\mu$  be the Carathéodory  $\sigma$ -algebra and measure for  $\mu^*$ , then  $(\mathcal{A}, \mu)$  extends  $(\mathcal{A}_0, \mu_0)$ . i.e.,

$$\mathcal{A} \supset \mathcal{A}_0, \quad \mu|_{\mathcal{A}_0} = \mu_0.$$

*Proof.* We prove this theorem in two parts.

- We first show  $\mathcal{A} \supset \mathcal{A}_0$ . Let  $A \in \mathcal{A}_0$ , we want to show  $A \in \mathcal{A}$ , i.e.,  $A$  is C-measurable, i.e.,

$$\forall E \subset X \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We first fix an  $E \subset X$ . From countable subadditivity of  $\mu^*$ , we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Hence, we only need to show another direction. If  $\mu^*(E) = \infty$ , then  $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$  clearly. So, assume  $\mu^*(E) < \infty$ .

Fix  $\epsilon > 0$ . By the Proposition 1.2 of  $\mu^*$ ,  $\exists B_1, B_2, \dots \in \mathcal{A}_0$ ,  $\bigcup_{n=1}^{\infty} B_n \supset E$  such that

$$\mu^*(E) + \epsilon \geq \sum_{n=1}^{\infty} \mu_0(B_n) = \sum_{n=1}^{\infty} \left( \underbrace{\mu_0(B_n \cap A)}_{\in \mathcal{A}_0} + \underbrace{\mu_0(B_n \cap A^c)}_{\in \mathcal{A}_0} \right)$$

by the [finite additivity](#) of  $\mu_0$ . Note that

$$\left\{ \begin{array}{l} \bigcup_{n=1}^{\infty} (B_n \cap A) \supset E \cap A \\ \bigcup_{n=1}^{\infty} (B_n \cap A^c) \subset E \cap A^c \end{array} \right. \implies \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

since

$$\mu^*(E \cap A) \leq \mu^*\left(\bigcup_{n=1}^{\infty} (B_n \cap A)\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A)$$

and

$$\mu^*(E \cap A^c) \leq \mu^*\left(\bigcup_{n=1}^{\infty} (B_n \cap A^c)\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n \cap A^c).$$

We then see that for any  $\epsilon > 0$ , the inequality

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

holds, hence so does

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

which implies  $\mathcal{A} \supset \mathcal{A}_0$ .

The proof will be [continued](#)...

## Lecture 6: Hahn-Kolmogorov Theorem and Extension.

18 Jan. 11:00

Let's continue the proof of [Theorem 1.3](#).

- Let  $A \in \mathcal{A}_0$ , we want to show that

$$\mu(A) = \mu_0(A).$$

– Firstly, let

$$B_i = \begin{cases} A, & \text{if } i = 1 \\ \emptyset, & \text{if } i \geq 2 \end{cases} \in \mathcal{A}_0,$$

hence  $\bigcup_{i=1}^{\infty} B_i = A$ , then we see that

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu_0(B_i) = \mu_0(A)$$

from the [definition](#) of  $\mu^*$  and [countable additivity within the algebra](#) of  $\mu_0$ .

- Secondly, let  $B_i \in \mathcal{A}_0$ ,  $\bigcup_{i=1}^{\infty} B_i \supset A$  be arbitrary. Let  $C_1 = A \cap B_1 \in \mathcal{A}_0$ ,  $C_i = A \cap B_i \setminus \left( \bigcup_{j=1}^{i-1} B_j \right) \in \mathcal{A}_0$  for  $i \geq 2$  since the operations are finite. Then we see

$$A = \bigcup_{i=1}^{\infty} C_i \in \mathcal{A}_0$$

are disjoint countable unions, by [countable additivity within the algebra](#), we therefore have

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(C_i) \leq \sum_{i=1}^{\infty} \mu_0(B_i) \implies \mu_0(A) \leq \mu^*(A)$$

by taking the infimum from the [definition](#) of  $\mu^*$ .

Combine these two inequality, we see that

$$\mu^*(A) = \mu_0(A),$$

for every  $A \in \mathcal{A}_0$ , which implies

$$\mu(A) = \mu_0(A)$$

for every  $A \in \mathcal{A}_0$  from [Theorem 1.2](#), where we extend  $\mu^*$  to  $\mu$  respect to  $\mathcal{A}_0$ . ■

**Definition 1.14 (HK extension).**  $(\mathcal{A}, \mu)$  obtained from [Theorem 1.3](#) is the *Hahn-Kolmogorov extensions* of  $(\mathcal{A}_0, \mu_0)$ .

We can show the uniqueness of [HK extension](#).

**Theorem 1.4 (Uniqueness of HK extension).** Let  $\mathcal{A}_0$  be an [algebra](#) on  $X$ ,  $\mu_0$  be a [pre-measure](#) on  $\mathcal{A}_0$ . Let  $(\mathcal{A}, \mu)$  be the [HK extension](#) of  $(\mathcal{A}_0, \mu_0)$ . Let  $(\mathcal{A}', \mu')$  be another extension of  $(\mathcal{A}_0, \mu_0)$ . Then if  $\mu_0$  is  [\$\sigma\$ -finite](#),  $\mu = \mu'$  on  $\mathcal{A} \cap \mathcal{A}'$ .

**Note.** Notice that  $\mathcal{A}_0 \subset \mathcal{A}, \mathcal{A}'$  since they both extend  $\mathcal{A}_0$ .

*Proof.* Let  $A \in \mathcal{A} \cap \mathcal{A}'$ , we need to show

$$\underbrace{\mu(A)}_{\mu^*(A)} = \mu'(A).$$

Firstly, it's easy to show that  $\mu^*(A) \geq \mu'(A)$  by choosing the arbitrary cover of  $A$  and using the [definition](#) of  $\mu^*$ .

Secondly, we will show that  $\mu(A) \leq \mu'(A)$ .

- Assume  $\mu(A) < \infty$ , and fix  $\epsilon > 0$ . Then there exists  $B_i \in \mathcal{A}_0$  with  $B := \bigcup_{i=1}^{\infty} B_i \supset A$  such that

$$\mu(A) + \epsilon = \mu^*(A) + \epsilon \stackrel{!}{\geq} \sum_{i=1}^{\infty} \mu_0(B_i) \stackrel{!}{=} \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(B).$$

This implies that

$$\mu(B \setminus A) = \mu(B) - \mu(A) \leq \epsilon$$

where the first equality comes from  $A \subset B$  and  $\mu(A) < \infty$ . On the other hand,

$$\mu(B) = \lim_{N \rightarrow \infty} \mu\left(\bigcup_{i=1}^N B_i\right) \stackrel{8}{=} \lim_{N \rightarrow \infty} \mu'\left(\bigcup_{i=1}^N B_i\right) = \mu'(B),$$

hence,

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \stackrel{9}{\leq} \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon$$

for arbitrary  $\epsilon$ , so we conclude  $\mu(A) \leq \mu'(A)$ .

- Assume  $\mu(A) = \infty$ . Since  $\mu_0$  is  $\sigma$ -finite, so we know  $X = \bigcup_{n=1}^{\infty} X_n$  for some  $X_n \in \mathcal{A}_0$  such that

$$\mu_0(X_n) < \infty.$$

Replacing  $X_n$  by  $X_1 \cup \dots \cup X_n \in \mathcal{A}_0$ , we may assume that

$$X_1 \subset X_2 \subset \dots$$

Then,

$$\bigvee_{n \in \mathbb{N}} \mu(A \cap X_n) < \infty \stackrel{!}{\implies} \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

From the continuity of [measure](#), we then have

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A \cap X_n) \leq \lim_{n \rightarrow \infty} \mu'(A \cap X_n) = \mu'(A).$$

■

---

<sup>8</sup> $\mu = \mu'$  on  $\mathcal{A}_0$ .

<sup>9</sup>From the first part.

**Corollary 1.1.** Let  $\mu_0$  be a [pre-measure](#) on [algebra](#)  $\mathcal{A}_0$  on  $X$ . Suppose  $\mu_0$  is  [\$\sigma\$ -finite](#), then

$\exists!$  [measure](#)  $\mu$  on  $\langle \mathcal{A}_0 \rangle$  that extends  $\mathcal{A}_0$ .

Furthermore,

- The completion of  $(X, \langle \mathcal{A}_0 \rangle, \mu)$  is the [HK extension](#) of  $(\mathcal{A}_0, \mu_0)$ .

- 

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) \mid B_i \in \mathcal{A}_0, \forall_{i \in \mathbb{N}} \bigcup_{i=1}^{\infty} B_i \supset A \right\}$$

for all  $A \in \langle \bar{\mathcal{A}}_0 \rangle$ .

## Lecture 7: Borel Measures

21 Jan. 11:00

### 1.5 Borel Measures on $\mathbb{R}$

We first introduce so-called *distribution function*.

**Definition 1.15 (Distribution function).** An [increasing](#)<sup>a</sup> function

$$F: \mathbb{R} \rightarrow \mathbb{R}$$

and [right-continuous](#).  $F$  is then a *distribution function*.

<sup>a</sup>Here, increasing means  $F(x) \leq F(y)$  for  $x < y$ .

**Example.** Here are some examples of right-continuous functions.

1.  $F(x) = x$ .

2.  $F(x) = e^x$ .

3. Define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

4. Let  $\mathbb{Q} := \{r_1, r_2, \dots\}$ . Define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n \\ 0, & \text{if } x < r_n, \end{cases}$$

and

$$F(x) := \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}.$$

Then  $F$  is a distribution function (hence right-continuous). This is shown in [Lemma A.1](#).

**Note.** If  $F$  is increasing, and

$$F(\infty) := \lim_{x \nearrow \infty} F(x), \quad F(-\infty) := \lim_{x \searrow -\infty} F(x)$$

exist in  $[-\infty, \infty]$ .

In probability theory, cumulative distribution function (CDF) is a distribution function with  $F(\infty) = 1$ ,  $F(-\infty) = 0$ .<sup>10</sup>

Now, we can define a *Borel measure* on  $(X, \mathcal{B}(\mathbb{R}))$ .

**Definition 1.16 (Borel measure).** A *Borel measure* is any [measure](#)  $\mu$  defined on the  [\$\sigma\$ -algebra](#) of [Borel sets](#).

**Definition 1.17 (Locally finite).** Let  $X$  be a Hausdorff topological space,  $\mu$  on  $(X, \mathcal{B}(X))$  is called *locally finite* if  $\mu(K) < \infty$  for every compact set  $K \subset X$ .

**Note.** Some authors will require a [Borel measure](#) equipped with the [locally finite](#) property. But formally, this is not so common.

**Lemma 1.6.** Let  $\mu$  be a [locally finite Borel measure](#) on  $\mathbb{R}$ , then

$$F_\mu(x) = \begin{cases} \mu((0, x]), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu((x, 0]), & \text{if } x < 0 \end{cases}$$

is a [distribution function](#).

*Proof.* To show  $F_\mu$  is increasing, consider  $x < y$  such that

$$F_\mu(x) \leq F_\mu(y)$$

by considering

- $x > 0$ : Then  $F_\mu(x) = \mu((0, x])$  and

$$F_\mu(y) = \mu((0, y]) = \mu((0, x] \cup (x, y]) \geq \mu((0, x]) = F_\mu(x).$$

- $x = 0$ : Then  $F_\mu(x) = 0$  and

$$F_\mu(y) = \mu((0, y]) \geq 0 = F_\mu(0)$$

since  $y > 0$ .

- $x < 0$ : Follows the same argument with  $x > 0$ .

<sup>10</sup>There are [distributions](#) [FF99] Ch9., but these are different from distribution functions.

Now, we need to show  $F_\mu$  is right-continuous. Firstly, assume that  $x \geq 0$ , then we see that

$$F_\mu(x) = \mu((0, x]) = \mu((0, x^+])$$

from the fact that a measure is right-continuous.<sup>11</sup> Now, if  $x \leq 0$ , the same argument follows since multiplying  $-1$  will not change the fact that a measure is continuous. ■

**Definition 1.18 (Half intervals).** We call

$$\emptyset, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$$

*half-intervals*.

**Lemma 1.7.** Let  $\mathcal{H}$  be the collection of finite disjoint unions of [half-intervals](#). Then,  $\mathcal{H}$  is an [algebra](#) on  $\mathbb{R}$ .

*Proof.* We see that

- $\emptyset \in \mathcal{H}$ . Clearly.
- To show  $\mathcal{H}$  is closed under complements, we have
  - $\emptyset^c = \mathbb{R} = (-\infty, \infty) \in \mathcal{H}$ .
  - $(a, b]^c = (-\infty, a] \cup (a, \infty) \in \mathcal{H}$ .<sup>12</sup>
  - $(a, \infty)^c = (-\infty, a] \in \mathcal{H}$ .
  - $(-\infty, b]^c = (b, \infty) \in \mathcal{H}$ .
  - $(-\infty, \infty)^c = \emptyset \in \mathcal{H}$ .
- $\mathcal{H}$  is closed under finite unions, clearly.

<sup>11</sup>Actually, a measure is always continuous.

<sup>12</sup>Since it's a two disjoint union of half intervals.

**Proposition 1.3 (Distribution function defines a pre-measure).** Let  $F: \mathbb{R} \rightarrow \mathbb{R}$  be a [distribution function](#). For a [half interval](#)  $I$ , define

$$\ell(I) := \ell_F(I) = \begin{cases} 0, & \text{if } I = \emptyset; \\ F(b) - F(a), & \text{if } I = (a, b]; \\ F(\infty) - F(a), & \text{if } I = (a, \infty]; \\ F(b) - F(-\infty), & \text{if } I = (-\infty, b]; \\ F(\infty) - F(-\infty), & \text{if } I = (-\infty, \infty). \end{cases}$$

Define  $\mu_0 := \mu_{0,F}$  as

$$\mu_{0,F}: \mathcal{H} \rightarrow [0, \infty]$$

by

$$\mu_0(A) = \sum_{k=1}^N \ell(I_k) \text{ if } A = \bigcup_{k=1}^N I_k,$$

where  $A$  is a finite disjoint union of [half intervals](#)  $I_1, \dots, I_N$ . Then,  $\mu_0$  is a [pre-measure](#) on  $\mathcal{H}$ .

*Proof.* We see that

1.  $\mu_0$  is well-defined.
2.  $\mu_0(\emptyset) = 0$ .
3.  $\mu_0$  is finite additive.
4.  $\mu_0$  is countable additive within  $\mathcal{H}$ .

Suppose  $A \in \mathcal{H}$  where  $A = \bigcup_{i=1}^{\infty} A_i$  is a countable disjoint union. It is enough to consider the case that  $A = I$ ,  $A_k = I_k$  are all half-intervals.<sup>13</sup>

Focus on the case  $I = (a, b]$ . Let

$$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

which is a disjoint union. Then we only need to check

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

- Since  $(a, b] \supset \bigcup_{n=1}^N (a_n, b_n]$  for any fixed  $N \in \mathbb{N}$ , hence

$$\forall_{N \in \mathbb{N}} F(b) - F(a) \geq \sum_{n=1}^N (F(b_n) - F(a_n)).$$

<sup>13</sup>Since  $\mathcal{H}$  is only a collection of *finite* disjoint [half intervals](#), hence after considering  $A = I$ , we can apply the same argument iteratively and stop in finite steps. Formally, we can consider  $H \in \mathcal{H}$ ,  $H = \bigcup_{i=1}^{\infty} A^i$ , where  $A^i$  being a [half interval](#). Then by the above argument, we have  $A^i = I^i$  and so on.



By letting  $N \rightarrow \infty$ , we have

$$F(b) - F(a) \geq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

- Fix  $\epsilon > 0$ . Since  $F$  is right-continuous,  $\exists a' > a$  such that

$$F(a') - F(a) < \epsilon.$$

For each  $n \in \mathbb{N}$ ,  $\exists b'_n > b_n$  such that

$$F(b'_n) - F(b_n) < \frac{\epsilon}{2^n}.$$

Then, we have

$$[a', b] \subset \bigcup_{n=1}^{\infty} (a_n, b'_n),$$

hence

$$\exists_{N \in \mathbb{N}} [a', b] \subset \bigcup_{n=1}^N (a_n, b'_n),^{14}$$

which is only finitely many unions now. In this case, we have

$$F(b) - F(a') \leq \sum_{n=1}^N F(b'_n) - F(a_n).$$

Finally, we see that

$$\begin{aligned} F(b) - F(a) &\leq F(b) - F(a') + \epsilon \\ &\leq \sum_{n=1}^{\infty} (F(b'_n) - F(a_n)) + \epsilon \\ &\leq \sum_{n=1}^{\infty} \left( F(b_n) - F(a_n) + \frac{\epsilon}{2^n} \right) + \epsilon \\ &= \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) + 2\epsilon \end{aligned}$$

for any fixed  $\epsilon > 0$ , hence

$$F(b) - F(a) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n)).$$

Combine these two inequalities, we have

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

as we desired.

---

<sup>14</sup>This essentially follows from the fact that open sets are closed under countable unions, hence the equality will not hold, even after taking the limit.



**Remark.** It's again the  $\frac{\epsilon}{2^n}$  trick we saw before!

## Lecture 8: Lebesgue-Stieltjes Measure on $\mathbb{R}$

24 Jan. 11:00

To classify all measures, we now see this last theorem to complete the task.

**Theorem 1.5 (Locally finite Borel measures on  $\mathbb{R}$ ).** We have

1.  $F: \mathbb{R} \rightarrow \mathbb{R}$  a **distribution function**, then there exists a **unique locally finite Borel measure**  $\mu_F$  on  $\mathbb{R}$  satisfying

$$\mu_F((a, b]) = F(b) - F(a)$$

for every  $a < b$ .

2. Suppose  $F, G: \mathbb{R} \rightarrow \mathbb{R}$  are **distribution functions**. Then,

$$\mu_F = \mu_G$$

on  $\mathcal{B}(\mathbb{R})$  if and only if  $F - G$  is a constant function.

*Proof.*



HW.

**Remark.** **Theorem 1.5** simply states that given a **distribution function**, if we restrict our attention on **locally finite** measures on  $\mathbb{R}$  following our usual convention, then it defines the **measure** on  $\mathcal{B}(\mathbb{R})$  uniquely up to a *constant shift*.

### 1.6 Lebesgue-Stieltjes Measure on $\mathbb{R}$

We see that

$F$  **distribution function**  $\xRightarrow{!} \mu_F$  on Carathéodory  $\sigma$ -algebra  $\mathcal{A}_{\mu_F} \supset \mathcal{B}(\mathbb{R})$ .

Furthermore, we have

$$(\mathcal{A}_{\mu_F}, \mu_F) = \overline{(\mathcal{B}(\mathbb{R}), \mu_F)}.$$

**Definition 1.19 (Lebesgue-Stieltjes measure).** Given a **distribution function**  $F$ , we say  $\mu_F$  on  $\mathcal{A}_{\mu_F}$  is called the *Lebesgue-Stieltjes measure* corresponding to  $F$ .

**Definition 1.20 (Lebesgue measure).** From **Definition 1.19**, if  $F(x) = x$ , then the induced  $(\mathcal{A}_{\mu_F}, \mu_F)$  is denoted as  $(\mathcal{L}, m)$ , where  $\mathcal{L}$  is called *Lebesgue  $\sigma$ -algebra*, and  $m$  is called *Lebesgue measure*.

**Remark.** Recall that  $\mathcal{L}$  is induced by [Theorem 1.2](#), namely given  $m$ , for all  $A \subset \mathbb{R}$ , we have

$$\mathcal{L} := \left\{ A \subset \mathbb{R} \mid \forall_{E \subset \mathbb{R}} m(A) = m(A \cap E) + m(A \setminus E) \right\}$$

**Note.** We see that since  $F$  is right-continuous and increasing, hence

$$F(x^-) \leq F(x) = F(x^+).^{15}$$

**Example.** We first see some examples.

1.  $\mu_F((a, b]) = F(b) - F(a)$ . Then

- $\mu_F(\{a\}) = F(a) - F(a^-)$
- $\mu_F([a, b]) = F(b) - F(a^-)$
- $\mu_F((a, b)) = F(b^-) - F(a)$

2. We define

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Then

- $\mu_F(\{0\}) = 1$
- $\mu_F(\mathbb{R}) = 1$
- $\mu_F(\mathbb{R} \setminus \{0\}) = 0$ . This is easy to see since  $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ , hence

$$\begin{aligned} \mu_F(\mathbb{R} \setminus \{0\}) &= \mu_F((-\infty, 0) \cup (0, \infty)) \\ &= \underbrace{\mu_F((-\infty, 0))}_{0-0^{16}} + \underbrace{\mu_F((0, \infty))}_{1-1^{17}} = 0. \end{aligned}$$

We call that  $\mu_F$  is the *Dirac measure* at 0.

3. Denote  $\mathbb{Q} = \{r_1, r_2, \dots\}$ , and we define

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ where } F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n. \end{cases}$$

Then

- $\mu_F(\{r_i\}) > 0$  for all  $r_i \in \mathbb{Q}$ .
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$ .

<sup>15</sup>Some text will use  $x^-$  and  $x^+$  instead of  $x^-$  and  $x^+$ , respectively.

<sup>16</sup>It follows from  $F(0^-) - F(-\infty) = 0 - 0 = 0$ .

<sup>17</sup>It follows from  $F(\infty) - F(0) = 1 - 1 = 0$ .

This is shown in [Lemma A.2](#).

4. If  $F$  is continuous at  $a$ , then  $\mu_F(\{a\}) = 0$ .
5.  $F(x) = x$ , then recall that we denote  $\mu_F := m$ , and we have
  - $m((a, b]) = m((a, b)) = m([a, b]) = b - a$ .
6.  $F(x) = e^x$ 
  - $\mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$ .

**Remark.** We see that the first two examples are *discrete measures*.

**Example (Middle thirds Cantor set).** Let  $C := \bigcap_{n=1}^{\infty} K_n$ , where we have

$$\begin{aligned} K_0 &:= [0, 1] \\ K_1 &:= K_0 \setminus \left( \frac{1}{3}, \frac{2}{3} \right) \\ K_2 &:= K_1 \setminus \left( \frac{1}{9}, \frac{2}{9} \right) \cup \left( \frac{7}{9}, \frac{8}{9} \right) \\ &\vdots \\ K_n &:= K_{n-1} \setminus \bigcup_{k=1}^{3^{n-1}} \left( \frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right). \end{aligned}$$

We see that  $C$  is uncountable and with  $m(C) = 0$ . And observe that  $x \in C$  if and only if  $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$  for some  $a_n \in \{0, 2\}$ . Hence, we can instead formulate  $K_n$  by

$$K_n = \bigcup_{\substack{a_i \in \{0, 2\} \\ 1 \leq i \leq n}} \left[ \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n} \right].$$



Figure 1: The top line corresponds to  $K_0$ , and then  $K_1$ , etc.

The proof of  $m(C) = 0$  is given in [Lemma A.3](#).

### 1.6.1 Cantor Function

Consider  $F$  as follows. We define a function  $F$  to be 0 to the left of 0, and 1 to the right of 1. Then, define  $F$  to be  $\frac{1}{2}$  on  $(\frac{1}{3}, \frac{2}{3})$ ,  $\frac{1}{4}$  on  $(\frac{1}{9}, \frac{2}{9})$ ,  $\frac{3}{4}$  on  $(\frac{7}{9}, \frac{8}{9})$  and so on. This is so-called *Cantor Function*. We can show  $F$  is continuous and increasing, which makes  $F$  a distribution function. Also, we see that the measure this  $F$  induced is called *Cantor measure*.



Figure 2: Cantor Function (Devil's Staircase).

We see that  $F$  is *continuous* and increasing. Furthermore,

| Cantor Measure $\mu_F$              |        | Lebesgue Measure $m$                     |
|-------------------------------------|--------|--|
| $\mu_F(\mathbb{R} \setminus C) = 0$ |        | $m(\mathbb{R} \setminus C) = \infty > 0$ |
| $\mu_F(C) = 1$                      | $\iff$ | $m(C) = 0$                               |
| $\mu_F(\{a\}) = 0$                  |        | $m(\{a\}) = 0$                           |

**Remark.**  $\mu_F$  and  $m$  are said to be **singular** to each other.

## 1.7 Regularity Properties of Lebesgue-Stieltjes Measures

We first see a lemma.

**Lemma 1.8.** Let  $\mu$  be **Lebesgue-Stieltjes measure** on  $\mathbb{R}$ . Then we have

$$\begin{aligned} \mu(A) &\stackrel{!}{=} \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i]) \mid \bigcup_{i=1}^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

for every  $A \in \mathcal{A}_\mu$

*Proof.* The second equality follows from the continuity of the **measure**. ■

**Remark.** This is similar to

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - 1/n], \quad (a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n].$$

## Lecture 9: Properties of Lebesgue-Stieltjes measure

26 Jan. 11:00

**As previously seen.** Let  $X \subset [0, \infty]$ . Recall that

•

$$\alpha = \sup X < \infty \iff \begin{cases} \forall_{x \in X} \alpha \geq x \\ \forall_{\epsilon > 0} \exists_{x \in X} x + \epsilon \geq \alpha. \end{cases}$$

•

$$\alpha = \sup X = \infty \iff \forall_{L > 0} \exists_{x \in X} x \geq L.$$

This should be useful latter on.

**Theorem 1.6 (Regularity).** Let  $\mu$  be [Lebesgue-Stieltjes measure](#). Then, for all  $A \in \mathcal{A}_\mu$ ,

1. (outer regularity)  $\mu(A) = \inf\{\mu(O) \mid O \supset A, O \text{ is open}\}$
2. (inner regularity)  $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$

*Proof.* We check them separately.

1.

DIY

2. Let  $s := \sup\{\mu(K) \mid K \subset A, K \text{ is compact}\}$ , then by [monotonicity](#), we have  $\mu(A) \geq s$ . To show the other direction, we consider

- $A$  is a bounded set.

Then  $\bar{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$ ,  $\bar{A}$  is also bounded  $\implies \mu(\bar{A}) < \infty$ . Fix  $\epsilon > 0$ , then by [outer regularity](#), there exists an open  $O \supset \bar{A} \setminus A$ , and  $\mu(O) - \mu(\bar{A} \setminus A) = \mu(O \setminus (\bar{A} \setminus A)) \leq \epsilon$ . Let  $K := \underbrace{A \setminus O}_{K \subset A} = \underbrace{\bar{A} \setminus O}_{\text{compact}}$ , we

show that

$$\mu(K) \geq \mu(A) - \epsilon.$$

DIY

- $A$  is an unbounded set with  $\mu(A) < \infty$ .

Let  $A = \bigcup_{n=1}^{\infty} A_n$ ,  $A_n = A \cap [-n, n]$  where  $A_1 \subset A_2 \subset \dots$ , then

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) < \infty.$$

- $A$  is an unbounded set with  $\mu(A) = \infty$ .

We can show that

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) = \infty.$$

Fix  $L > 0$ , then  $\exists N$  such that  $\mu(A_N) \geq L$ .

■

**Definition 1.21** ( $G_\delta$ -set,  $F_\sigma$ -set). Let  $X$  be a topological space. Then

- A  $G_\delta$ -set is  $G = \bigcap_{i=1}^{\infty} O_i$ ,  $O_i$  open.
- A  $F_\sigma$ -set is  $F = \bigcup_{i=1}^{\infty} F_i$ ,  $F_i$  closed.

**Theorem 1.7.** Let  $\mu$  be a Lebesgue-Stieltjes measure. Then  $TFAE^a$ :

1.  $A \in \mathcal{A}_\mu$
2.  $A = G \setminus M$ ,  $G$  is a  $G_\delta$ -set,  $M$  is a  $\mu$ -null set.
3.  $A = F \setminus N$ ,  $F$  is a  $F_\sigma$ -set,  $N$  is a  $\mu$ -null set.

<sup>a</sup>TFAE: The following are equivalent.

*Proof.* We see that (2.)  $\implies$  (1.) and (3.)  $\implies$  (1.) are clear.

- (1.)  $\implies$  (3.)

– Assume  $\mu(A) < \infty$ . From the inner regularity, we have

$$\forall n \in \mathbb{N} \exists \text{compact } K_n \subset A \text{ such that } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

Let  $F = \bigcup_{n=1}^{\infty} K_n$ , then  $N = A \setminus F$  is  $\mu$ -null.

Check!

– Assume  $\mu(A) = \infty$ . Let  $A = \bigcup_{k \in \mathbb{Z}} A_k$ ,  $A_k = A \cap (k, k+1]$ . From what we have just shown above,

$$\forall k \in \mathbb{Z} \ A_k = F_k \cup N_k, \quad A = \underbrace{\left( \bigcup_k F_k \right)}_{F_\sigma\text{-set}} \cup \underbrace{\left( \bigcup_k N_k \right)}_{\mu\text{-null}}.$$

- (1.)  $\implies$  (2.)

We see that

$$A^c = F \cup N, \quad A = F^c \cap N^c = F^c \setminus N.$$

■

**Proposition 1.4.** Let  $\mu$  be a Lebesgue-Stieltjes measure, and  $A \in \mathcal{A}_\mu$ ,  $\mu(A) < \infty$ . Then we have

$$\forall \epsilon > 0 \exists I = \bigcup_{i=1}^{N(\epsilon)} I_i$$

disjoint open intervals such that  $\mu(A \triangle I) \leq \epsilon$ .

*Proof.* Using [outer regularity](#) and the fact that every open set is  $\bigcup_{i=1}^{\infty} I_i$ , where  $I_i$  are disjoint open intervals. ■ [DIY](#)

We now see some properties of [Lebesgue measure](#).

**Theorem 1.8.** Let  $A \in \mathcal{L}$ , then we have  $A + s \in \mathcal{L}$ ,  $rA \in \mathcal{L}$  for all  $r, s \in \mathbb{R}$ .  
i.e.,

$$m(A + s) = m(A), \quad m(rA) = |r| \cdot m(A).$$

*Proof.* ■ [DIY](#)

**Example.** We now see some examples.

1. Let  $\mathbb{Q} = \{r_i\}_{i=1}^{\infty}$  which is dense in  $\mathbb{R}$ . Let  $\epsilon > 0$ , and

$$O = \bigcup_{i=1}^{\infty} \left( r_i - \frac{\epsilon}{2^i}, r_i + \frac{\epsilon}{2^i} \right).$$

We see that  $O$  is open and dense<sup>18</sup> in  $\mathbb{R}$ . But we see

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon.$$

Furthermore,  $\partial O = \overline{O} \setminus O$ ,  $m(\partial O) = \infty$

2. There exists uncountable set  $A$  with  $m(A) = 0$ .
3. There exists  $A$  with  $m(A) > 0$  but  $A$  contains no non-empty open intervals.
4. There exists  $A \notin \mathcal{L}$ . e.g. Vitali set.<sup>19</sup>
5. There exists  $A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$ .

## Lecture 10: Integration

26 Jan. 11:00

## 2 Integration

### 2.1 Measurable Function

We start with a definition.

**Definition 2.1 (Measurable space).** A *measurable space* or *Borel space* is a tuple of a set  $X$  and a  $\sigma$ -algebra  $\mathcal{A}$  on  $X$ , denoted by  $(X, \mathcal{A})$ .

<sup>18</sup>[https://en.wikipedia.org/wiki/Dense\\_set](https://en.wikipedia.org/wiki/Dense_set)

<sup>19</sup>[https://en.wikipedia.org/wiki/Vitali\\_set](https://en.wikipedia.org/wiki/Vitali_set)



**Definition 2.2 (Measurable function).** Suppose  $(X, \mathcal{A}), (Y, \mathcal{B})$  are measurable spaces. Then we say  $f: X \rightarrow Y$  is  $(\mathcal{A}, \mathcal{B})$ -measurable if

$$\forall_{B \in \mathcal{B}} f^{-1}(B) \in \mathcal{A}.$$

**Remark.** If  $\mathcal{A}$  and  $\mathcal{B}$  are given, we'll sometimes say  $f$  is measurable if it'll not cause any confusions.

**Lemma 2.1.** Given two measurable spaces  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$ , and suppose  $\mathcal{B} = \langle \mathcal{E} \rangle$  for some  $\mathcal{E} \subset Y$ . Then,

$$f: X \rightarrow Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall_{E \in \mathcal{E}} f^{-1}(E) \in \mathcal{A}.$$

*Proof.* We see that the *only if* part ( $\implies$ ) is clear. On the other direction, we consider the following. Let  $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$ , then

- $\mathcal{E} \subset \mathcal{D}$  by assumption
- $\mathcal{D}$  is a  $\sigma$ -algebra

Check!

hence, we see that  $\langle \mathcal{E} \rangle = \mathcal{B} \subset \mathcal{D}$  from Lemma 1.2. The result then follows from the definition of  $(\mathcal{A}, \mathcal{B})$ -measurable. ■

**Note.** Recall that

- $f^{-1}(E^c) = f^{-1}(E)^c$
- $f^{-1}\left(\bigcup_{\alpha} E_{\alpha}\right) = \bigcup_{\alpha} f^{-1}(E_{\alpha})$

**Definition 2.3 ( $\mathcal{A}$ -measurable).** Let  $(X, \mathcal{A})$  be a measurable space. Then,

$$\left. \begin{array}{l} f: X \rightarrow \mathbb{R} \\ f: X \rightarrow \overline{\mathbb{R}} \\ f: X \rightarrow \mathbb{C} \end{array} \right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{ \begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f: X \rightarrow \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{array} \right.$$

**Notation.** Notice that

- $\overline{\mathbb{R}} = [-\infty, \infty]$
- $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}.$
- $\Re f$  is the real part of  $f$ , while  $\Im f$  is the imaginary part of  $f$ .

**Example.** We see that

- $\mathcal{A} = \mathcal{P}(X) \implies$  Every function is  $\mathcal{A}$ -measurable.
- $\mathcal{A} = \{\emptyset, X\} \implies$  The only  $\mathcal{A}$ -measurable functions are constant functions.

**Definition 2.4 (Lebesgue measurable).** A *Lebesgue measurable function*  $f$  is a **measurable** function

$$f: (\mathbb{R}, \mathcal{L}) \rightarrow (\mathbb{C}, \mathcal{B}(\mathbb{C})).$$

**Lemma 2.2.** Given  $f: X \rightarrow \mathbb{R}$ , *TFAE*.

1.  $f$  is  **$\mathcal{A}$ -measurable**
2.  $\forall a \in \mathbb{R}, f^{-1}((a, \infty)) \in \mathcal{A}$
3.  $\forall a \in \mathbb{R}, f^{-1}([a, \infty)) \in \mathcal{A}$
4.  $\forall a \in \mathbb{R}, f^{-1}((-\infty, a)) \in \mathcal{A}$
5.  $\forall a \in \mathbb{R}, f^{-1}((-\infty, a]) \in \mathcal{A}$

*Proof.* The result follows from **Lemma 2.1** we just saw. ■

**Remark (Operations preserve  $\mathcal{A}$ -measurability).** Given  $f, g: X \rightarrow \mathbb{R}$  and  $f$  is  **$\mathcal{A}$ -measurable**, then

1.  $\phi: \mathbb{R} \rightarrow \mathbb{R}$ ,  **$\mathcal{A}$ -measurable<sup>20</sup>**, then

$$\phi \circ f: X \rightarrow \mathbb{R}$$

is  **$\mathcal{A}$ -measurable**.

2.  $-f, 3f, f^2, |f|$  are all  **$\mathcal{A}$ -measurable**, and  $\frac{1}{f}$  is  **$\mathcal{A}$ -measurable** if  $f(x) \neq 0, \forall x \in X$ .
3.  $f + g$  is  **$\mathcal{A}$ -measurable**. We see this from

$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} (f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty)))$$

with **Lemma 2.2**.

4.  $f \cdot g$  is  **$\mathcal{A}$ -measurable**. We see this from

$$f(x)g(x) = \frac{1}{2} ((f(x) + g(x))^2 - f(x)^2 - g(x)^2).$$

5. We see that

$$(f \vee g)(x) := \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) := \min\{f(x), g(x)\}$$

are  **$\mathcal{A}$ -measurable**.

6. Let  $f_n: X \rightarrow \overline{\mathbb{R}}$  be  **$\mathcal{A}$ -measurable**. Then

$$\sup_{n \in \mathbb{N}} f_n, \inf_{n \in \mathbb{N}} f_n, \limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n$$

are  **$\mathcal{A}$ -measurable**.

---

<sup>20</sup>In this case, we also call it *Borel measurable*.

*Proof.* Consider  $\sup_{n \in \mathbb{N}} f_n =: g$ , then

$$g^{-1}((a, \infty]) = \bigcup_{n \in \mathbb{N}} f_n^{-1}((a, \infty])$$

for  $\sup_{n \in \mathbb{N}} f_n(x) = g(x) > a$ . A similar argument can prove the case of  $\inf_{n \in \mathbb{N}} f_n$ . check

And notice that  $\limsup_{n \rightarrow \infty} f_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n$ , then the similar argument also proves this case. ■

7. If  $\lim_{n \rightarrow \infty} f_n(x)$  converges for every  $x \in X$ , then  $f$  is  $\mathcal{A}$ -measurable.

8. If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous

$\implies f$  is Borel measurable

$\implies f$  is Lebesgue measurable

since the preimage of an open set of a continuous function is open, then we consider  $f^{-1}((a, \infty))$ .

**Definition 2.5** ( $f^+$ ,  $f^-$ ). For  $f: X \rightarrow \overline{\mathbb{R}}$ , let  $f^+ := f \vee 0$  and  $f^- := (-f) \vee 0$ .<sup>a</sup>

<sup>a</sup>i.e.,  $f^+(x) = \max\{f(x), 0\}$ ,  $f^-(x) = \min\{-f(x), 0\}$

**Remark.** If  $\text{supp } f^+ \cap \text{supp } f^- = \emptyset$  and  $f(x) = f^+(x) - f^-(x)$ , then

$f$  is  $\mathcal{A}$ -measurable  $\iff f^+, f^-$  are  $\mathcal{A}$ -measurable.

**Notation.**  $\text{supp } f$  means the support of  $f$ , which is the set of domain which makes  $f$  being non-zero.

**Definition 2.6 (Characteristic (Indicator) function).** For  $E \subset X$ , the *characteristic (indicator) function* of  $E$  is

$$\chi_E(x) = \mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \in E^c. \end{cases}$$

**Remark.** We see that  $\mathbb{1}_E$  is  $\mathcal{A}$ -measurable  $\iff E \in \mathcal{A}$ .

**Definition 2.7 (Simple function).** Let  $(X, \mathcal{A})$  be a measurable space. Then a *simple function*  $\phi: X \rightarrow \mathbb{C}$  that is  $\mathcal{A}$ -measurable and takes only finitely many values.

**Remark.** We see that if

$$\phi(X) = \{c_1, \dots, c_N\},$$

then

$$E_i = \phi^{-1}(\{c_i\}) \in \mathcal{A} \implies \phi = \sum_{i=1}^N \underbrace{c_i}_{\neq \pm\infty} \mathbb{1}_{\underbrace{E_i}_{\in \mathcal{A}}}.$$

## Lecture 11: Integration of nonnegative functions

31 Jan. 11:00

As previously seen. For a [simple function](#)  $\phi$ ,  $c_i$  can actually be in  $\mathbb{C}$ .

**Theorem 2.1.** Given a [measurable space](#)  $(X, \mathcal{A})$  and let  $f: X \rightarrow [0, \infty]$ , the following is equivalent.

1.  $f$  is [mathcal{A}-measurable function.](#)
2. There exists [simple functions](#)  $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$  such that

$$\forall_{x \in X} \lim_{n \rightarrow \infty} \phi_n(x) = f(x)$$

i.e.,  $f$  is a pointwise upward limit of [simple functions](#).

*Proof.* We'll prove both directions.

- It's clear that (2.)  $\implies$  (1.) from the fact that  $f(x) = \sup_n \phi_n(x)$  and [the remark](#).
- We want to show that (1.)  $\implies$  (2.). Assume  $f$  is [mathcal{A}-measurable, and fix  \$n \in \mathbb{N}\$ .](#)

Let  $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$ . Also, for  $0 \leq k \leq 2^{2^n} - 1$ ,  $E_{n,k} = f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}]) \in \mathcal{A}$ .

Then, define  $\phi_n$  be

$$\phi_n = \sum_{k=0}^{2^{2^n}-1} \frac{k}{2^n} \mathbb{1}_{E_{n,k}} + 2^n \mathbb{1}_{F_n},$$

we have

- $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$  for every  $x \in X$
- $\forall x \in X \setminus F_n$ , we have  $0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$

Furthermore, we see that

$$F_1 \supset F_2 \supset \dots, \quad \bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\}),$$

then

- $x \in f^{-1}([0, \infty]) = X \setminus \bigcap_{n=1}^{\infty} F_n \implies \lim_{n \rightarrow \infty} \phi_n(x) = f(x)$
- $x \in f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} F_n \implies f_n(x) \geq 2^n \implies \lim_{n \rightarrow \infty} \phi_n(x) = \infty = f(x)$

■

**Corollary 2.1.** If  $f$  is bounded on a set  $A \subset \mathbb{R}$ , i.e.,  $\exists b > 0$  such that

$$\forall_{x \in A} |f(x)| \leq L,$$

then  $\phi_n \rightarrow f$  uniformly on  $A$ .

*Proof.*

■

DIY

**Corollary 2.2.** If  $f: X \rightarrow \mathbb{C}$  is a **measurable function** if and only if there exists **simple functions**  $\phi_n: X \rightarrow \mathbb{C}$  such that

$$0 \leq |\phi_1(x)| \leq |\phi_2(x)| \leq \dots \leq |f(x)|$$

with

$$\forall_{x \in X} \lim_{n \rightarrow \infty} \phi_n(x) = f(x).$$

*Proof.*

■

DIY

## 2.2 Integration of Nonnegative Functions

We start with our first definition about integral.

**Definition 2.8 (Integration of nonnegative function).** Let  $(X, \mathcal{A}, \mu)$  be a **measure space**, and  $\phi: X \rightarrow [0, \infty]$  such that

$$\phi = \sum_{i=1}^N c_i \mathbb{1}_{E_i}$$

be a **simple function**. Define

$$\int \phi = \int \phi \, d\mu = \int_X \phi \, d\mu = \sum_{i=1}^N c_i \mu(E_i).$$

Furthermore, for  $A \in \mathcal{A}$ ,

$$\int_A \phi = \int_A \phi \, d\mu = \int \phi \mathbb{1}_A \, d\mu.$$

**Note.** Note that

- In the expression  $\sum_{i=1}^N c_i \mu(E_i)$ , we're using the convention  $0 \cdot \infty = 0$ .
- The function  $\phi \mathbb{1}_A$  is also a **simple function** since both  $\phi$  and  $\mathbb{1}_A$  are **simple function**.

**Proposition 2.1.** Suppose we have  $\phi, \psi \geq 0$  be two **simple functions**. Then,

- **Definition 2.8** is well-defined.
- $\int c\phi = c \int \phi$  for  $c \in [0, \infty)$ .
- $\int \phi + \psi = \int \phi + \int \psi$ .
- $\phi(x) \geq \psi(x)$  for all  $x \implies \int \phi \geq \int \psi$ .
- $\nu(A) = \int_A \phi d\mu$  is a **measure** on  $(X, \mathcal{A})$ .

*Proof.* \_\_\_\_\_



DIY

**Definition 2.9 (Generalization of Integration of nonnegative function).** Given  $(X, \mathcal{A}, \mu)$  with  $f: X \rightarrow [0, \infty]$  be  **$\mathcal{A}$ -measurable**. Define

$$\int f = \int f d\mu = \sup \left\{ \int \phi : 0 \leq \phi \leq f \text{ such that } \phi \text{ is simple} \right\}.$$

**Note.** Note that

- If  $f$  is a **simple function**, the **Definition 2.8** and **Definition 2.9** of  $\int f$  are the same.
- $\int cf = c \int f$  for  $c \in [0, \infty)$ .
- If  $f \geq g \geq 0 \implies \int f \geq \int g$ .
- But  $\int f + g = \int f + \int g$  is not trivial.

**Theorem 2.2 (Monotone Convergence Theorem (MCT)).** Given  $(X, \mathcal{A}, \mu)$  be a **measure space**. Then if

- $f_n: X \rightarrow [0, \infty]$  be  **$\mathcal{A}$ -measurable** for every  $n \in \mathbb{N}$ ;
- $0 \leq f_1(x) \leq f_2(x) \leq \dots$  for every  $x \in X$ ;
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for every  $x \in X$ ,

we have

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

*Proof.* Note that if  $\lim_{n \rightarrow \infty} \int f_n$  exists, then it's equal to  $\sup_n \int f_n$ .

Then

- $f_n \leq f \implies \int f_n \leq \int f \implies \lim_{n \rightarrow \infty} \int f_n \leq \int f$ .
- Fix a **simple function**  $0 \leq \phi \leq f$ , then it's enough to show  $\lim_{n \rightarrow \infty} \int f_n \geq \int \phi$ .

We first fix  $\alpha = (0, 1)$ , then it's also enough to show

$$\lim_{n \rightarrow \infty} \int f_n \geq \alpha \int \phi.$$

Let  $A_n := \{x \in X \mid f_n(x) \geq \alpha \phi(x)\}$ , then since  $f_n$  is measurable,

- $A_n \in \mathcal{A}$
- $A_1 \subset A_2 \subset A_3 \subset \dots$
- $\bigcup_{n=1}^{\infty} A_n = X$

Check!

We then have

$$\int f_n \geq \int f_n \mathbb{1}_{A_n} \geq \int \alpha \phi \mathbb{1}_{A_n} = \alpha \int_{A_n} \phi = \alpha \nu(A_n)$$

where  $\nu(A) = \int_A \phi$  is a measure. This implies

$$\lim_{n \rightarrow \infty} \int f_n \geq \alpha \lim_{n \rightarrow \infty} \nu(A_n) \stackrel{21}{=} \alpha \nu(X) = \alpha \int \phi.$$

■

**Corollary 2.3 (Linearity of nonnegative integral).** Let  $f, g \geq 0$  be measurable, then

$$\int f + g = \int f + \int g.$$

*Proof.* There exists simple functions  $\phi_n$  and  $\psi_n$  such that

- $0 \leq \phi_1 \leq \phi_2 \leq \dots$  and  $\phi_n \rightarrow f$  pointwise
- $0 \leq \psi_1 \leq \psi_2 \leq \dots$  and  $\psi_n \rightarrow g$  pointwise

Then,

$$\int (f + g) \stackrel{!}{=} \lim_{n \rightarrow \infty} \int (\phi_n + \psi_n) = \lim_{n \rightarrow \infty} \int \phi_n + \int \psi_n \stackrel{!}{=} \int f + \int g.$$

■

## Lecture 12: Fatou's Lemma

2 Feb. 11:00

We start with a useful corollary.

**Corollary 2.4 (Tonelli's theorem for nonnegative series and integrals).** Given  $g_n \geq 0$  for every  $n \in \mathbb{N}$  and let  $g_n$  be measurable, then

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

<sup>21</sup>This follows from the continuity of measure from below

**Remark.** Recall that we have seen [two series case](#) before. We'll later see two integrals cases.

*Proof.* Let  $f_N := \sum_{n=1}^N g_n$  such that  $\lim_{N \rightarrow \infty} f_N = f$ , then since  $g_n \geq 0$ , we have  $0 \leq f_1 \leq f_2 \leq \dots$  with

$$\lim_{N \rightarrow \infty} f_N(x) = \sum_{n=1}^{\infty} g_n(x).$$

By [Theorem 2.2](#), we have

$$\lim_{N \rightarrow \infty} \int \underbrace{\sum_{n=1}^N g_n}_{f_N} = \int \underbrace{\sum_{n=1}^{\infty} g_n}_f.$$

Now, since the terms in the limit on the left-hand side is just a finite sum, by [Corollary 2.3](#), we have

$$\lim_{N \rightarrow \infty} \underbrace{\sum_{n=1}^N}_{\sum_{n=1}^{\infty}} \int g_n = \lim_{N \rightarrow \infty} \int \sum_{n=1}^N g_n = \int \sum_{n=1}^{\infty} g_n,$$

hence

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n.$$

■

**Theorem 2.3 (Fatou's Lemma).** Suppose  $f_n \geq 0$  and [measurable](#), then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

**Remark.** Recall that

$$\liminf_{n \rightarrow \infty} f_n := \lim_{k \rightarrow \infty} \inf_{n \geq k} f_n = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f_n$$

and

$$\exists \lim_{n \rightarrow \infty} a_n \iff \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

*Proof.* Let  $g_k = \inf_{n \geq k} f_n$ , then  $g_k$  is [measurable](#) and  $0 \leq g_1 \leq g_2 \leq \dots$ . Now, from [Theorem 2.2](#), we have

$$\int \lim_{k \rightarrow \infty} g_k = \lim_{k \rightarrow \infty} \int g_k.$$



Notice that the left-hand side is just  $\int \liminf_{n \rightarrow \infty} f_n$ , while the right-hand side is just  $\lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n$ , i.e.,

$$\int \liminf_{n \rightarrow \infty} f_n = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n.$$

We see that we want to take the inf outside the integral on the right-hand side. Observe that

$$\forall_{m \geq k} \inf_{n \geq k} f_n \leq f_m \implies \forall_{m \geq k} \int \inf_{n \geq k} f_n \leq \int f_m \implies \int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m.$$

Then, we have

$$\int \liminf_{n \rightarrow \infty} f_n = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n \leq \lim_{k \rightarrow \infty} \inf_{m \geq k} \int f_m = \liminf_{m \rightarrow \infty} \int f_m.$$

■

**Example.** Given  $(\mathbb{R}, \mathcal{L}, m)$ .

1. **Escape to horizontal infinity.** Let  $f_n := \mathbb{1}_{(n, n+1)}$ . We immediately see that

- $f_n \rightarrow 0$  pointwise
- $\int f_n = 1$  for every  $n$
- $\int f = 0$

From [Theorem 2.3](#), we have a strict inequality

$$0 = \int \liminf_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} \int f_n = 1.$$

2. **Escape to width infinity.** Let  $f_n := \frac{1}{n} \mathbb{1}_{(0, n)}$ .
3. **Escape to vertical infinity.** Let  $f_n := n \mathbb{1}_{(0, \frac{1}{n})}$ .

**Lemma 2.3 (Markov's inequality).** Let  $f \geq 0$  be [measurable](#). Then

$$\forall_{c \in (0, \infty)} \mu(\{x \mid f(x) \geq c\}) \leq \frac{1}{c} \int f.$$

*Proof.* Denote  $\{x \mid f(x) \geq c\} =: E$ , then

$$f(x) \geq c \mathbb{1}_E(x) \implies \int f \geq c \int \mathbb{1}_E = c \cdot \mu(E).$$

■

**Remark.** Notice that  $E = f^{-1}([c, \infty))$ , hence  $E$  is [measurable](#).

**Proposition 2.2.** Let  $f \geq 0$  be measurable. Then,

$$\int f = 0 \iff f = 0 \text{ a.e.}$$

i.e.,

$$\int f \, d\mu = 0 \iff \begin{cases} \mu(A) = 0 \\ A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty]). \end{cases}$$

*Proof.* Firstly, assume that  $f = \phi$  is a simple function. We may write

$$\phi = \sum_{i=1}^N c_i \mathbb{1}_{E_i}$$

where  $E_i$  are disjoint and  $c_i \in (0, \infty)$ . Then,

$$\begin{aligned} \int \phi &= \sum_{i=1}^N c_i \mu(E_i) = 0 \\ \iff \mu(E_1) = \dots = \mu(E_N) &= 0 \\ \iff \mu(A) = 0, \quad A = \bigcup_{i=1}^N E_i. \end{aligned}$$

Now, assume that  $f$  is a general function where  $f \geq 0$  is the only constraint.

1. Assume  $\mu(A) = 0$  (i.e.,  $f = 0$  a.e.). Let  $0 \leq \phi \leq f$ , where  $\phi$  is simple. Then

$$\forall_{x \in A^c} \phi(x) = 0$$

since  $f(x) = 0, \forall x \in A^c$ . This implies that  $\phi = 0$  a.e. since  $\mu(A) = 0$ , so  $\int \phi = 0$ . We then have

$$\int f = 0$$

from Definition 2.9.

2. Assume  $\int f = 0$ . Let  $A_n = f^{-1}([\frac{1}{n}, \infty])$ . Then we see that

- $A_1 \subset A_2 \subset \dots$
- $\bigcup_{n=1}^{\infty} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} [\frac{1}{n}, \infty]\right) = f^{-1}((0, \infty)) = A$ .

We then have

$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \stackrel{!}{\leq} n \int f = 0,$$

which further implies

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n) = 0$$

from the continuity of measure from below. ■

**Corollary 2.5.** If  $f, g \geq 0$  are both measurable and  $f = g$  a.e., then

$$\int f = \int g.$$

*Proof.* Let  $A = \{x \mid f(x) \neq g(x)\}$ <sup>22</sup>. Then by assumption,  $\mu(A) = 0$ , hence

$$f \mathbb{1}_A = 0 \text{ a.e.}, \quad g \mathbb{1}_A = 0 \text{ a.e.}$$

This further implies that

$$\begin{aligned} \int f &= \int f(\mathbb{1}_A + \mathbb{1}_{A^c}) \\ &\stackrel{!}{=} \int f \mathbb{1}_A + \int f \mathbb{1}_{A^c} \\ &= \int f \mathbb{1}_{A^c} = \int g \mathbb{1}_{A^c} \\ &= \int g \mathbb{1}_{A^c} + \int g \mathbb{1}_A = \int g. \end{aligned}$$

■

**Corollary 2.6.** Let  $f_n \geq 0$  be measurable. Then

1. 
$$\left. \begin{array}{l} 0 \leq f_1 \leq f_2 \leq \dots \leq f \text{ a.e.} \\ \lim_{n \rightarrow \infty} f_n = f \text{ a.e.} \end{array} \right\} \implies \lim_{n \rightarrow \infty} \int f_n = \int f.$$
2. 
$$\lim_{n \rightarrow \infty} f_n = f \text{ a.e.} \implies \int f \leq \liminf_{n \rightarrow \infty} \int f_n.$$

*Proof.*

■

DIY

**Remark.** Almost all the theorems we've proved can be replaced by theorems dealing with almost every condition.

## Lecture 13: Integration of Complex Functions

4 Feb. 11:00

### 2.3 Integration of Complex Functions

As usual, we start from a definition.

<sup>22</sup> $A$  is measurable indeed.

**Definition 2.10 (Integrable).** Let  $(X, \mathcal{A}, \mu)$  be a [measure space](#) and let  $f: X \rightarrow \overline{\mathbb{R}}$  and  $g: X \rightarrow \mathbb{C}$  be [measurable](#).<sup>a</sup>

Then  $f, g$  are called *integrable* if  $\int |f| < \infty$ , and we define

$$\int f = \int f^+ - \int f^-, \quad \int g = \int \Re g + i \int \Im g.$$

Furthermore, for  $f: X \rightarrow \overline{\mathbb{R}}$ , we define

$$\int f = \begin{cases} \infty, & \text{if } \int f^+ = \infty, \int f^- < \infty; \\ -\infty, & \text{if } \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

<sup>a</sup>Recall that for a complex-valued function like  $g$ , this means that both  $\Re g$  and  $\Im g$  are [measurable](#).

We now see a lemma.

**Lemma 2.4.** Let  $f, g: X \rightarrow \overline{\mathbb{R}}$  or  $\mathbb{C}$  [integrable](#). Assume that  $f(x) + g(x)$  is well-defined for all  $x \in X$ .<sup>a</sup>

Then we have

1.  $f + g, cf$  for all  $c \in \mathbb{C}$  are [integrable](#).
2.  $\int f + g = \int f + \int g$ . This is not trivial since  $(f + g)^+ \neq f^+ + g^+$ .
3.  $|\int f| \leq \int |f|$ .

<sup>a</sup>That is, we never see  $\infty + (-\infty)$  or  $(-\infty) + \infty$ .

*Proof.* Check [FF99] page 53. ■

**Lemma 2.5.** Let  $(X, \mathcal{A}, \mu)$  be a [measure space](#) and let  $f$  be an [integrable](#) function on  $X$ . Then

1.  $f$  is finite [a.e.](#) i.e.,  $\{x \in X \mid |f(x)| = \infty\}$  is a [null set](#).
2. The set  $\{x \in X \mid f(x) \neq 0\}$  is [σ-finite](#).

*Proof.* ■

HW 5  
Q8 by  
[Lemma 2.3](#)

**Proposition 2.3.** Let  $(X, \mathcal{A}, \mu)$  be a **measure space**, then

1. If  $h$  is **integrable** on  $X$ , then

$$\forall_{E \in \mathcal{A}} \int_E h = 0 \iff \int |h| = 0 \iff h = 0 \text{ a.e.}$$

2. If  $f, g$  are **integrable** on  $X$ , then

$$\forall_{E \in \mathcal{A}} \int_E f = \int_E g \iff f = g \text{ a.e.}$$

*Proof.* We prove this one by one.

1. We see that the second equivalence is done in **Proposition 2.2**, hence we prove the first equivalence only. Since we have

$$\int |h| = 0 \implies \left| \int_E h \right| \leq \int_E |h| \leq \int |h| = 0,$$

which shows one implication. Now assume that  $\int_E h = 0$  for all  $E \in \mathcal{A}$ , then we can write  $h$  as

$$h = u + iv = (u^+ - u^-) + i(v^+ - v^-).$$

Let  $B := \{x \in X \mid u^+(x) > 0\}$ , then by assumption, we have

$$0 = \int_B h = \Re \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+,$$

hence  $u^+ = 0$  **almost everywhere**. Similarly, we have  $u^-, v^+, v^-$  are all zero **almost everywhere**. This gives us that  $h$  is zero **almost everywhere** as desired.

2. DIY

■

**Theorem 2.4 (Dominated Convergence Theorem).** Let  $(X, \mathcal{A}, \mu)$  be a **measure space**, and

- Let  $f_n$  be **integrable** on  $X$ .
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  **almost everywhere**.
- There is a  $g: X \rightarrow [0, \infty]$  such that  $g$  is **integrable** and

$$\forall_{n \in \mathbb{N}} |f_n(x)| \leq g(x) \text{ a.e.}$$

Then we have

$$\lim_{n \rightarrow \infty} \int f_n = \int f = \int \lim_{n \rightarrow \infty} f_n.$$

*Proof.* Let  $F$  be the countable union of [null set](#) on which the three conditions may fail. Then we see that after modifying the definition of  $f_n, f$  and  $g$  on  $F$ , we may assume that all three conditions hold everywhere since modifying on a [null set](#) does not change the integral.

We now consider the  $\mathbb{R}$ -valued case only. Note that the second and the third conditions imply that  $f$  is [integrable](#) since  $|f| \leq g(x)$ . We then see that  $g + f_n \geq 0$  and  $g - f_n \geq 0$  because  $-g \leq f_n \leq g$ . From [Theorem 2.3](#), we have

$$\int g + f \leq \liminf_{n \rightarrow \infty} \int g + f_n, \quad \int g - f \leq \liminf_{n \rightarrow \infty} \int g - f_n.$$

From the [linearity of integral](#), we have

$$\int g + \int f \leq \int g + \liminf_{n \rightarrow \infty} \int f_n, \quad \int g - \int f \leq \int g - \liminf_{n \rightarrow \infty} \int f_n.$$

Now, since  $\int g < \infty$ , we can cancel it, which gives

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n, \quad -\int f \leq \liminf_{n \rightarrow \infty} \int -f_n = -\limsup_{n \rightarrow \infty} \int f_n,$$

which implies

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n \leq \limsup_{n \rightarrow \infty} \int f_n \leq \int f.$$

This shows that the limit exists, and the desired result indeed holds.  $\blacksquare$

**Corollary 2.7 (Tonelli's theorem for series and integrals).** Suppose  $f_n$  are [integrable](#) functions such that

$$\sum_{n=1}^{\infty} \int |f_n| < \infty,$$

then we have

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

*Proof.* Take  $G(x)$  to be

$$G(x) := \sum_{n=1}^{\infty} |f_n(x)|,$$

then we see

$$G(x) \geq |F_N(x)|$$

where

$$F_N(x) := \sum_{n=1}^N f_n(x).$$

By [Corollary 2.4](#), we have

$$\int G(x) = \sum_{n=1}^{\infty} \int |f_n(x)| < \infty.$$

Lastly, from [Theorem 2.4](#), the result follows.  $\blacksquare$

Check  
C-valued  
case

**Remark.** Compare to [Corollary 2.4](#), we see that we further generalize the result!

## Lecture 14: $L^1$ Space

7 Feb. 11:00

### 2.4 $L^1$ Space

**Definition 2.11 (Seminorm).** Let  $V$  be a vector space over field  $\mathbb{R}$  or  $\mathbb{C}$ . A *seminorm* on  $V$  is

$$\|\cdot\| : V \rightarrow [0, \infty)$$

such that

- $\|cv\| = |c| \|v\|$  for every  $v \in V$  and every scalar  $c$ .
- $\|v + w\| \leq \|v\| + \|w\|$  for every  $v, w \in V$ .

**Definition 2.12 (Norm).** A *norm* is a [seminorm](#) with

- $\|v\| = 0 \iff v = 0$ .

**Lemma 2.6.** A [normed](#) vector space is a metric space with metric

$$\rho(v, w) = \|v - w\|.$$

*Proof.*



DIY

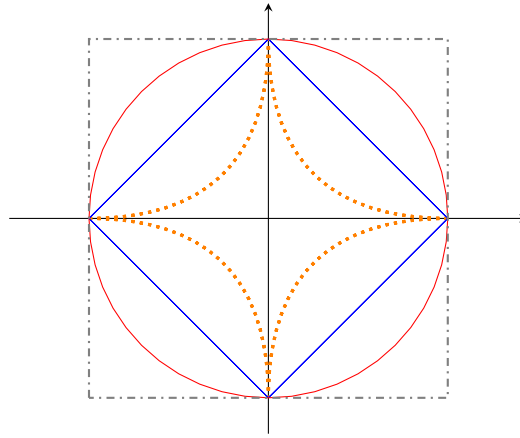
**Example.**  $V = \mathbb{R}^d$  with

$$\|x\|_p = \begin{cases} \left( \sum_{i=1}^d |x_i|^p \right)^{1/p}, & \text{if } p \in [0, \infty); \\ \max_{1 \leq i \leq d} |x_i|, & \text{if } p = \infty \end{cases}$$

is a [normed](#) vector space. The unit ball

$$\{x \in \mathbb{R}^d \mid \|x\|_p \leq 1\}$$

for different  $p$  has the following figures.



**Remark.** All  $\|\cdot\|_p$  norms induce the same topology. i.e., if  $U$  is open in  $p$ -norm, it is open in  $p'$ -norm as well.

**Note.** Recall that we say  $f$  is **integrable** means

$$\int |f| < \infty,$$

and if  $f = g$  **a.e.**, then

$$\int f = \int g$$

**Definition 2.13 ( $L^1$  Space).** Given  $(X, \mathcal{A}, \mu)$ ,

$$f \in L^1(X, \mathcal{A}, \mu) (= L^1(X, \mu) = L^1(X) = L^1(\mu))$$

means that  $f$  is an **integrable** function on  $X$ .

**Lemma 2.7.**  $L^1(X, \mathcal{A}, \mu)$  is a vector space with **seminorm**

$$\|f\|_1 = \int |f|.$$

**Definition 2.14 ( $L^1$  Space with equivalence class).** Define  $f \sim g$  if  $f = g$  **a.e.**

$$L^1(X, \mathcal{A}, \mu) / \sim = L^1(X, \mathcal{A}, \mu),$$

i.e., we simply denote the collection of equivalence classes by itself.<sup>a</sup>

<sup>a</sup>By some abusing of notation of  $L^1$ .

**Remark.** We have

- With **Definition 2.14**,  $L^1(X, \mathcal{A}, \mu)$  is a normed vector space.



- We say that the  $L^1$ -metric  $\rho(f, g)$  is simply

$$\rho(f, g) = \int |f - g|.$$

#### 2.4.1 Dense Subsets of $L^1$

**Note.** Recall the definition of a *dense set*<sup>23</sup>.

**Definition 2.15 (Step function).** A *step function* on  $\mathbb{R}$  is

$$\psi = \sum_{i=1}^N c_i \mathbb{1}_{I_i},$$

where  $I_i$  is an interval.

**Notation.** We denote the collection of continuous functions with compact support by  $C_c(\mathbb{R})$ .

**Theorem 2.5.** We have the following.

1. {integrable simple functions} is dense in  $L^1(X, \mathcal{A}, \mu)$  (with respect to  $L^1$ -metric).
2.  $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$ , where  $\mu$  is a Lebesgue-Stieltjes-measure. Then {integrable simple functions} is dense in  $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu)$ .
3.  $C_c(\mathbb{R})$  is dense in  $L^1(\mathbb{R}, \mathcal{L}, m)$ .

*Proof.* We prove this one by one.

1. Since there exists simple functions  $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|$ , where  $\phi_n \rightarrow f$  pointwise. Then by Theorem 2.4, we have

$$\lim_{n \rightarrow \infty} \int \underbrace{|f_n - f|}_{\leq |\phi_n| + |f| \leq 2|f|} = 0$$

where  $2|f|$  is in  $L^1$ .

2. Let  $\mathbb{1}_E$  approximate by  $\sum_{i=1}^{\infty} c_i \mathbb{1}_{I_i}$ . From Theorem 1.6 for Lebesgue-Stieltjes-measure,

$$\forall \epsilon' > 0 \exists I = \bigcup_{i=1}^N I_i \text{ such that } \mu(E \triangle I) \leq \epsilon'.$$

3. To approximate  $\mathbb{1}_{(a,b)}$ , we simply consider function  $g \in C_c(\mathbb{R})$  such that

$$\int |\mathbb{1}_{(a,b)} - g| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

■

<sup>23</sup>[https://en.wikipedia.org/wiki/Dense\\_set](https://en.wikipedia.org/wiki/Dense_set)

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## 2.5 Riemann Integrability

We are now working in  $(\mathbb{R}, \mathcal{L}, m)$ .

**Theorem 2.6.** Let  $f: [a, b] \rightarrow \mathbb{R}$  be a bounded function.

1. If  $f$  is Riemann integrable, then  $f$  is Lebesgue measurable. Thus,  $f$  is Lebesgue integrable and

$$\int_a^b f(x) \, dx = \int_{[a,b]} f \, dm.$$

2.  $f$  is Riemann integrable if and only if  $f$  is continuous Lebesgue a.e.

# Appendix

## A Additional Proofs

### A.1 Measure

This section gives all additional proofs in [Section 1](#).

**Theorem A.1 (Theorem 1.2 3.).** Under the setup of [Theorem 1.2](#),  $(X, \mathcal{A}, \mu)$  is a [complete measure space](#).

*Proof.* We see this in two parts.

1. **Claim:** If  $A \subset X$  satisfies  $\mu^*(A) = 0$ , then  $A$  is [Carathéodory measurable](#) with respect to  $\mu^*$ .

*Proof.* If  $A \subset X$  and  $\mu^*(A) = 0$ , where  $\mu^*$  is an outer measure on  $X$ , we'll show that  $A$  is [Carathéodory measurable](#) with respect to  $\mu^*$ .

Equivalently, we want to show that for any  $E \subset X$ ,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Firstly, noting that  $(E \cap A) \subset A$ , and by [monotonicity](#) of  $\mu^*$ , we see that

$$\mu^*(E \cap A) \leq \mu^*(A) = 0,$$

and since  $\mu^* \geq 0$ , hence  $\mu^*(E \cap A) = 0$ . Now, we only need to show that

$$\mu^*(E) = \mu^*(E \setminus A).$$

Since  $E \setminus A = E \cap A^c$ , and hence we have  $E \cap A^c \subset E$ , so

$$\mu^*(E) \geq \mu^*(E \setminus A).$$

To show another direction, we note that

$$\mu^*(E) \leq \mu^*(E \cup A) = \mu^*((E \setminus A) \cup A) \leq \mu^*(E \setminus A),$$

hence we conclude that  $A$  is [Carathéodory measurable](#) with respect to  $\mu^*$  if  $\mu^*(A) = 0$ . ■

2. **Claim:** If  $A$  is [μ-subnull](#), then  $A \in \mathcal{A}$ .

*Proof.* Let  $\mathcal{A}$  denotes the [Carathéodory σ-algebra](#), and  $\mu := \mu^*|_{\mathcal{A}}$ . We want to show if  $A$  is [μ-subnull](#), then  $A \in \mathcal{A}$ .

Firstly, if  $A$  is [μ-subnull](#), then there exists a  $B \in \mathcal{A}$  such that  $A \subset B$  and  $\mu(B) = 0$ . But since from the [monotonicity](#) of  $\mu^*$ , we further have

$$0 = \mu(B) = \mu^*(B) \geq \mu^*(A),$$

hence  $\mu^*(A) = 0$ .

From the first claim, we immediately see that  $A$  is [Carathéodory measurable](#) with respect to  $\mu^*$ , which implies  $A$  is in [Carathéodory σ-algebra](#), hence  $A \in \mathcal{A}$ . ■

We see that the second claim directly proves that  $(X, \mathcal{A}, \mu)$  is a [complete measure space](#). ■

**Lemma A.1.** The function  $F$  defined in [this example](#) is a [distribution function](#)

*Proof.* We define

$$F_n(x) = \begin{cases} 1, & \text{if } x \geq r_n; \\ 0, & \text{if } x < r_n \end{cases}$$

where  $\{r_1, r_2, \dots\} = \mathbb{Q}$ , and

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} = \sum_{n; r_n \leq x} \frac{1}{2^n}$$

is both increasing and right-continuous.

- Increasing. Consider  $x < y$ . We see that

$$F(x) = \sum_{n; r_n \leq x} \frac{1}{2^n} \leq \sum_{n; r_n \leq y} \frac{1}{2^n} = F(y)$$

clearly.<sup>[24](#)</sup>

- Right-continuous. We want to show  $F(x^+) = F(x)$ . Let  $x^+(\epsilon) := x + \epsilon$  with  $\epsilon > 0$ , we'll show that

$$\lim_{\epsilon \rightarrow 0} F(x^+(\epsilon)) = \lim_{\epsilon \rightarrow 0} F(x + \epsilon) = F(x).$$

Firstly, we have

$$F(x^+(\epsilon)) - F(x) = \sum_{n; r_n \leq x+\epsilon} \frac{1}{2^n} - \sum_{n; r_n \leq x} \frac{1}{2^n} = \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n},$$

and we want to show

$$\lim_{\epsilon \rightarrow 0} F(x^+(\epsilon)) - F(x) = \lim_{\epsilon \rightarrow 0} \sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} = 0.$$

Before we show how we choose  $\epsilon$ ,<sup>[26](#)</sup> we see that

$$\sum_{n=k}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

<sup>24</sup>This is trivial since we're always going to sum more strictly positive terms in  $F(y)$  than in  $F(x)$ .

<sup>25</sup>The strict is crucial to show the result, since if  $x = r_k$  for some fixed  $k$ , then we can't make the summation arbitrarily small.

<sup>26</sup>To be precise, how  $\epsilon$  depends on  $r_n$ .

Now, we observe that

$$\sum_{n; x < r_n \leq x+\epsilon} \frac{1}{2^n} \leq \sum_{n=\arg \min_k x < r_k \leq x+\epsilon}^{\infty} \frac{1}{2^n} = 2^{1-k}.$$

With this observation, it should be fairly easy to see that we can choose  $\epsilon$  based on how small we want to make  $2^{1-k}$  be,<sup>27</sup> and we indeed see that

$$\lim_{k \rightarrow \infty} 2^{1-k} = 0,$$

which implies that  $F$  is right-continuous by squeeze theorem. ■

**Lemma A.2.** The function  $F$  defined in [this example](#) satisfies

- $\mu_F(\{r_i\}) > 0$  for all  $r_i \in \mathbb{Q}$ .
- $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$

given in [this example](#).

*Proof.* We prove them one by one. And notice that  $F$  is indeed a distribution function as we proved in [Lemma A.1](#).

1. To show  $\mu_F(\{r\}) > 0$  for every  $r \in \mathbb{Q}$ , we first note that  $\{r\} = \bigcap_{a-1 \leq x < r} (x, r]$ .

Then, we see that

$$\mu_F(\{r\}) = \mu_F \left( \bigcap_{a-1 \leq x < a} (x, r] \right),$$

where each  $(x, r] \in \mathcal{A}$  and  $(x, r] \supset (y, r]$  whenever  $r-1 \leq x \leq y < r$ . Notice that we implicitly assign the order of the index by the order of  $x$ . Then, we see that  $\mu_F(r-1, r] < \infty$ .<sup>28</sup> Then, from continuity from above, we see that

$$\mu_F(\{r\}) = \lim_{i \rightarrow \infty} \mu_F((x_i, r]),$$

where we again implicitly assign an order to  $x$  as the usual order on  $\mathbb{R}$  by given index  $i$ . It's then clear that as  $i \rightarrow \infty$ ,  $x_i \rightarrow r$ . From the definition of  $F$ , we see that

$$F((x_i, r]) = F(r) - F(x_i) = \sum_{n; r_n \leq r} \frac{1}{2^n} - \sum_{n; r_n \leq x_i} \frac{1}{2^n} = \sum_{n; x_i < r_n \leq r} \frac{1}{2^n}.$$

It's then clear that since  $r \in \mathbb{Q}$ , there exists an  $i'$  such that  $r_{i'} = r$ . Then, we immediately see that no matter how close  $x_i \rightarrow r$ , this sum is at least

$$\frac{1}{2^{i'}}$$

for a fixed  $i'$ . Hence, we conclude that  $\mu_F(\{r\}) > 0$  for every  $r \in \mathbb{Q}$ .

<sup>27</sup>We're referring to  $\epsilon - \delta$  proof approach.

<sup>28</sup>This will be  $\mu(A_1)$  in the condition of continuity from above. Furthermore, since  $\mathbb{Q}$  is countable, hence  $F(x) < \infty$  is promised.

---

2. Now, we show  $\mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$ . Firstly, we claim that

$$\mu_F(\mathbb{Q}) = 1$$

and

$$\mu_F(\mathbb{R}) = 1$$

as well. Since  $\mu_F(\mathbb{Q}) = 1$  is clear, we note that the second one essentially follows from the fact that we can write

$$\mathbb{R} = \lim_{N \rightarrow \infty} \bigcup_{i=1}^N (a - i, a + i]$$

for any  $a \in \mathbb{R}$ , say 0. From continuity from below, we have

$$\mu_F\left(\bigcup_{i=1}^{\infty} (-i, +i]\right) = \lim_{n \rightarrow \infty} \mu_F((-n, n]) = \sum_{n; r_n \in \mathbb{Q}} \frac{1}{2^n} = 1.$$

Given the above, from countable additivity of  $\mu_F$ , we have

$$\mu_F(\mathbb{R} \setminus \mathbb{Q}) + \underbrace{\mu_F(\mathbb{Q})}_1 = \underbrace{\mu_F(\mathbb{R})}_1 \implies \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0$$

as we desired. ■

**Lemma A.3 (Cantor set has measure 0).** Let  $C$  denotes the [middle thirds Cantor set](#), then the [Lebesgue measure](#) of  $C$  is 0. i.e.,

$$m(C) = 0.$$

*Proof.* Since we're removing  $\frac{1}{3}$  of the whole interval at each  $n$ , we see that the measure of those removing parts, denoted by  $r$ , is

$$m(r) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 1.$$

Then, by [countable additivity](#) of  $m$ , we see that

$$m(C) = m([0, 1]) - m(r) = 1 - 1 = 0. \quad \blacksquare$$

## A.2 [Integration](#)

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