MATH592
Introduction to Algebraic Topology

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January 31, 2022

Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

Contents

Lecture 8: The Fundamental Group π_1

24 Jan. 10:00

Example. In category Ab free Abelian group on a set S is

$$\bigoplus_{S} \mathbb{Z}.$$

In category of fields, no such thing as **free field on S** .

0.0.1 Constructing the Free Groups F_S

Proposition 0.1. The free group defined by the universal property exists.

Proof. We'll just give a construction below. First, we see the definition.

Definition 0.1. Fix a set S, and we define a word as a finite sequence (possibly $\varnothing$) in the formal symbols

$$\left\{ s, s^{-1} \mid s \in S \right\}.$$

Then we see that elements in F_S are equivalence classes of words with the equivalence relation being

- delete ss^{-1} or $s^{-1}s$. i.e.,

$$vs^{-1}sw \sim vw$$
$$vss^{-1}w \sim vw$$

  for every word $v, w, s \in S$,

with the group operation being concatenation. ■

Example. Given words ab^{-1}, bba, their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

Exercise. There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that F_S satisfies the universal property with respect to the map

$$S \to F_S, \quad s \mapsto s.$$

1 The Fundamental Group

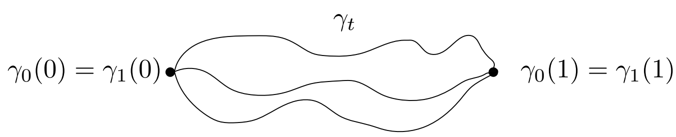
We start with the definition.

Definition 1.1 (Path). A *path* in a space X is a continuous map

$$\gamma \colon I \to X$$

where $I = [0, 1]$.

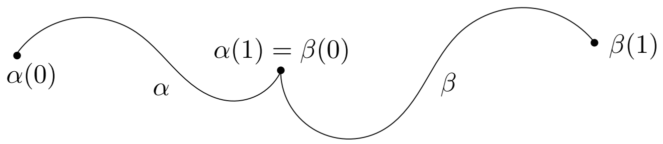
Definition 1.2 (Homotopy path). A *homotopy of paths* γ_0, γ_1 is a homotopy from γ_0 to γ_1 rel$\{0, 1\}$.



Example. Fix $x_1, x_0 \in X$, then $\exists$ homotopy of paths is an equivalence relation on paths from x_0 to x_1 (i.e., γ with $\gamma(0) = x_0, \gamma(1) = x_1$).

Definition 1.3 (Path composition). For paths α, β in X with $\alpha(1) = \beta(0)$, the *composition*[a] $\alpha \cdot \beta$ is

$$(\alpha \cdot \beta)(t) \coloneqq \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



[a] Also named *product*, *concatenation*.

Remark. By the pasting lemma, this is continuous, hence $\alpha \cdot \beta$ is actually a path from $\alpha(0)$ to $\beta(1)$.

Definition 1.4 (Reparameterization). Let $\gamma \colon I \to X$ be a path, then a *reparameterization* of γ is a path

$$\gamma' \colon I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where φ is <u>continuous</u> and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

Exercise. A path γ is homotopic rel$\{0, 1\}$ to all of its reparameterizations.

Proof. We show that γ and $\gamma \circ \phi$ are homotopic rel$\{0, 1\}$ by showing that there exists a continuous F_t such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since ϕ is continuous, so we define

$$F_t(x) = (1-t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all $x, t \in I$.

Now, we check that F_t really gives us a homotopic rel$\{0,1\}$. We have

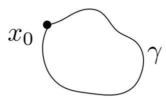
$$F_t(0) = (1-t)\gamma(0) + t\cdot\gamma\circ\phi(0) = (1-t)\gamma(0) + t\cdot\gamma(\underbrace{\phi(0)}_{0}) = \gamma(0),$$
$$F_t(1) = (1-t)\gamma(1) + t\cdot\gamma\circ\phi(1) = (1-t)\gamma(1) + t\cdot\gamma(\underbrace{\phi(1)}_{1}) = \gamma(1),$$

which shows that 0 and 1 are independent of t, hence γ and $\gamma\circ\phi$ are homotopic rel$\{0,1\}$. ∎

Exercise. Fix $x_1, x_1 \in X$. Then Homotopy of paths (relative $\{0,1\}$) is an equivalence relation on paths from x_0 to x_1.

Definition 1.5 (Fundamental Group). Let X denotes the space and let $x_0 \in X$ be the base point. The *fundamental group of X based at x_0*, denoted by $\pi_1(X, x_0)$, is a group such that

- Elements: Homotopy classes rel$\{0,1\}$ of paths $[\gamma]$ where γ is a **loop** with $\gamma(0) = \gamma(1) = x_0$[a]

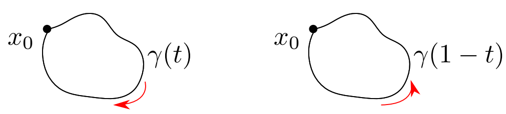


- Operation: Composition of paths.
- Identity: Constant loop γ based at x_0 such that

$$\gamma\colon I \to X, \quad t \mapsto x_0$$

- Inverses: The inverse $[\gamma]^{-1}$ of $[\gamma]$ is represented by the loop $\overline{\gamma}$ such that

$$\overline{\gamma}(t) = \gamma(1-t).$$



[a]We say γ is **based** at x_0.

Proof. We prove that

- Associativity: $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$. We break this down into

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t-2), & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t-3), & \text{if } t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t-1), & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, we define $\phi \colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \frac{1}{2}\right], & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ t + \frac{1}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right], & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{t+1}{2} \in \left[\frac{3}{4}, 1\right], & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$, which implies that the associativity holds.

- Identity: We want to show that $[\gamma \cdot c] = [\gamma]$. Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, consider $\phi \colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t, & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ 1, & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$.

- Inverses: We want to show that $\gamma \cdot \overline{\gamma} \simeq c$, where $\overline{\gamma}(t) = \gamma(1-t)$. Firstly, we have

$$(\gamma \cdot \overline{\gamma})(t) = \begin{cases} \gamma(2t), & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2t), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider F_t given by

$$F_t(x) = \begin{cases} \gamma(2xt), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2xt), & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

If $t = 0$, we have

$$F_0(x) = \begin{cases} \gamma(0), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0$$

for all $x \in I$, namely $F_0 = c$, while when $t = 1$, we have

$$F_1(x) = \begin{cases} \gamma(2x), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2x), & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \overline{\gamma})(x),$$

and we see that F_t is continuous since at $x = \frac{1}{2}$, we have

$$\gamma(2x) = \gamma(1) = \overline{\gamma}(0) = \overline{\gamma}(1-2x),$$

hence we see that F_t is the homotopy between $\gamma \cdot \overline{\gamma}$ and c.

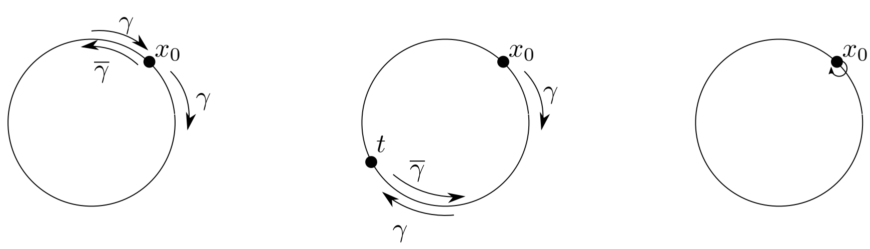


Figure 1: Illustration of F_t. Intuitively, the path $\gamma \cdot \overline{\gamma}$ is $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\overline{\gamma}} x_0$. But now, F_t is $x_0 \xrightarrow{\gamma} t \xrightarrow{\overline{\gamma}} x_0$. We can think of this homotopy is *pulling back* the turning point along the original path.

■

Theorem 1.1. If X is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Remark. We often write $\pi_1(X)$ up to isomorphism.

Proof. To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if $[h \cdot \gamma \cdot \overline{h}] = [h \cdot \gamma' \cdot \overline{h}]$, then since we know that $h^{-1} = \overline{h}$, hence in the fundamental group $\pi_1(X, x_0)$, we see that

  $$\overline{h} \cdot h \cdot \gamma \cdot \overline{h} \cdot h = \overline{h} \cdot h \cdot \gamma' \cdot \overline{h} \cdot h. \implies \gamma = \gamma'$$

  as we desired.
- onto. We see that for every $\alpha \in \pi_1(X, x_0)$, there exists a $\gamma \in \pi_1(X, x_0)$ such that

  $$\gamma = \overline{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)^1$$

  since $h \cdot \gamma \cdot \overline{h} = \alpha$.

We then see that the fundamental group of X does not depend on the choice of basepoint, only on the choice of the path component of the basepoint. If X is path-connected, it now makes sense to refer to *the* fundamental group of X and write $\pi_1(X)$ for the abstract group (up to isomorphism).

■

Exercise. Composition of paths is well-defined on homotopy classes rel$\{0, 1\}$.

Exercise. If X is a contractible space, then X is path connected and $\pi_1(X)$ is trivial.

Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

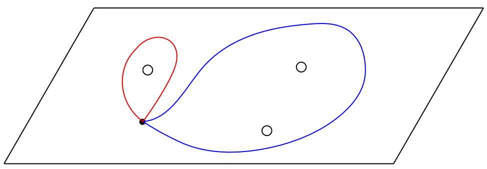


Figure 2: Fundamental Group is basically a *hole detector*!

[1] Notice that this is indeed the case, one can verify this by the fact that $h\colon x_0 \to x_1$ and $\overline{h}\colon x_1 \to x_0$.

Theorem 1.2. Given (X, x_0) and (Y, y_9), then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\begin{bmatrix} r\colon I \to X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{bmatrix} \mapsto (r_X, r_Y),$$

where γ is continuous $\iff$ f_X, f_Y are continuous.

Proof. Let $Z \to X \times Y$ with $z \mapsto (f_X(z), f_Y(z))$. Then we have

$$\text{continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply to

- $I \to X \times Y$.
- $I \times I \to X \times Y$.

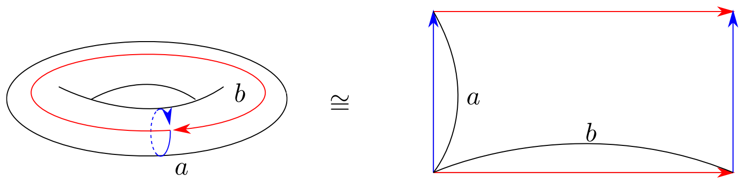
■

Corollary 1.1. Torus $T \cong S^1 \times S^1$. Additionally,

$$\pi_1\left((S^1)^k\right) \cong \mathbb{Z}^k.$$

Proof. Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

Example. We now see some examples.

1. $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2. $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$ since

$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R}.$$

Theorem 1.3. Let π_1 is a functor

$$\begin{aligned}\pi_1\colon \underline{\mathrm{Top}_*} &\to \underline{\mathrm{Gp}}\\ (X, x_0) &\mapsto \pi_1(X, x_0).\end{aligned}$$

A map $f\colon X \to Y$ taking base point x_0 to y_0 induces a map

$$\begin{aligned}f_*\colon \pi_1(X, x_0) &\to \pi_1(Y, y_0)\\ [\gamma] &\mapsto [f\circ\gamma]\end{aligned}$$

i.e.,

$$[f\colon X \to Y] \mapsto [f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))]\,.$$

Notation. We usually write f_* if it's a covariant functor, while writing f^* if it's an contravariant functor.

Proof. We need to check

- well-defined on path homotopy classes.
- f_* is a group homomorphism.

$$f_*(\alpha\cdot\beta) = f_*(\alpha)\cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s\in\left[0,\dfrac{1}{2}\right]\\ f(\beta(1-2s)), & \text{if } s\in\left[\dfrac{1}{2},1\right]\end{cases}.$$

- $\left(\mathrm{id}_{(X,x_0)}\right)_* = \mathrm{id}_{\pi_1(X,x_0)}$
- $(f_*\circ g_*) = (f\circ g)_*$

$$(f\circ g)_*[\gamma] = [f\circ g\circ\gamma] = [f\circ(g\circ\gamma)] \implies f_*(\gamma_*(\gamma)).$$

DIY

■

Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

The goal is to compute $\pi_1(X)$ where $X = A\cup B$ using the data

$$\pi_1(A), \pi_1(B), \pi_1(A\cap B).$$

We first introduce a definition.

Definition 1.6 (Free product with amalgamation). Given some collections of groups $\{G_\alpha\}_\alpha$, the *free product*, denoted by $\underset{\alpha}{*} G_\alpha$ is a group such that

- Elements: Words in $\{g \colon g \in G_\alpha \text{ for any } \alpha\}$ modulo by the equivalence relation generated by
$$wg_ig_jv \sim w(g_ig_j)v$$
when both $g_i, g_j \in G_\alpha$. Also, for the identity element $\mathrm{id} = e_\alpha \in G_\alpha$ for any α such that
$$we_\alpha v \sim wv.$$
- Operation: Concatenation of words.

Furthermore, if two groups G_α and G_β have a common subgroup $S_{\{\alpha,\beta\}}$[a], given two inclusion maps[b] $i_{\alpha\beta} \colon S_{\{\alpha,\beta\}} \to G_\alpha$ and $i_{\beta\alpha} \colon S_{\{\alpha,\beta\}} \to G_\beta$, the *free product with amalgamation* ${}_{\alpha}{*}_S G_\alpha$ is defined as $\underset{\alpha}{*} G_\alpha$ modulo the normal subgroup generated by

$$\left\{ i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \mid s_{\alpha\beta} \in S_{\{\alpha,\beta\}} \right\},$$

Namely[c],

$${}_{\alpha}{*}_S\, G_\alpha = \underset{\alpha}{*} G_\alpha \Big/ \left\langle i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \right\rangle$$

and satisfies the universal property

$$\begin{tikzcd}
S \arrow[r, "i_{\alpha\beta}"] \arrow[d, "i_{\beta\alpha}"'] & G_\alpha \arrow[d] \arrow[ddr, bend left] & \\
G_\beta \arrow[r] \arrow[drr, bend right] & G_\alpha *_S G_\beta \arrow[dr, dashed, "\exists!"] & \\
 & & X
\end{tikzcd}$$

[a] In general, we don't need $S_{\{\alpha,\beta\}}$ to be a subgroup.
[b] We don't actually need $i_{\alpha\beta}, i_{\beta\alpha}$ to be inclusive as well.
[c] Namely, $i_\alpha(s)$ and $i_\beta(s)$ will be identified in the quotient.

Remark. We see that

- We can then write out words such as $g_1g_2sg_3$ for $s \in S$, and view s as an element of G_α or G_β. In fact, we can do this construction even when i_α and i_β are not injective, though this means we are not working with a subgroup.

- Aside, in Top, the same universal property defines union

$$\begin{tikzcd} A \cap B \arrow[r, "i_\alpha"] \arrow[d, "i_\beta"'] & A \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right] & A \cup B \arrow[dr, dashed, "\exists !"] & \\ & & X \end{tikzcd}$$

for A, B are open subsets and the inclusion of intersection.

Theorem 1.4 (Seifert-Van Kampen Theorem). Given (X, x_0) such that $X = \bigcup\limits_{\alpha} A_\alpha$ with

- A_α are open and path-connected and $\forall \alpha\ x_0 \in A_\alpha$
- $A_\alpha \cap A_\beta$ is path-connected for all α, β.

Then there exists a surjective group homomorphism

$$\underset{\alpha}{*}\colon \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0).$$

If we additionally have $A_\alpha \cap A_\beta \cap A_\gamma$ where they are all path-connected for every α, β, γ, then

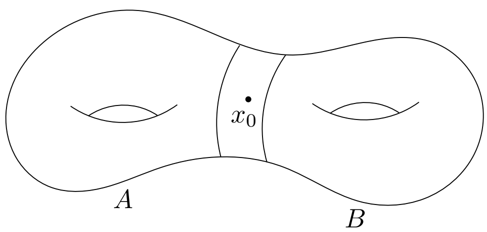
$$\pi_1(X, x_0) \cong_\alpha *_{\pi_1(A_\alpha \cap A_\beta, x_0)} \pi_1(A_\alpha, x_0)$$

associated to all maps $\pi_a(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha), \pi_1(A_\beta)$ induced by inclusions of spaces. i.e., $\pi_1(X, x_0)$ is a quotient of the free product $*_\alpha \pi_1(A_\alpha)$ where we have

$$(i_{\alpha\beta})_*\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A + \alpha)$$

which is induced by the inclusion $i_{\alpha\beta}\colon A_\alpha \cap A_\beta \to A_\alpha$. We then take the quotient by the normal subgroup generated by

$$\left\{ (i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_\alpha \cap A_\beta) \right\}.$$



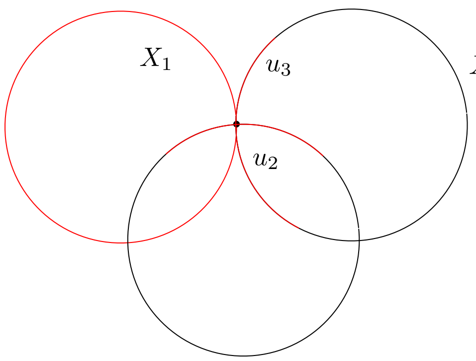
Lecture 11

31 Jan. 10:00

Example. We now see some applications. Given spaces $\{X_\alpha\}$ $\overline{w}$ basepoints x_α. Now, consider the wedge sum $\bigvee_\alpha X_\alpha$. Suppose $\forall \alpha$, x_α is a deformation retract of some neighborhood u_α of x_α. Then,

$$\pi_1 \left(\bigvee_\alpha X_\alpha, x_\alpha \right) \cong \underset{\alpha}{*} \pi_1 \left(X_\alpha, x_\alpha \right).$$

In particular, if we denote



as C_n, then $\pi_1(C_n) \cong F_n$. Then we apply Theorem 1.4 to $A_\alpha = X_\alpha \cup_\beta u_\beta$

1.1 Group Presentation

In order to go further, we introduce the concept of *group presentation.*

Definition 1.7 (Group presentation). A *presentation* $\langle S \mid R \rangle$ of a group G is

- S: set of generators
- R: set of relaters (words in a generator and inverses)

such that

$$G \cong {}^{F_S} \Big/ {}_{\langle R \rangle},$$

where $\langle R \rangle$ is a subgroup normally generated.

Notice that $\langle S \mid R \rangle$ is finite if S, R are, and G is *finitely presented* if there exists a finite presentation.

Example. We see that

1. $F_2 = \langle a, b \mid \; \rangle$

2. $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle$
3. $\mathbb{Z} / 3\mathbb{Z} = \left\langle a \mid a^3 \right\rangle$
4. $S_3 = \left\langle a, b \mid a^2, b^2, (ab)^3 \right\rangle$

Theorem 1.5. Any group G has a presentation.

Proof. We first choose a generating set S for G. From the universal property of free group, we see that there exists a surjective map $\varphi \colon F_S \to G, s \mapsto s$. Now, let R be the generating set for $\ker(\varphi)$, $G = \langle S \mid R \rangle$. ■

Remark. The advantages are that given $\langle S \mid R \rangle$, it's now easy to define a homomorphism $\psi \colon G \to H$ given a map $\psi \colon S \to H$, ψ extends to a group homomorphism $G \to H$ if and only if ψ vanishes on R.

Example. Given $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle$, let

$$\psi \colon \{a, b\} \to H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1 \in H.$$

It's sometimes easy to calculate G^{Ab}

$$G^{\text{Ab}} = \langle S \mid R, \text{commutators in } S \rangle .$$

The disadvantages are that, the computationally **very difficult**.

1.2 Presentations for π_1

Appendix

References

[HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com/books?id=BjKs86kosqgC.