

MATH635  
Riemannian Geometry

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## Abstract

This is the advanced graduate-level differential geometry course focused on Riemannian geometry taught by [Lydia Bieri](#). Topics include local and global aspects of differential geometry and the relation with the underlying topology. We'll use do Carmo's *Riemannian Geometry* [\[FC13\]](#) as our reference; while not required, but highly recommended have on.

# Contents

<b>1</b>	<b>Manifolds</b>	<b>2</b>
1.1	Differentiable Manifolds . . . . .	2
1.2	Tangent Vectors . . . . .	9

# Chapter 1

# Manifolds

## Lecture 1: A Foray to Smooth Manifolds

### 1.1 Differentiable Manifolds

5 Jan. 14:30

#### 1.1.1 Topological Manifolds

Let's start with a common definition.

**Definition 1.1.1 (Topological manifold).** A *topological manifold*  $\mathcal{M}$  of dimension  $n$  is a (topological) Hausdorff space such that each point  $p \in \mathcal{M}$  has a neighborhood  $U$  homeomorphic via  $\varphi: U \rightarrow U'$  to an open subset  $U' \subseteq \mathbb{R}^n$ .

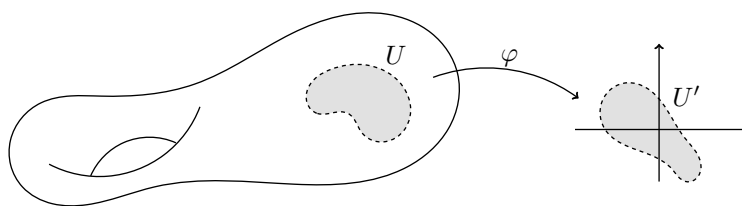
**Definition 1.1.2 (Local coordinate map).** For every  $p \in \mathcal{M}$ , the corresponding homeomorphism  $\varphi$  is called the *local coordinate map*.

**Definition 1.1.3 (Local coordinate).** The pull-back  $(x^1, \dots, x^n)$  of the *local coordinate map*  $\varphi$  from  $\mathbb{R}^n$  is called the *local coordinates* on  $U$ , given by

$$\varphi(p) = (x^1(p), \dots, x^n(p)).$$

**Definition 1.1.4 (Coordinate chart).** The pair  $(U, \varphi)$  is called a *(coordinate) chart* on  $M$ .

In other words, a *topological manifold* can be thought of as a space such that it looks like  $\mathbb{R}^n$  locally.



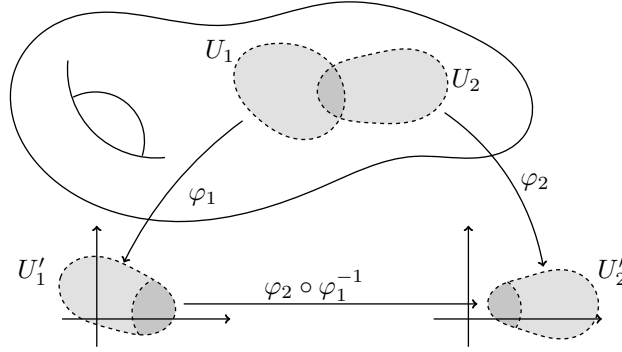
**Definition 1.1.5 (Atlas).** An *atlas*  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$  for a *manifold*  $\mathcal{M}$  is a collection of *charts* such that  $\{U_\alpha \subseteq \mathcal{M} \mid U_\alpha \text{ open}\}_\alpha$  are an open covering of  $\mathcal{M}$ , i.e.,  $\mathcal{M} = \bigcup_\alpha U_\alpha$ .

In other words, for all  $p \in \mathcal{M}$ , there exists a neighborhood  $U \subseteq \mathcal{M}$  and homeomorphism  $h: U \rightarrow U' \subseteq \mathbb{R}^n$  open.

**Definition 1.1.6 (Locally finite).** An *atlas* is said to be *locally finite* if each point  $p \in \mathcal{M}$  is contained in only a finite collection of its open sets.

Clearly, without any help of ambient space such as  $\mathbb{R}^n$ , there's no clear way to make sense of differentiability of a **manifold**. But thankfully, we now have an explicit relation to the ambient space  $\mathbb{R}^n$  via  $\varphi_\alpha$ . To formalize, let  $\mathcal{A}$  be an **atlas** for a **manifold**  $\mathcal{M}$ , and assume that  $(U_1, \varphi_1), (U_2, \varphi_2)$  are 2 elements of  $\mathcal{A}$ . Then clearly, the map  $\varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$  is a homeomorphism between 2 open sets of Euclidean spaces since both  $\varphi_1$  and  $\varphi_2$  are homeomorphism. Due to this map's importance, it has its own name.

**Definition 1.1.7 (Coordinate transition).** The map  $\varphi_2 \circ \varphi_1^{-1}$  is called the *coordinate transition* of  $\mathcal{A}$  for the pair of **charts**  $(U_1, \varphi_1), (U_2, \varphi_2)$ .



### 1.1.2 Differentiable Structures

Notice that the **coordinate transitions** are from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ ; hence differentiability makes sense now, which induces the following.

**Definition 1.1.8 (Differentiable atlas).** The **atlas**  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$  is *differentiable* if all **transitions** are differentiable.

**Remark.** Here, the differentiability depends on the content. Sometimes, we may want it to be  $C^\infty$ , and sometimes may be  $C^k$  for some finite  $k$ . On the other hand, smooth always refers to  $C^\infty$ . We'll use them interchangeably if it's clear which case we're referring to.

**Definition 1.1.9 (Equivalence atlas).** Two **atlases**  $\mathcal{U}, \mathcal{V}$  of a **manifold** are equivalent if for every  $(U, \varphi) \in \mathcal{U}, (V, \psi) \in \mathcal{V}$ ,

$$\varphi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \varphi(U \cap V)$$

and

$$\psi \circ \varphi^{-1}: \varphi(U \cap V) \rightarrow \psi(U \cap V)$$

are diffeomorphisms between subsets of Euclidean spaces.

Notably, we have the following notation.

**Notation (Smoothly compatible).** Two **charts**  $(U, \varphi)$  and  $(V, \psi)$  are *smoothly compatible* if either  $U \cap V = \emptyset$  or  $\psi \circ \varphi^{-1}$  is a diffeomorphism.

This suggests the following.

**Definition 1.1.10 (Smooth structure).** A *smooth structure* on  $\mathcal{M}$  is an equivalence class  $\mathcal{U}$  of **coordinate atlas** with the property that all **transition functions** are diffeomorphisms.

**Remark.** We can also use the *maximal differentiable atlas* to be our differentiable structure.

**Definition 1.1.11 (Smooth manifold).** A *smooth manifold* is a manifold  $\mathcal{M}$  with a smooth structure.

In this way, we can do calculus on smooth manifolds! Furthermore, it now makes sense to say that a function  $f: \mathcal{M} \rightarrow \mathbb{R}$  is differentiable (or  $C^\infty$ ) by considering differentiability of  $f \circ \varphi^{-1}$  around  $p$ .

**Notation.** The collection of smooth functions on smooth manifold  $\mathcal{M}$  is denoted by  $C^\infty(\mathcal{M}, \mathbb{R})$ , or  $C^k(\mathcal{M}, \mathbb{R})$ .

**Remark.** The class  $C^\infty(\mathcal{M}, \mathbb{R})$  consists of functions with property is well-defined.

**Proof.** Let  $\mathcal{A}$  be any given atlas from equivalence class that defines the smooth structure, and as we have shown, if  $(U, \varphi) \in \mathcal{A}$ , then  $f \circ \varphi^{-1}$  is a smooth function on  $\mathbb{R}^n$ . This requirement defines the same set of smooth functions no matter the choice of representative atlas by the nature of Definition 1.1.9 requirement that defines the equivalent manifolds.  $\circledast$

### 1.1.3 Orientation

Another essential property of a manifold is its orientability.

**Definition.** Consider an atlas  $\mathcal{A}$  for a differentiable manifold  $\mathcal{M}$ .

**Definition 1.1.12 (Oriented).**  $\mathcal{A}$  is *oriented* if all transitions have positive functional determinant.

**Definition 1.1.13 (Orientable).**  $\mathcal{M}$  is *orientable* if  $\mathcal{A}$  is an oriented atlas.

Motivated by the above definitions, we see that we can actually use an atlas to define an orientation.

**Definition 1.1.14 (Orientation).** Let  $\mathcal{M}$  be an orientable manifold. Then a oriented differentiable structure is called an *orientation* of  $\mathcal{M}$ .

If  $\mathcal{M}$  possesses an orientation, we can also say that it's *oriented*. But we don't bother to make a new definition to confuse ourselves with Definition 1.1.12.

**Remark.** Two differentiable structures obeying Definition 1.1.12 determine the same orientation if the union again satisfying Definition 1.1.12.

**Remark.** If  $\mathcal{M}$  is orientable and connected, then there exists exactly 2 distinct orientations on  $\mathcal{M}$ .

Now, we can see some examples of smooth manifolds.

**Example (Sphere).** The sphere  $S^n \subseteq \mathbb{R}^{n+1}$  given by

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Consider  $U_i^+ = \{x \in S^n \mid x_i > 0\}$ ,  $U_i^- = \{x \in S^n \mid x_i < 0\}$  for  $i = 1, \dots, n+1$ , and  $h_i^\pm: U_i^\pm \rightarrow \mathbb{R}^n$  such that

$$h_i^\pm(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1}).$$

Note that the minimum charts needed to cover  $S^n$  is 2.

**Example.** Let  $\mathcal{M} = U \subseteq \mathbb{R}^n$ , then  $\{(U, \varphi)\}$  is a smooth structure with  $\varphi = \text{id}$ .

**Example.** Open sets of  $C^\infty$ -manifolds are  $C^\infty$ -manifolds.

**Example (General linear group).**  $\text{GL}(n) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ , open.

**Example (Real projective space).**  $\mathbb{R}P^n = S^n / \sim$  where  $x \sim -x$  with  $\pi: S^n \rightarrow \mathbb{R}P^n$ ,  $x \mapsto [x]$ .

**Proof.**  $\pi$  is a homeomorphism on each  $U_i^+$  for  $i = 1, \dots, n+1$ , with

$$\{(\pi(U_i^+), \varphi_i^+ \circ \pi^{-1}), i = 1, \dots, n+1\}$$

is a  $C^\infty$ -atlas for  $\mathbb{R}P^n$ . \*

**Note.** Observe that  $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ .

## Lecture 2: Maps Between Smooth Manifolds

### 1.1.4 Smooth Maps

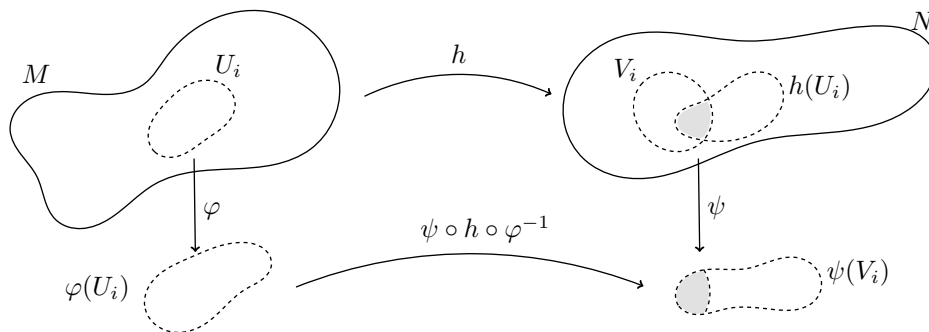
10 Jan. 14:30

We can now consider the maps between manifolds, specifically, the smooth manifolds.

**Definition 1.1.15 (Smooth function).** Let  $M, N$  be two smooth manifolds, and let  $\mathcal{U}$  be locally finite atlas from the equivalence class that gives the smooth structure on  $M$ , and let  $\mathcal{V}$  be the corresponding for  $N$ . A map  $h: M \rightarrow N$  is said to be smooth if each map in the collection

$$\{\psi \circ h \circ \varphi^{-1}: \varphi(U) \cap \psi(V) \neq \emptyset\},$$

where  $(U, \varphi) \in \mathcal{U}$ ,  $(V, \psi) \in \mathcal{V}$  is  $C^\infty$ -differentiable as a map from one Euclidean space to another.



**Remark.** Equivalence relation guarantees that Definition 1.1.15 depends only on the smooth structure of  $M, N$ , but not on the chosen representative coordinate atlas.

**Definition.** Consider two smooth manifolds  $M, N$  and a smooth homeomorphism  $h: M \rightarrow N$  with smooth inverse.

**Definition 1.1.16 (Diffeomorphic).** The two manifolds  $M, N$  are said to be diffeomorphic.

**Definition 1.1.17 (Diffeomorphism).** The map  $h$  is said to be a diffeomorphism.

Let  $M_1, M_2$  be two smooth manifolds, and let  $\varphi: M_1 \rightarrow M_2$  be a diffeomorphism. Then the following hold.

- $M_1$  is orientable if and only if  $M_2$  is orientable.
- If in addition,  $M_1$  and  $M_2$  are both connected and oriented, then  $\varphi$  induces an orientation on  $M_2$  that may or may not coincide with the initial orientation of  $M_2$ .

Check

If the induced **orientation** coincides, then we say  $\varphi$  preserves the **orientation**, otherwise  $\varphi$  reverses the **orientation**.

### 1.1.5 Grassmannian Manifold

Before proceeding, let's consider an interesting **smooth manifold**.

**Definition 1.1.18 (Grassmannian manifold).** Given  $m, n \in \mathbb{N}$ , the so-called *Grassmannian manifold*  $G(n, m)$  is the set of all  $n$ -dimensional subspaces of  $\mathbb{R}^{n+m}$ .

**Note.**  $G(1, m)$  is just  $\mathbb{R}P^m$ , and  $G(0, m)$ ,  $G(n, 0)$  are one-point sets.

As we will soon see,  $G(n, m)$  has the **smooth structure** of an  $mn$ -dimensional **manifold**.

**Intuition.** We obtain the **structure** by exhibiting an **atlas** whose **transitions** are **diffeomorphisms**.

Firstly, we give  $G(n, m)$  a suitable topology, i.e., the metric topology. Let  $\Pi \in G(n, m)$ , and let  $\mathcal{L}(\Pi, \Pi^\perp)$  denote the  $mn$ -dimensional space of linear maps from  $\Pi$  to  $\Pi^\perp$ . Define the map

$$\varphi_\Pi: \mathcal{L}(\Pi, \Pi^\perp) \rightarrow G(n, m), \quad \varphi_\Pi(\alpha) = (\mathbb{1}_\Pi \oplus \alpha)(\Pi)$$

where  $\mathbb{1}_\Pi \oplus \alpha$  is regarded as a map  $\Pi \rightarrow \Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$ .<sup>1</sup> Clearly,  $\varphi_\Pi$  is injective, and thus,  $(\mathcal{L}(\Pi, \Pi^\perp), \varphi_\Pi)$  is an  $mn$ -dimensional **chart** of  $G(n, m)$ .

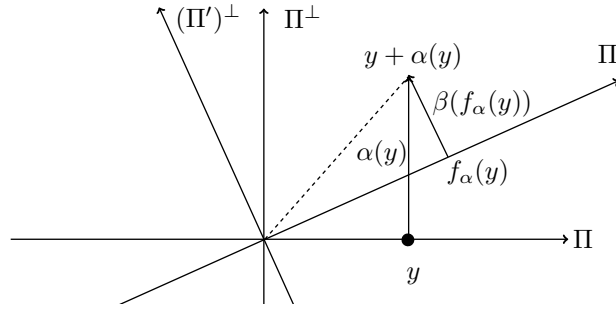
**Remark.** The images  $\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$  cover  $G(n, m)$ .

**Example.**  $\Pi = \varphi_\Pi(0) \in \varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp))$ .

We can now prove that these **charts** are mutually **compatible**. Let  $\Pi, \Pi' \in G(n, m)$ , and let  $P, P'$  be orthogonal projections from  $\mathbb{R}^{n+m}$  onto  $\Pi, \Pi'$  respectively. Firstly,

$$F = \varphi_{\Pi'}^{-1} \varphi_\Pi: \varphi_\Pi^{-1}(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp))) \rightarrow \varphi_{\Pi'}^{-1}(\varphi_\Pi(\mathcal{L}(\Pi, \Pi^\perp)))$$

is smooth.



Consider  $\alpha \in \mathcal{L}(\Pi, \Pi^\perp)$ , and  $\beta \in \mathcal{L}(\Pi', (\Pi')^\perp)$ , then for  $\alpha, \beta$ , the equality  $F(\alpha) = \beta$  means that  $\varphi_\Pi(\alpha) = \varphi_{\Pi'}(\beta)$ . Let  $f_\alpha: \Pi \rightarrow \Pi'$  be defined by

$$f_\alpha = P' \circ (\mathbb{1}_\Pi \oplus \alpha).$$

We need to check

- (a)  $f_\alpha$  is invertible, and
- (b)  $\forall y \in \Pi, y + \alpha(y) = f_\alpha(y) + \beta(f_\alpha(y))$ .

<sup>1</sup>In other words,  $\varphi_\Pi(\alpha)$  is the graph of  $\alpha$  in  $\Pi \oplus \Pi^\perp = \mathbb{R}^{n+m}$ .



**Note.** The condition that  $\det f_\alpha \neq 0$  gives an exact description of the subset

$$\varphi_{\Pi^{-1}}(\varphi_{\Pi'}(\mathcal{L}(\Pi', (\Pi')^\perp)))$$

of  $\mathcal{L}(\Pi, \Pi^\perp)$ , which is therefore open.

For  $\beta$ , it is  $(\mathbb{1}_{\Pi'} \oplus \beta) \circ f_\alpha = \mathbb{1}_\Pi \oplus \alpha$ , and hence

$$\beta = F(\alpha) = (\mathbb{1}_\Pi \oplus \alpha) \circ f_\alpha^{-1} - \mathbb{1}_{\Pi'}.$$

It follows by the construction that the image of  $\beta$  is contained in  $(\Pi')^\perp$ .

**Remark.** We obtain an infinite atlas for  $G(n, m)$  with charts labeled by  $\Pi \in G(n, m)$ . But it suffices to consider only  $\binom{n+m}{n}$  charts corresponding to subspaces  $\Pi$  spanned with  $n$  coordinate axes.

### 1.1.6 Manifolds with Boundary

We first introduce two notions.

**Definition 1.1.19** (Closed manifold). A manifold is *closed* if it is compact and without boundary.

**Definition 1.1.20** (Open manifold). A manifold is *open* if it has only non-compact components without boundary.

**Lemma 1.1.1.** If  $M$  can be covered by two coordinate neighborhoods  $V_1, V_2$  such that  $V_1 \cap V_2$  is connected, then  $M$  is *orientable*.

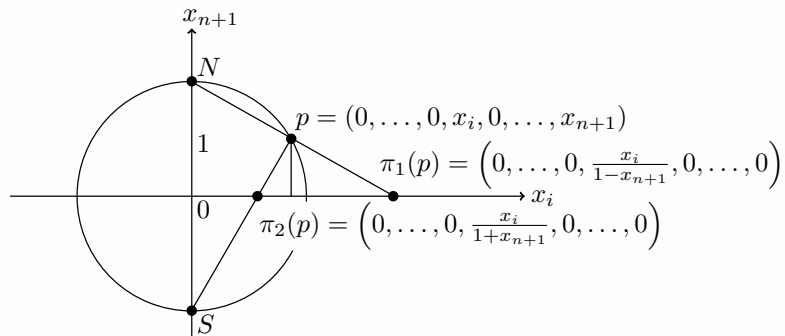
**Proof.** The determinant of the differential of the coordinate change  $\neq 0$ , so it does not change sign in  $V_1 \cap V_2$ . If it's negative at a single point, it's enough to change the sign of one of the coordinates to make it positive at that point, hence on  $V_1 \cap V_2$ . ■

**Example.** Let  $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$  is *orientable*.

**Proof.** Let  $N = (0, \dots, 0, 1)$  and  $S = (0, \dots, 0, -1)$ , consider given  $p = (0, \dots, 0, x_i, 0, \dots, x_{n+1})$ , then  $\pi_1: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  given by

$$\pi_1(p) = \left(0, \dots, 0, \frac{x_i}{1 - x_{n+1}}, 0, \dots, 0\right)$$

to be the stereographic projection from the north pole  $N$ .



More generally, it takes  $p(x_1, \dots, x_{n+1}) \in S^n - \{N\}$  into the intersection at the hyperplane

$x_{n+1} = 0$  with the line passing through  $p$  and  $N$ . In this way, we have

$$\pi_1(x_1, \dots, x_n) = \left( \frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right),$$

hence  $\pi_1: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  is differentiable, and is injective. Similarly,  $\pi_2: S^n \setminus \{S\} \rightarrow \mathbb{R}^n$  for  $S$  can also be defined and everything holds similarly. We see that these two parametrizations  $(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})$  cover  $S^n$ . The change of coordinate is given by

$$y_j = \frac{x_j}{1 - x_{n+1}} \leftrightarrow y'_j = \frac{x_j}{1 + x_{n+1}}, \quad (y_1, \dots, y_n) \in \mathbb{R}^n, \quad j = 1, \dots, n,$$

where

$$y'_j = \frac{y_j}{\sum_{i=1}^n y_i^2}.$$

This implies that  $\{(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})\}$  is a **differentiable structure** for  $S^n$ . Now, consider  $\pi_1^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^n) = S^n \setminus \{N \cup S\}$ , which is connected, and hence  $S^n$  is **orientable**, and the above **structure** gives an **orientation** of  $S^n$ .  $\otimes$

## Lecture 3: Complex Manifolds, Tangent Spaces and Bundles

Let's look at two more examples about **orientation**.

12 Jan. 14:30

**Example.** Let  $A: S^n \rightarrow S^n$  be the antipodal map given by  $A(p) = -p$  for  $p \in \mathbb{R}^{n+1}$ . It's easy to see that  $A$  is differentiable with  $A^2 = \mathbb{1}$ . Furthermore,  $A$  is **diffeomorphism** of  $S^n \subseteq \mathbb{R}^{n+1}$ . We see that

- if  $n$  is even,  $A$  reverses the **orientation**;
- if  $n$  is odd,  $A$  preserves the **orientation**.

**Example.**  $G(k, n)$  is **orientable** if and only if  $n$  is even or  $n = 1$ .

### 1.1.7 Complex Manifolds

Here we introduce the notion of **complex manifold**.

**Definition 1.1.21 (Complex manifold).** A *complex manifold*  $\mathcal{M}$  of complex dimension  $d$  ( $\dim_{\mathbb{C}} \mathcal{M} = d$ ) is a **differentiable manifold** of (real) dimension  $2d$  ( $\dim_{\mathbb{R}} \mathcal{M} = 2d$ ) whose **charts** take values in open subsets of  $\mathbb{C}^d$  with holomorphic **chart transitions**.

**As previously seen.** The **chart transitions**  $z_\beta \circ z_\alpha^{-1}: z_\alpha(U_\alpha \cap U_\beta) \rightarrow z_\beta(U_\alpha \cap U_\beta)$  is holomorphic if  $\partial z_\beta^j / \partial \bar{z}_\alpha^k = 0$  for all  $j, k$  where

$$\frac{\partial}{\partial \bar{z}^k} = \frac{1}{2} \left( \frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k} \right).$$

**Remark.** **Complex Grassmannians**  $G_{\mathbb{C}}(k, n)$  are all **orientable**. More generally, **complex manifolds** are always **orientable** because holomorphic maps always have positive functional determinant.

### 1.1.8 Partition of Unity

We state, without proof, of an important lemma about the **partition of unity**.

**Definition 1.1.22 (Partition of unity).** Let  $\mathcal{M}$  be a **differentiable manifold**, and let  $(U_\alpha)_{\alpha \in \mathcal{A}}$  be an open covering of  $\mathcal{M}$ . Then a *partition of unity* is a **locally finite** refinement  $(V_\beta)_{\beta \in \mathcal{B}}$  of  $(U_\alpha)$  and

$C^\infty$ -functions  $\varphi_\beta: \mathcal{M} \rightarrow \mathbb{R}$  with

- (a)  $\text{supp}(\varphi_\beta) \subseteq V_\beta$  for all  $\beta \in \mathcal{B}$ ;
- (b)  $0 \leq \varphi_\beta(x) \leq 1$  for all  $x \in \mathcal{M}$ ,  $\beta \in \mathcal{B}$ ;
- (c)  $\sum_{\beta \in \mathcal{B}} \varphi_\beta = 1$  for all  $x \in \mathcal{M}$ .<sup>a</sup>

<sup>a</sup>There are only finitely many non-vanishing summands of each point, since only finitely many  $\varphi_\beta$  are non-zero of any given point as the covering  $(V_\beta)$  is [locally finite](#).

**Lemma 1.1.2 (Partition of unity).** Let  $\mathcal{M}$  be a [differentiable manifold](#), and let  $(U_\alpha)_{\alpha \in \mathcal{A}}$  be an open covering of  $\mathcal{M}$ . Then there exists a [partition of unity](#) subordinate to  $(U_\alpha)$ ,

## 1.2 Tangent Vectors

### 1.2.1 Tangent Vectors in Euclidean Spaces

To discuss the concept of calculus between [manifolds](#) formally, we start with our discussion in Euclidean spaces, where we naturally have the coordinates for every point.

**Definition.** Let  $\mathcal{M}$  be a Euclidean [manifold](#) of dimension  $d$ ,  $x = (x^1, \dots, x^d)$  be Euclidean coordinates of  $\mathbb{R}^d$ , and  $x_0 \in \Omega \subseteq \mathbb{R}^d$  where  $\Omega$  is open.

**Definition 1.2.1 (Tangent space of Euclidean space).** The *tangent space*  $T_{x_0}\Omega$  of  $\Omega$  at  $x_0$  is the vector space  $\{x_0\} \times E^a$  spanned by the basis  $(\partial/\partial x^1, \dots, \partial/\partial x^d)$ .

<sup>a</sup> $E$  is a  $d$ -dimensional Euclidean space.

**Definition 1.2.2 (Tangent vector of Euclidean space).** The elements in the [tangent space of Euclidean spaces](#) is called *tangent vectors*.

Before proceeding, we introduce a shorthand notation.

**Notation (Einstein notation).** The *Einstein notation* abbreviates the summation  $\sum_i v^i x_i$  as  $v^i x_i$ , where we implicitly sum over the upper and lower index.

**Definition 1.2.3 (Differential of Euclidean space).** If  $\Omega \subseteq \mathbb{R}^d$ ,  $\Omega' \subseteq \mathbb{R}^d$  are open, and  $f: \Omega \rightarrow \Omega'$  is differentiable, then the *differential*  $df(x_0)$  for  $x_0 \in \Omega$  is the induced linear map between [tangent spaces](#)

$$df(x_0): T_{x_0}\Omega \rightarrow T_{f(x_0)}\Omega', \quad v = v^i \frac{\partial}{\partial x^i} \mapsto v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}.$$

**Definition 1.2.4 (Tangent bundle of Euclidean space).** The *tangent bundle* is defined as  $T\Omega := \bigsqcup_{x \in \Omega} T_x\Omega \cong \Omega \times E \cong \Omega \times \mathbb{R}^d$ , which is an open subset of  $\mathbb{R}^d \times \mathbb{R}^d$ .

**Note (Total space).**  $T\Omega$  is also called the *total space*.

**Remark.** Given a [tangent bundle](#)  $T\Omega$ , we define  $\pi$  to be the projection  $\pi: T\Omega \rightarrow \Omega$  given by  $\pi(x, v) = x$ . This makes  $T\Omega$  naturally a [differentiable manifold](#).

With the notion of [tangent bundle](#), given  $f: \Omega \rightarrow \Omega'$ , we can also define  $df: T\Omega \rightarrow T\Omega'$  as

$$\left(x, v^i \frac{\partial}{\partial x^i}\right) \mapsto \left(f(x), v^i \frac{\partial f^j}{\partial x^i}(x) \frac{\partial}{\partial f^j}\right).$$

**Notation.** We often write  $df(x)(v)$  instead of  $df(x, v)$  to coincide with the notation of [differential](#).

In particular, for  $v = v^i \partial / \partial x^i$ , we have

$$df(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R},$$

and we write  $v(f)(x)$  for  $df(x)(v)$ .

### 1.2.2 Tangent Vectors in Manifolds

We can now formally define the [tangent space](#) on a [smooth manifold](#).

**Definition.** Let  $\mathcal{M}$  be a [differentiable manifold](#) of dimension  $d$ ,  $x: U \subseteq \mathcal{M} \rightarrow \mathbb{R}^d$  be a [chart](#), and  $p \in U \subseteq \mathcal{M}$  where  $U$  is open.

**Definition 1.2.5 (Tangent space).** The *tangent space*  $T_p\mathcal{M}$  of  $\mathcal{M}$  at  $p$  is represented in the [chart](#)  $x$  by  $T_{x(p)}x(U)$ .

**Definition 1.2.6 (Tangent vector).** The elements in the [tangent space](#) is called *tangent vectors*.

To see that the above are well-defined, i.e.,  $T_p\mathcal{M}$  are independent of the choice of [charts](#), let  $x': U' \rightarrow \mathbb{R}^d$  to be another [chart](#) with  $p \in U' \subseteq \mathcal{M}$  where  $U'$  is also open. Denote  $\Omega := x(U)$ , and  $\Omega' := x'(U')$ , then the transition map

$$x' \circ x^{-1}: x(U \cap U') \rightarrow x'(U \cap U')$$

induces a vector space isomorphism

$$L := d(x' \circ x^{-1})(x(p)): T_{x(p)}\Omega \rightarrow T_{x'(p)}\Omega',$$

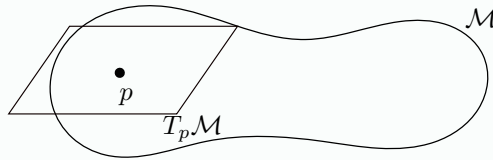
such that  $v \in T_{x(p)}\Omega$  and  $L(v) \in T_{x'(p)}\Omega'$  represent the same [tangent vector](#) in  $T_p\mathcal{M}$ .

**Remark.** A [tangent vector](#) in  $T_p\mathcal{M}$  is given by the family of the [coordinate representations](#).

Now, we want to define the similar notion of [differential of Euclidean spaces](#). Let consider a simple case first, where we let  $f: \mathcal{M} \rightarrow \mathbb{R}$  to be a differentiable function, and assume that the [tangent vector](#)  $w \in T_p\mathcal{M}$  is represented by  $v \in T_{x(p)}x(U)$ .

**Intuition.** We want to define  $df(p)$  as a linear map from  $T_p\mathcal{M} \rightarrow \mathbb{R}$ . In [chart](#)  $x$ , let  $w \in T_p\mathcal{M}$  be given as  $v = v^i \partial / \partial x^i \in T_{x(p)}x(U)$ . Say that  $df(p)(w)$  in this chart represented by

$$d(f \circ x^{-1})(x(p))(v).$$



**Remark.**  $T_p\mathcal{M}$  is a vector space of dimension  $d$  isomorphic to  $\mathbb{R}^d$ , where the isomorphism depends on choice of [chart](#).

**Intuition.** Pull functions on  $\mathcal{M}$  back by a [chart](#) to an open subset of  $\mathbb{R}^d$ , differentiate there.

In order to obtain a [tangent space](#) which does not depend on [charts](#), we need to have transformation behavior under change of [charts](#). Let  $F: \mathcal{M}^d \rightarrow \mathcal{N}^c$  be a differentiable map where  $\mathcal{M}, \mathcal{N}$  are [smooth](#)

**manifolds.** Then we want to represent  $dF$  in **local charts**  $x: U \subseteq \mathcal{M} \rightarrow \mathbb{R}^d, y: V \subseteq \mathcal{N} \rightarrow \mathbb{R}^c$  by  $d(y \circ F \circ x^{-1})$ . The **local coordinates** on  $U$  is given by  $(x^1, \dots, x^d)$ , and on  $V$  is  $(F^1, \dots, F^c)$  such that

$$F(x) = (F^1(x^1, \dots, x^d), \dots, F^c(x^1, \dots, x^d)).$$

Then,  $dF$  induces a linear map  $dF: T_p\mathcal{M} \rightarrow T_{F(x)}\mathcal{N}$  which in our **coordinate representation** is given by the matrix

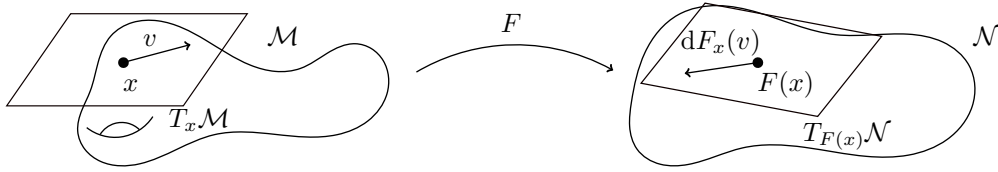
$$\left( \frac{\partial F^\alpha}{\partial x^i} \right)_{\substack{\alpha=1, \dots, c \\ i=1, \dots, d}},$$

and a change of **charts** is then just the base change at **tangent spaces**: if

$$\begin{aligned} (x^1, \dots, x^d) &\mapsto (\xi^1, \dots, \xi^d) \\ (F^1, \dots, F^c) &\mapsto (\phi^1, \dots, \phi^c) \end{aligned}$$

are **coordinate changes**, then  $dF$  represented in the new **coordinates** is given by

$$\left( \frac{\partial \phi^\beta}{\partial \xi^j} \right) = \left( \frac{\partial \phi^\beta}{\partial F^\alpha} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial x^i}{\partial \xi^j} \right).$$



# Appendix

# Bibliography

- [FC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013. ISBN: 9780817634902. URL: <https://books.google.com/books?id=ct91XCWkWEUC>.