MATH592
Introduction to Algebraic Topology

Pingbang Hu

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Abstract

This course will use Hatcher[HPM02] as the main text, but the order may differ here and there. Enjoy this fun course! In particular, I add some extra content which is not covered in lectures, things like groupoid, fibered coproduct, feel free to skip these content.

Note that I reference all definitions in the text as much as possible, but I may still miss some.

Contents

Lecture 1: Homotopies of Maps

05 Jan. 10:00

1 Foundation of Algebraic Topology

1.1 Homotopy

We start with the most important and fundamental concept, homotopy.

Definition 1.1 (Homotopy, homotopic, nullhomotopic). Let X, Y be topological spaces. Let $f, g\colon X \to Y$ continuous maps. Then a *homotopy* from f to g is a $1-$parameter family of maps that continuously deforms f to g, i.e., it's a continuous function $F\colon X \times I \to Y$, where $I = [0,1]$, such that

$$F(x,0) = f(x), \quad F(x,1) = g(x).$$

We often write $F_t(x)$ for $F(x,t)$.

If a homotopy exists between f and g, we say they are *homotopic* and write

$$f \simeq g.$$

If f is homotopic to a constant map, we call it *nullhomotopic*.

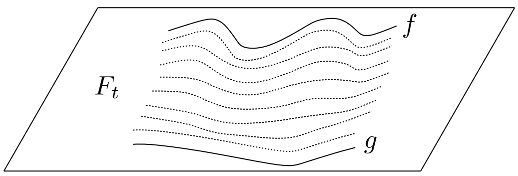


Figure 1: The continuous deforming from f to g described by F_t

Remark. Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

Example. We first see some examples.

1. Any two (continuous) maps with specification

   $$f,\ g\colon X \to \mathbb{R}^n$$

   are homotopic by considering

   $$F_t(x) = (1-t)f(x) + tg(x).$$

   We call it *the straight line homotopy.*

2. Let S^1 denotes the unit circle in $\mathbb{R}^2$, and D^2 denotes the unit disk in $\mathbb{R}^2$. Then the inclusion $f\colon S^1 \hookrightarrow D^2$ is nullhomotopic by considering

   $$F_t(x) = (1-t)f(x) + (t \cdot 0).$$

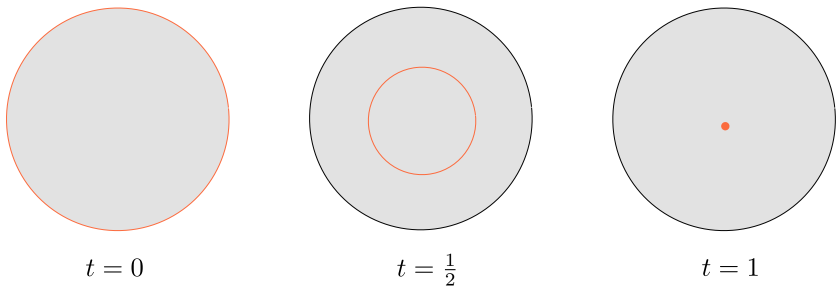
   

   Figure 2: The illustration of $F_t(x)$

   We see that there is a homotopy from $f(x)$ to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a nullhomotopy.

3. The maps

   $$\begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \to & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

   are **not** homotopy.

   Remark. It will essentially **flip** the orientation, hence we can't deform one to another continuously.

Exercise. We first see some exercises.

1. A subset $S \subseteq \mathbb{R}^n$ is star-shaped if

   $$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from x_0 to x lies in S.

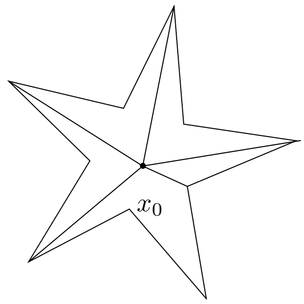


Figure 3: Star-shaped illustration

Show that $\mathrm{id}\colon S \to S$ is nullhomotopic.

Answer. Consider
$$F_t(x) \coloneqq (1-t)x + tx_0,$$
which essentially just concentrates all points x to x_0. ■

2. Suppose
$$X \underset{f_0}{\overset{f_1}{\rightrightarrows}} Y \underset{g_0}{\overset{g_1}{\rightrightarrows}} Z$$
where
$$f_0 \underset{F_t}{\simeq} f_1, \quad g_0 \underset{G_t}{\simeq} g_1.$$
Show
$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

Answer. Consider $I \times X \to Z$, where
$$\begin{array}{ccccc} X \times I & \to & Y \times I & \to & Z \\ (x,t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$
■

Remark. Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** homotopic?

Answer. We'll see! ■

Lecture 2: Homotopy Equivalence

07 Jan. 10:00

As previously seen. Two maps $f, g\colon X \to Y$ is homotopy if there exists a map
$$F_t(x)\colon X \times I \to Y$$
with the properties

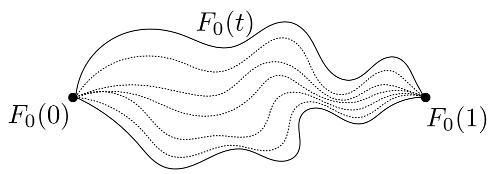
1. Continuous
2. $F_0(x) = f(x)$
3. $F_1(x) = g(x)$

Remark. The continuity of F_t is an even stronger condition for the continuity of F_t for a fixed t.

We now introduce another concept.

Definition 1.2 (Homotopy relative). Given two spaces X, Y, and let $B \subseteq X$. Then a homotopy $F_t(x)\colon X \to Y$ is called *homotopy relative* B (denotes rel$B$) if $F_t(b)$ is independent of t for all $b \in B$.

Example. Given X and $B = \{0, 1\}$. Then the homotopy of paths from $[0, 1] \to X$ is rel$\{0, 1\}$.



1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

Definition 1.3 (Homotopy equivalence, homotopy inverse). A map $f\colon X \to Y$ is a *homotopy equivalence* if $\exists g\colon Y \to X$ such that

$$f \circ g \simeq \mathrm{id}_Y, \quad g \circ f \simeq \mathrm{id}_X.$$

We say that X, Y are *homotopy equivalent*, and g is called *homotopy inverse* of f.

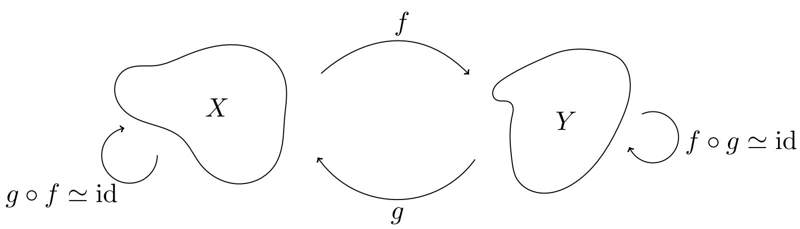
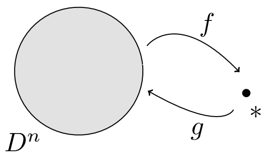


Figure 4: Homotopy Equivalence

If X, Y are homotopy equivalent, then we say that they have the same *homotopy type*.

Notation. We denote a closed n-disk as D^n.

Example. D^n is homotopy equivalent to a point.



We see that $f \circ g = \mathrm{id}_*$ and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is homotopic to id_{D^n} by straight line homotopy $F_t(x) = tx$. Specifically, we see that this holds for any convex set.

> **Definition 1.4 (Contractible).** We say that a space X is *contractible* if X is homotopy equivalent to a point.

The following proposition is added much after, which may uses some concepts not yet covered.

> **Proposition 1.1.** The followings are equivalent.
>
> 1. X is contractible.
> 2. $\forall x \in X, \mathrm{id}_X \simeq c_x$.
> 3. $\exists x \in X, \mathrm{id}_X \simeq c_x$.

Remark. Note that the above notation c_x is introduced at here.

Proof. We see that 2. $\implies$ 3. is obvious. We consider 3. $\implies$ 2. This follows the following general lemma.

> **Lemma 1.1.** Given a topological space X such that $\exists x \in X, \mathrm{id}_X \simeq c_p$, with $f, g \colon Y \to X$, then $f \simeq g$.

Proof. Let $x \in X$ such that $\mathrm{id}_X \simeq c_x$. Then

$$f = \mathrm{id}_X \circ f \simeq c_x \circ f = c_x \circ g \simeq \mathrm{id}_X \circ g = g.$$

■

Then, from this Lemma 1.1, we see that assuming $x_0 \in X$ such that $\mathrm{id}_X \simeq c_{x_0}$, then consider c_x for all $x \in X$, then from Lemma 1.1, we see that $c_x \simeq \mathrm{id}_X$.

To show 3. $\implies$ 1., we let $x_0 \in X$ such that $\mathrm{id}_X \simeq c_{x_0}$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & \{*\} \arrow[l, bend left, "g"] \end{tikzcd}$$

Since $g(*) = x_0$, and

$$\begin{aligned} g \circ f \colon X &\to X \\ x &\mapsto x_0, \end{aligned}$$

which is just c_{x_0}, from the assumption we're done.

Now, we show 1. $\implies$ 3. Let

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & \{*\} \arrow[l, bend left, "g"] \end{tikzcd}$$

be a homotopy equivalent, let $g(*) = x_0$. We see that $c_{x_0} \simeq \mathrm{id}_X$ since

$$g \circ f = c_{x_0} \simeq \mathrm{id}_X.$$

■

Before doing exercises, we introduce two new concepts.

Definition 1.5 (Retraction, retract). Given $B \subseteq X$, a *retraction* from X to B is a map $f \colon X \to X$ (or $X \to B$) such that $\forall b \in B\ f(b) = b$, namely $r|_B = \mathrm{id}_B$. Or one can see this from

$$\begin{tikzcd} B \arrow[r, hook, "i"] \arrow[rr, bend right, "r \circ i"'] & X \arrow[r, "r"] & B \end{tikzcd}$$

where r is a retraction if and only if $r \circ i = \mathrm{id}_B$, where i is an inclusion identity.

If r exists, B is a *retract* of X.

Definition 1.6 (Deformation retraction). Given X and $B \subseteq X$, a *(strong) deformation retraction* $F_t \colon X \to X$ onto B is a homotopy rel$B$ from the id_X to a *retraction* from X to B. i.e.,

$$\begin{aligned} F_0(x) &= x && \forall x \in X \\ F_1(x) &\in B && \forall x \in X \\ F_t(b) &= b && \forall t\ \forall b \in B. \end{aligned}$$

Exercise. We now see some problems.

1. Let $X \simeq Y$. Show X is path-connected if and only if Y is.

   Answer. Suppose X is path-connected. Then we see that given two points x_1 and x_2 in X, there exists a path $\gamma(t)$ with

   $$\gamma\colon [0,1] \to X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

   Since $X \simeq Y$, then there exists a pair of f and g such that $f\colon X \to Y$ and $g\colon Y \to X$ with

   $$f \circ g \underset{F}{\simeq} \mathrm{id}_Y, \quad g \circ f \underset{G}{\simeq} \mathrm{id}_X.$$

   (Notice the abuse of notation)

   For any two y_1 and $y_2 \in Y$, we want to construct a path $\gamma'(t)$ such that

   $$\gamma'\colon [0,1] \to Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

   Firstly, we let $g(y_1) =: x_1$ and $g(y_2) =: x_2$. From the argument above, we know there exists such a γ starting at $x_1 = g(y_1)$ ending at $x_2 = g(y_2)$. Now, consider $f(\gamma(t)) = (f \circ \gamma)(t)$ such that

   $$f \circ \gamma\colon I \to Y, \quad f \circ \gamma(0) = y_1', \quad f \circ \gamma(1) = y_2',$$

   we immediately see that y_1' and y_2' is path connected. Now, we claim that y_1 and y_1' are path connected in Y, hence so are y_2 and y_2'. To see this, note that

   $$f \circ g \underset{F}{\simeq} \mathrm{id}_Y,$$

   which means that there exists $F\colon Y \times I \to Y$ such that

   $$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y_1' \\ F(y_1, 1) = \mathrm{id}_Y(y_1) = y_1. \end{cases}$$

   Since F is continuous in I, we see that there must exist a path connects y_1 and y_1'. The same argument applies to y_2 and y_2'. Now, we see that the path

   $$y_1 \to y_1' \to y_2' \to y_2$$

   is a path in Y for any two y_1 and y_2, which shows Y is path-connected.

   ■

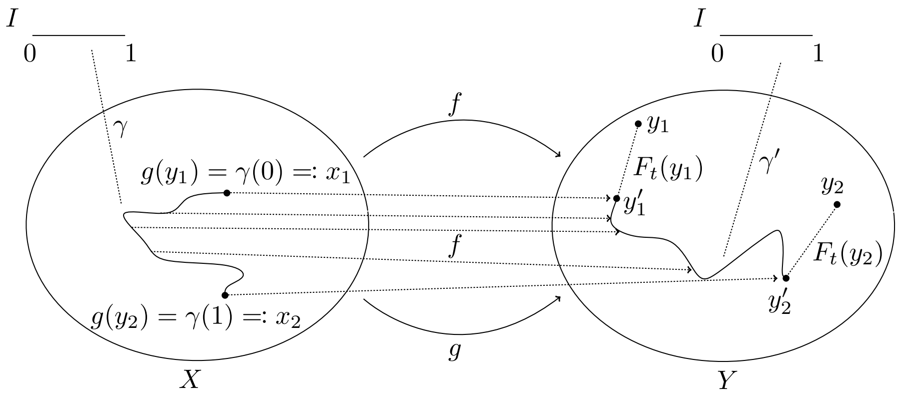


Figure 5: Demonstration of the proof.

Challenge: One can further show that the connectedness is also preserved by any homotopy equivalence.

Corollary 1.1. A contractible space is path-connected.

2. Show that if there exists deformation retraction from X to $B \subseteq X$, then $X \simeq B$.

Lecture 3: Deformation Retraction

10 Jan. 10:00

As previously seen. A *deformation retraction* is a homotopy of maps rel$B$ $X \to X$ from id_X to a retraction from X to B. Then B is a *deformation retract*.

Example. We can also show

1. S^1 is a deformation retraction of $D^2 \setminus \{0\}$. Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$

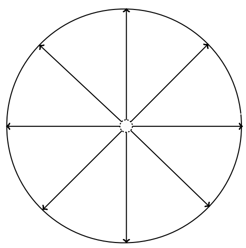


Figure 6: The deformation retraction of $D^{2\setminus\{0\}}$ is just to *enlarge* that hole and push all the interior of D^2 to the boundary, which is S^1.

2. $\mathbb{R}^n$ *deformation retracts* to 0. Indeed, since
$$F_t(x). = (1-t)x.$$
This implies that $\mathbb{R}^n \simeq *$, hence we see that
   - dimension
   - compactness
   - etc.

   are <u>not</u> homotopy invariants.
3. S^1 is a deformation retract of a cylinder and a Möbius band.

   For a cylinder, consider $X \times I \to X$. Define homotopy on a closed rectangle, then verify it induces map on quotient.

   For a Möbius band, we define a homotopy on a closed rectangle, then verify that it respect the equivalence relation.

   Finally, we use the universal property of quotient topology to argue that we get a homotopy on Möbius band.

   Upshot: Möbius band $\simeq S^1 \simeq$ cylinder, hence the orientability is <u>not</u> homotopy invariant.

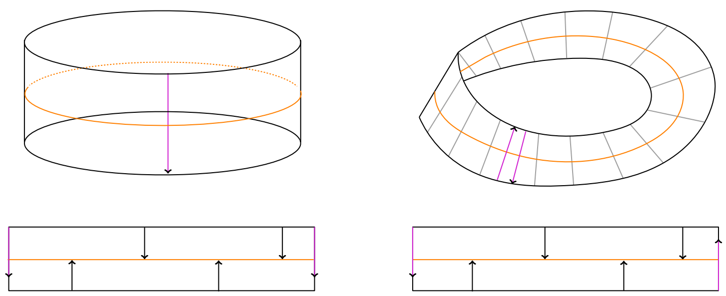


Figure 7: The deformation retraction for Cylinder and Möbius band

Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

As previously seen. We saw that
- homotopy equivalence
- homotopy invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

1.3 CW Complexes

Example. Let's start with a few examples.

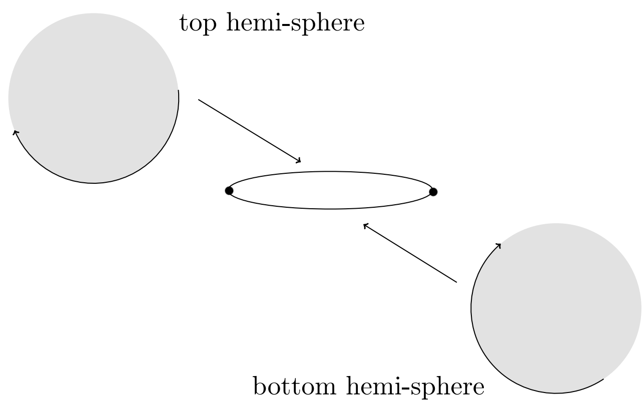
1. Constructing spheres:
   - S^1 (up to homeomorphism[1])

   - S^2
     - glue boundary of 2-disk to a point
     - glue 2 disks onto a circle

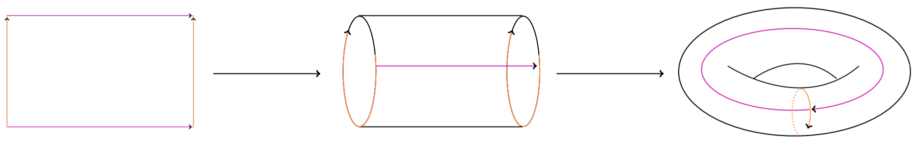
Figure 8: **Left**: Glue a 2-disk to a point along its boundary. **Right**: Glue 2 disks to S^1.

The gluing instruction to construct S^2 in the right-hand side can be demonstrated as follows.

[1] This is just the term for isomorphism in topology.



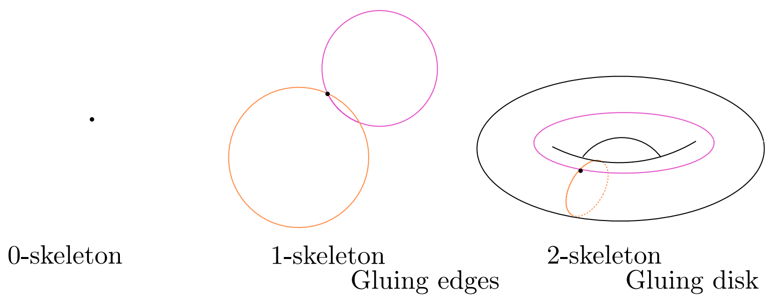
- $T = S^1 \times S^1$



view as gluing instructions

$$\text{vertex } + 2 \text{ edges } + 2\text{-disks.}$$

Specifically, we have



Formally, we have the following definition.

Notation. Let D^n denotes a closed n-disk (or n-ball)

$$D^n \simeq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

And let S^n denotes an n-sphere

$$S^n \simeq \left\{x \in \mathbb{R}^{n+1} \colon \|x\| = 1\right\}.$$

Lastly, we call a point as a *0-cell*, and the interior of D^n $\mathrm{int}(D^n)$ for $n \geq 1$ as a *n-cell*.

Definition 1.7 (CW Complex). A *CW Complex* is a topological space constructed inductively as

1. X^0 (the 0-skeleton) is a set of discrete points.
2. We inductively construct the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α, where α is the index.

   The gluing instructions glued by an attaching map is that $\forall \alpha$, $\exists$ continuous map φ_α

   $$\varphi_\alpha \colon \partial D^n_\alpha \to X^{n-1},$$

   then

   $$X^n = \left(X^{n-1} \coprod_\alpha D^n_\alpha\right) \Big/ x \sim \varphi_\alpha(x)$$

   with identification $x \sim \varphi_\alpha(x)$ for all $x \in \partial D^n_\alpha$ with quotient topology.
3. We let X be defined as

   $$X = \bigcup_{n=0} X^n,$$

   and let $\overline{w}$ denotes weak topology such that

   $$u \subseteq X \text{ is open } \iff \forall n \; u \cap X^n \text{ is open }.$$

   If all cells have dimension less than N and a $\exists N$-cell, then $X = X^N$ and we call it *N-dimensional CW complex*.

Remark. We write $X^{(n)}$ for n-skeleton if we need to distinguish from the Cartesian product.

Example. Let's look at some examples.

1. 0-dim CW complex is a discrete space.
2. 1-dim CW complex is a graph.
3. A CW complex X is finite if it has finitely many cells.

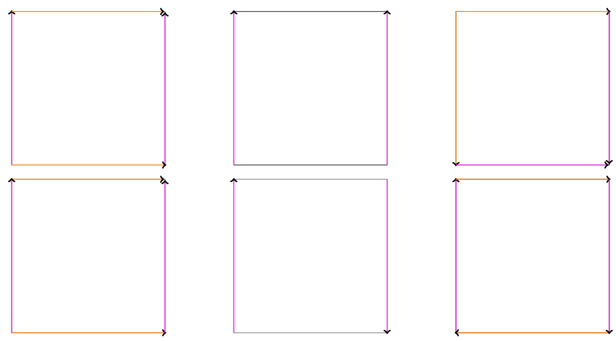
Definition 1.8 (CW subcomplex). A *CW subcomplex* $A \subseteq X$ is a closed subset equal to a union of cells

$$e^n_\alpha = \mathrm{int}\left(D^n_\alpha\right).$$

Remark. This inherits a CW complex structure.

Check the images of attaching maps.

Exercise. Given the following gluing instruction:



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere, $\mathbb{R}P$.

Answer. We see that

1. Torus
2. Cylinder
3. 2-sphere
4. Klein bottle
5. Möbius band
6. $\mathbb{R}P$

Notation. We call the real projection space as $\mathbb{R}P$, and we also have so-called complex projection space, denote as $\mathbb{C}P$.

Lecture 5: Operation on Spaces

14 Jan. 10:00

1.4 Operations on CW Complexes

1.4.1 Products

We can consider the product of two CW complex given by a CW complex structure. Namely, given X and Y two CW complexes, we can take two cells e^n_α from X and e^m_β from Y and form the product space $e^n_\alpha \times e^m_\beta$, which is homeomorphic to an $(n+m)$-cell. We then take these products as the cells for $X \times Y$.

Specifically, given X, Y are CW complexes, then $X \times Y$ has a cell structure

$$\{e^m_\alpha \times e^n_\alpha \colon e^m_\alpha \text{ is a } m\text{-cell on } X, e^n_\alpha \text{ is an } n\text{-cell on } Y\}.$$

Remark. The product topology may not agree with the weak topology on the $X \times Y$. However, they do agree if X or Y is locally compact <u>or</u> if X and Y both have at most countably many cells.

1.4.2 Wedge Sum

Given X, Y are CW complexes, and $x_0 \in X^0$, $y_0 \in Y^0$ (only points). Then we define

$$X \vee Y = X \coprod Y$$

with quotient topology.

Remark. $X \vee Y$ is a CW complex.

1.4.3 Quotients

Let X be a CW complex, and $A \subseteq X$ subcomplex (closed union of cells), then

$$X \big/ A$$

is a quotient space collapse A to one point and inherits a CW complex structure.

Remark. $X \,/\, A$ is a CW complex.

0-skeleton

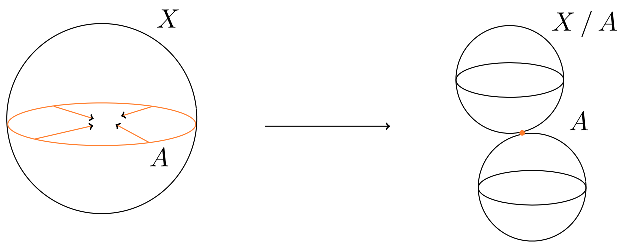
$$(X^0 - A^0) \coprod *$$

where $*$ is a point for A. Each cell of $X - A$ is attached to $(X \,/\, A)^n$ by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n \,/\, A^n$$

Example. Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a wedge of two spheres.



2. We can take the torus and squish down a ring around the hole.

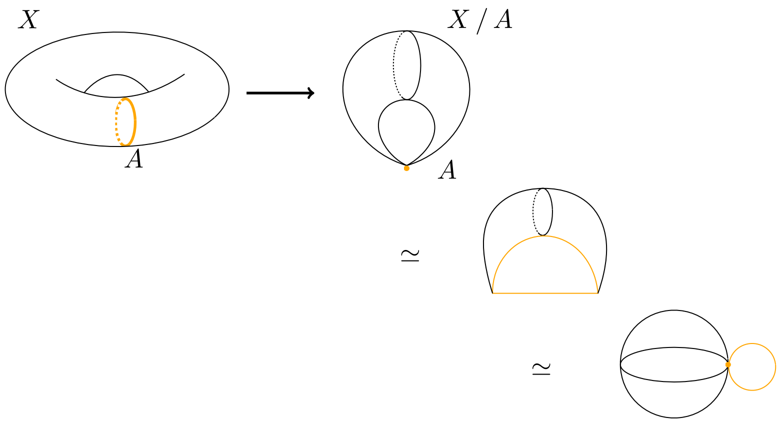


Figure 9: We see that $X\,/\,A$ is homotopy equivalent to a 2-sphere wedged with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other points, forming a circle.

Lecture 6: A Foray into Category Theory

19 Jan. 10:00

1.5 Category Theory

We start with a definition.

Definition 1.9 (Category, object, morphism). A *category* $\mathscr{C}$ is 3 pieces of data

- A class of *objects* $\mathrm{Ob}(\mathscr{C})$
- $\forall X, Y \in \mathrm{Ob}(\mathscr{C})$ a class of *morphisms* or arrows, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- $\forall X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, there exists a composition law

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z), \quad (f, g) \mapsto g \circ f$$

and 2 axioms

- Associativity. $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms f, g, h where composites are defined.
- Identity. $\forall X \in \mathrm{Ob}(\mathscr{C})$ $\exists \mathrm{id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g$$

for all f, g where this makes sense.

Let's see some examples.

Example. We introduce some common category.

$\mathcal{C}$	$\mathrm{Ob}(\mathcal{C})$	$\mathrm{Mor}(\mathcal{C})$
$\underline{\mathrm{set}}$	Sets X	All maps of sets
$\underline{\mathrm{fset}}$	Finite sets	All maps
$\underline{\mathrm{Gp}}$	Groups	Group Homomorphisms
$\underline{\mathrm{Ab}}$	Abelian groups	Group Homomorphisms
$\underline{k-\mathrm{vect}}$	Vector spaces over k	k-linear maps
$\underline{\mathrm{Rng}}$	Rings	Ring Homomorphisms
$\underline{\mathrm{Top}}$	Topological spaces	Continuous maps
$\underline{\mathrm{Haus}}$	Hausdorff Spaces	Continuous maps
$\underline{\mathrm{hTop}}$	Topological spaces	Homotopy classes of continuous maps
$\underline{\mathrm{Top}^*}$	Based topological spaces[2]	Based maps[3]

Remark. Any **diagram** plus composition law.

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[r] & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd}.$$

Definition 1.10 (Monic, epic). A morphism $f\colon M \to N$ is *monic* if

$$\forall g_1, g_2 \; f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$\begin{tikzcd} A \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

Dually, f is *epic* if

$$\forall g_1, g_2 \; g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & B \end{tikzcd}$$

Lemma 1.2. In $\underline{\mathrm{set}}$, $\underline{\mathrm{Ab}}$, $\underline{\mathrm{Top}}$, $\underline{\mathrm{Gp}}$, a map is monic if and only if f is injective, and epic if and only if f is surjective.

Proof. In $\underline{\mathrm{set}}$, we prove that f is monic if and only if f is injective. Suppose $f \circ g_1 = f \circ g_2$ and f is injective, then for any a,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence $g_1 = g_2$.

[2] Topological spaces with a distinguished base point $x_0 \in X$
[3] Continuous maps that presence base point $f\colon (x, x_0) \to (y, y_0)$ such that

$$f\colon X \to Y, \quad f(x_0) = y_0$$

is continuous.

Now we prove another direction, with contrapositive. Namely, we assume that f is not injective and show that f is not monic. Suppose $f(a) = f(b)$ and $a \neq b$, we want to show such g_i exists. This is easy by considering

$$g_1 \colon * \mapsto a, \quad g_2 \colon * \mapsto b.$$

■

1.5.1 Functor

After introducing the category, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

Definition 1.11 (Functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A (covariant) *functor* $F \colon \mathscr{C} \to \mathscr{D}$ is

1. a map on objects

$$\begin{aligned} F \colon \operatorname{Ob}(\mathscr{C}) &\to \operatorname{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}}(X, Y) &\to \operatorname{Hom}_{\mathscr{D}}(F(X), F(Y)) \\ [f \colon X \to Y] &\mapsto [F(f) \colon F(X) \to F(Y)] \end{aligned}$$

such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
- $F(f \circ g) = F(f) \circ F(g)$

Lecture 7: Functors

21 Jan. 10:00

As previously seen. Assume that we initially have a commutative diagram in $\mathscr{C}$ as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

After applying F, we'll have

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "F(g \circ f) = F(g) \circ F(f)"'] & F(Y) \arrow[d, "F(g)"] \\ & F(Z) \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

We can also have a so-called contravariant functor.

Definition 1.12 (Contravariant functor). Given $\mathscr{C}, \mathscr{D}$ be two categories. A *contravariant functor*

$$F\colon \mathscr{C} \to \mathscr{D}$$

is

1. a map on objects

$$\begin{aligned} F\colon \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X)) \\ [f\colon X \to Y] &\mapsto [F(f)\colon F(Y) \to F(X)] \end{aligned}$$

   such that

   - $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$
   - $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a contravariant functor F, the diagram becomes

$$\begin{tikzcd} F(X) & F(Y) \arrow[l, "F(f)"'] \\ & F(Z) \arrow[u, "F(g)"'] \arrow[ul, "F(g\circ f)=F(f)\circ F(g)"] \end{tikzcd}$$

which is a commutative diagram in $\mathscr{D}$.

Example. Let see some examples.

1. Identity functor.

$$I\colon \mathscr{C} \to \mathscr{C}.$$

2. Forgetful functor.

   -
   $$F\colon \underline{\mathrm{Gp}} \to \underline{\mathrm{set}}, \quad G \mapsto G^{4}$$
   such that
   $$[f\colon G \to H] \mapsto [f\colon G \to H].$$

   -
   $$F\colon \underline{\mathrm{Top}} \to \underline{\mathrm{set}}, \quad X \mapsto X^{5}$$
   such that
   $$[f\colon X \to Y] \mapsto [f\colon X \to Y].$$

[4] G is now just the underlying set of the group G.
[5] X is now just the underlying set of the topological space X.

3. Free functor.

$$\begin{aligned} \underline{\text{set}} &\to \underline{k\text{−vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

   i.e., vector space with basis s such that

$$[f\colon A \to B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

4.

$$\begin{aligned} \underline{k\text{−vect}} &\to \underline{k\text{−vect}} \\ V &\mapsto V^* = \mathrm{Hom}_k(V, k) \end{aligned}$$

   If we are working on a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functors.

1.

$$\begin{aligned} \underline{\text{Top}^*} &\to \underline{\text{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0) \end{aligned}$$

   where π_1 is so-called *fundamental group*.

2.

$$\begin{aligned} \underline{\text{Top}} &\to \underline{\text{Ab}} \\ X &\mapsto H_p(X) \end{aligned}$$

   where H_p is so-called p^{th} *homology*.

Let's see the formal definition.

1.6 Free Groups

Definition 1.13 (Free group). Given a set S, the *free group* is a group F_S on S with a map $S \to F_S$ satisfying the universal property.

If G is any group, $f\colon S \to G$ is any map of sets, f extends uniquely to group homomorphism $\overline{f}\colon F_S \to G$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\overline{f}\colon \text{gp hom}"] \\ & G \end{tikzcd}$$

Note. This defines a *natural bijection*

$$\mathrm{Hom}_{\underline{\mathrm{set}}}(S, \mathscr{U}(G)) \cong \mathrm{Hom}_{\underline{\mathrm{Gp}}}(F_S, G),$$

where $\mathscr{U}(G)$ is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are adjoint; specifically that the free functor is the left adjoint (appears on the left in the Hom above).

> **Definition 1.14 (Adjoints functor).** A free and forgetful functor is *adjoints.*

Remark. Whenever we state a universal property for an object (plus a map), an object (plus a map) may or may not exist. If such object exists, then it defines the object **uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the object (plus a map).

> **Lemma 1.3.** Universal property defines F_S (plus a map $S \to F(S)$) uniquely up to unique isomorphism.

Proof. Fix S. Suppose

$$S \to F_S, \quad S \to \widetilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, dashed, "\exists!\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\exists!\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

We'll show φ and ψ are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{tikzcd} & F_S \arrow[dd, "\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[dd, "\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & \widetilde{F}_S \end{tikzcd}$$

Hence, we see that

$$\begin{tikzcd} & F_S \arrow[d, "\psi"] \arrow[dd, bend left=50, "\varphi\circ\psi=\mathrm{id}_{F_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & \widetilde{F}_S \arrow[d, "\varphi"] \\ & F_S \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{F}_S \arrow[d, "\varphi"] \arrow[dd, bend left=50, "\psi\circ\varphi=\mathrm{id}_{\widetilde{F}_S}"] \\ S \arrow[ur, "f"] \arrow[r] \arrow[dr, "f"'] & F_S \arrow[d, "\psi"] \\ & \widetilde{F}_S \end{tikzcd}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \mathrm{id}_{F_S}, \qquad \psi \circ \varphi = \mathrm{id}_{\widetilde{F}_S},$$

so φ and ψ are inverses (thus group isomorphism). ■

Lecture 8: The Fundamental Group π_1

24 Jan. 10:00

Example. In category Ab free Abelian group on a set S is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on S** .

1.6.1 Constructing the Free Groups F_S

Proposition 1.2. The free group defined by the universal property exists.

Proof. We'll just give a construction below. First, we see the definition.

Definition 1.15 (Word). Fix a set S, and we define a *word* as a finite sequence (possibly $\varnothing$) in the formal symbols

$$\left\{s, s^{-1} \mid s \in S\right\}.$$

Then we see that elements in F_S are equivalence classes of words with the equivalence relation being

- deleted ss^{-1} or $s^{-1}s$. i.e.,

$$vs^{-1}sw \sim vw$$
$$vss^{-1}w \sim vw$$

  for every word $v, w, s \in S$,

with the group operation being concatenation. ■

Example. Given words ab^{-1}, bba, their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

Exercise. There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that F_S satisfies the universal property with respect to the map

$$S \to F_S, \quad s \mapsto s.$$

2 The Fundamental Group

2.1 Path

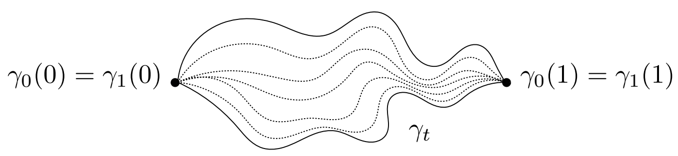
We start with the definition.

Definition 2.1 (Path). A *path* in a space X is a continuous map

$$\gamma\colon I \to X$$

where $I = [0, 1]$.

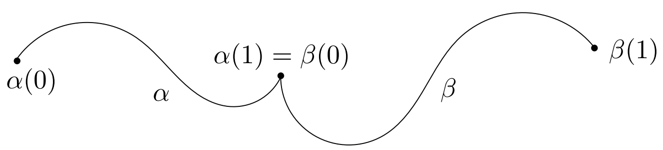
Definition 2.2 (Homotopy path). A *homotopy of paths* γ_0, γ_1 is a homotopy from γ_0 to γ_1 rel$\{0, 1\}$.



Example. Fix $x_1, x_0 \in X$, then $\exists$ homotopy of paths is an equivalence relation on paths from x_0 to x_1 (i.e., γ with $\gamma(0) = x_0, \gamma(1) = x_1$).

Definition 2.3 (Path composition). For paths α, β in X with $\alpha(1) = \beta(0)$, the *composition*[a] $\alpha \cdot \beta$ is

$$(\alpha\cdot\beta)(t) \coloneqq \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \dfrac{1}{2}\right] \\ \beta(2t-1), & \text{if } t \in \left[\dfrac{1}{2}, 1\right] \end{cases}.$$



[a]Also named *product*, *concatenation*.

Remark. By the pasting lemma, this is continuous, hence $\alpha \cdot \beta$ is actually a path from $\alpha(0)$ to $\beta(1)$.

Definition 2.4 (Reparameterization). Let $\gamma\colon I \to X$ be a path, then a *reparameterization* of γ is a path

$$\gamma'\colon I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where φ is continuous and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

Exercise. A path γ is homotopic rel$\{0, 1\}$ to all of its reparameterizations.

Proof. We show that γ and $\gamma \circ \phi$ are homotopic rel$\{0, 1\}$ by showing that there exists a continuous F_t such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since ϕ is continuous, so we define

$$F_t(x) = (1-t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all $x, t \in I$.

Now, we check that F_t really gives us a homotopic rel$\{0, 1\}$. We have

$$\begin{aligned} F_t(0) &= (1-t)\gamma(0) + t \cdot \gamma \circ \phi(0) = (1-t)\gamma(0) + t \cdot \gamma(\underbrace{\phi(0)}_{0}) = \gamma(0), \\ F_t(1) &= (1-t)\gamma(1) + t \cdot \gamma \circ \phi(1) = (1-t)\gamma(1) + t \cdot \gamma(\underbrace{\phi(1)}_{1}) = \gamma(1), \end{aligned}$$

which shows that 0 and 1 are independent of t, hence γ and $\gamma \circ \phi$ are homotopic rel$\{0, 1\}$. ∎

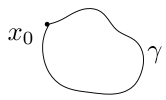
Exercise. Fix $x_1, x_1 \in X$. Then homotopy of paths (relative $\{0, 1\}$) is an equivalence relation on paths from x_0 to x_1.

2.2 Fundamental Group and Groupoid

2.2.1 Fundamental Group

Definition 2.5 (Fundamental Group). Let X denotes the space and let $x_0 \in X$ be the base point. The *fundamental group of X based at x_0*, denoted by $\pi_1(X, x_0)$, is a group such that

- Elements: Homotopy classes rel$\{0,1\}$ of paths $[\gamma]$ where γ is a **loop** with $\gamma(0) = \gamma(1) = x_0$[a]

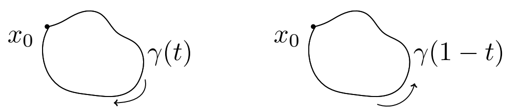


- Operation: Composition of paths.
- Identity: Constant loop γ based at x_0 such that

$$\gamma\colon I \to X, \quad t \mapsto x_0$$

- Inverses: The inverse $[\gamma]^{-1}$ of $[\gamma]$ is represented by the loop $\overline{\gamma}$ such that

$$\overline{\gamma}(t) = \gamma(1-t).$$



[a]We say γ is **based** at x_0.

Proof. We prove that

Associativity. $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$. We break this down into

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t), & t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t-2), & t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t-3), & t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t-1), & t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t-1), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, we define $\phi\colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \frac{1}{2}\right], & t \in \left[0, \frac{1}{4}\right]; \\ t + \frac{1}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right], & t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{t+1}{2} \in \left[\frac{3}{4}, 1\right], & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$, which implies that the associativity holds.

Identity. We want to show that $[\gamma \cdot c] = [\gamma]$. Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & t \in \left[0, \frac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, consider $\phi\colon I \to I$ such that

$$\phi(t) = \begin{cases} 2t, & t \in \left[0, \frac{1}{2}\right]; \\ 1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$.

Inverses. We want to show that $\gamma \cdot \overline{\gamma} \simeq c$, where $\overline{\gamma}(t) = \gamma(1-t)$. Firstly, we have

$$(\gamma \cdot \overline{\gamma})(t) = \begin{cases} \gamma(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2t), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider F_t given by

$$F_t(x) = \begin{cases} \gamma(2xt), & x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2xt), & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

If $t = 0$, we have

$$F_0(x) = \begin{cases} \gamma(0), & x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1), & x \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0$$

for all $x \in I$, namely $F_0 = c$, while when $t = 1$, we have

$$F_1(x) = \begin{cases} \gamma(2x), & x \in \left[0, \frac{1}{2}\right]; \\ \overline{\gamma}(1-2x), & x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \overline{\gamma})(x),$$

and we see that F_t is continuous since at $x = \frac{1}{2}$, we have

$$\gamma(2x) = \gamma(1) = \overline{\gamma}(0) = \overline{\gamma}(1-2x),$$

hence we see that F_t is the homotopy between $\gamma \cdot \overline{\gamma}$ and c.

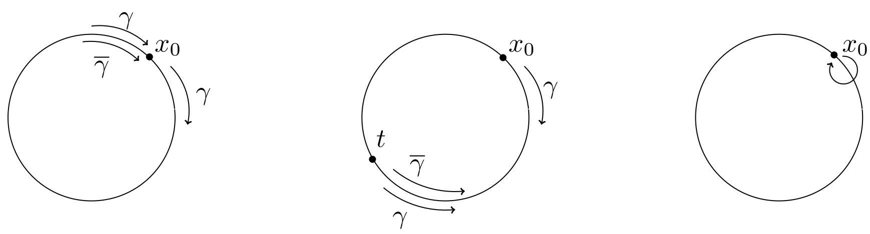


Figure 10: Illustration of F_t. Intuitively, the path $\gamma \cdot \overline{\gamma}$ is $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\overline{\gamma}} x_0$. But now, F_t is $x_0 \xrightarrow{\gamma} t \xrightarrow{\overline{\gamma}} x_0$. We can think of this homotopy is *pulling back* the turning point along the original path.

■

Theorem 2.1. If X is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Remark. We see that we can write $\pi_1(X)$ up to isomorphism given this result.

Proof. To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if $[h \cdot \gamma \cdot \overline{h}] = [h \cdot \gamma' \cdot \overline{h}]$, then since we know that $h^{-1} = \overline{h}$, hence in the fundamental group $\pi_1(X, x_0)$, we see that

  $$\overline{h} \cdot h \cdot \gamma \cdot \overline{h} \cdot h = \overline{h} \cdot h \cdot \gamma' \cdot \overline{h} \cdot h. \implies \gamma = \gamma'$$

  as we desired.

- onto. We see that for every $\alpha \in \pi_1(X, x_0)$, there exists a $\gamma \in \pi_1(X, x_0)$ such that

  $$\gamma = \overline{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)$$ [6]

  since $h \cdot \gamma \cdot \overline{h} = \alpha$.

We then see that the fundamental group of X does not depend on the choice of basepoint, only on the choice of the path component of the basepoint. If X is path-connected, it now makes sense to refer to *the* fundamental group of X and write $\pi_1(X)$ for the abstract group (up to isomorphism). ■

Exercise. Composition of paths is well-defined on homotopy classes rel$\{0,1\}$.

Exercise. If X is a contractible space, then X is path-connected and $\pi_1(X)$ is trivial.

The followings are the properties about homotopy path. They are useful when we introduce fundamental groupoid.

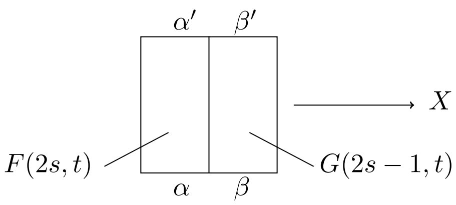
Lemma 2.1. Given $x_0, x_1, x_2 \in X$, α, α' are two paths from x_0 to x_1, and β, β' are two paths from x_1 to x_2. If $\langle\alpha\rangle = \langle\alpha'\rangle$, $\langle\beta\rangle = \langle\beta'\rangle$, then $\langle\alpha \cdot \beta\rangle = \langle\alpha' \cdot \beta'\rangle$.

Proof. Given $\alpha \underset{F}{\simeq} \alpha'$ rel$\{0,1\}$, $\beta \underset{G}{\simeq} \beta'$ rel$\{0,1\}$, then we want to prove

$$\alpha \cdot \beta \simeq \alpha' \cdot \beta' \text{ rel}\{0,1\}.$$

This is done by using homotopy $H\colon I \times I \to X$ such that it combines $F(2s,t)$ and $G(2s-1,t)$.

$$\begin{tikzcd} x_0 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha'"'] & x_1 \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\beta'"'] & x_2 \end{tikzcd}$$



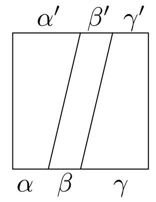
■

[6] Notice that this is indeed the case, one can verify this by the fact that $h\colon x_0 \to x_1$ and $\overline{h}\colon x_1 \to x_0$.

Lemma 2.2. Let $x_0, x_1, x_2, x_3 \in X$, α is a path from x_0 to x_1, β is a path from x_1 to x_2, γ is a path from x_2 to x_3. Then

$$\langle(\alpha \cdot \beta) \cdot \gamma\rangle = \langle\alpha \cdot (\beta \cdot \gamma)\rangle .$$

Proof. We can write out the homotopy by the following diagram.



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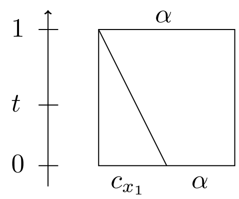
Lemma 2.3. Let X be a topological space, and $x_0 \in X$. Then for every path homotopy $\langle\alpha\rangle$ from x_1 to x_2, we have

$$\langle c_{x_1} \cdot \alpha\rangle = \langle\alpha\rangle = \langle\alpha \cdot c_{x_2}\rangle .$$

Proof. We only need to prove $c_{x_1} \cdot \alpha \simeq \alpha \ \mathrm{rel}\{0, 1\}$. The homotopy can be written out explicitly by the following diagram.

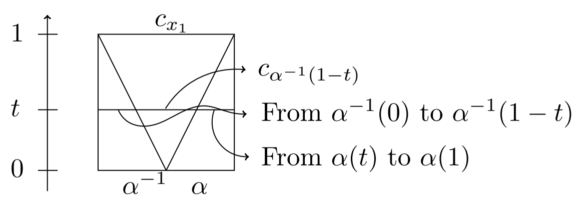


■

Lemma 2.4. For every path homotopy $\langle\alpha\rangle$ from x_1 to x_2, then

$$\left\langle\alpha \cdot \alpha^{-1}\right\rangle = \langle c_{x_1}\rangle , \qquad \left\langle\alpha^{-1} \cdot \alpha\right\rangle = \langle c_{x_2}\rangle .$$

Proof. For the first case, we have the following diagram.



The second case follows similarly. ■

2.2.2 Fundamental Groupoid

This section is not covered in class, but it's a useful concept. The idea is that after giving Definition 2.5, we see that we actually create a fundamental group at **every** point in X, furthermore, when we use Theorem 2.1 if X is path-connected, we actually **lose** some information about this space. Here is how we can store all the information.

Notation (Constant loop). We denote c_x, where $x \in X$ such that

$$\begin{aligned} c_x\colon [0,1] &\to X \\ t &\mapsto x \end{aligned}$$

as a *constant loop*.

Definition 2.6 (Groupoid). A category $\mathscr{C}$ is a *groupoid* if any morphisms in $\mathscr{C}$ is and isomorphism.

Remark. We'll soon see that for any topological space x, Definition 2.5 defines a groupoid, denoted by $\Pi(X)$.

Definition 2.7 (Fundamental groupoid). Let X denotes the space, then the category $\Pi(X)$ is a *fundamental groupoid of* X such that

- $\mathrm{Ob}(\Pi(X)) \coloneqq X$
- $\mathrm{Hom}(\Pi(X))\colon \forall p, q \in \mathrm{Ob}(\Pi(X)) = X$,

$$\mathrm{Hom}_{\Pi(X)}(p, q) \coloneqq \{\text{Paths from } p \text{ to } q\} \big/ \sim .$$

- Composition: For every $p, q, r \in \mathrm{Ob}(\Pi(X)) = X$,

$$\begin{aligned} \circ\colon \mathrm{Hom}_{\Pi(X)}(p, q) \times \mathrm{Hom}_{\Pi(X)}(q, r) &\to \mathrm{Hom}_{\Pi(X)}(p, r) \\ (\langle\alpha\rangle, \langle\beta\rangle) &\mapsto \langle\beta\rangle \circ \langle\alpha\rangle \coloneqq \langle\alpha \cdot \beta\rangle . \end{aligned}$$

- Identity: For every $p \in \mathrm{Ob}(\Pi(X)) = X$, we define $1_p \coloneqq \langle c_p\rangle \in \mathrm{Hom}_{\Pi(X)}(p, p)$ be the constant loop based at p such that for every $\langle\alpha\rangle \in \mathrm{Hom}_{\Pi(X)}(p, q)$,

$$\langle\alpha\rangle \circ \mathrm{id}_p = \mathrm{id}_q \circ \langle\alpha\rangle = \langle\alpha\rangle .$$

- Associativity: Given $p, q, r, s \in \mathrm{Ob}(\Pi(X)) = X$, with the paths

$$\begin{tikzcd} p \arrow[r, bend left, "\langle\alpha\rangle"] & q \arrow[r, bend right, "\langle\beta\rangle"] & r \arrow[r, bend left, "\langle\gamma\rangle"] & s \end{tikzcd}$$

Then

$$\langle\gamma\rangle \circ (\langle\beta\rangle \circ \langle\alpha\rangle) = (\langle\gamma\rangle \circ \langle\beta\rangle) \circ \langle\alpha\rangle .$$

Proof. Note that in Definition 2.7, we need to show some of the definitions is indeed well-defined, and we also need to show that $\Pi(X)$ is actually a groupoid.

- Composition: Since if $\alpha \simeq \alpha', \beta \simeq \beta'$, we have

$$\alpha \cdot \beta \simeq \alpha' \cdot \beta'$$

from Lemma 2.1.

- Identity: It follows that

$$\langle\alpha\rangle \circ \mathrm{id}_p = \langle c_p \cdot \alpha\rangle = \langle\alpha\rangle$$

from Lemma 2.3. The left identity can be shown similarly.

- Associativity: It's trivial in the sense that all the homotopy can be easily derived from Lemma 2.2.

Additionally, from Lemma 2.4, we see that given α is a path from p to q, then

$$\begin{cases} \langle\alpha^{-1} \cdot \alpha\rangle &= \langle c_q\rangle \eqqcolon \mathrm{id}_q \\ \langle\alpha \cdot \alpha^{-1}\rangle &= \langle c_p\rangle \eqqcolon \mathrm{id}_p . \end{cases}$$

Furthermore, since $\langle\alpha^{-1} \cdot \alpha\rangle = \langle\alpha\rangle \circ \langle\alpha^{-1}\rangle$ and $\langle\alpha \cdot \alpha^{-1}\rangle = \langle\alpha^{-1}\rangle \circ \langle\alpha\rangle$, hence this means $\Pi(X)$ is indeed a groupoid. ∎

Remark. Assume $\mathscr{C}$ is a groupoid, then for every $x \in \mathrm{Ob}(\mathscr{C})$, we can define

$$\cdot\colon \mathrm{Hom}_{\mathscr{C}}(x,x) \times \mathrm{Hom}_{\mathscr{C}}(x,x) \to \mathrm{Hom}_{\mathscr{C}}(x,x)$$

such that

$$(f,g) \mapsto f \cdot g \coloneqq g \circ f.$$

We can prove that

$$(\mathrm{Hom}_{\mathscr{C}}(x,x), \cdot)$$

defines a group $\mathrm{Aut}_{\mathscr{C}}(x)$ called the *isotropy group* of $\mathscr{C}$ at x.

Exercise. For every $x, y \in \mathrm{Ob}(\mathscr{C})$, if there exists $f \in \mathrm{Hom}_{\mathscr{C}}(x,y)$, then f induces

$$f_*\colon \mathrm{Aut}_{\mathscr{C}}(x) \overset{\cong}{\to} \mathrm{Aut}_{\mathscr{C}}(y),$$

where f_* is a group homomorphism.

Remark. For every $p \in X = \mathrm{Ob}(\Pi(X))$, we have

$$\mathrm{Aut}_{\Pi(X)}(p) = \pi_1(X,p).$$

Firstly, since they're the same in the sense of **set**:

$$\mathrm{Aut}_{\Pi(X)}(p) = \mathrm{Hom}_{\Pi(X)}(p,p) = {\{\text{Loops in } X \text{ based at } p\}}\big/{\sim} = \pi_1(X,p).$$

Hence, we only need to verify their group composition agrees. But this is trivial, since for every two $\langle \alpha \rangle, \langle \beta \rangle \in \mathrm{Aut}_{\Pi(X)}(p)$,

$$\underbrace{\langle \alpha \rangle \cdot \langle \beta \rangle}_{\text{Composition from } \mathrm{Aut}_{\Pi(X)}} = \langle \beta \rangle \circ \langle \alpha \rangle = \underbrace{\langle \alpha \cdot \beta \rangle}_{\text{Composition from } \pi_1}.$$

This implies that Theorem 2.1 is just a particular example as a groupoid.

Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

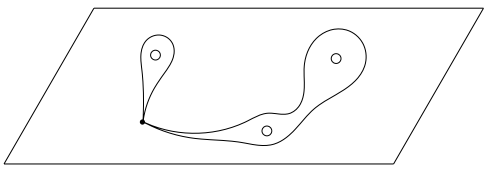


Figure 11: Fundamental Group is basically a *hole detector*!

2.3 Calculations with $\pi_1(S^n)$

Let's start with a basic but important theorem.

Theorem 2.2 (The fundamental group of S^1). The fundamental group of S^1 is

$$\pi_1(S^1) \cong \mathbb{Z},$$

and this identification is given by the paths

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))].$$

Remark. Intuitively, this winds around S^1 n times. The key to this proof was to understand S^1 via the covering space $\mathbb{R} \to S^1$. We will talk about covering spaces much later.

Proof. With the help of covering spaces and the theorems build around which, we can define

$$\begin{aligned} p &\colon \mathbb{R} \to S^1, & x &\mapsto e^{2\pi i x}, \\ \varphi &\colon \mathbb{Z} \to \pi_1(S^1, 1), & n &\mapsto \langle p \circ \gamma_n \rangle, \end{aligned}$$

where p defined above is a covering map. We need to show that this is well-defined.

From the definition of φ, we see that it's a homomorphism. But we also need to show

- φ is a surjection. This is shown by Corollary 3.1, specifically in the case of path.
- φ is an injection. This is shown by Corollary 3.1, specifically in the case of homotopy of paths.

■

Theorem 2.3. Given (X, x_0) and (Y, y_0), then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\begin{bmatrix} r \colon I \to X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{bmatrix} \mapsto (r_X, r_Y).$$

Proof. Let $Z \xrightarrow{f} X \times Y$ with $z \overset{f}{\mapsto} (f_X(z), f_Y(z))$. Then we have

$$f \text{ continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply above to

- Paths $I \to X \times Y$.
- Homotopies of paths $I \times I \to X \times Y$.

■

Corollary 2.1 (The fundamental group of S^k). The torus $T \cong S^1 \times S^1$ has fundamental group $\pi_1(T) \cong \mathbb{Z}^2$. Additionally, for a k-torus

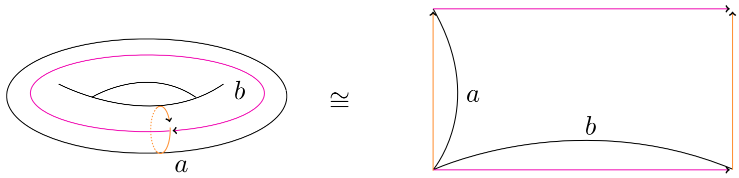
$$\underbrace{S^1 \times S^1 \times \ldots \times S^1}_{k \text{ times}} = (S^1)^k,$$

the fundamental group is then $\mathbb{Z}^k$, i.e.

$$\pi_1\left((S^1)^k\right) \cong \mathbb{Z}^k.$$

Proof. Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

Remark. One way to think of the k-torus is as a k-dimensional cube with opposite $(k-1)$-dimensional faces identified by translation.

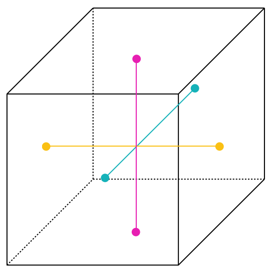


Figure 12: 3-torus with cube identified with parallel sides.

Lemma 2.5. Let $f, g\colon X \to Y$ such that $f \underset{F}{\simeq} g$. Let $x_0 \in X$, then given

$$\begin{aligned} f_*\colon \pi_1(X, x_0) &\to \pi_1(Y, f(x_0)) \\ g_*\colon \pi_1(X, x_0) &\to \pi_1(Y, g(x_0)) \end{aligned}$$

with $\gamma\colon [0,1] \to Y$, $t \mapsto F(x_0, t)$,

$$\begin{aligned} \gamma_*\colon \pi_1(Y, f(x_0)) &\to \pi_1(Y, g(x_0)) \\ \langle \alpha \rangle &\mapsto \left\langle \gamma^{-1} \cdot \alpha \cdot \gamma \right\rangle, \end{aligned}$$

the following diagram commutes.

$$\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[dd, "\gamma_*", "\cong"'] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(Y, g(x_0))
\end{tikzcd}$$

Proof. We want to prove that for any $\langle \alpha \rangle \in \pi_1(X, x_0)$, we have

$$\gamma_* \circ f_*(\langle \alpha \rangle) = g_*(\langle \alpha \rangle).$$

The left-hand side is just

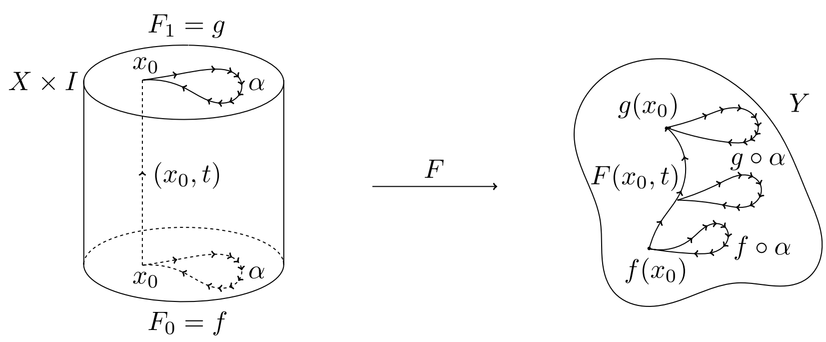
$$\gamma_* \circ f_*(\langle \alpha \rangle) = \gamma_*\left(\langle f \circ \alpha \rangle\right) = \left\langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \right\rangle,$$

while the right-hand side is just

$$g_*(\langle \alpha \rangle) = \langle g \circ \alpha \rangle.$$

That is, we now want to show

$$\left\langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \right\rangle = \langle g \circ \alpha \rangle.$$



We see that we can obtain a homotopy $G\colon I\times I\to Y$ such that

$$G \coloneqq F\circ(\alpha\times\mathrm{id}),$$

where we define $\alpha\times\mathrm{id}$ by

$$\alpha\times\mathrm{id}\colon I\times I\to X\times I,\quad (s,t)\mapsto(\alpha(s),t).$$

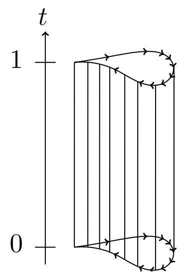
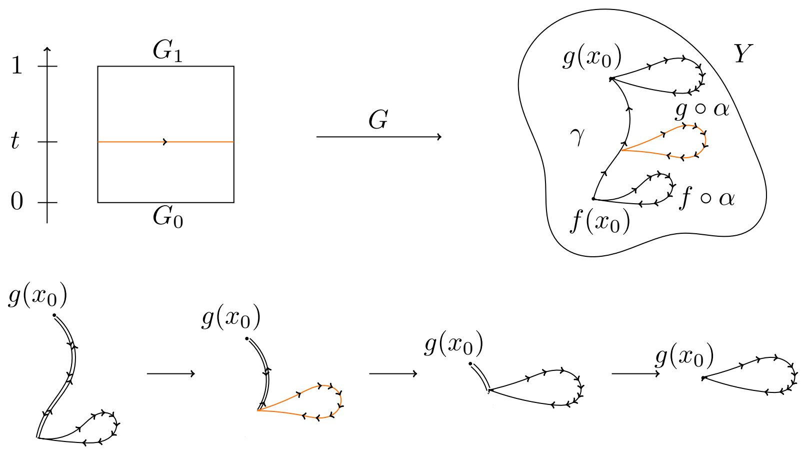
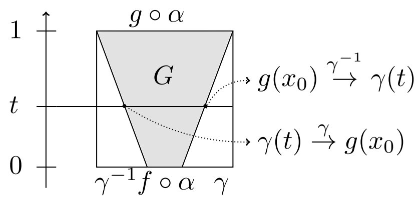


Figure 13: $\alpha\times\mathrm{id}$'s image.

We see that by defining such G, we have the following.



To write out this homotopy explicitly, we see the following diagram.



■

Theorem 2.4 (Fundamental group is a homotopy invariant). If X, Y are homotopy equivalent, then their fundamental groups are isomorphic.

Proof. ■ HW.

Remark. This gives us a powerful tool to calculate π_1.

Example. We now see some examples.

1. $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2. $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong 0 \times \mathbb{Z} = \mathbb{Z}$ since
$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R},$$
which means that the generators are just loops around the hole intuitively.

2.4 Fundamental Group and Groupoid Define Functors

Theorem 2.5 (Fundamental group defines a functor). π_1 is a functor such that

$$\begin{aligned} \pi_1 \colon \underline{\mathrm{Top}_*} &\to \underline{\mathrm{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0). \end{aligned}$$

While on a map $f\colon X \to Y$ taking base point x_0 to y_0, π_1 induces a map

$$\begin{aligned} f_* \colon \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

i.e.,
$$[f\colon X \to Y] \mapsto [f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))]\,.$$

Notation. We usually write f_* if it's a covariant functor, while writing f^* if it's a contravariant functor.

Proof. We need to check

- well-defined on path homotopy classes.
- f_* is a group homomorphism.
$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \dfrac{1}{2}\right] \\ f(\beta(1-2s)), & \text{if } s \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$
- $\left(\mathrm{id}_{(X,x_0)}\right)_* = \mathrm{id}_{\pi_1(X,x_0)}$

- $(f_* \circ g_*) = (f \circ g)_*$

$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \implies f_*(\gamma_*(\gamma)).$$

DIY

$$\begin{tikzcd}
(X, x_0) \arrow[r, squiggly] \arrow[d, "f"'] & \pi_1(X, x_0) \arrow[d, "f_*"] \\
(Y, y_0) \arrow[r, squiggly] & \pi_1(Y, y_0)
\end{tikzcd}$$

■

Remark. We see that the construction of fundamental group is actually constructing a functor. Specifically,

$$\pi_1 : \underline{\mathrm{Top}_*} \to \underline{\mathrm{Gp}}$$

such that

- on objects:

$$\forall (X, x_0) \in \mathrm{Ob}(\underline{\mathrm{Top}_*}), \qquad \pi_1(X, x_0) = \text{fundamental group based at } x_0.$$

- on morphisms:

$$\forall f : (X, x_0) \to (Y, y_0), \qquad \pi_1(f) = f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

Our initial motivation is to construct a topological invariant, but we see that using π_1, we need an additional **base point**. But as you already imagined, the fundamental groupoid actually is a functor as well.

Before we proceed further, we need to see the category of groupoid, denoted by $\underline{\mathrm{Gpd}}$.

Definition 2.8 (Category of groupoid). The *category of groupoid*, denoted as $\underline{\mathrm{Gpd}}$, contains the following data.

- $\mathrm{Ob}(\underline{\mathrm{Gpd}})$: groupoids.
- $\mathrm{Hom}(\underline{\mathrm{Gpd}})$: functors between groupoids.
- Composition: For every $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \in \mathrm{Ob}(\underline{\mathrm{Gpd}})$,

$$\mathfrak{X} \xrightarrow{F} \mathfrak{Y} \xrightarrow{G} \mathfrak{Z}$$

  then $G \circ F \colon \mathfrak{X} \to \mathfrak{Z}$ is a functor defined as

  - on objects: $\forall X \in \mathrm{Ob}(\mathfrak{X})$,

$$G \circ F(X) \coloneqq G(F(X)).$$

  - on morphisms: $\forall X, Y \in \mathrm{Ob}(\mathfrak{X})$ and $f \colon X \to Y$,

$$G \circ F(f) \coloneqq G(F(f)).$$

- Identity. For every groupoid $\mathfrak{X}$, we define $\mathrm{id}_{\mathfrak{X}} \colon \mathfrak{X} \to \mathfrak{X}$, where
  - $\forall X \in \mathrm{Ob}(\mathfrak{X})$, $\mathrm{id}_{\mathfrak{X}}(X) = X$
  - $\forall f \in \mathrm{Hom}(\mathfrak{X})$, $\mathrm{id}_{\mathfrak{X}}(f) = f$.
- Associativity. Since the composition is defined based on two functors (given $\mathfrak{X} \xrightarrow{F} \mathfrak{Y} \xrightarrow{G} \mathfrak{Z}$), this holds trivially.

Proof. We need to show that the composition is well-defined. Specifically, we need to check

- $G \circ F(\mathrm{id}_X) = \mathrm{id}_{G \circ F(X)}$, since

$$G \circ F(\mathrm{id}_X) = G(F(\mathrm{id}_X)) = G(\mathrm{id}_{F(X)}) = \mathrm{id}_{G(F(X))} = \mathrm{id}_{G \circ F(X)}.$$

- Given $X_1, X_2, X_3 \in \mathrm{Ob}(\mathfrak{X})$ and

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

  we want to show $G \circ F(g \circ f) = G \circ F(g) \circ G \circ F(f)$. Firstly, since G is a functor, hence

$$G \circ F(g) \circ G \circ F(f) = G(F(g)) \circ G(F(f)) = G\left(F(g) \circ F(f)\right).$$

  Again, since F is a functor, so we further have

$$G \circ F(g) \circ G \circ F(f) = G(F(g \circ f)) = G \circ F(g \circ f).$$

■

Theorem 2.6 (Fundamental groupoid defines a functor). Π is a functor such that

$$\Pi\colon \underline{\mathrm{Top}} \to \underline{\mathrm{Gpd}},$$

where

- on objects: For every $X \in \mathrm{Ob}(\underline{\mathrm{Top}})$,

$$X \mapsto \Pi(X).$$

- on morphisms: for every $X, Y \in \mathrm{Ob}(\underline{\mathrm{Top}})$, $f\colon X \to Y$, define a functor

$$\Pi(f)\colon \Pi(X) \to \Pi(Y)$$

  such that

  - on objects: For every $p \in \mathrm{Ob}(\Pi(X)) = X$, $\Pi(f)(p) = f(p)$. i.e.,

$$\Pi(f)\colon \underbrace{\mathrm{Ob}(\Pi(X))}_{X} \to \underbrace{\mathrm{Ob}(\Pi(Y))}_{Y}.$$

  - on morphisms: For every $\langle\alpha\rangle \in \mathrm{Hom}_{\Pi(X)}(p, q)$, define

$$\Pi(f)(\langle\alpha\rangle) := \langle f \circ \alpha\rangle \in \mathrm{Hom}_{\Pi(Y)}(f(p), f(q)).$$

Proof. We need to check that the defined functor $\Pi(f)$ satisfies

- $\Pi(f)(\mathrm{id}_p) = \mathrm{id}_{f(p)}$. Indeed, since

$$\Pi(f)(\mathrm{id}_p) = \Pi(f)(\langle c_p\rangle) = \langle f \circ d_p\rangle = \left\langle c_{f(p)}\right\rangle = \mathrm{id}_{f(p)}.$$

- For every $p, q, r \in X = \mathrm{Ob}(\Pi(X))$,

$$p \xrightarrow{\langle\alpha\rangle} q \xrightarrow{\langle\beta\rangle} r$$

  we want to show $\Pi(f)\left(\langle\beta\rangle \circ \langle\alpha\rangle\right) = \Pi(f)(\langle\beta\rangle) \circ \Pi(f)(\langle\alpha\rangle)$. Indeed, since

$$\Pi(f)\left(\langle\beta\rangle \circ \langle\alpha\rangle\right) = \Pi(f)(\langle\alpha \cdot \beta\rangle) = \langle f \circ (\alpha \cdot \beta)\rangle\,,$$

  and

$$\Pi(f)(\langle\beta\rangle) \circ \Pi(f)(\langle\alpha\rangle) = \langle f \circ \beta\rangle \circ \langle f \circ \alpha\rangle = \langle (f \circ \alpha) \cdot (f \circ \beta)\rangle\,.$$

  Since $\langle f \circ (\alpha \cdot \beta)\rangle = \langle (f \circ \alpha) \cdot (f \circ \beta)\rangle$, hence $\Pi(f)$ is well-defined.

Now, we need to prove the same thing for Π, namely Π satisfies

- $\Pi(\mathrm{id}_X) = \mathrm{id}_{\Pi(X)}$ for all $X \in \mathrm{Ob}(\underline{\mathrm{Top}})$. This is trivial since

$$\Pi(\mathrm{id}_X)\colon \Pi(X) \to \Pi(X),$$

  - on objects: $p \mapsto \mathrm{id}_X(p) = p$.

- on morphisms: $p \xrightarrow{\langle\alpha\rangle} q \mapsto \langle \mathrm{id}_X \circ \alpha \rangle = \langle \alpha \rangle$.

- For all $X, Y, Z \in \mathrm{Ob}(\underline{\mathrm{Top}})$,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

then $\Pi(g \circ f) = \Pi(g) \circ \Pi(f)$. The diagrams are as follows.

$$\Pi(g \circ f) \colon \Pi(X) \to \Pi(Z)$$

and

$$\Pi(X) \xrightarrow{\Pi(f)} \Pi(Y) \xrightarrow{\Pi(g)} \Pi(Z)$$

We see that this equality is in the sense of functor, hence we consider

  - on objects: For every $p \in \mathrm{Ob}(\Pi(X)) = X$, $\Pi(g \circ f)(p) = g \circ f(p)$ and

$$\Pi(g) \circ \Pi(f)(p) = \Pi(g)(\Pi(f)(p)) = \Pi(g)(f(p) = g(f(p))),$$

  hence they're the same.

  - on morphisms: For all $\langle\alpha\rangle \in \mathrm{Hom}_{\Pi(X)}(p, q)$,
    * $\Pi(g \circ f)(\langle\alpha\rangle) = \langle (g \circ f) \circ \alpha \rangle$.
    * $\Pi(g) \circ \Pi(f)(\langle\alpha\rangle) = \Pi(g)\left(\underbrace{\Pi(f)(\langle\alpha\rangle)}_{\langle f \circ \alpha \rangle}\right) = \langle g \circ (f \circ \alpha) \rangle$.

  We see that they're the same.

■

Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

The goal is to compute $\pi_1(X)$ where $X = A \cup B$ using the data

$$\pi_1(A), \pi_1(B), \pi_1(A \cap B).$$

2.5 Free Product

2.5.1 Free Product

We first introduce a definition.

Definition 2.9 (Free product). Given some collections of groups $\{G_\alpha\}_\alpha$, the *free product*, denoted by $\underset{\alpha}{*} G_\alpha$ is a group such that

- Elements: Words in $\{g \colon g \in G_\alpha \text{ for any } \alpha\}$ modulo by the equivalence relation generated by
$$wg_ig_jv \sim w(g_ig_j)v$$
when both $g_i, g_j \in G_\alpha$. Also, for the identity element $\mathrm{id} = e_\alpha \in G_\alpha$ for any α such that
$$we_\alpha v \sim wv.$$
Specifically,
$$*_\alpha G_\alpha := {\{\text{words in } \{G_\alpha\}_\alpha\}} \big/ {\sim}.$$
- Operation: Concatenation of words.

Remark. In particular, we have the following universal property of $*_\alpha G_\alpha$. For every α, there is a ι_α such that

$$\iota_\alpha \colon G_\alpha \to *_\alpha G_\alpha, \qquad g \mapsto \overline{g},$$

where ι_α is a group homomorphism obviously. Further, $(*_\alpha G_\alpha, \iota_\alpha)$ satisfies the following property: For every group H and a group homomorphism $\varphi_\alpha \colon G_\alpha \to G$ for all α, there exists an unique group homomorphism $\varphi \colon *_\alpha G_\alpha \to H$ such that $\varphi \circ \iota_\alpha = \varphi_\alpha$, i.e., the following diagram commutes.

$$\begin{tikzcd}
G_{\alpha_1} \arrow[dr, "\iota_{\alpha_1}"] \arrow[ddrr, "\varphi_{\alpha_1}"'] & & \\
G_{\alpha_2} \arrow[r, "\iota_{\alpha_2}"] \arrow[drr, "\varphi_{\alpha_2}"'] & *_\alpha G_\alpha \arrow[dr, dashed, "\exists!\varphi"] & \\
\vdots & & H
\end{tikzcd}$$

Proof. The proof is straightforward. Firstly, we define $w = \overline{g_1 g_2 \ldots g_n} \in *_\alpha G_\alpha$, $g_i \in G_{\alpha_i}$,

$$\varphi(w) := \varphi_{\alpha_1}(g_1) \ldots \varphi_{\alpha_n}(g_n).$$

Now, we just need to check

- It's well-defined, since φ_α is a group homomorphism.
- φ is a group homomorphism.
- $\varphi \circ \iota_\alpha = \varphi_\alpha$.
- Such φ is unique. Suppose there exists another $\psi \colon *_\alpha G_\alpha \to H$, then
$$\psi \circ \iota_\alpha = \varphi_\alpha \implies \underset{g \in G_\alpha}{\forall} \psi(\overline{g}) = \psi_\alpha(g),$$

But then for every $w = \overline{g_1 g_2 \dots g_n} \in *_\alpha G_\alpha$, $g_i \in G_{\alpha_i}$, we have

$$\psi(w) = \psi(\overline{g_1} \dots \overline{g_n}) = \psi(\overline{g_1}) \dots \psi(\overline{g_n}) = \psi_{\alpha_1}(\overline{g_1}) \dots \psi_{\alpha_n}(\overline{g_n}),$$

which is just φ.

■

Remark. We further claim that this universal property determines such free product uniquely. i.e., assume there are another group $\widetilde{G}$ and $\widetilde{\iota}_\alpha \colon G_\alpha \to \widetilde{G}$. Assume $(\widetilde{G}, \widetilde{\iota}_\alpha)$ also satisfies the following property: For every group H and group homomorphism $\varphi_\alpha \colon G_\alpha \to H$, then there exists a unique group homomorphism $\varphi \colon \widetilde{G} \to H$ such that the following diagram commutes.

$$\begin{tikzcd}
G_{\alpha_1} \arrow[rr, "\widetilde{\iota}_{\alpha_1}"] \arrow[rrrdd, "\varphi_{\alpha_1}"] & & *_\alpha \widetilde{G}_\alpha \arrow[rdd, dashed, "\exists!\varphi"] & \\
G_{\alpha_2} \arrow[rru, "\widetilde{\iota}_{\alpha_2}"'] \arrow[rrrd, "\varphi_{\alpha_2}"] & & & \\
\vdots & & & H
\end{tikzcd}$$

Then, $\widetilde{G} \cong *_\alpha G_\alpha$.

Proof. Assume $(\widetilde{G}, \widetilde{\iota}_\alpha)$ satisfies the universal property mentioned above. Then from the universal property and viewing $\widetilde{G}$ and $*_\alpha G_\alpha$ as H separately, we obtain the following diagram.

$$\begin{tikzcd}
\widetilde{G} \arrow[rr, dashed, shift right, "\exists!g"'] & & *_\alpha G_\alpha \arrow[ll, dashed, shift right, "\exists!f"'] \\
& G_{\alpha_1} \arrow[lu, "\widetilde{\iota}_{\alpha_1}"'] \arrow[ru, "\iota_{\alpha_1}"] & \\
& G_{\alpha_2} \arrow[luu, "\widetilde{\iota}_{\alpha_2}"] \arrow[ruu, "\iota_{\alpha_2}"'] & \\
& \vdots \arrow[luuu] \arrow[ruuu] &
\end{tikzcd}$$

We claim that

$$g \circ f = \mathrm{id}, \quad f \circ g = \mathrm{id}.$$

To see this, we simply apply the same observation, for example,

$$\begin{tikzcd}
G_{\alpha_1} \arrow[dr, "\widetilde{\iota}_{\alpha_1}"] \arrow[dddrr, "\widetilde{\iota}_{\alpha_1}"] & & \\
G_{\alpha_2} \arrow[r, "\widetilde{\iota}_{\alpha_2}"] \arrow[ddrr, "\widetilde{\iota}_{\alpha_2}"'] & \widetilde{G} \arrow[ddr, dashed, "\exists! g\circ f"] & \\
\vdots & & \\
& & \widetilde{G}
\end{tikzcd}$$

where $g\circ f$ comes from the previous diagram. But notice that id let the diagram commutes also, and since it's unique, hence $g \circ f = \mathrm{id}$. Similarily, we have $f \circ g = \mathrm{id}$. ■

If you're careful enough, you may find out that all we're doing is just writing out a specific example of Lemma 1.3! Indeed, this is exactly the construction of a free group.

Definition 2.10 (Fibered coproduct). Given a category $\mathscr{C}$, let $f\colon Z \to X$, $g\colon Z \to Y$. The *fibered coproduct between* f *and* g is the data (W, p_1, p_2), where $W \in \mathrm{Ob}(\mathscr{C})$, $;_1 : X \to W$, $p_2\colon Y \to W$ satisfy the following.

- The diagram commutes.

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "p_1"] \\
Y \arrow[r, "p_2"'] & W
\end{tikzcd}$$

- For every $u\colon X \to U$, $v\colon Y \to U$ such that the following diagram commutes

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "p_1"] \arrow[ddr, bend left, "u"] & \\
Y \arrow[r, "p_2"'] \arrow[drr, bend right, "v"'] & W \arrow[dr, dashed, "\exists! h"] & \\
& & U
\end{tikzcd}$$

there exists a unique $h\colon W \to U$ such that $h \circ p = u$, $h \circ p_2 = v$.

We say

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & W
\end{tikzcd}$$

is a *Cocartesian* diagram.

Exercise. Prove that in a category $\mathscr{C}$, if the fibered coproduct of f and g exists

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$

then such fibered coproduct is unique up to isomorphism.

Remark. If we reverse all the directions of morphism, then we have so-called *fibered product*.

Example. Let's see some example.

1. Let $\mathscr{C} = \underline{\mathrm{Top}}$, and let $X \in \mathrm{Ob}(\underline{\mathrm{Top}})$. Given $X_0, X_1 \in X$, and $\mathrm{int}(X_0) \cup \mathrm{int}(X_1) = X$, if we have

$$\begin{aligned} i_0 &\colon X_0 \hookrightarrow X, & i_1 &\colon X_1 \hookrightarrow X \\ j_0 &\colon X_0 \cap X_1 \hookrightarrow X_0, & j_1 &\colon X_0 \cap X_1 \hookrightarrow X_1, \end{aligned}$$

then

$$\begin{tikzcd} X_0 \cap X_1 \arrow[r, hook, "j_0"] \arrow[d, hook, "j_1"'] & X_0 \arrow[d, hook, "i_0"] \\ X_1 \arrow[r, hook, "i_1"'] & X \end{tikzcd}$$

is a cocartesian diagram.

Proof. All we need to show is that given a topological space $Y \in \underline{\mathrm{Top}}$ and $f \colon X_0 \to Y$, $g \colon X_1 \to Y$ in $\underline{\mathrm{Top}}$, we have

$$f \circ j_0 = g \circ j_1.$$

$$\begin{tikzcd} X_0 \cap X_1 \arrow[r, hook, "j_0"] \arrow[d, hook, "j_1"'] & X_0 \arrow[d, hook, "i_0"] \arrow[ddr, bend left, "f"] & \\ X_1 \arrow[r, hook, "i_1"'] \arrow[drr, bend right, "g"'] & X \arrow[dr, dashed, "\exists! h"] & \\ & & Y \end{tikzcd}$$

We simply define $h \colon X \to Y$, $x \mapsto h(x)$ such that

$$h(x) = \begin{cases} f(x), & \text{if } x \in X_0; \\ g(x), & \text{if } x \in X_1. \end{cases}$$

h is clearly well-defined since the diagram commutes, so if $x \in X_0 \cap X_1$, then $f(x) = g(x)$. The only thing we need to show is that h is continuous. But this is obvious too since $X = \mathrm{int}(X_0) \cup \mathrm{int}(X_1)$, and

$$h|_{\mathrm{int}(X_0)} = f|_{\mathrm{int}(X_0)}, \quad h|_{\mathrm{int}(X_1)} = g|_{\mathrm{int}(X_1)}.$$

The uniqueness is trivial, hence this is indeed a cocartesian diagram. ■

2. Let $\mathscr{C} = \underline{\mathrm{Top}_*}$. Given $p \in X_0 \cap X_1$, where all other data are the same with the above example, we see that

$$\begin{tikzcd}
(X_0 \cap X_1, p) \arrow[r, hook, "j_0"] \arrow[d, hook, "j_1"'] & (X_0, p) \arrow[d, hook, "i_0"] \\
(X_1, p) \arrow[r, hook, "i_1"'] & (X, p)
\end{tikzcd}$$

is a cocartesian diagram.

3. Let $\mathscr{C} = \underline{\mathrm{Gp}}$. Given $P, G, H \in \mathrm{Ob}(\underline{\mathrm{Gp}})$, we claim that the fibered coproduct of i and j exists.

$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[d, "j"'] & G \\
H &
\end{tikzcd}$$

Consider $G * H$ be the free product between G and H, with two inclusions

$$\iota_1 \colon G \hookrightarrow G * H, \quad \iota_2 \colon H \hookrightarrow G * H.$$

$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[d, hook, "\iota_1"] \\
H \arrow[r, hook, "\iota_2"'] & G * H
\end{tikzcd}$$

Let

$$N := \left\langle \left\{ \iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1} \mid x \in P \right\} \right\rangle,$$

we define

$$G *_p H = {G * H} \big/ {N}.$$

$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[d, hook, "\iota_1"] \arrow[ddr, bend left, "\tau"] & \\
H \arrow[r, hook, "\iota_2"'] \arrow[drr, bend right, "\nu"'] & G * H \arrow[dr, "\pi"] & \\
& & G *_p H
\end{tikzcd}$$

We claim that

$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[d, "\tau"] \\
H \arrow[r, "\nu"'] & G *_p H
\end{tikzcd}$$

is a cocartesian diagram in $\underline{\mathrm{Gp}}$.

Proof. Firstly, since it's just an outer diagram from above, hence it commutes. So we only need to prove this diagram satisfies the second diagram.

Given any group K, for every $f\colon G \to K$, $g\colon H \to K$ such that the following diagram commutes.

$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[d, "\tau"] \arrow[ddr, bend left, "f"] & \\
H \arrow[r, "\nu"'] \arrow[drr, bend right, "g"'] & G *_p H \arrow[dr, dashed, "h"] & \\
& & K
\end{tikzcd}$$

We want to prove that there exists a unique $h\colon G *_p H \to K$ such that this diagram still commutes. The idea is simple, from the universal property of $G * H$, we see that there exists a unique $\widetilde{h}\colon G * H \to K$ such that

$$\widetilde{h} \circ \iota_1 = f, \quad \widetilde{h} \circ \iota_2 = g.$$

$$\begin{tikzcd}
P \arrow[rr, "i"] \arrow[dd, "j"'] & & G \arrow[dl, "\iota_1"'] \arrow[dd, "\tau"] \arrow[dddr, bend left, "f"] & \\
& G * H \arrow[dr, "\pi"] \arrow[ddrr, bend left, "\exists!\widetilde{h}"] & & \\
H \arrow[ur, "\iota_2"] \arrow[rr, "\nu"'] \arrow[drrr, bend right, "g"'] & & G * H \arrow[dr, dashed, "h"] & \\
& & & G *_p H
\end{tikzcd}$$

We see that we can actually factor $\widetilde{h}$ through π, as long as $\ker(\widetilde{h}) \supset \ker(\pi)$. Now, since

$$\ker(\pi) = \left\langle \left\{ \iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1} \mid x \in p \right\} \right\rangle,$$

we see that the kernel of π is indeed in the kernel of $\widetilde{h}$ since for every $x \in P$,

$$\widetilde{h}\left(\iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1}\right) = \underbrace{\widetilde{h} \circ \tau_1}_{f} \circ i(x) \cdot \underbrace{\widetilde{h} \circ \iota_2}_{g} \circ j(x^{-1}) = 1,$$

which implies $\ker(\widetilde{h}) \supset \ker(\pi)$.

$$\begin{tikzcd}
G * H \arrow[r, "\pi"] \arrow[d, "\widetilde{h}"'] & K \\
G *_p H &
\end{tikzcd}$$

We then see that there exists a unique $h\colon G *_p H \to K$ such that the above diagram commutes. ■

2.5.2 Free Product with Amalgamation

After seeing the above examples, the following definition should make sense.

Definition 2.11 (Free product with amalgamation). If two groups G_α and G_β have a common subgroup $S_{\{\alpha,\beta\}}$[a], given two inclusion maps[b] $i_{\alpha\beta}\colon S_{\{\alpha,\beta\}} \to G_\alpha$ and $i_{\beta\alpha}\colon S_{\{\alpha,\beta\}} \to G_\beta$, the *free product with amalgamation* ${}_\alpha{*}_S\, G_\alpha$ is defined as $\underset{\alpha}{*} G_\alpha$ modulo the normal subgroup generated by

$$\left\{ i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \mid s_{\{\alpha,\beta\}} \in S_{\{\alpha,\beta\}} \right\},$$

Namely[c],

$${}_\alpha{*}_S\, G_\alpha = \underset{\alpha}{*} G_\alpha \Big/ \left\langle i_{\alpha\beta}(s_{\{\alpha,\beta\}}) i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \right\rangle$$

and satisfies the universal property

$$\begin{tikzcd}
S \arrow[r, "i_{\alpha\beta}"] \arrow[d, "i_{\beta\alpha}"'] & G_\alpha \arrow[d] \arrow[ddr, bend left] & \\
G_\beta \arrow[r] \arrow[drr, bend right] & G_\alpha *_S G_\beta \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}$$

[a]In general, we don't need $S_{\{\alpha,\beta\}}$ to be a subgroup.
[b]We don't actually need $i_{\alpha\beta}, i_{\beta\alpha}$ to be inclusive as well.
[c]i.e., $i_{\alpha\beta}(s)$ and $i_{\beta\alpha}(s)$ will be identified in the quotient.

Remark. We see that

- We can then write out words such as $g_\alpha \cdot s \cdot g_\beta$ for $s \in S$, and view s as an element of G_α or G_β. In fact, we can do this construction even when i_α and i_β are not injective, though this means we are not working with a subgroup.
- Aside, in Top, the same universal property defines union

$$\begin{tikzcd}
A \cap B \arrow[r, hook, "i_\alpha"] \arrow[d, hook, "i_\beta"'] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \cup B \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}$$

for A, B are open subsets and the inclusion of intersection.

2.6 Seifert-Van Kampen Theorem

With Definition 2.11, we can now see the important theorem.

Theorem 2.7 (Seifert-Van Kampen Theorem). Given (X, x_0) such that $X = \bigcup_\alpha A_\alpha$ with

- A_α are open and path-connected and $\forall \alpha\ x_0 \in A_\alpha$
- $A_\alpha \cap A_\beta$ is path-connected for all α, β.

Then there exists a surjective group homomorphism

$$\mathop{*}_\alpha \colon \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0).$$

If we additionally have $A_\alpha \cap A_\beta \cap A_\gamma$ where they are all path-connected for every α, β, γ, then

$$\pi_1(X, x_0) \cong_\alpha *_{\pi_1(A_\alpha \cap A_\beta, x_0)} \pi_1(A_\alpha, x_0)$$

associated to all maps $\pi_a(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha), \pi_1(A_\beta)$ induced by inclusions of spaces. i.e., $\pi_1(X, x_0)$ is a quotient of the free product $*_\alpha \pi_1(A_\alpha)$ where we have

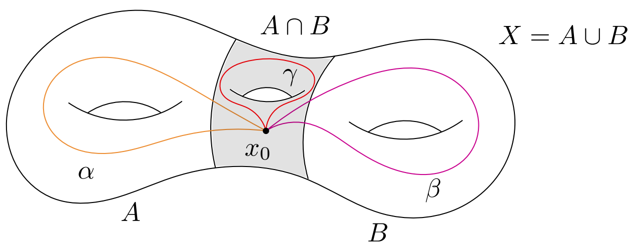
$$(i_{\alpha\beta})_* \colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$$

which is induced by the inclusion $i_{\alpha\beta} \colon A_\alpha \cap A_\beta \to A_\alpha$. We then take the quotient by the normal subgroup generated by

$$\{(i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_\alpha \cap A_\beta)\}.$$

We'll defer the proof of Theorem 2.7 until we get familiar with this theorem.

Example. We first see a great visualization of the Theorem 2.7.



Intuitively we see the fundamental group of X, which is built by gluing A and B along their intersection. As the fundamental group of A and B glued along the fundamental group of their intersection. In essence, $\pi_1(X, x_0)$ is the quotient of $\pi_1(A) * \pi_1(B)$ by relations to impose the condition that loops like γ lying in $A \cap B$ can be viewed as elements of either $\pi_1(A)$ or $\pi_1(B)$.

Remark. We can use a more abstract way to describe Theorem 2.7. Specifically, in the case that $n = 2$, i.e., $X = \bigcup_{i=1}^{2} A_i$, we let $A_i =: X_i$, then we have the following. The functor $\pi_1 \colon \underline{\mathrm{Top}_*} \to \underline{\mathrm{Gp}}$ maps the cocartesian diagram in $\underline{\mathrm{Top}_*}$ to a cocartesian diagram in $\underline{\mathrm{Gp}}$ as follows.

$$\begin{tikzcd} (X_0 \cap X_1, x_0) \arrow[r, "j_0"] \arrow[d, "j_1"'] & (X_0, x_0) \arrow[d, "i_0"] \\ (X_1, x_0) \arrow[r, "i_1"'] & (X, x_0) \end{tikzcd} \overset{\pi_1}{\longmapsto} \begin{tikzcd} \pi_1(X_0 \cap X_1, x_0) \arrow[r, "(j_0)_*"] \arrow[d, "(j_1)_*"'] & \pi_1(X_0, x_0) \arrow[d, "(i_0)_*"] \\ \pi_1(X_1, x_0) \arrow[r, "(i_1)_*"'] & \pi_1(X, x_0) \end{tikzcd}$$

Then, simply from the property of cocartesian diagram, we see that

$$\pi_1(X, x_0) \cong \pi_1(X_0, x_0) *_{\pi_1(X_0 \cap X_1, x_0)} \pi_1(X_1, x_0).$$

Additionally, there is a more general version of Theorem 2.7, which is defined on groupoid. The theorem is stated in Appendix A.1 with the proof.

With this more general version and the proof of which, we can apply it to Theorem 2.7. But one question is that, the above proof works in $\underline{\mathrm{Gpd}}$ rather than in $\underline{\mathrm{Gp}}$. We now see how to generalize a group to a groupoid.

For any group G, we can define a groupoid, denoted as G also, as follows.

- $\mathrm{Ob}(G) = \{\mathrm{pt}\}$, a one point set.
- $\mathrm{Hom}(G) = \{g \in G\}$.
- Composition: We define
$$g \circ h := h \cdot g.$$

We see that the associativity of group elements implies the associativity of composition defined above, and since there is an identity element in G, hence we also have an identity morphism, these two facts ensure that G is an category.

Furthermore, since for every $g \in G$, there is a $g^{-1} \in G$, hence every morphism is an isomorphism, which implies G is a groupoid.

With this, we see that we can view the following diagram in the category of groupoid $\underline{\mathrm{Gpd}}$.

$$\begin{tikzcd} \pi_1(X_0 \cap X_1, x_0) \arrow[r, "(j_0)_*"] \arrow[d, "(j_1)_*"'] & \pi_1(X_0, x_0) \arrow[d, "(i_0)_*"] \\ \pi_1(X_1, x_0) \arrow[r, "(i_1)_*"'] & \pi_1(X, x_0) \end{tikzcd}$$

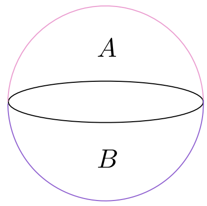
And to prove Theorem 2.7, we only need to show this diagram is cocartesian. This version of proof is given in Appendix A.2.

Lecture 11: Group Presentations

31 Jan. 10:00

Example. We now see some applications of Theorem 2.7.

1. We can use Seifert Van Kampen Theorem to compute the fundamental group of S^2. We see that



We see that $\pi_1(S^2)$ must be a quotient of $\pi_1(A) * \pi_1(B)$, but since $A, B \simeq D^2$, we know that $\pi_1(A)$ and $\pi_1(B)$ are both zero groups, thus $\pi_1(A) * \pi_1(B)$ is the zero group, and $\pi_1(S^2)$ is also the zero group.

Remark. Note that the inclusion of $A \cap B \to A$ induces the zero map $\pi_1(A \cap B) \to \pi_1(A)$, which cannot be an injection. In fact, we know that $\pi_1(A \cap B) \cong \mathbb{Z}$ since $A \cap B \simeq S^1$.

2. In the case of torus, consider the following.

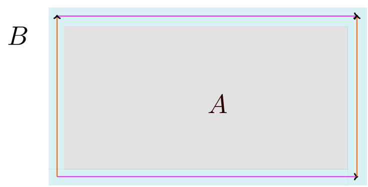


Figure 14: A is the interior, while B is the neighborhood of the boundary.

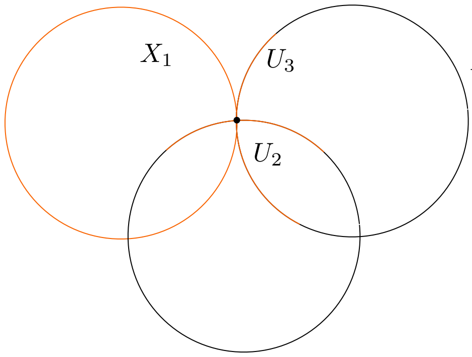
Now note that $A \simeq D^2$ and $B \simeq S^1 \vee S^1$, and since it's a thickening of the two loops around the torus in both ways, this suggests the question of how do we find $\pi_1(B)$? We grab a bit of knowledge from Seifert Van Kampen Theorem before we continue.

Exercise. Suppose we have path-connected spaces (X_α, x_α), and we take their wedge sum $\bigvee_\alpha X_\alpha$ by identifying the points x_α to a single point x. We also suppose a mild condition for all α, the point x_α is a deformation retract of some neighborhood of x_α.

For example, this doesn't work if we choose the *bad point* on the Hawaiian earring. Then we can use Seifert Van Kampen Theorem to show that

$$\pi_1\left(\bigvee_\alpha X_\alpha, x\right) \cong \mathop{*}_\alpha \pi_1\left(X_\alpha, x_\alpha\right).$$

Proof. If we denote



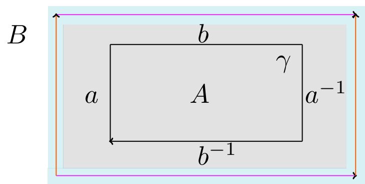
as C_n, then $\pi_1(C_n) \cong F_n$. Then we apply Theorem 2.7 to $A_\alpha = X_\alpha \cup_\beta U_\beta$ Specifically, take $A_\alpha = X_\alpha \cup_\beta U_\beta \simeq X_\alpha$, where U_β is a neighborhood of x_β which deformation retracts to x_β. This makes A_α open as desired. ■

Corollary 2.2. The wedge sum of circles $\pi_1(\bigvee_{\alpha \in A} S^1) = *_\alpha \mathbb{Z}$ is a free group on A. In particular, when A is finite, the fundamental group of a bouquet of circles is the free group on $|A|$.

Returning to the example of torus, we see that

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$

Further, we know that $\pi_1(A \cap B) \to \pi_1(A)$ is the zero map. We need to understand $\pi_1(A \cap B) \to \pi_1(B)$. To do so we need to understand how we're able to identify $\pi_1(S^1 \vee S_1)$ with F_2 and how we identify $\pi_1(S^1)$ with $\mathbb{Z}$. We update our Figure 14 to talk about this.



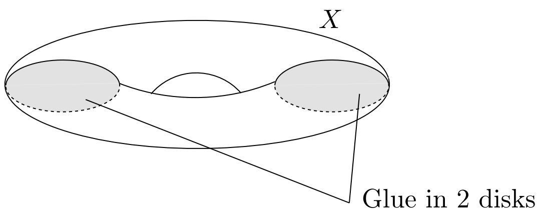
From this, we have

$$\begin{aligned} \pi_1(A \cap B) &\to \pi_1(B) \cong F_{a,b} \\ \gamma &\mapsto aba^{-1}b^{-1}. \end{aligned}$$

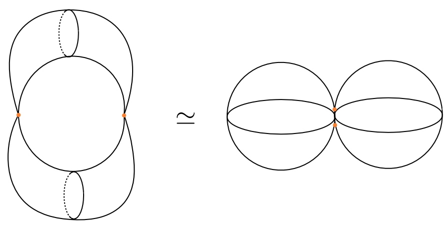
By Seifert Van Kampen Theorem, we identify the image of γ in $\pi_1(B)[aba^{-1}b^{-1}]$ with its image in $\pi_1(A)$, which is just trivial. Therefore, we have

$$\pi_1(T^2) = {F_{a,b}}\big/\langle aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2.$$

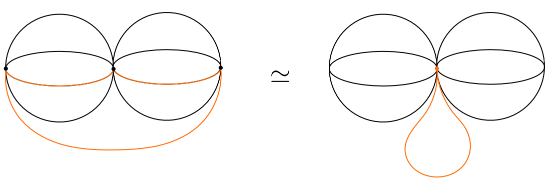
3. Let's see the last example which illustrate the power of Seifert Van Kampen Theorem. Start with a torus, and we glue in two disks into the hollow inside.



We'll call this space X, and out goal is to find $\pi_1(X)$. We can place a CW complex structure on this space so that each disk is a subcomplex. Then, we take quotient of each disk to a point without changing the homotopy type, hence X is homotopy to



By the same property, we can expand one of those points into an interval, and then contract the red path as follows.



This is exactly $S^2 \vee S^2 \vee S^1$. With Seifert Van Kampen Theorem, we have

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}.$$

Exercise. Consider $\mathbb{R}^2 \setminus \{x_1, \ldots, x_n\}$, that is the plane punctured at n points. Then $X \simeq \bigvee_n S^1$, so then

$$\pi_1(X) \simeq F_n.$$

One way to do this is to convince yourself that you can do a deformation retract the plane onto the following wedge.

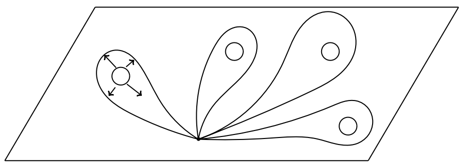


Figure 15: Deformation retract X onto wedge.

2.7 Group Presentation

In order to go further, we introduce the concept of *group presentation*.

Definition 2.12 (Group presentation). A *presentation* $\langle S \mid R \rangle$ of a group G is

- S: set of *generators*
- R: set of *relaters* (words in a generator and inverses)

such that

$$G \cong {}^{F_S}\big/_{\langle R \rangle},$$

where $\langle R \rangle$ is a subgroup normally generated by the elements of R.

Definition 2.13 (Finite presentation). If S and R are both finite, then $G = \langle S \mid R \rangle$ is a *finite presentation* if S, R are, and we say that G is *finitely presented*.

Note. One way to think about whether G is finitely presented is that if r is a word in R then $r = 1$, where 1 is the identity of G.

Example. We see that

1. $F_2 = \langle a, b \mid \; \rangle$
2. $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle = \langle a, b \rangle \,/\, \overline{\{aba^{-1}b^{-1}\}}$
3. $\mathbb{Z} \,/\, 3\mathbb{Z} = \left\langle a \mid a^3 \right\rangle$
4. $S_3 = \left\langle a, b \mid a^2, b^2, (ab)^3 \right\rangle$

Theorem 2.8. Any group G has a presentation.

Proof. We first choose a generating set S for G. Notice that we can even choose $S = G$ directly. From the universal property of free group, we see that there exists a surjective map $\varphi \colon F_S \to G, s \mapsto s$. Now, let R be the generating set for $\ker(\varphi)$, by the first isomorphism theorem[7], $G \cong F_S \,/\, \ker\varphi$. In fact, we have $G = \langle S \mid R \rangle$.

Specifically, $i \colon S \to G$ with $\iota \colon S \to F_S$, we have $\varphi \circ \iota = i$.

$$
\begin{tikzcd}
S \arrow[r, "\iota"] \arrow[dr, "i"'] & F_S \arrow[d, dashed, "\exists!\varphi"] \\
& G
\end{tikzcd}
$$

∎

Remark. The advantages of using group presentation are that given $G = \langle S \mid R \rangle$, it's now easy to define a homomorphism $\psi \colon G \to H$ given a map $\varphi \colon S \to H$, ψ extends to a group homomorphism $G \to H$ if and only if ψ vanishes on R, i.e., $\psi(r) = 0$ for all $r \in R$. We see an example to illustrate this.

Example. If we have $G = \langle a, b \mid aba \rangle$, a map $\varphi \colon \{a, b\} \to H$ gives a group homomorphism if and only if

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H.$$

This essentially uses the universal property of quotients.

Remark. It's sometimes easy to calculate G^{Ab}

$$G^{\mathrm{Ab}} = \langle S \mid R, \text{commutators in } S \rangle .$$

Example. Suppose all relations in R are commutators, so $R \subseteq [G, G]$. Then,

$$G^{\mathrm{Ab}} = (F_S)^{\mathrm{Ab}} = \bigoplus_S \mathbb{Z}.$$

Remark. The disadvantages are that this is computationally **very difficult**.

Example. Given $\mathbb{Z}^2 = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle$, let

$$\psi \colon \{a, b\} \to H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1_H \in H.$$

Namely, this is a presentation of the trivial group, but this is entirely unclear.

Lecture 12: Presentations for π_1 of CW Complexes

2 Feb. 10:00

Let's first see an exercise.

Exercise. Consider $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\} \rangle$
- $G_1 *_H G_2$ where $f_1 \colon H \to G_1$ and $f_2 \colon H \to G_2$. Then we have
$$G_1 *_H G_2 = \left\langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \left\{ f_1(h) f_2(h)^{-1} \mid h \in H \right\} \right\rangle .$$

2.7.1 Presentations for π_1 of CW Complexes

For X a CW complex, we have

1. A 1-dimensional CW complex has free π_1 (call its generators as $a_1, \ldots, a_n$).
2. Gluing a 2-disk by its boundary along a word w in the generators *kills* w in π_1. We then get a presentation for $\pi_1(X^2)$ given by
$$\langle a_1, \ldots, a_n \mid w \text{ for each 2-cell in } X_2 \rangle .$$
3. Gluing in any higher dimensional cells along their boundary will not change π_1. That is, in a CW complex, we have $\pi_1(X) = \pi_1(X^2)$.

Remark. We can write the above more precise.

1. Find free generators $\{a_i\}_{i \in I}$ for $\pi_1(X^1)$.
2. For each 2-disk D^2_α, write attaching map as word w_α in a_i. i.e.,
$$\pi_1(X^2) = \langle a_i \mid w_\alpha \rangle .$$
3. $\pi_1(X) = \pi_1(X^2)$.

Example. Given $G = \mathbb{Z} / n\mathbb{Z} = \langle a \mid a^n \rangle$, then we take a loop and then wind a 2-disk around the loop a for n times.

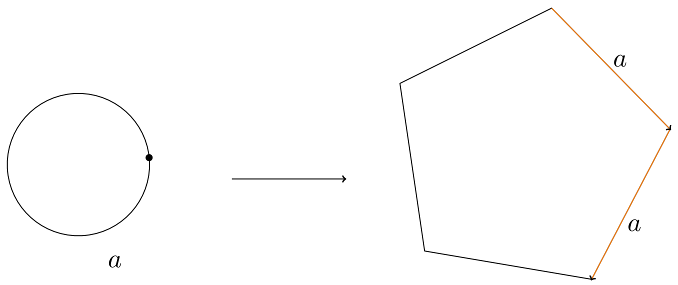


Figure 16: For $G = \mathbb{Z} / n\mathbb{Z} = \langle a \mid a^n \rangle$, we wind the boundary around a for n times.

[7] https://en.wikipedia.org/wiki/Isomorphism_theorems

We then see that given a group G with presentation $\langle S \mid R\rangle$, one can construct a 2-dimensional CW complex with $\pi_1 = G$ by

- Set $X^1 = \bigvee_{s\in S} S^1$
- For each relation $r \in R$, glue in a 2-disk along loops specified by the word r.

Every group is then π_1 of some space.

Theorem 2.9. If X is a CW complex and $\iota_1 : X^1 \hookrightarrow X$ and $\iota_2 : X^2 \hookrightarrow X$, then $(\iota_1)_*$ surjects onto π_1 and $(\iota_2)_*$ is an isomorphism on π_1.

Proof. ■ HW

Definition 2.14 (Graph, subgraph, tree, maximal tree). We import some topological definitions of graph theoretic concepts.

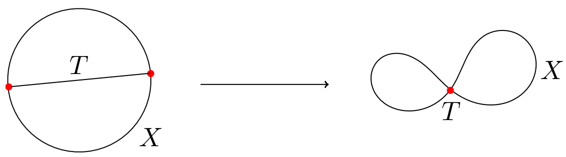
- A *graph* is a 1-dimensional CW complex.
- A *subgraph* is a subcomplex.
- A *tree* is a contractible graph.
- A tree in graph X (necessarily a subgraph) is *maximal* or *spanning* if it contains all the vertices.

Theorem 2.10. Every connected graph has a maximal tree. Every tree is contained in a maximal tree.

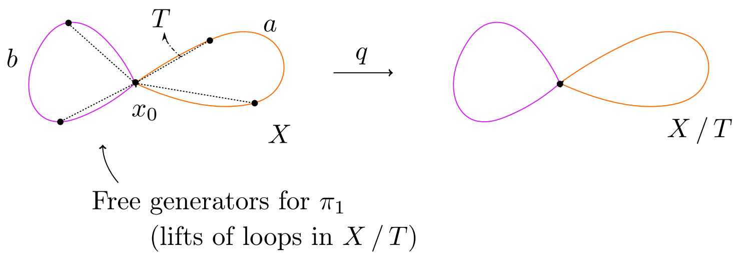
Corollary 2.3. Suppose X is a connected graph with basepoint x_0. Then $\pi_1(X, x_0)$ is a free group.

Furthermore, we can give a presentation for $\pi_1(X, x_0)$ by finding a spanning tree T in X. The generators of π_1 will be indexed by cells $e_\alpha \in X - T$, and e_α will correspond to a loop that passes through T, traverses e_α once, then returns to the basepoint x_0 through T.

Proof. The idea is simple. X is homotopy equivalent to $X \,/\, T$ via previous work on the homework, T contains all the vertices, so the quotient has a single vertex. Thus, it is a wedge of circles, and each e_α projects to a loop in $X \,/\, T$.



The current plan is to calculate the fundamental group of CW complexes. For now, we need to see that the fundamental group of a 1-skeleton (a graph) can be found by taking a maximal tree, and then quotienting out the space by the tree to get a wedge of circles.



We now prove that the maximal trees exist. Recall that X is a quotient of

$$X^0 \coprod_\alpha I_\alpha.$$

Each subset U is open if and only if it intersects each edge $\overline{e}_\alpha$ in an open subset. A map $X \to Y$ if and only if its restriction to each edge $\overline{e}_\alpha$ is continuous. Now, take X_0 to be a subgraph. Our goal is to construct a subgraph Y with

- $X_0 \subset Y \subset X$
- Y deformation retracts to X_0
- Y contains all vertices of X.

So if we take X_0 to be a vertex, then Y is our tree and we're done!

Our strategy now is to build a sequence $X_0 \subset X_1 \subset \dots$ and correspondingly, $Y_0 \subset Y_1 \subset \dots$. We start with X_0 and inductively define

$$X_i \coloneqq X_{i-1} \bigcup \text{all edges } \overline{e}_\alpha \text{ with one or both vertices in } X_{i-1}\ .$$

We then see that $X = \bigcup_i X_i$.[8] Now, let $Y_0 = X_0$. By induction, we'll assume that Y_i is a subgraph of X_i such that Check.

- Y_i contains all vertices of X_i.
- Y_i deformation retracts to Y_{i-1}.

We can then construct Y_{i+1} by taking Y_i and adding to it one edge to adjoin every vertex of X_{i+1}, namely

$$Y_{i+1} \coloneqq Y_i \bigcup \text{one edge to adjoint every vertex of } X_i{}^{9}$$

We then see that Y_{i+1} deformation retracts to Y_i by just smashing down each edge. Now, we can show that Y deformation retracts to $Y_0 = X_0$ by performing the deformation retraction from Y_i to Y_{i-1} during the time interval $[1/2^i, 1/2^{i-1}]$. ■

[8] [HPM02] do this by arguing the union on the right is both open and closed.
[9] This is possible if we assume Axiom of Choice.

Example. Let

- S^n: decompose into 2 open disks
- A_1: neighborhood of top hemisphere
- A_2: neighborhood of lower hemisphere

We see that $A_1 \cap A_2 \simeq S^{n-1}$, where we need $n \geq 2$ to let S^{n-1} be connected. We then have

$$\pi_1(S^n) \cong 0 \underset{\pi_1(A_1 \cap A_2)}{*} 0 = 0.$$

On the other hand, if $n \geq 3$, then we see that

$$S^n = {D^n \cup *}\big/{\sim}.$$

Since 2-skeleton is a point, thus $\pi_1(S^n) = 0$.

Lecture 13: Proof of Seifert-Van-Kampen Theorem

4 Feb. 10:00

2.8 Proof of Seifert-Van-Kampen Theorem

Let's start to prove Theorem 2.7.

Proof. The outline of the proof is the following. Let $X = \bigcup\limits_{\alpha} A_\alpha$ where A_α are open, path-connected and contain the bluepoint x_0. We also must guarantee that $A_\alpha \cap A_\beta$ is path-connected.

1. Since we have a map induced by the inclusions:

$$\Phi \colon \underset{\alpha}{*}\, \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0).$$

We want to show that ϕ is surjective. Take some $\gamma \colon I \to X$, then by the compactness of the interval I, we can show that there is a partition I with $s_1 < \ldots < s_n$ so that

$$\alpha|_{s_i, s_{i+1}} =: \alpha_i$$

has image in A_{α_i} for some α_i.[10] Specifically, since

- A_α is open for all α
- I is compact,

then for all i, we choose a path h_i from x_0 to $\gamma(s_i)$ in $A_{\sigma_{i-1}} \cap A_{\alpha_i}$, using path-connectedness of the pairwise intersections. Now, take γ and write it as

$$\gamma = (\gamma_1 \cdot \overline{h}_1) \cdot (\overline{h}_1 \cdot \gamma_2) \cdot \ldots \cdot (\gamma_{n-1} \cdot \overline{h}_{n-1}) \cdot (h_{n-1} \cdot \gamma_n).$$

Observe that each of these paths is fully contained in A_{α_i}, so this implies that $\gamma \in \operatorname{Im}(\Phi)$, therefore Φ is surjective.

[10]This is a good exercise for point-set topology.

2. For the next step, we'll show that the second part of Theorem 2.7. Assume that our triple intersections are path-connected. We want to show that $\ker(\Phi)$ is generated by
$$(i_{\alpha\beta})_*(\omega)(i_{\beta\alpha})_*(\omega)^{-1},$$
where
$$i_{\alpha\beta}\colon A_\alpha \cap A_\beta \hookrightarrow A_\alpha$$
for all loops $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$.

Before we go further, we'll need some definition.

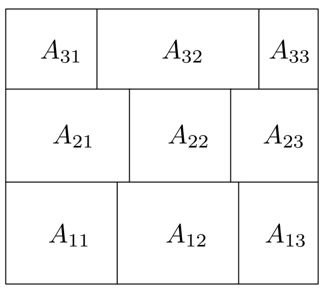
Definition 2.15 (Factorization). A *factorization* of a homotopy class $[f] \in \pi_1(X, x_0)$ is a formal product
$$[f_1][f_2]\dots[f_\ell]$$
with $[f_i] \in \pi_1(A_\alpha, x_0)$ such that
$$f \simeq f_1 \cdot f_2 \cdot \dots \cdot f_\ell.$$

We showed that every $[f]$ has a factorization in step 1 already. Now we want to show taht two factorizations
$$[f_1] \cdot \dots \cdot [f_\ell] \text{ and } [f_1'] \cdot \dots \cdot [f_{\ell'}']$$
of $[f]$ must be related by two moves:

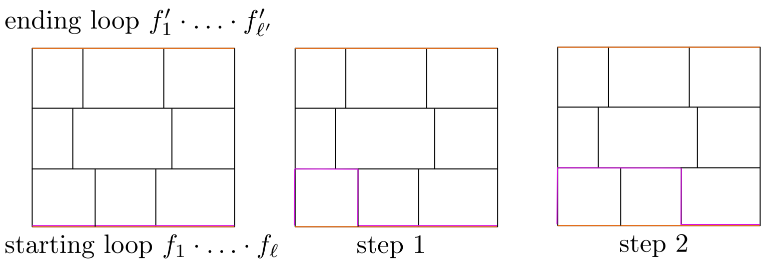
(a) $[f_i] \cdot [f_{i+1}] = [f_i \cdot f_{i+1}]$ if $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$. Namely, the reaction defining the free product of groups.

(b) $[f_i]$ can be viewed as an element of $\pi_1(A_\alpha, x_0)$ or $\pi_1(A_\beta, x_0)$ whenever
$$[f_i] \in \pi_1(A_\alpha \cap A_\beta, x_0).$$
This is the relation defining the amalgamated free product.

Now, let $F_t\colon I \times I \to X$ be a homotopy from $f_1 \dots f_\ell$ to $f_1' \dots f_{\ell'}'$, since they both represent $[f]$. We subdivide $I \times I$ into rectangles R_{ij} so that
$$F(R_{ij}) \subseteq A_{\alpha_{ij}} =: A_{ij}$$
for some α_{ij} using compactness. We also argue that we can perturb the corners of the squares so that a corner lies only in three of the A_α's indexed by adjacent rectangles.

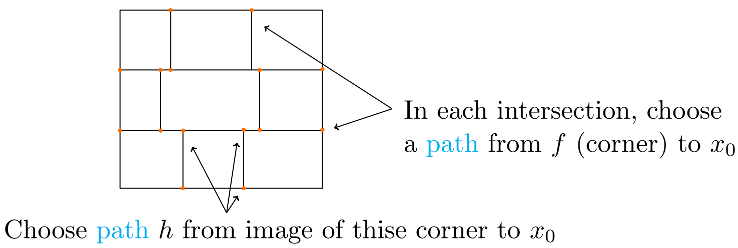


We also argue that we can set up our subdivision so that the partition of the top and bottom intervals must correspond with the two factorizations of $[f]$. We then perform our homotopy one rectangle at a time.



Idea: Argue that homotoping over a single rectangle has the effect of using allowable moves to modify the factorization.

At each triple intersection, choose a path from f (corner) to x_0 which lies in the triple intersection, so we use the assumption that the triple intersections are path-connected.



Along the top and bottom, we make choices compatible with the two factorizations. It's now an exercise to check that these choices result in homotoping across a rectangle gives a new factorization related by an allowable move.

■

Lecture 14: Covering Spaces Theory

7 Feb. 10:00

3 Covering Spaces

Lack of content... Things are in Hw.

3.1 Lifting Properties

As always, we start with a definition.

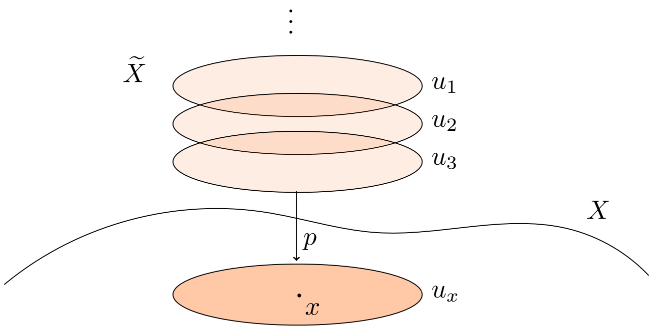
Definition 3.1 (Covering space). A *covering space* $\widetilde{X}$ of X is a space $\widetilde{X}$ and a map $p\colon \widetilde{X} \to X$ such that $\forall x \in X$ $\exists$ neighborhood u_x with $p^{-1}(u_x)$ the disjoint union of open sets

$$\coprod_\alpha u_\alpha$$

such that

$$p|_{u_\alpha} : u_x \to u_x$$

is a homeomorphism for every α.



We sometimes call p as *covering map*.

Although we already investigate into covering spaces quite a lot in homework, but a terminology is still worth mentioning.

Definition 3.2 (Evenly covered). Let $p\colon \widetilde{X} \to X$ be a continuous map of spaces. Then an open subset $U \subseteq X$ is called *evenly covered by p* if

$$p|_{V_i} : V_i \to U$$

is a homeomorphism.

We call the parts V_i of the partition $\coprod_i V_i$ of $p^{-1}(U)$ *slices*.

Remark. We see that p is a covering map if and only if every point $x \in X$ has a neighborhood which is evenly covered.

We immediately have the following proposition.

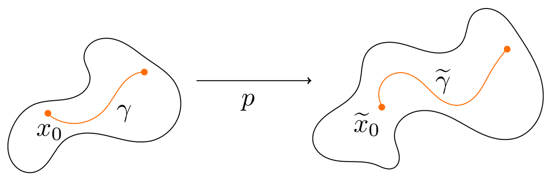
Proposition 3.1 (Homotopy lifting property). The covering spaces satisfy the *homotopy lifting property* such that the following diagram commutes.

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widetilde{F}_0"] \arrow[d, hook] & \widetilde{Y} \arrow[d, "p"] \\
X \times I \arrow[r, "F_t"'] \arrow[ur, "\exists!\widetilde{F}_t"] & Y
\end{tikzcd}$$

Proof. We already proved this in homework! ■

Corollary 3.1 (Path lifting property). For each path $\gamma \colon I \to X$ in X, $\widetilde{x}_0 \in p^{-1}(\gamma(0))$ such that there exists a unique lift $\widetilde{\gamma}$ starting at $\widetilde{x}_0$.

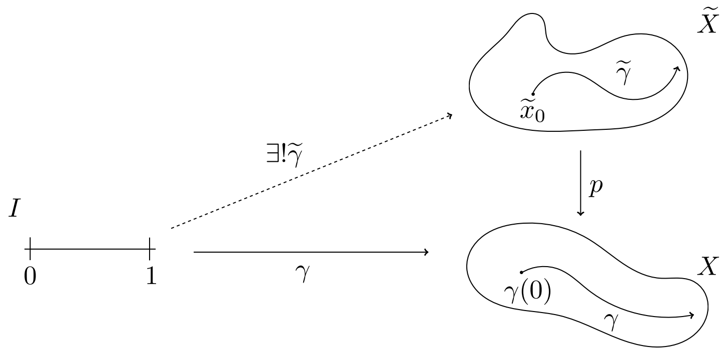
And for each path homotopy $I \times I \to X$, there exists a unique path homotopy $\widetilde{\gamma} \colon I \times I \to \widetilde{X}$ starting at $\widetilde{x}_0$.



Though we can directly use Proposition 3.1 to prove this, but we can see some insight by directly proving this.

Proof. We prove them separately.

Lifting a path. Assume that we have the following lift.



We first prove that a path will be lifted uniquely to a path $\widetilde{\gamma}$ from $\widetilde{x}_0$. For every

$x \in X$, there exists an open neighborhood U_x such that

$$p^{-1}(U_x) = \coprod_{\alpha} U_{x_\alpha},$$

where for every α,

$$p|_{U_{x_\alpha}} : U_{x_\alpha} \to U_x$$

is a homeomorphism. We see that $\{U_x \mid x \in X\}$ is an open cover of X, hence

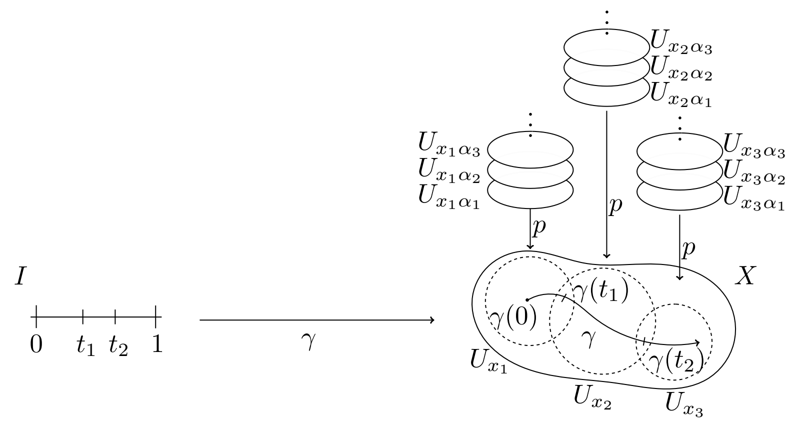
$$\{\gamma^{-1}(U_x) \mid x \in X\}$$

is an open cover of $[0,1]$. Note that since $[0,1]$ is a compact metric space, from Lebesgue Lemma[11], there exists a partition of $[0,1]$ such that

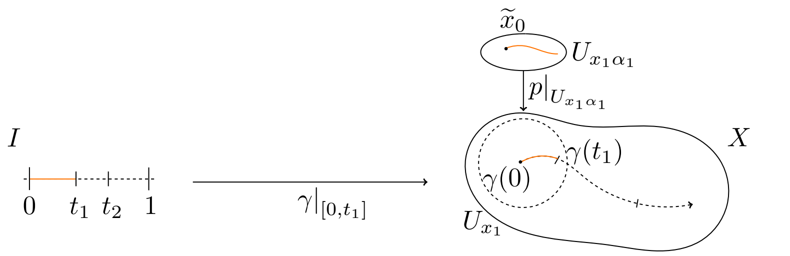
$$0 = t_0 < t_1 < \ldots < t_k = 1$$

such that for every i, $[t_i, t_{i+1}] \subset \gamma^{-1}(U_x)$ for some x. Without loss of generality, we assume that $[t_i, t_{i+1}] \subset \gamma^{-1}(U_{x_i})$, i.e.,

$$\gamma([t_i, t_{i+1}]) \subset U_{x_i}.$$



Now, since $p(\widetilde{x}_0) = \gamma(0)$ for $\gamma_0 \in U_{x_1}$ and $\widetilde{x}_0 \in p^{-1}(U_{x_1})$, we may assume $\widetilde{x}_0 \in U_{x_1\alpha_1}$. Consider lifting the first segment, namely $\gamma([0, t_1])$.



[11] https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma

Specifically, let $\widetilde{\gamma}_1(t) = \left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ \gamma(t)$ for $0 \leq t \leq t_1$, we see that

$$\widetilde{\gamma}_1 \colon [0, t_1] \to \widetilde{X}$$

is a lift of $\gamma|_{[0,t_1]}$ from $\widetilde{x}_0$. We claim that this lift is unique. Consider there exists another lift from $\widetilde{x}_0$ $\widetilde{\widetilde{\gamma}}_1 \colon [0, t_1] \to \widetilde{X}$, then since

- $\widetilde{\widetilde{\gamma}}_1(0) = \widetilde{x}_0$
- $\widetilde{\widetilde{\gamma}}_1$ is continuous
- $\widetilde{x}_0 \in U_{x_1\alpha_1}$,

we see that $\widetilde{\widetilde{\gamma}}_1(0, t_1) \subset U_{x_1\alpha_1}$, which implies

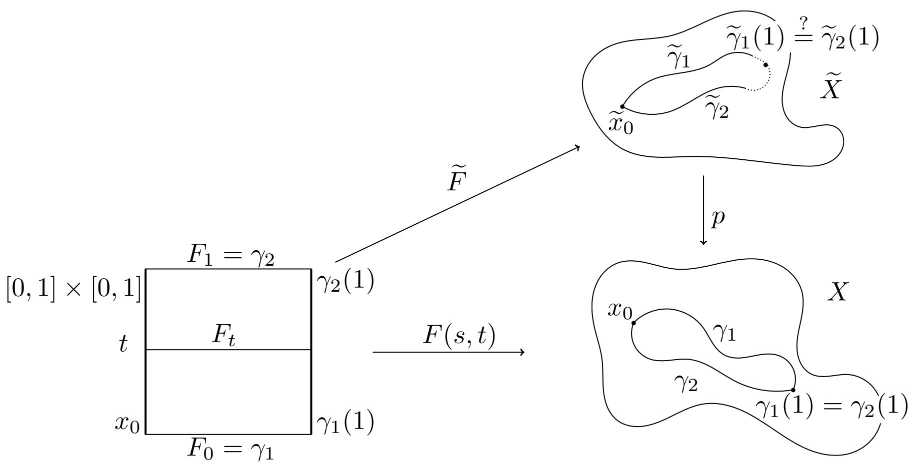
$$\begin{tikzcd} {[0,t_1]} \arrow[r, "\widetilde{\widetilde{\gamma}}_1"] \arrow[dr, "\gamma|_{[0,t_1]}"'] & U_{x_1\alpha_1} \arrow[d, "p|_{U_{x_1\alpha_1}}"] \\ & U_{x_1} \end{tikzcd} \implies \widetilde{\widetilde{\gamma}}_1 = \left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ \gamma|_{[0,t_1]} = \widetilde{\gamma}_1,$$

hence this lift is unique. Now, we see that we can simply repeat this argument, namely replacing t_i by t_{i+1}, $\widetilde{\gamma}_i(t_i)$ by $\widetilde{\gamma}_{i+1}(t_{i+1})$ and so on. Since this partition is finite, hence in finitely many steps, we obtain a unique path homotopy $\widetilde{\gamma}$ by concatenating all $\widetilde{\gamma}_i$ starting at $\widetilde{x}_0$.

Lifting a path homotopy. We now consider lifting a path homotopy. Consider

$$\gamma_1 \underset{F}{\simeq} \gamma_2 \text{ rel}\{0, 1\}$$

we'll show that $\widetilde{\gamma}_1 \underset{\widetilde{F}}{\simeq} \widetilde{\gamma}_2$ rel$\{0, 1\}$ where $p \circ \widetilde{F} = F$. Firstly, we denote $x_0 := \gamma_1(0) = \gamma_2(0)$, such that



We claim that it's sufficient to show that there exists a continuous $\widetilde{F}\colon I\times I\to X$ such that $p\circ\widetilde{F}=F$, and $\widetilde{F}(\{0\}\times I)=x_0$. It's because

$$p\circ\widetilde{F}_0=F_0=\gamma_1,\quad p\circ\widetilde{F}_1=F_1=\gamma_2$$

where $\widetilde{F}_0,\widetilde{F}_1$ is γ_1,γ_2's lifting starting at $\widetilde{x}_0$, respectively. And since $p\circ\widetilde{F}=F$, we have

$$p\left(\widetilde{F}(\{1\}\times I)\right)=x_1\implies\widetilde{F}(\{1\}\times I)\subset p^{-1}(\{x_1\}),$$

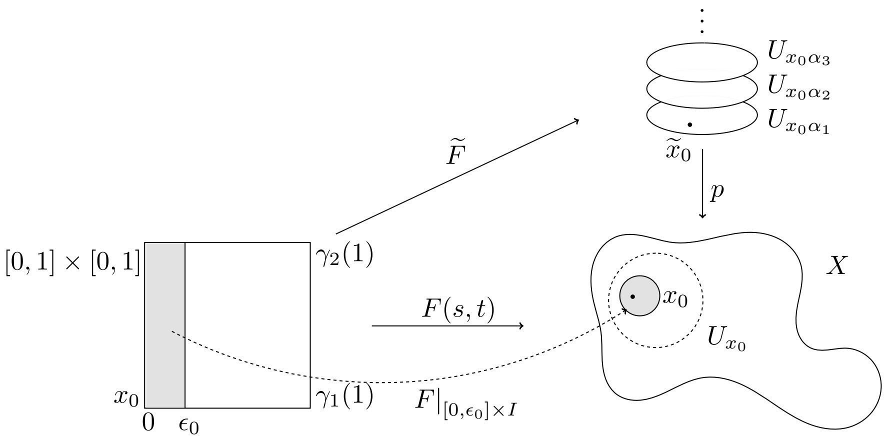
which implies $\exists\widetilde{x}_1\in p^{-1}(\{x_1\})$ such that $\widetilde{F}(\{1\}\times I)=\widetilde{x}_1$ since we know that $p^{-1}(\{x_1\})$ is a discrete points set and $\widetilde{F}$ is assumed to be continuous, and $\{1\}\times I$ is connected. We now show $\widetilde{F}$ exists.

We define

$$\begin{aligned}\widetilde{F}\colon I\times I&\to X\\(s,t)&\mapsto\widetilde{F}_t(s),\end{aligned}$$

where $\widetilde{F}_t\colon[0,1]\to\widetilde{X}$ is a lift starting at $\widetilde{x}_0$ of $F_t\colon[0,1]\to X, s\mapsto F(s,t)$. Obviously, $p\circ\widetilde{F}=F$ from the uniqueness of the lift of a path, and also, $\widetilde{F}(\{0\}\times I)=\widetilde{x}_0$ holds trivially, hence we only need to show $\widetilde{F}$ is continuous.

1. We show that $\exists\epsilon_0>0$ such that $\widetilde{F}\big|_{[0,\epsilon_0]\times I}$ is continuous.



Since F is continuous, we see that there exists an open neighborhood U_{x_0} of x_0 such that $p^{-1}(U_{x_0})=\coprod_\alpha U_{x_0\alpha}$, where

$$p|_{U_{x_0\alpha}}:U_{x_0\alpha}\overset{\cong}{\to}U_{x_0}.$$

Since $F^{-1}(U_{x_0})$ is an open set contain $\{0\}\times I$, there exists a $\epsilon_0>0$ such that $[0,\epsilon_0]\times I\subset F^{-1}(U_{x_0})$,[12] which implies

$$F\left([0,\epsilon_0]\times I\right)\subset U_{x_0}.$$

[12]Notice that we're working on product topology here.

Note that $x_0 \in U_{x_0}$ and $p(\widetilde{x}_0) = x_0$, we may assume $\widetilde{x}_0 \in U_{x_0\alpha_1}$. Consider $\left(p|_{U_{x_0\alpha_1}}\right)^{-1} \circ F|_{[0,\epsilon_0]\times I}$, which is a lift of $F|_{[0,\epsilon_0]\times I}$. We claim that

$$\left(p|_{U_{x_0\alpha_1}}\right)^{-1} \circ F|_{[0,\epsilon_0]\times I} = \widetilde{F}\Big|_{[0,\epsilon_0]\times I}.$$

This is because for every $t \in I$,

$$s \mapsto \left(p|_{U_{x_0\alpha_1}}\right)^{-1} \circ F|_{[0,\epsilon_0]\times I}(s,t)$$

is a lift starting at $\widetilde{x}_0$; also, for every $t \in I$,

$$s \mapsto \widetilde{F}\Big|_{[0,\epsilon_0]\times I}(s,t)$$

is a lift of F_t starting at $\widetilde{x}_0$. From the uniqueness of the lift of paths, we see that they're equal. Note that this implies $\widetilde{F}$ is now continuous at $[0,\epsilon_0] \times I$, since F is continuous and $p|_{U_{x_0\alpha_1}}$ is a homeomorphism, hence continuous, then from

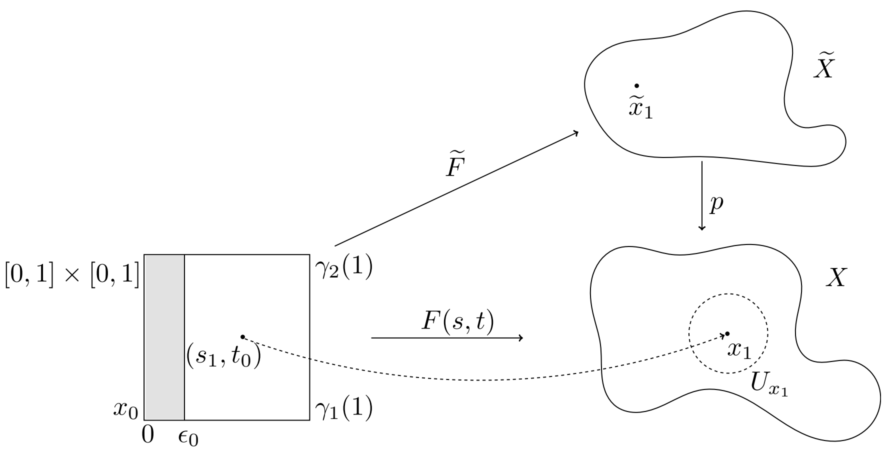
$$\widetilde{F}\Big|_{[0,\epsilon_0]\times I} = \underbrace{\left(p|_{U_{x_0\alpha_1}}\right)^{-1}}_{\text{continuous}} \circ \underbrace{F|_{[0,\epsilon_0]\times I}}_{\text{continuous}},$$

we see that $\widetilde{F}$ is indeed continuous at $[0,\epsilon_0] \times I$.

2. We now prove that $\widetilde{F} \colon I \times I \to \widetilde{X}$ is continuous. Assume there exists $(s_0, t_0) \in I \times I$ such that $\widetilde{F}$ is discontinuous at (s_0, t_0). Then consider

$$0 < \epsilon_0 \leq \inf \underbrace{\left\{ s \mid \widetilde{F} \text{ is discontinuous at } s, t_0 \right\}}_{\exists s_0 \implies \neq \varnothing} =: s_1,$$

where the first inequality is from the first step.

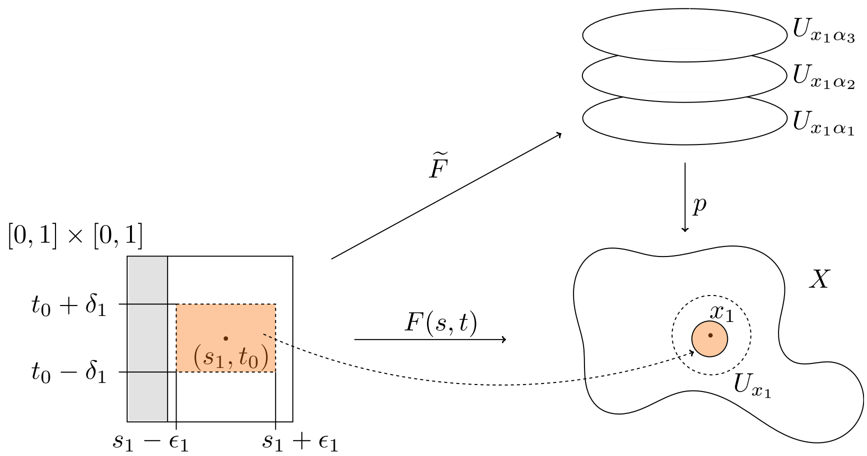


Let $x_1 \coloneqq F(s_1, t_0)$, $\widetilde{x}_1 \coloneqq \widetilde{F}(s_1, t_0)$, then there exists an open neighborhood U_{x_1} in X such that $x_1 \in U_{x_1} = \coprod_\alpha U_{x_1\alpha}$, where

$$p|_{U_{x_1\alpha}} : U_{x_1\alpha} \overset{\cong}{\rightarrow} U_{x_1}.$$

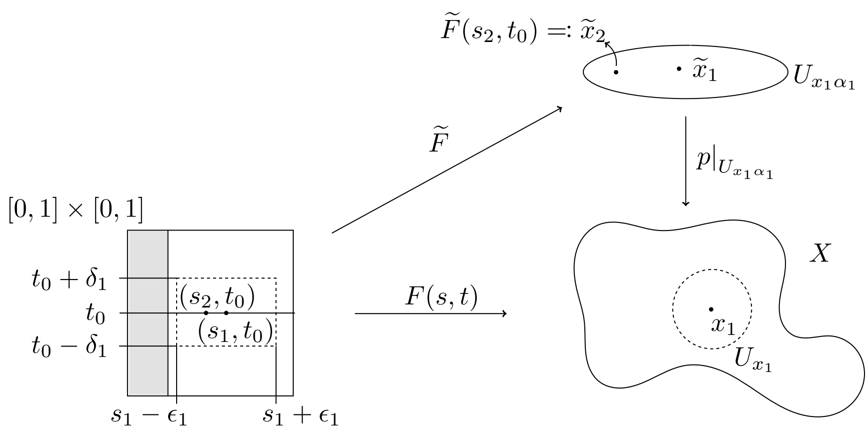
Since F is continuous, there exists an $\epsilon_1 > 0$, $\delta_1 > 0$ such that

$$F\left((s_1 - \epsilon_1, s_1 + \epsilon_1) \times (t_0 - \delta_1, t_0 + \delta_1)\right)^{13} \subset U_{x_1}.$$



We may assume $\widetilde{x}_1 \in U_{x_1\alpha_1}$. Then, we see that $\widetilde{F}_{t_0}$ is a lift of F_{t_0}, which means $\widetilde{F}_{t_0}$ is continuous, hence there exists an s_2 such that $s_1 - \epsilon_1 < s_2 < s_1$ such that

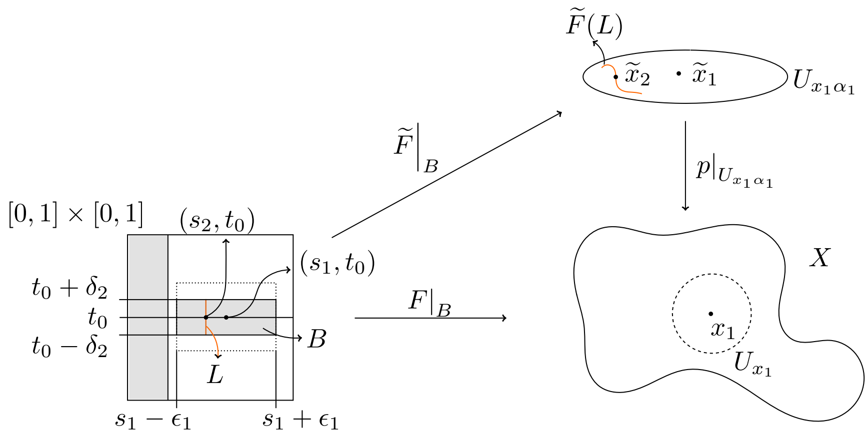
$$\widetilde{F}(s_2, t_0) \in U_{x_1\alpha_1}.$$



[13] Notice that here we're considering **open** box.

We see that $\widetilde{F}$ is continuous at (s_2, t_0), hence there exists a $\delta_2 > 0$ such that

$$\widetilde{F}\left(\{s_2\} \times (t_0 - \delta_2, t_0 + \delta_2)\right) \subset U_{x_1\alpha_1}.$$ [14]



Now, observe that $\widetilde{F}(B) \subset U_{x_1\alpha_1}$. To see this, consider a fixed $t \in (t_0 + \delta_2, t_0 - \delta_2)$, then the map $\widetilde{F}$ is

$$[s_1 - \epsilon_1, s_1 + \epsilon_1] \to \widetilde{X}, \quad s \mapsto \widetilde{F}(s,t) = \widetilde{F}_t(s).$$

Specifically,

$$\widetilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1]) \subset p^{-1}(U_{x_1}) = \coprod_{\alpha} U_{x_1\alpha},$$

with the fact that $\widetilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1])$ is connected, and $\widetilde{F}_t(s_2) \in U_{x_1\alpha_1}$ with $\widetilde{F}_t$ is a lift of F_t, hence continuous, so

$$\widetilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1]) \subset U_{x_1\alpha_1}.$$

This is true for every $t \in [t_0 - \delta_2, t_0 + \delta_2]$, hence $\widetilde{F}\Big|_B \subset U_{x_1\alpha_1}$. Now, since

$$p|_{U_{x_1\alpha_1}} \circ \widetilde{F}\Big|_B = F|_B\,,$$

and

$$\left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ F|_B : B \to U_{x_1\alpha_1},$$

so

$$p|_{U_{x_1\alpha_1}} \circ \left(\left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ F|_B\right) = F|_B$$

[14] Note that here we can also consider a closed interval, which matches what we're going to do. Namely, we're going to construct a **closed** box B. But this is just a technical detail.

obviously. Since $p|_{U_{x_1\alpha_1}}$ is a homeomorphism, we have

$$\widetilde{F}\Big|_B = \underbrace{\left(p|_{U_{x_1\alpha_1}}\right)^{-1}}_{\text{continuous}} \circ \underbrace{F|_B}_{\text{continuous}},$$

hence we have $\widetilde{F}\Big|_B$ is continuous, which leads to a contradiction since

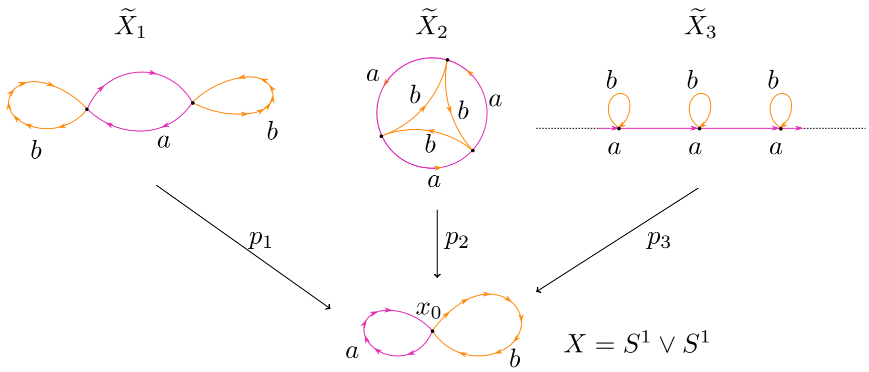
$$s_1 = \inf\left\{ s \mid \widetilde{F} \text{ is discontinuous at } s, t_0 \right\},$$

while $\widetilde{F}$ is continuous for all B, hence we see that $\widetilde{F} : I \times I \to \widetilde{X}$ is continuous.[15]

■

Example. Let see some examples.

1. Covers of $S^1 \vee S^1$.



Note that in each cover (those three on the top), the black dot is the preimage of $\{x_0\}$, namely $p_i^{-1}(\{x_0\})$.

Remark. We see that for each $p_i^{-1}(\{x_0\})$, there are exactly

- one a edge goes out
- one b edge goes out
- one a edge goes in
- one b edge goes in

It turns out that there are much more covers of $S^1 \vee S^1$, as long as this main property is satisfied.

[15]There is a tricky situation, namely while $s_1 = 1$. But this can be considered also.

Proposition 3.2. Let

$$p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$$

be a covering map. Then

1. $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.
2. $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$, which picks out the subset

$$\{[\gamma] \mid \text{Lift } \widetilde{\gamma} \text{ starting at } \widetilde{x}_0 \text{ is a loop.}\} \,.$$

Proof. We prove this one by one.

1. Suppose $\widetilde{\gamma} \in \pi_1(\widetilde{X}, \widetilde{x}_0)$ is in $\ker(p_*)$. Then

$$[\gamma] = p_*([\widetilde{\gamma}]) = [p \circ \widetilde{\gamma}] \,.$$

Let γ_t be a nullhomotopy from γ to the constant loop c_{x_0} rel$\{0, 1\}$. We can then lift γ_t to $\widetilde{\gamma}_t$ where $\widetilde{\gamma}_0 = \widetilde{\gamma}$. Now, we claim that

- $\widetilde{\gamma}$ is a homotopy rel$\{0, 1\}$.
- $\widetilde{\gamma}_1$ is the constant loop $c_{\widetilde{x}_0}$.

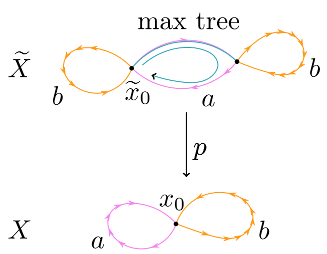
$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ I \arrow[ur, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ I \times I \arrow[ur, "\widetilde{\gamma}_t"] \arrow[r, "\gamma_t"'] & X \end{tikzcd}$$

We see that the above diagrams prove the first claim, since we know that the left and right edge of $I \times I$ maps to x_0 under γ_t, and $c_{\widetilde{x}_0}$ lifts this, so by uniqueness $t \mapsto \widetilde{\gamma}_t(0)$ and $t \mapsto \widetilde{\gamma}_t(1)$ must be constant paths at $\widetilde{x}_0$ as desired.

Then the lift $\widetilde{\gamma}_t$ is a homotopy of paths to the constant loop, so $[\widetilde{\gamma}] = 1$.

2. Let see an example to show the idea of the proof.

Example. Given



Then

$$p_*\pi_1 = \left\langle b, a^2, ab\overline{a} \right\rangle \subseteq \pi_1(X) = \langle a, b \mid \rangle \,.$$

$\blacksquare$

Proposition 3.3 (Lifting criterion). Let $p\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$ be a covering map. Given

- $f\colon (X, x_0) \to (Y, y_0)$;
- X is path-connected, locally path-connected,

then a lift

$$\widetilde{f}\colon (X, x_0) \to (\widetilde{Y}, y_0)$$

exists if and only if

$$f_*\left(\pi_1(X, x_0)\right) \subseteq p_*(\pi_1(\widetilde{Y}, \widetilde{y}_0)).$$

In diagram, we have

$$\begin{tikzcd}
 & (\widetilde{Y}, \widetilde{y}_0) \arrow[d, "p"] \\
(X, x_0) \arrow[ur, dashed, "\exists \widetilde{f}"] \arrow[r, "f"'] & (Y, y_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(\widetilde{Y}, \widetilde{y}_0) \arrow[d, "p_*"] \\
\pi_1(X, x_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(Y, y_0)
\end{tikzcd}$$

Lecture 15: Lifting

9 Feb. 10:00

Before proving Proposition 3.3, we first see an application.

Example. Prove that every continuous map $f\colon \mathbb{R}P^2 \to S^1$ is nullhomotopic.

Answer. If we can show that there is a lift $\widetilde{f}\colon \mathbb{R}P^2 \to \mathbb{R}$ of f, then we're done since we can apply the straight line nullhomotopy on $\mathbb{R}$, i.e.,

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, "p"] \\
\mathbb{R}P^2 \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & S^1
\end{tikzcd}$$

and consider $f = p \circ \widetilde{f}$ compose nullhomotopy with p, so $f \simeq$ constant map. Specifically, since $\pi_1(\mathbb{R}P^2) = \mathbb{Z} / 2\mathbb{Z}$ and $\pi_1(S^1) = \mathbb{Z}$, hence

$$f_*(\pi_1(\mathbb{R}P^2)) = 0$$

since $\mathbb{Z}$ has no (nonzero) torsion. So it lifts by Proposition 3.3.

Now we can proof Proposition 3.3.

Proof. We prove two directions as follows.

Necessary. We see that we can factorize f_* as

$$f_* = p_* \circ \widetilde{f}_*$$

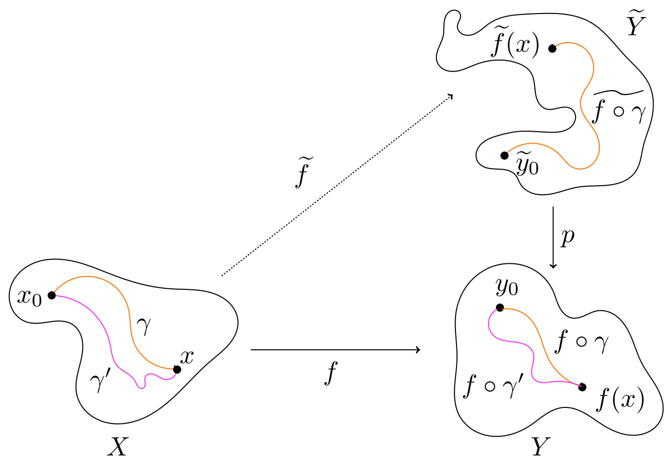
follows from the functoriality of π_1.

Sufficient. Let $x \in X$. Choose a path γ from x_0 to x by the assumption that X is path-connected. Then, $f\gamma$ has a unique lift starting at $\widetilde{y}_0$, denote by $\widetilde{f\gamma}$. Now, define

$$\widetilde{f}(x) = \widetilde{f\gamma}(1).$$

Then, we need to check

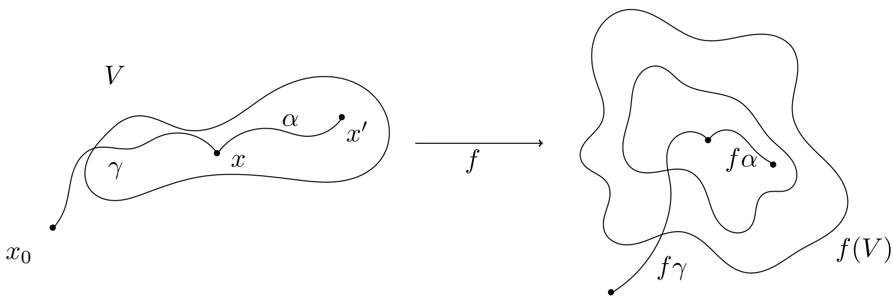
1. $\widetilde{f}$ is well-defined. Suppose γ, γ' are paths in X from x_0 to x. We want to show
$$\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1).$$
Since $\gamma \cdot \overline{\gamma'}$ is a loop in X at x_0, we know that $[(f\gamma) \cdot (\overline{f\gamma'})]$ is a class of loops in Y in $\operatorname{Im}(f_*)$. By hypothesis, this class of loops is in $\operatorname{Im}(p_*)$. It lifts to a loop which is based at $\widetilde{y}_0$. By uniqueness of lifts, this loop lifting $(f\gamma) \cdot \overline{(f\gamma')}$ to $\widetilde{Y}$ must be equal to the lifts $\widetilde{f\gamma} \cdot \overline{\widetilde{f\gamma'}}$ with a common value at $t = 1/2$. Hence, $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$ as desired, namely the endpoints agree.



Lecture 16: Proving Proposition 3.3

11 Feb. 10:00

2. $\widetilde{f}$ is continuous. Choose $x \in X$ and a neighborhood $\widetilde{U}$ of $\widetilde{f}(x)$ in $\widetilde{Y}$. Note that we can choose $\widetilde{U}$ small enough to $p|_{\widetilde{U}}$ is homeomorphism to U in Y. Now, there exists a neighborhood V of x in X with $f(V) \subseteq U$.



The goal is $\widetilde{f}(V) \subseteq \widetilde{U}$. Without loss of generality, we can assume that V is path-connected. Then,

$$\widetilde{f\gamma} \cdot \widetilde{f\alpha} = [\widetilde{f\gamma \cdot f\alpha}].$$

Hence,

$$\widetilde{f\alpha} = (p|_{\widetilde{U}})^{-1} \circ f \circ \alpha,$$

where $(p|_{\widetilde{U}})^{-1}$'s image is in $\widetilde{U}$, so

$$\widetilde{f}(x') = f\gamma \cdot f\alpha(1) \in \widetilde{U},$$

which implies

$$\widetilde{f}(V) \subseteq \widetilde{U}.$$

■

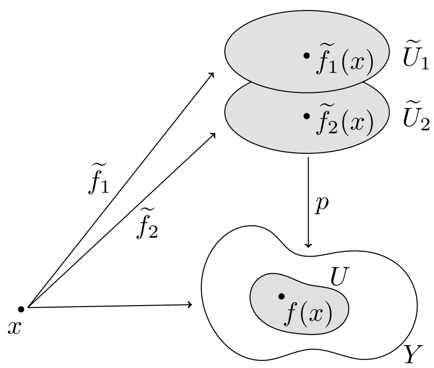
Proposition 3.4 (Uniqueness of lifts). Let $p\colon \widetilde{Y} \to Y$ be a covering map with X is a connected space. If two lifts $\widetilde{f}_1, \widetilde{f}_2$ of the same map f agree at a single point, then they agree everywhere.

$$\begin{tikzcd}
 & \widetilde{Y} \arrow[d, "p"] \\
X \arrow[ur, bend left, "\widetilde{f}_1"] \arrow[ur, "\widetilde{f}_2"'] \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proof. Let S being

$$S := \left\{ x \in X \mid \widetilde{f}_1(x) = \widetilde{f}_2(x) \right\}.$$

We want to show that S is both closed and open, so if S is nonempty, $S = X$.



We see that $\widetilde{U}_1$ and $\widetilde{U}_2$ are slices of $p^{-1}(U)$, where U is evenly covered neighborhood of $f(x)$.

1. If $\widetilde{f}_1(x) \neq \widetilde{f}_2(x)$. Then $\widetilde{U}_1, \widetilde{U}_2$ are disjoint. Since $\widetilde{f}_1, \widetilde{f}_2$ are continuous, there exists a neighborhood N of x with

$$\widetilde{f}_1(N) \subseteq \widetilde{U}_1, \quad \widetilde{f}_2(N) \subseteq \widetilde{U}_2,$$

with the fact that they're disjoint, so x is an interior point of S^c.

2. If $\widetilde{f}_1(x) = \widetilde{f}_2(x)$. Then $\widetilde{U}_1 = \widetilde{U}_2$. Choose N as before, then we have

$$\widetilde{f}_1(n) = (p|_{\widetilde{u}_1})^{-1}(f(n)) = \widetilde{f}_2(n),$$

hence $x \in \mathrm{int}(S)$.

■

3.2 Deck Transformation

We now want to introduce a special kind of transformation.

Definition 3.3 (Isomorphism of covers). Given covering maps

$$p_1 \colon \widetilde{X}_1 \to X, \qquad p_2 \colon \widetilde{X}_2 \to X,$$

an *isomorphism of covers* is a homeomorphism

$$f \colon \widetilde{X}_1 \to \widetilde{X}_2$$

such that $p_1 = p_2 \circ f$.

$$\begin{tikzcd}
\widetilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

Exercise. This defines equivalent relation on covers of X.

Definition 3.4 (Deck transformation). Given a covering map $p \colon \widetilde{X} \to X$, the isomorphisms of covers $\widetilde{X} \to \widetilde{X}$ are called *deck transformation*.

Furthermore, we'll let $G(\widetilde{X})$ denotes the *set of deck transformations*.

Note. Note that we've suppressed the data of p in the notation, but this data is essential to what a deck transformation is, when this is unclear we write $G(\widetilde{X}, p)$.

Lecture 17: Deck Transformation

14 Feb. 10:00

Example. Let's see some examples.

1. Deck transformations $G(\widetilde{X})$ are a subgroup of the group of homeomorphisms of $\widetilde{X}$.

2. Given the cover $p\colon \mathbb{R} \to S^1$.
    - Deck maps: translation by $n \in \mathbb{Z}$ units.
    - $G(\mathbb{R}) \cong \mathbb{Z}$
3. Given the cover $p_n\colon S^1 \to S^1$ be an n-sheeted cover.
    - Deck maps: rotation by $2\pi/n$.
    - $G(S^1, p_n) \cong \mathbb{Z} \,/\, n\mathbb{Z}$

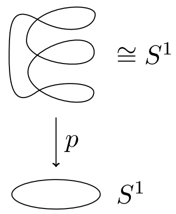


Figure 17: $p_n\colon S^1 \to S^1$ be an n-sheeted cover, where $n = 3$.

Exercise (Deck Transformation is determined by the image of one point). Given $X, \widetilde{X}$ are path-connected, locally path-connected, deck map is determined by the image of any one point.

Answer.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ \widetilde{X} \arrow[ur, "f"] \arrow[r, "p"'] & X \end{tikzcd}$$

> **Corollary 3.2.** If a deck transformation has a fixed point, it is the identity transformation.

Exercise. Let X be connected. Given a deck transformation $\tau\colon \widetilde{X} \to \widetilde{X}$, τ defines a permutation of $p^{-1}(\{x_0\})$. If this permutation has a fixed point, then it is the identity.

> **Definition 3.5 (Regular (normal) cover).** A covering space $p\colon \widetilde{X} \to X$ is *regular* or *normal* if $\forall x_0 \in X$, $\forall \widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(\{x_0\})$, there exists a deck transformation such that
> $$\widetilde{x}_0 \mapsto \widetilde{x}_1.$$

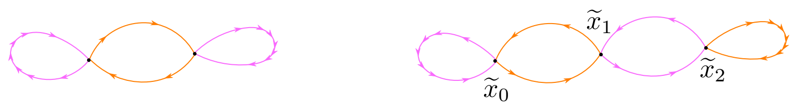


Figure 18: Covers of $S^1 \vee S^1$. The left one is regular, while the right one is not since there is no automorphism from $\widetilde{x}_0$ to $\widetilde{x}_1$ or $\widetilde{x}_2$.

Remark. A regular cover is *as symmetric as possible.*

Exercise. Regular means that the group $G(\widetilde{X})$ acts transitively on $p^{-1}(\{x_0\})$. Explain why we cannot ask for more than this:

> $G(\widetilde{X})$ cannot induce the full symmetric group on $p^{-1}(\{x_0\})$ provided that $\left|p^{-1}(\{x_0\})\right| > 2$.

Answer. The key is uniqueness.

Definition 3.6 (Normalizer). Given G as a group, $H \subseteq G$ is a subgroup of G. Then the *normalizer* of H, denoted by $N(H)$, is defined as

$$N(H) \coloneqq \{g \in G \mid gH = Hg\}.$$

Exercise. We can prove the followings.

1. $N(H)$ is a subgroup.
2. $H \leq N(H)$.
3. H is <u>normal</u> in $N(H)$.
4. If $H \leq G$ is normal, $N(H) = G$.
5. $N(H)$ is the largest subgroup (under containment) of G containing H as normal subgroup.

Proposition 3.5. Given $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a cover, and $\widetilde{X}, X$ are path-connected, locally path-connected. Let

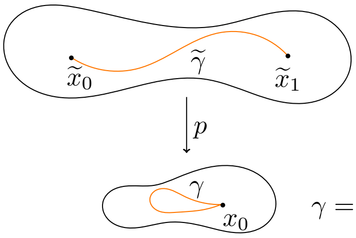
$$H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0).$$

Then

1. p is normal if and only if $H \subseteq \pi_1(X, x_0)$ is normal.
2. We have
$$G(\widetilde{X}) \cong {}^{N(H)}\big/_{H},$$
where $G(\widetilde{X})$ are deck maps, and $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

Remark. A fact is worth noting is the following. Let $\widetilde{\gamma}$ be a path $\widetilde{x}_1$ to $\widetilde{x}_0$. Then

$$p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = [\gamma] p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))[\gamma^{-1}].$$



$\gamma = p \circ \widetilde{\gamma}$

Lecture 18: Proving Proposition 3.5

16 Feb. 10:00

Now let's prove Proposition 3.5

Proof. Let X, x_0 be the base space and $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(\{x_0\})$ where $p\colon \widetilde{X} \to X$ is a covering map. Further, let $H \coloneqq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

In homework, given $(X, x_0), \widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(\{x_0\})$ if we change the basepoint from $\pi_1(\widetilde{X}, \widetilde{x}_0)$ to $\pi_1(\widetilde{X}, \widetilde{x}_1)$, then we have the induced subgroups of the base spaces fundamental group are conjugate by some loop $[\gamma] \in \pi_1(X, x_0)$, i.e.,

$$p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = [\gamma] \cdot p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \cdot [\gamma]^{-1}$$

where γ is lifted to a path from $\widetilde{x}_0$ to $\widetilde{x}_1$.

Therefore, $[\gamma] \in N(H)$ if and only if $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, and this holds if and only if there is a deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$ by the classification of based covering spaces in the homework.[16] This shows that p is a normal cover if and only if H is normal, which proves the first claim.

We then define a map Φ such that

$$\Phi\colon N(H) \to G(\widetilde{X})[\gamma], \quad \cdot \mapsto \tau$$

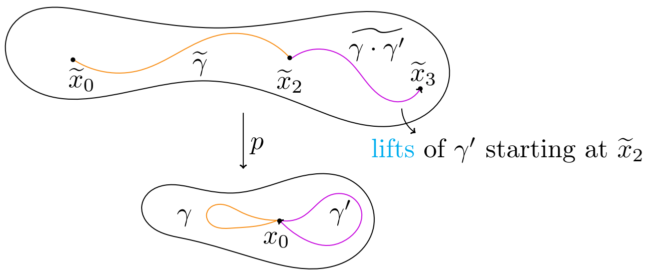
where τ lifts to a path from $\widetilde{x}_0$ to $\widetilde{x}_1$ and τ is a deck transformation mapping $\widetilde{x}_0$ to $\widetilde{x}_1$, which will be uniquely defined by the uniqueness of lifts with specified base points. We now need to check

1. Φ is surjective.
2. $\ker(\Phi) = H$.
3. Φ is a group homomorphism.

If we can prove all the above, then, from the result follows directly from the first isomorphism theorem.[17]

[16] Alternatively, we can use the lifting criterion.
[17] https://en.wikipedia.org/wiki/Isomorphism_theorems

1. We've proved that Φ is surjective before in our work above.
2. $\Phi([\gamma])$ is the identity if and only if τ sends $\widetilde{x}_0$ to $\widetilde{x}_0$, meaning that $[\gamma]$ lifts to a loop. Then by our characterization of the fundamental group downstairs:
$$\ker(\Phi) = \{[\gamma] \mid [\gamma] \text{ lifts to a loop}\} = H.$$
3. Suppose we have loops $[\gamma_1] \overset{\Phi}{\mapsto} \tau_1$ and $[\gamma_2] \overset{\Phi}{\mapsto} \tau_2$. We claim that $\gamma_1 \cdot \gamma_2$ lifts to $\widetilde{\gamma}_1 \cdot \tau(\widetilde{\gamma}_2)$.



It's an exercise to check that the lift of γ_2 starting at $\widetilde{x}_1$ is exactly $\phi_1(\widetilde{\gamma}_2)$, where $\widetilde{\gamma}_2$ is a lift starting at $\widetilde{x}_0$.

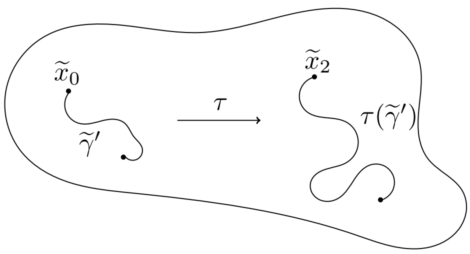


Figure 19: Must be lift of γ' starting at $\widetilde{x}_2$

The idea is that by uniqueness of lifts we'll have the desired claim. We then just observe that this path $\widetilde{\gamma}_1 \cdot \tau_1(\widetilde{\gamma}_2)$ is a path from $\widetilde{x}_0$ to $\gamma_1(\widetilde{\gamma}_2(1)) = \tau_1(\tau_2(\widetilde{x}_0))$, so the image must be a deck transformation sending $\widetilde{x}_0$ to $\tau_1(\tau_2(\widetilde{x}_0))$. But then $\tau_1 \circ \tau_2$ maps $\widetilde{x}_0$ to this same point, and from this exercise, we know that the deck transformations are determined by where they send a single point, hence we're done.

■

Corollary 3.3. If p is a normal covering, then $G(\widetilde{X}) \cong \pi_1(X, x_0) \,/\, H$.

Definition 3.7 (Universal covering). A cover $p\colon \widetilde{X} \to X$ is called a *universal covering* if $\widetilde{X}$ is simply connected.

Corollary 3.4. If $\widetilde{X}$ is the universal cover, then $G(\widetilde{X}) \cong \pi_1(X, x_0)$.

Exercise. Whether $\mathrm{Im}(p_*)$ is normal is independent of the basepoint in $\widetilde{X}$ and X.

So, p is normal if and only if $G(\widetilde{X})$ is transitive on $p^{-1}(x_0)$ for at least one $x_0 \in X$.

Exercise. Let Σg be the genus g surface. Prove that Σg has a normal n-sheeted path-connected cover for every n.

Lecture 19: Simplex

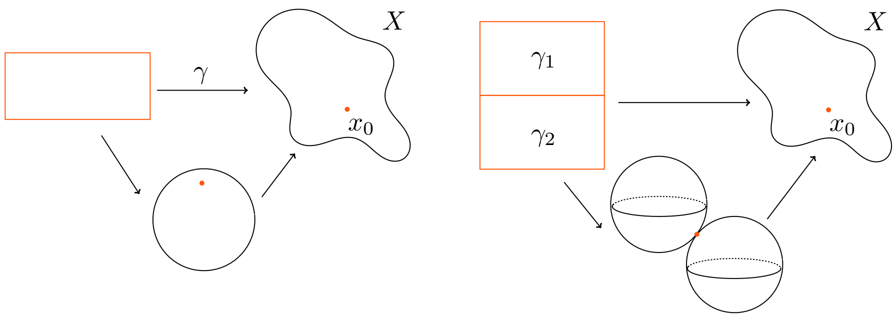
18 Feb. 10:00

4 Homology

4.1 Motivation for Homology

Informally, the higher homotopy groups is defined as

$$\pi_n(X, x_0)\colon I^n_* \to (X, x_0), \quad \partial I^n \mapsto x_0.$$



We see that it's extremely hard to compute higher fundamental group. Hence instead, we will study the higher dimensional structure of X via *homology*.

- **Cons.**
  - The definition is more opaque at first encounter.
- **Pros.**
  - Lots of computational tools
  - Functional
  - Abelian Groups

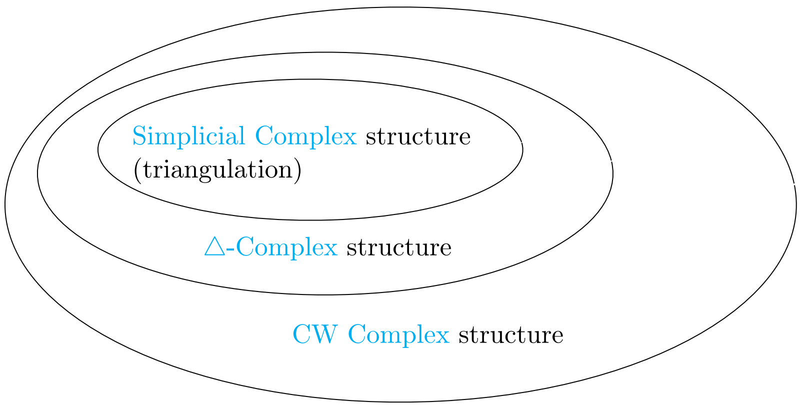
Remark. More like π_n for $n > 1$.

- No basepoints
- Can compute using CW structure.
- Good properties. For example, $H_n = 0$ if $n > \dim X$

4.2 Simplicial Homology

4.2.1 Δ-Simplex

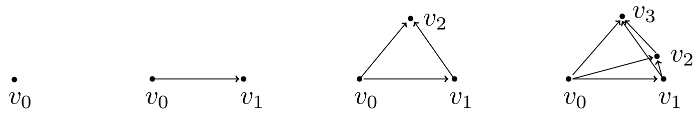
This is a stricter version of a CW complex which allows us to decompose our spaces into cells. In terms of how things fit together, we have the following diagram.



Now we try to give the definition.

Definition 4.1 (Simplex). We see that

- 0-*simplex.* A point.
- 1-*simplex.* Interval.
- 2-*simplex.* Triangle.
- 3-*simplex.* Tetrahedron.
- n-*simplex.* The convex hull of $(n+1)$-points position in $\mathbb{R}^n$.



Remark. We see that

- The top of which is the 2-disk and remember cell structure (edges and vertices) and remember orientation (ordering on vertices).
- The top of which is the 3-disk and cells and the orientation.

Further,

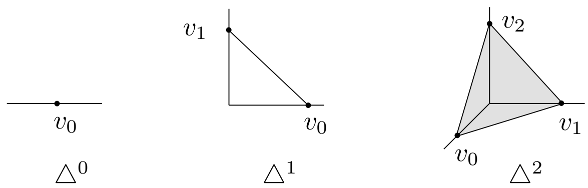
- We can view simplices as both *combinatorial* and *topological* objects.

An alternative definition can be done.

Definition 4.2 (Standard simplex). We say that an n-dimensional *standard simplex*, denoted by Δ^n is

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \ \sum_i t_i = 1 \right\}.$$

We'll call such a simplex as *standard n-simplex*.



Remark. In our definition, the simplices will implicitly come with a choice of ordering of the vertices as

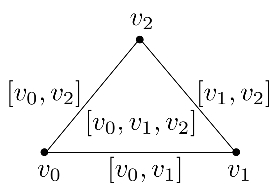
$$\Delta^n = [v_0, v_1, \dots, v_n]$$

such that the convex hull of these points is taken with this ordering.

Lecture 20: Simplicial Complex

21 Feb. 10:00

Definition 4.3 (Subsimplex). A *subsimplex* of a simplex σ combinatorially, it's a subset of the vertices; while topologically, it's the convex hull of the subset of vertices.



Definition 4.4 (Face). A *face* of a simplex Δ^n is a subsimplex of 1 dimensional lower than Δ^n (codimension 1).

Definition 4.5 (Boundary). The *boundary* $\partial\sigma$ of a simplex σ is the union of its faces.

Definition 4.6 (Open simplex). The *open simplex* of Δ is defined as

$$\mathring{\Delta}^n := \Delta^n - \partial\Delta^n.$$

Definition 4.7 (Δ-Complex). A Δ-*complex* structure on X is a collection of maps

$$\sigma_\alpha \colon \Delta^n \to X$$

such that

1. $\sigma_\alpha|_{\mathring{\Delta}^n}$ injective, each point of X is in the image of exactly one such map.
2. Each restriction of σ_α to a face coincides with a map

$$\sigma_\beta \colon \Delta^{n-1} \to X.$$

3. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in $\mathring{\Delta}^n$ for all σ_α, i.e., X is a quotient

$$\coprod_{n,\alpha} \Delta_\alpha^n \xrightarrow{\coprod \sigma_\alpha} X.$$

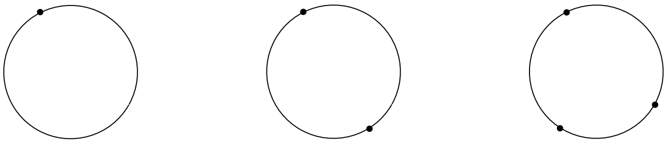
Exercise. A Δ-complex X is a CW complex W with characteristic maps σ_α with extra constraints on the attaching maps.

Note. We see that the second condition of Definition 4.7 implies that attaching maps injective on interior of faces.

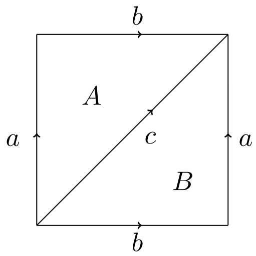
Definition 4.8 (Simplicial complex). A *simplicial complex* is a Δ-complex such that

- σ_α must map every face to a <u>different</u> $(n-1)$-simplex.
- Every simplex is uniquely determined by its vertex set.
- Any $(n+1)$ vertices in X^0 is the vertex set of at most 1 n-simplex.

Remark. With Definition 4.8, we see the followings.



Example. The torus with the following edges, a, b, c and the gluing in triangles A and B can be seen as follows.



For this Δ-complex, notice that we've glued down a triangle whose vertices are all identified. This is not allowed in a simplicial complex / triangulation.

Remark. The minimum number of triangles in a simplicial complex structure is **14**.

Lecture 21: Simplicial Homology

23 Feb. 10:00

To demonstrate how the definition of homology arise, we first see the idea behind it. Fix a space X which equips with the Δ-complex structure. Then, we define $C_n(X)$ to be the free Abelian group on the n-simplices of X. That is,

$$C_n(X) = \left\{ \text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha \colon \Delta^n \to X \right\}.$$

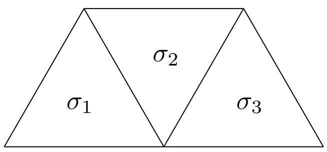
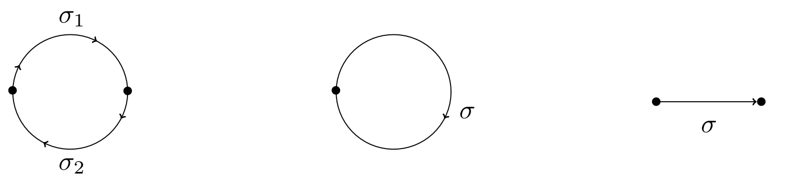


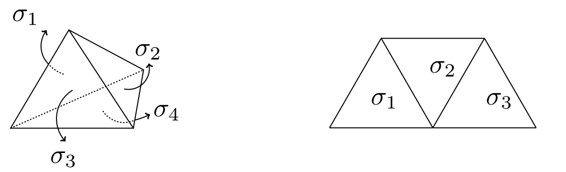
Figure 20: $C_2(X) = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \oplus \mathbb{Z}\sigma_3$.

Then, the n-th homology group will be a subquotient of $C_n(X)$, where the heuristic/imprecise idea is

- Take subgroup of C_n of *cycles*. These are sums of simplices satisfying a combinatorial condition on the boundary gluing maps to ensure that they *close up*, i.e., they have no boundary.

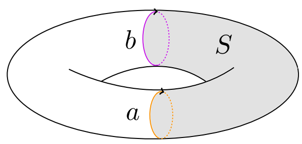


$\sigma_1 + \sigma_2$ cycles σ cycle σ not a cycle

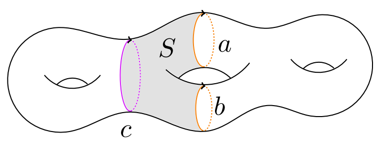


$\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$ cycles $\sigma_1 + \sigma_2 + \sigma_3$ not a cycles

- To take the quotient, we consider two cycles to be equivalent if their difference is a boundary. For example, in the case of torus, a is homologous to b since $a - b$ is the boundary of the shaded subsurface S on of the torus below.



In fact, a and b are homotopic (which will imply they're homologous essentially), but two loops do not need to be homotopic to be homologous. For example, in the figure below, $a + b$ is homologous to c, since $a + b - c$ is the boundary of S ($a + b$[18] and c are not homotopic).



Let's now see the formal definition.

[18] Which isn't even a loop

Definition 4.9 (Simplicial chain group). We define the *simplicial chain group* $C_n(X)$ of order n to be the free Abelian group on the n-simplices of X such that

$$C_n(X) := \left\{ \text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha \colon \Delta^n \to X \right\}.$$

Definition 4.10 (Cycles). Given any chain group $C_n(X)$, a *cycle* of $C_n(X)$ is those chains $\sum m_\alpha \sigma_\alpha$ with no boundaries.

Definition 4.11 (Boundary homomorphism). A map $\partial_n \colon C_n(X) \to C_{n-1}(X)$ is called a *boundary homomorphism* such that

$$\begin{aligned} \partial_n \colon C_n(X) &\to C_{n-1}(X) \\ [\sigma_\alpha] &\mapsto \sum_{i=1}^{n} (-1)^i \, \sigma_\alpha|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}, \end{aligned}$$

which defines the map on the basis, and we extend it linearly.

Remark. We see that the definition of boundary homomorphism indeed coincides with the definition of boundary when considering either Δ-complex or simplicial complex structure.

Example. We give some lower dimensions examples of Definition 4.11 to motivate the general definition.

- For $n = 1$, $\partial_1 \colon C_1(X) \to C_0(X)$ such that
$$[\sigma_\alpha \colon [v_0, v_1] \to X] \mapsto \sigma_\alpha|_{[v_1]} - \sigma_\alpha|_{[v_0]}.$$
- For $n = 2$, $\partial_2 \colon C_2(X) \to C_1(X)$ such that
$$[\sigma_\alpha \colon [v_0, v_1, v_2] \to X] \mapsto \sigma_\alpha|_{[v_1, v_2]} - \sigma_\alpha|_{[v_0, v_2]} + \sigma_\alpha|_{[v_0, v_1]}.$$

Lemma 4.1. For any $n \geq 2$, we have

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_n"] \arrow[rr, bend right, "\partial_{n-1} \circ \partial_n = 0"'] & C_{n-1}(X) \arrow[r, "\partial_{n-1}"] & C_{n-2}(X) \end{tikzcd}$$

Proof. Since all C_i are free Abelian group, hence we only need to consider $\partial_{n-1} \circ \partial_n(\sigma) = 0$ for a generator σ. Given a generator σ, the result follows from directly applying the definition and with some calculation. ∎

Definition 4.12 (Chain complex). A *chain complex* (C_*, d_*) is a collection of maps such that

$$\dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

of Abelian groups and group homomorphism such that

$$d_{n-1} \circ d_n = 0.$$

We call C_n the *n-th chain group* and d_n the *n-th differential.*

Note. Note that Definition 4.12 is purely *abstract*, namely we can put different chain group structure on C_n. We'll see what this means later.[19] But for now, C_n can be equipped with the definition we gave for simplicial chain group.

Remark. We see that

- Lemma 4.1 guarantees that our simplicial chain groups form a chain complex.
- Definition 4.12 means that $\ker(d_n)$ contains $\operatorname{Im}(d_{n+1})$, since $d_n \circ d_{n+1} = 0$.

Definition 4.13 (Exact). We say that the sequence is *exact at C_n* provided that $\ker(d_n) = \operatorname{Im}(d_{n+1})$. A chain complex is *exact* if it is *exact at each point.*

Definition 4.14 (Homology group). The n^{th} *homology group of a chain complex* (C_*, d_*), denoted as H_n or $H_n(C_*)$, is the quotient

$$H_n := \ker(d_n) \big/ \operatorname{Im}(d_{n+1}).$$

Remark. The homology group measures how far the chain complex is from being exact at C_n.

With what we have just defined, it's natural to define homology groups of space X with a Δ-complex structure.

[19] Spoiler: It just means we can give different definition about the map σ.

Definition 4.15 (Homology class). We say $\ker(\partial_n)$ is the subgroup of cycles is $C_n(X)$, and $\mathrm{Im}(\partial_{n+1})$ is the subgroup of boundaries in $C_n(X)$. We then set

$$H_n(X) := {}^{\ker(\partial_n)}\Big/_{\mathrm{Im}(\partial_{n+1})} = {}^{\{\text{cycles}\}}\Big/_{\{\text{boundaries}\}}.$$

In other words, it's the homology of the chain complex

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots$$

where we take it to be 0 in all negative indices, namely

$$\ldots \xrightarrow{\partial_3} C_{n+1} \xrightarrow{\partial_2} C_n \xrightarrow{\partial_1} C_{n-1} \xrightarrow{\partial_0} 0$$

We then call the elements of $H_n(X)$ as *homology classes*.

Definition 4.16 (Simplicial homology group). By considering the chain complex with simplicial chain group, we have so-called *simplicial homology group* induced by Definition 4.14.

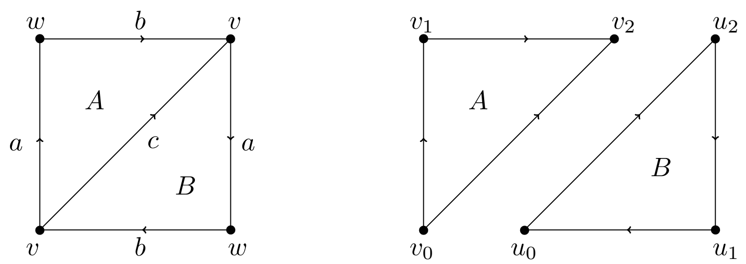
Lecture 22: Calculation of Homology

25 Feb. 10:00

4.2.2 Calculation of Homology

We start from some calculation about homology group of some spaces.

Example. Let $X = \mathbb{R}P^2$.



We see that we have

- $C_0 = \mathbb{Z}\langle v, w\rangle$
- $C_1 = \mathbb{Z}\langle a, b, c\rangle$
- $C_2 = \mathbb{Z}\langle A, B\rangle = \mathbb{Z}A \oplus \mathbb{Z}B$

The chain complex is then

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Where we let $A = [v_0, v_1, v_2]$ and $B = [u_0, u_1, u_2]$, then

$$\partial_2 \colon \begin{cases} A & \mapsto b - c + a \\ B & \mapsto -a - c - b \end{cases}, \qquad \partial_1 \colon \begin{cases} a & \mapsto w - v \\ b & \mapsto v - w \\ c & \mapsto v - v = 0 \end{cases}$$

We can also calculate the image and the kernel at C_i, i.e.,

$$\begin{aligned} C_2 &\colon \operatorname{Im} \partial_3 = 0, & \ker \partial_2 &= 0, \\ C_1 &\colon \operatorname{Im} \partial_2 = \langle 2c, b - c + a \rangle, & \ker \partial_1 &= \langle b + a, c \rangle, \\ C_0 &\colon \operatorname{Im} \partial_1 = \langle v - w \rangle, & \ker \partial_0 &= \langle v, w \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} &H_0 \cong \mathbb{Z}\langle v, w\rangle \big/ \mathbb{Z}\langle v - w \rangle \cong \mathbb{Z} \\ &H_1 \cong \mathbb{Z}\langle b + a, c\rangle \big/ \mathbb{Z}\langle 2c, b + a - c\rangle \cong \mathbb{Z}\langle b + a - c, c\rangle \big/ \mathbb{Z}\langle 2c, b + a - c\rangle \cong \mathbb{Z} \big/ 2\mathbb{Z} \\ &H_2 = 0 \end{aligned}$$

Remark. Given a basis for a free Abelian group $\langle b_1, \ldots, b_n \rangle$ we can replace b_i with

$$b_i \pm m_1 b_1 \pm \cdots \pm \widehat{m_i b_i} \pm \cdots \pm m_n b_n.$$

Remark. Warning! Care is needed when doing *change of bases* over $\mathbb{Z}$. For example, if b_1, b_2 is a basis for $A \subseteq \mathbb{Z}^n$, then $b_1 - b_2, b_1 + b_2$ is <u>not</u> a basis, it is an index-2 subgroup. The key to this is that $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ has determinant -2 (<u>not</u> unit in $\mathbb{Z}$).

We can transform a basis for a free group into a different basis by applying a matrix of determinant ± 1. If we apply a matrix of determinant D we will obtain generators for a subgroup of index $|D|$.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \pm m_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots 0 & \pm m_2 & 0 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \pm m_{i-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \pm m_{i+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \pm m_n & 0 & \cdots & 1 \end{bmatrix}$$

As a summary, we have the following procedures to compute $H_n(X)$.

1. Choose Δ-complex structure on X. (We will prove $H_*(X)$ is independent of the choice of Δ-complex structure)
2. Choose orientations on each simplex (Any choice is okay, but you must commit to a choice, or you will make a sign error!)

3. For each n-simplex σ compute $\partial_n(\sigma)$ (careful with signs!)
4. $\operatorname{Im} \partial_n = \langle \partial_n(\sigma) \mid \sigma \text{ an } n\text{-simplex} \rangle$. Use linear algebra to compute $\ker(\partial_n)$.
5. For each n compute $H_n(X) = \ker \partial_n \,/\, \operatorname{Im} \partial_{n+1}$. Be careful that any change-of-variables map you apply is invertible over $\mathbb{Z}$.

Lecture 23: Singular Homology

07 Mar. 10:00

4.3 Singular Homology

As we noted before, we can give a different structure of chain complex, which shall induces a different homology group compare to simplicial homology group.

We now see one abstract way to define σ, which will give us so-called singular homology group.

Definition 4.17 (Singular simplex). A *singular n-simplex* in a space X is a continuous map

$$\sigma \colon \Delta^n \to X.$$

Definition 4.18 (Singular chain). Let $C_n(X)$ be the free group on singular n-simplices in X, which we call it the *singular n-chains.*

Definition 4.19 (Singular chain complex). The singular chains with boundary maps

$$\begin{aligned} \partial_n \colon C_n(X) &\to C_{n-1}(X) \\ \sigma &\mapsto \sum_{i=1}^{n} (-1)^i \, \sigma|_{[v_0, \ldots, \widehat{v}_i, \ldots, v_n]} \end{aligned}$$

induces a *singular chain complex.*

Definition 4.20 (Singular homology group). The *singular homology groups* are the homology groups of this singular chain complex given as

$$H_n(X) = {}^{\textstyle \ker \partial_n} \Big/ {}_{\textstyle \operatorname{Im} \partial_{n+1}}.$$

Remark. We now see that from the definition of homology group, we can put different structure on which. But the idea is the same, namely we are taking $H_n(X)$ being

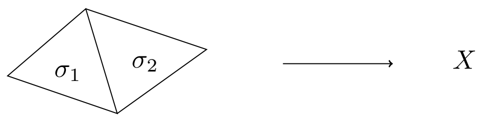
$$H_n(X) := {}^{\textstyle \ker \partial_n} \Big/ {}_{\textstyle \operatorname{Im} \partial_{n+1}},$$

where the difference is what structure we put on X which induces different chain complex $C_n(X)$. In this case, we have singular homology group since we are considering singular chain complex, while we can also have simplicial homology group.

Since the generating sets for $C_n(X)$ when considering singular chain complex are almost always hugely uncountable from its definition, it's almost impossible to compute with these. However, it does give us a definition that does not depend on any other structure than the topology of X, making it useful for developing theory.

Note. The heuristic is that, we interpret a chain $\sigma_1 \pm \sigma_2 \pm \cdots \pm \sigma_k$ as a map from a Δ-complex to X.

For example, with $\sigma_1 + \sigma_2$ as below,



where we've glued $[v_1, v_2]$ of σ_1 to $[v_0, v_2]$ of σ_2 if $\sigma_1|_{[v_1,v_2]}$ and $\sigma_{[v_0,v_2]}$ are the same singular n-chain with opposite signs.

With what we have defined, we now have some *goals*.

- Singular homology is a homotopy invariant. (Theorem 4.2)
- Singular and simplicial homology groups are isomorphic. (Theorem 4.9)

Exercise. We see some exercises.

1. Check that if X has path components $\{X_\alpha\}$ then
$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha).$$
2. If $X = \{*\}$, then
$$H_n(X) = \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$
3. If X is path-connected, then $H_0(X) \cong \mathbb{Z}$.

4.4 Functoriality and Homotopy Invariance

Definition 4.21 (Induced map on chains). For a given continuous map $f\colon X \to Y$, we can consider the *map $f_\#$ induced by chains* as

$$\begin{aligned} f_\# \colon C_n(X) &\to C_n(Y) \\ [\sigma\colon \Delta^n \to X] &\mapsto [f \circ \sigma\colon \Delta^n \to Y]. \end{aligned}$$

Remark. We see that the functoriality doesn't depend on any kind of Δ-complex structure.

Definition 4.22 (Chain map). Given two chain complexes (C_*, ∂_*) and (D_*, δ_*), a *chain map* between them is a collection of group homomorphisms $f_n\colon C_n \to D_n$ such that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\delta_{n+2}"] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots
\end{tikzcd}$$

i.e. we have that $\delta_n \circ f_n = f_{n-1} \circ \partial_n$.

Exercise. We see that

1. We have that $f_\# \partial = \partial f_\#$. In other words, $f_\#$ is a chain map. Thus, by the homework $f_\#$ induces a group homomorphism on the homology groups. We write this as $f_*\colon H_n(X) \to H_n(Y)$ for all n.
2. We have functoriality, i.e. $(f \circ g)_* = f_* \circ g_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$.

Theorem 4.1 (Homology group defines a functor). The n-th homology group $H_n\colon X \mapsto H_n(X)$ gives a functor from Top to Ab.

Proof. This follows from the two exercises above. ∎

Theorem 4.2 (Functoriality is homotopy invariant). If $f, g\colon X \to Y$ are homotopic, then they will induce the same map on homology

$$f_* = g_*\colon H_n(X) \to H_n(Y).$$

The proof of Theorem 4.2 can be found here.

Exercise. Theorem 4.1 and Theorem 4.2 imply that H_n is a homotopy invariant.

Lecture 24: Chain Homotopy

09 Mar. 10:00

To prove Theorem 4.2, we introduce some homological algebra.

Definition 4.23 (Chain homotopy). Given chain complexes (A_*, ∂_*^A) and (B_*, ∂_*^B) and chain maps $f_\#, g_\# \colon A_* \to B_*$. A *chain homotopy* from $f_\#$ to $g_\#$ is a sequence of group homomorphisms $\psi_n \colon A_n \to B_{n+1}$ such that

$$f_n - g_n = \partial_{n+1}^B \circ \psi_n + \psi_{n-1} \circ \partial_n^A.$$

In diagram, letting $h_n \coloneqq f_n - g_n$, we have the following.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}^A"] & A_{n+1} \arrow[r, "\partial_{n+1}^A"] \arrow[d, "h_{n+1}"] & A_n \arrow[r, "\partial_n^A"] \arrow[d, "h_n"] \arrow[dl, "\psi_n"'] & A_{n-1} \arrow[r, "\partial_{n-1}^A"] \arrow[d, "h_{n-1}"] \arrow[dl, "\psi_{n-1}"'] & \cdots \\
\cdots \arrow[r, "\partial_{n+2}^B"] & B_{n+1} \arrow[r, "\partial_{n+1}^B"] & B_n \arrow[r, "\partial_n^B"] & B_{n-1} \arrow[r, "\partial_{n-1}^B"] & \cdots
\end{tikzcd}$$

This diagram does **not** commute, however, the **red** map is the sum of the **green** maps composed up, so it shows everything that is going on.

Theorem 4.3. If there is a chain homotopy ψ from $f_\#$ to $g_\#$, then the induced maps f_*, g_* on homology are equal.

Proof. Let $\sigma \in A_n$ be an n-cycle, i.e. $\partial_n^A \sigma = 0$. Then we compute that:

$$(f_n - g_n)(\sigma) = \partial_{n+1}^B(\psi_n(\sigma)) + \psi_{n-1}(\partial_n^A(\sigma)) = \partial_{n+1}^B(\psi_n(\sigma)) \in \operatorname{Im}(\partial_{n+1}^B).$$

This tells us that $(f_n - g_n)(\sigma)$ is a boundary, and so $(f_n - g_n)(\sigma) = 0$ when considered as an element of the homology group (with degree n). Thus, $f_n(\sigma) = g_n(\sigma)$ in the homology group, and so f, g induce the same map as desired. ■

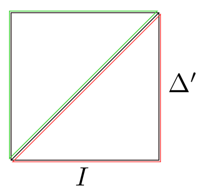
We now sketch the proof of Theorem 4.2 given in Hatcher[HPM02]. From this point in the course many of the theorems require much more algebraic work than we are interested in. We instead want to learn how to use the computational tools.

Proof. Suppose we have some homotopy $F \colon I \times X \to Y$ from f to g. The most difficulty in this proof is the combinatorial difficulty involved in the fact that the product of a simplex in X and I is not a simplex.

We now consider

1. Subdivide $\Delta^n \times I$ into $(n+1)$-dimensional subsimplices.[20]

[20] We want to do this since the product between two simplices is not a simplex, as we just note.



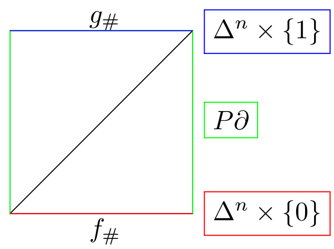
2. We define the prism operator:

$$\begin{aligned} P_n \colon C_n(X) &\to C_{n+1}(Y) \\ [\sigma \colon \Delta^n \to X] &\mapsto \begin{bmatrix} \text{alternating sums of restrictions} \\ \Delta^n \times I \xrightarrow{\sigma \times \mathrm{id}} X \times I \xrightarrow{F} Y \\ \text{to each simplex in our subdivision} \end{bmatrix} \end{aligned}$$

3. We now need to check that

$$\partial^Y_{n+1} P_n = \boxed{g_\#} - \boxed{f_\#} - \boxed{P_{n-1}\partial^X_n}.$$

We have the following diagram.



Thus P is a chain homotopy, and we're done.

■

Lecture 25: Relative Homology

11 Mar. 10:00

We are now interested in the relationship between $H_n(X), H_n(A), H_n(X / A)$.

4.5 Relative Homology

Definition 4.24 (Reduced homology group). The *reduced homology groups* $\widetilde{H}_n(X) = H_n(X)$ when $n > 0$. When $n = 0$ we have that:

$$\widetilde{H}_0(X) \oplus \mathbb{Z} = H_0(X).$$

Remark. The usefulness of this is that for path-connected space X we have $\widetilde{H}_0(X) = 0$, and for contractible spaces X we have $\widetilde{H}_n(X) = 0$.

Definition 4.25 (Good pair). Let X be a space, and $A \subseteq X$. Then (X, A) is a *good pair* if A is closed and nonempty, and also it is a deformation retract of a neighborhood in X.

Example. Let's see some examples.

1. If X is a CW complex and A is a nonempty subcomplex, then (X, A) is a good pair. The proof is given in the Appendix of Hatcher[HPM02] and requires some point-set topology.
2. If M is a smooth manifold, and $N \subseteq M$ is a smooth submanifold which is nonempty, then (M, N) is a good pair.
3. (Hawaiian earring, bad point) is not a good pair.
4. ($\mathbb{R}^n$, proper open set) is not a good pair.

Theorem 4.4 (Long exact sequence of a good pair). If (X, A) is a good pair, then there exists a long exact sequence (exact at every n) on reduced homology groups given by the following commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}_n(A) \arrow[r, "i_*"] & \widetilde{H}_n(X) \arrow[r, "j_*"] & \widetilde{H}_n(X\,/\,A) \arrow[dll, "\delta"'] \\
 & \widetilde{H}_{n-1}(A) \arrow[r, "i_*"] & \widetilde{H}_{n-1}(X) \arrow[r, "j_*"] & \widetilde{H}_{n-1}(X\,/\,A) \arrow[dll, "\delta"'] \\
 & \cdots \arrow[r, "i_*"] & \widetilde{H}_0(X) \arrow[r, "j_*"] & \widetilde{H}_0(X\,/\,A) \arrow[r] & 0
\end{tikzcd}$$

where $i\colon A \hookrightarrow X$ is the inclusion and $j\colon X \to X\,/\,A$ is the quotient map.

We see that both i_* and j_* is naturally induced, but not for δ. In fact, we'll construct δ in the proof! Specifically, we'll see that Theorem 4.4 is just a special case of Theorem 4.6, hence rather than proof Theorem 4.4 directly, we will prove Theorem 4.6 instead later.

Remark. The fact that this sequence is exact often means that if we know the homology groups of two of the spaces we can compute the homology of the remaining space.

Before we see the proof of Theorem 4.4, we see one application.

Proposition 4.1. We have that:

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & \text{if } i = n; \\ 0, & \text{if } i \neq n. \end{cases}$$

Exercise. Verify Proposition 4.1 in the case $n = 0$, so S^0 is just 2 points.

Proof. Some facts we need:

- $(D^n, \partial D^n)$ is a good pair (since it is a CW complex and a subcomplex)
- $D^n/\partial D^n \cong S^n$.
- $\widetilde{H}_n(D^n) = 0$ for all n since D^n is contractible.
- $\partial D^n \cong S^{n-1}$.

We then proceed by induction on n. Using the long exact sequence, we have

$$\begin{tikzcd} \dots \arrow[r] & \widetilde{H}_n(\partial D^n) \arrow[r, "i_*"] & \widetilde{H}_n(D^n) \arrow[r, "j_*"] & \widetilde{H}_n(S^n) \arrow[dll, "\delta"'] \\ & \widetilde{H}_{n-1}(\partial D^n) \arrow[r, "i_*"] & \widetilde{H}_{n-1}(D^n) \arrow[r, "j_*"] & \widetilde{H}_{n-1}(S^n) \arrow[dll, "\delta"'] \\ & \dots \arrow[r, "i_*"] & \widetilde{H}_0(D^n) \arrow[r, "j_*"] & \widetilde{H}_0(S^n) \arrow[r] & 0 \end{tikzcd}$$

By induction, we have $\widetilde{H}_{n-1}(\partial D^n) = \widetilde{H}_{n-1}(S^{n-1}) = \mathbb{Z}$, hence we can fill in some of these groups as follows.

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r, "i_*"] & 0 \arrow[r, "j_*"] & \widetilde{H}_n(S^n) \arrow[dll, "\delta"'] \\ & \mathbb{Z} \arrow[r, "i_*"] & 0 \arrow[r, "j_*"] & \widetilde{H}_{n-1}(S^n) \arrow[dll, "\delta"'] \\ & \dots \arrow[r, "i_*"] & 0 \arrow[r, "j_*"] & \widetilde{H}_0(S^n) \arrow[r] & 0 \end{tikzcd}$$

In all, we have an exact sequence:

$$0 \longrightarrow \widetilde{H}_n(S^n) \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0$$

By exactness, δ is an isomorphism, thus $\widetilde{H}_n(S^n) \cong \mathbb{Z}$. Now we must verify $\widetilde{H}_i(S^n) = 0$ when $i \neq n$. In that case the exact sequence looks like:

$$\longrightarrow \widetilde{H}_i(D^n) \longrightarrow \widetilde{H}_i(S^n) \longrightarrow \widetilde{H}_{i-1}(\partial D^n)$$

$$\longrightarrow 0 \longrightarrow \widetilde{H}_i(S^n) \longrightarrow 0$$

Exactness then tells us that $\widetilde{H}_i(S^n) = 0$. ∎

Theorem 4.5 (Brouwer's fixed point theorem). ∂D^n is not a retract of D^n. Hence, every continuous map $f\colon D^n \to D^n$ has a fixed point.

Proof. If $r\colon D^n \to \partial D^n$ were a retraction, then by definition this would give us

$$\begin{tikzcd} \partial D^n \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{id}_{\partial D^n}"'] & D^n \arrow[r, "r"] & \partial D^n \end{tikzcd}$$

Functoriality of homology implies

$$\begin{tikzcd} \widetilde{H}_{n-1}(\partial D^n) \arrow[r, "i_*"] \arrow[rr, bend right, "\mathrm{id}"'] & \widetilde{H}_{n-1}(D^n) \arrow[r, "r_*"] & \widetilde{H}_{n-1}(\partial D^n) \end{tikzcd}$$

So then:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "\mathrm{id}"'] & 0 \arrow[r, "r_*"] & \mathbb{Z} \end{tikzcd}$$

which is impossible since the map $\mathrm{id}_{\mathbb{Z}}$ can't be factored through 0. ■

Exercise. As with D^2, if $f\colon D^n \to D^n$ had no fixed point, we could build a retraction.

In order to proof Theorem 4.4, we introduce the concept of *diagram chase*.

Lemma 4.2 (The short five lemma). Suppose we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\psi'"] & B \arrow[r, "\varphi'"] & C' \arrow[r] & 0 \end{tikzcd}$$

so that the rows are exact. Then:

1. If α, γ are injective then β is injective.
2. If α, γ are surjective then β is surjective.
3. If α, γ are isomorphisms then β is an isomorphism

Proof. 1. and 2. imply 3. We leave 2. as an exercise. We fix $b \in B$ such that $\beta(b) = 0$. We want to show that $\beta = 0$. Well, we draw a diagram chase as

$$\begin{tikzcd} 0 \arrow[r, mapsto] & \bullet \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "\alpha"] & b \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\beta"] & \varphi(b) \arrow[r, mapsto] \arrow[d, mapsto, "\gamma"] & 0 \\ 0 \arrow[r, mapsto] & \bullet \arrow[r, mapsto, "\psi'"] & 0 \arrow[r, mapsto, "\varphi'"] & 0 \arrow[r, mapsto] & 0 \end{tikzcd}$$

And thus by injectivity of γ we know $\varphi(b) = 0$. By exactness, $b \in \operatorname{Im}\psi$. We then may write for some $a \in A$ such that the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r, mapsto] & a \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "\alpha"] & b \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\beta"] & 0 \arrow[r, mapsto] \arrow[d, mapsto, "\gamma"] & 0 \\ 0 \arrow[r, mapsto] & \alpha(a) \arrow[r, mapsto, "\psi'"] & 0 \arrow[r, mapsto, "\varphi'"] & 0 \arrow[r, mapsto] & 0 \end{tikzcd}$$

Therefore $\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) = 0$ by commutativity. By exactness of the bottom row we know that ψ' is an injection.

Thus, $\alpha(a) = 0$, so since α is injective, $a = 0$. With this $b = \psi(a) = \psi(0) = 0$. Great! With this $\ker(\beta) = 0$, and β injects. ■

Lecture 26: Continue on Relative Homology

14 Mar. 10:00

We start from a definition.

Definition 4.26 (Relative chain complex). Let X be a space and let $A \subseteq X$ be a subspace. Then we define the *relative chain complex*

$$C_n(X, A) = {}^{C_n(X)}\big/_{C_n(A)},$$

which is a quotient of Abelian groups of the singular chain groups.

Remark. We can indeed adapt Definition 4.26 by either singular chain complex structure or simplicial chain complex structure.

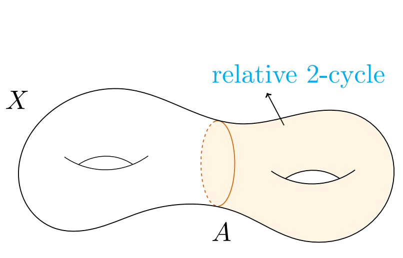
Exercise. Since $\partial_n^*(C_n(A)) \subseteq C_{n-1}(A)$, hence there exists a well-defined map

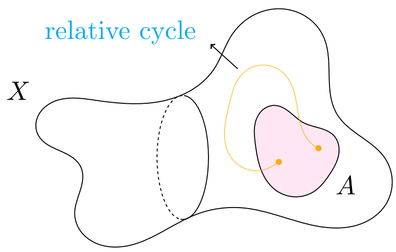
$$\partial_n \colon {}^{C_n(X)}\big/_{C_n(A)} \to {}^{C_{n-1}(X)}\big/_{C_{n-1}(A)}.$$

We can verify that $\partial^2 = 0$. Then, since $\partial^2 = 0$ we can conclude that these groups will in fact form a chain complex $(C_*(X, A), \partial)$.

Definition 4.27 (Relative homology). The homology groups of the chain complex $(C_*(X, A), \partial)$ are denoted by $H_n(X, A)$, and they are called *relative homology groups*.

(Relative cycle). Elements in $\ker \partial_n$ are called *relative n-cycles*. These are elements $\alpha \in C_n(X)$ such that $\partial_n \alpha \in C_{n-1}(A)$.





(Relative boundary). Likewise, elements in $\operatorname{Im} \partial_{n+1}$ are called *relative n-boundaries*. This means that $\alpha = \partial\beta + \gamma$ where $\beta \in C_n(X)$ and $\gamma \in C_{n-1}(A)$.

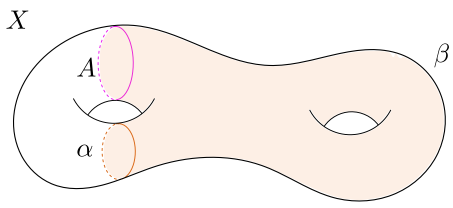


Figure 21: We see that we have $\alpha + \gamma = \partial\beta$, where α is a relative boundary, and $\gamma \in C_{n-1}(A)$.

Theorem 4.6 (Long exact sequence of a pair). Let $A \subseteq X$ be spaces, then there exists a long exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & \widetilde{H}_n(A) \arrow[r, "i_*"] & \widetilde{H}_n(X) \arrow[r, "q"] & \widetilde{H}_n(X, A) \arrow[dll, "\partial"'] & \\
& \widetilde{H}_{n-1}(A) \arrow[r, "i_*"] & \dots \arrow[r, "q"] & \widetilde{H}_0(X, A) \arrow[r] & 0
\end{tikzcd}$$

where i_* is induced by $A \hookrightarrow X$, and q is induced by $C_n(X) \twoheadrightarrow C_n(X) \,/\, C_n(A)$.

We will prove that when (X, A) is a good pair, then $H_n(X, A) \cong \widetilde{H}_n(X \,/\, A)$. Then Theorem 4.4 is a special case of Theorem 4.6. The key to the proof of Theorem 4.6 above is the following slogan.

Remark. A short exact sequence of chain complexes gives rise to a long exact sequence of homology groups. Namely, given a short exact sequence of chain complexes $(A_*, \partial^A), (B_*, \partial^B), (C_*, \partial^C)$ such that

$$0 \longrightarrow A_* \xrightarrow{\iota} B_* \xrightarrow{q} C_* \longrightarrow 0$$

where ι, q are chain maps such that

$$0 \longrightarrow A_n \xrightarrow{\iota_n} B_n \xrightarrow{q_n} C_n \longrightarrow 0$$

is exact for all n. Then Theorem 4.4 will follow from a short exact sequence

$$0 \longrightarrow \widetilde{C}_*(A) \longrightarrow \widetilde{C}_*(X) \longrightarrow \widetilde{C}_*(X, A) \longrightarrow 0$$

where $\widetilde{C}_*$ denotes the *augmented chain complex* (the one with $\mathbb{Z}$ after it).

Exercise. If A is a single point in X, then $H_n(X, A) = \widetilde{H}_n(X \,/\, A) = \widetilde{H}_n(X)$.

Lecture 27: Excision

16 Mar. 10:00

Let's start with a theorem.

Theorem 4.7 (Excision). Suppose we have subspace $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq \mathrm{Int}(A)$. Then the inclusion

$$(X - Z, A - Z) \hookrightarrow (X, A)$$

induces isomorphisms

$$H_n(X - Z, A - Z) \xrightarrow{\cong} H_n(X, A).$$

Exercise. Equivalently for subspaces $A, B \subseteq X$ whose interiors cover X, the inclusion

$$(B, A \cap B) \hookrightarrow (X, A)$$

induces an isomorphism

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

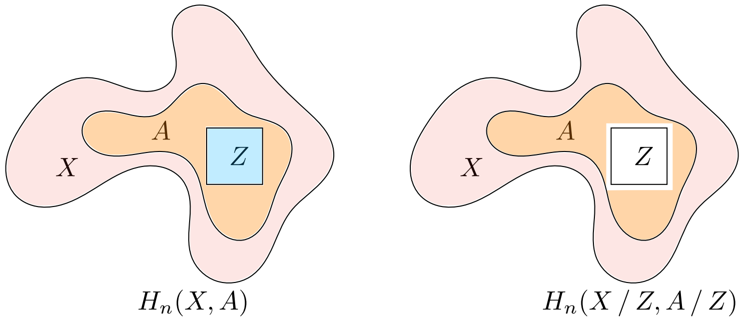
Answer. We see that this follows from

$$B \coloneqq X \setminus Z, \quad Z = X \setminus B,$$

then we see that $A \cap B = A - Z$ and the condition requires from Theorem 4.7, $\overline{Z} \subseteq \mathrm{Int}(A)$ is then equivalent to

$$X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$$

since $X \setminus \mathrm{Int}(B) = \overline{Z}$.



Proof Sketch. We sketch the proof here, which is notorious for being hairy.

- Given a relative cycle x in (X, A), subdivide the simplices to make x a linear combination of chains on *smaller simplices*, each contained in $\mathrm{Int}(A)$ or $X \setminus Z$.

Figure 22: $\Delta^n \to X$ subdivide into subsimplices with images in.

  This means x is homologous to sum of subsimplices with images in $\mathrm{Int}(A)$ or $X \setminus Z$. One of the things we use is that simplices are compact, so this process takes finite time.

  The key is that the Subdivision operator is chain homotopic to the identity.

- Since we are working relative to A, the chains with image in A are zero, thus we have a relative cycle homologous to x with all simplices contained in $X \setminus Z$.

■

Exercise. $H_*(Y, y_0) \cong \widetilde{H}(Y)$.

Theorem 4.8. For good pairs (X, A), the quotient map $q\colon (X, A) \to (X\,/\,A, A\,/\,A)$ induces isomorphisms

$$q_*\colon H_n(X, A) \xrightarrow{\cong} H_n(X\,/\,A, A\,/\,A) \cong \widetilde{H}_n(X\,/\,A)$$

for all n.

Remark. The last equality is from the exercise since $A\,/\,A = \{*\}$.

Proof Sketch. Let $A \subseteq V \subseteq X$ where V is a neighborhood of A that deformation retracts onto A. Using excision, we obtain a commutative diagram

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\cong"] \arrow[d, "q_*"] & H_n(X, V) \arrow[d, "q_*"] & H_n(X - A, V - A) \arrow[l, "\cong"'] \arrow[d, "\cong", "q_*"'] \\
H_n(X / A, A / A) \arrow[r, "\cong"] & H_n(X / A, V / A) & H_n(X / A - A / A, V / A - A / A) \arrow[l, "\cong"']
\end{tikzcd}$$

Done if we can prove all the colored isomorphisms.

- $\cong$ is an isomorphism by excision.
- $\cong$ is an isomorphism by direct calculation (since q is a homeomorphism on the complement of A).
- $\cong$ on Homework, since V deformation retracts to A.

■

Lecture 28: Singular Homology v.s. Simplicial Homology

18 Mar. 10:00

Remark. If M is a smooth manifold and N is an embedded smooth closed submanifold, then (M, N) is a good pair. Why? Well this follows from the tubular neighborhood theorem, which should be proven in a course like 591. We will only use the result in obvious cases, and simply assert that certain pairs are good pairs.

With pairs like $(\mathbb{R}^{n+1}, S^n)$, you can just assert that this is a good pair (and do not need to prove that S^n is a smooth submanifold of $\mathbb{R}^{n+1}$). Another good example is manifolds and their boundary always form a good pair.

Theorem 4.9 (Singular homology agrees with simplicial homology). Let X be a Δ-complex. We use $\Delta_n(X)$ to represent the simplicial chain groups on X, and $C_n(X)$ to denote the singular chain groups. Likewise, we denote

$$\Delta_n(X, A) = {}^{\Delta_n(X)} \big/ {}_{\Delta_n(A)}$$

and

$$C_n(X, A) = {}^{C_n(X)} \big/ {}_{C_n(A)}.$$

The inclusion $\Delta_*(X, A) \hookrightarrow C_*(X, A)$ given by

$$[\sigma \colon \Delta^n \to X] \mapsto [\sigma : \Delta^n \to X]$$

induces an isomorphism on homology such that

$$H_n^{\Delta}(X, A) \cong H_n(X, A).$$

If we consider the case that $A = \varnothing$, we recover the case of absolute homology

$$H_n^{\Delta}(X) \cong H_n(X).$$

The proof of Theorem 4.9 uses the following lemma.

Lemma 4.3 (The five lemma). If we have a commutative diagram with exact rows as following,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "\ell"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' \arrow[r, "k'"'] & D' \arrow[r, "\ell'"'] & E' \end{tikzcd}$$

If $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then so is γ.

Proof. Diagram chase! ∎

Lecture 29: Proof of Theorem 4.9

21 Mar. 10:00

Proof Sketch. The idea is as follows.

- We can use the long exact sequence of a pair and the Lemma 4.3 to reduce to proving the result for absolute homology groups (and we will recover the general result).
- Because the image $\Delta^n \to X$ is *compact*, it is contained in some finite skeleton X^k. Use this to reduce the proof to the finite skeleton X^k of X, namely we can use induction.

From the long exact sequence of a pair we get

$$\begin{tikzcd} H^{\Delta}_{n+1}(X^k, X^{k-1}) \arrow[r] \arrow[d, "\alpha"] & H^{\Delta}_n(X^{k-1}) \arrow[r] \arrow[d, "\beta"] & H^{\Delta}_n(X^k) \arrow[r] \arrow[d, "\gamma"] & H^{\Delta}_n(X^k, X^{k-1}) \arrow[r] \arrow[d, "\delta"] & H^{\Delta}_{n-1}(X^{k-1}) \arrow[d, "\epsilon"] \\ H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1}) \end{tikzcd}$$

The Goal is to prove γ is an isomorphism using the Lemma 4.3.

We assume that β, ϵ are isomorphisms by induction, checking the case manually for X^0 (which will be a discrete set of points). It remains to show that α, δ are isomorphisms.

We know then that

$$\Delta_n(X^k, X^{k-1}) = \begin{cases} \mathbb{Z}[k\text{-simplices}], & \text{if } k = n; \\ 0, & \text{otherwise} \end{cases} \cong H^{\Delta}_n(X^k, X^{k-1}).$$

We claim that $H_n(X^k, X^{k-1})$ are also free Abelian on the singular k-simplices defined by the characteristic maps $\Delta^k \to X^k$ when $n = k$, and 0 otherwise. Consider the map

$$\Phi\colon \coprod_{\alpha} (\Delta^k_\alpha, \partial\Delta^k_\alpha) \to (X^k, X^{k-1})$$

defined by the characteristic map. This induces an isomorphism on homology since

$$\coprod_{\alpha} \Delta^k_\alpha \Big/ \coprod \partial\Delta^k_\alpha \xrightarrow{\cong} X^k \Big/ X^{k-1}.$$

This reduces to check that

$$H_n(\Delta^k, \partial\Delta^k) = \begin{cases} 0, & \text{if } n \neq k; \\ \mathbb{Z}, & \text{if } n = k \end{cases}$$

generated by the identity map $\Delta^k \to \Delta^k$. ■

Corollary 4.1. If X has a Δ-complex structure (or is homotopy equivalent to one), then we have the followings.

1. If the dimension is $\leq d$, then $H_n(X) = 0$ for all $n > d$.
2. If $\overline{X}$ has no cells of dimension p, then $H_p(X) = 0$.
3. If $\overline{X}$ has no cells of dimension p, then $H_{p-1}(X)$ is free Abelian.

Corollary 4.2. Given a singular homology class on X, without loss of generality we can choose a Δ-complex structure on X, and we then we can assume the class is represented by a simplicial n-cycle.

4.6 Degree

Definition 4.28 (Degree). Let $f \colon S^n \to S^n$, then

$$f_* \colon \mathbb{Z} \cong H_n(S^n) \to H_n(S^n) \cong \mathbb{Z}.$$

From group theory, this map must be multiplication by some integer $d \in \mathbb{Z}$, which we call it as the *degree*, denotes as $\deg(f)$ of f.

Remark (Properties of Degree). We first see some properties of degree.

1. $\deg(\mathrm{id}_{S^n}) = 1$ since $(\mathrm{id}_{S_n})_* = \mathrm{id}_{\mathbb{Z}}$.
2. If $f \colon S^n \to S^n$, $n \geq 0$ is not surjective, then $\deg(f) = 0$. To see this, we know that f_* factors as

$$\begin{tikzcd} H_n(S^n) \arrow[r] \arrow[rr, bend right, "f_*"'] & H_n(S^n - \{*\}) = 0 \arrow[r] & H_n(S^n) \end{tikzcd}$$

   And since the middle group is zero, $f_* = 0$.
3. If $f \simeq g$, then $f_* = g_*$, so $\deg(f) = \deg(g)$.

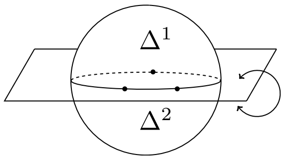
   Note. The converse is true! We'll see this later.
4. $(f \circ g)_* = f_* \circ g_*$, and so $\deg(f \circ g) = \deg(f)\deg(g)$.

   Consequently, if f is a homotopy equivalence then $\deg f = \pm 1$.

   Exercise. It is possible to put a Δ-complex structure with 2 n-cells, Δ_1 and Δ_2 glued together along their boundary ($\cong S^{n-1}$), and

$$H_n(S^n) = \langle \Delta_1 - \Delta_2 \rangle.$$

If f is a reflection fixing the equator, and swapping the 2-cells, then $\deg f = -1$.



5. We now have the following linear algebra exercise.

   Exercise. The map $S^{n+1} \to S^{n+1}$ given by $x \mapsto -x$ is the composite of $(n+1)$ reflections.

   So the antipodal map $S^n \to S^n$ given by $x \mapsto -x$ has degree which is the product of $n+1$ copies of (-1), and so it has degree $(-1)^{n+1}$. (i.e., since the $(n+1) \times (n+1)$ scalar matrix (-1) is composition of $(n+1)$ reflections.)

6. We see the following.

   Exercise. If $f\colon S^n \to S^n$ has no fixed points, then we can homotope f to the antipodal map via

$$f_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}.$$

   Therefore, $\deg f = (-1)^{n+1}$.

Lecture 30: Degree

23 Mar. 10:00

With the definition of degree and some of its properties, we have the following theorems.

Theorem 4.10 (Hairy ball theorem). The sphere S^n admits a nonvanishing continuous tangent vector field if and only if n is odd.

Proof. Recall that a tangent vector field to the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a continuous map

$$v\colon S^n \to \mathbb{R}^{n+1}$$

such that $v(x)$ is tangent to S^n at x, i.e., $v(x)$ is perpendicular to the vector x for each x. Let $v(x)$ be a nonvanishing tangent vector field on the sphere S^n, then we define

$$f_t(x) \coloneqq \cos(\pi t) + \sin(\pi t)\left(\frac{v(x)}{\|v(x)\|}\right),$$

which is a homotopy from the identity map $\mathrm{id}_{S^n}\colon S^n \to S^n$ to the antipodal map $-\mathrm{id}_{S^n}\colon S^n \to S^n$. This simply follows from varying t from 0 to 1, where we have

$$f_0(x) = \cos(0)x + \sin(0)\left(\frac{v(x)}{\|v(x)\|}\right) = x \implies f_0 = \mathrm{id}_{S^n},$$

while

$$f_1(x) = \cos(\pi)x + \sin(\pi)\left(\frac{v(x)}{\|v(x)\|}\right) = -x \implies f_1 = -\mathrm{id}_{S^n}.$$

The last thing needs to be verified is that $f_t(x)$ is continuous, but this is trivial.

From the property of degree, we know that it's a homotopy invariant, hence

$$\deg(-\mathrm{id}_{S^n}) = \deg(\mathrm{id}_{S^n}),$$

which implies

$$(-1)^{n+1} = 1,$$

so n must be odd.

Conversely, if n is odd, say $n = 2k-1$, we can define $v(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$. Then $v(x)$ is orthogonal to x, so v is a tangent vector field on S^n, and $|v(x)| = 1$ for all $x \in S^n$. ■

Theorem 4.11 (Groups acting on S^{2n}). If G acts on S^{2n} freely, then

$$G = {}^{\mathbb{Z}}\big/_{2\mathbb{Z}} \text{ or } 1.$$

Proof. There exists a homomorphism given by

$$\begin{aligned} G &\to \{\pm 1\} \\ g &\mapsto \deg(\tau_g) \end{aligned}$$

Where τ_g is the action of $g \in G$ on S^{2n} as a map $S^{2n} \to S^{2n}$. We know this map is well-defined since τ_g is invertible (simply take $\tau_{g^{-1}}$) for each $g \in G$. Our note on composites shows this is a homomorphism.

We want to show that the kernel is trivial, since then by the first isomorphism theorem $G \cong \mathrm{Im}$, and the image is either trivial or $\mathbb{Z} / 2\mathbb{Z}$. Suppose that g is a nontrivial element of G, then since G acts freely we know that τ_g has no fixed points. With this in mind we have

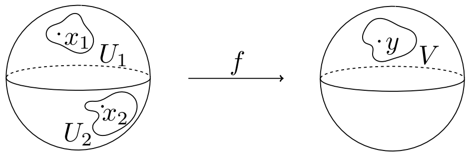
$$\deg \tau_g = (-1)^{2n+1} = -1.$$

Thus, $g \notin \ker$, hence the kernel is trivial as desired. ■

Corollary 4.3. S^{2n} has only the trivial cover $S^{2n} \to S^{2n}$ or degree 2 cover (for example, $S^{2n} \to \mathbb{R}P^{2n}$).

Proof. This follows since any covering space action acts freely. ■

Definition 4.29 (Local degree). Let $f\colon S^n \to S^n$ $(n > 0)$. Suppose there exists $y \in S^n$ such that $f^{-1}(y)$ is finite, say, $\{x_1, \dots, x_m\}$. Then let $U_1, \dots, U_m$ be disjoint neighborhoods of $x_1, \dots, x_m$ that are mapped by f to some neighborhood V of y.



The *local degree* of f at x_i, denote as $\deg f|_{x_i}$, is the degree of the map

$$f_*\colon \mathbb{Z} \cong H_n(U_i, U_i - \{x_i\}) \to H_n(V, V - \{y\}) \cong \mathbb{Z}.$$

Remark. The homomorphism f_* is a multiplication by an integer, which is the local degree as we just defined, arises from the following natural diagram.

$$\begin{tikzcd}
 & H_n(U_i, U_i - \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, hook, "k_i"] & H_n(V, V - \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n - \{x_i\}) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, hook', "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - \{y\}) \\
 & H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

The two isomorphisms in the upper half come from excision, and the lower two isomorphisms come from exact sequences of pairs.

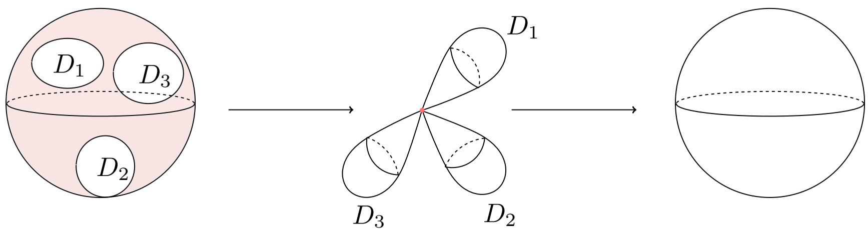
Theorem 4.12. Let $f\colon S^n \to S^n$ with $f^{-1}(y) = \{x_1, \dots, x_m\}$ as in Definition 4.29, then we have

$$\deg f = \sum_{i=1}^{m} \deg f|_{x_i}.$$

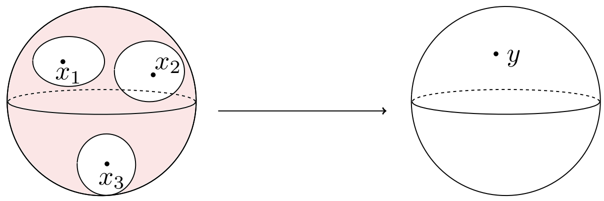
Remark. Thus, we can compute the degree of f by computing these local degrees.

Let's work with some examples for our edification.

Example. Consider S^n and choose m disks in S^n. Namely, we first collapse the complement of the m disks to a point, and then we identify each of the wedged n-spheres with the n-sphere itself.



The result will be a map of degree m. We can see this by computing local degree.



By choosing a good point in the codomain, we get one point for each disk in the preimage, and the map is a local homeomorphism around these points which is orientation preserving. We could likewise compose the maps to S^n from the wedge with a reflection to construct a map of degree $-m$.

Remark. We see that from the above construction, we can produce a map $S^n \to S^n$ in any degree.

Lecture 31: Local Degree and Local Homology

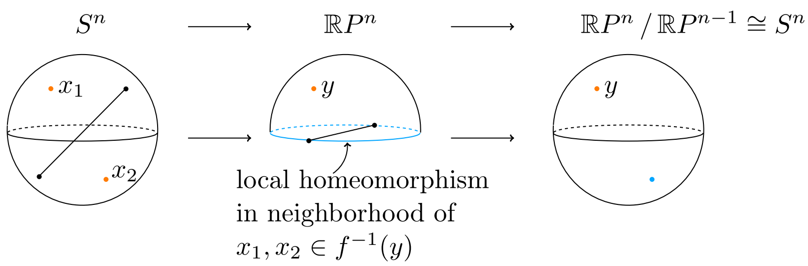
25 Mar. 10:00

We first see another example of the application of Theorem 4.12.

Example. Consider the composition of the quotient maps below

$$\begin{tikzcd} S^n \arrow[r] \arrow[rr, bend right, "f"'] & \mathbb{R}P^n \arrow[r] & \mathbb{R}P^n / \mathbb{R}P^{n-1} \cong S^n \end{tikzcd}$$

We want to compute the degree of this map.



Note that this restricts to a homeomorphism on each component of $S^n \setminus$ equator as a map to $\mathbb{R}P^n / \mathbb{R}P^{n-1}$. Suppose we've oriented our copies of S^n in such a way that the homeomorphism on the top hemisphere is orientation-preserving. The homeomorphism on the bottom hemisphere is given by taking the antipodal map and composing with the homeomorphism of the top hemisphere

$$\deg = \deg(\mathrm{id}) = \deg(\text{antipodal}) = 1 + (-1)^{n+1} = \begin{cases} 0, & \text{if } n \text{ even;} \\ 2, & \text{if } n \text{ odd.} \end{cases}$$

We can now prove Theorem 4.12.

Proof. If $f\colon S^n \to S^n$ and we have some $y \in S^n$ with $f^{-1}(\{y\}) = \{x_1, \dots, x_m\}$, then we have a nice commutative diagram as follows.

$$\begin{tikzcd}
H_n(S^n) \arrow[rr, "f_*"] \arrow[d, "\text{LES of pair}"'] & & H_n(S^n) \arrow[ddd, "\cong"', "\text{LES of a pair}"] \\
\bigoplus_{i=1}^m \mathbb{Z} = H_n(S^n, S^n - \{x_1, \dots, x_m\}) & & \\
H_n\left(\coprod_{i=1}^m U_i, \coprod_{i=1}^m (U_i - \{x_i\})\right) \arrow[u, "\text{excision}", "\cong"'] & & \\
& & H_n(S^n, S^n - \{y\}) \\
\bigoplus_i H_n(U_i, U_i - \{x_i\}) \arrow[uu, "\text{homology of disjoint union}", "\cong"'] \arrow[rr] & & H_n(V, V - \{y\}) \arrow[u, "\cong", "\text{excision}"']
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[d, mapsto] & \deg(f_*) \arrow[d, mapsto] \\
(1, 1, \dots, 1) \arrow[r, mapsto] & \deg f = \sum \deg f|_{x_i}
\end{tikzcd}$$

where we trace around the outside of the diagram at the bottom, which just proves the result. ■

With degree, we have a very efficient way for computing the homology groups of CW complexes, which is so-called cellular homology. But before we dive into this, we first grab some intuition about the essential of which, namely,

what really is local homology?

By excision, there is an isomorphism $H_n(S^n, S^n \setminus \{x_i\}) \cong H_n(U, U \setminus \{x_i\})$ for any open neighborhood U of x_i.

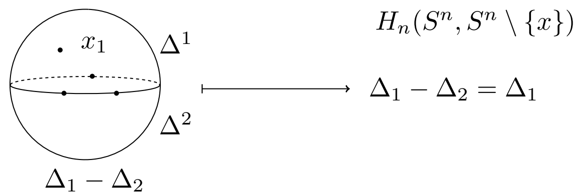
The long exact sequence of a pair also gives us

$$\ldots \longrightarrow H_k(S^n \setminus \{x_i\}) \longrightarrow H_k(S^n) \longrightarrow H_k(S^n, S^n \setminus \{x_i\}) \longrightarrow H_{k-1}(S^n \setminus \{x_i\}) \longrightarrow \ldots$$

Since $S^n \setminus \{x_i\}$ is homeomorphic to an open n-ball, we see that $H_k(S^n \setminus \{x_i\}) = H_{k-1}(S^n \setminus \{x_i\}) = 0$. With this in mind, j_* is an isomorphism.

We want to think about what j_* does when $k = n$, i.e., when this is an isomorphism $\mathbb{Z} \cong H_n(S^n) \to H_n(S^n, S^n \setminus \{x_i\}) \cong \mathbb{Z}$.

We see that $\Delta_1 - \Delta_2$ generate $H_n(S^n)$, where Δ_1, Δ_2 are the top and bottom hemisphere indicated below.



We then understand that $j_*(\Delta_1 - \Delta_2) = \Delta_1 - \Delta_2 = \Delta_1$ since $\Delta_2 = 0$ in $C_n(S^n)/C_n(S^n \setminus \{x_i\})$.

The upshot is that $H_n(S^n, S^n \setminus \{x\})$ is generated by an n-simplex with x in its interior.

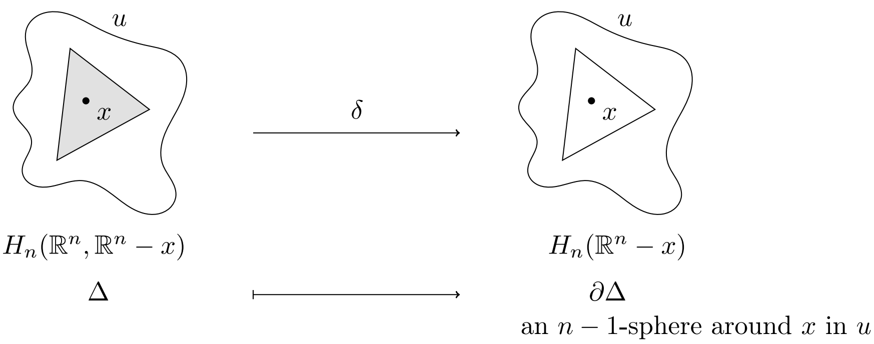
Suppose M is an n-manifold. Then $H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$, where U is a small ball around x. Because U is a ball homeomrphic to $\mathbb{R}^n$, we see that

$$H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}).$$

By the long exact sequence of a pair

$$0 = H_n(\mathbb{R}^n) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \longrightarrow H_{n-1}(\mathbb{R}^n \setminus \{x\}) \longrightarrow H_{n-1}(\mathbb{R}^n) = 0$$

And since $\mathbb{R}^n \setminus \{x\}$ is homotopy equivalent to an $n-1$ sphere, this means that $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}$. By homework, this connecting homomorphism is given by taking the boundary of a relative cycle as below.



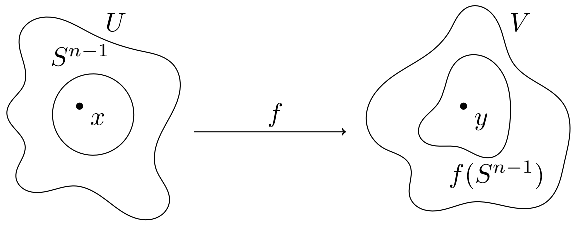
We intuitively want to use this idea to compute degree using this idea. We use naturality of the long exact sequence, namely the fact that where $f\colon (U_i, U_i \setminus \{x_i\}) \to (V, y)$ is a map of pairs, then the following diagram commutes.

$$\begin{tikzcd}
\dots \arrow[r] & H_n(U_i, U_i \setminus \{x_i\}) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(U_i, U_i \setminus \{x_i\}) \arrow[r] \arrow[d, "f_*"] & \dots \\
\dots \arrow[r] & H_n(V, V \setminus \{y\}) \arrow[r] & H_{n-1}(V, V \setminus \{y\}) \arrow[r] & \dots
\end{tikzcd}$$

By naturality of the long exact sequence and the isomorphism discussed above, we can compute the local degree of a map $S^n \to S^n$ at a point x by computing the degree of the map

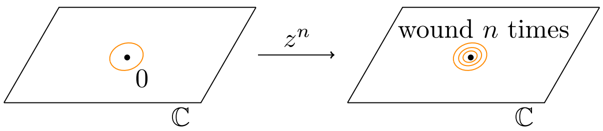
$$H_{n-1}(U \setminus \{x\}) \longrightarrow H_{n-1}(V - \{y\})$$

In fact the local degree will be the degree restricted to a small S^{n-1} n the neighborhood U.



Example. Consider

$$\begin{aligned} \hat{\mathbb{C}} &\to \hat{\mathbb{C}} \\ z &\mapsto z^n. \end{aligned}$$



We see that

$$\deg f|_0 = n.$$

Lecture 32: Cellular Homology

28 Mar. 10:00

4.7 Cellular Homology

Suppose that X is a CW complex, then (X^n, X^{n-1}) is a good pair for all $n > 1$, and $X^n \,/\, X^{n-1}$ is a wedge of n-spheres, one for each n-cell e_α^n. Hence,

$$H_k(X^n, X^{n-1}) \cong \begin{cases} 0, & \text{if } k \neq n; \\ \langle e_\alpha^n \mid e_\alpha^n \text{ is an } n\text{-cell} \rangle, & \text{if } k = n. \end{cases}$$

Definition 4.30 (Cellular chain complex). The *cellular chain complex* on X, denoted as $\overline{\omega}$, has

(Chain groups). The chain groups $C_n(X)$ are defined as

$$C_n(X) \coloneqq \mathbb{Z}\left\langle e_\alpha^n \mid e_\alpha^n \text{ an } n\text{-cell of } X \right\rangle (\cong H_n(X^n, X^{n-1}))$$

with $X^{-1} = \varnothing$.

(Boundary maps). For $n = 0$, we have

$$\begin{aligned} \partial_1 \colon C_1(X) &\to C_0(X) \\ \langle 1\text{-cells} \rangle &\to \langle 0\text{-cells} \rangle, \end{aligned}$$

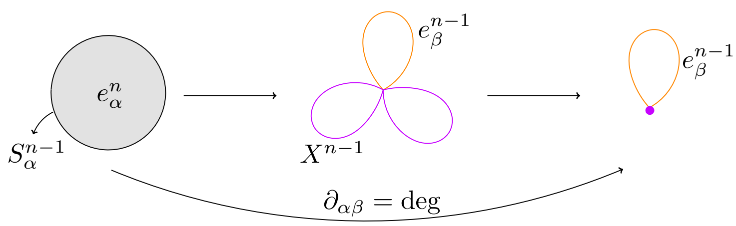
which is the usual simplicial boundary map.[a] For $n > 1$, the boundary map ∂_n are defined as

$$\partial_n(e_\alpha^n) = \sum_\beta \partial_{\alpha\beta} e_\beta^{n-1}$$

where $\partial_{\alpha\beta}$ is the degree of the map

$$\partial e_\alpha^n = S_\alpha^{n-1} \xrightarrow[\text{map}]{\text{attaching}} X^{n-1} \xrightarrow[X^{n-1} \setminus e_\beta^{n-1}]{\text{quotient by}} S_\beta^{n-1}$$

In pictures, this is given as the following.



[a] i.e., $\partial_1 \colon C_1(X) = H_1(X^1, X^0) \to C_0(X) = H_0(X^0)$ is just $\Delta_1(X) \to \Delta_0(X)$.

Remark. We see that

$$C_n(X) \cong H_n(X^n, X^{n-1})$$

since (X^n, X^{n-1}) is a good pair, so $H_n(X^n, X^{n-1}) \cong H_n(X^n \,/\, X^{n-1})$, which is just the wedge of 1 n-sphere for each n-cell of X.

Furthermore, the orientations on spheres are defined by identifying the domains of characteristic maps $D_\alpha^n \to X$ with an (oriented) disk in $\mathbb{R}^n$. i.e., we need to choose a generator of

$$H_{n-1}(\partial D_\alpha^n) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

Note. In Hatcher[HPM02], the approach of the definition of cellular chain complex is a bit different, especially for how we define the boundary maps. Here we simply define $\partial_n(e^n_\alpha) \coloneqq \sum_\beta \partial_{\alpha\beta} e^{n-1}_\beta$, where this is so-called *cellular boundary formula* in Hatcher[HPM02]. Here, we just defined ∂_n in this way instead, but we should still check that this is well-defined of this definition. The proof is given in Appendix A.3.

Definition 4.31 (Cellular homology group). We define the so-called *cellular homology group* by cellular chain complex in our usual way of defining homology group.

Remark. We sometimes denote the cellular homology group as $H_n^{\mathrm{CW}}(X)$ if it causes confusion.

Theorem 4.13. Definition 4.30 indeed forms a chain complex.

Proof. We need to check two things, namely the chain group $H_n(X^n, X^{n-1})$ defined in Definition 4.30 is indeed free Abelian with basis in each n-cell. But this is trivial since we have an one-to-one correspondence with the n-cells of X as we have shown, and we can think of elements of $H_n(X^n, X^{n-1})$ as linear combinations of n-cells of X.

The fact that the boundary map defined in Definition 4.30 has the property $\partial^2 = 0$ will be proved in Theorem 4.14. ■

Theorem 4.14 (Cellular homology agrees with singular homology). The cellular homology groups coincide with the singular homology groups, i.e.,

$$H_n^{\mathrm{CW}}(X) \cong H_n(X).$$

Note. i.e., the isomorphism commutes ϖf_* for all continuous $f\colon X \to Y$.

Theorem 4.14 implies the following.

Corollary 4.4. We have the followings.

- $H_n(X) = 0$ if X has a CW complex structure with no n-cells.
- If X has a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements.
- If $H_n(X)$ is a group with a minimum of k generators, then any CW complex structure on X must have at least k n-cells.
- If X has a CW complex with no n-cells, then

$$H_{n-1}(X) = \ker(\partial_{n-1}),$$

which is free Abelian.
- If X has a CW complex with no cells in consecutive dimensions, then all $\partial_n = 0$. Its homology are free Abelian on its n-cells, namely the cellular chain groups.

Example. The last point in Corollary 4.4 is quite useful, as the following examples will show.

1. $S^n, n \geq 2$. Since if we have S^n with $n \geq 2$, using the CW complex structure of e^n attached to a single point x_0. The cellular chain complex is given as

$$0 \longrightarrow 0 \longrightarrow \langle e^n \rangle \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow \langle x_0 \rangle \longrightarrow 0$$

So then all the boundary maps are zero, and we see that

$$H_k(S^n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

2. $\mathbb{C}P^n, \forall n$. In this case, we can let $\mathbb{C}P^n$ equipped with a CW complex structure with one cell of each even dimension $2k \leq 2n$, thus

$$H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \ldots, 2n; \\ 0, & \text{otherwise.} \end{cases}$$

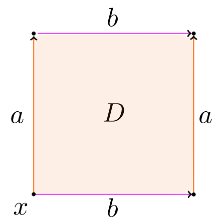
3. $S^n \times S^n, n > 1$. We let $S^n \times S^n$ has the product CW structure consisting of a 0-cell, two n-cells, and a $2n$-cell.

Exercise. Redo this calculation with other CW complex structure on S^n, e.g. glue 2 n-cells onto S^{n-1} and proceed inductively.

Lecture 33: Cellular Homology Examples

30 Mar. 10:00

Example. Let's do this with the torus



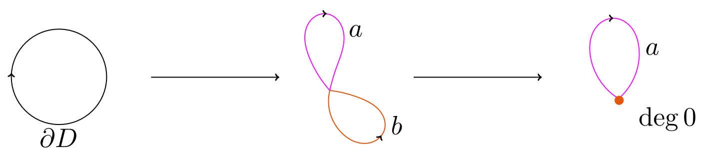
The cellular chain complex looks like

$$0 \longrightarrow \langle D \rangle \longrightarrow \langle a, b \rangle \longrightarrow \langle x \rangle \longrightarrow 0$$

where we choose x as a base point (i.e. the 0-cell).

For ∂_1, since this is defined as the same as the usual simplicial boundary map, hence by $a \mapsto x - x = 0$ and $b \mapsto x - x = 0$, we have $\partial_1 = 0$.

Now for ∂_2, since D is glued along $aba^{-1}b^{-1}$, so we look at the composed up maps



We wind forwards then backwards around a,[21] so the degree is zero. The same thing happens for b, so

$$\partial_2 D = \underbrace{0 \cdot a}_{\partial_{\alpha\beta_a} a} + \underbrace{0 \cdot b}_{\partial_{\alpha\beta_b} b} = 0.$$[22]

This gives a nice **principle**, namely if a 2-cell D is glued down via some words w (this only makes sense for 2-cells), then the coefficient[23] to a letter a in $\partial_2 D$ is the sum of the exponents of a in w. In this case, for both a and b, the coefficients for are both $1 + (-1) = 0$.

Now we just have that the homology groups are equal to the chain groups because the boundary maps are all zero. Hence, we have

$$H_k(T) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2; \\ \mathbb{Z}^2, & \text{if } k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Example. A genus g surface Σ_g has the CW complex structure as

[21] Intuitively, since we quotient out b, hence the gluing map is homotopy to constant maps.
[22] We assume that α is the index of D, and β_a is the index of a and same for b.
[23] i.e. $\partial_{\alpha\beta}(a)$ where α is the index of a.

- 1 0-cell x.
- $2g$ 1-cells $a_1, b_1, a_2, b_2, \ldots$.
- 1 2-cell D glued along $[a_1, b_2][a_2, b_2] \cdots [a_g, b_g]$ (a product of commutators)

For ∂_1, we have

$$\partial_1(a_i) = \partial_1(b_i) = x - x = 0.$$

Furthermore, by the principle discussed above, we know that every 1-cell appears once in the word, and its inverse appears once, so all the coefficients of 1-cells in $\partial_2(D)$ are zero, so $\partial_2(D) = 0$. This means we have a chain complex

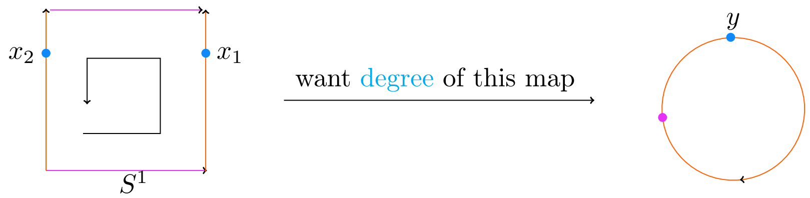
$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

And so then we have that

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2; \\ \mathbb{Z}^{2g}, & \text{if } k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Exercise. Calculate the cellular homology group of $\mathbb{R}P^n$.

Example (Torus example: ∂_2 in more detail). We're going to work through this example a bit more carefully.



Let's zoom in on these two preimage points and use *local homology* to compute this:

Fill this up!

Lecture 34: Proof of Theorem 4.14

1 Apr. 10:00

We're now going to work towards proving that cellular homology agrees with singular homology. First we need some nontrivial preliminaries.

Lemma 4.4. We have that

1. $H_k(X^n, X^{n-1}) = \begin{cases} 0, & \text{if } k \neq n; \\ \langle n\text{-cells} \rangle, & \text{if } k = n. \end{cases}$
2. $H_k(X^n) = 0$ for all $k > n$. If X is finite dimensional, then $H_k(X^n) = 0$ for all $k > \dim X$.
3. The inclusion $X^n \hookrightarrow X$ induces $H_k(X^n) \to H_k(X)$. Then this map is
    - an isomorphism for $k < n$
    - surjective for $k = n$
    - zero for $k > n$.

Exercise. Check 2. and 3. directly in the case that the CW complex structure is a Δ-complex structure using simplicial chains.

Proof. For 1., we see that

$$X^n \big/ X^{n-1} \cong \text{wedge of one } n\text{-sphere for each } n\text{-cell.}$$

The result then follows from Theorem 4.8 and its immediately corollary, namely

$$\bigoplus_\alpha \widetilde{H}_n(X_\alpha) \cong \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right)$$

provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that (X_α, x_α) are good, and then we simply consider $(X, A) = (\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\})$.

Now we prove 2. and 3., We consider the long exact sequence of a pair for fixed n,

$$\begin{tikzcd} \dots \arrow[r] & H_{k+1}(X^n, X^{n-1}) \arrow[r] & H_k(X^{n-1}) \arrow[dl, "\cong"'] & \\ & H_k(X^n) \arrow[r] & H_k(X^n, X^{n-1}) \arrow[r] & \dots \end{tikzcd}$$

When $k+1 < n$ or $k > n$ then $H_{k+1}(X^n, X^{n-1}) = 0$ and $H_k(X^n, X^{n-1}) = 0$, so the above map $H_k(X^{n-1}) \to H_k(X^n)$ is an isomorphism. We also get sequences telling us the injective and surjective maps when $k = n$ or $k = n - 1$,

$$\begin{tikzcd} \dots \arrow[r] & 0 = H_{n+1}(X^n, X^{n-1}) \arrow[r] & H_n(X^{n-1}) \arrow[r] & H_n(X^n) \arrow[dll] \\ & H_n(X^n, X^{n-1}) \arrow[r] & H_{n-1}(X^{n-1}) \arrow[r] & H_{n-1}(X^n) \arrow[dll] \\ & H_{n-1}(X^n, X^{n-1}) = 0 \arrow[r] & \dots & \end{tikzcd}$$

So the maps $H_n(X^{n-1}) \to H_n(X^n)$ is injective, and the map $H_{n-1}(X^{n-1}) \to H_{n-1}(X^n)$ is surjective.

Fix k, then we get a pile of maps induced by the inclusions $X^n \hookrightarrow X^{n+1}$

$$\begin{tikzcd}
H_k(X^0) \arrow[r, "\cong"] & H_k(X^1) \arrow[r, "\cong"] & H_k(X^2) \arrow[r, "\cong"] & \dots \arrow[dll, "\cong"] \\
& H_k(X^{k-1}) \arrow[r, "\text{inj.}"] & H_k(X^k) \arrow[r, "\text{surj.}"] & H_k(X^{k+1}) \arrow[dll, "\cong"] \\
& H_k(X^{k+2}) \arrow[r, "\cong"] & H_k(X^{k+3}) \arrow[r, "\cong"] & \dots
\end{tikzcd}$$

Note. This sequence is not exact. Descriptions of maps (in red) follow from our analysis of the long exact sequence of a pair above.

To prove 2.,

- $k = 0$, we do this by hand.
- $k \geq 1$, then $H_k(X^0) = 0$, so we have that $H_k(X^0), \dots, H_k(X^{k-1})$ are all zero from the isomorphisms above. That is the k-th homology $H_k(X^n) = H_k(X^n)$ is zero for every n-skeleton where $n < k$, just as desired.

We also have the following collection of maps for fixed k

$$H_k(X^k) \xrightarrow{\text{surj.}} H_k(X^{k+1}) \xrightarrow{\cong} H_k(X^{k+2}) \xrightarrow{\cong} \dots$$

This implies 3. when X is finite dimensional. For general X, we use the fact that every simplex has image contained in some finite skeleton (since image is compact). ■

Proof of Theorem 4.14. We get some exact sequences from our preliminaries,

$$0 = H_{n+1}(X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^n, X^{n+1}) \longrightarrow H_{n-1}(X^{n,-1})$$

$$H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) = 0$$

These come from the long exact sequences of a pair combined with the things we've deduced in the preliminaries. We can paste these together into a diagram, we have

$$\begin{tikzcd}
& & & & & & 0 & \\
& 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[urrr] & & & & \\
& & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & & \\
\dots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \dots & \\
& & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & & \\
& & & 0 \arrow[ur] & & & &
\end{tikzcd}$$

Hatcher[HPM02] tells us this diagram commutes, and what we've done here tells us that the two red diagonal pieces crossing at $H_n(X^n)$ are exact. We also have exactness of the bottom right diagonal by just going down a degree.

Then the horizontal row has to at least be a chain complex since the diagram commutes, and we have

$$d_n \circ d_{n+1} = (j_{n-1} \circ \underbrace{\partial_n) \circ (j_n}_{0} \circ \partial_{n+1}) = 0.$$[24]

By exactness, we know that if $\iota_* \colon H_n(X^n) \to H_n(X^{n+1})$, then using the first isomorphism theorem,

$$H_n(X) \cong H_n(X^{n+1}) = \operatorname{Im} \iota_* \cong H_n(X^n) \big/ \ker \iota_* = H_n(X^n) \big/ \operatorname{Im} \partial_{n+1}.$$

Since j_n injects by exactness,

$$\begin{aligned} j_n : H_n(X^n) &\xrightarrow{\cong} j_n(H_n(X^n)) \\ \operatorname{Im} \partial_{n+1} &\xrightarrow{\cong} \operatorname{Im}(j_n \circ \partial_{n+1}) = \operatorname{Im} d_{n+1}, \end{aligned}$$

so j_{n-1} must also inject by exactness, and by applying exactness, we have

$$\ker d_n = \ker \partial_n = \operatorname{Im} j_n.$$

Then we just do some group theory, the n-th cellular homology group is

$$\ker d_n \big/ \operatorname{Im} d_{n+1} \cong \operatorname{Im} j_n \big/ \operatorname{Im}(j_n \circ \partial_{n+1}) \cong H_n(X^n) \big/ \operatorname{Im} \partial_{n+1} \cong H_n(X).$$

There is one thing left to show, namely commutativity of this map. We claim that the differentials $d_n = j_n \circ \partial_{n+1}$ satisfy the formula (in terms of degree) that we stated. This is done by direct analysis of definitions of maps; details in Hatcher[HPM02]. ■

Lecture 35: Eilenberg-Steenrod Axioms

4 Apr. 10:00

4.8 The Formal Viewpoint: Eilenberg-Steenrod Axioms

We can approach the homology theory in an **axiomatic** way. Specifically, we're interested in the Eilenberg-Steenrod axioms. To start with, we first see some definitions.

[24] This is the missing part of the proof of Theorem 4.13.

Definition 4.32 (Natural transformation). Given two functors

$$F, G\colon \mathscr{C} \to \mathscr{D},$$

a *natural transformation* $\eta\colon F \to G$ is a collection of maps $\eta_X\colon F(X) \to G(X)$ lying in $\mathscr{D}$ for every $X \in \mathscr{C}$ so that for any map $f\colon X \to Y$, we have a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta Y"'] & G(Y) \end{tikzcd}$$

Definition 4.33 (Homology theory). A *homology theory* is a sequence of functors

$$H_n\colon \text{pairs } (X, A) \text{ of spaces} \to \text{Abelian groups}$$

equipped with natural transformations $\partial\colon H_n(X, A) \to H_{n-1}(A)$, where $H_{n-1}(A) \coloneqq H_{n-1}(A, \varnothing)$, is called the boundary map.

Naturality here means that for any map $f\colon (X, A) \to (Y, B)$ we have a commutative diagram

$$\begin{tikzcd} H_n(X, A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(A) \arrow[d, "f_*"] \\ H_n(Y, B) \arrow[r, "\partial"'] & H_{n-1}(B) \end{tikzcd}$$

These must satisfy the following 5 axioms.

1. (Homotopy) If $f, g\colon (X, A) \to (Y, B)$ and $f \simeq g$, then $f_* = g_*$.
2. (Excision) If $U \subseteq A \subseteq X$ such that $\overline{U} \subseteq \operatorname{Int}(A)$, then

$$\iota\colon (X \setminus U, A \setminus U) \hookrightarrow (X, A)$$

   induces isomorphisms on H_n.
3. (Dimension) $H_n(*) = 0$ for all $n \neq 0$.
4. (Additivity) $H_n\left(\coprod_\alpha X_\alpha\right) = \bigoplus_\alpha H_n(X_\alpha)$.
5. (Exactness) If we have an inclusion $\iota\colon A \hookrightarrow X$[a] and $j\colon X \to (X, A)$ induces a long exact sequence

$$\ldots \longrightarrow H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \ldots$$

[a]Note that we use $X \coloneqq (X, \varnothing)$ for every space X.

Definition 4.34 (Extraordinary homology theory). If H_* satisfies all axioms but dimension, it is called an *extraordinary homology theory*.

Example. Topological K-theory, bordism, and cobordism.[25]

Theorem 4.15. If $H_n\colon \text{CW pairs} \to \underline{\text{Ab}}$ is a homology theory and $H_0(*) = \mathbb{Z}$, then H_n are exactly the singular homology functors up to a natural isomorphism of functors.

More generally, if $H_0(*) = G$, then H_n are exactly the singular homology functors with coefficients in the Abelian group G.

Proof. Given H_*, reconstruct the cellular chain groups $H_n(X^n, X^{n-1})$ using the axioms.

- Show the homology of this chain complex are the cellular homology groups of X.
- Show these agree with $H_n(X^n, X^{n-1})$. The exact same argument in Theorem 4.14 applies.

We then check that the cellular homology groups we just constructed satisfies the degree formula as in our last step. This is a bit more difficult, but we won't get into it. ■

5 Lefschetz Fixed Point Theorem

Definition 5.1 (Trace). Let $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be a group homomorphism, we may represent this with a matrix $A = [a_{ij}]_{i,j}$ has *trace*

$$\operatorname{tr} A \coloneqq a_{11} + \ldots + a_{nn}.$$

For a group homomorphism $\varphi\colon M \to M$ where M is a finitely generated Abelian group, we define the *trace* of φ to be the *trace* of the induced map $\overline{\varphi}\colon M \,/\, M_T \to M \,/\, M_T$, where M_T is the torsion subgroup of M.

Exercise. We have

1. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
2. $\operatorname{tr}(A) = \operatorname{tr}(BAB^{-1})$.

Thus, trace is independent of change of basis of $\mathbb{Z}^n$.

Lecture 36: Lefschetz Fixed Point Theorem

6 Apr. 10:00

[25] https://en.wikipedia.org/wiki/Cobordism

Definition 5.2 (Lefschetz number). Let X be a space with the assumption that $\bigoplus_k H_k(X)$ is finitely generated. That is, each homology group is finitely generated, and there are finitely many nonzero homology groups. For example X could be a finite CW complex.

The *Lefschetz number* $\tau(f)$ of a map $f\colon X \to X$ is

$$\tau(f) := \sum_k (-1)^k \operatorname{tr}(f_*\colon H_k(X) \to H_k(X)).$$

Example. When $f \simeq \mathrm{id}_X$. Then $f_* = \mathrm{id}_{H_k(X)}$ for all k. Then $\operatorname{tr}(f_*\colon H_k(X) \to H_k(X)) = \operatorname{rank}(H_k(X))$. Therefore,

$$\tau(f) = \sum_k \operatorname{rank}(H_k(X)) = \chi(X),$$

where $\chi(X)$ is the *Euler characteristic*.

Theorem 5.1 (Lefschetz Fixed Point Theorem). Suppose X admits a finite triangulation (i.e. a finite simplicial complex structure). Or more generally, X is a retract of a finite simplicial complex.

Then if $f\colon X \to X$ is a map with $\tau(f) \neq 0$, then f has a fixed point.

Note. Note that the converse does not hold.

Theorem 5.2. If X is a compact, locally contractible space that can embed in $\mathbb{R}^n$ for some n, then X is a retract of a finite simplicial complex.

This includes

- Compact Manifolds.
- Finite CW complexes.

Definition 5.3. Let $\mathbb{F}$ be a field, and let $H_k(X;\mathbb{F})$ be the k-th homology of X with coefficients in $\mathbb{F}$. Then $H_k(X;\mathbb{F})$ is always a vector space over $\mathbb{F}$. Define $\tau^{\mathbb{F}}(X)$ be

$$\sum_k (-1)^k \operatorname{tr}(f_*\colon H_k(X;\mathbb{F}) \to H_k(X;\mathbb{F})).$$

The Lefschetz fixed point theorem still holds if we replace $\tau(x) \neq 0$ with $\tau^{\mathbb{F}} \neq 0$.

Example. Let $f\colon S^n \to S^n$ be a degree d map. Then $\tau(f)$ is

$$(-1)^0 \operatorname{tr}(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \operatorname{tr}(f_* : H_n(S^n) \to H_n(S^n)).$$

Then $f_*\colon H_0(S^n) \to H_0(S^n)$ is the identity, and $f_*\colon H_n(S^n) \to H_n(S^n)$ is given by the 1×1 matrix with entry d. And then we have

$$\tau(f) = 1 + (-1)^n d.$$

Corollary 5.1. f has a fixed point whenever $1 + (-1)^n \neq 0$. Namely, whenever $d \neq (-1)^{n+1}$. That is f has a fixed point if its degree is not equal to the degree of the antipodal map.

Exercise. If $f\colon X \to X$, then $\operatorname{tr}(f_*\colon H_0(X) \to H_0(X))$ is equal to the $\#$ of path-components of X mapped to themselves.

Exercise. If X is contractible, then its homology is concentrated in degree zero, so $\tau(f) = 1$.

If X is a compact manifold or finite CW complex, every f has a fixed point (in particular, this recovers Brouwer's Fixed Point Theorem).

Example. If we consider the map $f\colon \mathbb{R} \to \mathbb{R}$ given by translation by $x \neq 0$, then $\tau(f) = 1$, but f does not have a fixed point. The key here is that $\mathbb{R}$ is not compact.

Example (Qual, May 2016). Let X be a finite, connected CW complex. $\widetilde{X}$ is its universal cover, and $\widetilde{X}$ is compact. Show that $\widetilde{X}$ cannot be contractible unless X is contractible.

Answer. We actually have two different approaches.

1. By homework, we then know that, since $\widetilde{X}$ is contractible and $\widetilde{X}$ has finitely many sheets d over X,

   $$1 = \chi(\widetilde{X}) = d \cdot \chi(X).$$

   Therefore, $\chi(X) = d = 1$, and so $p\colon \widetilde{X} \to X$ is a 1-sheeted cover, so it is a homeomorphism. Therefore, X is contractible.

2. Since $\widetilde{X}$ is contractible, $\tau(f) = 1$ for all $f\colon \widetilde{X} \to \widetilde{X}$. Furthermore, because $\widetilde{X}$ is compact and covers a finite CW complex, it is a finite CW complex. Therefore, the Lefschetz fixed point theorem applies, so any such map has a fixed point. If f is a deck map, then that means that $f = \mathrm{id}_{\widetilde{X}}$ from our covering space theory.

   We have proved then that $X \cong \widetilde{X} \,/\, G(\widetilde{X})$ because $p\colon \widetilde{X} \to X$ is normal, but then the deck group $G(\widetilde{X})$ is trivial, so $X \cong \widetilde{X}$, and we are done.

Exercise. A 1-sheeted cover is always injective and surjective. Furthermore, it's a local homeomorphism. This suffices to show that a 1-sheeted cover is a homeomorphism.

Theorem 5.3. If X is a finite CW complex, with cellular chain groups $H_n(X^n, X^{n-1})$. If we have a cellular map $f\colon X \to X$, so f induces maps $f_*\colon H_n(X^n, X^{n-1}) \to H_n(X^n, X^{n-1})$. Then

$$\tau(f) = \sum_n (-1)^n \operatorname{tr}(f_* : H_n(X^n, X^{n-1}) \to H_n(X^n, X^{n-1})).$$

Proof. Do some algebra! This is a purely algebraic fact

Exercise. Given a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

then $\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma)$.

Using the above result, the theorem follows by an argument analogous to the argument for Euler Characteristic in Homework. ■

Appendix

A Additional Proofs

A.1 Seifert-Van Kampen Theorem on Groupoid

Theorem A.1 (Seifert-Van Kampen Theorem on groupoid). Given X_0, X_1, X as topological spaces with $X_0 \cup X_1 = X$. Then the functor $\Pi\colon \underline{\mathrm{Top}} \to \underline{\mathrm{Gpd}}$ maps the cocartesian diagram in $\underline{\mathrm{Top}_*}$ to a cocartesian diagram in $\underline{\mathrm{Gp}}$ as follows.

$$\begin{tikzcd}
(X_0 \cap X_1, x_0) \arrow[r, "j_0"] \arrow[d, "j_1"'] & (X_0, x_0) \arrow[d, "i_0"] & & \Pi(X_0 \cap X_1) \arrow[r, "\Pi(j_0)"] \arrow[d, "\Pi(j_1)"'] & \Pi(X_0) \arrow[d, "\Pi(i_0)"] \\
(X_1, x_0) \arrow[r, "i_1"'] & (X, x_0) \arrow[r, phantom, "\overset{\Pi}{\longmapsto}"] & & \Pi(X_1) \arrow[r, "\Pi(i_1)"'] & \Pi(X)
\end{tikzcd}$$

Note. Notice that X_0, X_1, X don't need to be path-connected in particular.

Surprisingly, the proof of Appendix A.1 is much more elegant with the elementary proof of Theorem 2.7, hence we give the proof here.

Proof. Let $\mathscr{G} \in \mathrm{Ob}(\underline{\mathrm{Gpd}})$ a groupoid, and given functors

$$F\colon \Pi(X_0) \to \mathscr{G}, \quad G\colon \Pi(X_1) \to \mathscr{G}$$

such that

$$F \circ \Pi(j_0) = G \circ \Pi(j_1).$$

$$\begin{tikzcd}
\Pi(X_0 \cap X_1) \arrow[r, "\Pi(j_0)"] \arrow[d, "\Pi(j_1)"'] & \Pi_1(X_0) \arrow[d, "\Pi(i_0)"] \arrow[ddr, bend left, "F"] & \\
\Pi_1(X_1) \arrow[r, "\Pi(i_1)"'] \arrow[drr, bend right, "G"'] & \Pi_1(X) \arrow[dr, dashed, "\exists! K"] & \\
& & \mathscr{G}
\end{tikzcd}$$

We now only need to prove that there exists a unique functor $K\colon \Pi(X) \to \mathscr{G}$ such that the above diagram commutes.

We can define K as

- on objects: For all $x \in \mathrm{Ob}(\Pi(X)) = X$,

$$K(x) = \begin{cases} F(x), & \text{if } x \in X_0; \\ G(x), & \text{if } x \in X_1. \end{cases}$$

  This is well-defined since the diagram (without K) commutes.

- on morphisms: For every $p, q \in X$, $\langle\gamma\rangle : p \to q$ in $\mathrm{Hom}_{\Pi(X)}(p, q)$, we need to define $K(\langle\gamma\rangle) \in \mathrm{Hom}_{\mathscr{G}}(K(p), K(q))$. Our strategy is for every path γ

from p to q, we define $\widetilde{K}(\gamma) \in \mathrm{Hom}_{\mathscr{G}}(K(p), K(q))$. Then if we also have $\widetilde{K}(\gamma) = \widetilde{K}(\gamma')$ for $\gamma \simeq \gamma'$ rel$\{0,1\}$, then we can just let

$$K(\langle\gamma\rangle) \coloneqq \widetilde{K}(\gamma).$$

Now we start to construct $\widetilde{K}$.

Given a path $\gamma\colon [0,1] \to X$, $\gamma(0) = p, \gamma(1) = q$. Since $\mathrm{int}(X_0) \cup \mathrm{int}(X_1) = X$, we see that

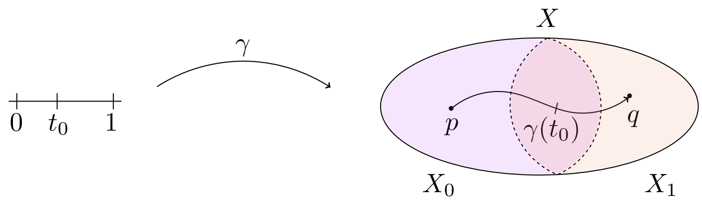
$$\gamma^{-1}(\mathrm{int}(X_0)) \cup \gamma^{-1}(\mathrm{int}(X_1)) = [0,1].$$

From Lebesgue Lemma[26], there exists a finite partition

$$0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 1$$

such that for every i,

$$\gamma([t_{i-1}, t_i]) \subset \mathrm{int}(X_0) \text{ or } \mathrm{int}(X_1).$$



Now, let $\gamma_i\colon [0,1] \to X, t \mapsto \gamma((1-t)t_{i-1} + t \cdot t_i)$, we see that γ_i is either a path in X_0 or X_1. We then define $\widetilde{K}(\gamma) \coloneqq \widetilde{K}(\gamma_m) \circ \widetilde{K}(\gamma_{m-1}) \circ \ldots \circ \widetilde{K}(\gamma_1) \in \mathrm{Hom}_{\mathscr{G}}(K(P), K(q))$ such that

$$\widetilde{K}(\gamma_i) = \begin{cases} F(\langle\gamma_i\rangle), & \text{if } \gamma_i \subset X_0; \\ G(\langle\gamma_i\rangle), & \text{if } \gamma_i \subset X_1. \end{cases}$$

We need to prove that $\widetilde{K}(\gamma)$ does not depend on the partition. It's sufficient to prove that for any partition

$$0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 1,$$

we consider any **finer** partition

$$0 = t_0 = t_{10} < t_{11} < \ldots < t_{1K_1} = t_1 = t_{20} < t_{21} < \ldots < t_{mK_m} = t_m = 1.$$

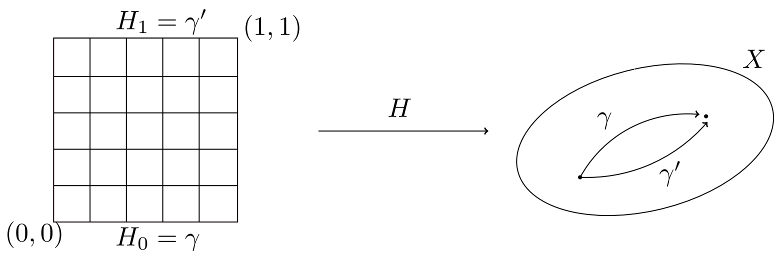
As before, we denote $\gamma_{ij}\colon [0,1] \to X, t \mapsto \gamma((1-t)t_{ij-1} + t \cdot t_{ij})$. It's clear that as long as

$$\widetilde{K}(\gamma_i) = \widetilde{K}(\gamma_{iK_i}) \circ \widetilde{K}(\gamma_{iK_i-1}) \circ \ldots \circ \widetilde{K}(\gamma_{i0}),$$

[26] https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma

then our claim is proved. But this is immediate since F and G are functor and for any i, we only use either F or G all the time.

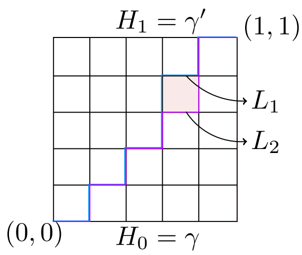
Now we prove $\gamma \underset{H}{\simeq} \gamma'$ rel$\{0,1\}$, then $\widetilde{K}(\gamma) = \widetilde{K}(\gamma')$. This is best shown by some diagram.



The left-hand side represents a partition $\mathcal{P}$ of $[0,1] \times [0,1]$ such that every small square's image in X under H is either entirely in X_0 or in x_1. Consider all paths from $(0,0)$ to $(1,1)$ such that it only goes right or up. We see that for any such path L, consider

$$\gamma_L \colon [0,1] \to L, \quad t \mapsto \gamma_L(t).$$

We let $\Gamma_L \colon H|_L \circ \gamma_L \colon [0,1] \to X$, we see that Γ_L is a path from p to q. Now, if for two paths L_1 and L_2 such that they only differ from a square.



We claim that $\gamma_{L_1}, \Gamma_{L_2}$ are two paths from p to q, and $\widetilde{K}(\Gamma_{L_1}) = \widetilde{K}(\Gamma_{L_2})$. Now, we denote Γ_0 and Γ_1 as follows.

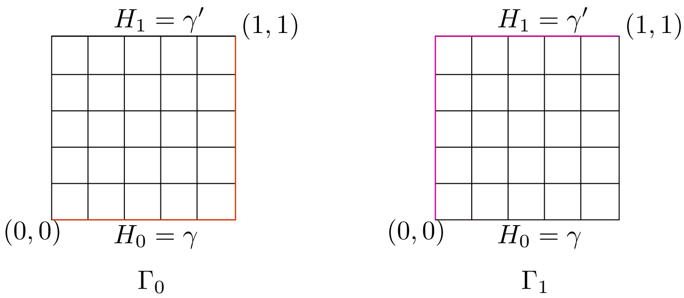


Figure 23: The definition of Γ_0 and Γ_1.

It's clearly that by only finitely many steps, we can transform Γ_0 to Γ_1, hence

$$\widetilde{K}(\Gamma_0) = \widetilde{K}(\Gamma_1).$$

Finally, we observe that

$$\widetilde{K}(\gamma_0) = \widetilde{K}(\Gamma_0) = \widetilde{K}(\Gamma_1) = \widetilde{K}(\gamma_1).$$

If we now define $K(\langle\gamma\rangle) = \widetilde{K}(\gamma)$, then $K\colon \mathrm{Mor}(\Pi(X)) \to \mathrm{Mor}(\mathscr{G})$, then it's well-defined.

We now prove $K\colon \Pi(X) \to \mathscr{G}$ is indeed a functor. But this is immediate from the definition of K, namely it'll send identity to identity and the composition associates.

Also, we need to prove that the following diagram commutes.

$$\begin{tikzcd}
\Pi(X_0 \cap X_1) \arrow[r, "\Pi(j_0)"] \arrow[d, "\Pi(j_1)"'] & \Pi_1(X_0) \arrow[d, "\Pi(i_0)"] \arrow[ddr, bend left, "F"] & \\
\Pi_1(X_1) \arrow[r, "\Pi(i_1)"'] \arrow[drr, bend right, "G"'] & \Pi_1(X) \arrow[dr, dashed, "K"] & \\
 & & \mathscr{G}
\end{tikzcd}$$

But this is again trivial.

Finally, we need to show that such K is unique. This is the same as the proof of Lemma 1.3, hence the proof is done. ■

A.2 An alternative proof of Seifert Van-Kampen Theorem

Theorem A.2. We claim that the diagram

$$\begin{tikzcd}
\pi_1(X_0 \cap X_1, x_0) \arrow[r, "(j_0)_*"] \arrow[d, "(j_1)_*"'] & \pi_1(X_0, x_0) \arrow[d, "(i_0)_*"] \\
\pi_1(X_1, x_0) \arrow[r, "(i_1)_*"'] & \pi_1(X, x_0)
\end{tikzcd}$$

is cocartesian.

Proof. The basic idea is that, for this diagram,

$$\begin{tikzcd}
\Pi(X_0 \cap X_1) \arrow[r] \arrow[d] & \Pi(X_0) \arrow[d] \\
\Pi(X_1) \arrow[r] & \Pi(X)
\end{tikzcd}$$

we want to construct a morphism $r\colon \Pi(Z) \to \pi_1(Z, p)$ in $\underline{\text{Gpd}}$ such that $Z = X_0 \cap X_1, X_0, X_1, X$. For every $x \in Z$, we fix a path γ_x such that it connects p and x and satisfies

1. If $x \in X_0 \cap X_1$, then $\text{Im}(\gamma_x) \subset X_0 \cap X_1$
2. If $x \in X_0$, then $\text{Im}(\gamma_x) \subset X_0$
3. If $x \in X_1$, then $\text{Im}(\gamma_x) \subset X_1$
4. $\gamma_p = c_p$

The proof is given in https://www.bilibili.com/video/BV1P7411N7fW?p=38&spm_id_from=pageDriver. ■

If have time.

A.3 Cellular Boundary Formula in Definition 4.30

Theorem A.3. For $n > 1$, the boundary maps ∂_n of cellular chain complex given by

$$\partial_n(e^n_\alpha) = \sum_\beta \partial_{\alpha\beta} e^{n-1}_\beta$$

is well-defined.

Proof. Here we are identifying the cells e^n_α and e^{n-1}_β with generators of the corresponding summands of the cellular chain groups, namely $C_n(X)$. The summation in the formula contains only finitely many terms since the attaching map of e^n_α has compact image, so this image meets only finitely many cells e^{n-1}_β. To derive the cellular boundary formula, consider the following commutative diagram.

$$
\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi_{\alpha *}"] & \widetilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1} / X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong"] & H_{n-1}(X^{n-1} / X^{n-2}, X^{n-2} / X^{n-2})
\end{tikzcd}
$$

where

- Φ_α is the characteristic map of the cell e^n_α and φ_α is its attaching map.
- $q\colon X^{n-1} \to X^{n-1} / X^{n-2}$ is the quotient map.
- $q_\beta\colon X^{n-1} / X^{n-2} \to S^{n-1}_\beta$ collapses the complement of the cell e^{n-1}_β to a point, the resulting quotient sphere being identified with S^{n-1}_β[27] via the characteristic map Φ_β.
- $\Delta_{\alpha\beta}\colon \partial D^n_\alpha \to S^{n-1}_\beta$ is the composition $q_\beta q \varphi_\alpha$, i.e., the attaching map of e^n_α followed by the quotient map $X^{n-1} \to S^{n-1}_\beta$ collapsing the complement of e^{n-1}_β in X^{n-1} to a point.

The map $\Phi_{\alpha *}$ takes a chosen generator $[D^n_\alpha] \in H_n(D^n_\alpha, \partial D^n_\alpha)$ to a generator of the $\mathbb{Z}$ summand of $H_n(X^n, X^{n-1})$ corresponding to e^n_α. Letting e^n_α denote this generator, commutativity of the left half of the diagram then gives

$$\partial_n(e^n_\alpha) = j_{n-1}\varphi_{\alpha *}\partial[D^n_\alpha].$$

In terms of the basis for $H_{n-1}(X^{n-1}, X^{n-2})$ corresponding to the cells e^{n-1}_β, the map $q_{\beta *}$ is the projection of $\widetilde{H}_{n-1}(X^{n-1} / X^{n-2})$ onto its $\mathbb{Z}$ summand corresponding to e^{n-1}_β. Commutativity of the diagram then yields the formula for ∂_n given above. ∎

B Abelian Group

This section aims to give some reference about Abelian groups, specifically for free Abelian group, which is used heavily when discuss homology.

B.1 Abelian Group

Definition B.1 (Abelian group). A group $(G, \cdot)$ is an *Abelian group* if for every $a, b \in G$, we have

$$a \cdot b = b \cdot a.$$

We often denote $\cdot$ as $+$ if $(G, \cdot)$ is a Abelian group.

[27] Which is just $D^{n-1}_\beta / \partial D^{n-1}_\beta$.

Definition B.2 (Product of groups). Given two groups $(G, \cdot), (H, \cdot)$, the *product of G and H*, denoted by $G \times H$ is defined as

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

and

$$(g_1, h_1) \cdot (g_2, h_2) \coloneqq (g_1 \cdot g_2, h_1 \cdot h_2).$$

Notation. For simplicity, given an index set I, we'll denote the order pair $(g_{\alpha_1}, g_{\alpha_2}, \ldots)$ as $(g_\alpha)_{\alpha \in I}$. Note that the latter notation can handle the case that I is either countable or uncountable, while the former can only handle the countable case.

Definition B.3 (Direct product). Given $(G_\alpha, +)$, $\alpha \in I$ as a collection of Abelian group, we define their *direct product* as

$$\left(\prod_{\alpha \in I} G_\alpha, + \right),$$

where

$$\prod_{\alpha \in I} G_\alpha = \{(g_\alpha)_{\alpha \in I} \mid g_\alpha \in G_\alpha\}$$

and $\forall (g_\alpha), (h_\alpha) \in \prod\limits_{\alpha \in I} G_\alpha$

$$(g_\alpha) + (h_\alpha) \coloneqq g_\alpha + h_\alpha$$

for all $\alpha \in I$.

Specifically, if I is finite, namely there are only finely many Abelian groups $(G_1, +), \ldots, (G_n, +)$, and $\left(\prod\limits_{i=1}^{n} G_i, + \right)$ can be denoted as

$$(G_1 \times \ldots \times G_n, +).$$

Definition B.4 (External direct sum). Given a collection of Abelian groups $\{G_\alpha\}_{\alpha \in I}$, the *external direct sum* of them, denoted as $\left(\bigoplus_{\alpha \in I} G_\alpha, +\right)$ as

$$\bigoplus_{\alpha \in I} G_\alpha \coloneqq \left\{ (g_\alpha)_{\alpha \in I} \mid \underset{\alpha \in I}{\forall} g_\alpha \in G_\alpha, \# \text{ non-zero elements in } g_\alpha < \infty \right\}.$$

And for every $(g_\alpha), (h_\alpha) \in \bigoplus_{\alpha \in I} G_\alpha$,

$$(g_\alpha) + (h_\alpha) \coloneqq g_\alpha + h_\alpha$$

for all $\alpha \in I$.[a]

[a] This may not be the best notation: What we're really trying to say is $(g_\alpha)_{\alpha \in I} + (h_\alpha)_{\alpha \in I} \coloneqq g_i + h_i$ for all $i \in I$.

Note. We see that

$$\bigoplus_{\alpha\in I} G_\alpha \subset \prod_{\alpha\in I} G_\alpha.$$

Additionally, we also have

$$\left(\bigoplus_{\alpha\in I} G_\alpha, +\right) < \left(\prod_{\alpha\in I} G_\alpha, +\right).$$

Remark. We see that the operation $+$ is indeed closed since the sum of $g, g' \in \bigoplus_{\alpha\in I} G_\alpha$ will have only finitely non-zero elements if g, g' both have only finitely many non-zero elements.

We see that if I is a finite index set, given a collection of Abelian group $\{G_\alpha\}_{\alpha\in I}$, then

$$G_1 \times \ldots \times G_n = G_1 \oplus \ldots \oplus G_n.$$

Definition B.5 (Internal direct sum). Given an Abelian group G, and a collection of the subgroups $\{G_\alpha\}_{\alpha\in I}$ of G, we say G is an *internal direct sum* of $\{G_\alpha\}_{\alpha\in I}$ if for any $g \in G$, we can write

$$g = \sum_{\alpha\in I} g_\alpha$$

uniquely, where $g_\alpha \in G_\alpha$ has only finitely many non-zero elements. In this case, we denote

$$G = \bigoplus_{\alpha\in I} G_\alpha.$$

Intuitively, the external direct sum is to build a new group based on the given collection of groups $\{G_\alpha\}_{\alpha\in I}$, while the internal direct sum is to express an **already known** group G with an **already known** collection of groups $\{G_\alpha\}_{\alpha\in I}$.

Remark (Relation between Internal and External direct sum). Given an Abelian group G and its internal direct sum decomposition $\bigoplus_{\alpha\in I} G_\alpha$, G is isomorphic to the external direct sum of $\{G_\alpha\}_{\alpha\in I}$. We see this from the following group homomorphism:

$$\forall_{g\in G}\ g = \sum_{\alpha\in I} g_\alpha \mapsto (g_\alpha)_{\alpha\in I}.$$

Conversely, given a collection of Abelian group $\{G_\alpha\}_{\alpha\in I}$, and let $G = \bigoplus_{\alpha\in I} G_\alpha$ as the external direct sum of $\{G_\alpha\}$, denote $i_{\alpha_0} \colon G_{\alpha_0} \to \bigoplus_{\alpha\in I} G_\alpha$ as a canonical embedding

$$g_{\alpha_0} \mapsto i_{\alpha_0}(g_{\alpha_0}) = (h_\alpha)_{\alpha\in I},$$

where

$$h_\alpha = \begin{cases} g_{\alpha_0}, & \text{if } \alpha_0 = \alpha; \\ 0, & \text{if } \alpha_0 \neq \alpha \end{cases}$$

given α_0. Then

$$G'_{\alpha_0} := i_{\alpha_0}(G_{\alpha_0}) < \bigoplus_{\alpha\in I} G_\alpha$$

and G is the internal direct sum of G'_{α_0}, $\alpha_0 \in I$. This is because $\forall g = (g_\alpha)_{\alpha\in I} \in G(= \bigoplus_{\alpha\in I} G_\alpha)$, we have

$$g = \sum_{\alpha\in I} i_\alpha(g_\alpha).$$

Note that the above sum is well-defined since there are only finitely many non-zero elements for each g_α. And additionally, we can see the uniqueness of this decomposition by defining π_{α_0} such that

$$\pi_{\alpha_0} \colon \bigoplus_{\alpha\in I} G_\alpha \to G_{\alpha_0}, \quad (g_\alpha)_{\alpha\in I} \mapsto g_{\alpha_0},$$

then $\pi_\alpha \circ i_\alpha = \mathrm{id}_{G_\alpha}$, $\pi_\alpha \circ i_\beta = 0$ for all $\beta \neq \alpha$ and

$$\pi_\beta(g) = \pi_\beta\left(\sum_{\alpha\in I} i_\alpha(g_\alpha)\right) = \sum_{\alpha\in I} \pi_\beta \circ i_\alpha(g_\alpha) = \pi_\beta \circ i_\beta(g_\beta) = g_\beta$$

for all $\beta \in I$, where the second equality is because this summation is finite. Hence, we have

$$g = \sum_{\alpha\in I} i_\alpha(\pi_\alpha(g)).$$

Definition B.6. Given two Abelian groups G, H, we define $\mathrm{Hom}(G, H)$ as

$$\mathrm{Hom}(G, H) := \{f \colon G \to H \mid f \text{ is a group homomorphism}\},$$

then we can define

$$\begin{aligned} + \colon \mathrm{Hom}(G, H) \times \mathrm{Hom}(G, H) &\to \mathrm{Hom}(G, H) \\ (\varphi, \psi) &\mapsto \varphi + \psi, \end{aligned}$$

where

$$(\varphi + \psi)(g) := \varphi(g) + \psi(g).$$

Remark (Relation between direct sum and direct product). Given a collection of Abelian groups $\{G_\alpha\}_{\alpha\in I}$, and another Abelian group H, there exists a φ such that

$$\begin{aligned} \varphi \colon \mathrm{Hom}\left(\bigoplus_{\alpha\in I} G_\alpha, H\right) &\to \prod_{\alpha\in I} \mathrm{Hom}(G_\alpha, H) \\ f &\mapsto \varphi(f) := (f_\alpha)_{\alpha\in I} \end{aligned}$$

where $f_\alpha = f \circ i_\alpha$, where i_α is the canonical embedding from G_α to $\bigoplus_{\alpha\in I} G_\alpha$. We claim that φ is an isomorphism.

- φ is injective. This is obvious since $\ker(\varphi) = 0$ from the fact that if $\varphi(f) = 0$, then $f_\alpha = 0$ for all α, hence f is 0.

- φ is surjective. For every $(f_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} \mathrm{Hom}(G_\alpha, H)$, we define

$$\begin{aligned} f\colon \bigoplus_{\alpha\in I} G_\alpha &\to H \\ \sum_{\alpha\in I} g_\alpha &\mapsto \sum_{\alpha\in I} f_\alpha(g_\alpha). \end{aligned}$$

  We see that $f \in \mathrm{Hom}\left(\bigoplus_{\alpha\in I} G_\alpha, H\right)$ and $\varphi(f) = (f_\alpha)_{\alpha\in I}$.

This shows that

$$\mathrm{Hom}\left(\bigoplus_{\alpha\in I} G_\alpha, H\right) \cong \prod_{\alpha\in I} \mathrm{Hom}(G_\alpha, H).$$

Exercise. We can show that

$$\mathrm{Hom}\left(H, \prod_{\alpha\in I} G_\alpha\right) \cong \prod_{\alpha\in I} \mathrm{Hom}(H, G_\alpha).$$

Note the order in the Hom matters.

B.2 Free Abelian Group

Definition B.7 (Free Abelian group). Given an Abelian group $(G, +)$, we say G is a *free Abelian group* if there exists a collection of elements $\{g_\alpha\}_{\alpha\in J}$ in G such that $\{g_\alpha\}_{\alpha\in J}$ forms a **basis** of G, i.e., for all $g \in G$, $\exists! n_\alpha \in \mathbb{Z}$ for all $\alpha \in J$ such that

$$g = \sum_{\alpha\in J} n_\alpha g_\alpha$$

with finitely many non-zero n_α.

Remark. If G is a free Abelian group, and $\{g_\alpha\}_{\alpha\in J}$ is a basis, then for every $\alpha \in J$, $\langle g_\alpha \rangle$ is an infinite cyclic group since

$$n \cdot g_\alpha = 0 = 0 \cdot g_\alpha \implies n = 0.$$

And from Definition B.7, we have

$$G = \bigoplus_{\alpha\in J} \langle g_\alpha \rangle .$$

Conversely, assume there are a collection of infinite cyclic group $\langle g_\alpha \rangle$ for $\alpha \in I$ in G such that

$$G = \bigoplus_{\alpha\in I} \langle g_\alpha \rangle ,$$

then $\{g_\alpha\}_{\alpha\in I}$ is a basis of G, hence G is a free Abelian group.

Proposition B.1. If G is an Abelian group, then the following are equivalent.

1. G is a free Abelian group.
2. G is an internal direct sum of some infinite cyclic groups.
3. G is isomorphic to the external direct sum of some additive groups of integers $\mathbb{Z}$.

Proof. We see that 1. $\iff$ 2. is already proved. And for 2. $\iff$ 3., this follows directly from the relation between internal and external direct sum. ∎

Now, consider G as a free Abelian group, then

$$u\colon\ G \xrightarrow{\cong} \bigoplus_{\alpha\in I}\mathbb{Z}$$

for some I. Denote $e_\alpha \coloneqq i_\alpha(1) \in \bigoplus_{\alpha\in I}\mathbb{Z}$, where $i_\alpha\colon \mathbb{Z} \to \bigoplus_{\alpha\in I}\mathbb{Z}$ is the canonical embedding, i.e., $e_\alpha = (g_\alpha)_{\alpha\in I} \in \bigoplus_{\alpha\in I}\mathbb{Z}$, where

$$g_\beta = \begin{cases} 1, & \text{if } \beta = \alpha; \\ 0, & \text{if } \beta \neq \alpha. \end{cases}$$

Moreover, denote ϵ_α as the image of e_α under the isomorphism u, namely $\epsilon_\alpha = u^{-1}(e_\alpha)$, then $\{\epsilon_\alpha\}_{\alpha\in I}$ is a basis of G.

Now, for every Abelian group H, we have

$$\begin{tikzcd}
\mathrm{Hom}(G,H) \arrow[rrdd, dashed, "\cong"'] & \mathrm{Hom}\left(\bigoplus_{\alpha\in I}\mathbb{Z}, H\right) \arrow[l, "\circ u"', "\cong"] \arrow[d, "\varphi", "\cong"'] & \\
& \prod_{\alpha\in I}\mathrm{Hom}(\mathbb{Z},H) \arrow[d, "\cong"] & \\
& \prod_{\alpha\in I} H &
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, mapsto] \arrow[rdd, dashed] & f\circ u^{-1} \arrow[d, mapsto] \\
& (f\circ u^{-1}\circ i_\alpha)_{\alpha\in I} \arrow[d, mapsto] \\
& \left(f\circ u^{-1}\circ i_\alpha(1)\right)_{\alpha\in I}
\end{tikzcd}$$

where φ is the homeomorphism defined in here, and the homeomorphism

$$\prod_{\alpha\in I}\mathrm{Hom}(\mathbb{Z},H) \xrightarrow{\cong} \prod_{\alpha\in I} H$$

is trivial since every $f \in \prod\limits_{\alpha\in I}\mathrm{Hom}(\mathbb{Z},H)$ corresponds to $f(1)\in H$ uniquely. We see that

$$f\circ u^{-1}\circ i_\alpha(1) = f\circ u^{-1}(e_\alpha) = f(\epsilon_\alpha).$$

In other words, for all Abelian group H, a morphism from the set $\{\epsilon_\alpha\}_{\alpha\in I}$ to H can be uniquely extended to the group a homomorphism from G to H.

Remark. This means, to determine $\mathrm{Hom}(G, H)$, we only need to determine where each base element in G will map to in H, and this is why it's *free*.

We now want to generate free Abelian group by a set. Roughly speaking, given a set S, we can generate a free Abelian group Z by defining

$$Z := \left\{ \sum_{x \in S} n_x x \mid n_x \in \mathbb{Z}, \# \text{ non-zero elements in } n_x < \infty \right\}$$

with the naturally defined $+$. Formally, we have the following.

Definition B.8 (Free Abelian group generated by a set). Given a set S, the *free Abelian group generated by* S $(Z, +)$ is defined as

$$Z := \{ f : S \to \mathbb{Z} \mid \text{only finitely many } x \in S \text{ such that } f(x) \neq 0 \},$$

with

$$\begin{aligned} + : Z \times Z &\to Z \\ (f, g) &\mapsto f + g. \end{aligned}$$

Remark. $\{\phi_x \mid x \in S\}$ forms a basis of Z, where $\phi_x : S \to \mathbb{Z}$ such that

$$y \mapsto \phi_x(y) = \begin{cases} 1, & \text{if } y = x; \\ 0, & \text{if } y \neq x \end{cases}$$

is the characteristic function at x. We see this by for all $f \in S$, $f = \sum_{x \in S} f(x)\phi_x$, which is uniquely defined. Hence, $(Z, +)$ is a free Abelian group.

Note. Note that

$$\begin{aligned} S &\xleftrightarrow{1:1} \{\phi_x \mid x \in S\} \\ x &\mapsto \phi_x. \end{aligned}$$

Hence, we often denote the element $\sum_{x \in S} \underbrace{n_x}_{f(x)} \phi_x$ in Z as

$$\sum_{x \in S} n_x \cdot x.$$

Theorem B.1 (The universal property of free Abelian group generated by a set). Denote a canonical embedding $i : S \to Z$, $x \mapsto \phi_x$. Then for all Abelian group H and $f : S \to H$, there exists a unique group homomorphism

$$\widetilde{f} : Z \to H$$

such that $\widetilde{f} \circ i = f$.

Proof. We define

$$\widetilde{f}\left(\sum_{x\in S} n_x \cdot x\right) := \sum_{x\in S} n_x f(x),$$

and the uniqueness is obvious. ∎

Note that we can use the above universal property to describe a free Abelian group since we have the following.

Proposition B.2. Given Z' as another Abelian group and $i'\colon S \to Z'$ as another canonical embedding such that for all Abelian group H and $f\colon S \to H$, there exists a unique group homomorphism $\widetilde{f}\colon Z' \to H$ such that $\widetilde{f} \circ i' = f$, then

$$Z' \cong Z.$$

Namely, we can describe a free Abelian group by its universal property uniquely up to isomorphism.

Theorem B.2. Assume G is a free Abelian group. Assume there exists a finite basis $\{g_1, \dots, g_n\}$ of G, and also assume that there exists another basis $\{h_\alpha\}_{\alpha\in I}$. Then we have

$$\operatorname{card}(I) < \infty,$$

specifically, we have

$$\operatorname{card}(I) = n.$$

Proof. Firstly, we observe that if we can show

$$\operatorname{card}(I) \leq n,$$

then by swapping $\{h_\alpha\}_{\alpha\in I}$ and $\{g_\alpha\}_{\alpha\in I}$, we will have $\operatorname{card}(I) = n$.

Suppose I is an infinite set, then we can find $h_{\alpha_1}, \dots, h_{\alpha_m}$ such that $m > n$ and $h_{\alpha_i} \neq h_{\alpha_j}$ for $i \neq j$. Then since $\{g_\alpha\}_{\alpha\in I}$ is a basis, we have

$$h_{\alpha_i} = \sum_{j=1}^{n} k_i^j g_j, \forall i = 1, \dots, m.$$

Specifically, we have

$$\begin{pmatrix} h_{\alpha_1} \\ \vdots \\ h_{\alpha_m} \end{pmatrix} = \underbrace{\begin{pmatrix} k_1^1 & k_1^2 & \dots & k_1^n \\ \vdots & & \ddots & \vdots \\ k_m^1 & k_m^2 & \dots & k_m^n \end{pmatrix}}_{K\in M_{m\times n}(\mathbb{Z})\subset M_{m\times n}(\mathbb{Q})} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix},$$

where $k_i^j \in \mathbb{Z}$. From linear algebra, we know that there exists $(r_1, \dots, r_m) \in \mathbb{Q}^m \setminus \{\vec{0}\}$ such that

$$(r_1, \dots, r_m)K = (0, \dots, 0).$$

Multiplying both sides with the common multiple of the denominator of r_i, we see that there exists $(\ell_1, \dots \ell_m) \in \mathbb{Z}^m \setminus \{\vec{0}\}$ such that

$$\begin{aligned}
&(\ell_1, \dots \ell_m)K = (0, \dots, 0) \\
\implies &(\ell_1, \dots, \ell_m)\begin{pmatrix} h_{\alpha_1} \\ \vdots \\ h_{\alpha_m} \end{pmatrix} = (\ell_1, \dots, \ell_m)K\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = (0, \dots, 0) \\
\implies &\sum_{i=1}^{m} \ell_i h_{\alpha_i} = \vec{0} \text{ for } (\ell_1, \dots, \ell_m) \in \mathbb{Z}^m \setminus \{\vec{0}\} \lightning \\
\implies &\operatorname{card}(I) < \infty.
\end{aligned}$$

From the same argument, we see that $\operatorname{card}(I) \leq n \implies \operatorname{card}(I) = n$. ■

Remark. Furthermore, one can prove that if G is a free Abelian group, then we can prove that any two bases of G are equinumerous, which handle the case that the basis is an infinite set.

This induces the following definition.

Definition B.9 (Rank). Let G ba a free Abelian group, the *rank* of G is the cardinality of any basis of G.

B.3 Finitely Generated Abelian Group

Since we're going to encounter some group as

$$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

so it's useful to look into those finitely generated Abelian group.

Let's start with a definition.

Definition B.10 (Torsion subgroup). Given an Abelian group G, we say that $g \in G$ has finite order if $\exists n \in \mathbb{Z}$ such that $n \cdot g = 0$. Specifically, we say that

$$T := \{g \in G \mid g \text{ has finite order}\}$$

is a *torsion subgroup.*

If $T = 0$ given G, we say that G is *torsion free.*

Note. Note that T is indeed a subgroup, since for any $g_1, g_2 \in T$, $g_1 + g_2 \in T$ from the fact that it still has finite order.

Remark. If G is a free Abelian group, then G is torsion free. Conversely, if G is torsion free, we can't deduce G is a free Abelian group. We see this from $(\mathbb{Q}, +)$. Firstly, we see that $\mathbb{Q}$ is torsion free, Now, suppose $\mathbb{Q}$ is a free Abelian group, then there exists a basis $\{r_\alpha\}_{\alpha \in I}$ of $\mathbb{Q}$ such that $|I| > 1$. Now, consider

$\alpha_1, \alpha_2 \in I$ such that $\alpha_1, \alpha_2 \in I$, for $r_{\alpha_1}, r_{\alpha_2}$, there exists $n, m \in \mathbb{Z}$ and $n, m \neq 0$ such that

$$nr_{\alpha_1} + mr_{\alpha_2} = 0 \implies n = m = 0 \lightning$$

B.3.1 Classification of Finitely generated Abelian Group

Given a finitely generated Abelian group G, we may assume its generators are $g_1, \ldots, g_n$. Let F be

$$F \coloneqq \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n \text{ times}},$$

then there are a natural surjective homomorphism

$$\varphi \colon F \to G, \quad e_i \mapsto g_i$$

where $e_i = (0, \ldots, 0, \underset{i^{th}}{1}, 0, \ldots, 0)$. Now, let $K \coloneqq \ker \varphi$, we have

$$G \cong {}^{F}\big/_{K}.$$

Then we have the following lemma.

Lemma B.1. *K is a finitely generated Abelian group.*

Proof.

$$\begin{aligned} & \mathbb{Z} \text{ is Noetherian}, F \text{ is a finitely generated } \mathbb{Z}\text{-module} \\ \implies & F \text{ is Noetherian module} \\ \implies & K \text{ as a submodule of } F \text{ is a finitely generated } \mathbb{Z}\text{-module} \\ \implies & K \text{ is a finitely generated Abelian group.} \end{aligned}$$

Please refer all the concepts above from [AM94]. ■

Hence, we may assume the generators of K as $b_1, \ldots, b_m$. From the definition of K, we can further express b_i as

$$b_i = (b_{i1}, b_{i2}, \ldots, b_{in}) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}_{n \times n}$$

for all $i = 1, \ldots, m$. Denote all such row vectors b_i in a matrix B, namely

$$B \coloneqq \begin{pmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \ldots & b_{mn} \end{pmatrix} \in M_{m \times n}(\mathbb{Z}),$$

then we have

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = B \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Multiply a matrix on the right-hand side. Now, consider a $p \in \mathrm{GL}(n;\mathbb{Z})$, then

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = B \cdot P \underbrace{P^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}}_{\text{new basis}} = (BP) \cdot \begin{pmatrix} e_1' \\ \vdots \\ e_n' \end{pmatrix},$$

where

$$P^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} =: \begin{pmatrix} e_1' \\ \vdots \\ e_n' \end{pmatrix}.$$

We see that $B \cdot P$ is the coefficient matrix of generators $b_1, \dots, b_m$ under the new basis $e_1', \dots, e_n'$.

Multiply a matrix on the left-hand side. For a $A \in \mathrm{GL}(m;\mathbb{Z})$, then

$$\begin{pmatrix} b_1' \\ \vdots \\ b_m' \end{pmatrix} = Q \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = QB \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix},$$

since Q is invertible, hence $b_1', \dots, b_m'$ are also generators of K. We see that QB is the coefficient matrix of new generators $b_1', \dots, b_m'$ under basis $e_1, \dots, e_n$.

Generally $Q \cdot B \cdot P$ is the matrix representation of a particular set of F's generators under a particular basis.

Proposition B.3. There exists $P \in \mathrm{GL}(n;\mathbb{Z})$ and $Q \in \mathrm{GL}(m;\mathbb{Z})$ such that

$$Q \cdot B \cdot P = \begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

where $d_i \in \mathbb{Z}^+$ and $d_1 \mid d_2 \mid \dots \mid d_k$.

Proof. In fact, P, Q can be taken as the multiplication of the following three types of square matrices:

- P_{ij}:

$$P_{ij} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1_{(ij)} & & \\ & & & \ddots & & \\ & & 1_{(ji)} & & 0 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

where the effect of multiplying P_{ij} from the right is *swapping column* i, j.

- $P_i(c)$, where c is the identity in $\mathbb{Z}$, i.e., $c = \pm 1$:

$$P_i(c) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c_{(ii)} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

  where the effect of multiplying $P_i(c)$ from the right is *multiplying* c *to column* i.

- $P_{ij}(a)$, $a \in \mathbb{Z}$:

$$P_{ij} = \begin{pmatrix} 1 & & & \\ & \ddots & a_{(ij)} & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

  where the effect of multiplying $P_{ij}(a)$ from the right is *adding* a *times column* i *to column* j.

We see that these are *elementary column transformations* in linear algebra. In particular, if we multiply these matrices from the left, then it's called *elementary row transformations.*

That is to say, we're going to show

$$B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix}$$

can become

$$\begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

$d_i \in \mathbb{Z}^+$, $d_1 \mid d_2 \mid \dots \mid d_k$ from *elementary column/row transformations.*

We now show the steps to make this happens.

Step 1. Using elementary transformations, we make $b_{11} > 0$.

Step 2. Using elementary transformations, we make b_{11} become a divisor of all elements in the first column and row.

We see that if $b_{11} \nmid b_{1i}$ for $i \neq 1$, we have $b_{1i} = r \cdot b_{11} + s$ where $0 < s < b_{11}$. Then we add $(-r)$ times the 1^{th} column to the i^{th} column and swapping the 1^{th} and the i^{th} column, which makes B becomes

$$\begin{pmatrix} s & \dots \\ \vdots & \ddots \end{pmatrix},$$

for $0 < s < b_{11}$. Since $\operatorname{card}(\{n \in \mathbb{Z} \mid 0 < n < b_{11}\}) < \infty$, hence in finitely many steps we can make B becomes

$$\begin{pmatrix} d_1 & \cdots \\ \vdots & \ddots \end{pmatrix},$$

where d_1 is a divisor of all other elements in the first column and row.

Step 3. Using elementary transformations, we can multiply the first row by a proper integer and add it to the other rows, do the same but for columns also, then we can make B becomes

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B_1 & \\ 0 & & & \end{pmatrix}.$$

Step 4. We iteratively apply Step 1. to step 3., we make B into

$$\begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

where $d_i \in \mathbb{Z}^+$.

Step 5. Using elementary transformations, by swapping columns and rows, we may assume $d_1 \leq d_2 \leq \ldots \leq d_k$.

Step 6. Using elementary transformations, we can make B into

$$\begin{pmatrix} d'_1 & & & & \\ & \ddots & & & \\ & & d'_\ell & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

such that $0 < d'_1 \leq \ldots \leq d'_\ell$, $d'_1 \mid d'_2 \mid \ldots \mid d'_\ell$ since if $d_1 \nmid d_i$ for some $i \in \{2, \ldots, k\}$, then

$$\begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \to \begin{pmatrix} d_1 & d_i & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

then from Step 2., we have

$$\begin{pmatrix} s & \cdots \\ \vdots & \ddots \end{pmatrix}$$

where $0 < s < d_1$ and s is a divisor of all other elements in the first row and column. Now, we repeat Step 3. to Step 5., we obtain

$$\begin{pmatrix} \widetilde{d_1} & & & & \\ & \ddots & & & \\ & & \widetilde{d_j} & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

where $\widetilde{d_1} \leq \ldots \leq \widetilde{d_j}$ such that $\widetilde{d_1} < d_1$. Since there are only finitely many integers which is smaller than d_1, we see that by repeating these steps, we can always make

$$\begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

into

$$\begin{pmatrix} \widetilde{d_1} & & & & \\ & \ddots & & & \\ & & \widetilde{d_p} & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

such that $d_1' \mid d_i'$ for all $i \neq 1$ and $d_1' \leq d_2' \leq \ldots \leq d_p'$. By the same idea of Step 3., we have the desired matrix.

Since all operations are elementary and there are only finitely many of them, hence the result follows. ∎

From the definition of $Q \cdot B \cdot P$ and Proposition B.3, there exists a basis $e_1', \ldots, e_n'$ of F such that K has finitely many generators $d_1 e_1', \ldots, d_k e_k'$, hence

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_k\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-k \text{ times}}.$$

This leads to the following important theorem.

Theorem B.3 (Fundamental theorem of finitely generated Abelian group). Given a finitely generated Abelian group, either G is a free Abelian group, or there exists a unique set of $\{m_i \in \mathbb{Z} \mid m_i > 1, i = 1, \ldots, t\}$ such that $m_1 \mid m_2 \mid \ldots \mid m_t$ and a unique non-negative integer s such that

$$G \cong \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{s \text{ times}}.$$

Proof. We need to show both uniqueness and existence.

Existence. From Proposition B.3, we obtain a basis $e'_1, \ldots, e'_n$ of F and a basis $d_1 e'_1, \ldots, d_k e'_k$ in K such that $d_1 \mid \ldots \mid d_k$. Let

$$(d_1, \ldots, d_k) = (1, \ldots, 1, m_1, \ldots, m_t),$$

which implies

$$\begin{aligned} G &\cong F \big/ K \\ &\cong \mathbb{Z} \big/ d_1\mathbb{Z} \oplus \mathbb{Z} \big/ d_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \big/ d_k\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \\ &= \underline{\mathbb{Z} \big/ 1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \big/ 1\mathbb{Z}} \oplus \mathbb{Z} \big/ m_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \big/ m_t\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \\ &= \mathbb{Z} \big/ m_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \big/ m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\exists! s \text{ times}}. \end{aligned}$$

Uniqueness. Under the isomorphism $\mathbb{Z} / m_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} / m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{s \text{ times}}$, we see that

$$\mathbb{Z} \big/ m_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \big/ m_t\mathbb{Z}$$

corresponds to G's torsion subgroup T, which implies

$$G \big/ T \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{s \text{ times}},$$

which further implies G / T is a free Abelian group with

$$\operatorname{rk}\left(G \big/ T\right) = s,$$

which proves the uniqueness of s.

The proof of the uniqueness of m_i are long and tedious, we refer to [Arm13]. ■

Definition B.11 (Invariant factor). We call $m_1, \ldots, m_t$ obtained from Theorem B.3 the *invariant factor*.

Lemma B.2. Given a positive integer m such that

$$m = p_1^{n_1} \cdot \ldots \cdot p_s^{n_s}$$

where $p \in \mathcal{P}$ are all prime and $p_i \neq p_j$ for $i \neq j$, with $n_i \in \mathbb{Z}^+$ for all i. Then

$$\mathbb{Z} \big/ m\mathbb{Z} \cong \mathbb{Z} \big/ p_1^{n_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \big/ p_s^{n_s}\mathbb{Z}.$$

Proof. We define ϕ as

$$\begin{aligned} \phi \colon \mathbb{Z} \big/ m\mathbb{Z} &\to \mathbb{Z} \big/ p_1^{n_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \big/ p_s^{n_s}\mathbb{Z} \\ \overline{n} &\mapsto (n + \langle p_1^{n_1} \rangle, \ldots, n + \langle p_s^{n_s} \rangle). \end{aligned}$$

Then $\overline{n} \in \ker\phi \iff \underset{i}{\forall}\, p_i^{n_i} \mid n \iff m \mid n \iff \overline{n} = \overline{0}$. This means $\ker\phi = 0$, hence ϕ is an injection.

We now prove ϕ is a surjection. It's sufficient to prove that for all i,

$$(0, \ldots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \ldots, 0) \in \mathbb{Z}\big/p_1^{n_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}\big/p_s^{n_s}\mathbb{Z},$$

there exists an $\overline{n}$ such that

$$\phi(\overline{n}) = (0, \ldots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \ldots, 0).$$

Notice that for all $i \neq j$, $\langle p_i^{n_i} \rangle + \langle p_j^{n_j} \rangle \in \mathbb{Z}$, hence there exists $u_j \in \langle p_i^{n_i} \rangle$ and $v_j \in \langle p_j^{n_j} \rangle$ such that $u_j + v_j = 1$. Let n as

$$n = \prod_{i \neq j}(1 - u_j),$$

then

$$n + \langle p_i^{n_i} \rangle = 1 + \langle p_i^{n_i} \rangle, \quad n + \langle p_j^{n_j} \rangle = 0 + \langle p_j^{n_j} \rangle.$$

Above implies

$$\phi(\overline{n}) = (0, \ldots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \ldots, 0),$$

hence ϕ surjects, so

$$\mathbb{Z}\big/m\mathbb{Z} \cong \mathbb{Z}\big/p_1^{n_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}\big/p_s^{n_s}\mathbb{Z}.$$

■

Combine Theorem B.3 and Lemma B.2, we see that we now only have

$$G \cong \mathbb{Z}\big/m_1\mathbb{Z} \oplus \mathbb{Z}\big/m_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}\big/m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{s \text{ times}},$$

we can further decompose G into

$$G \cong \mathbb{Z}\big/p_1^{s_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}\big/p_k^{s_k}\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{s \text{ times}},$$

where $p_1, \ldots, p_k$ are primes (which may includes repeated terms), $s_i \in \mathbb{Z}^+$ for all i.

Definition B.12 (Elementary divisors). The set

$$\{p_1^{s_1}, \ldots, p_k^{s_k}\}$$

are called *elementary divisors* of G.

Theorem B.4 (Uniqueness of elementary divisors). Elementary divisors of a group G is unique.

Proof. Please refer to [Arm13]. ■

C Homological Algebra

As previously seen. Given two Abelian groups A, B and the group homomorphism $\varphi\colon A \to B$, then we have

- $\ker\varphi = \{x \in A \mid \varphi(x) = 0\}$
- $\operatorname{Im}\varphi = \{\varphi(x) \mid x \in A\}$
- $\operatorname{coker}\varphi := B \,/\, \operatorname{Im}\varphi$
- $\operatorname{coIm}\varphi := A \,/\, \ker\varphi$

Consider a sequence of Abelian group homomorphism

$$\ldots \longrightarrow A_{i-1} \xrightarrow{\phi_{i-1}} A_i \xrightarrow{\phi_i} A_{i+1} \longrightarrow \ldots$$

We denote this sequence as S.

Definition C.1 (Exact). We say S is *exact* at A_i if

$$\operatorname{Im}\phi_{i-1} = \ker\phi_i.$$

Remark. Definition C.1 is same as Definition 4.13.

Definition C.2 (Exact sequence). We call S is an *exact sequence* if it's exact at A_i for all i.

Remark. Specifically, consider the following two situations.

- We say
$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \ldots$$
is an exact sequence if it's exact at A_i for all $i \geq 1$.
- We say
$$\ldots \longrightarrow A_{-2} \longrightarrow A_{-1} \longrightarrow A_0$$
is an exact sequence if it's exact at A_i for all $i \leq -1$.

Remark. Denote $\circ$ as a trivial Abelian group, then

$$A \xrightarrow{\phi} B \longrightarrow \circ \quad \text{is an exact sequence} \iff \phi \text{ is a surjective homomorphism;}$$

conversely,

$$\circ \longrightarrow B \xrightarrow{\phi} A \quad \text{is an exact sequence} \iff \phi \text{ is an injective homomorphism.}$$

Definition C.3 (Short exact sequence). A *short exact sequence* is an exact sequence such that it has the following form

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ\ .$$

Remark. Let $B \xrightarrow{\psi} C$ as a surjective homomorphism and $K = \ker \psi$, and we denote $K \xrightarrow{i} B$ as an injection. Then

$$\circ \longrightarrow K \xrightarrow{i} B \xrightarrow{\psi} C \longrightarrow \circ$$

is a short exact sequence. Conversely, if

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

is a short exact sequence, then ϕ is an injective homomorphism since it is exact at A, and ψ is a surjective homomorphism since it is exact at C, and $\phi(A) = \ker \psi$ since it is exact at B. This implies $\phi\colon A \to \phi(A) = \ker \psi$ is a group homeomorphism.

Example. We see some examples.

1. Given A, B as Abelian groups, then

$$\begin{aligned} &\circ \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\mathrm{Proj}_2} B \longrightarrow \circ \\ &a \xmapsto{i} (a, 0) \\ &(a, b) \xmapsto{\mathrm{Proj}_2} b \end{aligned}$$

   is a short exact sequence.

2. We see that

$$\begin{aligned} &\circ \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\mathrm{Proj}_2} \mathbb{Z} / n\mathbb{Z} \longrightarrow \circ \\ &k \longmapsto k \cdot n \end{aligned}$$

   for $n \in \mathbb{Z}_{\geq 1}$ is a short exact sequence.

Definition C.4 (Isomorphism between sequences). Given $A.$ and $B.$ defined as two sequences of Abelian group homomorphisms

$$A_\bullet : \ \dots \longrightarrow A_i \xrightarrow{\phi_i} A_{i+1} \longrightarrow \dots$$

and

$$B_\bullet : \ \dots \longrightarrow B_i \xrightarrow{\psi_i} B_{i+1} \longrightarrow \dots$$

And we say a morphism α from $A_\bullet$ to $B_\bullet$ is a series of group homomorphisms $\alpha_i \colon A_i \to B_i$ for all $i \in \mathbb{Z}$ such that the following diagram commutes.

$$\begin{tikzcd}
\dots \arrow[r] & A_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"] & A_{i+1} \arrow[r] \arrow[d, "\alpha_{i+1}"] & \dots \\
\dots \arrow[r] & B_i \arrow[r, "\psi_i"] & B_{i+1} \arrow[r] & \dots
\end{tikzcd}$$

Additionally, if for all i, α_i is a group homeomorphism, then we say $\alpha \colon A_\bullet \to B_\bullet$ is a homeomorphism.

Definition C.5 (Split short exact sequence). Given a short exact sequence

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

we say it is *split* if there exists a group homeomorphism $\theta \colon B \to A \oplus C$ such that

$$\begin{tikzcd}
\circ \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "\psi"] \arrow[d, "\theta"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & \circ \\
\circ \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & \circ
\end{tikzcd}$$

is the isomorphism between these two short exact sequences.

Remark. Given split short exact sequence

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

and θ defined in Definition C.5, let $i \colon A \to A \oplus C$, $a \mapsto (a, 0)$ and $j \colon C \to A \oplus C$, $c \mapsto (0, c)$ are two canonical embeddings, then we have

$$A \oplus C = i(A) \oplus j(C).$$

Consider $\theta^{-1} \colon A \oplus C \xrightarrow{\cong} B$, then

$$B = \theta^{-1}(i(A)) \oplus \theta^{-1}(j(C)).$$

Since the diagram in Definition C.5 commutes, hence

$$\theta^{-1}(i(A)) = \theta^{-1} \circ i(A) = \phi(A),$$

hence

$$B = \phi(A) \oplus \underbrace{\theta^{-1}(j(C))}_{D},$$

which implies $\psi|_D : D \to C$ is a group homeomorphism. We see that

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

split implies $B = \phi(A) \oplus D$ and $\psi|_D : D \xrightarrow{\cong} C$.

Conversely, if $B = \phi(A) \oplus D$ and $\psi|_D : D \xrightarrow{\cong} C$, then there exists a θ

$$\begin{aligned} \theta \colon B &\to A \oplus C \\ \phi(a) + d &\mapsto (a, \psi(d)) \end{aligned}$$

for $a \in A, d \in D$ such that

$$\begin{tikzcd} \circ \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "\psi"] \arrow[d, "\theta"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & \circ \\ \circ \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & \circ \end{tikzcd}$$

$$\begin{tikzcd} \phi(a) + d \arrow[r, mapsto] \arrow[d, mapsto] & \psi(d) \arrow[d, mapsto] \\ (a, \psi(d)) \arrow[r, mapsto] & \psi(d) \end{tikzcd}$$

commutes.

Hence, for a short exact sequence $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$ is split if and only if $B = \phi(A) \oplus D$ and $\psi|_D : D \xrightarrow{\cong} C$.

Remarkably, let $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$ is a split short exact sequence, then D constructed above is not unique. To see this, consider

$$\circ \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{Proj}_2} \mathbb{Z} \longrightarrow \circ$$

$$n \longmapsto (n, 0)$$

$$(n, m) \longmapsto m$$

We have $\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus j(\mathbb{Z})$ where $j \colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $n \mapsto (0, n)$. We see that we can let $D := j(\mathbb{Z})$. Meanwhile, we can also let

$$D := \{(n, n) \mid n \in \mathbb{Z}\} < \mathbb{Z} \oplus \mathbb{Z}$$

such that $\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus D$.

Example (Non-split short exact sequence). We see that

$$
\begin{tikzcd}
\circ \arrow[r] & \mathbb{Z} \arrow[r, "i"] & \mathbb{Z} \arrow[r, "\mathrm{Proj}_2"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & \circ
\end{tikzcd}
$$

$$
k \longmapsto k \cdot n
$$

is not a split short exact sequence, since if it is, then

$$
\begin{aligned}
\mathbb{Z} \oplus {}^{\mathbb{Z}}\big/_{n\mathbb{Z}} &\cong \mathbb{Z} \\
(0, 1) &\mapsto k,
\end{aligned}
$$

which is a contradiction since $\mathbb{Z}$ is torsion-free while $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ is not.

Lemma C.1 (Splitting lemma). If $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$ is a short exact sequence, then the following are equivalent.

1. This short exact sequence split.
2. $\exists p \colon B \to A$ such that $p \circ \phi = \mathrm{id}_A$.
3. $\exists q \colon C \to B$ such that $\psi \circ q = \mathrm{id}_C$.

Proof.

- 1. $\implies$ 2. Let $\theta \colon B \xrightarrow{\cong} A \oplus C$ such that it's the isomorphism which makes the following diagram commutes.

$$
\begin{tikzcd}
\circ \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "\psi"] \arrow[d, "\theta"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & \circ \\
\circ \arrow[r] & A \arrow[r, "i"] & A \oplus C \arrow[l, bend left, "\mathrm{Proj}_1"] \arrow[r] & C \arrow[r] & \circ
\end{tikzcd}
$$

Then we let $p \coloneqq \mathrm{Proj}_1 \circ \theta$, then

$$
p \circ \phi = \mathrm{Proj}_1 \circ \theta \circ \phi = \mathrm{Proj}_1 \circ i = \mathrm{id}_A.
$$

- 1. $\implies$ 3. Let $\theta \colon B \xrightarrow{\cong} A \oplus C$ such that it's the isomorphism which makes the following diagram commutes.

$$
\begin{tikzcd}
\circ \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "\psi"] \arrow[d, "\theta"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & \circ \\
\circ \arrow[r] & A \arrow[r] & A \oplus C \arrow[r, "\mathrm{Proj}_2"] & C \arrow[l, bend left, "j"] \arrow[r] & \circ
\end{tikzcd}
$$

Then we let $q \coloneqq \theta^{-1} \circ j$, then for all $c \in C$, we have

$$
\psi \circ q(c) = \psi\left(\theta^{-1}(j(c))\right) = \mathrm{Proj}_2 \circ \theta\left(\theta^{-1}(j(c))\right) = \mathrm{Proj}_2(j(c)) = c,
$$

hence $\psi \circ q = \mathrm{id}_C$.

- 2. $\implies$ 1. We have

$$\circ \longrightarrow A \underset{p}{\overset{\phi}{\rightleftarrows}} B \xrightarrow{\psi} C \longrightarrow \circ$$

where $p \circ \phi = \mathrm{id}_A$. We claim that $B = \phi(A) \oplus \ker(p)$ since for every $b \in B$, $\phi(p(b)) \in \phi(A)$, and

$$b = \underbrace{\phi(p(b))}_{\in \phi(A)} + \underbrace{(b - \phi(p(b)))}_{\in \ker(p)}$$

from the fact that

$$p(b - \phi(p(b))) = p(b) - p \circ \phi(p(b)) = p(b) - p(b) = 0.$$

We need to show the uniqueness also. Suppose $b = \phi(a_1) + d_1 = \phi(a_2) + d_2$, $a_1, a_2 \in A$, $d_1, d_2 \in \ker(p)$. We see that

$$\phi(a_1 - a_2) = d_2 - d_1 \implies p(\phi(a_1 - a_2)) = 0 \implies a_1 = a_2 \implies d_1 = d_2.$$

Finally, we claim that

$$\psi|_{\ker(p)} : \ker(p) \to C$$

is a group homeomorphism. But it's obvious that $\psi|_{\ker(p)}$ are both surjective and injective.

- 3. $\implies$ 1. We have

$$\circ \longrightarrow A \xrightarrow{\phi} B \underset{q}{\overset{\psi}{\rightleftarrows}} C \longrightarrow \circ$$

where $\psi \circ q = \mathrm{id}_C$. We claim that $B = \phi(A) \oplus q(C)$ since for every $b \in B$,

$$b = \underbrace{(b - q(\psi(b)))}_{\in \ker(\psi) = \mathrm{Im}(\phi)} + \underbrace{q(\psi(b))}_{\in q(C)},$$

which implies $B = \phi(A) + q(C)$. We can also prove that

$$B = \phi(A) \oplus q(C)$$

similarly.

$\blacksquare$

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