

MATH592

Introduction to Algebraic Topology

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Abstract

This course will use Hatcher[HPM02] as the main text, but the order may differ here and there. Enjoy this fun course! In particular, I add some extra content which is not covered in lectures, things like [groupoid](#), [fibered coproduct](#), feel free to skip these content.

Note that I reference all definitions in the text as much as possible, but I may still miss some.

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Lecture 1: Homotopies of Maps

05 Jan. 10:00

1 Foundation of Algebraic Topology

1.1 Homotopy

We start with the most important and fundamental concept, [homotopy](#).

Definition 1.1 (Homotopy, homotopic, nullhomotopic). Let X, Y be topological spaces. Let $f, g: X \rightarrow Y$ continuous maps. Then a *homotopy* from f to g is a 1-parameter family of maps that continuously deforms f to g , i.e., it's a continuous function $F: X \times I \rightarrow Y$, where $I = [0, 1]$, such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x).$$

We often write $F_t(x)$ for $F(x, t)$.

If a homotopy exists between f and g , we say they are *homotopic* and write

$$f \simeq g.$$

If f is homotopic to a constant map, we call it *nullhomotopic*.

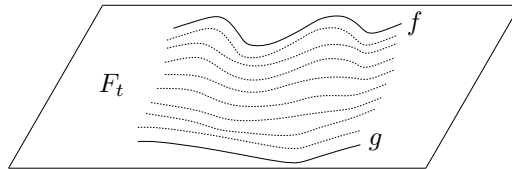


Figure 1: The continuous deforming from f to g described by F_t

Remark. Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

Example. We first see some examples.

- Any two (continuous) maps with specification

$$f, g: X \rightarrow \mathbb{R}^n$$

are **homotopic** by considering

$$F_t(x) = (1 - t)f(x) + tg(x).$$

We call it *the straight line homotopy*.

- Let S^1 denotes the unit circle in \mathbb{R}^2 , and D^2 denotes the unit disk in \mathbb{R}^2 . Then the inclusion $f: S^1 \hookrightarrow D^2$ is **nullhomotopic** by considering

$$F_t(x) = (1 - t)f(x) + (t \cdot 0).$$

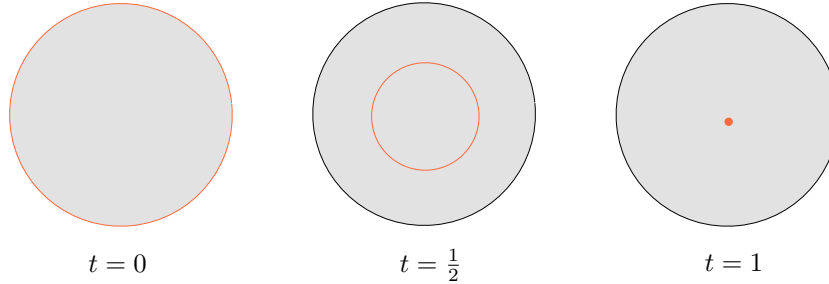


Figure 2: The illustration of $F_t(x)$

We see that there is a **homotopy** from $f(x)$ to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a **nullhomotopy**.

- The maps

$$\begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not homotopy**.

Remark. It will essentially **flip** the orientation, hence we can't deform one to another continuously.

Exercise. We first see some exercises.

- A subset $S \subseteq \mathbb{R}^n$ is star-shaped if

$$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from x_0 to x lies in S .

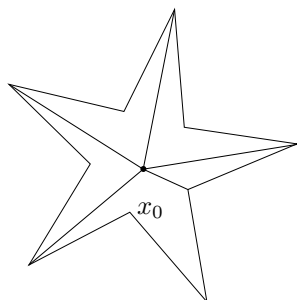


Figure 3: Star-shaped illustration

Show that $\text{id}: S \rightarrow S$ is **nullhomotopic**.

Answer. Consider

$$F_t(x) := (1 - t)x + tx_0,$$

which essentially just concentrates all points x to x_0 . ■

2. Suppose

$$X \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_0} \end{array} Y \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_0} \end{array} Z$$

where

$$f_0 \simeq_{F_t} f_1, \quad g_0 \simeq_{G_t} g_1.$$

Show

$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

Answer. Consider $I \times X \rightarrow Z$, where

$$\begin{array}{ccccc} X \times I & \rightarrow & Y \times I & \rightarrow & Z \\ (x, t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

■

Remark. Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** **homotopic**?

Answer. We'll see! ■

Lecture 2: Homotopy Equivalence

07 Jan. 10:00

As previously seen. Two maps $f, g: X \rightarrow Y$ is **homotopy** if there exists a map

$$F_t(x): X \times I \rightarrow Y$$

with the properties

1. Continuous

$$2. F_0(x) = f(x)$$

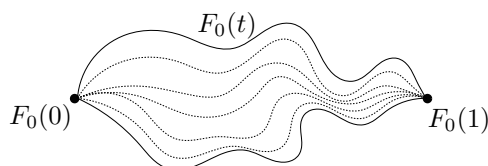
$$3. F_1(x) = g(x)$$

Remark. The continuity of F_t is an even stronger condition for the continuity of F_t for a fixed t .

We now introduce another concept.

Definition 1.2 (Homotopy relative). Given two spaces X, Y , and let $B \subseteq X$. Then a [homotopy](#) $F_t(x): X \rightarrow Y$ is called *homotopy relative B* (denotes $\text{rel}B$) if $F_t(b)$ is independent of t for all $b \in B$.

Example. Given X and $B = \{0, 1\}$. Then the [homotopy](#) of paths from $[0, 1] \rightarrow X$ is $\text{rel}\{0, 1\}$.



1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

Definition 1.3 (Homotopy equivalence, homotopy inverse). A map $f: X \rightarrow Y$ is a *homotopy equivalence* if $\exists g: Y \rightarrow X$ such that

$$f \circ g \simeq \text{id}_Y, \quad g \circ f \simeq \text{id}_X.$$

We say that X, Y are *homotopy equivalent*, and g is called *homotopy inverse* of f .

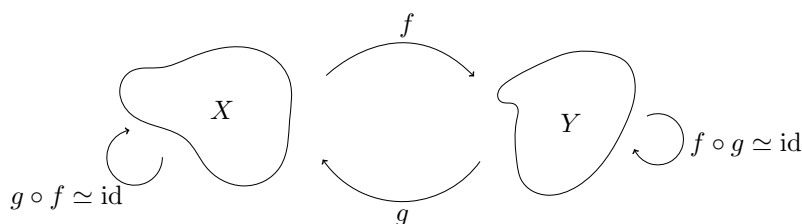
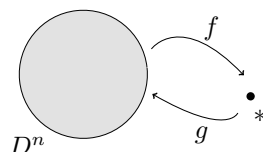


Figure 4: [Homotopy Equivalence](#)

If X, Y are [homotopy equivalent](#), then we say that they have the same *homotopy type*.

Notation. We denote a closed n -disk as D^n .

Example. D^n is [homotopy equivalent](#) to a point.



We see that $f \circ g = \text{id}_*$ and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is [homotopic](#) to id_{D^n} by [straight line homotopy](#) $F_t(x) = tx$. Specifically, we see that this holds for any convex set.

Definition 1.4 (Contractible). We say that a space X is *contractible* if X is [homotopy equivalent](#) to a point.

The following proposition is added much after, which may use some concepts not yet covered.

Proposition 1.1. The followings are equivalent.

1. X is [contractible](#).
2. $\forall x \in X, \text{id}_X \simeq c_x$.
3. $\exists x \in X, \text{id}_X \simeq c_x$.

Remark. Note that the above notation c_x is introduced at [here](#).

Proof. We see that 2. \implies 3. is obvious. We consider 3. \implies 2. This follows the following general lemma.

Lemma 1.1. Given a topological space X such that $\exists x \in X, \text{id}_X \simeq c_x$, with $f, g: Y \rightarrow X$, then $f \simeq g$.

Proof. Let $x \in X$ such that $\text{id}_X \simeq c_x$. Then

$$f = \text{id}_X \circ f \simeq c_x \circ f = c_x \circ g \simeq \text{id}_X \circ g = g.$$

■

Then, from this [Lemma 1.1](#), we see that assuming $x_0 \in X$ such that $\text{id}_X \simeq c_{x_0}$, then consider c_x for all $x \in X$, then from [Lemma 1.1](#), we see that $c_x \simeq \text{id}_X$.

To show 3. \implies 1., we let $x_0 \in X$ such that $\text{id}_X \simeq c_{x_0}$.

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} \{*\}$$

Since $g(*) = x_0$, and

$$g \circ f: X \rightarrow X \\ x \mapsto x_0,$$

which is just c_{x_0} , from the assumption we're done.

Now, we show 1. \implies 3. Let

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} \{*\}$$

be a [homotopy equivalent](#), let $g(*) = x_0$. We see that $c_{x_0} \simeq \text{id}_X$ since

$$g \circ f = c_{x_0} \simeq \text{id}_X.$$

■

Before doing exercises, we introduce two new concepts.

Definition 1.5 (Retraction, retract). Given $B \subseteq X$, a *retraction* from X to B is a map $f: X \rightarrow B$ (or $X \rightarrow B$) such that $\forall b \in B$ $f(b) = b$, namely $r|_B = \text{id}_B$. Or one can see this from

$$B \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{r} \end{array} X \begin{array}{c} \xrightarrow{r} \\ \xleftarrow{r \circ i} \end{array} B$$

where r is a retraction if and only if $r \circ i = \text{id}_B$, where i is an inclusion identity.

If r exists, B is a *retract* of X .

Definition 1.6 (Deformation retraction). Given X and $B \subseteq X$, a *(strong) deformation retraction* $F_t: X \rightarrow X$ onto B is a [homotopy](#) rel B from the id_X to a [retraction](#) from X to B . i.e.,

$$\begin{aligned} F_0(x) &= x & \forall x \in X \\ F_1(x) &\in B & \forall x \in X \\ F_t(b) &= b & \forall t \forall b \in B. \end{aligned}$$

Exercise. We now see some problems.

1. Let $X \simeq Y$. Show X is path-connected if and only if Y is.

Answer. Suppose X is path-connected. Then we see that given two points x_1 and x_2 in X , there exists a path $\gamma(t)$ with

$$\gamma: [0, 1] \rightarrow X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Since $X \simeq Y$, then there exists a pair of f and g such that $f: X \rightarrow Y$ and $g: Y \rightarrow X$ with

$$f \circ g \underset{F}{\simeq} \text{id}_Y, \quad g \circ f \underset{G}{\simeq} \text{id}_X.$$

(Notice the abuse of notation)

For any two y_1 and $y_2 \in Y$, we want to construct a path $\gamma'(t)$ such that

$$\gamma': [0, 1] \rightarrow Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

Firstly, we let $g(y_1) =: x_1$ and $g(y_2) =: x_2$. From the argument above, we know there exists such a γ starting at $x_1 = g(y_1)$ ending at $x_2 = g(y_2)$. Now, consider $f(\gamma(t)) = (f \circ \gamma)(t)$ such that

$$f \circ \gamma: I \rightarrow Y, \quad f \circ \gamma(0) = y'_1, \quad f \circ \gamma(1) = y'_2,$$

we immediately see that y'_1 and y'_2 is path connected. Now, we claim that y_1 and y'_1 are path connected in Y , hence so are y_2 and y'_2 . To see this, note that

$$f \circ g \underset{F}{\simeq} \text{id}_Y,$$

which means that there exists $F: Y \times I \rightarrow Y$ such that

$$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y'_1 \\ F(y_1, 1) = \text{id}_Y(y_1) = y_1. \end{cases}$$

Since F is continuous in I , we see that there must exist a path connects y_1 and y'_1 . The same argument applies to y_2 and y'_2 . Now, we see that the path

$$y_1 \rightarrow y'_1 \rightarrow y'_2 \rightarrow y_2$$

is a path in Y for any two y_1 and y_2 , which shows Y is path-connected. ■



Figure 5: Demonstration of the proof.

Challenge: One can further show that the connectedness is also preserved by any [homotopy equivalence](#).

Corollary 1.1. A [contractible](#) space is [path](#)-connected.

2. Show that if there exists [deformation retraction](#) from X to $B \subseteq X$, then $X \simeq B$.

Lecture 3: Deformation Retraction

10 Jan. 10:00

As previously seen. A [deformation retraction](#) is a [homotopy](#) of maps $\text{rel} B$ $X \rightarrow X$ from id_X to a [retraction](#) from X to B . Then B is a [deformation retract](#).

Example. We can also show

1. S^1 is a [deformation retraction](#) of $D^2 \setminus \{0\}$. Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$



Figure 6: The [deformation retraction](#) of $D^2 \setminus \{0\}$ is just to *enlarge* that hole and push all the interior of D^2 to the boundary, which is S^1 .

2. \mathbb{R}^n *deformation retracts* to 0. Indeed, since

$$F_t(x) = (1-t)x.$$

This implies that $\mathbb{R}^n \simeq *$, hence we see that

- dimension
- compactness
- etc.

are not *homotopy* invariants.

3. S^1 is a *deformation retract* of a cylinder and a Möbius band.

For a cylinder, consider $X \times I \rightarrow X$. Define *homotopy* on a closed rectangle, then verify it induces map on quotient.

For a Möbius band, we define a *homotopy* on a closed rectangle, then verify that it respects the equivalence relation.

Finally, we use the universal property of quotient topology to argue that we get a *homotopy* on Möbius band.

Upshot: Möbius band $\simeq S^1 \simeq$ cylinder, hence the orientability is not *homotopy* invariant.

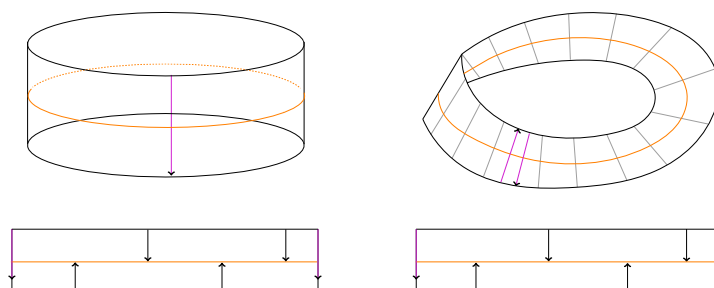


Figure 7: The *deformation retraction* for Cylinder and Möbius band

Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

As previously seen. We saw that

- *homotopy equivalence*
- *homotopy* invariants
 - path-connectedness
- not invariant
 - dimension
 - orientability
 - compactness

1.3 CW Complexes

Example. Let's start with a few examples.

1. Constructing spheres:

- S^1 (up to homeomorphism¹)



- S^2
 - glue boundary of 2-disk to a point
 - glue 2 disks onto a circle



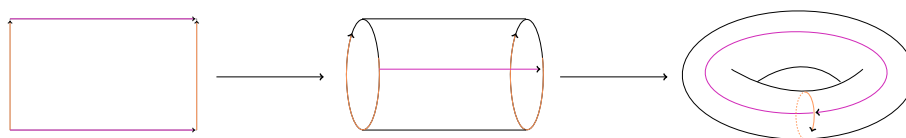
Figure 8: **Left:** Glue a 2-disk to a point along its boundary. **Right:** Glue 2 disks to S^1 .

The gluing instruction to construct S^2 in the right-hand side can be demonstrated as follows.

¹This is just the term for isomorphism in topology.



• $T = S^1 \times S^1$



view as gluing instructions

vertex + 2 edges + 2-disks.

Specifically, we have



Formally, we have the following definition.

Notation. Let D^n denotes a closed n-disk (or n-ball)

$$D^n \simeq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

And let S^n denotes an n -sphere

$$S^n \simeq \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

Lastly, we call a point as a 0 -cell, and the interior of D^n $\text{int}(D^n)$ for $n \geq 1$ as a n -cell.

Definition 1.7 (CW Complex). A *CW Complex* is a topological space constructed inductively as

1. X^0 (the 0-skeleton) is a set of discrete points.
2. We inductively construct the n -skeleton X^n from X^{n-1} by attaching n -cells e_α^n , where α is the index.

The gluing instructions glued by an attaching map is that $\forall \alpha, \exists$ continuous map φ_α

$$\varphi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1},$$

then

$$X^n = \left(X^{n-1} \amalg \coprod_\alpha D_\alpha^n \right) / x \sim \varphi_\alpha(x)$$

with identification $x \sim \varphi_\alpha(x)$ for all $x \in \partial D_\alpha^n$ with quotient topology.

3. We let X be defined as

$$X = \bigcup_{n=0} X^n,$$

and let \bar{w} denotes weak topology such that

$$u \subseteq X \text{ is open} \iff \forall n \ u \cap X^n \text{ is open}.$$

If all cells have dimension less than N and a $\exists N$ -cell, then $X = X^N$ and we call it N -dimensional *CW complex*.

Remark. We write $X^{(n)}$ for n -skeleton if we need to distinguish from the Cartesian product.

Example. Let's look at some examples.

1. 0-dim **CW complex** is a discrete space.
2. 1-dim **CW complex** is a graph.
3. A **CW complex** X is finite if it has finitely many cells.

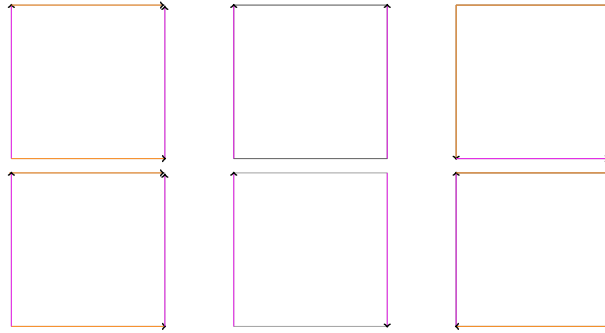
Definition 1.8 (CW subcomplex). A *CW subcomplex* $A \subseteq X$ is a closed subset equal to a union of cells

$$e_\alpha^n = \text{int}(D_\alpha^n).$$

Remark. This inherits a **CW complex** structure.

Check the images of attaching maps.

Exercise. Given the following gluing instruction:



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere, $\mathbb{R}P$.

Answer. We see that

1. Torus
2. Cylinder
3. 2-sphere
4. Klein bottle
5. Möbius band
6. $\mathbb{R}P$

Notation. We call the real projection space as $\mathbb{R}P$, and we also have so-called complex projection space, denote as $\mathbb{C}P$.

Lecture 5: Operation on Spaces

14 Jan. 10:00

1.4 Operations on CW Complexes

1.4.1 Products

We can consider the product of two **CW complex** given by a **CW complex** structure. Namely, given X and Y two **CW complexes**, we can take two cells e_α^n from X and e_β^m from Y and form the product space $e_\alpha^n \times e_\beta^m$, which is homeomorphic to an $(n+m)$ -cell. We then take these products as the cells for $X \times Y$.

Specifically, given X, Y are **CW complexes**, then $X \times Y$ has a cell structure

$$\{e_\alpha^m \times e_\beta^n : e_\alpha^m \text{ is a } m\text{-cell on } X, e_\beta^n \text{ is an } n\text{-cell on } Y\}.$$

Remark. The product topology may not agree with the weak topology on the $X \times Y$. However, they do agree if X or Y is locally compact or if X and Y both have at most countably many cells.

1.4.2 Wedge Sum

Given X, Y are **CW complexes**, and $x_0 \in X^0, y_0 \in Y^0$ (only points). Then we define

$$X \vee Y = X \amalg Y$$

with quotient topology.

Remark. $X \vee Y$ is a **CW complex**.

1.4.3 Quotients

Let X be a **CW complex**, and $A \subseteq X$ **subcomplex** (closed union of cells), then

$$X / A$$

is a quotient space collapse A to one point and inherits a **CW complex** structure.

Remark. X / A is a **CW complex**.

0-skeleton

$$(X^0 - A^0) \coprod *$$

where $*$ is a point for A . Each cell of $X - A$ is attached to $(X / A)^n$ by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n / A^n$$

Example. Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a **wedge** of two spheres.



2. We can take the torus and squish down a ring around the hole.



Figure 9: We see that X / A is **homotopy equivalent** to a 2-sphere **wedged** with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other points, forming a circle.

Lecture 6: A Foray into Category Theory

19 Jan. 10:00

1.5 Category Theory

We start with a definition.

Definition 1.9 (Category, object, morphism). A *category* \mathcal{C} is 3 pieces of data

- A class of *objects* $\text{Ob}(\mathcal{C})$
- $\forall X, Y \in \text{Ob}(\mathcal{C})$ a class of *morphisms* or arrows, $\text{Hom}_{\mathcal{C}}(X, Y)$.
- $\forall X, Y, Z \in \text{Ob}(\mathcal{C})$, there exists a composition law

$$\text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z), \quad (f, g) \mapsto g \circ f$$

and 2 axioms

- Associativity. $(f \circ g) \circ h = f \circ (g \circ h)$ for all **morphisms** f, g, h where composites are defined.
- Identity. $\forall X \in \text{Ob}(\mathcal{C}) \exists \text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$ such that

$$f \circ \text{id}_X = f, \quad \text{id}_X \circ g = g$$

for all f, g where this makes sense.

Let's see some examples.

Example. We introduce some common **category**.

\mathcal{C}	$\text{Ob}(\mathcal{C})$	$\text{Mor}(\mathcal{C})$
$\underline{\text{set}}$	Sets X	All maps of sets
$\underline{\text{fset}}$	Finite sets	All maps
$\underline{\text{Gp}}$	Groups	Group Homomorphisms
$\underline{\text{Ab}}$	Abelian groups	Group Homomorphisms
$\underline{k\text{-vect}}$	Vector spaces over k	k -linear maps
$\underline{\text{Rng}}$	Rings	Ring Homomorphisms
$\underline{\text{Top}}$	Topological spaces	Continuous maps
$\underline{\text{Haus}}$	Hausdorff Spaces	Continuous maps
$\underline{\text{hTop}}$	Topological spaces	Homotopy classes of continuous maps
$\underline{\text{Top}^*}$	Based topological spaces ²	Based maps ³

Remark. Any **diagram** plus composition law.

$$\text{id}_A \hookrightarrow A \longrightarrow B \hookleftarrow \text{id}_B .$$

Definition 1.10 (Monic, epic). A **morphism** $f: M \rightarrow N$ is *monic* if

$$\forall g_1, g_2 \quad f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$A \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} M \xrightarrow{f} N$$

Dually, f is *epic* if

$$\forall g_1, g_2 \quad g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$M \xrightarrow{f} N \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} B$$

Lemma 1.2. In $\underline{\text{set}}, \underline{\text{Ab}}, \underline{\text{Top}}, \underline{\text{Gp}}$, a map is **monic** if and only if f is injective, and **epic** if and only if f is surjective.

Proof. In $\underline{\text{set}}$, we prove that f is **monic** if and only if f is injective. Suppose $f \circ g_1 = f \circ g_2$ and f is injective, then for any a ,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence $g_1 = g_2$.

²Topological spaces with a distinguished base point $x_0 \in X$

³Continuous maps that presence base point $f: (x, x_0) \rightarrow (y, y_0)$ such that

$$f: X \rightarrow Y, \quad f(x_0) = y_0$$

is continuous.

Now we prove another direction, with contrapositive. Namely, we assume that f is not injective and show that f is not **monic**. Suppose $f(a) = f(b)$ and $a \neq b$, we want to show such g_i exists. This is easy by considering

$$g_1: * \mapsto a, \quad g_2: * \mapsto b.$$

■

1.5.1 Functor

After introducing the **category**, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

Definition 1.11 (Functor). Given \mathcal{C}, \mathcal{D} be two **categories**. A (covariant) *functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ is

1. a map on **objects**

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of **morphisms**

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(X) \rightarrow F(Y)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(f) \circ F(g)$

Lecture 7: Functors

21 Jan. 10:00

As previously seen. Assume that we initially have a commutative diagram in \mathcal{C} as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & Z \end{array}$$

After applying F , we'll have

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ & \searrow F(g \circ f) = F(g) \circ F(f) & \downarrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in \mathcal{D} .

We can also have a so-called contravariant **functor**.

Definition 1.12 (Contravariant functor). Given \mathcal{C}, \mathcal{D} be two categories. A *contravariant functor*

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is

1. a map on objects

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(Y), F(X)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(Y) \rightarrow F(X)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a *contravariant functor* F , the diagram becomes

$$\begin{array}{ccc} F(X) & \xleftarrow{F(f)} & F(Y) \\ & \nwarrow F(g \circ f) = F(f) \circ F(g) & \uparrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in \mathcal{D} .

Example. Let see some examples.

1. Identity functor.

$$I: \mathcal{C} \rightarrow \mathcal{C}.$$

2. Forgetful functor.

•

$$F: \underline{\text{Gp}} \rightarrow \underline{\text{set}}, \quad G \mapsto G^4$$

such that

$$[f: G \rightarrow H] \mapsto [f: G \rightarrow H].$$

•

$$F: \underline{\text{Top}} \rightarrow \underline{\text{set}}, \quad X \mapsto X^5$$

such that

$$[f: X \rightarrow Y] \mapsto [f: X \rightarrow Y].$$

⁴ G is now just the underlying set of the group G .

⁵ X is now just the underlying set of the topological space X .

3. Free functor.

$$\begin{aligned} \underline{\text{set}} &\rightarrow \underline{k\text{-vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

i.e., vector space with basis s such that

$$[f: A \rightarrow B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

4.

$$\begin{aligned} \underline{k\text{-vect}} &\rightarrow \underline{k\text{-vect}} \\ V &\mapsto V^* = \text{Hom}_k(V, k) \end{aligned}$$

If we are working on a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functors.

1.

$$\begin{aligned} \underline{\text{Top}}^* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0) \end{aligned}$$

where π_1 is so-called *fundamental group*.

2.

$$\begin{aligned} \underline{\text{Top}} &\rightarrow \underline{\text{Ab}} \\ X &\mapsto H_p(X) \end{aligned}$$

where H_p is so-called p^{th} *homology*.

Let's see the formal definition.

1.6 Free Groups

Definition 1.13 (Free group). Given a set S , the *free group* is a group F_S on S with a map $S \rightarrow F_S$ satisfying the universal property.

If G is any group, $f: S \rightarrow G$ is any map of sets, f extends uniquely to group homomorphism $\bar{f}: F_S \rightarrow G$.

$$\begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \bar{f}: \text{gp hom} \\ & & G \end{array}$$

Note. This defines a *natural bijection*

$$\mathrm{Hom}_{\mathrm{set}}(S, \mathcal{U}(G)) \cong \mathrm{Hom}_{\mathrm{Grp}}(F_S, G),$$

where $\mathcal{U}(G)$ is the **forgetful functor** from the **category** of groups to the **category** of sets. This is the statement that the **free functor** and the forgetful functor are **adjoint**; specifically that the **free functor** is the left **adjoint** (appears on the left in the Hom above).

Definition 1.14 (Adjoint functor). A **free** and **forgetful functor** is *adjoints*.

Remark. Whenever we state a universal property for an **object** (plus a map), an **object** (plus a map) may or may not exist. If such **object** exists, then it defines the **object uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the **object** (plus a map).

Lemma 1.3. Universal property defines F_S (plus a map $S \rightarrow F(S)$) uniquely up to unique isomorphism.

Proof. Fix S . Suppose

$$S \rightarrow F_S, \quad S \rightarrow \tilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{array}{ccc} S & \longrightarrow & \tilde{F}_S \\ & \searrow f & \downarrow \exists! \varphi \\ & & F_S \end{array} \quad \begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \psi \\ & & \tilde{F}_S \end{array}$$

We'll show φ and ψ are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \mathrm{id}_{F_S} & \searrow f \\ S & & \\ f \searrow & & \end{array} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \mathrm{id}_{\tilde{F}_S} & \searrow f \\ S & & \\ f \searrow & & \end{array}$$

Hence, we see that

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \psi & \searrow f \\ S & \longrightarrow & \tilde{F}_S \\ f \searrow & \downarrow \varphi & \nearrow f \\ & F_S & \end{array} \quad \varphi \circ \psi = \mathrm{id}_{F_S} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \varphi & \searrow f \\ S & \longrightarrow & F_S \\ f \searrow & \downarrow \psi & \nearrow f \\ & \tilde{F}_S & \end{array} \quad \psi \circ \varphi = \mathrm{id}_{\tilde{F}_S}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \text{id}_{F_S}, \quad \psi \circ \varphi = \text{id}_{\tilde{F}_S},$$

so φ and ψ are inverses (thus group isomorphism). ■

Lecture 8: The Fundamental Group π_1

24 Jan. 10:00

Example. In [category](#) [Ab](#) [free](#) Abelian group on a set S is

$$\bigoplus_S \mathbb{Z}.$$

In [category](#) of fields, no such thing as [free field on \$S\$](#) .

1.6.1 Constructing the Free Groups F_S

Proposition 1.2. The [free group](#) defined by the universal property exists.

Proof. We'll just give a construction below. First, we see the definition.

Definition 1.15 (Word). Fix a set S , and we define a *word* as a finite sequence (possibly \emptyset) in the formal symbols

$$\{s, s^{-1} \mid s \in S\}.$$

Then we see that elements in F_S are equivalence classes of [words](#) with the equivalence relation being

- deleted ss^{-1} or $s^{-1}s$. i.e.,

$$vs^{-1}sw \sim vw$$

$$vss^{-1}w \sim vw$$

for every [word](#) $v, w, s \in S$,

with the group operation being concatenation. ■

Example. Given [words](#) ab^{-1}, bba , their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

Exercise. There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of [words](#) has a unique *reduced form*, namely the representation.
3. Check that F_S satisfies the universal property with respect to the map

$$S \rightarrow F_S, \quad s \mapsto s.$$

2 The Fundamental Group

2.1 Path

We start with the definition.

Definition 2.1 (Path). A *path* in a space X is a continuous map

$$\gamma: I \rightarrow X$$

where $I = [0, 1]$.

Definition 2.2 (Homotopy path). A *homotopy of paths* γ_0, γ_1 is a *homotopy* from γ_0 to γ_1 rel $\{0, 1\}$.



Example. Fix $x_1, x_0 \in X$, then \exists *homotopy of paths* is an equivalence relation on *paths* from x_0 to x_1 (i.e., γ with $\gamma(0) = x_0, \gamma(1) = x_1$).

Definition 2.3 (Path composition). For *paths* α, β in X with $\alpha(1) = \beta(0)$, the *composition*^a $\alpha \cdot \beta$ is

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



^aAlso named *product*, *concatenation*.

Remark. By the pasting lemma, this is continuous, hence $\alpha \cdot \beta$ is actually a *path* from $\alpha(0)$ to $\beta(1)$.

Definition 2.4 (Reparameterization). Let $\gamma: I \rightarrow X$ be a [path](#), then a *reparameterization* of γ is a [path](#)

$$\gamma': I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where φ is [continuous](#) and

$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

Exercise. A [path](#) γ is [homotopic rel \$\{0, 1\}\$](#) to all of its [reparameterizations](#).

Proof. We show that γ and $\gamma \circ \phi$ are [homotopic rel \$\{0, 1\}\$](#) by showing that there exists a continuous F_t such that

$$F_0 = \gamma, \quad F_1 = \gamma \circ \phi.$$

Notice that since ϕ is continuous, so we define

$$F_t(x) = (1 - t)\gamma(x) + t \cdot \gamma \circ \phi(x).$$

We see that

$$F_0(x) = \gamma(x), \quad F_1(x) = \gamma \circ \phi(x),$$

and also, we have

$$F_t(x) \in X$$

for all $x, t \in I$.

Now, we check that F_t really gives us a [homotopic rel \$\{0, 1\}\$](#) . We have

$$\begin{aligned} F_t(0) &= (1 - t)\gamma(0) + t \cdot \gamma \circ \phi(0) = (1 - t)\gamma(0) + t \cdot \underbrace{\gamma(\phi(0))}_0 = \gamma(0), \\ F_t(1) &= (1 - t)\gamma(1) + t \cdot \gamma \circ \phi(1) = (1 - t)\gamma(1) + t \cdot \underbrace{\gamma(\phi(1))}_1 = \gamma(1), \end{aligned}$$

which shows that 0 and 1 are independent of t , hence γ and $\gamma \circ \phi$ are [homotopic rel \$\{0, 1\}\$](#) . ■

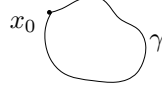
Exercise. Fix $x_0, x_1 \in X$. Then [homotopy of paths](#) ([relative \$\{0, 1\}\$](#)) is an equivalence relation on [paths](#) from x_0 to x_1 .

2.2 Fundamental Group and Groupoid

2.2.1 Fundamental Group

Definition 2.5 (Fundamental Group). Let X denotes the space and let $x_0 \in X$ be the base point. The *fundamental group of X based at x_0* , denoted by $\pi_1(X, x_0)$, is a group such that

- Elements: **Homotopy** classes $\text{rel}\{0, 1\}$ of **paths** $[\gamma]$ where γ is a **loop** with $\gamma(0) = \gamma(1) = x_0$ ^a



- Operation: **Composition of paths**.
- Identity: Constant loop γ based at x_0 such that

$$\gamma: I \rightarrow X, \quad t \mapsto x_0$$

- Inverses: The inverse $[\gamma]^{-1}$ of $[\gamma]$ is represented by the loop $\bar{\gamma}$ such that

$$\bar{\gamma}(t) = \gamma(1 - t).$$



^aWe say γ is **based** at x_0 .

Proof. We prove that

Associativity. $[\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)] = [(\gamma_1 \cdot \gamma_2) \cdot \gamma_3]$. We break this down into

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t), & t \in \left[0, \frac{1}{2}\right]; \\ (\gamma_2 \cdot \gamma_3)(2t - 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(4t - 2), & t \in \left[\frac{1}{2}, \frac{3}{4}\right]; \\ \gamma_3(4t - 3), & t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} (\gamma_1 \cdot \gamma_2)(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \gamma_1(4t), & t \in \left[0, \frac{1}{4}\right]; \\ \gamma_2(4t - 1), & t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \gamma_3(2t - 1), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, we define $\phi: I \rightarrow I$ such that

$$\phi(t) = \begin{cases} 2t \in \left[0, \frac{1}{2}\right], & t \in \left[0, \frac{1}{4}\right]; \\ t + \frac{1}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right], & t \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{t+1}{2} \in \left[\frac{3}{4}, 1\right], & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \circ \phi(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$, which implies that the associativity holds.

Identity. We want to show that $[\gamma \cdot c] = [\gamma]$. Again, we consider

$$(\gamma \cdot c)(t) = \begin{cases} \gamma(2t), & t \in \left[0, \frac{1}{2}\right]; \\ c(2t-1) = c = x_0 = \gamma(0), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, consider $\phi: I \rightarrow I$ such that

$$\phi(t) = \begin{cases} 2t, & t \in \left[0, \frac{1}{2}\right]; \\ 1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We easily see that

$$(\gamma \cdot c)(t) = (\gamma \circ \phi)(t)$$

and $\phi(t)$ is continuous and satisfied $\phi(0) = 0$ and $\phi(1) = 1$.

Inverses. We want to show that $\gamma \cdot \bar{\gamma} \simeq c$, where $\bar{\gamma}(t) = \gamma(1-t)$. Firstly, we have

$$(\gamma \cdot \bar{\gamma})(t) = \begin{cases} \gamma(2t), & t \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2t), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We consider F_t given by

$$F_t(x) = \begin{cases} \gamma(2xt), & x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2xt), & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

If $t = 0$, we have

$$F_0(x) = \begin{cases} \gamma(0), & x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1), & x \in \left[\frac{1}{2}, 1\right] \end{cases} = x_0$$

for all $x \in I$, namely $F_0 = c$, while when $t = 1$, we have

$$F_1(x) = \begin{cases} \gamma(2x), & x \in \left[0, \frac{1}{2}\right]; \\ \bar{\gamma}(1-2x), & x \in \left[\frac{1}{2}, 1\right] \end{cases} = (\gamma \cdot \bar{\gamma})(x),$$

and we see that F_t is continuous since at $x = \frac{1}{2}$, we have

$$\gamma(2x) = \gamma(1) = \bar{\gamma}(0) = \bar{\gamma}(1-2x),$$

hence we see that F_t is the **homotopy** between $\gamma \cdot \bar{\gamma}$ and c .

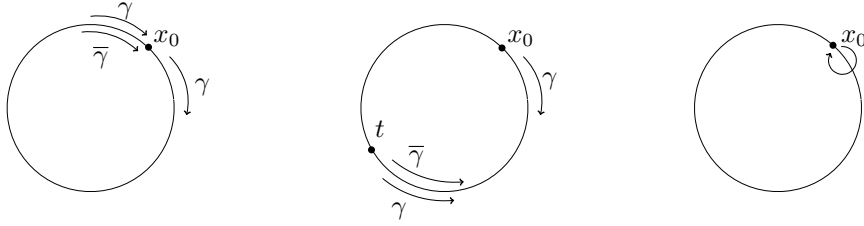


Figure 10: Illustration of F_t . Intuitively, the **path** $\gamma \cdot \bar{\gamma}$ is $x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\bar{\gamma}} x_0$. But now, F_t is $x_0 \xrightarrow{\gamma} t \xrightarrow{\bar{\gamma}} x_0$. We can think of this **homotopy** is *pulling back* the turning point along the original **path**.

■

Theorem 2.1. If X is **path**-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Remark. We see that we can write $\pi_1(X)$ up to isomorphism given this result.

Proof. To show that the *change-of-basepoint map* is isomorphism, we show that it's one-to-one and onto.

- one-to-one. Consider that if $[h \cdot \gamma \cdot \bar{h}] = [h \cdot \gamma' \cdot \bar{h}]$, then since we know that $h^{-1} = \bar{h}$, hence in the **fundamental group** $\pi_1(X, x_0)$, we see that

$$\bar{h} \cdot h \cdot \gamma \cdot \bar{h} \cdot h = \bar{h} \cdot h \cdot \gamma' \cdot \bar{h} \cdot h. \implies \gamma = \gamma'$$

as we desired.

- onto. We see that for every $\alpha \in \pi_1(X, x_0)$, there exists a $\gamma \in \pi_1(X, x_0)$ such that

$$\gamma = \bar{h} \cdot \alpha \cdot h \in \pi_1(X, x_1)^6$$

since $h \cdot \gamma \cdot \bar{h} = \alpha$.

We then see that the **fundamental group** of X does not depend on the choice of basepoint, only on the choice of the **path** component of the basepoint. If X is **path-connected**, it now makes sense to refer to *the fundamental group* of X and write $\pi_1(X)$ for the abstract group (up to isomorphism). ■

Exercise. Composition of **paths** is well-defined on **homotopy** classes $\text{rel}\{0, 1\}$.

Exercise. If X is a contractible space, then X is **path-connected** and $\pi_1(X)$ is trivial.

The followings are the properties about **homotopy path**. They are useful when we introduce **fundamental groupoid**.

Lemma 2.1. Given $x_0, x_1, x_2 \in X$, α, α' are two **paths** from x_0 to x_1 , and β, β' are two **paths** from x_1 to x_2 . If $\langle \alpha \rangle = \langle \alpha' \rangle$, $\langle \beta \rangle = \langle \beta' \rangle$, then $\langle \alpha \cdot \beta \rangle = \langle \alpha' \cdot \beta' \rangle$.

Proof. Given $\alpha \simeq_F \alpha' \text{ rel}\{0, 1\}$, $\beta \simeq_G \beta' \text{ rel}\{0, 1\}$, then we want to prove

$$\alpha \cdot \beta \simeq \alpha' \cdot \beta' \text{ rel}\{0, 1\}.$$

This is done by using **homotopy** $H: I \times I \rightarrow X$ such that it combines $F(2s, t)$ and $G(2s - 1, t)$.



■

⁶Notice that this is indeed the case, one can verify this by the fact that $h: x_0 \rightarrow x_1$ and $\bar{h}: x_1 \rightarrow x_0$.

Lemma 2.2. Let $x_0, x_1, x_2, x_3 \in X$, α is a path from x_0 to x_1 , β is a path from x_1 to x_2 , γ is a path from x_2 to x_3 . Then

$$\langle (\alpha \cdot \beta) \cdot \gamma \rangle = \langle \alpha \cdot (\beta \cdot \gamma) \rangle.$$

Proof. We can write out the homotopy by the following diagram.



■

Lemma 2.3. Let X be a topological space, and $x_0 \in X$. Then for every path homotopy $\langle \alpha \rangle$ from x_1 to x_2 , we have

$$\langle c_{x_1} \cdot \alpha \rangle = \langle \alpha \rangle = \langle \alpha \cdot c_{x_2} \rangle.$$

Proof. We only need to prove $c_{x_1} \cdot \alpha \simeq \alpha \text{ rel } \{0, 1\}$. The homotopy can be written out explicitly by the following diagram.



■

Lemma 2.4. For every path homotopy $\langle \alpha \rangle$ from x_1 to x_2 , then

$$\langle \alpha \cdot \alpha^{-1} \rangle = \langle c_{x_1} \rangle, \quad \langle \alpha^{-1} \cdot \alpha \rangle = \langle c_{x_2} \rangle.$$

Proof. For the first case, we have the following diagram.



The second case follows similarly. ■

2.2.2 Fundamental Groupoid

This section is not covered in class, but it's a useful concept. The idea is that after giving [Definition 2.5](#), we see that we actually create a [fundamental group](#) at **every** point in X , furthermore, when we use [Theorem 2.1](#) if X is [path-connected](#), we actually **lose** some information about this space. Here is how we can store all the information.

Notation (Constant loop). We denote c_x , where $x \in X$ such that

$$\begin{aligned} c_x : [0, 1] &\rightarrow X \\ t &\mapsto x \end{aligned}$$

as a *constant loop*.

Definition 2.6 (Groupoid). A [category](#) \mathcal{C} is a *groupoid* if any [morphisms](#) in \mathcal{C} is and isomorphism.

Remark. We'll soon see that for any topological space x , [Definition 2.5](#) defines a [groupoid](#), denoted by $\Pi(X)$.

Definition 2.7 (Fundamental groupoid). Let X denotes the space, then the [category](#) $\Pi(X)$ is a *fundamental groupoid of X* such that

- $\text{Ob}(\Pi(X)) := X$
- $\text{Hom}(\Pi(X)) : \forall p, q \in \text{Ob}(\Pi(X)) = X,$

$$\text{Hom}_{\Pi(X)}(p, q) := \{\text{Paths from } p \text{ to } q\} / \sim.$$

- Composition: For every $p, q, r \in \text{Ob}(\Pi(X)) = X,$

$$\begin{aligned} \circ : \text{Hom}_{\Pi(X)}(p, q) \times \text{Hom}_{\Pi(X)}(q, r) &\rightarrow \text{Hom}_{\Pi(X)}(p, r) \\ (\langle \alpha \rangle, \langle \beta \rangle) &\mapsto \langle \beta \rangle \circ \langle \alpha \rangle := \langle \alpha \cdot \beta \rangle. \end{aligned}$$

- Identity: For every $p \in \text{Ob}(\Pi(X)) = X,$ we define $1_p := \langle c_p \rangle \in \text{Hom}_{\Pi(X)}(p, p)$ be the constant loop based at p such that for every $\langle \alpha \rangle \in \text{Hom}_{\Pi(X)}(p, q),$

$$\langle \alpha \rangle \circ \text{id}_p = \text{id}_q \circ \langle \alpha \rangle = \langle \alpha \rangle.$$

- Associativity: Given $p, q, r, s \in \text{Ob}(\Pi(X)) = X,$ with the [paths](#)

$$p \xrightarrow{\langle \alpha \rangle} q \xrightarrow{\langle \beta \rangle} r \xrightarrow{\langle \gamma \rangle} s$$

Then

$$\langle \gamma \rangle \circ (\langle \beta \rangle \circ \langle \alpha \rangle) = (\langle \gamma \rangle \circ \langle \beta \rangle) \circ \langle \alpha \rangle.$$

Proof. Note that in [Definition 2.7](#), we need to show some of the definitions is indeed well-defined, and we also need to show that $\Pi(X)$ is actually a [groupoid](#).

- Composition: Since if $\alpha \simeq \alpha', \beta \simeq \beta',$ we have

$$\alpha \cdot \beta \simeq \alpha' \cdot \beta'$$

from [Lemma 2.1](#).

- Identity: It follows that

$$\langle \alpha \rangle \circ \text{id}_p = \langle c_p \cdot \alpha \rangle = \langle \alpha \rangle$$

from [Lemma 2.3](#). The left identity can be shown similarly.

- Associativity: It's trivial in the sense that all the [homotopy](#) can be easily derived from [Lemma 2.2](#).

Additionally, from [Lemma 2.4](#), we see that given α is a [path](#) from p to q , then

$$\begin{cases} \langle \alpha^{-1} \cdot \alpha \rangle &= \langle c_q \rangle =: \text{id}_q \\ \langle \alpha \cdot \alpha^{-1} \rangle &= \langle c_p \rangle =: \text{id}_p. \end{cases}$$

Furthermore, since $\langle \alpha^{-1} \cdot \alpha \rangle = \langle \alpha \rangle \circ \langle \alpha^{-1} \rangle$ and $\langle \alpha \cdot \alpha^{-1} \rangle = \langle \alpha^{-1} \rangle \circ \langle \alpha \rangle,$ hence this means $\Pi(X)$ is indeed a [groupoid](#). \blacksquare

Remark. Assume \mathcal{C} is a [groupoid](#), then for every $x \in \text{Ob}(\mathcal{C})$, we can define

$$\cdot : \text{Hom}_{\mathcal{C}}(x, x) \times \text{Hom}_{\mathcal{C}}(x, x) \rightarrow \text{Hom}_{\mathcal{C}}(x, x)$$

such that

$$(f, g) \mapsto f \cdot g := g \circ f.$$

We can prove that

$$(\text{Hom}_{\mathcal{C}}(x, x), \cdot)$$

defines a group $\text{Aut}_{\mathcal{C}}(x)$ called the *isotropy group* of \mathcal{C} at x .

Exercise. For every $x, y \in \text{Ob}(\mathcal{C})$, if there exists $f \in \text{Hom}_{\mathcal{C}}(x, y)$, then f induces

$$f_* : \text{Aut}_{\mathcal{C}}(x) \xrightarrow{\sim} \text{Aut}_{\mathcal{C}}(y),$$

where f_* is a group homomorphism.

Remark. For every $p \in X = \text{Ob}(\Pi(X))$, we have

$$\text{Aut}_{\Pi(X)}(p) = \pi_1(X, p).$$

Firstly, since they're the same in the sense of **set**:

$$\text{Aut}_{\Pi(X)}(p) = \text{Hom}_{\Pi(X)}(p, p) = \{\text{Loops in } X \text{ based at } p\} / \sim = \pi_1(X, p).$$

Hence, we only need to verify their group composition agrees. But this is trivial, since for every two $\langle \alpha \rangle, \langle \beta \rangle \in \text{Aut}_{\Pi(X)}(p)$,

$$\underbrace{\langle \alpha \rangle \cdot \langle \beta \rangle}_{\text{Composition from } \text{Aut}_{\Pi(X)}} = \langle \beta \rangle \circ \langle \alpha \rangle = \underbrace{\langle \alpha \cdot \beta \rangle}_{\text{Composition from } \pi_1}.$$

This implies that [Theorem 2.1](#) is just a particular example as a [groupoid](#).

Lecture 9: Calculate Fundamental Group

26 Jan. 10:00

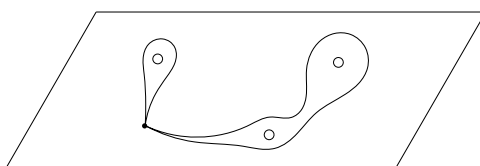


Figure 11: [Fundamental Group](#) is basically a *hole detector*!

2.3 Calculations with $\pi_1(S^n)$

Let's start with a basic but important theorem.

Theorem 2.2 (The fundamental group of S^1). The fundamental group of S^1 is

$$\pi_1(S^1) \cong \mathbb{Z},$$

and this identification is given by the [paths](#)

$$n \leftrightarrow [\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))].$$

Remark. Intuitively, this winds around S^1 n times. The key to this proof was to understand S^1 via the [covering space](#) $\mathbb{R} \rightarrow S^1$. We will talk about [covering spaces](#) much later.

Proof. With the help of [covering spaces](#) and the theorems build around which, we can define

$$\begin{aligned} p: \mathbb{R} &\rightarrow S^1, & x &\mapsto e^{2\pi i x}, \\ \varphi: \mathbb{Z} &\rightarrow \pi_1(S^1, 1), & n &\mapsto \langle p \circ \gamma_n \rangle, \end{aligned}$$

where p defined above is a [covering map](#). We need to show that this is well-defined.

From the definition of φ , we see that it's a homomorphism. But we also need to show

- φ is a surjection. This is shown by [Corollary 3.1](#), specifically in the case of [path](#).
- φ is an injection. This is shown by [Corollary 3.1](#), specifically in the case of [homotopy of paths](#).

■

Theorem 2.3. Given (X, x_0) and (Y, y_0) , then

$$\pi(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

such that

$$\left[\begin{array}{l} r: I \rightarrow X \times Y \\ r(t) = (r_X(t), r_Y(t)) \end{array} \right] \mapsto (r_X, r_Y).$$

Proof. Let $Z \xrightarrow{f} X \times Y$ with $z \mapsto (f_X(z), f_Y(z))$. Then we have

$$f \text{ continuous} \iff f_X, f_Y \text{ are continuous.}$$

Now, apply above to

- [Paths](#) $I \rightarrow X \times Y$.
- [Homotopies of paths](#) $I \times I \rightarrow X \times Y$.

■

Corollary 2.1 (The fundamental group of S^k). The torus $T \cong S^1 \times S^1$ has fundamental group $\pi_1(T) \cong \mathbb{Z}^2$. Additionally, for a k -torus

$$\underbrace{S^1 \times S^1 \times \dots \times S^1}_{k \text{ times}} = (S^1)^k,$$

the fundamental group is then \mathbb{Z}^k , i.e.

$$\pi_1((S^1)^k) \cong \mathbb{Z}^k.$$

Proof. Since

$$\pi_1 \cong \mathbb{Z}^2 \cong \mathbb{Z}_a \oplus \mathbb{Z}_b.$$



■

Remark. One way to think of the k -torus is as a k -dimensional cube with opposite $(k - 1)$ -dimensional faces identified by translation.

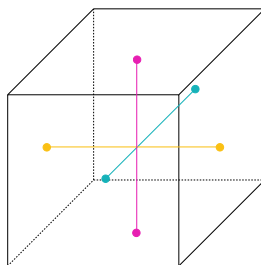


Figure 12: 3-torus with cube identified with parallel sides.

Lemma 2.5. Let $f, g: X \rightarrow Y$ such that $f \simeq_F g$. Let $x_0 \in X$, then given

$$f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$$

$$g_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, g(x_0))$$

with $\gamma: [0, 1] \rightarrow Y$, $t \mapsto F(x_0, t)$,

$$\begin{aligned} \gamma_*: \pi_1(Y, f(x_0)) &\rightarrow \pi_1(Y, g(x_0)) \\ \langle \alpha \rangle &\mapsto \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle, \end{aligned}$$

the following diagram commutes.

$$\begin{array}{ccc} & \pi_1(Y, f(x_0)) & \\ f_* \nearrow & \downarrow \gamma_* & \searrow g_* \\ \pi_1(X, x_0) & & \pi_1(Y, g(x_0)) \end{array}$$

Proof. We want to prove that for any $\langle \alpha \rangle \in \pi_1(X, x_0)$, we have

$$\gamma_* \circ f_*(\langle \alpha \rangle) = g_*(\langle \alpha \rangle).$$

The left-hand side is just

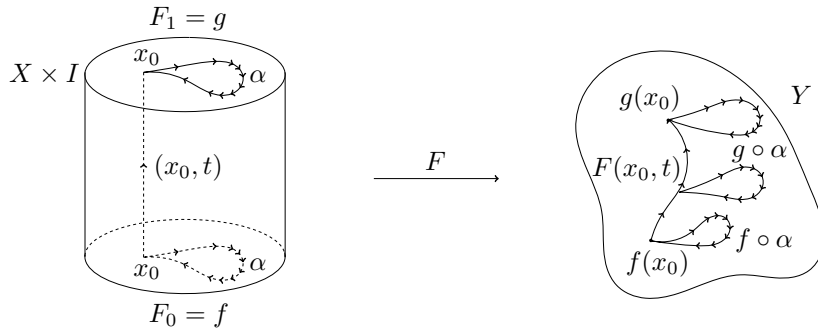
$$\gamma_* \circ f_*(\langle \alpha \rangle) = \gamma_*(\langle f \circ \alpha \rangle) = \langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \rangle,$$

while the right-hand side is just

$$g_*(\langle \alpha \rangle) = \langle g \circ \alpha \rangle.$$

That is, we now want to show

$$\langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \rangle = \langle g \circ \alpha \rangle.$$



We see that we can obtain a [homotopy](#) $G: I \times I \rightarrow Y$ such that

$$G := F \circ (\alpha \times \text{id}),$$

where we define $\alpha \times \text{id}$ by

$$\alpha \times \text{id}: I \times I \rightarrow X \times I, \quad (s, t) \mapsto (\alpha(s), t).$$



Figure 13: $\alpha \times \text{id}$'s image.

We see that by defining such G , we have the following.



To write out this [homotopy](#) explicitly, we see the following diagram.



■

Theorem 2.4 (Fundamental group is a homotopy invariant). If X, Y are homotopy equivalent, then their fundamental groups are isomorphic.

Proof.

■

HW.

Remark. This gives us a powerful tool to calculate π_1 .

Example. We now see some examples.

1. $\pi_1(S^\infty \times S^1) \cong \mathbb{Z}$
2. $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong 0 \times \mathbb{Z} = \mathbb{Z}$ since

$$\mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R},$$

which means that the generators are just loops around the hole intuitively.

2.4 Fundamental Group and Groupoid Define Functors

Theorem 2.5 (Fundamental group defines a functor). π_1 is a functor such that

$$\begin{aligned} \pi_1: \underline{\text{Top}}_* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \pi_1(X, x_0). \end{aligned}$$

While on a map $f: X \rightarrow Y$ taking base point x_0 to y_0 , π_1 induces a map

$$\begin{aligned} f_*: \pi_1(X, x_0) &\rightarrow \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

i.e.,

$$[f: X \rightarrow Y] \mapsto [f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))].$$

Notation. We usually write f_* if it's a covariant functor, while writing f^* if it's a contravariant functor.

Proof. We need to check

- well-defined on path homotopy classes.
- f_* is a group homomorphism.

$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot f_*(\beta) = \begin{cases} f(\alpha(2s)), & \text{if } s \in \left[0, \frac{1}{2}\right] \\ f(\beta(1-2s)), & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

- $(\text{id}_{(X, x_0)})_* = \text{id}_{\pi_1(X, x_0)}$

- $(f_* \circ g_*) = (f \circ g)_*$

$$(f \circ g)_*[\gamma] = [f \circ g \circ \gamma] = [f \circ (g \circ \gamma)] \implies f_*(\gamma_*(\gamma)).$$

DIY

$$\begin{array}{ccc} (X, x_0) & \rightsquigarrow & \pi_1(X, x_0) \\ f \downarrow & & \downarrow f_* \\ (Y, y_0) & \rightsquigarrow & \pi_1(Y, y_0) \end{array}$$

■

Remark. We see that the construction of **fundamental group** is actually constructing a **functor**. Specifically,

$$\pi_1: \underline{\text{Top}}_* \rightarrow \underline{\text{Gp}}$$

such that

- on **objects**:

$$\forall (X, x_0) \in \text{Ob}(\underline{\text{Top}}_*), \quad \pi_1(X, x_0) = \text{fundamental group based at } x_0.$$

- on **morphisms**:

$$\forall f: (X, x_0) \rightarrow (Y, y_0), \quad \pi_1(f) = f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).$$

Our initial motivation is to construct a topological invariant, but we see that using π_1 , we need an additional **base point**. But as you already imagined, the **fundamental groupoid** actually is a **functor** as well.

Before we proceed further, we need to see the **category** of **groupoid**, denoted by $\underline{\text{Gpd}}$.

Definition 2.8 (Category of groupoid). The *category of groupoid*, denoted as $\underline{\text{Gpd}}$, contains the following data.

- $\text{Ob}(\underline{\text{Gpd}})$: **groupoids**.
- $\text{Hom}(\underline{\text{Gpd}})$: **functors** between **groupoids**.
- Composition: For every $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \in \text{Ob}(\underline{\text{Gpd}})$,

$$\mathfrak{X} \xrightarrow{F} \mathfrak{Y} \xrightarrow{G} \mathfrak{Z}$$

then $G \circ F: \mathfrak{X} \rightarrow \mathfrak{Z}$ is a **functor** defined as

- on **objects**: $\forall X \in \text{Ob}(\mathfrak{X})$,

$$G \circ F(X) := G(F(X)).$$

- on **morphisms**: $\forall X, Y \in \text{Ob}(\mathfrak{X})$ and $f: X \rightarrow Y$,

$$G \circ F(f) := G(F(f)).$$

- Identity. For every **groupoid** \mathfrak{X} , we define $\text{id}_{\mathfrak{X}}: \mathfrak{X} \rightarrow \mathfrak{X}$, where
 - $\forall X \in \text{Ob}(\mathfrak{X})$, $\text{id}_{\mathfrak{X}}(X) = X$
 - $\forall f \in \text{Hom}(\mathfrak{X})$, $\text{id}_{\mathfrak{X}}(f) = f$.
- Associativity. Since the composition is defined based on two **functors** (given $\mathfrak{X} \xrightarrow{F} \mathfrak{Y} \xrightarrow{G} \mathfrak{Z}$), this holds trivially.

Proof. We need to show that the composition is well-defined. Specifically, we need to check

- $G \circ F(\text{id}_X) = \text{id}_{G \circ F(X)}$, since

$$G \circ F(\text{id}_X) = G(F(\text{id}_X)) = G(\text{id}_{F(X)}) = \text{id}_{G(F(X))} = \text{id}_{G \circ F(X)}.$$

- Given $X_1, X_2, X_3 \in \text{Ob}(\mathfrak{X})$ and

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

we want to show $G \circ F(g \circ f) = G \circ F(g) \circ G \circ F(f)$. Firstly, since G is a **functor**, hence

$$G \circ F(g) \circ G \circ F(f) = G(F(g)) \circ G(F(f)) = G(F(g) \circ F(f)).$$

Again, since F is a functor, so we further have

$$G \circ F(g) \circ G \circ F(f) = G(F(g \circ f)) = G \circ F(g \circ f).$$

■

Theorem 2.6 (Fundamental groupoid defines a functor). Π is a **functor** such that

$$\Pi: \underline{\text{Top}} \rightarrow \underline{\text{Gpd}},$$

where

- on **objects**: For every $X \in \text{Ob}(\underline{\text{Top}})$,

$$X \mapsto \Pi(X).$$

- on **morphisms**: for every $X, Y \in \text{Ob}(\underline{\text{Top}})$, $f: X \rightarrow Y$, define a **functor**

$$\Pi(f): \Pi(X) \rightarrow \Pi(Y)$$

such that

- on **objects**: For every $p \in \text{Ob}(\Pi(X)) = X$, $\Pi(f)(p) = f(p)$. i.e.,

$$\Pi(f): \underbrace{\text{Ob}(\Pi(X))}_X \rightarrow \underbrace{\text{Ob}(\Pi(Y))}_Y.$$

- on **morphisms**: For every $\langle \alpha \rangle \in \text{Hom}_{\Pi(X)}(p, q)$, define

$$\Pi(f)(\langle \alpha \rangle) := \langle f \circ \alpha \rangle \in \text{Hom}_{\Pi(Y)}(f(p), f(q)).$$

Proof. We need to check that the defined **functor** $\Pi(f)$ satisfies

- $\Pi(f)(\text{id}_p) = \text{id}_{f(p)}$. Indeed, since

$$\Pi(f)(\text{id}_p) = \Pi(f)(\langle c_p \rangle) = \langle f \circ d_p \rangle = \langle c_{f(p)} \rangle = \text{id}_{f(p)}.$$

- For every $p, q, r \in X = \text{Ob}(\Pi(X))$,

$$p \xrightarrow{\langle \alpha \rangle} q \xrightarrow{\langle \beta \rangle} r$$

we want to show $\Pi(f)(\langle \beta \rangle \circ \langle \alpha \rangle) = \Pi(f)(\langle \beta \rangle) \circ \Pi(f)(\langle \alpha \rangle)$. Indeed, since

$$\Pi(f)(\langle \beta \rangle \circ \langle \alpha \rangle) = \Pi(f)(\langle \alpha \cdot \beta \rangle) = \langle f \circ (\alpha \cdot \beta) \rangle,$$

and

$$\Pi(f)(\langle \beta \rangle) \circ \Pi(f)(\langle \alpha \rangle) = \langle f \circ \beta \rangle \circ \langle f \circ \alpha \rangle = \langle (f \circ \alpha) \cdot (f \circ \beta) \rangle.$$

Since $\langle f \circ (\alpha \cdot \beta) \rangle = \langle (f \circ \alpha) \cdot (f \circ \beta) \rangle$, hence $\Pi(f)$ is well-defined.

Now, we need to prove the same thing for Π , namely Π satisfies

- $\Pi(\text{id}_X) = \text{id}_{\Pi(X)}$ for all $X \in \text{Ob}(\underline{\text{Top}})$. This is trivial since

$$\Pi(\text{id}_X): \Pi(X) \rightarrow \Pi(X),$$

- on **objects**: $p \mapsto \text{id}_X(p) = p$.

– on **morphisms**: $p \xrightarrow{\langle \alpha \rangle} q \mapsto \langle \text{id}_X \circ \alpha \rangle = \langle \alpha \rangle$.

- For all $X, Y, Z \in \text{Ob}(\underline{\text{Top}})$,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

then $\Pi(g \circ f) = \Pi(g) \circ \Pi(f)$. The diagrams are as follows.

$$\Pi(g \circ f): \Pi(X) \rightarrow \Pi(Z)$$

and

$$\Pi(X) \xrightarrow{\Pi(f)} \Pi(Y) \xrightarrow{\Pi(g)} \Pi(Z)$$

We see that this equality is in the sense of **functor**, hence we consider

– on **objects**: For every $p \in \text{Ob}(\Pi(X)) = X$, $\Pi(g \circ f)(p) = g \circ f(p)$ and

$$\Pi(g) \circ \Pi(f)(p) = \Pi(g)(\Pi(f)(p)) = \Pi(g)(f(p) = g(f(p))),$$

hence they're the same.

– on **morphisms**: For all $\langle \alpha \rangle \in \text{Hom}_{\Pi(X)}(p, q)$,

$$* \Pi(g \circ f)(\langle \alpha \rangle) = \langle (g \circ f) \circ \alpha \rangle.$$

$$* \Pi(g) \circ \Pi(f)(\langle \alpha \rangle) = \Pi(g) \left(\underbrace{\Pi(f)(\langle \alpha \rangle)}_{\langle f \circ \alpha \rangle} \right) = \langle g \circ (f \circ \alpha) \rangle.$$

We see that they're the same.

■

Lecture 10: Seifert-Van Kampen Theorem

26 Jan. 10:00

The goal is to compute $\pi_1(X)$ where $X = A \cup B$ using the data

$$\pi_1(A), \pi_1(B), \pi_1(A \cap B).$$

2.5 Free Product

2.5.1 Free Product

We first introduce a definition.

Definition 2.9 (Free product). Given some collections of groups $\{G_\alpha\}_\alpha$, the *free product*, denoted by $*_\alpha G_\alpha$ is a group such that

- Elements: **Words** in $\{g: g \in G_\alpha \text{ for any } \alpha\}$ modulo by the equivalence relation generated by

$$wg_i g_j v \sim w(g_i g_j)v$$

when both $g_i, g_j \in G_\alpha$. Also, for the identity element $\text{id} = e_\alpha \in G_\alpha$ for any α such that

$$we_\alpha v \sim wv.$$

Specifically,

$$*_\alpha G_\alpha := \{\text{words in } \{G_\alpha\}_\alpha\} / \sim.$$

- Operation: Concatenation of **words**.

Remark. In particular, we have the following universal property of $*_\alpha G_\alpha$. For every α , there is a ι_α such that

$$\iota_\alpha: G_\alpha \rightarrow *_\alpha G_\alpha, \quad g \mapsto \bar{g},$$

where ι_α is a group homomorphism obviously. Further, $(*_\alpha G_\alpha, \iota_\alpha)$ satisfies the following property: For every group H and a group homomorphism $\varphi_\alpha: G_\alpha \rightarrow H$ for all α , there exists an unique group homomorphism $\varphi: *_\alpha G_\alpha \rightarrow H$ such that $\varphi \circ \iota_\alpha = \varphi_\alpha$, i.e., the following diagram commutes.



Proof. The proof is straightforward. Firstly, we define $w = \overline{g_1 g_2 \dots g_n} \in *_\alpha G_\alpha$, $g_i \in G_{\alpha_i}$,

$$\varphi(w) := \varphi_{\alpha_1}(g_1) \dots \varphi_{\alpha_n}(g_n).$$

Now, we just need to check

- It's well-defined, since φ_α is a group homomorphism.
- φ is a group homomorphism.
- $\varphi \circ \iota_\alpha = \varphi_\alpha$.
- Such φ is unique. Suppose there exists another $\psi: *_\alpha G_\alpha \rightarrow H$, then

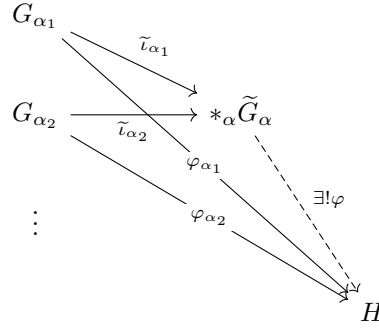
$$\varphi \circ \iota_\alpha = \varphi_\alpha \implies \forall_{g \in G_\alpha} \psi(\bar{g}) = \varphi_\alpha(g),$$

But then for every $w = \overline{g_1 g_2 \dots g_n} \in *_\alpha G_\alpha$, $g_i \in G_{\alpha_i}$, we have

$$\psi(w) = \psi(\overline{g_1} \dots \overline{g_n}) = \psi(\overline{g_1}) \dots \psi(\overline{g_n}) = \psi_{\alpha_1}(\overline{g_1}) \dots \psi_{\alpha_n}(\overline{g_n}),$$

which is just φ . ■

Remark. We further claim that this universal property determines such [free product](#) uniquely. i.e., assume there are another group \tilde{G} and $\tilde{\iota}_\alpha: G_\alpha \rightarrow \tilde{G}$. Assume $(\tilde{G}, \tilde{\iota}_\alpha)$ also satisfies the following property: For every group H and group homomorphism $\varphi_\alpha: G_\alpha \rightarrow H$, then there exists a unique group homomorphism $\varphi: \tilde{G} \rightarrow H$ such that the following diagram commutes.



Then, $\tilde{G} \cong *_\alpha G_\alpha$.

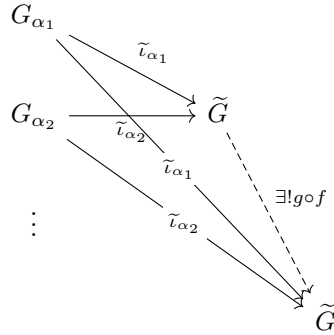
Proof. Assume $(\tilde{G}, \tilde{\iota}_\alpha)$ satisfies the universal property mentioned above. Then from the universal property and viewing \tilde{G} and $*_\alpha G_\alpha$ as H separately, we obtain the following diagram.



We claim that

$$g \circ f = \text{id}, \quad f \circ g = \text{id}.$$

To see this, we simply apply the same observation, for example,



where $g \circ f$ comes from the previous diagram. But notice that id let the diagram commutes also, and since it's unique, hence $g \circ f = \text{id}$. Similarly, we have $f \circ g = \text{id}$. ■

If you're careful enough, you may find out that all we're doing is just writing out a specific example of [Lemma 1.3](#)! Indeed, this is exactly the construction of a [free group](#).

Definition 2.10 (Fibred coproduct). Given a [category](#) \mathcal{C} , let $f: Z \rightarrow X$, $g: Z \rightarrow Y$. The *fibred coproduct* between f and g is the data (W, p_1, p_2) , where $W \in \text{Ob}(\mathcal{C})$, $p_1: X \rightarrow W$, $p_2: Y \rightarrow W$ satisfy the following.

- The diagram commutes.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \downarrow p_1 \\ Y & \xrightarrow{p_2} & W \end{array}$$

- For every $u: X \rightarrow U$, $v: Y \rightarrow U$ such that the following diagram commutes

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \downarrow p_1 \\ Y & \xrightarrow{p_2} & W \end{array} \quad \begin{array}{c} \searrow u \\ \vdots \\ \xrightarrow{\exists! h} \\ \searrow v \end{array} \quad \begin{array}{c} \\ \\ U \end{array}$$

there exists a unique $h: W \rightarrow U$ such that $h \circ p_1 = u$, $h \circ p_2 = v$.

We say

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & W \end{array}$$

is a *Cocartesian* diagram.

Exercise. Prove that in a category \mathcal{C} , if the **fibred coproduct** of f and g exists

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \\ Y & & \end{array}$$

then such **fibred coproduct** is unique up to isomorphism.

Remark. If we reverse all the directions of **morphism**, then we have so-called *fibred product*.

Example. Let's see some example.

1. Let $\mathcal{C} = \underline{\text{Top}}$, and let $X \in \text{Ob}(\underline{\text{Top}})$. Given $X_0, X_1 \in X$, and $\text{int}(X_0) \cup \text{int}(X_1) = X$, if we have

$$\begin{aligned} i_0: X_0 &\hookrightarrow X, & i_1: X_1 &\hookrightarrow X \\ j_0: X_0 \cap X_1 &\hookrightarrow X_0, & j_1: X_0 \cap X_1 &\hookrightarrow X_1, \end{aligned}$$

then

$$\begin{array}{ccc} X_0 \cap X_1 & \xrightarrow{j_0} & X_0 \\ j_1 \downarrow & & \downarrow i_0 \\ X_1 & \xrightarrow{i_1} & X \end{array}$$

is a **cocartesian** diagram.

Proof. All we need to show is that given a topological space $Y \in \underline{\text{Top}}$ and $f: X_0 \rightarrow Y, g: X_1 \rightarrow Y$ in $\underline{\text{Top}}$, we have

$$f \circ j_0 = g \circ j_1.$$

$$\begin{array}{ccc} X_0 \cap X_1 & \xrightarrow{j_0} & X_0 \\ j_1 \downarrow & & \downarrow i_0 \\ X_1 & \xrightarrow{i_1} & X \end{array} \quad \begin{array}{c} \nearrow f \\ \searrow g \\ \exists! h \end{array}$$

We simply define $h: X \rightarrow Y, x \mapsto h(x)$ such that

$$h(x) = \begin{cases} f(x), & \text{if } x \in X_0; \\ g(x), & \text{if } x \in X_1. \end{cases}$$

h is clearly well-defined since the diagram commutes, so if $x \in X_0 \cap X_1$, then $f(x) = g(x)$. The only thing we need to show is that h is continuous. But this is obvious too since $X = \text{int}(X_0) \cup \text{int}(X_1)$, and

$$h|_{\text{int}(X_0)} = f|_{\text{int}(X_0)}, \quad h|_{\text{int}(X_1)} = g|_{\text{int}(X_1)}.$$

The uniqueness is trivial, hence this is indeed a **cocartesian** diagram. \blacksquare

2. Let $\mathcal{C} = \mathbf{Top}_*$. Given $p \in X_0 \cap X_1$, where all other data are the same with the above example, we see that

$$\begin{array}{ccc} (X_0 \cap X_1, p) & \xrightarrow{j_0} & (X_0, p) \\ j_1 \downarrow & & \downarrow i_0 \\ (X_1, p) & \xrightarrow{i_1} & (X, p) \end{array}$$

is a **cocartesian** diagram.

3. Let $\mathcal{C} = \mathbf{Gp}$. Given $P, G, H \in \mathbf{Ob}(\mathbf{Gp})$, we claim that the **fibered coproduct** of i and j exists.

$$\begin{array}{ccc} P & \xrightarrow{i} & G \\ j \downarrow & & \\ H & & \end{array}$$

Consider $G * H$ be the **free product** between G and H , with two inclusions

$$\iota_1: G \hookrightarrow G * H, \quad \iota_2: H \hookrightarrow G * H.$$

$$\begin{array}{ccc} P & \xrightarrow{i} & G \\ j \downarrow & & \downarrow \iota_1 \\ H & \xrightarrow{\iota_2} & G * H \end{array}$$

Let

$$N := \langle \{ \iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1} \mid x \in P \} \rangle,$$

we define

$$G *_p H = G * H / N.$$

$$\begin{array}{ccccc} P & \xrightarrow{i} & G & & \\ j \downarrow & & \downarrow \iota_1 & \searrow \tau & \\ H & \xrightarrow{\iota_2} & G * H & \xrightarrow{\pi} & G *_p H \\ & \searrow \nu & & & \end{array}$$

We claim that

$$\begin{array}{ccc} P & \xrightarrow{i} & G \\ j \downarrow & & \downarrow \tau \\ H & \xrightarrow{\nu} & G *_p H \end{array}$$

is a **cocartesian** diagram in \mathbf{Gp} .

Proof. Firstly, since it's just an outer diagram from above, hence it commutes. So we only need to prove this diagram satisfies the second diagram.

Given any group K , for every $f: G \rightarrow K$, $g: H \rightarrow K$ such that the following diagram commutes.

$$\begin{array}{ccc}
 P & \xrightarrow{i} & G \\
 j \downarrow & & \downarrow \tau \\
 H & \xrightarrow{\nu} & G *_p H
 \end{array}
 \begin{array}{c}
 \nearrow f \\
 \searrow g \\
 \text{---} h \text{---}
 \end{array}
 \rightarrow K$$

We want to prove that there exists a unique $h: G *_p H \rightarrow K$ such that this diagram still commutes. The idea is simple, from the universal property of $G * H$, we see that there exists a unique $\tilde{h}: G * H \rightarrow K$ such that

$$\tilde{h} \circ \iota_1 = f, \quad \tilde{h} \circ \iota_2 = g.$$

$$\begin{array}{ccc}
 P & \xrightarrow{i} & G \\
 j \downarrow & & \downarrow \tau \\
 H & \xrightarrow{\nu} & G *_p H
 \end{array}
 \begin{array}{c}
 \nearrow f \\
 \searrow g \\
 \text{---} h \text{---}
 \end{array}
 \rightarrow G *_p H$$

$\begin{array}{ccc} & G * H & \\ \iota_1 \swarrow & & \searrow \iota_2 \\ H & & G \end{array}$

We see that we can actually factor \tilde{h} through π , as long as $\ker(\tilde{h}) \supset \ker(\pi)$. Now, since

$$\ker(\pi) = \langle \{ \iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1} \mid x \in p \} \rangle,$$

we see that the kernel of π is indeed in the kernel of \tilde{h} since for every $x \in P$,

$$\tilde{h}(\iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1}) = \underbrace{\tilde{h} \circ \iota_1}_{f} \circ i(x) \cdot \underbrace{\tilde{h} \circ \iota_2}_{g} \circ j(x)^{-1} = 1,$$

which implies $\ker(\tilde{h}) \supset \ker(\pi)$.

$$\begin{array}{ccc}
 G * H & \xrightarrow{\pi} & K \\
 \tilde{h} \downarrow & & \\
 G *_p H & &
 \end{array}$$

We then see that there exists a unique $h: G *_p H \rightarrow K$ such that the above diagram commutes. ■

2.5.2 Free Product with Amalgamation

After seeing the above examples, the following definition should make sense.

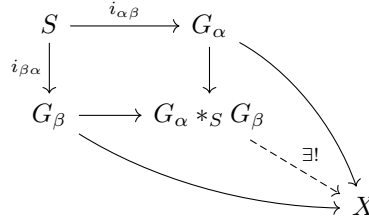
Definition 2.11 (Free product with amalgamation). If two groups G_α and G_β have a common subgroup $S_{\{\alpha,\beta\}}$ ^a, given two inclusion maps^b $i_{\alpha\beta}: S_{\{\alpha,\beta\}} \rightarrow G_\alpha$ and $i_{\beta\alpha}: S_{\{\alpha,\beta\}} \rightarrow G_\beta$, the *free product with amalgamation* ${}_\alpha *_S G_\alpha$ is defined as ${}^*_\alpha G_\alpha$ modulo the normal subgroup generated by

$$\{i_{\alpha\beta}(s_{\{\alpha,\beta\}})i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \mid s_{\{\alpha,\beta\}} \in S_{\{\alpha,\beta\}}\},$$

Namely^c,

$${}_\alpha *_S G_\alpha = {}^*_\alpha G_\alpha / \langle i_{\alpha\beta}(s_{\{\alpha,\beta\}})i_{\beta\alpha}(s_{\{\alpha,\beta\}})^{-1} \rangle$$

and satisfies the universal property



^aIn general, we don't need $S_{\{\alpha,\beta\}}$ to be a subgroup.

^bWe don't actually need $i_{\alpha\beta}, i_{\beta\alpha}$ to be inclusive as well.

^ci.e., $i_{\alpha\beta}(s)$ and $i_{\beta\alpha}(s)$ will be identified in the quotient.

Remark. We see that

- We can then write out words such as $g_\alpha \cdot s \cdot g_\beta$ for $s \in S$, and view s as an element of G_α or G_β . In fact, we can do this construction even when i_α and i_β are not injective, though this means we are not working with a subgroup.
- Aside, in Top, the same universal property defines union



for A, B are open subsets and the inclusion of intersection.

2.6 Seifert-Van Kampen Theorem

With Definition 2.11, we can now see the important theorem.

Theorem 2.7 (Seifert-Van Kampen Theorem). Given (X, x_0) such that $X = \bigcup_{\alpha} A_{\alpha}$ with

- A_{α} are open and **path**-connected and $\forall \alpha \ x_0 \in A_{\alpha}$
- $A_{\alpha} \cap A_{\beta}$ is **path**-connected for all α, β .

Then there exists a surjective group homomorphism

$$*_\alpha: \pi_1(A_{\alpha}, x_0) \rightarrow \pi_1(X, x_0).$$

If we additionally have $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$ where they are all **path**-connected for every α, β, γ , then

$$\pi_1(X, x_0) \cong *_\alpha \pi_1(A_{\alpha} \cap A_{\beta}, x_0) \pi_1(A_{\alpha}, x_0)$$

associated to all maps $\pi_a(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha}), \pi_1(A_{\beta})$ induced by inclusions of spaces. i.e., $\pi_1(X, x_0)$ is a quotient of the **free product** $*_{\alpha} \pi_1(A_{\alpha})$ where we have

$$(i_{\alpha\beta})_*: \pi_1(A_{\alpha} \cap A_{\beta}) \rightarrow \pi_1(A_{\alpha})$$

which is induced by the inclusion $i_{\alpha\beta}: A_{\alpha} \cap A_{\beta} \rightarrow A_{\alpha}$. We then take the quotient by the normal subgroup generated by

$$\{(i_{\alpha\beta})_*(\gamma)(i_{\beta\alpha})_* \mid \gamma \in \pi_1(A_{\alpha} \cap A_{\beta})\}.$$

We'll defer the **proof** of **Theorem 2.7** until we get familiar with this theorem.

Example. We first see a great visualization of the **Theorem 2.7**.



Intuitively we see the **fundamental group** of X , which is built by gluing A and B along their intersection. As the **fundamental group** of A and B glued along the **fundamental group** of their intersection. In essence, $\pi_1(X, x_0)$ is the quotient of $\pi_1(A) * \pi_1(B)$ by relations to impose the condition that loops like γ lying in $A \cap B$ can be viewed as elements of either $\pi_1(A)$ or $\pi_1(B)$.

Remark. We can use a more abstract way to describe **Theorem 2.7**. Specifically, in the case that $n = 2$, i.e., $X = \bigcup_{i=1}^2 A_i$, we let $A_i =: X_i$, then we have the

following. The functor $\pi_1: \underline{\text{Top}}_* \rightarrow \underline{\text{Gp}}$ maps the [cocartesian](#) diagram in $\underline{\text{Top}}_*$ to a [cocartesian](#) diagram in $\underline{\text{Gp}}$ as follows.

$$\begin{array}{ccc}
 (X_0 \cap X_1, x_0) & \xrightarrow{j_0} & (X_0, x_0) \\
 \downarrow j_1 & & \downarrow i_0 \\
 (X_1, x_0) & \xrightarrow{i_1} & (X, x_0)
 \end{array}
 \xrightarrow{\pi_1}
 \begin{array}{ccc}
 \pi_1(X_0 \cap X_1, x_0) & \xrightarrow{(j_0)_*} & \pi_1(X_0, x_0) \\
 \downarrow (j_1)_* & & \downarrow (i_0)_* \\
 \pi_1(X_1, x_0) & \xrightarrow{(i_1)_*} & \pi_1(X, x_0)
 \end{array}$$

Then, simply from the property of [cocartesian](#) diagram, we see that

$$\pi_1(X, x_0) \cong \pi_1(X_0, x_0) *_{\pi_1(X_0 \cap X_1, x_0)} \pi_1(X_1, x_0).$$

Additionally, there is a more general version of [Theorem 2.7](#), which is defined on [groupoid](#). The theorem is stated in [Appendix A.1](#) with the proof.

With this more general version and the proof of which, we can apply it to [Theorem 2.7](#). But one question is that, the above proof works in $\underline{\text{Gpd}}$ rather than in $\underline{\text{Gp}}$. We now see how to generalize a group to a [groupoid](#).

For any group G , we can define a [groupoid](#), denoted as G also, as follows.

- $\text{Ob}(G) = \{\text{pt}\}$, a one point set.
- $\text{Hom}(G) = \{g \in G\}$.
- Composition: We define

$$g \circ h := h \cdot g.$$

We see that the associativity of group elements implies the associativity of composition defined above, and since there is an identity element in G , hence we also have an identity [morphism](#), these two facts ensure that G is an [category](#).

Furthermore, since for every $g \in G$, there is a $g^{-1} \in G$, hence every [morphism](#) is an isomorphism, which implies G is a [groupoid](#).

With this, we see that we can view the following diagram in the [category](#) of [groupoid](#) $\underline{\text{Gpd}}$.

$$\begin{array}{ccc}
 \pi_1(X_0 \cap X_1, x_0) & \xrightarrow{(j_0)_*} & \pi_1(X_0, x_0) \\
 \downarrow (j_1)_* & & \downarrow (i_0)_* \\
 \pi_1(X_1, x_0) & \xrightarrow{(i_1)_*} & \pi_1(X, x_0)
 \end{array}$$

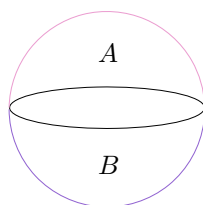
And to prove [Theorem 2.7](#), we only need to show this diagram is [cocartesian](#). This version of proof is given in [Appendix A.2](#).

Lecture 11: Group Presentations

31 Jan. 10:00

Example. We now see some applications of [Theorem 2.7](#).

1. We can use [Seifert Van Kampen Theorem](#) to compute the [fundamental group](#) of S^2 . We see that



We see that $\pi_1(S^2)$ must be a quotient of $\pi_1(A) * \pi_1(B)$, but since $A, B \simeq D^2$, we know that $\pi_1(A)$ and $\pi_1(B)$ are both zero groups, thus $\pi_1(A) * \pi_1(B)$ is the zero group, and $\pi_1(S^2)$ is also the zero group.

Remark. Note that the inclusion of $A \cap B \rightarrow A$ induces the zero map $\pi_1(A \cap B) \rightarrow \pi_1(A)$, which cannot be an injection. In fact, we know that $\pi_1(A \cap B) \cong \mathbb{Z}$ since $A \cap B \simeq S^1$.

2. In the case of torus, consider the following.

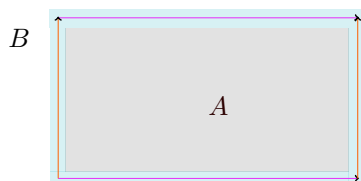


Figure 14: A is the interior, while B is the neighborhood of the boundary.

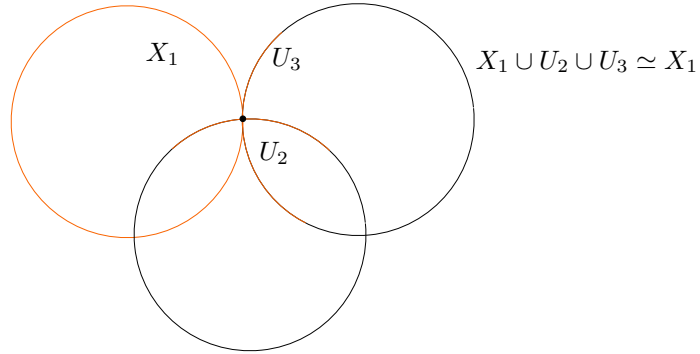
Now note that $A \simeq D^2$ and $B \simeq S^1 \vee S^1$, and since it's a thickening of the two loops around the torus in both ways, this suggests the question of how do we find $\pi_1(B)$? We grab a bit of knowledge from [Seifert Van Kampen Theorem](#) before we continue.

Exercise. Suppose we have [path](#)-connected spaces (X_α, x_α) , and we take their [wedge sum](#) $\bigvee_\alpha X_\alpha$ by identifying the points x_α to a single point x . We also suppose a mild condition for all α , the point x_α is a [deformation retract](#) of some neighborhood of x_α .

For example, this doesn't work if we choose the *bad point* on the Hawaiian earring. Then we can use [Seifert Van Kampen Theorem](#) to show that

$$\pi_1 \left(\bigvee_\alpha X_\alpha, x \right) \cong \ast_\alpha \pi_1 (X_\alpha, x_\alpha).$$

Proof. If we denote



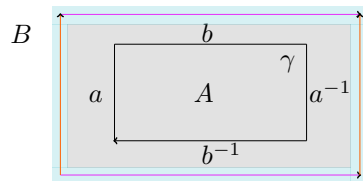
as C_n , then $\pi_1(C_n) \cong F_n$. Then we apply [Theorem 2.7](#) to $A_\alpha = X_\alpha \cup_\beta U_\beta$. Specifically, take $A_\alpha = X_\alpha \cup_\beta U_\beta \simeq X_\alpha$, where U_β is a neighborhood of x_β which [deformation retracts](#) to x_β . This makes A_α open as desired. ■

Corollary 2.2. The [wedge sum](#) of circles $\pi_1(\bigvee_{\alpha \in A} S^1) = *_\alpha \mathbb{Z}$ is a [free group](#) on A . In particular, when A is finite, the [fundamental group](#) of a bouquet of circles is the [free group](#) on $|A|$.

Returning to the [example of torus](#), we see that

- $\pi_1(A) = 0$
- $\pi_1(B) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$
- $\pi_1(A \cap B) = \pi_1(S^1) = \mathbb{Z}$

Further, we know that $\pi_1(A \cap B) \rightarrow \pi_1(A)$ is the zero map. We need to understand $\pi_1(A \cap B) \rightarrow \pi_1(B)$. To do so we need to understand how we're able to identify $\pi_1(S^1 \vee S^1)$ with F_2 and how we identify $\pi_1(S^1)$ with \mathbb{Z} . We update our [Figure 14](#) to talk about this.



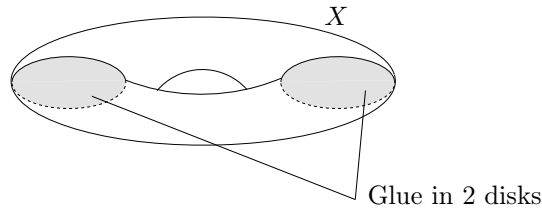
From this, we have

$$\begin{aligned} \pi_1(A \cap B) &\rightarrow \pi_1(B) \cong F_{a,b} \\ \gamma &\mapsto aba^{-1}b^{-1}. \end{aligned}$$

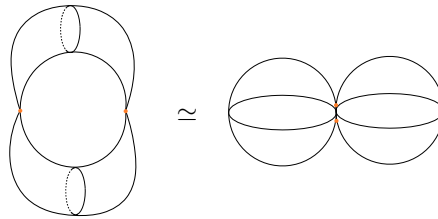
By [Seifert Van Kampen Theorem](#), we identify the image of γ in $\pi_1(B)[aba^{-1}b^{-1}]$ with its image in $\pi_1(A)$, which is just trivial. Therefore, we have

$$\pi_1(T^2) = F_{a,b} / \langle aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2.$$

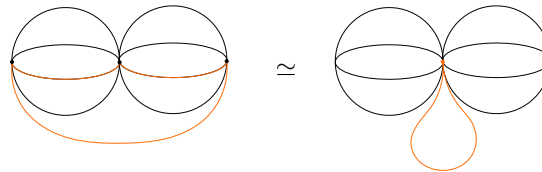
- Let's see the last example which illustrate the power of [Seifert Van Kampen Theorem](#). Start with a torus, and we glue in two disks into the hollow inside.



We'll call this space X , and our goal is to find $\pi_1(X)$. We can place a [CW complex](#) structure on this space so that each disk is a [subcomplex](#). Then, we take quotient of each disk to a point without changing the [homotopy type](#), hence X is [homotopy](#) to



By the same property, we can expand one of those points into an interval, and then contract the red [path](#) as follows.



This is exactly $S^2 \vee S^2 \vee S^1$. With [Seifert Van Kampen Theorem](#), we have

$$\pi_1(X) = \pi_1(S^2 \vee S^2 \vee S^1) = 0 * 0 * \mathbb{Z} \cong \mathbb{Z}.$$

Exercise. Consider $\mathbb{R}^2 \setminus \{x_1, \dots, x_n\}$, that is the plane punctured at n points. Then $X \simeq \bigvee_n S^1$, so then

$$\pi_1(X) \simeq F_n.$$

One way to do this is to convince yourself that you can do a [deformation retract](#) the plane onto the following [wedge](#).



Figure 15: [Deformation retract](#) X onto [wedge](#).

2.7 Group Presentation

In order to go further, we introduce the concept of *group presentation*.

Definition 2.12 (Group presentation). A *presentation* $\langle S \mid R \rangle$ of a group G is

- S : set of *generators*
- R : set of *relators* ([words](#) in a generator and inverses)

such that

$$G \cong F_S / \langle R \rangle,$$

where $\langle R \rangle$ is a subgroup normally generated by the elements of R .

Definition 2.13 (Finite presentation). If S and R are both finite, then $G = \langle S \mid R \rangle$ is a *finite presentation* if S, R are, and we say that G is *finitely presented*.

Note. One way to think about whether G is [finitely presented](#) is that if r is a [word](#) in R then $r = 1$, where 1 is the identity of G .

Example. We see that

1. $F_2 = \langle a, b \mid \rangle$
2. $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid \overline{aba^{-1}b^{-1}} \rangle$
3. $\mathbb{Z}/3\mathbb{Z} = \langle a \mid a^3 \rangle$
4. $S_3 = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$

Theorem 2.8. Any group G has a [presentation](#).

Proof. We first choose a generating set S for G . Notice that we can even choose $S = G$ directly. From the universal property of [free group](#), we see that there exists a surjective map $\varphi: F_S \rightarrow G, s \mapsto s$. Now, let R be the generating set for $\ker(\varphi)$, by the first isomorphism theorem⁷, $G \cong F_S / \ker \varphi$. In fact, we have $G = \langle S \mid R \rangle$.

Specifically, $i: S \rightarrow G$ with $\iota: S \rightarrow F_S$, we have $\varphi \circ \iota = i$.

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F_S \\ & \searrow i & \downarrow \exists! \varphi \\ & & G \end{array}$$

■

Remark. The advantages of using [group presentation](#) are that given $G = \langle S \mid R \rangle$, it's now easy to define a homomorphism $\psi: G \rightarrow H$ given a map $\varphi: S \rightarrow H$, ψ extends to a group homomorphism $G \rightarrow H$ if and only if ψ vanishes on R , i.e., $\psi(r) = 1$ for all $r \in R$. We see an example to illustrate this.

Example. If we have $G = \langle a, b \mid aba \rangle$, a map $\varphi: \{a, b\} \rightarrow H$ gives a group homomorphism if and only if

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) = 1_H.$$

This essentially uses the universal property of quotients.

Remark. It's sometimes easy to calculate G^{Ab}

$$G^{\text{Ab}} = \langle S \mid R, \text{commutators in } S \rangle.$$

Example. Suppose all relations in R are commutators, so $R \subseteq [G, G]$. Then,

$$G^{\text{Ab}} = (F_S)^{\text{Ab}} = \bigoplus_S \mathbb{Z}.$$

Remark. The disadvantages are that this is computationally **very difficult**.

Example. Given $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$, let

$$\psi: \{a, b\} \rightarrow H$$

extends to a homomorphism if and only if

$$\psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} = 1_H \in H.$$

Namely, this is a [presentation](#) of the trivial group, but this is entirely unclear.

Lecture 12: Presentations for π_1 of CW Complexes

2 Feb. 10:00

Let's first see an exercise.

Exercise. Consider $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then we have

- $G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$
- $G_1 \oplus G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{[g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2\} \rangle$
- $G_1 *_H G_2$ where $f_1: H \rightarrow G_1$ and $f_2: H \rightarrow G_2$. Then we have

$$G_1 *_H G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{f_1(h)f_2(h)^{-1} \mid h \in H\} \rangle.$$

2.7.1 Presentations for π_1 of CW Complexes

For X a **CW complex**, we have

1. A 1-dimensional **CW complex** has free π_1 (call its generators as a_1, \dots, a_n).
2. Gluing a 2-disk by its boundary along a word w in the generators *kills* w in π_1 . We then get a **presentation** for $\pi_1(X^2)$ given by

$$\langle a_1, \dots, a_n \mid w \text{ for each 2-cell in } X_2 \rangle.$$

3. Gluing in any higher dimensional cells along their boundary will not change π_1 . That is, in a **CW complex**, we have $\pi_1(X) = \pi_1(X^2)$.

Remark. We can write the above more precise.

1. Find free generators $\{a_i\}_{i \in I}$ for $\pi_1(X^1)$.
2. For each 2-disk D_α^2 , write attaching map as word w_α in a_i . i.e.,

$$\pi_1(X^2) = \langle a_i \mid w_\alpha \rangle.$$

3. $\pi_1(X) = \pi_1(X^2)$.

Example. Given $G = \mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$, then we take a loop and then wind a 2-disk around the loop a for n times.



Figure 16: For $G = \mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$, we wind the boundary around a for n times.

⁷https://en.wikipedia.org/wiki/Isomorphism_theorems

We then see that given a group G with [presentation](#) $\langle S \mid R \rangle$, one can construct a 2-dimensional [CW complex](#) with $\pi_1 = G$ by

- Set $X^1 = \bigvee_{s \in S} S^1$
- For each relation $r \in R$, glue in a 2-disk along loops specified by the [word](#) r .

Every group is then π_1 of some space.

Theorem 2.9. If X is a [CW complex](#) and $\iota_1: X^1 \hookrightarrow X$ and $\iota_2: X^2 \hookrightarrow X$, then $(\iota_1)_*$ surjects onto π_1 and $(\iota_2)_*$ is an isomorphism on π_1 .

Proof.

HW

Definition 2.14 (Graph, subgraph, tree, maximal tree). We import some topological definitions of graph theoretic concepts.

- A *graph* is a 1-dimensional [CW complex](#).
- A *subgraph* is a [subcomplex](#).
- A *tree* is a contractible [graph](#).
- A [tree](#) in [graph](#) X (necessarily a [subgraph](#)) is *maximal* or *spanning* if it contains all the vertices.

Theorem 2.10. Every connected [graph](#) has a [maximal tree](#). Every [tree](#) is contained in a [maximal tree](#).

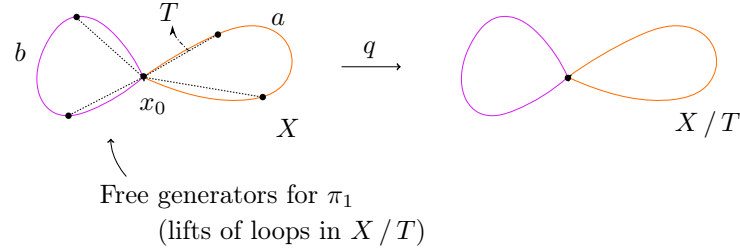
Corollary 2.3. Suppose X is a connected [graph](#) with basepoint x_0 . Then $\pi_1(X, x_0)$ is a [free group](#).

Furthermore, we can give a [presentation](#) for $\pi_1(X, x_0)$ by finding a [spanning tree](#) T in X . The generators of π_1 will be indexed by cells $e_\alpha \in X - T$, and e_α will correspond to a loop that passes through T , traverses e_α once, then returns to the basepoint x_0 through T .

Proof. The idea is simple. X is [homotopy equivalent](#) to X/T via previous work on the homework, T contains all the vertices, so the quotient has a single vertex. Thus, it is a [wedge](#) of circles, and each e_α projects to a loop in X/T .



The current plan is to calculate the **fundamental group** of **CW complexes**. For now, we need to see that the **fundamental group** of a 1-skeleton (a graph) can be found by taking a **maximal tree**, and then quotienting out the space by the **tree** to get a **wedge** of circles.



We now prove that the **maximal trees** exist. Recall that X is a quotient of

$$X^0 \coprod_{\alpha} I_{\alpha}.$$

Each subset U is open if and only if it intersects each edge \bar{e}_{α} in an open subset. A map $X \rightarrow Y$ if and only if its restriction to each edge \bar{e}_{α} is continuous. Now, take X_0 to be a **subgraph**. Our goal is to construct a **subgraph** Y with

- $X_0 \subset Y \subset X$
- Y **deformation retracts** to X_0
- Y contains all vertices of X .

So if we take X_0 to be a vertex, then Y is our **tree** and we're done!

Our strategy now is to build a sequence $X_0 \subset X_1 \subset \dots$ and correspondingly, $Y_0 \subset Y_1 \subset \dots$. We start with X_0 and inductively define

$$X_i := X_{i-1} \cup \text{all edges } \bar{e}_{\alpha} \text{ with one or both vertices in } X_{i-1}.$$

We then see that $X = \bigcup_i X_i$.⁸ Now, let $Y_0 = X_0$. By induction, we'll assume that Y_i is a **subgraph** of X_i such that

- Y_i contains all vertices of X_i .
- Y_i **deformation retracts** to Y_{i-1} .

We can then construct Y_{i+1} by taking Y_i and adding to it one edge to adjoin every vertex of X_{i+1} , namely

$$Y_{i+1} := Y_i \cup \text{one edge to adjoin every vertex of } X_i^9$$

We then see that Y_{i+1} **deformation retracts** to Y_i by just smashing down each edge. Now, we can show that Y **deformation retracts** to $Y_0 = X_0$ by performing the **deformation retraction** from Y_i to Y_{i-1} during the time interval $[1/2^i, 1/2^{i-1}]$. ■

⁸[HPM02] do this by arguing the union on the right is both open and closed.

⁹This is possible if we assume Axiom of Choice.

Example. Let

- S^n : decompose into 2 open disks
- A_1 : neighborhood of top hemisphere
- A_2 : neighborhood of lower hemisphere

We see that $A_1 \cap A_2 \simeq S^{n-1}$, where we need $n \geq 2$ to let S^{n-1} be connected. We then have

$$\pi_1(S^n) \cong 0 \underset{\pi_1(A_1 \cap A_2)}{*} 0 = 0.$$

On the other hand, if $n \geq 3$, then we see that

$$S^n = D^n \cup * / \sim.$$

Since 2-skeleton is a point, thus $\pi_1(S^n) = 0$.

Lecture 13: Proof of Seifert-Van-Kampen Theorem

4 Feb. 10:00

2.8 Proof of Seifert-Van-Kampen Theorem

Let's start to prove Theorem 2.7.

Proof. The outline of the proof is the following. Let $X = \bigcup_{\alpha} A_{\alpha}$ where A_{α} are open, path-connected and contain the bluepoint x_0 . We also must guarantee that $A_{\alpha} \cap A_{\beta}$ is path-connected.

1. Since we have a map induced by the inclusions:

$$\Phi: \underset{\alpha}{*} \pi_1(A_{\alpha}, x_0) \rightarrow \pi_1(X, x_0).$$

We want to show that ϕ is surjective. Take some $\gamma: I \rightarrow X$, then by the compactness of the interval I , we can show that there is a partition I with $s_1 < \dots < s_n$ so that

$$\alpha|_{s_i, s_{i+1}} =: \alpha_i$$

has image in A_{α_i} for some α_i .¹⁰ Specifically, since

- A_{α} is open for all α
- I is compact,

then for all i , we choose a path h_i from x_0 to $\gamma(s_i)$ in $A_{\sigma_{i-1}} \cap A_{\alpha_i}$, using path-connectedness of the pairwise intersections. Now, take γ and write it as

$$\gamma = (\gamma_1 \cdot \bar{h}_1) \cdot (\bar{h}_1 \cdot \gamma_2) \cdot \dots \cdot (\gamma_{n-1} \cdot \bar{h}_{n-1}) \cdot (h_{n-1} \cdot \gamma_n).$$

Observe that each of these paths is fully contained in A_{α_i} , so this implies that $\gamma \in \text{Im}(\Phi)$, therefore Φ is surjective.

¹⁰This is a good exercise for point-set topology.

2. For the next step, we'll show that the second part of [Theorem 2.7](#). Assume that our triple intersections are [path-connected](#). We want to show that $\ker(\Phi)$ is generated by

$$(i_{\alpha\beta})_*(\omega)(i_{\beta\alpha})_*(\omega)^{-1},$$

where

$$i_{\alpha\beta}: A_\alpha \cap A_\beta \hookrightarrow A_\alpha$$

for all loops $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$.

Before we go further, we'll need some definition.

Definition 2.15 (Factorization). A *factorization* of a [homotopy](#) class $[f] \in \pi_1(X, x_0)$ is a formal product

$$[f_1][f_2] \dots [f_\ell]$$

with $[f_i] \in \pi_1(A_\alpha, x_0)$ such that

$$f \simeq f_1 \cdot f_2 \cdot \dots \cdot f_\ell.$$

We showed that every $[f]$ has a [factorization](#) in step 1 already. Now we want to show that two [factorizations](#)

$$[f_1] \cdot \dots \cdot [f_\ell] \text{ and } [f'_1] \cdot \dots \cdot [f'_{\ell'}]$$

of $[f]$ must be related by two moves:

- (a) $[f_i] \cdot [f_{i+1}] = [f_i \cdot f_{i+1}]$ if $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$. Namely, the relation defining the [free product](#) of groups.
- (b) $[f_i]$ can be viewed as an element of $\pi_1(A_\alpha, x_0)$ or $\pi_1(A_\beta, x_0)$ whenever

$$[f_i] \in \pi_1(A_\alpha \cap A_\beta, x_0).$$

This is the relation defining the [amalgamated free product](#).

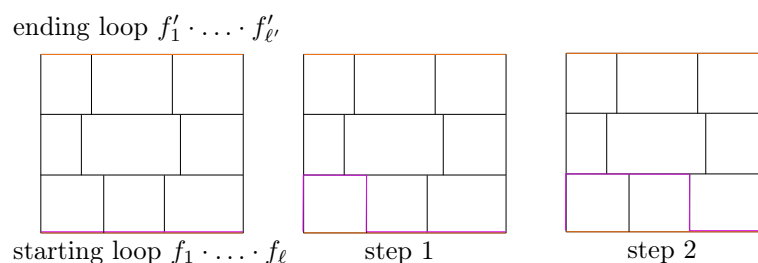
Now, let $F_t: I \times I \rightarrow X$ be a [homotopy](#) from $f_1 \dots f_\ell$ to $f'_1 \dots f'_{\ell'}$, since they both represent $[f]$. We subdivide $I \times I$ into rectangles R_{ij} so that

$$F(R_{ij}) \subseteq A_{\alpha_{ij}} =: A_{ij}$$

for some α_{ij} using compactness. We also argue that we can perturb the corners of the squares so that a corner lies only in three of the A_α 's indexed by adjacent rectangles.

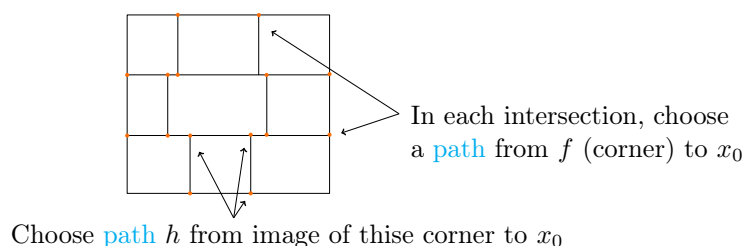
A_{31}	A_{32}	A_{33}
A_{21}	A_{22}	A_{23}
A_{11}	A_{12}	A_{13}

We also argue that we can set up our subdivision so that the partition of the top and bottom intervals must correspond with the two **factorizations** of $[f]$. We then perform our **homotopy** one rectangle at a time.



Idea: Argue that **homotoping** over a single rectangle has the effect of using allowable moves to modify the **factorization**.

At each triple intersection, choose a **path** from f (corner) to x_0 which lies in the triple intersection, so we use the assumption that the triple intersections are **path-connected**.



Along the top and bottom, we make choices compatible with the two **factorizations**. It's now an exercise to check that these choices result in **homotoping** across a rectangle gives a new **factorization** related by an allowable move.

■

Lecture 14: Covering Spaces

7 Feb. 10:00

3 Covering Spaces

3.1 Lifting Properties

As always, we start with a definition.

Definition 3.1 (Covering space). A *covering space* \tilde{X} of X is a space \tilde{X} and a map $p: \tilde{X} \rightarrow X$ such that $\forall x \in X \exists$ neighborhood u_x with $p^{-1}(u_x)$ the disjoint union of open sets

$$\coprod_{\alpha} u_{\alpha}$$

such that

$$p|_{u_{\alpha}} : u_{\alpha} \rightarrow u_x$$

is a homeomorphism for every α .



We sometimes call p as *covering map*.

Although we already investigate into [covering spaces](#) quite a lot in homework, but a terminology is still worth mentioning.

Definition 3.2 (Evenly covered). Let $p: \tilde{X} \rightarrow X$ be a continuous map of spaces. Then an open subset $U \subseteq X$ is called *evenly covered by p* if

$$p|_{V_i} : V_i \rightarrow U$$

is a homeomorphism.

We call the parts V_i of the partition $\coprod_i V_i$ of $p^{-1}(U)$ *slices*.

Remark. We see that p is a [covering map](#) if and only if every point $x \in X$ has a neighborhood which is [evenly covered](#).

We immediately have the following proposition.

Proposition 3.1 (Homotopy lifting property). The covering spaces satisfy the *homotopy lifting property* such that the following diagram commutes.

$$\begin{array}{ccc}
 X \times \{0\} & \xrightarrow{\tilde{F}_0} & \tilde{Y} \\
 \downarrow & \nearrow \exists! \tilde{F}_t & \downarrow p \\
 X \times I & \xrightarrow{F_t} & Y
 \end{array}$$

Proof. We already proved this in homework! ■

Corollary 3.1 (Path lifting property). For each path $\gamma: I \rightarrow X$ in X , $\tilde{x}_0 \in p^{-1}(\gamma(0))$ such that there exists a unique lift $\tilde{\gamma}$ starting at \tilde{x}_0 .

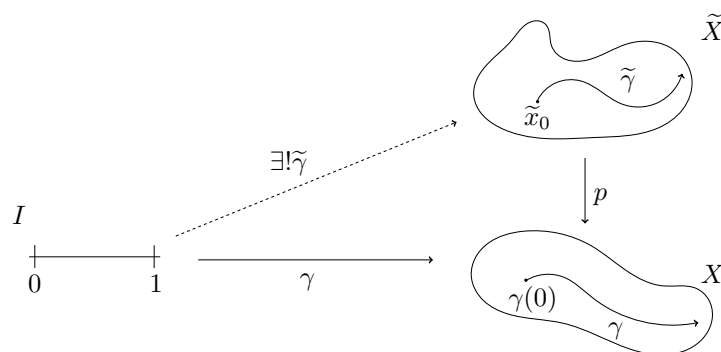
And for each path homotopy $I \times I \rightarrow X$, there exists a unique path homotopy $\tilde{\gamma}: I \times I \rightarrow \tilde{X}$ starting at \tilde{x}_0 .



Though we can directly use Proposition 3.1 to prove this, but we can see some insight by directly proving this.

Proof. We prove them separately.

Lifting a path. Assume that we have the following lift.



We first prove that a path will be lifted uniquely to a path $\tilde{\gamma}$ from \tilde{x}_0 . For every

$x \in X$, there exists an open neighborhood U_x such that

$$p^{-1}(U_x) = \coprod_{\alpha} U_{x_\alpha},$$

where for every α ,

$$p|_{U_{x_\alpha}} : U_{x_\alpha} \rightarrow U_x$$

is a homeomorphism. We see that $\{U_x \mid x \in X\}$ is an open cover of X , hence

$$\{\gamma^{-1}(U_x) \mid x \in X\}$$

is an open cover of $[0, 1]$. Note that since $[0, 1]$ is a compact metric space, from Lebesgue Lemma¹¹, there exists a partition of $[0, 1]$ such that

$$0 = t_0 < t_1 < \dots < t_k = 1$$

such that for every i , $[t_i, t_{i+1}] \subset \gamma^{-1}(U_x)$ for some x . Without loss of generality, we assume that $[t_i, t_{i+1}] \subset \gamma^{-1}(U_{x_i})$, i.e.,

$$\gamma([t_i, t_{i+1}]) \subset U_{x_i}.$$



Now, since $p(\tilde{x}_0) = \gamma(0)$ for $\gamma_0 \in U_{x_1}$ and $\tilde{x}_0 \in p^{-1}(U_{x_1})$, we may assume $\tilde{x}_0 \in U_{x_1\alpha_1}$. Consider **lifting** the first segment, namely $\gamma([0, t_1])$.



¹¹https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma

Specifically, let $\tilde{\gamma}_1(t) = \left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ \gamma(t)$ for $0 \leq t \leq t_1$, we see that

$$\tilde{\gamma}_1: [0, t_1] \rightarrow \tilde{X}$$

is a **lift** of $\gamma|_{[0, t_1]}$ from \tilde{x}_0 . We claim that this **lift** is unique. Consider there exists another **lift** from \tilde{x}_0 $\tilde{\tilde{\gamma}}_1: [0, t_1] \rightarrow \tilde{X}$, then since

- $\tilde{\tilde{\gamma}}_1(0) = \tilde{x}_0$
- $\tilde{\tilde{\gamma}}_1$ is continuous
- $\tilde{x}_0 \in U_{x_1\alpha_1}$,

we see that $\tilde{\tilde{\gamma}}_1([0, t_1]) \subset U_{x_1\alpha_1}$, which implies

$$\begin{array}{ccc} [0, t_1] & \xrightarrow{\tilde{\tilde{\gamma}}_1} & U_{x_1\alpha_1} \\ & \searrow \gamma|_{[0, t_1]} & \downarrow p|_{U_{x_1\alpha_1}} \\ & & U_{x_1} \end{array} \implies \tilde{\tilde{\gamma}}_1 = \left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ \gamma|_{[0, t_1]} = \tilde{\gamma}_1,$$

hence this **lift** is unique. Now, we see that we can simply repeat this argument, namely replacing t_i by t_{i+1} , $\tilde{\gamma}_i(t_i)$ by $\tilde{\gamma}_{i+1}(t_{i+1})$ and so on. Since this partition is finite, hence in finitely many steps, we obtain a unique **path homotopy** $\tilde{\gamma}$ by concatenating all $\tilde{\gamma}_i$ starting at \tilde{x}_0 .

Lifting a path homotopy. We now consider lifting a **path homotopy**. Consider

$$\gamma_1 \underset{\tilde{F}}{\simeq} \gamma_2 \text{ rel } \{0, 1\}$$

we'll show that $\tilde{\gamma}_1 \underset{\tilde{F}}{\simeq} \tilde{\gamma}_2 \text{ rel } \{0, 1\}$ where $p \circ \tilde{F} = F$. Firstly, we denote $x_0 := \gamma_1(0) = \gamma_2(0)$, such that



We claim that it's sufficient to show that there exists a continuous $\tilde{F}: I \times I \rightarrow X$ such that $p \circ \tilde{F} = F$, and $\tilde{F}(\{0\} \times I) = x_0$. It's because

$$p \circ \tilde{F}_0 = F_0 = \gamma_1, \quad p \circ \tilde{F}_1 = F_1 = \gamma_2$$

where \tilde{F}_0, \tilde{F}_1 is γ_1, γ_2 's **lifting** starting at \tilde{x}_0 , respectively. And since $p \circ \tilde{F} = F$, we have

$$p(\tilde{F}(\{1\} \times I)) = x_1 \implies \tilde{F}(\{1\} \times I) \subset p^{-1}(\{x_1\}),$$

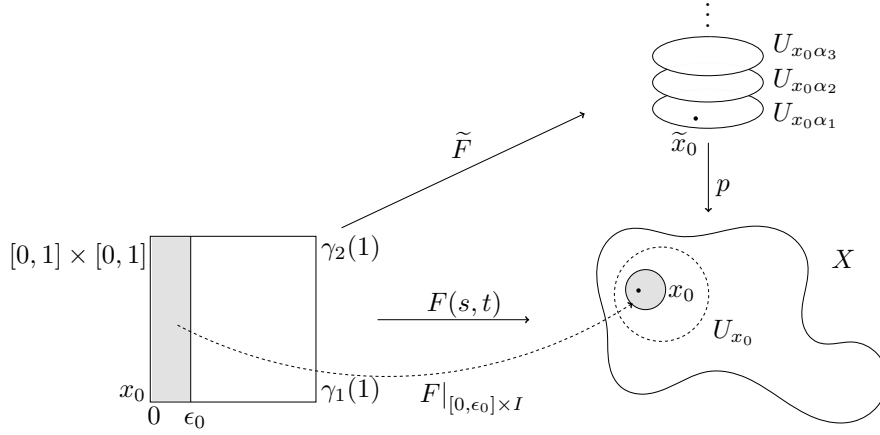
which implies $\exists \tilde{x}_1 \in p^{-1}(\{x_1\})$ such that $\tilde{F}(\{1\} \times I) = \tilde{x}_1$ since we know that $p^{-1}(\{x_1\})$ is a discrete points set and \tilde{F} is assumed to be continuous, and $\{1\} \times I$ is connected. We now show \tilde{F} exists.

We define

$$\begin{aligned} \tilde{F}: I \times I &\rightarrow X \\ (s, t) &\mapsto \tilde{F}_t(s), \end{aligned}$$

where $\tilde{F}_t: [0, 1] \rightarrow \tilde{X}$ is a **lift** starting at \tilde{x}_0 of $F_t: [0, 1] \rightarrow X, s \mapsto F(s, t)$. Obviously, $p \circ \tilde{F} = F$ from the uniqueness of the **lift** of a path, and also, $\tilde{F}(\{0\} \times I) = \tilde{x}_0$ holds trivially, hence we only need to show \tilde{F} is continuous.

1. We show that $\exists \epsilon_0 > 0$ such that $\tilde{F}|_{[0, \epsilon_0] \times I}$ is continuous.



Since F is continuous, we see that there exists an open neighborhood U_{x_0} of x_0 such that $p^{-1}(U_{x_0}) = \coprod_{\alpha} U_{x_0 \alpha}$, where

$$p|_{U_{x_0 \alpha}}: U_{x_0 \alpha} \xrightarrow{\cong} U_{x_0}.$$

Since $F^{-1}(U_{x_0})$ is an open set contain $\{0\} \times I$, there exists a $\epsilon_0 > 0$ such that $[0, \epsilon_0] \times I \subset F^{-1}(U_{x_0})$,¹² which implies

$$F([0, \epsilon_0] \times I) \subset U_{x_0}.$$

¹²Notice that we're working on product topology here.

Note that $x_0 \in U_{x_0}$ and $p(\tilde{x}_0) = x_0$, we may assume $\tilde{x}_0 \in U_{x_0\alpha_1}$. Consider $\left(p|_{U_{x_0\alpha_1}}\right)^{-1} \circ F|_{[0,\epsilon_0] \times I}$, which is a **lift** of $F|_{[0,\epsilon_0] \times I}$. We claim that

$$\left(p|_{U_{x_0\alpha_1}}\right)^{-1} \circ F|_{[0,\epsilon_0] \times I} = \tilde{F}|_{[0,\epsilon_0] \times I}.$$

This is because for every $t \in I$,

$$s \mapsto \left(p|_{U_{x_0\alpha_1}}\right)^{-1} \circ F|_{[0,\epsilon_0] \times I}(s, t)$$

is a **lift** starting at \tilde{x}_0 ; also, for every $t \in I$,

$$s \mapsto \tilde{F}|_{[0,\epsilon_0] \times I}(s, t)$$

is a **lift** of F_t starting at \tilde{x}_0 . From the uniqueness of the **lift** of **paths**, we see that they're equal. Note that this implies \tilde{F} is now continuous at $[0, \epsilon_0] \times I$, since F is continuous and $p|_{U_{x_0\alpha_1}}$ is a homeomorphism, hence continuous, then from

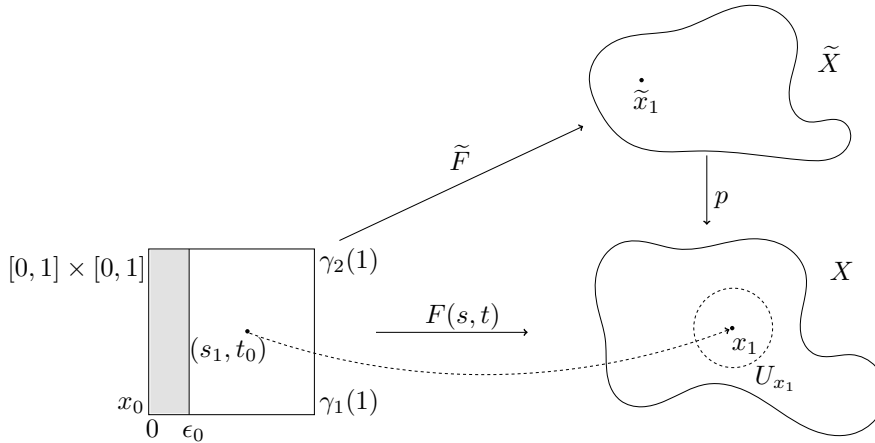
$$\tilde{F}|_{[0,\epsilon_0] \times I} = \underbrace{\left(p|_{U_{x_0\alpha_1}}\right)^{-1}}_{\text{continuous}} \circ \underbrace{F|_{[0,\epsilon_0] \times I}}_{\text{continuous}},$$

we see that \tilde{F} is indeed continuous at $[0, \epsilon_0] \times I$.

2. We now prove that $\tilde{F}: I \times I \rightarrow \tilde{X}$ is continuous. Assume there exists $(s_0, t_0) \in I \times I$ such that \tilde{F} is discontinuous at (s_0, t_0) . Then consider

$$0 < \epsilon_0 \leq \underbrace{\inf \left\{ s \mid \tilde{F} \text{ is discontinuous at } s, t_0 \right\}}_{\exists s_0 \implies \neq \emptyset} =: s_1,$$

where the first inequality is from the first step.

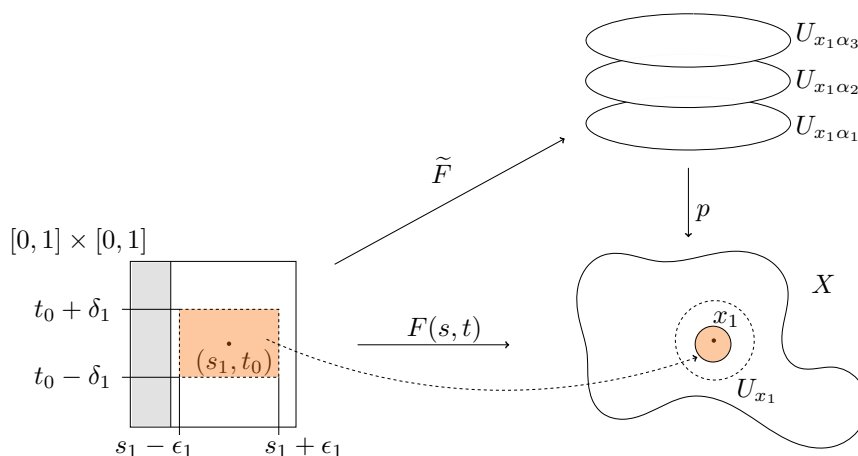


Let $x_1 := F(s_1, t_0)$, $\tilde{x}_1 := \tilde{F}(s_1, t_0)$, then there exists an open neighborhood U_{x_1} in X such that $x_1 \in U_{x_1} = \coprod_{\alpha} U_{x_1\alpha}$, where

$$p|_{U_{x_1\alpha}} : U_{x_1\alpha} \xrightarrow{\cong} U_{x_1}.$$

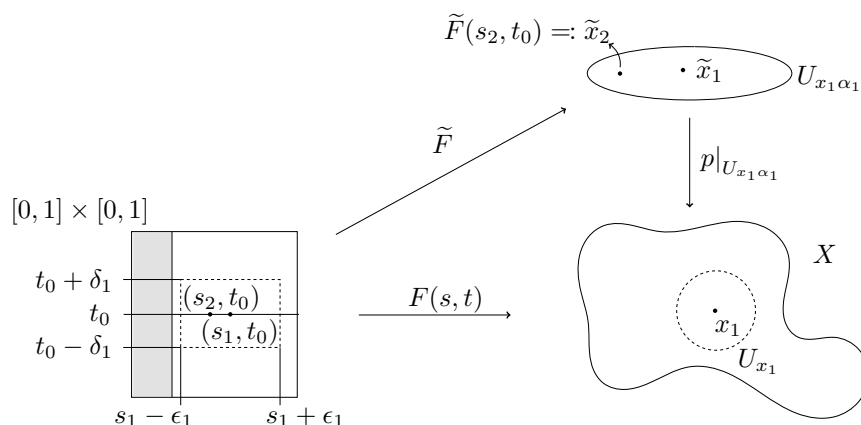
Since F is continuous, there exists an $\epsilon_1 > 0$, $\delta_1 > 0$ such that

$$F((s_1 - \epsilon_1, s_1 + \epsilon_1) \times (t_0 - \delta_1, t_0 + \delta_1))^{13} \subset U_{x_1}.$$



We may assume $\tilde{x}_1 \in U_{x_1\alpha_1}$. Then, we see that \tilde{F}_{t_0} is a **lift** of F_{t_0} , which means \tilde{F}_{t_0} is continuous, hence there exists an s_2 such that $s_1 - \epsilon_1 < s_2 < s_1$ such that

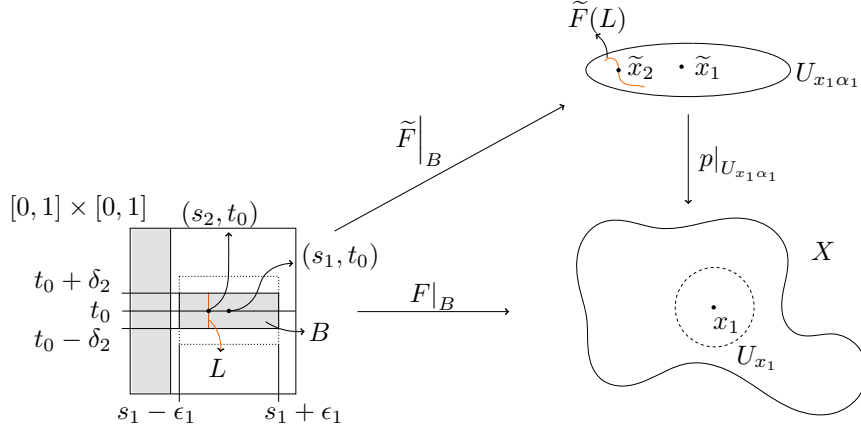
$$\tilde{F}(s_2, t_0) \in U_{x_1\alpha_1}.$$



¹³Notice that here we're considering **open** box.

We see that \tilde{F} is continuous at (s_2, t_0) , hence there exists a $\delta_2 > 0$ such that

$$\tilde{F}(\{s_2\} \times (t_0 - \delta_2, t_0 + \delta_2)) \subset U_{x_1\alpha_1}.^{14}$$



Now, observe that $\tilde{F}(B) \subset U_{x_1\alpha_1}$. To see this, consider a fixed $t \in (t_0 - \delta_2, t_0 + \delta_2)$, then the map \tilde{F} is

$$[s_1 - \epsilon_1, s_1 + \epsilon_1] \rightarrow \tilde{X}, \quad s \mapsto \tilde{F}(s, t) = \tilde{F}_t(s).$$

Specifically,

$$\tilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1]) \subset p^{-1}(U_{x_1}) = \coprod_{\alpha} U_{x_1\alpha},$$

with the fact that $\tilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1])$ is connected, and $\tilde{F}_t(s_2) \in U_{x_1\alpha_1}$ with \tilde{F}_t is a [lift](#) of F_t , hence continuous, so

$$\tilde{F}_t([s_1 - \epsilon_1, s_1 + \epsilon_1]) \subset U_{x_1\alpha_1}.$$

This is true for every $t \in [t_0 - \delta_2, t_0 + \delta_2]$, hence $\tilde{F}|_B \subset U_{x_1\alpha_1}$. Now, since

$$p|_{U_{x_1\alpha_1}} \circ \tilde{F}|_B = F|_B,$$

and

$$\left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ F|_B : B \rightarrow U_{x_1\alpha_1},$$

so

$$p|_{U_{x_1\alpha_1}} \circ \left(\left(p|_{U_{x_1\alpha_1}}\right)^{-1} \circ F|_B\right) = F|_B$$

¹⁴Note that here we can also consider a closed interval, which matches what we're going to do. Namely, we're going to construct a **closed** box B . But this is just a technical detail.

obviously. Since $p|_{U_{x_1\alpha_1}}$ is a homeomorphism, we have

$$\tilde{F}|_B = \underbrace{\left(p|_{U_{x_1\alpha_1}}\right)^{-1}}_{\text{continuous}} \circ \underbrace{F|_B}_{\text{continuous}},$$

hence we have $\tilde{F}|_B$ is continuous, which leads to a contradiction since

$$s_1 = \inf \left\{ s \mid \tilde{F} \text{ is discontinuous at } s, t_0 \right\},$$

while \tilde{F} is continuous for all B , hence we see that $\tilde{F} : I \times I \rightarrow \tilde{X}$ is continuous.¹⁵

■

Example. Let see some examples.

1. Covers of $S^1 \vee S^1$.



Note that in each cover (those three on the top), the black dot is the preimage of $\{x_0\}$, namely $p_i^{-1}(\{x_0\})$.

Remark. We see that for each $p_i^{-1}(\{x_0\})$, there are exactly

- one a edge goes out
- one b edge goes out
- one a edge goes in
- one b edge goes in

It turns out that there are much more covers of $S^1 \vee S^1$, as long as this main property is satisfied.

¹⁵There is a tricky situation, namely while $s_1 = 1$. But this can be considered also.

Proposition 3.2. Let

$$p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a **covering map**. Then

1. $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is injective.
2. $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$, which picks out the subset

$$\{[\gamma] \mid \text{Lift } \tilde{\gamma} \text{ starting at } \tilde{x}_0 \text{ is a loop.}\}.$$

Proof. We prove this one by one.

1. Suppose $\tilde{\gamma} \in \pi_1(\tilde{X}, \tilde{x}_0)$ is in $\ker(p_*)$. Then

$$[\gamma] = p_*([\tilde{\gamma}]) = [p \circ \tilde{\gamma}].$$

Let γ_t be a **nullhomotopy** from γ to the constant loop $c_{x_0} \text{ rel}\{0, 1\}$. We can then **lift** γ_t to $\tilde{\gamma}_t$ where $\tilde{\gamma}_0 = \tilde{\gamma}$. Now, we claim that

- $\tilde{\gamma}$ is a **homotopy rel** $\{0, 1\}$.
- $\tilde{\gamma}_1$ is the constant loop $c_{\tilde{x}_0}$.

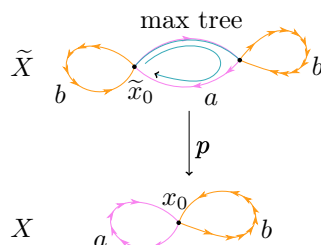
$$\begin{array}{ccc} & \tilde{X} & \\ \tilde{\gamma} \nearrow & \downarrow p & \\ I & \xrightarrow{\gamma} & X \end{array} \quad \begin{array}{ccc} & \tilde{X} & \\ \tilde{\gamma}_t \nearrow & \downarrow p & \\ I \times I & \xrightarrow{\gamma_t} & X \end{array}$$

We see that the above diagrams prove the first claim, since we know that the left and right edge of $I \times I$ maps to x_0 under γ_t , and $c_{\tilde{x}_0}$ **lifts** this, so by uniqueness $t \mapsto \tilde{\gamma}_t(0)$ and $t \mapsto \tilde{\gamma}_t(1)$ must be constant **paths** at \tilde{x}_0 as desired.

Then the **lift** $\tilde{\gamma}_1$ is a **homotopy of paths** to the constant loop, so $[\tilde{\gamma}] = 1$.

2. Let see an example to show the idea of the proof.

Example. Given



Then

$$p_*\pi_1 = \langle b, a^2, ab\bar{a} \rangle \subseteq \pi_1(X) = \langle a, b \mid \rangle.$$



Proposition 3.3 (Lifting criterion). Let $p: (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$ be a **covering map**. Given

- $f: (X, x_0) \rightarrow (Y, y_0)$;
- X is **path-connected**, locally **path-connected**,

then a **lift**

$$\tilde{f}: (X, x_0) \rightarrow (\tilde{Y}, \tilde{y}_0)$$

exists if and only if

$$f_* (\pi_1(X, x_0)) \subseteq p_*(\pi_1(\tilde{Y}, \tilde{y}_0)).$$

In diagram, we have

$$\begin{array}{ccc} & (\tilde{Y}, \tilde{y}_0) & \\ \nearrow \exists \tilde{f} & \downarrow p & \\ (X, x_0) & \xrightarrow{f} & (Y, y_0) \end{array} \quad \begin{array}{ccc} & \pi_1(\tilde{Y}, \tilde{y}_0) & \\ \nearrow \tilde{f}_* & \downarrow p_* & \\ \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \end{array}$$

Lecture 15: Lifting

9 Feb. 10:00

Before proving **Proposition 3.3**, we first see an application.

Example. Prove that every continuous map $f: \mathbb{R}P^2 \rightarrow S^1$ is **nullhomotopic**.

Answer. If we can show that there is a **lift** $\tilde{f}: \mathbb{R}P^2 \rightarrow \mathbb{R}$ of f , then we're done since we can apply the **straight line nullhomotopy** on \mathbb{R} , i.e.,

$$\begin{array}{ccc} & \mathbb{R} & \\ \nearrow \tilde{f} & \downarrow p & \\ \mathbb{R}P^2 & \xrightarrow{f} & S^1 \end{array}$$

and consider $f = p \circ \tilde{f}$ compose **nullhomotopy** with p , so $f \simeq$ constant map. Specifically, since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(S^1) = \mathbb{Z}$, hence

$$f_*(\pi_1(\mathbb{R}P^2)) = 0$$

since \mathbb{Z} has no (nonzero) torsion. So it **lifts** by **Proposition 3.3**.

Now we can proof **Proposition 3.3**.

Proof. We prove two directions as follows.

Necessary. We see that we can **factorize** f_* as

$$f_* = p_* \circ \tilde{f}_*$$

follows from the **functoriality** of π_1 .

Sufficient. Let $x \in X$. Choose a path γ from x_0 to x by the assumption that X is path-connected. Then, $f\gamma$ has a unique lift starting at \tilde{y}_0 , denote by $\tilde{f}\gamma$. Now, define

$$\tilde{f}(x) = \tilde{f}\gamma(1).$$

Then, we need to check

1. \tilde{f} is well-defined. Suppose γ, γ' are paths in X from x_0 to x . We want to show

$$\tilde{f}\gamma'(1) = \tilde{f}\gamma(1).$$

Since $\gamma \cdot \overline{\gamma'}$ is a loop in X at x_0 , we know that $[(f\gamma) \cdot (f\overline{\gamma'})]$ is a class of loops in Y in $\text{Im}(f_*)$. By hypothesis, this class of loops is in $\text{Im}(p_*)$. It lifts to a loop which is based at \tilde{y}_0 . By uniqueness of lifts, this loop lifting $(f\gamma) \cdot (f\overline{\gamma'})$ to \tilde{Y} must be equal to the lifts $\tilde{f}\gamma \cdot \overline{\tilde{f}\gamma'}$ with a common value at $t = 1/2$. Hence, $\tilde{f}\gamma(1) = \tilde{f}\gamma'(1)$ as desired, namely the endpoints agree.



Lecture 16: Proving Proposition 3.3

11 Feb. 10:00

2. \tilde{f} is continuous. Choose $x \in X$ and a neighborhood \tilde{U} of $\tilde{f}(x)$ in \tilde{Y} . Note that we can choose \tilde{U} small enough to $p|_{\tilde{U}}$ is homeomorphism to U in Y . Now, there exists a neighborhood V of x in X with $f(V) \subseteq U$.



The goal is $\tilde{f}(V) \subseteq \tilde{U}$. Without loss of generality, we can assume that V is path-connected. Then,

$$\widetilde{f\gamma} \cdot \widetilde{f\alpha} = [\widetilde{f\gamma \cdot f\alpha}].$$

Hence,

$$\widetilde{f\alpha} = (p|_{\tilde{U}})^{-1} \circ f \circ \alpha,$$

where $(p|_{\tilde{U}})^{-1}$'s image is in \tilde{U} , so

$$\tilde{f}(x') = f\gamma \cdot f\alpha(1) \in \tilde{U},$$

which implies

$$\tilde{f}(V) \subseteq \tilde{U}.$$

■

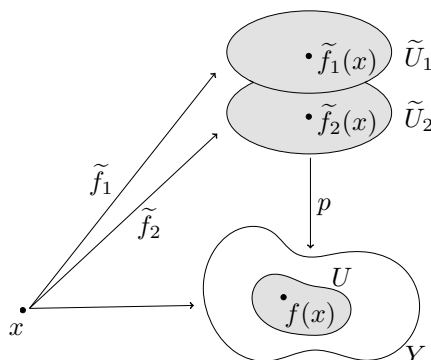
Proposition 3.4 (Uniqueness of lifts). Let $p: \tilde{Y} \rightarrow Y$ be a covering map with X is a connected space. If two lifts \tilde{f}_1, \tilde{f}_2 of the same map f agree at a single point, then they agree everywhere.



Proof. Let S being

$$S := \{x \in X \mid \tilde{f}_1(x) = \tilde{f}_2(x)\}.$$

We want to show that S is both closed and open, so if S is nonempty, $S = X$.



We see that \tilde{U}_1 and \tilde{U}_2 are slices of $p^{-1}(U)$, where U is evenly covered neighborhood of $f(x)$.

1. If $\tilde{f}_1(x) \neq \tilde{f}_2(x)$. Then \tilde{U}_1, \tilde{U}_2 are disjoint. Since \tilde{f}_1, \tilde{f}_2 are continuous, there exists a neighborhood N of x with

$$\tilde{f}_1(N) \subseteq \tilde{U}_1, \quad \tilde{f}_2(N) \subseteq \tilde{U}_2,$$

with the fact that they're disjoint, so x is an interior point of S^c .

2. If $\tilde{f}_1(x) = \tilde{f}_2(x)$. Then $\tilde{U}_1 = \tilde{U}_2$. Choose N as before, then we have

$$\tilde{f}_1(n) = (p|_{\tilde{U}_1})^{-1}(f(n)) = \tilde{f}_2(n),$$

hence $x \in \text{int}(S)$.

■

3.2 Deck Transformation

We now want to introduce a special kind of transformation.

Definition 3.3 (Isomorphism of covers). Given [covering maps](#)

$$p_1: \tilde{X}_1 \rightarrow X, \quad p_2: \tilde{X}_2 \rightarrow X,$$

an *isomorphism of covers* is a homeomorphism

$$f: \tilde{X}_1 \rightarrow \tilde{X}_2$$

such that $p_1 = p_2 \circ f$.

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{f} & \tilde{X}_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

Exercise. This defines equivalent relation on [covers](#) of X .

Definition 3.4 (Deck transformation). Given a [covering map](#) $p: \tilde{X} \rightarrow X$, the [isomorphisms of covers](#) $\tilde{X} \rightarrow \tilde{X}$ are called *deck transformation*.

Furthermore, we'll let $G(\tilde{X})$ denotes the *set of deck transformations*.

Note. Note that we've suppressed the data of p in the notation, but this data is essential to what a [deck transformation](#) is, when this is unclear we write $G(\tilde{X}, p)$.

Lecture 17: Deck Transformation

14 Feb. 10:00

Example. Let's see some examples.

1. [Deck transformations](#) $G(\tilde{X})$ are a subgroup of the group of homeomorphisms of \tilde{X} .

2. Given the **cover** $p: \mathbb{R} \rightarrow S^1$.
 - **Deck maps**: translation by $n \in \mathbb{Z}$ units.
 - $G(\mathbb{R}) \cong \mathbb{Z}$
3. Given the **cover** $p_n: S^1 \rightarrow S^1$ be an n -sheeted cover.
 - **Deck maps**: rotation by $2\pi/n$.
 - $G(S^1, p_n) \cong \mathbb{Z} / n\mathbb{Z}$



Figure 17: $p_n: S^1 \rightarrow S^1$ be an n -sheeted **cover**, where $n = 3$.

Exercise (Deck Transformation is determined by the image of one point). Given X, \tilde{X} are **path**-connected, locally **path**-connected, **deck map** is determined by the image of any one point.

Answer.

$$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow f & \downarrow p \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

Corollary 3.2. If a **deck transformation** has a fixed point, it is the identity transformation.

Exercise. Let X be connected. Given a **deck transformation** $\tau: \tilde{X} \rightarrow \tilde{X}$, τ defines a permutation of $p^{-1}(\{x_0\})$. If this permutation has a fixed point, then it is the identity.

Definition 3.5 (Regular (normal) cover). A **covering space** $p: \tilde{X} \rightarrow X$ is *regular* or *normal* if $\forall x_0 \in X, \forall \tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$, there exists a **deck transformation** such that

$$\tilde{x}_0 \mapsto \tilde{x}_1.$$



Figure 18: Covers of $S^1 \vee S^1$. The left one is **regular**, while the right one is not since there is no automorphism from \tilde{x}_0 to \tilde{x}_1 or \tilde{x}_2 .

Remark. A **regular cover** is *as symmetric as possible*.

Exercise. **Regular** means that the group $G(\tilde{X})$ acts transitively on $p^{-1}(\{x_0\})$. Explain why we cannot ask for more than this:

$G(\tilde{X})$ cannot induce the full symmetric group on $p^{-1}(\{x_0\})$ provided that $|p^{-1}(\{x_0\})| > 2$.

Answer. The key is uniqueness.

Definition 3.6 (Normalizer). Given G as a group, $H \subseteq G$ is a subgroup of G . Then the *normalizer* of H , denoted by $N(H)$, is defined as

$$N(H) := \{g \in G \mid gH = Hg\}.$$

Exercise. We can prove the followings.

1. $N(H)$ is a subgroup.
2. $H \leq N(H)$.
3. H is normal in $N(H)$.
4. If $H \leq G$ is normal, $N(H) = G$.
5. $N(H)$ is the largest subgroup (under containment) of G containing H as normal subgroup.

Proposition 3.5. Given $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a **cover**, and \tilde{X}, X are **path**-connected, locally **path**-connected. Let

$$H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0).$$

Then

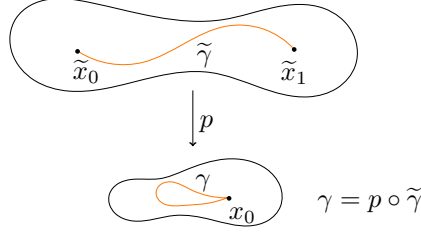
1. p is **normal** if and only if $H \subseteq \pi_1(X, x_0)$ is **normal**.
2. We have

$$G(\tilde{X}) \cong N(H) / H,$$

where $G(\tilde{X})$ are **deck maps**, and $N(H)$ is the **normalizer** of H in $\pi_1(X, x_0)$.

Remark. A fact is worth noting is the following. Let $\tilde{\gamma}$ be a path \tilde{x}_0 to \tilde{x}_1 . Then

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = [\gamma] p_*(\pi_1(\tilde{X}, \tilde{x}_1)) [\gamma^{-1}].$$



Lecture 18: Proving Proposition 3.5

16 Feb. 10:00

Now let's prove Proposition 3.5

Proof. Let X, x_0 be the base space and $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$ where $p: \tilde{X} \rightarrow X$ is a covering map. Further, let $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

In homework, given $(X, x_0), \tilde{x}_0, \tilde{x}_1 \in p^{-1}(\{x_0\})$ if we change the basepoint from $\pi_1(\tilde{X}, \tilde{x}_0)$ to $\pi_1(\tilde{X}, \tilde{x}_1)$, then we have the induced subgroups of the base spaces fundamental group are conjugate by some loop $[\gamma] \in \pi_1(X, x_0)$, i.e.,

$$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = [\gamma] \cdot p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cdot [\gamma]^{-1}$$

where γ is lifted to a path from \tilde{x}_0 to \tilde{x}_1 .

Therefore, $[\gamma] \in N(H)$ if and only if $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, and this holds if and only if there is a deck transformation taking \tilde{x}_0 to \tilde{x}_1 by the classification of based covering spaces in the homework.¹⁶ This shows that p is a normal cover if and only if H is normal, which proves the first claim.

We then define a map Φ such that

$$\Phi: N(H) \rightarrow G(\tilde{X})[\gamma], \quad \cdot \mapsto \tau$$

where τ lifts to a path from \tilde{x}_0 to \tilde{x}_1 and τ is a deck transformation mapping \tilde{x}_0 to \tilde{x}_1 , which will be uniquely defined by the uniqueness of lifts with specified base points. We now need to check

1. Φ is surjective.
2. $\ker(\Phi) = H$.
3. Φ is a group homomorphism.

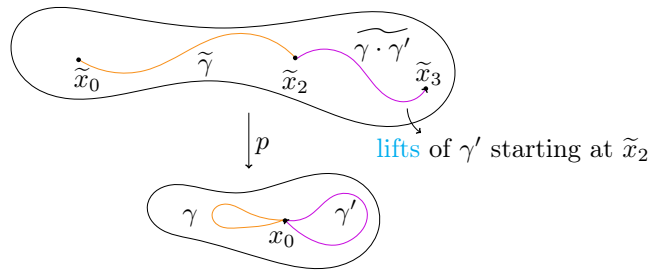
If we can prove all the above, then, from the result follows directly from the first isomorphism theorem.¹⁷

¹⁶Alternatively, we can use the lifting criterion.

¹⁷https://en.wikipedia.org/wiki/Isomorphism_theorems

1. We've proved that Φ is surjective before in our work above.
2. $\Phi([\gamma])$ is the identity if and only if τ sends \tilde{x}_0 to \tilde{x}_0 , meaning that $[\gamma]$ **lifts** to a loop. Then by our characterization of the **fundamental group** downstairs:

$$\ker(\Phi) = \{[\gamma] \mid [\gamma] \text{ lifts to a loop}\} = H.$$
3. Suppose we have loops $[\gamma_1] \xrightarrow{\Phi} \tau_1$ and $[\gamma_2] \xrightarrow{\Phi} \tau_2$. We claim that $\gamma_1 \cdot \gamma_2$ **lifts** to $\tilde{\gamma}_1 \cdot \tau(\tilde{\gamma}_2)$.



It's an exercise to check that the **lift** of γ_2 starting at \tilde{x}_1 is exactly $\phi_1(\tilde{\gamma}_2)$, where $\tilde{\gamma}_2$ is a **lift** starting at \tilde{x}_0 .

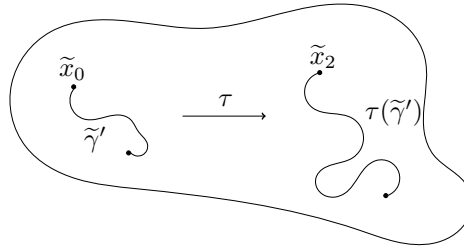


Figure 19: Must be **lift** of γ' starting at \tilde{x}_2

The idea is that by uniqueness of **lifts** we'll have the desired claim. We then just observe that this **path** $\tilde{\gamma}_1 \cdot \tau_1(\tilde{\gamma}_2)$ is a **path** from \tilde{x}_0 to $\tau_1(\tilde{\gamma}_2(1)) = \tau_1(\tau_2(\tilde{x}_0))$, so the image must be a **deck transformation** sending \tilde{x}_0 to $\tau_1(\tau_2(\tilde{x}_0))$. But then $\tau_1 \circ \tau_2$ maps \tilde{x}_0 to this same point, and from **this exercise**, we know that the **deck transformations** are determined by where they send a single point, hence we're done.

■

Corollary 3.3. If p is a **normal covering**, then $G(\tilde{X}) \cong \pi_1(X, x_0) / H$.

Definition 3.7 (Universal covering). A [cover](#) $p: \tilde{X} \rightarrow X$ is called a *universal covering* if \tilde{X} is simply connected.

Corollary 3.4. If \tilde{X} is the [universal cover](#), then $G(\tilde{X}) \cong \pi_1(X, x_0)$.

Exercise. Whether $\text{Im}(p_*)$ is normal is independent of the basepoint in \tilde{X} and X .

So, p is normal if and only if $G(\tilde{X})$ is transitive on $p^{-1}(x_0)$ for at least one $x_0 \in X$.

Exercise. Let Σg be the genus g surface. Prove that Σg has a normal n -sheeted [path-connected cover](#) for every n .

Lecture 19: Simplex

18 Feb. 10:00

4 Homology

4.1 Motivation for Homology

Informally, the higher [homotopy](#) groups is defined as

$$\pi_n(X, x_0): I^n \rightarrow (X, x_0), \quad \partial I^n \mapsto x_0.$$



We see that it's extremely hard to compute higher [fundamental group](#). Hence instead, we will study the higher dimensional structure of X via *homology*.

- **Cons.**
 - The definition is more opaque at first encounter.
- **Pros.**
 - Lots of computational tools
 - Functional
 - Abelian Groups

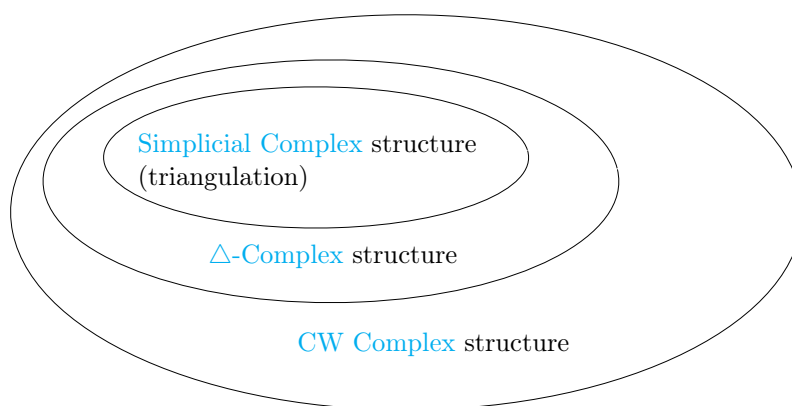
Remark. More like π_n for $n > 1$.

- No basepoints
- Can compute using **CW** structure.
- Good properties. For example, $H_n = 0$ if $n > \dim X$

4.2 Simplicial Homology

4.2.1 Δ -Simplex

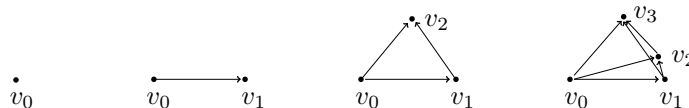
This is a stricter version of a **CW complex** which allows us to decompose our spaces into cells. In terms of how things fit together, we have the following diagram.



Now we try to give the definition.

Definition 4.1 (Simplex). We see that

- *0-simplex.* A point.
- *1-simplex.* Interval.
- *2-simplex.* Triangle.
- *3-simplex.* Tetrahedron.
- *n -simplex.* The convex hull of $(n + 1)$ -points position in \mathbb{R}^n .



Remark. We see that

- The top of which is the 2-disk and remember cell structure (edges and vertices) and remember orientation (ordering on vertices).
- The top of which is the 3-disk and cells and the orientation.

Further,

- We can view [simplices](#) as both *combinatorial* and *topological* objects.

An alternative definition can be done.

Definition 4.2 (Standard simplex). We say that an n -dimensional *standard simplex*, denoted by Δ^n is

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1 \right\}.$$

We'll call such a simplex as *standard n -simplex*.



Remark. In our definition, the [simplices](#) will implicitly come with a choice of ordering of the vertices as

$$\Delta^n = [v_0, v_1, \dots, v_n]$$

such that the convex hull of these points is taken with this ordering.

Lecture 20: Simplicial Complex

21 Feb. 10:00

Definition 4.3 (Subsimplex). A *subsimplex* of a [simplex](#) σ combinatorially, it's a subset of the vertices; while topologically, it's the convex hull of the subset of vertices.



Definition 4.4 (Face). A *face* of a [simplex](#) Δ^n is a [subsimplex](#) of 1 dimensional lower than Δ^n (codimension 1).

Definition 4.5 (Boundary). The *boundary* $\partial\sigma$ of a **simplex** σ is the union of its **faces**.

Definition 4.6 (Open simplex). The *open simplex* of Δ is defined as

$$\mathring{\Delta}^n := \Delta^n - \partial\Delta^n.$$

Definition 4.7 (Δ -Complex). A Δ -*complex* structure on X is a collection of maps

$$\sigma_\alpha: \Delta^n \rightarrow X$$

such that

1. $\sigma_\alpha|_{\mathring{\Delta}^n}$ injective, each point of X is in the image of exactly one such map.
2. Each restriction of σ_α to a **face** coincides with a map

$$\sigma_\beta: \Delta^{n-1} \rightarrow X.$$

3. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in $\mathring{\Delta}^n$ for all σ_α , i.e., X is a quotient

$$\coprod_{n,\alpha} \Delta_\alpha^n \xrightarrow{\coprod \sigma_\alpha} X.$$

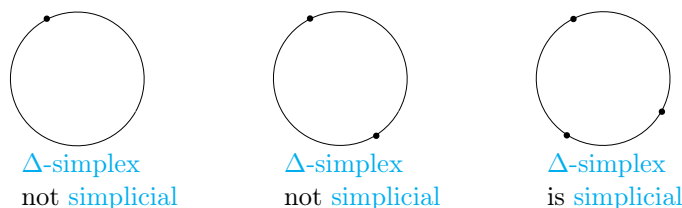
Exercise. A Δ -**complex** X is a **CW complex** W with characteristic maps σ_α with extra constraints on the attaching maps.

Note. We see that the second condition of **Definition 4.7** implies that attaching maps injective on interior of **faces**.

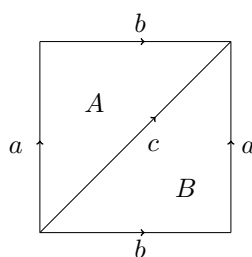
Definition 4.8 (Simplicial complex). A *simplicial complex* is a Δ -**complex** such that

- σ_α must map every **face** to a different $(n-1)$ -**simplex**.
- Every **simplex** is uniquely determined by its vertex set.
- Any $(n+1)$ vertices in X^0 is the vertex set of at most 1 n -**simplex**.

Remark. With **Definition 4.8**, we see the followings.



Example. The torus with the following edges, a, b, c and the gluing in triangles A and B can be seen as follows.



For this Δ -complex, notice that we've glued down a triangle whose vertices are all identified. This is not allowed in a **simplicial complex** / triangulation.

Remark. The minimum number of triangles in a **simplicial complex** structure is 14.

Lecture 21: Simplicial Homology

23 Feb. 10:00

To demonstrate how the definition of homology arise, we first see the idea behind it. Fix a space X which equips with the Δ -complex structure. Then, we define $C_n(X)$ to be the **free Abelian group** on the n -simplices of X . That is,

$$C_n(X) = \left\{ \text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha: \Delta^n \rightarrow X \right\}.$$

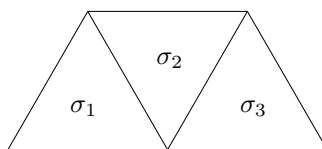
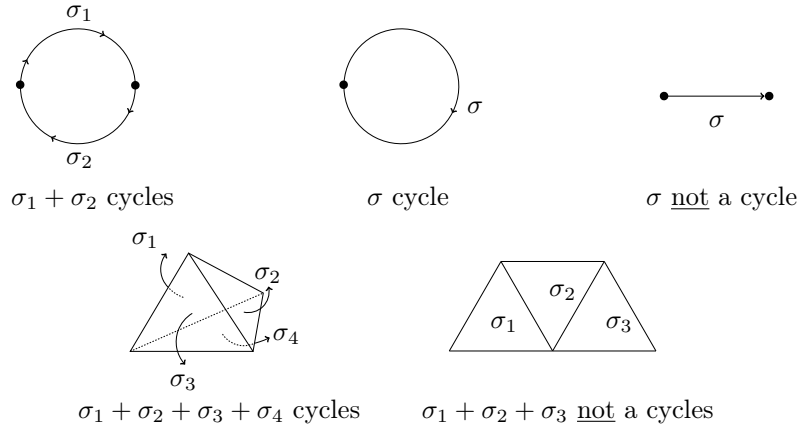


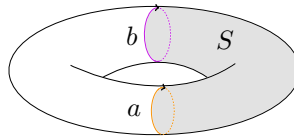
Figure 20: $C_2(X) = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \oplus \mathbb{Z}\sigma_3$.

Then, the n -th homology group will be a subquotient of $C_n(X)$, where the heuristic/imprecise idea is

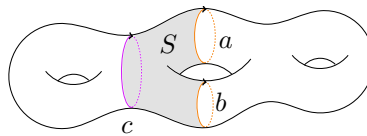
- Take subgroup of C_n of *cycles*. These are sums of **simplices** satisfying a combinatorial condition on the boundary gluing maps to ensure that they *close up*, i.e., they have no **boundary**.



- To take the quotient, we consider two cycles to be equivalent if their difference is a **boundary**. For example, in the case of torus, a is homologous to b since $a - b$ is the **boundary** of the shaded subsurface S on of the torus below.



In fact, a and b are **homotopic** (which will imply they're homologous essentially), but two loops do not need to be **homotopic** to be homologous. For example, in the figure below, $a + b$ is homologous to c , since $a + b - c$ is the **boundary** of S ($a + b$ ¹⁸ and c are not **homotopic**).



Let's now see the formal definition.

¹⁸Which isn't even a loop

Definition 4.9 (Simplicial chain group). We define the *simplicial chain group* $C_n(X)$ of order n to be the **free Abelian group** on the n -simplices of X such that

$$C_n(X) := \left\{ \text{finite sums } \sum m_\alpha \sigma_\alpha \mid m_\alpha \in \mathbb{Z}, \sigma_\alpha: \Delta^n \rightarrow X \right\}.$$

Definition 4.10 (Cycles). Given any chain group $C_n(X)$, a *cycle* of $C_n(X)$ is those chains $\sum m_\alpha \sigma_\alpha$ with no **boundaries**.

Definition 4.11 (Boundary homomorphism). A map $\partial_n: C_n(X) \rightarrow C_{n-1}(X)$ is called a *boundary homomorphism* such that

$$\begin{aligned} \partial_n: C_n(X) &\rightarrow C_{n-1}(X) \\ [\sigma_\alpha] &\mapsto \sum_{i=1}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}, \end{aligned}$$

which defines the map on the basis, and we extend it linearly.

Remark. We see that the definition of **boundary homomorphism** indeed coincides with the definition of **boundary** when considering either **Δ -complex** or **simplicial complex** structure.

Example. We give some lower dimensions examples of **Definition 4.11** to motivate the general definition.

- For $n = 1$, $\partial_1: C_1(X) \rightarrow C_0(X)$ such that

$$[\sigma_\alpha: [v_0, v_1] \rightarrow X] \mapsto \sigma_\alpha|_{[v_1]} - \sigma_\alpha|_{[v_0]}.$$

- For $n = 2$, $\partial_2: C_2(X) \rightarrow C_1(X)$ such that

$$[\sigma_\alpha: [v_0, v_1, v_2] \rightarrow X] \mapsto \sigma_\alpha|_{[v_1, v_2]} - \sigma_\alpha|_{[v_0, v_2]} + \sigma_\alpha|_{[v_0, v_1]}.$$

Lemma 4.1. For any $n \geq 2$, we have

$$\begin{array}{ccccc} C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \xrightarrow{\partial_{n-1}} & C_{n-2}(X) \\ & \searrow & \xrightarrow{\partial_{n-1} \circ \partial_n = 0} & & \end{array}$$

Proof. Since all C_i are **free Abelian group**, hence we only need to consider $\partial_{n-1} \circ \partial_n(\sigma) = 0$ for a generator σ . Given a generator σ , the result follows from directly applying the **definition** and with some calculation. ■

Definition 4.12 (Chain complex). A *chain complex* (C_*, d_*) is a collection of maps such that

$$\dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

of **Abelian groups** and group homomorphism such that

$$d_{n-1} \circ d_n = 0.$$

We call C_n the *n-th chain group* and d_n the *n-th differential*.

Note. Note that **Definition 4.12** is purely *abstract*, namely we can put different chain group structure on C_n . We'll see what this means later.¹⁹ But for now, C_n can be equipped with the definition we gave for **simplicial chain group**.

Remark. We see that

- **Lemma 4.1** guarantees that our **simplicial chain groups** form a **chain complex**.
- **Definition 4.12** means that $\ker(d_n)$ contains $\text{Im}(d_{n+1})$, since $d_n \circ d_{n+1} = 0$.

Definition 4.13 (Exact). We say that the sequence is *exact at C_n* provided that $\ker(d_n) = \text{Im}(d_{n+1})$. A **chain complex** is *exact* if it is *exact at each point*.

Definition 4.14 (Homology group). The n^{th} *homology group* of a chain complex (C_*, d_*) , denoted as H_n or $H_n(C_*)$, is the quotient

$$H_n := \ker(d_n) / \text{Im}(d_{n+1}).$$

Remark. The **homology group** measures how far the **chain complex** is from being **exact** at C_n .

With what we have just defined, it's natural to define **homology groups** of space X with a **Δ -complex** structure.

¹⁹Spoiler: It just means we can give different definition about the map σ .

Definition 4.15 (Homology class). We say $\ker(\partial_n)$ is the subgroup of **cycles** in $C_n(X)$, and $\text{Im}(\partial_{n+1})$ is the subgroup of **boundaries** in $C_n(X)$. We then set

$$H_n(X) := \ker(\partial_n) / \text{Im}(\partial_{n+1}) = \{\text{cycles}\} / \{\text{boundaries}\}.$$

In other words, it's the **homology** of the **chain complex**

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

where we take it to be 0 in all negative indices, namely

$$\dots \xrightarrow{\partial_3} C_{n+1} \xrightarrow{\partial_2} C_n \xrightarrow{\partial_1} C_{n-1} \xrightarrow{\partial_0} 0$$

We then call the elements of $H_n(X)$ as *homology classes*.

Definition 4.16 (Simplicial homology group). By considering the **chain complex** with **simplicial chain group**, we have so-called *simplicial homology group* induced by Definition 4.14.

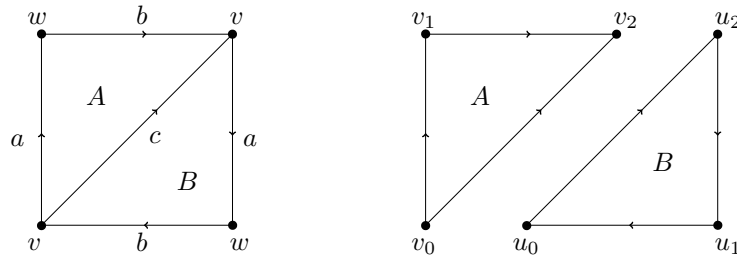
Lecture 22: Calculation of Homology

25 Feb. 10:00

4.2.2 Calculation of Homology

We start from some calculation about **homology group** of some spaces.

Example. Let $X = \mathbb{R}P^2$.



We see that we have

- $C_0 = \mathbb{Z}\langle v, w \rangle$
- $C_1 = \mathbb{Z}\langle a, b, c \rangle$
- $C_2 = \mathbb{Z}\langle A, B \rangle = \mathbb{Z}A \oplus \mathbb{Z}B$

The **chain complex** is then

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Where we let $A = [v_0, v_1, v_2]$ and $B = [u_0, u_1, u_2]$, then

$$\partial_2: \begin{cases} A & \mapsto b - c + a \\ B & \mapsto -a - c - b \end{cases}, \quad \partial_1: \begin{cases} a & \mapsto w - v \\ b & \mapsto v - w \\ c & \mapsto v - v = 0 \end{cases}$$

We can also calculate the image and the kernel at C_i , i.e.,

$$\begin{aligned} C_2: \operatorname{Im} \partial_3 &= 0, & \ker \partial_2 &= 0, \\ C_1: \operatorname{Im} \partial_2 &= \langle 2c, b - c + a \rangle, & \ker \partial_1 &= \langle b + a, c \rangle, \\ C_0: \operatorname{Im} \partial_1 &= \langle v - w \rangle, & \ker \partial_0 &= \langle v, w \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} H_0 &\cong \mathbb{Z} \langle v, w \rangle / \mathbb{Z} \langle v - w \rangle \cong \mathbb{Z} \\ H_1 &\cong \mathbb{Z} \langle b + a, c \rangle / \mathbb{Z} \langle 2c, b + a - c \rangle \cong \mathbb{Z} \langle b + a - c, c \rangle / \mathbb{Z} \langle 2c, b + a - c \rangle \cong \mathbb{Z} / 2\mathbb{Z} \\ H_2 &= 0 \end{aligned}$$

Remark. Given a basis for a [free Abelian group](#) $\langle b_1, \dots, b_n \rangle$ we can replace b_i with

$$b_i \pm m_1 b_1 \pm \dots \pm \widehat{m_i b_i} \pm \dots \pm m_n b_n.$$

Remark. Warning! Care is needed when doing *change of bases* over \mathbb{Z} . For example, if b_1, b_2 is a basis for $A \subseteq \mathbb{Z}^n$, then $b_1 - b_2, b_1 + b_2$ is not a basis, it is an index-2 subgroup. The key to this is that $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ has determinant -2 (not unit in \mathbb{Z}).

We can transform a basis for a [free group](#) into a different basis by applying a matrix of determinant ± 1 . If we apply a matrix of determinant D we will obtain generators for a subgroup of index $|D|$.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \pm m_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \pm m_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \pm m_{i-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \pm m_{i+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \pm m_n & 0 & \cdots & 1 \end{bmatrix}$$

As a summary, we have the following procedures to compute $H_n(X)$.

1. Choose [Δ-complex](#) structure on X . (We will prove $H_*(X)$ is independent of the choice of [Δ-complex](#) structure)
2. Choose orientations on each [simplex](#) (Any choice is okay, but you must commit to a choice, or you will make a sign error!)

3. For each n -simplex σ compute $\partial_n(\sigma)$ (careful with signs!)
4. $\text{Im } \partial_n = \langle \partial_n(\sigma) \mid \sigma \text{ an } n\text{-simplex} \rangle$. Use linear algebra to compute $\ker(\partial_n)$.
5. For each n compute $H_n(X) = \ker \partial_n / \text{Im } \partial_{n+1}$. Be careful that any change-of-variables map you apply is invertible over \mathbb{Z} .

Lecture 23: Singular Homology

07 Mar. 10:00

4.3 Singular Homology

As we noted before, we can give a different structure of **chain complex**, which shall induce a different **homology group** compare to **simplicial homology group**.

We now see one abstract way to define σ , which will give us so-called **singular homology group**.

Definition 4.17 (Singular simplex). A *singular n -simplex* in a space X is a continuous map

$$\sigma: \Delta^n \rightarrow X.$$

Definition 4.18 (Singular chain). Let $C_n(X)$ be the **free group** on **singular n -simplices** in X , which we call it the *singular n -chains*.

Definition 4.19 (Singular chain complex). The **singular chains** with **boundary maps**

$$\begin{aligned} \partial_n: C_n(X) &\rightarrow C_{n-1}(X) \\ \sigma &\mapsto \sum_{i=1}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v}_i, \dots, v_n]} \end{aligned}$$

induces a *singular chain complex*.

Definition 4.20 (Singular homology group). The *singular homology groups* are the **homology groups** of this **singular chain complex** given as

$$H_n(X) = \ker \partial_n / \text{Im } \partial_{n+1}.$$

Remark. We now see that from the definition of **homology group**, we can put different structure on which. But the idea is the same, namely we are taking $H_n(X)$ being

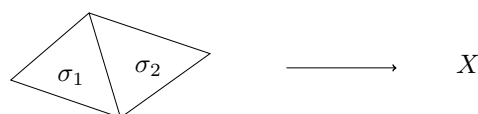
$$H_n(X) := \ker \partial_n / \text{Im } \partial_{n+1},$$

where the difference is what structure we put on X which induces different **chain complex** $C_n(X)$. In this case, we have **singular homology group** since we are considering **singular chain complex**, while we can also have **simplicial homology group**.

Since the generating sets for $C_n(X)$ when considering [singular chain complex](#) are almost always hugely uncountable from its definition, it's almost impossible to compute with these. However, it does give us a definition that does not depend on any other structure than the topology of X , making it useful for [developing theory](#).

Note. The heuristic is that, we interpret a [chain](#) $\sigma_1 \pm \sigma_2 \pm \cdots \pm \sigma_k$ as a map from a [\$\Delta\$ -complex](#) to X .

For example, with $\sigma_1 + \sigma_2$ as below,



where we've glued $[v_1, v_2]$ of σ_1 to $[v_0, v_2]$ of σ_2 if $\sigma_1|_{[v_1, v_2]}$ and $\sigma_2|_{[v_0, v_2]}$ are the same [singular \$n\$ -chain](#) with opposite signs.

With what we have defined, we now have some *goals*.

- [Singular homology](#) is a [homotopy](#) invariant. ([Theorem 4.2](#))
- [Singular](#) and [simplicial homology groups](#) are isomorphic. ([Theorem 4.9](#))

Exercise. We see some exercises.

1. Check that if X has [path](#) components $\{X_\alpha\}$ then

$$H_n(X) \cong \bigoplus_{\alpha} H_n(X_\alpha).$$

2. If $X = \{*\}$, then

$$H_n(X) = \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$

3. If X is [path](#)-connected, then $H_0(X) \cong \mathbb{Z}$.

4.4 Functoriality and Homotopy Invariance

Definition 4.21 (Induced map on chains). For a given continuous map $f: X \rightarrow Y$, we can consider the *map $f_\#$ induced by [chains](#)* as

$$\begin{aligned} f_\# : C_n(X) &\rightarrow C_n(Y) \\ [\sigma : \Delta^n \rightarrow X] &\mapsto [f \circ \sigma : \Delta^n \rightarrow Y]. \end{aligned}$$

Remark. We see that the functoriality doesn't depend on any kind of [\$\Delta\$ -complex](#) structure.

Definition 4.22 (Chain map). Given two [chain complexes](#) (C_*, ∂_*) and (D_*, δ_*) , a *chain map* between them is a collection of group homomorphisms $f_n: C_n \rightarrow D_n$ such that the following diagram commutes.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \xrightarrow{\partial_{n-1}} \dots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \dots & \xrightarrow{\delta_{n+2}} & D_{n+1} & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} \xrightarrow{\delta_{n-1}} \dots \end{array}$$

i.e. we have that $\delta_n \circ f_n = f_{n-1} \circ \partial_n$.

Exercise. We see that

1. We have that $f_{\#}\partial = \partial f_{\#}$. In other words, $f_{\#}$ is a [chain map](#). Thus, by the homework $f_{\#}$ induces a group homomorphism on the [homology groups](#). We write this as $f_*: H_n(X) \rightarrow H_n(Y)$ for all n .
2. We have [functoriality](#), i.e. $(f \circ g)_* = f_* \circ g_*$ and $(\text{id}_X)_* = \text{id}_{H_n(X)}$.

Theorem 4.1 (Homology group defines a functor). The n -th [homology group](#) $H_n: X \mapsto H_n(X)$ gives a [functor](#) from $\underline{\text{Top}}$ to $\underline{\text{Ab}}$.

Proof. This follows from the two exercises above. ■

Theorem 4.2 (Functoriality is homotopy invariant). If $f, g: X \rightarrow Y$ are [homotopic](#), then they will induce the same map on [homology](#)

$$f_* = g_*: H_n(X) \rightarrow H_n(Y).$$

The proof of [Theorem 4.2](#) can be found [here](#).

Exercise. [Theorem 4.1](#) and [Theorem 4.2](#) imply that H_n is a [homotopy](#) invariant.

Lecture 24: Chain Homotopy

09 Mar. 10:00

To prove [Theorem 4.2](#), we introduce some [homological algebra](#).

Definition 4.23 (Chain homotopy). Given chain complexes (A_*, ∂_*^A) and (B_*, ∂_*^B) and chain maps $f_\#, g_\#: A_* \rightarrow B_*$. A chain homotopy from $f_\#$ to $g_\#$ is a sequence of group homomorphisms $\psi_n: A_n \rightarrow B_{n+1}$ such that

$$f_n - g_n = \partial_{n+1}^B \circ \psi_n + \psi_{n-1} \circ \partial_n^A.$$

In diagram, letting $h_n := f_n - g_n$, we have the following.

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{\partial_{n+2}^A} & A_{n+1} & \xrightarrow{\partial_{n+1}^A} & A_n & \xrightarrow{\partial_n^A} & A_{n-1} \xrightarrow{\partial_{n-1}^A} \dots \\
 & & \downarrow h_{n+1} & \swarrow \psi_n & \downarrow h_n & \swarrow \psi_{n-1} & \downarrow h_{n-1} \\
 \dots & \xrightarrow{\partial_{n+2}^B} & B_{n+1} & \xrightarrow{\partial_{n+1}^B} & B_n & \xrightarrow{\partial_n^B} & B_{n-1} \xrightarrow{\partial_{n-1}^B} \dots
 \end{array}$$

This diagram does **not** commute, however, the **red** map is the sum of the **green** maps composed up, so it shows everything that is going on.

Theorem 4.3. If there is a chain homotopy ψ from $f_\#$ to $g_\#$, then the induced maps f_*, g_* on homology are equal.

Proof. Let $\sigma \in A_n$ be an n -cycle, i.e. $\partial_n^A \sigma = 0$. Then we compute that:

$$(f_n - g_n)(\sigma) = \partial_{n+1}^B(\psi_n(\sigma)) + \psi_{n-1}(\partial_n^A(\sigma)) = \partial_{n+1}^B(\psi_n(\sigma)) \in \text{Im}(\partial_{n+1}^B).$$

This tells us that $(f_n - g_n)(\sigma)$ is a boundary, and so $(f_n - g_n)(\sigma) = 0$ when considered as an element of the homology group (with degree n). Thus, $f_n(\sigma) = g_n(\sigma)$ in the homology group, and so f, g induce the same map as desired. ■

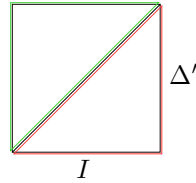
We now sketch the proof of Theorem 4.2 given in Hatcher[HPM02]. From this point in the course many of the theorems require much more algebraic work than we are interested in. We instead want to learn how to use the computational tools.

Proof. Suppose we have some homotopy $F: I \times X \rightarrow Y$ from f to g . The most difficulty in this proof is the combinatorial difficulty involved in the fact that the product of a simplex in X and I is not a simplex.

We now consider

1. Subdivide $\Delta^n \times I$ into $(n+1)$ -dimensional subsimplices.²⁰

²⁰We want to do this since the product between two simplices is not a simplex, as we just note.



2. We define the prism operator:

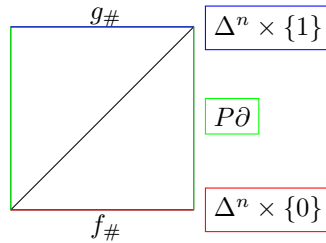
$$P_n: C_n(X) \rightarrow C_{n+1}(Y)$$

$$[\sigma: \Delta^n \rightarrow X] \mapsto \left[\begin{array}{l} \text{alternating sums of restrictions} \\ \Delta^n \times I \xrightarrow{\sigma \times \text{id}} X \times I \xrightarrow{F} Y \\ \text{to each simplex in our subdivision} \end{array} \right]$$

3. We now need to check that

$$\partial_{n+1}^Y P_n = g_{\#} - f_{\#} - P_{n-1} \partial_n^X.$$

We have the following diagram.



Thus P is a chain homotopy, and we're done. ■

Lecture 25: Relative Homology

11 Mar. 10:00

We are now interested in the relationship between $H_n(X), H_n(A), H_n(X/A)$.

4.5 Relative Homology

Definition 4.24 (Reduced homology group). The *reduced homology groups* $\tilde{H}_n(X) = H_n(X)$ when $n > 0$. When $n = 0$ we have that:

$$\tilde{H}_0(X) \oplus \mathbb{Z} = H_0(X).$$

Remark. The usefulness of this is that for path-connected space X we have $\tilde{H}_0(X) = 0$, and for contractible spaces X we have $\tilde{H}_n(X) = 0$.

Definition 4.25 (Good pair). Let X be a space, and $A \subseteq X$. Then (X, A) is a *good pair* if A is closed and nonempty, and also it is a *deformation retract* of a neighborhood in X .

Example. Let's see some examples.

1. If X is a *CW complex* and A is a nonempty *subcomplex*, then (X, A) is a *good pair*. The proof is given in the Appendix of Hatcher[HPM02] and requires some point-set topology.
2. If M is a smooth manifold, and $N \subseteq M$ is a smooth submanifold which is nonempty, then (M, N) is a *good pair*.
3. (Hawaiian earring, bad point) is not a *good pair*.
4. $(\mathbb{R}^n, \text{proper open set})$ is not a *good pair*.

Theorem 4.4 (Long exact sequence of a good pair). If (X, A) is a *good pair*, then there exists a long *exact* sequence (*exact* at every n) on *reduced homology groups* given by the following commutative diagram.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \tilde{H}_n(A) & \xrightarrow{i_*} & \tilde{H}_n(X) & \xrightarrow{j_*} & \tilde{H}_n(X/A) \\
 & & & & \searrow \delta & & \\
 & & \tilde{H}_{n-1}(A) & \xleftarrow{i_*} & \tilde{H}_{n-1}(X) & \xrightarrow{j_*} & \tilde{H}_{n-1}(X/A) \\
 & & & & \searrow \delta & & \\
 & & \dots & \xleftarrow{i_*} & \tilde{H}_0(X) & \xrightarrow{j_*} & \tilde{H}_0(X/A) \longrightarrow 0
 \end{array}$$

where $i: A \hookrightarrow X$ is the inclusion and $j: X \rightarrow X/A$ is the quotient map.

We see that both i_* and j_* is naturally induced, but not for δ . In fact, we'll construct δ in the proof! Specifically, we'll see that [Theorem 4.4](#) is just a special case of [Theorem 4.6](#), hence rather than proof [Theorem 4.4](#) directly, we will prove [Theorem 4.6](#) instead later.

Remark. The fact that this sequence is *exact* often means that if we know the *homology groups* of two of the spaces we can compute the *homology* of the remaining space.

Before we see the proof of [Theorem 4.4](#), we see one application.

Proposition 4.1. We have that:

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & \text{if } i = n; \\ 0, & \text{if } i \neq n. \end{cases}$$

Exercise. Verify [Proposition 4.1](#) in the case $n = 0$, so S^0 is just 2 points.

Proof. Some facts we need:

- $(D^n, \partial D^n)$ is a **good pair** (since it is a **CW complex** and a **subcomplex**)
- $D^n / \partial D^n \cong S^n$.
- $\tilde{H}_n(D^n) = 0$ for all n since D^n is **contractible**.
- $\partial D^n \cong S^{n-1}$.

We then proceed by induction on n . Using the long **exact** sequence, we have

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \tilde{H}_n(\partial D^n) & \xrightarrow{i_*} & \tilde{H}_n(D^n) & \xrightarrow{j_*} & \tilde{H}_n(S^n) \\
 & & & & \delta \nearrow & & \\
 & & \tilde{H}_{n-1}(\partial D^n) & \xrightarrow{i_*} & \tilde{H}_{n-1}(D^n) & \xrightarrow{j_*} & \tilde{H}_{n-1}(S^n) \\
 & & & & \delta \nearrow & & \\
 \dots & \xleftarrow{i_*} & \tilde{H}_0(D^n) & \xrightarrow{j_*} & \tilde{H}_0(S^n) & \longrightarrow & 0
 \end{array}$$

By induction, we have $\tilde{H}_{n-1}(\partial D^n) = \tilde{H}_{n-1}(S^{n-1}) = \mathbb{Z}$, hence we can fill in some of these groups as follows.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & 0 & \xrightarrow{i_*} & 0 & \xrightarrow{j_*} & \tilde{H}_n(S^n) \\
 & & & & \delta \nearrow & & \\
 & & \mathbb{Z} & \xleftarrow{i_*} & 0 & \xrightarrow{j_*} & \tilde{H}_{n-1}(S^n) \\
 & & & & \delta \nearrow & & \\
 \dots & \xleftarrow{i_*} & 0 & \xrightarrow{j_*} & \tilde{H}_0(S^n) & \longrightarrow & 0
 \end{array}$$

In all, we have an **exact** sequence:

$$0 \longrightarrow \tilde{H}_n(S^n) \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0$$

By **exactness**, δ is an isomorphism, thus $\tilde{H}_n(S^n) \cong \mathbb{Z}$. Now we must verify $\tilde{H}_i(S^n) = 0$ when $i \neq n$. In that case the **exact** sequence looks like:

$$\begin{array}{ccccccc}
 \longrightarrow & \tilde{H}_i(D^n) & \longrightarrow & \tilde{H}_i(S^n) & \longrightarrow & \tilde{H}_{i-1}(\partial D^n) \\
 & & & & & & \\
 \longrightarrow & 0 & \longrightarrow & \tilde{H}_i(S^n) & \longrightarrow & 0
 \end{array}$$

Exactness then tells us that $\tilde{H}_i(S^n) = 0$. ■

Theorem 4.5 (Brouwer's fixed point theorem). ∂D^n is not a **retract** of D^n . Hence, every continuous map $f: D^n \rightarrow D^n$ has a fixed point.

Proof. If $r: D^n \rightarrow \partial D^n$ were a [retraction](#), then by definition this would give us

$$\begin{array}{ccccc} \partial D^n & \xrightarrow{i} & D^n & \xrightarrow{r} & \partial D^n \\ & \searrow & \text{id}_{\partial D^n} & \nearrow & \\ & & & & \end{array}$$

[Functoriality](#) of [homology](#) implies

$$\begin{array}{ccccc} \tilde{H}_{n-1}(\partial D^n) & \xrightarrow{i_*} & \tilde{H}_{n-1}(D^n) & \xrightarrow{r_*} & \tilde{H}_{n-1}(\partial D^n) \\ & \searrow & \text{id} & \nearrow & \\ & & & & \end{array}$$

So then:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{i_*} & 0 & \xrightarrow{r_*} & \mathbb{Z} \\ & \searrow & \text{id} & \nearrow & \\ & & & & \end{array}$$

which is impossible since the map $\text{id}_{\mathbb{Z}}$ can't be factored through 0. ■

Exercise. As with D^2 , if $f: D^n \rightarrow D^n$ had no fixed point, we could build a [retraction](#).

In order to proof [Theorem 4.4](#), we introduce the concept of *diagram chase*.

Lemma 4.2 (The short five lemma). Suppose we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{\psi'} & B & \xrightarrow{\varphi'} & C' & \longrightarrow & 0 \end{array}$$

so that the rows are [exact](#). Then:

1. If α, γ are injective then β is injective.
2. If α, γ are surjective then β is surjective.
3. If α, γ are isomorphisms then β is an isomorphism

Proof. 1. and 2. imply 3. We leave 2. as an exercise. We fix $b \in B$ such that $\beta(b) = 0$. We want to show that $\beta = 0$. Well, we draw a diagram chase as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bullet & \xrightarrow{\psi} & b & \xrightarrow{\varphi} & \varphi(b) & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & \bullet & \xrightarrow{\psi'} & 0 & \xrightarrow{\varphi'} & 0 & \longrightarrow & 0 \end{array}$$

And thus by injectivity of γ we know $\varphi(b) = 0$. By [exactness](#), $b \in \text{Im } \psi$. We then may write for some $a \in A$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & a & \xrightarrow{\psi} & b & \xrightarrow{\varphi} & 0 & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & \alpha(a) & \xrightarrow{\psi'} & 0 & \xrightarrow{\varphi'} & 0 & \longrightarrow & 0 \end{array}$$

Therefore $\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) = 0$ by commutativity. By [exactness](#) of the bottom row we know that ψ' is an injection.

Thus, $\alpha(a) = 0$, so since α is injective, $a = 0$. With this $b = \psi(a) = \psi(0) = 0$. Great! With this $\ker(\beta) = 0$, and β injects. ■

Lecture 26: Continue on Relative Homology

14 Mar. 10:00

We start from a definition.

Definition 4.26 (Relative chain complex). Let X be a space and let $A \subseteq X$ be a subspace. Then we define the *relative chain complex*

$$C_n(X, A) = C_n(X) / C_n(A),$$

which is a quotient of [Abelian groups](#) of the [singular chain groups](#).

Remark. We can indeed adapt [Definition 4.26](#) by either [singular chain complex](#) structure or [simplicial chain complex](#) structure.

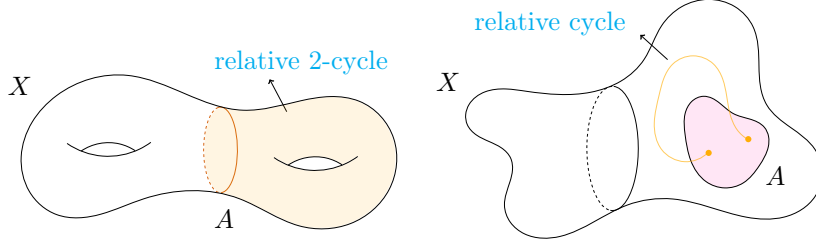
Exercise. Since $\partial_n^*(C_n(A)) \subseteq C_{n-1}(A)$, hence there exists a well-defined map

$$\partial_n: C_n(X) / C_n(A) \rightarrow C_{n-1}(X) / C_{n-1}(A).$$

We can verify that $\partial^2 = 0$. Then, since $\partial^2 = 0$ we can conclude that these groups will in fact form a [chain complex](#) $(C_*(X, A), \partial)$.

Definition 4.27 (Relative homology). The homology groups of the chain complex $(C_*(X, A), \partial)$ are denoted by $H_n(X, A)$, and they are called *relative homology groups*.

(Relative cycle). Elements in $\ker \partial_n$ are called *relative n -cycles*. These are elements $\alpha \in C_n(X)$ such that $\partial_n \alpha \in C_{n-1}(A)$.



(Relative boundary). Likewise, elements in $\text{Im } \partial_{n+1}$ are called *relative n -boundaries*. This means that $\alpha = \partial \beta + \gamma$ where $\beta \in C_n(X)$ and $\gamma \in C_{n-1}(A)$.

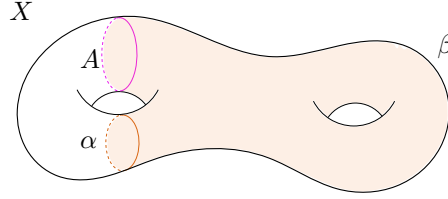


Figure 21: We see that we have $\alpha + \gamma = \partial \beta$, where α is a *relative boundary*, and $\gamma \in C_{n-1}(A)$.

Theorem 4.6 (Long exact sequence of a pair). Let $A \subseteq X$ be spaces, then there exists a long *exact* sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & \tilde{H}_n(A) & \xrightarrow{i_*} & \tilde{H}_n(X) & \xrightarrow{q} & \tilde{H}_n(X, A) \\ & & & & \searrow \partial & & \\ & & \tilde{H}_{n-1}(A) & \xleftarrow{i_*} & \dots & \xrightarrow{q} & \tilde{H}_0(X, A) \longrightarrow 0 \end{array}$$

where i_* is induced by $A \hookrightarrow X$, and q is induced by $C_n(X) \twoheadrightarrow C_n(X) / C_n(A)$.

We will prove that when (X, A) is a *good pair*, then $H_n(X, A) \cong \tilde{H}_n(X/A)$. Then Theorem 4.4 is a special case of Theorem 4.6. The key to the proof of Theorem 4.6 above is the following slogan.

Remark. A [short exact sequence](#) of [chain complexes](#) gives rise to a long [exact sequence](#) of [homology groups](#). Namely, given a [short exact sequence](#) of [chain complexes](#) $(A_*, \partial^A), (B_*, \partial^B), (C_*, \partial^C)$ such that

$$0 \longrightarrow A_* \xrightarrow{\iota} B_* \xrightarrow{q} C_* \longrightarrow 0$$

where ι, q are [chain maps](#) such that

$$0 \longrightarrow A_n \xrightarrow{\iota_n} B_n \xrightarrow{q_n} C_n \longrightarrow 0$$

is [exact](#) for all n . Then [Theorem 4.4](#) will follow from a [short exact sequence](#)

$$0 \longrightarrow \tilde{C}_*(A) \longrightarrow \tilde{C}_*(X) \longrightarrow \tilde{C}_*(X, A) \longrightarrow 0$$

where \tilde{C}_* denotes the *augmented chain complex* (the one with \mathbb{Z} after it).

Exercise. If A is a single point in X , then $H_n(X, A) = \tilde{H}_n(X / A) = \tilde{H}_n(X)$.

Lecture 27: Excision

16 Mar. 10:00

Let's start with a theorem.

Theorem 4.7 (Excision). Suppose we have subspace $Z \subseteq A \subseteq X$ such that $\bar{Z} \subseteq \text{Int}(A)$. Then the inclusion

$$(X - Z, A - Z) \hookrightarrow (X, A)$$

induces isomorphisms

$$H_n(X - Z, A - Z) \xrightarrow{\cong} H_n(X, A).$$

Exercise. Equivalently for subspaces $A, B \subseteq X$ whose interiors cover X , the inclusion

$$(B, A \cap B) \hookrightarrow (X, A)$$

induces an isomorphism

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

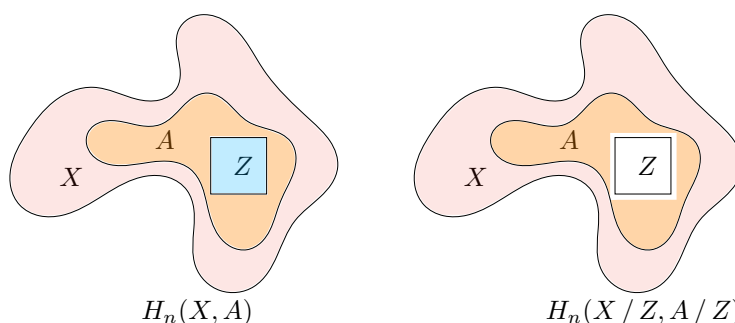
Answer. We see that this follows from

$$B := X \setminus Z, \quad Z = X \setminus B,$$

then we see that $A \cap B = A - Z$ and the condition requires from [Theorem 4.7](#), $\bar{Z} \subseteq \text{Int}(A)$ is then equivalent to

$$X = \text{Int}(A) \cup \text{Int}(B)$$

since $X \setminus \text{Int}(B) = \bar{Z}$.



Proof Sketch. We sketch the proof here, which is notorious for being hairy.

- Given a **relative cycle** x in (X, A) , subdivide the **simplices** to make x a linear combination of chains on *smaller simplices*, each contained in $\text{Int}(A)$ or $X \setminus Z$.



Figure 22: $\Delta^n \rightarrow X$ subdivide into **subsimpllices** with images in.

This means x is homologous to sum of **subsimpllices** with images in $\text{Int}(A)$ or $X \setminus Z$. One of the things we use is that **simplices** are compact, so this process takes finite time.

The key is that the Subdivision operator is chain **homotopic** to the identity.

- Since we are working relative to A , the chains with image in A are zero, thus we have a relative cycle homologous to x with all **simplices** contained in $X \setminus Z$.

■

Exercise. $H_*(Y, y_0) \cong \tilde{H}(Y)$.

Theorem 4.8. For **good pairs** (X, A) , the quotient map $q: (X, A) \rightarrow (X/A, A/A)$ induces isomorphisms

$$q_*: H_n(X, A) \xrightarrow{\cong} H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$$

for all n .

Remark. The last equality is from the exercise since $A/A = \{*\}$.

Proof Sketch. Let $A \subseteq V \subseteq X$ where V is a neighborhood of A that deformation retracts onto A . Using excision, we obtain a commutative diagram

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\cong} & H_n(X, V) & \xleftarrow{\cong} & H_n(X - A, V - A) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(X/A, A/A) & \xrightarrow{\cong} & H_n(X/A, V/A) & \xleftarrow{\cong} & H_n(X/A - A/A, V/A - A/A) \end{array}$$

Done if we can prove all the colored isomorphisms.

- \cong is an isomorphism by excision.
- \cong is an isomorphism by direct calculation (since q is a homeomorphism on the complement of A).
- \cong on Homework, since V deformation retracts to A .

■

Lecture 28: Singular Homology v.s. Simplicial Homology

18 Mar. 10:00

Remark. If M is a smooth manifold and N is an embedded smooth closed submanifold, then (M, N) is a good pair. Why? Well this follows from the tubular neighborhood theorem, which should be proven in a course like 591. We will only use the result in obvious cases, and simply assert that certain pairs are good pairs.

With pairs like (\mathbb{R}^{n+1}, S^n) , you can just assert that this is a good pair (and do not need to prove that S^n is a smooth submanifold of \mathbb{R}^{n+1}). Another good example is manifolds and their boundary always form a good pair.

Theorem 4.9 (Singular homology agrees with simplicial homology). Let X be a Δ -complex. We use $\Delta_n(X)$ to represent the simplicial chain groups on X , and $C_n(X)$ to denote the singular chain groups. Likewise, we denote

$$\Delta_n(X, A) = \Delta_n(X) / \Delta_n(A)$$

and

$$C_n(X, A) = C_n(X) / C_n(A).$$

The inclusion $\Delta_*(X, A) \hookrightarrow C_*(X, A)$ given by

$$[\sigma : \Delta^n \rightarrow X] \mapsto [\sigma : \Delta^n \rightarrow X]$$

induces an isomorphism on homology such that

$$H_n^\Delta(X, A) \cong H_n(X, A).$$

If we consider the case that $A = \emptyset$, we recover the case of absolute homology

$$H_n^\Delta(X) \cong H_n(X).$$

The proof of Theorem 4.9 uses the following lemma.

Lemma 4.3 (The five lemma). If we have a commutative diagram with **exact** rows as following,

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E' \end{array}$$

If $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then so is γ .

Proof. Diagram chase! ■

Lecture 29: Proof of Theorem 4.9

21 Mar. 10:00

Proof Sketch. The idea is as follows.

- We can use the long **exact sequence** of a pair and the **Lemma 4.3** to reduce to proving the result for **absolute homology groups** (and we will recover the general result).
- Because the image $\Delta^n \rightarrow X$ is *compact*, it is contained in some finite skeleton X^k . Use this to reduce the proof to the finite skeleton X^k of X , namely we can use induction.

From the long **exact sequence** of a **pair** we get

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^k, X^{k-1}) & \rightarrow & H_n^\Delta(X^{k-1}) & \rightarrow & H_n^\Delta(X^k) & \rightarrow & H_n^\Delta(X^k, X^{k-1}) & \rightarrow & H_{n-1}^\Delta(X^{k-1}) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ H_{n+1}(X^k, X^{k-1}) & \rightarrow & H_n(X^{k-1}) & \rightarrow & H_n(X^k) & \rightarrow & H_n(X^k, X^{k-1}) & \rightarrow & H_{n-1}(X^{k-1}) \end{array}$$

The Goal is to prove γ is an isomorphism using the **Lemma 4.3**.

We assume that β, ϵ are isomorphisms by induction, checking the case manually for X^0 (which will be a discrete set of points). It remains to show that α, δ are isomorphisms.

We know then that

$$\Delta_n(X^k, X^{k-1}) = \begin{cases} \mathbb{Z}[k\text{-simplices}], & \text{if } k = n; \\ 0, & \text{otherwise} \end{cases} \cong H_n^\Delta(X^k, X^{k-1}).$$

We claim that $H_n(X^k, X^{k-1})$ are also **free Abelian** on the **singular k -simplices** defined by the characteristic maps $\Delta^k \rightarrow X^k$ when $n = k$, and 0 otherwise. Consider the map

$$\Phi: \coprod_{\alpha} (\Delta_{\alpha}^k, \partial \Delta_{\alpha}^k) \rightarrow (X^k, X^{k-1})$$

defined by the characteristic map. This induces an isomorphism on **homology** since

$$\coprod_{\alpha} \Delta_{\alpha}^k / \coprod_{\alpha} \partial \Delta_{\alpha}^k \xrightarrow{\cong} X^k / X^{k-1}.$$

This reduces to check that

$$H_n(\Delta^k, \partial\Delta^k) = \begin{cases} 0, & \text{if } n \neq k; \\ \mathbb{Z}, & \text{if } n = k \end{cases}$$

generated by the identity map $\Delta^k \rightarrow \Delta^k$. ■

Corollary 4.1. If X has a Δ-complex structure (or is homotopy equivalent to one), then we have the followings.

1. If the dimension is $\leq d$, then $H_n(X) = 0$ for all $n > d$.
2. If \bar{X} has no cells of dimension p , then $H_p(X) = 0$.
3. If \bar{X} has no cells of dimension p , then $H_{p-1}(X)$ is free Abelian.

Corollary 4.2. Given a singular homology class on X , without loss of generality we can choose a Δ-complex structure on X , and we then we can assume the class is represented by a simplicial n -cycle.

4.6 Degree

Definition 4.28 (Degree). Let $f: S^n \rightarrow S^n$, then

$$f_*: \mathbb{Z} \cong H_n(S^n) \rightarrow H_n(S^n) \cong \mathbb{Z}.$$

From group theory, this map must be multiplication by some integer $d \in \mathbb{Z}$, which we call it as the *degree*, denotes as $\deg(f)$ of f .

Remark (Properties of Degree). We first see some properties of degree.

1. $\deg(\text{id}_{S^n}) = 1$ since $(\text{id}_{S^n})_* = \text{id}_{\mathbb{Z}}$.
2. If $f: S^n \rightarrow S^n$, $n \geq 0$ is not surjective, then $\deg(f) = 0$. To see this, we know that f_* factors as

$$H_n(S^n) \xrightarrow{\quad} H_n(S^n - \{*\}) = 0 \xrightarrow{\quad} H_n(S^n)$$

$\searrow f_* \quad \nearrow$

And since the middle group is zero, $f_* = 0$.

3. If $f \simeq g$, then $f_* = g_*$, so $\deg(f) = \deg(g)$.

Note. The converse is true! We'll see this later.

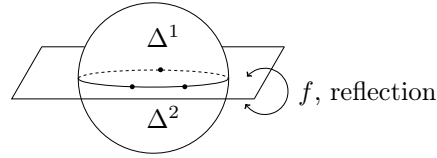
4. $(f \circ g)_* = f_* \circ g_*$, and so $\deg(f \circ g) = \deg(f) \deg(g)$.

Consequently, if f is a homotopy equivalence then $\deg f = \pm 1$.

Exercise. It is possible to put a Δ-complex structure with 2 n -cells, Δ_1 and Δ_2 glued together along their boundary ($\cong S^{n-1}$), and

$$H_n(S^n) = \langle \Delta_1 - \Delta_2 \rangle.$$

If f is a reflection fixing the equator, and swapping the 2-cells, then $\deg f = -1$.



5. We now have the following linear algebra exercise.

Exercise. The map $S^{n+1} \rightarrow S^{n+1}$ given by $x \mapsto -x$ is the composite of $(n+1)$ reflections.

So the antipodal map $S^n \rightarrow S^n$ given by $x \mapsto -x$ has degree which is the product of $n+1$ copies of (-1) , and so it has [degree](#) $(-1)^{n+1}$. (i.e., since the $(n+1) \times (n+1)$ scalar matrix (-1) is composition of $(n+1)$ reflections.)

6. We see the following.

Exercise. If $f: S^n \rightarrow S^n$ has no fixed points, then we can homotope f to the antipodal map via

$$f_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}.$$

Therefore, $\deg f = (-1)^{n+1}$.

Lecture 30: Degree

23 Mar. 10:00

With the definition of [degree](#) and some of its [properties](#), we have the following theorems.

Theorem 4.10 (Hairy ball theorem). The sphere S^n admits a nonvanishing continuous tangent vector field if and only if n is odd.

Proof. ■ HW.

Theorem 4.11 (Groups acting on S^{2n}). If G acts on S^{2n} [freely](#), then

$$G = \mathbb{Z}/2\mathbb{Z} \text{ or } 1.$$

Proof. There exists a homomorphism given by

$$\begin{aligned} G &\rightarrow \{\pm 1\} \\ g &\mapsto \deg(\tau_g) \end{aligned}$$

Where τ_g is the action of $g \in G$ on S^{2n} as a map $S^{2n} \rightarrow S^{2n}$. We know this map is well-defined since τ_g is invertible (simply take $\tau_{g^{-1}}$) for each $g \in G$. Our note on composites shows this is a homomorphism.

We want to show that the kernel is trivial, since then by the first isomorphism theorem $G \cong \text{Im}$, and the image is either trivial or $\mathbb{Z}/2\mathbb{Z}$. Suppose that g is a nontrivial element of G , then since G acts **freely** we know that τ_g has no fixed points. With this in mind we have

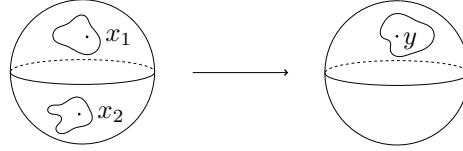
$$\deg \tau_g = (-1)^{2n+1} = -1.$$

Thus, $g \notin \ker$, hence the kernel is trivial as desired. ■

Corollary 4.3. S^{2n} is only the trivial cover $S^{2n} \rightarrow S^{2n}$ or **degree 2** cover (for example, $S^{2n} \rightarrow \mathbb{R}P^{2n}$).

Proof. This follows since any **covering space** action acts **freely**. ■

Definition 4.29 (Local degree). Let $f: S^n \rightarrow S^n$ ($n > 0$). Suppose there exists $y \in S^n$ such that $f^{-1}(y)$ is finite, say, $\{x_1, \dots, x_m\}$. Then let U_1, \dots, U_m be disjoint neighborhoods of x_1, \dots, x_m that are mapped by f to some neighborhood V of y .



The *local degree* of f at x_i , denote as $\deg f|_{x_i}$, is the **degree** of the map

$$f_*: \mathbb{Z} \cong H_n(U_i, U_i - \{x_i\}) \rightarrow H_n(V, V - \{y\}) \cong \mathbb{Z}.$$

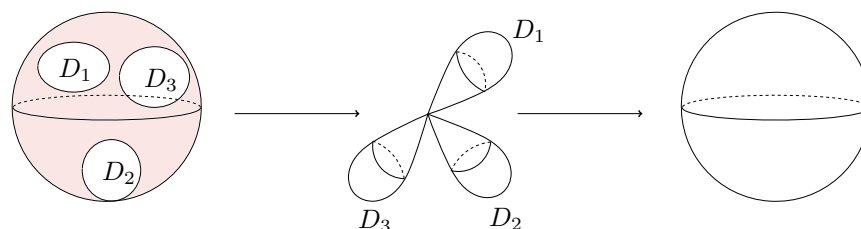
Theorem 4.12. Let $f: S^n \rightarrow S^n$ with $f^{-1}(y) = \{x_1, \dots, x_m\}$ as in **Definition 4.29**, then we have

$$\deg f = \sum_{i=1}^m \deg f|_{x_i}.$$

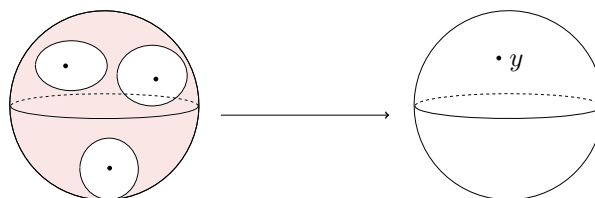
Remark. Thus, we can compute the **degree** of f by computing these **local degrees**.

Let's work with some examples for our edification.

Example. Consider S^n and choose m disks in S^n . Namely, we first collapse the complement of the m disks to a point, and then we identify each of the [wedged \$n\$ -spheres](#) with the n -sphere itself.



The result will be a map of [degree \$m\$](#) . We can see this by computing [local degree](#).



By choosing a good point in the codomain, we get one point for each disk in the preimage, and the map is a local homeomorphism around these points which is orientation preserving. We could likewise compose the maps to S^n from the [wedge](#) with a reflection to construct a map of [degree \$-m\$](#) .

Lecture 31: Local Degree and Local Homology

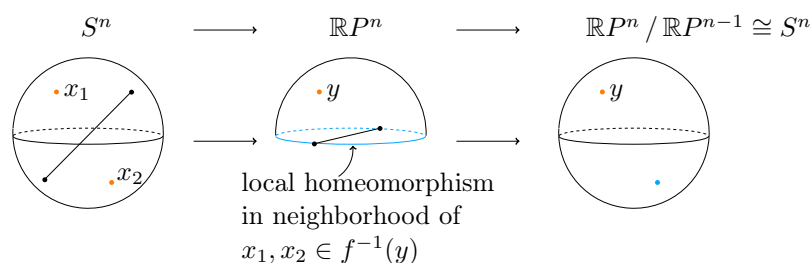
25 Mar. 10:00

Example. Consider the composition of the quotient maps below

$$S^n \xrightarrow{\quad} \mathbb{R}P^n \xrightarrow{\quad} \mathbb{R}P^n / \mathbb{R}P^{n-1} \cong S^n$$

$\searrow \quad \quad \quad \nearrow$
 f

We want to compute the [degree](#) of this map.



Note that this restricts to a homeomorphism on each component of $S^n \setminus \text{equator}$ as a map to $\mathbb{R}P^n / \mathbb{R}P^{n-1}$. Suppose we've oriented our copies of S^n in such a

way that the homeomorphism on the top hemisphere is orientation-preserving. The homeomorphism on the bottom hemisphere is given by taking the antipodal map and composing with the homeomorphism of the top hemisphere

$$\deg = \deg(\text{id}) = \deg(\text{antipodal}) = 1 + (-1)^{n+1} = \begin{cases} 0, & \text{if } n \text{ even;} \\ 2, & \text{if } n \text{ odd.} \end{cases}$$

We can now prove [Theorem 4.12](#).

Proof. If $f: S^n \rightarrow S^n$ and we have some $y \in S^n$ with $f^{-1}(\{y\}) = \{x_1, \dots, x_m\}$, then we have a nice commutative diagram as follows.

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \text{LES of pair} \downarrow & & \downarrow \cong \text{LES of a pair} \\ \bigoplus_{i=1}^m \mathbb{Z} = H_n(S^n, S^n - \{x_1, \dots, x_m\}) & & H_n(S^n, S^n - \{y\}) \\ \text{excision} \uparrow \cong & & \uparrow \cong \text{excision} \\ H_n\left(\coprod_{i=1}^m U_i, \coprod_{i=1}^m (U_i - \{x_i\})\right) & & H_n(S^n, S^n - \{y\}) \\ \text{homology of disjoint union} \uparrow \cong & & \uparrow \cong \text{excision} \\ \bigoplus_i H_n(U_i, U_i - \{x_i\}) & \longrightarrow & H_n(V, V - \{y\}) \end{array} \quad \bullet$$

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & \deg(f_*) \\ \downarrow & & \downarrow \\ (1, 1, \dots, 1) & \xrightarrow{\quad} & \deg f = \sum \deg f|_{x_i} \end{array}$$

where we trace around the outside of the diagram at the bottom, which just proves the result. \blacksquare

Let's grab some intuition. *What really is local homology?*

By [excision](#), there is an isomorphism $H_n(S^n, S^n \setminus \{x_i\}) \cong H_n(U, U \setminus \{x_i\})$ for any open neighborhood U of x_i .

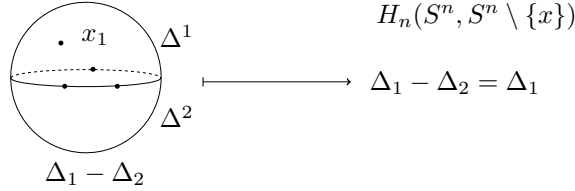
The long [exact sequence](#) of a [pair](#) also gives us

$$\dots \rightarrow H_k(S^n \setminus \{x_i\}) \rightarrow H_k(S^n) \rightarrow H_k(S^n, S^n \setminus \{x_i\}) \rightarrow H_{k-1}(S^n \setminus \{x_i\}) \rightarrow \dots$$

Since $S^n \setminus \{x_i\}$ is homeomorphic to an open n -ball, we see that $H_k(S^n \setminus \{x_i\}) = H_{k-1}(S^n \setminus \{x_i\}) = 0$. With this in mind, j_* is an isomorphism.

We want to think about what j_* does when $k = n$, i.e., when this is an isomorphism $\mathbb{Z} \cong H_n(S^n) \rightarrow H_n(S^n, S^n \setminus \{x_i\}) \cong \mathbb{Z}$.

We see that $\Delta_1 - \Delta_2$ generate $H_n(S^n)$, where Δ_1, Δ_2 are the top and bottom hemisphere indicated below.



We then understand that $j_*(\Delta_1 - \Delta_2) = \Delta_1 - \Delta_2 = \Delta_1$ since $\Delta_2 = 0$ in $C_n(S^n)/C_n(S^n \setminus \{x_i\})$.

The upshot is that $H_n(S^n, S^n \setminus \{x\})$ is generated by an n -simplex with x in its interior.

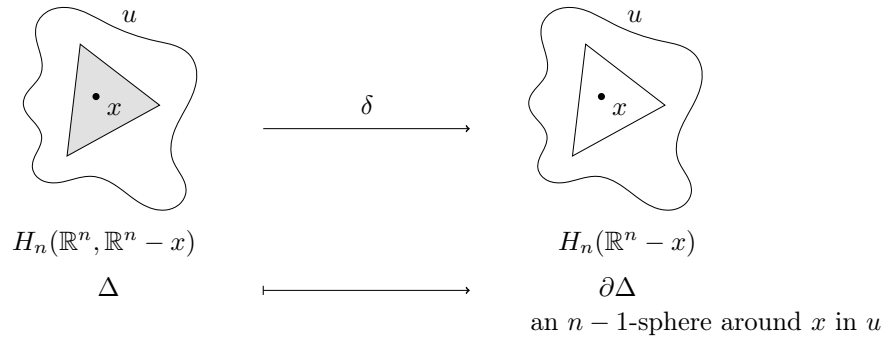
Suppose M is an n -manifold. Then $H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$, where U is a small ball around x . Because U is a ball homeomorphic to \mathbb{R}^n , we see that

$$H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}).$$

By the long exact sequence of a pair

$$0 = H_n(\mathbb{R}^n) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \rightarrow H_{n-1}(\mathbb{R}^n \setminus \{x\}) \rightarrow H_{n-1}(\mathbb{R}^n) = 0$$

And since $\mathbb{R}^n \setminus \{x\}$ is homotopy equivalent to an $n-1$ sphere, this means that $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}$. By homework, this connecting homomorphism is given by taking the boundary of a relative cycle as below.



We intuitively want to use this idea to compute degree using this idea. We use naturality of the long exact sequence, namely the fact that where $f: (U_i, U_i \setminus$

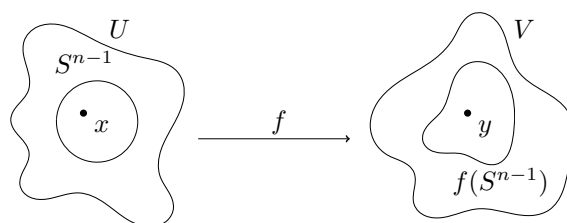
$\{x_i\} \rightarrow (V, y)$ is a map of pairs, then the following diagram commutes.

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_n(U_i, U_i \setminus \{x_i\}) & \longrightarrow & H_{n-1}(U_i, U_i \setminus \{x_i\}) & \longrightarrow & \dots \\ & & \downarrow f_* & & \downarrow f_* & & \\ \dots & \longrightarrow & H_n(V, V \setminus \{y\}) & \longrightarrow & H_{n-1}(V, V \setminus \{y\}) & \longrightarrow & \dots \end{array}$$

By naturality of the long exact sequence and the isomorphism discussed above, we can compute the local degree of a map $S^n \rightarrow S^n$ at a point x by computing the degree of the map

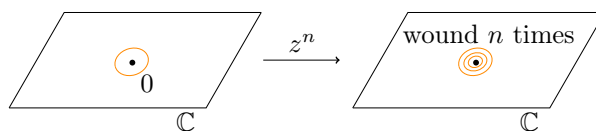
$$H_{n-1}(U \setminus \{x\}) \longrightarrow H_{n-1}(V \setminus \{y\})$$

In fact the local degree will be the degree restricted to a small S^{n-1} in the neighborhood U .



Example. Consider

$$\begin{aligned} \hat{\mathbb{C}} &\rightarrow \hat{\mathbb{C}} \\ z &\mapsto z^n. \end{aligned}$$



We see that

$$\deg f|_0 = n.$$

Lecture 32: Cellular Homology

28 Mar. 10:00

4.7 Cellular Homology

Suppose that X is a CW complex. Then (X^n, X^{n-1}) is a good pair for all $n > 1$, and X^n / X^{n-1} is a wedge of n -spheres, one for each n -cell e_α^n . Hence,

$$H_k(X^n, X^{n-1}) \cong \begin{cases} 0, & \text{if } k \neq n; \\ \langle e_\alpha^n \mid e_\alpha^n \text{ is an } n\text{-cell} \rangle, & \text{if } k = n. \end{cases}$$

Definition 4.30 (Cellular chain complex). The *cellular chain complex* on X , denoted as \bar{w} , has

(Chain groups). The chain groups $C_n(X)$ are defined as

$$C_n(X) := \mathbb{Z} \langle e_\alpha^n \mid e_\alpha^n \text{ an } n\text{-cell of } X \rangle (\cong H_n(X^n, X^{n-1}))$$

with $X^{-1} = \emptyset$.

(Boundary maps). For $n = 0$, we have

$$\begin{aligned} \partial_1: C_1(X) &\rightarrow C_0(X) \\ \langle 1\text{-cells} \rangle &\rightarrow \langle 0\text{-cells} \rangle, \end{aligned}$$

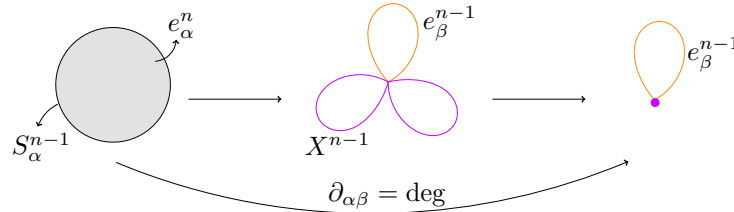
which is the usual [simplicial boundary map](#). For $n > 1$, the boundary map ∂_n are defined as

$$\partial_n(e_\alpha^n) = \sum_\beta \partial_{\alpha\beta} e_\beta^{n-1}$$

where $\partial_{\alpha\beta}$ is the [degree](#) of the map

$$\partial e_\alpha^n = S_\alpha^{n-1} \xrightarrow{\text{attaching map}} X^{n-1} \xrightarrow{\text{quotient by } X^{n-1} \setminus e_\beta^{n-1}} S_\beta^{n-1}$$

In pictures, this is given as the following.



Remark. We see that

$$C_n(X) \cong H_n(X^n, X^{n-1})$$

since (X^n, X^{n-1}) is a [good pair](#), so $H_n(X^n, X^{n-1}) \cong H_n(X^n / X^{n-1})$, which is just the [wedge](#) of 1 n -sphere for each n -cell of X .

Furthermore, the orientations on spheres are defined by identifying the domains of characteristic maps $D_\alpha^n \rightarrow X$ with an (oriented) disk in \mathbb{R}^n . i.e., we need to choose a generator of

$$H_{n-1}(\partial D_\alpha^n) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

Definition 4.31 (Cellular homology group). We define the so-called *cellular homology group* by [cellular chain complex](#) in our usual way of defining [homology group](#).

Theorem 4.13. [Definition 4.30](#) indeed forms a [chain complex](#).

Theorem 4.14 (Cellular homology agrees with singular homology). The [cellular homology groups](#) coincide with the [singular homology groups](#).

Note. i.e., the isomorphism commutes $\bar{w}f_*$ for all continuous $f: X \rightarrow Y$.

[Theorem 4.14](#) implies the following.

Corollary 4.4. We have the followings.

- $H_n(X) = 0$ if X has a [CW complex](#) structure with no n -cells.
- If X has a [CW complex](#) with k n -cells, then $H_n(X)$ is generated by at most k elements.
- If $H_n(X)$ is a group with a minimum of k generators, then any [CW complex](#) structure on X must have at least k n -cells.
- If X has a [CW complex](#) with no n -cells, then

$$H_{n-1}(X) = \ker(\partial_{n-1}),$$

which is [free Abelian](#).

- If X has a [CW complex](#) with no cells in consecutive dimensions, then all $\partial_n = 0$. Its [homology](#) are [free Abelian](#) on its n -cells, namely the [cellular chain groups](#).

For example

1. $S^n, n \geq 2$.
2. $\mathbb{C}P^n, \forall n$.

Example. S^n with $n \geq 2$, using the [CW complex](#) structure of e^n attached to a single point x_0 . The [cellular chain complex](#) is given as

$$0 \longrightarrow 0 \longrightarrow \langle e^n \rangle \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \langle x_0 \rangle \longrightarrow 0$$

So then all the boundary maps are zero, and we see that

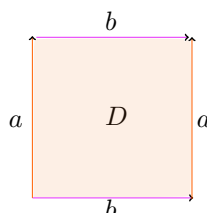
$$H_k(S^n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

Exercise. Redo this calculation with other [CW complex](#) structure on S^n , e.g. glue 2 n -cells onto S^{n-1} and proceed inductively.

Lecture 33: Cellular Homology Examples

30 Mar. 10:00

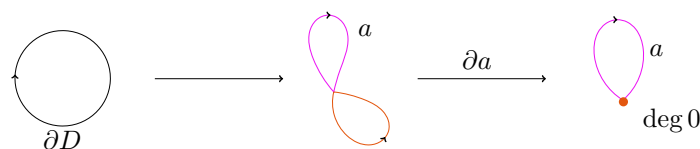
Example. Let's do this with the torus



The **chain complex** looks like

$$0 \longrightarrow \langle D \rangle \longrightarrow \langle a, b \rangle \longrightarrow \langle x \rangle \longrightarrow 0$$

Note that $a \mapsto x - x = 0$ and $b \mapsto x - x = 0$ and so $\partial_1 = 0$. Now D is glued along $aba^{-1}b^{-1}$, so we look at the composed up map



We wind forwards then backwards around a , so the **degree** is zero. The same thing happens for b so

$$\partial_2 D = 0 \cdot a + 0 \cdot b = 0.$$

This gives a nice **principle**, namely if a 2-cell D is glued down via some **words** w (this only makes sense for 2-cells), then the coefficient to a letter b in $\partial_2 D$ is the sum of the exponents of b in w .

Now we just have that the **homology groups** are equal to the **chain groups** because the boundary maps are all zero. Hence, we have

Example. A genus g surface Σ_g has the **CW complex** structure as

- 1 0-cell x .
- $2g$ 1-cells $a_1, b_1, a_2, b_2, \dots$
- 1 2-cell D glued along $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g]$ (a product of commutators)

We obtain the result

$$\partial_1(a_i) = \partial_1(b_i) = x - x = 0.$$

Furthermore, by the principle discussed above, we know that every 1-cell appears once in the **word**, and its inverse appears once, so all the coefficients of 1-cells

in $\partial_2(D)$ are zero, so $\partial_2(D) = 0$. This means we have a [chain complex](#)

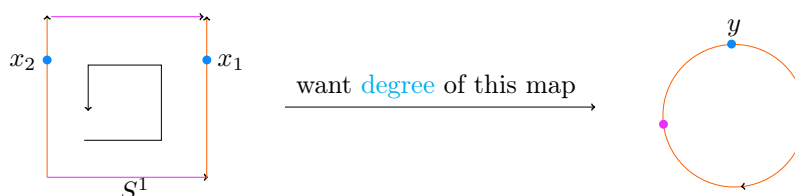
$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

And so then we have that

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2; \\ \mathbb{Z}^{2g}, & \text{if } k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Exercise. Calculate the [cellular homology group](#) of $\mathbb{R}P^n$.

Example (Torus example: ∂_2 in more detail). We're going to work through this example a bit more carefully.



Let's zoom in on these two preimage points and use *local homology* to compute this:

Lecture 34: Proof of Theorem 4.14

1 Apr. 10:00

We're now going to work towards proving that [cellular homology agrees with singular homology](#). First we need some nontrivial preliminaries.

Lemma 4.4. We have that

1. $H_k(X^n, X^{n-1}) = \begin{cases} 0, & \text{if } k \neq n; \\ \langle n\text{-cells} \rangle, & \text{if } k = n. \end{cases}$
2. $H_k(X^n) = 0$ for all $k > n$. If X is finite dimensional, then $H_k(X^n) = 0$ for all $k > \dim X$.
3. The inclusion $X^n \hookrightarrow X$ induces $H_k(X^n) \rightarrow H_k(X)$. Then this map is
 - an isomorphism for $k < n$
 - surjective for $k = n$
 - zero for $k > n$.

Exercise. Check 2. and 3. directly in the case that the [CW complex](#) structure is a [\$\Delta\$ -complex](#) structure using [simplicial chains](#).

Proof. For 1., we see that

$$X^n / X^{n-1} \cong \text{wedge of one } n\text{-sphere for each } n\text{-cell.}$$

Now we prove 2. and 3., We consider the long exact sequence of a pair for fixed n ,

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_{k+1}(X^n, X^{n-1}) & \longrightarrow & H_k(X^{n-1}) & & \\ & & & \searrow \cong & & & \\ & & H_k(X^n) & \longrightarrow & H_k(X^n, X^{n-1}) & \longrightarrow & \dots \end{array}$$

When $k+1 < n$ or $k > n$ then $H_{k+1}(X^n, X^{n-1}) = 0$ and $H_k(X^n, X^{n-1}) = 0$, so the above map $H_k(X^{n-1}) \rightarrow H_k(X^n)$ is an isomorphism. We also get sequences telling us the injective and surjective maps when $k = n$ or $k = n-1$,

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 = H_{n+1}(X^n, X^{n-1}) & \longrightarrow & H_n(X^{n-1}) & \longrightarrow & H_n(X^n) \\ & & & & \searrow & & \\ & & H_n(X^n, X^{n-1}) & \longrightarrow & H_{n-1}(X^{n-1}) & \longrightarrow & H_{n-1}(X^n) \\ & & & & \searrow & & \\ & & H_{n-1}(X^n, X^{n-1}) = 0 & \longrightarrow & \dots & & \end{array}$$

So the maps $H_n(X^{n-1}) \rightarrow H_n(X^n)$ is injective, and the map $H_{n-1}(X^{n-1}) \rightarrow H_{n-1}(X^n)$ is surjective.

Fix k , then we get a pile of maps induced by the inclusions $X^n \hookrightarrow X^{n+1}$

$$\begin{array}{ccccccc} H_k(X^0) & \xrightarrow{\cong} & H_k(X^1) & \xrightarrow{\cong} & H_k(X^2) & \xrightarrow{\cong} & \dots \\ & & & & \searrow \cong & & \\ & & H_k(X^{k-1}) & \xrightarrow{\text{inj.}} & H_k(X^k) & \xrightarrow{\text{surj.}} & H_k(X^{k+1}) \\ & & & & \searrow \cong & & \\ & & H_k(X^{k+2}) & \xrightarrow{\cong} & H_k(X^{k+3}) & \xrightarrow{\cong} & \dots \end{array}$$

Note. This sequence is not exact. Descriptions of maps (in red) follow from our analysis of the long exact sequence of a pair above.

To prove 2.,

- $k = 0$, we do this by hand.
- $k \geq 1$, then $H_k(X^0) = 0$, so we have that $H_k(X^0), \dots, H_k(X^{k-1})$ are all zero from the isomorphisms above. That is the k -th homology $H_k(X^n) = H_k(X^n)$ is zero for every n -skeleton where $n < k$, just as desired.

We also have the following collection of maps for fixed k

$$H_k(X^k) \xrightarrow{\text{surj.}} H_k(X^{k+1}) \xrightarrow{\cong} H_k(X^{k+2}) \xrightarrow{\cong} \dots$$

This implies 3. when X is finite dimensional. For general X , we use the fact that every [simplex](#) has image contained in some finite skeleton (since image is compact). ■

Proof of Theorem 4.14. We get some [exact sequences](#) from our [preliminaries](#),

$$0 = H_{n+1}(X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^n, X^{n+1}) \longrightarrow H_{n-1}(X^n, -1)$$

$$H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) = 0$$

These come from the long exact sequences of a pair combined with the things we've deduced in the [preliminaries](#). We can paste these together into a diagram, we have

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & \nearrow & \\
 & & & & H_n(X^{n+1}) \cong H_n(X) & & \\
 & & & \nearrow & & & \\
 0 & & & H_n(X^n) & & & \\
 \nearrow & & \searrow & \nearrow & \searrow & & \\
 \partial_{n+1} & & j_n & & & & \\
 \nearrow & & \searrow & & & & \\
 \dots \rightarrow H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) & \rightarrow \dots \\
 & & \searrow & \nearrow & & & \\
 & & \partial_n & j_{n-1} & & & \\
 & & \nearrow & & & & \\
 & & 0 & & & &
 \end{array}$$

Hatcher[HPM02] tells us this diagram commutes, and what we've done here tells us that the two red diagonal pieces crossing at $H_n(X^n)$ are [exact](#). We also have [exactness](#) of the bottom right diagonal by just going down a degree.

Then this has to at least be a [chain complex](#) since the diagram commutes because of Hatcher[HPM02]. We then know that

$$d_n \circ d_{n+1} = (j_{n-1} \circ \underbrace{\partial_n \circ j_n}_0 \circ \partial_{n+1}) = 0.$$

By [exactness](#), we know that if $\iota_* : H_n(X^n) \rightarrow H_n(X^{n+1})$ then using the first isomorphism theorem,

$$H_n(X) \cong H_n(X^{n+1}) = \text{Im } \iota_* \cong H_n(X^n) / \ker \iota_* = H_n(X^n) / \text{Im } \partial_{n+1}.$$

Since j_n injects by [exactness](#),

$$\begin{aligned}
 j_n : H_n(X^n) &\xrightarrow{\cong} j_n(H_n(X^n)) \\
 \text{Im } \partial_{n+1} &\xrightarrow{\cong} \text{Im}(j_n \circ \partial_{n+1}) = \text{Im } d_{n+1}
 \end{aligned}$$

j_{n-1} must also inject by [exactness](#), and so applying [exactness](#):

$$\ker d_n = \ker \partial_n = \text{Im } j_n.$$

Then we just do some group theory, the n -th [cellular homology group](#) is

$$\ker d_n / \operatorname{Im} d_{n+1} \cong \operatorname{Im} j_n / \operatorname{Im}(j_n \circ \partial_{n+1}) \cong H_n(X^n) / \operatorname{Im} \partial_{n+1} \cong H_n(X).$$

There is one thing left to show, namely commutativity of this map. We claim that the differentials $d_n = j_n \circ \partial_{n+1}$ satisfy the formula (in terms of degree) that we stated. This is done by direct analysis of definitions of maps; details in Hatcher[\[HPM02\]](#). ■

Appendix

A Additional Proofs

A.1 Seifert-Van Kampen Theorem on Groupoid

Theorem A.1 (Seifert-Van Kampen Theorem on groupoid). Given X_0, X_1, X as topological spaces with $X_0 \cup X_1 = X$. Then the functor $\Pi: \underline{\text{Top}} \rightarrow \underline{\text{Gpd}}$ maps the [cocartesian](#) diagram in $\underline{\text{Top}}_*$ to a [cocartesian](#) diagram in $\underline{\text{Gp}}$ as follows.

$$\begin{array}{ccccc} (X_0 \cap X_1, x_0) & \xrightarrow{j_0} & (X_0, x_0) & & \Pi(X_0 \cap X_1) \xrightarrow{\Pi(j_0)} \Pi(X_0) \\ j_1 \downarrow & & \downarrow i_0 & \xrightarrow{\Pi} & \Pi(j_1) \downarrow \quad \downarrow \Pi(i_0) \\ (X_1, x_0) & \xrightarrow{i_1} & (X, x_0) & & \Pi(X_1) \xrightarrow{\Pi(i_1)} \Pi(X) \end{array}$$

Note. Notice that X_0, X_1, X don't need to be [path](#)-connected in particular.

Surprisingly, the proof of [Appendix A.1](#) is much more elegant with the elementary proof of [Theorem 2.7](#), hence we give the proof here.

Proof. Let $\mathcal{G} \in \text{Ob}(\underline{\text{Gpd}})$ a [groupoid](#), and given [functors](#)

$$F: \Pi(X_0) \rightarrow \mathcal{G}, \quad G: \Pi(X_1) \rightarrow \mathcal{G}$$

such that

$$\begin{array}{ccc} \Pi(X_0 \cap X_1) & \xrightarrow{\Pi(j_0)} & \Pi_1(X_0) \\ \Pi(j_1) \downarrow & & \downarrow \Pi(i_0) \\ \Pi_1(X_1) & \xrightarrow{\Pi(i_1)} & \Pi_1(X) \end{array} \quad \begin{array}{c} \xrightarrow{F} \\ \searrow \exists! K \\ \xrightarrow{G} \end{array} \mathcal{G}$$

We now only need to prove that there exists a unique [functor](#) $K: \Pi(X) \rightarrow \mathcal{G}$ such that the above diagram commutes.

We can define K as

- on [objects](#): For all $x \in \text{Ob}(\Pi(X)) = X$,

$$K(x) = \begin{cases} F(x), & \text{if } x \in X_0; \\ G(x), & \text{if } x \in X_1. \end{cases}$$

This is well-defined since the diagram (without K) commutes.

- on [morphisms](#): For every $p, q \in X$, $\langle \gamma \rangle : p \rightarrow q$ in $\text{Hom}_{\Pi(X)}(p, q)$, we need to define $K(\langle \gamma \rangle) \in \text{Hom}_{\mathcal{G}}(K(p), K(q))$. Our strategy is for every path γ

from p to q , we define $\tilde{K}(\gamma) \in \text{Hom}_{\mathcal{G}}(K(p), K(q))$. Then if we also have $\tilde{K}(\gamma) = \tilde{K}(\gamma')$ for $\gamma \simeq \gamma' \text{ rel}\{0, 1\}$, then we can just let

$$K(\langle \gamma \rangle) := \tilde{K}(\gamma).$$

Now we start to construct \tilde{K} .

Given a path $\gamma: [0, 1] \rightarrow X$, $\gamma(0) = p, \gamma(1) = q$. Since $\text{int}(X_0) \cup \text{int}(X_1) = X$, we see that

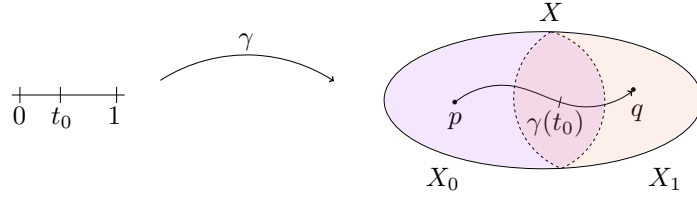
$$\gamma^{-1}(\text{int}(X_0)) \cup \gamma^{-1}(\text{int}(X_1)) = [0, 1].$$

From Lebesgue Lemma²¹, there exists a finite partition

$$0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1$$

such that for every i ,

$$\gamma([t_{i-1}, t_i]) \subset \text{int}(X_0) \text{ or } \text{int}(X_1).$$



Now, let $\gamma_i: [0, 1] \rightarrow X, t \mapsto \gamma((1-t)t_{i-1} + t \cdot t_i)$, we see that γ_i is either a [path](#) in X_0 or X_1 . We then define $\tilde{K}(\gamma) := \tilde{K}(\gamma_m) \circ \tilde{K}(\gamma_{m-1}) \circ \dots \circ \tilde{K}(\gamma_1) \in \text{Hom}_{\mathcal{G}}(K(p), K(q))$ such that

$$\tilde{K}(\gamma_i) = \begin{cases} F(\langle \gamma_i \rangle), & \text{if } \gamma_i \subset X_0; \\ G(\langle \gamma_i \rangle), & \text{if } \gamma_i \subset X_1. \end{cases}$$

We need to prove that $\tilde{K}(\gamma)$ does not depend on the partition. It's sufficient to prove that for any partition

$$0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1,$$

we consider any **finer** partition

$$0 = t_0 = t_{10} < t_{11} < \dots < t_{1K_1} = t_1 = t_{20} < t_{21} < \dots < t_{mK_m} = t_m = 1.$$

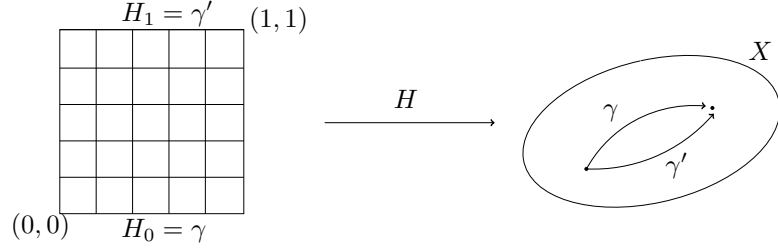
As before, we denote $\gamma_{ij}: [0, 1] \rightarrow X, t \mapsto \gamma((1-t)t_{ij-1} + t \cdot t_{ij})$. It's clear that as long as

$$\tilde{K}(\gamma_i) = \tilde{K}(\gamma_{iK_i}) \circ \tilde{K}(\gamma_{iK_i-1}) \circ \dots \circ \tilde{K}(\gamma_{i0}),$$

²¹https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma

then our claim is proved. But this is immediate since F and G are [functor](#) and for any i , we only use either F or G all the time.

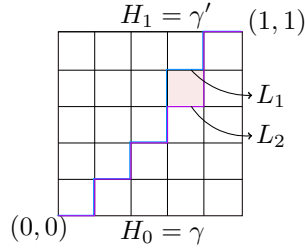
Now we prove $\gamma \underset{H}{\simeq} \gamma' \text{ rel}\{0, 1\}$, then $\tilde{K}(\gamma) = \tilde{K}(\gamma')$. This is best shown by some diagram.



The left-hand side represents a partition \mathcal{P} of $[0, 1] \times [0, 1]$ such that every small square's image in X under H is either entirely in X_0 or in x_1 . Consider all paths from $(0, 0)$ to $(1, 1)$ such that it only goes right or up. We see that for any such path L , consider

$$\gamma_L: [0, 1] \rightarrow L, \quad t \mapsto \gamma_L(t).$$

We let $\Gamma_L: H|_L \circ \gamma_L: [0, 1] \rightarrow X$, we see that Γ_L is a [path](#) from p to q . Now, if for two paths L_1 and L_2 such that they only differ from a square.



We claim that $\gamma_{L_1}, \gamma_{L_2}$ are two [paths](#) from p to q , and $\tilde{K}(\Gamma_{L_1}) = \tilde{K}(\Gamma_{L_2})$. Now, we denote Γ_0 and Γ_1 as follows.

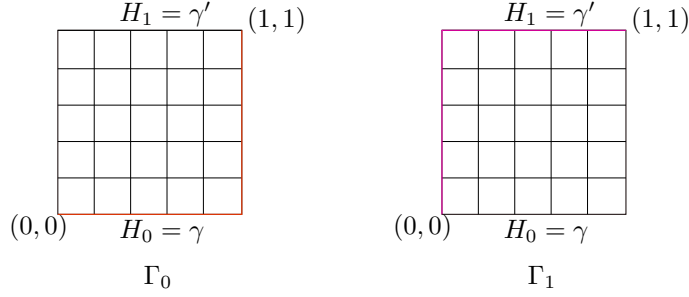


Figure 23: The definition of Γ_0 and Γ_1 .

It's clearly that by only finitely many steps, we can transform Γ_0 to Γ_1 , hence

$$\tilde{K}(\Gamma_0) = \tilde{K}(\Gamma_1).$$

Finally, we observe that

$$\tilde{K}(\gamma_0) = \tilde{K}(\Gamma_0) = \tilde{K}(\Gamma_1) = \tilde{K}(\gamma_1).$$

If we now define $K(\langle \gamma \rangle) = \tilde{K}(\gamma)$, then $K: \text{Mor}(\Pi(X)) \rightarrow \text{Mor}(\mathcal{G})$, then it's well-defined.

We now prove $K: \Pi(X) \rightarrow \mathcal{G}$ is indeed a **functor**. But this is immediate from the definition of K , namely it'll send identity to identity and the composition associates.

Also, we need to prove that the following diagram commutes.

$$\begin{array}{ccc}
 \Pi(X_0 \cap X_1) & \xrightarrow{\Pi(j_0)} & \Pi_1(X_0) \\
 \Pi(j_1) \downarrow & & \downarrow \Pi(i_0) \\
 \Pi_1(X_1) & \xrightarrow{\Pi(i_1)} & \Pi_1(X) \\
 & \searrow G & \downarrow K \\
 & & \mathcal{G}
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow F \\
 \searrow K
 \end{array}$$

But this is again trivial.

Finally, we need to show that such K is unique. This is the same as the proof of [Lemma 1.3](#), hence the proof is done. ■

A.2 An alternative proof of **Seifert Van-Kampen Theorem**

Theorem A.2. We claim that the diagram

$$\begin{array}{ccc} \pi_1(X_0 \cap X_1, x_0) & \xrightarrow{(j_0)_*} & \pi_1(X_0, x_0) \\ (j_1)_* \downarrow & & \downarrow (i_0)_* \\ \pi_1(X_1, x_0) & \xrightarrow{(i_1)_*} & \pi_1(X, x_0) \end{array}$$

is [cocartesian](#).

Proof. The basic idea is that, for this diagram,

$$\begin{array}{ccc} \Pi(X_0 \cap X_1) & \longrightarrow & \Pi(X_0) \\ \downarrow & & \downarrow \\ \Pi(X_1) & \longrightarrow & \Pi(X) \end{array}$$

we want to construct a [morphism](#) $r: \Pi(Z) \rightarrow \pi_1(Z, p)$ in \mathbf{Gpd} such that $Z = X_0 \cap X_1, X_0, X_1, X$. For every $x \in Z$, we fix a [path](#) γ_x such that it connects p and x and satisfies

1. If $x \in X_0 \cap X_1$, then $\text{Im}(\gamma_x) \subset X_0 \cap X_1$
2. If $x \in X_0$, then $\text{Im}(\gamma_x) \subset X_0$
3. If $x \in X_1$, then $\text{Im}(\gamma_x) \subset X_1$
4. $\gamma_p = c_p$

The proof is given in https://www.bilibili.com/video/BV1P7411N7fW?p=38&spm_id_from=pageDriver. ■

If have time.

B Abelian Group

This section aims to give some reference about [Abelian groups](#), specifically for [free Abelian group](#), which is used heavily when discuss homology.

B.1 Abelian Group

Definition B.1 (Abelian group). A group (G, \cdot) is an *Abelian group* if for every $a, b \in G$, we have

$$a \cdot b = b \cdot a.$$

We often denote \cdot as $+$ if (G, \cdot) is a [Abelian group](#).

Definition B.2 (Product of groups). Given two groups $(G, \cdot), (H, \cdot)$, the *product of G and H* , denoted by $G \times H$ is defined as

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

and

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot g_2, h_1 \cdot h_2).$$

Notation. For simplicity, given an index set I , we'll denote the order pair $(g_{\alpha_1}, g_{\alpha_2}, \dots)$ as $(g_{\alpha})_{\alpha \in I}$. Note that the latter notation can handle the case that I is either countable or uncountable, while the former can only handle the countable case.

Definition B.3 (Direct product). Given $(G_{\alpha}, +)$, $\alpha \in I$ as a collection of [Abelian group](#), we define their *direct product* as

$$\left(\prod_{\alpha \in I} G_{\alpha}, + \right),$$

where

$$\prod_{\alpha \in I} G_{\alpha} = \{(g_{\alpha})_{\alpha \in I} \mid g_{\alpha} \in G_{\alpha}\}$$

and $\forall (g_{\alpha}), (h_{\alpha}) \in \prod_{\alpha \in I} G_{\alpha}$

$$(g_{\alpha}) + (h_{\alpha}) := g_{\alpha} + h_{\alpha}$$

for all $\alpha \in I$.

Specifically, if I is finite, namely there are only finitely many [Abelian groups](#)

$(G_1, +), \dots, (G_n, +)$, and $\left(\prod_{i=1}^n G_i, + \right)$ can be denoted as

$$(G_1 \times \dots \times G_n, +).$$

Definition B.4 (External direct sum). Given a collection of [Abelian groups](#) $\{G_{\alpha}\}_{\alpha \in I}$, the *external direct sum* of them, denoted as $(\bigoplus_{\alpha \in I} G_{\alpha}, +)$ as

$$\bigoplus_{\alpha \in I} G_{\alpha} := \left\{ (g_{\alpha})_{\alpha \in I} \mid \forall_{\alpha \in I} g_{\alpha} \in G_{\alpha}, \# \text{ non-zero elements in } g_{\alpha} < \infty \right\}.$$

And for every $(g_{\alpha}), (h_{\alpha}) \in \bigoplus_{\alpha \in I} G_{\alpha}$,

$$(g_{\alpha}) + (h_{\alpha}) := g_{\alpha} + h_{\alpha}$$

for all $\alpha \in I$.^a

^aThis may not be the best notation: What we're really trying to say is $(g_{\alpha})_{\alpha \in I} + (h_{\alpha})_{\alpha \in I} := g_i + h_i$ for all $i \in I$.

Note. We see that

$$\bigoplus_{\alpha \in I} G_\alpha \subset \prod_{\alpha \in I} G_\alpha.$$

Additionally, we also have

$$\left(\bigoplus_{\alpha \in I} G_\alpha, + \right) < \left(\prod_{\alpha \in I} G_\alpha, + \right).$$

Remark. We see that the operation $+$ is indeed closed since the sum of $g, g' \in \bigoplus_{\alpha \in I} G_\alpha$ will have only finitely non-zero elements if g, g' both have only finitely many non-zero elements.

We see that if I is a finite index set, given a collection of [Abelian group](#) $\{G_\alpha\}_{\alpha \in I}$, then

$$G_1 \times \dots \times G_n = G_1 \oplus \dots \oplus G_n.$$

Definition B.5 (Internal direct sum). Given an [Abelian group](#) G , and a collection of the subgroups $\{G_\alpha\}_{\alpha \in I}$ of G , we say G is an *internal direct sum* of $\{G_\alpha\}_{\alpha \in I}$ if for any $g \in G$, we can write

$$g = \sum_{\alpha \in I} g_\alpha$$

uniquely, where $g_\alpha \in G_\alpha$ has only finitely many non-zero elements. In this case, we denote

$$G = \bigoplus_{\alpha \in I} G_\alpha.$$

Intuitively, the [external direct sum](#) is to build a new group based on the given collection of groups $\{G_\alpha\}_{\alpha \in I}$, while the internal direct sum is to express an **already known** group G with an **already known** collection of groups $\{G_\alpha\}_{\alpha \in I}$.

Remark (Relation between Internal and External direct sum). Given an [Abelian group](#) G and its [internal direct sum](#) decomposition $\bigoplus_{\alpha \in I} G_\alpha$, G is isomorphic to the [external direct sum](#) of $\{G_\alpha\}_{\alpha \in I}$. We see this from the following group homomorphism:

$$\forall_{g \in G} \quad g = \sum_{\alpha \in I} g_\alpha \mapsto (g_\alpha)_{\alpha \in I}.$$

Conversely, given a collection of [Abelian group](#) $\{G_\alpha\}_{\alpha \in I}$, and let $G = \bigoplus_{\alpha \in I} G_\alpha$ as the [external direct sum](#) of $\{G_\alpha\}$, denote $i_{\alpha_0} : G_{\alpha_0} \rightarrow \bigoplus_{\alpha \in I} G_\alpha$ as a canonical embedding

$$g_{\alpha_0} \mapsto i_{\alpha_0}(g_{\alpha_0}) = (h_\alpha)_{\alpha \in I},$$

where

$$h_\alpha = \begin{cases} g_{\alpha_0}, & \text{if } \alpha_0 = \alpha; \\ 0, & \text{if } \alpha_0 \neq \alpha \end{cases}$$

given α_0 . Then

$$G'_{\alpha_0} := i_{\alpha_0}(G_{\alpha_0}) < \bigoplus_{\alpha \in I} G_{\alpha}$$

and G is the **internal direct sum** of G'_{α_0} , $\alpha_0 \in I$. This is because $\forall g = (g_{\alpha})_{\alpha \in I} \in G (= \bigoplus_{\alpha \in I} G_{\alpha})$, we have

$$g = \sum_{\alpha \in I} i_{\alpha}(g_{\alpha}).$$

Note that the above sum is well-defined since there are only finitely many non-zero elements for each g_{α} . And additionally, we can see the uniqueness of this decomposition by defining π_{α_0} such that

$$\pi_{\alpha_0}: \bigoplus_{\alpha \in I} G_{\alpha} \rightarrow G_{\alpha_0}, \quad (g_{\alpha})_{\alpha \in I} \mapsto g_{\alpha_0},$$

then $\pi_{\alpha} \circ i_{\alpha} = \text{id}_{G_{\alpha}}$, $\pi_{\alpha} \circ i_{\beta} = 0$ for all $\beta \neq \alpha$ and

$$\pi_{\beta}(g) = \pi_{\beta} \left(\sum_{\alpha \in I} i_{\alpha}(g_{\alpha}) \right) = \sum_{\alpha \in I} \pi_{\beta} \circ i_{\alpha}(g_{\alpha}) = \pi_{\beta} \circ i_{\beta}(g_{\beta}) = g_{\beta}$$

for all $\beta \in I$, where the second equality is because this summation is finite. Hence, we have

$$g = \sum_{\alpha \in I} i_{\alpha}(\pi_{\alpha}(g)).$$

Definition B.6. Given two **Abelian groups** G, H , we define $\text{Hom}(G, H)$ as

$$\text{Hom}(G, H) := \{f: G \rightarrow H \mid f \text{ is a group homomorphism}\},$$

then we can define

$$\begin{aligned} +: \text{Hom}(G, H) \times \text{Hom}(G, H) &\rightarrow \text{Hom}(G, H) \\ (\varphi, \psi) &\mapsto \varphi + \psi, \end{aligned}$$

where

$$(\varphi + \psi)(g) := \varphi(g) + \psi(g).$$

Remark (Relation between direct sum and direct product). Given a collection of **Abelian groups** $\{G_{\alpha}\}_{\alpha \in I}$, and another **Abelian group** H , there exists a φ such that

$$\begin{aligned} \varphi: \text{Hom} \left(\bigoplus_{\alpha \in I} G_{\alpha}, H \right) &\rightarrow \prod_{\alpha \in I} \text{Hom}(G_{\alpha}, H) \\ f &\mapsto \varphi(f) := (f_{\alpha})_{\alpha \in I} \end{aligned}$$

where $f_{\alpha} = f \circ i_{\alpha}$, where i_{α} is the canonical embedding from G_{α} to $\bigoplus_{\alpha \in I} G_{\alpha}$. We claim that φ is an isomorphism.

- φ is injective. This is obvious since $\ker(\varphi) = 0$ from the fact that if $\varphi(f) = 0$, then $f_{\alpha} = 0$ for all α , hence f is 0.

- φ is surjective. For every $(f_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} \text{Hom}(G_\alpha, H)$, we define

$$f: \bigoplus_{\alpha \in I} G_\alpha \rightarrow H$$

$$\sum_{\alpha \in I} g_\alpha \mapsto \sum_{\alpha \in I} f_\alpha(g_\alpha).$$

We see that $f \in \text{Hom}(\bigoplus_{\alpha \in I} G_\alpha, H)$ and $\varphi(f) = (f_\alpha)_{\alpha \in I}$.

This shows that

$$\text{Hom}\left(\bigoplus_{\alpha \in I} G_\alpha, H\right) \cong \prod_{\alpha \in I} \text{Hom}(G_\alpha, H).$$

Exercise. We can show that

$$\text{Hom}\left(H, \prod_{\alpha \in I} G_\alpha\right) \cong \prod_{\alpha \in I} \text{Hom}(H, G_\alpha).$$

Note the order in the Hom matters.

B.2 Free Abelian Group

Definition B.7 (Free Abelian group). Given an Abelian group $(G, +)$, we say G is a *free Abelian group* if there exists a collection of elements $\{g_\alpha\}_{\alpha \in J}$ in G such that $\{g_\alpha\}_{\alpha \in J}$ forms a **basis** of G , i.e., for all $g \in G$, $\exists! n_\alpha \in \mathbb{Z}$ for all $\alpha \in J$ such that

$$g = \sum_{\alpha \in J} n_\alpha g_\alpha$$

with finitely many non-zero n_α .

Remark. If G is a free Abelian group, and $\{g_\alpha\}_{\alpha \in J}$ is a basis, then for every $\alpha \in J$, $\langle g_\alpha \rangle$ is an infinite cyclic group since

$$n \cdot g_\alpha = 0 = 0 \cdot g_\alpha \implies n = 0.$$

And from Definition B.7, we have

$$G = \bigoplus_{\alpha \in J} \langle g_\alpha \rangle.$$

Conversely, assume there are a collection of infinite cyclic group $\langle g_\alpha \rangle$ for $\alpha \in I$ in G such that

$$G = \bigoplus_{\alpha \in I} \langle g_\alpha \rangle,$$

then $\{g_\alpha\}_{\alpha \in I}$ is a basis of G , hence G is a free Abelian group.

Proposition B.1. If G is an [Abelian group](#), then the following are equivalent.

1. G is a [free Abelian group](#).
2. G is an [internal direct sum](#) of some infinite cyclic groups.
3. G is isomorphic to the [external direct sum](#) of some additive groups of integers \mathbb{Z} .

Proof. We see that 1. \iff 2. is already proved. And for 2. \iff 3., this follows directly from the [relation between internal and external direct sum](#). ■

Now, consider G as a [free Abelian group](#), then

$$u: G \xrightarrow{\cong} \bigoplus_{\alpha \in I} \mathbb{Z}$$

for some I . Denote $e_\alpha := i_\alpha(1) \in \bigoplus_{\alpha \in I} \mathbb{Z}$, where $i_\alpha: \mathbb{Z} \rightarrow \bigoplus_{\alpha \in I} \mathbb{Z}$ is the canonical embedding, i.e., $e_\alpha = (g_\alpha)_{\alpha \in I} \in \bigoplus_{\alpha \in I} \mathbb{Z}$, where

$$g_\beta = \begin{cases} 1, & \text{if } \beta = \alpha; \\ 0, & \text{if } \beta \neq \alpha. \end{cases}$$

Moreover, denote ϵ_α as the image of e_α under the isomorphism u , namely $\epsilon_\alpha = u^{-1}(e_\alpha)$, then $\{\epsilon_\alpha\}_{\alpha \in I}$ is a basis of G .

Now, for every [Abelian group](#) H , we have

$$\begin{array}{ccc} \text{Hom}(G, H) & \xleftarrow[\cong]{\circ u} & \text{Hom}\left(\bigoplus_{\alpha \in I} \mathbb{Z}, H\right) \\ & \searrow \cong & \downarrow \varphi \\ & & \prod_{\alpha \in I} \text{Hom}(\mathbb{Z}, H) \\ & & \downarrow \cong \\ & & \prod_{\alpha \in I} H \end{array} \quad \begin{array}{ccc} f & \xrightarrow{\quad} & f \circ u^{-1} \\ & \searrow & \downarrow \\ & & (f \circ u^{-1} \circ i_\alpha)_{\alpha \in I} \\ & & \downarrow \\ & & (f \circ u^{-1} \circ i_\alpha(1))_{\alpha \in I} \end{array}$$

where φ is the homeomorphism defined in [here](#), and the homeomorphism

$$\prod_{\alpha \in I} \text{Hom}(\mathbb{Z}, H) \xrightarrow{\cong} \prod_{\alpha \in I} H$$

is trivial since every $f \in \prod_{\alpha \in I} \text{Hom}(\mathbb{Z}, H)$ corresponds to $f(1) \in H$ uniquely. We see that

$$f \circ u^{-1} \circ i_\alpha(1) = f \circ u^{-1}(e_\alpha) = f(\epsilon_\alpha).$$

In other words, for all [Abelian group](#) H , a morphism from the set $\{\epsilon_\alpha\}_{\alpha \in I}$ to H can be uniquely extended to the group a homomorphism from G to H .

Remark. This means, to determine $\text{Hom}(G, H)$, we only need to determine where each base element in G will map to in H , and this is why it's *free*.

We now want to generate [free Abelian group](#) by a set. Roughly speaking, given a set S , we can generate a [free Abelian group](#) Z by defining

$$Z := \left\{ \sum_{x \in S} n_x x \mid n_x \in \mathbb{Z}, \# \text{ non-zero elements in } n_x < \infty \right\}$$

with the naturally defined $+$. Formally, we have the following.

Definition B.8 (Free Abelian group generated by a set). Given a set S , the [free Abelian group](#) generated by S $(Z, +)$ is defined as

$$Z := \{f: S \rightarrow \mathbb{Z} \mid \text{only finitely many } x \in S \text{ such that } f(x) \neq 0\},$$

with

$$\begin{aligned} +: Z \times Z &\rightarrow Z \\ (f, g) &\mapsto f + g. \end{aligned}$$

Remark. $\{\phi_x \mid x \in S\}$ forms a basis of Z , where $\phi_x: S \rightarrow \mathbb{Z}$ such that

$$y \mapsto \phi_x(y) = \begin{cases} 1, & \text{if } y = x; \\ 0, & \text{if } y \neq x \end{cases}$$

is the characteristic function at x . We see this by for all $f \in S$, $f = \sum_{x \in S} f(x) \phi_x$, which is uniquely defined. Hence, $(Z, +)$ is a [free Abelian group](#).

Note. Note that

$$\begin{aligned} S &\xrightarrow{1:1} \{\phi_x \mid x \in S\} \\ x &\mapsto \phi_x. \end{aligned}$$

Hence, we often denote the element $\sum_{x \in S} \underbrace{n_x}_{f(x)} \phi_x$ in Z as

$$\sum_{x \in S} n_x \cdot x.$$

Theorem B.1 (The universal property of free Abelian group generated by a set). Denote a canonical embedding $i: S \rightarrow Z$, $x \mapsto \phi_x$. Then for all [Abelian group](#) H and $f: S \rightarrow H$, there exists a unique group homomorphism

$$\tilde{f}: Z \rightarrow H$$

such that $\tilde{f} \circ i = f$.

Proof. We define

$$\tilde{f}\left(\sum_{x \in S} n_x \cdot x\right) := \sum_{x \in S} n_x f(x),$$

and the uniqueness is obvious. ■

Note that we can use the above [universal property](#) to describe a [free Abelian group](#) since we have the following.

Proposition B.2. Given Z' as another [Abelian group](#) and $i': S \rightarrow Z'$ as another canonical embedding such that for all [Abelian group](#) H and $f: S \rightarrow H$, there exists a unique group homomorphism $\tilde{f}: Z' \rightarrow H$ such that $\tilde{f} \circ i' = f$, then

$$Z' \cong Z.$$

Namely, we can describe a [free Abelian group](#) by its [universal property](#) uniquely up to isomorphism.

Theorem B.2. Assume G is a [free Abelian group](#). Assume there exists a finite basis $\{g_1, \dots, g_n\}$ of G , and also assume that there exists another basis $\{h_\alpha\}_{\alpha \in I}$. Then we have

$$\text{card}(I) < \infty,$$

specifically, we have

$$\text{card}(I) = n.$$

Proof. Firstly, we observe that if we can show

$$\text{card}(I) \leq n,$$

then by swapping $\{h_\alpha\}_{\alpha \in I}$ and $\{g_\alpha\}_{\alpha \in I}$, we will have $\text{card}(I) = n$.

Suppose I is an infinite set, then we can find $h_{\alpha_1}, \dots, h_{\alpha_m}$ such that $m > n$ and $h_{\alpha_i} \neq h_{\alpha_j}$ for $i \neq j$. Then since $\{g_\alpha\}_{\alpha \in I}$ is a basis, we have

$$h_{\alpha_i} = \sum_{j=1}^n k_i^j g_j, \forall i = 1, \dots, m.$$

Specifically, we have

$$\begin{pmatrix} h_{\alpha_1} \\ \vdots \\ h_{\alpha_m} \end{pmatrix} = \underbrace{\begin{pmatrix} k_1^1 & k_1^2 & \dots & k_1^n \\ \vdots & & \ddots & \vdots \\ k_m^1 & k_m^2 & \dots & k_m^n \end{pmatrix}}_{K \in M_{m \times n}(\mathbb{Z}) \subset M_{m \times n}(\mathbb{Q})} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix},$$

where $k_i^j \in \mathbb{Z}$. From linear algebra, we know that there exists $(r_1, \dots, r_m) \in \mathbb{Q}^m \setminus \{0\}$ such that

$$(r_1, \dots, r_m)K = (0, \dots, 0).$$

Multiplying both sides with the common multiple of the denominator of r_i , we see that there exists $(\ell_1, \dots, \ell_m) \in \mathbb{Z}^m \setminus \{\vec{0}\}$ such that

$$\begin{aligned} (\ell_1, \dots, \ell_m)K &= (0, \dots, 0) \\ \implies (\ell_1, \dots, \ell_m) \begin{pmatrix} h_{\alpha_1} \\ \vdots \\ h_{\alpha_m} \end{pmatrix} &= (\ell_1, \dots, \ell_m)K \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = (0, \dots, 0) \\ \implies \sum_{i=1}^m \ell_i h_{\alpha_i} &= \vec{0} \text{ for } (\ell_1, \dots, \ell_m) \in \mathbb{Z}^m \setminus \{\vec{0}\} \not\Rightarrow \\ \implies \text{card}(I) &< \infty. \end{aligned}$$

From the same argument, we see that $\text{card}(I) \leq n \implies \text{card}(I) = n$. ■

Remark. Furthermore, one can prove that if G is a [free Abelian group](#), then we can prove that any two bases of G are equinumerous, which handle the case that the basis is an infinite set.

This induces the following definition.

Definition B.9 (Rank). Let G be a [free Abelian group](#), the *rank* of G is the cardinality of any basis of G .

B.3 Finitely Generated Abelian Group

Since we're going to encounter some group as

$$\mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z},$$

so it's useful to look into those finitely generated [Abelian group](#).

Let's start with a definition.

Definition B.10 (Torsion subgroup). Given an [Abelian group](#) G , we say that $g \in G$ has finite order if $\exists n \in \mathbb{Z}$ such that $n \cdot g = 0$. Specifically, we say that

$$T := \{g \in G \mid g \text{ has finite order}\}$$

is a *torsion subgroup*.

If $T = 0$ given G , we say that G is *torsion free*.

Note. Note that T is indeed a subgroup, since for any $g_1, g_2 \in T$, $g_1 + g_2 \in T$ from the fact that it still has finite order.

Remark. If G is a [free Abelian group](#), then G is [torsion free](#). Conversely, if G is [torsion free](#), we can't deduce G is a [free Abelian group](#). We see this from $(\mathbb{Q}, +)$. Firstly, we see that \mathbb{Q} is [torsion free](#). Now, suppose \mathbb{Q} is a [free Abelian group](#), then there exists a basis $\{r_\alpha\}_{\alpha \in I}$ of \mathbb{Q} such that $|I| > 1$. Now, consider

$\alpha_1, \alpha_2 \in I$ such that $\alpha_1, \alpha_2 \in I$, for $r_{\alpha_1}, r_{\alpha_2}$, there exists $n, m \in \mathbb{Z}$ and $n, m \neq 0$ such that

$$nr_{\alpha_1} + mr_{\alpha_2} = 0 \implies n = m = 0 \not\downarrow$$

B.3.1 Classification of Finitely generated Abelian Group

Given a finitely generated [Abelian group](#) G , we may assume its generators are g_1, \dots, g_n . Let F be

$$F := \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n \text{ times}},$$

then there are a natural surjective homomorphism

$$\varphi: F \rightarrow G, \quad e_i \mapsto g_i$$

where $e_i = (0, \dots, 0, \underset{i^{th}}{1}, 0, \dots, 0)$. Now, let $K := \ker \varphi$, we have

$$G \cong F / K.$$

Then we have the following lemma.

Lemma B.1. K is a finitely generated [Abelian group](#).

Proof.

\mathbb{Z} is Noetherian, F is a finitely generated \mathbb{Z} -module
 $\implies F$ is Noetherian module
 $\implies K$ as a submodule of F is a finitely generated \mathbb{Z} -module
 $\implies K$ is a finitely generated [Abelian group](#).

Please refer all the concepts above from [\[AM94\]](#). ■

Hence, we may assume the generators of K as b_1, \dots, b_m . From the definition of K , we can further express b_i as

$$b_i = (b_{i1}, b_{i2}, \dots, b_{in}) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}_{n \times n}$$

for all $i = 1, \dots, m$. Denote all such row vectors b_i in a matrix B , namely

$$B := \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix} \in M_{m \times n}(\mathbb{Z}),$$

then we have

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = B \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Multiply a matrix on the right-hand side. Now, consider a $p \in \text{GL}(n; \mathbb{Z})$, then

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = B \cdot \underbrace{P P^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}}_{\text{new basis}} = (BP) \cdot \begin{pmatrix} e'_1 \\ \vdots \\ e'_n \end{pmatrix},$$

where

$$P^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} =: \begin{pmatrix} e'_1 \\ \vdots \\ e'_n \end{pmatrix}.$$

We see that $B \cdot P$ is the coefficient matrix of generators b_1, \dots, b_m under the new basis e'_1, \dots, e'_n .

Multiply a matrix on the left-hand side. For a $A \in \text{GL}(m; \mathbb{Z})$, then

$$\begin{pmatrix} b'_1 \\ \vdots \\ b'_m \end{pmatrix} = Q \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = QB \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix},$$

since Q is invertible, hence b'_1, \dots, b'_m are also generators of K . We see that QB is the coefficient matrix of new generators b'_1, \dots, b'_m under basis e_1, \dots, e_n .

Generally $Q \cdot B \cdot P$ is the matrix representation of a particular set of F 's generators under a particular basis.

Proposition B.3. There exists $P \in \text{GL}(n; \mathbb{Z})$ and $Q \in \text{GL}(m; \mathbb{Z})$ such that

$$Q \cdot B \cdot P = \begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

where $d_i \in \mathbb{Z}^+$ and $d_1 \mid d_2 \mid \dots \mid d_k$.

Proof. In fact, P, Q can be taken as the multiplication of the following three types of square matrices:

- P_{ij} :

$$P_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1_{(ij)} \\ & & & \ddots & \\ & 1_{(ji)} & & 0 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

where the effect of multiplying P_{ij} from the right is *swapping column i, j* .

- $P_i(c)$, where c is the identity in \mathbb{Z} , i.e., $c = \pm 1$:

$$P_i(c) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c_{(ii)} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

where the effect of multiplying $P_i(c)$ from the right is *multiplying c to column i* .

- $P_{ij}(a)$, $a \in \mathbb{Z}$:

$$P_{ij} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & a_{(ij)} & \\ & & & \ddots \\ & & & & 1 \end{pmatrix},$$

where the effect of multiplying $P_{ij}(a)$ from the right is *adding a times column i to column j* .

We see that these are *elementary column transformations* in linear algebra. In particular, if we multiply these matrices from the left, then it's called *elementary row transformations*.

That is to say, we're going to show

$$B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix}$$

can become

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 \\ & & & & \ddots \end{pmatrix},$$

$d_i \in \mathbb{Z}^+$, $d_1 \mid d_2 \mid \dots \mid d_k$ from *elementary column/row transformations*.

We now show the steps to make this happens.

Step 1. Using elementary transformations, we make $b_{11} > 0$.

Step 2. Using elementary transformations, we make b_{11} become a divisor of all elements in the first column and row.

We see that if $b_{11} \nmid b_{1i}$ for $i \neq 1$, we have $b_{1i} = r \cdot b_{11} + s$ where $0 < s < b_{11}$. Then we add $(-r)$ times the 1^{th} column to the i^{th} column and swapping the 1^{th} and the i^{th} column, which makes B becomes

$$\begin{pmatrix} s & \dots \\ \vdots & \ddots \end{pmatrix},$$

for $0 < s < b_{11}$. Since $\text{card}(\{n \in \mathbb{Z} \mid 0 < n < b_{11}\}) < \infty$, hence in finitely many steps we can make B becomes

$$\begin{pmatrix} d_1 & \cdots \\ \vdots & \ddots \end{pmatrix},$$

where d_1 is a divisor of all other elements in the first column and row.

Step 3. Using elementary transformations, we can multiply the first row by a proper integer and add it to the other rows, do the same but for columns also, then we can make B becomes

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B_1 & \\ 0 & & & \end{pmatrix}.$$

Step 4. We iteratively apply Step 1. to step 3., we make B into

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

where $d_i \in \mathbb{Z}^+$.

Step 5. Using elementary transformations, by swapping columns and rows, we may assume $d_1 \leq d_2 \leq \dots \leq d_k$.

Step 6. Using elementary transformations, we can make B into

$$\begin{pmatrix} d'_1 & & & \\ & \ddots & & \\ & & d'_\ell & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

such that $0 < d'_1 \leq \dots \leq d'_\ell$, $d'_1 \mid d'_2 \mid \dots \mid d'_\ell$ since if $d_1 \nmid d_i$ for some $i \in \{2, \dots, k\}$, then

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \rightarrow \begin{pmatrix} d_1 & d_i & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

then from Step 2., we have

$$\begin{pmatrix} s & \cdots \\ \vdots & \ddots \end{pmatrix}$$

where $0 < s < d_1$ and s is a divisor of all other elements in the first row and column. Now, we repeat Step 3. to Step 5., we obtain

$$\begin{pmatrix} \tilde{d}_1 & & & \\ & \ddots & & \\ & & \tilde{d}_j & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

where $\tilde{d}_1 \leq \dots \leq \tilde{d}_j$ such that $\tilde{d}_1 < d_1$. Since there are only finitely many integers which is smaller than d_1 , we see that by repeating these steps, we can always make

$$\begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_k & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

into

$$\begin{pmatrix} \tilde{d}_1 & & & \\ & \ddots & & \\ & & \tilde{d}_p & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

such that $d'_1 \mid d'_i$ for all $i \neq 1$ and $d'_1 \leq d'_2 \leq \dots \leq d'_p$. By the same idea of Step 3., we have the desired matrix.

Since all operations are elementary and there are only finitely many of them, hence the result follows. \blacksquare

From the definition of $Q \cdot B \cdot P$ and [Proposition B.3](#), there exists a basis e'_1, \dots, e'_n of F such that K has finitely many generators $d_1 e'_1, \dots, d_k e'_k$, hence

$$G \cong \mathbb{Z} / d_1 \mathbb{Z} \oplus \mathbb{Z} / d_2 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / d_k \mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n-k \text{ times}}.$$

This leads to the following important theorem.

Theorem B.3 (Fundamental theorem of finitely generated Abelian group). Given a finitely generated [Abelian group](#), either G is a [free Abelian group](#), or there exists a unique set of $\{m_i \in \mathbb{Z} \mid m_i > 1, i = 1, \dots, t\}$ such that $m_1 \mid m_2 \mid \dots \mid m_t$ and a unique non-negative integer s such that

$$G \cong \mathbb{Z} / m_1 \mathbb{Z} \oplus \mathbb{Z} / m_2 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}}.$$

Proof. We need to show both uniqueness and existence.

Existence. From [Proposition B.3](#), we obtain a basis e'_1, \dots, e'_n of F and a basis $d_1 e'_1, \dots, d_k e'_k$ in K such that $d_1 \mid \dots \mid d_k$. Let

$$(d_1, \dots, d_k) = (1, \dots, 1, m_1, \dots, m_t),$$

which implies

$$\begin{aligned} G &\cong F / K \\ &\cong \mathbb{Z} / d_1 \mathbb{Z} \oplus \mathbb{Z} / d_2 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / d_k \mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \\ &= \mathbb{Z} / 1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / 1 \mathbb{Z} \oplus \mathbb{Z} / m_1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \\ &= \mathbb{Z} / m_1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{\exists! s \text{ times}}. \end{aligned}$$

Uniqueness. Under the isomorphism $\mathbb{Z} / m_1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}}$,

we see that

$$\mathbb{Z} / m_1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / m_t \mathbb{Z}$$

corresponds to G 's [torsion subgroup](#) T , which implies

$$G / T \cong \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}},$$

which further implies G / T is a [free Abelian group](#) with

$$\text{rk} \left(G / T \right) = s,$$

which proves the uniqueness of s .

The proof of the uniqueness of m_i are long and tedious, we refer to [\[Arm13\]](#). ■

Definition B.11 (Invariant factor). We call m_1, \dots, m_t obtained from [Theorem B.3](#) the *invariant factor*.

Lemma B.2. Given a positive integer m such that

$$m = p_1^{n_1} \cdot \dots \cdot p_s^{n_s}$$

where $p \in \mathcal{P}$ are all prime and $p_i \neq p_j$ for $i \neq j$, with $n_i \in \mathbb{Z}^+$ for all i . Then

$$\mathbb{Z} / m \mathbb{Z} \cong \mathbb{Z} / p_1^{n_1} \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / p_s^{n_s} \mathbb{Z}.$$

Proof. We define ϕ as

$$\begin{aligned} \phi: \mathbb{Z} / m \mathbb{Z} &\rightarrow \mathbb{Z} / p_1^{n_1} \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / p_s^{n_s} \mathbb{Z} \\ \bar{n} &\mapsto (n + \langle p_1^{n_1} \rangle, \dots, n + \langle p_s^{n_s} \rangle). \end{aligned}$$

Then $\bar{n} \in \ker \phi \iff \forall_i p_i^{n_i} \mid n \iff m \mid n \iff \bar{n} = \bar{0}$. This means $\ker \phi = 0$, hence ϕ is an injection.

We now prove ϕ is a surjection. It's sufficient to prove that for all i ,

$$(0, \dots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \dots, 0) \in \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_s^{n_s}\mathbb{Z},$$

there exists an \bar{n} such that

$$\phi(\bar{n}) = (0, \dots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \dots, 0).$$

Notice that for all $i \neq j$, $\langle p_i^{n_i} \rangle + \langle p_j^{n_j} \rangle \in \mathbb{Z}$, hence there exists $u_j \in \langle p_i^{n_i} \rangle$ and $v_j \in \langle p_j^{n_j} \rangle$ such that $u_j + v_j = 1$. Let n as

$$n = \prod_{i \neq j} (1 - u_j),$$

then

$$n + \langle p_i^{n_i} \rangle = 1 + \langle p_i^{n_i} \rangle, \quad n + \langle p_j^{n_j} \rangle = 0 + \langle p_j^{n_j} \rangle.$$

Above implies

$$\phi(\bar{n}) = (0, \dots, 0, 1 + \langle p_i^{n_i} \rangle, 0, \dots, 0),$$

hence ϕ surjects, so

$$\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_s^{n_s}\mathbb{Z}.$$

■

Combine [Theorem B.3](#) and [Lemma B.2](#), we see that we now only have

$$G \cong \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}},$$

we can further decompose G into

$$G \cong \mathbb{Z}/p_1^{s_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_k^{s_k}\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{s \text{ times}},$$

where p_1, \dots, p_k are primes (which may includes repeated terms), $s_i \in \mathbb{Z}^+$ for all i .

Definition B.12 (Elementary divisors). The set

$$\{p_1^{s_1}, \dots, p_k^{s_k}\}$$

are called *elementary divisors* of G .

Theorem B.4 (Uniqueness of elementary divisors). [Elementary divisors](#) of a group G is unique.

Proof. Please refer to [\[Arm13\]](#).

■

C Homological Algebra

As previously seen. Given two [Abelian groups](#) A, B and the group homomorphism $\varphi: A \rightarrow B$, then we have

- $\ker \varphi = \{x \in A \mid \varphi(x) = 0\}$
- $\operatorname{Im} \varphi = \{\varphi(x) \mid x \in A\}$
- $\operatorname{coker} \varphi := B / \operatorname{Im} \varphi$
- $\operatorname{coIm} \varphi := A / \ker \varphi$

Consider a sequence of [Abelian](#) group homomorphism

$$\dots \longrightarrow A_{i-1} \xrightarrow{\phi_{i-1}} A_i \xrightarrow{\phi_i} A_{i+1} \longrightarrow \dots$$

We denote this sequence as S .

Definition C.1 (Exact). We say S is *exact* at A_i if

$$\operatorname{Im} \phi_{i-1} = \ker \phi_i.$$

Remark. [Definition C.1](#) is same as [Definition 4.13](#).

Definition C.2 (Exact sequence). We call S is an *exact sequence* if it's [exact](#) at A_i for all i .

Remark. Specifically, consider the following two situations.

- We say

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots$$

is an [exact sequence](#) if it's [exact](#) at A_i for all $i \geq 1$.

- We say

$$\dots \longrightarrow A_{-2} \longrightarrow A_{-1} \longrightarrow A_0$$

is an [exact sequence](#) if it's [exact](#) at A_i for all $i \leq -1$.

Remark. Denote \circ as a trivial [Abelian group](#), then

$A \xrightarrow{\phi} B \longrightarrow \circ$ is an [exact sequence](#) $\iff \phi$ is a surjective homomorphism;

conversely,

$\circ \longrightarrow B \xrightarrow{\phi} A$ is an [exact sequence](#) $\iff \phi$ is an injective homomorphism.

Definition C.3 (Short exact sequence). A *short exact sequence* is an [exact sequence](#) such that it has the following form

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ.$$

Remark. Let $B \xrightarrow{\psi} C$ as a surjective homomorphism and $K = \ker \psi$, and we denote $K \xrightarrow{i} B$ as an injection. Then

$$\circ \longrightarrow K \xrightarrow{i} B \xrightarrow{\psi} C \longrightarrow \circ$$

is a [short exact sequence](#). Conversely, if

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

is a [short exact sequence](#), then ϕ is an injective homomorphism since it is [exact](#) at A , and ψ is a surjective homomorphism since it is [exact](#) at C , and $\phi(A) = \ker \psi$ since it is [exact](#) at B . This implies $\phi: A \rightarrow \phi(A) = \ker \psi$ is a group homeomorphism.

Example. We see some examples.

1. Given A, B as [Abelian groups](#), then

$$\circ \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\text{Proj}_2} B \longrightarrow \circ$$

$$a \xrightarrow{i} (a, 0)$$

$$(a, b) \xrightarrow{\text{Proj}_2} b$$

is a [short exact sequence](#).

2. We see that

$$\circ \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\text{Proj}_2} \mathbb{Z}/n\mathbb{Z} \longrightarrow \circ$$

$$k \longmapsto k \cdot n$$

for $n \in \mathbb{Z}_{\geq 1}$ is a [short exact sequence](#).

Definition C.4 (Isomorphism between sequences). Given A_\bullet and B_\bullet defined as two sequences of [Abelian group](#) homomorphisms

$$A_\bullet : \dots \longrightarrow A_i \xrightarrow{\phi_i} A_{i+1} \longrightarrow \dots$$

and

$$B_\bullet : \dots \longrightarrow B_i \xrightarrow{\psi_i} B_{i+1} \longrightarrow \dots$$

And we say a morphism α from A_\bullet to B_\bullet is a series of group homomorphisms $\alpha_i : A_i \rightarrow B_i$ for all $i \in \mathbb{Z}$ such that the following diagram commutes.

$$\begin{array}{ccccccc} \dots & \longrightarrow & A_i & \xrightarrow{\phi_i} & A_{i+1} & \longrightarrow & \dots \\ & & \downarrow \alpha_i & & \downarrow \alpha_{i+1} & & \\ \dots & \longrightarrow & B_i & \xrightarrow{\psi_i} & B_{i+1} & \longrightarrow & \dots \end{array}$$

Additionally, if for all i , α_i is a group homeomorphism, then we say $\alpha : A_\bullet \rightarrow B_\bullet$ is a homeomorphism.

Definition C.5 (Split short exact sequence). Given a [short exact sequence](#)

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

we say it is *split* if there exists a group homeomorphism $\theta : B \rightarrow A \oplus C$ such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus C & \longrightarrow & C \longrightarrow 0 \end{array}$$

is the [isomorphism](#) between these two [short exact sequences](#).

Remark. Given [split short exact sequence](#)

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

and θ defined in [Definition C.5](#), let $i : A \rightarrow A \oplus C$, $a \mapsto (a, 0)$ and $j : C \rightarrow A \oplus C$, $c \mapsto (0, c)$ are two canonical embeddings, then we have

$$A \oplus C = i(A) \oplus j(C).$$

Consider $\theta^{-1} : A \oplus C \xrightarrow{\cong} B$, then

$$B = \theta^{-1}(i(A)) \oplus \theta^{-1}(j(C)).$$

Since the diagram in [Definition C.5](#) commutes, hence

$$\theta^{-1}(i(A)) = \theta^{-1} \circ i(A) = \phi(A),$$

hence

$$B = \phi(A) \oplus \underbrace{\theta^{-1}(j(C))}_D,$$

which implies $\psi|_D : D \rightarrow C$ is a group homeomorphism. We see that

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

[split](#) implies $B = \phi(A) \oplus D$ and $\psi|_D : D \xrightarrow{\cong} C$.

Conversely, if $B = \phi(A) \oplus D$ and $\psi|_D : D \xrightarrow{\cong} C$, then there exists a θ

$$\begin{aligned} \theta : B &\rightarrow A \oplus C \\ \phi(a) + d &\mapsto (a, \psi(d)) \end{aligned}$$

for $a \in A, d \in D$ such that

$$\begin{array}{ccccccc} \circ & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow \circ \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ \circ & \longrightarrow & A & \longrightarrow & A \oplus C & \longrightarrow & C \longrightarrow \circ \end{array}$$

$$\begin{array}{ccc} \phi(a) + d & \longmapsto & \psi(d) \\ \downarrow & & \downarrow \\ (a, \psi(d)) & \longmapsto & \psi(d) \end{array}$$

commutes.

Hence, for a [short exact sequence](#) $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$ is [split](#) if and only if $B = \phi(A) \oplus D$ and $\psi|_D : D \xrightarrow{\cong} C$.

Remarkably, let $\circ \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$ is a [split short exact sequence](#), then D constructed above is not unique. To see this, consider

$$\circ \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\text{Proj}_2} \mathbb{Z} \longrightarrow \circ$$

$$n \longmapsto (n, 0)$$

$$(n, m) \longmapsto m$$

We have $\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus j(\mathbb{Z})$ where $j : \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}, n \mapsto (0, n)$. We see that we can let $D := j(\mathbb{Z})$. Meanwhile, we can also let

$$D := \{(n, n) \mid n \in \mathbb{Z}\} < \mathbb{Z} \oplus \mathbb{Z}$$

such that $\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus D$.

Example (Non-split short exact sequence). We see that

$$\begin{aligned} 0 &\longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\text{Proj}_2} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \\ k &\longmapsto k \cdot n \end{aligned}$$

is not a [split short exact sequence](#), since if it is, then

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z} / n\mathbb{Z} &\cong \mathbb{Z} \\ (0, 1) &\mapsto k, \end{aligned}$$

which is a contradiction since \mathbb{Z} is [torsion-free](#) while $\mathbb{Z} \oplus \mathbb{Z} / n\mathbb{Z}$ is not.

Lemma C.1 (Splitting lemma). If $0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$ is a [short exact sequence](#), then the following are equivalent.

1. This [short exact sequence](#) [splits](#).
2. $\exists p: B \rightarrow A$ such that $p \circ \phi = \text{id}_A$.
3. $\exists q: C \rightarrow B$ such that $\psi \circ q = \text{id}_C$.

Proof. • 1. \implies 2. Let $\theta: B \xrightarrow{\cong} A \oplus C$ such that it's the [isomorphism](#) which makes the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \xrightarrow{i} & A \oplus C & \longrightarrow & C \longrightarrow 0 \\ & & & \swarrow \text{Proj}_1 & & & \end{array}$$

Then we let $p := \text{Proj}_1 \circ \theta$, then

$$p \circ \phi = \text{Proj}_1 \circ \theta \circ \phi = \text{Proj}_1 \circ i = \text{id}_A.$$

- 1. \implies 3. Let $\theta: B \xrightarrow{\cong} A \oplus C$ such that it's the [isomorphism](#) which makes the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus C & \xrightarrow{\text{Proj}_2} & C \longrightarrow 0 \\ & & & & \swarrow j & & \end{array}$$

Then we let $q := \theta^{-1} \circ j$, then for all $c \in C$, we have

$$\psi \circ q(c) = \psi(\theta^{-1}(j(c))) = \text{Proj}_2 \circ \theta(\theta^{-1}(j(c))) = \text{Proj}_2(j(c)) = c,$$

hence $\psi \circ q = \text{id}_C$.

- 2. \implies 1. We have

$$\circ \longrightarrow A \xrightleftharpoons[p]{\phi} B \xrightarrow{\psi} C \longrightarrow \circ$$

where $p \circ \phi = \text{id}_A$. We claim that $B = \phi(A) \oplus \ker(p)$ since for every $b \in B$, $\phi(p(b)) \in \phi(A)$, and

$$b = \underbrace{\phi(p(b))}_{\in \phi(A)} + \underbrace{(b - \phi(p(b)))}_{\in \ker(p)}$$

from the fact that

$$p(b - \phi(p(b))) = p(b) - p \circ \phi(p(b)) = p(b) - p(b) = 0.$$

We need to show the uniqueness also. Suppose $b = \phi(a_1) + d_1 = \phi(a_2) + d_2$, $a_1, a_2 \in A$, $d_1, d_2 \in \ker(p)$. We see that

$$\phi(a_1 - a_2) = d_2 - d_1 \implies p(\phi(a_1 - a_2)) = 0 \implies a_1 = a_2 \implies d_1 = d_2.$$

Finally, we claim that

$$\psi|_{\ker(p)} : \ker(p) \rightarrow C$$

is a group homeomorphism. But it's obvious that $\psi|_{\ker(p)}$ are both surjective and injective.

- 3. \implies 1. We have

$$\circ \longrightarrow A \xrightarrow{\phi} B \xrightleftharpoons[q]{\psi} C \longrightarrow \circ$$

where $\psi \circ q = \text{id}_C$. We claim that $B = \phi(A) \oplus q(C)$ since for every $b \in B$,

$$b = \underbrace{(b - q(\psi(b)))}_{\in \ker(\psi) = \text{Im}(\phi)} + \underbrace{q(\psi(b))}_{\in q(C)},$$

which implies $B = \phi(A) + q(C)$. We can also prove that

$$B = \phi(A) \oplus q(C)$$

similarly. ■

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