

# MATH592

## Introduction to Algebraic Topology

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### Abstract

This course will use [HPM02] as the main text, but the order may differ here and there. Enjoy this fun course!

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## Lecture 1: Homotopies of Maps

05 Jan. 10:00

### 1 Foundation of Algebraic Topology

#### 1.1 Homotopy

**Definition 1.1 (Homotopy).** Let  $X, Y$  be topological spaces. Let  $f, g: X \rightarrow Y$  continuous maps. Then a *homotopy* from  $f$  to  $g$  is a 1-parameter family of maps that continuously deforms  $f$  to  $g$ , i.e., it's a continuous function  $F: X \times I \rightarrow Y$ , where  $I = [0, 1]$ , such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x).$$

We often write  $F_t(x)$  for  $F(x, t)$ .

If a homotopy exists between  $f$  and  $g$ , we say they are *homotopic* and write

$$f \simeq g.$$

If  $f$  is homotopic to a constant map, we call it *nullhomotopic*.

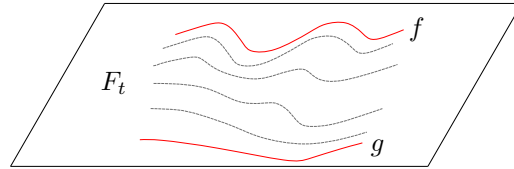


Figure 1: The continuous deforming from  $f$  to  $g$  described by  $F_t$

**Remark.** Later, we'll not state that a map is continuous explicitly since we almost always assume this in this context.

**Example.** We first see some examples.

1. Any two maps (continuous) with specification

$$f, g: X \rightarrow \mathbb{R}^n$$

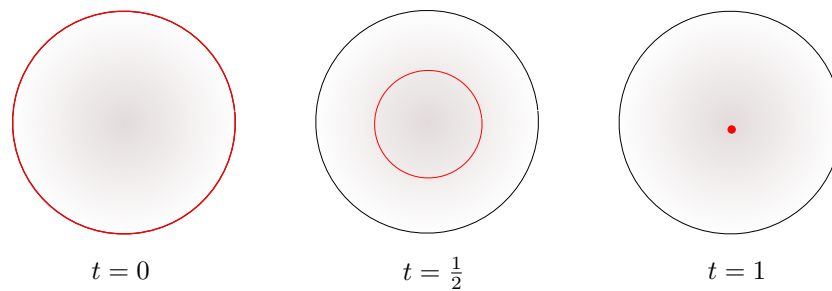
are homotopic by considering

$$F_t(x) = (1 - t)f(x) + tg(x).$$

We call it *the straight line homotopy*.

2. Let  $S^1$  denotes the unit circle in  $\mathbb{R}^2$ , and  $D^2$  denotes the unit disk in  $\mathbb{R}^2$ . Then the inclusion  $f: S^1 \hookrightarrow D^2$  is nullhomotopic by considering

$$F_t(x) = (1 - t)f(x) + (t \cdot 0).$$


 Figure 2: The illustration of  $F_t(x)$ 

We see that there is a homotopy from  $f(x)$  to 0 (the zero map which maps everything to 0), and since 0 is a constant map, hence it's actually a nullhomotopy.

3. The maps

$$\begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & S^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \rightarrow & S^1 \\ \Theta & \mapsto & -\Theta \end{array}$$

are **not** homotopy.

**Remark.** It will essentially **flip** the orientation, hence we can't deform one to another continuously.

**Exercise.** We first see some exercises.

1. A subset  $S \subseteq \mathbb{R}^n$  is star-shaped if

$$\exists x_0 \in S \text{ s.t. } \forall x \in S,$$

the line from  $x_0$  to  $x$  lies in  $S$ .

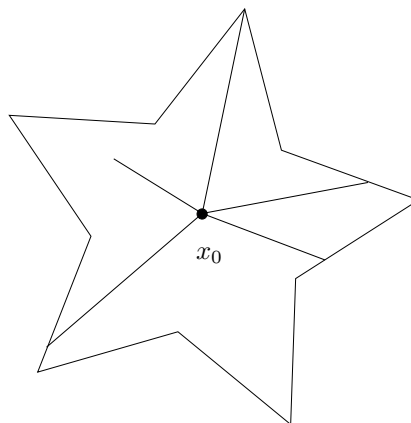


Figure 3: Star-shaped illustration

Show that  $\text{id}: S \rightarrow S$  is nullhomotopic.

**Answer.** Consider

$$F_t(x) := (1-t)x + tx_0,$$

which essentially just concentrates all points  $x$  to  $x_0$ .

2. Suppose

$$X \xrightarrow[f_0]{f_1} Y \xrightarrow[g_0]{g_1} Z.$$

where

$$f_0 \simeq_{F_t} f_1, \quad g_0 \simeq_{G_t} g_1.$$

Show

$$g_0 \circ f_0 \simeq g_1 \circ f_1.$$

**Answer.** Consider  $I \times X \rightarrow Z$ . Then

$$\begin{array}{ccccc} X \times I & \rightarrow & Y \times I & \rightarrow & Z \\ (x, t) & \mapsto & (F_t(x), t) & \mapsto & G_t(F_t(x)). \end{array}$$

**Remark.** Noting that if one wants to be precise, you need to check the continuity of this construction.

3. How could you show 2 maps are **not** homotopic?

**Answer.**

## Lecture 2: Homotopy Equivalence

07 Jan. 10:00

**As previously seen.** Two maps  $f, g: X \rightarrow Y$  is homotopy if there exists a map

$$F_t(x): X \times I \rightarrow Y$$

with the properties

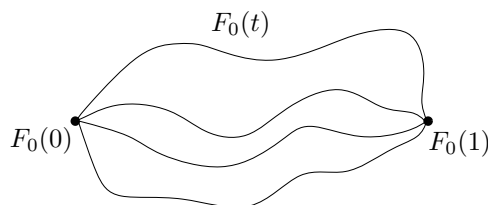
1. Continuous
2.  $F_0(x) = f(x)$
3.  $F_1(x) = g(x)$

**Remark.** The continuity of  $F_t$  is an even stronger condition for the continuity of  $F_t$  for a fixed  $t$ .

We now introduce another concept.

**Definition 1.2 (Homotopy relative).** Given two spaces  $X, Y$ , and let  $B \subseteq X$ . Then a homotopy  $F_t(x): X \rightarrow Y$  is called *homotopy relative B* (denotes  $\text{rel}B$ ) if  $F_t(b)$  is independent of  $t$  for all  $b$ .

**Example.** Let  $X = [0, 1]$  and  $B = \{0, 1\}$ . Then the homotopy of paths from  $[0, 1] \rightarrow X$  is  $\text{rel}\{0, 1\}$ .



## 1.2 Homotopy Equivalence

With this, we can introduce the concept of *homotopy equivalence*.

**Definition 1.3 (Homotopy Equivalence).** A map  $f: X \rightarrow Y$  is a *homotopy equivalence* if  $\exists g: Y \rightarrow X$  such that

$$f \circ g \simeq \text{id}_Y, \quad g \circ f \simeq \text{id}_X.$$

We say that  $X, Y$  are *homotopy equivalent*, and  $g$  is called *homotopy inverse* of  $f$ .

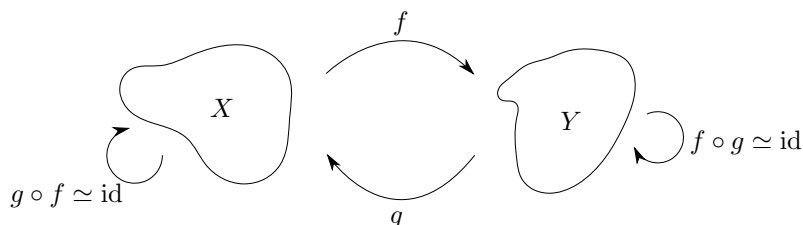


Figure 4: Homotopy Equivalence

If  $X, Y$  are called *homotopy equivalent*, then we say that they have the same *homotopy type*.

**Notation.** We denote a closed  $n$ -disk as  $D^n$ .

**Example.**  $D^n$  is homotopy equivalent to a point.



We see that  $f \circ g = \text{id}_*$  and

$$g \circ f = \text{constant map at } \underbrace{0}_{g(*)},$$

which is homotopic to  $\text{id}_{D^n}$  by straight-line homotopy  $F_t(x) = tx$ .

**Note.** We say that a space is *contractible* if  $H$  is homotopy equivalent to a point.

Before doing exercises, we introduce two new concepts.

**Definition 1.4 (Retraction).** Given  $B \subseteq X$ , a *retraction* from  $X$  to  $B$  is a map  $f: X \rightarrow X$  (or  $X \rightarrow B$ ) such that  $\forall b \in B$   $f(b) = b$ , namely  $r|_B = \text{id}_B$ . Or one can see this from

$$\begin{array}{ccc} B & \xrightarrow{i} & X \\ & \searrow r \circ i & \nearrow r \\ & & B \end{array}$$

where  $r$  is a retraction if and only if  $r \circ i = \text{id}_B$ , where  $i$  is an inclusion identity. If  $r$  exists,  $B$  is a retract of  $X$ .

**Definition 1.5 (Deformation retraction).** Given  $X$  and  $B \subseteq X$ , a *(strong) deformation retraction*  $F_t: X \rightarrow X$  onto  $B$  is a homotopy  $\text{rel} B$  from the  $\text{id}_X$  to a *retraction* from  $X$  to  $B$ . i.e.,

$$\begin{aligned} F_0(x) &= x & \forall x \in X \\ F_1(x) &\in B & \forall x \in X \\ F_t(b) &= b & \forall t \forall b \in B. \end{aligned}$$

**Exercise.** We now see some problems.

1. Let  $X \simeq Y$ . Show  $X$  is path-connected if and only if  $Y$  is.

**Answer.** Suppose  $X$  is path-connected. Then we see that given two points  $x_1$  and  $x_2$  in  $X$ , there exists a path  $\gamma(t)$  with

$$\gamma: [0, 1] \rightarrow X, \quad \gamma(0) = x_1, \quad \gamma(1) = x_2.$$

Since  $X \simeq Y$ , then there exists a pair of  $f$  and  $g$  such that  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  with

$$f \circ g \underset{F}{\simeq} \text{id}_Y, \quad g \circ f \underset{G}{\simeq} \text{id}_X.$$

(Notice the abuse of notation)

For any two  $y_1$  and  $y_2 \in Y$ , we want to construct a path  $\gamma'(t)$  such that

$$\gamma': [0, 1] \rightarrow Y, \quad \gamma'(0) = y_1, \quad \gamma'(1) = y_2.$$

Firstly, we let  $g(y_1) =: x_1$  and  $g(y_2) =: x_2$ . From the argument above, we know there exists such a  $\gamma$  starting at  $x_1 = g(y_1)$  ending at  $x_2 = g(y_2)$ . Now, consider  $f(\gamma(t)) = (f \circ \gamma)(t)$  such that

$$f \circ \gamma: I \rightarrow Y, \quad f \circ \gamma(0) = y'_1, \quad f \circ \gamma(1) = y'_2,$$

we immediately see that  $y'_1$  and  $y'_2$  is path connected. Now, we claim that  $y_1$  and  $y'_1$  are path connected in  $Y$ , hence so are  $y_2$  and  $y'_2$ . To see this, note that

$$f \circ g \underset{F}{\simeq} \text{id}_Y,$$

which means that there exists  $F: Y \times I \rightarrow Y$  such that

$$\begin{cases} F(y_1, 0) = f \circ g(y_1) = f(x_1) = f(\gamma(0)) = (f \circ \gamma)(0) = y'_1 \\ F(y_1, 1) = \text{id}_Y(y_1) = y_1. \end{cases}$$

Since  $F$  is continuous in  $I$ , we see that there must exist a path connects  $y_1$  and  $y'_1$ . The same argument applies to  $y_2$  and  $y'_2$ . Now, we see that the path

$$y_1 \rightarrow y'_1 \rightarrow y'_2 \rightarrow y_2$$

is a path in  $Y$  for any two  $y_1$  and  $y_2$ , which shows  $Y$  is path-connected.



Figure 5: Demonstration of the proof

**Challenge:** One can further show that the connectedness is also preserved by any homotopy equivalence.

2. Show that if there exists deformation retraction from  $X$  to  $B \subseteq X$ , then  $X \simeq B$ .

### Lecture 3: Deformation Retraction

10 Jan. 10:00

**As previously seen.** A *deformation retraction* is a homotopy of maps  $\text{rel} B$   $X \rightarrow X$  from  $\text{id}_X$  to a retraction from  $X$  to  $B$ . Then  $B$  is a *deformation retract*.

**Example.** We can also show

1.  $S^1$  is a deformation retraction of  $D^2 \setminus \{0\}$ . Indeed, since

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)x.$$

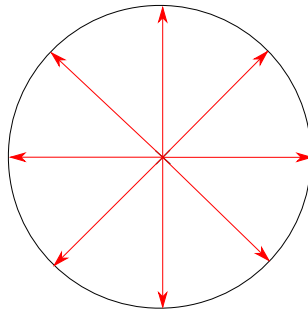


Figure 6: The deformation retraction of  $D^2 \setminus \{0\}$  is just to *enlarge* that hold and push all the interior of  $D^2$  to the boundary, which is  $S^1$

2.  $\mathbb{R}^n$  deformation retracts to 0. Indeed, since

$$F_t(x) = (1-t)x.$$

This implies that  $\mathbb{R}^n \simeq *$ , hence we see that

- dimension
- compactness
- etc.

are not homotopy invariants.

3.  $S^1$  is a deformation retract of a cylinder and a Möbius band.

For a cylinder, consider  $X \times I \rightarrow X$ . Define homotopy on a closed rectangle, then verify it induces map on quotient.

For a Möbius band, we define a homotopy on a closed rectangle, then verify that it respect the equivalence relation.



Finally, we use the universal property of quotient topology to argue that we get a homotopy on Möbius band.

**Upshot:** Möbius band  $\simeq S^1 \simeq$  cylinder, hence the orientability is not homotopy invariant.

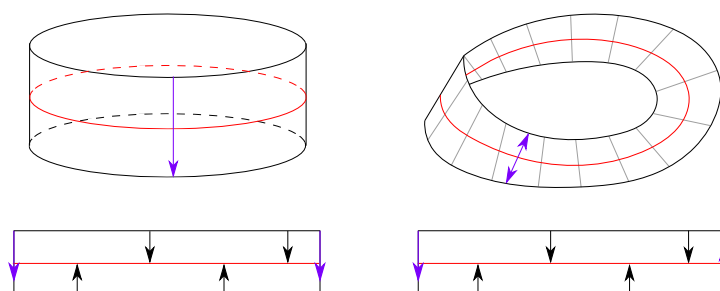


Figure 7: The deformation retraction for Cylinder and Möbius band

## Lecture 4: Cell Complex (CW Complex)

12 Jan. 10:00

**As previously seen.** We saw that

- homotopy equivalence
- homotopy invariants
  - path-connectedness
- not invariant
  - dimension
  - orientability
  - compactness

### 1.3 CW Complexes

**Example.** Let's start with a few examples.

1. Constructing spheres:

- $S^1$  (up to homeomorphism)

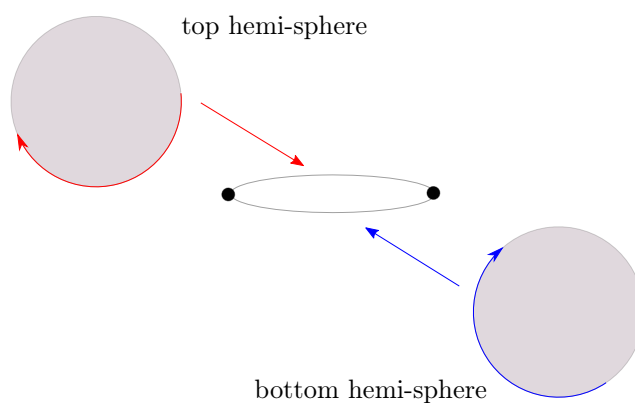


- $S^2$ 
  - glue boundary of 2-disk to a point
  - glue 2 disks onto a circle



Figure 8: **Left:** Glue a 2-disk to a point along its boundary. **Right:** Glue 2 disks to  $S^1$ .

The gluing instruction to construct  $S^2$  in the right-hand side can be demonstrated as follows.



- $T = S^1 \times S^1$



view as gluing instructions

vertex + 2 edges + 2-disks.

Specifically, we have




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Formally, we have the following definition.

**Notation.** Let  $D^n$  denotes a closed  $n$ -disk (or  $n$ -ball)

$$D^n \simeq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

And let  $S^n$  denotes an  $n$ -sphere

$$S^n \simeq \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

Lastly, we call a point as a  $0$ -cell, and the interior of  $D^n$   $\text{int}(D^n)$  for  $n \geq 1$  as a  $n$ -cell.

**Definition 1.6 (CW Complex).** A *CW Complex* is a topological space constructed inductively as

1.  $X^0$  (the 0-skeleton) is a set of discrete points.
2. We inductively construct the  $n$ -skeleton  $X^n$  from  $X^{n-1}$  by attaching  $n$ -cells  $e_\alpha^n$ , where  $\alpha$  is the index.

The gluing instructions glued by an attaching map is that  $\forall \alpha, \exists$  continuous map  $\varphi_\alpha$

$$\varphi_\alpha: \partial D_\alpha^n \rightarrow X^{n-1},$$

then

$$X^n = \left( X^{n-1} \coprod_\alpha D_\alpha^n \right) / x \sim \varphi_\alpha(x)$$

with identification  $x \sim \varphi_\alpha(x)$  for all  $x \in \partial D_\alpha^n$  with quotient topology.

- 3.

$$X = \bigcup_{n=0} X^n,$$

and let  $\bar{w}$  denotes weak topology. Then

$$u \subseteq X \text{ is open} \iff \forall n \ u \cap X^n \text{ is open}.$$

If all cells have dimension less than  $N$  and a  $\exists N$ -cell, then  $X = X^N$  and we call it  $N$ -dim CW complex.

**Remark.** We write  $X^{(n)}$  for  $n$ -skeleton if we need to distinguish from the Cartesian product.

**Example.** Let's look at some examples.

1. 0-dim CW complex is a discrete space.
2. 1-dim CW complex is a graph.
3. A CW complex  $X$  is finite if it has finitely many cells.

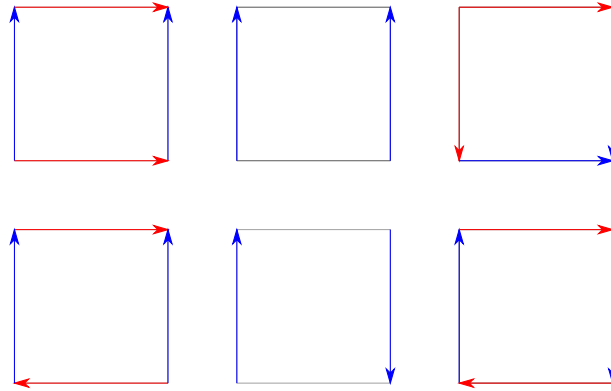
**Definition 1.7 (CW subcomplex).** A *CW subcomplex*  $A \subseteq X$  is a closed subset equal to a union of cells

$$e_\alpha^n = \text{int}(D_\alpha^n).$$

**Remark.** This inherits a CW complex structure.

**Exercise.** Given the following gluing instruction:

Check the images of attaching maps.



identify Torus, Klein bottle, Cylinder, Möbius band, 2-sphere,  $\mathbb{R}P$ .

**Answer.** We see that

1. Torus
2. Cylinder
3. 2-sphere
4. Klein bottle
5. Möbius band
6.  $\mathbb{R}P$

**Notation.** We call the real projection space as  $\mathbb{R}P$ , and we also have so-called complex projection space, denote as  $\mathbb{C}P$ .

## Lecture 5: Operation on Spaces

14 Jan. 10:00

### 1.4 Operations on CW Complexes

#### 1.4.1 Products

We can consider the product of two CW complexes given by a CW complex structure. Namely, given  $X$  and  $Y$  two CW complexes, we can take two cells  $e_\alpha^n$  from  $X$  and  $e_\beta^m$  from  $Y$  and form the product space  $e_\alpha^n \times e_\beta^m$ , which is homeomorphic to an  $n + m$ -cell. We then take these products as the cells for  $X \times Y$ .

Specifically, given  $X, Y$  are CW complexes, then  $X \times Y$  has a cell structure

$$\{e_\alpha^m \times e_\alpha^n : e_\alpha^m \text{ is a } m\text{-cell on } X, e_\alpha^n \text{ is a } n\text{-cell on } Y\}.$$

**Remark.** The product topology may not agree with the weak topology on the  $X \times Y$ . However, they do agree if  $X$  or  $Y$  is locally compact or if  $X$  and  $Y$  both have at most countably many cells.

**Note.** Notice that if the product is wild enough, then the product topology may not agree with the weak topology.

### 1.4.2 Wedge Sum

Given  $X, Y$  are CW complexes, and  $x_0 \in X^0, y_0 \in Y^0$  (only points). Then we define

$$X \vee Y = X \amalg Y$$

with quotient topology.

**Remark.**  $X \vee Y$  is a CW complex.

### 1.4.3 Quotients

Let  $X$  be a CW complex, and  $A \subseteq X$  subcomplex (closed union of cells), then

$$X / A$$

is a quotient space collapse  $A$  to one point and inherits a CW complex structure.

**Remark.**  $X / A$  is a CW complex.

0-skeleton

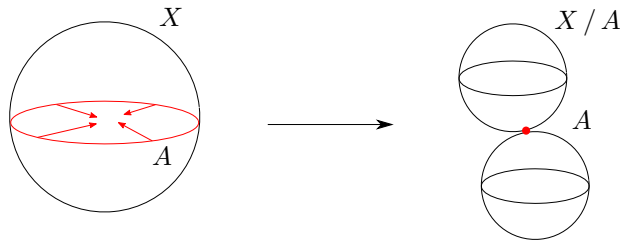
$$(X^0 - A^0) \amalg *$$

where  $*$  is a point for  $A$ . Each cell of  $X - A$  is attached to  $(X / A)^n$  by attaching map

$$S^n \xrightarrow{\phi_\alpha} X^n \xrightarrow{\text{quotient}} X^n / A^n$$

**Example.** Here is some interesting examples.

1. We can take the sphere and squish the equator down to form a wedge of two spheres.



2. We can take the torus and squish down a ring around the hole.



Figure 9: We see that  $X / A$  is homotopy equivalent to a 2-sphere wedged with a 1-sphere via extending the red point into a line, and then sliding the left point to the line along the 2-sphere towards the other point, forming a circle.

## Lecture 6: A Foray into Category Theory

19 Jan. 10:00

### 1.5 Category Theory

We start with a definition.

**Definition 1.8 (Object, Morphism).** A category  $\mathcal{C}$  is 3 pieces of data

- A class of objects  $\text{Ob}(\mathcal{C})$
- $\forall X, Y \in \text{Ob}(\mathcal{C})$  a class of morphisms or arrows,  $\text{Hom}_{\mathcal{C}}(X, Y)$ .
- $\forall X, Y, Z \in \text{Ob}(\mathcal{C})$ , there exists a composition law

$$\begin{aligned} \text{Hom}(X, Y) \times \text{Hom}(Y, Z) &\rightarrow \text{Hom}(X, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

and 2 axioms

- Associativity.  $(f \circ g) \circ h = f \circ (g \circ h)$  for all morphisms  $f, g, h$  where composites are defined.
- Identity.  $\forall X \in \text{Ob}(\mathcal{C}) \exists \text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$  such that

$$f \circ \text{id}_X = f, \quad \text{id}_X \circ g = g$$

for all  $f, g$  where this makes sense.

Let's see some examples.

**Example.** We introduce some common category.

$\mathcal{C}$	$\text{Ob}(\mathcal{C})$	$\text{Mor}(\mathcal{C})$
$\underline{\text{set}}$	Sets $X$	All maps of sets
$\underline{\text{fset}}$	Finite sets	All maps
$\underline{\text{Gp}}$	Groups	Group Homomorphisms
$\underline{\text{Ab}}$	Abelian groups	Group Homomorphisms
$\underline{k\text{-vect}}$	Vector spaces over $k$	$k$ -linear maps
$\underline{\text{Rng}}$	Rings	Ring Homomorphisms
$\underline{\text{Top}}$	Topological spaces	Continuous maps
$\underline{\text{Haus}}$	Hausdorff Spaces	Continuous maps
$\underline{\text{hTop}}$	Topological spaces	Homotopy classes of continuous maps
$\underline{\text{Top}^*}$	Based topological spaces <sup>1</sup>	Based maps <sup>2</sup>

**Remark.** Any **diagram** plus composition law.

$$\text{id}_A \hookrightarrow A \longrightarrow B \hookleftarrow \text{id}_B .$$

**Definition 1.9 (monic, epic).** A morphism  $f: M \rightarrow N$  is *monic* if

$$\forall g_1, g_2 \quad f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$A \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} M \xrightarrow{f} N$$

Dually,  $f$  is *epic* if

$$\forall g_1, g_2 \quad g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$M \xrightarrow{f} N \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} B$$

**Lemma 1.1.** In  $\underline{\text{set}}, \underline{\text{Ab}}, \underline{\text{Top}}, \underline{\text{Gp}}$ , a map is monic if and only if  $f$  is injective, and epic if and only if  $f$  is surjective.

*Proof.* In  $\underline{\text{set}}$ , we prove that  $f$  is monic if and only if  $f$  is injective. Suppose  $f \circ g_1 = f \circ g_2$  and  $f$  is injective, then for any  $a$ ,

$$f(g_1(a)) = f(g_2(a)) \implies g_1(a) = g_2(a),$$

hence  $g_1 = g_2$ .

<sup>1</sup>Topological spaces with a distinguished base point  $x_0 \in X$

<sup>2</sup>Continuous maps that presence base point  $f: (x, x_0) \rightarrow (y, y_0)$  such that

$$f: X \rightarrow Y, \quad f(x_0) = y_0$$

is continuous.



Now we prove another direction, with contrapositive. Namely, we assume that  $f$  is not injective and show that  $f$  is not monic. Suppose  $f(a) = f(b)$  and  $a \neq b$ , we want to show such  $g_i$  exists. This is easy by considering

$$g_1: * \mapsto a, \quad g_2: * \mapsto b.$$

■

### 1.5.1 Functor

After introducing the category, we then see the most important concept we'll use, a *functor*. Again, we start with the definition.

**Definition 1.10 (Functor).** Given  $\mathcal{C}, \mathcal{D}$  be two categories. A (covariant) *functor*

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is

1. a map on objects

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(X) \rightarrow F(Y)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(f) \circ F(g)$

## Lecture 7: Functors

21 Jan. 10:00

**As previously seen.** Assume that we initially have a commutative diagram in  $\mathcal{C}$  as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & Z \end{array}$$

After applying  $F$ , we'll have

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ & \searrow F(g \circ f) = F(g) \circ F(f) & \downarrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in  $\mathcal{D}$ .

We can also have a so-called contravariant functor.

**Definition 1.11 (Contravariant functor).** Given  $\mathcal{C}, \mathcal{D}$  be two categories. A contravariant functor

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is

1. a map on objects

$$\begin{aligned} F: \text{Ob}(\mathcal{C}) &\rightarrow \text{Ob}(\mathcal{D}) \\ X &\mapsto F(X). \end{aligned}$$

2. maps of morphisms

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(Y), F(X)) \\ [f: X \rightarrow Y] &\mapsto [F(f): F(Y) \rightarrow F(X)] \end{aligned}$$

such that

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(g) \circ F(f)$

Then, we see that in this case, when we apply a contravariant functor  $F$ , the diagram becomes

$$\begin{array}{ccc} F(X) & \xleftarrow{F(f)} & F(Y) \\ & \nwarrow F(g \circ f) = F(f) \circ F(g) & \uparrow F(g) \\ & & F(Z) \end{array}$$

which is a commutative diagram in  $\mathcal{D}$ .

**Example.** Let see some examples.

1. Identity functor.

$$I: \mathcal{C} \rightarrow \mathcal{C}.$$

2. Forgetful functors.

•

$$\begin{aligned} F: \underline{\text{Gp}} &\rightarrow \underline{\text{set}} \\ G &\mapsto G^3 \\ [f: G \rightarrow H] &\mapsto [f: G \rightarrow H] \end{aligned}$$

•

$$\begin{aligned} F: \underline{\text{Top}} &\rightarrow \underline{\text{set}} \\ X &\mapsto X^4 \\ [f: X \rightarrow Y] &\mapsto [f: X \rightarrow Y] \end{aligned}$$

<sup>3</sup> $G$  is now just the underlying set of the group  $G$ .

## 3. Free functors.

$$\begin{aligned} \underline{\text{set}} &\rightarrow \underline{k\text{-vect}} \\ s &\mapsto \text{"free" } k\text{-vector space on } s \end{aligned}$$

i.e., vector space with basis  $s$

$$[f: A \rightarrow B] \mapsto [\text{unique } k\text{-linear map extending } f]$$

## 4.

$$\begin{aligned} \underline{k\text{-vect}} &\rightarrow \underline{k\text{-vect}} \\ V &\mapsto V^* = \text{Hom}_k(V, k) \end{aligned}$$

If we are working in a basis, then we have

$$A \mapsto A^T.$$

Specifically, we care about two functors.

## 1.

$$\begin{aligned} \underline{\text{Top}}^* &\rightarrow \underline{\text{Gp}} \\ (X, x_0) &\mapsto \Pi_1(X, x_0) \end{aligned}$$

where  $\Pi_1$  is so-called *fundamental group*.

## 2.

$$\begin{aligned} \underline{\text{Top}} &\rightarrow \underline{\text{Ab}} \\ X &\mapsto \text{Hp}(X) \end{aligned}$$

where  $\text{Hp}$  is so-called  $p^{\text{th}}$  *homology*.

Let see the formal definition.

## 1.6 Free Groups

**Definition 1.12 (Free group).** Given a set  $S$ , the *free group* is a group  $F_S$  on  $S$  with a map  $S \rightarrow F_S$  satisfying the universal property.

If  $G$  is any group,  $f: S \rightarrow G$  is any map of sets,  $f$  extends uniquely to group homomorphism  $\bar{f}: F_S \rightarrow G$ .

$$\begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \bar{f}: \text{gp hom} \\ & & G \end{array}$$

<sup>4</sup> $X$  is now just the underlying set of the topological space  $X$ .

**Note.** This defines a *natural bijection*

$$\mathrm{Hom}_{\mathrm{set}}(S, \mathcal{U}(G)) \cong \mathrm{Hom}_{\mathrm{Grp}}(F_S, G),$$

where  $\mathcal{U}(G)$  is the forgetful functor from the category of groups to the category of sets. This is the statement that the free functor and the forgetful functor are **adjoint**; specifically that the free functor is the left adjoint (appears on the left in the Hom's above).

**Definition 1.13 (Adjoint functor).** A free and forgetful functors are *adjoints*.

**Remark.** Whenever we state a universal property for an object (plus a map), an object (plus a map) may or may not exist. If such object exists, then it defines the object **uniquely up to unique isomorphism**, so we can use the universal property as the *definition* of the object (plus a map).

**Lemma 1.2.** Universal property defines  $F_S$  (plus a map  $S \rightarrow F(S)$ ) uniquely up to unique isomorphism.

*Proof.* Fix  $S$ . Suppose

$$S \rightarrow F_S, \quad S \rightarrow \tilde{F}_S$$

both satisfy the unique property. By universal property, there exist maps such that

$$\begin{array}{ccc} S & \longrightarrow & \tilde{F}_S \\ & \searrow f & \downarrow \exists! \varphi \\ & & F_S \end{array} \quad \begin{array}{ccc} S & \longrightarrow & F_S \\ & \searrow f & \downarrow \exists! \psi \\ & & \tilde{F}_S \end{array}$$

We'll show  $\varphi$  and  $\psi$  are inverses (and the unique isomorphism making above commute). Since we must have the following two commutative graphs.

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \mathrm{id}_{F_S} & \nwarrow f \\ S & & \\ f \searrow & & \end{array} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \mathrm{id}_{\tilde{F}_S} & \nwarrow f \\ S & & \\ f \searrow & & \end{array}$$

Hence, we see that

$$\begin{array}{ccc} & F_S & \\ f \nearrow & \downarrow \psi & \nwarrow \varphi \\ S & \longrightarrow & \tilde{F}_S \\ f \searrow & \downarrow \varphi & \nearrow \psi \\ & F_S & \end{array} \quad \varphi \circ \psi = \mathrm{id}_{F_S} \quad \begin{array}{ccc} & \tilde{F}_S & \\ f \nearrow & \downarrow \varphi & \nwarrow \psi \\ S & \longrightarrow & F_S \\ f \searrow & \downarrow \psi & \nearrow \varphi \\ & \tilde{F}_S & \end{array} \quad \psi \circ \varphi = \mathrm{id}_{\tilde{F}_S}$$

where the identity makes these outer triangles commute, then by the uniqueness in universal property, we must have

$$\varphi \circ \psi = \text{id}_{F_S}, \quad \psi \circ \varphi = \text{id}_{\tilde{F}_S},$$

so  $\varphi$  and  $\psi$  are inverses (thus group isomorphism). ■

## Lecture 8: The Fundamental Group $\pi_1$

24 Jan. 10:00

**Example.** In category Ab free Abelian group on a set  $S$  is

$$\bigoplus_S \mathbb{Z}.$$

In category of fields, no such thing as **free field on  $S$** .

### 1.6.1 Constructing the Free Groups $F_S$

**Proposition 1.1.** The free group defined via the universal property before exists.

*Proof.* We'll just give a construction below. First, we see the definition.

**Definition 1.14.** Fix a set  $S$ , and we define a word as a finite sequence (possibly  $\emptyset$ ) in the formal symbols

$$\{s, s^{-1} \mid s \in S\}.$$

Then we see that elements in  $F_S$  are equivalence classes of words with the equivalence relation being

- delete  $ss^{-1}$  or  $s^{-1}s$ . i.e.,

$$vs^{-1}sw \sim vw$$

$$vss^{-1}w \sim vw$$

for every word  $v, w, s \in S$ ,

with the group operation being concatenation. ■

**Example.** Given words  $ab^{-1}, bba$ , their product is

$$ab^{-1} \cdot bba = ab^{-1}bba = aba.$$

**Exercise.** There are something we can check.

1. This product is well-defined on equivalence classes.
2. Every equivalence class of words has a unique *reduced form*, namely the representation.
3. Check that  $F_S$  satisfies the universal property with respect to the map

$$S \rightarrow F_S, \quad s \mapsto s.$$

## 2 The Fundamental Group $\pi_1$

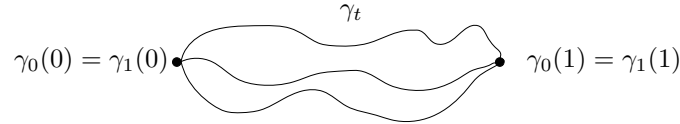
We start with the definition.

**Definition 2.1 (Path).** A *path* in a space  $X$  is a continuous map

$$\gamma: I \rightarrow X$$

where  $I = [0, 1]$ .

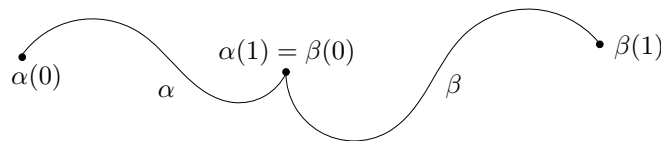
**Definition 2.2 (Homotopy path).** A *homotopy of paths*  $\gamma_0, \gamma_1$  is a homotopy from  $\gamma_0$  to  $\gamma_1$  rel $\{0, 1\}$ .



**Example.** Fix  $x_1, x_0 \in X$ , then  $\exists$  homotopy of paths is an equivalence relation on paths from  $x_0$  to  $x_1$  (i.e.,  $\gamma$  with  $\gamma(0) = x_0, \gamma(1) = x_1$ ).

**Definition 2.3 (Path composition).** For paths  $\alpha, \beta$  in  $X$  with  $\alpha(1) = \beta(0)$ , the *composition*<sup>a</sup>  $\alpha \cdot \beta$  is

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



<sup>a</sup>Also named *product*, *concatenation*.

**Remark.** By the pasting lemma, this is continuous, hence  $\alpha \cdot \beta$  is actually a path from  $\alpha(0)$  to  $\beta(1)$ .

**Definition 2.4 (Reparameterization).** Let  $\gamma: I \rightarrow X$  be a path, then a *reparameterization* of  $\gamma$  is a path

$$\gamma': I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

where  $\varphi$  is continuous and

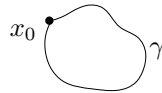
$$\varphi(0) = 0, \quad \varphi(1) = 1.$$

**Exercise.** A path  $\gamma$  is homotopic  $\text{rel}\{0, 1\}$  to all of its reparameterizations. HW

**Exercise.** Fix  $x_0, x_1 \in X$ . Then Homotopy of paths (relative  $\{0, 1\}$ ) is an equivalence relation on paths from  $x_0$  to  $x_1$ .

**Definition 2.5 (Fundamental Group).** Let  $X$  denotes the space and let  $x_0 \in X$  be the base point. The *fundamental group of  $X$  based at  $x_0$* , denoted by  $\pi_1(X, x_0)$ , is a group such that

- Elements: Homotopy classes  $\text{rel}\{0, 1\}$  of paths  $[\gamma]$  where  $\gamma$  is a **loop** with  $\gamma(0) = \gamma(1) = x_0$ <sup>a</sup>

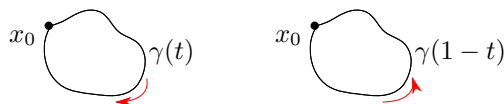


- Operation: [Composition of paths](#).
- Identity: Constant loop  $\gamma$  based at  $x_0$  such that

$$\gamma: I \rightarrow X, \quad t \mapsto x_0$$

- Inverses: The inverse  $[\gamma]^{-1}$  of  $[\gamma]$  is represented by the loop  $\bar{\gamma}$  such that

$$\bar{\gamma}(t) = \gamma(1 - t).$$



<sup>a</sup>We say  $\gamma$  is **based** at  $x_0$ .

*Proof.* We need to prove that the above define a group. ■ HW.

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**Theorem 2.1.** If  $X$  is path-connected, then

$$\forall x_0, x_1 \in X \quad \pi_1(X, x_0) \cong \pi_1(X, x_1).$$

**Remark.** We often write  $\pi_1(X)$  up to isomorphism.

*Proof.*



HW.

**Exercise.** Composition of paths is well-defined on homotopy classes  $\text{rel}\{0, 1\}$ .

**Exercise.** If  $X$  is a contractible space, then  $X$  is path connected and  $\pi_1(X)$  is trivial.



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## Appendix

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## References

- [HPM02] A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: <https://books.google.com/books?id=BjKs86kosqC>.