

# Generative Adventures in Advanced Calculus

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# Full Disclaimer

I am not an AI expert, my background is firmly in math.

I am an AI chatbot experimenter, especially when it comes to its use in calculus math classrooms.

Today I'm sharing experiments specifically in undergraduate courses where it is critical for students to understand proofs.

In my capacity as a service-course instructor I have seen AI's typical use-case to be similar to a glorified calculator: input problem, output solution.

Perhaps AI can help our students understand what they're learning rather than generate answers for their homework. :)

Main question in my mind when designing AI activities: "What do my students do when they encounter a concept they do not understand?"

# Broad Demographics

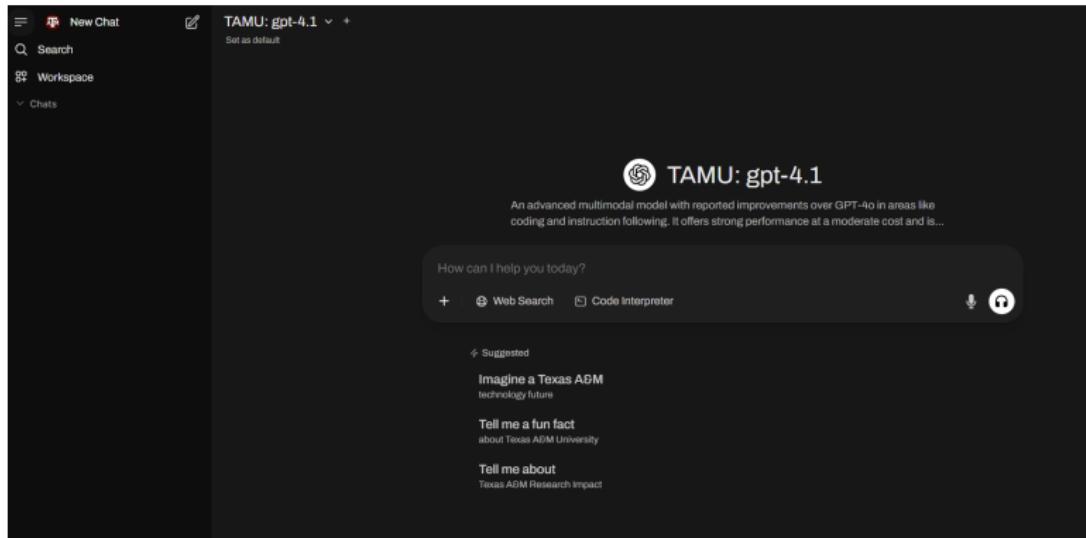
Texas A&M University - large R1 institution

Advanced Calculus - required for all majors in math, applied math, and math education

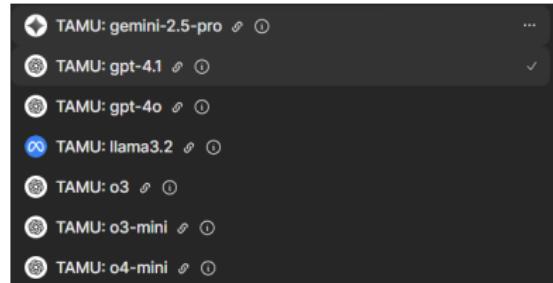
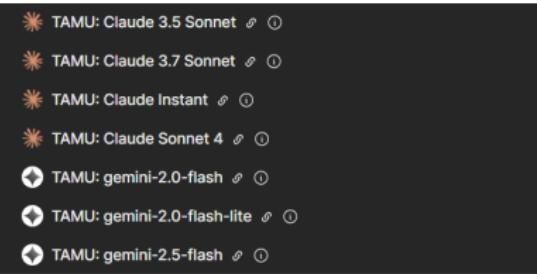
My course (Summer 2025) had 21 students, mostly undergraduate junior and seniors (1 freshman, 2 sophomores, 2 graduate)

In mid-June, all faculty/students/staff received an email from university IT announcing the beta for a new AI tool launching campuswide 2025

# New Developments



A new Open WebUI tool made available to all Texas A&M students and faculty launching campuswide Fall 2025



Unlimited browser-based chatbot access to all models above

Hi GPT! I am trying to solve  $\lim_{n \rightarrow \infty} \int_a^{\infty} \frac{1}{1+n^2x^2} dx$  for various real values of  $a$ . Would you please start by giving a detailed solution for this problem? Please write your answer in LaTeX then prettyprint the solution in your output. Ready when you are!



TAMU: gpt-5

**Solution.**  $I_n(a) = \int_a^{\infty} \frac{n}{1+n^2x^2} dx.$

Make the substitution  $u = nx$  (so  $du = n dx$ ). When  $x = a$ ,  $u = na$ ; when  $x \rightarrow \infty$ ,  $u \rightarrow \infty$ . Then

$$I_n(a) = \int_{na}^{\infty} \frac{1}{1+u^2} du = [\arctan u]_{na}^{\infty} = \frac{\pi}{2} - \arctan(na).$$

Therefore,

$$\lim_{n \rightarrow \infty} I_n(a) = \lim_{n \rightarrow \infty} \left( \frac{\pi}{2} - \arctan(na) \right) = \begin{cases} 0, & a > 0, \\ \frac{\pi}{2}, & a = 0, \\ \pi, & a < 0. \end{cases}$$

Yes, this does include GPT-5 (with a reasoning toggle)!  
AI is making very fast advances into our workplace and will soon be accessible to all students on campus who seek it out.

# Opinions on AI from Math Faculty - Generally Negative

Most common exposure to AI is from student use, typically unprompted, typically on homework assignments and take-home assessments.

Teacher tools using AI receive pushback (e.g. quiz-writing tools).

Incredible use cases for AI in making digital mathematics content accessible (OpenAI Whisper for captioning and transcriptions, Mathpix for converting LaTeX PDFs to Word documents).

## Limit Laws

Let's return to a limit we have worked with before:

$$\lim_{n \rightarrow \infty} \frac{n^2}{2n^2 + 1} = \frac{1}{2}.$$

How does calculus tell us to solve this limit?

Indeed! Calculus first tells us to divide both numerator and denominator by  $n^2$ :

$$\lim_{n \rightarrow \infty} \frac{n^2}{2n^2 + 1} = \lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^2}}{\frac{2n^2 + 1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{2 + \frac{1}{n^2}}.$$



All of the steps we've taken so far are entirely based on when we say two rational numbers are the same ( $\frac{p}{q} = \frac{r}{s} \iff ps = qr$ ). But the next step to calculate the limit requires a stronger result: we need to be able to say that *the limit of a quotient is a quotient of the limits*.

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## Student Opinions on AI - Somewhat Uninformed

Some students largely avoid use of AI due to teacher censure.

Some students see it as a tool for collecting answers for assessments.

Some students are largely enthusiastic about AI but are unclear on how to use it.

Math majors take an introductory course in Python or C as part of their degree program, but use of AI is generally discouraged in these courses.

My read: a majority of mathematics students are taking jobs that will be transformed by AI within the next decade, but they are largely unaware of what AI is capable of!

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# The In-Class Activity

Students were assigned to read the following tract from Thomson, Bruckner, and Bruckner's *Elementary Real Analysis* the night before the lecture over the Quotient of Limits Law for Sequences.

**Theorem 2.17 (Quotients of Limits)** Suppose that  $\{s_n\}$  and  $\{t_n\}$  are convergent sequences. Suppose further that  $t_n \neq 0$  for all  $n$  and that the limit

$$\lim_{n \rightarrow \infty} t_n \neq 0.$$

Then

$$\lim_{n \rightarrow \infty} \left( \frac{s_n}{t_n} \right) = \frac{\lim_{n \rightarrow \infty} s_n}{\lim_{n \rightarrow \infty} t_n}.$$

# The Proof

**Theorem 2.17 (Quotients of Limits)** Suppose that  $\{s_n\}$  and  $\{t_n\}$  are convergent sequences. Suppose further that  $t_n \neq 0$  for all  $n$  and that the limit

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Then

$$\lim_{n \rightarrow \infty} \left( \frac{s_n}{t_n} \right) = \frac{\lim_{n \rightarrow \infty} s_n}{\lim_{n \rightarrow \infty} t_n}.$$

*Proof.* Rather than prove the theorem at once as it stands let us prove just a special case of the theorem, namely that

$$\lim_{n \rightarrow \infty} \left( \frac{1}{t_n} \right) = \frac{1}{\lim_{n \rightarrow \infty} t_n}.$$

Let  $T = \lim_{n \rightarrow \infty} t_n$ . We need to show that no matter what positive number  $\epsilon$  is given we can find an integer  $N$  so that

$$\left| \frac{1}{t_n} - \frac{1}{T} \right| < \epsilon$$

if  $n \geq N$ . To work with this inequality requires us to consider

$$\left| \frac{1}{t_n} - \frac{1}{T} \right| = \frac{|t_n - T|}{|t_n| |T|}.$$

It is only the  $|t_n|$  in the denominator that offers any trouble since if it is too small we cannot control the size of the fraction. This explains the first step in the proof that we now give, which otherwise might have seemed strange.

Suppose that  $\epsilon > 0$ . Choose  $N_1$  so that

$$|t_n - T| < |T|/2$$

if  $n \geq N_1$  and also choose  $N_2$  so that

$$|t_n - T| < \epsilon |T|^2/2$$

if  $n \geq N_2$ . From the first inequality we see that

$$|T| - |t_n| \leq |T - t_n| < |T|/2$$

and so

$$|t_n| \geq |T|/2$$

if  $n \geq N_1$ . Set  $N = \max\{N_1, N_2\}$  and note that if  $n \geq N$  we must have

$$\begin{aligned} \left| \frac{1}{t_n} - \frac{1}{T} \right| &= \frac{|t_n - T|}{|t_n| |T|} \\ &< \frac{\epsilon |T|^2/2}{|T|^2/2} = \epsilon. \end{aligned}$$

This is precisely the statement that  $\lim_{n \rightarrow \infty} (1/t_n) = 1/T$ .

We now complete the proof of the theorem by applying the product theorem along with what we have just proved to obtain

$$\lim_{n \rightarrow \infty} \left( \frac{s_n}{t_n} \right) = \left( \lim_{n \rightarrow \infty} s_n \right) \left( \lim_{n \rightarrow \infty} \frac{1}{t_n} \right) = \frac{\lim_{n \rightarrow \infty} s_n}{\lim_{n \rightarrow \infty} t_n}$$

as required. ■

While this proof appears straightforward to students at first, students fail to understand why two bounds for  $N$  are chosen.

Suppose that  $\epsilon > 0$ . Choose  $N_1$  so that

$$|t_n - T| < |T|/2$$

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$$|t_n| \geq |T|/2$$

In lecture students were given the reading again as well as an opportunity to discuss it in groups.

We then discussed some of the major issues that arose with understanding the proof.

Then I handed out a printout of a chatbot conversation! Students remained in groups and discussed how the chatbot was helpful.

## Quotient of Limits Theorem Explained

<https://g.co/gemini/share/9dcf8fd09b>  

Created with 2.5 Pro (preview) June 16, 2025 at 01:37 PM • Published June 16, 2025 at 01:41 PM

tbb-quotie...itstheorem

 PDF

Hi Gemini! I am trying to understand the quotient of limits theorem. Can you summarize the key points of this theorem and help explain some of the parts that I might find confusing? I may have a few questions too if that's alright! Thanks :)

## Understanding the Proof

The proof provided can seem a bit tricky at first glance. Let's break down the two main parts that might be confusing.

### Why prove a special case first?

Instead of tackling the full  $\frac{s_n}{t_n}$  formula at once, the proof first proves a simpler, special case: finding the limit of the reciprocal,  $\lim_{n \rightarrow \infty} \left(\frac{1}{t_n}\right) = \frac{1}{T}$ .  $\heartsuit$

This is a strategic move. By proving this special case, the main theorem can then be proven very easily by combining this result with the Product of Limits Theorem. The final step looks like this:

$$\lim_{n \rightarrow \infty} \left(\frac{s_n}{t_n}\right) = \lim_{n \rightarrow \infty} (s_n \cdot \frac{1}{t_n}) = (\lim_{n \rightarrow \infty} s_n) \cdot (\lim_{n \rightarrow \infty} \frac{1}{t_n}) = \frac{\lim_{n \rightarrow \infty} s_n}{\lim_{n \rightarrow \infty} t_n}. \quad \heartsuit$$

This simplifies the overall logic by breaking the problem into smaller, more manageable parts.

### What is the purpose of $|T|/2$ ?

The most challenging part of the proof is handling the  $|t_n|$  in the denominator of the expression  $\left|\frac{1}{t_n} - \frac{1}{T}\right|$  [*cite\_start*]  $= \frac{|t_n - T|}{|t_n||T|}$ . We need to make sure this fraction doesn't blow up, which could happen if  $|t_n|$  gets very close to zero.  $\heartsuit$   $\heartsuit$

The chatbot excelled with detecting some of where students typically struggle - why is there a special case? What is the purpose of  $|T|/2$  as a bound?

The proof cleverly prevents this with its very first step. Let's call the limit of the sequence  $\{t_n\}$ ,  $T$ . Since we know  $T \neq 0$ , the value  $|T|/2$  is a positive number that is smaller than  $|T|$ .

The proof establishes that for a large enough  $n$  (specifically, for all  $n \geq N_1$ ), the distance between a term  $t_n$  and the limit  $T$  is less than  $|T|/2$ . This forces  $t_n$  to be closer to  $T$  than to 0, which in turn guarantees that  $|t_n|$  is always greater than  $|T|/2$ .

By ensuring the denominator  $|t_n|$  cannot be smaller than the fixed positive number  $|T|/2$ , we can control the size of the fraction and complete the proof.

I hope this helps clarify the theorem! Feel free to ask any more questions you have.

During the discussion following AI conversation reading, one student declared, "Now I understand! Because the function  $\frac{s_n}{t_n}$  can blow up as it approaches  $\frac{S}{T}$ , we need to bound the value of  $|t_n - T|$  away from 0." This was a crucial moment for that student and their group.

## A Few Observations

When in groups, discussion about the proof was lively.

When the chatbot conversation was handed out, the room was quieter, and there wasn't as much conversation.

Students started asking me more questions - perhaps they viewed the conversation in a way that could help them *individually* understand the proof.

The general consensus was that the chatbot conversation was “kinda helpful”.

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# Incorporating Chatbot Conversations Into Reading

From the semester's beginning students are assigned reading in Perusall (online annotation platform where multiple readings can be assigned at once).

After our reading on L'Hôpital's Rule I assigned students a reading as part of their Perusall assignment.

Students were prompted to discuss the reading with one another by highlighting text and making comments.

**Example 2.3:** The rational numbers can also be listed but this is quite remarkable, for at first sight no reasonable way of ordering them into a sequence seems likely to be possible. The usual order of the rationals in the reals is of little help.

To find such a scheme define the "rank" of a rational number  $m/n$  in its lowest terms (with  $n \geq 1$ ) to be  $|m| + n$ . Now begin making a finite list of all the rational numbers at each rank; list these from smallest to largest. For example, at rank 1 we would have only the rational number  $0/1$ . At rank 2 we would have only the rational numbers  $-1/1, 1/1$ . At rank 3 we would have only the rational numbers  $-2/1, -1/2, 1/2, 2/1$ . Carry on in this fashion through all the ranks. Now construct the final list by concatenating these shorter lists in order of the ranks:

$$0/1, -1/1, 1/1, -2/1, -1/2, 1/2, 2/1, \dots$$

The range of this sequence is the set of all rational numbers.

AF: I get how the ranking system helps list them, but how do we make sure we don't repeat equivalent fractions like  $1/2$  and  $2/4$ ? Are we just assuming lowest terms fixes that?

SR: i think it assumed that the numbers are listed in their lowest terms to avoid this redundancy

## Prompt for GenAI

"Hi Gemini! I am attaching a couple sections on L'Hopital's Rule from Thomson, Bruckner, and Bruckner's Elementary Real Analysis. I am finding the proofs in these sections difficult to understand. Would you help walk me through them?

For each of the sections titled "7.11.1", "7.11.2", and "7.11.3", would you provide a summary of the proof? If a step in the proof seems difficult to you, please pause in your summary and provide an explanation. You may find the examples in the text provided to be helpful.

Before beginning your summary, please give a brief explanation of Cauchy's Mean Value Theorem, as it seems crucial to understand for this proof."

## Example Prompt for Students to Respond

1. Did this explanation make this more confusing than the reading did?
2. Did this explanation seem to restate exactly what the text said without adding any extra explanation?
3. Was this explanation helpful? Did it add more context where it was useful to do so? Did it help make difficult steps easier to work through?
4. Is something missing from the explanation it gave? Remember, we concluded that AI is only “kinda helpful” - we need to help each other to truly understand these concepts!

## Testimonials from Students

[From A Student Who Was Struggling in the Course at the Time]

"I just wanted to say that the way you structured **this AI prompt has really helped me** learn how to ask better questions and **use AI more effectively for learning**. It's making a big difference in how I understand and engage with the material. Thank you!"

## Testimonials from Students

[From A Student with a High Grade in the Course at the Time]

"I thought these were pretty helpful. A challenge I'm starting to have with the TBB [Thomson, Bruckner, and Bruckner] text is that it tends to make internal references like calling theorems X.Y.Z instead of their names and the labeling of the equations can be tough to follow in areas. This is a bit more **clear/succinct** to me. That probably comes at the cost of some of the rigor; these don't work as standalone proofs because they omit some of the details ensuring that all the prerequisite conditions for the theorems/limit rules are satisfied."

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# One Last Example: A Video

# Student Testimonials

"It's really interesting to see how AI can be used to break down these proofs and provide another way of understanding the math."

This is where the cleverness of the proof comes in. To find out more about M, we need to use a theorem that relates the values of a function to the value of its derivative. The most famous one is Rolle's Theorem, which says that if a differentiable function has the same value at two different points, its derivative must be zero somewhere between them.

"This is a pretty good description of Rolle's Theorem. I also like how the AI is trying to explain why the theorem is being used."

## 2) What Is the Use of This Theorem?

Your intuition here is spot on. The main power of Taylor's Theorem is exactly what you described:

it allows us to approximate a complicated function with a simple polynomial and gives us a way to calculate the worst-case error for that approximation. ☰

"I like that we can use AI for explanations of every step. Books sometimes keep explanations very brief or leave it to the reader. It's ok to do that because that is part of learning, but AI is nice because it break down a problem to a level that is very accessible."

# My Takeaways

AI is an incredible tool for individual study, but quite a few students don't know how to use it well enough

AI only gives limited options if students don't know how to clarify what they're asking about

Instructors should consider giving students access to instructive AI conversations to help explain intricate concepts (in advanced calculus and beyond)