Separation Algebras

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A separation algebra (SA) is a structure $(M, \circ, U)$ such that for some $\top \notin M$:

- $\circ \in M^\top \times M^\top \to M^\top$
- $U \subseteq M$
- $\forall x \in M^\top.\, \top \circ x = \top$
- $\forall m \in M.\, \exists u \in U.\, u \circ m = m$
- $\forall x, y, z \in M^\top.\, (x \circ y) \circ z = x \circ (y \circ z)$
- $\forall x, y \in M^\top.\, x \circ y = y \circ z$
- $\forall x_1, x_2, y \in M^\top.\, x_1 \circ y = x_2 \circ y \neq \top \Rightarrow x_1 = x_2$

Adapted from Dockins et al.:

Definition 1. Given a set M we can construct the flat SA $M_\perp$ like so:

$$M_\perp \stackrel{\text{def}}{=} (M \uplus \{\perp\}, \circ, \{\perp\})$$

where:

$$m \circ m \stackrel{\text{def}}{=} \begin{cases} m' & \text{if } m = \perp \\ m & \text{if } m' = \perp \\ \top & \text{otherwise} \end{cases}$$

Definition 2. Given an SA $S = (M, \circ_S, U)$, we can construct the discrete SA $S_=$ like so:

$$S_= \stackrel{\text{def}}{=} (M, \circ, M)$$

where

$$m \circ_= m' \stackrel{\text{def}}{=} \begin{cases} m & \text{if } m = m' \\ \top & \text{otherwise} \end{cases}$$

Definition 3. Given SAs $S_1 = (M, \circ_1, U_1)$ and $S_2 = (N, \circ_2, U_2)$ we can construct the product SA $S_1 \times S_2$ like so:

$$S_1 \times S_2 \stackrel{\text{def}}{=} (M \times N, \circ, U_1 \times U_2)$$

where

$$(m, n) \circ (m', n') \stackrel{\text{def}}{=} \begin{cases} (m \circ_1 m', n \circ_2 n') & \text{if } m \circ_1 m' \neq \top \text{ and } n \circ_2 n' \neq \top \\ \top & \text{otherwise} \end{cases}$$

Definition 4. Given a set M and an SA $S = (N, \circ_S, U_S)$ we can construct the finite function SA $M \rightarrow_{\text{fin}} S = (M \rightarrow_{\text{fin}} N, \circ, U)$, where

$$\begin{aligned} M \rightarrow_{\text{fin}} N &\stackrel{\text{def}}{=} \{f \in M \rightarrow N \mid \text{finite}(f^{-1}\, U_N)\} \\ f_1 \circ f_2 &\stackrel{\text{def}}{=} \begin{cases} \lambda m \in M.\, f_1\, m \circ_S f_2\, m & \text{if } \forall m \in M.\, f_1\, m \circ_S f_2\, m \neq \top \\ \top & \text{otherwise} \end{cases} \\ U &\stackrel{\text{def}}{=} \{f \mid \forall m \in M.\, f\, m \in U_S\} \end{aligned}$$

A state is an element of a finite function SA from naturals to the flat SA of integers. A world is an element of the product SA comprising a state together with a discretised finite function SA from region names to states.

$$\begin{aligned} l, s &\in \mathsf{State} \stackrel{\text{def}}{=} \mathbb{N} \rightarrow_{\text{fin}} \mathbb{Z}_\perp \\ w &\in \mathsf{World} \stackrel{\text{def}}{=} \mathsf{State} \times (\mathsf{RName} \rightarrow_{\text{fin}} \mathsf{State})_= \end{aligned}$$

Remark 5. This isn't quite right – in fact the local state and each of the shared states must be pairwise disjoint.