Dérivées successives

$p \in \mathbb{N}, \quad f^{(p)} = (f')^{(p-1)} = (f^{(p-1)})'$

$f^{(2)} = (f')' \quad f^{(3)} = (f')^{(2)} = (f^{(2)})'$

1) $\forall x \in \mathbb{R}, \quad p \in \mathbb{N}, \quad (e^x)^{(p)} = e^x$

$(e^x)' = e^x$

$(e^x)^{(2)} = ((e^x)')' = (e^x)' = e^x$

$(e^x)^{(3)} = ((e^x)^{(2)})' = (e^x)' = e^x$

Ex : $f(x) = e^{2x}, \quad f^{(n)}(x) = ?$

$f'(x) = 2e^{2x}, \quad f''(x) = 2 \cdot 2 e^{2x} = 2^2 e^{2x}, \quad f^{(3)}(x) = 2 \times 2 \times 2 e^{2x}, \quad f^{(3)}(x) = 2^3 e^{2x}$

$\boxed{f^{(n)}(x) = 2^n e^{2x}}$

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2) Dérivée p-ème de x^n, $n, p \in \mathbb{N}$

| $p > n$
| $p = n$
| $p < n$

$(x^2)^{(3)} = (2x)^{(2)} = (2)^{(1)} = 0$

$(x^3)^{(4)} = (3x^2)^{(3)} = (6x)^{(2)} = (6)^{(1)} = 0$

$\boxed{\text{(*) Si } n < p \quad (x^n)^{(p)} = 0}$

$(x^3)^{(3)} = (3x^2)^{(2)} = (3 \times 2x)^{(1)} = 3 \times 2 \times 1 = 3!$

$(x^4)^{(4)} = (4x^3)^{(3)} = (4 \times 3x^2)^{(2)} = (4 \times 3 \times 2x)^{(1)}$

$= 4 \times 3 \times 2 \times 1 = 4!$

(**) Si $p = n$, $(x^n)^{(p)} = n!$

$$(x^4)^{(3)} = (4x^3)^{(2)} = (4\times 3x^2)^{(1)} = 4\times 3\times 2\, x^1$$

$$(x^7)^{(3)} = (7x^6)^{(2)} = (7\times 6x^5)^{(1)}$$

$$= \boxed{7\times 6\times 5}\, x^{4} \quad 4 = 7-3$$

$$(x^8)^{(3)} = 8\times 7\times 6\, x^5$$

$$7! = \boxed{7\times 6\times 5}\times \boxed{4\times 3\times 2\times 1}$$

$$= 7\times 6\times 5\times 4! \Rightarrow 7\times 6\times 5 = \frac{7!}{4!} = \frac{7!}{(7-3)!}$$

$$(x^7)^{(3)} = \frac{7!}{(7-3)!}\, x^{7-3}$$

(***) Si $n > p$,

$$\boxed{(x^n)^{(p)} = \frac{n!}{(n-p)!}\, x^{n-p}}$$

Résumé

$p, n \in \mathbb{N}$,

$$(x^n)^{(p)} = \begin{cases} 0 & \text{si } n < p \\ n! & \text{si } n = p \\ \dfrac{n!}{(n-p)!}\, x^{n-p} & \text{si } n > p \end{cases}$$

3) $f(x) = \frac{1}{x}$, $f^{(n)}(x)$

$x \neq 0$, $f'(x) = \frac{-1}{x^2}$, $f^{(2)}(x) = \left(\frac{-1}{x^2}\right)' = \frac{(-1) \times 2x}{x^4}$

$f^{(2)} = \frac{2}{x^3}$

$$f^{(3)} = \left(\frac{2}{x^3}\right)' = \frac{-2 \times 3x^2}{x^6} = \frac{-1 \times 1 \times 2 \times 3}{x^4}$$

$$f^{(4)}(x) = \left(\frac{-2 \times 3}{x^4}\right)' = \frac{2 \times 3 \times 4\, x^3}{x^8} = \frac{+1 \times 2 \times 3 \times 4}{x^5}$$

$$\boxed{f^{(p)}(x) = \left(\frac{1}{x}\right)^{(p)} = \frac{(-1)^p\, p!}{x^{p+1}}}$$

(1)

EX : $f(x) = \ln x$, $f^{(n)}(x) = \left(f'(x)\right)^{(n-1)}$

$f'(x) = \left(\frac{1}{x}\right)$ $= \left(\frac{1}{x}\right)^{(n-1)}$

$$\boxed{f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}}$$

$$\text{F.B.N :} \quad (f \cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} \cdot g^{(n-k)}$$

EX : $f(x) = \underline{x^{n-1}}\ \underline{\ln x}$, $g(x) = \underline{x^{n-1}}\ \underline{e^{2x}}$

Calculer $f^{(n)}(x)$ et $g^{(n)}$

Posons $u = x^{n-1}$ et $v = \ln x$

$$\rightarrow f^{(n)}(x) = (u \cdot v)^{(n)} = \sum_{p=0}^{n} C_n^p\, u^{(p)} \cdot v^{(n-p)}$$

$* \quad u^{(p)} = \left(x^{n-1}\right)^{(p)} = ? \qquad p \in [\![0, n]\!]$

Si $p = n$, $u^{(p)} = \left(x^{n-1}\right)^{(n)} = 0$

Si $p = n-1$ $u^{(p)} = \left(x^{n-1}\right)^{(p)} = (n-1)!$

Si $p < n-1$ $u^{(p)} = \left(x^{n-1}\right)^{(p)} = \frac{(n-1)!}{(n-1-p)!} x^{n-1-p}$

$* \quad v^{(n-p)} = (\ln x)^{(n-p)} = \frac{(-1)^{n-p}(n-p-1)!}{x^{n-p}}$

$$f^{(n)}(x) = \sum_{p=0}^{n} C_n^p u^{(p)} \cdot v^{(n-p)}$$

$$v^{(1)} = \frac{(-1)^{n-(n-1)} \cdot (\cancel{n}-(\cancel{n}-1)-1)!}{x^{n-(n-1)}} = \frac{-1 \cdot 0}{\cancel{x}}$$

$$= \underbrace{C_n^n u^{(n)} v^{(0)}}_{0} + \underbrace{C_n^{n-1} u^{(n-1)} \cdot v^{(1)}}_{0} + \sum_{p=0}^{n-2} C_n^p u^{(p)} \cdot v^{(n-p)}$$

$$= \sum_{p=0}^{n-2} C_n^p \frac{(n-1)!}{\cancel{(n-1-p)!}} x^{n-1-p} \cdot \frac{(-1)^{n-p} \cancel{(n-1-p)!}}{x^{n-p}}$$

$$= \sum_{p=0}^{n-2} C_n^p (n-1)! \cdot \frac{x^{n-p-1}}{x^{n-p}} (-1)^{n-p}$$

$$= (n-1)! \sum_{p=0}^{n-2} C_n^p x^{-1} \cdot (-1)^{n-p}$$

$$= \frac{(n-1)!}{x} \cdot \sum_{p=0}^{n-2} C_n^p (-1)^{n-p}$$

$$= \frac{(n-1)!}{x} \cdot \underbrace{\sum_{p=0}^{n-2} \boxed{C_n^p 1^p \cdot (-1)^{n-p}}}_{n-1}$$

$$\boxed{\sum_{p=0}^{(n)} C_n^p a^p \cdot b^{n-p}} = (a+b)^n$$

$$\sum_{p=0}^{n-2} C_{n-2}^p a^p \cdot b^{n-2-p} = (a+b)^{n-2}$$

$$\boxed{f^{(n)}(x) = (n-1) \cdot \frac{(n-1)!}{x}}$$

$$\sum_{p=0}^{n} C_n^p \underline{a}^p \underline{b}^{n-p} = \boxed{\sum_{p=0}^{n-2} C_n^p a^p b^{n-p}} + C_n^{n-1} a^{n-1} b^1 + C_n^n a^n b^0$$

$$\sum_{p=0}^{n-2} C_n^p \underline{a}^p \underline{b}^{n-p} = \underbrace{\boxed{\sum_{p=0}^{n} C_n^p a^p b^{n-p}}}_{(a+b)^n} - C_n^{n-1} a^{n-1} b^1 - C_n^n a^n b^0$$

$$\boxed{\sum_{p=0}^{n-2} C_n^p 1^p \cdot (-1)^{n-p}} = \sum_{p=0}^{n} C_n^p \underline{1}^p \cdot \underline{(-1)}^{n-p} - \boxed{C_n^{n-1}} 1^{n-1} (-1)^1 - C_n^n 1^n \cdot (-1)^0$$

$$= (1+(-1))^n + n - 1$$

$$= n-1$$

$$C_n^p = \frac{n!}{p!\,(n-p)!} \qquad 0! = 1$$

$$C_n^{(n-1)} = \frac{n!}{(n-1)!\,(1)!} = \frac{n!}{(n-1)!}$$

$$= \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$$

$$C_n^{(n-1)} \qquad C_4^3 = 4$$

$$g^{(n)}(x) = ? \qquad g(x) = x^{n-1} e^{2x}.$$
