STAT4106 Midterm 1

John W Smith Jr

Choose any 5 questions. Problems are weighted equally. You are allowed one formula sheet and are to be given a table of distributions.

Problem 1: Poisson probabilities and more

Suppose $Y_1, ..., Y_n \sim$ i.i.d. $Poisson(\lambda)$.

Part A

What is the exact distribution for $T = \sum_{i=1}^{n} Y_i$? Prove this result. Any required properties of the Poisson (expected value, variance, MGF, etc) do not need to be derived.

Part B

Suppose that we observed $X_1, .., X_n \sim Pois(2\lambda)$ and (independently) $Y_1, ..., Y_n \sim Pois(3\lambda)$. Let $c \in [0, 1], \hat{\lambda}_1 = \frac{\bar{X}}{2}, \hat{\lambda}_2 = \frac{\bar{Y}}{3}$. Let $\hat{\lambda}_3 = c\hat{\lambda}_1 + (1-c)\hat{\lambda}_2$ Find the c that minimizes $MSE(\hat{\theta}_3)$.

Problem 2: Building distributions

Suppose $Y_1, ..., Y_6 \sim N(0, 1)$. Let $\bar{Y}$ be the sample mean of the **first 5 observations**, $W = \sum_{i=1}^{5} Y_i^2$, and $U = \sum_{i=1}^{5} (Y_i - \bar{Y})^2$.

Part 1

What is the distribution of $\sqrt{5}Y_6/\sqrt{W}$? Why?

Part 2

What is the distribution of $2Y_6/\sqrt{U}$. Why?

Part 3

What is the distribution of $2(5\bar{Y}^2 + Y_6^2)/U$? Why?

Problem 3: Order statistics as estimators

Suppose that $X_1, X_2, ..., X_n \sim$ i.i.d. $Unif(0, \theta)$.

Part A

Consider the estimator $\hat{\theta}_1 = 2 \cdot \bar{X}$. Find $MSE(\hat{\theta}_1)$. You do not need to derive the variance of the uniform distribution.

Part B

Find the distribution of the n^{th} order statistic, $X_{(n)}$.

Part C

Suppose we want to use the n^{th} order statistic as our estimator, i.e. $\hat{\theta}_2 = X_{(n)}$. Find $MSE(\hat{\theta}_2)$. Are we able to use this estimator to create an interval estimate of the form $\hat{\theta}_2 \pm k\sigma_{\hat{\theta}_2}$? Justify your answer.

Part D

Find c such that $\hat{\theta}_3 = c\hat{\theta}_2$ is unbiased.

Problem 4: The best scaled S^2 estimator for σ^2

Suppose that $Y_1, Y_2, ..., Y_n \sim$ i.i.d. $N(\mu, \sigma^2)$. Consider all estimators of the form $\hat{\sigma}^2 = cS^2$, where S^2 is the standard sample variance estimate. Find the value for c that minimizes the MSE.

Problem 5: Exponential inference

Suppose $X_1, X_2, ..., X_n \sim$ i.i.d. $exp(\lambda)$, with pdf $f(x) = \lambda^{-1}\exp(-x/\lambda)$. Then, we know that $U = \sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$

Part A

Using the result from above and properties of the MGF, find the sampling distribution for $\bar{X}$.

Part B

Consider $T = \frac{X_1}{\lambda}$. Show that T is a pivotal quantity for λ.

Problem 6: True or false

Part A

The only condition required to use the Central Limit Theorem is that $n \geq 30$.

Part B

$\hat{\theta}^*$ is a minimum MSE estimator for a certain class of θ estimators. $\hat{\theta}^*$ is unbiased.

Part C

There can exist a random variable X such that $\mathbb{E}[X^3] = \infty$ but $\mathbb{E}[X^4] < \infty$.

Part D

$X_1, X_2, X_3, X_4 \sim N(0, 1)$. Then $Q = \frac{(X_1^2+X_2^2+X_3^2)/3}{(X_3^2+X_4^2)/2} \sim F_{3,2}$

Extra Credit (10 Points)

Suppose that X, Y are independent. Using only properties of the MGF, show that

$$\mathbb{V}[X+Y] = \mathbb{V}[X] + \mathbb{V}[Y]$$