STAT4106 Exam 1 Study Guide: Due October 1^{st}

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This is the study guide for Exam 1. All problems on here have the solutions provided as well as hints to help to get you there. This assignment can be handed in at any time before Exam 1 and will be graded based on completion. This is meant to help you practice. You are free to work on these problems with your classmates, though everyone must turn in their own copy of the assignment. Note that most of these problems should be relatively challenging.

Problem 1: MGFs and Transforms

Consider $X_1, .., X_n \sim Bern(p)$, and X_i independent.

Part 1

Compute the Moment Generating Function for a given X_i.

Part 2

Using properties of the MGF, show that $U = \sum_{i=1}^{n} X_i$ follows a Binomial distribution. Hint: the MGF of a binomial is given by $M_Y(t) = (1 - p + p\exp(t))^n$.

Part 3

Using properties of the MGF, show that if X, Y are independent, then $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. Hint: what is the "$0^{th}$" moment of a distribution? Using this alongside some elementary calculus will help you arrive at the solution.

Problem 2: Linear Combinations of Poisson R.V.'s

Suppose that $Y_1, ..., Y_N \sim Pois(\lambda_i)$. Let $a_i, i = 1, ..., n \in \mathbb{R}$.

Part 1

Show that $\sum_{i=1}^{N} a_i Y_i$ is still a Poisson distribution, and give the new value for λ. Hint: use the moment generating function. The resulting distribution should have $\lambda^* = \sum_{i=1}^{N} a_i \lambda_i$.

Part 2

Suppose that $X_1, ..., X_n \sim Pois(\lambda)$ are independent and $Y_1, ..., Y_n \sim Pois(2\lambda)$ are independent, with the $X_i's$ and $Y_i's$ also independent. Let $\hat{\lambda}_1 = \bar{X}$ and $\hat{\lambda}_2 = \frac{1}{2}\bar{Y}$. Show that $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are unbiased.

Part 3

Consider the estimator $\hat{\lambda}_3 = c\hat{\lambda}_1 + (1-c)\hat{\lambda}_2$. Show that this is also an unbiased estimator.

Part 4

Find c such that $\mathbb{V}[\hat{\lambda}_3]$ is minimized.

Problem 3: Uniform Order Stats as Estimators

Suppose that $X_1, .., X_n \sim Unif(0, \theta)$ are independent. Consider $\hat{\theta}_1 = 2\bar{X}$.

Part 1

Find $MSE(\hat{\theta}_1)$.

Part 2

Consider an estimator of the form $\hat{\theta}_2 = cX_{(n)}$. Find c such that $B(\hat{\theta}_2) = 0$. Then, find $MSE(\hat{\theta}_2)$. Which estimator is better?

Part 3

In class we briefly discussed that a property we wanted for $\hat{\theta}$ was that $\hat{\theta} \in (0, \theta)$. If we restrict $\hat{\theta}$ to being a random variable, we can show that there is no unbiased estimator of θ with this property. Prove this fact. Hint: we don't even need to use the fact that we are using an estimator. We can look at properties of the expectation simply by looking at $\mathbb{E}[X] = \int_0^\theta xf(x)dx$ through integration by parts.

Problem 4: Tricky True False

Answer the following with true/false.

- If $p(x)$ is a probability mass function for a discrete random variable X, then $p(x) \leq 1$ always.

- If $f(x)$ is a probability density function for a continuous random variable X, then $f(x) \leq 1$ always.
- The sample mean is **always** an unbiased estimator of the mean.
- For any distribution, $\frac{(n-1)S^2}{\sigma^2}$ follows a χ^2 distribution with $n-1$ degrees of freedom.
- The minimum MSE estimator for a class of estimators of a certain form must always be unbiased.

Problem 5: Moment Sequences in Lieu of MGFs

The moment generating function is a useful tool in looking at distributions of statistics, but it is not always the most practical when we are looking to compute moments themselves. Derivatives of higher orders get particularly messy when we want to look at high order moments. Some distributions exhibit **conjugacy** when we look at their moments - the integrand of the expectation has the same kernel as the original distribution. This allows us to find expressions for higher order moments without having to mess around with the MGF.

Part 1: Gamma Moment Sequences

Suppose that $X \sim Gamma(\alpha, \beta)$. Find a general expression to compute $\mathbb{E}[X^n]$. Hint: results from Homework 2 will be useful, as will $\Gamma(n+1) = n\Gamma(n)$. The expression that you derive should be simplified so that you do not have any auxiliary gamma functions except the one that you start with (i.e. it should be a function only of $\Gamma(\alpha)$, not of $\Gamma(\alpha+n)$ etc.)

Part 2: Beta Moment Sequences

The Beta distribution has a particularly ugly moment generating function. If $X \sim Beta(\alpha, \beta)$, then $M_X(t) = 1 + \sum_{k=0}^{\infty}(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r})\frac{t^k}{k!}$. The true form of the beta distribution is hidden by the normalizing constant, $\beta(\alpha, \beta)$. Using the definition of this normalizing constant, we can write the pdf as

$$f(x|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Using this unmasked pdf, compute a general expression for $\mathbb{E}[X^n]$. You should follow the same simplifcation rules as above. Hint: you should once again use results from last time. Remember that the kernel of the integral will once again follow a beta distribution. Use the fact that $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.