STAT4106 Exam 2 Study Guide

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This is the study guide for Exam 1. All problems on here have the solutions provided as well as hints to help to get you there. This assignment can be handed in at any time before Exam 1 and will be graded based on completion. This is meant to help you practice. You are free to work on these problems with your classmates, though everyone must turn in their own copy of the assignment.

Problem 1: Geometric Distribution Estimators

Suppose that $X_1, .., X_n \sim Geom(p)$, with pmf

$$p(X|p) = p(1-p)^{X-1}, p \in (0, 1)$$

Part A

Compute the maximum likelihood estimator for p.

Part B

The mean and variance of the geometric distribution are given by

$$\mathbb{E}[X] = \frac{1}{p}, \mathbb{V}[X] = \frac{1-p}{p^2}$$

Give a maximum likelihood estimator for the mean and variance of a geometric distribution.

Part C

Is the estimator for the mean from Part B consistent? Justify your answer.

Part D

Suppose that we want to perform a Bayesian analysis, with $\pi(p) \sim Beta(a, b)$. Compute the posterior distribution, $\pi(p|\mathbf{X})$. Is the beta distribution the conjugate prior?

Part E

Using your result from Part D, compute a Bayes estimator for p.

Problem 2

Suppose that $X_1, ..., X_n \sim$ iid $Normal(\mu, \sigma^2)$.

Part A

Compute the Cramer-Rao Lower Bound for μ.

Part B

Use the Rao-Blackwell Theorem to find an MVUE for μ.

Part C

Compute the Cramer-Rao Lower Bound for σ^2

Part D

Use the Rao-Blackwell Theorem to find an MVUE for σ^2.

Part E

Compute the variance of our MVUE for σ^2. What do you notice?

Problem 3

Suppose that X is a random variable belonging to the exponential family, with parameterization

$$f(x|\Theta) = c(\Theta)h(x)\exp\left(\sum_{i=1}^{n} t_i(x)w_i(\Theta)\right)$$

Part A

Using the factorization theorem, show that if we observe $X_1, ..., X_n$, then

$$T(\mathbf{X}) = \left(\sum_{i=1}^{n} t_1(X_i), ..., \sum_{i=1}^{n} t_k(X_i)\right)$$

is a sufficient statistic for Θ.

Part B

Show that the Gamma distribution is a member of the exponential family.

Part C

Using the result from Part A, find a sufficient statistic for the situation where $X_1, .., X_n \sim$ iid $Gamma(\alpha, \beta)$.

Problem 4

Suppose that we observe $X_1, ..., X_n \sim$ iid $Unif(0, \theta)$. We estimate θ with estimators

$$\hat{\theta}_1 = 2\bar{X}, \hat{\theta}_2 = X_{(n)}, \hat{\theta}_3 = \frac{n+1}{n} X_{(n)}$$

Part A

Which estimator is the method of moments estimator? Which estimator is the MVUE? Which estimator is the maximum likelihood estimator?

Part B

Do any of these estimators dominate each other? Justify your answer.

Part C

Which of these estimators are (asymptotically) consistent?

Part D

Suppose we are interested in θ^2 instead. Which of these estimators converge in probability to $\hat{\theta}^2$? Justify your answer.