STAT4106 Homework 4: Due October 15^{th}

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All submitted work is expected to be your own. Violations to this will be at the mercy of the Virginia Tech Honor Court.

Submitted assignments should be very neat and easy to read. For this reason, LaTeX is preferred. Microsoft Word documents must be rendered as pdf or html files, or resubmission will be required with a loss of 10 points. Hand written assignments are accepted so long as they are neat and legible. Homeworks must be stapled or 10 points will be lost.

Problem 1: Efficient Exponential Expectation Estimators ($\mathbf{E}^4$)

Suppose that $Y_1, ..., Y_{15} \sim Exp(\lambda)$ are independent, with pdf

$$\frac{1}{\lambda} \exp\Big(-\frac{x}{\lambda}\Big), x \in (0, \infty)$$

Consider the following estimators:

$$\hat{\lambda}_1 = \bar{X},\ \hat{\lambda}_2 = \frac{1}{5}\sum_{i=1}^{5} X_i,\ \hat{\lambda}_3 = Y_7$$

Part A

Find $eff(\hat{\lambda}_1, \hat{\lambda}_2)$

Part B

Find $eff(\hat{\lambda}_1, \hat{\lambda}_3)$

Part C

Find $eff(\hat{\lambda}_2, \hat{\lambda}_3)$

Part D

Do any of these estimators dominate another? If so, state which ones and explain why.

Problem 2: Consistent Uniform Estimators

Suppose $X_1, ..., X_n \sim Unif(0, \theta)$ are independent, and suppose that n is an odd number. Consider the following estimators:

$$\hat{\theta}_1 = 2\bar{X},\ \hat{\theta}_2 = \frac{n+1}{n} X_{(n)}\ , \hat{\theta}_3 = median(X_1, ..., X_n)$$

Part A

Which of these estimators are consistent? Give full justification.

Part B

Suppose that instead we are interested in estimating θ^2, using $\hat{\theta}_1^2$, $\hat{\theta}_2^2$ and $\hat{\theta}_3^2$. Which of these estimators converge in probability to θ^2? Justify your answer.

Problem 3: Sufficient Estimators and the Likelihood Function

In this problem we will explore the likelihood and determine whether or not estimators are sufficient.

Part A

Suppose $X_1, ..., X_n \sim Pois(\lambda)$ are independent. Find $\mathscr{L}(\lambda|\mathbf{X})$. Is $\bar{X}$ a sufficient estimator for λ? Justify your answer using the factorization theorem.

Part B

Suppose $X_1, ..., X_n \sim N(\mu, \sigma^2)$ are independent, and σ^2. is known. Find $\mathscr{L}(\mu|\mathbf{X}, \sigma^2)$. Is $\bar{X}$ a sufficient estimator for λ? Justify your answer using the factorization theorem. (Hint: look for a way to write $\bar{X}$ inside of the likelihood...)

Part C

Suppose that we have n independent realizations from a $Unif(0, \theta)$ distribution. Is $\hat{\theta} = X_{(n)}$ a sufficient estimator for θ? You do not need to use the factorization theorem, but you must justify your answer.

Part D

Suppose that $X_1, ..., X_n \sim Bern(p)$ are independent. Find $\mathscr{L}(p|\mathbf{X})$.

Part E

So far in this course we have been treating parameters as unknown constants, and our data as random. Here we will take a look at what happens when we treat our parameters as random and our data as fixed. Recall $\mathscr{L}(p|\mathbf{X})$ from above. If we treat p as random and $\mathbf{X}$ as fixed, what distribution does p follow? Justify your answer. (Hint: remember where p lives, your resulting distribution should be defined over the same space).