STAT4106 Final Exam

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This exam contains 7 problems, equally weighted at 15 points per problem. **All problems must be completed**. There are a total of 105 points not including the extra credit. Scores will be taken out of 100 points. You are allowed one formula sheet and will be given a table of distributions.

Problem 1: Exponential Estimators

Suppose $X_1, ..., X_n \sim$ i.i.d. $Exp(\lambda)$, with pdf given by

$$f(x|\lambda) = \lambda^{-1} \exp(-\lambda^{-1}x), x \in (0, \infty), \lambda > 0$$

Part A

Find the distribution of $X_{(1)}$, the minimum order statistic. State the type of distribution that it follows and specify any parameters.

Part B

Suppose that we want to estimate λ using an estimator of the form $\hat{\lambda} = cX_{(1)}$. Find c such that $\hat{\lambda}$ is unbiased.

Part C

Is the estimator found in Part B consistent? Justify your answer.

Part D

Using the Method of Moments, compute an estimator $\hat{\lambda}_{MOM}$.

Part E

Using MSE as the comparison criterion, which estimator performs better? Justify your answer.

Problem 2: Geometric Estimators

Suppose $X_1, ..., X_n \sim$ iid $Geom(p)$, which pmf given by:

$$p(X|p) = p(1-p)^{X-1}, p \in (0,1)$$

Part A

Compute the maximum likelihood estimator for p.

Part B

The mean and variance of the geometric distribution are given by

$$\mathbb{E}[X] = \frac{1}{p}, \mathbb{V}[X] = \frac{1-p}{p^2}$$

Give a maximum likelihood estimator for the mean and variance of a geometric distribution.

Part C

Suppose that we want to perform a Bayesian analysis, with $\pi(p) \sim Beta(a,b)$. Compute the posterior distribution, $\pi(p|\mathbf{X})$. Is the beta distribution the conjugate prior?

Part D

Using your result from Part D, compute a Bayes estimator for p.

Problem 3: Poisson Exponential Families

Part A

Suppose that $X_1, .., X_n \sim$ iid $Pois(\lambda)$. Let U be a linear combination of these Poisson random variables, i.e.

$$U = \sum_{i=1}^{n} a_i X_i, a_i \in \mathbb{R}$$

Find the distribution of U. Any additional properties (mean, variance, MGF, etc) do not need to be derived.

Part B

Show that the Poisson distribution is a member of the exponential family.

Part C

Using the method of your choice, find a sufficient statistic for the Poisson distribution.

Part D

State the Rao-Blackwell Theorem. Using the sufficient statistic from Part C, find a minimum variance unbiased estimator $\hat{\lambda}_{MVUE}$.

Part E

Compute the Cramer-Rao Lower Bound. Does the estimator from Part D achieve this bound?

Problem 4: Normal Variance Estimation

Suppose $X_1, ..., X_n \sim$ iid $N(\mu, \sigma^2)$.

Part A

Find the maximum likelihood estimator for σ^2. You may use the fact that $\hat{\mu}_{MLE} = \bar{X}$

Part B

Recall the canonical variance estimator for the normal distribution,

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Show that $\frac{(n-1)S^2}{\sigma^2}$ is a pivotal quantity for σ^2.

(Hint: what equation relates S^2 to σ^2 and what is its distribution?)

Part C

Suppose that we wanted to invoke the Central Limit Theorem to build confidence intervals of the form

$$\hat{\sigma}^2 \pm k\sigma_{\hat{\sigma}}$$

Which of these estimators can be used to build these intervals? Justify your answer.

Problem 5: Power and LRTs

Part A

Suppose that $X_1, .., X_n \sim N(\mu, \sigma^2)$ with σ^2 known. Let $H_0 : \mu = \mu_0$ vs $H_a : \theta \neq \theta_0$. Show that the test based on the likelihood ratio test statistic is equivalent to the Normal two-sided z-test.

Part B

Suppose that a factory foreman is interested in testing whether or not one of his pieces of machinery needs to be calibrated. To do this, he looks at $n = 50$ items $(X_1, ..., X_n)$ and tests whether or not they are defective. He picks $H_0 : p = .005$ vs $H_a : p > .005$.

Suppose that the foreman decides to use $RR = \{x | \sum_{i=1}^{n} x \geq 2\}$. Compute α, the Type I error rate.

Part C

Suppose that the true value for p is $p = .01$. Compute β, the Type II error rate, as well as the power of the test. Note: you just need to write an expression, no actual computations are required.

Problem 6: Generating Random Samples

Suppose that $X \sim Beta(\alpha, 2)$.

Part A

Find the CDF, $F(x)$.

Part B

Find the inverse cdf, $F^{-1}(x)$

Part C

Write psuedo-code to outline how to generate a random sample from the pdf $f(x)$. Are these exact samples?

Part D

The beta distribution does not have a closed form moment generating function, but we can derive an expression for an arbitrary moment, $\mathbb{E}[X^k]$ using conjugacy. Using the pdf a $Beta(\alpha, 2)$, given by

$$f(x|\alpha) = \frac{x^{\alpha-1}(1-x)}{\beta(\alpha, 2)}, x \in (0, 1), \alpha > 0,$$

derive a general expression for $\mathbb{E}[X^k]$. You do not need to write the $B(\alpha, 2)$ functions as ratios of Gamma functions.

Problem 7: Tricky True False

Answer the following questions as True or False.

- As our confidence level α decreases, our confidence intervals become more narrow.
- Samples drawn from a t-distribution with 2 degrees of freedom does not obey the Central Limit Theorem
- In hypothesis testing, with all other things fixed, there is a trade-off between the Type I and Type II error of a test (i.e. as Type I error rate goes up, Type II error goes down and vice versa).
- If $\hat{\theta}$ is the MVUE for θ then it must achieve the Cramer-Rao Lower Bound
- If θ is a parameter for a distribution with $\theta > 0$ and $\hat{\theta}$ converges in probability to θ, then $1/\hat{\theta}$ converges in probability to $1/\theta$.