STAT4106 Homework 2: Due September 12^{th}

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All submitted work is expected to be your own. Violations to this will be at the mercy of the Virginia Tech Honor Court.

Submitted assignments should be very neat and easy to read. For this reason, LaTeX is preferred. Microsoft Word documents must be rendered as pdf or html files, or resubmission will be required with a loss of 10 points. Hand written assignments are accepted so long as they are neat and legible. Homeworks must be stapled or 10 points will be lost.

Problem 1: χ^2 Expectation - The Hard Way

The pdf of a χ^2_ν distribution is given by

$$f(x|\nu) = \frac{x^{\frac{k}{2}-1}\exp\big(-\frac{x}{2}\big)}{2^{\frac{k}{2}}\Gamma\big(\frac{k}{2}\big)}$$

We see that this is a special form of the Gamma distribution, with $\alpha = \nu/2, \beta = 2$. The general pdf of the Gamma distribution is written

$$f(x|\alpha,\beta) = \frac{x^{\alpha-1}\exp\big(-\frac{x}{\beta}\big)}{\beta^\alpha\Gamma(\alpha)}$$

Part 1

Using properties of the pdf, show that

$$\int_0^\infty x^{\alpha-1}\exp\big(-\frac{x}{\beta}\big)dx = \beta^\alpha\Gamma(\alpha).$$

Part 2

Using the fact proved above, compute the expected value of a χ^2_ν distribution.

Hint: The relation that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ will be helpful.

Part 3

Using only results from Part 1 and 2, compute the variance of a χ^2_ν distribution.

Problem 2: χ^2 Expectation - The Easy Way

Part 1

Let X be a random variable. Show that $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
Hint: write $\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$ and use laws of expectation.

Part 2

Recall that if $Z \sim N(0,1)$, then $Z^2 \sim \chi^2_1$. Using the relation derived in Part 1, find the expected value of a χ^2_1.

Part 3

Recall that we may build a χ^2_ν by summing together independent χ^2_1 random variables. Using only this fact and results from Part 1 and 2, compute the expectation of a χ^2_ν distribution.

Problem 3: Multiple Sample Means

Suppose $Y_1, ..., Y_N$ and $X_1, ..., X_M$ are independent random samples, with $X_i \sim N(\mu_1, \sigma_1^2)$ and $Y_i \sim N(\mu_2, \sigma_2^2)$.

Part 1

Argue that the distribution of $\bar{X} - \bar{Y}$ must also be normal. You may use results from previous homework.

Part 2

Using only the individual sampling distributions of $\bar{X}$ and $\bar{Y}$ and properties of the expectation operator, find $\mathbb{E}[\bar{X} - \bar{Y}]$.

Part 3

Using only the individual sampling distributions of $\bar{X}$ and $\bar{Y}$, problem specific assumptions, and properties of the expectation operator, find $\mathbb{V}[\bar{X} - \bar{Y}]$.

Problem 4: Determining Distributions

Suppose $Y_1, ..., Y_6 \sim N(0, 1)$. Let $\bar{Y}$ be the sample mean of the **first 5 observations**, $W = \sum_{i=1}^{5} Y_i^2$, and $U = \sum_{i=1}^{5}(Y_i - \bar{Y})^2$.

Part 1

What is the distribution of $\sqrt{5}Y_6/\sqrt{W}$? Why?

Part 2

What is the distribution of $2Y_6/\sqrt{U}$. Why?

Part 3

What is the distribution of $2(5\bar{Y}^2 + Y_6^2)/U$? Why?

Problem 5: Applicability of the CLT

Recall the definition of the **strong** central limit theorem that was proved in class. Determine whether the central limit theorem applies to each of the following distributions. If it does, explain why (with relevant computations to show that assumptions are met). If it does not, explain why.

- A random variable X with pdf $f(x) = \frac{1}{x}$, $x \in [1, e]$.
- A t random variable with 2 degrees of freedom
- **Extra Credit**: An *inverse gamma* random variable with $\alpha = 1/2, \beta = \beta$. The pdf is given by

  $$f(x) = \frac{\beta^{3/2}}{\Gamma[3/2]} \exp\Big(-\frac{\beta}{x}\Big)x^{-3/2}, x \in [0, \infty].$$

  Hint: Neither the variance or expectation integrals are tractable - they are referred to as *exponential integrals*. Without using these we are still able to show that one of the assumptions is violated...