STAT4106 Final Exam

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This exam contains 7 problems, equally weighted at 15 points per problem. **All problems must be completed**. There are a total of 105 points not including the extra credit. Scores will be taken out of 100 points. You are allowed one formula sheet and will be given a table of distributions.

Problem 1: Exponential Estimators

Suppose $X_1, ..., X_n \sim$ i.i.d. $Exp(\lambda)$, with pdf given by

$$f(x|\lambda) = \lambda^{-1} \exp(-\lambda^{-1} x), x \in (0, \infty), \lambda > 0$$

Part A

Find the distribution of $X_{(1)}$, the minimum order statistic. State the type of distribution that it follows and specify any parameters.

Part B

Suppose that we want to estimate λ using an estimator of the form $\hat{\lambda} = cX_{(1)}$. Find c such that $\hat{\lambda}$ is unbiased.

Part C

Is the estimator found in Part B consistent? Justify your answer.

Part D

Using the Method of Moments, compute an estimator $\hat{\lambda}_{MOM}$.

Part E

Using MSE as the comparison criterion, which estimator performs better? Justify your answer.

Problem 2: Bernoulli Estimators

Suppose $X_1, ..., X_n \sim$ iid $Bern(p)$

Part A

Find the method of moments estimator for p, $\hat{p}_{MOM}$.

Part B

Compute an MVUE estimator for p using the Rao-Blackwell Theorem.

Part C

Suppose that we want to perform a Bayesian analysis, and we pick $\pi(p) \sim Beta(\frac{1}{2}, \frac{1}{2})$. What distribution does the posterior follow? Give a name and values for the parameter(s).

Part D

Using your result from Part C, find a Bayes estimator for p.

Part E

Compute the Cramer-Rao Lower Bound for unbiased estimators. Do either of the estimators from Part A or Part B attain this lower bound?

Problem 3: Exponential Families

Part A

Suppose that $X_1, X_2, ..., X_n$ are i.i.d. observations from a random variable that belongs to the exponential family, with the canonical parameterization

$$f(x_j|\Theta) = h(x_j)c(\Theta)\exp\Big(\sum_{i=1}^{k} w_i(\Theta)t_i(x_j)\Big)$$

Prove that

$$T(\mathbf{X}) = \Big(\sum_{j=1}^{n} t_1(X_j), \sum_{j=1}^{n} t_2(X_j), ..., \sum_{j=1}^{n} t_k(X_j)\Big)$$

is a sufficient statistic for Θ.

Part B

Show that the Gamma distribution, with parameterization

$$f(x|\alpha,\beta) = \frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}\exp\Big(-\frac{x}{\beta}\Big), x \geq 0, \alpha > 0, \beta > 0$$

is a member of the exponential family.

Part C

Using the result from above, find a minimal sufficient statistic for $\{\alpha, \beta\}$.

Part D

Compute a method of moments estimator for α and β. Are the resulting estimators the MVUEs for α and β? Justify your answer.

Problem 4: Reparameterization and estimation

Suppose $X_1, ..., X_n \sim$ iid $Exp(\lambda)$, with parameterization:

$$f(x|\lambda) = \frac{1}{\lambda} \exp\Big(-\frac{x}{\lambda}\Big), x \geq 0, \lambda > 0$$

Part A

Find the maximum likelihood estimator for λ.

Part B

Suppose instead that we parameterize the exponential as

$$f(x|\lambda) = \theta \exp(-\theta x), x \geq 0, \theta > 0$$

Find the maximum likelihood estimator for θ.

Part C

Give the definition of a consistent estimator, and show that $\hat{\lambda}_{MLE}$ is a consistent estimator of λ.

Part D

Is $\hat{\theta}_{MLE}$ a consistent estimator of θ? Thoroughly justify your answer.

Problem 5: True or false

Part A

If $\hat{\theta}$ converges in probability to θ, then $\frac{1}{\hat{\theta}}$ always converges in probability to $\frac{1}{\theta}$.

Part B

If $\hat{\theta}$ converges in probability to θ, then $\hat{\theta}^2$ always converges in probability to θ^2.

Part C

Maximum likelihood estimators are always unbiased.

Part D

If $\hat{\theta}$ is an unbiased estimator, then the Cramer-Rao Lower Bound can always be achieved.

Part E

If $\mathbb{E}[X] = \mu$ and $\mathbb{V}[X] = \sigma^2 < \infty$, then $\bar{X}$ is always a consistent estimator of μ.

Extra Credit (10 Points)

In class, we proved a result for exponential families regarding their expectations. Here we will prove a similar identity regarding their variance.

Suppose X comes from a distribution that belongs to the exponential family. Show that

$$\mathbb{V}\Big[\sum_{i=1}^{k}\frac{dw_i(\Theta)}{d\theta_j}t_i(X)\Big] = -\frac{d^2}{d\theta_j^2}\log(c(\Theta)) - \mathbb{E}\Big[\sum_{i=1}^{k}\frac{d^2 w_i(\Theta)}{d\theta_j^2}t_i(X)\Big]$$

Hint: Apply a similar approach to the derivation from class, and use the fact that $\frac{d^2}{dx^2}\log(g(x)) = \frac{g''(x)}{g(x)} - \left(\frac{g'(x)}{g(x)}\right)^2$