# STAT4106 Homework 4: Due October $25^{th}$

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All submitted work is expected to be your own. Violations to this will be at the mercy of the Virginia Tech Honor Court.

Submitted assignments should be very neat and easy to read. For this reason, LaTeX is preferred. Microsoft Word documents must be rendered as pdf or html files, or resubmission will be required with a loss of 10 points. Hand written assignments are accepted so long as they are neat and legible. Homeworks must be stapled or 10 points will be lost.

## Problem 1: Exponential Families and Sufficiency

### Part A

The probability mass function for the Poisson distribution is

$$p(x|\lambda) = \exp(-\lambda)\frac{\lambda^x}{x!}, x = 0, 1, ...,$$

Show that the Poisson distribution is a member of the exponential family.

### Part B

Suppose that $X_1, ..., X_n \sim Pois(\lambda)$ are independent. Using Theorem 9.5, find a sufficient statistic for $\lambda$.

### Part C

Using the Rao-Blackwell Theorem, find the minimum variance unbiased estimator for $\lambda$.

### Part D

Compute the Cramer-Rao Lower Bound for the Poisson family. Does the estimator from above attain the bound?

# Problem 2: Ahead of the curve

A **curved exponential family** is a family of densities that can be written in the form of an exponential family for which the dimension of its unknown parameter space ($\Theta$) is less than $k$.

## Part A

Show that a Normal distribution with $\mu = \mu$ and $\sigma^2 = \mu^2$ is a member of the curved exponential family.

## Part B

Show that the Central Limit Theorem approximation for $\bar{X}$ from Problem 1 results in a curved expoential family, and give the new parameters $\mu^*$ and $\sigma^{2*}$

# Problem 3: Gamma MVUE

Suppose $X_1, ..., X_n \sim Gamma(\alpha, \beta)$ are iid, and $\alpha$ is known, with pdf

$$f(x|\alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp\Big(-\frac{x}{\beta}\Big), x \geq 0, \beta > 0$$

## Part A

Using the factorization theorem, find a sufficient statistic for $\beta$.

## Part B

Apply the Rao-Blackwell Theorem to obtain the MVUE for $\beta$

## Part C

Compute the Cramer-Rao Lower Bound for an estimator of $\beta$. Does our estimator from Part B attain the lower bound?

# Problem 4: Trouble in paradise

Suppose that $X_1, ..., X_n \sim Unif(0, \theta)$ are iid.

## Part A

Using the result that for unbiased estimators $W(X)$,

$$\mathbb{V}[W(X)] \geq \frac{1}{n\mathbb{E}_\theta[\left(\frac{d}{d\theta}\log(f(x|\theta))^2\right)]}$$

compute the theoretical Cramer-Rao Lower Bound for unbiased estimators for $\theta$.

## Part B

Compute the variance of the scaled maximum order statistic,

$$\hat{\theta} = \frac{n+1}{n}X_{(n)}.$$

What do you notice about the variance of the estimator compared to the theoretical lower bound?

## Part C

Show that the lower bound from Part A is not valid (hint: remember the assumptions required in order for the CRLB to be valid).

# Extra Credit: More Exponential Family Identities

In class, we proved a result for exponential families regarding their expectations. Here we will prove a similar identity regarding their variance.

Suppose $X$ comes from a distribution that belongs to the exponential family. Show that

$$\mathbb{V}\Big[\sum_{i=1}^{k} \frac{dw_i(\Theta)}{d\theta_j} t_i(X)\Big] = -\frac{d^2}{d\theta_j^2} \log(c(\Theta)) - \mathbb{E}\Big[\sum_{i=1}^{k} \frac{d^2 w_i(\Theta)}{d\theta_j^2} t_i(X)\Big]$$

Hint: Apply a similar approach to the derivation from class, and use the fact that $\frac{d^2}{dx^2} \log(g(x)) = \frac{g''(x)}{g(x)} - \left(\frac{g'(x)}{g(x)}\right)^2$