A walk through moduli space with SLags

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Chapter 1

Type IIA on the quintic

1.1 Motivation

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Why study SLags?

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In this section, we describe Fermat's quintic and study the properties of SLag three-cycles defined on this manifold.

Fermat's quintic

Chow's theorem asserts that any $(n-r)$-dimensional submanifold of $\mathbb{CP}^n$ can be realized as the zero locus of r-homogeneous polynomial equations. This suggests Calabi-Yau manifolds as submanifolds of a complex projective space

Motivate better

The complex projective space $\mathbb{CP}^n$ is defined considering the complex space minus the origin $\mathbb{C}^{n+1}\backslash\{0\}$ and establishing the equivalence relation $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$, $\lambda \in \mathbb{C}$. To emphasize that z_i are homogeneous coordinates, they are sometimes denoted as $[z_0 : \ldots : z_n]$. The local (non-homogeneous) coordinates ξ^i of a j-patch where $z_j \neq 0$ are obtained by choosing $\lambda = 1/z_j$ so

$$(\xi_j^1, \ldots, \xi_j^n) = \left(\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j} \right). \tag{1.1}$$

The complex projective space allows us to obtain lower dimensional manifolds as subspaces of $\mathbb{CP}^n$. Indeed, Chow's theorem states that any submanifold of complex dimension $n - r$ of $\mathbb{CP}^n$ can be realized as the zero locus of r-homogeneous polynomial equations. An homogeneous polynomial P of degree d satisfies

$$P(\lambda z_1, \ldots, \lambda z_n) = \lambda^d P(z_1, \ldots, z_n). \tag{1.2}$$

We are interested in calculating the number of inequivalent $(n - 1)$-dimensional submanifolds of $\mathbb{CP}^n$ that can be defined through a polynomial equation of degree d. The number of independent monomials of degree d in $n + 1$ variables is given by the binomial coefficient

$$\# \text{ indep. monomials} = \binom{d + (n+1) - 1}{(n+1) - 1} \tag{1.3}$$

But not all of these lead to different manifolds, since some of them can be related through coordinate transformations, which belong to complex general linear group $GL(\mathbb{C})_{n+1}$. Thus, the number of possible submanifolds is given by

$$\# \text{ submanifolds } \mathbb{CP}^n = \# \text{ indep. monomials} - \# \text{ components } GL(\mathbb{C})_{n+1} \tag{1.4}$$

Among of the $\binom{9}{4} - 5^5 = 101$ possible submanifolds of $\mathbb{CP}^4$ defined by a quintic polynomial, lies Fermat's quintic threefold defined by the polynomial P_5 as

$$P_5(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0. \tag{1.5}$$

Construction of SLags

A special-Lagrangian three-cycle is constructed by introducing an anti-holomorphic involution. In the case of $\mathbb{CP}^4$, there is only one consistent anti-holomorphic involution, which acts on the homogeneous coordinates as $\mathcal{R} : z_i \to \bar{z}_i$. A particular instance of SLag three-cycle is defined as the fixed loci under $\mathcal{R}$. In the case of Fermat's quintic leads it to the following SLag

$$\left\{ [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{RP}^4 | x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0 \right\}. \tag{1.6}$$

This is the space where O6-planes lie. Think dim O6-planes and sLags

This subspace is topologically equivalent to $\mathbb{RP}^3$, which can be noticed through the homeomorphism from $\mathbb{RP}^3$ to the SLag:

$$(u_0, u_1, u_3) \rightarrow (u_0, u_1, u_2, u_3, -(u_0^5 + u_1^5 + u_2^5 + u_3^5)^{1/5}). \tag{1.7}$$

In order to prove that the obtained subspace is indeed special-Lagrangian we must verify that the Kähler form vanishes at the SLag and that it is calibrated with respect to the same three-form as Ω_3.

Clarify

- The Kähler form k transforms under the anti-holomorphic involution as $k \rightarrow -k$, but since the pull-back of k onto the three-dimensional subspace must be real, the only possibility is $k|_{\text{SLag}} = 0$

- The holomorphic three-form in a non-homogeneous patch where $x_0 \neq 0$ is

Justify, what do we integrate and why

$$\Omega_3 = \frac{4}{2\pi i} \int \frac{x_0 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{P_5(x_i)} \tag{1.8}$$

if we interpret x_4 as a function of P_5 and integrate over a loop around $P_5 = 0$

$$\Omega_3 = \frac{x_0 dx_1 \wedge dx_2 \wedge dx_3}{x_4^4} = \frac{dy_1 \wedge dy_2 \wedge dy_3}{y_4^4} \tag{1.9}$$

where we have defined the local coordinates $y_i = x_i/x_0$. By taking the norm of the previous equation, we relate Ω_3 to the determinant of the six-dimensional metric

Clarify step

$$||\Omega_3||^2 = \frac{1}{\det g |y_4|^8}. \tag{1.10}$$

Since Ω_3 is covariantly constant, $||\Omega_3||$ must also be proportional to a constant defined as

$$||\Omega_3||^2 = 8e^{2\kappa}. \tag{1.11}$$

$$\det g = \frac{e^{-2\kappa}}{8|y_4|^8} \tag{1.12}$$

The pull-back of the six-dimensional metric onto the three-cycle is

$$h_{\alpha\beta} = 2\partial_\alpha X^i \partial_\beta X^{\bar{j}} g_{i\bar{j}} \tag{1.13}$$

So the volume form of the SLag three-cycle coincides with Ω_3

$$e^k \sqrt{\det h_{ab}} d\sigma^1 \wedge d\sigma^2 \wedge d\sigma^3 = e^k |\det \partial y| \frac{e^{-\kappa}}{|y_4|^4} d\sigma^1 \wedge d\sigma^2 \wedge d\sigma^3 = \Omega_3 \tag{1.14}$$

> Ask reasoning

It is possible to define different SLags by exploiting the $\mathbb{Z}_5^4$ symmetry of the quintic, which leads to a whole family of SLags

$$|0, k_1, k_2, k_3, k_4\rangle = \left[x_0 : \omega_1^k x_1 : \omega_2^k x_2 : \omega_3^k x_3 : \omega_4^k x_4 \right] \tag{1.15}$$

where $\omega = e^{i\frac{2\pi}{5}}$. The freedom to choose of $k_i \in \mathbb{Z}_5$ leads to $5^4 = 625$ SLag three-cycles, but not all of them are calibrated with respect to the same three-form as O6-plane $|0, 0, 0, 0, 0\rangle$. $\sum_i k_i^4 = 0$ mod 5 so only 125 SLags are calibrated with respect to the same three-form as the O6-plane.

Moduli space of SLags

> Not sure if the reasoning in this section is too sloppy.

We are not only interested in SLags themselves, but also in their moduli space. The reason is that deformations of SLag three-cycles codify the information of the open string excitations of the D6-branes that wrap around them. In particular, they determine de vev of the scalar field excitations and thus the position of the D6-branes and whether they are stable configurations or not.

A deformation of a reference SLag three-cycle Π_3 into another three-cycle (in the same homology class as the original SLag) can be parameterized by a normal vector field, which belongs to the normal bundle of Π_3. This deformation normal vector field can be written in a basis $\{s^i\}_i$ as

$$X = \phi_i s^i \tag{1.16}$$

where ϕ_i correspond to the scalar fields defined on the D6-brane worldvolume theory. The Kähler form introduces an isomorphism between the normal

bundle of Π_3 and the cotangent bundle of Π_3. According to McLean's theorem, harmonic one-forms are associated to a deformation vector field. Recalling that there is a single harmonic form in each cohomology class, we associate to every s^i a one-form $\xi_i \in H^1(\Pi)$. As a consequence, the moduli space of SLags is identified with $H^1(\Pi_3)$.

A gauge potential A on the D6-brane can be expanded in a basis of one-forms ξ_i as $A = a^i \xi_i$. In order to fill in the multiplets of the spectrum arising from open string excitation of D6-branes, we must arrange the a^i and ϕ_i into a complex scalar field

$$\Phi^i = \phi_i + ia^i. \tag{1.17}$$

We can now apply the previous reasoning to SLag three-cycles defined on Fermat's quintic. Since $H^1(\mathbb{RP}^3) \simeq H_1(\mathbb{RP}^3) \simeq \mathbb{Z}_2$, there are no continuous deformations of the SLags and they are guaranteed to remain stable.

Intersection numbers

We now study the intersection numbers between different SLags, which are used to determine the number of chiral fermions associated to the intersecting D6-branes.

First of all, we choose a coordinate patch of $\mathbb{CP}^4$ where $x_0 = 1$ and take one of the SLags to be $|0,0,0,0,0\rangle$ and the other SLag $|0, k_1, k_2, k_3, k_4\rangle = \omega^{k_1} \omega^{k_2} \omega^{k_3} \omega^{k_4} |0,0,0,0,0\rangle$, obtained by applying successive rotations ω associated to the $\mathbb{Z}_5^4$ symmetry. The intersection numbers in this patch are:

- If $\omega_i^{k_i} \neq 1$ for all i, there are no intersections in this patch. The justification is that the coordinates of one of the SLags $[1 : x_1 : x_2 : x_3 : x_4]$ where $x_i \in \mathbb{R}$ take only real values, while some of the coordinates of the SLag $[1 : \omega_1^{k_1} x_1 : \omega_2^{k_2} x_2 : \omega_3^{k_3} x_3 : \omega_4^{k_4} x_4]$ take always complex numbers.

- If $\omega_i^{k_i} \neq 1$ for only one i, which we set to be $i = 4$, the intersection is at a single point. In particular, the intersection of $[1 : x_1 : x_2 : x_3 : x_4]$ with $[1 : \omega_1^{k_1} x_1 : \omega_2^{k_2} x_2 : \omega_3^{k_3} x_3 : \omega_4^{k_4} x_4]$ is located at $(1, 0, 0, 0, x_4)$. When we restrict this line in $\mathbb{CP}^4$ to the quintic hypersurface, the intersection reduces to the point $(1, 0, 0, 0, -1)$. The signature of the intersection is $\text{sgn}(\text{Im}\,\omega^{k_1}\,\text{Im}\,\omega^{k_2}\,\text{Im}\,\omega^{k_3})$.

- If $\omega_i^{k_i} \neq 1$ for at least two different values of k_i, we have to deform one of the SLags.

  - $I_{\mathbb{RP}^3} = 0$

  - $I_{\mathbb{RP}^2} = 1$

  - $I_{\mathbb{RP}^3} = 0$

Volumes

$$\Omega_3 = \frac{dy_1 \wedge dy_2 \wedge dy_3}{5y_4^4} \tag{1.18}$$

$$\int \frac{dy_1 \wedge dy_2 \wedge dy_3}{5(1 + y_1^4 + y_2^4 + y_3^4)^{4/5}} \tag{1.19}$$

SM on the quintic

Bibliography