A walk through moduli space with SLags

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Chapter 1

Generalities of type IIA string theory

In this thesis we will only work with the supersymmetric type IIA theory. The study of string theory in Minkowski spacetime has lead to the identification of five consistent string theories, which all turn out to be supersymmetric and give rise to massless bosonic and fermionic excitations in their spectrum. The five string theories were given their name according to their own specificities: Type heterotic HE and HO, Type I, Type IIB and Type IIA string theory. In this thesis we will only concentrate on the last one in the list.

Type IIA spectrum

Type IIA string theory requires ten space-time dimensions to be consistent. Furthermore, it has a 10-dimensional supersymmetry with 32 supercharges, which corresponds to $\mathcal{N} = (1,1)$. The flat 10-dimensional space-time bosonic spectrum of type IIA can be classified according to the boundary conditions of the strings, whether we consider Ramond (R) or Neveu–Schwarz (NS) conditions. There would be an infinite tower of massive string states, but restrict the discussion to the massless states only. In the NS-NS sector, we find the dilaton ϕ, a two-form B_2 and a graviton $G_{\mu\nu}$, while in the R-R sector we identify the 1- and 3-forms c_1, c_3. The fermions, which belong to the NS-R and R-NS sectors, are two opposite-chirality gravitinos ψ and two opposite-chirality dilatinos λ.

Type IIA SUGRA

The low-energy theory of the ten-dimensional type IIA string theory is type IIA supergravity (SUGRA).

The spectrum of of type IIA SUGRA is the massless spectrum of type IIA string theory and has as effective action

$$2\kappa^2 S = \int d^{10}x\sqrt{-G}\left[e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\right) - \frac{1}{2}|F_2|^2 - \frac{1}{2}|\tilde{F}_4|^2\right] \tag{1.1}$$

$$-\frac{1}{2}\int B_2 \wedge F_4 \wedge F_4 \tag{1.2}$$

where we employ the following definitions: $H_3 = dB_2$, $F_2 = dC_1$, $F_4 = dC_3$, $\tilde{F}_4 = dC_3 - C_1 \wedge H_3$ and $2\kappa^2 = (2\pi)^7\alpha'^4$. α' is the Regge slope which is a free parameter of string theory of dimension length squared.

The D-brane

The two-dimensional strings can be generalized to $(p+1)$-dimensional extended object, which are called Dp-branes. Thus, a D1-brane would correspond to a D-string, a D2-brane would be a three-dimensional membrane and so on. The existence of Dp-branes can be motivated, in the weak coupling limit, as objects where open strings end, so they are a way to impose Dirichlet boundary conditions on open strings. In type IIA string theory, only even-dimensional Dp-branes are physical, which are: the D0, D2, D4, D6 and D8-branes.

We can study the dynamics of a Dp-branes in terms of the open string excitations with endpoints attached to the Dp-brane. Let us consider the open string excitations of a Dp-brane, the latter spanning $p+1$ dimensions and transverse to $d-p-1$ dimensions. The presence of the Dp-brane breaks the ten-dimensional Poincaré invariance of the theory, because open string excitations propagate on the $(p+1)$-dimensional volume of the Dp-brane only. This implies that massless particles must transform under irreducible representations of $SO(p-2)$, instead of $SO(d-2)$. The massless spectrum in $(p+1)$ dimensions of the open string theory is composed by a gauge boson A^μ ($\mu = 0, \ldots, p$) (corresponding to longitudinal oscillations to the brane), $9-p$ real scalars ϕ^i

(corresponding to transverse oscillations to the brane) and some fermions [1] λ_a. This particle content can be arranged into a vector supermultiplet of $U(1)$ with 16 supersymmetries in $(p+1)$ dimensions. Thus, a Dp-brane reduces the degree of supersymmetry of the type IIA theory by half.

In order to find out the action of a Dp-brane we must realize that it corresponds to the $(p+1)$-dimensional effective action of the massless open string excitations of the Dp-brane. As an illustration of this, a Dp-brane breaks the translational symmetry of the vacuum, which allows us to conclude that the ϕ^i scalar fields are the Goldstone bosons associated to the broken symmetry. The vev of these scalar fields determine the position of the Dp-brane in the transverse space, and the fluctuations of the scalar fields determine the evolution of the Dp-brane worldvolume W_{p+1} (the generalization of the particle worldline to the case of higher-dimensional branes). The resulting action of the bosonic sector of the Dp-brane is the sum of a Dirac-Born-Infeld term S_{DBI} and a Chern-Simons term S_{CS}.

The DBI term carries the information of how a Dp-brane interacts with the NSNS fields. It takes the form

$$S_{DBI} = -\mu_p \int d^{p+1}x e^{-\phi} \sqrt{-\det(P[G+B] - 2\pi\alpha' F)} \tag{1.3}$$

where the space-time gauge field strength is $F = dA_1$, the coefficient μ_p is

$$\mu_p = \frac{(\alpha')^{-(p+1)/2}}{(2\pi)^p} \tag{1.4}$$

and we define the pullback of a tensor G into the brane worldvolume as

$$P[G]_{\mu\nu} = G_{\mu\nu} + G_{\mu i}\partial_\nu \phi^i + \partial_\mu \phi^i G_{i\nu} + \partial_\mu \phi^i \partial_\nu \phi^j G_{ij}. \tag{1.5}$$

The form is this action can be justified in the following way:

Complete details.

- due to the pull-back of the ten-dimensional metric to the D-brane worldvolume, there is a natural metric on the D-brane worldvolume , which can be used to compute to set up a volume-element for the D-brane; integrating over the entire worldvolume then gives you part of the DBI-action

[1] These $p+1$ fermions correspond to the decomposition of a ten-dimensional Majorana fermion along the D-brane. They are the gaugini (superpartners of the gauge bosons) and the superpartners of the scalar fields.

- the pull-back of the NS two-form represents a first generalization of the D-brane volume-element and requires also the inclusion of the U(1) gauge field associated to the open string excitations (to preserve gauge invariance)
- the coupling of the D-brane to the NS-fields is proportional to $e^{-\phi}$, which is the characteristic coupling for open string interactions

If we expand the DBI action in powers of α', we obtain the Yang-Mills term

$$S_{YM} = \frac{\alpha'^{-(p-3)/2}}{4g_s(2\pi)^{p-2}} \int d^{p+1}x\sqrt{-g}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} \tag{1.6}$$

which allows us to identify the Yang-Mills coupling as

$$g_{YM}^2 = g_s\alpha'^{(p-3)/2}(2\pi)^{p-2} \tag{1.7}$$

The Chern-Simons term is topological in nature and describes how Dp-branes interact with RR-fields. It is given by

$$S_{CS} = \mu_p \int P\left[\sum_q c_q\right] \wedge e^{2\pi\alpha F_2 - B_2} \wedge \hat{A}(R) \tag{1.8}$$

where we integrate only over the $(p+1)$-forms of the integrand. The first terms of the A-roof polynomial are $\hat{A}(R) = 1 - \frac{1}{24(8\pi)}\,\mathrm{Tr}\,R^2 + \cdots$ and R is the space-time curvature two-form.

Multiple D-branes

It is convenient to generalize the single Dp-brane configuration to N parallel Dp-branes. In order to determine the spectrum of a stack of Dp-branes, we consider open strings with endpoints attached to either a single brane or two distinct ones.

If all branes are separated from each other, strings that stretch from a brane to itself correspond to massless gauge bosons that belong to $U(1)^N$. In contrast, strings that stretch from one brane A to another brane B lead to massive particles whose masses increases with the distance between branes. The lightest of these particles have opposite charge $(1,-1)$ under $U(1)_A \times U(1)_B$. Since Type IIA strings carry an orientation, a string stretching B to A would have opposite charges.

In the case of N coincident Dp-branes, strings that stretch between any two Dp-branes (possibly the same) give rise to massless states, so the gauge symmetry enhances from $U(1)^N$ to $U(N)$. The massless spectrum is composed of $(p-1)$-dimensional $U(N)$ gauge bosons, $(9-p)$ real scalars in the adjoint representation of $U(N$ and several fermions in the adjoint representation.

Let us now suppose Dp-branes which are not parallel, so they can intersect each other. This situation is relevant as it can lead to four-dimensional chiral fermions in the case of intersecting D6-branes. We are interested in describing the open string spectrum of two stacks of D6-branes that intersect over a 4-dimensional subspace of their volumes.

Strings that stretch from a coincident stack of N D6-branes to itself lead to 7-dimensional $U(N)$ gauge bosons, three real adjoint scalars and their fermion superpartners.

String that stretch from a stack of N_1 D6-branes to another stack of N_2 D6-branes are localized at the intersection, in order to minimize their energy. They lead to a 4-dimensional fermion charged in the $(\mathbf{N_1}, \mathbf{N_2})$ of $U(N_1) \times U(N_2)$ or its conjugate, depending on the orientation of the intersection.

Not all geometric configurations preserve supersymmetry. Let us decompose space-time as $M_4 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. The D6-branes span all M_4 and a line in each $\mathbb{R}^2$ plane, such that the angle between two stacks is given by θ_i for each plane. It can be shown that the condition $\theta_1 \pm \theta_2 \pm \theta_3 = 0(\text{mod} 2\pi)$ implies $\mathcal{N} = 1$ supersymmetry in 4 dimensions, provided that no angle vanishes. If some of the angles vanish, the supersymmetry would we enhanced.

The reason we have used D6-branes and no other dimension of Dp-branes is that they would not lead to chiral fermions in 4 dimensions. Intuitively, two D6-branes allow to define an orientation in the transverse 6-dimensional space, which would not be possible with two other type of Dp-branes in type IIA string theory.

Chapter 2

Type IIA compactifications

In the following section we motivate the requirement that additional dimensions are compactified over a Calabi-Yau manifold.

As we have seen in the previous chapter, Type IIA superstring theory requires nine spatial dimensions and one time dimension for consistency, yet our universe only consists of a four-dimensional spacetime continuum. This implies that six spatial dimensions have to be compactified on an internal manifold with an unobservably small volume. We assume that the manifold M is factorizable into a four-dimensional maximally symmetric space-time T and a six-dimensional compact space K, $M = T \times K$.

Type IIA string theory on 10 dimensional flat space has a large degree of supersymmetry, but the compactification choice can either preserve some degree of supersymmetry in four dimensions or remove it completely. We will consider compactifications over an internal manifold that leave some supersymmetry in four dimensions intact. A historical motivation for this choice is that they provide a nice way to obtain realistic particle physics models. In particular, we will see that a $\mathcal{N} = 1$ supersymmetric theory allows for chiral fermions in four dimensions, while field theories with a higher number of supersymmetry in four dimensions do not. In addition, supersymmetric configurations are easier to study before tackling more general compactifications. The main reason is that supersymmetric compactifications of string theory allow for stable dimensional reductions, whose higher-dimensional corrections can be systematically studied.

The algebra of a $\mathcal{N} = 1$ supersymmetric theory in four-dimensional Minkowski spacetime is an extension of the Poincaré algebra by adding supersymmetry generators which satisfy specific anti-commutation relations, instead of commutation relations.

Reformulate. We want to obtain a condition for a susy vacuum in 4d. We can obtain a classical field theory constraint.

A conserved charge Q associated to an unbroken supersymmetry annihilates the vacuum $|\Omega\rangle$, so $Q\,|\Omega\rangle = 0$. This in turn means that for any operator U, $\langle\Omega|\{Q,U\}|\Omega\rangle = 0$. If U is a fermionic operator, we derive that the variation of the operator under the supersymmetry transformation is $\delta U = \{Q,U\}$. Taking this as the classical limit, $\delta U = \langle\Omega|\delta U|\Omega\rangle$. Thus, we conclude that at the classical level $\delta U = \langle\Omega|\{Q,U\}|\Omega\rangle = 0$ for any fermionic field U.

We now consider the SUGRA theory of type IIA string theory and the condition that some four-dimensional supersymmetry remains. In the same way that a translation generated by the momentum operator is parametrized by a vector and a rotation is a parametrized by an antisymmetric tensor, a supersymmetry transformation generated by Q_α is parametrized by a spinor η_α. The variation of the gravitino field under a supersymmetry transformation is

$$\delta\psi_M = D_M\eta + (\text{fluxes}). \tag{2.1}$$

Where D_M is the covariant derivative on M. Supersymmetry preservation means that all variations must be zero. Assuming that all fluxes vanish, this leads to the constraint that η is a covariantly constant ten-dimensional spinor

$$\delta\psi_M = D_M\eta = 0. \tag{2.2}$$

To study the implication of this equation to the four-dimensional space-time T, we employ the fact that T is maximally symmetric, so we can decompose the metric as

$$ds^2 = e^{2A(y)}\tilde{g}_{\mu\nu}dx^\mu dx^\nu + g_{mn}dy^m dy^n, \qquad \mu = 0,1,2,3 \quad m = 1,\dots,6 \tag{2.3}$$

where x^μ are the compact coordinates, y^m the internal coordinates and $\tilde{g}_{\mu\nu}$ can be either the de Sitter, anti-de Sitter or the Minkowski metric in four dimensions.

Particularizing to the space-time components, equation (2.2) can be written as

$$\widetilde{\nabla}_\mu \eta + \frac{1}{2}\left(\tilde{\gamma}_\mu \gamma_5 \otimes \nabla\!\!\!/ A\right)\eta = 0 \tag{2.4}$$

where $\widetilde{\nabla}$ and $\tilde{\gamma}_\mu$ are the covariant derivative and gamma matrix with respect $\tilde{g}_{\mu\nu}$. This equation leads to the integrability condition

$$\left[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu\right]\eta = \frac{1}{2}(\nabla_m A)(\nabla^m A)\gamma_{\mu\nu}\eta. \tag{2.5}$$

On the other hand, the definition of the Riemann tensor is

$$\left[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu\right]\eta = \frac{1}{4}\tilde{R}_{\mu\nu\lambda\rho}\gamma^{\lambda\rho}\eta. \tag{2.6}$$

In the case of a maximally symmetric space, the Riemann tensor is $R_{\mu\nu\lambda\rho} = k(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})$, where k is negative for anti-de Sitter, zero for Minkowski and positive for de Sitter. Combining equations (2.5) and (2.6), and inverting $\gamma^{\mu\nu}$, we obtain

$$k + \nabla_m A \nabla^m A = 0. \tag{2.7}$$

Owing to the fact that on a compact manifold the only constant value of $(\nabla A)^2$ is zero, we conclude that $k = 0$ and thus the four-dimensional space-time must be Minkowski space.

ten-dimensional covariantly constant spinor η into ξ χ

The existence of a covariantly constant spinor implies for a type IIA theory that there are two four-dimensional supersymmetry parameters and therefore, $\mathcal{N} = 2$.

Explain better.

2.1 Type IIA on Calabi-Yau manifolds

We examine more closely what the existence of a covariantly constant spinor field implies on the compact space.

Let us consider a Riemannian manifold K of dimension six with a spin connection ω, which is in general a $SO(6)$ gauge field. If we parallel transport a field ψ around a contractible closed curve γ, the field becomes $\psi' = U\psi$ where $U = \mathcal{P}e^{\int_\gamma dx\omega}$ and $\mathcal{P}$ denotes the path ordering of the exponential. The set of transformation matrices associated to all possible loops form the holonomy group of the manifold, which must be a subgroup of $SO(6)$.

A covariantly constant spinor is left unchanged when parallel transported along a contractible closed curve, so the holonomy matrices of a manifold that admits a covariantly constant spinor must satisfy $U\eta = \eta$. Taking into account the Lie algebra isomorphism $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$ we identify the positive (negative)-chirality spinors of $SO(6)$ with the fundamental $\mathbf{4}$ ($\bar{\mathbf{4}}$) of $SU(4)$. Let us consider that η is a positive chirality spinor, so it transforms according with the $\mathbf{4}$ of $SU(4)$. In order to have a covariantly constant spinor, the holonomy group must be such that the $\mathbf{4}$ representation decomposes into a singlet. This decomposition is achieved if the holonomy group is $SU(3)$ so that

$$SO(6) \to SU(3) \tag{2.8}$$

$$\mathbf{4} \to \mathbf{3} \oplus \mathbf{1} \tag{2.9}$$

Integrate better

The existence of a single covariantly constant spinor on the compact manifold can be reformulated as a topological condition, namely that the holonomy group of the compact manifold is $SU(3)$. A compact manifold of $SU(3)$ (local) holonomy is the definition of a Calabi-Yau manifold. The holonomy group being a proper subgroup of $SU(3)$ is equivalent to having more than one covariantly constant spinor, which would lead to a larger degree of supersymmetry preserved.

We can also check that the 2-form $\mathbf{15}$ and the 3-form $\mathbf{20}$ decompositions contain a singlet, $\mathbf{15} \to \mathbf{8} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}$ and $\mathbf{20} \to \mathbf{6} \oplus \bar{\mathbf{6}} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1} \oplus \mathbf{1}$, so they are globally well defined. We refer to the 2-form as J and the 3-form as the holomorphic three-form Ω. Raising an index of J we obtain an almost-complex structure, which satisfies $(J^2)^i_j = -\delta^i_j$. For a particular point of the manifold, we can form a basis of complex coordinates z^i from the real coordinates x^i, as $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$ and $z^3 = x^5 + ix^6$, in which $J = i dz^i \otimes dz^i - i d\bar{z}^{\bar{i}} \otimes d\bar{z}^{\bar{i}}$. If we can extend this particular form of J to the neighborhood of any point, J is said to be integrable and the manifold is complex. An integrable almost-complex structure is referred to as a complex structure. The integrability condition is equivalent to the Nijenhuis tensor

$$N^k_{ij} = J^l_i(\partial_l J^k_j - \partial_j J^k_l) - J^l_j(\partial_l J^k_i - \partial_i J^k_l) \tag{2.10}$$

vanishing everywhere.

It is useful to define with the aid of the metric the form $k = g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$. A manifold is Kähler if $dk = 0$ and k is then called the Kähler form. It can be shown that the holonomy group being contained in $U(N)$ implies that the manifold is Kähler.

Cohomology

It is useful to introduce some algebraic topology tools which we will use later on.

Let us consider a smooth manifold of dimension d. A differential p-form ω_p is $(0,p)$-rank tensor which has completely anti-symmetric components. A p-form is expanded as a linear combination of the basis cotangent vectors $\{dx^\nu\}_{\nu=1\dots d}$ as

$$\omega_p = \frac{1}{p!}\omega_{\nu_1\dots\nu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p} = \frac{1}{p!}\omega_{\nu_1\dots\nu_p} dx^{[\nu_1} \otimes \cdots \otimes dx^{\nu_p]}, \tag{2.11}$$

where the square brackets denote antisymmetrization.

The wedge product of a p-form ω_p and a q-form α_q is a $(p+q)$-form

$$\omega_p \wedge \alpha_q = \frac{1}{p!q!}\omega_{\nu_1\dots\nu_p}\alpha_{\mu_1\dots\mu_q} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}. \tag{2.12}$$

The exterior derivative of a p-form yields a $(p+1)$-form

$$d\omega_p = \frac{1}{p!}\partial_\mu \omega_{\nu_1\dots\nu_p} dx^\mu \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p}. \tag{2.13}$$

A p-form whose exterior derivative vanishes is called closed and a p-form that is the exterior derivative of a $(p-1)$-form is exact.

A fundamental property of the exterior derivative is Poincare's lemma, which states that for any differential form α, $d(d\alpha) = 0$ holds. This can be rewritten as $d^2 = 0$. In other words, every exact form is closed. We could ask ourselves if the inverse statement is true: is every closed form exact? The answer for an arbitrary manifold is no.This information is encoded in the q-th deRham cohomology group, which is formed by considering the set of all closed q-forms defined on a manifold. Since given a closed form ω, we can always find another closed form by adding an exact form $\omega' = \omega + d\alpha$, we take the equivalence relation that two forms are equivalent if they differ by a closed form. The q-th deRham cohomology group of a manifold X is defined as the quotient

$$H^q_d(X, \mathbb{R}) = \{\omega | d\omega = 0\}/\{\alpha|\alpha = d\beta\}. \tag{2.14}$$

The dimension of $H^q_d(X,\mathbb{R})$ is the Betti number $b^q(X)$. Only when $b^q(X)=1$, all closed q-forms on X are exact.

We can easily make a generalization of the previous concepts to complex manifolds of complex dimension $n = d/2$. Complexifying the basis $\{dx_\mu\}_{\mu=1,\dots,d} \to \{dz_i, d\bar{z}_{\bar{j}}\}_{i,j=1,\dots,n}$, we can consider tensors $\omega_{r,s}$ with r holomorphic and s anti-holomorphic indices so they can be written as

$$\omega_{r,s} = \omega_{\mu_1,\dots\mu_r,\bar{\nu}_1,\dots,\bar{\nu}_s} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\bar{\nu}_1} \wedge \cdots \wedge d\bar{z}^{\bar{\nu}_s} \tag{2.15}$$

This allows to split the exterior derivative into holomorphic and anti-holomorphic derivatives $d = \partial + \bar{\partial}$. The complex equivalent of the deRham cohomology group is the Dolbeault cohomology group associated to $\bar{\partial}$ (it can be analogously be defined for ∂)

$$H^{r,s}_{\bar{\partial}}(X,\mathbb{C}) = \{\omega|\bar{\partial}\omega = 0\}/\{\alpha|\alpha = \bar{\partial}\beta\}. \tag{2.16}$$

We define the Hodge dual $\star$ of a p-form as the $(d-p)$-form

$$\star\omega = \frac{1}{(n-p)!p!}\epsilon_{\mu_1\dots\mu_n}\sqrt{|\det g|}g^{\mu_1\nu_1}\dots g^{\mu_p\dots\nu_p}\omega_{\nu_1\dots\nu_p}dx^{\mu_{p+1}}\wedge\dots\wedge dx^{\mu_d}. \tag{2.17}$$

This operation allows us to form the adjoint exterior derivative or codifferential $d^\dagger$, that maps p-forms into $(p-1)$-forms

$$d^\dagger = (-1)^{np+n+1} \star d \star . \tag{2.18}$$

The codifferential is the adjoint of the exterior derivative with respect to the inner product

$$\langle\omega,\omega'\rangle = \int_X \omega\wedge\omega', \tag{2.19}$$

so that given a p-form ω and a $(p-1)$-form σ

$$\langle\omega, d\sigma\rangle = \langle d^\dagger\omega,\sigma\rangle. \tag{2.20}$$

The Laplacian can be generalized as $\Delta = dd^\dagger + d^\dagger d$ and a harmonic form ω satisfies the Laplace equation $\Delta\omega = 0$.

An important theorem is Hodge's decomposition, which states that a p-form ω can be uniquely written in terms of a $(p-1)$-form β, a $(p+1)$-form γ and a harmonic p-form ω'

$$\omega = d\beta + d^\dagger\gamma + \omega' \tag{2.21}$$

If ω is a closed form, γ vanishes so

$$\omega = d\beta + \omega'. \tag{2.22}$$

Identifying $\omega - d\beta = \omega'$ as an element of a cohomology class in $H_d^p(X, \mathbb{R})$, we can conclude that for every class belonging to $H_d^p(X, \mathbb{R})$, there is an unique harmonic p-form.

The dimension of $H_{\bar{\partial}}^{r,s}(X, \mathbb{C})$ is known as the Hodge number, $h^{p,q}(X)$ and it is a topological invariant. Thanks to the Hodge star, there is a relation between Hodge numbers $h^{p,q} = h^{n-p,n-q}$. The fact that the manifold is Kähler also guarantees the symmetry $h^{p,q} = h^{q,p}$. The decomposition of the deRham cohomology into Dolbeault cohomologies is given by

$$H_d^p(X, \mathbb{R}) = \bigoplus_{r+s=p} H_{\bar{\partial}}^{r,s}(X, \mathbb{C}). \tag{2.23}$$

In the case of Calabi-Yau manifolds, it also holds that $h^{s,0} = 0$ if $1 < s < n$, $h^{n,0} = h^{0,n} = 1$. If the manifold is connected, then $h^{0,0} = 1$.

We can arrange the Hodge numbers into a Hodge diamond, which for a manifold of complex dimension three would be

$$\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{10} & & h^{01} & & \\
 & h^{20} & & h^{11} & & h^{02} & \\
h^{30} & & h^{21} & & h^{12} & & h^{03} \\
 & h^{20} & & h^{11} & & h^{02} & \\
 & & h^{10} & & h^{01} & & \\
 & & & h^{00} & & &
\end{array}$$

In the case of a Calabi-Yau three-fold

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & & h^{21} & & h^{21} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Mirror symmetry

Homology

A very related construction to cohomology is homology. The basic element of homology is the p-chain a_p, which in the simplest formulation is the formal sum of p-dimensional submanifolds N_p^k (possibly with boundary)

$$a_p = \sum_k c_k N_p^k \tag{2.24}$$

p-forms that have no boundary are called p-cycles. The boundary operator ∂ satisfies ∂^2. The homology group $H_q(X, \mathbb{R})$ is defined as the quotient space of q-cycles modulo q-dimensional boundaries

$$H_q(X, \mathbb{R}) = \{a|\partial a = 0\}/\{b|b = \partial a\} \tag{2.25}$$

The dimension of $H_q(X, \mathbb{R})$ is $h_q(X)$.

It can be seen that homology resembles cohomology replacing chains by forms, the boundary operator by the exterior derivative and cycles by closed forms. In fact, they are algebraic duals, in the sense that integration of a p-form over a p-chain defines and isomorphism between H_q and H_d^q in the case of compact manifolds. This implies that the dimensions of both group coincide $h_q = h^q$.

It is interesting to generalize the concept of how many times to lines intersect to the case of p-cycles. intersection number topological invariant should depend on the homology class only

$$[a_p][b_{d-p}] = \int_X \delta(a_p) \wedge \delta(b_{d-p}) \tag{2.26}$$

$$\int_{a_p} B_p = \int_X B_p \wedge \delta(a_p) \tag{2.27}$$

Complete. Notation.

Moduli space

Starting from a particular choice of metric g on a Calabi-Yau manifold X, we could try to determine which deformations of the metric still preserve the Calabi-Yau condition. These deformations of the metric are known as moduli and play an important role in the physics of compactifications. We will restrict our discussion to Calabi-Yau manifolds of complex dimension three. An arbitrary deformation of the metric will consist of those with pure indices $g_{ij}dz^i dz^j$ and those with mixed indices $g_{i\bar{j}}dz^i dz^{\bar{j}}$. In order to preserve the Calabi-Yau condition they must lead to a vanishing Ricci tensor, $R_{i\bar{j}} = 0$. This constraint implies that:

A deformation of the type $g_{ij}dz^i \wedge dz^j$ must be harmonic, so it can be identified with an unique element of a cohomology class in $H^{1,1}$, the Kähler form. If we write the Kähler form in terms of the basis elements $\{t_a\}_{a=1,\ldots,h_{1,1}}$

$$k = \sum_{a=1}^{h_{1,1}} t_a \omega_a \tag{2.28}$$

the $h_{1,1}$ real parameters t_a are the Kähler moduli of the manifold. The Kähler form is employed to calculate the volume of a Calabi-Yau manifold of complex dimension three as $\int k \wedge k \wedge k$, since $k \wedge k \wedge \wedge k$ has the same rank as the volume form, which is unique up to a proportionality constant.

Deformations of the type $\Omega_{ijk} g^{k\bar{k}} \delta_{\bar{k}\bar{l}} dz^i \wedge dz^j \wedge d\bar{z}^{\bar{l}}$ must be a harmonic form belonging to a cohomology class in $H^{2,1}$. These deformations correspond to deformations of the complex structure, since the choice of a complex structure is related to a $(2,1)$-form $J_{ij\bar{k}} = \Omega_{ijl} J^l_{\bar{k}}$ obtained from the holomorphic three-form. There are $h_{2,1}$ complex parameters associated to the choice of the complex structure, which are called the complex structure moduli of the manifold. They determine the volume of 3-cycles Π in the compact space through Ω_3

$$\mathrm{Vol}(\Pi) = \int_\Pi \Omega_3. \tag{2.29}$$

In conclusion, a Calabi-Yau metric is determined uniquely by the Kähler form and the holomorphic three-form. The former leads to $h_{1,1}$ real parameters while the latter requires $h_{2,1}$ complex parameters.

Type IIA spectrum on Calabi-Yau manifolds

In order to compute the 4-dimensional massless spectrum of type IIA theory on a Calabi-Yau, we consider the Kaluza-Klein dimensional reduction. This consists in choosing an energy scale at which the compactification resides (the KK-scale) and then studying the effective four-dimensional theory at energies below the

KK-scale. In practice, this corresponds to taking the KK-scale relatively large (or equivalently taking the associated radius of the compact space very small).

The simplest example of KK reduction is based on a free scalar field $\phi(x^M)$ in ten dimensions. We first apply its Fourier expansion in terms of the eigenvectors $\phi_k(x^m)$ of the Laplace operator in the internal space with eigenvalues λ_k

$$\phi(x^M) = \sum_k \phi_k(x^\mu)\phi_k(x^m) \tag{2.30}$$

where the dimension of the mode is determined by the argument, x^μ for the 4-dimensional Minkowski space and x^m for the compact space. The masslessness condition of $\phi(x^M)$ implies that

$$\Box\phi(x^\mu) - \lambda_k \phi(x^\mu) = 0 \tag{2.31}$$

This equation permits us to identify λ_k as the squared mass of the 4-dimensional $\phi(x^\mu)$ field. Thus, the number of massless scalar fields is given by the number of solutions of $\Box\phi(x^\mu) = 0$ which in the case of compact manifolds is one. We conclude that a 10-dimensional scalar field leads to a massless scalar field in 4-dimensions (in addition, the is a tower of KK modes).

Our next example is the KK reduction of a p-form C_p with the expansion

$$C_p = \sum_{k,q} c_q^k(x^m) \wedge C_{p-q}^k(x^\mu) \tag{2.32}$$

Massless 4-dimensional $(p-q)$-form fields correspond to internal modes that satisfy $dc_q = d^\dagger c_q = 0$, so c_q is a harmonic form. Since there is a single harmonic q-form in each q-cohomology class, the number of 4-dimensional massless $(p-q)$-forms arising from a p-form is the dimension of the H_q cohomology group, the Betti number b_q.

Check if correction is needed.

In the case of a Calabi-Yau manifold, from the relation of the Betti numbers with the Hodge numbers, we determine $b_0 = h_{0,0} = 1$, $b_1 = h_{1,0} + h_{0,1} = 0$, $b_2 = h_{1,1} + h_{2,0} + h_{0,2} = h_{1,1}$ and $b_3 = h_{3,0} + h_{0,3} + h_{2,1} + h_{1,2} = 2h_{2,1} + 1$. Thus, c_1 leads to a 4-dimensional 1-form, B_2 leads to a 2-form and $h_{1,1}$ scalar fields and c_3 leads to a 3-form (although it is not dynamical), $h_{1,1}$ 1-forms and $2h_{2,1} + 2$ scalar fields.

The KK reduction of the 10-dimensional metric is applied considering its components separately:

- The $G_{\mu\nu}$ components correspond to scalar fields in the internal space satisfying the Laplace equation and whose solution is unique for compact spaces. Thus, a 10-dimensional graviton reduces to a 4-dimensional graviton.

- The $G_{\mu m}$ components would correspond to 4-dimensional vector bosons, associated to 6-dimensional vector fields in the compact space. The masslessness condition of the 4-dimensional field would imply that the 6-dimensional vectors are Killing vectors associated to continuous isometries of the compact space, which in the case of Calabi-Yau manifolds are non-existent. As a consequence, the $G_{\mu m}$ components do not lead to any massless fields in 4 dimensions.

- The G_{mn} components reduce to 4-dimensional scalar fields associated to the moduli of the internal space, whose vev determine the geometry of the internal space. In the case of Calabi-Yau manifolds, we have seen that there are h_{21} real scalar fields and h_{11} complex scalar fields.

Having seen how the bosonic fields of type IIA behave under KK reduction, we proceed to describe the massless spectrum of type IIA theory compactified on a Calabi-Yau manifold.

In order to fill in the supermultiplets of 4-dimensional $\mathcal{N} = 2$ supersymmetry, we must combine scalar fields arising from the dilaton ϕ, p-forms and the geometric moduli into complex scalar fields. The spectrum is arranged as follows:

A single supergravity multiplet, composed of a graviton $G_{\mu\nu}$, a gauge boson arising from the KK reduction of B_2 and two gravitinos ψ with opposite chiralities.

$h_{1,1,}$ vector multiplets, composed of a gauge boson that arises from c_3, a complex scalar (obtained by combining the Kähler moduli t_a and the scalar field B_0 associated to B_2 into $B_2 + it_a$) and two Majorana fermions.

$h_{2,1}$ vector hypermultiplets composed of two complex scalars (obtained by combining the complex structure moduli with the scalar fields associated to mixed index components of c_3) and two left-handed fermions.

A single vector hypermultiplet composed of two complex scalars (obtained by combining the dilaton, a scalar field associated to B_2 [1] and the scalars that arise from pure index components of c_3) and two left-handed fermions.

2.2 Type IIA on orientifold projections

Generalities of orientifolds

If we compactify a type II string theory on a Calabi-Yau manifold, we obtain a four-dimensional $\mathcal{N} = 2$ supersymmetric theory. This degree of supersymmetry does not allow for chiral fermions, so Calabi-Yau compactifications of type II theories have no straightforward application in the context of model building. An option to reduce the supersymmetry to $\mathcal{N} = 1$ is to apply the orientifold projection, which consists in modding out the action of ΩR, where Ω is the worldsheet parity, so strings become unoriented, and R is a particular $\mathbb{Z}_2$ symmetry of the compact six-dimensional space In type IIA string theory we define $R = \mathcal{R}(-1)^{F_L}$. $\mathcal{R}$ satisfies the condition that it is an involution (squares to the identity) and acts anti-holomorphically on the complex coordinates of the internal space ($\mathcal{R} : z_i \to \bar{z}_i$). This implies that the Kähler and the holomorphic three-form transform as $J \to -J$ and $\Omega_3 \to \bar{\Omega}_3$. F_L is an operator that counts the number of left-moving fermions.

The fixed points under $\mathcal{R}$ define the orientifold planes in the model and are denoted as Op-planes, where p is the spatial dimension. In type IIA theory, the only consistent choice are O6-planes, which span the entire four-dimensional Minkowski space and wrap a compact 3-cycle on the internal space.

In order to have a stable compactification, we expect all RR and NSNS charges to vanish. Furthermore, RR tadpole cancellation implies that the 4-dimensional theory is free of non-abelian gauge anomalies. O6-planes carry RR charge, so in order to eliminate RR tadpoles we must also introduce D6-branes, which carry opposite charge. It is important to note that D6-branes do not need to wrap the same 3-cycles as the O6-planes to remove RR tadpoles.

[1] The KK reduction of the 10-dimensional 2-form B_2 leads to a 4-dimensional 2-form b_2. We can then define a scalar field $\tilde{b}$ as the dual $d\tilde{b} = \star db_2$.

D-branes on Calabi-Yau manifolds

In order to obtain stable D6-brane configurations on a type IIA theory compactified on a Calabi-Yau manifold, we impose that they wrap around volume minimizing 3-cycles on the compact space, so that their tension is minimized as well. The volume minimizing condition means that the branes must wrap special Lagrangian 3-cycles in the internal space. Special Lagrangian 3-cycles Π are defined by

$$k|_{\Pi} = 0, \qquad \mathrm{Im}(e^{-i\phi}\Omega_3)|_{\Pi} = 0 \tag{2.33}$$

for some real ϕ, where k is the Kähler two-form and Ω_3 the holomorphic three-form. The $e^{-i\phi}\Omega_3$ is referred to as a calibration and the special Lagrangian is calibrated with respect to it.The volume of the special Lagrangian 3-cycle is

$$\mathrm{Vol}(\Pi) = \int_{\Pi} \mathrm{Re}(e^{-i\phi}\Omega_3) \tag{2.34}$$

D6-branes wrapped around a special Lagrangian cycle are guaranteed to preserve 4-dimensional $\mathcal{N} = 1$ supersymmetry. This preserved supersymmetry coincides with the same supersymmetry preserved by the Op-planes only if $\phi = 0$.

The open string spectrum of stacks of N_a D6_a-branes wrapping special Lagrangian 3-cycles Π_a can be classified into two sectors: strings that stretch from one stack to itself and those that stretch between to different stacks, 6_a and 6_b.

Strings that stretch over 6_a lead to $U(N_a)$ vector multiplets of 4-dimensional $\mathcal{N} = 1$ supersymmetry. There are also $b_1(\Pi_a)$ chiral multiplets in the adjoint representation, which are composed of the internal components of the gauge fields along Π_a combined with the geometric moduli of the 3-cycle, and their fermion superpartners.

Strings that stretch between 6_a and 6_b lead to $I_{ab} = [\Pi_a][\Pi_b]$ chiral fermions, where I_{ab} is the intersection number between 3-cycles. These fermions transform in the $(\mathbf{N_a}, \bar{\mathbf{N}}_\mathbf{b})$ representation. There are also massless scalar fields if the intersection preserves supersymmetry.

Orientifold compactifications with intersecting D-branes

We consider N_a D6-branes that wrap 3-cycles Π_a and whose image under the orientifold projection wrap the 3-cycles $\Pi_{a'}$. The condition that D6-branes preserve the same $\mathcal{N}=1$ supersymmetry as the O6-planes Π_{O6} is that the local relative angles between them obey

$$\theta_1 + \theta_2 + \theta_3 = 0 \tag{2.35}$$

If D6-branes do not coincide with their mirror images, the light spectrum of the model consists of:

- $U(N_a)$ gauge bosons arising from non-intersecting D6-branes.
- I_{ab} fermions in the representation $(\mathbf{N_a}, \mathbf{\bar{N}_b})$ arising from the intersection of two different D6-branes.
- $I_{ab'}$ fermions in the representation $(\mathbf{N_a}, \mathbf{N_b})$ arising from the intersection of a D6-brane with the mirror of a different D6-brane.
- $1/2([\Pi_a][\Pi_{a'}] + [\Pi_a][\Pi_{O6}])$ fermions in the anti-symmetric representation $(\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}, \mathbf{1})$ and $1/2([\Pi_a][\Pi_{a'}] - [\Pi_a][\Pi_{O6}])$ fermions in the symmetric representation $(\Box\Box, \mathbf{1})$ which arise from the intersection of a D6-brane with its own mirror.

The condition for RR tadpole cancellation imposes a topological restriction, namely, that the sum of the three-cycles wrapped by the D-branes and their orientifold images has to combine with the O6-plane three-cycle into the trivial cycle in homology

$$\sum_a N_a([\Pi_a] + [\Pi_{a'}]) - 4[\Pi_{O6}] = 0. \tag{2.36}$$

Effective action of D-branes on Calabi-Yau orientifolds

We recall that the action of a Dp-brane contains the DBI term (1.3)

$$S_{DBI} = -\mu_p \int_{D_p} e^{-\phi} \sqrt{\det(G + B - 2\pi\alpha' F)} \tag{2.37}$$

which reduces to the Yang-Mills action for small values of α'. In the case of compactification on a Calabi-Yau orientifold, the gauge coupling constant is given in terms of the volume of the special-Lagrangian three-cycles along the internal space

$$\frac{1}{g^2} = e^{-\phi} \frac{(\alpha')^{-3/2}}{(2\pi)^4} \mathrm{Vol}(\Pi_3). \tag{2.38}$$

Chapter 3

Type IIA on the quintic

3.1 Motivation

Why the quintic?

Why study sLags?

3.2 sLags on Fermat's quintic

Definition

Construction of sLags

Moduli space of sLags

Intersection numbers

Volumes

SM on the quintic

3.3 sLags on the deformed quintic

Deformations of the quintic. Classification.

The consider the possible deformations of Fermat's quintic. There should be 101 independent deformations, since they correspond to different complex structures and are given be the Hodge number $h_{21} = 101$.

We can add terms of the following type to the quintic

$$x_i^5, x_i^4 x_j^1, x_i^3 x_j^2, x_i^3 x_j x_k, x_i^2 x_j x_k x_l, x_1 x_2 x_3 x_4 x_5 \tag{3.1}$$

Not all of these terms are independent, since a coordinate redefinition $GL(5, \mathbb{C})$.

Deformation classification

Coordinate redefinition freedom

As an example, we take as deformation $-5\phi z_1 z_2 z_3 z_4 z_5$

Change of variables to study geometry of the singularity

In order to determine the geometry near the singularity, we make the following change of variables

$$\begin{aligned}
x_1 &= 1 + y_1/\sqrt{10} + y_2/5 + y_4/\sqrt{50} \\
x_2 &= 1 + y_1/\sqrt{10} - y_2/5 + y_4/\sqrt{50} \\
x_3 &= 1 + y_1/\sqrt{10} + y_3/5 - y_4/\sqrt{50} \\
x_4 &= 1 + y_1/\sqrt{10} - y_3/5 - y_4/\sqrt{50}
\end{aligned} \tag{3.2}$$

In these coordinates, the quintic becomes

$$5(\psi - 1) = y_1^2 + y_2^2 + y_3^2 + y_4^2 + O(\psi - 1) \tag{3.3}$$

Something about the branch

Volume of cycles wrapping singularities

Three-form integration

$$\Omega = \frac{x_5 dx_1 \wedge dx_2 \wedge dx_3}{\frac{\partial p}{\partial x_4}} \tag{3.4}$$

$$\int_{A^2} \Omega = \int \dots \tag{3.5}$$

Bibliography