A walk through moduli space with SLags

John Xihua Liu Anta

Universidad Autónoma de Madrid
Instituto de Física Teórica

Adviser: Wieland Staessens

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Chapter 1

Generalities of type IIA string theory

The study of string theory in Minkowski spacetime has lead to the identification of five consistent string theories, which all turn out to be supersymmetric and give rise to massless bosonic and fermionic excitations in their spectrum. The five string theories were given their name according to their own specificities: Type heterotic HE and HO, Type I, Type IIB and Type IIA string theory. In this thesis we will only concentrate on the last one in the list.

Type IIA spectrum

Type IIA string theory requires ten space-time dimensions to be consistent. Furthermore, it has a 10-dimensional supersymmetry with 32 supercharges, which corresponds to $\mathcal{N} = (1,1)$ supersymmetry. The bosonic spectrum of Type IIA on a flat ten-dimensional space-time results upon the correct quantization of the two-dimensional string theory. The two-fold choice of boundary conditions in the supersymmetric string, Ramond (R) or Neveu-Schwarz (NS) conditions, allows for rich spectrum of states that needs to be classified in terms of representations under the space-time group $SO(9,1)$. There is an infinite tower of massive string states (whose mass is inversely proportional to the string length ℓ_s), but we restrict the discussion to the massless states only. In the NS-NS sector, we find the dilaton ϕ, a two-form B_2 and a graviton $G_{\mu\nu}$, while in the R-R sector we identify the 1- and 3-forms c_1, c_3. The fermions, which belong to the NS-R and R-NS sectors, are two opposite-chirality gravitinos ψ and two opposite-chirality dilatinos λ.

Type IIA SUGRA

The low-energy theory of the ten-dimensional type IIA string theory is type IIA supergravity (SUGRA). The spectrum of type IIA SUGRA is the massless spectrum of type IIA string theory and the effective action yields

$$2\kappa^2 S = \int d^{10}x\sqrt{-G}\Big[e^{-2\phi}\Big(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\Big) - \frac{1}{2}|F_2|^2 - \frac{1}{2}|\tilde{F}_4|^2\Big] \tag{1.1}$$

$$-\frac{1}{2}\int B_2 \wedge F_4 \wedge F_4 \tag{1.2}$$

where we employ the following definitions: $H_3 = dB_2$, $F_2 = dC_1$, $F_4 = dC_3$, $\tilde{F}_4 = dC_3 - C_1 \wedge H_3$ and $2\kappa^2 = (2\pi)^7\alpha'^4$. α' is known as the Regge slope which is a free parameter of string theory related to the string length as $\ell_s = 2\pi\sqrt{\alpha'}$.

The D-brane

The two-dimensional strings can be generalized to $(p+1)$-dimensional extended objects, which are called Dirichtlet p-branes or Dp-branes for short. Thus, a D1-brane would correspond to a D-string, a D2-brane would be a three-dimensional membrane and so on. The existence of Dp-branes can be motivated, in the weak string coupling limit, as objects on which open strings end, such that they correspond to physical configurations described in terms of Dirichlet boundary conditions for open strings. In type IIA string theory, only even-dimensional Dp-branes are physical: the D0, D2, D4, D6 and D8-branes.

We can study the dynamics of a Dp-branes in terms of the open string excitations with endpoints attached to the Dp-brane. Let us consider the open string excitations of a Dp-brane, the latter spanning $p+1$ dimensions and transverse to $d-p-1$ dimensions. The presence of the Dp-brane breaks the ten-dimensional Poincaré invariance of the theory, while open string excitations propagate on the $(p+1)$-dimensional volume of the Dp-brane only. This implies that massless particles must transform under irreducible representations of the D-brane worlvolume little group $SO(p-2)$ instead of the space-time little group $SO(d-2)$. The massless spectrum in $(p+1)$ dimensions of the open supersymmetric string is composed of a gauge boson A^{μ} $(\mu = 0, \ldots, p)$ (corresponding to longitudinal oscillations to the brane), $9-p$ real scalars ϕ^i (corresponding to transverse oscillations to the brane) and some fermions [1] λ_a. This particle content can be arranged into a vector supermultiplet of $U(1)$ with 16 supersymmetries in $(p+1)$ dimensions. Thus, a Dp-brane reduces the degree of supersymmetry of the type IIA theory by half.

In order to find out the action of a Dp-brane we must realize that it corresponds to the $(p+1)$-dimensional effective action of the massless open string excitations of the Dp-brane. As an illustration of this, a Dp-brane breaks the translational symmetry of the vacuum, which allows us to conclude that the ϕ^i scalar fields are the Goldstone bosons associated to the broken symmetry. The vev of these scalar fields determine the position of the Dp-brane in the transverse space, and the fluctuations of the scalar fields determine the evolution of the Dp-brane worldvolume W_{p+1} (the generalization of the particle worldline to the case of higher-dimensional branes). The resulting action of the bosonic sector of the Dp-brane is the sum of a Dirac-Born-Infeld term S_{DBI} and a Chern-Simons term S_{CS}.

The DBI term carries the information on how a Dp-brane interacts with the NSNS fields. It takes the form

$$S_{DBI} = -\mu_p \int d^{p+1}x e^{-\phi} \sqrt{-\det(P[G+B] - 2\pi\alpha' F)} \tag{1.3}$$

where the space-time gauge field strength is $F = dA_1$, the coefficient μ_p is

$$\mu_p = \frac{(\alpha')^{-(p+1)/2}}{(2\pi)^p} \tag{1.4}$$

and we define the pullback of a tensor G into the brane worldvolume as

$$P[G]_{\mu\nu} = G_{\mu\nu} + G_{\mu i}\partial_\nu \phi^i + \partial_\mu \phi^i G_{i\nu} + \partial_\mu \phi^i \partial_\nu \phi^j G_{ij}. \tag{1.5}$$

The form is this action can be motivated in the following way:

- Ignoring the contributions of F, B and ϕ, the action reduces to

$$S_{DBI} = -\mu_p \int d^{p+1}x \sqrt{-\det(P[G])}. \tag{1.6}$$

[1] These $p+1$ fermions correspond to the decomposition of a ten-dimensional Majorana fermion along the D-brane. They are the gaugini (superpartners of the gauge bosons) and the superpartners of the scalar fields.

$P[G]$ should be viewed as a metric defined on the D-brane worldvolume, which is inherited from the ambient ten-dimensional metric. Thus, integrating $\sqrt{-\det(P[G])}$ yields the total volume of the worldvolume.

- The pull-back of the NS two-form represents is a first generalization of the D-brane volume-element and requires also the inclusion of the $U(1)$ gauge field associated to the open string excitations (to preserve gauge invariance)

- The coupling of the D-brane to the NS-fields is proportional to $e^{-\phi}$, which is the characteristic coupling for open string interactions.

If we expand the DBI action in powers of α', we obtain the Yang-Mills term

$$S_{YM} = \frac{\alpha'^{-(p-3)/2}}{4g_s(2\pi)^{p-2}} \int d^{p+1}x\sqrt{-g}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} \tag{1.7}$$

which allows us to identify the Yang-Mills coupling as

$$g^2_{YM} = g_s\alpha'^{(p-3)/2}(2\pi)^{p-2}. \tag{1.8}$$

The Chern-Simons term is topological in nature and describes how Dp-branes interact with RR-fields. It is given by

$$S_{CS} = \mu_p \int_{W_{p+1}} P\left[\sum_q c_q\right] \wedge e^{2\pi\alpha F_2 - B_2} \wedge \hat{A}(R) \tag{1.9}$$

where we integrate only over the $(p+1)$-forms of the integrand. The first terms of the A-roof polynomial are $\hat{A}(R) = 1 - \frac{1}{24(8\pi)}\,\mathrm{Tr}\,R^2 + \cdots$ and R is the space-time curvature two-form.

Multiple D-branes

It is convenient to generalize the single Dp-brane configuration to N parallel Dp-branes. In order to determine the spectrum of a stack of Dp-branes, we consider open strings with endpoints attached to either a single brane or two distinct ones.

If all branes are separated from each other, strings that stretch from a brane to itself correspond to massless gauge bosons that belong to $U(1)^N$. In contrast, strings that stretch from one brane A to another brane B lead to massive particles whose masses increases with the distance between branes. The lightest of these particles have opposite charge $(1,-1)$ under $U(1)_A \times U(1)_B$. Since Type IIA strings carry an orientation, a string stretching B to A would have opposite charges.

In the case of N coincident Dp-branes, strings that stretch between any two Dp-branes (possibly the same) give rise to massless states, so the gauge symmetry enhances from $U(1)^N$ to $U(N)$. The massless spectrum is composed of $(p-1)$-dimensional $U(N)$ gauge bosons, $(9-p)$ real scalars in the adjoint representation of $U(N)$ and several fermions in the adjoint representation.

Let us now suppose Dp-branes which are not parallel, so they can intersect each other. This situation is relevant as it can lead to four-dimensional chiral fermions in the case of intersecting D6-branes. We are interested in describing the open string spectrum of two stacks of D6-branes that intersect over a 4-dimensional subspace of their volumes.

Strings that stretch from a coincident stack of N D6-branes to itself lead to 7-dimensional $U(N)$ gauge bosons, three real adjoint scalars and their fermion superpartners.

String that stretch from a stack of N_1 D6-branes to another stack of N_2 D6-branes are localized at the intersection, in order to minimize their energy. They lead to a 4-dimensional fermion charged in the $(\mathbf{N_1}, \mathbf{N_2})$ of $U(N_1)\times U(N_2)$ or its conjugate, depending on the orientation of the intersection.

Not all geometric configurations preserve supersymmetry. Let us decompose space-time as $M_4 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. The D6-branes span all M_4 and a line in each $\mathbb{R}^2$ plane, such that the angle between two stacks is given by θ_i for each plane. It can be shown that the condition $\theta_1 \pm \theta_2 \pm \theta_3 = 0(\mathrm{mod}2\pi)$ implies $\mathcal{N} = 1$ supersymmetry in 4 dimensions, provided that no angle vanishes. If some of the angles vanish, the supersymmetry would we enhanced.

The reason we have used D6-branes and no other dimension of Dp-branes is that they would not lead to chiral fermions in 4 dimensions. Intuitively, two D6-branes allow to define an orientation in the transverse 6-dimensional space, which would not be possible with two other type of Dp-branes in type IIA string theory.

Chapter 2

Type IIA compactifications

As we have seen in the previous chapter, Type IIA superstring theory requires nine spatial dimensions and one time dimension for consistency, yet our universe only consists of a four-dimensional spacetime continuum. This implies that six spatial dimensions have to be compactified on an internal manifold with an unobservably small volume. We assume that the manifold M is factorizable into a four-dimensional maximally symmetric space-time T and a six-dimensional compact space K, $M = T \times K$. In what follows, section we motivate the requirement that additional dimensions are compactified over a Calabi-Yau manifold.

Type IIA string theory on 10-dimensional flat space has a large degree of supersymmetry, but the compactification choice can either preserve some degree of supersymmetry in four dimensions or remove it completely. We will consider compactifications over an internal manifold that leave some supersymmetry in four dimensions intact. A historical motivation for this choice is that they provide a nice way to obtain realistic particle physics models. In particular, we will see that a $\mathcal{N} = 1$ supersymmetric theory allows for chiral fermions in four dimensions, while field theories with a higher number of supersymmetry in four dimensions do not. In addition, supersymmetric configurations are easier to study before tackling more general compactifications. The main reason is that supersymmetric compactifications of string theory allow for stable dimensional reductions, whose higher-dimensional corrections can be systematically studied.

The algebra of a $\mathcal{N} = 1$ supersymmetric theory in four-dimensional Minkowski spacetime is an extension of the Poincaré algebra by adding supersymmetry generators which satisfy specific anti-commutation relations, instead of commutation relations.

We now consider the SUGRA theory of type IIA string theory and the condition that some four-dimensional supersymmetry remains. In the same way that a translation generated by the momentum operator is parametrized by a vector and a rotation is a parametrized by an antisymmetric tensor, a supersymmetry transformation generated by Q_α is parametrized by a spinor η_α.

We try to obtain the classical implication that there is an unbroken susy.

A conserved charge Q associated to an unbroken supersymmetry must annihilate the vacuum $|\Omega\rangle$. This in turn means that for any operator U, $\langle\Omega|\{Q,U\}|\Omega\rangle = 0$. If U is a fermionic operator, $\{Q,U\}$ is precisely the variation δU. So in the classical limit, an unbroken supersymmetry means that $\delta U = 0$ for every fermionic field U. In particular, the variation of the two gravitino fields ψ^1_M, ψ^2_M must vanish

$$\delta\psi^a_M = \nabla_M \eta^a + (\text{fluxes}) = 0 \tag{2.1}$$

where ∇_M is the covariant derivative on M. Assuming that all fluxes vanish, this leads to the constraint that there are two covariantly constant ten-dimensional spinors η^1, η^2

$$\nabla_M \eta^a = 0. \tag{2.2}$$

To study the implication of this equation to the four-dimensional space-time T, we employ the fact that T is maximally symmetric, so we can decompose the metric as

$$ds^2 = e^{2A(y)}\tilde{g}_{\mu\nu}dx^\mu dx^\nu + g_{mn}dy^m dy^n, \qquad \mu = 0,1,2,3 \quad m = 1,\dots,6 \tag{2.3}$$

where x^μ are the compact coordinates, y^m the internal coordinates and $\tilde{g}_{\mu\nu}$ can be either the de Sitter, anti-de Sitter or the Minkowski metric in four dimensions.

Particularizing to the space-time components, equation (2.2) can be written as

$$\widetilde{\nabla}_\mu \eta + \frac{1}{2}(\tilde{\gamma}_\mu\gamma_5 \otimes \nabla\!\!\!\!/ A)\eta = 0 \tag{2.4}$$

where $\widetilde{\nabla}$ and $\tilde{\gamma}_\mu$ are the covariant derivative and gamma matrix with respect $\tilde{g}_{\mu\nu}$. This equation leads to the integrability condition

$$\left[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu\right]\eta = \frac{1}{2}(\nabla_m A)(\nabla^m A)\gamma_{\mu\nu}\eta. \tag{2.5}$$

On the other hand, the definition of the Riemann tensor is

$$\left[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu\right]\eta = \frac{1}{4}\tilde{R}_{\mu\nu\lambda\rho}\gamma^{\lambda\rho}\eta. \tag{2.6}$$

In the case of a maximally symmetric space, the Riemann tensor is $R_{\mu\nu\lambda\rho} = k(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})$, where k is negative for anti-de Sitter, zero for Minkowski and positive for de Sitter. Combining equations (2.5) and (2.6), and inverting $\gamma^{\mu\nu}$, we obtain

$$k + \nabla_m A \nabla^m A = 0. \tag{2.7}$$

Owing to the fact that on a compact manifold the only constant value of $(\nabla A)^2$ is zero, we conclude that $k = 0$ and thus the four-dimensional space-time must be Minkowski space.

If we rewrite the ten-dimensional covariantly constant spinors η^1, η^2 in terms of two four-dimensional spinors χ^1, χ^2 and a six-dimensional spinor ξ (and its conjugate), equation (2.2) implies that the internal spinor must be covariantly constant as well

$$\nabla_M \xi = 0. \tag{2.8}$$

We have assumed that η^1, η^2 can be written in terms a single six-dimensional spinor, so there are two four-dimensional supersymmetry parameters and therefore, $\mathcal{N} = 2$. Had we considered a decomposition in terms of additional spinors in the compact space, the supersymmetry would be enhanced.

2.1 Type IIA on Calabi-Yau manifolds

We examine more closely what the existence of an internal covariantly constant spinor ξ implies on the geometry compact space.

Let us consider a Riemannian manifold K of dimension six with a spin connection ω, which is in general a $SO(6)$ gauge field. If we parallel transport a field ψ around a contractible closed curve γ, the field becomes $\psi' = U\psi$ where $U = \mathcal{P}e^{\int_\gamma dx\omega}$ and $\mathcal{P}$ denotes the path ordering of the exponential. The set of transformation matrices associated to all possible loops form the holonomy group of the manifold, which must be a subgroup of $SO(6)$.

A covariantly constant spinor is left unchanged when parallel transported along a contractible closed curve, so the holonomy matrices of a manifold that admits a covariantly constant spinor must satisfy $U\xi = \xi$. Taking into account the Lie algebra isomorphism $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$ we identify the positive (negative)-chirality spinors of $SO(6)$ with the fundamental $\mathbf{4}$ ($\bar{\mathbf{4}}$) of $SU(4)$. Let us consider that ξ is a positive chirality spinor, so it transforms according with the $\mathbf{4}$ of $SU(4)$. In order to have a covariantly constant spinor, the holonomy group must be such that the $\mathbf{4}$ representation decomposes into a singlet. This decomposition is achieved if the holonomy group is $SU(3)$ so that

$$SO(6) \to SU(3) \tag{2.9}$$

$$\mathbf{4} \to \mathbf{3} \oplus \mathbf{1} \tag{2.10}$$

Thus, the existence of a single covariantly constant spinor on the compact manifold can be reformulated as a topological condition, namely that the holonomy group of the compact manifold is $SU(3)$. A compact manifold of $SU(3)$ (local) holonomy is the definition of a Calabi-Yau manifold. The holonomy group being a proper subgroup of $SU(3)$ is equivalent to having more than one internal covariantly constant spinor, which would lead to a larger degree of supersymmetry preserved.

We can also check that the 2-form $\mathbf{15}$ and the 3-form $\mathbf{20}$ decompositions contain a singlet, $\mathbf{15} \to \mathbf{8} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}$ and $\mathbf{20} \to \mathbf{6} \oplus \bar{\mathbf{6}} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1} \oplus \mathbf{1}$, so they are globally well defined. We refer to the 2-form as J and the 3-form as the holomorphic three-form Ω. Raising an index of J we obtain an almost-complex structure, which satisfies $(J^2)^i_j = -\delta^i_j$. For a particular point of the manifold, we can consider complex coordinates z^i from the real coordinates x^i, as $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$ and $z^3 = x^5 + ix^6$, in which $J = idz^i \otimes dz^i - id\bar{z}^{\bar{i}} \otimes d\bar{z}^{\bar{i}}$. If we can extend this particular form of J to the neighborhood of any point, J is said to be integrable and the manifold is complex. An integrable almost-complex structure is referred to as a complex structure. The integrability condition is equivalent to the Nijenhuis tensor

$$N^k_{ij} = J^l_i(\partial_l J^k_j - \partial_j J^k_l) - J^l_j(\partial_l J^k_i - \partial_i J^k_l) \tag{2.11}$$

vanishing everywhere.

It is useful to define with the aid of the metric the form $k = g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$. A manifold is Kähler if $dk = 0$ and k is then called the Kähler form. It can be shown that the holonomy group being contained in $U(N)$ implies that the manifold is Kähler.

Cohomology

It is useful to introduce some algebraic topology tools which we will use later on.

Let us consider a smooth manifold of dimension d. A differential p-form ω_p is $(0,p)$-rank tensor which has completely anti-symmetric components. A p-form is expanded as a linear combination of the basis cotangent vectors $\{dx^\nu\}_{\nu=1\dots d}$ as

$$\omega_p = \frac{1}{p!}\omega_{\nu_1\dots\nu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p} = \frac{1}{p!}\omega_{\nu_1\dots\nu_p} dx^{[\nu_1} \otimes \cdots \otimes dx^{\nu_p]}, \tag{2.12}$$

where the square brackets denote antisymmetrization.

The wedge product of a p-form ω_p and a q-form α_q is a $(p+q)$-form

$$\omega_p \wedge \alpha_q = \frac{1}{p!q!}\omega_{\nu_1\dots\nu_p}\alpha_{\mu_1\dots\mu_q} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}. \tag{2.13}$$

The exterior derivative of a p-form yields a $(p+1)$-form

$$d\omega_p = \frac{1}{p!}\partial_\mu \omega_{\nu_1\dots\nu_p} dx^\mu \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p}. \tag{2.14}$$

A p-form whose exterior derivative vanishes is called a closed form and a p-form that is the exterior derivative of a $(p-1)$-form is exact.

A fundamental property of the exterior derivative is Poincare's lemma, which states that for any differential form α, $d(d\alpha) = 0$ holds. This can be rewritten as $d^2 = 0$. In other words, every exact form is closed. We could ask ourselves if the inverse statement is true: is every closed form exact? The answer for an arbitrary manifold is no. This information is encoded in the q-th deRham cohomology group, which is formed by considering the set of all closed q-forms defined on a manifold. Since given a closed form ω, we can always find another closed form by adding an exact form $\omega' = \omega + d\alpha$, we consider the equivalence relation that two forms are equivalent if they differ by a closed form. The q-th deRham cohomology group of a manifold X is defined as the quotient

$$H^q_d(X,\mathbb{R}) = \{\omega|d\omega = 0\}/\{\alpha|\alpha = d\beta\}. \tag{2.15}$$

The dimension of $H^q_d(X,\mathbb{R})$ is the Betti number $b^q(X)$. Only when $b^q(X) = 1$, all closed q-forms on X are exact.

We define the Hodge dual $\star$ of a p-form as the $(d-p)$-form

$$\star\omega = \frac{1}{(d-p)!p!}\epsilon_{\mu_1\ldots\mu_n}\sqrt{|\det g|}g^{\mu_1\nu_1}\ldots g^{\mu_p\ldots\nu_p}\omega_{\nu_1\ldots\nu_p}dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_d}. \tag{2.16}$$

This operation allows us to define the adjoint exterior derivative or codifferential $d^\dagger$, that maps p-forms into $(p-1)$-forms

$$d^\dagger = (-1)^{dp+n+1}\star d\star. \tag{2.17}$$

The codifferential is the adjoint of the exterior derivative with respect to the inner product

$$\langle\omega,\omega'\rangle = \int_X \omega\wedge\omega', \tag{2.18}$$

so that given a p-form ω and a $(p-1)$-form σ

$$\langle\omega, d\sigma\rangle = \langle d^\dagger\omega,\sigma\rangle. \tag{2.19}$$

The codifferential is used to generalize the Laplacian as $\Delta = dd^\dagger + d^\dagger d$ and a harmonic form ω satisfies the Laplace equation $\Delta\omega = 0$. An important theorem involving harmonic forms is Hodge's decomposition, which states that a p-form ω can be uniquely written in terms of a $(p-1)$-form β, a $(p+1)$-form γ and a harmonic p-form ω'

$$\omega = d\beta + d^\dagger\gamma + \omega' \tag{2.20}$$

If ω is a closed form, γ vanishes so

$$\omega = d\beta + \omega'. \tag{2.21}$$

Identifying $\omega - d\beta = \omega'$ as an element of a cohomology class in $H^p_d(X,\mathbb{R})$, we can conclude that for every class belonging to $H^p_d(X,\mathbb{R})$, there is an unique harmonic p-form.

We can easily make a generalization of the previous concepts to complex manifolds of complex dimension $n = d/2$. Complexifying the basis

$\{dx_\mu\}_{\mu=1,\dots,d} \to \{dz_i, d\bar{z}_{\bar{j}}\}_{i,j=1,\dots,n}$, we can consider tensors $\omega_{r,s}$ with r holomorphic and s anti-holomorphic indices so they can be written as

$$\omega_{r,s} = \omega_{\mu_1,\dots\mu_r,\bar{\nu}_1,\dots,\bar{\nu}_s} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_r} \wedge d\bar{z}^{\bar{\nu}_1} \wedge \dots \wedge d\bar{z}^{\bar{\nu}_s} \tag{2.22}$$

This allows us to split the exterior derivative into holomorphic and anti-holomorphic derivatives $d = \partial + \bar{\partial}$. The complex equivalent of the deRham cohomology group is the Dolbeault cohomology group associated to $\bar{\partial}$ (it can be analogously defined for ∂)

$$H^{r,s}_{\bar{\partial}}(X, \mathbb{C}) = \{\omega | \bar{\partial}\omega = 0\} / \{\alpha | \alpha = \bar{\partial}\beta\}. \tag{2.23}$$

The dimension of $H^{r,s}_{\bar{\partial}}(X, \mathbb{C})$ is known as the Hodge number $h^{p,q}(X)$ and it is a topological invariant. Thanks to the Hodge star operation, there is a relation between Hodge numbers $h^{p,q} = h^{n-p,n-q}$. The fact that the manifold is Kähler also guarantees the symmetry $h^{p,q} = h^{q,p}$. The decomposition of the deRham cohomology into Dolbeault cohomologies is given by

$$H^p_d(X, \mathbb{R}) = \bigoplus_{r+s=p} H^{r,s}_{\bar{\partial}}(X, \mathbb{C}). \tag{2.24}$$

In the case of Calabi-Yau manifolds, it also holds that $h^{s,0} = 0$ if $1 < s < n$, $h^{n,0} = h^{0,n} = 1$. If the manifold is connected, then $h^{0,0} = 1$.

We can arrange the Hodge numbers into a Hodge diamond, which for a manifold of complex dimension three would be

$$\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{10} & & h^{01} & & \\
 & h^{20} & & h^{11} & & h^{02} & \\
h^{30} & & h^{21} & & h^{12} & & h^{03} \\
 & h^{20} & & h^{11} & & h^{02} & \\
 & & h^{10} & & h^{01} & & \\
 & & & h^{00} & & &
\end{array}$$

In the case of a Calabi-Yau three-fold

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11} & & 0 & \\ 1 & & h^{21} & & h^{21} & & 1 \\ & 0 & & h^{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

A very fruitful relation in physics and mathematics is mirror symmetry, which relates the topological properties between different Calabi-Yau manifold. A realization of mirror symmetry for a Calabi-Yau three-fold X is that there exists a mirror manifold Y such that

$$h_{1,1}(X) = h_{2,1}(Y) \qquad \text{and} \qquad h_{2,1}(X) = h_{1,1}(Y) \tag{2.25}$$

Thus, we can calculate some properties on X and immediately obtain information on Y.

Homology

A related construction to cohomology is homology. The basic element of homology is the p-chain a_p, which in the simplest formulation is the formal sum of p-dimensional submanifolds N_p^k (possibly with boundary)

$$a_p = \sum_k c_k N_p^k \tag{2.26}$$

where c_k are real numbers (or in some cases integers). p-chains that have no boundary are called p-cycles. Given that p-cycles do not have a boundary and that not all p-cycles form the boundary of $(p+1)$-chains, it makes sense to introduce the homology group. The homology group $H_q(X, \mathbb{R})$ is defined as the quotient space of q-cycles modulo q-dimensional boundaries

$$H_q(X, \mathbb{R}) = \{a | \partial a = 0\} / \{b | b = \partial c\}. \tag{2.27}$$

The dimension of $H_q(X, \mathbb{R})$ is $h_q(X)$. It is sometimes convenient to consider the coefficients c_k of the expansion (2.26) to be integers, in that case the associated homology group is $H_p(X, \mathbb{Z})$.

It can be seen that homology structure resembles cohomology structure by replacing p-chains with p-forms, the boundary operator with the exterior derivative, boundaries with exact forms and cycles with closed forms. In fact, they are algebraic duals, in the sense that integration of a p-form over a p-chain defines and isomorphism between H_q and H^q_d in the case of compact manifolds. This implies that the dimensions of both groups coincide $h_q = h^q$.

It will useful in our study to generalize the concept of how many times two lines intersect to the case of p-cycles. Given a p-cycle a_p and a $(d-p)$-cycle b_{d-p} which admit an expansion following equation (2.26) with integer c_k, the intersection number is defined as

$$I_{ab} = [a_p][b_{d-p}] = \int_X \delta(a_p) \wedge \delta(b_{d-p}) \tag{2.28}$$

where the square brackets denote the cohomology class associated to the cycle and the Dirac delta function δ satisfies

$$\int_{a_p} B_p = \int_X B_p \wedge \delta(a_p). \tag{2.29}$$

Since the intersection number depends on the homology classes of $H(X, \mathbb{Z})$ only, it is a topological invariant.

Moduli space

Starting from a particular choice of metric g on a Calabi-Yau manifold X, we could try to determine which deformations of the metric still preserve the Calabi-Yau condition. These deformations of the metric are known as moduli and play an important role in the physics of compactifications. We will restrict our discussion to Calabi-Yau manifolds of complex dimension three. An arbitrary deformation of the metric will consist of those with pure indices $g_{ij}dz^i dz^j$ and those with mixed indices $g_{i\bar{j}}dz^i dz^{\bar{j}}$. In order to preserve the Calabi-Yau condition they must lead to a vanishing Ricci tensor, $R_{i\bar{j}} = 0$. This constraint implies that:

A deformation of the type $g_{ij}dz^i \wedge dz^j$ must be harmonic, so it can be identified with an unique element of a cohomology class in $H^{1,1}$, the Kähler form. If we write the Kähler form in terms of the basis elements $\{t_a\}_{a=1,\ldots,h_{1,1}}$

$$k = \sum_{a=1}^{h_{1,1}} t_a \omega_a \tag{2.30}$$

the $h_{1,1}$ real parameters t_a are the Kähler moduli of the manifold. The Kähler form is employed to calculate the volume of a Calabi-Yau manifold of complex dimension three as $\int k \wedge k \wedge k$, since $k \wedge k \wedge \wedge k$ has the same rank as the volume form, which is unique up to a proportionality constant.

Deformations of the type $\Omega_{ijk} g^{k\bar{k}} \delta_{\bar{k}\bar{l}} dz^i \wedge dz^j \wedge d\bar{z}^{\bar{l}}$ must be a harmonic form belonging to a cohomology class in $H^{2,1}$. These deformations correspond to deformations of the complex structure, since the choice of a complex structure is related to a $(2,1)$-form $J_{ij\bar{k}} = \Omega_{ijl} J^l_{\bar{k}}$ obtained from the holomorphic three-form. There are $h_{2,1}$ complex parameters associated to the choice of the complex structure, which are called the complex structure moduli of the manifold. They determine the volume of 3-cycles Π in the compact space through Ω_3

$$\mathrm{Vol}(\Pi) = \int_{\Pi} \Omega_3. \tag{2.31}$$

In conclusion, a Calabi-Yau metric is determined uniquely by the Kähler form and the holomorphic three-form. The former leads to $h_{1,1}$ real parameters while the latter requires $h_{2,1}$ complex parameters.

Type IIA spectrum on Calabi-Yau manifolds

In order to compute the 4-dimensional massless spectrum of type IIA theory on a Calabi-Yau, we consider the Kaluza-Klein dimensional reduction. This consists in choosing an energy scale at which the compactification resides (the KK-scale) and then studying the effective four-dimensional theory at energies below the KK-scale. In practice, this corresponds to taking the KK-scale relatively large (or equivalently taking the associated radius of the compact space very small).

The simplest example of KK reduction is based on a free scalar field $\phi(x^M)$ in ten dimensions. We first apply its Fourier expansion in terms of the eigenvectors $\phi_k(x^m)$ of the Laplace operator in the internal space with eigenvalues λ_k

$$\phi(x^M) = \sum_k \phi_k(x^\mu)\phi_k(x^m) \tag{2.32}$$

where the dimension of the mode is determined by the argument, x^μ for the 4-dimensional Minkowski space and x^m for the compact space. The masslessness condition of $\phi(x^M)$ implies that

$$\Box\phi(x^\mu) - \lambda_k \phi(x^\mu) = 0 \tag{2.33}$$

This equation permits us to identify λ_k as the squared mass of the 4-dimensional $\phi(x^\mu)$ field. Thus, the number of massless scalar fields is given by the number of solutions of $\Box\phi(x^\mu) = 0$ which in the case of compact manifolds is one. We conclude that a 10-dimensional scalar field leads to a massless scalar field in 4-dimensions (in addition, the is a tower of KK modes).

Our next example is the KK reduction of a p-form C_p with the expansion

$$C_p = \sum_{k,q} c_q^k(x^m) \wedge C_{p-q}^k(x^\mu) \tag{2.34}$$

Massless 4-dimensional $(p-q)$-form fields are in one-to-one correspondence to internal modes that satisfy $dc_q = d^\dagger c_q = 0$, or in other words, to harmonic forms c_q. Since there is a single harmonic q-form in each q-cohomology class, the number of 4-dimensional massless $(p-q)$-forms arising from a p-form is the dimension of the H^q cohomology group, the Betti number b^q.

In the case of a Calabi-Yau manifold, from the relation of the Betti numbers with the Hodge numbers, we determine $b^0 = h_{0,0} = 1$, $b^1 = h_{1,0} + h_{0,1} = 0$, $b^2 = h_{1,1} + h_{2,0} + h_{0,2} = h_{1,1}$ and $b^3 = h_{3,0} + h_{0,3} + h_{2,1} + h_{1,2} = 2h_{2,1} + 1$. Thus, c_1 leads to a 4-dimensional 1-form, B_2 leads to a 2-form and $h_{1,1}$ scalar fields and c_3 leads to a 3-form (although it is not dynamical), $h_{1,1}$ 1-forms and $2h_{2,1} + 2$ scalar fields.

The KK reduction of the 10-dimensional metric is applied considering its components separately:

- The $G_{\mu\nu}$ components correspond to scalar fields in the internal space satisfying the Laplace equation and whose solution is unique for compact spaces. Thus, a 10-dimensional graviton reduces to a 4-dimensional graviton.

- The $G_{\mu m}$ components would correspond to 4-dimensional vector bosons, associated to 6-dimensional vector fields in the compact space. The masslessness condition of the 4-dimensional field would imply that the 6-dimensional vectors are Killing vectors associated to continuous isometries of the compact space, which in the case of Calabi-Yau manifolds are non-existent. As a consequence, the $G_{\mu m}$ components do not lead to any massless fields in 4 dimensions.

- The G_{mn} components reduce to 4-dimensional scalar fields associated to the moduli of the internal space, whose vev determine the geometry of the internal space. In the case of Calabi-Yau manifolds, we have seen that there are h_{21} real scalar fields and h_{11} complex scalar fields.

Having seen how the bosonic fields of type IIA behave under KK reduction, we proceed to describe the massless spectrum of type IIA theory compactified on a Calabi-Yau manifold.

In order to fill in the supermultiplets of 4-dimensional $\mathcal{N} = 2$ supersymmetry, we must combine scalar fields arising from the dilaton ϕ, p-forms and the geometric moduli into complex scalar fields. The spectrum is arranged as follows:

A single supergravity multiplet, composed of a graviton $G_{\mu\nu}$, a gauge boson arising from the KK reduction of B_2 and two gravitinos ψ with opposite chiralities.

$h_{1,1,}$ vector multiplets, composed of a gauge boson that arises from c_3, a complex scalar (obtained by combining the Kähler moduli t_a and the scalar field B_0 associated to B_2 into $B_2 + it_a$) and two Majorana fermions.

$h_{2,1}$ hypermultiplets composed of two complex scalars (obtained by combining the complex structure moduli with the scalar fields associated to mixed index components of c_3) and two left-handed fermions.

A single hypermultiplet composed of two complex scalars (obtained by combining the dilaton, a scalar field associated to B_2 [1] and the scalars that arise from pure index components of c_3) and two left-handed fermions.

2.2 Type IIA on Calabi-Yau orientifolds

Generalities of orientifolds

If we compactify a type II string theory on a Calabi-Yau manifold, we obtain a four-dimensional $\mathcal{N} = 2$ supersymmetric theory. This degree of supersymmetry does not allow for chiral fermions, so Calabi-Yau compactifications of type II theories have no straightforward application in the context of model building.

[1] The KK reduction of the 10-dimensional 2-form B_2 leads to a 4-dimensional 2-form b_2. We can then define a scalar field $\tilde{b}$ as the dual $d\tilde{b} = \star db_2$.

An option to reduce the supersymmetry to $\mathcal{N} = 1$ is to apply the orientifold projection, which consists in modding out the action of ΩR, where Ω is the worldsheet parity, so strings become unoriented, and R is a particular $\mathbb{Z}_2$ symmetry of the compact six-dimensional space In type IIA string theory we define $R = \mathcal{R}(-1)^{F_L}$. $\mathcal{R}$ satisfies the condition that it is an involution (squares to the identity) and acts anti-holomorphically on the complex coordinates of the internal space ($\mathcal{R} : z_i \to \bar{z}_i$). This implies that the Kähler and the holomorphic three-form transform as $J \to -J$ and $\Omega_3 \to \bar{\Omega}_3$. F_L is an operator that counts the number of left-moving fermions.

The fixed points under $\mathcal{R}$ define the orientifold planes in the model and are denoted as Op-planes, where p is the spatial dimension. In type IIA theory, the relevant choice are O6-planes, which span the entire four-dimensional Minkowski space and wrap a compact 3-cycle on the internal space.

In order to have a stable compactification, we expect all RR and NSNS charges to vanish. Furthermore, RR tadpole cancellation implies that the 4-dimensional theory is free of non-abelian gauge anomalies. O6-planes carry RR charge, so in order to eliminate RR tadpoles we must also introduce D6-branes, which carry opposite charge. It is important to note that D6-branes do not need to wrap the same 3-cycles as the O6-planes to remove RR tadpoles.

D-branes on Calabi-Yau manifolds

In order to obtain stable D6-brane configurations on a type IIA theory compactified on a Calabi-Yau manifold, we impose that they wrap around volume minimizing 3-cycles on the compact space, so that their tension is minimized as well. The volume minimizing condition means that the branes must wrap special Lagrangian 3-cycles in the internal space. Special Lagrangian 3-cycles Π are defined by

$$k|_\Pi = 0, \qquad \mathrm{Im}(e^{-i\phi}\Omega_3)|_\Pi = 0 \tag{2.35}$$

for some real ϕ, where k is the Kähler two-form and Ω_3 the holomorphic three-form. The $e^{-i\phi}\Omega_3$ is referred to as a calibration and the special Lagrangian is calibrated with respect to it.The volume of the special Lagrangian 3-cycle is

$$\mathrm{Vol}(\Pi) = \int_\Pi \mathrm{Re}(e^{-i\phi}\Omega_3) \tag{2.36}$$

D6-branes wrapped around a special Lagrangian cycle are guaranteed to preserve 4-dimensional $\mathcal{N}=1$ supersymmetry. This preserved supersymmetry coincides with the same supersymmetry preserved by the Op-planes only if $\phi = 0$.

The open string spectrum of stacks of N_a D6_a-branes wrapping special Lagrangian 3-cycles Π_a can be classified into two sectors: strings that stretch from one stack to itself and those that stretch between to different stacks, 6_a and 6_b.

Strings that stretch over 6_a lead to $U(N_a)$ vector multiplets of 4-dimensional $\mathcal{N}=1$ supersymmetry. There are also $b_1(\Pi_a)$ chiral multiplets in the adjoint representation, which are composed of the internal components of the gauge fields along Π_a combined with the geometric moduli of the 3-cycle, and their fermion superpartners.

Strings that stretch between 6_a and 6_b lead to $I_{ab} = [\Pi_a][\Pi_b]$ chiral fermions, where I_{ab} is the intersection number between 3-cycles. These fermions transform in the $(\mathbf{N_a}, \bar{\mathbf{N}}_\mathbf{b})$ representation. There are also massless scalar fields if the intersection preserves supersymmetry.

Orientifold compactifications with intersecting D-branes

We consider N_a D6-branes that wrap 3-cycles Π_a and whose image under the orientifold projection wrap the 3-cycles $\Pi_{a'}$. The condition that D6-branes preserve the same $\mathcal{N}=1$ supersymmetry as the O6-planes Π_{O6} is that the local relative angles between them obey

$$\theta_1 + \theta_2 + \theta_3 = 0 \tag{2.37}$$

If D6-branes do not coincide with their mirror images, the light spectrum of the model consists of:

- $U(N_a)$ gauge bosons arising from non-intersecting D6-branes.
- I_{ab} fermions in the representation $(\mathbf{N_a}, \bar{\mathbf{N}}_\mathbf{b})$ arising from the intersection of two different D6-branes.
- $I_{ab'}$ fermions in the representation $(\mathbf{N_a}, \mathbf{N_b})$ arising from the intersection of a D6-brane with the mirror of a different D6-brane.

- $1/2([\Pi_a][\Pi_{a'}] + [\Pi_a][\Pi_{O6}])$ fermions in the anti-symmetric representation $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \mathbf{1})$ and $1/2([\Pi_a][\Pi_{a'}] - [\Pi_a][\Pi_{O6}])$ fermions in the symmetric representation $(\square\square, \mathbf{1})$ which arise from the intersection of a D6-brane with its own mirror.

The condition for RR tadpole cancellation imposes a topological restriction, namely, that the sum of the three-cycles wrapped by the D-branes and their orientifold images has to combine with the O6-plane three-cycle into the trivial cycle in homology

$$\sum_a N_a([\Pi_a] + [\Pi_{a'}]) - 4[\Pi_{O6}] = 0. \tag{2.38}$$

Effective action of D-branes on Calabi-Yau orientifolds

We recall that the action of a Dp-brane contains the DBI term (1.3)

$$S_{DBI} = -\mu_p \int_{D_p} e^{-\phi} \sqrt{\det(G + B - 2\pi\alpha' F)} \tag{2.39}$$

which reduces to the Yang-Mills action for small values of α'. In the case of compactification on a Calabi-Yau orientifold, the gauge coupling constant is given in terms of the volume of the special-Lagrangian three-cycles along the internal space

$$\frac{1}{g^2} = e^{-\phi} \frac{(\alpha')^{-3/2}}{(2\pi)^4} \mathrm{Vol}(\Pi_3). \tag{2.40}$$

In terms of model-building, this relation implies that knowledge about the coupling strength of a gauge theory in four dimensions can be translated in geometric terms to the volume of a wrapped special-Lagrangian three-cycle. In terms of mathematics this relation challenges us to understand how to compute the volumes of special Lagrangian three-cycles on Calabi-Yau manifolds.

Chapter 3

Type IIA on the quintic

3.1 Motivation

Why the quintic?

Why study SLags?

3.2 SLags on Fermat's quintic

In this section, we describe Fermat's quintic and study the properties of SLag three-cycles defined on this manifold.

Fermat's quintic

Chow's theorem asserts that any $(n-r)$-dimensional submanifold of $\mathbb{CP}^n$ can be realized as the zero locus of r-homogeneous polynomial equations. This suggests Calabi-Yau manifolds as submanifolds of a complex projective space

Motivate better

The complex projective space $\mathbb{CP}^n$ is defined considering the complex space minus the origin $\mathbb{C}^{n+1}\backslash\{0\}$ and establishing the equivalence relation $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$, $\lambda \in \mathbb{C}$. To emphasize that z_i are homogeneous coordinates, they are sometimes denoted as $[z_0 : \ldots : z_n]$. The local (non-homogeneous) coordinates ξ^i of a j-patch where $z_j \neq 0$ are obtained by choosing $\lambda = 1/z_j$ so

$$(\xi^1_j, \ldots, \xi^n_j) = (\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}). \tag{3.1}$$

It can be shown that $\mathbb{CP}^n$ is a Kähler manifold. *PGL*

Projective group see how it acts, coordinates

The complex projective space allows us to obtain lower dimensional manifolds as subspaces of $\mathbb{CP}^n$. Indeed, Chow's theorem states that any submanifold of complex dimension $n-r$ of $\mathbb{CP}^n$ can be realized as the zero locus of r-homogeneous polynomial equations. An homogeneous polynomial P of degree d satisfies

$$P(\lambda z_1, \ldots, \lambda z_n) = \lambda^d P(z_1, \ldots, z_n). \tag{3.2}$$

We are interested in calculating the number of inequivalent $(n-1)$-dimensional submanifolds of $\mathbb{CP}^n$ that can be defined through a polynomial equation of degree d. The number of independent monomials of degree d in $n+1$ variables is given by the binomial coefficient

$$\#\ \text{indep. monomials} = \binom{d+(n+1)-1}{(n+1)-1} \tag{3.3}$$

But not all of these lead to different manifolds, since some of them can be related through coordinate transformations, which belong to complex general linear group $GL(\mathbb{C})_{n+1}$. Thus, the number of the number of possible submanifolds is given by

$$\#\ \text{submanifolds}\ \mathbb{CP}^n = \#\ \text{indep. monomials} - \#\ \text{components}\ GL(\mathbb{C})_{n+1} \tag{3.4}$$

Among of the $\binom{9}{4} - 5^5 = 101$ possible submanifolds of $\mathbb{CP}^4$ defined by a quintic polynomial, lies Fermat's quintic threefold defined by the polynomial P_5 as

$$P_5(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0. \tag{3.5}$$

Construction of SLags

A special-Lagrangian three-cycle is constructed introducing an anti-holomorphic involution. In the case of $\mathbb{CP}^4$, there is only one consistent anti-holomorphic, that acts on the homogeneous coordinates as $\mathcal{R} : z_i \to \bar{z}_i$. SLags three-cycles are then defined as the fixed loci under $\mathcal{R}$, which in the case of Fermat's quintic leads to the following SLag

$$\left\{[x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{RP}^4 | x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\right\} \tag{3.6}$$

This subspace is topologically equivalent to $\mathbb{RP}^3$, which can be noticed through the homeomorphism from $\mathbb{RP}^3$ to the SLag:

$$(u_0, u_1, u_3) \to (u_0, u_1, u_2, u_3, -(u_0^5 + u_1^5 + u_2^5 + u_3^5)^{1/5} \tag{3.7}$$

Show that the obtained subspace is special-Lagrangian $\mathcal{R} : k \to -k$ calibrated with respect Ω_3. The three-form in a local coordinate patch where $z_0 \neq 0$ is

$$\Omega_3 = \frac{4}{2\pi i} \int \frac{x_0 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{P_5(x_i)} \tag{3.8}$$

if we interpret x_4 as a function of P_5 and integrate over a loop around $P_5 = 0$

$$\Omega_3 = \frac{x_0 dx_1 \wedge dx_2 \wedge dx_3}{x_4^4} = \frac{dy_1 \wedge dy_2 \wedge dy_3}{y_4^4} \tag{3.9}$$

where we have defined the local coordinates $y_i = x_i/x_0$. By taking the norm of the previous equation, we relate Ω_3 to the determinant of the six-dimensional metric

$$||\Omega_3||^2 = \frac{1}{\det g |y_4|^8} \tag{3.10}$$

Since Ω_3 is covariantly constant, $||\Omega_3||$ must also be proportional to a constant defined as

$$||\Omega_3||^2 = 8e^{2\kappa} \tag{3.11}$$

$$\det g = \frac{e^{-2\kappa}}{8|y_4|^8} \tag{3.12}$$

The pull-back of the six-dimensional metric onto the three-cycle is

$$h_{\alpha\beta} = 2\partial_\alpha X^i \partial_\beta X^{\bar{j}} g_{i\bar{j}} \tag{3.13}$$

$$e^k \sqrt{\det h_{ab}} d\sigma^1 \wedge d\sigma^2 \wedge d\sigma^3 = e^k |\det \partial y| \frac{e^{-\kappa}}{|y_4|^4} d\sigma^1 \wedge d\sigma^2 \wedge d\sigma^3 = \Omega_3 \tag{3.14}$$

It is possible to define different SLags by exploiting the $\mathbb{Z}_5^4$ symmetry of the quintic, which are rotations

$$|0, k_1, k_2, k_3, k_4\rangle = \left[x_0 : \omega_1^k x_1 : \omega_2^k x_2 : \omega_3^k x_3 : \omega_4^k x_4\right] \tag{3.15}$$

where $\omega = e^{i\frac{2\pi}{5}}$ calibrated O6-plane $|0, 0, 0, 0, 0\rangle$. The choice of k_i leads to $5^4 = 625$ SLag three-cycles, but not all of them are calibrated This now!

Moduli space of SLags

Hard

Intersection numbers

spectrum of intersecting D6-branes

Volumes

$$\Omega_3 = \frac{dy_1 \wedge dy_2 \wedge dy_3}{5y_4^4} \tag{3.16}$$

$$\int \frac{dy_1 \wedge dy_2 \wedge dy_3}{5(1 + y_1^4 + y_2^4 + y_3^4)^{4/5}} \tag{3.17}$$

SM on the quintic

3.3 SLags on the deformed quintic

Deformations of the quintic

A generalization of Fermat's quintic consists in considering hypersurfaces defined by a generic polynomial of degree five

$$P_5(z) = \sum_{n_0+n_1+n_2+n_3+n_4=5} a_{n_0n_1n_2n_3n_4} z_0^{n_0} z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} = 0 \tag{3.18}$$

This construction reduces to Fermat's quintic when $a_{50000} = a_{50000} = a_{50000} = a_{50000} = a_{50000} = 1$ and all other coefficients are zero. A well-know example is

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0 \tag{3.19}$$

where the parameter ψ can take three values of particular relevance

- $\psi = 1$: the Fermat (or Gepner point).
- $\psi = 0$: the conifold point.
- $\psi = \infty$: the large complex structure limit.

We can deform Fermat's quintic by adding a monomial to the defining polynomial. The possible monomials are of the type:

1. $z_0z_1z_2z_3z_4$, 1 deformation.
2. $z_iz_jz_k(z_l)^2$, $\binom{5}{3}\binom{2}{1}$= 20 deformations.
3. $z_i(z_j)^k(z_l)^2$, $\binom{5}{1}\binom{4}{2}$= 30 deformations.
4. $z_iz_j(z_k)^3$, $\binom{5}{2}\binom{3}{1}$= 30 deformations.
5. $(z_i)^2(z_j)^3$, $\binom{5}{1}\binom{4}{1}$= 20 deformations.
6. $z_i(z_j)^4$, $\binom{5}{3}\binom{2}{1}$= 20 deformations.

In total there are 126 deformations, but they are not all independent, since they can be related through coordinate transformations. Subtracting 25 we 101 deformation parameters. This is precisely the Hodge number $h_{2,1} = 101$, since the defining homogeneous polynomial represents a particular choice of the complex structure, which is given by harmonic $(2,1)$-form in $H^{2,1}(X,\mathbb{C})$. The deformations of Kähler structure is given by the $h^{1,1}$ and is precisely one.

Fermat's quintic enjoys the $\mathbb{Z}_5^4$ symmetry acting on the homogeneous coordinates

$$z_i \to \omega_i^k z_i, \qquad \omega_i = e^{i\frac{2\pi}{5}} \tag{3.20}$$

which can be broken into a smaller subgroup when turning on deformations. We should also examine whether deformations introduce any singularities, which are defined as the points

$$P_5(X) = 0, \qquad dP_5 = 0 \tag{3.21}$$

The nature of the singularity is obtained by evaluating the Hessian at the singularity. That is calculating d^2P_5 in a local coordinate patch, where one of the homogeneous coordinate is non-zero.

Complete

We proceed to examine the singularities and associated symmetry subgroups of the deformations.

1. No deformations.

   $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$ The only point where $dP_5 = 0$ is $(0,0,0,0,0)$, which is not part of $\mathbb{CP}^4$, so

2. $5\psi z_0 z_1 z_2 z_3 z_4$

   $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0$

   The singularities are located at:

$$z_i^5 = \psi x_0 x_1 x_2 x_3 x_4 \tag{3.22}$$

$$\Pi z_i^5 = \psi^5 (x_0 x_1 x_2 x_3 x_4)^5 \tag{3.23}$$

   so in order to obtain a singular point, $\psi^5 = 1$. Taking $\psi = 1$, there a single singular point $(1,1,1,1,1)$ which is a node, since the Hessian doesn't vanish. Locally, the node can be recast into a conifold singularity.

3. $5\psi z_0 z_1 z_2 (z_3)^2$

   $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 (z_3)^2 = 0$

   Singularities in the coordinate patch where $z_3 = 1$

$$(\frac{1}{\sqrt[5]{2}}, \frac{1}{\sqrt[5]{2}}, \frac{1}{\sqrt[5]{2}}, 1, 0) \tag{3.24}$$

   $\psi = \frac{1}{\sqrt[5]{2}}$

4. $5\psi z_0 (z_1)^2 (z_2)^2$

   $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 (z_3)^2 = 0$

$$(1, \sqrt[5]{2}, \sqrt[5]{2}, 0, 0) \tag{3.25}$$

   $\psi = \frac{1}{\sqrt[5]{2^4}}$

5. $5\psi z_0 z_1 (z_2)^3$

$$(\sqrt[5]{3}, \sqrt[5]{3}, 1, 0, 0) \tag{3.26}$$

   $\psi = \frac{1}{\sqrt[5]{3^3}}$

6. $5\psi (z_0)^2 (z_1)^3$

$$(1, \sqrt[5]{\frac{3}{2}}, 0, 0, 0) \tag{3.27}$$

   $\psi = \frac{1}{\sqrt[5]{3^3}}$

7. $5\psi z_0(z_1)^4$

$$(1, \sqrt[5]{4}, 0, 0, 0) \tag{3.28}$$

$\psi = \frac{1}{\sqrt[5]{4^4}}$

Check values, add words

SLags on the deformed quintic

Generalities. Special case. Generalization

3.4 Conclusions and outlook

model building obstruction

for especific def, vol def param psi is not easy to calculate, region proxy

check same study for other deformations study type singularities

Bibliography