A walk through moduli space with SLags

John Xihua Liu Anta

Universidad Autónoma de Madrid
Instituto de Física Teórica

Adviser: Wieland Staessens

Madrid, 2017

Contents

Chapter 1

Type IIA on the quintic

1.1 Motivation

1.2 SLags on Fermat's quintic

In this section, we describe Fermat's quintic and study the properties of SLag three-cycles defined on this manifold.

Fermat's quintic

The complex projective space $\mathbb{CP}^n$ is defined considering the complex space minus the origin $\mathbb{C}^{n+1}\backslash\{0\}$ and establishing the equivalence relation $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$, $\lambda \in \mathbb{C}$. The local (non-homogeneous) coordinates ξ^i of a j-patch where $z_j \neq 0$ are obtained in terms of the homogeneous coordinates z_i by choosing $\lambda = 1/z_j$ so

$$(\xi_j^1, \ldots, \xi_j^n) = \left(\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}\right). \tag{1.1}$$

The complex projective space allows us to obtain lower dimensional manifolds as subspaces of $\mathbb{CP}^n$. Indeed, Chow's theorem states that any submanifold of complex dimension $n-r$ of $\mathbb{CP}^n$ can be realized as the zero locus of r-homogeneous polynomial equations. An homogeneous polynomial P of degree d satisfies

$$P(\lambda z_1, \ldots, \lambda z_n) = \lambda^d P(z_1, \ldots, z_n). \tag{1.2}$$

We are interested in calculating the number of inequivalent $(n-1)$-dimensional submanifolds of $\mathbb{CP}^n$ that can be defined through a polynomial equation of degree d. The number of independent monomials of degree d in $n+1$ variables is given by the binomial coefficient

$$\# \text{ indep. monomials} = \binom{d+(n+1)-1}{(n+1)-1} \tag{1.3}$$

Not all of these lead to different manifolds, since some of them can be related through coordinate transformations, which belong to complex general linear group $GL(\mathbb{C})_{n+1}$. Thus, the number of possible submanifolds is given by

$$\# \text{ submanifolds } \mathbb{CP}^n = \# \text{ indep. monomials} - \# \text{ components } GL(\mathbb{C})_{n+1} \tag{1.4}$$

Among of the $\binom{9}{4} - 5^5 = 101$ possible submanifolds of $\mathbb{CP}^4$ defined by a quintic polynomial, lies Fermat's quintic threefold defined by the polynomial P_5 as

$$P_5(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0. \tag{1.5}$$

Construction of SLags

A special-Lagrangian three-cycle is constructed by introducing an anti-holomorphic involution. In the case of $\mathbb{CP}^4$, there is only one consistent anti-holomorphic involution, which acts on the homogeneous coordinates as $\mathcal{R} : z_i \to \bar{z}_i$. A particular instance of SLag three-cycle is defined as the fixed loci under $\mathcal{R}$. In the case of Fermat's quintic leads it to the following SLag

$$\big\{[x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{RP}^4 | x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\big\}. \tag{1.6}$$

This subspace is topologically equivalent to $\mathbb{RP}^3$, which can be noticed through the homeomorphism from $\mathbb{RP}^3$ to the SLag:

$$(u_0, u_1, u_3) \to (u_0, u_1, u_2, u_3, -(u_0^5 + u_1^5 + u_2^5 + u_3^5)^{1/5}). \tag{1.7}$$

In order to prove that the obtained subspace is indeed special-Lagrangian we must verify that the Kähler form vanishes at the SLag and that it is calibrated with respect to the same three-form as the Ω_3 of the quintic.

- The Kähler form k transforms under the anti-holomorphic involution as $k \to -k$, but since the pull-back of k onto the three-dimensional subspace must be real, the only possibility is $k|_{\text{SLag}} = 0$

- The holomorphic three-form in a non-homogeneous patch where $x_0 \neq 0$ is

$$\Omega_3 = \frac{4}{2\pi i} \int \frac{x_0 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{P_5(x_i)}. \tag{1.8}$$

If we interpret x_4 as a function of P_5 and integrate over a loop around $P_5 = 0$

$$\Omega_3 = \frac{x_0 dx_1 \wedge dx_2 \wedge dx_3}{x_4^4} = \frac{dy_1 \wedge dy_2 \wedge dy_3}{y_4^4} \tag{1.9}$$

where we have defined the local coordinates $y_i = x_i/x_0$. By taking the norm of the previous equation, we relate Ω_3 to the determinant of the six-dimensional metric

Why metric appears?

$$||\Omega_3||^2 = \frac{1}{\det g |y_4|^8}. \tag{1.10}$$

Since Ω_3 is covariantly constant, $||\Omega_3||$ must also be proportional to a constant defined as

$$||\Omega_3||^2 = 8e^{2\kappa}. \tag{1.11}$$

The pull-back of the six-dimensional metric onto the three-cycle is

$$h_{\alpha\beta} = 2\partial_\alpha X^i \partial_\beta X^{\bar{j}} g_{i\bar{j}} \tag{1.12}$$

So the volume form of the SLag three-cycle coincides with Ω_3

$$e^k \sqrt{\det h_{ab}} d\sigma^1 \wedge d\sigma^2 \wedge d\sigma^3 = e^k |\det \partial y| \frac{e^{-\kappa}}{|y_4|^4} d\sigma^1 \wedge d\sigma^2 \wedge d\sigma^3 = \Omega_3 \tag{1.13}$$

Why lhs has that form?

It is possible to define different SLags by exploiting the $\mathbb{Z}_5^4$ symmetry of the quintic, which leads to a whole family of SLags

$$|0, k_1, k_2, k_3, k_4\rangle = \left[x_0 : \omega_1^k x_1 : \omega_2^k x_2 : \omega_3^k x_3 : \omega_4^k x_4\right] \tag{1.14}$$

where $\omega = e^{i\frac{2\pi}{5}}$. The freedom to choose of $k_i \in \mathbb{Z}_5$ leads to $5^4 = 625$ SLag three-cycles, but not all of them are calibrated with respect to the same three-form as O6-plane $|0, 0, 0, 0, 0\rangle$. Concretely, only 125 SLags are calibrated with respect to the same three-form as the O6-plane.

Moduli space of SLags

Not sure if the reasoning in this section is too sloppy.

We are not only interested in SLags themselves, but also in their moduli space. The reason is that deformations of SLag three-cycles codify the information of the open string excitations of the D6-branes that wrap around them. In particular, they determine de vev of the scalar field excitations and thus the position of the D6-branes and whether they are stable configurations or not.

A deformation of a reference SLag three-cycle Π_3 into another three-cycle (in the same homology class as the original SLag) can be parameterized by a normal vector field, which belongs to the normal bundle of Π_3. This deformation normal vector field can be written in a basis $\{s^i\}_i$ as

$$X = \phi_i s^i \tag{1.15}$$

where ϕ_i correspond to the scalar fields defined on the D6-brane worldvolume theory. The Kähler form introduces an isomorphism between the normal bundle of Π_3 and the cotangent bundle of Π_3. According to McLean's theorem, harmonic one-forms are associated to a deformation vector field. Recalling that there is a single harmonic form in each cohomology class, we associate to every s^i a one-form $\xi_i \in H^1(\Pi)$. As a consequence, the moduli space of SLags is identified with $H^1(\Pi_3)$.

A gauge potential A on the D6-brane can be expanded in a basis of one-forms ξ_i as $A = a^i \xi_i$. In order to fill in the multiplets of the spectrum arising from open string excitation of D6-branes, we must arrange the a^i and ϕ_i into a complex scalar field

$$\Phi^i = \phi_i + i a^i. \tag{1.16}$$

We can now apply the previous reasoning to SLag three-cycles defined on Fermat's quintic. Since $H^1(\mathbb{RP}^3) \simeq H_1(\mathbb{RP}^3) \simeq \mathbb{Z}_2$, there are no continuous deformations of the SLags and the SLags are guaranteed to remain stable.

Intersection numbers

We now study the intersection numbers between different SLags, which are used to determine the number of chiral fermions associated to the intersecting D6-branes.

First of all, we choose a coordinate patch of $\mathbb{CP}^4$ where $x_0 = 1$ and take one of the SLags to be $|0,0,0,0,0\rangle$ and the other SLag $|0,k_1,k_2,k_3,k_4\rangle = \omega^{k_1}\omega^{k_2}\omega^{k_3}\omega^{k_4}\,|0,0,0,0,0\rangle$, obtained by applying successive rotations ω associated to the $\mathbb{Z}_5^4$ symmetry. The intersection numbers in this patch are:

- If $\omega_i^{k_i} \neq 1$ for all i, there are no intersections in this patch. The justification is that the coordinates of one of the SLags $[1 : x_1 : x_2 : x_3 : x_4]$ where $x_i \in \mathbb{R}$ take only real values, while some of the coordinates of the SLag $[1 : \omega_1^{k_1} x_1 : \omega_2^{k_2} x_2 : \omega_3^{k_3} x_3 : \omega_4^{k_4} x_4]$ take always complex numbers.
- If $\omega_i^{k_i} \neq 1$ for only one i, which we set to be $i = 4$, the intersection is at a single point. In particular, the intersection of $[1 : x_1 : x_2 : x_3 : x_4]$ with $[1 : \omega_1^{k_1} x_1 : \omega_2^{k_2} x_2 : \omega_3^{k_3} x_3 : \omega_4^{k_4} x_4]$ is located at $(1,0,0,0,x_4)$.When we restrict this line in $\mathbb{CP}^4$ to the quintic hypersurface, the intersection reduces to the point $(1,0,0,0,-1)$. The signature of the intersection is $\mathrm{sgn}(\mathrm{Im}\,\omega^{k_1}\,\mathrm{Im}\,\omega^{k_2}\,\mathrm{Im}\,\omega^{k_3})$.

To compute the total intersection number, we would have to repeat this procedure in all patches and the add all the obtained intersection numbers. The total intersection number of $|0,0,0,0,0\rangle$ with

Volumes

We now try to compute the volume of SLag three-cycles, as the carry information of the four-dimensional gauge theory supported by the wrapping D6-branes, such as the gauge coupling.

Complete

$|0,0,0,0,0\rangle$, since a generic SLag can be obtained through $\mathbb{Z}_4$ rotations. The holomorphic three-form on a patch $x_0 \neq 0$ in terms of the local coordinates y_i is

$$\Omega_3 = \frac{dy_1 \wedge dy_2 \wedge dy_3}{5y_4^4} \tag{1.17}$$

The volume of the SLag is obtained integrating Ω_3 over the three-cycle

$$\mathrm{Vol} = \int \frac{dy_1 \wedge dy_2 \wedge dy_3}{5(1+y_1^4+y_2^4+y_3^4)^{4/5}} \tag{1.18}$$

It is not evident how to compute this integral, due to the non-trivial integration domain.

In principle, the calculation of the volume of a SLag can be reduced to the

We can tran

Alternative

SLag three-cycle is homeomorphic to the $\mathbb{RP}^3$, which in turn is diffeomorphic to the the rotation group $SO(3)$. Considering that we there is a map from S^3 with antipodal points identified onto $SO(3)$, we conclude that the volume of a SLag is half the volume of a corresponding three-sphere on the quintic. In practice, we can make no further progress through this approach, since we do not know the Calabi-Yau metric of the quintic.

SM on the quintic

1.3 SLags on the deformed quintic

Deformations of the quintic

A generalization of Fermat's quintic consists in considering hypersurfaces defined by a generic polynomial of degree five

$$P_5(z) = \sum_{n_0+n_1+n_2+n_3+n_4=5} a_{n_0 n_1 n_2 n_3 n_4} z_0^{n_0} z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} = 0 \tag{1.19}$$

This construction reduces to Fermat's quintic when $a_{50000} = a_{50000} = a_{50000} = a_{50000} = a_{50000} = 1$ and all other coefficients are zero. A well-know example is

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0 \tag{1.20}$$

where the parameter ψ can take three values of particular relevance

- $\psi = 1$: the Fermat (or Gepner point).
- $\psi = 0$: the conifold point.
- $\psi = \infty$: the large complex structure limit.

We can deform Fermat's quintic by adding a monomial to the defining polynomial. The possible monomials are of the type:

1. $z_0 z_1 z_2 z_3 z_4$, 1 deformation.
2. $z_i z_j z_k (z_l)^2$, $\binom{5}{3}\binom{2}{1}$= 20 deformations.

3. $z_i(z_j)^k(z_l)^2$, $\binom{5}{1}\binom{4}{2}$= 30 deformations.

4. $z_iz_j(z_k)^3$, $\binom{5}{2}\binom{3}{1}$= 30 deformations.

5. $(z_i)^2(z_j)^3$, $\binom{5}{1}\binom{4}{1}$= 20 deformations.

6. $z_i(z_j)^4$, $\binom{5}{3}\binom{2}{1}$= 20 deformations.

In total there are 126 deformations, but they are not all independent, since they can be related through coordinate transformations. Subtracting 25 we 101 deformation parameters. This is precisely the Hodge number $h_{2,1} = 101$, since the defining homogeneous polynomial represents a particular choice of the complex structure, which is given by harmonic $(2,1)$-form in $H^{2,1}(X, \mathbb{C})$. The deformations of Kähler structure is given by the $h^{1,1}$ and is precisely one.

Fermat's quintic enjoys the $\mathbb{Z}_5^4$ symmetry acting on the homogeneous coordinates

$$z_i \to \omega_i^k z_i, \qquad \omega_i = e^{i\frac{2\pi}{5}} \tag{1.21}$$

which can be broken into a smaller subgroup when turning on deformations. We should also examine whether deformations introduce any singularities, which are defined as the points

$$P_5(X) = 0, \qquad dP_5 = 0 \tag{1.22}$$

The nature of the singularity is obtained by evaluating the Hessian at the singularity. That is calculating d^2P_5 in a local coordinate patch, where one of the homogeneous coordinate is non-zero.

Complete

We proceed to examine the singularities and associated symmetry subgroups of the deformations.

1. No deformations.

   $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$ The only point where $dP_5 = 0$ is $(0,0,0,0,0)$, which is not part of $\mathbb{CP}^4$, so

2. $5\psi z_0z_1z_2z_3z_4$

   $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0z_1z_2z_3z_4 = 0$

The singularities are located at:

$$z_i^5 = \psi x_0 x_1 x_2 x_3 x_4 \tag{1.23}$$

$$\Pi z_i^5 = \psi^5 (x_0 x_1 x_2 x_3 x_4)^5 \tag{1.24}$$

so in order to obtain a singular point, $\psi^5 = 1$. Taking $\psi = 1$, there a single singular point $(1, 1, 1, 1, 1)$ which is a node, since the Hessian doesn't vanish. Locally, the node can be recast into a conifold singularity.

3. $5\psi z_0 z_1 z_2 (z_3)^2$

   $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 (z_3)^2 = 0$

   Singularities in the coordinate patch where $z_3 = 1$

$$(\frac{1}{\sqrt[5]{2}}, \frac{1}{\sqrt[5]{2}}, \frac{1}{\sqrt[5]{2}}, 1, 0) \tag{1.25}$$

   $\psi = \frac{1}{\sqrt[5]{2}}$

4. $5\psi z_0 (z_1)^2 (z_2)^2$

   $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 (z_3)^2 = 0$

$$(1, \sqrt[5]{2}, \sqrt[5]{2}, 0, 0) \tag{1.26}$$

   $\psi = \frac{1}{\sqrt[5]{2^4}}$

5. $5\psi z_0 z_1 (z_2)^3$

$$(\sqrt[5]{3}, \sqrt[5]{3}, 1, 0, 0) \tag{1.27}$$

   $\psi = \frac{1}{\sqrt[5]{3^3}}$

6. $5\psi (z_0)^2 (z_1)^3$

$$(1, \sqrt[5]{\frac{3}{2}}, 0, 0, 0) \tag{1.28}$$

   $\psi = \frac{1}{\sqrt[5]{3^3}}$

7. $5\psi z_0(z_1)^4$

$$(1, \sqrt[5]{4}, 0, 0, 0) \tag{1.29}$$

$\psi = \frac{1}{\sqrt[5]{4^4}}$

Check values, add words

SLags on the deformed quintic

Generalities. Special case. Generalization

Bibliography