Navigation Functions for Focally Admissible Surfaces

Ioannis Filippidis and Kostas J. Kyriakopoulos

***Abstract*— This work presents a sharper condition for the applicability of Navigation Functions (NF). The condition depends on the placement of the destination with respect to the focal surfaces of obstacles. The focal surface is the locus of centers of principal curvatures. If each obstacle encompasses at least one of its focal surfaces, then the world is navigable using a Koditschek-Rimon NF (KRNF). Moreover, the Koditschek-Rimon (KR) potential is non-degenerate for all destinations which are not on a focal surface. So, for almost all destinations there exists a non-degenerate KR potential. This establishes a link between the differential geometry of obstacle surfaces and KRNFs. Channel surfaces (e.g. Dupin cyclides) and certain Boolean operations between shapes are examples of admissible obstacles. We also prove a weak converse result about the inexistence of a KRNF for obstacles with some concave point, for large tuning parameters. Finally, our results support non-trivial simulations in a forest, a pipeline and a cynlinder rig, with some notes about allowable types of non-smoothness.**

I. INTRODUCTION

Multiple different methods have been proposed for motion planning [1], [2], from sampling-based ones like rapidly exploring random trees [3], [4] to combinatorial as Cell Decompositions [5]. Firstly proposed by Khatib [6], Artificial Potential Fields work by constructing a scalar function which is maximal on obstacle boundaries and minimal at the destination. An agent driven by its negated gradient avoids collisions and moves to the desired destination, unless any other local minima exist and attract it instead.

Koditschek and Rimon [7] introduced Navigation Functions (NF) which are free of other local minima, proved they exist for any n-dimensional Riemannian manifold with boundary and proposed a particular construction for sphere worlds [8], [9], although efficiently constructing a NF remains an open problem in general. Sphere world solutions rely on diffeomorphisms to pull-back the solution from the spheres to more realistic obstacles [10]. Using diffeomorphisms [11], [12], [13] greatly extends NF applicability, but constructing the diffeomorphisms is challenging, besides they can map only topologically equivalent obstacles, e.g., not a genus-1 torus to a genus-0 sphere.

To avoid the need for diffeomorphisms, the authors proposed an extension in [14] which introduced a second-order geometric condition characterizing the surfaces for which a Koditschek-Rimon NF (KRNF) exists.

In this work we extend this approach further to consider in more detail the first-order structure of obstacle surfaces. In other words, taking into account both the curvature and normal bundle of the obstacle surfaces, we are led to a sharper admissibility criterion. This is obtained by studying the contact points between the obstacle surface and the spherical level sets of the attractive field, in two steps.

Firstly, critical points can only arise close to first-order contact points. The higher we choose the tuning parameter, the closer the critical set is to the first-order contact locus. The Hessian eigenvalues at critical points distinguish saddles from minima. These eigenvalues can be determined from the principal curvatures at their nearby first-order contact points. This leads to a navigability criterion concerning the inclusion of one focal surface within the obstacle.

Secondly, a critical point which is not near a second-order contact point is non-degenerate. Second-order contact points arise only when the destination is on a focal surface of an obstacle. A focal surface is the locus of centers of principal curvature. As a result, only a measure zero set of destinations (the focal surfaces) can cause degeneracy. Furthermore, a converse result is also proved, indicating that concave geometries cannot be navigated with KR potentials.

The rest of this paper is organized as following: Navigation Functions are reviewed in § II, the problem is defined in § III, focal admissibility in § IV, the relation of critical points to first-order contact points is studied in § V, and to second-order contact points in § VI. Supporting simulations are presented in § VIII and future work considered in § IX.

II. NAVIGATION FUNCTIONS

We define $M \in \mathbb{N}$ obstacles in n-dimensional Euclidean space E^n. Each obstacle is a connected (non-empty) subset of E^n implicitly defined as $\mathscr{O}_i \triangleq \{q \in E^n | \ \beta_i(q) < 0\}$. We assume that each $\partial\mathscr{O}_i \neq \emptyset$ is compact, and $\overline{\mathscr{O}_i} \cap \overline{\mathscr{O}_j} = \emptyset$. Assuming that $E^n \setminus \mathscr{O}_i$ is connected, then the free space $\mathscr{F}$ is connected (no single $\mathscr{O}_i$ can disconnect it). This ensures that a solution exists, because initial condition and destination belong to the same connected component. In addition, we assume that $\beta_i \in C^2(E^n, \mathbb{R})$, for some $\rho_r > 0$, $|\beta_i(q)| > a > 0$ for $\|q\| > \rho_r$ (radially lower bounded) and $\nabla\beta_i \neq 0$ on $\partial\mathscr{O}_i$. Radial lower boundedness ensures that there exists some $\bar{\varepsilon}_i > 0$, such that the preimage $\beta_i^{-1}([0, \bar{\varepsilon}_i])$ is compact [15]. Define the ε-neighborhood of an obstacle as $\mathscr{B}_i(\varepsilon) \triangleq \{q \in E^n | \ 0 < \beta_i(q) < \varepsilon\}$. Let $I_0 \triangleq \mathbb{N}_{\leq M}$. The whole world is defined as $\mathscr{W} \triangleq E^n \setminus \mathscr{O}_0$. Assuming that $E^n \setminus \mathscr{O}_0$ is bounded, then $\mathscr{W}$ is compact, otherwise technical machinery from [16] is needed to ensure radial unboundedness of $\hat{\varphi}$ (defined later). Obstacle $\mathscr{O}_0$ surrounds internal obstacles $\mathscr{O}_i, i \neq 0$. The free space $\mathscr{F} \triangleq \mathscr{W} \setminus \bigcup_{i \in I_0} \mathscr{O}_i \subset E^n$ is a compact connected C^2 Riemannian manifold with boundary.

Ioannis Filippidis is currently with the Control and Dynamical Systems Dept., California Institute of Technology, Pasadena, CA, 91125, USA and was previously with CSL, NTUA. Kostas J. Kyriakopoulos is with the Control Systems Lab, Dept. of Mechanical Engineering, National Technical University of Athens, 9 Heroon Polytechniou Street, Zografou 15780, Greece. E-mail: `ifilippi@caltech.edu`, `kkyria@mail.ntua.gr`

Definition 1 (Navigation Function (NF) [8]): An NF on $\mathscr{F}$ is defined as a map $\varphi : \mathscr{F} \to [0,1]$ which is

1) C^2-smooth on $\mathscr{F}$ (gradient flow exists & unique);
2) Admissible on $\mathscr{F}$: uniformly maximal on $\partial\mathscr{F}$ (safety);
3) Polar on $\mathscr{F}$: unique minimum at q_d (convergence);
4) Morse (non-degenerate critical points).

An efficient method to construct a closed-form NF for the general case is currently unknown.

Definition 2 (Koditschek-Rimon function (KRf) [8]): The KRf $\varphi_{\mathrm{KR}} \triangleq \gamma_d \left(\gamma_d^k + \beta\right)^{-\frac{1}{k}}$ defines a potential field, where $\beta \triangleq \prod_{i \in I_0} \beta_i$ the obstacle, $\gamma_d(q) \triangleq \|q - q_d\|^2$ the destination attractive field and $k \in \mathbb{N}_{\geq 2}$, $\bar{\beta}_i \triangleq \prod_{j \in I_0 \setminus \{i\}} \beta_j$. For φ_{KR} to acquire NF properties and become a KRNF useful for navigation, we need to increase k above some threshold $k_{\min}$. This is possible for certain geometries explored in [14] and enriched here to focally admissible surfaces. We also prove that for almost all destinations q_d, the KRf is Morse, so generic φ are non-degenerate. This also allows for multiple non-convex principal curvatures κ_{ij} and multiple "small" $\kappa_{ij} > 0$ provided the latter remain unequal within free space, to avoid double degeneracy. In any case, only q_d at the intersection of 2 focal surface sheets can cause double degeneracy, so it is "multiply" non-generic. So, a much wider class of geometries is allowable, especially in multiple dimensions. For the measure zero set of q_d causing degeneracy the Extended NF still applies [14]. We will use φ in place of φ_{KR}.

III. PROBLEM DEFINITION

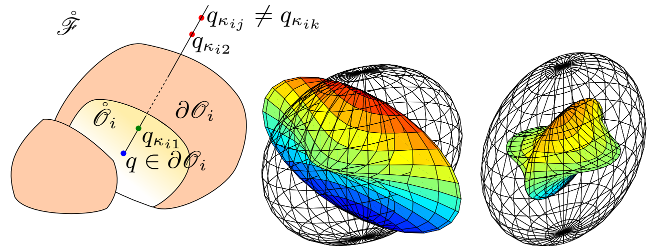
We are interested in determining for which geometries and destinations there exists a KRNF. The motion planning problem concerns finding a path from the initial condition $x(0)$ to the desired destination $q_d \in \mathring{\mathscr{F}}$. Given some world $\mathscr{F}$ and a KRNF φ as in Def. 2, then the holonomic robot with state $x \in E^n$ can be controlled by $\frac{\partial x}{\partial t}(t) = -\nabla_q \varphi(x(t))$ to safely reach q_d in a provably correct way, from almost all initial conditions $x(0)$ (from all is impossible [8]). We prove that for focally admissible geometries the above is possible for almost all destinations $q_d \in \mathscr{F}$, i.e., a KRNF exists for a dense subset of destinations. Here we outline the main steps of the proof in [15].

Morse functions form an open dense subset of smooth functions (Theorem 1.21 [17]), but genericity of Morse KRfs does not follow from this fact. Fixing q_d and considering k as the only parameter, degeneracy can be persistent. In other words, for a given q_d the resulting KRfs may form a section of smooth functions non-transverse to the set of degenerate functions. However, KRfs parameterized by q_d form a section transverse to the set of degenerate functions.

IV. FOCAL ADMISSIBILITY

Focal admissibility comprises of two conditions concerning principal curvatures κ_{ij} [18]. We first need to define the *relative curvature* between the boundary $\partial\mathscr{O}_i$ of an obstacle and the spheres centered at the destination q_d

$$\nu_i(q, \hat{t}_i) \triangleq \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) + \nu_{i4}(q, \hat{t}_i)\right), \tag{1}$$



(a) Focal conditions. (b) Focal sheet 1. (c) Sheet 2.

Fig. 1: Focally admissible ellipsoid: at least one focal surface sheet within the obstacle at every point (ensuring navigability) and no focal sheet intersection within free space $\mathring{\mathscr{F}}$ (for at most simple degeneracy).

where $\nu_{i3} \triangleq \frac{2\hat{\psi}_i}{\|\nabla\gamma_d\|}$, $\theta_i \triangleq (\widehat{\nabla\beta_i, \nabla\gamma_d})$ so $\hat{\psi}_i \triangleq \cos\theta_i$ measures the angle between $\nabla\beta_i$ and $\nabla\gamma_d$, and

$$\nu_{i4}(q, \hat{t}_i) \triangleq -\frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i}{\|\nabla\beta_i(q)\|} = -\kappa_i(q, \hat{t}_i), \tag{2}$$

where κ_i is the normal curvature of the level set of β_i at q and $\hat{t}_i$ is any unit tangent vector of this level set. This measures the curvature differences between the obstacle β_i and the attractive function γ_d. The *principal relative curvatures* ν_{ij} are defined as the eigenvalues of $Q(\hat{t}_i) = \nu_i(q, \hat{t}_i)$. As proved in [15] that the principal directions of the obstacle level set are the eigenvectors of ν_i, with eigenvalues $\nu_{ij}(q) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_{ij}(q)\right)$, where κ_{ij} are the principal curvatures. The locus of principal curvature centers is called *focal surface* and comprises of $n-1$ sheets. For compact surfaces, the focal surface is of measure zero [19].

Points on $\partial\mathscr{O}_i$ where $\nabla\beta_i$ and $\nabla\gamma_d$ are parallel ($\nabla\beta_i \times \nabla\gamma_d = 0$ and $\nabla\beta_i \cdot \nabla\gamma_d > 0$) are called *first-order contact points* and their locus is denoted by C_i^1. On C_i^1 it is $\hat{\psi}_i = \cos\theta_i = 1$. We have assumed that $\partial\mathscr{O}_i$ is compact, so C_i^1 is compact. *Second-order contact points* are first-order contact points q where $\kappa_{ij}(q) = \frac{1}{\|q - q_d\|} \iff R_{ij}(q) = \|q - q_d\|$ for some principal curvature κ_{ij} at q. The second-order contact locus is C_i^2.

The next follows from substituting $\cos\theta_i = 1$ on C_i^1 in $\nu_{ij}(q)$ and comparing this with the condition on C_i^2 that $R_{ij} = \|q - q_d\|$, noting that $\|q - q_d\| = \frac{1}{2}\|\nabla\gamma_d\|$.

Theorem 3 ([15]): Select any $q \in C_i^1$ and consider any center of principal curvature $q_{\kappa_{ij}}$. If $\kappa_{ij}(q) > 0$, then

- $\nu_{ij}(q) > 0$ if and only if (iff) $\|q_d - q\| < \|q_{\kappa_{ij}} - q\|$.
- $\nu_{ij}(q) = 0$ iff $q_d = q_{\kappa_{ij}}$.
- $\nu_{ij}(q) < 0$ iff $\|q_d - q\| > \|q_{\kappa_{ij}} - q\|$.

If $\kappa_{ij}(q) \leq 0$, then $\nu_{ij}(q) > 0$ for all q_d.
This implies that $\nu_{ij} = 0$ only on C_i^2. If $C_i^2 = \emptyset$, then all $\nu_{ij} \neq 0$ on C_i^1.

Assumption 4 (Navigability): Assume that at every $q \in \partial\mathscr{O}_i$, at least one principal radius of curvature $R_{ij}(q) \subseteq \mathscr{O}_i$. Here $R_{ij}(q)$ refers to the *line segment* connecting q to its center of principal curvature $q_{\kappa_{ij}}$. By Assumption 4 it is $\kappa_{ij}(q) > 0$, because otherwise $R_{ij}(q)$ would be outwardly oriented, so its intersection with $\mathring{\mathscr{F}}$ would be nonempty, which is a contradiction. Then, given some q_d, Theorem 3 implies that at each $q \in C_i^1$ at least one $\nu_{ij}(q) < 0$.

Assumption 5 (Limited degeneracy): Assume that the focal surface sheets of $\mathscr{O}_i$ do not intersect in $\mathring{\mathscr{F}}$.
This ensures that any destination $q_d \in \mathring{\mathscr{F}}$ can be on at most one focal surface sheet. So $q_d = q_{\kappa_{ij}}$ for at most one j. Then, given some q_d, Theorem 3 implies that at each $q \in C_i^1$ at most one $\nu_{ij}(q) = 0$. Any obstacle whose boundary $\partial\mathscr{O}_i$ satisfies both Assumptions 4 and 5 is called *focally admissible*, illustrated in Fig. 1a, Fig. 1b, Fig. 1c.

V. FIRST ORDER CONTACT POINTS

In the previous section we defined focally admissible obstacles. We showed that at least one $\nu_{ij}(q) < 0$ and at most one $\nu_{ij}(q) = 0$ on them. So we now know something about the signs of principal relative curvatures ν_{ij} on C_i^1.

In this section, we bring critical points q_c "close" to C_i^1. In the next section, we establish a relation between the signs of ν_{ij} on C_i^1 and the eigenvalues of the Hessian $D^2\hat{\varphi}\left(q_c\right)$ at critical points. Overall, this determines the Hessian eigenvalue signs at critical points, based on properties of C_i^1.

By Prop.2.7 [8] functions φ and $\hat{\varphi} \triangleq \frac{\gamma_d^k}{\beta}$ have the same critical points in $\mathring{\mathscr{F}}$. No critical points exist on $\partial\mathscr{F}$ (Prop.3.3 [8]) and q_d is non-degenerate (Prop.3.2 [8]). So we can study the critical set $\mathscr{C}_{\hat{\varphi}}$ of $\hat{\varphi}$, instead of φ. For convenience, define $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \triangleq \mathscr{B}_i\left(\varepsilon_i\right) \cap \mathscr{C}_{\hat{\varphi}}$. The set $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ contains only those critical points which are in $\mathscr{B}_i\left(\varepsilon_i\right)$.

Consider $\nabla\hat{\varphi} = \frac{k\gamma_d^{k-1}}{\beta}\nabla\gamma_d + \frac{\gamma_d^k}{\beta^2}(-\nabla\beta)$. If $\nabla\beta(q)$ and $\nabla\gamma_d\left(q\right)$ are not parallel, then $\nabla\hat{\varphi}$ cannot be 0. Close to obstacles $\nabla\beta_i$ dominates $\nabla\beta$, so it is the angle between $\nabla\beta_i$ and $\nabla\gamma_d$ that determines if q is a critical point.

Proposition 6: Select any $\lambda \in (0,1)$. Then, there exists an $\bar{\varepsilon}_i > 0$ such that for $\varepsilon_i \in (0, \bar{\varepsilon}_i)$ and $q \in \overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ function $\cos\theta_i$ is well-defined and if $\cos\theta_i(q) < \lambda$, then q is not a critical point.

Proof: By radial lower boundedness and continuity there exists an $\bar{\varepsilon}_{i1} > 0$ such that if $\varepsilon_i < \bar{\varepsilon}_{i1}$, then $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ is compact and $\nabla\beta_i, \nabla\gamma_d \neq 0$ in it [15]. Disjointness of obstacles ensures that there exists some $\bar{\varepsilon}_{i2} > 0$ such that $\bar{\beta}_i > 0$ in $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ for $\varepsilon_i < \bar{\varepsilon}_{i2}$. By hypothesis $\cos\theta_i < \lambda$, so

$$\begin{aligned}&\bar{\beta}_i \nabla\beta_i \cdot \nabla\gamma_d < \lambda\bar{\beta}_i \left\|\nabla\beta_i\right\| \left\|\nabla\gamma_d\right\| \implies \\ &\qquad \nabla\beta \cdot \nabla\gamma_d < \lambda\bar{\beta}_i \left\|\nabla\beta_i\right\| \left\|\nabla\gamma_d\right\| + \beta_i \left\|\nabla\bar{\beta}_i\right\| \left\|\nabla\gamma_d\right\|,\end{aligned}$$

because $\beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d \leq \beta_i \left\|\nabla\bar{\beta}_i\right\| \left\|\nabla\gamma_d\right\|$. Define

$$\bar{\varepsilon}_{i4} \triangleq \frac{(1-\lambda)\min_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\bar{\beta}_i \left\|\nabla\beta_i\right\|\}}{\max_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\left\|\nabla\bar{\beta}_i\right\|\}} \tag{3}$$

where $\bar{\varepsilon}_{i3} \triangleq \min\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i2}\}$. If $\beta_i < \min\{\bar{\varepsilon}_{i3}, \bar{\varepsilon}_{i4}\}$ then $\lambda\bar{\beta}_i \left\|\nabla\beta_i\right\| + \beta_i \left\|\nabla\bar{\beta}_i\right\| < \bar{\beta}_i \left\|\nabla\beta_i\right\| - \beta_i \left\|\nabla\bar{\beta}_i\right\|$. Since $\bar{\beta}_i \left\|\nabla\beta_i\right\| - \beta_i \left\|\nabla\bar{\beta}_i\right\| \leq \left\|\nabla\beta\right\|$, the previous ensures that $\lambda\bar{\beta}_i \left\|\nabla\beta_i\right\| + \beta_i \left\|\nabla\bar{\beta}_i\right\| < \left\|\nabla\beta\right\|$ which is equivalent to $\lambda\bar{\beta}_i \left\|\nabla\beta_i\right\| \left\|\nabla\gamma_d\right\| + \beta_i \left\|\nabla\bar{\beta}_i\right\| \left\|\nabla\gamma_d\right\| < \left\|\nabla\beta\right\| \left\|\nabla\gamma_d\right\|$. This proves the claim that $\nabla\beta \cdot \nabla\gamma_d < \left\|\nabla\beta\right\| \left\|\nabla\gamma_d\right\|$. ■

Definition 7: For some $\lambda \in (0,1)$ and $\varepsilon > 0$, the sets "away" from and "near" to first-order contact points are $L_{i1}(\varepsilon, \lambda) \triangleq \{q \in \mathscr{B}_i\left(\varepsilon\right) | \; \hat{\psi}_i(q) < \lambda\}$, and $L_{i2}(\varepsilon, \lambda) \triangleq \mathscr{B}_i\left(\varepsilon\right) \setminus L_{i1}(\varepsilon, \lambda)$, where $\hat{\psi}_i = \cos\theta_i$.

By Prop. 6 we conclude that for each arbitrary $\lambda \in (0,1)$ there exists an $\bar{\varepsilon}_i > 0$ such that for $0 < \varepsilon_i < \bar{\varepsilon}_i$ there are no critical points in $L_{i1}(\varepsilon_i, \lambda)$. In that case, all critical points of $\mathscr{B}_i\left(\varepsilon_i\right)$ are in $L_{i2}(\varepsilon_i, \lambda)$. So $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq L_{i2}(\varepsilon_i, \lambda)$.

Definition 8 (Uniform Tube): Select arbitrary $\delta > 0$. The *uniform tube* of a set $S \subseteq E^n$ is $T_\delta(S) \triangleq \bigcup_{x \in S} B_r(x)$, where $B_r(x) \triangleq \{y \in E^n | \; \|y - x\| < \delta\}$.
The set "near" contact points $L_{i2}(\varepsilon_i, \lambda)$ can be "shrunk" within the uniform tube $T_\delta(C_i^1)$.

Lemma 9: Select any $\delta > 0$. Then, there exist $\hat{\varepsilon} > 0$ and $\lambda \in (0,1)$ such that for all $\varepsilon_i \in (0, \hat{\varepsilon})$, $L_{i2}(\varepsilon_i, \lambda) \subseteq T_\delta\left(C_i^1\right)$.

Proof: The proof is detailed in [15]. In summary, it is based on the continuity of β_i and $\hat{\psi}_i$ in a regular uniform tube of $\partial\mathscr{O}_i$. If the claim does not hold, then there exists a sequence of points q_j with $\lim_{j\to\infty}\beta_i(q_j) = 0^+$ and $\lim_{j\to\infty}\hat{\psi}_i(q_j) = 1$, which tend to some boundary point $q_0 \notin C_i^1$, so $\hat{\psi}_i(q_0) < 1$. This contradicts $\hat{\psi}_i$ continuity. ■

Combining Prop. 6 and Lemma 9 provides a way to bring critical points "close" to the first-order contact locus C_i^1:

Theorem 10 ([15]): Select any $\delta > 0$. Then, there exists $\bar{\varepsilon}_i > 0$ such that for all $\varepsilon_i \in (0, \bar{\varepsilon}_i), \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq T_\delta\left(C_i^1\right)$.

VI. HESSIAN EIGENVALUES

In Theorem 10 we proved that the critical set $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ "close" to an obstacle is also δ-"close" to the first-order contact locus C_i^1. We now want to choose the δ. This choice is determined in two steps.

Firstly, the principal relative curvature functions ν_{ij} are continuous on the compact C_i^1, so ν_{ij} are uniformly continuous on C_i^1. This yields a $\delta > 0$ such that the numbers of zero and negative ν_{ij} in $T_\delta\left(C_i^1\right)$ follow from what happens on C_i^1. Then, Theorem 10 ensures that for some $\bar{\varepsilon}_i(\delta) > 0$ the numbers of zero and negative ν_{ij} at each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ are known, provided $\varepsilon_i < \bar{\varepsilon}_i$.

Secondly, we want our knowledge about the signs of ν_{ij} to extend to the eigenvalues of $D^2\hat{\varphi}\left(q_c\right)$. This holds for some other upper bound $\bar{\varepsilon}_{i2} > 0$, as has been proved in [14] and is treated in more detail in [15].

Take $\varepsilon_i < \min\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i2}\}$. Then, the number of zero and negative eigenvalues of $D^2\hat{\varphi}\left(q_c\right)$ in $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ are determined from the signs of ν_{ij} over C_i^1. The latter can be specified by Assumptions 4 and 5 which define focal admissibility.

A. Principal Relative Curvature near $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$

Near the first-order contact locus C_i^1, the principal relative curvatures ν_{ij} depend on ν_{ij} on C_i^1. This is a consequence of β_i C^2-continuity and eigenvalue continuity of continuous symmetric matrix functions [20], [21], [22], [23], [18], so:

Theorem 11 ([15]): Select arbitrary $q_d \in \mathring{\mathscr{F}}$. Assume that at each first-order contact point in C_i^1, at least m principal relative curvatures are negative and at most N_0 are zero.

Then, there exists a $\delta > 0$ and an $L > 0$ with the following properties. For each point $q \in T_\delta(C_i^1)$ there exist two numbers $N_n \geq m$ and N_p with $N_n + N_p \geq (n-1) - N_0$ such that for N_n functions $\nu_{ij_1}(q) \leq \nu_{ij_2}(q) \leq \ldots \leq \nu_{ij_{N_n}}(q) \leq -L < 0$, and for N_p functions $0 < L \leq \nu_{ij_{N_n+1}}(q) \leq \ldots \leq \nu_{ij_{N_n+N_p}}(q)$.

A detailed proof can be found in [15].

Using Theorems 10 and 11 leads to

Proposition 12 ([15]): Select any $q_d \in \mathring{\mathscr{F}}$. Assume that at each first-order contact point $q \in C_i^1$, at least m principal relative curvatures $\nu_{ij}(q) < 0$ and at most N_0 are 0.

Then, there exist $L > 0$ and $\bar{\varepsilon}_i > 0$ such that for all $\varepsilon_i \in (0, \bar{\varepsilon}_i)$, for each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ there exist $N_n \geq m$ and N_p with $N_n + N_p \geq (n-1) - N_0$ such that N_n functions $\nu_{ij}(q_c) \leq -L$ and N_p functions $\nu_{ij}(q_c) \geq L$.

B. Hessian quadratic form and relative curvature

The Hessian form $\hat{v}^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{v}$ is related to the relative curvature quadratic form $Q(\hat{t}_i) \triangleq \nu_i(q_c, \hat{t}_i)$. The sign-definiteness of the Hessian form depends on Q. To establish this relation, we need to work on the unit $(n-2)$-sphere within the tangent space $T_q B_i$ of the level set B_i of β_i.

Let the principal directions be $\hat{p}_{ij}(q), I \triangleq \mathbb{N}^*_{\leq n-1}$ and the principal eigensystem $P_i(q) \triangleq \{\hat{p}_{ij}(q)\}_{j \in I}$ (orthonormal but may be non-unique at umbilics) and $\mathscr{P}_i(q) \triangleq \operatorname{span}\{P_i(q)\}$. Let $U\mathscr{P}_i(q) \triangleq \mathscr{P}_i(q) \cap UT_q B_i$ be the unit $(n-2)$-sphere in $T_q B_i$. Given any $L \in \mathbb{R}$ let $I_i^-(q, L) \triangleq \{j \in I |\ \nu_{ij}(q) \leq L\}$, $P_i^-(q, L) \triangleq \{\hat{p}_{ij}\}_{j \in I_i^-(q,L)}$, $\mathscr{P}_i^-(q, L) \triangleq \operatorname{span}\left\{P_i^-(q, L)\right\}$, $R_i^-(q, L) \triangleq \left\{\hat{t}_i \in U\mathscr{P}_i(q) |\ \nu_i(q, \hat{t}_i) \leq L\right\}$, $\mathscr{R}_i^-(q, L) \triangleq \operatorname{cone}\left\{R_i^-(q, L)\right\}$, where the multiplicative span $\operatorname{cone}\{U\} \triangleq \{v \in V | \exists u \in U. \exists \lambda \in \mathbb{R} : v = \lambda u\}$, for a vector space V. The set $\mathscr{R}_i^+(q, L)$ is defined similarly to $\mathscr{R}_i^-(q, L)$, using $\geq L$. Let $\xi_i \triangleq \left(1 - \frac{1}{k}\right) \bar{\beta}_i \|\nabla \beta_i\|^2$.

Based on these, the following was introduced in [14] and proved in detail in [15].

Theorem 13 ([15]): Select arbitrary $L_1 > 0$ and $L_2 > 0$. Assume that there exists an $\bar{\varepsilon}_{i1} > 0$ such that $\nabla \beta_i \neq 0$ and $\nabla \gamma_d \neq 0$ in $\mathscr{B}_i(\varepsilon_i)$ for all $\varepsilon_i \in (0, \bar{\varepsilon}_{i1})$. Further, assume that $\xi_i(q) \geq L_2$ for all $q \in \mathscr{B}_i(\varepsilon_i)$.

Then, there exists an $\bar{\varepsilon}_i > 0$ such that if $\varepsilon_i < \bar{\varepsilon}_i$, then the following hold. At each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ the quadratic form associated with the Hessian $D^2 \hat{\varphi}(q_c)$ is negative definite on $\mathscr{R}_i^-(q_c, -L_1)$ and positive definite on $\bigcup_{t_i \in \mathscr{R}_i^+(q_c, L_1)} \operatorname{span}\{t_i, \hat{r}_i\}$.

C. Hessian eigenvalues and principal relative curvatures

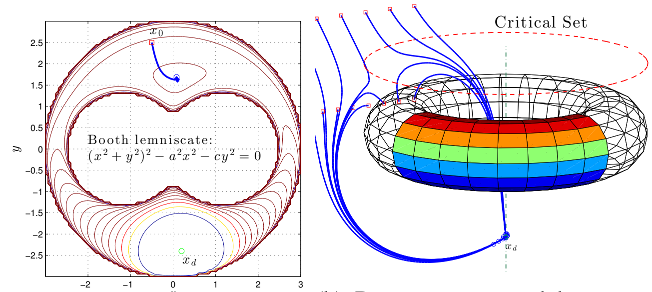
We now relate the signs of $\nu_{ij}(q_c)$ with the eigenvalues of the Hessian $D^2 \hat{\varphi}(q_c)$. This follows as a combination of Prop. 12 and Theorem 13. In order to combine them, a relation is needed between principal directions and the sets $\mathscr{R}_i^-(q_c, -L_1)$ and $\bigcup_{t_i \in \mathscr{R}_i^+(q_c, L_1)} \operatorname{span}\{t_i, \hat{r}_i\}$. This is provided by the following Lemma.

Lemma 14 ([15]): Select an $L \in \mathbb{R}$. It is $\mathscr{P}_i^-(q, L) \subseteq \mathscr{R}_i^-(q, L), \mathscr{P}_i^+(q, L) \subseteq \mathscr{R}_i^+(q, L), \operatorname{span}\left\{\mathscr{P}_i^+(q, L), \hat{r}_i\right\} \subseteq \bigcup_{t_i \in \mathscr{R}_i^+(q, L)} \operatorname{span}\{t_i, \hat{r}_i\}$.

Based on these, we can prove the main result of this work.

Theorem 15 ([15]): Select arbitrary $q_d \in \mathring{\mathscr{F}}$. Assume that at each first-order contact point $q \in C_i^1$, at least m principal relative curvatures $\nu_{ij}(q) < 0$ and at most N_0 are zero.

Then, there exists an $\bar{\varepsilon}_i > 0$ such that for all $\varepsilon_i \in (0, \bar{\varepsilon}_i)$, for each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ there exist $N_n \geq m$ and N_p with $N_n + N_p \geq n - N_0$ such that the Hessian $D^2 \hat{\varphi}(q_c)$ has at least N_n negative and at least N_p positive eigenvalues.



(a) Concave counterexample. (b) Degeneracy caused by q_d on symmetry axis (non-generic).

Fig. 2: Converse theorems.

D. KRNF Existence and Non-degeneracy

For a focally admissible obstacle $m = 1$ and $N_0 = 1$ in Theorem 15. So all critical points "near" obstacles, in $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ have at least $N_n \geq 1$ negative eigenvalues and at most $n - N_n + N_p \leq 1$ zero eigenvalues of $D^2 \hat{\varphi}(q_c)$. Also, Theorem 15 provides us with some upper bounds $\bar{\varepsilon}_0, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_M > 0$. Observe that these $\bar{\varepsilon}_i$ are independent of k (it has not been involved in deriving them, only the obstacle geometry has been involved). Let $\bar{\varepsilon}_{I_0} \triangleq \{\bar{\varepsilon}_i\}_{i \in I_0}$.

Finally, by Proposition 3.4 [8] if we choose

$$k > \hat{k}_{\min}(\bar{\varepsilon}_{I_0}) \triangleq \max_{\mathscr{W}} \{\sqrt{\gamma_d}\} \sum_{i \in I_0} \max_{\mathscr{W}} \{\|\nabla \beta_i\|\} (2\bar{\varepsilon}_i)^{-1},$$

then no critical points (apart from q_d) exist outside any $\mathscr{B}_i(\bar{\varepsilon}_i)$. So $\mathscr{C}_{\hat{\varphi}} = \bigcup_{i \in I_0} \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. This implies that *any* critical point is within some $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. Therefore, all critical points of $\hat{\varphi}$ have at least one negative and at most one zero eigenvalues. So the Koditschek-Rimon function φ is an Extended Navigation Function.

Moreover, since the zero eigenvalue results only for a measure zero set of destinations q_d (the intersection of the focal surfaces with the free space interior), for almost all destinations the Koditschek-Rimon function is a Navigation Function according to the original definition, i.e., Morse.

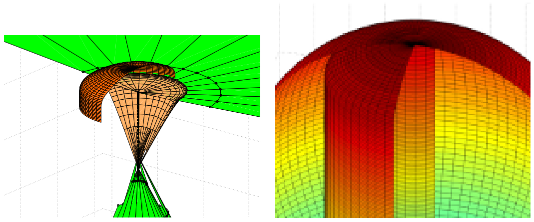
VII. Converse Results

For a fixed description of obstacles β and attraction γ_d, the set of KR functions Φ on $\mathscr{F}$ can be parameterized either as Φ_k with respect to the tuning parameter k, with q_d fixed, or with respect to the destination $q_d \in \mathring{\mathscr{F}} \subset E^n$.

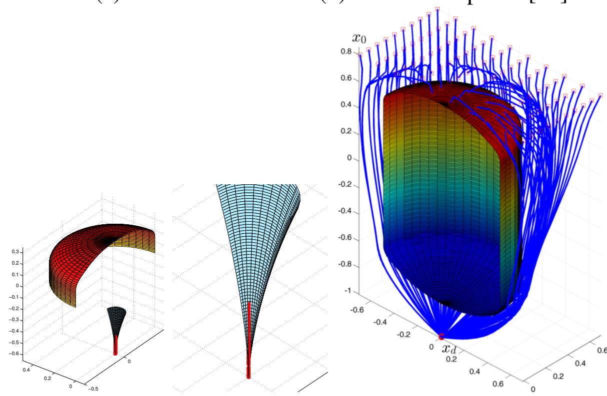
Lemma 16: A Koditschek-Rimon function may be degenerate for all values of the tuning parameter k.

Proof: For any q_d on the axis of symmetry of the 2-torus, there arise second-order contact points on the torus and near them critical points Fig. 2b. By rotational symmetry, a circle of critical points results, which are non-isolated, so degenerate. This degeneracy persists for any value of k. ■

Theorem 17: The set Φ_k is not always transverse to the set of all degenerate functions D, as implied by Lemma 16. In other words, for some (general) world and a given destination q_d, there may not exist a non-degenerate Koditschek-Rimon function. An obstacle boundary point where all principal curvatures are non-positive is here called a *concave* point (completely non-convex).



(a) Focal sheets. (b) Curvature sphere [14].



(c) Focal sheet detail. (d) Trajectories.

Fig. 3: Smooth blending of cylinder with two ellipsoids.

Theorem 18 (KR potential Limitation): Assume that M is a world with some concave point $q_0 \in \partial\mathscr{O}_i$ and the normal space $N_q\partial\mathscr{O}_i \cap \mathring{\mathscr{F}} \neq \emptyset$.

Then, there exist some destination q_d, a $\bar{\varepsilon}_i > 0$ and a $k_{\min} \geq 2$, such that for all $\varepsilon_i \in (0, \bar{\varepsilon}_i)$, for all $k > k_{\min}$, if a critical point arises in $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, then it is a local minimum of the Koditschek-Rimon function φ, Fig. 2a.

Proof: We can choose a $q_d \in N_q\partial\mathscr{O}_i \cap \mathring{\mathscr{F}} \neq \emptyset$, so that $q_0 \in C_i^1$. Point $q_0 \in C_i^1$ is concave, so by Theorem 3 it is $\nu_{ij}(q) > 0$ for all $j \in \mathbb{N}^*_{\leq n-1}$.

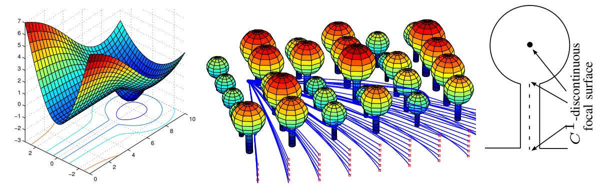
Suppose that there exists a critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. Then, using Prop. 12, Theorem 13, and Lemma 14 it can be proved that $D^2\hat{\varphi}\,(q_c) < 0$, so q_c is a local minimum. ■

VIII. Simulation Results

A. Navigability

The first example compares the relaxed condition of Assumption 4 with the much stronger one in [14], which required that at each $q_0 \in \partial\mathscr{O}_i$ some principal *curvature sphere* $B_{ij}(q_0) \subseteq \overline{\mathscr{O}_i}$, where $B_{ij}(q_0) \triangleq \left\{q \in E^n \middle| \left\| q - q_0 + \frac{\nabla\beta_i(q_0)}{\|\nabla\beta_i(q_0)\|}\rho_{ij}(q_0)\right\| \leq \rho_{ij}(q_0)\right\}$, where $\rho_{ij} \triangleq \frac{1}{2}R_{ij}(q_0)$. Now it suffices that at least one principal radius of curvature (line segment) be within the obstacle.

Consider the smooth blending of a cylinder tapering into two ellipsoids of Fig. 3d, $\beta_i(x,y,z) \triangleq \frac{x^2}{a^2} + \frac{y^2}{b^2} + \sigma_{\delta,L_1}(z)\frac{(z-L_1)^2}{c^2} + \sigma_{\delta,L_2}(-z)\frac{(-z-L_2)^2}{c^2} - 1$, where L_1, L_2 are the distances from $z = 0$ where the transition from cylinder to ellipsoid starts. whose focal surfaces are shown in Fig. 3. The C^∞-smooth switch function used for blending is $\sigma_{\delta,x_0}(x) \triangleq \frac{s\left(\frac{x-x_0}{\delta}\right)}{s\left(\frac{x-x_0}{\delta}\right)+s\left(1-\frac{x-x_0}{\delta}\right)}$, where $s(t) \triangleq e^{-\frac{1}{t}}, t > 0$, $s(t) \triangleq 0, t \leq 0$.



(a) Tree β_i. (b) Trajectories. (c) Foci.

Fig. 4: KRNF trajectories in a forest world.

The blending is a surface of revolution, so one of its focal surface sheets degenerates to a line segment on the z-axis, the intersections of the surface normal with the axis of revolution. Away from the blending, the other focal sheet can be determined by revolving the evolute of an ellipse. Its maximal curvature occurs at the vertex on the minor semi-axes, with curvature radius $\frac{a^2}{b}$. By rotationaly symmetry both of the ellipsoid's top curvature radii are equal to $R_{ij} = \frac{a^2}{b}$ and for $a = 0.5, b = 0.25$ it is $R_{ij} = 1$ (marked in red in Fig. 3c). The focal sheet associated to the other κ_{ij} comes from infinity Fig. 3a, due to the cylinder's flatness transitioning to ellipsoid's finite curvature. This is not contained in the obstacle. If the cylinder is longer than $2R_{ij}$, then it contains the z-axis focal sheet, which suffices for navigation, fulfilling Assumption 4.

On the contrary, [14] would require that the whole curvature balls B_{ij} be inside the obstacle which does not hold, as Fig. 3b shows, because it ignores that placing q_d, e.g., below the obstacle cannot cause a critical point "away" from the z-axis, because $\nabla\beta_i$ is not opposite to $\nabla\gamma_d$ there.

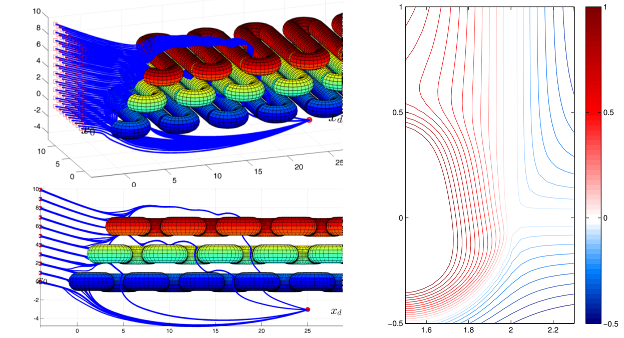
B. Simulations

Using Rvachev functions described in section VIII-C cylinders, spheres and hyperplanes are combined through Boolean operations to represent trees navigated in Fig. 4. The C^1 discontinuities are discussed in section VIII-D. The intersection of ground plane with barks creates non-smooth non-convexities. Nonetheless, the destination can only be on one side of the ground plane, so $\nabla\beta_i \cdot \nabla\gamma_d > 0$ for all the possible destinations (β_i is the ground). In other words $N_q\partial\mathscr{O}_i \cap \mathring{\mathscr{F}} = \emptyset$, so Theorem 18 does not apply.

Pipe networks Fig. 5 and structures with circular cross-sections Fig. 6 can also be navigated with a properly tuned KRNF. More generally, for any channel (canal) surface [24] there exists a KRNF (or extended KRNF for q_d on a focal surface). The C^2 discontinuity between tori and cylinders and C^1 at cylinder intersections are discussed in section VIII-D. This can prove useful for navigating ropes and chains, for example by an underwater vehicle in a harbor.

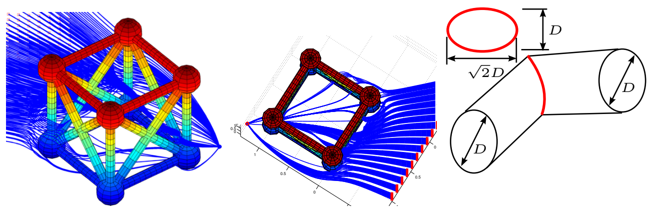
C. Constructive Solid Geometry with Rvachev functions

Complicated obstacles can be constructed using Boolean operations over continuous functions performed by p-Rvachev functions [25] ($+$ for $\wedge$ and $-$ for $\vee$) $R_p(u,v) \triangleq u + v \pm (u^p + v^p)^{\frac{1}{p}}$, where u and v are implicit functions. The R_p function is not differentiable at corner points, but the magnitude of its gradient is bounded from below and above everywhere near the corner point (i.e., in the interior of the



(a) Trajectories. (b) Level sets.

Fig. 5: KRNF trajectories avoiding pipe obstacles.



(a) Trajectories. (b) Intersection.

Fig. 6: Navigating a rig structure.

free space). So despite loss of differentiability at corners, $\beta_i = R_p$ are regular abritrarily close to the corners. This is required to enable a relaxation of assumptions in section II to allow boundary discontinuities. Discontinuities are allowable because critical points arise only in the free space interior, so $\min_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\bar{\beta}_i \left\|\nabla\beta_i\right\|\}$ in Prop. 6 can be replaced by $\inf_{\mathscr{B}_i(\bar{\varepsilon}_{i3})}\{\bar{\beta}_i \left\|\nabla\beta_i\right\|\}$. For example, if c is a cylinder, p a half-space and b a ball, then a tree is $(c \vee p) \wedge b$, Fig. 4a.

D. Allowable discontinuities

Modeling obstacles using Boolean functions in the framework of Constructive Solid Geometry (CSG) can create discontinuities, some of which are admissible here. An obstacle resulting from a Boolean combination of multiple others is considered a single function β_i. So intersecting Boolean primitives are not intersecting different obstacles. However, the intersection must sufficiently preserve the properties critical to navigation, as now outlined.

In the case of C^2 (curvature/Hessian) discontinuity, the principal curvature κ_{ij} associated with $\nu_{ij} < 0$ must be continuous (the others can be discontinuous), as in the pipe example. Also, level sets are always C^∞ in free space and they become firstly non-convex and then convex, so that they can follow the discontinuous level set, as shown in Fig. 5b.

In the case of C^1 discontinuity, the intersecting surfaces must share the tangent direction along which navigation is possible ($\nu_i < 0$). If the intersection manifold is focally admissible, then there exists a level set near enough to the surface, which is focally admissible, which suffices for navigation. This is the case between tree bark and top in Fig. 4c and marginally so for cylinder-cylinder intersection in the rig of Fig. 6. These intersections shown in Fig. 6b form ellipses whose foci are on the obstacle boundary. The destination can only be in the free space interior, so it cannot be on the focal surfaces and so this example is still navigable.

IX. Conclusions and Future Work

It has been proved that Koditschek-Rimon Navigation Functions can work for any obstacle which encompasses at least one focal surface sheet and whose focal surface sheets do not intersect in free space. This improves our previous extension of KRNFs which allowed degeneracy, showing that non-degeneracy is generic, resulting for almost all destinations. Future work concerns tuning mechanisms and the critical set structure in degenerate cases.

References

[1] S. LaValle, *Planning Algorithms*. Cambridge Univ. Press, 2006.
[2] J.-C. Latombe, *Robot Motion Planning*. Kluwer Acad. Pub., 1991.
[3] S. LaValle and J. Kuffner, "Randomized kinodynamic planning," *The Int. J. of Rob. Research*, vol. 20, no. 5, pp. 378–400, 2001.
[4] S. Karaman and E. Frazzoli, "Sampling-based algorithms for optimal motion planning," *Intl. J. Rob. Res.*, vol. 30, no. 7, pp. 846–894, 2011.
[5] J. Canny, *The complexity of robot motion planning*. MIT, 1988.
[6] O. Khatib, "Real-time obstacle avoidance for manipulators and mobile robots," *Int. J. Rob. Research*, vol. 5, no. 1, pp. 90–98, 1986.
[7] E. Rimon and D. E. Koditschek, "Exact robot navigation using cost functions: The case of distinct spherical boundaries in E^n," in *Proc. IEEE Int. Conf. on Rob. and Aut.*, vol. 3, 1988, pp. 1791–1796.
[8] D. Koditschek and E. Rimon, "Robot navigation functions on manifolds with boundary," *Adv. in App. Math.*, vol. 11, pp. 412–442, 1990.
[9] E. Rimon and D. E. Koditschek, "Exact robot navigation using artificial potential functions," *IEEE Trans. on Rob. and Aut.*, vol. 8, no. 5, pp. 501–518, 1992.
[10] ——, "The construction of analytic diffeomorphisms for exact robot navigation on star worlds," *Trans. Am. Math. Soc.*, vol. 327, no. 1, pp. 71–116, 1991.
[11] H. Tanner, S. Loizou, and K. Kyriakopoulos, "Nonholonomic navigation and control of cooperating mobile manipulators," *IEEE Trans. on Rob. and Aut.*, vol. 19, no. 1, pp. 53 – 64, 2003.
[12] G. Lionis, X. Papageorgiou, and K. J. Kyriakopoulos, "Towards locally computable polynomial navigation functions for convex obstacle workspaces," in *IEEE Int. Conf. on Rob. & Aut.*, 2008, pp. 3725 –3730.
[13] S. Loizou, "Navigation functions in topologically complex 3-d workspaces," in *Proc. of the Amer. Control Conf.*, 2012, pp. 4861–4866.
[14] I. Filippidis and K. J. Kyriakopoulos, "Navigation functions for everywhere partially sufficiently curved worlds," in *IEEE Intl. Conf. on Rob. & Autom.*, St. Paul, Minnesota, USA, 2012, pp. 2115–2120.
[15] I. Filippidis, "Navigation functions for focally admissible surfaces," CSL, NTUA, Technical Report, 2012.
[16] I. Filippidis and K. J. Kyriakopoulos, "Adjustable navigation functions for unknown sphere worlds," in *Proc. 50^{th} IEEE Conf. on Dec. & Contr.*, 2011, pp. 4276–4281.
[17] L. Nicolaescu, *An Invitation to Morse Theory*. Springer, 2007.
[18] D. Struik, *Lectures on Classical Differential Geometry*. Dover, 1988.
[19] M. Morse, "Focal sets of regular manifolds M_{n-1} in E_n," *J. Diff. Geometry*, vol. 1, p. 309, 1967.
[20] T. E. Cecil and P. J. Ryan, "The principal curvatures of the monkey saddle," *Amer. Math. Monthly*, vol. 93, no. 5, pp. 380–382, 1986.
[21] K. Nomizu, "Characteristic roots and vectors of a differentiable family of symmetric matrices," *Linear and Multilinear Algebra*, vol. 1, no. 2, pp. 159–162, 1973.
[22] P. J. Ryan, "Homogeneity and some curvature conditions for hypersurfaces," *Tôhoku Math. J. (2)*, vol. 21, pp. 363–388, 1969.
[23] D. H. Singley, "Smoothness theorems for the principal curvatures and principal vectors of a hypersurface," *Rocky Mountain Journal Math.*, vol. 5, pp. 135–144, 1975.
[24] D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination*. Chelsea, 1952.
[25] V. Shapiro and I. Tsukanov, "Implicit functions with guaranteed differential properties," in *Proc. of 5^{th} ACM Symp. on Solid Mod. and App.*, 1999, pp. 258–269.