Technical Report

Navigation Functions for Focally Admissible Surfaces

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Chapter 1

Problem Definition

1.1 Introduction

1.1.1 Motivation

Navigation functions have been introduced by Koditschek and Rimon [1] to solve the motion planning problem for sphere worlds. All the obstacles in a sphere world are spheres. However, obstacles in real world problems are usually more complicated. In order to bridge this gap, [1] uses diffeomorphisms to deform a sphere world into the shape of the real world problem. Diffeomorphisms are difficult to construct and may affect adversely the solution's shape. For these reasons avoiding them is desirable.

Moreover, sphere worlds can model only simply connected obstacles. A multiply connected obstacle like a torus cannot be diffeomorphically mapped to a sphere. This leads us to ask if shapes different than spheres can be used as model obstacles.

The previous thoughts motivate the following question: For which geometries does there exist a Koditschek-Rimon Navigation Function? This question is answered to a large extent in what follows. The proof extends the work of Koditschek and Rimon [1] to the case of general obstacles β_i. General destination functions γ_d are initially considered. However, considerations about symmetry suggest choosing a quadratic γ_d, which is used in the remainder of this work.

1.1.2 Main result

The main result we obtain is a sufficient condition for the existence of a Koditschek-Rimon Navigation Function for a particular geometry. This condition specifies which obstacle geometries can be navigated and which not. It is a geometric condition depending on a comparison of curvatures between two functions. These two functions are the attractive and repulsive fields used in the Koditschek-Rimon function. The curvature of interest is the normal curvature of level sets associated with each of these two functions. If the attractive function has more curved level sets than the repulsive function, then a stable equilibrium (i.e., local minimum) can arise. This minimum can entrap the agent.

If a quadratic γ_d is used, then the relative curvature condition obtains a simpler form. This simpler form suits analysis and has an interesting and intuitive geometric interpretation. As shown later, it is also related to concepts involved in Meusnier's Theorem [2, 3].

The relative curvature condition depends on the choice of destination $q_d \in \mathscr{F}$. Every destination in free space is allowable. For this reason, the condition is equivalent to requesting that the principal curvature spheres[1] $\mathscr{S}_{cij}(q)$ be proper subsets of obstacle sets[2] $\mathscr{O}_i$, i.e. $\mathscr{S}_{cij}(q) \subseteq \mathscr{O}_i \cup \{q\}$.

1.1.3 Necessity of the condition

Moreover, the condition is necessary in the following sense. If all principal directions at a point are not sufficiently curved, then two alternatives exist. The first alternative is when all principal curvatures are non-convex. In this case, there exists a $k_{\min}$, such that for every $k \geq k_{\min}$ if a critical point arises there, it is a local minimum. This precludes use of the same proof procedure. Additionally, it indicates why k tuning alone cannot, in general, make a Kodistchek-Rimon function a Navigation Function in worlds with full non-convexities. For a more detailed discussion, see chapter 8.

On the other hand, the second alternative is when there exists some sufficient principal curvature. Then the NF Hessian has at least one negative eigenvalue. In this case, the critical point is not a local minimum, even if it is degenerate. It is either a saddle, or a local maximum.

Therefore, existence of at least one sufficient principal curvature suffices to ensure that the Navigation Function is Polar (single global minimum at destination). The Polarity property is additional to Analyticity and Admissibility (uniformly maximal on free space boundary), both of which are ensured by construction. Note that Propositions 2.7, 3.2 and 3.3 [1] still hold, allowing us to work with the diffeomorphic $\hat{\varphi}$ in $\mathscr{F} \setminus (\partial\mathscr{F} \cup \{q_d\})$.

Furthermore, the existence of at least one sufficiently curved tangent direction suffices to ensure that at least one sufficient principal curvature exists. As a result, if the sufficient curvature condition holds for *at least* one tangent direction[3], then any critical point there can be turned into a saddle. Intuitively this corresponds to at least one direction of escape.

1.1.4 Degeneracy considerations

But this is not enough to ensure non-degeneracy. Although the result about positive definiteness along $\nabla\beta_i$ of Proposition 3.9 [1] in the case of spheres is extended in section 6.2 to the general case, combining it with negative definiteness along at least one tangential direction is not strong enough. The Hessian may be degenerate.

In the original proof the condition of sufficient curvature is required to hold for all the tangent space. This leads to a direct sum decomposition to two subspaces. In the tangent space negative definiteness is ensured, while in the radial positive definiteness. These suffice by Lemma 3.8 [1] to ensure Hessian non-degeneracy. This is equivalent to local quadratic behavior, so the quadratic form defined by the Hessian can be used to categorize the type of critical point. Considering that the associated quadratic form is continuous in the set $\{\hat{v} \in E^n : \|\hat{v}\| = 1\}$ and assumes both negative and positive values, its minima and maxima (which are eigenvalues of the Hessian) are negative and positive respectively, so the critical

[1]A curvature sphere is defined in (**??**). It is tangent to the obstacle boundary $\partial\mathscr{O}_i$ at a point q. Its center is in the direction of the inward normal of the level set $\beta_i^{-1}(\beta(q))$ at q. The diameter is equal to the radius of normal curvature at q in the selected tangent direction. The principal curvature spheres defined in (**??**) correspond to selecting the principal directions as the tangent directions.

[2]Note that obstacle sets $\mathscr{O}_i$ are defined as open sets, which do not include their boundary $\partial\mathscr{O}_i$.

[3]Tangency is relative to the obstacle level sets implicitly defined by function β_i.

point is a saddle point.

It is worth noting that existence of at least one direction of negative definiteness and one direction of positive definiteness of the Hessian quadratic form suffice to prove that the critical point is a saddle, even if degenerate [4]. This means that in the general case, sufficient curvature for at least one tangent direction ensures all critical points other than the destination are (possibly degenerate) saddles. Degeneracy is the remaining problem.

Degeneracy means that the function's behavior at a critical point is more complicated than quadratic. Continuity of critical points is possible [4], forming critical sets[5]. Critical sets may be smooth and nondegenerate, in which case Morse-Bott theory applies to them, or non-smooth and possibly degenerate, in which case more general theorems are needed. Another possibility is existence of isolated degenerate critical points, such as a monkey saddle[6], which is illustrated in Fig. 1.1.

Then Morse-Bott theory [7, 8] in combination with Thom's Splitting Lemma [9, 10] can be used to examine the dimensionality of stable sets of degenerate saddle points. In the next chapter it will be proved that if the function has at most one degenerate eigenvalue, then these sets are still of Lebesgue measure zero.

Let us return to the generalization that we make in this chapter. The sufficient curvature condition is less strict than working only with spheres. Spheres satisfy this condition. But other obstacle shapes do so as well.

Requiring that this condition holds along all directions of the tangent space leads to a Navigation Function[7]. This way we can allow obstacle shapes which contain the associated curvature sphere, at every boundary point.

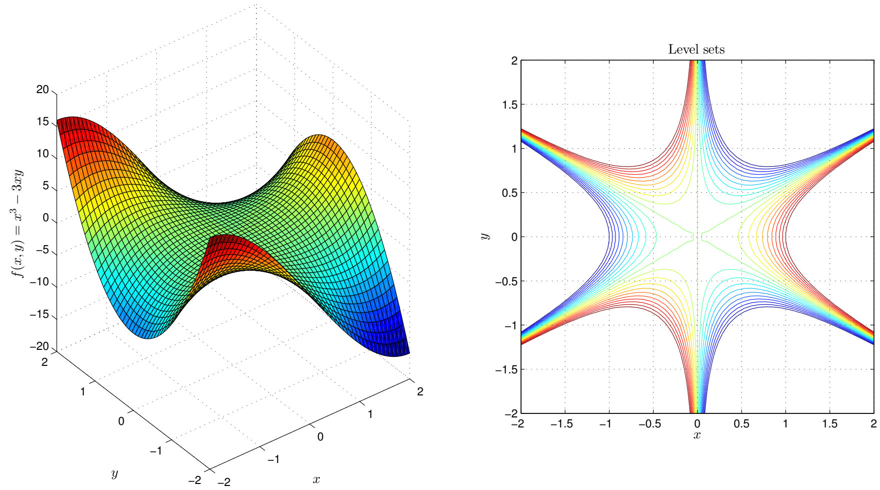
Examples of such shapes are n-dimensional ellipsoids with an upper bound on eccentricity. The example of ellipses is used here as a demonstration of the theoretical results developed. For eccentricities $e < \sqrt{\frac{1}{2}}$ ellipses satisfy the relative curvature condition. But for greater eccentricities they do not. This also provides an example of shapes that are not acceptable.

[4] For example due to symmetry, as in the case of a torus. Note that a torus is topologically different from a sphere. This is an important aspect justifying interest in (degenerate) Navigation Functions. More details regarding the thinking behind the original derivation of [1] can be found in [5].

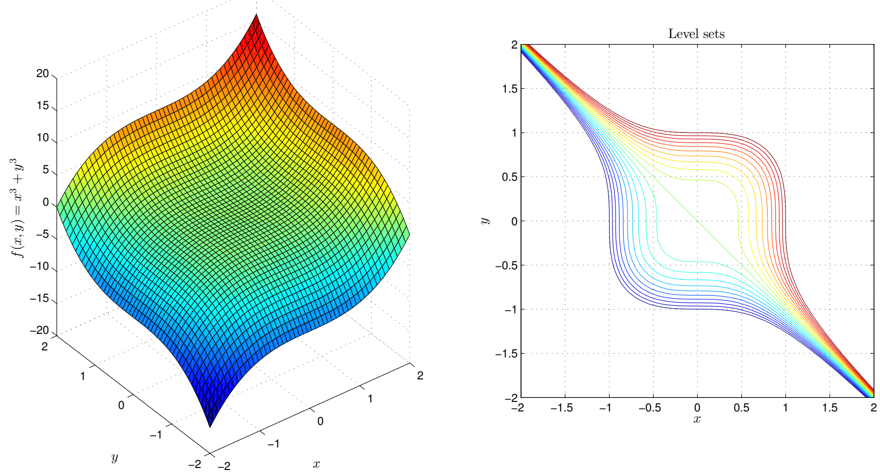
[5] Critical sets are not always submanifolds.

[6] [6], pp. 183-204.

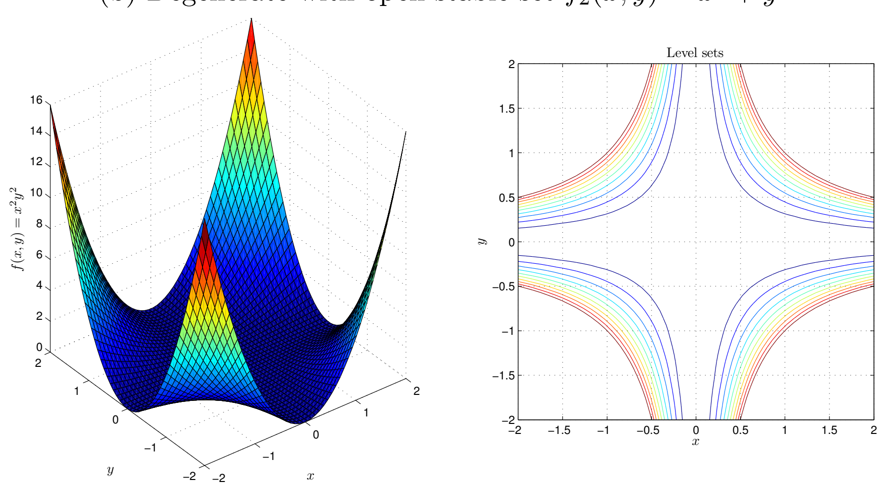
[7] Ensuring non-degeneracy, in addition to polarity, analyticity and admissibility.



(a) Monkey saddle $f_1(x,y) = x^3 - 3xy^2$



(b) Degenerate with open stable set $f_2(x,y) = x^3 + y^3$



(c) Crossed trough $f_3(x,y) = x^2y^2$

Figure 1.1: All of the above scalar functions f_1, f_2, f_3 have a critical point at the origin and their Hessian matrix is *fully* degenerate there $(D^2z)\left([0\,0]^{\mathrm{T}}\right) = 0_{2\times 2} \in \mathbb{R}^{2\times 2}$. In the first and second cases, the origin is a saddle point, whereas in the third one it is a minimum. But we cannot distinguish between saddle point and minimum based on the Hessian eigenvalues, due to full degeneracy. Also, note that although both f_1 and f_2 are saddle points, f_1 has a stable manifold of Lebesgue measure zero, whereas f_2 has open stable sets.

1.2 World definition

1.2.1 Obstacle definitions

Let E^n be n-dimensional Euclidean space. There are $M \in \mathbb{N}$ obstacles in E^n. Each obstacle $\mathscr{O}_i$ is an open (nonempty) subset of E^n, $\mathscr{O}_i \subset E^n$. All configurations in it $q \in \mathscr{O}_i$ are unsafe for the system[8]. Obstacle sets are defined implicitly, using *obstacle functions*

$$\beta_i : E^n \to \mathbb{R}, \quad i \in I_0 \triangleq \mathbb{N}_{\leq M} \tag{1.1}$$

as follows

$$\mathscr{O}_i \triangleq \{q \in E^n |\ \beta_i(q) < 0\}, \quad i \in I_0 \tag{1.2}$$

So functions β_i indicate set membership for points $q \in E^n$. Obstacles are the negative coset preimages of β_i functions.

If $M = 0$ there are no obstacles and the space can be navigated following, for example, the gradient of the quadratic potential function $\|q - q_d\|^2$. This is in agreement with the definition of the Koditschek-Rimon potential function introduced later

$$\varphi : E^n \to \mathbb{R} \quad q \mapsto \varphi(q) \triangleq \frac{\gamma_d(q)}{\beta(q)}, \tag{1.3}$$

because in this case there are no obstacles at all, i.e., $I_0 = \emptyset$. Then, by definition the empty product is $\prod_\emptyset = 1$. So the aggregate obstacle function β becomes

$$\beta(q) = \prod_{i \in I_0 = \emptyset} \beta_i(q) = 1, \tag{1.4}$$

for all $q \in E^n$. So, in this case

$$\varphi(q) = \frac{\gamma_d(q)}{\beta(q)} = \frac{\gamma_d(q)}{1} = \gamma_d(q) \tag{1.5}$$

and if the destination functions γ_d is selected to be the quadratic function $\|q - q_d\|^2$, then $\varphi(q) = \|q - q_d\|^2$.

1.2.2 Obstacle names

We use some naming conventions for obstacles. Obstacle $\mathscr{O}_0$ is called the zero$^{\text{th}}$ obstacle. The $M \in \mathbb{N}$ obstacles

$$\mathscr{O}_i, \quad i \in I_1 \triangleq \mathbb{N}^*_{\leq M} \tag{1.6}$$

are called *internal obstacles*[9]. In the sequel we will refer to both the sets $\mathscr{O}_0$ and their defining functions β_i as "obstacles" interchangeably.

The zero level set of β_i defines the obstacle's boundary

$$\partial\mathscr{O}_i \triangleq \{q \in E^n |\ \beta_i(q) = 0\}, \quad i \in I_0 \tag{1.7}$$

From the range $\mathbb{R}$ of β_i and the definition of an obstacle set and its boundary, it follows that

$$\beta_i(q) > 0, \quad \forall q \notin \mathscr{O}_i \cup \partial\mathscr{O}_i \tag{1.8}$$

[8]The interpretation in motion planning terms is that these configurations cause collisions between the agent and obstacles in its workspace.

[9]This naming convention makes sense only when $\mathscr{O}_0$ is defined and the world $\mathscr{W}$ (as defined later) is a bounded set. For this to hold, some requirements should be placed on $\mathscr{O}_0$.

1.2.3 Obstacle properties

We place some requirements on them. They need to be at least twice continuously differentiable everywhere[10]

$$\beta_i \in C^2(E^n, \mathbb{R}) \tag{1.9}$$

Note that C^2 continuity suffices for the geometric Propositions. Nonetheless, for directly applying Morse-Bott Theory and Thom's Lemma in the next chapter, C^∞ continuity is assumed[11]

Assumption 1. *Functions β_i should be such, that*

1. *Each obstacle set be nonempty $\mathscr{O}_i \neq \emptyset$.*
2. *Each obstacle set $\mathscr{O}_i$ be connected*[a]*.*
3. *Each obstacle boundary $\partial\mathscr{O}_i$ be compact (and nonempty)*[b] *(and connected)*[c]*.*
4. *The obstacle set closures be disjoint*[d]

$$\overline{\mathscr{O}_i} \cap \overline{\mathscr{O}_j} = \emptyset, \quad \forall j \in I_0 \setminus \{i\}, \quad \forall i \in I_0 \tag{1.10}$$

5. *No critical points of β_i arise on any obstacle boundary $\partial\mathscr{O}_i$*

$$\nabla\beta_i(q) \neq 0, \quad \forall q \in \partial\mathscr{O}_i, \quad \forall i \in I_0 \tag{1.11}$$

6. *The complement $E^n \setminus \mathscr{O}_i$ of every obstacle be connected.*

[a] Obstacles are not constrained to be simply connected (which is the only possibility in 2 dimensions), but are allowed to be multiply connected.

[b] Combining this with $\mathscr{O}_i \neq \emptyset$ ensures that obstacle boundaries are nonempty sets

[c] Assume that $\mathscr{O}_i$ is such, that its boundary comprises of multiple components. Then, these isolate subsets of the free space. An assumption later asserts that the free space is connected. This will imply that $\partial\mathscr{O}_i$ is connected.

[d] This means that different obstacles do not "touch" each other.

Some remarks about these requirements follow.

1.2.4 Discussion of obstacle properties

Connectedness places no restriction on obstacles. If an obstacle set is disconnected, then we can always separate it into its connected components. Then, each connected component can be represented by a different obstacle.

Compact obstacle boundary implies that the level sets close to the obstacle are also compact. To prove this, the regularity condition $\nabla\beta_i \neq 0$ is essential. The proof involves a Theorem due to Hirsch, using the gradient flow of β_i to define a diffeomorphism between level sets and invocation of conservation of topological invariants under diffeomorphism (compactness here).

[10] We require C^2 properties everywhere to ensure φ is C^2 everywhere, whereas absence of critical points of its gradient $\nabla\beta_i$ and positive definiteness of its Hessian matrix $D^2\beta_i$ in a neighborhood of $\mathscr{O}_i$ suffices.

[11] Relaxing this is related to the technical details of these Lemmas.

Note that the obstacles themselves need not be bounded. For example, the zero$^{\text{th}}$ obstacle is later required to have a bounded complement, so it is unbounded.

However, the obstacle boundaries must be compact. Since every obstacle boundary is closed by definition (because it is a level set of β_i and this function is C^0), the additional requirement is boundedness[12]. This is an essential property for the proofs. It prevents searching along an infinite boundary for a way to "go around" it. Whether the obstacle set is unbounded does not affect what is happening to the dynamical system in the free space. The dynamical system is affected only by the properties of the interface between the free space $\mathscr{F}$ and the unsafe set (obstacles) $\mathscr{O}$. This interface comprises of the obstacle boundaries only $\partial\mathscr{O} = \bigcup_{i \in I_0} \partial\mathscr{O}_i$.

Disjointness of obstacle closures does not place any restriction on obstacles. It means that in this framework intersecting obstacles are perceived as a single obstacle. In other words, they are treated as a single entity. This ismotivated by the fact that they form a distict topological component of E^n (they are topologically indistinguishable). If two (or more) obstacle sets intersect, the only requirement is that they be described using a single obstacle function.

The **regularity condition** $\nabla\beta_i(q) \neq 0$ on $\partial\mathscr{O}_i$ ensures each obstacle boundary $\partial\mathscr{O}_i$ is a regular $(n-1)$-dimensional (hyper)surface in E^n. Due to the C^2 property of β_i, this regularity condition extends to a neighborhood of the obstacle (i.e., "close to the osbtacle"). This is proved later (and its converse is also true). To prove this, firstly the notion of neighborhood needs to be defined. A neighborhood is defined in terms of β_i being close to 0.

Connentedness of the complement $E^n \setminus \mathscr{O}_i$ ensures that no obstacle can disconnect the free space $\mathscr{F}$ later. This condition is equivalent to requiring that the free space $\mathscr{F}$ be connected. The equivalence follows by observing that if any obstacle has a disconnected complement, then, because all obstacles are pairwise disjoint (and therefore the disconnected "hole" cannot be covered completely by any other obstacle), the free space will be disconnected. So, if the free space is connected, no obstacle complement can be disconnected, hence all complements should be connected.

It is not essential, in the following sense. If $\mathscr{F}$ is disconnected, then by the previous argument there are one or more obstacles whose complement is disconnected. In this case, we can assume that both initial condition and destination are in the same connected component of $\mathscr{F}$. Then, all obstacles which are in this connected component should have connected complements.

1.2.5 Neighborhood Problem

However, the following observation is important. In the proof, it is essential to use the following property about a compact neighborhood of $\partial\mathscr{O}_i$. Each level set within this neighborhood is diffeomorphic to $\partial\mathscr{O}_i$. This follows from the hypothesis that 0 is a regular value of β_i, by application of the Regular Interval Theorem[13] which exploits the flow of $\nabla\beta_i$ as shown in Fig. 1.2.

[12] When a boundary is not bounded, then the set is also unbounded. When a boundary is bounded, then the set may be unbounded. When a set is bounded, then so is its boundary (by the first property). When a set is unbounded, then so is its boundary. Therefore, requiring that the boundary be bounded is less restrictive than requiring that the set be bounded (which is equivalent to requiring that the set closure be bounded).

[13] Theorem 2.2, p.153, [11].

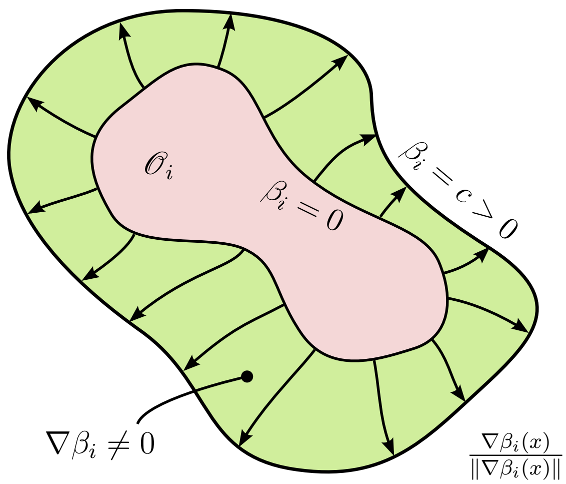


Figure 1.2: On obstacle boundary $\beta_i^{-1}(0) = \partial\mathscr{O}_i$ it is $\nabla\beta_i \neq 0$, so $\partial\mathscr{O}_i$ is a regular surface. Continuity of $\nabla\beta_i$ implies that there exists an ε_i-neighborhood of $\partial\mathscr{O}_i$ in which $\nabla\beta_i \neq 0$. As a result, level sets $\beta^{-1}(c)$ near 0 are diffeomorphic to $\partial\mathscr{O}_i$. The (re-parameterized) gradient flow $\frac{\nabla\beta_i}{\|\nabla\beta_i\|}$ provides such a diffeomorphism.

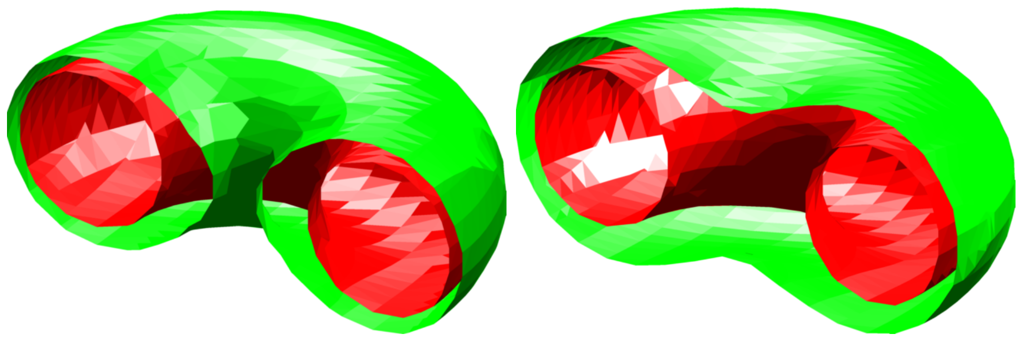


Figure 1.3: Level sets close to the torus are diffeomorphic to it (left). Level sets away from the torus may not be diffeomorphic to it (right). In the case that $\beta_i \to 0^+$ away from the torus, then the green level set on the right will belong to the preimage of $(0, \alpha]$. But it will not be diffeomorphic to the torus.

However, note that we want to refer to the previously mentioned neighborhood as the preimage

$$\{q \in E^n |\, 0 \leq \beta_i(q) \leq \alpha\} \tag{1.12}$$

This is not always correct. The reason is that some diffeomorphic level sets around $\partial\mathscr{O}_i$ have $\beta_i \in (0, \alpha]$. But, there may be other level sets, far away from $\partial\mathscr{O}_i$ and not diffeomorphic to it, which also have $\beta_i \in (0, \alpha]$. This is illustrated in Fig. 1.3.

Depending on how small the values of function β_i become away from $\partial\mathscr{O}_i$, these sets may include any point away from $\partial\mathscr{O}_i$. These sets may be disconnected, as the union of an obstacle neighborhood and another subset of E^n, away from the obstacle boundary. This other subset may even be unbounded, depending on the behavior of function β_i at infinity (consider, for example, the case $\lim_{\|q\|\to\infty} \beta_i(q) = 0$). This is shown in Fig. 1.4.

This is an issue about how to refer to the desired neighborhoods later. There are two ways to avoid it.

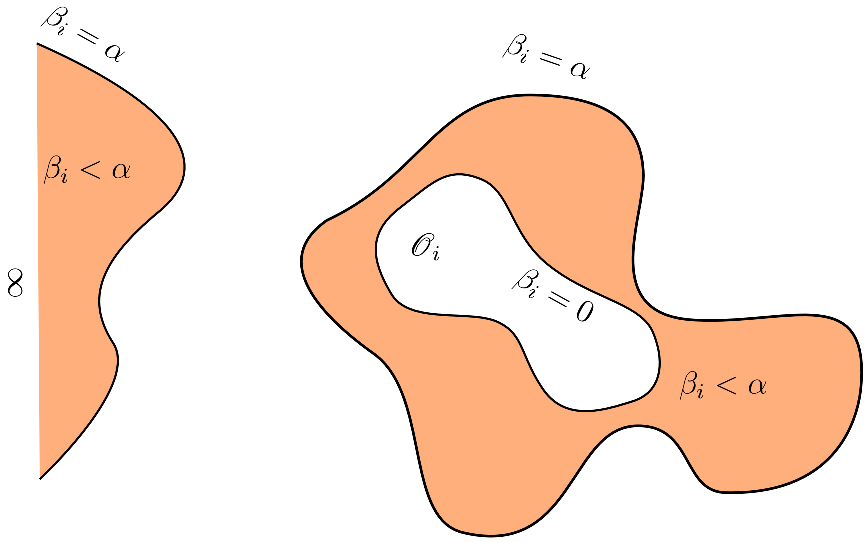


Figure 1.4: Sublevel sets $(0, \alpha)$ may be disconnected. The one unrelated to $\partial\mathscr{O}_i$ may even be unbounded.

The first is to always denote the desired neighborhood by $\mathscr{B}_i$. Then, we make clear that by $\mathscr{B}_i$ we do not refer to the preimage of $(0, \alpha)$. This is not an acceptable approach, because of how the proof works. It "pushes" critical points close to obstacle boundaries and this is only possible if $\beta_i \to 0^+$ only with limit points on $\partial\mathscr{O}_i$ (and not at infinity). This is not possible if β_i is arbitrarily small away from $\partial\mathscr{O}_i$.

The second way is to place further restrictions on β_i. The requirement is that $|\beta_i(q)|$ be *radially lower bounded*. Before using this additional assumption, some preliminary proofs are useful.

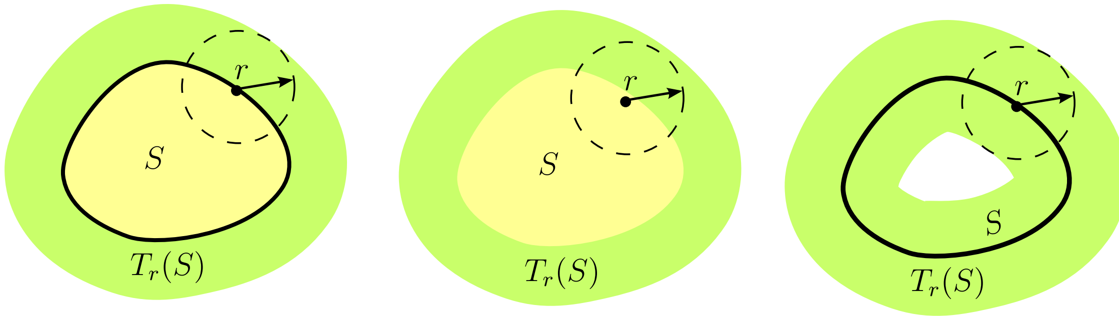


Figure 1.5: The definition of a uniform neighborhood. (a) Set S is compact, so $T_r(S)$ includes S. Moreover, $T_r(S)$ is open, because the balls are open. (b) Set S is open, nonetheless $T_r(S)$ remains the same. (c) Set S is of measure zero and $T_r(S)$ extends around it, like a tube. Essentially it is the set of points closer to S than a given distance r.

1.2.6 Uniform Tubes

Definition 2 (Uniform Tube of a Set). *Select arbitrary $r > 0$. Let $B_r(x) \triangleq \{y \in E^n | \; \|y - x\| < r\}$ be the closed ball of radius r centered on point x. Define the* uniform tube *of a set $S \in E^n$ as the union of balls centered on points of S, i.e.,*

$$T_r(S) \triangleq \bigcup_{x \in S} B_r(x) \tag{1.13}$$

The definition of a uniform tube is shown for three different sets S in Fig. 1.5.

The following proves that a continuous function which is positive on a compact set, is positive also in some uniform tube of that set.

Proposition 3. *Assume set $S \subset E^n$ is compact. Select arbitrary $\delta > 0$. Then, the uniform tube $T_\delta(S)$ is a bounded set.*

Proof. By hypothesis set S is compact in E^n, so it is bounded. There exists an $x_0 \in E^n$ and an $R > 0$ ($R < \infty$), such that $\|y - x_0\| < R$ for any $y \in S$, because S is bounded. For any $z \in T_\delta(S)$ apply the triangular inequality

$$\|z - x_0\| = \|z - y + y - x_0\| \leq \|z - y\| + \|y - x_0\| \leq \|z - y\| + R \tag{1.14}$$

and by definition of the uniform tube

$$\|z - y\| + R \leq \delta + R \tag{1.15}$$

Define $\delta' = \delta + R > 0$. Both $\delta < \infty$ and $R < \infty$, so $\delta' < \infty$. Therefore $\|z - x_0\| < \delta'$ for any $z \in T_\delta(S)$, hence the uniform tube is bounded. □

Corollary 4. *Assume set $S \subset E^n$ is compact. Select arbitrary $\delta > 0$. Then, the uniform tube closure $\bar{T}_\delta(S)$ is a compact set.*

Proof. By the premises and Proposition 3 the uniform tube is a bounded set. Therefore, its closure $\bar{T}_\delta(S) \subset E^n$ is closed by definition and bounded, so it is compact. □

Proposition 5. *Select $\delta > 0$. Let $T_\delta(S)$ be the uniform tube of set S. Then, for every point y on the boundary $\partial T_\delta(S)$ and every point $x \in S$ it is*

$$\delta \leq \|x - y\| \tag{1.16}$$

Proof. Assume the contrary, that for some boundary point $y \in \partial T_\delta(S)$ and some $x \in S$ it is

$$\|x - y\| < \delta \tag{1.17}$$

Then, point y is in the interior of a ball $B_\delta(x)$ centered on $x \in S$. By definition of $T_\delta(S)$ point y is in the interior of $T_r(S)$. Therefore, y cannot be a boundary point of $T_r(S)$. This contradicts the assumption that y is a boundary point of $T_r(S)$. □

The following will prove useful for compact sets. Note that it does not hold for open sets. In that case $\|x - y\| < \delta$ for all boundary points y of a set S.

Proposition 6. *Select $\delta > 0$. Let $T_\delta(S)$ be the uniform tube of a compact set S. If for a point $y \in E^n$ and every point $x \in S$ it is*

$$\delta < \|x - y\| , \tag{1.18}$$

then y is an exterior point of $T_\delta(S)$.

In other words, y is not an interior nor a boundary point of $T_\delta(S)$.

Proof. Assume S is connected. If S has multiple connected components, then the proof is similar, defining a separate continuous function on each connected component. Define the continuous function $f : S \to \mathbb{R}$ as

$$f(x) \triangleq \|x - y\| \tag{1.19}$$

Function f is continuous on the compact set S, so by the Extreme Value Theorem it attains its minimum on S. So there exists an $x_0 \in S$ such that

$$\min_S\{f\} = f(x_0) = \|x_0 - y\| \tag{1.20}$$

By hypothesis $\delta < \|x_0 - y\|$, hence $\delta < \min_S\{f\}$.

Select $\rho = \min_S\{f\} - \delta > 0$. Consider the open ball $B_\rho(y)$ of radius ρ centered at y. Let $z \in B_\rho(y)$. By the triangular inequality, for every $x \in S$

$$\|z - x\| \geq \|y - x\| - \|z - y\| \geq \min_S\{f\} - \|z - y\| > \min_S\{f\} - (\min_S\{f\} - \delta) = \delta \tag{1.21}$$

so z cannot be in the interior of any ball $B_\delta(x)$ for $x \in S$. As a result, z is not contained in $T_\delta(S)$.

This proves that there exists an open neighborhood of y containing points $z \notin T_\delta(S)$. Therefore, point y is an exterior point of $T_\delta(S)$. □

Lemma 7. *Select $\delta > 0$. Let $T_\delta(S)$ be the uniform tube of a compact set S. Then, for every point y on the boundary $\partial T_\delta(S)$ the following holds. There exist points $x \in S$, such that $\|x - y\| = \delta$. Furthermore, for every $x \in S$ for which $\|x - y\| \neq \delta$, it is $\|x - y\| > \delta$.*

Proof. Assume $y \in \partial T_\delta(S)$. By Proposition 5 it is $\delta \leq \|y - x\|$, for every $x \in S$. Suppose that there does not exist $x_0 \in S$ for which $\|y - x_0\| = \delta$. Then, by Proposition 6 y cannot be a boundary point of $T_\delta(S)$. This contradicts the assumption. Therefore, if y is a boundary point of $T_\delta(S)$, there exist $x \in S$ for which $\|y - x\| = \delta$. This completes the proof of this lemma. □

The boundary points of a uniform tube are not in the compact set S.

Corollary 8. *Let $T_\delta(S)$ be the uniform tube of a compact set S. Then $\partial T_\delta(S) \cap S = \emptyset$.*

This can be proved even for non-compact S.

Proposition 9. *Select $\delta > 0$. Let $T_\delta(S)$ be the uniform tube of set S. Then, $\partial T_\delta(S) \cap S = \emptyset$.*

Proof. Suppose the contrary, that there exists some point $x \in \partial T_\delta(S) \cap S$. Then $x \in S$ and by definition of $T_\delta(S)$ the open ball $B_\delta(x) \subseteq T_\delta(S)$. This implies that x is an interior point of $T_\delta(S)$, which contradicts the assumption that $x \in \partial T_\delta(S)$. As a result, $\partial T_\delta(S) \cap S = \emptyset$. □

Proposition 10. *Select arbitrary $\delta_1 > 0$ and $\delta_2 > 0$, such that $\delta_1 < \delta_2$. Then, for any set S, it is $T_{\delta_1}(S) \subseteq T_{\delta_2}(S)$.*

Proof. By definition , if $y \in T_{\delta_1}(S)$, then there exists an $x \in S$ such that $\|x - y\| < \delta_1$. By hypothesis $\delta_1 < \delta_2$, so

$$\|x - y\| < \delta_1 < \delta_2 \implies \|x - y\| < \delta_2 \implies y \in B_{\delta_2}(x) \implies y \in T_{\delta_2}(S). \tag{1.22}$$

Therefore, for every $y \in T_{\delta_1}(S)$, it is also $y \in T_{\delta_2}(S)$. In other words $T_{\delta_1}(S) \subseteq T_{\delta_2}(S)$. □

The next claim follows directly from the definition of a uniform tube.

Proposition 11. *Let $T_\delta(S)$ be a uniform tube of set S. For every point $y \in T_\delta(S)$ there exists at least one point $x \in S$ with $\|x - y\| < \delta$.*

1.2.7 Continuous functions on uniform tubes

The following is illustrated in Fig. 1.6.

Proposition 12. *Assume function f is continuous on E^n and set $S \subset E^n$ is compact. In addition, assume $f > 0$ on S. Then, there exists a $\delta > 0$, such that $f > 0$ on the uniform tube $T_\delta(S)$.*

Proof. By hypothesis, set S is compact, so there exists a closed ball B^n of finite radius containing S in its interior. Then, f is continuous on the compact set B^n. By the Heine-Cantor theorem f is uniformly continuous on B^n.

Moreover, f is continuous on the compact set S. By the Extreme Value theorem, f attains its minimum on S. Denote this minimum by $\min_S\{f\}$. The previous argument ensures that $\min_S\{f\} = f(x)$ for some $x \in S$. Since by hypothesis $f > 0$ on S, it follows that $\min_S\{f\} = f(x) > 0$, because $x \in S$.

Select $\varepsilon = \frac{1}{2}\min_S\{f\} > 0$. Due to uniform continuity on B^n, there exists a $\delta_1 > 0$ with the following property. For all $x, y \in B^n$ with $\|y - x\| < \delta_1$ it is

$$|f(y) - f(x)| < \varepsilon = \frac{1}{2}\min_S\{f\}. \tag{1.23}$$

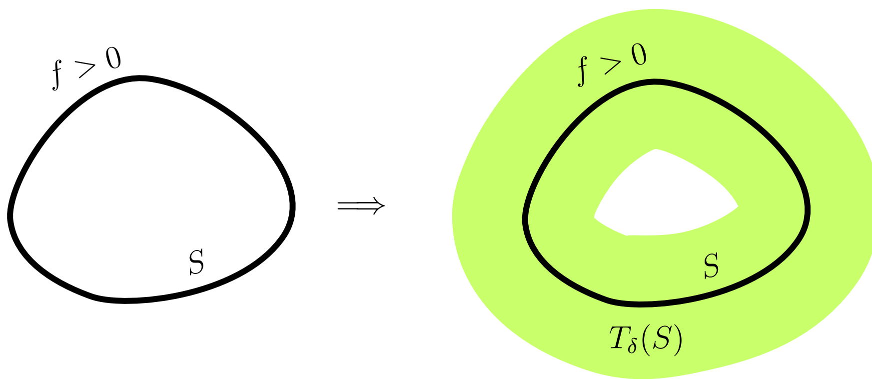


Figure 1.6: If $f > 0$ on compact set S, then there is a uniform tube $T_\delta(S)$ on which $f > 0$.

By definition B^n is compact and contains the compact set S in its interior. Therefore, there exists a $\delta_2 > 0$, such that the uniform tube $T_{\delta_2}(S) \subseteq B^n$. Define $\delta = \min\{\delta_1, \delta_2\} > 0$.

Consider the uniform tube $T_\delta(S)$. Let $y \in T_\delta(S)$. By definition of a uniform tube, there exists an $x \in S$ such that $\|x - y\| < \delta$. Both $x \in S \subset B^n$ and $y \in T_\delta(S) \subseteq B^n$. Then, because f is uniformly continuous on B^n, it follows that

$$|f(y) - f(x)| < \frac{1}{2}\min_S\{f\} \tag{1.24}$$

But $x \in S$, so that

$$\begin{aligned} \min_S\{f\} \leq f(x) \implies \\ \frac{1}{2}\min_S\{f\} \leq f(x) - \frac{1}{2}\min_S\{f\} \end{aligned} \tag{1.25}$$

and because $0 < \frac{1}{2}\min_S\{f\}$ it is $0 < f(x) - \frac{1}{2}\min_S\{f\}$. Therefore

$$-\frac{1}{2}\min_S\{f\} < f(y) - f(x) \implies f(x) - \frac{1}{2}\min_S\{f\} < f(y) \tag{1.26}$$

hence $0 < f(y)$. □

The following ensures that a regular set has a regular uniform tube around it. This is shown in Fig. 1.7, which is similar to Fig. 1.6.

Corollary 13. *Let f be twice continuously differentiable over E^n. Assume that $\nabla f \neq 0$ on a compact set S (f is regular on S). Then, there exists a uniform tube $T_\delta(S)$ in which $\nabla f \neq 0$.*

Proof. Function $\|\nabla f\|$ is continuous on E^n. By hypothesis $\|\nabla f\| \neq 0$ on the compact set S. Then, Proposition 12 implies that there exists a $\delta > 0$, such that $\|\nabla f\| > 0$ on the uniform tube $T_\delta(S)$.

If $\|\nabla f\| > 0$, then $\nabla f \neq 0$. As a result, $\nabla f \neq 0$ on the uniform tube $T_\delta(S)$. □

Proposition 14. *Select arbitrary $\delta > 0$. Assume set S is (path-)connected. Then its uniform tube $T_\delta(S)$ is (path-)connected.*

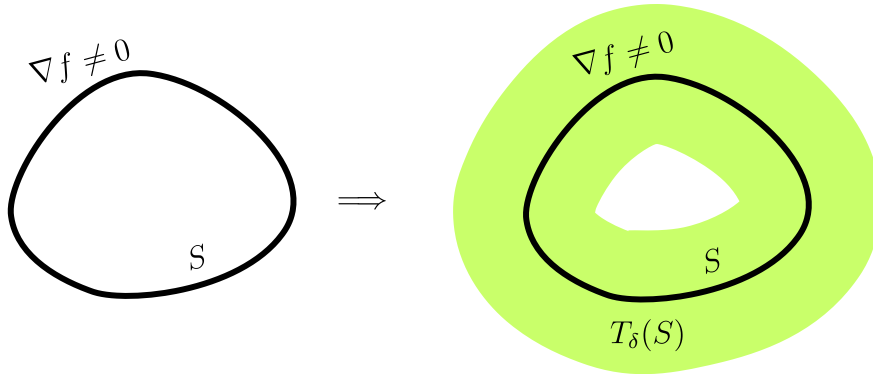


Figure 1.7: If $\nabla f \neq 0$ on a compact set S, then there is a uniform tube $T_\delta(S)$ on which $\nabla f \neq 0$.

Proof. For any two points q_1, q_2 in the uniform tube $T_\delta(S)$, there exist points $q_3, q_4 \in S$ with $\|q_1 - q_3\| < \delta$ and $\|q_2 - q_4\| < \delta$. All points in a δ-ball of q_3 and q_4 are in the uniform tube. Therefore, the linear segments connecting q_1 with q_3 and q_2 with q_4 are subsets of the uniform tube. By hypothesis set S is connected, so there exists a path from q_3 to q_4. This completes the proof that there exists a path from q_1 to q_2, hence that the uniform tube $T_\delta(S)$ is a connected set. $\square$

1.2.8 Neighborhood definition

The following requirement is needed.

Assumption 15. *We assume that each obstacle function β_i is* radially lower bounded. *In other words, we require that there exists some closed ball B^n centered at the origin, such that*[a]

$$\alpha < |\beta_i(q)| \tag{1.27}$$

for all q outside B^n.

[a] It is essential to constraint the absolute value $|\beta_i|$ instead of β_i itself. The reason is that an obstacle like $\mathscr{O}_0$ is unbounded and $\beta_i < 0$ as $\|q\| \to \infty$. Nevertheless, note that the radial lower boundedness contraint is redundant in this case, because $E^n \setminus \mathscr{O}_0$ is compact in this case.

Note that set S corresponds to $\partial\mathscr{O}_i$ in the following.

Remark 16. *The requirement that function β_i be radially lower bounded is not a requirement on the obstacle. It is the compactness of the obstacle's boundary that constitutes a constraint. Nonetheless, observe that radial (positive) lower boundedness implies compactness of the obstacle's boundary. This makes radial lower boundedness a sufficient condition which can replace boundary compactness.*

Theorem 17. *Select arbitrary $\delta > 0$. Assume set S is compact and the continuous function g is radially lower bounded by $a > 0$. Moreover, assume that $g(x) = 0$, for every $x \in S$.*

Then there exists an $\varepsilon > 0$ with the following property. For every $x \in E^n$ with $0 < g(x) < \varepsilon$, it is $x \in T_\delta(S)$.

Proof. We will work with function $f(x) \triangleq |g(x)|$. Function f is continuous, because by hypothesis function g is continuous. By definition $f(x) \geq 0$ for all $x \in E^n$. Also

$$0 < a < g(x) \implies 0 < a < |g(x)| \implies 0 < a < f(x)$$

outside B^n.

Proposition 18. *Set S is in the interior of ball B^n.*

Proof. By hypothesis $f(x) = 0$ for $x \in S$, but by the previous argument $f(y) \neq 0$ for $y \notin B^n$. Hence $S \subseteq B^n$, i.e., set S is within the ball. Moreover $f(x) > 0$ outside B^n. Therefore, if there exists $y \in \partial B^n$ such that $f(y) = 0$, then f cannot be continuous at y. As a result $f \neq 0$ on ∂B^n. This implies that $S \cap \partial B^n = \emptyset$, so set S is contained in the interior of ball B^n. Formally $S \subseteq \mathring{B}^n$. $\square$

Define the uniform tube $C \triangleq T_\delta(S)$ of set S. By Corollary 4 the compactness of S implies that $\bar{T}_\delta(S)$ is compact. Due to compactness, we can always select a large enough B^n to include $\bar{T}_\delta(S)$ in its interior. For this reason, we assume that $T_\delta(S) \subseteq \mathring{B}^n$ in what follows.

Define the subsets

$$A \triangleq E^n \setminus B^n \quad D \triangleq B^n \setminus T_\delta(S) = B^n \setminus C \tag{1.28}$$

By hypothesis $f(x) > a$ for every $x \in A$. By definition $S \subseteq T_\delta(S)$, so by construction

$$x \in D \implies x \notin S \implies f(x) \neq 0 \tag{1.29}$$

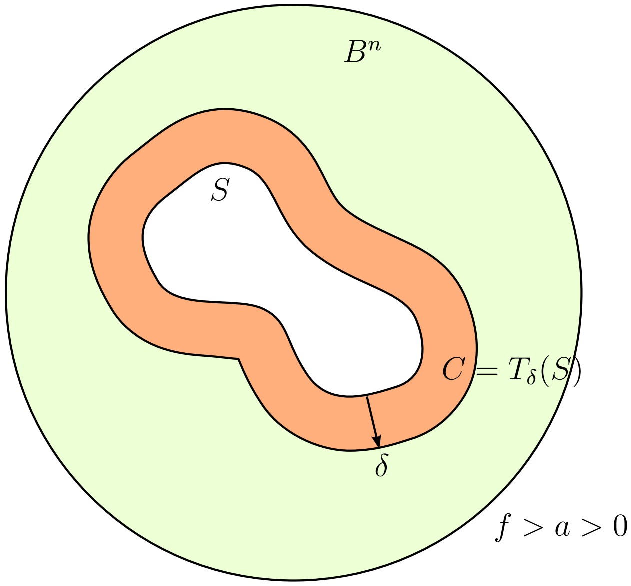


Figure 1.8: The definition of sets A, D.

By definition $f(x) \geq 0$ for all $x \in E^n$. Therefore $f(x) > 0$ over D.

Set D is bounded by construction, because it is a subset of ball B^n. Its complement

$$E^n \setminus D = E^n \setminus (B^n \setminus T_\delta(S)) = E^n \setminus (B^n \setminus C) = (E^n \setminus B^n) \cup C = A \cup C \tag{1.30}$$

is open, because by definition $E^n \setminus B^n$ (as the complement of a closed ball) and $T_\delta(S)$ (as the union of open balls) are open sets, and so is their union. Therefore set D is closed, so it is compact.

By construction set D includes the boundary ∂B^n, so it is non-empty. Function f is continuous on the compact set D, so by the Extreme Value Theorem it attains its infimum for some $x_0 \in D$. In other words, there exists an $x_0 \in D$ such that $f(x_0) = \inf_D\{f\}$. By construction of set D, it is $f(x) > 0$ over D, hence $f(x_0) > 0$. Therefore $\inf_D\{f\} = f(x_0) > 0$. As a result

$$f(x) \geq \inf_D\{f\} = f(x_0) > 0 \tag{1.31}$$

for every $x \in D$. Define

$$\eta \triangleq \min(f(x_0), a) > 0 \tag{1.32}$$

Select $\varepsilon = \eta > 0$.

Construct the two subsets

$$\begin{aligned} C_1 &\triangleq \{x \in C |\ f(x) < \eta\} \\ C_2 &\triangleq \{x \in C |\ \eta \leq f(x)\} \end{aligned} \tag{1.33}$$

By construction $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$.

Select any $x \in E^n$ with $f(x) < \eta$ (note that by definition $f(x) \geq 0$). Then $x \in C_1$. Assume the contrary, that $x \notin C_1$. Then $x \in E^n \setminus C_1 = C_2 \cup D \cup A$. By construction

$$f(y) \geq \inf_D\{f\} = f(x_0) \geq \eta \tag{1.34}$$

for all $y \in D$,

$$f(y) \geq \eta \tag{1.35}$$

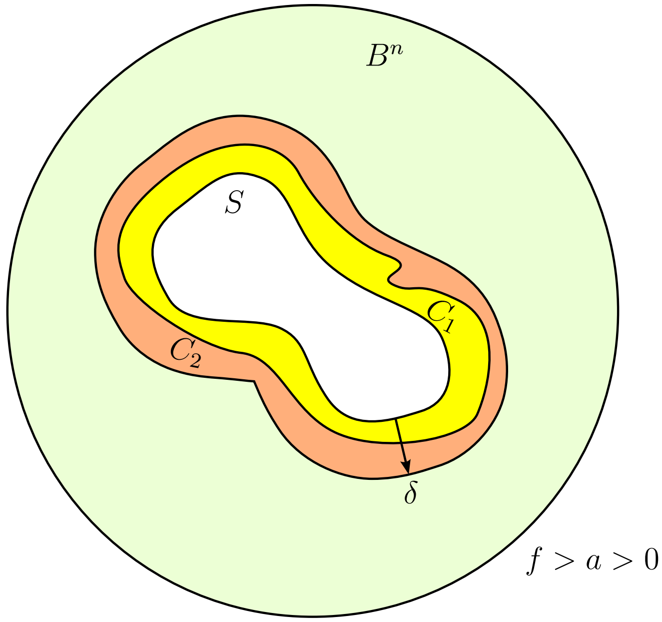


Figure 1.9: The two subsets of the uniform tube C_1 and C_2.

for all $y \in C_2$, and

$$f(y) \geq a \geq \eta \tag{1.36}$$

for every $y \in A$.

Therefore $x \notin D \wedge x \notin C_2 \wedge x \notin A$, which implies that $x \notin D \cup C_2 \cup A$. This contradicts the assumption. As a result $x \in C_1$. By construction $C_1 \subseteq C = T_\delta(S)$.

In summary, for every $x \in E^n$ with $0 \leq f(x) < \varepsilon$ it is $x \in T_\delta(S)$.

□

It is important to emphasize here that this is a restriction on the functions β_i which *describe* the problem and *not* on the obstacle sets. It results from the way our proof works.

Corollary 19. *Select arbitrary $\delta > 0$. Assume set S is compact and the continuous function β_i is radially lower bounded by $a > 0$. Moreover, assume that $\beta_i(x) = 0$, for every $x \in S$.*

Then there exists an $\bar{\varepsilon} > 0$ with the following property. For every $\varepsilon > 0$ with $\varepsilon < \bar{\varepsilon}$ it is

$$\mathscr{B}_i(\varepsilon) \subseteq T_\delta(S) \tag{1.37}$$

Proof. By the hypothesis and Theorem 17 there exists a $\bar{\varepsilon}$ such that

$$0 < \beta_i(x) < \bar{\varepsilon} \implies x \in T_\delta(S),$$

By definition

$$\mathscr{B}_i(\bar{\varepsilon}) = \{x \in E^n | \, 0 < \beta_i(x) < \bar{\varepsilon}\}$$

so

$$x \in \mathscr{B}_i(\bar{\varepsilon}) \implies 0 < \beta_i(x) < \bar{\varepsilon} \implies x \in T_\delta(S),$$

i.e.,

$$\mathscr{B}_i(\bar{\varepsilon}) \subseteq T_\delta(S).$$

The same holds for all $\varepsilon < \bar{\varepsilon}$, because then $\mathscr{B}_i(\varepsilon) \subset \mathscr{B}_i(\bar{\varepsilon})$. □

Proposition 20. *Select arbitrary $\varepsilon > 0$. Assume that function $f : E^n \to \mathbb{R}$ is continuous and $f(x) = 0$ for every $x \in S$, where S is a compact set.*

Then, there exists a $\delta > 0$ such that $|f(x)| < \varepsilon$ for all $x \in T_\delta(S)$.

Proof. By hypothesis S is compact, so by Corollary 4 every uniform tube closure $\bar{T}_\delta(S)$ is compact. Select arbitrary $\delta_1 > 0$. Function f is continuous on the compact set $|T|_{\delta_1}(S)$, hence uniformly continuous on it. This ensures that there exists a $\delta_2 > 0$ such that all $x, y \in |T|_{\delta_1}(S)$ with

$$\|x - y\| < \delta_2$$

have

$$|f(x) - f(y)| < \varepsilon.$$

Define $\delta \triangleq \min\{\delta_1, \delta_2\}$. By Proposition 10 $\delta \leq \delta_1$ implies $T_\delta(S) \subseteq T_{\delta_1}(S) \subset \bar{T}_{\delta_1}(S)$. So the same uniform continuity under distance δ_2 applies in $T_\delta(S)$.

By Proposition 11, for any $x \in T_\delta(S)$ there exists a $y \in S$ with $\|x - y\| < \delta$. So

$$\|x - y\| < \delta_2,$$

because by definition $\delta \leq \delta_2$. This leads to

$$\|x - y\| < \delta_2 \implies |f(x) - f(y)| < \varepsilon.$$

It is $f(y) = 0$, because $y \in S$, hence

$$|f(x)| < \varepsilon. \tag{1.38}$$

□

Corollary 21. *Select arbitrary $\varepsilon > 0$. Then there exists a $\delta > 0$, such that[a] $T_\delta(S) \cap \mathring{\mathscr{F}} \subseteq \mathscr{B}_i(\varepsilon)$.*

[a] It holds that $T_\delta(S) \subseteq \{x \in E^n | \; -\varepsilon < \beta_i(x) < \varepsilon\}$. Therefore, the claim holds for the union $\{x \in E^n | \; -\varepsilon < \beta_i(x) < 0\} \cup \{0\} \cup \mathscr{B}_i(\varepsilon)$. Because by definition of $\mathring{\mathscr{F}}$ it is $\mathring{\mathscr{F}} \subseteq \{x \in E^n | \; 0 < \beta_i(x)\}$, so $T_\delta(S) \cap \mathring{\mathscr{F}} \subseteq (\{x \in E^n | \; -\varepsilon < \beta_i(x) < 0\} \cup \{0\} \cup \mathscr{B}_i(\varepsilon)) \cap \{x \in E^n | \; 0 < \beta_i(x)\} = \mathscr{B}_i(\varepsilon)$.

Corollary 22. *Select arbitrary $\delta > 0$. Assume the continuous function β_i is radially lower bounded by $a > 0$. Moreover, assume that $\beta_i(x) = 0$ for every $x \in S$.*

Then, there exists an $\varepsilon > 0$ and a $\delta_1 > 0$ (where $\delta_1 < \delta$) such that

$$T_{\delta_1}(S) \cap \mathscr{F} \subseteq \mathscr{B}_i(\varepsilon) \subseteq T_\delta(S). \tag{1.39}$$

Proof. By Corollary 19 there exists an $\bar{\varepsilon} > 0$ such that

$$\mathscr{B}_i(\varepsilon') \subseteq T_\delta(S)$$

for every $0 < \varepsilon' < \bar{\varepsilon}$.

Select some $\varepsilon > 0$ such that $\varepsilon < \bar{\varepsilon}$. By Corollary 21 there exists a $\delta_1 > 0$ such that

$$T_{\delta_1}(S) \cap \mathscr{F} \subseteq \mathscr{B}_i(\varepsilon)$$

Since $\varepsilon < \bar{\varepsilon}$ it follows that

$$\mathscr{B}_i(\varepsilon) \subseteq T_\delta(S)$$

therefore

$$T_{\delta_1}(S) \cap \mathscr{F} \subseteq \mathscr{B}_i(\varepsilon) \subseteq T_\delta(S) \tag{1.40}$$

The previous implies $T_{\delta_1}(S) \subseteq T_\delta(S)$, so by Proposition 10 it is[14] $\delta_1 < \delta$. □

Corollary 23. *Let $f : E^n \to \mathbb{R}$ be twice continuously differentiable. Assume that $\nabla f \neq 0$ on the compact set S.*

Then, there exists a $\bar{\varepsilon} > 0$ with the following property. For all $\varepsilon > 0$ with $\varepsilon < \bar{\varepsilon}$, it is $\nabla f \neq 0$ in $\mathscr{B}_i(\varepsilon)$.

Proof. By Corollary 13 there exists some $\delta > 0$ such that $\nabla f \neq 0$ in the uniform tube $T_\delta(S)$. By Corollary 19 there exists an $\bar{\varepsilon} > 0$ such that

$$\mathscr{B}_i(\varepsilon) \subseteq T_\delta(S) \tag{1.41}$$

for every $0 < \varepsilon < \bar{\varepsilon}$. As a result $\nabla f \neq 0$ in $\mathscr{B}_i(\varepsilon)$. □

Proposition 24. *There exists an $\bar{\varepsilon}_i > 0$, such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ the following hold. Set $\overline{\mathscr{B}_i(\varepsilon_i)}$ is compact. Functions β_i and γ_d are regular in $\overline{\mathscr{B}_i(\varepsilon_i)}$.*

Proof. By Corollary 23 there exists an $\bar{\varepsilon}_{i1} > 0$ such that for all $\varepsilon_i' > 0$ with $\varepsilon_i' < \bar{\varepsilon}_{i1}$ both[15]

$$\mathscr{B}_i(\varepsilon_i') \subseteq T_\delta(\partial \mathscr{O}_i). \tag{1.42}$$

and

$$\nabla \beta_i(q) \neq 0 \tag{1.43}$$

for all $q \in \mathscr{B}_i(\varepsilon_i')$. The set $\partial\mathscr{O}_i$ is compact, so by Corollary 4 the uniform tube $T_\delta(\partial\mathscr{O}_i)$ is bounded and hence $\mathscr{B}_i(\varepsilon_i')$ is bounded[16], so its closure $\overline{\mathscr{B}_i(\varepsilon_i')}$ is compact.

Let $\bar{\varepsilon}_{i2} \triangleq \beta_i(q_d)$. Define

$$\bar{\varepsilon}_i \triangleq \frac{1}{2}\min\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i2}\} > 0. \tag{1.44}$$

Every $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ enjoys the following properties. Every point $q \in \overline{\mathscr{B}_i(\varepsilon_i)}$ has

$$\beta_i(q) \leq \varepsilon_i < \frac{1}{2}\bar{\varepsilon}_{i1}.$$

So it is in the set $\mathscr{B}_i(\varepsilon_i')$, where $\varepsilon_i' = \frac{\bar{\varepsilon}_{i1}}{2} < \bar{\varepsilon}_{i1}$. Therefore, $\nabla\beta_i(q) \neq 0$ for all $q \in \overline{\mathscr{B}_i(\varepsilon_i)}$. Also, $\beta_i(q) \leq \varepsilon < \bar{\varepsilon}_{i2}$ for every $q \in \overline{\mathscr{B}_i(\varepsilon_i)}$. Therefore, $q \neq q_d$, so $\nabla\gamma_d \neq 0$ in $\overline{\mathscr{B}_i(\varepsilon_i)}$. □

The regularity of β_i and γ_d ensures later that function $\hat{\psi}_i$ is well-defined in $\overline{\mathscr{B}_i(\varepsilon_i)}$. Therefore, the sets L_{i1} and L_{i2} are well-defined subsets of $\mathscr{B}_i(\varepsilon_i)$.

---

[14]To prove assume the opposite, which leads to a contradiction.

[15]This originates from Corollary 19. So we first choose an appropriate $\delta > 0$ to get a regular uniform tube. Then, we choose sufficiently small ε_i' to ensure $\mathscr{B}_i(\varepsilon_i')$ is in that uniform tube. In this way we make sure $\mathscr{B}_i(\varepsilon_i')$ is both bounded and that β_i is regular everywhere in it.

[16]Note that the assumption of radial lower boundedness of β_i and its continuity (and the proof of Corollary 23) ensure that the range of function β_i includes the interval $[0, a]$. This is important to ensure that if $\varepsilon_i' \in [0, a]$, then the boundary of $\mathscr{B}_i(\varepsilon_i')$ comprises of the level sets $0, \varepsilon_i'$ and that points with $\beta_i \in (0, \varepsilon_i')$ are interior points of $\overline{\mathscr{B}_i(\varepsilon_i')}$. These can be proved by using the continuity of β_i.

1.2.9 Bundle of level sets

The following ensures that a regular submanifold S has a bundle of level sets near it which are *contained* within a uniform tube $T_\delta(S)$. This is another proof (using the theory of differential equations) for what has already been proved in the previous section. Nonetheless it is interesting. In addition, it also proves that the level sets in the neighborhood are diffeomorphic to S.

Theorem 25. *Select arbitrary $\delta > 0$.*

Assume function f is twice continuously differentiable over E^n and 0 is a regular value of f. Assume $S = f^{-1}(\{0\})$ is a compact subset of E^n.

Then, there exists an $\bar{\varepsilon} > 0$ and a $\delta_1 > 0$, with the following properties. For any $x \in T_{\delta_1}(S)$ with

$$0 < f(x) < \bar{\varepsilon}, \tag{1.45}$$

there exists a $y \in S$, such that

$$\|x - y\| < \delta. \tag{1.46}$$

Moreover, the set $\{x \in T_{\delta_1}(S) |\ f(x) = \varepsilon\}$, where $0 < \varepsilon < \bar{\varepsilon}$, is diffeomorphic to S. In other words $\mathscr{B}_i(\bar{\varepsilon}) \cap T_{\delta_1}(S) \subseteq T_\delta(S)$.

In other words, any level set $\beta^{-1}(\varepsilon_0)$ with $0 < \varepsilon_0 < \varepsilon$ is contained in $T_\delta(S) = T_\delta(f^{-1}(\{0\}))$.

Proof. By Corollary 13 and the hypothesis ($f \in C^2(E^n)$ and compact S), there exists some $\delta > 0$ such that $\nabla f \neq 0$ in the uniform tube $T_\delta(S)$. Let $W \triangleq T_{\delta_1}(S)$. Within the uniform tube W we can define the map

$$\begin{aligned} g &: W \to E^n \\ g(x) &\mapsto \frac{\nabla f(x)}{\|\nabla f(x)\|^2}. \end{aligned} \tag{1.47}$$

Map g is continuously differentiable in W, because function f is twice continuously differentiable, $\|z\|$ is continuously differentiable for $z \neq 0_{n\times 1}$ and $\nabla f \neq 0$.

By definition of its uniform tube $S \subseteq T_{\delta_1} = W$. By definition the uniform tube W is open as a union of open balls and S is in the interior of W. The ordinary differential equation

$$\frac{dx}{dt}(t) = g\left(x(t)\right) = \frac{\nabla f\left(x(t)\right)}{\|\nabla f\left(x(t)\right)\|^2} \tag{1.48}$$

with initial condition $x(0) \in S \subseteq W$ is well-defined on the opens et W. Then, by the Fundamental Theorem of Ordinary Differential Equations (Theorem 1, pp.162–163 [12]), there exist some $a(x(0)) < 0$ and $b(x(0)) > 0$, such that the solution $x(t) \in W$ of (1.48) exists for all $t \in (a(x(0)), b(x(0)))$ and the interval $(a(x(0)), b(x(0)))$ is the maximal interval of existence of a solution.

Function ∇f is continuously differentiable over E^n. Within U we can define the map

$$g(x) \triangleq \frac{\nabla f(x)}{\|\nabla f(x)\|^2} \tag{1.49}$$

because $\nabla f \neq 0$ over U. Map g is continuously differentiable in U.

By hypothesis, the set S is contained in the interior $\mathring{U}$, so there is an open neighborhood W of $x(0)$ in which map g is C^1.

By Theorem 2.1, p.94 [?] the end-point $b(x(0))$ is a lower semi-continuous function of $x(0) \in S$. By Theorem 3, p.361 [13] the lower semi-continuous function b on the compact[17] set S attains its infimum on S. In other words, there exists an $x_0 \in S$ such that $b(x) \geq \min_S\{b\} = b(x_0) > 0$.

Observe that

$$\frac{d}{dt} f\left(x(t)\right) = \nabla_x f\left(x(t)\right) \cdot \frac{dx}{dt}(t) = \nabla_x f\left(x(t)\right) \cdot \frac{\nabla_x f\left(x(t)\right)}{\left\|\nabla_x f\left(x(t)\right)\right\|^2} = \frac{\left\|\nabla_x f\left(x(t)\right)^2\right\|}{\left\|\nabla_x f\left(x(t)\right)\right\|^2} = 1 \quad (1.50)$$

Therefore

$$f\left(x(t)\right) - f\left(x(0)\right) = \int_0^t \frac{d}{dt} f\left(x(\tau)\right) = \int_0^t 1 d\tau = t \quad (1.51)$$

As a result, the positive flow from $x(0) \in S$ has $f\left(x(t)\right) > 0$ for $t > 0$ and $f\left(x(t)\right) < 0$ for $t < 0$.

Proposition 26. *Every trajectory in W tends to its boundary in finite time.*

Proof. Suppose that there exists some initial condition $x_0 \in W$ with $b(x(0)) =_\infty$. Then, the solution $x(t)$ is defined on $[0, +\infty)$ which implies that $f\left(x(t)\right) = t$ is unbounded for $x(t) \in W$.

However, by Corollary 4 the closure $\bar{W} = \bar{T}_{\delta_1}(S)$ is compact, because by hypothesis S is compact. This implies that the continuous function f is bounded above on S. This contradicts $f\left(x(t)\right) = t$ on $[0, +\infty)$. So the upper end-point $b(x(0)) < +\infty$ for every initial condition $x(0) \in W$.

By the Theorem of p.171 [12] for every compact set $K \subset W$ there is some $t \in (a(x(0)), b(x(0)))$ with $x(t) \notin K$. Therefore, as $t \to b^-$, either

- $x(t)$ tends to the boundary of W, or
- $|x(t)|$ tends to $+\infty$.

(See the Remarks in p.172 of [12].)

Set $\bar{W}$ is compact, so $|x(t)|$ is bounded. Therefore $x(t)$ tends to the boundary of W as $t \to b^- < +\infty$ (i.e., in finite time). □

Proposition 27. *The boundary of W is not empty. Furthermore, it comprises of at least two disjoint components:*

1. *one in which $f > 0$, and*
2. *one in which $f < 0$.*

Proof. Suppose ∂W is empty. Then $\mathring{W} = \bar{W}$ is compact. But for every solution $x(t)$ there exists a $t > 0$, such that $x(t)$ escapes any compact set in W. This is a contradiction.

Another way to prove this claim is to assume that no ∂W with $f > 0$ exists. Then $f^{-1}\left([0, +\infty)\right) \cap W$ is compact, because it includes its boundary $f^{-1}(\{0\}) = S$. Then $g(x) = \dot{x}$ is a gradient system on a compact set on which the flow is transeverse to the boundary. As a result, by Theorem 4, p.203 [12], the flow tends to some equilibrium in W.

At an equilibrium of g it is $g(x) = 0$, which is not possible in W, because $\nabla f \neq 0$ in W. This is a contradiction. The case $f < 0$ is similar. □

[17] Quasi-compact in the terminology of Bourbaki.

The boundary ∂W is non-empty and comprises of two subsets $\partial W_1, \partial W_2$ with $f > 0$ and $f < 0$, respectively. Set $\bar{W}$ is compact, hence ∂W_1 is compact. Moreover, by Corollary 8

$$\partial T_\delta(S) \cap S = \emptyset.$$

Therefore $f > 0$ on $\partial T_\delta(S)$ because $S = f^{-1}(\{0\})$ hence $f \neq 0$ on $E^n \setminus S$. Function f is continuous on the compact set ∂W_1 so it attains its infimum for some $x_0 \in \partial W_1$

$$f(x) \geq \inf_{\partial W_1}\{f\} = f(x_0) \tag{1.52}$$

Define $\eta \triangleq f(x_0) > 0$.

We can now partition W into four sets

$$\begin{aligned} C_1 &\triangleq \{x \in W |\ 0 < f(x) < \eta\} \\ C_2 &\triangleq \{x \in W |\ \eta \leq f(x)\} \\ S &\triangleq \{x \in W |\ f(x) = 0\} \\ C_3 &\triangleq \{x \in W |\ f(x) < 0\} \end{aligned} \tag{1.53}$$

If the initial condition $y(0) \in C_1$, then the negative flow from $y(0)$ has strictly decreasing $f(y(t)) - f(y(0)) = t$, so $f(y(t)) < f(y(0))$ for all $t < 0$. By the previous argument, this flow tends to ∂W. But $f(y(t)) < f(y(0)) < \eta$ and $f(\partial W_1) \geq \eta$. Hence, this flow can only tend to $\partial W_@$ with $f(\partial W_2) < 0$.

By continuity of f on the connected space $y(t), t \in (a, 0]$ and Theorem, **??**, [**?**], there exists $t_1 \in (a, 0]$ such that $f(y(t_1)) = 0$ hence $y(t_1) \in S$. As a result, there exists a solution of (1.48) with initial condition $x(0) = y(t_1) \in S$, such that $x(t) = y(0)$ for some $t > 0$. Therefore $\underline{b} \geq \eta$.

If $x(0) \in S$ and $t < \frac{\eta}{2}$, then solutions to $\dot{x}(t) = g\left(x(t)\right)$ exist and

$$\begin{aligned} \|x(t) - x(0)\| &\leq \left\| \int_0^t \frac{\nabla f(x(s))}{\|\nabla f(x(s))\|^2} \right\| ds \leq \int_0^t \frac{\|\nabla f(x(s))\|}{\|\nabla f(x(s))\|^2} ds \\ &= \int_0^t \frac{1}{\|\nabla f(x(s))\|} ds \leq \int_0^t \frac{1}{\inf_U\{\|\nabla f(q)\|\}} ds \\ &= \frac{1}{\inf_U\{\|\nabla f(q)\|\}} \int_0^t ds = \frac{t}{\inf_U\{\|\nabla f(q)\|\}} \end{aligned} \tag{1.54}$$

where $U \triangleq \mathscr{B}_i\left(\frac{\eta}{2}\right) \cap T_{\delta_1}(S)$. (points in U are interior points of $\bar{T}_{\delta_1}(S)$, because of continuity, Bourbaki and $f(\partial \bar{T}_{\delta_1}(S)) \geq \eta$.

Remark 28. *Note that we have used the inequality*

$$\frac{1}{\|\nabla f(x(t))\|} \leq \frac{1}{\inf_W\{\|\nabla f(q)\|\}}$$

for $t \in [0, \frac{\eta, 2}{)}$, *because both sides are well-defined.*

Firstly, the Fyndamental Theorem ensures that for $t \in [0, \frac{\eta}{2})$ *it is* $x(t) \in W \implies \nabla f\left(x(t)\right) \neq 0$. *So* $\frac{1}{\|\nabla f(x(t))\|}$ *is finite for* $t \in [0, \frac{\eta}{2})$.

Secondly, by hypothesis $\|\nabla f\|$ *is continuous on the compact set* $\overline{\mathscr{B}_i\left(\frac{\eta}{2}\right)}$ *and by the Extreme Value Theorem*

$$\inf_U \{\|\nabla f\|\} = \|\nabla f(x_1)\| > 0$$

for some $x_1 \in W$. *This ensures that* $\frac{1}{\inf_U\{\|\nabla f(q)\|\}}$ *is bounded.*

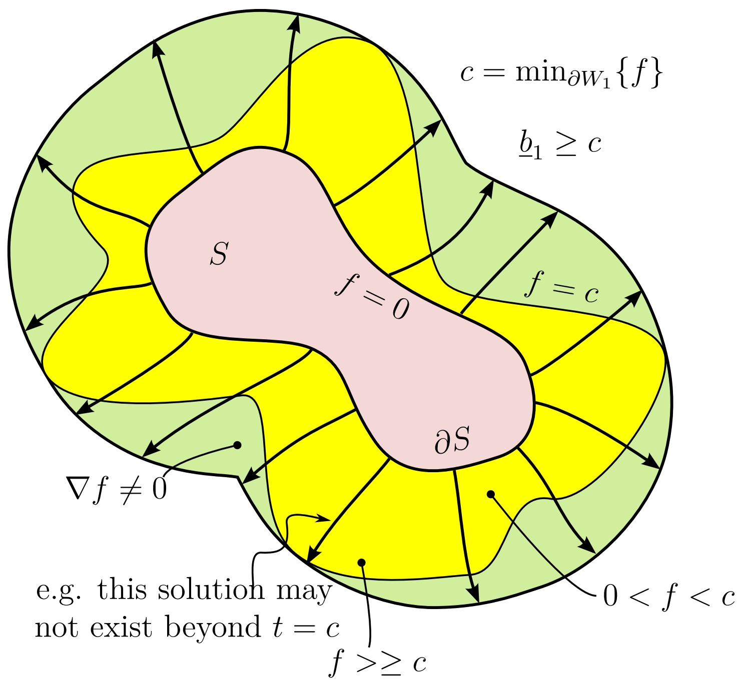


Figure 1.10: The upper end-point of the solutions is lower semi-continuous. So it attains its minimum on a compact set of initial conditions. This means that there exists some minimal time, for which every solution on S "lives" before exiting the neighborhood were the flow is well-defined.

Select

$$
\begin{aligned}
\varepsilon_1 &= \delta \inf_U \{\|\nabla f\|\} > 0 \\
\varepsilon_2 &= \delta_1 \inf_U \{\|\nabla f\|\} > 0 \\
\varepsilon_3 &= \frac{\eta}{2}
\end{aligned}
\tag{1.55}
$$

which are positive because δ, δ_1 and $\inf_U \{\|\nabla f\|\}$ are positive. Using the previous, define

$$\varepsilon \triangleq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \tag{1.56}$$

Then, for every $y \in C_4 = f^{-1}((0,\varepsilon)) \cap W$, there exists (by construction of C_4) a trajectory with $x(t) = y$ and $t \in \left(0, \frac{\eta}{2}\right)$ from $x(0) \in S$ to y, so

$$\|x(t) - x(0)\| \leq \frac{t}{\inf_U \{\|\nabla f\|\}} = \frac{f\left(x(y)\right)}{\inf_U \{\|\nabla f\|\}} = \frac{f(y)}{\inf_U \{\|\nabla f\|\}} \leq \frac{\varepsilon}{\inf_U \{\|\nabla f\|\}} \leq \delta \tag{1.57}$$

Every $x \in W$ with $f(x) < \varepsilon$ is is C_4, because $W \setminus C_4 = C_2 \cup S \cup C_3 \cup (C_1 \setminus C_4)$ and by definition $x \notin C_2 \wedge x \notin S \wedge x \notin C_3$, because $f(x) < \varepsilon < \eta$. Also $y \in C_1 \setminus C_4 \implies f(y) \in \left[\frac{\eta}{2}, \eta\right)$.

The set C_4 comprises of level sets which are diffeomorphic to S by the Regular Interval Theorem 2.2, p.153 [11]. An illustration for the proof is shown in Fig. 1.10.

□

Corollary 29. *Select arbitrary $\delta > 0$. There exists an $\hat{\varepsilon}_i > 0$, such that every ε_i with $0 < \varepsilon_i < \hat{\varepsilon}_i$ has the following property. Suppose q is in an ε_i-neighborhood of $\partial\mathscr{O}_i$. Then, there exists a $q' \in \partial\mathscr{O}_i$, such that $\|q - q'\| < \delta$.*

In other words, if $0 < \varepsilon_i < \hat{\varepsilon}_i$, then the ε_i-neighborhood is contained in the uniform tube of the obstacle boundary, i.e., $\mathscr{B}_i\left(\varepsilon_i\right) \subseteq T_\delta(\partial\mathscr{O}_i)$.

Corollary 30. *Select arbitrary $\delta > 0$. There exists an $\hat{\varepsilon}_i > 0$, such that every ε_i with $0 < \varepsilon_i < \hat{\varepsilon}_i$ has the following property. The ε_i-neighborhood is included in the uniform δ-tube of the obstacle boundary*

$$\mathscr{B}_i\left(\varepsilon_i\right) \subseteq T_\delta(\partial\mathscr{O}_i) \tag{1.58}$$

By hypothesis set $\partial\mathscr{O}_i$ is compact. By Corollary 4 the uniform δ-tube of any compact set is bounded[18]. This implies that $\mathscr{B}_i\left(\varepsilon_i\right)$ is bounded.

Corollary 31. *Select arbitrary $\delta > 0$. There exists an $\hat{\varepsilon}_i > 0$, such that every ε_i with $0 < \varepsilon_i < \hat{\varepsilon}_i$ has the following property. The ε_i-neighborhood $\mathscr{B}_i\left(\varepsilon_i\right)$ is bounded.*

Note that set $\mathscr{B}_i\left(\varepsilon_i\right)$ is not compact, because it is open.

Moreover, applying the Regular Interval theorem proves that $\partial\mathscr{O}_i$ is diffeomorphic to level sets within the uniform tube $T_\delta(\partial\mathscr{O}_i)$.

1.2.10 Other sets

Some more sets will be useful in our discussion. The whole world is

$$\mathscr{W} \triangleq E^n \setminus \mathscr{O}_0 = \left\{q \in E^n \middle|\, 0 \leq \beta_0(q)\right\}, \tag{1.59}$$

bounded by the zero$^{\text{th}}$ obstacle $\mathscr{O}_0$. The union of all obstacles is the *unsafe space*

$$\mathscr{O} \triangleq \bigcup_{i \in I_0} \mathscr{O}_i \tag{1.60}$$

The free space is

$$\mathscr{F} \triangleq \mathscr{W} \setminus \mathscr{O} \implies \mathscr{F} = E^n \setminus \mathscr{O} = E^n \setminus \bigcup_{i \in I_0} \mathscr{O}_i \tag{1.61}$$

As a result, the free space is by definition a closed subset of E^n (which inherits the Euclidean metric and becomes Riemannian).

The free space $\mathscr{F}$ is not necessarily connected though. Different obstacles are disjoint by hypothesis, so they cannot intersect and disconnect $\mathscr{F}$. But, a single obstacle can disconnect $\mathscr{F}$. For example, an obstacle which is a spherical annulus disconnects $\mathscr{F}$ into two sets. In this case, there may exist no path between the initial condition and the destination.

For this reason, we *assume* that $\mathscr{F}$ is connected.

If this does not hold, then the NF methodology cannot find a path. Nonetheless, assume that the obstacles have particular geometric properties discussed later. Furthermore, assume that we can find the tuning that makes the potential have a unique local minimum on every connected component of $\mathscr{F}$. As proved in this work, such a tuning always exists for obstacles with these particular geometric properties. Then, if the agent converges to a local minimum other than the destination, then there does not exist a path and we can conclude that the space is disconnected. However, this decision process involves convergence, which is difficult

[18] Its closure is compact.

to decide in finite time and in a numerical implementation. Therefore, it is not a practical approach to be implemented.

The free space $\mathscr{F}$ is not necessarily bounded, because this depends on whether $\mathscr{O}_0$ has been defined to have the appropriate properties, enclosing the whole world $\mathscr{W}$ and making it a bounded set. Therefore, $\mathscr{F}$ is not necessarily compact.

In the present study we will *assume* that $\mathscr{O}_0$ is indeed defined appropriately to render $\mathscr{F}$ bounded. This assumption can be removed, by applying the appropriate technical machinery from [14], which was developed for unboundedness occurring due to incomplete world knowledge during exploration (when $\mathscr{O}_0$ may exist but remain presently unknown). Hence, $\mathscr{F}$ will be closed and bounded, therefore compact.

Finally, $\mathscr{F}$ is (at least) a C^2-differentiable manifold, because its boundary is defined by the functions β_i, which are by hypothesis (at least) twice continuously differentiable[19]

As a result, $\mathscr{F}$ is by definition a Riemannian manifold with boundary. Due to obstacle functions β_i being C^2, the free space $\mathscr{F}$ is a C^2 manifold. The additional assumptions introduced here are

1. $\mathscr{F}$ is connected. This is equivalent to assuming that for every obstacle, its complement is connected.

2. There exists an obstacle $\mathscr{O}_i$ (we index it as the zero$^{\text{th}}$ one $\mathscr{O}_0$), whose complement is bounded[20].

and ensure that the free space $\mathscr{F}$ is connected and compact. Compactness in this case can be proved by the hypothesis as follows. By hypothesis, $E^n \setminus \mathscr{O}_0$ is bounded and by definition closed, so $E^n \setminus \mathscr{O}_0$ is compact. All other obstacles $\mathscr{O}_i, i \neq 0$ are disjoint from $\mathscr{O}_0$, therefore they are contained in its complement. So $E^n \setminus \mathscr{O}_0$ contains all other obstacles and is compact. By definition, the world $\mathscr{W} = E^n \setminus \mathscr{O}_0$, so by the previous argument $\mathscr{W}$ is compact. Therefore, the free space $\mathscr{F} \subseteq \mathscr{W}$ is compact. We conclude that the free space $\mathscr{F}$ is a compact connected C^2 Riemannian manifold with boundary.

Note, that using the notion of free space, (1.8) becomes

$$0 < \beta_i(q), \quad \forall q \in \mathscr{F} \setminus \mathscr{O}_i, \quad \forall i \in I_0 \tag{1.62}$$

1.2.11 Destination function

Assumption 32. *Select a destination $q_d \in \mathring{\mathscr{F}}$. The function $\gamma_d \in C^2\left(E^n, [0, +\infty)\right)$ is the destination attractive effect. It should have the first-order properties*

$$\nabla \gamma_d\left(q\right) \neq 0, \tag{1.63}$$

for all $q \in E^n \setminus \{q_d\}$, and

$$\nabla \gamma_d\left(q_d\right) = 0. \tag{1.64}$$

It should also have the second-order property

$$D^2 \gamma_d(q) > 0, \tag{1.65}$$

for all $q \in E^n$.

[19] This can also be extended to piecewise functions, as discussed in [15].

[20] This hypothesis also implies that there exists at least one obstacle, i.e., $1 \leq M$.

Besides, the specific form of a paraboloid γ_d, which satisfies these conditions, is selected in the course of derivation due to symmetry considerations and in order to enable complete geometric interpretation of the condition.

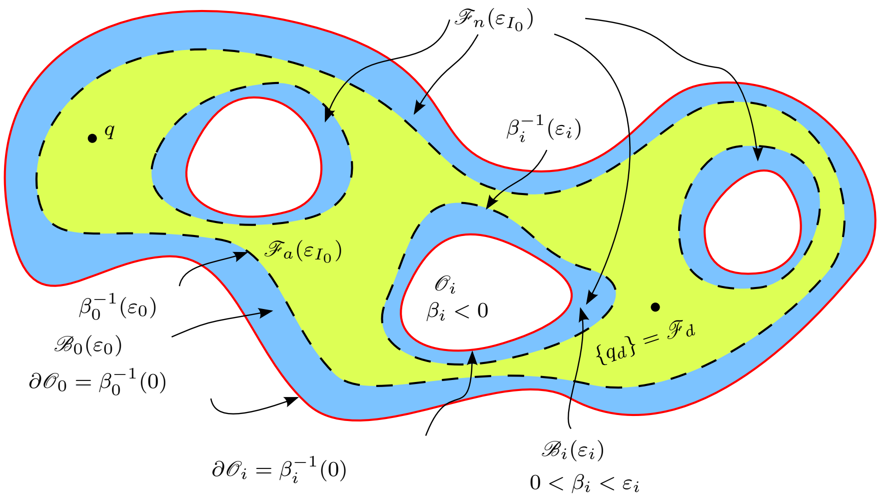


Figure 1.11: Sets defined on a general world.

1.3 Navigation Function

The Navigation Function $\varphi : \mathscr{F} \to [0,1]$ considered here is of the form

$$\varphi \triangleq \frac{\gamma_d}{\left(\gamma_d^k + \beta\right)^{\frac{1}{k}}} \tag{1.66}$$

where $\beta \triangleq \prod_{i\in I_0} \beta_i$ is the aggregate obstacle function and $k \in \mathbb{N} \cap [2, +\infty)$ a tuning parameter. The proof establishes the existence of a sufficient lower bound on k for φ to be a Navigation Function.

Additionally, the following function is defined

$$\hat{\varphi} : \mathscr{F} \setminus \partial\mathscr{F} \to [0, +\infty) \quad \hat{\varphi} \triangleq \frac{\gamma_d^k}{\beta} \tag{1.67}$$

and called the “un-squashed” Navigation Function, defined in the free space interior $\mathscr{F} \setminus \partial\mathscr{F}$.

1.4 Definition of world subsets

The following sets are used and illustrated in Fig. 1.11:

1. Destination point
$$\mathscr{F}_d \triangleq \{q_d\}; \tag{1.68}$$

2. Free space boundary
$$\partial\mathscr{F} \triangleq \beta^{-1}(0) = \bigcup_{i\in I_0} \beta_i^{-1}(0); \tag{1.69}$$

3. i^{th} obstacle neighborhood

$$\mathscr{B}_i\left(\varepsilon_i\right) \triangleq \{q \in E^n |\ 0 < \beta_i < \varepsilon_i\}, \quad i \in I_0 \tag{1.70}$$

and we also require that $\mathscr{B}_i(\varepsilon_i)$ are pairwise disjoint[21]

$$\begin{aligned} \mathscr{B}_i(\varepsilon_i) \cap \mathscr{B}_j(\varepsilon_j) = \emptyset, \quad \forall j \in I_0 \setminus \{i\}, \quad \forall i \in I_0 \iff \\ \beta_j(q) \geq \varepsilon_j, \quad \forall q \in \mathscr{B}_i(\varepsilon_i), \quad \forall j \in I_0 \setminus \{i\}, \quad \forall i \in I_0 \end{aligned} \tag{1.71}$$

Since obstacle sets $\mathscr{O}_i$ have been defined as pairwise disjoint in (1.10), there always exists a set ε_{I_0} of $0 < \varepsilon_i, i \in I_0$, such that the neighborhoods $\mathscr{B}_i(\varepsilon_i)$ be pairwise disjoint. In the proof this is addressed by placing the appropriate requirement on the selection of ε_{i3j};

4. "Near" all obstacles (i.e., internal and zero$^{\text{th}}$)

$$\mathscr{F}_n(\varepsilon_{I_0}) \triangleq \left(\bigcup_{i \in I_0} \mathscr{B}_i(\varepsilon_i) \right) \setminus \{q_d\}; \tag{1.72}$$

5. Set "away" from all obstacles (i.e., internal and zero$^{\text{th}}$)

$$\mathscr{F}_a(\varepsilon_{I_0}) \triangleq \mathscr{F} \setminus (\mathscr{F}_d(\varepsilon_{I_0}) \cup \partial\mathscr{F} \cup \mathscr{F}_n(\varepsilon_{I_0})) \tag{1.73}$$

where $\varepsilon_{I_0} \triangleq \{\varepsilon_i\}_{i \in I_0}$.

In consequence of the above definitions, there are two alternatives for defining sets $\mathscr{B}_i, \mathscr{F}_n, \mathscr{F}_a$ as either functions of a *single global* "width" $\varepsilon \triangleq \min_{i \in I_0}\{\varepsilon_i\}$, or as functions of the *set* ε_{I_0} of "widths" ε_i. Here the sets are functions of $M+1$ parameters ε_{I_0} defined as $\mathscr{B}_i(\varepsilon_i), i \in I_0, \mathscr{F}_n(\varepsilon_{I_0}), \mathscr{F}_a(\varepsilon_{I_0})$. Note that the above definitions differ from those in [1]. Hereafter sets $\mathscr{F}_i$ are denoted omitting their arguments. Let $\mathscr{C}_f \triangleq \{q_c \in E^n | \nabla f = 0\}$ the critical set of a function f.

We can define the restriction of the critical set of function $\hat{\varphi}$ to the ε-neighborhood of an obstacle.

Definition 33. *Let the set of critical points in the ε-neighborhood of obstacle $\mathscr{O}_i$ be*

$$\mathscr{C}_{\hat{\varphi},i}(\varepsilon) \triangleq \mathscr{C}_{\hat{\varphi}} \cap \mathscr{B}_i(\varepsilon) = \{q \in \mathscr{B}_i(\varepsilon) \mid \nabla\hat{\varphi}(q) = 0\} \tag{1.74}$$

Note that $\mathscr{C}_{\hat{\varphi},i}(\varepsilon)$ is *not* necessarily closed.

[21] Note that Koditschek and Rimon enforce this only between their $\mathscr{F}_0$ and the rest $\mathscr{B}_i(\varepsilon_i), i \in I_1$, by appropriately removing them from $\mathscr{F}_0$ in its definition. This was used in Proposition 3.7, p.432, [1], ensuring that $\beta_i \geq \varepsilon, \forall q \in \mathscr{F}_1(\varepsilon), \forall i \in \{1, \ldots, M\}$. Here this Proposition 3.7 is replaced by a general Proposition which applies to subsets of the neighborhoods of *all* obstacles. This is the reason for which we place this requirement on all obstacles and not only between internal obstacles and the zero$^{\text{th}}$ one.

Chapter 2

Relative Curvature

2.1 (Unit) Tangent and Normal bundles

Let $T_q\mathscr{F}$ denote the tangent space of $\mathscr{F}$ at point q. Then, the unit tangent space $UT_q\mathscr{F}$ of $\mathscr{F}$ at point q can be defined as

$$UT_q\mathscr{F} \triangleq \{u \in T_q\mathscr{F}| \ \|u\| = 1\} \tag{2.1}$$

which is the set of all unit vectors in the tangent space $T_q\mathscr{F}$ at q.

Set $c = \beta_i(q)$ and let

$$B_i(q) \triangleq \beta_i^{-1}(\{c\}) \tag{2.2}$$

denote the level set of the implicit obstacle function β_i to which point q belongs. If the level set $\beta_i^{-1}(\beta_i(q))$ is disconnected, then $B_i(q)$ is defined as that connected component of this level set, to which point q belongs.

Let

$$\mathscr{N}_i(q) \triangleq \text{span}\{\nabla\beta_i(q)\} \subset T_q\mathscr{F} = N_qB_i(q) \tag{2.3}$$

be the normal subspace at q spanned by $\nabla\beta_i(q)$.

Define the orthogonal complement of N_qB_i in the tangent space $T_q\mathscr{F}$ as

$$\mathscr{T}_i(q) \triangleq \{u \in T_q\mathscr{F}| \quad u \cdot \nabla\beta_i(q) = 0\} \subset T_q\mathscr{F}. \tag{2.4}$$

This is equal to the tangent space of level set $B_i(q)$, i.e.,

$$\mathscr{T}_i(q) = T_qB_i(q). \tag{2.5}$$

Also, note that $\mathscr{N}_i(q)$ and $\mathscr{T}_i(q)$ comprise a direct sum decomposition of $T_q\mathscr{F}$, i.e.,

$$T_q\mathscr{F} = \mathscr{R}_i(q) \oplus \mathscr{T}_i(q). \tag{2.6}$$

Moreover, let us define the corresponding unit normal space as

$$\begin{aligned} U\mathscr{N}_i(q) &\triangleq \{u \in \mathscr{N}_i(q)| \ \|u\| = 1\} = \{\hat{v} \in UT_q\mathscr{F}| \ \hat{v} \cdot \nabla\beta_i(q) = \|\nabla\beta_i\|\} \\ &= \mathscr{N}_i(q) \cap UT_q\mathscr{F} \subset UT_q\mathscr{F} \end{aligned} \tag{2.7}$$

and the unit tangent space of $B_i(q)$ as

$$\begin{aligned} U\mathscr{T}_i(q) &\triangleq \{u \in \mathscr{T}_i(q)| \ \|u\| = 1\} = \{\hat{v} \in UT_q\mathscr{F}| \ \hat{v} \cdot \nabla\beta_i(q) = 0\} \\ &= \mathscr{T}_i(q) \cap UT_q\mathscr{F} \subset UT_q\mathscr{F} \end{aligned} \tag{2.8}$$

Then, let us define the unit vectors

$$\hat{r}_i \triangleq \frac{\nabla \beta_i(q)}{\|\nabla \beta_i(q)\|} \in U\mathscr{N}_i(q), \qquad \hat{t}_i \in U\mathscr{T}_i(q). \tag{2.9}$$

These are the outward unit normal vector $\hat{r}_i$ parallel to $\nabla \beta_i$ and the unit tangent vector $\hat{t}_i$, which is orthogonal to $\hat{r}_i$. Vector $\hat{t}_i$ is tangent to the i^{th} obstacle level set $B_i(q)$ at point q.

We will now consider those values of its range, for which β_i is regular on the whole corresponding level set. Provided that regularity holds, we can define the associated normal and tangent bundles of the level set, as well as their unit counterparts. Let

$$\begin{aligned} NB_i(q_0) &= \bigsqcup_{q \in B_i(q_0)} N_q B_i(q) = \bigcup_{q \in B_i(q_0)} \left(\{q\} \times N_q B_i(q)\right) = \bigcup_{q \in B_i(q_0)} \left(\{q\} \times \mathscr{N}_i(q)\right), \\ TB_i(q_0) &= \bigsqcup_{q \in B_i(q_0)} T_q B_i(q). \end{aligned} \tag{2.10}$$

be the normal and tangent bundles of $B_i(q_0)$, respectively. Furthermore, let

$$\begin{aligned} UNB_i(q_0) &\triangleq \bigsqcup_{q \in B_i(q_0)} U\mathscr{N}_i(q) \\ &= \bigsqcup_{q \in B_i(q_0)} \{u \in N_q B_i(q) | \ \|u\| = 1\} = \bigsqcup_{q \in B_i(q_0)} \{u \in \mathscr{N}_i(q) | \ \|u\| = 1\}, \\ UTB_i(q_0) &\triangleq \bigsqcup_{q \in B_i(q_0)} U\mathscr{T}_i(q) \\ &= \bigsqcup_{q \in B_i(q_0)} \{u \in T_q B_i(q) | \ \|u\| = 1\} = \bigsqcup_{q \in B_i(q_0)} \{u \in \mathscr{T}_i(q) | \ \|u\| = 1\}. \end{aligned} \tag{2.11}$$

denote the unit tangent bundle of $B_i(q_0)$.

We are interested in a family of tangent bundles associated with a range of level sets. For this reason, the disjoint unions above will be taken over free space $\mathscr{F}$, instead of a single level set $B_i(q)$.

$$\begin{aligned} NB_i &\triangleq \bigsqcup_{q \in \mathscr{F}} N_q B_i(q) \qquad UNB_i \triangleq \bigsqcup_{q \in \mathscr{F}} UN_q B_i(q) \\ TB_i &\triangleq \bigsqcup_{q \in \mathscr{F}} T_q B_i(q) \qquad UTB_i \triangleq \bigsqcup_{q \in \mathscr{F}} UT_q B_i(q). \end{aligned} \tag{2.12}$$

2.2 Relative Curvature Function (any γ_d)

Decomposing the Hessian quadratic form into two parts in **??** reveals that on a critical point which is close to an obstacle, its sign behaves as the function ν_i. This indicates that ν_i is an important function to study for our problem.

Definition 34 (Relative Curvature Function $\nu_i(q, \hat{t}_i)$)**.** *The* relative curvature *function $\nu_i : UTB_i \to \mathbb{R}$ is defined as*

$$\nu_i(q, \hat{t}_i) \triangleq \frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|^2} \hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i - \hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i \tag{2.13}$$

where the unit vector $\hat{t}_i \in U\mathscr{T}_i(q)$, the point $q \in \mathscr{F}$ and the obstacle index $i \in I_0$. It is well-defined where $\nabla\gamma_d(q) \neq 0$.

As will be shown later, the relative curvature function ν_i compares the curvature of level sets of the attractive function γ_d, with the curvature of level sets of the obstacle function β_i. In this respect, it is a comparison function.

Definition 35. *Let the angle between the two gradients $\nabla\gamma_d(q)$ and $\nabla\beta_i(q)$ be denoted by*

$$\begin{aligned} \theta_i(q) &\triangleq (\nabla\gamma_d \widehat{(q), \nabla}\beta_i(q)) \implies \\ \cos\theta_i(q) &= \frac{\nabla\gamma_d(q) \cdot \nabla\beta_i(q)}{\|\nabla\gamma_d(q)\| \, \|\nabla\beta_i(q)\|}. \end{aligned} \tag{2.14}$$

Functions θ_i and $\cos\theta_i$ are well-defined at a point q only if $\nabla\gamma_d(q) \neq 0$ and $\nabla\beta_i(q) \neq 0$. For brevity, we can define

$$\begin{aligned} \psi_i(q) &\triangleq \nabla\gamma_d(q) \cdot \nabla\beta_i(q) \\ \hat{\psi}_i(q) &\triangleq \cos\theta_i(q) = \frac{\nabla\gamma_d(q) \cdot \nabla\beta_i(q)}{\|\nabla\gamma_d(q)\| \, \|\nabla\beta_i(q)\|} = \frac{\psi_i(q)}{\|\nabla\gamma_d(q)\| \, \|\nabla\beta_i(q)\|}. \end{aligned} \tag{2.15}$$

Function $\hat{\psi}_i$ is well-defined at a point q only when both $\nabla\gamma_d(q) \neq 0$ and $\nabla\beta_i(q) \neq 0$. Note that

$$\hat{\psi}_i = \frac{\nabla\beta_i}{\|\nabla\beta_i\|} \cdot \frac{\nabla\gamma_d}{\|\nabla\gamma_d\|} = \hat{r}_i \cdot \hat{r}_d \tag{2.16}$$

where $\hat{r}_i$ and $\hat{r}_d$ are the unit vectors in the direction of $\nabla\beta_i$ and $\nabla\gamma_d$.

The relative curvature function decomposes into two parts, as the following proves.

Proposition 36 (Relative curvature function decomposition)**.** *Assume that $\nabla\gamma_d(q) \neq 0$. Then, the relative curvature function $\nu_i(q, \hat{t}_i)$ is equal to the sum of two functions*

$$\nu_{i1} : UTB_i \to \mathbb{R} \qquad \nu_{i2} : UTB_i \to \mathbb{R} \tag{2.17}$$

which are defined as

$$\nu_{i1}(q, \hat{t}_i) \triangleq \frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|^2} \hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i \qquad \nu_{i2}(q, \hat{t}_i) \triangleq -\hat{t}_i D^2\beta_i(q)\hat{t}_i. \tag{2.18}$$

The above functions are well-defined where $\nabla\gamma_d(q) \neq 0$.

Moreover, assume that $\nabla\beta_i(q) \neq 0$. Then, these functions can be expressed as

$$\begin{aligned} \nu_{i1}(q, \hat{t}_i) &= \|\nabla\beta_i(q)\| \frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\|\nabla\beta_i(q)\| \, \|\nabla\gamma_d(q)\|} \frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i}{\|\nabla\gamma_d(q)\|} \\ &= \|\nabla\beta_i(q)\| \cos\theta_i(q) \frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i}{\|\nabla\gamma_d(q)\|} \end{aligned} \tag{2.19}$$

and

$$\nu_{i2}(q, \hat{t}_i) = \|\nabla\beta_i(q)\| \left(-\frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i}{\|\nabla\beta_i(q)\|} \right). \tag{2.20}$$

The proof follows directly by observing the expression of $\nu_i(q, \hat{t}_i)$.

The relative curvature function is well-defined only if $\nabla\gamma_d \neq 0$. So are its component functions ν_{i1} and ν_{i2}. Functions ν_{i1}, ν_{i2} can be expressed in terms of $\hat{\psi}_i = \cos\theta_i$ only if $\nabla\beta_i \neq 0$. We can introduce the functions ν_{i3}, ν_{i4} only if $\nabla\beta_i \neq 0$. This presents no problem, because the functions ν_{i3} and ν_{i4} are of interest only "near" the obstacles and on their boundaries.

By hypothesis, the destination q_d is in $\mathring{\mathscr{F}}$, so it cannot be on an obstacle boundary. By definition, the set "near" obstacles $\mathscr{F}_n$ excludes q_d. Therefore, $\nabla\gamma_d \neq 0$ on $\mathscr{F}_n \cap \mathscr{C}_{\hat{\varphi}}$. By hypothesis the obstacles are regular surfaces, so by Corollary 13 there exist regular neighborhoods around them, in which $\nabla\beta_i \neq 0$.

Note that the unit tangent bundle UTB_i mentioned earlier is associated with level sets "close" to the obstacles. At other points within $\mathscr{F}$ it may be $\nabla\beta_i = 0$ or $\nabla\gamma_d = 0$. This is not of concern and UTB_i is undefined there.

In what follows, the analogous restriction applies to the domain $\mathscr{F}$ of ν_{i1} and ν_{i3} in section 2.3, where they become functions of only $q \in \mathscr{F}$. In other words, the domain $\mathscr{F}$ refers to the subset of $\mathscr{F}$ within which these functions are well-defined. This is the set $\{q \in \mathscr{F} | \, \nabla\beta_i(q) \neq 0 \wedge \nabla\gamma_d(q) \neq 0\}$.

According to Proposition 36, the relative curvature function can be written in the form

$$\begin{aligned} \nu_i(q, \hat{t}_i) &= \frac{\nabla\beta_i \cdot \nabla\gamma_d}{\|\nabla\gamma_d\|^2} \hat{t}_i^{\mathrm{T}} D^2\gamma_d\hat{t}_i - \hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i \overset{\|\nabla\beta_i\| \neq 0}{=} \\ &= \|\nabla\beta_i\| \left(\frac{\nabla\beta_i}{\|\nabla\beta_i\|} \cdot \frac{\nabla\gamma_d}{\|\nabla\gamma_d\|} \frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d\hat{t}_i}{\|\nabla\gamma_d\|} - \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i}{\|\nabla\beta_i\|} \right) \\ &= \|\nabla\beta_i\| \left(\cos(\theta_i) \frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d\hat{t}_i}{\|\nabla\gamma_d\|} - \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i}{\|\nabla\beta_i\|} \right), \end{aligned} \tag{2.21}$$

provided that $\nabla\beta_i(q) \neq 0$. This factorization of ν_i makes $\cos\theta_i$ and $\frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i}{\|\nabla\beta_i\|}$ appear. Both are desired. The cosine because it has much clearer geometric meaning and behavior than the inner product. The fraction of the Hessian quadratic form divided by the gradient norm because it is the normal curvature of a level set. Apart from the geometric insight this facilitates, it is additionally motivated by the similar term $\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i}{\|\nabla\gamma_d\|}$ when distributing the powers of $\|\nabla\gamma_d\|$ in the denominator.

This leads us to define two new functions ν_{i3}, ν_{i4}, which are proportional to ν_{i1} and ν_{i2}, respectively. The proportionality factor is $\|\nabla\beta_i\|$, so the functions are

$$\begin{aligned}\nu_{i3}(q,\hat{t}_i) &\triangleq \frac{\nabla\beta_i(q)\cdot\nabla\gamma_d(q)}{\|\nabla\beta_i(q)\|\,\|\nabla\gamma_d(q)\|}\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i}{\|\nabla\gamma_d(q)\|}\\ &= \cos\theta_i(q)\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i}{\|\nabla\gamma_d(q)\|}\\ &= \hat{\psi}_i(q)\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i}{\|\nabla\gamma_d(q)\|}\end{aligned} \tag{2.22}$$

and

$$\nu_{i4}(q,\hat{t}_i) \triangleq -\frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i}{\|\nabla\beta_i(q)\|}. \tag{2.23}$$

From the definitions of $\nu_{i1}, \nu_{i2}, \nu_{i3}, \nu_{i4}$ it follows that

$$\left\{\begin{array}{l}\nu_{i1}(q,\hat{t}_i) = \|\nabla\beta_i(q)\|\,\nu_{i3}(q,\hat{t}_i)\\ \nu_{i2}(q,\hat{t}_i) = \|\nabla\beta_i(q)\|\,\nu_{i4}(q,\hat{t}_i)\end{array}\right\} \iff \left\{\begin{array}{l}\nu_{i3}(q,\hat{t}_i) = \frac{\nu_{i1}(q,\hat{t}_i)}{\|\nabla\beta_i(q)\|}\\ \nu_{i4}(q,\hat{t}_i) = \frac{\nu_{i2}(q,\hat{t}_i)}{\|\nabla\beta_i(q)\|}\end{array}\right\}. \tag{2.24}$$

This leads us to substitute the above relations in the results of Proposition 36.

Proposition 37 (Relative curvature function decomposition proportionally)**.** *Assume that* $\nabla\beta_i(q) \neq 0$ *and* $\nabla\gamma_d(q) \neq 0$. *Define two functions*

$$\nu_{i3} : UTB_i \to \mathbb{R} \qquad \nu_{i4} : UTB_i \to \mathbb{R} \tag{2.25}$$

as

$$\begin{aligned}\nu_{i3}(q,\hat{t}_i) &\triangleq \frac{\nabla\beta_i(q)\cdot\nabla\gamma_d(q)}{\|\nabla\beta_i(q)\|\,\|\nabla\gamma_d(q)\|}\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i}{\|\nabla\gamma_d(q)\|}\\ \nu_{i4}(q,\hat{t}_i) &\triangleq -\frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i}{\|\nabla\beta_i(q)\|}.\end{aligned} \tag{2.26}$$

Then, the relative curvature function can be decomposed as

$$\nu_i(q,\hat{t}_i) = \|\nabla\beta_i(q)\|\left(\nu_{i3}(q,\hat{t}_i) + \nu_{i4}(q,\hat{t}_i)\right) \tag{2.27}$$

where $\hat{t}_i \in U\mathscr{T}_i(q)$, $q \in \mathscr{F}$ *and* $i \in I_0$.

Proof. According to Proposition 36 the relative curvature function can be decomposed as $\nu_i(q,\hat{t}_i) = \nu_{i1}(q,\hat{t}_i) + \nu_{i2}(q,\hat{t}_i)$. Replacing ν_{i1}, ν_{i2} by $\|\nabla\beta_i\|\,\nu_{i3}$ and $\|\nabla\beta_i\|\,\nu_{i4}$, respectively, leads to

$$\begin{aligned}\nu_i(q,\hat{t}_i) &= \nu_{i1}(q,\hat{t}_i) + \nu_{i2}(q,\hat{t}_i)\\ &= \|\nabla\beta_i(q)\|\,\nu_{i3}(q,\hat{t}_i) + \|\nabla\beta_i(q)\|\,\nu_{i4}(q,\hat{t}_i)\\ &= \|\nabla\beta_i(q)\|\left(\nu_{i3}(q,\hat{t}_i) + \nu_{i4}(q,\hat{t}_i)\right)\end{aligned} \tag{2.28}$$

□

Observe that this result is another way of writing (2.21).

2.3 Relative Curvature Function (paraboloid γ_d)

If $\gamma_d(q) = \|q - q_d\|^2$, then the quadratic form $\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i$ becomes radially symmetric. In this case, function $\nu_{i1}(q, \hat{t}_i)$ is independent of $\hat{t}_i$ and becomes a function of only q. This is formally proved by the following.

Proposition 38 (First component symmetry for paraboloid γ_d)**.** *Assume that $\nabla\gamma_d(q) \neq 0$. If $\gamma_d(q) = \|q - q_d\|^2$, then function $\nu_{i1} : \mathscr{F} \to \mathbb{R}$ becomes*

$$\nu_{i1}(q) = 2\frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|^2} \tag{2.29}$$

which is a function of only q.

Moreover, assume that $\nabla\beta_i(q) \neq 0$. Then, function $\nu_{i1}(q)$ can be expressed as

$$\nu_{i1}(q) = \|\nabla\beta_i(q)\| \cos\theta_i(q) \frac{2}{\|\nabla\gamma_d(q)\|} \tag{2.30}$$

Proof. If $\gamma_d(q) = \|q - q_d\|^2$, then $D^2\gamma_d(q) = 2I$. As a result, if γ_d is paraboloid, then $\hat{t}_i^{\mathrm{T}} D^2 \gamma_d(q_c)\hat{t}_i = \hat{t}_i^{\mathrm{T}} 2I \hat{t}_i = 2$. Substituting this into the definition of $\nu_{i1}(q, \hat{t}_i)$ in (2.18), it follows that

$$\begin{aligned} \nu_{i1}(q, \hat{t}_i) &= \frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\nabla\gamma_d(q)} \frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i}{\|\nabla\gamma_d(q)\|} = \frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\nabla\gamma_d(q)} \frac{2}{\|\nabla\gamma_d(q)\|} \\ &= 2\frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\nabla\gamma_d(q)^2}. \end{aligned} \tag{2.31}$$

This shows that in this case ν_{i1} does not depend on $\hat{t}_i$. As a result, we can write $\nu_{i1}(q)$.

The second claim follows directly

$$\begin{aligned} \nu_{i1}(q) &= 2\frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\nabla\gamma_d(q)^2} = \|\nabla\beta_i(q)\| \frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\|\nabla\beta_i(q)\| \nabla\gamma_d(q)} \frac{2}{\|\nabla\gamma_d(q)\|} \\ &= \|\nabla\beta_i(q)\| \cos\theta_i(q) \frac{2}{\|\nabla\gamma_d(q)\|} \end{aligned} \tag{2.32}$$

□

The same observation extends directly to function ν_{i3}, because it is proportional to ν_{i1}.

Corollary 39 (Symmetry extends to first proportional component ν_{i3})**.** *Assume that $\nabla\gamma_d(q) \neq 0$. If $\gamma_d(q) = \|q - q_d\|^2$, then function $\nu_{i3} : \mathscr{F} \to \mathbb{R}$ becomes*

$$\nu_{i3}(q) = 2\frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|^2} = \cos\theta_i(q) \frac{2}{\|\nabla\gamma_d(q)\|} = \hat{\psi}_i(q) \frac{2}{\|\nabla\gamma_d(q)\|} \tag{2.33}$$

which is a function of only q.

We can then state further propositions which follow from Proposition 38 and Corollary 39.

Proposition 40 (Relative curvature function ν_i decomposition for paraboloid γ_d)**.** *Assume that $\nabla\gamma_d(q) \neq 0$. If $\gamma_d(q) = \|q - q_d\|^2$, then the relative curvature function ν_i is equal to the sum of two functions*

$$\nu_{i1} : \mathscr{F} \to \mathbb{R} \qquad \nu_{i2} : UTB_i \to \mathbb{R}, \tag{2.34}$$

which are defined as

$$\nu_{i1}(q) \triangleq 2\frac{\nabla\beta_i(q) \cdot \nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|^2} \qquad \nu_{i2}(q, \hat{t}_i) \triangleq -\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i \tag{2.35}$$

so that

$$\nu_i(q, \hat{t}_i) = \nu_{i1}(q) + \nu_{i2}(q, \hat{t}_i) \tag{2.36}$$

for $\hat{t}_i \in U\mathscr{T}_i(q)$, $q \in \mathscr{F}$ and $i \in I_0$.

Note that for paraboloid γ_d the $\nu_{i1}(q)$ is a function *only* of q (i.e., independent of tangent direction $\hat{t}_i$), therefore common for all tangent directions at q. On the contrary, $\nu_{i2}(q, \hat{t}_i)$ is a function of *both* q and $\hat{t}_i$. But, actually $\nu_{i2}(q, \hat{t}_i)$ is the curvature of level set $\beta_i^{-1}(c)$, scaled by the gradient norm $\|\nabla\beta_i(q)\|$, which is constant for all directions at q.

Similarly, the following is the special case of Proposition 37 when $\gamma_d(q) = \|q - q_d\|^2$.

Proposition 41 (Relative curvature function decomposition proportionally when γ_d is paraboloid)**.** *Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d(q) \neq 0$. Define*

$$\nu_{i3} : \mathscr{F} \to \mathbb{R} \qquad \nu_{i4} : UTB_i \to \mathbb{R} \tag{2.37}$$

as

$$\nu_{i3}(q) \triangleq \frac{2\cos\theta_i(q)}{\|\nabla\gamma_d(q)\|} \qquad \nu_{i4}(q, \hat{t}_i) \triangleq -\frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i}{\|\nabla\beta_i(q)\|}. \tag{2.38}$$

Then, the relative curvature function can be decomposed as

$$\nu_i(q, \hat{t}_i) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) + \nu_{i4}(q, \hat{t}_i)\right) \tag{2.39}$$

where $\hat{t}_i \in U\mathscr{T}_i(q)$, $q \in \mathscr{F}$ and $i \in I_0$.

As already analyzed, the first term on the right-hand side of (2.21) should be strictly negative. So, an upper bound constraint can be specified, without the need to explicitly find actual extremal values of the two terms on the right-side. Therefore the general [1] condition which leads to the modified constraint $\varepsilon_i < \varepsilon'_{i0}$ is

$$\begin{aligned} \nu_i(q) = \frac{\nabla\beta_i \cdot \nabla\gamma_d}{\|\nabla\gamma_d\|^2}\hat{t}_i^{\mathrm{T}} D^2\gamma_d\hat{t}_i - \hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i < 0 \overset{\|\nabla\beta_i\|\neq 0, \forall q\in\mathscr{F}}{\iff} \\ \cos(\theta_i)\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d\hat{t}_i}{\|\nabla\gamma_d\|} < \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i}{\|\nabla\beta_i\|} \end{aligned} \tag{2.40}$$

[1] General here refers to any choice of γ_d, β_i.

and when $\left(\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i\right)\left(\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right)\cos(\theta_i) > 0$

$$\frac{\|\nabla\beta_i\|}{\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i} < \frac{\frac{\|\nabla\gamma_d\|}{\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i}}{\cos\theta_i} \tag{2.41}$$

Note that in more detail this is required to hold at a critical point q_c (and not at every point) confined within obstacle free space neighborhood $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$

$$\frac{\|\nabla\beta_i(q_c)\|}{\hat{t}_i(q_c)^{\mathrm{T}} D^2 \beta_i(q_c) \hat{t}_i(q_c)} < \frac{\frac{\|\nabla\gamma_d(q_c)\|}{\hat{t}_i(q_c)^{\mathrm{T}} D^2 \gamma_d(q_c) \hat{t}_i(q_c)}}{\cos\theta_i(q_c)}, \quad q_c \in \overline{\mathscr{B}_i\left(\varepsilon_i\right)} \tag{2.42}$$

Proposition 42. *Assume that $\nabla\beta_i \neq 0$ and $\nabla\gamma_d \neq 0$. Then, functions ψ_i and $\hat{\psi}_i$ have the same sign.*

Proof. By definition

$$\psi_i(q) = \nabla\beta_i \cdot \nabla\gamma_d \tag{2.43}$$

By hypothesis

$$\left.\begin{array}{r}\nabla\beta_i \neq 0\\ \nabla\gamma_d \neq 0\end{array}\right\} \implies 0 < \|\nabla\beta_i\|\,\|\nabla\gamma_d\| \tag{2.44}$$

Divide ψ_i by $\|\nabla\beta_i\|\,\|\nabla\gamma_d\| > 0$ to obtain

$$\begin{gathered}\frac{\psi_i(q)}{\|\nabla\beta_i(q)\|\,\|\nabla\gamma_d\,(q)\|} = \frac{\nabla\beta_i \cdot \nabla\gamma_d}{\|\nabla\beta_i(q)\|\,\|\nabla\beta_i(q)\|} = \hat{\psi}_i(q) \implies \\ \psi_i(q) = \hat{\psi}_i(q)\underbrace{\|\nabla\beta_i(q)\|\,\|\nabla\gamma_d\,(q)\|}_{>0}\end{gathered} \tag{2.45}$$

Therefore ψ_i and $\hat{\psi}_i$ have the same sign. □

Proposition 43. *Assume $\gamma_d = \|q - q_d\|^2$ and $\nabla\beta_i \neq 0$, as well as $\nabla\gamma_d \neq 0$. Then, the functions $\hat{\psi}_i, \nu_{i1}$ and ν_{i3} have the same sign.*

Proof. By definition

$$\hat{\psi}_i = \frac{\nabla\beta_i \cdot \nabla\gamma_d}{\|\nabla\beta_i\|\,\|\nabla\gamma_d\|} \tag{2.46}$$

By Proposition 38 it is

$$\nu_{i1}(q) = \|\nabla\beta_i(q)\|\cos\theta_i(q)\frac{2}{\|\nabla\gamma_d\,(q)\|} = \|\nabla\beta_i(q)\|\,\hat{\psi}_i(q)\frac{2}{\|\nabla\gamma_d\,(q)\|} \tag{2.47}$$

By hypothesis, both $\nabla\beta_i \neq 0$ and $\nabla\gamma_d \neq 0$, hence $0 < \frac{2\|\nabla\beta_i(q)\|}{\|\nabla\gamma_d(q)\|}$, which is equi-signed to $\hat{\psi}_i$. By Corollary 39 it is

$$\nu_{i3}(q) = \hat{\psi}_i(q)\frac{2}{\|\nabla\gamma_d\,(q)\|} \tag{2.48}$$

which is equi-signed to $\hat{\psi}_i$. □

Definition 44. *The angle function $\theta : \mathscr{F} \to [0, \pi]$ is defined as*

$$\begin{aligned} \theta(q) &\triangleq (\widehat{\nabla \gamma_d (q) , \nabla \beta(q)}) \implies \\ \cos\theta(q) &\triangleq \frac{\nabla \beta(q) \cdot \nabla \gamma_d (q)}{\|\nabla \beta(q)\| \, \|\nabla \gamma_d (q)\|} \end{aligned} \tag{2.49}$$

For convenience we define the functions

$$\begin{aligned} \psi(q) &\triangleq \nabla \beta(q) \cdot \nabla \gamma_d (q) \\ \hat{\psi}(q) &\triangleq \frac{\nabla \beta(q) \cdot \nabla \gamma_d (q)}{\|\nabla \beta(q)\| \, \|\nabla \gamma_d (q)\|}, \end{aligned} \tag{2.50}$$

similarly to ψ_i and $\hat{\psi}_i$. Note that functions $\theta, \cos\theta, \psi, \hat{\psi}$ are well-defined only if both $\nabla\beta \neq 0$ and $\nabla\gamma_d \neq 0$.

Proposition 45. *If $\nabla\beta(q) \neq 0$ and $\nabla\gamma_d (q) \neq 0$, then function $\hat{\psi}$ is well-defined and C^1 at q.*

Proposition 46. *If $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d (q) \neq 0$, then function $\hat{\psi}_i$ is well-defined and C^1 at q.*

Proof. Function $\hat{\psi}_i$ is defined as

$$\hat{\psi}_i \triangleq \frac{\nabla \beta_i \cdot \nabla \gamma_d}{\|\nabla \beta_i\| \, \|\nabla \gamma_d\|} \tag{2.51}$$

Both β_i and γ_d are twice continuously differentiable. Therefore, their gradients $\nabla\beta_i$ and $\nabla\gamma_d$ are continuously differentiable functions. By hypothesis $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d (q) \neq 0$. Since $\nabla\beta_i, \nabla\gamma_d$ are C^1, they are also continuous. So continuity ensures that $\nabla\beta_i \neq 0$ and $\nabla\gamma_d \neq 0$ in a neighborhood of q. The norm $\|\cdot\|$ and ratio functions are C^∞ on $E^n \setminus \{0\}$. The inner product function is C^∞. Moreover, by hypothesis $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d (q) \neq 0$. As a result, $\hat{\psi}_i$ is continuously differentiable, as the composition of the previous functions.

Note that $\hat{\psi}_i$ is undefined at points q where $\nabla\beta_i(q) = 0$ or $\nabla\gamma_d (q) = 0$. □

Proposition 47. *Select $\gamma_d = \|q - q_d\|^2$ and assume $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d (q) \neq 0$. Then, function $\nu_{i3}(q)$ is continuously differentiable at q.*

Proof. Function $\nu_{i3}(q)$ is defined as

$$\nu_{i3}(q) = \frac{\nabla \beta_i \cdot \nabla \gamma_d}{\|\nabla \beta_i\| \, \|\nabla \gamma_d\|} \frac{2}{\|\nabla \gamma_d\|} = \hat{\psi}_i \frac{2}{\|\nabla \gamma_d\|} \tag{2.52}$$

By Proposition 46 and the premises, function $\hat{\psi}_i$ is C^1 at q. Function $\frac{2}{\nabla\gamma_d}$ is C^1 in a neighborhood of q, because γ_d is C^∞ and $\nabla\gamma_d (q) \neq 0$. Therefore, ν_{i3} is continuously differentiable at q. □

Proposition 48. *Assume $\nabla\gamma_d \neq 0$ and $\nabla\beta_i \neq 0$. Then, function $\nu_i(q, \hat{t}_i)$ is continuously differentiable.*

Proof. By definition

$$\nu_i(q,\hat{t}_i) = \|\nabla\beta_i(q)\| \left(\hat{\psi}_i(q) \frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i}{\|\nabla\gamma_d\,(q)\|} - \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i}{\|\nabla\beta_i(q)\|} \right) \tag{2.53}$$

which is not obviously continuously differentiable. The reason is that $\hat{t}_i \in T_qB_i(q)$. Hence, it is necessary that the tangent space be well-defined and continuously varying. In other words, the level set through q should be a regular surface at q. Regularity is equaivalent to $\nabla\beta_i(q) \neq 0$.

We will show that regularity is a sufficient by using the matrix

$$P_n = I_n - \frac{\nabla\beta_i \nabla\beta_i^{\mathrm{T}}}{\|\nabla\beta_i\|^2} \tag{2.54}$$

of the projector operator from the tangent space of the ambient space T_qE^n to the tangent space of the level set $T_qB_i(q)$.

Then, we can use unit vectors of the tangent space $\hat{v} \in T\mathscr{F}$ of the ambient space. Their projections defined as $t_i = P_n\hat{v}$ are in $T_qB_i(q)$. Normalize to obtain a unit tangent vector in the tangent space $\hat{t}_i = \frac{t_i}{\|t_i\|} = \frac{P_n\hat{v}}{\|P_n\hat{v}\|}$.

Based on the previous, we can express ν_i as

$$\nu_i(q,\hat{t}_i) = \|\nabla\beta_i(q)\| \left(\hat{\psi}_i(q) \frac{\frac{(P_n\hat{v})^{\mathrm{T}}}{\|P_n\hat{v}\|} D^2\gamma_d(q) \frac{P_n\hat{v}}{\|P_n\hat{v}\|}}{\|\nabla\gamma_d\,(q)\|} - \frac{\frac{(P_n\hat{v})^{\mathrm{T}}}{\|P_n\hat{v}\|} D^2\beta_i(q) \frac{P_n\hat{v}}{\|P_n\hat{v}\|}}{\|\nabla\beta_i(q)\|} \right) \tag{2.55}$$

This expression is continuously differentiable on the (ambient) tangent bundle $T\mathscr{F}$. Observe how we can now use $T\mathscr{F}$ with this expression, instead of $TB_i(q)$. The latter involves $T_qB_i(q)$ which is a function of q, whereas the tangent spaces $T_q\mathscr{F}$ in $T\mathscr{F}$ are not. $\square$

2.4 Principal Relative Curvature Functions

Definition 49. *Let $\nu_{ij} : \mathscr{F} \to \mathbb{R}$ be defined as*

$$\nu_{ij}(q) \triangleq \nu_i\left(q,\ \hat{p}_{ij}(q)\right) \tag{2.56}$$

where $\hat{p}_{ij}(q)$ is an eigenvector of the Weingarten map of $B_i(q)$ at q.

In other words, the principal relative curvature functions are the relative curvature evaluated for $\hat{p}_{ij}(q)$, the principal directions of $B_i(q)$. These are the eigenvectors of the *restricted* quadratic form $D^2\beta_i(q)|_{T_qB_i(q)}$. Note that the vectors $\hat{p}_{ij}(q)$ are chosen in Definition 174 to form an orthonormal basis of the tangent space. The unit length $\|\hat{p}_{ij}(q)\| = 1$ is important later.

Proposition 50. *Select $\gamma_d \triangleq \|q - q_d\|^2$. Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d\left(q\right) \neq 0$. Then*

$$\nu_i(q, \hat{t}_i) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_i(q, \hat{t}_i)\right). \tag{2.57}$$

Proof. By hypothesis and Proposition 41 function ν_i can be decomposed as

$$\nu_i(q, \hat{t}_i) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) + \nu_{i4}(q, \hat{t}_i)\right)$$

By the definition of ν_{i4} in Proposition 37 and the expression from Proposition 169 for the normal curvature of an implicitly defined surface in terms of its defining implicit function, it is $\nu_{i4}(q, \hat{t}_i) = -\kappa_i(q, \hat{t}_i)$. Substitute to obtain

$$\nu_i(q, \hat{t}_i) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_i(q, \hat{t}_i)\right) \tag{2.58}$$

□

Corollary 51. *Select $\gamma_d \triangleq \|q - q_d\|^2$. Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d\left(q\right) \neq 0$. Then*

$$\nu_{ij}(q) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_{ij}(q)\right) \tag{2.59}$$

Proof. By Proposition 50 it is

$$\nu_i(q, \hat{t}_i) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_i(q, \hat{t}_i)\right) \tag{2.60}$$

Substitute the principal directions $\hat{p}_{ij}$ for $\hat{t}_i$ to obtain

$$\nu_i(q, \hat{p}_{ij}) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_i(q, \hat{p}_{ij})\right) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_{ij}(q)\right) \tag{2.61}$$

By Definition 49 it is $\nu_{ij}(q) = \nu_i(q, \hat{p}_{ij})$, which completes the claim's proof. □

Theorem 52. *Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d\left(q\right) \neq 0$. Let the quadratic form $Q(\hat{t}_i) \triangleq \nu_i(q, \hat{t}_i)$. Then, Q and the Weingarten map of $B_i(q)$ have the same eigenvectors (the principal directions). The eigenvalues of Q are*

$$\lambda_i = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_{ij}(q)\right) \tag{2.62}$$

Proof. By Proposition 50 it is

$$\nu_i(q, \hat{t}_i) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_i(q, \hat{t}_i)\right). \tag{2.63}$$

The symmetric quadratic form $\kappa_i(q, \hat{t}_i)$ can be expressed in terms of the matrix of the Weingarten map W as $\kappa_i(q, \hat{t}_i) = \hat{t}_i^{\mathrm{T}} W \hat{t}_i$. (In particular $W = \frac{1}{\|\nabla\beta_i\|} D^2\beta_i$.) So

$$\begin{aligned}
\nu_i(q, \hat{t}_i) &= \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \hat{t}_i^{\mathrm{T}} W \hat{t}_i\right) = \|\nabla\beta_i(q)\| \left(\hat{t}_i^{\mathrm{T}} \hat{t}_i \nu_{i3}(q) - \hat{t}_i^{\mathrm{T}} W \hat{t}_i\right) \\
&= \|\nabla\beta_i(q)\| \left(\hat{t}_i^{\mathrm{T}} \nu_{i3}(q) \hat{t}_i - \hat{t}_i^{\mathrm{T}} W \hat{t}_i\right) = \|\nabla\beta_i(q)\| \, \hat{t}_i^{\mathrm{T}} \left(\nu_{i3}(q) - W\right) \hat{t}_i \\
&= \hat{t}_i^{\mathrm{T}} \left(\|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - W\right)\right) \hat{t}_i
\end{aligned} \tag{2.64}$$

because $\hat{t}_i^{\mathrm{T}} \hat{t}_i = \left\|\hat{t}_i\right\|^2 = 1$. This proves that the matrix associated with the quadratic form $Q(\hat{t}_i) = \nu_i(q, \hat{t}_i)$ is $P = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - W\right)$. Assume that x is an eigenvector of $\kappa_i(q, \hat{t}_i)$, i.e., of matrix W (i.e., $x = \hat{p}_{ij}$) with eigenvalue κ_{ij}. Then

$$\begin{aligned}
\left(\|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - W\right)\right) x &= \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) x - W x\right) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) x - \kappa_{ij} x\right) \\
&= \left(\|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_{ij}\right)\right) x
\end{aligned} \tag{2.65}$$

so x is also an eigenvector of matrix P with eigenvalue $\|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_{ij}\right)$ and therefore an eigenvector of $Q(\hat{t}_i)$.

Solving for κ_i, it is [2] $\kappa_i(q, \hat{t}_i) = \frac{1}{\|\nabla\beta_i(q)\|} \hat{t}_i^{\mathrm{T}} P \hat{t}_i - c$. So it can be similarly proved that every eigenvector of ν_i is an eigenvector of κ_i, with the inverse relation for their associated eigenvalues. □

Corollary 53. *Select $\gamma_d(q) \triangleq \|q - q_d\|^2$. Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d(q) \neq 0$. Then, the functions ν_{ij} are the eigenvalues of the quadratic form $Q(\hat{t}_i) \triangleq \nu_i(q, \hat{t}_i)$ with associated eigenvectors $\hat{p}_{ij}$.*

Proof. By the hypotheses and Corollary 51 it holds that

$$\nu_{ij}(q) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_{ij}(q)\right)$$

Then Theorem 52 implies that the eigenvalues of $Q(\hat{t}_i)$ are

$$\lambda_i = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \kappa_{ij}(q)\right)$$

Combining the two previous equations yields that

$$\nu_{ij}(q) = \lambda_i \tag{2.66}$$

and proves the claim that the functions ν_{ij} equal the eigenvalues of $\nu_i(q, \hat{t}_i)$.

By Theorem 52 the Weingarten map and Q share the same eigenvectors, these are $\hat{p}_{ij}$. Since by Definition 49 it is $Q(\hat{p}_{ij}) = \nu_{ij}$, we can associate each eigenvalue ν_{ij} with a unique eigenvector $\hat{p}_{ij}$ corresponding to it. □

This justifies calling the functions ν_{ij} *principal relative curvatures* or principal relative curvature functions.

[2] By hypothesis function ν_{i3} is well-defined at q, which ensures that $\nabla\beta_i(q) \neq 0$. Therefore we can divide by $\|\nabla\beta_i(q)\|$.

Lemma 54. *Select $\gamma_d(q) \triangleq \|q - q_d\|^2$. Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d(q) \neq 0$. Then, the functions $\nu_{ij}(q)$ are real and continuous over $\mathscr{F}$.*

Proof. By the hypotheses and Proposition 50 it is

$$\nu_{ij}(q) = \|\nabla\beta_i(q)\| \left(\nu_{i3}(q) - \hat{\kappa}_{ij}(q)\right) = \|\nabla\beta_i(q)\| \left(\hat{\psi}_i(q)\frac{2}{\|\nabla\gamma_d(q)\|} - \hat{\kappa}_{ij}(q)\right) \tag{2.67}$$

By Lemma 110 the principal curvature functions $\hat{\kappa}_{ij}(q)$ are continuous over the points of E^n where β_i is regular. (A regular uniform tube of $\partial\mathscr{O}_i$ exists as already proved in Corollary 13). Also by assumption β_i is at least twice continuously differentiable over E^n and by hypothesis $\nabla\beta_q \neq 0$, so $\|\nabla\beta_i\|$ is continuous at q. By hypothesis and Proposition 47 function ν_{i3} is continuous.

So functions $\nu_{ij}(q)$ are continuous, as the sum of products of continuous functions $\|\nabla\beta_i\|$, ν_{i3} and κ_{ij}.

The principal curvatures are the eigenvalues of a symmetric matrix representing the Weingarten map, so they are real. The principal relative curvature functions are affinely related to the principal curvatures, so they are real. □

It is important that the eigenvalues ν_{ij} are all real, because it ensures that they can always be ordered.

Remark 55. *As remarked by Cecil (1986), the eigenvectors may be discontinuous functions and this happends at umbilic points. For this reason, it is not obvious that ν_{ij} are continuous functions at all.*

At this points, it is interesting to note that, according to Proposition 48 is continuously differentiable, hence continuous. Essentially, this relies on the fact that surface B_i is regular. This ensures that the tangent space T_qB_i is continuously smoothly mapped to the ambient space, by a smooth projection.

However, the same need not *be true for ν_{ij} as well, because it is a restriction of ν_i. It is a fiber section, in more detail the principal subbundle.*

In order to ensure that ν_{ij} is continuous, this restriction should be continuous. As noted by Cecil and proved by others, the restriction p_{ij} may be discontinuous.

Therefore, the principal subbundle may be discontinuous. How can it be that ν_{ij} is a continuous function? A continuous function defined in terms of a discontinuous section of the tangent bundle?

As it turns out, a generic continuous function would become discontinuous when one of its arguments becomes discontinuous. Nonetheless, this does not happen with ν_i.

The reason is that ν_i is not a continuous functions unrelated to $\hat{t}_i$ and the discontinuity of the principal subbundle. In particular, ν_i includes the Weingarten map and $\hat{\mathrm{p}}_{ij}$ are its eigenvectors - discontinuous though they may be at umbillics.

For this reason, it turns out that ν_i "absorbs" the discontinuity of $\hat{\mathrm{p}}_{ij}(q)$ by reducing to a function of the principal curvatures, which are continuous functions.

Chapter 3

Critical point-free neighborhoods

3.1 Acute or orthogonal gradients

We now show that obstacle parts whose outward normal $\hat{r}_i$ "looks" towards the destination q_d, do not obstruct the agent. Consider the negated gradient of $\hat{\varphi}$

$$-\nabla\hat{\varphi} = -\frac{\beta k \gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k \nabla\beta}{\beta^2} = \frac{k\gamma_d^{k-1}}{\beta}(-\nabla\gamma_d) + \frac{\gamma_d^k}{\beta^2}\nabla\beta \tag{3.1}$$

Observe that $-\nabla\hat{\varphi}$ is a linear combination of $(-\nabla\gamma_d)$ and $\nabla\beta$ with positive coefficients [1] .

Start by examining the components of $\nabla\beta_i$. Close enough to obstacle $\partial\mathscr{O}_i$, its repulsion $\nabla\beta_i$ dominates over $\nabla\bar{\beta}_i$. To illustrate this, consider

$$\nabla\beta = \beta\sum_{i\in I_0}\frac{\nabla\beta_i}{\beta_i} = \beta\left(\frac{\nabla\beta_i}{\beta_i} + \sum_{j\in I_0\setminus i}\frac{\nabla\beta_j}{\beta_j}\right) = \beta\left(\frac{1}{\beta_i}\nabla\beta_i + \frac{1}{\bar{\beta}_i}\nabla\bar{\beta}_i\right) \tag{3.2}$$

Select arbitrary $\varepsilon_i > 0$. The obstacles are disjoint. For this reason [2] , in an ε_i-neighborhood of $\mathscr{O}_i$, every other obstacle $\beta_j, j \neq i$ is bounded from below. In other words, there exist $\varepsilon_j > 0, j \neq i$, such that $\beta_j \geq \varepsilon_j$ for all $j \in I_0 \setminus i$. As a result, the product of $\beta_j, j \neq i$ is bounded from below

$$\bar{\beta}_i = \prod_{j\in I_0\setminus i}\beta_j \geq \prod_{j\in I_0\setminus i}\varepsilon_j > 0 \tag{3.3}$$

To make notation more convenient, let

$$\bar{\varepsilon}_i \triangleq \prod_{j\in I_0\setminus i}\varepsilon_j > 0 \tag{3.4}$$

By definition, $\partial\mathscr{O}_i$ is compact. Therefore, any ε_i-neighborhood is bounded. This implies that within an ε_i-neighborhood $\left\|\nabla\bar{\beta}_i\right\|$ is bounded from above, i.e., $\left\|\nabla\bar{\beta}_i\right\| \leq M < \infty$. In addition, by definition $\partial\mathscr{O}_i$ is a regular surface. Therefore, $\|\nabla\beta_i\|$ is bounded from below within a small enough ε_i-neighborhood, i.e., $0 < m \leq \|\nabla\beta_i\|$.

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[1]By definition, $0 < \beta$ and $0 < \gamma_d$ on the free space interior and away from the destination.

[2]Provided a sufficiently small $\varepsilon_i > 0$ is chosen.

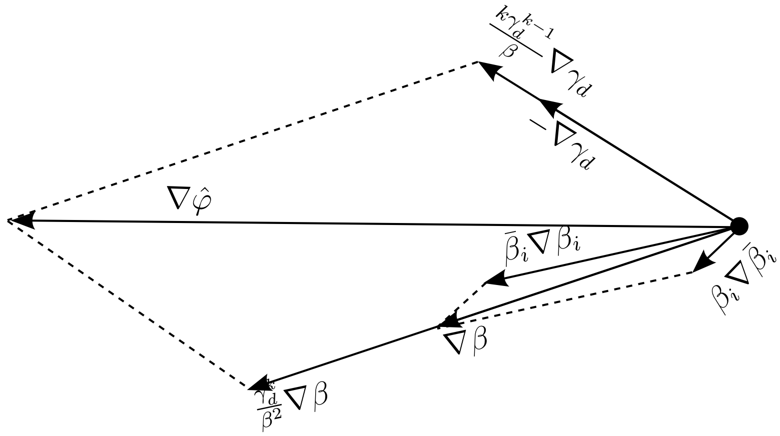

Figure 3.1: The vector $-\nabla\hat{\varphi}$ lies in the convex hull of $-\nabla\gamma_d$ and $\nabla\beta$. The vector $\nabla\beta$ depends on both $\nabla\beta_i$ and $\nabla\bar{\beta}_i$. But close to obstacle $\partial\mathscr{O}_i$, $\nabla\beta$ tends to $\nabla\beta_i$. The contribution of $\nabla\bar{\beta}_i$ to $\nabla\beta$ vanishes on $\partial\mathscr{O}_i$.

Using these observations it follows that in an ε_i-neighborhood of $\partial\mathscr{O}_i$

$$\left.\begin{array}{c}\|\nabla\bar{\beta}_i\| \leq M \\ \bar{\varepsilon}_i \leq \bar{\beta}_i\end{array}\right\} \implies \frac{\|\nabla\bar{\beta}_i\|}{\bar{\beta}_i} \leq \frac{M}{\bar{\varepsilon}_i}$$
$$m \leq \|\nabla\beta_i\| \implies \frac{m}{\beta_i} \leq \frac{\|\nabla\beta_i\|}{\beta_i} \tag{3.5}$$

Within an ε_i-neighborhood it is

$$\beta_i < \varepsilon_i \implies \frac{m}{\varepsilon_i} < \frac{m}{\beta_i} \leq \frac{\|\nabla\beta_i\|}{\beta_i} \tag{3.6}$$

For a small enough $\varepsilon_i > 0$, $\frac{\|\nabla\beta_i\|}{\beta_i}$ can be made arbitrarily larger than $\frac{\|\nabla\bar{\beta}_i\|}{\bar{\beta}_i}$, because the latter is bounded from above. Returning to (3.2), this illustrates why $\nabla\beta_i$ dominates $\nabla\bar{\beta}_i$ close to the obstacle. This can be observed also by considering that

$$\beta\left(\frac{1}{\beta_i}\nabla\beta_i + \frac{1}{\bar{\beta}_i}\nabla\bar{\beta}_i\right) = \bar{\beta}_i\nabla\beta_i + \beta_i\nabla\bar{\beta}_i \tag{3.7}$$

in combination with the preceding arguments.

For this reason, close to $\partial\mathscr{O}_i$, the existence of critical points depends on $-\nabla\gamma_d$ and only $\nabla\beta_i$ (instead of $\nabla\beta$). When $\nabla\beta_i$ and $-\nabla\gamma_d$ make an acute angle or are orthogonal, then they cannot annihilate. Their angle is at most $\frac{\pi}{2}$ when

$$0 \leq (-\nabla\gamma_d)\cdot\nabla\beta_i \iff \nabla\gamma_d\cdot\nabla\beta_i \leq 0 \tag{3.8}$$

So $\nabla\hat{\varphi}(q) \neq 0$ when $\nabla\gamma_d(q)\cdot\nabla\beta_i(q) \leq 0$. In other words, q cannot be a critical point.

By "pushing" the critical points very close to $\partial\mathscr{O}_i$, $\nabla\beta_i$ dominates $\nabla\bar{\beta}_i$ and we can apply the previous conclusions. This was an informal explanation. The formal proof reveals more. Provided $\mathscr{B}_i(\varepsilon_i)$ are pairwise disjoint, the thershold of (7.26) for k is sufficient to ensure that no "acute/orthogonal" point is a critical point.

We need to define two half-spaces [3], separated by the tangent plane T_qB_i of B_i at q.

[3] More accurately, the intersection of each half-space with the free space.

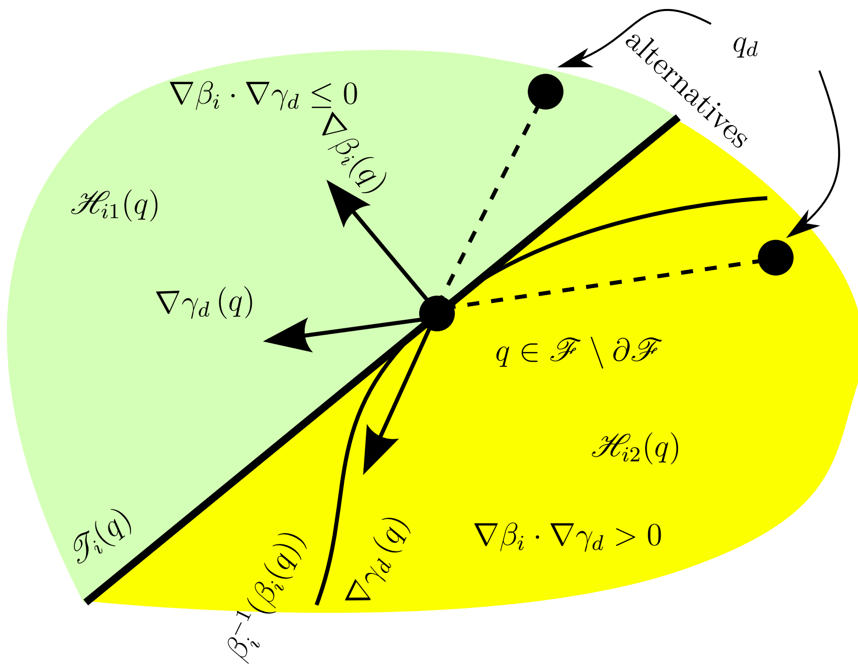

Figure 3.2: Positive/nonpositive inner product half-spaces, depending on q_d.

Definition 56 (Good and bad Half-spaces)**.** *The "good" half-space is*

$$\begin{aligned}\mathscr{H}_{i1}(q) &\triangleq \{q_d \in \mathscr{F} \setminus (\partial\mathscr{F} \cup \{q\}) \mid \nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) \leq 0\} \\ &= \left\{q_d \in \mathring{\mathscr{F}} \setminus \{q\} \middle| \, \psi_i(q) \leq 0\right\}\end{aligned} \tag{3.9}$$

For a fixed point q, it is the half space of possible q_d which render the inner product $\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q)$ nonpositive.

The "bad" half-space is

$$\begin{aligned}\mathscr{H}_{i2}(q) &\triangleq \{q_d \in \mathscr{F} \setminus (\partial\mathscr{F} \cup \{q\}) \mid 0 < \nabla\beta_i(q) \cdot \nabla\gamma_d\,(q)\} \\ &= \left\{q_d \in \mathring{\mathscr{F}} \setminus \{q\} \middle| \, \psi_i(q) \leq 0\right\}\end{aligned} \tag{3.10}$$

For destinations $q_d \in \mathscr{H}_{i1}(q)$ it is $\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) \leq 0$, so q is not a critical point, provided a large enough k has been selected. If k is sufficiently large, then all the critical points near obstacles are in $\mathscr{H}_{i1}(q)$.

For a given destination q_d, for each q, either

$$\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) \leq 0 \iff \psi_i(q) \leq 0 \tag{3.11}$$

or

$$0 < \nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) \iff 0 < \psi_i(q) \tag{3.12}$$

This leads us to define two disjoint and complementary subsets of $\mathscr{F}_n$.

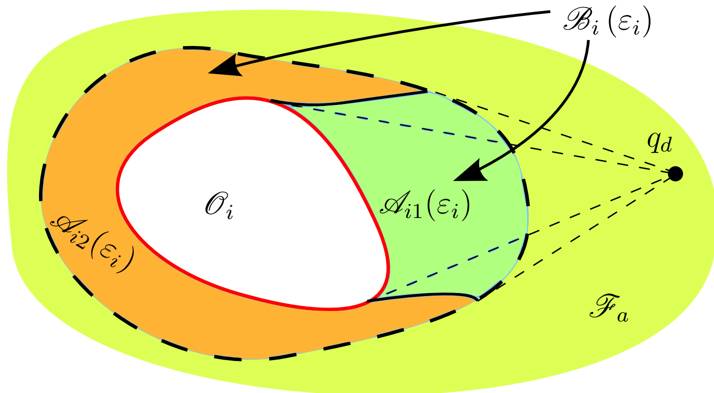

Figure 3.3: Good and bad neighborhoods.

Definition 57 (Subsets of set "near" obstacles)**.** *Let*

$$\begin{aligned}\mathscr{A}_{i1}(\varepsilon_i) &\triangleq \{q \in \mathscr{B}_i(\varepsilon_i) \mid \nabla\beta_i(q)\cdot\nabla\gamma_d(q) \leq 0\}\\ &= \{q \in \mathscr{B}_i(\varepsilon_i) \mid \psi_i(q) \leq 0\}\\ \mathscr{A}_{i2}(\varepsilon_i) &\triangleq \{q \in \mathscr{B}_i(\varepsilon_i) \mid 0 < \nabla\beta_i(q)\cdot\nabla\gamma_d(q)\}\\ &= \{q \in \mathscr{B}_i(\varepsilon_i) \mid 0 < \psi_i(q)\}\end{aligned} \tag{3.13}$$

where $i \in I_0$. Their unions are defined as

$$\mathscr{A}_1(\varepsilon_{I_0}) \triangleq \bigcup_{i\in I_0} \mathscr{A}_{i1}(\varepsilon_i) \qquad \mathscr{A}_2(\varepsilon_{I_0}) \triangleq \bigcup_{i\in I_0} \mathscr{A}_{i2}(\varepsilon_i) \tag{3.14}$$

It follows that

$$\mathscr{B}_i(\varepsilon_i) = \mathscr{A}_{i1}(\varepsilon_i) \cup \mathscr{A}_{i2}(\varepsilon_i) \tag{3.15}$$

and

$$\mathscr{F}_n = \mathscr{A}_1(\varepsilon_{I_0}) \cup \mathscr{A}_2(\varepsilon_{I_0}) \tag{3.16}$$

Note how these are related to $\mathscr{H}_{i1}(q), \mathscr{H}_{i2}(q)$. The sets $\mathscr{H}_{i1}(q)$ and $\mathscr{H}_{i2}(q)$ are defined for fixed q and concern all possible q_d selections. On the contrary, sets $\mathscr{A}_1, \mathscr{A}_2$ are defined for a given q_d. The proof assumes a q_d has been selected. For this reason we are going to work with sets $\mathscr{A}_1, \mathscr{A}_2$. The sets $\mathscr{H}_{i1}(q)$ and $\mathscr{H}_{i2}(q)$ are still useful for a better understanding of the problem's geometry. In addition, they we will prove useful later, when we explore what type of geometries are tractable by KRNFs [4]

Corollary 58. $\mathscr{A}_{i1}(\varepsilon_i) = \left\{q \in \mathscr{B}_i(\varepsilon_i) \mid \hat{\psi}_i(q) \leq 0\right\}, \mathscr{A}_{i2}(\varepsilon_i) = \left\{q \in \mathscr{B}_i(\varepsilon_i) \mid 0 < \hat{\psi}_i(q)\right\}$.

Recall the definition of $N(\varepsilon_{I_0})$ in (7.26).

Proposition 59 (No critical points in good subset "near" obstacles)**.** *Select a $q_d \in \mathring{\mathscr{F}}$. If $k \geq N(\varepsilon_{I_0})$ then $\mathscr{A}_1(\varepsilon_{I_0})$ contains no critical points. Formally*

$$k \geq N(\varepsilon_{I_0}) \implies \mathscr{C}_{\hat{\varphi}} \cap \mathscr{A}_1(\varepsilon_{I_0}) = \emptyset \tag{3.17}$$

[4] It turns out that the tractable geometries are common for any potential field-based method whose potential field is constructed as the synthesis of an attractive and a repulsive component.

This means that by setting $k \geq N(\varepsilon_{I_0})$ we confine critical points not just in $\bigcup_{i \in I_0} \mathscr{B}_i$, but in $\bigcup_{i \in I_0} (\mathscr{B}_i \cap \mathscr{A}_2(\varepsilon_{I_0}))$. The proof is inspired by Proposition 3.7, pp. 432-433, [1]. It generalizes it, from "inward" spheres, to any obstacle boundary part which "looks" towards the destination.

Proof. By definition of set $\mathscr{A}_1(\varepsilon_{I_0})$

$$\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) \leq 0, \quad \forall q \in \mathscr{A}_1(\varepsilon_{I_0}) \tag{3.18}$$

The inner product of $\nabla\gamma_d$ with $\nabla\hat{\varphi}$ is (Lemma 3.1 [1])

$$\begin{aligned}
\nabla\hat{\varphi} \cdot \nabla\gamma_d &= \frac{\gamma_d^k}{\beta^2}\left(4k\beta - \nabla\beta \cdot \nabla\gamma_d\right) \\
&= \frac{\gamma_d^k}{\beta^2}\Big(4k\beta - \big(\beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d + \bar{\beta}_i \nabla\beta_i \cdot \nabla\gamma_d\big)\Big) \\
&= \frac{\gamma_d^k}{\beta^2}\big(4k\beta - \beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d - \bar{\beta}_i \nabla\beta_i \cdot \nabla\gamma_d\big) \\
&= \frac{\gamma_d^k}{\beta^2}\big(4k\beta - \beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d\big) + \underbrace{\frac{\gamma_d^k \bar{\beta}_i}{\beta^2}\left(-\nabla\beta_i \cdot \nabla\gamma_d\right)}_{0\leq}
\end{aligned} \tag{3.19}$$

and, since from (3.18)

$$0 \leq -\nabla\beta_i \cdot \nabla\gamma_d \tag{3.20}$$

it follows that

$$\nabla\hat{\varphi} \cdot \nabla\gamma_d \geq \beta_i \frac{\gamma_d^k}{\beta^2}\big(4k\bar{\beta}_i - \nabla\bar{\beta}_i \cdot \nabla\gamma_d\big) \tag{3.21}$$

A sufficient condition for this inner product to be positive is

$$0 < 4k\bar{\beta}_i - \nabla\bar{\beta}_i \cdot \nabla\gamma_d \iff \nabla\bar{\beta}_i \cdot \nabla\gamma_d < 4k\bar{\beta}_i \iff \frac{1}{4}\frac{\nabla\bar{\beta}_i \cdot \nabla\gamma_d}{\bar{\beta}_i} < k \tag{3.22}$$

Therefore, if the tuning parameter is large enough

$$k > \frac{1}{4}\frac{\nabla\bar{\beta}_i(q) \cdot \nabla\gamma_d\,(q)}{\bar{\beta}_i(q)}, \quad \forall q \in \mathscr{A}_1(\varepsilon_{I_0}) \tag{3.23}$$

then the inner product

$$\nabla\hat{\varphi}\,(q) \cdot \nabla\gamma_d\,(q) > 0, \quad \forall q \in \mathscr{A}_1(\varepsilon_{I_0}) \tag{3.24}$$

and q cannot be a critical point. If q was a critical point, then $\nabla\hat{\varphi}\,(q) = 0 \implies \nabla\hat{\varphi}\,(q) \cdot \nabla\gamma_d\,(q) = 0$, which contradicts the previous inequality.

But this is satisfied by $k > N(\varepsilon_{I_0})$, because

$$\frac{1}{4}\frac{\nabla\bar{\beta}_i \cdot \nabla\gamma_d}{\bar{\beta}_i} \leq \frac{1}{2}\sqrt{\gamma_d}\frac{\left\|\nabla\bar{\beta}_i\right\|}{\bar{\beta}_i} \tag{3.25}$$

and

$$\frac{\left\|\nabla\bar{\beta}_i\right\|}{\bar{\beta}_i} \leq \sum_{j \in I_0 \setminus i} \frac{\left\|\nabla\beta_j\right\|}{\beta_j} \leq \sum_{j \in I_0 \setminus i} \frac{\left\|\nabla\beta_j\right\|}{\varepsilon_j} \tag{3.26}$$

because $\varepsilon_j \leq \beta_j(q)$ for all $q \in \mathscr{B}_i$, where $j \in I_0 \setminus i$. Then

$$\begin{aligned}\sum_{j\in I_0\setminus i} \frac{\|\nabla\beta_j\|}{\varepsilon_j} &= -\frac{\|\nabla\beta_i\|}{\varepsilon_i} + \frac{\|\nabla\beta_i\|}{\varepsilon_i} + \sum_{j\in I_0\setminus i} \frac{\|\nabla\beta_j\|}{\varepsilon_j} \\ &= -\frac{\|\nabla\beta_i\|}{\varepsilon_i} + \sum_{j\in I_0} \frac{\|\nabla\beta_j\|}{\varepsilon_j} < \sum_{j\in I_0} \frac{\|\nabla\beta_j\|}{\varepsilon_j}\end{aligned} \tag{3.27}$$

Also, it is

$$\frac{1}{2}\sqrt{\gamma_d(q)}\sum_{j\in I_0} \frac{\|\nabla\beta_j(q)\|}{\varepsilon_j(q)} \leq N(\varepsilon_{I_0}) < k \tag{3.28}$$

for all $q \in \mathscr{B}_i$.

As a result, combining the previous yields

$$\frac{1}{4}\frac{\nabla\bar{\beta}_i(q)\cdot\nabla\gamma_d(q)}{\bar{\beta}_i(q)} < N(\varepsilon_{I_0}) \leq k, \quad \forall q \in \mathscr{B}_i \tag{3.29}$$

By definition, for all $q \in \mathscr{A}_1(\varepsilon_{I_0})$ there exists an $i \in I_0$, such that $q \in \mathscr{B}_i$. It follows that $\frac{1}{4}\frac{\nabla\bar{\beta}_i(q)\cdot\nabla\gamma_d(q)}{\bar{\beta}_i(q)} < k$ and using (3.21) and (3.22) this ensures that $0 < \nabla\hat{\varphi}(q)\cdot\nabla\gamma_d(q)$. □

Lemma 60 (Critical points remain only in $\mathscr{A}_2(\varepsilon_{I_0})$)**.** *Select arbitrary $\varepsilon_i > 0$, where $i \in I_0$. For all $k \geq N(\varepsilon_{I_0})$ all critical points apart from the destination are in $\mathscr{A}_2(\varepsilon_{I_0})$. Formally*

$$\forall\varepsilon_i > 0.\ \exists\delta: \quad q_c \in \mathscr{A}_2(\varepsilon_{I_0}), \quad \forall q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}, \quad \forall k \geq \delta \tag{3.30}$$

where $\delta = N(\varepsilon_{I_0})$.

Proof. By Propositions 2.7, 3.2, 3.3 [1] there are no critical points on $\partial\mathscr{F}$. Moreover, critical points $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}$ different than the destination q_d cannot arise in $\mathscr{F}_d$. So the union $\partial\mathscr{F} \cup \mathscr{F}_d$ does not contain critical points other than q_d.

By Proposition 3.4 [1] for any $\varepsilon_i > 0$, where $i \in I_0$, if $k \geq N(\varepsilon_{I_0})$ then $\mathscr{F}_a$ is free of critical points other than q_d. So far $\mathscr{F}_a \cup \partial\mathscr{F} \cup \mathscr{F}_d$ does not contain critical points other than q_d.

Then, all critical points $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}$ can arise only in the set "near" obstacles

$$q_c \in \mathscr{F}_n = \mathscr{A}_1(\varepsilon_{I_0}) \cup \mathscr{A}_2(\varepsilon_{I_0}) \tag{3.31}$$

By Proposition 59 if $k \geq N(\varepsilon_{I_0})$ then the set $\mathscr{A}_1(\varepsilon_{I_0})$ is clear of critical points. It follows that

$$q_c \in \mathscr{F}_n \setminus \mathscr{A}_1(\varepsilon_{I_0}) = \mathscr{A}_2(\varepsilon_{I_0}) \tag{3.32}$$

This is equivalent to

$$\mathscr{C}_{\hat{\varphi}} \setminus \{q_d\} \subseteq \mathscr{A}_2(\varepsilon_{I_0}) \tag{3.33}$$

provided $k \geq N(\varepsilon_{I_0})$. □

Proposition 61 (Positive $\nu_{i1}(q)$ in $\mathscr{A}_2(\varepsilon_{I_0})$)**.** *Assume $\gamma_d(q) = \|q - q_d\|^2$. For any point $q \in \mathscr{A}_2(\varepsilon_{I_0})$ it is $0 < \nu_{i1}(q)$.*

Proof. For $\gamma_d = \|q - q_d\|^2$ by Proposition 40, provided $\nabla\gamma_d \neq 0$ then

$$\nu_{i1}(q) = 2\frac{\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q)}{\|\nabla\gamma_d\,(q)\|^2} \tag{3.34}$$

which has the same sign as $\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q)$. By definition, for every $q \in \mathscr{A}_2(\varepsilon_{I_0})$

$$0 < \nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) \implies 0 < \nu_{i1}(q) \tag{3.35}$$

because ν_{i1} has the same sign with $\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q)$. □

Corollary 62 (Remaining critical points have $0 < \nu_{i1}(q)$)**.** *Select arbitrary $\varepsilon_i > 0$, where $i \in I_0$. Assume $\gamma_d(q) = \|q - q_d\|^2$. If $k \geq N(\varepsilon_{I_0})$, then all critical points q_c other than the destination have $0 < \nu_{i1}(q_c)$.*

Proof. By hypothesis $k \geq N(\varepsilon_{I_0})$. Then, by Lemma 60, all critical points other than the destination are in $\mathscr{A}_2(\varepsilon_{I_0})$. By Proposition 61 it is $0 < \nu_{i1}(q)$ for any $q \in \mathscr{A}_2(\varepsilon_{I_0})$. As a result, $0 < \nu_{i1}(q_c)$ for all critical points other than the destination, because $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\} \subseteq \mathscr{A}_2(\varepsilon_{I_0})$. □

Corollary 63. *Select arbitrary $\varepsilon_i > 0$, where $i \in I_0$. Assume $\gamma_d = \|q - q_d\|^2$ and $\|\nabla\beta_i\|, \|\nabla\gamma_d\| \neq 0$. Then, for all $q \in \mathscr{A}_2(\varepsilon_{I_0})$ it is $0 < \nu_{i3}(q)$.*

Proof. By Proposition 61 function ν_{i1} is positive in $\mathscr{A}_2(\varepsilon_{I_0})$. Then, the hypothesis ensures that Proposition 43 applies and ν_{i1} is equi-signed to ν_{i3}. □

Proposition 64. *Select $\gamma_d = \|q - q_d\|^2$. Assume $\nabla\gamma_d\,(q), \nabla\beta_i(q) \neq 0$ for a $q \in U$. If $0 < a \leq \hat{\psi}_i(q)$ over a compact set U, then there exists a $b > 0$, such that $b \leq \nu_{i3}(q)$ over U.*

Proof. Function γ_d is C^∞, hence $\|\nabla\gamma_d\|$ is continuous on set U. Set U is compact by hypothesis, therefore $\|\nabla\gamma_d\|$ attains its maximum in U, so that

$$0 < \frac{2}{\max_U\{\|\nabla\gamma_d\,(q)\|\}} \leq \frac{2}{\|\nabla\gamma_d\,(q)\|} \tag{3.36}$$

for all $q \in U$. The maximum is positive because by hypothesis there exists a $q \in U$ such that $\|\nabla\gamma_d\,(q)\| \neq 0$.

By hypothesis $0 < a \leq \hat{\psi}_i(q)$ and multiplying with the previous inequality

$$\frac{2a}{\max_U\{\|\nabla\gamma_d\,(q)\|\}} \leq \frac{2\hat{\psi}_i(q)}{\|\nabla\gamma_d\,(q)\|} = \nu_{i3}(q) \tag{3.37}$$

Set $b = \frac{2a}{\max_U\{\|\nabla\gamma_d(q)\|\}} > 0$ to obtain the claim

$$0 < b \leq \nu_{i3}(q) \tag{3.38}$$

□

Note that the previous does not follow directly from Proposition 43. This is because it does not only refer to signs, i.e., that $0 < \psi_i(q)$ over U implies $0 < \nu_{i3}(q)$ over U. It concerns lower bounds on ψ_i, ν_{i3}. In order to derive the above result, the assumption of compactness for set U is essential.

Proposition 65. *Select* $\gamma_d = \|q - q_d\|^2$. *Assume* $\nabla\gamma_d(q), \nabla\beta_i(q) \neq 0$ *for all* $q \in U$. *If* $0 < a \leq \nu_{i3}(q)$ *over a compact set* U*, then there exists a* $b > 0$*, such that* $b \leq \nu_{i3}(q)$ *over* U.

Proof. As proved in (2.24) it is $\nu_{i1} = \|\nabla\beta_i\| \, \nu_{i3}$. By assumption $\nabla\beta_i(q) \neq 0$ over the compact set U, hence it attains its minimum in U and $0 < \min_U\{\|\nabla\beta_i\|(q)\} \leq \|\nabla\beta_i\|(q)$ for all $q \in U$. Combining this with the hypothesis $0 < a \leq \nu_{i3}(q)$, it follows that

$$0 < a \min_U\{\|\nabla\beta_i\|(q)\} \leq \|\nabla\beta_i\|(q)\nu_{i3}(q) = \nu_{i1}(q) \tag{3.39}$$

for all $q \in U$. $\square$

3.2 Obtuse gradients

The following is the formal counterpart of the discussion concerning (3.1). It proves that a critical point can arise only when $\nabla\beta$ has opposite direction from $\nabla\gamma_d$.

Proposition 66. *If $\nabla\gamma_d(q)\cdot\nabla\beta(q) < \|\nabla\gamma_d(q)\|\,\|\nabla\beta(q)\|$, then $\nabla\hat{\varphi}(q) \neq 0$ and hence q is not a critical point.*

Proof. It is

$$\begin{aligned} \nabla\hat{\varphi} &= \frac{\beta\nabla\left(\gamma_d^k\right) - \gamma_d^k\nabla\beta}{\beta^2} = \frac{\beta k\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla\beta}{\beta^2} \implies \\ \|\nabla\varphi\|^2 = \nabla\varphi\cdot\nabla\varphi &= \left(\frac{\beta k\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla\beta}{\beta^2}\right)\cdot\left(\frac{\beta k\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla\beta}{\beta^2}\right) \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} &\left(\beta k\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla\beta\right)\cdot\left(\beta k\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla\beta\right) \\ &= \left(\beta k\gamma_d^{k-1}\nabla\gamma_d\right)\cdot\left(\beta k\gamma_d^{k-1}\nabla\gamma_d\right) + \left(\beta k\gamma_d^{k-1}\nabla\gamma_d\right)\cdot\left(-\gamma_d^k\nabla\beta\right) \\ &\quad + \left(-\gamma_d^k\nabla\beta\right)\cdot\left(\beta k\gamma_d^{k-1}\nabla\gamma_d\right) + \left(-\gamma_d^k\nabla\beta\right)\cdot\left(-\gamma_d^k\nabla\beta\right) \\ &= \left(\beta k\gamma_d^{k-1}\|\nabla\gamma_d\|\right)^2 + \left(\gamma_d^k\|\nabla\beta\|\right)^2 - 2\left(\beta k\gamma_d^{k-1}\right)\gamma_d^k\nabla\gamma_d\cdot\nabla\beta \end{aligned} \tag{3.41}$$

and because by hypothesis

$$\begin{aligned} \nabla\gamma_d(q)\cdot\nabla\beta(q) &< \|\nabla\gamma_d(q)\|\,\|\nabla\beta(q)\| \implies \\ -\nabla\gamma_d(q)\cdot\nabla\beta(q) &< -\|\nabla\gamma_d(q)\|\,\|\nabla\beta(q)\| \end{aligned} \tag{3.42}$$

it follows that

$$\begin{aligned} &\left(\beta k\gamma_d^{k-1}\|\nabla\gamma_d\|\right)^2 + \left(\gamma_d^k\|\nabla\beta\|\right)^2 - 2\left(\beta k\gamma_d^{k-1}\right)\gamma_d^k\nabla\gamma_d\cdot\nabla\beta \\ &> \left(\beta k\gamma_d^{k-1}\|\nabla\gamma_d\|\right)^2 + \left(\gamma_d^k\|\nabla\beta\|\right)^2 - 2\left(\beta k\gamma_d^{k-1}\right)\gamma_d^k\|\nabla\gamma_d\|\,\|\nabla\beta\| \\ &= \left(\beta k\gamma_d^{k-1}\|\nabla\gamma_d\|\right)^2 + \left(\gamma_d^k\|\nabla\beta\|\right)^2 - 2\left(\beta k\gamma_d^{k-1}\|\nabla\gamma_d\|\right)\left(\gamma_d^k\|\nabla\beta\|\right) \\ &= \left(\beta k\gamma_d^{k-1}\|\nabla\gamma_d\| - \gamma_d^k\|\nabla\beta\|\right)^2 \geq 0 \end{aligned} \tag{3.43}$$

Therefore

$$\begin{aligned} \|\nabla\varphi(q)\|^2 &= \frac{1}{\beta^4}\Big(\left(\beta k\gamma_d^{k-1}\|\nabla\gamma_d\|\right)^2 + \left(\gamma_d^k\|\nabla\beta\|\right)^2 - 2\left(\beta k\gamma_d^{k-1}\right)\gamma_d^k\nabla\gamma_d\cdot\nabla\beta\Big) \\ &> \frac{\left(\beta k\gamma_d^{k-1}\|\nabla\gamma_d\| - \gamma_d^k\|\nabla\beta\|\right)^2}{\beta^4} \geq 0 \end{aligned} \tag{3.44}$$

which implies $\|\nabla\hat{\varphi}(q)\| > 0$, hence q is not a critical point, i.e., $q \notin \mathscr{C}_{\hat{\varphi}}$. □

Using Definition 44 and Proposition 45, Proposition 66 can be re-stated in terms of the cosine. This offers a clear geometric viewpoint.

Corollary 67. *Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d(q) \neq 0$, which ensures function $\cos\theta$ is well-defined at q. If $\cos\theta(q) < 1$, then q is not a critical point.*

Proof. By hypothesis and Proposition 45 function $\hat{\psi}(q)$ is well-defined. This means that the condition

$$\cos\theta(q) = \frac{\nabla\beta(q)\cdot\nabla\gamma_d\left(q\right)}{\left\|\nabla\beta(q)\right\|\left\|\nabla\gamma_d\left(q\right)\right\|} < 1 \iff \nabla\beta(q)\cdot\nabla\gamma_d\left(q\right) < \left\|\nabla\beta(q)\right\|\left\|\nabla\gamma_d\left(q\right)\right\| \tag{3.45}$$

By Proposition 66 this inequality implies that q is not a critical point. □

Proposition 68. *Select an arbitrary $\lambda \in (0,1)$. Then, there exists an $\hat{\varepsilon}_i > 0$ such that all $0 < \varepsilon_i < \hat{\varepsilon}_i$ have the following property. For every $q \in \overline{\mathscr{B}_i\left(\varepsilon_i\right)}$, function $\cos\theta_i$ is well-defined and if $\cos\theta_i(q) < \lambda$, then $\cos\theta(q) < 1$.*

The above is equivalent to

$$\cos\theta(q) = \frac{\nabla\beta(q)\cdot\nabla\gamma_d\left(q\right)}{\left\|\nabla\beta(q)\right\|\left\|\nabla\gamma_d\left(q\right)\right\|} < 1 \iff \nabla\beta(q)\cdot\nabla\gamma_d\left(q\right) < \left\|\nabla\beta(q)\right\|\left\|\nabla\gamma_d\left(q\right)\right\| \tag{3.46}$$

provided $\nabla\gamma_d\left(q\right), \left\|\nabla\beta(q)\right\| \neq 0$. Since the hypothesis does not ensure that $\cos\theta, \cos\theta_i$ are well-defined, we need to prove this. For this reason, it is first shown that $\nabla\beta(q)\cdot\nabla\gamma_d\left(q\right) < \left\|\nabla\beta(q)\right\|\left\|\nabla\gamma_d\left(q\right)\right\|$. This ensures that both $\nabla\gamma_d\left(q\right) \neq 0$ and $\nabla\beta(q) \neq 0$, so $\cos\theta$ is well-defined and we can divide by $\left\|\nabla\beta(q)\right\|\left\|\nabla\gamma_d\left(q\right)\right\|$ to obtain $\cos\theta(q) < 1$.

Proof. By Proposition 24 there exists an $\bar{\varepsilon}_{i1} > 0$ such that for all $0 < \varepsilon_i < \bar{\varepsilon}_{i1}$ set $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ is compact and $\nabla\beta_i, \nabla\gamma_d \neq 0$ in it. This ensures that function $\bar{\psi}_i = \cos\theta_i$ is well-defined in $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$.

For any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i1}$ the ratio $\frac{\nabla\gamma_d\cdot\nabla\beta_i}{\|\nabla\beta_i\|\|\nabla\gamma_d\|}$ is well-defined for every $q \in \overline{\mathscr{B}_i\left(\varepsilon_i\right)}$. So there exists an $\bar{\varepsilon}_{i1} > 0$ for which the criterion $\cos\theta_i(q) < \lambda$ is well-defined in $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ for all $\varepsilon_i < \bar{\varepsilon}_{i1}$.

The assumption that obstacles are disjoint ensures there exists a $\bar{\varepsilon}_{i2} > 0$ such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i2}$, it is $\bar{\beta}_i > 0$ in $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$.

Define $\bar{\varepsilon}_{i3} \triangleq \min\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i2}\}$ In what follows we take $\varepsilon_i < \bar{\varepsilon}_{i3}$.

It is

$$\nabla\beta = \bar{\beta}_i\nabla\beta_i + \beta_i\nabla\bar{\beta}_i \tag{3.47}$$

By hypothesis we are concerned with all points for which

$$\begin{aligned} &\frac{\nabla\gamma_d\cdot\nabla\beta_i}{\left\|\nabla\beta_i\right\|\left\|\nabla\gamma_d\right\|} < \lambda \iff \nabla\gamma_d\cdot\nabla\beta_i < \lambda\left\|\nabla\beta_i\right\|\left\|\nabla\gamma_d\right\| \overset{0<\bar{\beta}_i}{\iff} \\ &\bar{\beta}_i\nabla\beta_i\cdot\nabla\gamma_d < \lambda\bar{\beta}_i\left\|\nabla\beta_i\right\|\left\|\nabla\gamma_d\right\| \end{aligned} \tag{3.48}$$

Also

$$\nabla\gamma_d\cdot\nabla\bar{\beta}_i \leq \left\|\nabla\gamma_d\right\|\left\|\nabla\bar{\beta}_i\right\| \overset{0<\bar{\beta}_i}{\iff} \beta_i\nabla\bar{\beta}_i\cdot\nabla\gamma_d \leq \beta_i\left\|\nabla\bar{\beta}_i\right\|\left\|\nabla\gamma_d\right\| \tag{3.49}$$

Combining the previous two inequalities yields

$$\begin{aligned} \bar{\beta}_i\nabla\beta_i\cdot\nabla\gamma_d + \beta_i\nabla\bar{\beta}_i\cdot\nabla\gamma_d < \lambda\bar{\beta}_i\left\|\nabla\beta_i\right\|\nabla\gamma_d + \beta_i\left\|\nabla\bar{\beta}_i\right\|\left\|\nabla\gamma_d\right\| &\iff \\ \left(\bar{\beta}_i\nabla\beta_i + \beta_i\nabla\bar{\beta}_i\right)\cdot\nabla\gamma_d < \lambda\bar{\beta}_i\left\|\nabla\beta_i\right\|\nabla\gamma_d + \beta_i\left\|\nabla\bar{\beta}_i\right\|\nabla\gamma_d &\iff \\ \nabla\beta\cdot\nabla\gamma_d < \lambda\bar{\beta}_i\left\|\nabla\beta_i\right\|\left\|\nabla\gamma_d\right\| + \beta_i\left\|\nabla\bar{\beta}_i\right\|\left\|\nabla\gamma_d\right\| & \end{aligned} \tag{3.50}$$

We want to prove that $\nabla\beta \cdot \nabla\gamma_d < \|\nabla\beta\| \|\nabla\gamma_d\|$. Taking into account (3.50), a sufficient condition for this inequality to hold is

$$\begin{aligned} \lambda\bar{\beta}_i \|\nabla\beta_i\| \|\nabla\gamma_d\| + \beta_i \left\|\nabla\bar{\beta}_i\right\| \|\nabla\gamma_d\| < \|\nabla\beta\| \|\nabla\gamma_d\| \overset{q\neq q_d \implies \nabla\gamma_d \neq 0}{\iff} \\ \lambda\bar{\beta}_i \|\nabla\beta_i\| + \beta_i \left\|\nabla\bar{\beta}_i\right\| < \|\nabla\beta\| \end{aligned} \tag{3.51}$$

Since

$$\begin{aligned} \nabla\beta = \bar{\beta}_i\nabla\beta_i + \beta_i\nabla\bar{\beta}_i \implies \\ \|\nabla\beta\| = \left\|\bar{\beta}_i\nabla\beta_i + \beta_i\nabla\bar{\beta}_i\right\| \geq \left\|\bar{\beta}_i\nabla\beta_i\right\| - \left\|\beta_i\nabla\bar{\beta}_i\right\| \overset{0<\bar{\beta}_i,\beta_i}{=} \\ = \bar{\beta}_i \|\nabla\beta_i\| - \beta_i \left\|\nabla\bar{\beta}_i\right\| \implies \\ \bar{\beta}_i \|\nabla\beta_i\| - \beta_i \left\|\nabla\bar{\beta}_i\right\| \leq \|\nabla\beta\| \end{aligned} \tag{3.52}$$

A sufficient condition for this to hold is

$$\begin{aligned} \lambda\bar{\beta}_i \|\nabla\beta_i\| + \beta_i \left\|\nabla\bar{\beta}_i\right\| < \bar{\beta}_i \|\nabla\beta_i\| - \beta_i \left\|\nabla\bar{\beta}_i\right\| \iff \\ \beta_i \left\|\nabla\bar{\beta}_i\right\| + \beta_i \left\|\nabla\bar{\beta}_i\right\| < \bar{\beta}_i \|\nabla\beta_i\| - \lambda\bar{\beta}_i \|\nabla\beta_i\| \iff \\ 2\beta_i \left\|\nabla\bar{\beta}_i\right\| < (1-\lambda)\bar{\beta}_i \|\nabla\beta_i\| \end{aligned} \tag{3.53}$$

If $\nabla\bar{\beta}_i = 0$, then no restriction applies to β. In case $\nabla\bar{\beta}_i \neq 0$, then the above is equaivalent to the requirement

$$\beta_i < \frac{(1-\lambda)\bar{\beta}_i \|\nabla\beta_i\|}{\left\|\nabla\bar{\beta}_i\right\|} \tag{3.54}$$

A sufficient condition for this to hold is

$$0 < \beta_i < \varepsilon_i \tag{3.55}$$

where $0 < \varepsilon_i < \bar{\varepsilon}_i$ and

$$\bar{\varepsilon}_i \triangleq \min\{\bar{\varepsilon}_{i3}, \bar{\varepsilon}_{i4}\} \tag{3.56}$$

where

$$\bar{\varepsilon}_{i4} \triangleq \frac{(1-\lambda)\min_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\bar{\beta}_i \|\nabla\beta_i\|\}}{\max_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\left\|\nabla\bar{\beta}_i\right\|\}} \tag{3.57}$$

Note that $\max_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\left\|\nabla\bar{\beta}_i\right\|\} > 0$ because we need to define $\bar{\varepsilon}_i$ only in the case that $\nabla\bar{\beta}_i \neq 0$ at some point in $\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}$. Also $\min_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\bar{\beta}_i\} > 0$ because $\bar{\varepsilon}_{i3} \leq \bar{\varepsilon}_{i2}$, which ensures that $\bar{\beta}_i > 0$ in $\overline{\mathscr{B}_i(\varepsilon_{i3})}$. It is $\min_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\|\nabla\beta_i\|\} > 0$ because $\bar{\varepsilon}_{i3} \leq \bar{\varepsilon}_{i1}$, which ensures that $\nabla\beta_i \neq 0$ in $\overline{\mathscr{B}_i(\varepsilon_{i3})}$ (a regular neighborhood).

This proves the claim that

$$\nabla\beta(q) \cdot \nabla\gamma_d(q) < \|\nabla\beta(q)\| \|\nabla\gamma_d(q)\|.$$

In turn, this ensures that $\nabla\gamma_d(q) \neq 0$ and $\nabla\beta(q) \neq 0$, because otherwise equality would hold. So $\cos\theta$ is well-defined at point q. We can divide both sides of the inequality to obtain

$$\frac{\nabla\beta(q) \cdot \nabla\gamma_d(q)}{\|\nabla\beta(q)\| \|\nabla\gamma_d(q)\|} < 1 \iff \cos\theta(q) < 1 \tag{3.58}$$

$\square$

Corollary 69. *Select arbitrary $\lambda \in (0,1)$. There exists an $\hat{\varepsilon}_i > 0$, such that every $0 < \varepsilon_i < \hat{\varepsilon}_i$ has the following property. For every $q \in \overline{\mathscr{B}_i(\varepsilon_i)}$, function $\cos\theta_i$ is well-defined and if $\cos\theta_i(q) < \lambda$, then q is not a critical point.*

Proof. By Proposition 68, there exists an $\hat{\varepsilon}_i > 0$, such that for all $0 < \varepsilon_i < \hat{\varepsilon}_i$ the following holds. For every $q \in \overline{\mathscr{B}_i(\varepsilon_i)}$ with $\cos\theta_i(q) < \lambda$, it holds that

$$\cos\theta(q) < 1 \tag{3.59}$$

Then, by Corollary 67 it follows that q is not a critical point. □

In the following definition "contact" refers to first order contact between the (regular) level sets of β_i and γ_d. Note that sets L_{i1}, L_{i2} are well-defined only if $\varepsilon_i < \bar{\varepsilon}_i$, where $\bar{\varepsilon}_i$ is the one provided by Proposition 24. This restriction is needed to ensure that $\hat{\psi}_i$ is well-defined. This requirement is ensured by Proposition 68.

Definition 70 (Set "near" contact points)**.** *For some $\lambda \in (0,1)$ and $\varepsilon > 0$, the set "away" from first-order contact points is*

$$L_{i1}(\varepsilon, \lambda) \triangleq \left\{ q \in \mathscr{B}_i(\varepsilon) \mid \hat{\psi}_i(q) < \lambda \right\} \tag{3.60}$$

and the set "near" to first order contact-points is

$$L_{i2}(\varepsilon, \lambda) \triangleq \left\{ q \in \mathscr{B}_i(\varepsilon) \mid \lambda \leq \hat{\psi}_i(q) \right\}, \tag{3.61}$$

where $i \in I_0$.

These definitions lead us to define the unions

$$L_1(\varepsilon_{I_0}, \lambda) \triangleq \bigcup_{i \in I_0} L_{i1}(\varepsilon_i, \lambda) \qquad L_2(\varepsilon_{I_0}, \lambda) \triangleq \bigcup_{i \in I_0} L_{i2}(\varepsilon_i, \lambda) \tag{3.62}$$

Observe that $\mathscr{F}_n = L_1(\varepsilon_{I_0}, \lambda) \cup L_2(\varepsilon_{I_0}, \lambda)$ and $L_1(\varepsilon_{I_0}, \lambda) \cap L_2(\varepsilon_{I_0}, \lambda) = \emptyset$.

The following proves that the set "away" from first order contact points does not contain any critical points. In other words $L_{i1}(\varepsilon_i, \lambda) \cap \mathscr{C}_{\hat{\varphi}} = \emptyset$.

Proposition 71. *Select arbitrary $\lambda \in (0,1)$. Then, there exists an $\hat{\varepsilon}_i > 0$, such that all $0 < \varepsilon_i < \hat{\varepsilon}_i$ have the following property. There are no critical points in $L_{i1}(\varepsilon_i, \lambda)$.*

Proof. By Definition 70, for every $q \in L_{i1}(\varepsilon_i, \lambda)$

$$\hat{\psi}_i(q) < \lambda \iff \cos\theta_i(q) < \lambda \tag{3.63}$$

Select the $\hat{\varepsilon}_i > 0$ defined in Corollary 69. Applying Corollary 69 implies that q is not a critical point. To summarize, no point in the set $L_{i1}(\varepsilon_i, \lambda)$ can be a critical point, provided $0 < \varepsilon_i < \hat{\varepsilon}_i$. □

Corollary 72. *Select arbitrary $\lambda \in (0,1)$. Then, there exists an $\hat{\varepsilon}_i > 0$, such that all $0 < \varepsilon_i < \hat{\varepsilon}_i$*

$$\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq L_{i2}(\varepsilon_i, \lambda) \tag{3.64}$$

In other words, all critical points in the set $\mathscr{B}_i(\varepsilon)$ are within its subset $L_{i2}(\varepsilon_i, \lambda)$.

Proof. Consider the set $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ of critical points in $\mathscr{B}_i(\varepsilon_i)$. By Proposition 71 there exists an $\hat{\varepsilon}_i > 0$, such that all $0 < \varepsilon_i < \hat{\varepsilon}_i$ there are no critical points in $L_{i1}(\varepsilon_i, \lambda)$. Definition 70 implies that $\mathscr{B}_i(\varepsilon_i) \setminus L_{i1}(\varepsilon_i, \lambda) = L_{i2}(\varepsilon_i, \lambda)$. So all critical points in the set $\mathscr{B}_i(\varepsilon_i)$ are within $L_{i2}(\varepsilon_i, \lambda)$. □

Lemma 73. *Select arbitrary $\lambda \in (0, 1)$. There exist $\hat{\varepsilon}_i > 0$, where $i \in I_0$, such that for any set ε_{I_0} of $0 < \varepsilon_i < \hat{\varepsilon}_i$, the following holds. If $k > N(\varepsilon_{I_0})$, then all critical points, other than the destination, arise in $L_2(\varepsilon_i, \lambda)$.*

Proof. By Propositions 2.7, 3.2, 3.3, 3.4 [1], all critical points apart from the destination are in $\mathscr{F}_n$. This has been discussed previously in Lemma 60.

For every $i \in I_0$ select the $\hat{\varepsilon}_i > 0$ as defined in Proposition 71. Then, by Proposition 71 for any $0 < \varepsilon_i < \hat{\varepsilon}_i$ there are no critical points in $L_{i1}(\varepsilon_i, \lambda)$. Therefore, there are no critical points in $L_1(\varepsilon_{I_0}, \lambda) = \bigcup_{i \in I_0} L_{i1}(\varepsilon_i, \lambda)$.

As a result, all critical points apart from the destination are in $\mathscr{F}_n \setminus L_1(\varepsilon_{I_0}, \lambda)$. By definition $\mathscr{F}_n(\varepsilon_{I_0}) = L_1(\varepsilon_{I_0}, \lambda) \cup L_1(\varepsilon_{I_0}, \lambda)$ and $L_1(\varepsilon_{I_0}, \lambda) \cap L_2(\varepsilon_{I_0}, \lambda) = \emptyset$. Therefore, $\mathscr{F}_n \setminus L_1(\varepsilon_{I_0}, \lambda) = L_2(\varepsilon_{I_0}, \lambda)$.

In sum, all critical points other than the destination arise in $L_2(\varepsilon_{I_0}, \lambda)$, provided $0 < \varepsilon_i < \hat{\varepsilon}_i$ and $k > N(\varepsilon_{I_0})$. □

Chapter 4

Critical points tend to contact points

In this chapter we are going to show that critical points tend closer to first order contact points between β_i and γ_d. At first order contact points some level set of γ_d has a common tangent space (in the ambient space) with the 0 level set of some obstacle β_i (for some $i \in I_0$.

4.1 Contact points

Definition 74 (First-Order Contact Point). *A point $q \in \partial\mathscr{O}_i$ for which $\nabla\beta_i(q) \neq 0$, $\nabla\gamma_d(q) \neq 0$,*

$$\nabla\beta_i(q) \times \nabla\gamma_d(q) = 0 \tag{4.1}$$

and

$$\nabla\beta_i(q) \cdot \nabla\gamma_d(q) > 0 \tag{4.2}$$

is called a first-order contact point *of $\partial\mathscr{O}_i$ with γ_d.*

Note that the definition requires that level sets of both β_i and γ_d be well-defined at q. Definition 74 concerns only boundary points, so well-definiteness is ensured by the next Proposition.

Proposition 75. *The functions β_i and γ_d are regular on the obstacle boundary $\partial\mathscr{O}_i$.*

Proof. Assumption 1 ensures that $\nabla\beta_i(q) \neq 0$ and Assumption 32 that $\nabla\gamma_d(q) \neq 0$ on $\partial\mathscr{O}_i$. □

Therefore, the criterion for first-order contact is well-defined on all of the obstacle boundary $\partial\mathscr{O}_i$.

Definition 76 (First-order Contact Locus). *The set of first-order contact points between $\partial\mathscr{O}_i$ and γ_d is denoted by $C_i^1(\partial\mathscr{O}_i, \gamma_d)$ (C_i^1 for brevity).*

The set C_i^1 is the locus of points where $\nabla\beta_i(q)$ and $\nabla\gamma_d(q)$ become parallel.

Proposition 77. *Assume q is a (first-order) contact point of $\partial\mathscr{O}_i$ with γ_d. Then $\hat{\psi}_i(q) = 1$.*

Proof. By definition $\hat{\psi}_i(q) = \cos\theta_i(q)$. By definition of a contact point, it is $\nabla\beta_i(q) \times \nabla\gamma_d(q) = 0$, which implies $\theta_i(q) = 0$. Therefore, $\cos\theta_i(q) = \cos(0) = 1$, so $\hat{\psi}_i(q) = 1$. □

Proposition 78. *Assume that $\hat{\psi}_i(q) = 1$ for some $q \in \partial\mathscr{O}_i$. Then q is a first-order contact point of $\partial\mathscr{O}_i$ with γ_d.*

Proof. By Proposition 75 it is $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d\,(q) \neq 0$ for all $q \in \partial\mathscr{O}_i$. So the function $\hat{\psi}_i$ is well-defined on all of $\partial\mathscr{O}_i$. Therefore the hypothesis is always well-defined.

It is

$$\hat{\psi}_i(q) = 1 \iff \frac{\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q)}{\|\nabla\beta_i(q)\|\,\|\nabla\gamma_d\,(q)\|} = 1$$

so because $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d\,(q) \neq 0$

$$\frac{\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q)}{\|\nabla\beta_i(q)\|\,\|\nabla\gamma_d\,(q)\|} = 1 \iff \nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) = \|\nabla\beta_i(q)\|\,\|\nabla\gamma_d\,(q)\| > 0,$$

which is the second property in Definition 74.

The first property of Definition 74 follows from the identity

$$\begin{aligned}\|\nabla\beta_i(q) \times \nabla\gamma_d\,(q)\| &= \|\nabla\beta_i(q)\|^2\,\|\nabla\gamma_d\,(q)\|^2 - (\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q))^2 \\ &= \|\nabla\beta_i(q)\|^2\,\|\nabla\gamma_d\,(q)\|^2 - (\|\nabla\beta_i(q)\|\,\|\nabla\gamma_d\,(q)\|)^2 = 0.\end{aligned}$$

□

Lemma 79. *Any boundary point $q \in \partial\mathscr{O}_i$ is a first-order contact point if and only if $\hat{\psi}_i(q) = 1$.*

Proof. Follows directly from Proposition 77 and Proposition 78. □

Proposition 80. *The first-order contact locus C_i^1 is compact.*

Proof. By Assumption 1 the obstacle level set $\partial\mathscr{O}_i$ is compact. By Definition 74 it is $C_i^1 \subseteq \partial\mathscr{O}_i$, so C_i^1 is bounded. Also by its definition, C_i^1 is closed, hence C_i^1 is a compact set. □

Let $\kappa_d(q, \hat{v}), \hat{v} \in T_q E^n$ be the normal curvature of γ_d and $\kappa_{dw}(q), w \in \mathbb{N}^*_{\leq n}$ be the principal curvatures of γ_d at q.

Definition 81. *Let q be a first-order contact point. Assume that $\kappa_{ij}(q)$ and $\kappa_{dw}(q)$ are principal curvatures associated with the same direction (which is a principal direction for both β_i and γ_d). If*

$$\kappa_{ij}(q) = \kappa_{dw}(q), \tag{4.3}$$

then q is called a second-order contact point *of $\partial\mathscr{O}_i$ with γ_d.*

Corollary 82. *All second-order contact points are first-order contact points.*

Note that the converse is not true. Not all first-order contact points are second-order contact points. In fact, it can be the case that no second-order contact points exist, whereas it can be proved that first-order contact points always do exist.

Define the principal curvature center as

$$\eta_{ij}(q) \triangleq q - R_{ij}(q)\frac{\nabla\beta_i(q)}{\|\nabla\beta_i(q)\|}. \tag{4.4}$$

Note that Definition 81 is equivalent to the following.

Definition 83 (Second-order Contact Point)**.** *A point $q \in \partial\mathscr{O}_i$ for which*

$$\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) > 0 \tag{4.5}$$

and some principal curvature center of β_i coincides with some principal curvature center of γ_d for the same tangent direction, i.e.,

$$\eta_{ij}(q) = \eta_{dw}(q) \quad \textit{and} \quad \hat{p}_{ij} = \hat{p}_{dw} \tag{4.6}$$

is called a second-order contact point *of $\partial\mathscr{O}_i$ with γ_d.*

Let us prove their equivalence.

Proposition 84. *Definition 81 is equivalent to Definition 83.*

Proof. Assume that q is a first-order contact point with $\kappa_{ij}(q) = \kappa_{dr}(q)$. By Lemma 79 at a first-order contact point it is

$$\cos\theta_i(q) = 1$$

so there exists some $a \in \mathbb{R}$ such that

$$\nabla\beta_i(q) = a\nabla\gamma_d\,(q) \implies \frac{\nabla\beta_i(q)}{\|\nabla\beta_i(q)\|} = \frac{\nabla\gamma_d\,(q)}{\|\nabla\gamma_d\,(q)\|}$$

By hypothesis $\kappa_{ij}(q) = \kappa_{dr}(q)$, so $\eta_{ij}(q) = \eta_{dr}(q)$.

Assume now that $\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) > 0$ and $\eta_{ij}(q) = \eta_{dw}(q)$. Then,

$$\kappa_{ij}(q)\frac{\nabla\beta_i(q)}{\|\nabla\beta_i(q)\|} = \kappa_{dw}(q)\frac{\nabla\gamma_d\,(q)}{\|\nabla\gamma_d\,(q)\|}.$$

This is possible only if $\frac{\nabla\beta_i(q)}{\|\nabla\beta_i(q)\|}$ and $\frac{\nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|}$ are parallel *or* anti-parallel. Since $\nabla\beta_i(q) \cdot \nabla\gamma_d\,(q) > 0$, they can only be parallel. So q is a first-order contact point and

$$\frac{\nabla\beta_i(q)}{\|\nabla\beta_i(q)\|} = \frac{\nabla\gamma_d\,(q)}{\|\nabla\gamma_d\,(q)\|}.$$

Therefore

$$\kappa_{ij}(q) = \kappa_{dw}(q).$$

$\square$

Remark 85. *The important point to observe in the previous two definitions is the orientation they explicitly incorporate. At a first-order contact point, it readily follows from Definition 74 that the normals $\nabla\beta_i(q)$ and $\nabla\gamma_d\,(q)$ have the same orientation. However, if we only assumed that $\eta_{ij}(q) = \eta_{dw}(q)$, then either*

$$\frac{\nabla\beta_i(q)}{\|\nabla\beta_i(q)\|} = \frac{\nabla\gamma_d\,(q)}{\|\nabla\gamma_d\,(q)\|} \quad \textit{and} \quad \kappa_{ij}(q) = \kappa_{dw}(q), \tag{4.7}$$

or

$$\frac{\nabla\beta_i(q)}{\|\nabla\beta_i(q)\|} = -\frac{\nabla\gamma_d\,(q)}{\|\nabla\gamma_d\,(q)\|} \quad \textit{and} \quad \kappa_{ij}(q) = -\kappa_{dw}(q). \tag{4.8}$$

But in the later case $\kappa_{ij}(q)$ is convex and $\kappa_{dw}(q)$ concave. So these principal curvatures of β_i and γ_d are qualitatively different. For this reason, the orientation constraint $\nabla\beta_i(q)\cdot\nabla\gamma_d\,(q) > 0$ is not omitted.

So, the fact that a second-order contact point is also a first-order contact point is almost explicitly stated in both definitions of a second-order contact point. It may appear superfluous, but it is not. The reason is that the principal curvature $\kappa_{ij}(q)$ defines the relative orientation of $\eta_{ij}(q) - q$ and $\nabla\beta_i(q)$.

Therefore, equality of principal curvatures ensures that $\eta_{ij}(q) - q$ is parallel to $\nabla\beta_i(q)$ and $\eta_{dw}(q) - q$ is parallel to $\nabla\gamma_d\,(q)$. But it does not tell us anything about the relative directions of $\nabla\beta_i(q)$ and $\nabla\gamma_d\,(q)$. So the principal curvature centers $\eta_{ij}(q)$ and $\eta_{dw}(q)$ need not coincide. Therefore, it is important to add the condition that $\nabla\beta_i(q)$ and $\nabla\gamma_d\,(q)$ are parallel, i.e., that q is a first-order contact point. Then $\eta_{ij}(q) - q$ is parallel to $\nabla\beta_i(q)$, which is parallel to $\eta_{dw}(q) - q$, which is parallel to $\eta_{dw}(q) - q$.

Conversely, if the principal curvature centers coincide, then $\nabla\beta_i(q)$ and $\nabla\gamma_d\,(q)$ may be parallel, in which case $\eta_{ij}(q) - q$ and $\eta_{dw}(q) - q$ are parallel as well, so the principal curvatures are equal. However, if $\nabla\beta_i(q)$ and $\nabla\gamma_d\,(q)$ are anti-parallel, then $\eta_{ij}(q) - q$ and $\eta_{dw}(q) - q$ would be anti-parallel, so the principal curvatures would have equal absolute values, but opposite sign.

Proposition 86. *Define $\gamma_d(q) = \|q - q_d\|^2$.*

Then, eq. (4.3) in Definition 81 becomes

$$\kappa_{ij}(q) = \frac{1}{\|q - q_d\|} \iff R_{ij}(q) = \|q - q_d\|\,. \tag{4.9}$$

Proof. By hypothesis $\gamma_d(q) = \|q - q_d\|^2$, so

$$\kappa_d(q, \hat{t}_i) = \frac{1}{\|q - q_d\|}, \tag{4.10}$$

for all unit tangent vectors $\hat{t}_i$. Then, every unit tangent direction can be chosen as a principal direction for $\kappa_d(q, \hat{t}_i)$. This implies that eq. (4.3) becomes

$$\kappa_{ij}(q) = \frac{1}{\|q - q_d\|}. \tag{4.11}$$

Note the equivalent form of eq. (4.3) in this case

$$R_{ij}(q) = \|q - q_d\|\,. \tag{4.12}$$

□

Corollary 87. *Define $\gamma_d(q) = \|q - q_d\|^2$. Assume that q is a second-order contact point with respect to $\kappa_{ij}(q)$.*

Then, $\kappa_{ij}(q) > 0$.

Corollary 88. *Define $\gamma_d(q) = \|q - q_d\|^2$. Assume that at point $q \in \partial\mathscr{O}_i$ it is $\kappa_{ij}(q) \leq 0$ (non-convex). Then q cannot be a second-order contact point with respect to $\kappa_{ij}(q)$.*

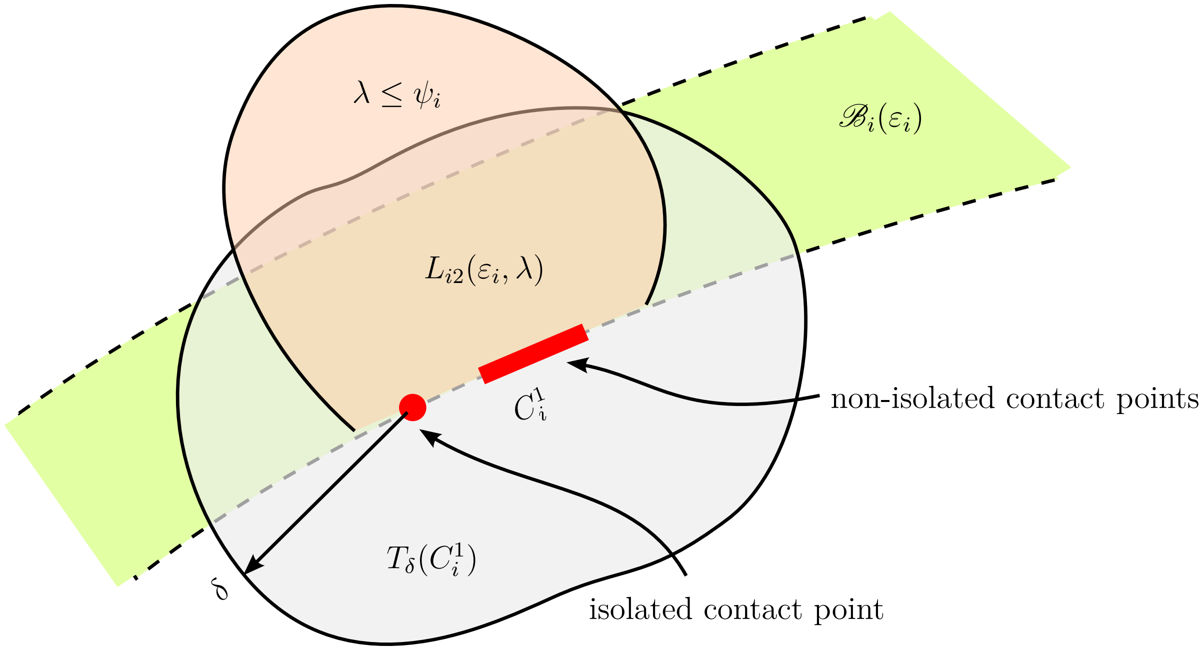

Figure 4.1: Select arbitrary $\delta > 0$. For some $\hat{\varepsilon} > 0$ and $\lambda \in (0,1)$, it is $L_{i2}(\varepsilon_i, \lambda) \subseteq T_\delta(C_i^1)$, for every $\varepsilon_i \in (0, \hat{\varepsilon})$.

Recall that all principal curvatures are non-positive at a concave point, defined in Definition 173. The next Corollary is a direct consequence of Corollary 88.

Corollary 89. *Define $\gamma_d(q) = \|q - q_d\|^2$. A concave point cannot be a second-order contact point.*

Definition 90 (Second-order Contact Locus)**.** *The set of second-order contact points between $\partial\mathscr{O}_i$ and γ_d is defined as the* second-order contact locus *and denoted by $C_i^2\left(\partial\mathscr{O}_i, \gamma_d\right)$.*

4.2 Critical points and Contact points

Lemma 91. *Select arbitrary $\delta > 0$. Then, there exist $\hat{\varepsilon} > 0$ and $\lambda \in (0,1)$, with the following property. For all $\varepsilon_i > 0$ with $\varepsilon_i < \hat{\varepsilon}$*

$$L_{i2}(\varepsilon_i, \lambda) \subseteq T_\delta\left(C_i^1\right) \tag{4.13}$$

In other words, the set "near" contact points $L_{i2}(\varepsilon_i, \lambda)$ is contained in the uniform tube $T_\delta(C_i^1)$.

Proof. Assume the contrary. For all $\varepsilon_i > 0$ and all $\lambda \in (0,1)$ there exists some point in $L_{i2}(\varepsilon_i, \lambda)$ which is not in $T_\delta(C_i^1)$. Denote this point using q_0. Let q_{id} be any contact point. By assumption, q_0 is not within an open ball $B_\delta(q_{id})$ of some contact point. Equivalently, no contact point is within an open ball $B_\delta(q_0)$ with center q_0. Therefore [1] , every $q \in B_\delta(q_0) \cap \partial\mathscr{O}_i$ has $\psi_i(q) < 1$.

[1] Formally, the claim is

$$\forall \delta > 0.\ \exists \hat{\varepsilon}_i > 0.\ \exists \lambda \in (0,1):$$
$$\forall \varepsilon_i \in (0, \hat{\varepsilon}_i)\ \forall q_0 \in L_{i2}(\varepsilon_i, \lambda).\ \exists q_{id} \in B_\delta(q_0) \cap \partial\mathscr{O}_i:\ \psi_i(q_{id}) = 1$$

We are going to show that this claim is absurd, given the continuity of function ψ_i.

Observe that it is not guaranteed that the neighborhood $\mathscr{B}_i(\varepsilon_i)$ will necessarily include points with $\lambda \leq \psi_i(q)$. In other words, there may exist a $\lambda_0 \in (0,1)$, such that

$$\exists \varepsilon_i^* > 0: \; L_{i2}(\varepsilon_i^*, \lambda_0) = \emptyset \tag{4.14}$$

which implies that

$$\begin{aligned} &\forall \varepsilon_i \in (0, \varepsilon_i^*] \quad \forall \lambda \in [\lambda_0, 1). \\ &\mathscr{B}_i(\varepsilon_i) \subseteq \mathscr{B}_i(\varepsilon_i^*) \quad \wedge \quad \{q \in E^n |\, \lambda \leq \psi_i(q)\} \subseteq \{q \in E^n |\, \lambda_0 \leq \psi_i(q)\} \end{aligned} \tag{4.15}$$

so that

$$\forall \varepsilon_i \in (0, \varepsilon_i^*] \quad \forall \lambda \in [\lambda_0, 1). \qquad L_{i2}(\varepsilon_i, \lambda) \subseteq L_{i2}(\varepsilon_i^*, \lambda_0) = \emptyset \tag{4.16}$$

which implies that

$$\forall \varepsilon_i \in (0, \varepsilon_i^*] \quad \forall \lambda \in [\lambda_0, 1). \qquad L_{i2}(\varepsilon_i, \lambda) = \emptyset \tag{4.17}$$

Therefore, if we take an ε_i sufficiently small, according to Lemma 73 any critical points $q_c \neq q_d$ are in $L_2(\varepsilon_{I_0}, \lambda)$. But $L_{i2}(\varepsilon_i, \lambda) = \emptyset$. As a result, there are no critical points in $L_{i2}(\varepsilon_i, \lambda)$, so this subset is of no concern. We do not need to prove anything in this case about $\mathscr{B}_i(\varepsilon_i)$, because it is free of critical points.

Note that this case cannot arise for closed surfaces. The reason is that a critical point appears in the obstacle's neighborhood due to topological and smoothness considerations. Nonetheless, it ahs been anlyzed here to observe that this case is not of concern.

Suppose now that for every $\epsilon_i > 0$ and every $\lambda \in (0,1)$ set $L_{i2}(\varepsilon_i, \lambda)$ is not empty. According to the assumption, we can construct an infinite sequence of points outside the tube $T_\delta(C_i^1)$, in progressively smaller neighborhoods L_{i2}.

Let $\{q_m\}, m \in \mathbb{N}^*$ be this sequence. For $m \in \mathbb{N}^*$, select $\varepsilon_i^m > 0$, such that

$$\varepsilon_i^m = \frac{1}{2}\varepsilon_i^{m-1}.$$

This implies that

$$\varepsilon_i^{m-1} > \varepsilon_i^m,$$

for every $m \in \mathbb{N} \setminus \{0, 1\}$ and

$$\varepsilon_i^m \to 0^+ \quad \text{as} \quad m \to \infty.$$

Select $\lambda_m \in (0,1)$, such that

$$\lambda_m = \frac{1}{2}(1 - \lambda_{m-1})$$

This implies that

$$\lambda_{m-1} < \lambda_m,$$

The above does not hold if we assume that the converse holds, which is the claim

$$\begin{aligned} &\forall \delta > 0.\, \forall \varepsilon_i > 0.\, \forall \lambda \in (0,1). \\ &\exists q_0 \in L_{i2}(\varepsilon_i, \lambda): \; \nexists q_{id} \in B_\delta(q_0) \cap \partial \mathscr{O}_i: \; \psi_i(q_{id}) = 1 \end{aligned}$$

which is equivalent to the following, because by definition $\psi_i \in [-1, 1]$ (it is a cosine)

$$\begin{aligned} &\forall \delta > 0.\, \forall \varepsilon_i > 0.\, \forall \lambda \in (0,1). \\ &\exists q_0 \in L_{i2}(\varepsilon_i, \lambda): \; \forall q \in B_\delta(q_0) \cap \partial \mathscr{O}_i: \; \psi_i(q) < 1 \end{aligned}$$

for every $m \in \mathbb{N} \setminus \{0, 1\}$ and

$$\lambda_m \to 1^- \quad \text{as} \quad m \to \infty.$$

To every pair $\varepsilon_i^m, \lambda_m$, there corresponds a different set $L_{i2}(\varepsilon_i^m, \lambda_m)$.

By assumption, in every $L_{i2}(\varepsilon_i^m, \lambda_m)$ there exists at least one point q_0^m outside the uniform tube $T_\delta(C_i^1)$. Define $q_m \triangleq q_0^m$.

Proposition 92. *The sequence $\{q_m\}_{m \in \mathbb{N}^*}$ has a convergent subsequence.*

Proof. Both $\varepsilon_i^{m-1} > \varepsilon_i^m$ and $\lambda_{m-1} < \lambda_m$. This implies the subset relation

$$\begin{aligned} L_{i2}(\varepsilon_i^m, \lambda_m) &= \mathscr{B}_i\left(\varepsilon_i^m\right) \cap \{q \in E^n | \lambda_m \leq \psi_i(q)\} \\ &\subseteq \mathscr{B}_i\left(\varepsilon_i^{m-1}\right) \cap \{q \in E^n | \lambda_{m-1} \leq \psi_i(q)\} \\ &= L_{i2}(\varepsilon_i^{m-1}, \lambda_{m-1}) \end{aligned} \tag{4.18}$$

for every $m \in \mathbb{N} \setminus \{0, 1\}$. So we have defined a decreasing sequence of sets

$$L_{i2}(\varepsilon_i^1, \lambda_1) \supseteq L_{i2}(\varepsilon_i^2, \lambda_2) \supseteq \ldots \tag{4.19}$$

By Corollary 31, if $\varepsilon_i^1 < \hat{\varepsilon}_i$, then $\mathscr{B}_i\left(\varepsilon_i^1\right)$ is bounded. Therefore, if $\varepsilon_i^1 < \hat{\varepsilon}_i$, then set $L_{i2}(\varepsilon_i^1, \lambda_1)$ is bounded, as a subset of $\mathscr{B}_i\left(\varepsilon_i^1\right)$.

This implies that every point

$$q_m \in L_{i2}(\varepsilon_i^m, \lambda_m) \subseteq L_{i2}(\varepsilon_i^1, \lambda_1) \tag{4.20}$$

is in the same bounded set $L_{i2}(\varepsilon_i^1, \lambda_1)$. The closure of the bounded set $L_{i2}(\varepsilon_i^1, \lambda_1)$ is bounded. By Theorem 3.12.6, p.104 [16] the closure is closed. The ambient space is Euclidean, so the closed and bounded closure is compact by an extension to arbitrary dimension of Theorem 3.17.14, p.146 [16]. By Theorem 3.17.13, p.146 [16] any compact subset of a metric space is sequentially compact. So the set $L_{i2}(\varepsilon_i^1, \lambda_1)$ has sequentially compact closure. Then, by Lemma 3.17.4, p.143 [16] every sequence in $L_{i2}(\varepsilon_i^1, \lambda_1)$ has a subsequence that converges to some point in its closure. So the sequence $\{q_m\}_{m \in \mathbb{N}^*}$ has a convergent subsequence. This subsequence has a limit point q_a. □

Denote the index set of this convergent subsequence by $J \subset \mathbb{N}^*$. Then, the convergent subsequence referenced above is $\{q_m\}_{m \in J}$.

The limit point q_a can be either in the interior of some $L_{i2}(\varepsilon_i^m, \lambda_m)$, or on its boundary $\partial L_{i2}(\varepsilon_i^m, \lambda_m)$. It cannot be an exterior point for all sets $\partial L_{i2}(\varepsilon_i^m, \lambda_m)$, because in that case it cannot be a limit point of a sequence of points in these sets.

The subset relation between the non-empty sets

$$L_{i2}(\varepsilon_i^m, \lambda_m) \subseteq L_{i2}(\varepsilon_i^{m-1}, \lambda_{m-1})$$

where $m \in \mathbb{N} \setminus \{0, 1\}$, ensures that their intersection

$$\bigcap_{m \in \mathbb{N}^*} L_{i2}(\varepsilon_i^m, \lambda_m)$$

is non-empty. Therefore, its closure is non-empty. The closure of the intersection is a subset of the intersection of the closures

$$\overline{\bigcap_{m \in \mathbb{N}^*} L_{i2}(\varepsilon_i^m, \lambda_m)} \subseteq \bigcap_{m \in \mathbb{N}^*} \overline{L_{i2}(\varepsilon_i^m, \lambda_m)}$$

As a result, the intersection of the closures is non-empty.

As already noted, the limit point

$$q_a \in \bigcap_{m \in \mathbb{N}^*} \overline{L_{i2}(\varepsilon_i^m, \lambda_m)}$$

We will now prove that the limit point is on the obstacle boundary $\partial \mathscr{O}_i$.

Proposition 93. *The limit point q_a of the convergent subsequence belongs to the obstacle boundary $q_a \in \partial \mathscr{O}_i$.*

Proof. Assume the contrary, that $q_a \notin \partial \mathscr{O}_i$. Define the distance function $d(x) = \|x - q_a\|$ on set $\partial \mathscr{O}_i$. Since $q_a \notin \partial \mathscr{O}_i$, it is $d(x) > 0$ for every $x \in \partial \mathscr{O}_i$. By hypothesis $\partial \mathscr{O}_i$ is compact, so function d attains its infimum for some $x_0 \in \partial \mathscr{O}_i$

$$d(x) \geq \inf_{\partial \mathscr{O}_i} \{d\} = d(x_0) > 0 \tag{4.21}$$

As a result, every point $x \in \partial \mathscr{O}_i$ is at least $d(x_0)$ away from q_a. Define $\delta_0 \triangleq d(x_0)$, then $\delta_0 \leq \|q_a - x\|$ for every $x \in \partial \mathscr{O}_i$.

By construction, $\varepsilon_i^m \to 0^+$. Select $\delta_1 = \frac{\delta_0}{2} > 0$. By Corollary 30, there exists some $\hat{\varepsilon}_i > 0$, such that $\mathscr{B}_i(\varepsilon_i) \subseteq T_{\delta_1}(\partial \mathscr{O}_i)$ for every $\varepsilon_i < \hat{\varepsilon}_i$.

Point q_a is at least $\delta_0 > \delta_1$ away from every point in $\partial \mathscr{O}_i$. Therefore, by Proposition 6, q_a is an exterior point of $T_{\delta_1}(\partial \mathscr{O}_i)$. This implies it is an exterior point of $\mathscr{B}_i(\varepsilon_i)$.

By definition $L_{i2}(\varepsilon_i, \lambda) \subseteq \mathscr{B}_i(\varepsilon_i)$, for every ε_i. Therefore, q_a is an exterior point of $L_{i2}(\varepsilon_i, \lambda)$. As a result, it cannot be the limit of a convergent sequence with terms in $L_{i2}(\varepsilon_i, \lambda)$.

By construction $\varepsilon_i^m \to 0^+$, so for some $M \in \mathbb{N}^*$ it is

$$\varepsilon_i^m < \hat{\varepsilon}_i$$

for all $m > M$. Then, because $L_{i2}(\varepsilon_i^m, \lambda_m) \subseteq \mathscr{B}_i(\hat{\varepsilon}_i)$ for every $\varepsilon_i^m < \hat{\varepsilon}_i$, then q_a is an exterior point of all these $L_{i2}(\varepsilon_i^m, \lambda_m)$. As a result, it cannot be a limit point of a convergent sequence with terms in them (because all of them are contained in $L_{i2}(\varepsilon_i^M, \lambda_M)$, which is bounded, so the convergent sequence is also in this set).

This contradicts the hypothesis that q_a is a limit point of a convergent (sub)sequence. Hence, $q_a \in \partial \mathscr{O}_i$. □

Proposition 94. *Assume $\{q_n\}_{n \in K}$, with $K \subseteq \mathbb{N}^*$, is a convergent subsequence of $\{q_m\}_{m \in \mathbb{N}^*}$. Then, it is $\lim_{n \to \infty} \psi_i(q_n) = 1$.*

Proof. Relabel $\{q_n\}$ so that it is indexed by $\mathbb{N}^*$. Since $K \subseteq \mathbb{N}^*$ is infinite, this is always possible.

Sequence $\{q_n\}_{n \in \mathbb{N}^*}$ is a subsequence of $\{q_m\}$, therefore

$$1 \geq \cos \theta_i(q_n) = \psi_i(q_n) \geq \lambda_n$$

because $q_m \in L_{i2}(\varepsilon_i^m, \lambda_m)$ for every m (note the relabeling of λ_n, where $n \geq m$).

By construction $\lim_{n \to \infty} \lambda_n = 1$, so the Sandwich Theorem for limits implies that

$$\lim_{n \to \infty} \psi_i(q_n) = 1 \tag{4.22}$$

□

Proposition 95. *By the assumption we want to disprove, it is $\psi_i(q_a) < 1$.*

Proof. Point q_a is a limit point of a subsequence of $\{q_m\}_{m\in\mathbb{N}^*}$. Distance $\delta > 0$ is arbitrary but fixed by hypothesis. Therefore, for some $M_0 \in \mathbb{N}^*$ it is

$$\|q_a - q_m\| < \delta \tag{4.23}$$

for every $m > M_0$. Therefore $q_a \in B_\delta(q_m)$ for all $m > M_0$. As we have proved, aslo $q_a \in \partial\mathscr{O}_i$. Therefore $q_a \in B_\delta(q_m) \cap \partial\mathscr{O}_i$ for some $q_m \in L_{i2}(\varepsilon_i^m, \lambda_m)$.

By the assumption we want to disprove, it is $\psi_i(q_a) < 1$. □

Proposition 96. *By the assumption we want to disprove, it is $\lim_{n\to\infty} \psi_i(q_n) < 1$.*

Proof. Function ψ_i is continuous where $\nabla\beta_i \neq 0$ and $\nabla\gamma_d \neq 0$. We have selected all ε_i^m sufficiently small to ensure it is well-defined in our case. This is always possible when $\partial\mathscr{O}_i$ is a compact regular surface, by Corollary 13.

Because ψ_i is continuous and the subsequence $\{q_m\}_{m\in J}$ converges to q_a, it follows that

$$\lim_{J\ni m\to\infty} \psi_i(q_m) = \psi_i(q_a) < 1 \tag{4.24}$$

□

This is in contradiction with Proposition 94. Therefore, the initial assumption contrary to the claim does not hold. This proves the claim. □

Theorem 97. *Select arbitrary $\delta > 0$. There exists an $\bar{\varepsilon}_i > 0$ such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ it is*

$$\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq T_\delta(C_i^1). \tag{4.25}$$

Proof. For the selected $\delta > 0$ by Lemma 91 there exist $\bar{\varepsilon}_{i1} > 0$ and $\lambda \in (0,1)$ such that

$$L_{i2}(\varepsilon_i, \lambda) \subseteq T_\delta(C_i^1) \tag{4.26}$$

for all $\varepsilon_i < \bar{\varepsilon}_{i1}$.

For the above λ by Corollary 72 there exists an $\bar{\varepsilon}_{i2} > 0$ such that for all $\varepsilon_i < \bar{\varepsilon}_{i2}$ it is

$$\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq L_{i2}(\varepsilon_i, \lambda) \tag{4.27}$$

Define $\bar{\varepsilon}_i \triangleq \min\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i2}\}$. Then, for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ both (4.26) and (4.27) hold, hence

$$\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq L_{i2}(\varepsilon_i, \lambda) \subseteq T_\delta(C_i^1) \implies \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq T_\delta(C_i^1). \tag{4.28}$$

□

Remark 98. *Observe that λ appears to be freely selectable within the interval $(0,1)$. So are there better and worse selections for λ, without any cost? In fact, the larger we choose λ, the larger the associated $\bar{\varepsilon}_{i1}$ becomes. At first consideration this appears to be good, because the larger $\bar{\varepsilon}_{i1}$, the farther away from obstacles we need to "push" critical points. So the agent would "turn" farther away from obstacles (and avoid coming very close to them). Would it not then be the case, that the closer we choose λ to 1, the better we achieve the previous objective?*

This is not the case. The reason for this becomes apparent by observing the subsequent definition of $\bar{\varepsilon}_{i2}$ *as*

$$\frac{(1-\lambda)\min_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\bar{\beta}_i\,\|\nabla\beta_i\|\}}{\max_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}}\{\left\|\nabla\bar{\beta}_i\right\|\}}$$

The larger we select λ*, the smaller* $\bar{\varepsilon}_{i2}$ *becomes, because of the multiplicative factor* $(1-\lambda)$*. Then,* $\bar{\varepsilon}_i$ *becomes smaller, because it is the minimum of both* $\bar{\varepsilon}_{i1}$ *and* $\bar{\varepsilon}_{i2}$*.*

Lemma 99. *Select arbitrary* $\delta > 0$*. Every critical point* $q_c \neq q_d$ *can be confined within a* δ*-neighborhood of some first order contact point in* $C_i^1(\partial\mathscr{O}_i, \gamma_d)$*.*

Proof. Denote the first order contact point by $q_{id} \in \partial\mathscr{O}_i$. Define $\hat{\varepsilon}_i = \min\{\varepsilon_{i0}, \varepsilon_{i1}(\lambda)\}$. Formally, we claim that

$$\begin{gathered}\forall\delta > 0\ \exists\varepsilon_i \in (0,\hat{\varepsilon}_i) : \\ \forall q_c \in \mathscr{B}_i(\varepsilon_i) \cap (\mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}).\exists q_{id} \in B(q,\delta) \cap \partial\mathscr{O}_i : \psi_i(q_{id}) = 1\end{gathered} \tag{4.29}$$

Taking into account Lemma 73, according to which if $\varepsilon_i < \min\{\varepsilon_{i0}, \varepsilon_{i1}\}$, for a selected λ, then

$$q_c \in \mathscr{B}_i(\varepsilon_i) \cap (\mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}) \implies q_c \in L_{i2}(\varepsilon_i, \lambda) \tag{4.30}$$

it suffices to show the claim for the superset $L_{i2}(\varepsilon_i, \lambda)$ of set $\mathscr{B}_i(\varepsilon_i) \cap (\mathscr{C}_{\hat{\varphi}} \setminus \{q_d\})$. Therefore, the claim to prove becomes

$$\begin{gathered}\forall\delta > 0.\exists\varepsilon_i \in (0,\hat{\varepsilon}_i)\exists\lambda \in (0,1) : \\ \forall q \in L_{i2}(\varepsilon_i,\lambda).\exists q_{id} \in B(q,\delta) \cap \partial\mathscr{O}_i : \psi_i(q_{id}) = 1\end{gathered} \tag{4.31}$$

$\square$

Chapter 5

Relative Curvature near Obstacle

The principal relative curvatures are the eigenvalues of the (continuous) relative curvature quadratic form. So, at any point, we expect that the range of relative curvatures can be determined from the principal relative curvatures there. In more detail, the dimensions of the null, positive-definite and negative-definite subspaces of relative curvature can be readily obtained from its eigenvalues.

Moreover, the first-order contact locus C_i^1 is compact, because the obstacle boundaries $\partial\mathscr{O}_i$ have been assumed to be compact. The relative curvature form $\nu_i(q, \hat{t}_i)$ is continuous in q, so its eigenvalues $\nu_{ij}(q)$ are uniformly continuous on the first-order contact locus C_i^1. This enables the uniform[1] extension of lower- and upper-bounds on the principal relative curvatures ν_{ij} from the first-order contact locus C_i^1, to a uniform tube (δ-neighborhood) $T_\delta(C_i^1)$ around it. Here $\delta > 0$ denotes the modulus of continuity applicable to all ν_{ij}. This is schematically depicted in Fig. 5.1.

Combining the previous two arguments, we conclude that the relative curvature form in $T_\delta(C_i^1)$ obeys the bounds inherited from C_i^1. This will be derived in chapter 7.

5.1 Ordered permutations of continuous functions

Proposition 100. *Assume function f is continuous at point $x \in E^n$. If $f(x) \neq a$, then there exists an open neighborhood $U \subset E^n$, such that function $f - a$ has a constant sign over U.*

Formally, we claim that

$$\left.\begin{array}{c} f(x) \neq a \\ f \in C^0(\{x\}) \end{array}\right\} \implies \exists U \subset E^n : (f(y) - a)(f(x) - a) > 0, \tag{5.1}$$

for all $y \in U$.

Proof. Function f is continuous at x, so function $f - a$ is continuous at x. For any $\varepsilon > 0$ there exists an open neighborhood $U(\varepsilon)$ of x, such that

$$|f(y) - f(x)| < \varepsilon \iff -\varepsilon < f(y) - f(x) < \varepsilon \implies f(x) - a - \varepsilon < f(y) - a \tag{5.2}$$

for all $y \in U$.

[1] In the sense of uniform continuity.

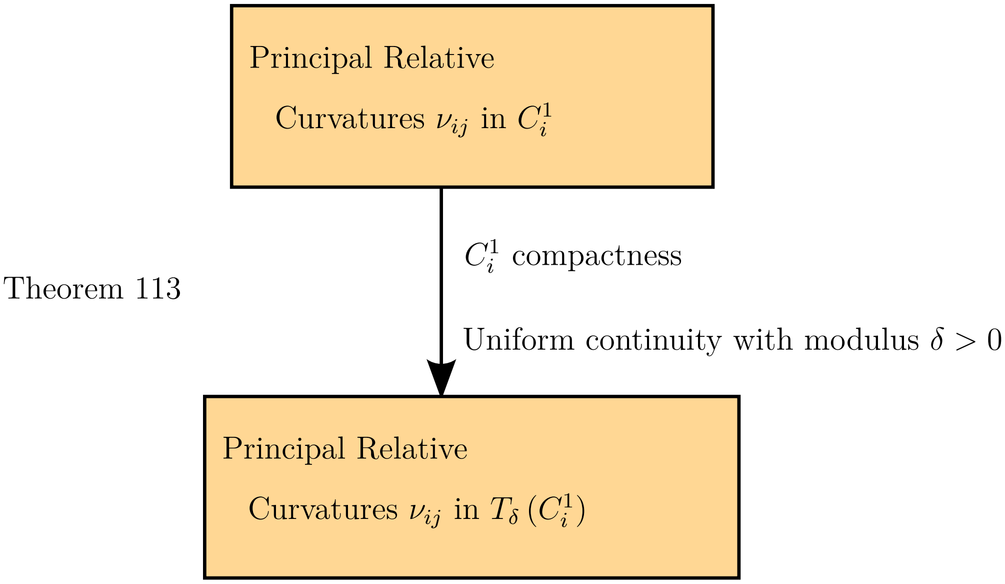

Figure 5.1: The relative curvature $\nu_i(q, \hat{t}_i)$ near the first-order contact locus C_i^1 depends uniformly on the principal relative curvatures $\nu_{ij}(q)$ on the locus C_i^1.

By hypothesis $f(x) - a \neq 0$. Assume $f(x) - a > 0$, proving the case $f(x) - a < 0$ is similar. Choose $\varepsilon = \frac{1}{2}(f(x) - a)$. It follows that

$$-\frac{1}{2}(f(x) - a) = -\varepsilon \iff \frac{1}{2}(f(x) - a) = f(x) - a - \varepsilon \tag{5.3}$$

By assumption $0 < f(x) - a$, therefore $0 < f(x) - a - \varepsilon$.

As a result, there exists an open neighborhood U of x, such that

$$0 < f(x) - a - \varepsilon < f(y) - a \implies 0 < f(y) - a \tag{5.4}$$

for every $y \in U$. This implies that $(f(x) - a)(f(y) - a) > 0$ for every $y \in U$. □

Corollary 101. *Assume $\nu_{i3}(q_0) \neq a$. Assume $\gamma_d = \|q - q_d\|^2$ and $\nabla\beta_i, \nabla\gamma_d \neq 0$ in an open neighborhood of q_0. Then, there exists an open neighborhood U of q_0 in which $\nu_{i3} - a$ has constant sign.*

Proof. By hypothesis $\nabla\beta_i, \nabla\gamma_d \neq 0$ and $\gamma_d = \|q - q_d\|^2$, so function $\nu_{i3} - a$ is defined and continuous over an open neighborhood of q_0, by Proposition 47. Then, by Proposition 100 there exists a neighborhood U of q_0, within which $\nu_{i3} - a$ maintains its sign. THis implies that $(\nu_{i3}(q) - a)(\nu_{i3}(q_0) - a) > 0$ for all $q \in U$. □

In other words, within the open neighborhood U of q_0

$$\begin{aligned} \nu_{i3}(q_0) > a &\implies \nu_{i3}(q) > a, \quad \forall q \in U \\ \nu_{i3}(q_0) < a &\implies \nu_{i3}(q) < a, \quad \forall q \in U \end{aligned} \tag{5.5}$$

A special case is when $a = 0$.

Corollary 102. *Assume $\nu_{i3}(q_0) \neq 0$. Assume $\gamma_d = \|q - q_d\|^2$ and $\nabla\beta_i, \nabla\gamma_d \neq 0$ in an open neighborhood of q_0.*
Then, there exists an open neighborhood U of q_0 in which ν_{i3} maintains its sign.

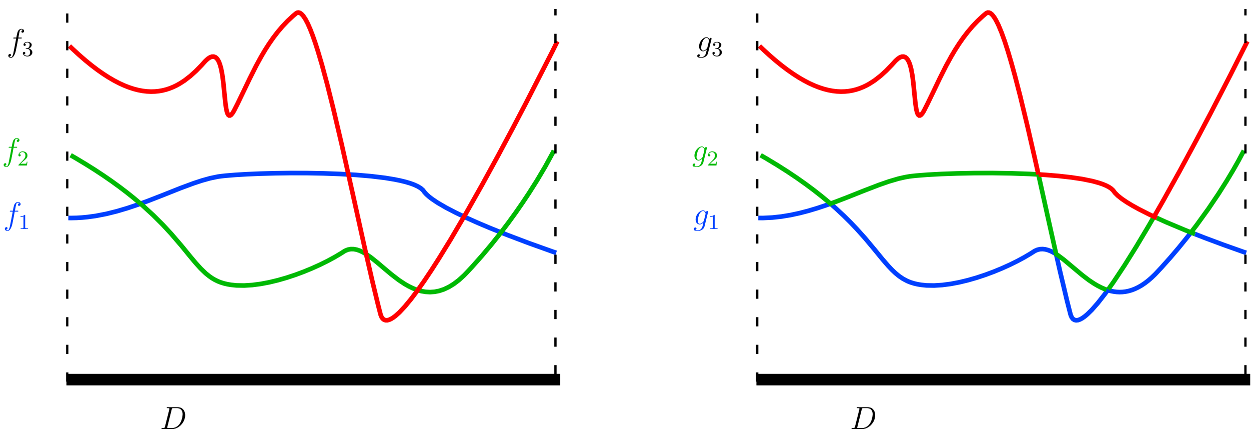

Figure 5.2: Creating new continuous functions g_j with constant ordering, which equal the ordered permutation of the original functions f_i at each $x \in D$.

The following is schematically illustrated in Fig. 5.2

Proposition 103. *Assume n continuous functions $f_1, f_2, \dots, f_n$ are defined over the connected set D.*

Then, we can define n continuous functions $g_1, g_2, \dots, g_n$ over D with the following properties. At each point $x \in D$, for every function f_i there exists at least one function g_j, such that $f_i(x) = g_j(x)$. At each point $x \in D$, for every function g_j, there exists at least one function f_i, such that $g_j(x) = f_i(x)$.

It is[a]

$$g_1(x) \leq g_2(x) \leq \cdots \leq g_n(x)$$

for all $x \in D$.

[a]In other words, the ordering of functions $\{g_j\}$ remains contant over D.

Proof. Define the set $I \triangleq \mathbb{N}^*_{\leq n}$ to condense notation. Select a point $x \in D$. The function values at x

$$f_1(x),\ f_2(x),\ \dots,\ f_n(x)$$

can be permuted in increasing order

$$f_{i_1}(x) \leq f_{i_2}(x)\ \leq \dots\ \leq f_{i_n}(x)$$

This is a permutation of the indices of functions f_i, from the set I

$$1,\ 2,\ \dots,\ n$$

to the set

$$i_1,\ i_2,\ \dots,\ i_n$$

which is again I. A permutation is by definition a bijective function.

If all function values $f_i(x)$ are pairwise unequal, then a unique permutation into increasing order exists at x. If there are function values $f_i(x) = f_p(x)$ for different functions $i \neq p$, then multiple different permutations yield a tuple of increasing order[2] Any n-tuple of real numbers can always be permuted to another n-tuple with elements in increasing order.

[2]Because the sub-tuples i, p and p, i are different but equivalent in terms of increasing order.

At each x, select exactly one of the permutations of I which result into an ordered set of function values $f_{i_j}(x) \leq f_{i_{j+1}}(x)$ at x. Define function $h_x : I \to I$ as this permutation of the function indices. By definition, function h_x is a bijection. Therefore, its inverse h_x^{-1} exists and is a bijection.

For every $j \in I$, define

$$g_j(x) \triangleq f_{i_j}(x) \tag{5.6}$$

where $i_j \triangleq h_x^{-1}(j)$. Note that the bijection between i and j indices may change for different points $x, y \in D$.

By construction

$$g_j(x) = f_{i_j}(x) \leq f_{i_{j+1}}(x) = g_{j+1}(x)$$

for every $x \in D$ and $j \in I$. This proves the claim that the ordering

$$g_1 \ \leq g_2 \ \leq \ldots \ \leq g_n$$

remains the same over D.

By construction, at each point $x \in D$, h_x is a function. So for every function f_i, the function g_j with $j = h_x(i)$, has $f_i(x) = g_j(x)$. At each point $x \in D$, h_x^{-1} is a function. So for every g_j the function f_i, with $i = h_x^{-1}(j)$, has $g_j(x) = f_i(x)$.

It remains to prove that functions g_j are continuous over D. Select some $x_0 \in D$ and some $i \in I$. Define $j = h(i, x)$. There are two cases, either $g_j(x_0)$ is different from all other $g_k(x_0)$, or it is equal to some of them.

In more detail, the first case is

$$g_j(x_0) \neq g_k(x_0) \tag{5.7}$$

for every $k \neq j$. Function h_{x_0} is a bijection, so the previous is equivalent to

$$f_{i_j}(x_0) \neq f_{i_k}(x_0) \tag{5.8}$$

for every $k \neq j$.

The second case is

$$g_j(x_0) = g_{j+1}(x_0) = \cdots = g_{j+m}(x_0) \neq g_k(x_0) \tag{5.9}$$

for every $k \notin \{j, j+1, \ldots, j+m\}$. Function h_{x_0} is a bijection, so the previous is equivalent to

$$f_{i_j}(x_0) = f_{i_{j+1}}(x_0) = \cdots = f_{i_{j+m}}(x_0) \neq f_{i_k}(x_0) \tag{5.10}$$

for every $k \notin \{j, j+1, \ldots, j+m\}$.

(Only consecutively ordered functions can be equal, no index jumps can happen between equal functions, omitting a function with an intermediate index Note that it may be $j - m, j-m+1, \ldots, j-1, j, j+1, \ldots, j+p$, but this is equivalent to the above. If we prove the above, we have proved also that case, because it is equivalent to considering $j - m$ first). We prove continuity for each case separately.

Suppose that for some arbitrary $j \in I$, it is

$$g_j(x_0) \neq g_k(x_0)$$

for all $k \neq j$. Define

$$i_0 \triangleq h_{x_0}^{-1}(j).$$

Then $f_{i_0}(x_0)$ is isolated from the values of other functions $f_{i_k}(x_0)$ there. This implies that

$$f_{i_0}(x_0) - f_{i_k}(x_0) \neq 0$$

for all $k \neq j$.

Each function $d_{ik} \triangleq f_{i_0} - f_{i_k}$ is continuous, as the difference of continuous functions f_{i_0} and f_{i_k}. Suppose $f_{i_0}(x_0) - f_{i_k}(x_0) > 0$ (the case $f_{i_0}(x_0) - f_{i_k}(x_0) < 0$ is similar). By Proposition 100 there exists an neighborhood $U_k \subseteq D$ of x_0 open in D, within which $f_{i_0} - f_{i_k} > 0$. Therefore, all functions f_{i_k} which are smaller than f_{i_0} at x_0, remain smaller in some open neighborhood of x_0. The same holds for all functions larger than f_{i_0} at x_0. This yields $n-1$ neighborhoods $U_{k_1}, U_{k_2}, \ldots, U_{k_{n-1}}$. Define their intersection

$$U \triangleq \bigcap_{k \in \mathbb{N}^*_{\leq n} \setminus \{j\}} U_k \subseteq D.$$

Set U is an open neighborhood of x_0, as the intersection of a finite number of open neighborhoods of x_0.

Both the number of functions larger than f_{i_0} and the number of functions smaller than f_{i_0} remains the same within U. In addition, no functions are equal to f_{i_0} in U. Therefore, f_{i_0} is ordered at the same place for every $x \in U$. It is the j-th function.

This implies that $g_j(x) = f_{i_0}(x)$ for the same i_0 over all of U. In other words, function $h_x(i_0)$ remains constant within U, because function f_{i_0} is ordered at the same plane (j-th) all over U.

Function f_{i_0} is continuous at x_0, so for every $\varepsilon > 0$ there exists a $\delta_1 > 0$, such that

$$\|x - x_0\| < \delta_1 \implies |f_{i_0}(x) - f_{i_0}(x_0)| < \varepsilon$$

for every $x \in D$. For an open neighborhood as U is, it is always possible to choose a $\delta > 0$, such that $\delta < \delta_1$, so that for every $x \in D$ with

$$\|x - x_0\| < \delta$$

is in U. It follows that for every $x \in U$ with

$$\begin{aligned} \|x - x_0\| < \delta \implies \|x - x_0\| < \delta_1 \implies |f_{i_0}(x) - f_{i_0}(x_0)| < \varepsilon \implies \\ |g_j(x) - g_j(x_0)| < \varepsilon \end{aligned}$$

because $g_j = f_{i_0}$ over U. Therefore, function g_j is continuous at point x_0. This case is shown in Fig. 5.3.

We now consider the second case. Suppose that for some $j, j+1, \ldots, j+m \in I$ it is

$$g_j(x_0) = g_{j+1}(x_0) = \cdots = g_{j+m}(x_0) \neq g_k(x_0)$$

for every $k \notin \{j, j+1, \ldots, j+m\}$. Define

$$i_0^0 \triangleq h_{x_0}^{-1}(j), \quad i_0^2 \triangleq h_{x_0}^{-1}(j+1), \cdots \quad i_0^m \triangleq h_{x_0}^{-1}(j+m)$$

It follows that values of functions $f_{i_0^0}(x_0), f_{i_0^1}(x_0), \ldots, f_{i_0^m}(x_0)$ are ordered

$$f_{i_0^0}(x_0) \leq f_{i_0^1}(x_0) \leq \cdots \leq f_{i_0^m}(x_0)$$

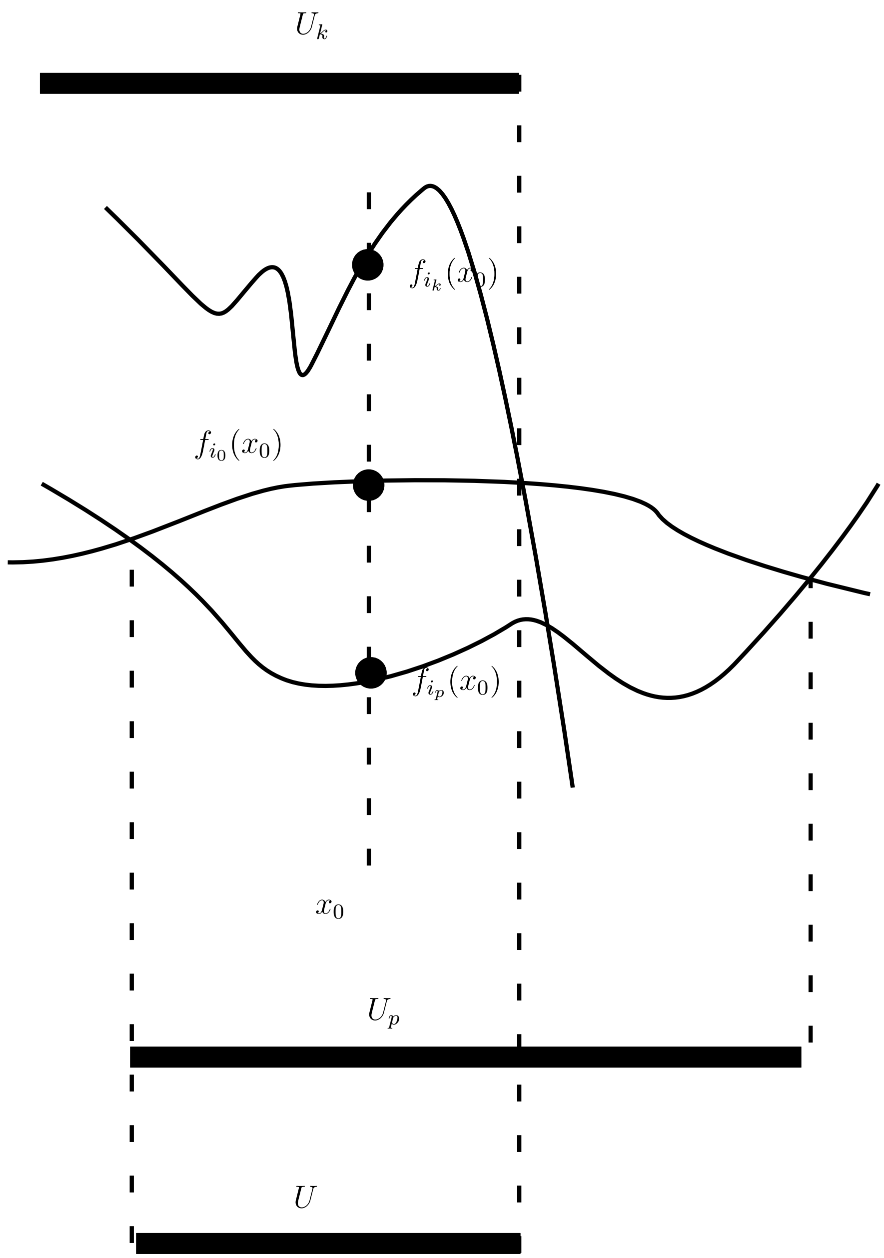

Figure 5.3: First case.

Consider the functions f_s with $s \in I_s$, which are smaller than $f_{i_0^0}(x_0)$ there, i.e.,

$$f_s(x_0) \leq f_{i_0^0}(x_0)$$

These are smaller than each of $f_{i_0^0}(x_0), f_{i_0^1}(x_0), \ldots, f_{i_0^m}(x_0)$. Each difference function $f_{i_0^r} - f_s$ is continuous and similarly to the first case, there exists a neighborhood U_{rs} of x_0 open in D, such that f_s remains smaller than $f_{i_0^r}$ within U_{rs}. Reapeating this for all combinations of r and s yields a finite number of such open neighborhoods U_{rs}. Their intersection

$$U \triangleq \bigcap_{r\in\mathbb{N}_{\leq m}, s\in I_s} U_{rs}$$

is an open neighborhood of x_0.

As a result, any $f_{i_0^r}$ remains larger than N_s functions and smaller than N_l functions in every point $x \in U$. Therefore, $f_{i_0^r}$ is ordered within the range of indices $j, j+1, \ldots, j+m$. For this reason, at each point in $x \in U$, function $f_{i_0^r}$ is equal to at least one function in the set $g_j, g_{j_1}, \ldots, g_{j+m}$. This assignment between $f_{i_0^r}$ and functions $g_j, g_{j_1}, \ldots, g_{j+m}$ may change over U. But no $f_{i_0^r}$ is equal to some g_k with $k \notin \{j, j+1, \ldots, j+m\}$.

Choose arbitrary $\varepsilon > 0$. Consider g_j, which we want to prove continuous. Then, by continuity there exist $\delta_1, \delta_2, \ldots, \delta_m > 0$, such that

$$\|x - x_0\| < \delta_r \implies \left|f_{i_0^r}(x) - f_{i_0^r}(x_0)\right| < \varepsilon. \tag{5.11}$$

Because U is open in D, it is always possible to choose a $\delta > 0$, such that $\delta \leq \delta_r$ for all $r \in \mathbb{N}_{\leq m}$ and for every $x \in D$ with

$$\|x - x_0\| < \delta \tag{5.12}$$

it is $x \in U$.

Then, for every $x \in U$ with

$$\|x - x_0\| < \delta \implies \|x - x_0\| < \delta_r \implies \left|f_{i_0^r}(x) - f_{i_0^r}(x_0)\right| < \varepsilon \tag{5.13}$$

for every $r \in \mathbb{N}_{\leq m}$. For every $x \in U$, there exists some $r \in \mathbb{N}_{\leq m}$ for which $g_j(x) = f_{i_0^r}(x)$. This enables substitution of g_j for $f_{i_0^r}$, so that

$$\left|f_{i_0^r}(x) - f_{i_0^r}(x_0)\right| < \varepsilon \implies |g_j(x) - g_j(x_0)| < \varepsilon \tag{5.14}$$

Hence, function g_j is continuous at x_0. This case is illustrated in Fig. 5.4.

□

Proposition 104. *Assume that n continuous functions $f_1, f_2, \ldots, f_n$ are defined over the connected compact set D. In addition, assume that at each $x \in D$, at least $p \geq 0$ of $f_i(x)$ are positive.*

Then, there exists a lower bound $L > 0$, with following property. At each point $x \in D$, the p largest values $f_i(x)$ are larger than or equal to L, i.e.,

$$0 < L \leq f_{j_1}(x) \ \leq f_{j_2}(x) \ \ldots \ \leq f_{j_p}(x). \tag{5.15}$$

(where the indices j_k may be different at different points $x_1 \neq x_2$.)

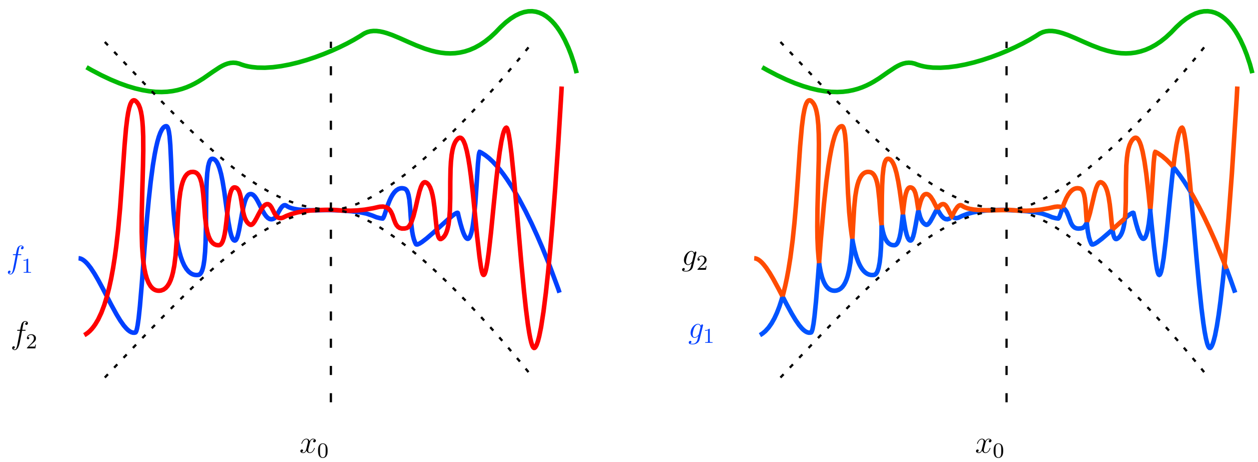

Figure 5.4: Continuity of two g_j, g_{j+1} when $g_j(x_0) = g_{j+1}(x_0)$. It is $f_{i_0^0}(x_0) = f_{i_0^1}(x_0)$ and f_j and f_{j+1} oscillate a lot near x_0. This results in the assignment between g_j, g_{j+1} and $f_{i_0^0}, f_{i_0^1}$ to vary within any neighborhood of x_0. Nonetheless, g_j, g_{j+1} are continuous, because to whichever of $f_{i_0^0}, f_{i_0^1}$ they are equal at some x near x_0, they tend continuously to the value $g_j(x_0)$.

In more detail, select any $x \in D$. Permute the values $f_1(x), \dots, f_n(x)$ in increasing order

$$f_{j_1}(x) \;\le f_{j_2}(x) \;\le \dots \;\le f_{j_{n-p}}(x) \underbrace{\le f_{j_{n-p+1}}(x) \;\le f_{j_{n-p+2}}(x) \;\le \cdots \;\le f_{j_n}(x)}_{p}. \tag{5.16}$$

Then, the last p terms of this ordered permutation are

$$f_{j_{n-p+1}}(x),\; f_{j_{n-p+2}}(x),\; \dots,\; f_{j_n}(x). \tag{5.17}$$

We claim that the p largest terms will be lower bounded by $L > 0$

$$0 < L \le f_{j_{n-p+1}}(x) \;\le f_{j_{n-p+2}}(x) \;\le \cdots \;\le f_{j_n}(x) \tag{5.18}$$

Note that the lower bound L is common for all $x \in D$.

Proof. By Proposition 103 there exist continuous functions $g_1, g_2, \dots, g_n$ equal at each point to the ordered permutation of the original functions $f_1, \dots, f_n$.

At each point at least p of $f_i(x)$ remain positive. It follows that at each point at least p of $g_j(x)$ remain positive.

Suppose that at some point $x \in D$ only p of the $g_j(x)$ are positive. Then, these are the p largest g_j, because g_j are indexed in order of increasing value, by definition.

Suppose that at some point $x \in D$ more than p of the $g_j(x)$ are positive. Let the number of positive $g_j(x)$ at x be M. The other $n - M$ of $g_j(x)$ are negative or zero. This implies that the M positive $g_j(x)$ are the largest M of all $g_j(x)$, because they are indexed by definition in order of increasing value.

Therefore, the largest $M > p$ of the $g_j(x)$ are all positive. The largest p of all $g_j(x)$ are a subset of these M values. Hence, again the largest p of $g_j(x)$ are positive.

By Proposition 103, the order of functions g_j remains the same over D. Therefore, the same p largest functions g_j are positive over D. Each of them is continuous on the compact set, so it attains its infimum for some $x_{j0} \in D$. Because $g_j(x_{j0}) > 0$, the infima of these p largest functions are all positive. Choose l to be the smallest of these infima

$$l \triangleq \min_{N-p+1 \leq j \leq N} \{g_j(x_{j0})\} \tag{5.19}$$

It is $l > 0$, because by the previous argument all infima $g_j(x_{j0})$ are positive.

Select any point $x \in D$. We have proved that the p largest g_j are $g_j(x) \geq l > 0$. By the inverse permutation h_x^{-1} of Proposition 103, we obtain that the p largest f_i will be lower bounded

$$0 < l \leq f_j(x) \tag{5.20}$$

This proves the claim. □

Proposition 105. *Assume n continuous functions $f_1, f_2, \ldots, f_n$ are defined over the connected compact set D. In addition, assume that at each $x \in D$ at most[a] $N_0 \geq 0$ of them are zero.*

Then, there exists a lower bound $l > 0$ with the following property. At each point $x \in D$ the $N_a = n - N_0$ functions have absolute value larger than L, i.e.,

$$0 \ < L \ \leq |f_{j_1}(x)| \ \leq |f_{j_2}(x)| \ \leq \ldots \ \leq \left|f_{j_{N_a}}(x)\right|. \tag{5.21}$$

[a] $N_0 \in \mathbb{N}_{\leq n}$.

Proof. Define the n functions

$$h_1(x) \triangleq |f_1(x)|\,, \quad h_2(x) \triangleq |f_2(x)|\,, \quad \ldots, \quad h_n(x) \triangleq |f_n(x)| \tag{5.22}$$

Then all h_i are continuous, because the functions f_i and the absolute value function are continuous. By definition $h_i(x) \geq 0$ for all $i \in \mathbb{N}^*_{\leq n}$.

By hypothesis at most $N_0 \in \mathbb{N}_{\leq n}$ function values in the set $\{f_i(x)\}_{i \in \mathbb{N}^*_{\leq n}}$ are equal to 0 at any point $x \in D$. So at most N_0 function values in the set $\{h_i(x)\}_{i \in \mathbb{N}^*_{\leq n}}$ are equal to 0 at any point $x \in D$. This implies that at least $N_a = n - N_0 \in \mathbb{N}_{\leq n}$ function values $h_i(x)$ are positive at any point $x \in D$.[3]

In summary, the n continuous functions h_i are defined over the compact connected set D. At every point $x \in D$, at least $N_a = n - N_0 \geq 0$ of the values $h_i(x)$ are positive.

Then, according to Proposition 104 there exists a lower bound $L > 0$ such that the N_a largest function values at every point are greater than L. □

Lemma 106. *Assume n continuous functions $f_1, f_2, \ldots, f_n$ are defined over the connected compact set D. In addition, assume that at each $x \in D$ at least p of them have positive values and at most N_0 zero values.*

Then, there exists an $L > 0$ with the following property. At each point $x \in D$ there are two numbers $N_p \geq p$ and N_n with $N_p + N_n \geq n - N_0$, such that N_n functions have values less than or equal to $-L$

$$f_{j_1}(x) \ \leq f_{j_2}(x) \ \ldots \ \leq f_{j_{N_n}}(x) \ \leq -L \ < 0 \tag{5.23}$$

and N_p functions have values larger than or equal to L

$$0 \ < L \leq f_{j_{N_n+1}}(x) \ \leq f_{j_{N_n+2}}(x) \ \leq \ldots \ \leq f_{j_{N_n+N_p}}(x) \tag{5.24}$$

[3] Formally $\left|\{i \in \mathbb{N}^*_{\leq n} |\ h_i(x) > 0\}\right| \geq n - N_0 \geq 0$.

Proof. By the hypotheses and Proposition 105 it follows that there exists a lower bound $L_1 > 0$ so that at every point the $N_a = n - N_0$ functions have absolute value larger than L

$$0 \ < L_1 \ \leq |f_{j_1}(x)| \ \leq |f_{j_2}(x)| \ \leq \ldots \ \leq \left|f_{j_{N_a}}(x)\right|. \tag{5.25}$$

Consider only the hypothesis that at every point $x \in D$, p functions have positive values. In other words, at each $x \in D$, at least p of $f_i(x)$ are positive. Then, by Proposition 104 there exists a lower bound $L_2 > 0$ such that at each point $x \in D$ the p largest values $f_i(x) \geq L_2$.

Define $L \triangleq \min\{L_1, L_2\} > 0$. As proved, there are N_a functions with values $f_i(x) \geq L_2 \geq L$. Each of the function values $f_{j_1}(x), f_{j_2}(x), \ldots, f_{j_{N_a}}(x)$ is non-zero, so it is either positive or negative. Suppose that N_1 are positive and N_2 are negative, so $N_1 + N_2 = N_a$. If positive, then by the previous inequality it is $f_{j_k}(x) \geq L_1 \geq L$. If negative, then by the previous inequality it is $f_{j_k}(x) \geq L_1 \leq -L$.

There cannot be any common functions between the sets of those p functions with positive values and those N_2 functions with negative values. Therefore, there are N_2 functions with negative values less than or equal to $-L$, i.e.,

$$f_{j_{k_1}}(x) \ \leq f_{j_{k_2}}(x) \ \leq \ldots \ \leq f_{j_{k_{N_2}}}(x) \ \leq -L < 0. \tag{5.26}$$

However, functions in the set of p function values and the set of N_1 function values are all positive. So there may be common functions in those two sets. Suppose that the maximal possible number of functions are common between those two sets. Then, the minimal possible number of functions with positive values is $N_p = \max\{p, N_1\} \geq p$. This ensures that there are $N_p \geq p$ functions with positive values greater than or equal to L, i.e.,

$$0 < L \ \leq f_{j_{k_{N_2+1}}}(x) \ \leq f_{j_{k_{N_2+2}}}(x) \ \leq \ldots \ \leq f_{j_{k_{N_2+N_p}}}(x) \tag{5.27}$$

To complete the proof observe that $N_p \geq N_1$, so that $N_p + N_2 \geq N_1 + N_2 = N_a = n - N_0$. □

Proposition 107 (Uniform positivity around compact set)**.** *Assume that for some $\delta_1 > 0$, the n continuous functions $f_1, f_2, \ldots, f_n$ are defined over the uniform tube $T_{\delta_1}(S)$ of a compact set S. In addition, assume that at each $x \in S$ at least $p \geq 0$ of them have positive values and at most N_0 zero values.*

Then, there exists a $\delta > 0$ ($\delta \leq \delta_1$) and an $L > 0$ with the following property. For each point $x \in T_\delta(S)$ there are two numbers $N_p \geq p$ and N_n with $N_p + N_n \geq n - N_0$ such that N_n functions have values less than or equal to $-L$

$$f_{j_1}(x) \ \leq f_{j_2}(x) \ \leq \ldots \ \leq f_{j_{N_n}}(x) \ \leq -L < 0, \tag{5.28}$$

and N_p functions have values larger than or equal to L

$$0 < L \ \leq f_{j_{N_n+1}}(x) \ \leq f_{j_{N_n+2}}(x) \ \leq \ldots \ \leq f_{j_{N_n+N_p}}(x). \tag{5.29}$$

(Note that we need the uniform lower bound to carry the positivity further, to the Hessian quadratic form of the Koditschek-Rimon function.)

Proof. By hypothesis the functions f_i are continuous over the compact set S and at most m function values are zero at any point $x \in S$. Then, by Lemma 106 there exists an $L_1 > 0$

such that At any point $x_2 \in S$ there are two numbers $N_p \geq p$ and N_n with $N_p + N_n \geq n - N_0$, such that N_n functions have values less than or equal to $-L_1$

$$f_{j_1}(x_2) \ \leq f_{j_2}(x_2) \ \ldots \ \leq f_{j_{N_n}}(x_2) \ \leq -L_1 \ < 0 \tag{5.30}$$

and N_p functions have values larger than or equal to L_1

$$0 \ < L_1 \leq f_{j_{N_n+1}}(x_2) \ \leq f_{j_{N_n+2}}(x_2) \ \leq \ldots \ \leq f_{j_{N_n+N_p}}(x_2) \tag{5.31}$$

(Note that the indices j_k may change for different $x_2 \in S$.)

Take $\delta_2 \triangleq \frac{1}{2}\delta_1 > 0$. Then each point in $\bar{T}_{\delta_2}(S)$ is also in $T_{\delta_1}(S)$, which ensures that all functions f_i are well-defined in $\bar{T}_{\delta_2}$. By hypothesis set S is compact, so by Proposition 3 the uniform tube closure $\bar{T}_{\delta_2}(S)$ is compact. Each of the n continuous functions f_i is defined over the compact set $\bar{T}_{\delta_2}(S)$, so it is uniformly continuous over $\bar{T}_{\delta_2}(S)$. Choose $L \triangleq \frac{1}{2}L_1$ For each function f_i, there exists a $\rho_i > 0$ such that if

$$\|x_3 - x_4\| < \rho_i$$

for some $x_3, x_4 \in \bar{T}_{\delta_2}(S)$, then

$$|f_i(x_3) - f_i(x_4)| < \varepsilon = \frac{1}{2}L_1.$$

Define $\delta_3 \triangleq \min_i\{\rho_i\} > 0$. Define $\delta \triangleq \min\{\delta_2, \delta_3\}$.

Let $x_1 \in T_\delta(S)$ by Proposition 10 $\delta \leq \delta_2$ implies that $T_\delta(S) \subset T_{\delta_2}(S)$, so $x_1 \in \bar{T}_{\delta_2}(S)$. Then, by Proposition 11 there exists at least one point $x_2 \in S$ such that $\|x_1 - x_2\| < \delta$.

$$\|x_1 - x_2\| < \delta$$

so it is

$$|f_i(x_1) - f_i(x_2)| < L$$

for all functions f_i.

By the definition of L and because $x_2 \in S$, there are N_p functions with values

$$0 < 2L \ \leq f_{j_1}(x_2) \ \leq f_{j_2}(x_2) \ \ldots \ \leq f_{j_{N_n}}(x_2) < 0.$$

So it is

$$\begin{aligned}
&0 < 2L \leq |f_{j_k}(x_2)| \implies \\
&0 < 2L \leq |f_{j_k}(x_2) - f_{j_k}(x_1) + f_{j_k}(x_1)| \implies \\
&0 < 2L \leq |f_{j_k}(x_2) - f_{j_k}(x_1)| + |f_{j_k}(x_1)|
\end{aligned}$$

By uniform continuity within the ball of radius δ and center x_2, it is $|f_{j_i}(x_1) - f_{j_i}(x_2)| < 2L$, substitute to obtain

$$0 < 2L < L + |f_{j_i}(x_1)| \implies 0 < L < |f_{j_i}(x_1)|$$

This holds for all x in a δ-ball around x_2. Take the linear segment with end-points x_1 and x_2. It is a subset of the δ-ball around x_2 and f_{j_i} is not zero over it. So f_{j_i} maintains its sign over the linear segment[4].

[4]Otherwise by continuity and the Bolzano theorem, the function has a root at some point of the linear segment. This is a contradiction, because its absolute value remains non-zero.

This ensures that if $|f_{j_i}(x_2)| >$, then $f_{j_i}(x_2)$ and $f_{j_i}(x_1)$ have the same sign. This implies that if $f_{j_i}(x_1)$ is positive, because $f_{j_i}(x_2)$ is positive. Moreover, $L < |f_{j_i}(x_1)|$ implies $L < f_{j_i}(x_1)$, given the positivity of $f_{j_i}(x_1)$. Similar arguments hold for the functions with negative values.

As a result, there are N_n functions which have values less than or equal to $-L$ at x_1, i.e.,

$$f_{j_1}(x_1) \ \leq f_{j_2}(x_1) \ \leq \ldots \ \leq f_{j_{N_n}}(x_1) \leq -L \ < 0$$

and N_p functions which have values larger than or equal to L at x_1, i.e.,

$$0 < L \ \leq f_{j_{N_n+1}}(x_1) \ \leq f_{j_{N_n+2}}(x_1) \ \leq \ldots \ \leq f_{j_{N_n+N_p}}(x_1).$$

□

Corollary 108. *Assume that for some $\delta_1 > 0$, the n continuous functions $f_1, f_2, \ldots, f_n$ are defined over the uniform tube $T_{\delta_1}(\partial\mathscr{O}_i)$. In addition, assume that at each $x \in S$ at least $p \geq 0$ of them have positive values and at most N_0 zero values.*

Then, there exists an $\bar{\varepsilon}_i > 0$, an $L > 0$ and a $\delta > 0$ with the following properties. For all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$, for each point $q_1 \in \mathscr{B}_i(\varepsilon_i)$, there are two numbers $N_p \geq p$ and N_n with $N_p + N_n \geq n - N_0$ such that N_n functions have values less than or equal to $-L$

$$f_{j_1}(q_1) \ \leq f_{j_2}(q_1) \ \leq \ldots \ \leq f_{j_{N_n}}(q_1) \ \leq -L < 0, \tag{5.32}$$

and N_p functions have values larger than or equal to L

$$0 < L \ \leq f_{j_{N_n+1}}(q_1) \ \leq f_{j_{N_n+2}}(q_1) \ \leq \ldots \ \leq f_{j_{N_n+N_p}}(q_1). \tag{5.33}$$

Proof. By Proposition 107 there exist a $\delta_2 > 0$ ($\delta_2 \leq \delta_1$) and an $L > 0$ with the following properties. For each point $q_1 \in T_{\delta_2}(\partial\mathscr{O}_i)$ there are two numbers $N_p \geq p$ and N_n with $N_p + N_n \geq n - N_0$ such that N_n functions have values less than or equal to $-L$

$$f_{j_1}(q_1) \ \leq f_{j_2}(q_1) \ \leq \ldots \ \leq f_{j_{N_n}}(q_1) \ \leq -L < 0, \tag{5.34}$$

and N_p functions have values larger than or equal to L

$$0 < L \ \leq f_{j_{N_n+1}}(q_1) \ \leq f_{j_{N_n+2}}(q_1) \ \leq \ldots \ \leq f_{j_{N_n+N_p}}(q_1). \tag{5.35}$$

By Corollary 19 there exists some $\bar{\varepsilon}_{i1} > 0$ such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i1}$ it is $\mathscr{B}_i(\varepsilon_i) \subseteq T_{\delta_2}(\partial\mathscr{O}_i)$.

This proves the claim for any point $q_1 \in \mathscr{B}_i(\varepsilon_i)$. □

5.2 Continuity of eigenvalue functions

Assume that function $f : E^n \to \mathbb{R}$ is twice continuously differentiable. Moreover, assume that it is regular at some point q. Then, the level sets of f are well-defined in an open ball around q. The level sets are characterized by their normal curvature. To each point q, there corresponds a set of principal curvatures κ_j. In this section we prove that there are $n-1$ continuous functions which are equal to the principal curvatures at every point. Therefore, the principal curvatures vary continuously[5] over the regular points of f.

The principal curvatures are the eigenvalues of the Weingarten map. The Weingarten map can be expressed in terms of a square matrix. In our case the ambient space is E^n, so the square matrix expressed in the tangent space would be of dimension $(n-1) \times (n-1)$. This matrix is a function of $x \in E^n$. So we are interested in the eigenvalues of $m \times m$ matrices whose elements are functions of $x \in E^n$, $n = m+1$. In particular, we want to prove the continuity of the eigenvalues as functions of x (after the are appropriately permuted at each point).

This problem has been treated in the literature and the results are quoted here with minor adaptations for our case. Using Rouché's Theorem, Ryan proves the continuity of the eigenvalues of a symmetric tensor field of type[6] (1, 1) in Lemma 2.1 [17], which is quoted here:

Lemma 109 (Continuity of (1,1)-tensor field eigenvalues [17])**.** *Let A be a symmetric tensor field of type $(1, 1)$ on a connected Riemannian manifold M^n. Then, there exist n continuous functions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that for each $x \in M^n$ the set of numbers $\{\lambda_i(x)\}_{i=1}^n$ is the set of eigenvalues of A_x.*

The tensor field in this case is of dimension n and is defined on a manifold of dimension n.

In [18] Nomizu considers the differentiability of eigenvalues with constant multiplicity. The work of [19] is cited for the case $n = 1$ and the work of Ryan [17] for the case $n = m$. It is also noted that the proof of [17] remains valid for arbitrary m and n. Following this suggestion, an adapted version of the proof is quoted here, in which m and n are considered arbitrary.

A good review of the previous works can be found in [20]. In Theorem 1 [21] Shigley proves the continuity of the eigenvalues using Rouché's Theorem and in Theorem 2 [21] their differentiability in an open, dense subset of the tensor field domain.

The above works assume that the tensor field is C^∞. However, we are interested only in continuity, so we assume that the tensor field is continuous. For a topological proof that the roots of a polynomial vary continuously as a function of the coefficients, see [22]. If the characteristic polynomial of a matrix B is considered, then the coefficients a_i are sums of products of matrix elements b_{ij}. If the matrix B is a continuous function of $x \in E^n$, so are the coeeficients a_i of its characteristic polynomial. By [22] the roots p_k of the characteristic polynomial are continuous functions of its coefficients a_i. In turn, the coefficients a_i are continuous functions of x. So the eigenvalues p_k are continuous functions of x, as the composition of the continuous functions $p_k(a_k)$ and $a_i(x)$.

The adapted version of Lemma 2.1 [17] (quoted previously as Lemma 109) with some details added from Theorem 1 [21] follows.

[5]Note that they are not differentiable functions everywhere (although this is of no concern here).
[6]Mixed with contravariant rank 1 and covariant rank 1.

Lemma 110 (Symmetric matrix eigenvalue continuity)**.** *Let $B : M^n \to \mathbb{R}^{m\times m}$ be a continuous symmetric matrix function on a connected Riemannian manifold M^n. Then, there exist m continuous functions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ such that for each $x \in M^n$ the set of numbers $\{\lambda_i(x)\}_{i=1}^n$ is the set of eigenvalues of $B(x)$.*

Proof. Let the characteristic polynomial of $B(x)$ be

$$f(t, x) = t^m + a_1(x)t^{m-1} + \cdots + a_m(x) \tag{5.36}$$

where $t \in \mathbb{C}$ and $x \in M^n$. Each coefficient a_i is a sum of products of elements of B. By hypothesis B is a continuous function of x, so each a_i is a continuous function of $x \in M^n$

$$a_i : M^n \to \mathbb{R} \ : \quad x \mapsto a_i(x). \tag{5.37}$$

Suppose that $\{\xi_i\}_{i=1}^r$ are the distinct eigenvalues of $B(x_0)$ and let $m_i \in \mathbb{N}^*$ be their respective multiplicities, i.e.,

$$\underbrace{\xi_1,\ \xi_1,\ \ldots,\ \xi_1}_{m_1},\quad \underbrace{\xi_2,\ \xi_2,\ \ldots,\ \xi_2}_{m_2},\quad \underbrace{\xi_3,\ \xi_3,\ \ldots,\ \xi_3}_{m_3},\quad \ldots,\quad \underbrace{\xi_r,\ \xi_r,\ \ldots,\ \xi_r}_{m_r}.$$

So by definition $f(\xi_i, x_0) = 0$ for all $i \in \mathbb{N}^*_{\leq r}$ and $\sum_{i\in\mathbb{N}^*_{\leq r}} = m$.

Denote the circle of radius ρ around ξ_i on the complex plane by

$$C(\xi_i, \rho) \triangleq \{z \in \mathbb{C}|\ |z - \xi_i| = \rho\} \tag{5.38}$$

We are going to prove that the eigenvalues are continuous functions. Choose arbitrary $\varepsilon > 0$. We will show that there exists a $\delta > 0$, such that there are m_i eigenvalues closer to ξ_i than ε.

Proposition 111. *There exists an $\varepsilon_1 > 0$ with $\varepsilon_1 \leq \varepsilon$ such that $f(z, x_0) \neq 0$ for all $z \in C(\xi_i, \varepsilon_1)$.*

Proof. Define

$$\varepsilon_1 \triangleq \min\left\{\varepsilon,\ \frac{1}{2},\ \frac{1}{2}\min_{i,j\in\mathbb{N}^*_{\leq r}}\{|\xi_i - \xi_j|\}\right\} \tag{5.39}$$

By the definition of ε_1, for all $z \in C(\xi_i, \varepsilon_1)$ it is

$$|z - \xi_i| = \varepsilon \leq \frac{1}{2}\min_{i,j\in\mathbb{N}^*_{\leq r}}\{|\xi_i - \xi_j|\} \implies |z - \xi_i| \leq \frac{1}{2}|\xi_i - \xi_j| \tag{5.40}$$

for all $j \neq i$ (i.e., $j \in \mathbb{N}^*_{\leq r} \setminus \{i\}$). By definition $\xi_i \neq \xi_j$, so $|\xi_i - \xi_j| > 0$ which ensures that $\frac{1}{2}|\xi_i - \xi_j| < |\xi_i - \xi_j|$. So

$$|z - \xi_i| < |\xi_i - \xi_j| \implies z \neq \xi_j. \tag{5.41}$$

Also

$$|z - \xi_i| = \varepsilon_0 \implies z \neq \xi_i. \tag{5.42}$$

So $z \neq \xi_k$ for all $k \in \mathbb{N}^*_{\leq r}$. By definition $\{\xi_i\}_{i=1}^r$ is the set of all the roots in t of $f(t, x_0)$. Therefore for all $z \in C(\xi_i, \varepsilon_1)$ it is $f(z, x_0) \neq 0$. $\square$

From now on we will denote the circle $C(\xi_i, \varepsilon_1)$ by C_i.

Define $\varepsilon_{f,i} \triangleq |f(z_{\inf,i}, x_0)| > 0$.

Let $d : M^n \times M^n \to [0, +\infty) : (x, y) \mapsto d(x, y)$ denote the distance function induced by the Riemannian metric g with which M^n is equipped. Define the length of a differentiable curve $\gamma : [a, b] \to M^n$ connecting x and y as

$$L(\gamma) \triangleq \int_a^b \|\gamma'(t)\| \, dt. \tag{5.43}$$

Then the distance function can be defined as the infimum of the length over all differentiable curves joining x and y

$$d(x, y) \triangleq \inf_{\gamma \in C^1} \{L(\gamma)\}. \tag{5.44}$$

Define $w(z) \triangleq f(z, x_0)$ and $g(z) \triangleq f(z, x)$.

Proposition 112 (Uniform continuity on C_i wrt x)**.** *Select arbitrary $\varepsilon_2 > 0$. Then, there exists a $\delta > 0$ such that if $d(x, x_0) < \delta$ then $|w(z) - g(z)| < |w(z)|$ for all $z \in C_i$.*

Proof. Function $|f(z, x_0)|$ is continuous on the compact set C_i. By the Extreme Value Theorem it attains its infimum for some $z_{\inf,i} \in C_i$. It is

$$\inf_{z \in C_i} \{|f(z, x_0)|\} = |f(z_{\inf,i}, x_0)| > 0, \tag{5.45}$$

because by Proposition 111 it is $f(z, x_0) \neq 0$ on C_i.

Function $|z|^j, j \in \mathbb{N}$ is continuous on the compact set C_i. So it attains its supremum for some point $z_{\sup,ij} \in C_i$. Let $\varepsilon_{z,i} \triangleq \max_{j \in \mathbb{N}_{\leq m}} \{z_{\sup,ij}\}$. It is $\varepsilon_{z,i} > 0$ because as previously shown $z \neq 0$ on C_i. Define $\varepsilon_1 \triangleq \frac{\varepsilon_{f,i}}{\varepsilon_{z,i}(m+1)}$. By hypothesis the coefficient functions a_k are continuous in x, so there exist m numbers $\delta_k > 0$ such that

$$d(x, x_0) < \delta_m \implies |a_k(x) - a_k(x_0)| < \varepsilon_1 \tag{5.46}$$

Take $\delta \triangleq \min_k\{\delta_k\}$. Then

$$d(x, x_0) < \delta \implies |a_k(x) - a_k(x_0)| < \varepsilon_1 \tag{5.47}$$

for all $k \in \mathbb{N}_{\leq m}$. It is

$$\begin{aligned} |f(z, x) - f(z, x_0)| &= \left|(a_1(x) - a_1(x_0))\, z^{m-1} + \cdots + (a_m(x) - a_m(x_0))\right| \\ &\leq \left|\varepsilon_{z,i} \sum_{k=1}^{m} (a_k(x) - a_k(x_0))\right| \\ &\leq \varepsilon_{z,i} \sum_{k=1}^{m} |a_k(x) - a_k(x_0)| \end{aligned} \tag{5.48}$$

because by definition $\varepsilon_{z,i} > 0$. So if $d(x, x_0) < \delta$, then

$$\begin{aligned} \varepsilon_{z,i} \sum_{k=1}^{m} |a_k(x) - a_k(x_0)| &\leq \varepsilon_{z,i} \sum_{k=1}^{m} \varepsilon_1 = \varepsilon_{z,i} m \varepsilon_1 = \varepsilon_{z,i} m \frac{\varepsilon_{f,i}}{\varepsilon_{z,i}(m+1)} = \frac{m}{m+1} \varepsilon_{f,i} \\ &< \varepsilon_{f,i} \leq |f(z, x_0)| \end{aligned} \tag{5.49}$$

on C_i. Combining the previous yields that if $d(x, x_0) < \delta$, then

$$|f(z,x) - f(z,x_0)| < \varepsilon_{f,i} \implies |f(z,x) - f(z,x_0)| < |f(z,x_0)|\,. \tag{5.50}$$

The inequality $|f(z,x) - f(z,x_0)| < \varepsilon_{f,i}$ implies that $f(z,x) \neq 0$ on C_i, provided $d(x,x_0) < \delta$. We will need this later. Nonetheless, observe[7] that $|f(z,x) - f(z,x_0)| < |f(z,x_0)|$ ensures that both $f(z,x) \neq 0$ and $f(z,x_0) \neq 0$ on C_i. Assume the contrary, if $f(z,x)$ becomes zero, then $|f(z,x) - f(z,x_0)| = |f(z,x_0)|$ which contradicts the inequality. If $f(z,x_0)$ becomes zero, then the inequality implies $|f(z,x)| < 0$ which is impossible. □

We can now use Rouché's (Theorem 6.2.5 p.388 [23]). Function f is a polynomial in z so it is holomorphic in z on all $\mathbb{C}$, which is a simply connected domain. As a polynomial, function f does not have any poles, so it is meromorphic. Circle C_i is a simple (i.e., homotopic to a point), closed and smooth oriented curve in $\mathbb{C}$ passing through no zero of w or g, as implied[8] by Proposition 112. By Proposition 112 it is $|w(z) - g(z)| < |w(z)|$ on C_i.

Then, according to Rouché's Theorem it is

$$Z_w - P_w = Z_g - P_g \tag{5.51}$$

where Z_w is the number of zeros of function w within the area enclosed by C_i, P_w is the number of poles and similarly for Z_g, P_g. Since $w(z) = f(z,x_0)$ and $g(z) = f(z,x)$, and f is a polynomial in z, it does not have any poles, so $P_w = 0$ and $P_g = 0$. This leads to $Z_w = Z_g$, so $f(z,x_0)$ and $f(z,x)$ have the same number of zeros within C_i. We have defined $Z_w = m_i$, so $Z_g = m_i$.

As a result, $f(z,x)$ has m_i zeros $\lambda_j(x)$ in the interior of circle C_i. Hence

$$|\lambda_j(x) - \xi_i| < \varepsilon_1 \leq \varepsilon, \tag{5.52}$$

because all points in the interior of C_i are closer to ξ_i than the radius of circle C_i and by the definition of C_i its radius is ε_1.

In summary, we have proved that for arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(x,x_0) < \delta \implies |\lambda_j(x) - \xi_i| < \varepsilon \tag{5.53}$$

for m_i functions $\lambda_j(x)$ equal to the eigenvalues of B, which expresses the continuity of the eigenvalues. This holds for all circles C_i, so for all eigenvalues. □

5.3 Continuity of principal relative curvature functions

This has been proved in Lemma 54 using Lemma 110.

[7] This is also a comment in p.389 of [23].

[8] A previous comment explains this in detail. The conclusion follows also directly from Proposition 111 and $|f(z,x) - f(z,x_0)| < \varepsilon_{f,i}$ in (5.50).

5.4 Principal Relative Curvatures near Obstacle

Theorem 113. *Select arbitrary $q_d \in \mathring{\mathscr{F}}$. Assume that at each first-order contact point in C_i^1, at least m principal relative curvatures are negative and at most N_0 are zero.*

Then, there exists a $\delta > 0$ and an $L > 0$ with the following properties. For each point $q \in T_\delta(C_i^1)$ there exist two numbers $N_n \geq m$ and N_p with $N_n + N_p \geq (n-1) - N_0$ such that N_n principal relative curvature functions have values less than or equal to $-L$

$$\nu_{ij_1}(q) \ \leq \nu_{ij_2}(q) \ \leq \ldots \ \leq \nu_{ij_{N_n}}(q) \ \leq -L < 0, \tag{5.54}$$

and N_p principal relative curvature functions have values larger than or equal to L

$$0 < L \ \leq \nu_{ij_{N_n+1}}(q) \ \leq \nu_{ij_{N_n+2}}(q) \ \leq \ldots \ \leq \nu_{ij_{N_n+N_p}}(q). \tag{5.55}$$

This reveals that the need for one negative principal relative curvature function reserves one dimension of the principal tangent bundle for that purpose (the "relative convexity" between β_i and γ_d). So to have a zero principal relative curvature and remain navigable, there should be a second dimension of the principal bundle available. This requires that the problem be at least 3-dimensional (the ambient E^3).

Proof. By Corollary 13 there exists a $\delta_1 > 0$ such that $\nabla\beta_i \neq 0$ in the uniform tube $T_{\delta_1}(\partial\mathscr{O}_i)$. (Since $q_d \notin \partial\mathscr{F}$ we can ensure $q_d \notin T_{\delta_1}(\partial\mathscr{O}_i)$ by taking δ_1 sufficiently small. Then $\nabla\gamma_d \neq 0$ within $T_{\delta_1}(\partial\mathscr{O}_i)$). So by Lemma 54 all the $(n-1)$ principal relative curvature functions ν_{ij} are well-defined and continuous over $T_{\delta_1}(C_i^1)$. By Proposition 80 the set C_i^1 is compact.

Then by Proposition 107, there exists a $\delta > 0$ ($\delta \leq \delta_1$) and an $L > 0$ with the following property. (Note that we have reversed the argument about the minimal number of positive values to negative values, but that is always possible by considering the negated functions $-\nu_{ij}$.) For each point $q \in T_\delta(C_i^1)$ there are two numbers $N_n \geq m$ and N_p with $N_n + N_p \geq (n-1) - N_0$ such that the N_n principal relative curvature functions have values less than or equal to $-L$

$$\nu_{ij_1}(q) \ \leq \nu_{ij_2}(q) \ \leq \ldots \ \leq \nu_{ij_{N_n}}(q) \ \leq -L < 0, \tag{5.56}$$

and the N_p principal relative curvature functions have values larger than or equal to L

$$0 < L \ \leq \nu_{ij_{N_n+1}}(q) \ \leq \nu_{ij_{N_n+2}}(q) \ \leq \ldots \ \leq \nu_{ij_{N_n+N_p}}(q). \tag{5.57}$$

□

If we make Assumptions 143 and 147, then $m = 1$ and $N_0 = 1$ and these ensure $N_n \geq 1$ and at most $N_0 = 1$ zero ν_{ij} on C_i^1. With these assumptions Theorem 113 ensures that at every point δ-near to a first-order contact point the above hold for $N_n + N_p = (n-1) - N_0 = n-2$. The case $N_n = 0$ is not navigable with Koditschek-Rimon functions, by the converse Theorem 165, so it must be $m > 0$.

The cases of degeneracy are caused by only a measure zero set of destinations q_d. For this reason, they comprise a non-generic situation and are separately studied. For $N_0 \leq 1$ the critical set comprises of 1-dimensional curves and is considered in **??**. The case $N_0 > 1$ allows higher-order degeneracy to occur, so the critical set may become complicated. This case is investigated in **??**.

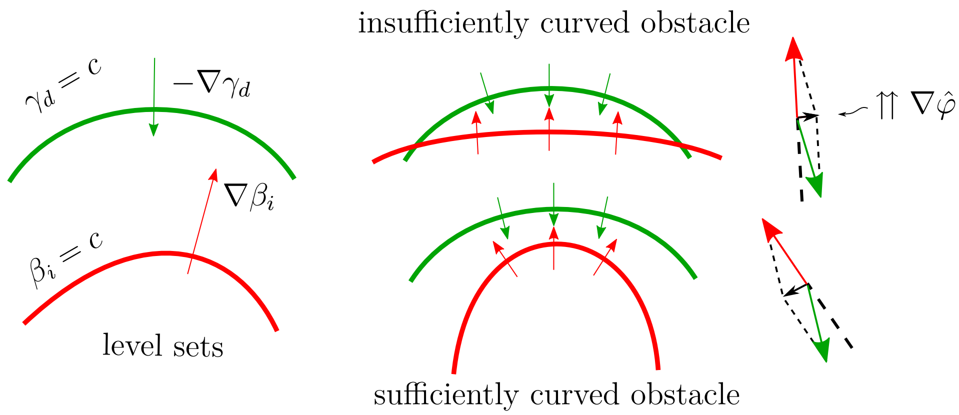

Figure 5.5: KRNF tuning mechanism geometry.

Note that Theorem 113 cannot be proved for the set $\partial\mathscr{O}_i$, despite its compactness, assumed in Assumption 1. For an arbitrary point[9] the Assumptions 143 and 147 about the principal curvatures $\kappa_{ij}(q)$ on $\partial\mathscr{O}_i$ do not imply the required conditions for the principal relative curvatures $\nu_{ij}(q)$ on $\partial\mathscr{O}_i$. Only on C_i^1.

We first consider what would happen with the first condition. Suppose that q is not a first-order contact point, i.e., $q \in \partial\mathscr{O}_i \setminus C_i^1$. Then $\hat{\psi}_i(q) < 1$ and the condition $\nu_i(q) < 0$ involves $\hat{\psi}_i(q) = \cos\theta_i(q)$. This leads to the "curvature sphere assumption" used in the earlier work [24]. The curvature sphere assumption is stronger than Assumption 143. Instead of inclusion of the principal curvature radius within the obstacle, it requires that the whole curvature sphere be included (i.e., every radius of one principal curvature sphere).

Now consider what would happen with the second condition. If $\hat{\psi}_i(q) < 1$, then $\nu_i(q) = 0$ can happen on the boundary of the whole curvature sphere, not only at the curvature center. So non-contact points appear to be causing degeneracy and need to be avoided, which is a stronger assumption.

In conclusion, referring to C_i^1 instead of $\partial\mathscr{O}_i$ is important in Theorem 113, because by Lemma 79 the equation $\hat{\psi}_i = 1$ holds if and only if $q \in C_i^1$.

5.5 Commentary: Connecting normal curvature to relative curvature signs: Past and Present

The question is: *How does the normal curvature sign affect the sign of relative curvature at obstacle boundary points close to critical points?* This relation changed from the Diploma thesis to the present treatment.

1. Previously, if the curvature sphere was in the obstacle, then everything was ok. It ensured that $\nu_i < 0$ at that boundary point.

   Now, if the curvature *center* is in the obstacle (and not the whole curvature sphere), then $\nu_i < 0$. Previously the requirement $\nu_{i3} > 0$ and *not* $\nu_{i3} = 1$ left the possibility of the whole curvature sphere, whereas now only one point on the curvature sphere can cause degeneracy.

[9] A point which may not be a first-order contact point.

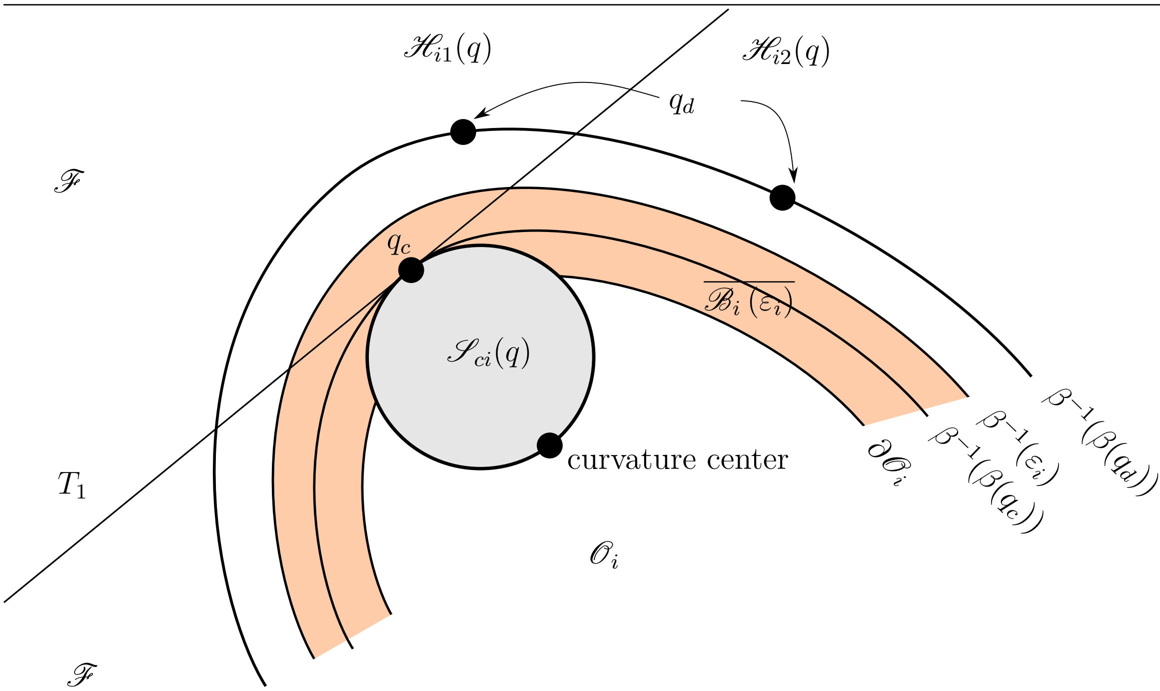

Figure 5.6: Sets involved.

2. Previously, if the curvature sphere was convex but not sufficiently small, degeneracy could happen, because the curvature sphere protruded from the obstacle.

   Previously, if q_d was on the curvature sphere (which "covered" the space between the surface and its focal surface as well), then it could cause degeneracy, even when $\nu_{i3} > 0$

   Now, if the curvature center is outside the obstacle and convex there, only if q_d is on the curvature center degeneracy is caused. If it is on some other point of the curvature sphere, then degeneracy is not caused. Then it is not a contact point and no critical point inherits the sign of ν_i from it (although $\nu_i = 0$ if q_d is on the curvature sphere, which is the same as before :). Some other boundary point, which is a contact point, determines what happens.

   Previously, a protruding convex curvature sphere and a q_d inside or outside it where inconclusive and they were considered as causing a possible degeneracy.

   Now, if the destination is further than a focal surface, the respective $\nu_{ij} < 0$. Now, if the destination is closer than a focal surface, the respective $\nu_{ij} > 0$.

3. Previously, if the curvature sphere was non-convex, then we infer $\nu_i > 0$ from $\nu_{i3} > 0$. Now, if the curvature sphere is non-convex, we infer $\nu_i > 0$ from $\hat{\psi}_i = 1$.

From the previous comments, $\nu_{i3} > 0$ is needed ONLY in the thesis proof to be inferred from critical points. For first-order contact points, it is not needed. The rest of the proof is the same. Therefore, the new proof has first-order contact points and the following part. The old proof has the ν_{i3} part fixed using the compact set based on angle and the following part as well.

Depending on old/new, the combination of $\nu_{i3} > 0$ and destination position wrt curvature spheres, or on the destination position wrt focal surfaces, it is ν_{ij} positive for at least N_p everywhere on the surface, zero for at most N_0 and negative for the rest.

5.6 Relative Curvature Sign-definite Subspaces depend on Principal Relative Curvatures

It is useful to define at a point q the spans of principal directions $\hat{p}_{ij}(q)$ whose associated principal relative curvatures $\nu_{ij(q)}$ satisfy a selected upper- or lower-bound L.

Definition 114. *Let $I \triangleq \mathbb{N}^*_{\leq n-1}$ and define*

$$P_i(q) \triangleq \{\hat{p}_{ij}(q)\}_{j\in I}, \quad \mathscr{P}_i(q) \triangleq \operatorname{span}\{P_i(q)\} \tag{5.58}$$

Given any $L \in \mathbb{R}$ let

$$\begin{aligned} I_i^-(q,L) &\triangleq \{j \in I |\ \nu_i(q,\hat{p}_{ij}(q)) \leq L\} = \{j \in I |\ \nu_{ij}(q) \leq L\}, \\ P_i^-(q,L) &\triangleq \{\hat{p}_{ij}(q)\}_{j\in I_i^-(q,L)}, \\ \mathscr{P}_i^-(q,L) &\triangleq \operatorname{span}\left\{P_i^-(q,L)\right\} \end{aligned} \tag{5.59}$$

and

$$\begin{aligned} I_i^+(q,L) &\triangleq \{j \in I |\ L \leq \nu_i(q,\hat{p}_{ij}(q))\} = \{j \in I |\ L \leq \nu_{ij}(q)\}, \\ P_i^+(q,L) &\triangleq \{\hat{p}_{ij}(q)\}_{j\in I_i^+(q,L)}, \\ \mathscr{P}_i^+(q,L) &\triangleq \operatorname{span}\left\{P_i^+(q,L)\right\} \end{aligned} \tag{5.60}$$

Also, let

$$\begin{aligned} I_i^\pm(q,L) &\triangleq I_i^-(q,L) \cup I_i^+(q,L) = \{j \in I |\ L \leq |\nu_{ij}(q)|\}, \\ P_i^\pm(q,L) &\triangleq P_i^-(q,L) \cup P_i^+(q,L), \\ \mathscr{P}_i^\pm(q,L) &\triangleq \mathscr{P}_i^-(q,L) \cup \mathscr{P}_i^+(q,L), \end{aligned} \tag{5.61}$$

and

$$\begin{aligned} I_i^0(q,L) &\triangleq I \setminus I_i^\pm(q,L), \\ P_i^0(q,L) &\triangleq P_i \setminus P_i^\pm(q,L), \\ \mathscr{P}_i^0(q,L) &\triangleq \mathscr{P}_i \setminus \mathscr{P}_i^\pm(q,L) \end{aligned} \tag{5.62}$$

and the corresponding unit subsets (restrictions of the previous subsets onto the unit sphere, they can be viewed also as the radial projections along the rays emanating from the origin of the previous subsets onto the unit sphere)

$$\begin{aligned} U\mathscr{P}_i(q) &\triangleq \mathscr{P}_i(q) \cap UT_qB_i, \\ U\mathscr{P}_i^-(q,L) &\triangleq \mathscr{P}_i^-(q,L) \cap UT_qB_i, \\ U\mathscr{P}_i^+(q,L) &\triangleq \mathscr{P}_i^+(q,L) \cap UT_qB_i, \\ U\mathscr{P}_i^\pm(q,L) &\triangleq \mathscr{P}_i^\pm(q,L) \cap UT_qB_i, \\ U\mathscr{P}_i^0(q,L) &\triangleq \mathscr{P}_i^0(q,L) \cap UT_qB_i. \end{aligned} \tag{5.63}$$

Observe that $\mathscr{P}_i^-(q,L)$, $\mathscr{P}_i^+(q,L)$, $\mathscr{P}_i^\pm(q,L)$ are proper subspaces of the tangent space T_qB_i.

For example, $\mathscr{P}_i^+(q,0)$ is the maximal positive-definite subspace of $Q(\hat{t}_i) = \nu_i(q,\hat{t}_i)$ and is spanned by $P_i^+(q,0)$. The maximal negative-definite subspace of $Q(\hat{t}_i) = \nu_i(q,\hat{t}_i)$ is $\mathscr{P}_i^+(q,0)$ and is spanned by $P_i^+(q,0)$.

Observe that the choice of eigenvectors may *not be unique*. Indeed this can happen if there exist multiple eigenvalues. The eigenvectors associated with the non-simple eigenvalues span a semi-umbilic subspace, so there is no unique selection of eigenvectors. Nevertheless, we can circumvent this by fixing one possible choice of eigenvectors for the umbilic subspace. This is formally ensured by the next proposition.

Hereafter we set $\gamma_d(q) = \|q - q_d\|^2$ and work in a regular neighborhood of $\partial\mathscr{O}_i$ to ensure $\nabla\beta_i(q) \neq 0$. Using the notions developed so far, it is now possible to generalize Proposition 3.6 [1].

Proposition 115. *Select $\gamma_d \in \|q - q_d\|^2$. Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d(q) \neq 0$. Then, there exists an orthonormal system of eigenvectors of $\nu_i(q, \hat{t}_i)$ at point q.*

It is useful to define the maximal subset of tangent vectors $t_i \in T_q B_i$ whose associated relative curvatures $\nu_i(q, t_i)$ satisfy an upper- or lower bound L. For this we first need to define the multiplicative span.

Definition 116. *Let U be a subset of the vector space V. The* multiplicative span *operator is defined as*

$$\operatorname{muspan}\{U\} \triangleq \{v \in V |\, \exists u \in U.\ \exists\lambda \in \mathbb{R}:\ v = \lambda u\} \tag{5.64}$$

The multiplicative span operator yields the multiplicative closure of a subset $U \subset V$.

Definition 117. *Given any $L \in \mathbb{R}$ define*

$$\begin{aligned} R_i^-(q, L) &\triangleq \left\{\hat{t}_i \in U\mathscr{P}_i(q) |\, \nu_i(q, \hat{t}_i) \leq L\right\}, \\ \mathscr{R}_i^-(q, L) &\triangleq \operatorname{muspan}\left\{R_i^-(q, L)\right\}, \end{aligned} \tag{5.65}$$

and

$$\begin{aligned} R_i^+(q, L) &\triangleq \left\{\hat{t}_i \in U\mathscr{P}_i(q) |\, L \leq \nu_i(q, \hat{t}_i)\right\}, \\ \mathscr{R}_i^+(q, L) &\triangleq \operatorname{muspan}\left\{R_i^+(q, L)\right\}, \end{aligned} \tag{5.66}$$

and

$$\begin{aligned} R_i^\pm(q, L) &\triangleq R_i^+(q, L) \cup R_i^-(q, L) = \left\{\hat{t}_i \in U\mathscr{P}_i(q) |\, L \leq \left|\nu_i(q, \hat{t}_i)\right|\right\}, \\ \mathscr{R}_i^\pm(q, L) &\triangleq \mathscr{R}_i^+(q, L) \cup \mathscr{R}_i^-(q, L), \end{aligned} \tag{5.67}$$

and

$$\mathscr{R}_i^0(q, L) \triangleq \mathscr{P}_i \setminus \mathscr{R}_i^\pm(q, L). \tag{5.68}$$

and the corresponding unit subsets (restrictions of the previous subsets onto the unit sphere, they can be viewed also as the radial projections along the rays emanating from the origin of the previous subsets onto the unit sphere)

$$\begin{aligned} U\mathscr{R}_i^-(q, L) &\triangleq \mathscr{R}_i^-(q, L) \cap UT_q B_i = R_i^-(q, L), \\ U\mathscr{R}_i^+(q, L) &\triangleq \mathscr{R}_i^+(q, L) \cap UT_q B_i = R_i^+(q, L), \\ U\mathscr{R}_i^\pm(q, L) &\triangleq \mathscr{R}_i^\pm(q, L) \cap UT_q B_i = R_i^\pm(q, L), \\ U\mathscr{R}_i^0(q, L) &\triangleq \mathscr{R}_i^0(q, L) \cap UT_q B_i = R_i^0(q, L). \end{aligned} \tag{5.69}$$

Remark 118. *Note that, in general, the subsets $\mathscr{R}_i^-(q, L), \mathscr{R}_i^+(q, L)$ and $\mathscr{R}_i^\pm(q, L)$ are* not *linear subspaces of the tangent space $T_q B_i$. This is in contrast to the subsets $\mathscr{P}_i^-(q, L)$, $\mathscr{P}_i^+(q, L)$ and $\mathscr{P}_i^\pm(q, L)$, which* are *linear subspaces of the tangent space.*

Hereafter the notation $\mathscr{R}_i^*(q,L)$, $\mathscr{P}_i^*(q,L)$ refers to the collections of subsets

$$\mathscr{R}_i^-(q,L), \mathscr{R}_i^+(q,L), \mathscr{R}_i^\pm(q,L)$$

and

$$\mathscr{P}_i^-(q,L), \mathscr{P}_i^+(q,L), \mathscr{P}_i^\pm(q,L)$$

respectively.

The following asserts that the relative curvature ν_i is upper-bounded and lower-bounded by the maximal and minimal principal relative curvatures on the unit span of any selected subset of principal directions.

Lemma 119. *Select some point q. Define $\gamma_d \in \|q-q_d\|^2$. Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d(q) \neq 0$. Choose $I_v \subseteq I$ and define the (sub)set $P \triangleq \{\hat{p}_{ij}(q)\}_{j\in I_v}$ of principal directions. Let $\mathscr{P} \triangleq \operatorname{span}\{P\}$ and $U\mathscr{P} \triangleq \mathscr{P} \cap UT_qB_i$.*

Then, the relative curvature ν_i is bounded by the minimal and maximal relative curvatures of the principal directions considered, i.e.,

$$\min_{j\in I_v}\{\nu_{ij}\} \leq \nu_i(q,\hat{t}_i) \leq \max_{j\in I_v}\{\nu_{ij}\}, \tag{5.70}$$

for all $\hat{t}_i \in U\mathscr{P}$.

Proof. By Corollary 53 the vectors $\hat{p}_{ij}$ are the eigenvectors of the quadratic form $Q(\hat{t}_i) = \nu_i(q,\hat{t}_i)$ at q. Their associated eigenvalues are $\nu_{ij}(q)$.

Then, by Proposition 166 the quadratic form $Q(\hat{t}_i)$ is bounded on $U\mathscr{P}$ by the minimal and maximal eigenvalues of the spanning eigenvectors in P. The minimal eigenvalue is $\min_{j\in I_v}\{\nu_i(q,\hat{p}_{ij}(q))\} = \min_{j\in I_v}\{\nu_{ij}(q)\}$. The maximal eigenvalue on this subspace is similarly obtained. □

The following asserts that relative curvature ν_i is upper bounded by L on the unit span of principal directions $\hat{p}_{ij}$ with associated principal relative curvatures ν_{ij} which are upper bounded by L.

Corollary 120. *Select $\gamma_d(q) = \|q-q_d\|^2$. Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d(q) \neq 0$.*

Then, $\nu_i(q,\hat{t}_i) \leq L$ for all $\hat{t}_i \in U\mathscr{P}_i^-(q,L)$.

Proof. By Definition 114 the set $P_i^-(q,L)$ includes only eigenvectors $\hat{p}_{ij}(q)$ with associated eigenvalues $\nu_{ij}(q) \geq L$. So

$$\max_{j\in I_v}\{\nu_{ij}\} = \max_{j\in I_i^-(q,L)}\{\nu_{ij}\} \leq L. \tag{5.71}$$

Choose $I_v = I^-(q,L)$. Then, by Lemma 119 it follows that

$$\nu_i(q,\hat{t}_i) \leq \max_{j\in I_v}\{\nu_{ij}\}, \tag{5.72}$$

for all $\hat{t}_i \in U\mathscr{P}_i^-(q,L)$. Combine (5.71) and (5.72) to obtain

$$\nu_i(q,\hat{t}_i) \leq L,$$

for all $\hat{t}_i \in U\mathscr{P}_i^-(q,L)$. □

It also follows that the relative curvature is negative on the span of negative principal relative curvatures (expected).

The following asserts that relative curvature ν_i is lower bounded by L on the unit span of principal directions $\hat{p}_{ij}$ with associated principal relative curvatures ν_{ij} which are lower bounded by L.

Corollary 121. *Select $\gamma_d(q) = \|q - q_d\|^2$. Assume that $\nabla\beta_i(q) \neq 0$ and $\nabla\gamma_d(q) \neq 0$. Then, $\nu_i(q, \hat{t}_i) \geq L$ for all $\hat{t}_i \in U\mathscr{P}_i^+(q, L)$.*

Proof. Similar to Corollary 120. □

Corollaries 120 and 121 imply that if N_n principal relative curvatures $\nu_{ij}(q) \leq -L$, then relative curvature in the whole subspace spanned by them is upper bounded by $-L$, i.e., $\nu_i(q, \hat{t}_i) \leq -L$ for all $\hat{t}_i$ is the unit subspace spanned by the corresponding $\hat{p}_{ij}$.

No relation exists between the dimensions of the subsets $\mathscr{R}_i^*(q, L)$ and $\mathscr{P}_i^*(q, L)$. The reason is that the subsets $\mathscr{R}_i^*(q, L)$ are not necessarily linear subspaces, see Remark 118. So the dimension of $\mathscr{R}_i^*(q, L)$ is not always defined. Nonetheless, the dimension of $\mathscr{P}_i^*(q, L)$ is always well-defined, because by Definition 114 it is a linear subspace.

Although we cannot compare their dimensions, the next lemma establishes a subset relation between them.

Lemma 122. *Select an $L \in \mathbb{R}$. It is*

$$\mathscr{P}_i^-(q, L) \subseteq \mathscr{R}_i^-(q, L), \quad \mathscr{P}_i^+(q, L) \subseteq \mathscr{R}_i^+(q, L). \tag{5.73}$$

Proof. Consider the first claim. Choose any $v \in \mathscr{P}_i^-(q, L)$. Project it onto the unit sphere $y \triangleq \frac{v}{\|v\|}$. It is $\|y\| = 1$. By Definition 114 and Corollary 120 it follows that $\nu_i(q, y) \leq L$. Therefore by Definition 117 it is $v \in \mathscr{R}_i^-(q, L)$, because there exists a multiplicative factor $\lambda = \|v\| \in \mathbb{R}$ such that $v = \lambda y$ and $\|y\| = 1$ and $\nu_i(q, y) \leq L$.

The proof of the second claim is similar and uses Corollary 121. □

Corollary 123. *Select an $L \in \mathbb{R}$. It is*

$$\bigcup_{t_i \in \mathscr{P}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\} \subseteq \bigcup_{t_i \in \mathscr{R}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\}. \tag{5.74}$$

Proof. Choose any $v \in \bigcup_{t_i \in \mathscr{P}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\}$. Then, there exist $t_i \in \mathscr{P}_i^-(q, L)$ and $a, b \in \mathbb{R}$ such that

$$v = at_i + b\hat{r}_i.$$

By Lemma 122 it is $\mathscr{P}_i^+(q, L) \subseteq \mathscr{R}_i^+(q, L)$ so $t_i \in \mathscr{R}_i^+(q, L)$. Therefore $v \in \operatorname{span}\{t_i, \hat{r}_i\}$ for some $t_i \in \mathscr{R}_i^+(q, L)$. This implies that $v \in \bigcup_{t_i \in \mathscr{R}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\}$. □

Proposition 124. *Select an $L \in \mathbb{R}$. It is*

$$\bigcup_{t_i \in \mathscr{P}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\} = \operatorname{span}\left\{\mathscr{P}_i^+(q, L), \hat{r}_i\right\}. \tag{5.75}$$

Proof. First we show that

$$\bigcup_{t_i \in \mathscr{P}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\} \subseteq \operatorname{span}\left\{\mathscr{P}_i^+(q,L), \hat{r}_i\right\}.$$

Choose any $v \in \bigcup_{t_i \in \mathscr{P}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\}$. Then, there exist $t_i \in \mathscr{P}_i^+(q,L)$ and $a, b \in \mathbb{R}$ such that v can be written as the linear combination

$$v = at_i + b\hat{r}_i.$$

It is $t_i \in \mathscr{P}_i^+(q,L)$ and $\hat{r}_i \in \{\hat{r}_i\}$. So $v \in \operatorname{span}\left\{\mathscr{P}_i^+(q,L), \hat{r}_i\right\}$.

We now show that

$$\operatorname{span}\left\{\mathscr{P}_i^+(q,L), \hat{r}_i\right\} \subseteq \bigcup_{t_i \in \mathscr{P}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\}.$$

Choose any $v \in \operatorname{span}\left\{\mathscr{P}_i^+(q,L), \hat{r}_i\right\}$. Then, there exist $w_j \in \mathscr{P}_i^+(q,L)$ and $a_j \in \mathbb{R}$ such that v can be written as the (finite) linear combination

$$v = b\hat{r}_i + \sum_{j=1}^{N} a_j w_j.$$

But by Definition 114 the subset $\mathscr{P}_i^+(q,L)$ is a linear subspace of $T_q B_i$. It follows that there exists some $t_i' \in \mathscr{P}_i^+(q,L)$ such that $t_i' = \sum_{j=1}^{N} a_j w_j$. Substitute to obtain

$$v = b\hat{r}_i + 1 \cdot t_i'.$$

This implies that $v \in \operatorname{span}\{t_i', \hat{r}_i\}$, so

$$v \in \bigcup_{t_i \in \mathscr{P}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\}. \tag{5.76}$$

□

Lemma 125. *Select an $L \in \mathbb{R}$. It is*

$$\operatorname{span}\left\{\mathscr{P}_i^+(q,L), \hat{r}_i\right\} \subseteq \bigcup_{t_i \in \mathscr{R}_i^+(q,L)} \operatorname{span}\{t_i, \hat{r}_i\}. \tag{5.77}$$

Proof. Combine Corollary 123 and Proposition 124. □

5.7 ν_{i3} near the critical set

If k has been selected larger than $N(\varepsilon_{I_0})$, then no critical points exist on the boundary of $\mathscr{B}_i(\varepsilon)$. In that case, the closure of set $\mathscr{C}_{\hat{\varphi},i}(\varepsilon)$ is again a subset of $\mathscr{B}_i(\varepsilon)$, so $\mathscr{C}_{\hat{\varphi},i}(\varepsilon) = \overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon)$ which ensures that $\mathscr{C}_{\hat{\varphi},i}(\varepsilon)$ is closed.

Proposition 126. *Choose arbitrary $\lambda \in (0,1)$.*

Then, there exists an $\bar{\varepsilon}_i > 0$, such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$, set $\overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i)$ is compact and

$$\hat{\psi}_i(q) \geq \lambda > 0 \tag{5.78}$$

for all $q \in \overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i)$.

Proof. Select arbitrary $\lambda \in (0,1)$. By Proposition 71 there exists an $\bar{\varepsilon}_{i1} > 0$ such that for all $0 < \varepsilon_i < \bar{\varepsilon}_{i1}$ no critical points are contained in $L_{i1}(\varepsilon_i, \lambda)$. So any critical points of $\mathscr{B}_i(\varepsilon_i)$ are in $L_{i2}(\varepsilon_i, \lambda)$

$$\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq L_{i2}(\varepsilon_i, \lambda). \tag{5.79}$$

The upper bound $\bar{\varepsilon}_{i1}$ is by construction[10] smaller than the one provided by Proposition 24, so $\overline{\mathscr{B}_i(\varepsilon_i)}$ is compact and β_i, γ_d are regular in $\overline{\mathscr{B}_i(\varepsilon_i)}$. It is

$$\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq \mathscr{B}_i(\varepsilon_i) \implies \overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i) \subseteq \overline{\mathscr{B}_i(\varepsilon_i)}.$$

So $\hat{\psi}_i$ and L_{i2} are well-defined in $\overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i)$. The compactness of $\overline{\mathscr{B}_i(\varepsilon_i)}$ implies that set $\overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i)$ is compact.

By Definition 70 it is $\hat{\psi}_i(q) \geq \lambda > 0$ for all $q \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. Function $\hat{\psi}_i$ is continuous as proved in Proposition 46, so $\hat{\psi}_i(q) \geq \lambda > 0$ also in the closure $\overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i)$. □

Proposition 127. *There exists an $\bar{\varepsilon}_i > 0$ and a $b > 0$, such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$, it holds that*

$$\nu_{i3}(q) \geq b > 0 \tag{5.80}$$

for all $q \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. (and $\overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i)$ as well)

Proof. Choose arbitrary $\lambda \in (0,1)$. By Proposition 126 there exists an $\bar{\varepsilon}_{i1} > 0$, such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i1}$ it holds that

$$\hat{\psi}_i(q) \geq \lambda > 0 \tag{5.81}$$

for all $q \in \overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i)$. Note that Proposition 126 ensures that $\hat{\psi}_i$ is well-defined in $\mathscr{B}_i(\varepsilon_i)$. Therefore, the function ν_{i3} is also well-defined in $\mathscr{B}_i(\varepsilon_i)$.

By Proposition 43 functions $\hat{\psi}_i$ and ν_{i3} have the same sign, so

$$\nu_{i3}(q) > 0$$

for all $q \in \overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i)$.

Select $\bar{\varepsilon}_i \triangleq \frac{1}{2}\bar{\varepsilon}_{i1} > 0$. By Proposition 47 function ν_{i3} is continuous. Set $\overline{\mathscr{C}_{\hat{\varphi},i}}(\bar{\varepsilon}_i)$ is compact, so function ν_{i3} attains its infimum for some q_0 in $\overline{\mathscr{C}_{\hat{\varphi},i}}(\bar{\varepsilon}_i)$

$$\nu_{i3}(q) \geq b = \inf_{q \in \overline{\mathscr{C}_{\hat{\varphi},i}}(\varepsilon_i)} \{\nu_{i3}(q)\} = \nu_{i3}(q_0) > 0 \implies \nu_{i3}(q) \geq b > 0 \tag{5.82}$$

[10] See the proof of Proposition 71.

for all $q \in \overline{\mathscr{C}_{\hat{\varphi},i}}\,(\bar{\varepsilon}_i)$ (hence also in $\mathscr{C}_{\hat{\varphi},i}\,(\bar{\varepsilon}_i)$).

All sets $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ with $\varepsilon_i < \bar{\varepsilon}_i$ are subsets of $\overline{\mathscr{C}_{\hat{\varphi},i}}\,(\bar{\varepsilon}_i)$. Therefore the same infimum b applies for all $\varepsilon_i < \bar{\varepsilon}_i$. □

Proposition 128. *There exists an $\bar{\varepsilon}_i > 0$ and a $b > 0$ such that all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ have the following properties. There exists a $\delta > 0$, such that for every $q_1 \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, it is*

$$\nu_{i3}(q_2) > \frac{b}{2} > 0 \tag{5.83}$$

for every $q_2 \in \overline{\mathscr{B}_i\,(\varepsilon_i)}$ with $\|q_1 - q_2\| < \delta$.

Proof. By Proposition 127 there exists an $\bar{\varepsilon}_i > 0$ and a $b > 0$, such that for every $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$, it is

$$\nu_{i3}(q) \geq b > 0$$

for all $q \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. Moreover, originating from Corollary 19, the set $\overline{\mathscr{B}_i\,(\varepsilon_i)}$ is compact.

By uniform continuity of the continuous function ν_{i3} over the compact set $\overline{\mathscr{B}_i\,(\varepsilon_i)}$, there exists a $\delta > 0$ such that

$$|\nu_{i3}(q_1) - \nu_{i3}(q_2)| < \frac{b}{2} \tag{5.84}$$

for all $q_1, q_2 \in \overline{\mathscr{B}_i\,(\varepsilon_i)}$ with $\|q_1 - q_2\| < \delta$. This implies that

$$-\frac{b}{2} < \nu_{i3}(q_2) - \nu_{i3}(q_1) < \frac{b}{2} \iff 0 < \frac{b}{2} < \nu_{i3}(q_2) + (b - \nu_{i3}(q_1) < \frac{3b}{2}$$

for $q_1 \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq \overline{\mathscr{B}_i\,(\varepsilon_i)}$ and $q_2 \in \overline{\mathscr{B}_i\,(\varepsilon_i)}$ with $\|q_1 - q_2\| < \delta$. It is

$$\nu_{i3}(q_1) \geq b \implies b - \nu_{i3}(q_1) \leq 0,$$

because $q_1 \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. This yields

$$0 < \frac{b}{2} < \nu_{i3}(q_2), \tag{5.85}$$

which proves the claim. □

Proposition 129. *Select arbitrary $\delta > 0$. There exists an $\bar{\varepsilon}_i > 0$ such that all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ have the following property. For every $q_1 \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, there exists a $q_2 \in \partial\mathscr{O}_i$ with*

$$\nu_{i3}(q_2) > 0 \tag{5.86}$$

and $\|q_1 - q_2\| < \delta$.

Proof. By Proposition 128 there exist an $\bar{\varepsilon}_{i1} > 0$ and a $b > 0$, such that for all $\varepsilon_i < \bar{\varepsilon}_{i1}$ there exists a $\delta_1 > 0$ so that for every $q_1 \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, it is

$$\nu_{i3}(q_2) > 0 \tag{5.87}$$

for every $q_2 \in \overline{\mathscr{B}_i\,(\varepsilon_i)}$ with distance $\|q_1 - q_2\| < \delta_1$.

Define $\delta_2 \triangleq \min\{\delta, \delta_1\}$. By Corollary 19 there exists an $\bar{\varepsilon}_{i2} > 0$, such that $\mathscr{B}_i\,(\varepsilon_i) \subseteq T_{\delta_2}(\partial \mathscr{O}_i)$ for every $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i2}$.

Define $\bar{\varepsilon}_i \triangleq \min\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i2}\}$. Then, for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$, the following holds. Select any $q_1 \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. Since $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq \mathscr{B}_i\,(\varepsilon_i)$ and $\mathscr{B}_i\,(\varepsilon_i) \subseteq T_{\delta_2}(\partial \mathscr{O}_i)$, there exists some $q_2 \in \partial \mathscr{O}_i$ with $\|q_1 - q_2\| < \delta_2 \leq \delta$.

It is $q_1 \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, $q_2 \in \partial \mathscr{O}_i \subseteq \overline{\mathscr{B}_i\,(\varepsilon_i)}$, $\|q_1 - q_2\| < \delta_2 \leq \delta_1$ and $\varepsilon_i < \bar{\varepsilon}_i \leq \bar{\varepsilon}_{i1}$. This implies that $\nu_{i3}(q_2) > 0$. $\square$

Chapter 6

Relative Curvature determines Hessian Quadratic Form on Subspaces

For easy reference throughout this chapter, we have collected here the following definitions.

Definition 130. *Let*

$$\begin{aligned} \nu_i(q,\hat{t}_i) &\triangleq \frac{\nabla\beta_i(q)\cdot\nabla\gamma_d(q)}{\left\|\nabla\gamma_d(q)\right\|^2}\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i - \hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i \\ \zeta_i(q,\hat{t}_i) &\triangleq \frac{1}{\bar{\beta}_i(q)}\left(\frac{\nabla\bar{\beta}_i(q)\cdot\nabla\gamma_d(q)}{\left\|\nabla\gamma_d(q)\right\|^2} + \hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i(q)\nabla\bar{\beta}_i(q)^{\mathrm{T}}}{\bar{\beta}_i(q)} - D^2\bar{\beta}_i(q)\right)\hat{t}_i\right), \end{aligned} \tag{6.1}$$

$$\begin{aligned} \xi_i(q) &\triangleq \left(1-\frac{1}{k}\right)\bar{\beta}_i(q)\left\|\nabla\beta_i(q)\right\|^2 \\ \eta_i(q) &\triangleq \eta_{i1}(q)\beta_i(q) + \eta_{i2}(q)\beta_i^2(q) \\ \eta_{i1}(q) &\triangleq -\bar{\beta}_i(q)\hat{r}_i^{\mathrm{T}} D^2\beta_i(q)\hat{r}_i \\ \eta_{i2}(q) &\triangleq -\hat{r}_i^{\mathrm{T}} D^2\bar{\beta}_i(q)\hat{r}_i \end{aligned} \tag{6.2}$$

$$\begin{aligned} \sigma_i(q,\hat{t}_i) &\triangleq -\frac{1}{k}\left\|\nabla\beta_i(q)\right\|\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i(q) - \bar{\beta}_i(q)\hat{r}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i \\ \tau_i(q,\hat{t}_i) &\triangleq \hat{r}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i(q)\nabla\bar{\beta}_i(q)^{\mathrm{T}}}{\bar{\beta}_i(q)} - D^2\bar{\beta}_i(q)\right)\hat{t}_i, \end{aligned} \tag{6.3}$$

and for any $\mu\in\mathbb{R}$, let

$$\begin{aligned} A_i(q,\hat{t}_i) &\triangleq \bar{\beta}_i(q)\left(\nu_i(q,\hat{t}_i) + \beta_i(q)\zeta_i(q,\hat{t}_i)\right), \\ B_i(q,\hat{t}_i) &\triangleq 2\mu\left(\sigma_i(q,\hat{t}_i) + \beta_i(q)\tau_i(q,\hat{t}_i)\right) \\ C_i(q) &\triangleq \mu^2\frac{1}{\bar{\beta}_i(q)}\left(\xi_i(q) + \beta_i(q)\eta_{i1}(q) + \beta_i^2(q)\eta_{i2}(q)\right). \end{aligned} \tag{6.4}$$

Observe that the functions $\eta_i, \eta_{i1}, \eta_{i2}, \sigma_i, \tau_i$ depend on $\hat{r}_i$, but it does not appear as one of their arguments. The reason is that $\hat{r}_i \triangleq \frac{\nabla\beta_i}{\|\nabla\beta_i\|}$ so the unit normal vector $\hat{r}_i$ is a function of q only, at points where the obstacle function β_i is regular. In contrast, the tangent vector $\hat{t}_i \in T_qB_i$ is not a function of q. The tangent space T_qB_i which contains it *is* a function of q.

In the following sections three decompositions of the quadratic form associated with the Hessian matrix $D^2\hat{\varphi}$ are proved. All of them are valid *only* at critical points. To emphasize this fact, we will usually explicitly write q_c as the argument of the functions. The first decomposition is valid for unit vectors $\hat{t}_i$ in the tangent space T_qB_i

$$\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{t}_i \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = \bar{\beta}_i(q_c)\left(\nu_i(q_c,\hat{t}_i) + \beta_i(q_c)\zeta_i(q_c,\hat{t}_i)\right) = A_i(q_c,\hat{t}_i).$$

The second decomposition is valid for the unit normal vector $\hat{r}_i$

$$\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{r}_i \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = \frac{1}{\beta_i(q_c)}\left(\xi_i(q_c) + \eta_{i1}(q_c)\beta_i(q_c) + \eta_{i2}(q_c)\beta_i^2(q_c)\right) = \frac{C(q)}{\mu^2}.$$

The third decomposition is valid on the span $\{\hat{r}_i, \hat{t}_i\}$ $(\hat{v} = \mu\hat{r}_i + \lambda\hat{t}_i)$

$$\hat{v}^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{v} \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = A_i(q_c,\hat{t}_i)\lambda^2 + B_i(q_c,\hat{t}_i)\lambda + C_i(q_c)$$

We are going to describe the usefulness of these decompositions by considering the first of them as an example. Observe that both $\nu_i(q_c,\hat{t}_i)$ and $\zeta_i(q_c,\hat{t}_i)$ are independent of $\beta_i(q_c)$ and depend only on the other obstacles, which are represented by function $\bar{\beta}_i$. Therefore, the second term vanishes to the first order in $\beta_i(q_c)$

$$\beta_i(q_c)\zeta_i(q_c,\hat{t}_i) = O\left(\beta_i(q_c)\right). \tag{6.5}$$

The multiplicative factors $\frac{\beta(q_c)^2}{\gamma_d(q_c)^k}$ and $\bar{\beta}_i(q_c)$ are positive, provided $q_c \in \mathring{\mathscr{F}} \setminus \{q_d\}$. So the decomposition implies that

$$\alpha(q_c)\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{t}_i = \nu_i(q_c,\hat{t}_i) + O\left(\beta_i(q_c)\right) \tag{6.6}$$

where $0 < \alpha(q_c) = \frac{\beta_i(q_c)\beta(q_c)}{\gamma_d(q_c)^k}$. Then, as we increase the tuning parameter k and "push" the critical points (including q_c) closer to obstacles, the obstacle value $\beta_i(q_c)$ tends to zero, canceling $\beta_i(q_c)\zeta_i(q_c,\hat{t}_i)$. This results in

$$\alpha(q_c)\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{t}_i \approx \nu_i(q_c,\hat{t}_i). \tag{6.7}$$

In other words, the quadratic form $\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{t}_i$ acquires the same sign as $\nu_i(q_c,\hat{t}_i)$.

6.1 Convex and Concave Tangent Space: Negative- and Positive-Definite

Note that for any critical point q_c in the set "near obstacles" $\mathscr{F}_n \cap \mathscr{C}_{\hat{\varphi}}$, there exists some $i \in \mathbb{N}_{\leq M}$ such that $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. The following Proposition and Lemma form an extended version of Proposition 3.6 [1] concerning arbitrary obstacles β_i and attractive functions γ_d.

Proposition 131 (KRF Hessian Quadratic Form Decomposition on Tangent Space)**.** *Assume that both $\nabla\beta_i(q_c) \neq 0$ and $\nabla\gamma_d\,(q_c) \neq 0$ at the critical point $q_c \in \mathscr{C}_{\hat{\varphi}}$.*

Then, the quadratic form associated with the Hessian $D^2\hat{\varphi}\,(q_c)$ can be decomposed as

$$\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{t}_i \frac{\beta_i(q_c)\beta(q_c)}{\gamma_d(q_c)^k} = \nu_i(q_c,\hat{t}_i) + \beta_i(q_c)\,\zeta_i(q_c,\hat{t}_i), \tag{6.8}$$

for unit tangent vectors $\hat{t}_i \in UT_{q_c}B_i$.

Note the equivalence

$$\begin{aligned}\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\frac{\beta_i(q_c)\beta(q_c)}{\gamma_d(q_c)^k} &= \nu_i(q_c,\hat{t}_i)+\beta_i(q_c)\zeta_i(q_c,\hat{t}_i) \iff \\ \hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\frac{\beta(q_c)^2}{\gamma_d(q_c)^{k-1}} &= \gamma_d(q_c)\bar{\beta}_i(q_c)\left(\nu_i(q_c,\hat{t}_i)+\beta_i(q_c)\zeta_i(q_c,\hat{t}_i)\right)\end{aligned} \tag{6.9}$$

Substituting ν_i and ζ_i above, the claim to prove is

$$\begin{aligned}\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\frac{\beta(q_c)^2}{\gamma_d(q_c)^{k-1}} &= \gamma_d(q_c)\bar{\beta}_i(q_c)\left(\frac{\nabla\beta_i(q_c)\cdot\nabla\gamma_d\left(q_c\right)}{\left\|\nabla\gamma_d\left(q_c\right)\right\|^2}\hat{t}_i^{\mathrm{T}}D^2\gamma_d(q_c)\hat{t}_i-\hat{t}_i^{\mathrm{T}}D^2\beta_i(q_c)\hat{t}_i\right)\\ &+\gamma_d(q_c)\beta_i(q_c)\left(\frac{\nabla\bar{\beta}_i(q_c)\cdot\nabla\gamma_d\left(q_c\right)}{\left\|\nabla\gamma_d\left(q_c\right)\right\|^2}+\hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i(q_c)\nabla\bar{\beta}_i(q_c)^{\mathrm{T}}}{\bar{\beta}_i(q_c)}-D^2\bar{\beta}_i(q_c)\right)\hat{t}_i\right)\end{aligned}$$

Proof. For

$$\rho\triangleq\frac{\nu}{\delta},\quad \nu,\delta\in C^2(E^n,\mathbb{R}) \implies D^2\rho|_{\mathscr{C}_\rho}=\frac{1}{\delta^2}\left[\delta D^2\nu-\nu D^2\delta\right]. \tag{6.10}$$

Here we have $\rho=\hat{\varphi},\nu=\gamma_d^k,\delta=\beta$ so that it follows (derivation of $D^2\left(\gamma_d^k\right)$ in **??**)

$$\begin{aligned}D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} &= \frac{1}{\beta^2}\left[\beta D^2\left(\gamma_d^k\right)-\gamma_d^kD^2\beta\right] \overset{\text{seederivationof}D^2\left(\gamma_d^k\right)}{=}\\ &= \frac{1}{\beta^2}\left[\beta\left(k\gamma_d^{k-1}\left(\frac{k-1}{\gamma_d}\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}}+D^2\gamma_d\right)\right)-\gamma_d^kD^2\beta\right]\\ &= \frac{1}{\beta^2}\left[k\beta\gamma_d^{k-2}\left(\left(k-1\right)\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}}+\gamma_dD^2\gamma_d\right)-\gamma_d^{k-2}\gamma_d^2D^2\beta\right]\\ &= \frac{\gamma_d^{k-2}}{\beta^2}\left[k\beta\left(\gamma_dD^2\gamma_d+\left(k-1\right)\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}}\right)-\gamma_d^2D^2\beta\right].\end{aligned} \tag{6.11}$$

At a critical point

$$\begin{aligned}\nabla\hat{\varphi}=0 \iff \nabla\left(\frac{\gamma_d^k}{\beta}\right)=0 \iff \frac{\beta\nabla\left(\gamma_d^k\right)-\gamma_d^k\nabla\beta}{\beta^2}=0 \overset{q\notin\partial\mathscr{F}\implies\beta\neq0}{\iff}\\ \beta\nabla\left(\gamma_d^k\right)-\gamma_d^k\nabla\beta=0 \iff \beta k\gamma_d^{k-1}\nabla\gamma_d-\gamma_d^k\nabla\beta=0 \overset{q\notin\{q_d\}\implies\gamma_d\neq0}{\iff} k\beta\nabla\gamma_d=\gamma_d\nabla\beta\end{aligned} \tag{6.12}$$

Taking the outer product of both sides

$$\begin{aligned}\left(k\beta\nabla\gamma_d\right)\left(k\beta\nabla\gamma_d\right)^{\mathrm{T}}=\left(\gamma_d\nabla\beta\right)\left(\gamma_d\nabla\beta\right)^{\mathrm{T}} \iff \left(k\beta\right)^2\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}}=\gamma_d^2\nabla\beta\nabla\beta^{\mathrm{T}} \overset{q\notin\partial\mathscr{F}\implies\beta\neq0}{\iff}\\ k\beta\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}}=\frac{\gamma_d^2}{k\beta}\nabla\beta\nabla\beta^{\mathrm{T}}\end{aligned} \tag{6.13}$$

and substitution in (6.11) yields

$$\begin{aligned}D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} &= \frac{\gamma_d^{k-2}}{\beta^2}\left[k\beta\gamma_dD^2\gamma_d+\left(k-1\right)\frac{\gamma_d^2}{k\beta}\nabla\beta\nabla\beta^{\mathrm{T}}-\gamma_d^2D^2\beta\right]\\ &= \frac{\gamma_d^{k-1}}{\beta^2}\left[k\beta D^2\gamma_d+\frac{k-1}{k}\frac{\gamma_d}{\beta}\nabla\beta\nabla\beta^{\mathrm{T}}-\gamma_dD^2\beta\right]\\ &= \frac{\gamma_d^{k-1}}{\beta^2}\left[k\beta D^2\gamma_d+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\nabla\beta\nabla\beta^{\mathrm{T}}-\gamma_dD^2\beta\right]\end{aligned} \tag{6.14}$$

Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Then its symmetric part is given by $\frac{1}{2}\left(A + A^{\mathrm{T}}\right) = A_{\text{symmetric}}$ abbreviated as A_s. Note that

$$\begin{aligned}\beta &= \beta_i\bar{\beta}_i \implies \nabla\beta \\ &= \beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i \implies \\ D^2\beta &= D\left[\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right] = \beta_i D^2\bar{\beta}_i + \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \bar{\beta}_i D^2\beta_i + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} \\ &= \beta_i D^2\bar{\beta}_i + \left[\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right] + \bar{\beta}_i D^2\beta_i\end{aligned} \tag{6.15}$$

but since

$$\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} = \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}} = A + A^{\mathrm{T}} = 2A_s \tag{6.16}$$

for $A = \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}$ so (6.15) can be written as

$$D^2\beta = \beta_i D^2\bar{\beta}_i + 2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2\beta_i \tag{6.17}$$

Also similarly

$$\begin{aligned}\nabla\beta\nabla\beta^{\mathrm{T}} &= \left(\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right)\left(\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right)^{\mathrm{T}} \\ &= \left(\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right)\left(\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} + \bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right) \\ &= \beta_i\nabla\bar{\beta}_i\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} + \beta_i\nabla\bar{\beta}_i\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \bar{\beta}_i\nabla\beta_i\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} + \bar{\beta}_i\nabla\beta_i\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} \\ &= \beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + \left(\beta_i\bar{\beta}_i\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \beta_i\bar{\beta}_i\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right) + \bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}}\end{aligned} \tag{6.18}$$

where

$$\beta_i\bar{\beta}_i\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \beta_i\bar{\beta}_i\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} = \beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right) \tag{6.19}$$

and since

$$\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} = \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}} = A + A^{\mathrm{T}} = 2A_s \tag{6.20}$$

again for $A = \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}$, it follows that

$$\nabla\beta\nabla\beta^{\mathrm{T}} = \beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + 2\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} \tag{6.21}$$

Then substitution of $D^2\beta$ from (6.17) and $\nabla\beta\nabla\beta^{\mathrm{T}}$ from (6.21) in (6.14) yields

$$\begin{aligned}D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} =& \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(k\beta D^2\gamma_d \\ &+ \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + 2\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}}\right) \\ &- \gamma_d\left(\beta_i D^2\bar{\beta}_i + 2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2\beta_i\right)\Bigg)\end{aligned} \tag{6.22}$$

Now we are going to evaluate the quadratic form associated with $D^2\hat{\varphi}(q_c)$ in the direction of the unit tangent vector

$$\hat{t}_i \triangleq \left(\frac{\nabla\beta_i(q_c)}{\|\nabla\beta_i(q_c)\|}\right)^{\perp} = \left(\nabla\beta_i(q_c)\right)^{\perp}\frac{1}{\|\nabla\beta_i(q_c)\|} \tag{6.23}$$

which, treating term by term the expression, yields ($q \neq q_d \implies \gamma_d \neq 0$ and $q \notin \partial\mathscr{F} \implies \beta \neq 0$)

$$
\begin{aligned}
\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i \frac{\beta^2}{\gamma_d^{k-1}} &= \hat{t}_i^{\mathrm{T}} \left(k\beta D^2 \gamma_d\right) \hat{t}_i \\
&+ \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i \\
&+ \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} 2\beta_i\bar{\beta}_i \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s\right) \hat{t}_i \\
&+ \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla\beta_i \nabla\beta_i^{\mathrm{T}}\right) \hat{t}_i \\
&- \hat{t}_i^{\mathrm{T}} \left(\gamma_d \beta_i D^2 \bar{\beta}_i\right) \hat{t}_i \\
&- \hat{t}_i^{\mathrm{T}} \left(\gamma_d 2 \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s\right) \hat{t}_i \\
&- \hat{t}_i^{\mathrm{T}} \left(\gamma_d \bar{\beta}_i D^2 \beta_i\right) \hat{t}_i
\end{aligned}
\tag{6.24}
$$

and the comprising terms are (term 1)

$$
\hat{t}_i^{\mathrm{T}} \left(k\beta D^2 \gamma_d\right) \hat{t}_i = k\beta \left(\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i\right) \tag{6.25}
$$

and (term 7)

$$
\hat{t}_i^{\mathrm{T}} \left(\gamma_d \bar{\beta}_i D^2 \beta_i\right) \hat{t}_i = \gamma_d \bar{\beta}_i \left(\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right) \tag{6.26}
$$

and (term 2)

$$
\hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} 2\beta_i\bar{\beta}_i \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s\right) \hat{t}_i = 2\left(1 - \frac{1}{k}\right) \gamma_d \hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s \hat{t}_i \tag{6.27}
$$

where

$$
\begin{aligned}
\hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s \hat{t}_i &= \frac{1}{2} \hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}} + \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}}\right) \hat{t}_i \\
&= \frac{1}{2} \hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}} + \nabla\beta_i \nabla\bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i \\
&= \frac{1}{2} \left(\hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right) \hat{t}_i + \hat{t}_i^{\mathrm{T}} \left(\nabla\beta_i \nabla\bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i\right) \\
&= \frac{1}{2} \left(\left(\hat{t}_i^{\mathrm{T}} \nabla\bar{\beta}_i\right) \cancelto{0}{\left(\nabla\beta_i^{\mathrm{T}} \hat{t}_i\right)} + \cancelto{0}{\left(\hat{t}_i^{\mathrm{T}} \nabla\beta_i\right)} \left(\nabla\bar{\beta}_i \hat{t}_i\right)\right) = 0
\end{aligned}
\tag{6.28}
$$

so that from (6.27)

$$
\hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} 2\beta_i\bar{\beta}_i \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s\right) \hat{t}_i = 0 \tag{6.29}
$$

and (term 4)

$$
\hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla\beta_i \nabla\beta_i^{\mathrm{T}}\right) \hat{t}_i = \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \hat{t}_i^{\mathrm{T}} \left(\nabla\beta_i \nabla\beta_i^{\mathrm{T}}\right) \hat{t}_i \tag{6.30}
$$

where

$$
\hat{t}_i^{\mathrm{T}} \left(\nabla\beta_i \nabla\beta_i^{\mathrm{T}}\right) \hat{t}_i = \cancelto{0}{\left(\hat{t}_i^{\mathrm{T}} \nabla\beta_i\right)} \cancelto{0}{\left(\nabla\beta_i^{\mathrm{T}} \hat{t}_i\right)} = 0 \tag{6.31}
$$

so that from (6.30)

$$
\hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla\beta_i \nabla\beta_i^{\mathrm{T}}\right) \hat{t}_i = 0 \tag{6.32}
$$

and (term 6)

$$\hat{t}_i^{\mathrm{T}} \left(\gamma_d 2 \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s\right) \hat{t}_i = 2\gamma_d \hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s \hat{t}_i \tag{6.33}$$

where

$$\begin{aligned}
\hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s \hat{t}_i &= \frac{1}{2} \hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}} + \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}}\right) \hat{t}_i \\
&= \frac{1}{2} \hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}} + \nabla\beta_i \nabla\bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i \\
&= \frac{1}{2} \left(\hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right) \hat{t}_i + \hat{t}_i^{\mathrm{T}} \left(\nabla\beta_i \nabla\bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i\right) \\
&= \frac{1}{2} \left(\left(\hat{t}_i^{\mathrm{T}} \nabla\bar{\beta}_i\right) \cancelto{0}{\left(\nabla\beta_i \hat{t}_i\right)} + \cancelto{0}{\left(\hat{t}_i^{\mathrm{T}} \nabla\beta_i\right)} \left(\nabla\bar{\beta}_i^{\mathrm{T}} \hat{t}_i\right)\right) = 0
\end{aligned} \tag{6.34}$$

for the same reason as before. The zero inner products are justified by normality of chosen direction $\hat{t}_i$ to gradient $\nabla\beta_i$ since $\hat{t}_i$ is tangent to level sets

$$\begin{aligned}
&\nabla\beta_i^{\mathrm{T}} \hat{t}_i = \nabla\beta_i \cdot \hat{t}_i = \left(\nabla\beta_i \nabla\beta_i^{\perp}\right) \frac{1}{\|\nabla\beta_i\|} = 0 \\
&\hat{t}_i^{\mathrm{T}} \nabla\beta_i = \hat{t}_i \cdot \nabla\beta_i = \nabla\beta_i \cdot \hat{t}_i = 0.
\end{aligned} \tag{6.35}$$

So substitution of these terms in (6.24) leads to

$$\begin{aligned}
&\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i \frac{\beta^2}{\gamma_d^{k-1}} \\
&= k\beta \left(\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i\right) - \gamma_d \bar{\beta}_i \left(\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right) + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}} - \gamma_d \beta_i D^2 \bar{\beta}_i\right) \hat{t}_i.
\end{aligned} \tag{6.36}$$

At critical point $q_c \in \mathscr{C}_{\hat{\varphi}}$ the following holds

$$\begin{aligned}
k\beta \nabla\gamma_d = \gamma_d \nabla\beta \implies k\beta \nabla\gamma_d \cdot \nabla\gamma_d &= \gamma_d \nabla\beta \cdot \nabla\gamma_d \iff \\
k\beta \left\|\nabla\gamma_d\right\|^2 &= \gamma_d \left(\nabla\left(\bar{\beta}_i \beta_i\right)\right) \cdot \nabla\gamma_d = \gamma_d \left(\bar{\beta}_i \nabla\beta_i + \beta_i \nabla\bar{\beta}_i\right) \cdot \nabla\gamma_d \\
&= \gamma_d \left(\bar{\beta}_i \nabla\beta_i \cdot \nabla\gamma_d + \beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d\right) \overset{q \neq q_d \iff \|\nabla\gamma_d\| \neq 0}{\iff} \\
k\beta &= \gamma_d \frac{\bar{\beta}_i \nabla\beta_i \cdot \nabla\gamma_d + \beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2}
\end{aligned} \tag{6.37}$$

then the condition for general γ_d, β_i results by substitution in (6.36)

$$\begin{aligned}
\hat{t}_i D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i \frac{\beta^2}{\gamma_d^{k-1}} =& \gamma_d \frac{\bar{\beta}_i \nabla\beta_i \cdot \nabla\gamma_d + \beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2} \left(\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i\right) - \gamma_d \bar{\beta}_i \left(\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right) \\
&+ \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}} - \gamma_d \beta_i D^2 \bar{\beta}_i\right) \hat{t}_i
\end{aligned} \tag{6.38}$$

and, rearranging terms

$$\begin{aligned}
\hat{t}_i D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i \frac{\beta^2}{\gamma_d^{k-1}} &= \gamma_d \bar{\beta}_i \left(\frac{\nabla\beta_i \cdot \nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2} \left(\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i\right) - \left(\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right)\right) \\
&+ \gamma_d \beta_i \left(\frac{\nabla\bar{\beta}_i \cdot \nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2} + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2 \bar{\beta}_i\right) \hat{t}_i\right)
\end{aligned} \tag{6.39}$$

which then becomes

$$
\begin{aligned}
\hat{t}_i D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i \frac{\beta^2}{\gamma_d^{k-1}} \frac{1}{\gamma_d \bar{\beta}_i} &= \frac{1}{\gamma_d \bar{\beta}_i} \gamma_d \bar{\beta}_i \left(\frac{\nabla \beta_i \cdot \nabla \gamma_d}{\|\nabla \gamma_d\|^2} \hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i - \hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i \right) \\
&+ \frac{1}{\gamma_d \bar{\beta}_i} \gamma_d \beta_i \left(\frac{\nabla \bar{\beta}_i \cdot \nabla \gamma_d}{\|\nabla \gamma_d\|^2} + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k} \right) \frac{\nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2 \bar{\beta}_i \right) \hat{t}_i \right) \implies \\
\hat{t}_i D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i \frac{\beta \beta}{\bar{\beta}_i \gamma_d^k} &= \frac{\nabla \beta_i \cdot \nabla \gamma_d}{\|\nabla \gamma_d\|^2} \hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i - \hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i \\
&+ \frac{\beta_i}{\bar{\beta}_i} \left(\frac{\nabla \bar{\beta}_i \cdot \nabla \gamma_d}{\|\nabla \gamma_d\|^2} + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k} \right) \frac{\nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2 \bar{\beta}_i \right) \hat{t}_i \right)
\end{aligned}
\tag{6.40}
$$

and replacing for the appropriate terms the functions ν_i, ζ_i from Definition 130, we obtain

$$
\hat{t}_i D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i \frac{\beta_i(q_c)\beta(q_c)}{\gamma_d(q_c)^k} = \nu_i(q_c, \hat{t}_i) + \beta_i(q_c)\zeta_i(q_c, \hat{t}_i) \tag{6.41}
$$

$\square$

In Proposition 131 the Hessian quadratic form $\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi} \hat{t}_i$ was decomposed at a critical point q_c for the case that the vector $\hat{t}_i$ belongs to the obstacle tangent space $T_q B_i$. Note that the decomposition is valid *only* at critical points. The following Lemma connects the sign of the Hessian quadratic form $\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}\,(q_c)\,\hat{t}_i$ to the sign of the relative curvature function $\nu_i(q_c, \hat{t}_i)$. The relation derived holds only at a critical point. The smoothness assumption $\beta_i, \gamma_d \in C^2(E^n, \mathbb{R})$ is not explicitly stated, because it has been introduced in earlier chapters.

Lemma 132. *Select arbitrary $L > 0$. Assume that there exists an $\bar{\varepsilon}_{i1} > 0$ such that $\nabla \beta_i \neq 0$ and $\nabla \gamma_d \neq 0$ in $\mathscr{B}_i\,(\varepsilon_i)$, for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i1}$.*

Then, there exists an $\bar{\varepsilon}_i > 0$ such that if $\varepsilon_i < \bar{\varepsilon}_i$, then the following holds. If at a critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ it is

$$
\nu_i(q_c, \hat{t}_i) < -L, \quad (> L)
$$

for the unit tangent vector $\hat{t}_i \in UT_{q_c} B_i$, then the Hessian quadratic form

$$
t_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c) t_i < 0 \quad (> 0),
$$

for any tangent vector $t_i = a\hat{t}_i \in T_{q_c} B_i$ $(a \neq 0)$.

Proof. Define $\bar{\varepsilon}_{i2} \triangleq \frac{1}{2}\bar{\varepsilon}_{i1} > 0$. By hypothesis $\nabla \beta_i \neq 0$ and $\nabla \gamma_d \neq 0$ in $\mathscr{B}_i\,(\bar{\varepsilon}_{i1})$, so the functions ζ_i and ν_i are well-defined in $\mathscr{B}_i\,(\bar{\varepsilon}_{i1})$. By the definition of $\mathscr{B}_i$, it is $\overline{\mathscr{B}_i\,(\bar{\varepsilon}_{i2})} \subseteq \mathscr{B}_i\,(\bar{\varepsilon}_{i1})$, so ζ_i and ν_i are well-defined in $\overline{\mathscr{B}_i\,(\bar{\varepsilon}_{i2})}$. For any critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ with $0 < \varepsilon_i < \bar{\varepsilon}_{i2}$ by Proposition 131 the Hessian quadratic form can be decomposed as

$$
\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}\,(q_c)\,\hat{t}_i \frac{\beta_i(q_c)\beta(q_c)}{\gamma_d(q_c)^k} = \nu_i(q_c, \hat{t}_i) + \beta_i(q_c)\zeta_i(q_c, \hat{t}_i). \tag{6.42}
$$

Let

$$
L_2 \triangleq \max_{(q,\hat{v}) \in UT\overline{\mathscr{B}_i(\varepsilon_{i2})}} \{|\zeta_i(q, \hat{v})|\} \tag{6.43}
$$

If $L_2 = 0$, then $\zeta_i(q, \hat{t}_i) = 0$ for all $\hat{t}_i \in UT_qB_i$ and $q \in \mathscr{B}_i(\bar{\varepsilon}_{i2})$. Define $\bar{\varepsilon}_i \triangleq \bar{\varepsilon}_{i2}$ in this case. Take any $\varepsilon_i < \bar{\varepsilon}_i$ and assume that $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. Then (6.42) is valid, so substitute $\zeta_i(q, \hat{t}_i) = 0$ to obtain

$$\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{t}_i \frac{\beta_i(q_c)\beta(q_c)}{\gamma_d(q_c)^k} = \nu_i(q_c, \hat{t}_i) + \beta_i(q_c)0 = \nu_i(q_c, \hat{t}_i)$$

for all $\hat{t}_i \in UT_{q_c}B_i$. This proves the claim for the case $L_2 = 0$.

If $L_2 > 0$, then define

$$\bar{\varepsilon}_i \triangleq \min\left\{\bar{\varepsilon}_{i2}, \frac{L}{2L_2}\right\} > 0. \tag{6.44}$$

Suppose that $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ for some $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$, and $\nu_i(q_c, \hat{t}_i) < -L$. Then by the assumption and the Definition 33 it follows that

$$\begin{aligned} q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \implies \beta_i(q_c) < \bar{\varepsilon}_i \implies \beta_i(q_c)\zeta_i(q_c, \hat{t}_i) < \bar{\varepsilon}_i \left|\zeta_i(q_c, \hat{t}_i)\right| \implies \\ \beta_i(q_c)\zeta_i(q_c, \hat{t}_i) < \frac{1}{2}L\frac{\left|\zeta_i(q_c, \hat{t}_i)\right|}{L_2}. \end{aligned} \tag{6.45}$$

By the definition of L_2 it is $\frac{\left|\zeta_i(q_c, \hat{t}_i)\right|}{L_2} \leq 1$, so

$$\begin{gathered} \beta_i(q_c)\zeta_i(q_c, \hat{t}_i) < \frac{1}{2}L \implies \\ \nu_i(q_c, \hat{t}_i) + \beta_i(q_c)\zeta_i(q_c, \hat{t}_i) < -L + \frac{1}{2}L \end{gathered}$$

and since $\beta_i(q_c) < \bar{\varepsilon}_{i2}$ we can use (6.42) to obtain

$$\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{t}_i \frac{\beta_i(q_c)\beta(q_c)}{\gamma_d(q_c)^k} < -\frac{1}{2}L < 0.$$

The quadratic form is positive-definite in the direction of the unit tangent vector $\hat{t}_i \in UT_{q_c}B_i$, which implies that it is positive-definite on the subspace spanned by $\hat{t}_i$. The case $\nu_i(q_c, \hat{t}_i) > L$ can be proved similarly. $\square$

6.2 Normal Space: Positive-Definite

This section the positive definite subspace part of Proposition 3.9 [1] is extended to the case of general β_i, γ_d.

Proposition 133 (KRF Hessian Quadratic Form Decomposition on Normal Space)**.** *Assume that both $\nabla\beta_i(q_c) \neq 0$ and $\nabla\gamma_d\,(q_c) \neq 0$ at the critical point $q_c \in \mathscr{C}_{\hat{\varphi}}$.*

Then, the quadratic form associated with the Hessian $D^2\hat{\varphi}\,(q_c)$ can be decomposed as

$$\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\,\hat{r}_i \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = \frac{1}{\beta_i(q_c)}\left(\xi_i(q_c) + \eta_{i1}(q_c)\beta_i(q_c) + \eta_{i2}(q_c)\beta_i^2(q_c)\right), \tag{6.46}$$

for the unit normal vector $\hat{r}_i \in N_q B_i$.

Proof. At a critical point of $\hat{\varphi}$ it holds[1] that

$$\begin{aligned} k\beta\nabla\gamma_d = \gamma_d\nabla\beta \implies (k\beta\nabla\gamma_d)\cdot(k\beta\nabla\gamma_d) &= (\gamma_d\nabla\beta)\cdot(\gamma_d\nabla\beta) \iff \\ (k\beta)^2\left(\nabla\gamma_d\cdot\nabla\gamma_d\right) &= \gamma_d^2\left(\nabla\beta\cdot\nabla\beta\right) \iff \\ (k\beta)^2\left\|\nabla\gamma_d\right\|^2 &= \gamma_d^2\left\|\nabla\beta\right\|^2 \overset{\beta\neq 0,\forall q\in\mathscr{B}_i(\varepsilon_i),k\geq 2,q\neq q_d \implies \gamma_d\neq 0,\forall q\in\mathscr{B}_i(\varepsilon_i)}{\Longrightarrow} \\ k\beta &= \frac{\gamma_d^2}{\left(2\sqrt{\gamma_d}\right)^2}\frac{1}{k\beta}\left\|\nabla\beta\right\|^2 \iff \\ k\beta &= \frac{\gamma_d^2}{k\beta}\frac{\left\|\nabla\beta\right\|^2}{\left\|\nabla\gamma_d\right\|^2} \end{aligned} \tag{6.47}$$

Taking into consideration that Equation 6.14 holds for any γ_d, β substitution of $k\beta$ from (6.47) in it yields

$$\left.\begin{aligned} D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} &= \tfrac{\gamma_d^{k-1}}{\beta^2}\left(k\beta D^2\gamma_d + \left(1-\tfrac{1}{k}\right)\tfrac{\gamma_d}{\beta}\nabla\beta\nabla\beta^{\mathrm{T}} - \gamma_d D^2\beta\right) \\ k\beta &= \tfrac{\gamma_d^2}{k\beta}\tfrac{\|\nabla\beta\|^2}{\|\nabla\gamma_d\|^2} \end{aligned}\right\} \implies$$

$$\begin{aligned} \frac{\beta^2}{\gamma_d^{k-1}}\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{r}_i &= \hat{r}_i^{\mathrm{T}}\left(k\beta D^2\gamma_d\right)\hat{r}_i + \hat{r}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\nabla\beta\nabla\beta^{\mathrm{T}}\right)\hat{r}_i - \hat{r}_i^{\mathrm{T}}\left(\gamma_d D^2\beta\right)\hat{r}_i \\ &= k\beta\left(\hat{r}_i^{\mathrm{T}} D^2\gamma_d\hat{r}_i\right) + \frac{\gamma_d}{\beta}\left(1-\frac{1}{k}\right)\left(\hat{r}_i^{\mathrm{T}}\left(\nabla\beta\nabla\beta^{\mathrm{T}}\right)\hat{r}_i\right) - \gamma_d\left(\hat{r}_i^{\mathrm{T}} D^2\beta\hat{r}_i\right) \\ &= \frac{\gamma_d^2}{\left\|\nabla\gamma_d\right\| k\beta}\left\|\nabla\beta\right\|^2\left(\hat{r}_i^{\mathrm{T}} D^2\gamma_d\hat{r}_i\right) \\ &\quad + \frac{\gamma_d}{\beta}\left(1-\frac{1}{k}\right)\left(\hat{r}_i^{\mathrm{T}}\left(\nabla\beta\nabla\beta^{\mathrm{T}}\right)\hat{r}_i\right) - \gamma_d\left(\hat{r}_i^{\mathrm{T}} D^2\beta\hat{r}_i\right) \end{aligned} \tag{6.48}$$

It is

$$\left\|\nabla\beta\right\|^2 = \left\|\nabla\left(\bar{\beta}_i\beta_i\right)\right\|^2 = \left\|\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right\|^2 \tag{6.49}$$

and

$$\begin{aligned} \hat{r}_i^{\mathrm{T}}\left(\nabla\beta\nabla\beta^{\mathrm{T}}\right)\hat{r}_i &= \left(\hat{r}_i^{\mathrm{T}}\right)\left(\nabla\beta^{\mathrm{T}}\hat{r}_i\right) = \left(\hat{r}_i\cdot\nabla\beta\right)\left(\nabla\beta\cdot\hat{r}_i\right) \\ &= \left(\nabla\beta\cdot\hat{r}_i\right)\left(\nabla\beta\cdot\hat{r}_i\right) = \left(\nabla\beta\cdot\hat{r}_i\right)^2 = \left(\nabla\left(\bar{\beta}_i\beta_i\right)\cdot\hat{r}_i\right)^2 \end{aligned} \tag{6.50}$$

[1] [1], Lemma 3.1, p.426.

Substitution of these in (6.48) yields

$$\begin{aligned}\frac{\beta^2}{\gamma_d^{k-1}}\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{r}_i =& \frac{\gamma_d^2}{k\beta\left\|\nabla\gamma_d\right\|^2}\left(\beta_i\nabla\bar{\beta}_i+\bar{\beta}_i\nabla\beta_i\right)\cdot\left(\beta_i\nabla\bar{\beta}_i+\bar{\beta}_i\nabla\beta_i\right)\left(\hat{r}_i^{\mathrm{T}} D^2\gamma_d\hat{r}_i\right)\\ &+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\left(\beta_i\nabla\bar{\beta}_i+\bar{\beta}_i\nabla\beta_i\right)\cdot\hat{r}_i\right)^2\\ &-\gamma_d\left(\hat{r}_i^{\mathrm{T}} D^2\beta\hat{r}_i\right)\end{aligned}\tag{6.51}$$

since

$$\begin{aligned}&\frac{\gamma_d^2}{\left\|\nabla\gamma_d\right\|^2 k\beta}\left(\left(\beta_i\nabla\bar{\beta}_i\right)\cdot\left(\beta_i\nabla\bar{\beta}_i\right)+\left(\beta_i\nabla\bar{\beta}_i\right)\cdot\left(\bar{\beta}_i\nabla\beta_i\right)+\left(\bar{\beta}_i\nabla\beta_i\right)\cdot\left(\beta_i\nabla\bar{\beta}_i\right)+\left(\bar{\beta}_i\nabla\beta_i\right)\cdot\left(\bar{\beta}_i\nabla\beta_i\right)\right)\\ &=\frac{\gamma_d^2}{\left\|\nabla\gamma_d\right\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\cdot\nabla\beta_i\right)+\beta_i\bar{\beta}_i\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\\ &=\frac{\gamma_d^2}{\left\|\nabla\gamma_d\right\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+2\beta_i\bar{\beta}_i\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\\ &=\frac{\gamma_d^2}{\left\|\nabla\gamma_d\right\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\end{aligned}\tag{6.52}$$

and

$$\begin{aligned}\left(\left(\beta_i\nabla\bar{\beta}_i+\bar{\beta}_i\nabla\beta_i\right)\cdot\hat{r}_i\right)^2&=\left(\beta_i\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)+\bar{\beta}_i\left(\nabla\beta_i\cdot\hat{r}_i\right)\right)^2\\ &=\beta_i^2\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)^2+2\beta_i\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\bar{\beta}_i\left(\nabla\beta_i\cdot\hat{r}_i\right)+\bar{\beta}_i^2\left(\nabla\beta_i\cdot\hat{r}_i\right)^2\\ &=\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)+2\beta\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\left(\nabla\beta_i\cdot\hat{r}_i\right)+\bar{\beta}_i^2\left(\nabla\beta_i\cdot\hat{r}_i\right)^2\end{aligned}\tag{6.53}$$

Note that

$$\nabla\beta_i\cdot\hat{r}_i=\nabla\beta_i\cdot\frac{\nabla\beta_i}{\left\|\nabla\beta_i\right\|}=\frac{\nabla\beta_i\cdot\nabla\beta_i}{\left\|\nabla\beta_i\right\|}=\frac{\left\|\nabla\beta_i\right\|^2}{\left\|\nabla\beta_i\right\|}=\left\|\nabla\beta_i\right\|\tag{6.54}$$

so substituting in (6.53) yields

$$\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+2\beta\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\left\|\nabla\beta_i\right\|+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\tag{6.55}$$

Substitution of these results in (6.51) leads to

$$\begin{aligned}&\frac{\beta^2}{\gamma_d^{k-1}}\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{r}_i\\ &\quad=\frac{\gamma_d^2}{\left\|\nabla\gamma_d\right\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\left(\hat{r}_i^{\mathrm{T}} D^2\gamma_d\hat{r}_i\right)\\ &\quad+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+2\beta\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\left\|\nabla\beta_i\right\|+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\\ &\quad-\gamma_d\left(\hat{r}_i^{\mathrm{T}} D^2\beta\hat{r}_i\right)\end{aligned}\tag{6.56}$$

But since

$$\begin{aligned}\left\|\nabla\beta_i\right\|\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)=\left\|\nabla\beta_i\right\|\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)&=\left(\left\|\nabla\beta_i\right\|\hat{r}_i\right)\cdot\nabla\bar{\beta}_i\\ &=\left(\left\|\nabla\beta_i\right\|\frac{\nabla\beta_i}{\left\|\nabla\beta_i\right\|}\right)\cdot\nabla\bar{\beta}_i=\nabla\beta_i\cdot\nabla\bar{\beta}_i\end{aligned}\tag{6.57}$$

and also

$$\begin{aligned}\hat{r}_i^{\mathrm{T}} D^2 \beta \hat{r}_i &= \hat{r}_i^{\mathrm{T}} \left(D^2 \left(\bar{\beta}_i \beta_i\right)\right) \hat{r}_i = \hat{r}_i^{\mathrm{T}} \left(D \left(\beta_i \nabla \bar{\beta}_i + \bar{\beta}_i \nabla \beta_i\right)\right) \hat{r}_i \\ &= \hat{r}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} + \beta_i D^2 \bar{\beta}_i + \nabla \beta_i \nabla \bar{\beta}_i^{\mathrm{T}} + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \\ &= \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} + \nabla \beta_i \nabla \bar{\beta}_i^{\mathrm{T}}\right) + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \\ &= \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + 2 \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i\end{aligned} \tag{6.58}$$

because $\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} + \nabla \beta_i \nabla \bar{\beta}_i^{\mathrm{T}} = \nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} + \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)^{\mathrm{T}} = 2 \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s$. Substitution in (6.56) yields

$$\begin{aligned}&\frac{\beta^2}{\gamma_d^{k-1}} \hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{r}_i \\ &\quad= \frac{\gamma_d^2}{\left\|\nabla \gamma_d\right\|^2 k\beta} \left(\beta_i^2 \left\|\nabla \bar{\beta}_i\right\|^2 + 2\beta \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right) + \bar{\beta}_i^2 \left\|\nabla \beta_i\right\|^2\right) \left(\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i\right) \\ &\qquad + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \left(\beta_i^2 \left(\hat{r}_i \cdot \nabla \bar{\beta}_i\right)^2 + \bar{\beta}_i^2 \left\|\nabla \beta_i\right\|^2\right) \\ &\qquad + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} 2\beta \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right) \\ &\qquad - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + 2 \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \\ &\quad= \frac{\gamma_d^2}{\left\|\nabla \gamma_d\right\|^2 k\beta} \left(\beta_i^2 \left\|\nabla \bar{\beta}_i\right\|^2 + 2\beta \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right) + \bar{\beta}_i^2 \left\|\nabla \beta_i\right\|^2\right) \left(\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i\right) \\ &\qquad + \gamma_d 2 \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right) - \frac{1}{k} \frac{\gamma_d}{\beta} 2\beta \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right) \\ &\qquad + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \left(\beta_i^2 \left(\hat{r}_i \cdot \nabla \bar{\beta}_i\right)^2 + \bar{\beta}_i^2 \left\|\nabla \beta_i\right\|^2\right) \\ &\qquad - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + 2 \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2 \beta\right) \hat{r}_i\end{aligned} \tag{6.59}$$

$$\begin{aligned}&= \left(\frac{\gamma_d^2}{\left\|\nabla \gamma_d\right\|^2 k\beta} \left(\beta_i^2 \left\|\nabla \bar{\beta}_i\right\|^2 + \bar{\beta}_i^2 \left\|\nabla \beta_i\right\|^2\right) + \frac{\gamma_d^2}{\left\|\nabla \gamma_d\right\|^2 k\beta} 2\beta \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right)\right) \left(\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i\right) \\ &\quad + (-1) \frac{\gamma_d^2}{\left\|\nabla \gamma_d\right\|^2 k\beta} 2\beta \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right) \left(\frac{1}{\gamma_d} \left\|\nabla \gamma_d\right\|^2 \frac{1}{\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i}\right) \left(\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i\right) \\ &\quad + 2\gamma_d \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right) \\ &\quad + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \left(\beta_i^2 \left(\hat{r}_i \cdot \nabla \bar{\beta}_i\right)^2 + \bar{\beta}_i^2 \left\|\nabla \beta_i\right\|^2\right) \\ &\quad - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + 2 \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \\ &= \frac{\gamma_d^2}{\left\|\nabla \gamma_d\right\|^2 k\beta} \left(\beta_i^2 \left\|\nabla \bar{\beta}_i\right\|^2 + \bar{\beta}_i^2 \left\|\nabla \beta_i\right\|^2\right) \left(\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i\right) \\ &\quad + \frac{\gamma_d^2}{\left\|\nabla \gamma_d\right\|^2 k\beta} 2\beta \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right) \left(1 - \frac{\frac{\left\|\nabla \gamma_d\right\|}{\gamma_d}}{\frac{\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i}{\left\|\nabla \gamma_d\right\|}}\right) \left(\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i\right) \\ &\quad + 2\gamma_d \left(\nabla \beta_i \cdot \nabla \bar{\beta}_i\right) \\ &\quad - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + 2 \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \\ &\quad + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \left(\beta_i^2 \left(\hat{r}_i \cdot \nabla \bar{\beta}_i\right)^2 + \bar{\beta}_i^2 \left\|\nabla \beta_i\right\|^2\right)\end{aligned} \tag{6.60}$$

Terms $2\gamma_d\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)$ and $-\gamma_d\hat{r}_i^{\mathrm{T}}2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s\hat{r}_i$ cancel because

$$2\gamma_d\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)=2\gamma_d\left\|\nabla\beta_i\right\|\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\tag{6.61}$$

and

$$\begin{aligned}-\gamma_d\hat{r}_i^{\mathrm{T}}2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s\hat{r}_i&=-\gamma_d\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}}\right)\hat{r}_i\\&=-\gamma_d\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{r}_i\\&=-\gamma_d\left(\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)\hat{r}_i+\hat{r}_i^{\mathrm{T}}\left(\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{r}_i\right)\\&=-\gamma_d\left(\left(\hat{r}_i^{\mathrm{T}}\nabla\bar{\beta}_i\right)\left(\nabla\beta_i^{\mathrm{T}}\hat{r}_i\right)+\left(\hat{r}_i^{\mathrm{T}}\nabla\beta_i\right)\left(\nabla\bar{\beta}_i^{\mathrm{T}}\hat{r}_i\right)\right)\\&=-\gamma_d\left(\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)\left(\nabla\beta_i\cdot\hat{r}_i\right)+\left(\hat{r}_i\cdot\nabla\beta_i\right)\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\right)\overset{\nabla\beta_i\cdot\hat{r}_i=\|\nabla\beta_i\|}{=}\\&=-\gamma_d\left(\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)\left\|\nabla\beta_i\right\|+\left\|\nabla\beta_i\right\|\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)\right)\\&=-\gamma_d2\left\|\nabla\beta_i\right\|\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)\\&=-2\gamma_d\left\|\nabla\beta_i\right\|\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)\end{aligned}\tag{6.62}$$

Hence (6.59) becomes

$$\begin{aligned}\frac{\beta^2}{\gamma_d^{k-1}}\hat{r}_i^{\mathrm{T}}D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{r}_i=\frac{\gamma_d^2\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i\right)}{k\beta\left\|\nabla\gamma_d\right\|^2}&\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2+\left(1-\frac{\frac{\|\nabla\gamma_d\|}{\gamma_d}}{\frac{\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i}{\|\nabla\gamma_d\|}}\right)\left(2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\right)\right)\\&+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\\&-\gamma_d\hat{r}_i^{\mathrm{T}}\left(\beta_iD^2\bar{\beta}_i+\bar{\beta}_iD^2\beta_i\right)\hat{r}_i\end{aligned}\tag{6.63}$$

To proceed further we select a symmetric attractive effect $\gamma_d=\|q-q_d\|^2$. It follows that

$$\frac{\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i\right)\gamma_d^2}{k\beta\left\|\nabla\gamma_d\right\|^2}=\frac{\left(\hat{r}_i^{\mathrm{T}}\left(2I\right)\hat{r}_i\right)\gamma_d^2}{k\beta\left(2\sqrt{\gamma_d}\right)^2}=\frac{2\gamma_d^2}{k\beta4\gamma_d}=\frac{\gamma_d}{2k\beta}\tag{6.64}$$

and

$$1-\frac{\frac{\|\nabla\gamma_d\|}{\gamma_d}}{\frac{\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i}{\|\nabla\gamma_d\|}}=1-\frac{\frac{\left(2\sqrt{\gamma_d}\right)^2}{\gamma_d}}{\hat{r}_i^{\mathrm{T}}\left(2I\right)\hat{r}_i}=1-\frac{4}{2}=1-2=-1\tag{6.65}$$

therefore (6.63) implies

$$\begin{aligned}\frac{\beta^2}{\gamma_d^{k-1}}\hat{r}_i^{\mathrm{T}}D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{r}_i&=\frac{\gamma_d}{2k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2-2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\right)\\&+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\\&-\gamma_d\hat{r}_i^{\mathrm{T}}\left(\beta_iD^2\bar{\beta}_i+\bar{\beta}_iD^2\beta_i\right)\hat{r}_i\\&=\frac{\gamma_d}{2k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2-2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\right)\\&+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\\&-\gamma_d\hat{r}_i^{\mathrm{T}}\left(\beta_iD^2\bar{\beta}_i+\bar{\beta}_iD^2\beta_i\right)\hat{r}_i\end{aligned}\tag{6.66}$$

Now note that

$$
\begin{aligned}
0 &\leq \left(\beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \left\|\nabla\beta_i\right\|\right)^2 = \left(\beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \nabla\beta_i\right)\left(\beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \nabla\beta_i\right) \\
&= \beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 - \bar{\beta}_i \left\|\nabla\beta_i\right\| \beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \left\|\nabla\beta_i\right\| \beta_i \left\|\nabla\bar{\beta}_i\right\| + \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 \\
&= \beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 - 2\beta \left\|\nabla\beta_i\right\| \left\|\nabla\bar{\beta}_i\right\| + \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 \\
&\leq \beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 - 2\beta \left(\nabla\beta_i \cdot \nabla\bar{\beta}_i\right) + \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2
\end{aligned}
\tag{6.67}
$$

because

$$
\nabla\beta_i \cdot \nabla\bar{\beta}_i \leq \left\|\nabla\beta_i\right\| \left\|\nabla\bar{\beta}_i\right\| \iff -\left\|\nabla\beta_i\right\| \left\|\nabla\bar{\beta}_i\right\| \leq -\nabla\beta_i \cdot \nabla\bar{\beta}_i \tag{6.68}
$$

and also note that $\beta_i^2 \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)^2 \geq 0$ so that (6.66) implies

$$
\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}} \hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{r}_i =& \left(\frac{\gamma_d}{2k\beta} \left(\beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 + \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 - 2\beta\left(\nabla\beta_i \cdot \nabla\bar{\beta}_i\right) + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)^2 \right) \right. \\
&\left. + \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \right) \right) \\
\geq& \left(\frac{\gamma_d}{2k\beta} \left(\beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \left\|\nabla\beta_i\right\|\right)^2 + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)^2 \right) \\
&+ \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \right) \\
\geq& \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i
\end{aligned}
\tag{6.69}
$$

$$
\begin{aligned}
&= \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta_i} \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \frac{\gamma_d}{\beta_i} \hat{r}_i^{\mathrm{T}} \beta_i^2 D^2 \bar{\beta}_i \hat{r}_i - \frac{\gamma_d}{\beta_i} \beta_i \bar{\beta}_i \hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i \\
&= \frac{\gamma_d}{\beta_i} \left(\left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i^2 \hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{r}_i - \beta_i \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right) \right) \\
&= \frac{\gamma_d}{\beta_i} \left(\left(\frac{1}{2} \left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right) \right) \right. \\
&\qquad \left. + \left(\frac{1}{2} \left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i^2 \left(\hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{r}_i\right) \right) \right).
\end{aligned}
\tag{6.70}
$$

Substitute above the functions ξ_i, η_{i1} and η_{i2} from Definition 130 to obtain

$$
\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{r}_i \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = \frac{1}{\beta_i(q_c)} \left(\frac{1}{2}\xi_i(q_c) + \beta_i(q_c)\eta_{i1}(q_c) + \frac{1}{2}\xi_i(q_c) + \beta_i(q_c)^2 \eta_{i2}(q_c) \right). \tag{6.71}
$$

$\square$

Lemma 134. *Assume that there exists an $\bar{\varepsilon}_{i1} > 0$ such that $\nabla\beta_i \neq 0$ and $\nabla\gamma_d \neq 0$ in $\mathscr{B}_i\left(\varepsilon_i\right)$, for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i1}$.*

Then, there exists an $\bar{\varepsilon}_i > 0$ such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ the following holds. At each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ the Hessian quadratic form is positive-definite on the normal space, i.e.,

$$
r_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) r_i > 0,
$$

for any normal vector $r_i = a\hat{r}_i \in N_{q_c} B_i$ $(a \neq 0)$.

Proof. Define $\bar{\varepsilon}_{i2} \triangleq \frac{1}{2}\bar{\varepsilon}_{i1} > 0$. By the definition of $\mathscr{B}_i$, it is $\overline{\mathscr{B}_i\left(\bar{\varepsilon}_{i2}\right)} \subseteq \mathscr{B}_i\left(\bar{\varepsilon}_{i1}\right)$, so again by hypothesis $\nabla\beta_i \neq 0$ in $\overline{\mathscr{B}_i\left(\bar{\varepsilon}_{i2}\right)}$. Therefore

$$L_1 \triangleq \min_{\overline{\mathscr{B}_i(\bar{\varepsilon}_{i2})}} \{\|\nabla\beta_i(q)\|\} > 0. \tag{6.72}$$

By the hypothesis and Proposition 133 the quadratic form associated with the Hessian matrix $D^2\hat{\varphi}\left(q_c\right)$ can be decomposed for the unit normal vector $\hat{r}_i$ as

$$\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{r}_i \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = \frac{1}{\beta_i(q_c)}\left(\xi_i(q_c) + \eta_{i1}(q_c)\beta_i(q_c) + \eta_{i2}(q_c)\beta_i^2(q_c)\right), \tag{6.73}$$

for any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i2}$. The right-hand side of the above equation can be separated in two terms, $\frac{1}{2}\xi(q_c) + \eta_{i1}(q_c)\beta_i(q_c)$ and $\frac{1}{2}\xi_i(q_c) + \eta_{i2}(q_c)\beta_i^2(q_c)$. The term which has changed compared to the sphere world case studied in [1] is $\frac{1}{2}\xi_i(q_c) + \eta_{i1}(q_c)\beta_i(q_c)$.

By definition of the Koditschek-Rimon function φ, it is $2 \leq k$ which implies that $\frac{1}{2} \leq 1 - \frac{1}{k}$. So

$$\xi_i(q_c) = \left(1 - \frac{1}{k}\right)\bar{\beta}_i(q_c)\left\|\nabla\beta_i(q_c)\right\|^2 \geq \frac{1}{2}\bar{\beta}_i(q_c)L_1^2 > 0 \tag{6.74}$$

Define

$$L_2 \triangleq \max_{q\in\overline{\mathscr{B}_i(\bar{\varepsilon}_{i2})}} \left\{\left|\hat{r}_i^{\mathrm{T}} D^2\beta_i(q)\hat{r}_i\right|\right\}. \tag{6.75}$$

(Observe that $|\eta_{i1}(q)| \leq \bar{\beta}_i(q)L_2$.) If $L_2 = 0$, then $\eta_{i1}(q_c) = 0$ for any $q_c \in \mathscr{B}_i\left(\varepsilon_i\right)$ with $\varepsilon_i < \bar{\varepsilon}_{i2}$. In this case, define $\bar{\varepsilon}_{i3} \triangleq \bar{\varepsilon}_{i2}$. Take any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i3}$. Assume that the critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, so

$$\frac{1}{2}\xi_i(q_c) + \eta_{i1}(q_c)\beta_i(q_c) = \frac{1}{2}\xi_i(q_c) + 0\beta_i(q_c) \geq \frac{1}{4}\bar{\beta}_i(q_c)L_1^2 = \frac{1}{4}\bar{\beta}_i(q_c)L_1^2 > 0, \tag{6.76}$$

which proves the claim in the case $L_2 = 0$.

If $L_2 > 0$, then define

$$\bar{\varepsilon}_{i3} \triangleq \min\left\{\bar{\varepsilon}_{i2},\ \frac{L_1^2}{8L_2}\right\}. \tag{6.77}$$

Take any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i3}$. Assume that the critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, so

$$\begin{gathered}\beta_i(q_c) < \bar{\varepsilon}_{i3} \implies -\bar{\varepsilon}_{i3}\left|\eta_{i1}(q_c)\right| < \beta_i(q_c)\eta_i(q_c) \implies -\bar{\varepsilon}_{i3}\bar{\beta}_i(q_c)L_2 < \beta_i(q_c)\eta_{i1}(q_c) \implies \\ -\frac{1}{8}\bar{\beta}_i(q_c)L_1^2 < \beta_i(q_c)\eta_{i1}(q_c) \implies \frac{1}{8}\bar{\beta}_i(q_c)L_1^2 + \frac{1}{4}\bar{\beta}_i(q_c)L_1^2 < \frac{1}{2}\xi_i(q_c) + \beta_i(q_c)\eta_{i1}(q_c) \implies \\ \frac{1}{8}\bar{\beta}_i(q_c)L_1^2 < \frac{1}{2}\xi_i(q_c) + \beta_i(q_c)\eta_{i1}(q_c)\end{gathered} \tag{6.78}$$

Define

$$L_3 \triangleq \max_{q\in\overline{\mathscr{B}_i(\bar{\varepsilon}_{i2})}} \{|\eta_{i2}(q_c)|\} = \max_{q\in\overline{\mathscr{B}_i(\bar{\varepsilon}_{i2})}} \left\{\left|\hat{r}_i^{\mathrm{T}} D^2\bar{\beta}_i(q)\hat{r}_i\right|\right\}. \tag{6.79}$$

By Assumption 1 the obstacles are disjoint sets. Therefore, there exists an $\bar{\varepsilon}_{i4} > 0$ such that $\overline{\mathscr{B}_i\left(\bar{\varepsilon}_{i4}\right)}$ is disjoint from all $\mathscr{O}_j, j \neq i$. This ensures that

$$L_4 \triangleq \min_{q\in\overline{\mathscr{B}_i(\bar{\varepsilon}_{i4})}} \{\bar{\beta}_i(q)\} > 0. \tag{6.80}$$

As a result

$$\xi_i(q_c) = \left(1 - \frac{1}{k}\right) \bar{\beta}_i(q_c) \left\|\nabla \beta_i(q_c)\right\|^2 \geq \frac{1}{2} L_4 L_1^2 \tag{6.81}$$

If $L_3 = 0$, then $\eta_{i2}(q_c) = 0$ for any $q_c \in \mathscr{B}_i(\varepsilon_i)$ with $\varepsilon_i < \bar{\varepsilon}_{i4}$. In this case, define $\bar{\varepsilon}_{i5} \triangleq \min\{\bar{\varepsilon}_{i2}, \bar{\varepsilon}_{i4}\}$. Take any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i5}$. Assume that the critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, so

$$\frac{1}{2}\xi_i(q_c) + \eta_{i2}(q_c)\beta_i(q_c) = \frac{1}{2}\xi_i(q_c) + 0\beta_i(q_c) \geq \frac{1}{4} L_4 L_1^2, \tag{6.82}$$

which proves the claim in the case $L_3 = 0$.

If $L_3 > 0$, the define

$$\bar{\varepsilon}_{i5} \triangleq \min\left\{\bar{\varepsilon}_{i2}, \bar{\varepsilon}_{i4}, \sqrt{\frac{L_1^2 L_4}{8L_3}}\right\}. \tag{6.83}$$

Take any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i5}$. Assume that the critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, so

$$\begin{aligned} \beta_i(q_c)^2 < \bar{\varepsilon}_{i5}^2 \implies -\bar{\varepsilon}_{i5}^2 \left|\eta_{i2}(q_c)\right| < \beta_i(q_c)^2 \eta_{i2}(q_c) \implies \\ -\frac{L_1^2 L_4}{8L_3} L_3 < \beta_i(q_c)^2 \eta_{i2}(q_c) \implies -\frac{1}{8} L_1^2 L_4 < \beta_i(q_c)^2 \eta_{i2}(q_c) \implies \\ \frac{1}{4} L_1^2 L_4 - \frac{1}{8} L_1^2 L_4 < \frac{1}{2}\xi_i(q_c) + \beta_i(q_c)^2 \eta_{i2}(q_c) \implies \frac{1}{8} L_1^2 L_4 < \frac{1}{2}\xi_i(q_c) + \beta_i(q_c)^2 \eta_{i2}(q_c). \end{aligned} \tag{6.84}$$

Define $\bar{\varepsilon}_i \triangleq \min\{\bar{\varepsilon}_{i3}, \bar{\varepsilon}_{i5}\}$. Take any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i5}$. Assume that the critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. It is $\beta_i(q_c) < \bar{\varepsilon}_{i2}$ so (6.73) yields (taking the lower bounds in both cases, for each of the two terms)

$$\begin{aligned} \hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{r}_i \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} &= \frac{1}{\beta_i(q_c)} \left(\xi_i(q_c) + \eta_{i1}(q_c)\beta_i(q_c) + \eta_{i2}(q_c)\beta_i^2(q_c)\right) \\ &\geq \frac{1}{\beta_i(q_c)} \left(\frac{1}{8}\bar{\beta}_i(q_c) L_1^2 + \frac{1}{8} L_1^2 L_4\right) > 0. \end{aligned} \tag{6.85}$$

$\square$

6.3 Span of Concave Tangent and Normal Space: Positive-Definite

We have shown how to ensure positive-definiteness of the Hessian form on the normal space. We have shown how to ensure sign-definiteness of the Hessian form given boundedness of the relative curvature in a tangent direction. Using these the sing-definiteness of the Hessian on the tangent space and the normal space can be derived. If the form maintains the same sign over the tangent space, then the tangent and normal space comprise a direct sum decomposition of the ambient space and the form is sign-definite on each, so Lemma 3.8 from [1] applied. However, if the form is negative-definite on a subspace of the tangent space and positive-definite on its complement (with respect to the tangent bundle), then we need to ensure that it is positive definite also on the span of the normal space with the directions in the positive-definite subspace of the tangent space (more accurately the subspace of the tangent space which is spanned by the principal-directions with positive principal relative curvatures - note that the relative curvature of any tangent direction in this span is bounded by the same bound which applies to the corresponding principal relative curvatures), before applying Lemma 3.8 from [1].

Proposition 135. *Assume that both $\nabla\beta_i(q_c) \neq 0$ and $\nabla\gamma_d(q_c) \neq 0$ at the critical point $q_c \in \mathscr{C}_{\hat{\varphi}}$.*

Then, the bilinear quadratic form associated with the Hessian $D^2\hat{\varphi}(q_c)$ can be decomposed as

$$\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = \sigma_i(q_c, \hat{t}_i) + \beta_i(q_c)\tau_i(q, \hat{t}_i) \tag{6.86}$$

for any unit tangent vector $\hat{t}_i \in UT_{q_c}B_i$ and where $\hat{r}_i$ is the unit normal vector.

Proof. Let us proceed for the third term in the same spirit

$$\begin{aligned}
\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i &= \hat{r}_i^{\mathrm{T}}\left(\frac{\gamma_d^{k-1}}{\beta^2}\left(k\beta D^2\gamma_d + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}\right.\right.\\
&\quad\left.\left. + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} - \frac{2}{k}\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s - \gamma_d\beta_i D^2\bar{\beta}_i - \gamma_d\bar{\beta}_i D^2\beta_i\right)\right)\hat{t}_i\\
\overset{\frac{\gamma_d^{k-1}}{\beta^2}>0}{\Longrightarrow}\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i\right)\frac{\beta^2}{\gamma_d^{k-1}} &= \hat{r}_i^{\mathrm{T}}\left(k\beta D^2\gamma_d + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}}\right.\\
&\quad\left. - \frac{2}{k}\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s - \gamma_d\beta_i D^2\bar{\beta}_i - \gamma_d\bar{\beta}_i D^2\beta\right)\hat{t}_i\\
&= \hat{r}_i^{\mathrm{T}}\left(k\beta D^2\gamma_d\right)\hat{t}_i + \hat{r}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i + \hat{r}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}}\right)\hat{t}_i\\
&\quad - \hat{r}_i^{\mathrm{T}}\left(\frac{2}{k}\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s\right)\hat{t}_i - \hat{r}_i^{\mathrm{T}}\left(\gamma_d\beta_i D^2\bar{\beta}_i\right)\hat{t}_i - \hat{r}_i^{\mathrm{T}}\left(\gamma_d\bar{\beta}_i D^2\beta_i\right)\hat{t}_i
\end{aligned}\tag{6.87}$$

Let us calculate each term separately

$$
\begin{aligned}
\hat{r}_i^{\mathrm{T}}\left(k\beta D^2\gamma_d\right)\hat{t}_i &= k\beta\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{t}_i\right) \overset{\gamma_d=\|q-q_d\|^2\Longrightarrow D^2\gamma_d=2I}{=} k\beta\left(\hat{r}_i^{\mathrm{T}}2I\hat{t}_i\right)\\
&= (2k\beta)\,\hat{r}_i^{\mathrm{T}}\left(I\hat{t}_i\right) \overset{I\hat{t}_i=\hat{t}_i}{=} (2k\beta)\,\hat{r}_i^{\mathrm{T}}\hat{t}_i = \hat{r}_i\cdot\hat{t}_i \overset{\hat{r}_i\cdot\hat{t}_i=\frac{\nabla\beta_i\cdot\nabla\beta_i^{\perp}}{\|\nabla\beta_i\|^2}}{=} 0
\end{aligned}
\tag{6.88}
$$

and

$$
\left(\hat{r}_i^{\mathrm{T}}\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\right)\left(\nabla\beta_i^{\mathrm{T}}\hat{t}_i\right) \overset{\nabla\beta_i^{\mathrm{T}}\hat{t}_i=\nabla\beta_i\cdot\hat{t}_i=\nabla\beta_i\cdot\frac{\nabla\beta_i^{\perp}}{\|\nabla\beta_i\|}}{=} 0
\tag{6.89}
$$

As a result

$$
\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}}\left(\hat{r}_i^{\mathrm{T}}D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right) &= \beta_i^2\left(\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i\right)-\beta_i\left(\gamma_d\left(\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{t}_i\right)\right)\right.\\
&\quad -\frac{2\gamma_d}{k}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s\hat{t}_i-\gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i\right)\\
&= \beta_i\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\bar{\beta}_i}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i-\gamma_d\left(\hat{r}_iD^2\bar{\beta}_i\hat{t}_i\right)\right)\\
&\quad -\frac{2\gamma_d}{k}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s\hat{t}_i-\gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i\right)
\end{aligned}
\tag{6.90}
$$

Now observe that

$$
\begin{aligned}
\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s\hat{t}_i &= \frac{1}{2}\hat{r}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}}\right)\hat{t}_i\\
&= \frac{1}{2}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i\\
&= \frac{1}{2}\left(\hat{r}_i^{\mathrm{T}}\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\hat{t}_i+\hat{r}_i^{\mathrm{T}}\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i\right)\\
&= \frac{1}{2}\hat{r}_i^{\mathrm{T}}\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i \overset{\hat{r}_i=\frac{\nabla\beta_i}{\|\nabla\beta_i\|}}{=} \overset{\|\nabla\beta_i\|\neq 0\text{close to}\mathscr{O}_i}{=} \frac{1}{2}\frac{\nabla\beta_i^{\mathrm{T}}\nabla\beta_i}{\|\nabla\beta_i\|}\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i\\
&= \frac{1}{2}\frac{\|\nabla\beta_i\|\,\|\nabla\beta_i\|}{\|\nabla\beta_i\|}\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i = \frac{1}{2}\|\nabla\beta_i\|\,\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i\\
&= \frac{1}{2}\|\nabla\beta_i\|\,\nabla\bar{\beta}_i\cdot\hat{t}_i = \frac{1}{2}\|\nabla\beta_i\|\,\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i
\end{aligned}
\tag{6.91}
$$

and therefore

$$
\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}}\left(\hat{r}_i^{\mathrm{T}}D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right) &= \beta_i\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\bar{\beta}_i}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i-\gamma_d\left(\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{t}_i\right)\right)\\
&\quad -\frac{\gamma_d\|\nabla\beta_i\|}{k}\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i-\gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i\right)\\
&= \gamma_d\Bigg(-\frac{1}{k}\|\nabla\beta_i\|\,\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i-\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i\right)\\
&\quad +\beta_i\hat{r}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i}-D^2\bar{\beta}_i\right)\hat{t}_i\Bigg).
\end{aligned}
\tag{6.92}
$$

Substitute σ_i and τ_i from Definition 130 to obtain

$$\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right) \hat{t}_i \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = \sigma_i(q_c, \hat{t}_i) + \beta_i(q_c)\tau_i(q, \hat{t}_i).$$

□

Proposition 136 (KRF Hessian Quadratic Form Decomposition on Span of Normal and Tangent)**.** *Assume that both $\nabla\beta_i(q_c) \neq 0$ and $\nabla\gamma_d\left(q_c\right) \neq 0$ at the critical point $q_c \in \mathscr{C}_{\hat{\varphi}}$.*

Then, the quadratic form associated with the Hessian $D^2\hat{\varphi}\left(q_c\right)$ can be decomposed as

$$\hat{v}^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right) \hat{v} \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = A_i(q_c, \hat{t}_i)\lambda^2 + B_i(q_c, \hat{t}_i)\lambda + C_i(q_c), \tag{6.93}$$

for any unit vector $\hat{v} = \mu\hat{r}_i + \lambda\hat{t}_i$, where $\hat{r}_i$ is the normal unit vector and $\hat{t}_i$ is any unit tangent vector in UT_qB_i.

Proof. Let the vector spanned by the radial $\hat{r}_i$ and tangential $\hat{t}_i$ vectors be denoted by

$$u_i = \mu\hat{r}_i + \lambda\hat{t}_i \tag{6.94}$$

where[2] $\mu, \lambda \in \mathbb{R} \setminus \{0\}$ are weighting coefficients and the radial and tangential unit vectors are defined with respect to the i^{th} obstacle $\mathscr{O}_i$ implicit function β_i gradient as

$$\hat{r}_i \triangleq \frac{\nabla\beta_i}{\|\nabla\beta_i\|}, \quad \hat{t}_i \triangleq \frac{\nabla\beta_i^{\perp}}{\|\nabla\beta_i\|} \tag{6.95}$$

Note that if $A \in \mathbb{R}^{n\times n}, a \in E^n$ a square real matrix and a euclidean vector respectively, and $b = ca \in E^n, c \in \mathbb{R} \setminus \{0\}$ a vector parallel to a, then for the quadratic form associated to A

$$\begin{gathered} b^{\mathrm{T}}Ab = (ca)^{\mathrm{T}} A\,(ca) = ca^{\mathrm{T}}Aca = ca^{\mathrm{T}}cAa = c^2a^{\mathrm{T}}Aa = c^2\left(a^{\mathrm{T}}Aa\right) \overset{c\in\mathbb{R}\setminus\{0\}\implies c^2>0}{\Longrightarrow} \\ \left\{\begin{array}{l} b^{\mathrm{T}}Ab > 0 \iff a^{\mathrm{T}}Aa > 0 \\ b^{\mathrm{T}}Ab = 0 \iff a^{\mathrm{T}}Aa = 0 \\ b^{\mathrm{T}}Ab < 0 \iff a^{\mathrm{T}}Aa < 0 \end{array}\right\} \end{gathered} \tag{6.96}$$

So it suffices to determine the quadratic form sign on a direction, and it is common for all vectors in that direction.

Let us now at a critical point q_c express the Hessian's associated quadratic form along the direction of u_i. The Hessian matrix at the critical point is

$$\begin{aligned} D^2\hat{\varphi}\left(q_c\right) = \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(& k\beta D^2\gamma_d + \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + 2\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}}\right) \\ & - \gamma_d\left(\beta_i D^2\bar{\beta}_i + 2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2\beta_i\right)\Bigg) \\ = \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(& k\beta D^2\gamma_d + \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{\beta}2\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s \\ & + \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} - \gamma_d\beta_i D^2\bar{\beta}_i - 2\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s - \gamma_d\bar{\beta}_i D^2\beta_i\Bigg) \end{aligned}$$

[2]For our purpose exclusion of 0 from $\mathbb{R}$ is not mandatory.

$$
\begin{aligned}
&= \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(k\beta D^2\gamma_d + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} \\
&\quad + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} + 2\gamma_d\left(\nabla\bar{\beta}_i\nabla\bar{\beta}_i\right)_s - 2\frac{1}{k}\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s \\
&\quad - \gamma_d\beta_i D^2\bar{\beta}_i - 2\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s - \gamma_d\bar{\beta}_i D^2\beta_i\Bigg) \\
&= \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(k\beta D^2\gamma_d + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} \\
&\quad - \frac{2}{k}\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s \gamma_d\beta_i D^2\bar{\beta}_i - \gamma_d\bar{\beta}_i D^2\beta_i\Bigg)
\end{aligned}
$$

At a critical point the quadratic form along u_i is

$$
\begin{aligned}
u_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\, u_i &= \left(\mu\hat{r}_i + \lambda\hat{t}_i\right)^{\mathrm{T}} D^2\hat{\varphi}(q_c)\left(\mu\hat{r}_i + \lambda\hat{t}_i\right) \\
&= \left(\left(\mu\hat{r}_i\right)^{\mathrm{T}} + \left(\lambda\hat{t}_i\right)^{\mathrm{T}}\right) D^2\hat{\varphi}(q_c)\left(\mu\hat{r}_i + \lambda\hat{t}_i\right) \\
&= \left(\mu\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c) + \lambda\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\right)\left(\mu\hat{r}_i + \lambda\hat{t}_i\right) \\
&= \left(\mu\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\mu\hat{r}_i\right) + \left(\mu\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\lambda\hat{t}_i\right) + \left(\lambda\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\mu\hat{r}_i\right) + \left(\lambda\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\lambda\hat{t}_i\right) \\
&= \mu^2\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{r}_i\right) + \mu\lambda\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i\right) + \mu\lambda\left(\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{r}_i\right) + \lambda^2\left(\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i\right)
\end{aligned} \tag{6.97}
$$

Note that by the Clairaut-Schwarz Theorem C^2 continuity of function $\hat{\varphi}$ implies symmetry of its Hessian matrix[3]

$$
\hat{\varphi} \in C^2\left(\mathring{\mathscr{F}}, \mathbb{R}_{\geq 0}\right) \implies D^2\hat{\varphi} = \left(D^2\hat{\varphi}\right)^{\mathrm{T}} \tag{6.98}
$$

and because $\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i\right) \in \mathbb{R}$, it is

$$
\begin{aligned}
\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i &= \left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i\right)^{\mathrm{T}} = \hat{t}_i^{\mathrm{T}}\left(\hat{r}_i D^2\hat{\varphi}(q_c)\right)^{\mathrm{T}} \\
&= \hat{t}_i^{\mathrm{T}}\left(D^2\hat{\varphi}(q_c)\right)^{\mathrm{T}}\left(\hat{r}_i^{\mathrm{T}}\right)^{\mathrm{T}} \overset{D^2\hat{\varphi}=\left(D^2\hat{\varphi}\right)^{\mathrm{T}}}{=} \hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{r}_i
\end{aligned} \tag{6.99}
$$

Using the previous result yields

$$
u_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\, u_i = \mu^2\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{r}_i\right) + \lambda^2\left(\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i\right) + 2\mu\lambda\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i\right) \tag{6.100}
$$

By Proposition 131 the term $\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i$ can be decomposed as

$$
\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i = \frac{\gamma_d(q_c)^k}{\beta(q_c)^2}\bar{\beta}_i(q_c)\left(\nu_i(q_c,\hat{t}_i) + \beta_i(q_c)\,\zeta_i(q_c,\hat{t}_i)\right), \tag{6.101}
$$

for unit tangent vectors $\hat{t}_i \in UT_{q_c}B_i$. By Proposition 133 the term $\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{r}_i$ can be decomposed as

$$
\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{r}_i = \frac{\gamma_d(q_c)^k}{\beta(q_c)^2}\frac{1}{\beta_i(q_c)}\left(\xi_i(q_c) + \eta_{i1}(q_c)\beta_i(q_c) + \eta_{i2}(q_c)\beta_i^2(q_c)\right), \tag{6.102}
$$

[3]In other words the order of partial derivation in mixed derivatives does not matter.

for the normal unit vector $\hat{r}_i \in N_q B_i$. By Proposition 135 the term $\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{t}_i$ can be decomposed as

$$\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{t}_i = \frac{\gamma_d(q_c)^k}{\beta(q_c)^2} \left(\sigma_i(q_c, \hat{t}_i) + \beta_i(q_c)\tau_i(q, \hat{t}_i)\right), \tag{6.103}$$

for any unit tangent vector $\hat{t}_i \in UT_{q_c} B_i$ and where $\hat{r}_i$ is the unit normal vector. So substitution of (6.101), (6.102) and (6.103) into (6.100) yields

$$\begin{aligned}
\hat{v}^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{v} &= \mu^2 \left(\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{r}_i\right) + \lambda^2 \left(\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{t}_i\right) + 2\mu\lambda \left(\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{t}_i\right) \\
&= \mu^2 \frac{\gamma_d(q_c)^k}{\beta(q_c)^2} \frac{1}{\beta_i(q_c)} \left(\xi_i(q_c) + \eta_{i1}(q_c)\beta_i(q_c) + \eta_{i2}(q_c)\beta_i^2(q_c)\right) \\
&+ \lambda^2 \frac{\gamma_d(q_c)^k}{\beta(q_c)^2} \bar{\beta}_i(q_c) \left(\nu_i(q_c, \hat{t}_i) + \beta_i(q_c)\, \zeta_i(q_c, \hat{t}_i)\right) \\
&+ 2\mu\lambda \frac{\gamma_d(q_c)^k}{\beta(q_c)^2} \left(\sigma_i(q_c, \hat{t}_i) + \beta_i(q_c)\tau_i(q, \hat{t}_i)\right) \implies \\
\hat{v}^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{v} \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} &= \lambda^2 \left(\bar{\beta}_i(q_c) \left(\nu_i(q_c, \hat{t}_i) + \beta_i(q_c)\, \zeta_i(q_c, \hat{t}_i)\right)\right) \\
&+ \lambda \left(2\mu \left(\sigma_i(q_c, \hat{t}_i) + \beta_i(q_c)\tau_i(q, \hat{t}_i)\right)\right) \\
&+ \left(\mu^2 \frac{1}{\beta_i(q_c)} \left(\xi_i(q_c) + \eta_{i1}(q_c)\beta_i(q_c) + \eta_{i2}(q_c)\beta_i^2(q_c)\right)\right)
\end{aligned} \tag{6.104}$$

so recalling Definition 130

$$\hat{v}^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{v} \frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = A_i(q_c, \hat{t}_i)\lambda^2 + B_i(q_c, \hat{t}_i)\lambda + C_i(q_c).$$

□

Expanded, the above is

$$\begin{aligned}
&u_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, u_i = \mu^2 \left(\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{r}_i\right) + \lambda^2 \left(\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{t}_i\right) + 2\mu\lambda \left(\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}(q_c)\, \hat{t}_i\right) \\
&= \mu^2 \frac{\gamma_d^{k-1}}{\beta^2} \frac{\gamma_d}{\beta_i} \left(\left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla \beta_i\right\|^2 - \beta_i^2 \left(\hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{r}_i\right) - \beta_i \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right)\right) \\
&\quad + \lambda^2 \frac{\gamma_d^{k-1}}{\beta^2} \left(\gamma_d \bar{\beta}_i \nu_i(q) + \beta_i \gamma_d \left(\frac{\nabla \bar{\beta}_i \cdot \nabla \gamma_d}{\left\|\nabla \gamma_d\right\|^2} + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2 \bar{\beta}_i\right) \hat{t}_i\right)\right) \\
&\quad + 2\mu\lambda \frac{\gamma_d^{k-1}}{\beta^2} \Bigg(\beta_i \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\bar{\beta}_i} \hat{r}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i - \gamma_d \left(\hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{t}_i\right)\right) \\
&\quad\quad - \frac{\gamma_d \left\|\nabla \beta_i\right\|}{k} \hat{t}_i^{\mathrm{T}} \nabla \bar{\beta}_i - \gamma_d \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right)\Bigg) \overset{\frac{\gamma_d^{k-1}}{\beta^2} > 0}{\Longrightarrow}
\end{aligned}$$

$$
\begin{aligned}
&\frac{\beta^2}{\gamma_d^{k-1}}\left(u_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right) u_i\right) = \lambda^2\left(\gamma_d\bar{\beta}_i\nu_i(q_c) + \beta_i\gamma_d\left(\frac{\nabla\bar{\beta}_i\cdot\nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2} + \hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right)\hat{t}_i\right)\right)\\
&+\lambda\left(2\mu\left(\beta_i\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\bar{\beta}_i}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i - \gamma_d\left(\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{t}_i\right)\right) - \frac{\gamma_d\left\|\nabla\beta_i\right\|}{k}\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i - \gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i\right)\right)\right)\\
&\quad+\left(\mu^2\frac{\gamma_d}{\beta_i}\left(\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2 - \beta_i^2\left(\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{r}_i\right) - \beta_i\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{r}_i\right)\right)\right)\\
&=\lambda^2\left(\gamma_d\bar{\beta}_i\nu_i(q_c) + O\left(\beta_i\right)\right)\\
&\quad+\lambda\left(2\mu\left(O\left(\beta_i\right) - \frac{\gamma_d\left\|\nabla\beta_i\right\|}{k}\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i - \gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i\right)\right)\right)\\
&\quad+\left(\mu^2\frac{\gamma_d}{\beta_i}\left(\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2 - O\left(\beta_i\right)\right)\right)
\end{aligned}
$$

The following proves that the Hessian form is positive-definite on the span of a positive-definite (lower bounded) tangent direction and the normal space, i.e., on span $\{\hat{r}_i, \hat{t}_i\}$, where $\hat{t}_i$ is a positive-definite tangent direction.

Lemma 137. *Select arbitrary $L_1 > 0$ and $L_2 > 0$. Assume that there exists an $\bar{\varepsilon}_{i1} > 0$ such that $\nabla\beta_i \neq 0$ and $\nabla\gamma_d \neq 0$ in $\mathscr{B}_i\left(\varepsilon_i\right)$ for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i1}$.*

Then, there exists an $\bar{\varepsilon}_i > 0$ such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ the following holds. If at a critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ it is

$$\nu_i(q_c, \hat{t}_i) \geq L_1 \quad \textit{and} \quad \xi_i(q_c) \geq L_2,$$

for the unit tangent vector $\hat{t}_i \in UT_{q_c}B_i$, then the Hessian quadratic form

$$u_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right) u_i > 0,$$

for any vector $u_i \in$ span $\{\hat{t}_i, \hat{r}_i\}$.

Proof. By Assumption 1 the obstacles are disjoint sets. Therefore, there exists an $\bar{\varepsilon}_{i4} > 0$ such that $\overline{\mathscr{B}_i\left(\bar{\varepsilon}_{i4}\right)}$ is disjoint from all $\mathscr{O}_j, j \neq i$. This ensures that

$$L_4 \triangleq \min_{q\in\overline{\mathscr{B}_i\left(\bar{\varepsilon}_{i4}\right)}}\{\bar{\beta}_i(q)\} > 0. \tag{6.105}$$

and

$$M_1 \triangleq \max_{q\in\overline{\mathscr{B}_i\left(\bar{\varepsilon}_{i4}\right)}}\{\bar{\beta}_i(q)\} > 0 \tag{6.106}$$

Define $\bar{\varepsilon}_{i2} \triangleq \frac{1}{2}\bar{\varepsilon}_{i1} > 0$. By the definition of $\mathscr{B}_i$, it is $\overline{\mathscr{B}_i\left(\bar{\varepsilon}_{i2}\right)} \subseteq \mathscr{B}_i\left(\bar{\varepsilon}_{i1}\right)$, so again by hypothesis $\nabla\beta_i \neq 0$ in $\overline{\mathscr{B}_i\left(\bar{\varepsilon}_{i2}\right)}$. Therefore

$$L_3 \triangleq \min_{\overline{\mathscr{B}_i\left(\bar{\varepsilon}_{i2}\right)}}\{\left\|\nabla\beta_i(q)\right\|\} > 0. \tag{6.107}$$

Also $\nabla\gamma_d \neq 0$ in $\mathscr{B}_i\left(\bar{\varepsilon}_{i1}\right)$ so both ν_i and ζ_i are well-defined in $\mathscr{B}_i\left(\bar{\varepsilon}_{i1}\right)$.

By Proposition 136 the quadratic form associated with the Hessian matrix $D^2\hat{\varphi}\left(q_c\right)$ can be decomposed for any unit vector $\hat{v} = \mu\hat{r}_i + \lambda\hat{t}_i$ as

$$\hat{v}^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{v}\frac{\beta(q_c)^2}{\gamma_d(q_c)^k} = A_i(q_c,\hat{t}_i)\lambda^2 + B_i(q_c,\hat{t}_i)\lambda + C_i(q_c), \tag{6.108}$$

We want to ensure that the form $\hat{v}^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{v}$ is positive-definite. It suffices to show that the quadratic polynomial $A_i(q_c, \hat{t}_i)\lambda^2 + B_i(q_c, \hat{t}_i)\lambda + C_i(q_c)$ is positive. This is true if $A_i(q_c, \hat{t}_i) > 0$ and the polynomial does not have any real roots.

Let us start by considering the coefficient $A_i(q_c, \hat{t}_i)$. Let

$$L_4 \triangleq \max_{(q,\hat{v}) \in UT\overline{\mathscr{B}_i(\bar{\varepsilon}_{i2})}} \left\{ |\zeta_i(q_c, \hat{v})| \right\}. \tag{6.109}$$

If $L_4 = 0$, then $\zeta_i(q_c, \hat{t}_i) = 0$. In this case define $\bar{\varepsilon}_{i3} \triangleq \min\{\bar{\varepsilon}_{i2}, \bar{\varepsilon}_{i4}\}$. Take any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i4}$. Assume that $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, then by hypothesis $\nu_i(q_c, \hat{t}_i) \geq L_1$, so

$$A_i(q_c, \hat{t}_i) = \bar{\beta}_i(q_c)\left(\nu_i(q_c, \hat{t}_i) + \beta_i(q_c)\zeta_i(q_c, \hat{t}_i)\right) = \bar{\beta}_i(q_c)\left(\nu_i(q_c, \hat{t}_i) + \beta_i(q_c)0\right) \geq \frac{L_1}{M_1}. \tag{6.110}$$

If $L_4 > 0$ define

$$\bar{\varepsilon}_{i3} \triangleq \min\left\{ \bar{\varepsilon}_{i2}, \bar{\varepsilon}_{i4}, \frac{L_1}{2M_1 L_4} \right\}. \tag{6.111}$$

Take any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i4}$. Assume that $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, then by hypothesis $\nu_i(q_c, \hat{t}_i) \geq L_1$, so

$$\begin{aligned} &\beta_i(q_c) < \bar{\varepsilon}_{i3} \implies -\bar{\varepsilon}_{i3}\left|\zeta_i(q_c, \hat{t}_i)\right| < \beta_i(q_c)\zeta_i(q_c, \hat{t}_i) \implies \\ -&\frac{L_1}{2M_1L_4}L_4 < \beta_i(q_c)\zeta_i(q_c, \hat{t}_i) \implies \frac{L_1}{M_1} - \frac{L_1}{2M_1} < \nu_i(q_c, \hat{t}_i) + \beta_i(q_c)\zeta_i(q_c, \hat{t}_i) \implies \\ &\frac{L_1}{2M_1} < A_i(q_c, \hat{t}_i). \end{aligned} \tag{6.112}$$

Similar reasoning can yield a lower bound on C_i. Let

$$\begin{aligned} L_5 &\triangleq \max_{q \in \overline{\mathscr{B}_i(\bar{\varepsilon}_{i2})}} \left\{ |\eta_{i1}(q)| \right\} \\ L_6 &\triangleq \max_{q \in \overline{\mathscr{B}_i(\bar{\varepsilon}_{i2})}} \left\{ |\eta_{i2}(q)| \right\} \end{aligned} \tag{6.113}$$

If either $L_5 = 0$ or $L_6 = 0$, then the corresponding term need not be included in the definition of $\bar{\varepsilon}_{i6}$. Define

$$\bar{\varepsilon}_{i6} \triangleq \min\left\{ \bar{\varepsilon}_{i2}, \bar{\varepsilon}_{i4}, \frac{L_2}{4L_5}, \sqrt{\frac{L_2}{4L_6}} \right\}. \tag{6.114}$$

Take any $\varepsilon_i > 0$ such that $\varepsilon_i < \bar{\varepsilon}_{i6}$. Assume that $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, then by hypothesis $\xi_i(q_c) \geq L_2$, so

$$\begin{aligned} &\beta_i(q_c) < \bar{\varepsilon}_{i6} \implies -\bar{\varepsilon}_{i6}\left|\eta_{i1}(q_c)\right| < \beta_i(q_c)\eta_{i1}(q_c) \implies \\ -&\frac{L_2}{4L_5}L_5 < \beta_i(q_c)\eta_{i1}(q_c) \implies \frac{1}{2}L_2 - \frac{1}{4}L_2 < \frac{1}{2}\xi_i(q_c) + \beta_i(q_c)\eta_{i1}(q_c) \implies \\ &\frac{1}{4}L_2 < \frac{1}{2}\xi_i(q_c) + \beta_i(q_c)\eta_{i1}(q_c) \end{aligned} \tag{6.115}$$

and

$$\begin{aligned} &\beta_i(q_c)^2 < \bar{\varepsilon}_{i6}^2 \implies -\bar{\varepsilon}_{i6}^2\left|\eta_{i2}(q_c)\right| < \beta_i(q_c)^2\eta_{i2}(q_c) \implies \\ -&\frac{L_2}{4L_6}L_6 < \beta_i(q_c)^2\eta_{i2}(q_c) \implies \frac{1}{2}L_2 - \frac{1}{4}L_2 < \frac{1}{2}\xi_i(q_c) + \beta_i(q_c)^2\eta_{i2}(q_c) \implies \\ &\frac{1}{4}L_2 < \frac{1}{2}\xi_i(q_c) + \beta_i(q_c)^2\eta_{i2}(q_c) \end{aligned} \tag{6.116}$$

Adding the previous two equations yields

$$\begin{aligned} &\frac{1}{2}L_2 < \xi_i(q_c) + \beta_i(q_c)\eta_{i1}(q_c) + \beta_i(q_c)^2\eta_{i2}(q_c) \implies \\ &\frac{L_2\mu^2}{\beta_i(q_c)} < \frac{\mu^2}{\beta_i(q_c)}\left(\xi_i(q_c) + \beta_i(q_c)\eta_{i1}(q_c) + \beta_i(q_c)^2\eta_{i2}(q_c)\right) \iff \\ &\frac{L_2\mu^2}{\beta_i(q_c)} < C_i(q_c) \end{aligned} \tag{6.117}$$

The second condition to ensure that the polynomial is positive without roots, is that its discriminant be negative. The discriminant is equal to

$$\begin{aligned} \Delta &= B_i(q_c,\hat{t}_i)^2 - 4A_i(q_c,\hat{t}_i)C_i(q_c) \le B_i(q_c,\hat{t}_i)^2 - 4\frac{L_1}{2M_1}\frac{L_2\mu^2}{\beta_i(q_c)} \\ &= 4\mu^2\left(\sigma_i(q_c,\hat{t}_i) + \beta_i(q_c)\tau(q_c,\hat{t}_i)^2\right) - 4\mu^2\frac{L_1L_2}{2M_1\beta_i(q_c)}. \end{aligned} \tag{6.118}$$

We want to ensure that ($\mu > 0$)

$$\Delta < 0 \iff \sigma_i(q_c,\hat{t}_i) + \beta_i(q_c)\tau(q_c,\hat{t}_i)^2 < \frac{L_1L_2}{2M_1\beta_i(q_c)}. \tag{6.119}$$

Define

$$L_7 \triangleq \max_{q\in\mathscr{B}_i(\bar{\varepsilon}_{i2})}\left\{\left|\sigma_i(q_c,\hat{t}_i) + \beta_i(q_c)\tau(q_c,\hat{t}_i)^2\right|\right\} \tag{6.120}$$

If $L_7 = 0$, then $\Delta < 0$ for $\varepsilon_i < \bar{\varepsilon}_i \triangleq \min\{\bar{\varepsilon}_{i3}, \bar{\varepsilon}_{i6}\}$. If $L_7 > 0$ (Note that $\sigma_i, \tau_i, \beta_i$ are upper-bounded in $\overline{\mathscr{B}_i(\bar{\varepsilon}_{i2})}$, so $L_7 < \infty$), then define

$$\bar{\varepsilon}_i \triangleq \min\left\{\bar{\varepsilon}_{i3}, \bar{\varepsilon}_{i6}, \frac{L_1L_2}{2M_1L_7}\right\}. \tag{6.121}$$

Take any $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$. Assume that $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, so

$$\beta_i(q_c) < \bar{\varepsilon}_i \implies L_7 < \frac{L_1L_2}{2M_1\beta_i(q_c)} \implies \sigma_i(q_c,\hat{t}_i) + \beta_i(q_c)\tau(q_c,\hat{t}_i)^2 < \frac{L_1L_2}{2M_1\beta_i(q_c)}, \tag{6.122}$$

because $\beta_i(q_c) > 0$. This completes the proof. $\square$

6.4 Hessian Quadratic Form dependence on Relative Curvature

Theorem 138 (Hessian Form relation to Relative Curvature in $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$)**.** *Select arbitrary $L_1 > 0$ and $L_2 > 0$. Assume that there exists an $\bar{\varepsilon}_{i1} > 0$ such that $\nabla\beta_i \neq 0$ and $\nabla\gamma_d \neq 0$ in $\mathscr{B}_i(\varepsilon_i)$ for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i1}$. Further, assume that $\xi_i(q) \geq L_2$ for all $q \in \mathscr{B}_i(\varepsilon_i)$.*

Then, there exists an $\bar{\varepsilon}_i > 0$ such that if $\varepsilon_i < \bar{\varepsilon}_i$, then the following hold. At each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ the quadratic form associated with the Hessian $D^2\hat{\varphi}(q_c)$ is

- *Negative definite on $\mathscr{R}_i^-(q_c, -L_1)$,*
- *Positive definite on $\bigcup_{\hat{t}_i \in R_i^+(q_c, L_1)} \operatorname{span}\{\hat{t}_i, \hat{r}_i\}$.*

Remark 139. *Note that $\mathscr{R}_i^+(q_c, L_1)$ can be used as an alternative to $R_i^+(q_c, L_1)$ above. In that case vectors in $\mathscr{R}_i^+(q_c, L_1)$ are not of unit norm. For this reason the notation t_i would be used instead of $\hat{t}_i$, which is reserved for unit tangent vectors. So*

$$\bigcup_{\hat{t}_i \in R_i^+(q_c, L_1)} \operatorname{span}\{\hat{t}_i, \hat{r}_i\} = \bigcup_{t_i \in \mathscr{R}_i^+(q_c, L_1)} \operatorname{span}\{t_i, \hat{r}_i\} \tag{6.123}$$

This is the subset of linear combinations between elements $\hat{t}_i$ in $R_i^+(q_c, L_1)$ and the singleton $\{\hat{r}_i\}$. These linear combinations are $a\hat{t}_i + b\hat{r}_i$. In other words, linear combinations are allowed only between an element in $R_i^+(q_c, L_1)$ and an element in $\{\hat{r}_i\}$. No linear combinations are allowed between multiple elements from $R_i^+(q_c, L_1)$.

Proof. Note that $\bar{\varepsilon}_i$ has not been defined yet, so we do not know the set $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ from which to choose the critical point q_c. Thus the critical point q_c referred to before defining $\bar{\varepsilon}_i$ is a dummy one, used only for the purpose of discussion (i.e., it represents the $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ which will be specific when $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ is fixed later in this proof).

By Definition 117 it is

$$\nu_i(q_c, \hat{t}_i) \leq -L_1$$

for each

$$\hat{t}_i \in R_i^-(q_c, -L_1).$$

Then by Lemma 132 there exists an $\bar{\varepsilon}_{i2} > 0$ with the following property. If $\varepsilon_i > 0$ and $\varepsilon_i < \bar{\varepsilon}_{i2}$, then for each $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ it is

$$t_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\, t_i < 0$$

for any tangent vector $t_i = a\hat{t}_i$ with $a \neq 0$, for all $\hat{t}_i \in R_i^-(q_c, -L_1)$. By Definition 117 we can choose t_i to be any element of the subset (not a subspace in general) $\mathscr{R}_i^-(q_c, -L_1)$. So for all $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ with $0 < \varepsilon_i < \bar{\varepsilon}_{i2}$ the Hessian $D^2\hat{\varphi}(q_c)$ is negative-definite on $\mathscr{R}_i^-(q_c, -L_1)$.

Consider the normal space $\operatorname{span}\{\hat{r}_i(q_c)\}$. By Lemma 134 there exists an $\bar{\varepsilon}_{i3} > 0$ such that for all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i3}$ the following holds. If $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ then

$$r_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\, \hat{r}_i > 0,$$

for any normal vector $r_i = a\hat{r}_i$, $a \neq 0$. This implies that for all $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ with $0 < \varepsilon_i < \bar{\varepsilon}_{i3}$ the Hessian $D^2\hat{\varphi}(q_c)$ is positive-definite on $\operatorname{span}\{\hat{r}_i(q_c)\}$.

We have shown that $D^2\hat{\varphi}\,(q_c)$ is positive-definite on span $\{\hat{r}_i(q_c)\}$. It remains to show that it is positive-definite on $\bigcup_{\hat{t}_i \in R_i^+(q_c,L_1)} \text{span}\left\{\hat{t}_i, \hat{r}_i\right\}$. If $R_i^+(q_c, L_1)$ is empty, then

$$\bigcup_{\hat{t}_i \in R_i^+(q_c,L_1)} \text{span}\left\{\hat{t}_i, \hat{r}_i\right\} = \text{span}\left\{\hat{r}_i(q_c)\right\}$$

and we have already shown that $D^2\hat{\varphi}\,(q_c)$ is positive-definite on span $\{\hat{r}_i(q_c)\}$. So we want to prove the case when $R_i^+(q_c, L_1)$ is not empty.

By Definition 117 it is

$$\nu_i(q_c, \hat{t}_i) \geq L_1$$

for each

$$\hat{t}_i \in R_i^+(q_c, L_1).$$

Then by Lemma 137 there exists an $\bar{\varepsilon}_{i4} > 0$ such that all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i4}$ have the following property. If $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ and $\xi_i(q_c) \geq L_2$ then

$$u_i^{\mathrm{T}} D^2\hat{\varphi}\,(q_c)\, u_i > 0$$

for any vector $u_i \in \text{span}\left\{\hat{t}_i, \hat{r}_i\right\}$, where $\hat{t}_i \in R^+(q_c, L_1)$. This ensures that if $\xi_i(q_c) \geq L_2$, then $D^2\hat{\varphi}\,(q_c)$ is positive-definite on $\bigcup_{\hat{t}_i \in R_i^+(q_c,L_1)} \text{span}\left\{\hat{t}_i, \hat{r}_i\right\}$.

By hypothesis $\xi_i(q) \geq L_2$ for all $q \in \mathscr{B}_i\,(\varepsilon_i)$ where $\varepsilon_i < \bar{\varepsilon}_{i1}$. Take

$$\bar{\varepsilon}_{i5} \triangleq \min\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i4}\} > 0. \tag{6.124}$$

Then every $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ where $0 < \varepsilon_i < \bar{\varepsilon}_{i5}$ is also in a $\mathscr{B}_i\,(\varepsilon_i)$ with $\varepsilon_i < \bar{\varepsilon}_{i1}$. By the hypothesis, this implies that $\xi_i(q_c) \geq L_2$ for all $q_c \in \mathscr{C}_{\hat{\varphi}}$ where $0 < \varepsilon_i < \bar{\varepsilon}_{i5}$. Therefore for all $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ with $0 < \varepsilon_i < \bar{\varepsilon}_{i5}$ the Hessian $D^2\hat{\varphi}\,(q_c)$ is positive-definite on $\bigcup_{\hat{t}_i \in R_i^+(q_c,L_1)} \text{span}\left\{\hat{t}_i, \hat{r}_i\right\}$.

Define

$$\bar{\varepsilon}_i \triangleq \min\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i2}, \bar{\varepsilon}_{i3}, \bar{\varepsilon}_{i4}\}. \tag{6.125}$$

Then for all $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ with $0 < \varepsilon_i < \bar{\varepsilon}_i$ the Hessian $D^2\hat{\varphi}\,(q_c)$ is

- Negative-definite on $\mathscr{R}_i^-(q_c, -L_1)$, and
- Positive-definite on $\bigcup_{\hat{t}_i \in R_i^+(q_c,L_1)} \text{span}\left\{\hat{t}_i, \hat{r}_i\right\}$.

□

Chapter 7

KRNF exist for generic destinations

7.1 Focally Admissible Principal Curvatures

The principal relative curvatures ν_{ij} are related to the principal curvatures κ_{ij} of the obstacle boundary $\partial\mathscr{O}_i$. By assuming that $\partial\mathscr{O}_i$ is focally admissible (see Definition 152), certain properties can be derived for the principal curvatures on the first-order contact locus C_i^1. This is outlined in Fig. 7.2.

Theorem 140. *Select any first-order contact point $q \in C_i^1$. Define $\gamma_d(q) \triangleq \|q - q_d\|^2$. Consider any center of principal curvature $q_{\kappa_{ij}}$ at q.*

If $\kappa_{ij}(q) > 0$ (convex principal curvature), then

- *$\nu_{ij}(q) > 0$ if and only if $\|q_d - q\| < \|q_{\kappa_{ij}} - q\|$.*
- *$\nu_{ij}(q) = 0$ if and only if $q_d = q$.*
- *$\nu_{ij}(q) < 0$ if and only if $\|q_d - q\| > \|q_{\kappa_{ij}} - q\|$.*

If $\kappa_{ij}(q) \leq 0$ (non-convex principal curvature), then $\nu_{ij}(q) > 0$ for all q_d.

Proof. By Assumption 1 it is $\nabla\beta_i \neq 0$ on $\partial\mathscr{O}_i$. By Assumption 32 it is $\nabla\gamma_d \neq 0$ on $\partial\mathscr{O}_i$. By Definition 74 $q \in \partial\mathscr{O}_i$. By the hypothesis $\gamma_d(q) \triangleq \|q - q_d\|^2$ and Proposition 41 the relative curvature function can be decomposed as

$$\nu_i(q, \hat{p}_{ij}(q)) = \|\nabla\beta_i(q)\| \left(\frac{2\cos\theta_i(q)}{\|\nabla\gamma_d(q)\|} - \kappa_i(q, \hat{p}_{ij}(q)) \right). \tag{7.1}$$

By hypothesis $q \in C_i^1$, so Lemma 79 implies $\cos\theta_i(q) = 1$. Substitute to obtain

$$\nu_i(q, \hat{p}_{ij}(q)) = \|\nabla\beta_i(q)\| \left(\frac{2}{\|\nabla\gamma_d(q)\|} - \kappa_i(q, \hat{p}_{ij}(q)) \right). \tag{7.2}$$

By Definition 49 the principal relative curvature $\nu_{ij}(q) = \nu_i(q, \hat{p}_{ij}(q))$. The obstacle boundary surface $\partial\mathscr{O}_i$ is regular, so $\|\nabla\beta_i(q)\| > 0$. Therefore, $\nu_{ij}(q)$ is equi-signed with

$$\frac{2}{\|\nabla\gamma_d(q)\|} - \kappa_i(q, \hat{p}_{ij}) = \frac{2}{\|\nabla\gamma_d(q)\|} - \frac{1}{R_i(q, \hat{p}_{ij})}. \tag{7.3}$$

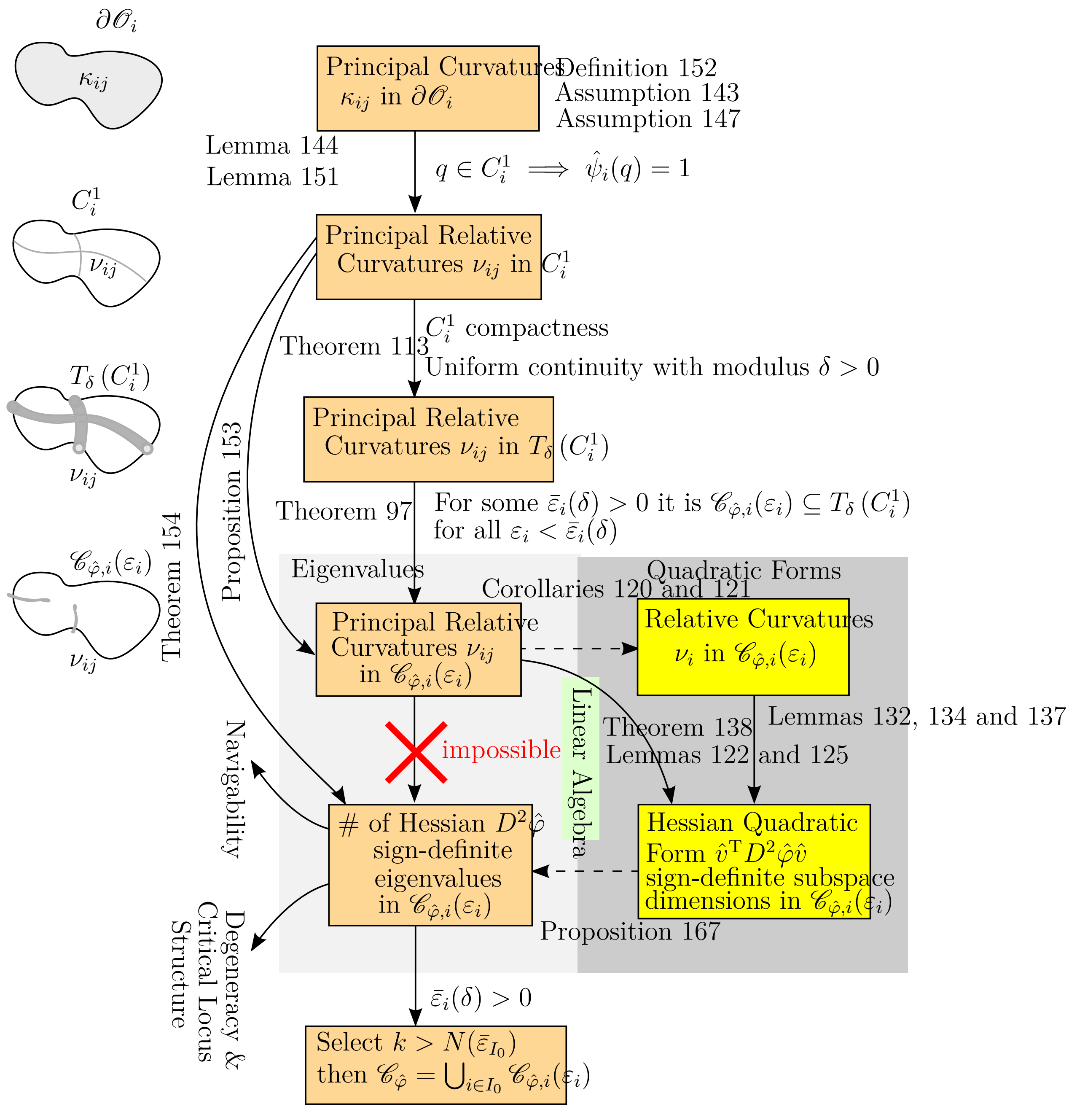

Figure 7.1: Proof structure.

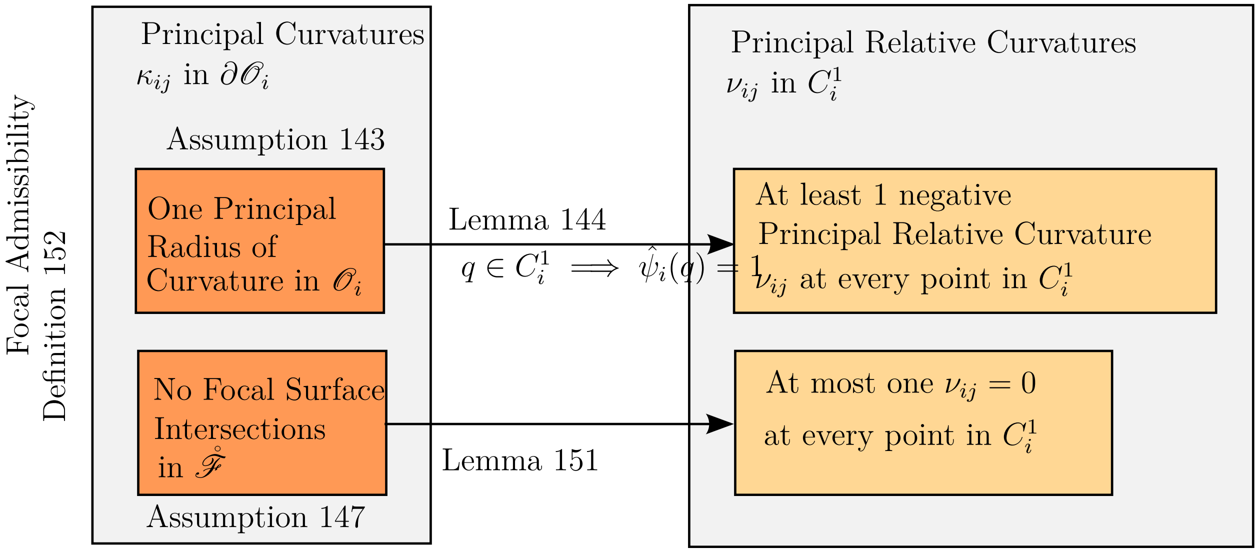

Figure 7.2: The proof is based on two assumptions which restrict the admissible principal curvatures κ_{ij} on the obstacle boundary $\partial\mathscr{O}_i$. These ensure conditions about the principal relative curvatures ν_{ij} on C_i^1.

If $\kappa_{ij}(q) > 0$ (convex) then $R_{ij} > 0$, so the previous is equi-signed with

$$R_i(q, \hat{p}_{ij}) - \frac{1}{2}\left\|\nabla\gamma_d(q)\right\|. \tag{7.4}$$

Since $\gamma_d(q) = \|q - q_d\|^2 \implies \|\nabla\gamma_d(q)\| = 2\|q - q_d\|$, we obtain

$$R_i(q, \hat{p}_{ij}) - \|q - q_d\|. \tag{7.5}$$

By definition of the principal curvature center, it is $|R_{ij}(q)| = \left\|q_{\kappa_{ij}} - q\right\|$. By hypothesis $R_{ij}(q) > 0$, so $R_{ij}(q) = \left\|q_{\kappa_{ij}} - q\right\|$. The three cases can now be derived

- If $\|q_d - q\| < \left\|q_{\kappa_{ij}} - q\right\| = R_i(q, \hat{p}_{ij})$, then $R_i(q, \hat{p}_{ij}) - \|q - q_d\| > 0$, so $\nu_{ij}(q) > 0$.
- If $\|q_d - q\| = \left\|q_{\kappa_{ij}} - q\right\| = R_i(q, \hat{p}_{ij})$, then $R_i(q, \hat{p}_{ij}) - \|q - q_d\| = 0$, so $\nu_{ij}(q) = 0$.
- If $\|q_d - q\| > \left\|q_{\kappa_{ij}} - q\right\| = R_i(q, \hat{p}_{ij})$, then $R_i(q, \hat{p}_{ij}) - \|q - q_d\| < 0$, so $\nu_{ij}(q) < 0$.

The converse statements are also true, due to equi-signedness.

If $\kappa_{ij}(q) \leq 0$, then $-\frac{1}{R_i(q,\hat{p}_{ij})} > 0$, so

$$\frac{2}{\|\nabla\gamma_d(q)\|} - \frac{1}{R_i(q, \hat{p}_{ij})} > 0 \tag{7.6}$$

which implies that $\nu_{ij}(q) > 0$. □

Corollary 141. *Define $\gamma_d(q) \triangleq \|q - q_d\|^2$. Select any first-order contact point q.*

A principal relative curvature function value $\nu_{ij}(q)$ is zero if and only if q is a second-order contact point.

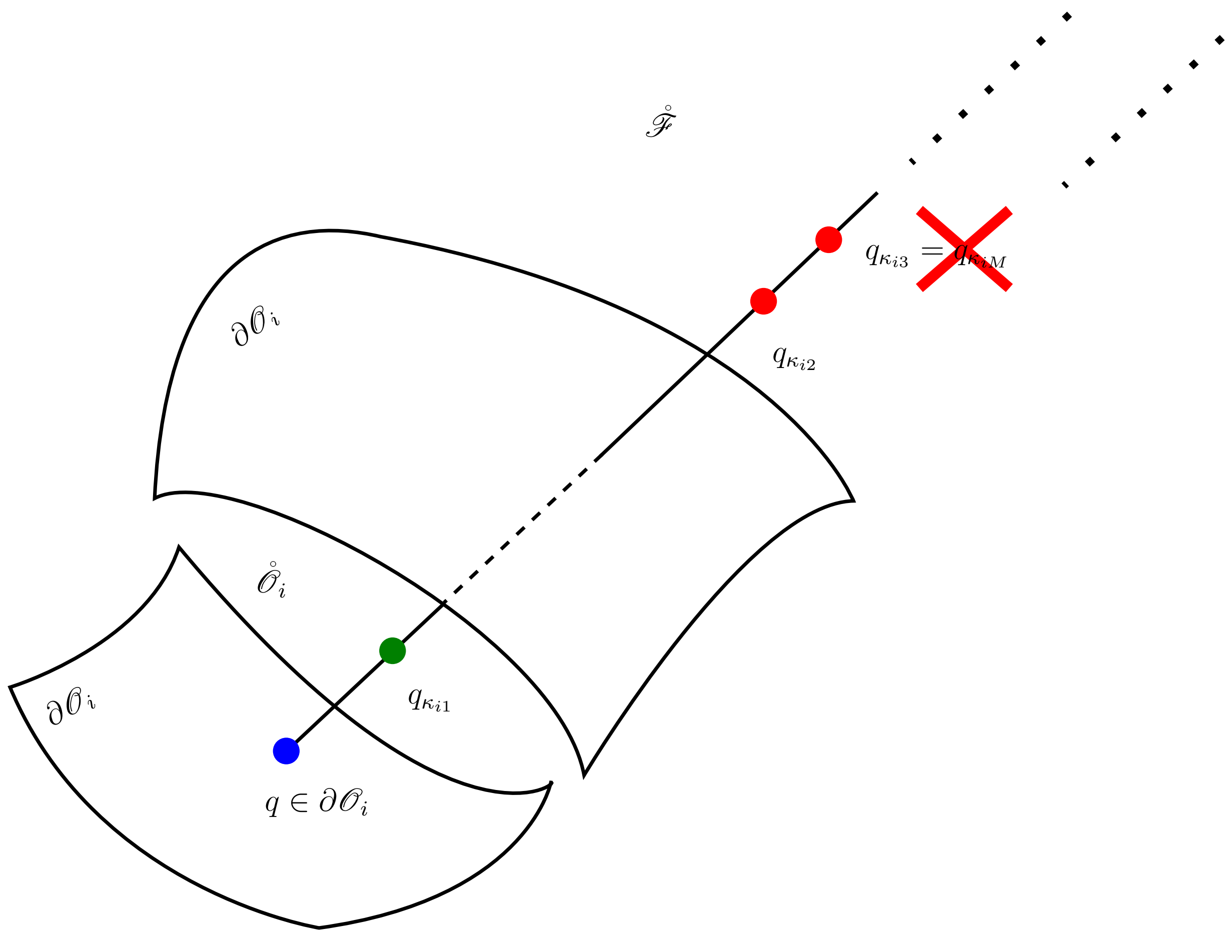

Figure 7.3: The two assumptions about the obstacle's focal surface sheets. Assumption 143 ensures that at each boundary point $q \in \partial\mathscr{O}_i$, at least one center of principal curvature $q_{\kappa_{i1}}$ is within $\mathscr{O}_i$. This ensures navigability of the geometry by use of Koditschek-Rimon functions (i.e., that there exists a KRNF). Assumption 147 ensures that at each boundary point $q \in \partial\mathscr{O}_i$, no two principal curvature centers coincide outside $\mathscr{O}_i$, as it happens with $q_{\kappa_{i3}}$ and $q_{\kappa_{i4}}$.

Proof. Assume that $\nu_{ij}(q) = 0$ for some $j \in \mathbb{N}^*_{\leq n-1}$. Then, there exists some $j \in \mathbb{N}^*_{\leq n-1}$ such that the principal curvature $\kappa_{ij}(q) > 0$ (convex). We will prove this by contradiction. Suppose that $\kappa_{ij}(q) \leq 0$ for all $j \in \mathbb{N}^*_{\leq n-1}$. Then, by Definition 173 point q is concave. By Theorem 140 this implies that $\nu_{ij}(q) > 0$ for all $j \in \mathbb{N}^*_{\leq n-1}$. This contradicts the hypothesis $\nu_{ij}(q) = 0$ for some j.

For this j, it is $\kappa_{ij}(q) > 0$ and $\nu_{ij}(q) = 0$. Therefore Theorem 140 implies that $\left\|q_{\kappa_{ij}} - q\right\| = \|q - q_d\|$. Since the definition of principal curvature implies that $|\kappa_{ij}(q)| = \frac{1}{\left\|q_{\kappa_{ij}} - q\right\|}$ and by hypothesis $\kappa_{ij}(q) > 0$, it follows that $\kappa_{ij}(q) = \frac{1}{\left\|q_{\kappa_{ij}} - q\right\|}$.

By hypothesis q is in C_i^1. We have shown that $\kappa_{ij}(q) = \frac{1}{\left\|q_{\kappa_{ij}} - q\right\|}$. So by Definition 81 and Proposition 86 it follows that point q is a second-order contact-point.

Now assume that q is a second-order contact point. Then, by Definition 81 and Proposition 86 it is $R_{ij}(q) = \|q - q_d\|$ for some j. This implies that $\kappa_{ij}(q) > 0$. By Theorem 140 it follows that $\nu_{ij}(q) = 0$. □

Corollary 142. *Define $\gamma_d(q) \triangleq \|q - q_d\|^2$. Assume that $C_i^2 = \emptyset$.*

Then, at every first-order contact point $q \in C_i^1$, all the principal relative curvature functions are nonzero.

Proof. By Corollary 141 all functions ν_{ij} are zero at any first-order contact point if and only if it is a second-order contact point. By hypothesis no second-order contact points exist on $\partial\mathscr{O}_i$, so all ν_{ij} are nonzero on it. □

7.1.1 Navigability

The term "radius of principal curvature" in the next statement refers to the line segment connecting the center of principal curvature with the surface point to which it corresponds.

Assumption 143. *Assume that at every point $q \in \partial\mathscr{O}_i$, at least one principal radius of curvature (associated with q) is within the obstacle set $\mathscr{O}_i$.*

In other words, at every point $q \in \partial\mathscr{O}_i$, at least one focal surface sheet is within $\mathscr{O}_i$, as illustrated in Fig. 7.3. Note that we do not require that "at least one focal surface sheet be within the obstacle", because this renders it unclear whether the "same" focal surface sheet is meant or not. Which focal surface sheet remains within the obstacle is allowed to change, so the selected phrasing avoids over-constraining the geometry.

Lemma 144. *Assume that at every point $q \in \partial\mathscr{O}_i$, at least one radius of principal curvature (associated with q) is within the obstacle set $\mathscr{O}_i$. Select arbitrary destination $q_d \in \mathring{\mathscr{F}}$ and define $\gamma_d(q) \triangleq \|q - q_d\|^2$.*

Then, at every point $q \in C_i^1$ at least one principal relative curvature function value $\nu_{ij}(q)$ is negative.

Proof. Choose any first-order contact point $q \in C_i^1$. By hypothesis there exists some (at least one) $j \in \mathbb{N}^*_{\leq n-1}$ such that the line segment connecting $q_{\kappa_{ij}}$ to q is a subset of $\mathscr{O}_i$. This implies that $\kappa_{ij}(q) > 0$. Suppose the opposite, that $\kappa_{ij}(q) \leq 0$. By the definition of the principal curvature center

$$q_{\kappa_{ij}} - q = -\frac{1}{\kappa_{ij}(q)} \frac{\nabla\beta_i(q)}{\|\nabla\beta_i(q)\|} = -\frac{1}{\kappa_{ij}(q)\,\|\nabla\beta_i(q)\|} \nabla\beta_i(q). \tag{7.7}$$

Since $\kappa_{ij}(q) \leq 0$, it is $-\frac{1}{\kappa_{ij}(q)\|\nabla\beta_i(q)\|} > 0$ (by Assumption 1 β_i is regular on $\partial\mathscr{O}_i$, which ensures $\|\nabla\beta_i(q)\| > 0$), so

$$(q_{\kappa_{ij}} - q) \cdot \nabla\beta_i(q) > 0. \tag{7.8}$$

This implies that there exists some point $q' = q + a(q_{\kappa_{ij}} - q)$ where $a \in (0,1)$, which is on the line segment connecting q to $q_{\kappa_{ij}}$, that is in $\mathring{\mathscr{F}}$. This contradicts the hypothesis. So $\kappa_{ij}(q) > 0$.

By Assumption 32 $q_d \in \mathring{\mathscr{F}}$, so q_d cannot be on this line segment. Point q is a first-order contact point, so q_d is on the half-line passing through q with direction $\nabla\beta_i(q)$. So it is on this half-line, but beyond point $q_{\kappa_{ij}}$. Then, it is $\|q_d - q\| > \left\|q_{\kappa_{ij}} - q\right\|$. By Theorem 140, $\kappa_{ij}(q) > 0$ and $\|q_d - q\| > \left\|q_{\kappa_{ij}} - q\right\|$ imply that $\nu_{ij} < 0$. □

Note that this assumption corresponds to the "at every obstacle boundary point, at least one sufficient principal curvature exists", but is less strict than it, because "sufficient principal relative curvature" refers to inclusion within $\mathscr{O}_i$ of the whole curvature sphere associated with it, not just of the center of principal curvature.

7.1.2 Limited degeneracy

Each obstacle boundary is a compact regular hypersurface without boundary. Therefore it is a C^2-differentiable manifold $M^{n-1} = \partial\mathscr{O}_i$.

Definition 145 (End-point map [25, 26])**.** *Assume M^{n-1} is a differentiable manifold. Furthermore, assume $f : M \to E^n$ is an embedding of M^{n-1} as a regular submanifold of n-dimensional Euclidean space. Let $N_{f(q)}f(M)$ be the normal space of $f(M)$ at $f(q)$ and $Nf(M)$ be the normal bundle of $f(M)$. The* end-point map *$\eta : M \times \mathbb{R}^n \to E^n$ is defined as*

$$\eta(q, v) \triangleq f(q) + v \tag{7.9}$$

for $v \in N_{f(q)}f(M)$.

The end-point map takes a point q on the manifold and a vector v in the normal space associated with that point to $\eta(q, v)$, which results after the translation of $f(q)$ by v.

Definition 146 (Focal Set)**.** *The critical set of η is the set of* focal points *of the embedding.*

The focal set can be obtained using the map $\eta(q, \kappa_j(q)^{-1}\hat{n})$ where κ_j is the jth principal curvature of $M(q)$ at q.

Assumption 147. *Assume that the focal surface sheets do not intersect outside obstacle $\mathscr{O}_i$.*

Therefore the focal surface sheets do not intersect in $\mathring{\mathscr{F}}$, an example is shown in Fig. 7.3.

Proposition 148. *Assume that the focal surface sheets do not intersect in $\mathring{\mathscr{F}}$.*

Then, any destination $q_d \in \mathring{\mathscr{F}}$ can be on at most one focal sheet.

Proposition 149. *Assume that the destination $q_d \in \mathring{\mathscr{F}}$ is on at most one focal sheet.*

Then, on every first-order contact point $q \in C_i^1$, at most one principal relative curvature function value $\nu_{ij}(q)$ can be zero.

Proof. Consider the centers of principal curvature $q_{\kappa_{ij}}$ associated with q. By hypothesis q_d can be equal to at most one $q_{\kappa_{ij}}$. Then, by Theorem 140 at most one principal relative curvature function value $\nu_{ij}(q)$ can be zero. All the other principal relative curvature function values at q are nonzero. □

Remark 150. *Note that if we consider also other boundary points q' of $\partial\mathscr{O}_i$, it may be $\nu_{ij}(q') = 0$. This can happen because q_d is on the boundary of the jth principal curvature sphere associated with q'. This shows why the new proof is stronger than the older.*

The following Lemma relates the number of principal relative curvatures which can be zero at any first-order contact point to the condition of Assumption 147.

Lemma 151. *Assume that the focal surface sheets do not intersect outside obstacle $\mathscr{O}_i$.*

Then, on every first-order contact point $q \in C_i^1$, at most one principal relative curvature function value $\nu_{ij}(q)$ can be zero.

Proof. The claim follows from the hypothesis by combining Proposition 148 and Proposition 149. □

Definition 152. *An obstacle β_i whose boundary $\partial\mathscr{O}_i$ satisfies Assumption 143 and Assumption 147 is called* focally admissible.

7.2 Relative Curvature in $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$

Proposition 153. *Select arbitrary $q_d \in \mathring{\mathscr{F}}$. Assume that at each first-order contact point in C_i^1, at least m principal relative curvatures are negative and at most N_0 are zero.*

Then, there exist an $L > 0$ and an $\bar{\varepsilon}_i > 0$ such that all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ have the following property. For each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ there exist two numbers $N_n \geq m$ and N_p with $N_n + N_p \geq (n-1) - N_0$ such that N_n principal relative curvature functions have values less than or equal to $-L$

$$\nu_{ij_1}(q_c) \ \leq \nu_{ij_2}(q_c) \ \leq \ldots \ \leq \nu_{ij_{N_n}}(q_c) \ \leq -L < 0, \tag{7.10}$$

and N_p principal relative curvature functions have values larger than or equal to L

$$0 < L \ \leq \nu_{ij_{N_n+1}}(q_c) \ \leq \nu_{ij_{N_n+2}}(q_c) \ \leq \ldots \ \leq \nu_{ij_{N_n+N_p}}(q_c). \tag{7.11}$$

Proof. By Theorem 113 there exists a $\delta > 0$ and an $L > 0$ with the following properties. For each point $q \in T_\delta\left(C_i^1\right)$ there exist two numbers $N_n \geq m$ and N_p with $N_n + N_p \geq (n-1) - N_0$ such that N_n principal relative curvature functions have values less than or equal to $-L$

$$\nu_{ij_1}(q) \ \leq \nu_{ij_2}(q) \ \leq \ldots \ \leq \nu_{ij_{N_n}}(q) \ \leq -L < 0, \tag{7.12}$$

and N_p principal relative curvature functions have values larger than or equal to L

$$0 < L \ \leq \nu_{ij_{N_n+1}}(q) \ \leq \nu_{ij_{N_n+2}}(q) \ \leq \ldots \ \leq \nu_{ij_{N_n+N_p}}(q). \tag{7.13}$$

For the above δ by Theorem 97 there exists an $\bar{\varepsilon}_i > 0$ such that for all $\varepsilon_i < \bar{\varepsilon}_i$ it is

$$\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i) \subseteq T_\delta\left(C_i^1\right). \tag{7.14}$$

As a result, for each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ there exist two numbers $N_n \geq m$ and N_p with $N_n + N_p \geq (n-1) - N_0$ such that N_n principal relative curvature functions have values $\nu_{ij}(q_c) \leq -L$ and N_p principal relative curvature functions have values $\nu_{ij}(q_c) \geq L$. □

7.3 Hessian Eigenvalues depend on Principal Relative Curvatures in $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$

Theorem 154. *Select arbitrary $q_d \in \mathring{\mathscr{F}}$. Assume that at each first-order contact point in C_i^1, at least m principal relative curvatures are negative and at most N_0 are zero.*

Then, there exists an $\bar{\varepsilon}_i > 0$ such that all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ have the following property. For each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ there exist two numbers $N_n \geq m$ and N_p with $N_n + N_p \geq n - N_0$ such that the Hessian matrix $D^2\hat{\varphi}(q_c)$ has at least N_n negative eigenvalues and at least N_p positive eigenvalues.

Proof. By the hypothesis and Proposition 153 there exist an $L > 0$ and an $\bar{\varepsilon}_{i1} > 0$ such that all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_{i1}$ have the following property. For each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ there exist two numbers $N_n \geq m$ and N_p' with

$$N_n + N_p' \geq (n-1) - N_0 \tag{7.15}$$

such that N_n principal relative curvature functions have values less than or equal to $-L$

$$\nu_{ij_1}(q_c) \ \leq \nu_{ij_2}(q_c) \ \leq \dots \ \leq \nu_{ij_{N_n}}(q_c) \ \leq -L < 0,$$

and N_p' principal relative curvature functions have values larger than or equal to L

$$0 < L \ \leq \nu_{ij_{N_n+1}}(q_c) \ \leq \nu_{ij_{N_n+2}}(q_c) \ \leq \dots \ \leq \nu_{ij_{N_n+N_p'}}(q_c).$$

Define $L_1 \triangleq \frac{L}{2} > 0$. By Definition 114 the set $P_i^-(q_c, -L_1)$ includes the principal directions

$$\hat{p}_{ij_1}, \hat{p}_{ij_2}, \dots, \hat{p}_{ij_{N_n}}, \tag{7.16}$$

because

$$\nu_{ij} \leq -L < -\frac{L}{2} = -L_1 \implies \nu_{ij} < -L_1,$$

for all the principal relative curvatures associated with them. So $\mathscr{P}_i^-(q_c, -L_1)$ has dimension at least N_n. It can be similarly shown that $\mathscr{P}_i^+(q_c, L_1)$ has dimension at least N_p'.

By Assumption 1 the obstacles are disjoint, so there exists an $\bar{\varepsilon}_{i2} > 0$ such that

$$\bar{\beta}_i(q) \geq l_1 \triangleq \min_{q \in \overline{\mathscr{B}_i(\bar{\varepsilon}_{i2})}} \{\bar{\beta}_i(q)\} > 0 \tag{7.17}$$

for all $q \in \mathscr{B}_i(\varepsilon_i)$ where $0 < \varepsilon_i < \bar{\varepsilon}_{i2}$. By Proposition 24 there exists an $\bar{\varepsilon}_{i3} > 0$ such that $\nabla\beta_i(q) \neq 0$ for all $q \in \overline{\mathscr{B}_i(\varepsilon_i)}$ where $0 < \varepsilon_i < \bar{\varepsilon}_{i3}$. Then, it is

$$\|\nabla\beta_i(q)\| \geq l_2 \triangleq \min_{q \in \overline{\mathscr{B}_i(\bar{\varepsilon}_{i3})}} \{\|\nabla\beta_i(q)\|\} > 0 \tag{7.18}$$

for all $q \in \mathscr{B}_i(\varepsilon_i)$ where $0 < \varepsilon_i < \bar{\varepsilon}_{i3}$. Take $\bar{\varepsilon}_{i4} \triangleq \min\{\bar{\varepsilon}_{i3}, \bar{\varepsilon}_{i4}\}$. Then ($k \geq 2 \implies 1 - \frac{1}{k} \geq \frac{1}{2}$)

$$\xi_i(q) \geq L_2 \triangleq \frac{1}{2} l_1 l_2^2 > 0 \tag{7.19}$$

for all $q \in \mathscr{B}_i(\varepsilon_i)$ where $0 < \varepsilon_i < \bar{\varepsilon}_{i4}$.

For the given $L_1 > 0$ and $L_2 > 0$ and the properties of $\bar{\varepsilon}_{i4}$, it follows from Theorem 138 that there exists an $\bar{\varepsilon}_{i5} > 0$ such that if $0 < \varepsilon_i < \bar{\varepsilon}_{i5}$, then the following hold. At each critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ the quadratic form associated with the Hessian $D^2\hat{\varphi}(q_c)$ is

- Negative-definite on $\mathscr{R}_i^-(q_c, -L_1)$, and
- Positive-definite on $\bigcup_{\hat{t}_i \in R_i^+(q_c, L_1)} \operatorname{span}\left\{\hat{t}_i, \hat{r}_i\right\}$.

By Remark 139 it follows that the Hessian $D^2\hat{\varphi}\,(q_c)$ is positive-definite on

$$\bigcup_{t_i \in \mathscr{R}_i^+(q_c, L_1)} \operatorname{span}\left\{t_i, \hat{r}_i\right\}. \tag{7.20}$$

Define

$$\bar{\varepsilon}_i \triangleq \min\{\bar{\varepsilon}_{i1}, \bar{\varepsilon}_{i5}\} > 0. \tag{7.21}$$

Let $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ for any $\varepsilon_i > 0$ such that $\varepsilon_i < \bar{\varepsilon}_i$.

By Lemma 122 it is

$$\mathscr{P}_i^-(q, -L_1) \subseteq \mathscr{R}_i^-(q, -L_1). \tag{7.22}$$

Therefore the Hessian $D^2\hat{\varphi}\,(q_c)$ is negative-definite on $\mathscr{P}_i^-(q, -L_1)$. It has been previously shown that the linear subspace $\mathscr{P}_i^-(q, -L_1)$ has dimension at least N_n. Therefore the Hessian $D^2\hat{\varphi}\,(q_c)$ is negative-definite on a subspace of dimension (at least) N_n.

By Lemma 125 it is

$$\operatorname{span}\left\{\mathscr{P}_i^+(q, L_1), \hat{r}_i\right\} \subseteq \bigcup_{t_i \in \mathscr{R}_i^+(q, L_1)} \operatorname{span}\left\{t_i, \hat{r}_i\right\}. \tag{7.23}$$

Therefore the Hessian $D^2\hat{\varphi}\,(q_c)$ is positive-definite on $\operatorname{span}\left\{\mathscr{P}_i^+(q, L_1), \hat{r}_i\right\}$. It has been previously shown that the linear subspace $\mathscr{P}_i^+(q, L_1)$ has dimension at least N_n. Since $\{\hat{r}_i\} \cap T_{q_c}B_i = \emptyset$ and $\mathscr{P}_i^+(q, L_1) \subseteq T_{q_c}B_i$ it follows that

$$\dim\left\{\operatorname{span}\left\{\mathscr{P}_i^+(q, L_1), \hat{r}_i\right\}\right\} = \dim\left\{\mathscr{P}_i^+(q, L_1)\right\} + \dim\left\{\{\hat{r}_i\}\right\} \geq N_p' + 1 \tag{7.24}$$

Therefore the Hessian $D^2\hat{\varphi}\,(q_c)$ is positive-definite on a subspace of dimension (at least) $N_p' + 1$.

By Proposition 167 it follows that the Hessian $D^2\hat{\varphi}\,(q_c)$ has at least N_n negative eigenvalues and at least $N_p' + 1$ positive eigenvalues. If we define $N_p \triangleq N_p' + 1$, then eq. (7.15) becomes

$$N_n + Np - 1 \geq (n-1) - N_0 \iff N_n + N_p \geq n - N_0. \tag{7.25}$$

$\square$

7.4 No equilibria away from obstacles

7.4.1 Tuning for quadratic γ_d

Propositions 2.7, 3.2 and 3.3 [1] are independent of β_i zero level set shape, i.e., obstacle type. So they are valid here and we can use them.

By Prop.2.7 [1] functions φ and $\hat{\varphi} \triangleq \frac{\gamma_d^k}{\beta}$ have the same critical points in $\mathring{\mathscr{F}}$. No critical points exist on $\partial\mathscr{F}$ (Prop.3.3 [1]) and q_d is non-degenerate (Prop.3.2 [1]). Therefore we can study the critical set $\mathscr{C}_{\hat{\varphi}}$ of $\hat{\varphi}$, instead of φ.

For a focally admissible obstacle $m = 1$ and $N_0 = 1$ in Theorem 154. So all critical points "near" obstacles, in $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$ have at least $N_n \geq 1$ negative eigenvalues and at most $n - N_n + N_p \leq 1$ zero eigenvalues of $D^2\hat{\varphi}\,(q_c)$. Also, Theorem 154 provides us with some upper bounds $\bar{\varepsilon}_0, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_M > 0$. Observe that these $\bar{\varepsilon}_i$ are independent of k (it has not been involved in deriving them, only the obstacle geometry has been involved).

Define $\gamma_d(q) = \|q - q_d\|^2$ Finally, by Proposition 3.4 [1] if we choose[1]

$$\begin{aligned} k &> \hat{k}_{\min}\left(\bar{\varepsilon}_0, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_M\right) \\ &\triangleq \frac{1}{2}\max_{\mathscr{W}}\{\sqrt{\gamma_d}\}\sum_{i\in I_0}\frac{\max_{\mathscr{W}}\{\|\nabla\beta_i\|\}}{\bar{\varepsilon}_i}, \end{aligned} \tag{7.26}$$

then no critical points (apart from q_d) exist outside any $\mathscr{B}_i\left(\bar{\varepsilon}_i\right)$. So that there are no critical points in $\mathscr{F}_a$, all critical points (excluding q_d) are in $\mathscr{F}_n$, i.e., "near" the obstacles. So $\mathscr{C}_{\hat{\varphi}} = \bigcup_{i\in I_0}\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. This implies that *any* critical point is within some $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. Therefore, all critical points of $\hat{\varphi}$ have at least one negative and at most one zero eigenvalues. So the Koditschek-Rimon function φ is an Extended Navigation Function.

Moreover, since the zero eigenvalue results only for a measure zero set of destinations q_d (the intersection of the focal surfaces with the free space interior), for almost all destinations the Koditschek-Rimon function is a Navigation Function according to the original definition, i.e., Morse.

7.4.2 Tuning parameter lower bound for any γ_d

In the case of a γ_d which is not $\|q - q_d\|^2$, but another function has been selected instead, (7.26) is different. For any γ_d, at a critical point the following holds

$$k\beta\nabla\gamma_d = \gamma_d\nabla\beta \iff k\beta\,\|\nabla\gamma_d\| = \gamma_d\,\left\|\nabla\bar{\beta}_i\right\| \tag{7.27}$$

A sufficient condition for the above equation not to hold is

$$\frac{\gamma_d\,\|\nabla\beta_i\|}{\beta\,\|\nabla\gamma_d\|} < k \tag{7.28}$$

An upper bound on the left side is given by

$$\frac{\gamma_d}{\|\nabla\gamma_d\|}\frac{\|\nabla\beta_i\|}{\beta} \leq \frac{\gamma_d}{\|\nabla\gamma_d\|}\sum_{i\in I_0}\frac{\|\nabla\beta_i\|}{\beta_i} \leq \max_{\mathscr{W}}\left\{\frac{\gamma_d}{\|\nabla\gamma_d\|}\right\}\sum_{i\in I_0}\frac{\max\{\|\nabla\beta_i\|\}}{\varepsilon_i} \tag{7.29}$$

[1]This Proposition "clears" the set away from obstacles from critical points.

since

$$\varepsilon_i \leq \beta_i \iff \frac{1}{\beta_i} \leq \frac{1}{\varepsilon_i}, \quad \forall i \in I_0 \tag{7.30}$$

for all $q \in \mathscr{F}_a$, i.e., “away” from obstacles.

7.5 KR functions are Morse for generic destinations

7.5.1 Non-degeneracy of Koditschek-Rimon functions

The following proves that the destination q_d should be on some focal surface for second-order contact points to exist. Otherwise, the set of second-order contact points is empty.

The set of first-order contact points is not empty. It can be proved that there always exists at least one first-order contact point. The proof is similar to the proof that the gauss map of any closed regular surface is surjective.

In what follows, we refer to the “Koditschek-Rimon function” and avoid using the term “Navigation Function”. The reason for this is that we do not consider the navigational properties of the KR function, only its degeneracy properties.

Proposition 155. *Assume that the destination q_d is not on a focal surface of $\partial\mathscr{O}_i$. Define $\gamma_d(q) \triangleq \|q - q_d\|^2$.*

Then, $C_i^2(\partial\mathscr{O}_i, \gamma_d) = \emptyset$.

In other words, there is no second-order contact point of γ_d with $\partial\mathscr{O}_i$.

Proof. Assume the contrary, that some point $q_0 \in \partial\mathscr{O}_i$ is a second-order contact point. By Corollary 82 point q_0 is also a first-order contact point. By hypothesis, $q_d \neq q_{\kappa_{ij}}$ for all the principal curvature centers. This is equivalent to $\|q - q_d\| \neq R_{ij}(q)$ for all $j \in \mathbb{N}^*_{\leq n-1}$. As a result, by Definition 81 the destination is not a second-order contact point. This contradicts the assumption. Therefore, no second-order contact point exists on the surface $\partial\mathscr{O}_i$. □

It readily follows that destinations on focal surfaces can cause degeneracy and no others.

Theorem 156. *If the destination is not on a focal set, then the Koditschek-Rimon function is a Morse function.*

Proof. If the destination is not on a focal surface, then by Proposition 155 there exist no second-order contact points. So by Corollary 142 all first-order contact points have sign-definite principal relative curvatures ν_{ij}. So at every point $q \in C_i^1$ at most $N_0 = 0$ principal relative curvature functions $\nu_{ij}(q)$ are zero.

Therefore, by **??** all critical points of $\hat{\varphi}$ have nonzero eigenvalues, provided $k > N(\bar{\varepsilon}_{I_0})$. This means that all critical points of φ will be non-degenerate. So the Koditschek-Rimon function is then non-degenerate. □

As a result, surfaces which are non-focal and their focal set is in the interior component of the ambient space, are focally admissible for navigation, i.e., there exist KRNF for them. Examples are channel surfaces, which have at least one focal sheet in their interior, see pp. 310–314 [27].

7.5.2 Degeneracy of Koditschek-Rimon functions

Consider what happens if the hypothesis of Theorem 156 does not hold. Assume the destination is on some focal surface. In that case, there exists some second-order contact point q_0. At least one ν_{ij} is degenerate at q_0. Therefore, the Koditschek-Rimon function may be degenerate.

Proposition 157. *The focal set of obstacle boundaries is a measure zero subset of E^n.*

Proof. By hypothesis about the obstacles, there is a finite total number of obstacles $M \in \mathbb{N}$. Therefore, it suffices to prove that the focal set of a single obstacle is of measure zero. Consider the focal set of a single obstacle. This comprises of $n-1$ sheets of focal surfaces. Each of them corresponds to one principal curvature of the obstacle boundary surface at that point of the surface. Therefore, it suffices to prove that each focal surface is of measure zero. The map $g : \partial\mathscr{O}_i \to E^n \cap \{\pm\infty\}^n$ defined as

$$g(q) = q + \frac{\nabla\beta_i}{\|\nabla\beta_i\|}\frac{\|\nabla\beta_i\|^2}{\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i} = q + \frac{\nabla\beta_i \,\|\nabla\beta_i\|}{\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i} \tag{7.31}$$

maps each obstacle boundary surface to its focal surface. The image of g can be at most $(n-1)$-dimensional. Therefore, each focal surface is of measure zero. This proves the claim. □

The complement in $\mathscr{F}$ of the focal set is a dense subset of $\mathscr{F}$.

Corollary 158. *If the destination is in a dense subset of $\mathscr{F}$, then the Koditschek-Rimon function is Morse.*

Proof. Assume the destination is not on a focal set. Then, it is on the complement in $\mathscr{F}$ of the focal set. According to Proposition 157, the focal set is of measure zero and its complement in $\mathscr{F}$ is dense. Therefore, the set of destinations not on a focal surface is a dense subset of $\mathscr{F}$.

By Theorem 156 for each such destination, the Koditschek-Rimon function is non-degenerate. In other words, it is a Morse function. □

Lemma 159. *If all (partially) umbilic points are contained in the interior of obstacles, then the Koditschek-Rimon function is at most singly-degenerate.*

Remark 160. *The previous form of the condition involved curvature spheres. It required checking whether curvature spheres are subsets of the obstacle. This is a problem concerning set membership of an $(n-1)$-dimensional sphere in an n-dimensional set. Since the obstacle is implicitly defined by β_i, it reduces to finding the maximum of β_i over the $(n-1)$-dimensional curvature sphere. So it is an optimization problem over an $(n-1)$-dimensional domain (the curvature sphere). But it should be repeated for all curvature spheres over $\partial\mathscr{O}_i$. These are parameterized by $(n-1)$ parameters. There are $(n-1)$ families of curvature spheres (as many as the principal curvatures). So we need to solve $(n-1)$ optimizations, each over an $(n-1)^2$-dimensional domain (the product of the curvature sphere dimension and the obstacle boundary dimension).*

On the contrary, the new condition concerns only the position of the centers of principal curvature with respect to the obstacle. These are $n-1$ numbers of β_i values at each boundary point. So it is equivalent to solving an optimization problem to find if two curvature centers coincide outside the obstacle over an $(n-1)$-dimensional domain (the obstacle's boundary).

Lemma 161. *A Koditschek-Rimon function is non-degenerate for a dense subset of the parameter space $\mathscr{F}$ of q_d.*

Proof. By Corollary 158 a dense subset of destinations q_d in $\mathscr{F}$ results in Koditschek-Rimon functions which are Morse. $\square$

For the previous follows directly the following.

Theorem 162. *Set Φ_d is transverse to D.*

This tells us that a generic Koditschek-Rimon function is Morse. Note that we refer to a generic selection of destination.

Chapter 8

Converse Theorems

Select an arbitrary (sufficiently smooth) Riemannian manifold with boundary $\mathscr{F}$. Let the set of Koditschek-Rimon functions on $\mathscr{F}$ be Φ. We are interested in the properties of Φ with respect to its parameterizations. Two parameterizations considered in this chapter.

The first is with respect to the tuning parameter k. This requires that some destination $q_d \in \mathscr{F}$ has been selected. Let Φ_k be the set of KRfs on $\mathscr{F}$ for the selected q_d, parameterized by the tuning parameter k.

The second is with respect to the destination position $q_d \in \mathscr{F} \subset E^n$. This investigates how the properties of a KRf depend on q_d. Let $\Phi_d(k > \hat{k}_{\min})$ be the set of tuned KRfs on $\mathscr{F}$, parameterized by the destination configuration $q_d \in \mathring{\mathscr{F}}$.

The set of all degenerate functions is denoted by D.

The following proves that Φ_k is not transverse to the set D.

Lemma 163. *A Koditschek-Rimon function may be degenerate for all values of the tuning parameter k.*

Proof. We prove the claim using a counterexample. The 2-torus provides this counterexample. When placing q_d on the axis of rotational symmetry (axis of revolution), then there are second-order contact points on the torus. In particular, there are second-order contact-points in the free space as well. On principal relative curvature ν_{i1} is zero at them. By symmetry around the axis of revolution, these critical points form a continuum (a circle). A calculation can prove this claim. This non-degenerate critical loop (transversely it is non-degenerate, in particular it is sufficiently curved and for this reason the torus navigable) persists for all values of k. If no other obstacle exists, then the degeneracy carries over directly to the NF. □

The following theorem follows directly from the previous lemma (like a corollary).

Theorem 164. *Set Φ_k is not always transverse to D.*

In other words, for some world and some destination, there may not exist a non-degenerate Koditschek-Rimon function.

An explanatory remark is due at this point. When we brake the symmetry by not placing the destination on the torus axis of symmetry, the first-order contact points move. One of them is closer and the other further away than before. This can be seen by an orthogonal projection of the lines connecting q_d to the first-order contact points. On the contrary, the point which is now further away is closer to the equatorial circle. Therefore, its principal radius of curvature is smaller.

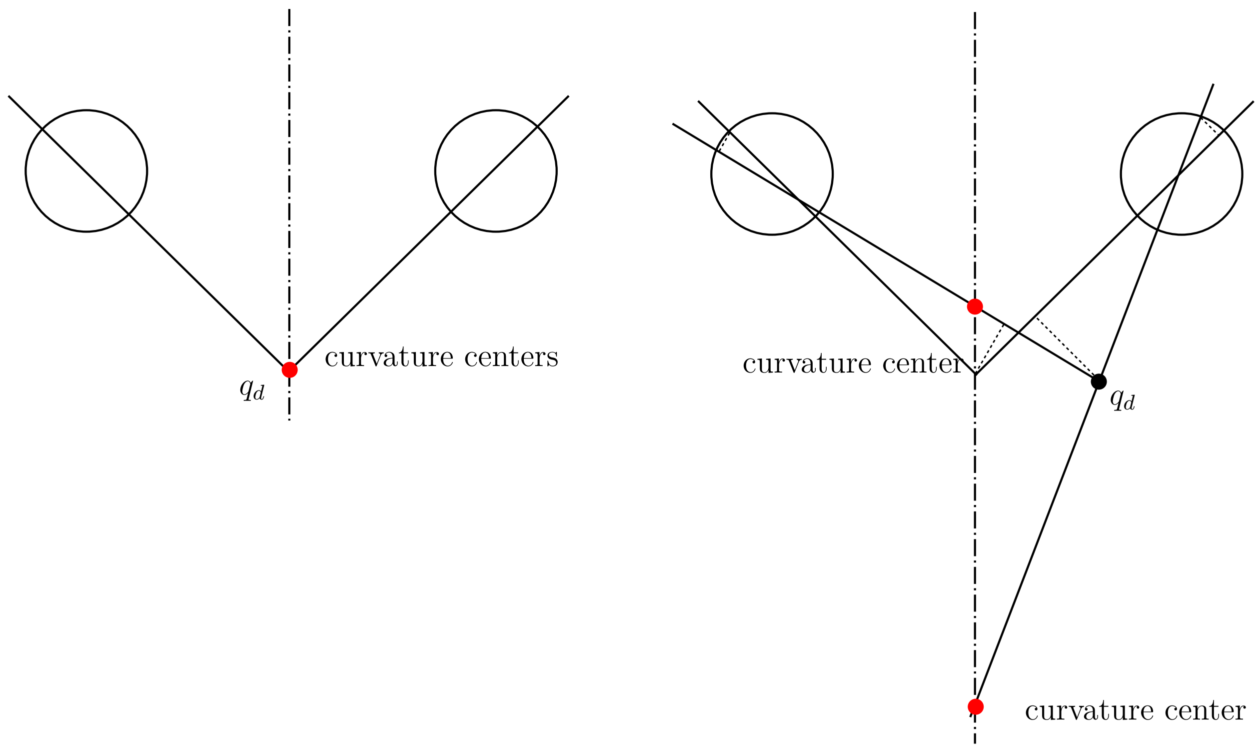

Figure 8.1: A geometric explanation of why symmetry breaking removes the degeneracy in the case of a torus.

This implies that it cannot be a second-order contact point. (it was before, so the principal radius of curvature was equal to the distance from the destination, whereas now the principal radius of curvature is smaller than before and the distance to the destination larger, so they are unequal).

The other point is now closer to the destination, but its principal radius of curvature is larger, because it has moved towards the polar circle of the torus. Therefore, again the distance from the destination is unequal to the principal radius of curvature there. As a result, the first-order contact point is not a second-order contact point.

To support the result more formally, using focal theorems, the focal set of a torus is a circle in its interior and its axis of revolution. The geometric argument to explain how symmetry breaking removes degeneracy is illustrated in Fig. 8.1.

An obstacle boundary point where all principal curvatures are non-positive is here called a *concave* point (completely non-convex).

Theorem 165 (KRf Limitation)**.** *Assume that M is a world with some concave point $q_0 \in \partial\mathscr{O}_i$. Moreover, assume that $N_q\partial\mathscr{O}_i \cap \mathring{\mathscr{F}} \neq \emptyset$.*

Then, there exist some destination q_d, a $\bar{\varepsilon}_i > 0$ and a $k_{\min} \geq 2$, such that all $\varepsilon_i > 0$ with $\varepsilon_i < \bar{\varepsilon}_i$ have the following property. For all $k > k_{\min}$, if a critical point arises in $\mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$, then it is a local minimum of the Koditschek-Rimon function φ.

Proof. We can choose a $q_d \in N_q\partial\mathscr{O}_i \cap \mathring{\mathscr{F}} \neq \emptyset$, so that $q_0 \in C_i^1$. Point $q_0 \in C_i^1$ is concave, so by Theorem 140 it is $\nu_{ij}(q) > 0$ for all $j \in \mathbb{N}^*_{\leq n-1}$.

Suppose that there exists a critical point $q_c \in \mathscr{C}_{\hat{\varphi},i}(\varepsilon_i)$. Then, using Proposition 153, Theorem 138, and Lemma 125 it can be proved that $D^2\hat{\varphi}\,(q_c) < 0$, so q_c is a local minimum. $\square$

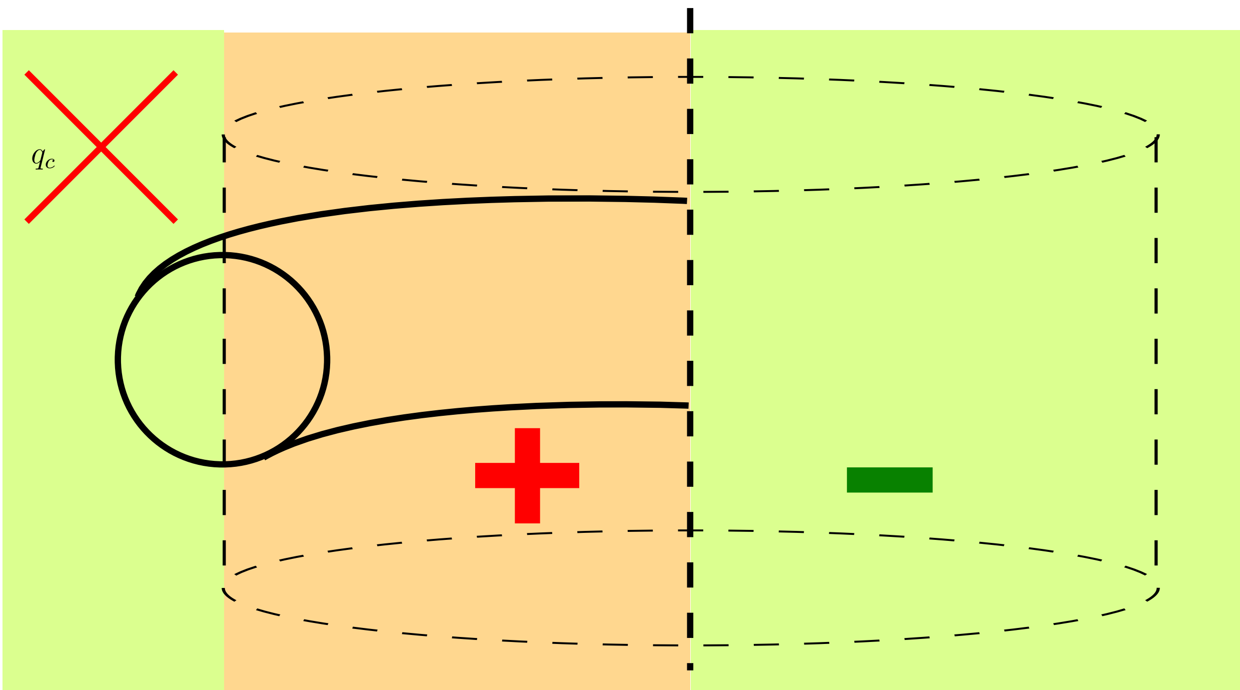

Figure 8.2: The different regions for placing the destination with respect to the axis of revolution of a torus. Consider points on the outer side of the circular section on the left. Placing q_d in the leftmost green region results in no critical point near that side of the torus (because $\nabla\gamma_d$ and $\nabla\beta_i$ have an acute angle there). Placing q_d in the rightmost region results in a negative eigenvalue. Placing it within the red region results in a positive eigenvalue. Finally, placing it on the axis of symmetry results in a possibly zero eigenvalue, because the axis of symmetry is a sheet of the focal surface of the torus.

Appendix A

Auxiliary Mathematical Facts

A.1 Eigenvalues of Quadratic Forms

Proposition 166 (Eigenvalue bounds on Quadratic form in eigen-subspace)**.** *Let $H = H^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ be a real symmetric matrix with real eigenvalues $\lambda_i, i \in \{1, 2, \ldots, n\}$ and associated eigenvectors $\delta_i, i \in \{1, 2, \ldots, n\}$. Consider a subset $\lambda_j, j \in \{1, 2, \ldots, l\}, l \in \mathbb{N}$ of its eigenvalues. Then the associated quadratic form $\hat{u}^{\mathrm{T}} H \hat{u}$ is bounded by the minimum and maximum eigenvalues of the selected subset*

$$\min_{j \in \{1,2,\ldots,l\}} \{\lambda_j\} \leq \hat{u}^{\mathrm{T}} H \hat{u} \leq \max_{j \in \{1,2,\ldots,l\}} \{\lambda_j\} \tag{A.1}$$

on the intersection

$$\hat{u} \in S \cap U \tag{A.2}$$

of the unit sphere

$$S \triangleq \{u \in \mathbb{R}^n | \quad \|u\| = 1\} \tag{A.3}$$

with the linear subspace spanned by those eigenvectors

$$U \triangleq \left\{u \in \mathbb{R}^n | \quad u \in \operatorname{span}\left\{\{\delta_j\}_{j \in \{1,2,\ldots,l\}}\right\}\right\} \tag{A.4}$$

Proof. Without loss of generality assume the eigenvalues λ_j are numbered in increasing order as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l \tag{A.5}$$

For each unit vector $\hat{u} \in \mathbb{R}^n$, $\|\hat{u}\| = 1$ in the linear span

$$\begin{aligned} \hat{u} \in \operatorname{span}\{\delta_1, \delta_2, \ldots, \delta_l\} \implies \\ \exists a_j \in \mathbb{R}, j \in \{1, 2, \ldots, l\} : \quad \hat{u} = \sum_{j=1}^{l} a_j \delta_j \end{aligned} \tag{A.6}$$

The quadratic form associated with H for $\hat{u}$ is

$$\begin{aligned}
\hat{u}^{\mathrm{T}} H \hat{u} &= \left(\sum_{j=1}^{l} a_j \delta_j\right)^{\mathrm{T}} H \left(\sum_{j=1}^{l} a_j \delta_j\right) = \left(\sum_{j=1}^{l} a_j \delta_j^{\mathrm{T}}\right) \left(\sum_{j=1}^{l} a_j H \delta_j\right) \\
&= \sum_{j=1}^{l} \left(a_j \delta_j^{\mathrm{T}} \sum_{p=1}^{l} (a_p \lambda_p \delta_p)\right) = \sum_{j=1}^{l} \sum_{p=1}^{l} \left(a_j a_p \lambda_p \delta_j^{\mathrm{T}} \delta_p\right) \overset{\delta_j^{\mathrm{T}}\delta_p = 0, \forall j \neq p}{=} \\
&= \sum_{j=1}^{l} \left(a_j^2 \lambda_j \delta_j^{\mathrm{T}} \delta_j\right) = \sum_{j=1}^{l} \left(a_j^2 \lambda_j \left\|\delta_j\right\|^2\right) \overset{\|\delta_j\|=1}{=} \\
&= \sum_{j=1}^{l} (a_j^2 \lambda_j)
\end{aligned} \tag{A.7}$$

since matrix H is symmetric so that its eigen-system is orthogonal, hence the zero inner products $\delta_j^{\mathrm{T}} \delta_p = 0, \forall j \neq p$. Taking into account that

$$\left.\begin{array}{c} \|\hat{u}\| = 1 \implies \hat{u}^{\mathrm{T}}\hat{u} = 1 \\ \hat{u} = \sum_{j=1}^{l} a_j \delta_j \end{array}\right\} \implies \left(\sum_{j=1}^{l} a_j \delta_j\right)^{\mathrm{T}} \left(\sum_{j=1}^{l} a_j \delta_j\right) = 1 \implies \sum_{j=1}^{l} a_j^2 = 1 \tag{A.8}$$

it follows that for all $a_j \neq 0 \implies 0 < a_j^2$ it is

$$\begin{gathered}
\lambda_1 \leq \lambda_j \leq \lambda_l, \quad \forall j \in \{1, 2, \ldots, l\} : a_j \neq 0 \overset{0 < a_j^2}{\Longrightarrow} \\
a_j^2 \lambda_1 \leq a_j^2 \lambda_j \leq a_j^2 \lambda_l, \quad \forall j \in \{1, 2, \ldots, l\} : a_j \neq 0
\end{gathered} \tag{A.9}$$

and for all $a_j = 0$ it is

$$0 = a_j^2 \lambda_1 = a_j^2 \lambda_2 = \cdots = a_j^2 \lambda_l, \quad \forall j \in \{1, 2, \ldots, l\} : a_j = 0 \tag{A.10}$$

therefore

$$\begin{gathered}
\sum_{j=1}^{l} (a_j^2 \lambda_1) \leq \sum_{j=1}^{l} (a_j^2 \lambda_j) \leq \sum_{j=1}^{l} (a_j^2 \lambda_l) \implies \\
\lambda_1 \sum_{j=1}^{l} a_j^2 \leq \sum_{j=1}^{l} (a_j^2 \lambda_j) \leq \lambda_l \sum_{j=1}^{l} a_j^2 \overset{\sum_{j=1}^{l} a_j^2 = 1}{\Longrightarrow} \\
\lambda_1 \leq \sum_{j=1}^{l} (a_j^2 \lambda_j) \leq \lambda_l
\end{gathered} \tag{A.11}$$

Substitution of (A.7) in the previous leads to

$$\lambda_1 \leq \hat{u}^{\mathrm{T}} H \hat{u} \leq \lambda_l, \quad \forall \hat{u} \in S \cap U \tag{A.12}$$

which is the desired result, since in (A.5) we have ordered the eigenvalues such that $\lambda_1 = \min_{j \in \{1,2,\ldots,l\}}\{\lambda_j\}$ and $\lambda_l = \lambda_l \max_{j \in \{1,2,\ldots,l\}}\{\lambda_j\}$. □

Proposition 167. *Let $Q(\hat{v}) \triangleq \hat{v}^{\mathrm{T}} A \hat{v}$ be a quadratic form. Assume that Q is positive-definite on the linear subspace $\mathcal{N}$ with $\dim\{\mathcal{N}\} = m$.*

Then, the quadratic form Q has at least m positive eigenvalues.

Proof. Let $\mathscr{D}$ be the domain of Q and $n = \dim\{\mathscr{D}\}$. The quadratic form has n eigenvalues. Suppose that the claim does not hold. Then, there are at most $p \leq m-1$ positive eigenvalues. By this assumption the other $n-p \geq n-m+1$ eigenvalues are either negative or zero. So the span of their eigenvectors is a semi-negative definite subspace $\mathscr{N}$ of dimension $n-p$, Proposition 166. This implies that the maximal positive-definite subspace is a subset of the complement $\mathscr{W} \triangleq \mathscr{D} \subset \mathscr{N}$. But $\dim\{\mathscr{W}\} = \dim\{\mathscr{D}\} + \dim\{\mathscr{N}\} = n - (n-p) = p \leq m-1$. This contradicts the hypothesis that Q is positive-definite on a linear subspace of dimension m. $\square$

Appendix B

Differential Geometry

B.1 Implicit Surfaces

We need to interpret term $\nu_{i2}(q, \hat{t}_i)$ in terms of differential geometry. This is provided in the work of Dombrowski [28], who treats the general n-dimensional case. This applies to Navigation Functions, which are defined over n-dimensional space. A simplified derivation for 3-dimensional space is provided by Hughes [29].

Let us denote the normal curvature of a surface along tangent unit vector $\hat{t}_i$ by $\kappa_{i,q}(\hat{t}_i)$. This is given by the second fundamental form II_q at q as

$$\kappa_{i,q}(\hat{t}_i) = \mathrm{II}_q(\hat{t}_i, \hat{t}_i) \tag{B.1}$$

Definition 168 (Weingarten map[30]). *Let the Weingarten map*[a] *(or shape operator) at q be*

$$\mathrm{L}_q : T_q B_i \to T_q B_i \tag{B.2}$$

Let $n_{B_i}(q) \perp B_i$ be the vector normal to B_i at point q. Suppose $\gamma : [-1, 1] \to B_i$ is a path on (hyper)surface B_i with $\gamma(0) = q$, which has tangent $t_i \in T_q B_i$. The Weingarten map is defined as

$$\mathrm{L}_q(t_i) \triangleq \frac{d\left(n_{B_i}\left(\gamma(t)\right)\right)}{dt}(0) \tag{B.3}$$

so it is the derivative of the surface normal $n_{B_i}(\gamma(t))$ at time $t = 0$, as $\gamma(t)$ passes through q in direction t_i.

[a] [30], § 4.7: The Second Fundamental Form and the Weingarten Map, pp.122-127.

Proposition 169 (Weingarten map for Implicit Surfaces [28, 29]). *For the* implicitly *defined surface B_i the Weingarten map at q is equal to the linear mapping*[a]

$$\mathrm{L}_q(\hat{t}_i) = \frac{1}{\|\nabla \beta_i\|}\left(D^2 \beta_i\right)(q)\hat{t}_i, \quad \hat{t}_i \in UT_q B_i \tag{B.4}$$

The Weingarten map is related to the second fundamental form by

$$\mathrm{II}_q(X, Y) = \mathrm{L}_q(X) \cdot Y = X \cdot \mathrm{L}_q(Y), X, Y \in T_q B_i \tag{B.5}$$

[a] [29], § 1.4: The relation between N and ∇G, p.6.

This leads to the following expression for the normal curvature of implicit surface B_i at q along $\hat{t}_i$

$$\begin{aligned}\kappa_{i,q}(\hat{t}_i) &= \mathrm{II}_q(\hat{t}_i,\hat{t}_i) = \hat{t}_i \cdot \mathrm{L}_q(\hat{t}_i)\\ &= \hat{t}_i^{\mathrm{T}} \frac{1}{\|(\nabla\beta_i)(q)\|}\left(D^2\beta_i\right)(q)\hat{t}_i \\ &= \frac{\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i}{\|(\nabla\beta_i)(q)\|} \in (-\infty,+\infty) = \mathbb{R}, \quad \hat{t}_i \in UT_qB_i\end{aligned} \tag{B.6}$$

This derivation of normal curvature $\kappa_{i,q}$ of an implicitly defined surface connects it to the implicit function β_i defining the surface. This reveals the role of the *restricted* quadratic form $\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i, \hat{t}_i \in UT_qB_i$. Restriction is with respect to the surface's unit tangent space UT_qB_i and is important to avoid misinterpretations. The principal directions are the eigenvectors of the Weingarten map. Hence, they are also the eigenvectors of the restricted quadratic form $\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i|_{\hat{t}_i\in UT_qB_i}$, but they are *not* (necessarily) eigenvectors of the Hessian matrix $\left(D^2\beta_i\right)(q)$.

Definition 170 (Radius of Normal Curvature)**.** *We can also define the* radius of normal curvature $R_{i,q}(\hat{t}_i)$ *along tangent direction* $\hat{t}_i$*, as the inverse of the normal curvature at the same point (allowing* $R_{i,q} = \pm\infty$ *and understanding that this means flatness of the implicit surface along* $\hat{t}_i$ *at point* q*)*

$$R_{i,q}(\hat{t}_i) \triangleq \frac{\|(\nabla\beta_i)(q)\|}{\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i} \in [-\infty,0)\cup(0,+\infty] = \bar{\mathbb{R}}\setminus\{0\}, \quad \hat{t}_i \in UT_qB_i \tag{B.7}$$

Definition 171 (Convex, Nonconvex)**.** *It follows that at* q*, in direction* $\hat{t}_i \in UT_qB_i$*, the surface* B_i *can be either*

1. Convex *if*
$$0 < \hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i \implies 0 < \kappa_{i,q}(\hat{t}_i); \tag{B.8}$$

2. Non-convex *if*
$$\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i \le 0 \implies \kappa_{i,q}(\hat{t}_i) \le 0. \tag{B.9}$$

Definition 172. *A point* q *where all principal curvatures* $\kappa_{ij}(q) > 0$ *is called* convex.

Definition 173. *A point* q *where all principal curvatures* $\kappa_{ij}(q) \le 0$ *is called* concave.

Definition 174 (Principal curvatures, principal directions)**.** *Let* $\kappa_{i,q}(\hat{t}_i)$ *be the normal curvature of surface* B_i *at point* q *along tangent direction* $\hat{t}_i \in UT_qB_i$*. The Weingarten map is represented in the tangent space by a linear symmetric operator, which has orthonormal eigenvectors*

$$p_{ij}(q) \in UT_qB_i, \quad i \in I_0, \quad j \in \{1,2,\ldots,n\} \tag{B.10}$$

and real eigenvalues

$$\kappa_{ij}(q) \in \mathbb{R}, \quad i \in I_0, \quad j \in \{1,2,\ldots,n\} \tag{B.11}$$

associated to them. These eigenvectors $\hat{p}_{ij}(q)$ *are called* principal directions *at* q *and their associated eigenvalues* $\kappa_{ij}(q)$ *are called* principal curvatures *at* q[a]*.*

[a][30], § 4.8: Principal, Gaussian, Mean, and Normal Curvatures, pp.128-141. In particular Definition: The principal curvatures of a surface M at a point p are the eigenvalues of L_q there. Corresponding unit eigenvectors are called principal directions at p.

From the definition of normal curvature and radius of normal curvature it follows that for an implicitly defined surface β_i, the principal curvatures and principal radii of curvature are related to their associated principal directions as follows

$$\begin{aligned} \kappa_{ij}(q) = \kappa_{i,q}(\hat{p}_{ij}) &= \frac{\hat{p}_{ij}^{\mathrm{T}} D^2 \beta_i(q) \hat{p}_{ij}}{\|\nabla \beta_i(q)\|} \\ R_{ij}(q) = R_{i,q}(\hat{p}_{ij}) &= \frac{\|\nabla \beta_i(q)\|}{\hat{p}_{ij}^{\mathrm{T}} D^2 \beta_i(q) \hat{p}_{ij}} \end{aligned} \tag{B.12}$$

B.2 Curvature of Principal Direction Spans

Proposition 175 (Curvature of subspace spanned by principal directions)**.** *Let $\hat{p}_{ij}$ be some principal directions at point q. Then every direction $\hat{t}_i$ in the subspace linearly spanned by these principal directions has normal curvature which is bounded by the minimal and maximal principal curvatures associated with those principal directions.*

Proof. The proof follows directly from the previous proposition, taking into account that principal directions are eigenvectors of the matrix form of the Weingarten map and normal curvature is the associated quadratic form of the Weingarten map in the tangent space at q. □

Lemma 176 (Span of convex principal directions is convex)**.** *Let $\hat{p}_{ij}$ be some principal directions at point q, which are convex. Then all the directions $\hat{t}_i$ in the subspace spanned by these principal directions are also convex.*

Lemma 177 (Span of non-convex principal directions is non-convex)**.** *Let $\hat{p}_{ij}$ be some principal directions at point q, which are non-convex. Then all the directions $\hat{t}_i$ in the subspace spanned by these principal directions are also non-convex.*

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