Navigation Functions for Everywhere Partially Sufficiently Curved Worlds

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***Abstract*— We extend Navigation Functions (NF) to worlds of more general geometry and topology. This is achieved without the need for diffeomorphisms, by direct definition in the geometrically complicated configuration space. Every obstacle boundary point should be partially sufficiently curved. This requires that at least one principal normal curvature be sufficient. A normal curvature is termed sufficient when the tangent sphere with diameter the associated curvature radius is a subset of the obstacle. Examples include ellipses with bounded eccentricity, tori, cylinders, one-sheet hyperboloids and others. Our proof establishes the existence of appropriate tuning for this purpose. Direct application to geometrically complicated cases is illustrated through nontrivial simulations.**

I. INTRODUCTION

Motion planning has been a central problem in robotics [1], [2]. Many different approaches have been proposed, ranging from combinatorial like Canny's roadmap algorithm [3] and sampling-based [4], to feedback methods in continuous spaces [5], and also combinations of them [6].

The basic motion planning problem over continuous space can be defined as finding a safe path from an initial to a desired configuration, avoiding collisions with obstacles.

Artificial Potential Fields are a class of methods introduced by Khatib [5] to solve the motion planning problem. They utilize a scalar potential field over the workspace. The negated gradient of this field repels from obstacles and attracts to the destination. A robot driven by gradient descent can safely reach the desired configuration. For certain obstacle worlds local minima arise, which trap the robot and prevent successful attainment of the desired configuration.

To overcome the problem of local minima, Rimon and Koditschek [7] have proposed Navigation Functions (NF). These are a family of scalar fields that achieve provably correct navigation from almost all initial conditions, apart from a set of Lebesgue measure zero.

The NF methodology has been primarily developed for n-dimensional sphere worlds. These serve as model worlds. To solve a problem defined in a geometrically complicated world, firstly it should be diffeomorphically transformed to a model sphere world. Diffeomorphisms are quite demanding mappings [8], which have been applied to star worlds, as well as forests of stars [9].

The original NF method has been extended to centralized [10] and decentralized [11] multi-agent systems with limited sensing range, non-holonomic systems [12] using dipolar NFs, incorporating prioritization [13] and multi-agent connectivity constraints [14], to partially known worlds by applying belt zones [15] and to unknown sphere worlds [16], as well as convex worlds using local diffeomorphisms [17]. Nonetheless, all of them involved either sphere worlds or diffeomorphisms, hence also simply connected obstacles.

The present contribution aims to extend this set of applications. We propose a proof that applies to general geometries and multiply connected obstacle topologies, not only to spheres. This is achieved by relating the desired NF properties with the obstacle principal curvatures. Firstly we treat worlds that are sufficiently curved at every obstacle boundary point. Then, we relax this to partially non-convex obstacles, whose convex principal curvatures should still all be sufficient. Finally, we extend this to worlds which are partially sufficiently curved at every point. Principal directions which are not sufficiently curved are allowed to be convex. To achieve this, we alter the NF definition to allow limited degeneracy at critical points, while preserving navigation properties. The form of NF proposed by Koditschek and Rimon (KRNF) require tuning of a parameter $k \geq \hat{k}_{\min}$ to guarantee the desired navigation properties. We extended KRNFs to unknown sphere worlds in [16] and the present work further extends them by proving that $\hat{k}_{\min}$ exists for more general worlds.

The rest of this paper is organized as following: NFs are defined in § II, the problem in § III, relative curvature in § IV, sufficient curvature in § V. This allows generalization to sufficiently curved worlds§ VI, partially non-convex worlds § VII, to prepare for the general case in § VIII. Simulation supporting this method and topological comments are provided in § IX and future work is considered in § X.

II. NAVIGATION FUNCTIONS

A. World and Function Definitions

Let E^n be n-dimensional Euclidean space. Obstacles are implicitly defined as $\mathscr{O}_i \triangleq \{q \in E^n | \beta_i(q) < 0\}, i \in I_0 \triangleq \mathbb{N}_{\leq M}, M \in \mathbb{N}$ where $\beta_i \in C^2(E^n, \mathbb{R})$. The zero$^{\text{th}}$ obstacle $\mathscr{O}_0$ bounds the whole world $\mathscr{W} \triangleq E^n \setminus \mathscr{O}_0$. Their closures should be disjoint $\overline{\mathscr{O}_i} \cap \overline{\mathscr{O}_j} = \emptyset, \forall i \neq j$ and boundaries $\partial\mathscr{O}_i$ compact. Let $\beta \triangleq \prod_{i \in I_0} \beta_i$. In the sequel we will refer to both sets $\mathscr{O}_i$ and their defining functions β_i as "obstacles" interchangeably. The free space is $\mathscr{F} \triangleq \mathscr{W} \setminus \bigcup_{i \in I_0} \mathscr{O}_i$.

Definition 1 (Navigation Function [7]): A NF on $\mathscr{F}$ is defined as a map $\varphi : \mathscr{F} \to [0, 1]$ which is:

1) C^2-smooth on $\mathscr{F}$: analyticity is not necessary [7] (existence and uniqueness of solutions of $\dot{x} = -\nabla_q \varphi(x)$);

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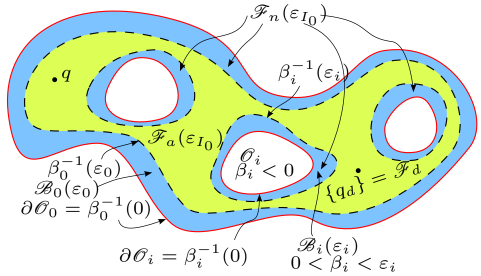


Fig. 1: Sets defined on a general world.

2) Admissible on $\mathscr{F}$: uniformly maximal on $\partial\mathscr{F}$ (safety);
3) Polar on $\mathscr{F}$: unique minimum at q_d (convergence);
4) Morse on $\mathscr{F}$: all critical points are non-degenerate.

This definition is used in § V, but later altered in § VII.

In [7] NFs were proved to exist for *any* analytic manifold with boundary $\mathscr{F}$. A *particular type* of NF proposed there by Koditschek and Rimon (hereafter denoted by KRNF) is

$$\varphi \triangleq \frac{\gamma_d}{\left(\gamma_d^k + \beta\right)^{\frac{1}{k}}} \tag{1}$$

where $\gamma_d \in C^2\left(E^n, [0, +\infty)\right), \|\nabla\gamma_d\| > 0, \forall q \neq q_d$ is the destination attractive effect, specified later. The KRNF φ can be made into a NF when $\mathscr{F}$ is a sphere world [7]. The means to achieve this is by selecting $k \geq \hat{k}_{\min}$. The existence of $k_{\min}$ is what we want to prove.

The detailed sphere world proof [7], [9], [8] is essential here. We extend it to more general worlds, assuming *general* obstacles β_i and in doing so we derive a geometric condition, termed *sufficient curvature*.

B. Useful World Subsets

We use the following sets, which are illustrated in Fig. 1:

1) Destination point $\mathscr{F}_d \triangleq \{q_d\}$
2) Free space boundary $\partial\mathscr{F} \triangleq \beta^{-1}(0) = \bigcup_{i\in I_0} \beta_i^{-1}(0)$
3) i^{th} obstacle neighborhood
   $\mathscr{B}_i\left(\varepsilon_i\right) \triangleq \{q \in E^n |\ 0 < \beta_i < \varepsilon_i\}, i \in I_0$
4) "Near" all obstacles $\mathscr{F}_n\left(\varepsilon_{I_0}\right) \triangleq \bigcup_{i\in I_0} \mathscr{B}_i\left(\varepsilon_i\right) \setminus \{q_d\}$
5) Set "away" from obstacles
   $\mathscr{F}_a\left(\varepsilon_{I_0}\right) \triangleq \mathscr{F} \setminus \left(\mathscr{F}_d \cup \partial\mathscr{F} \cup \mathscr{F}_n\left(\varepsilon_{I_0}\right)\right)$,

where $\varepsilon_{I_0} \triangleq \{\varepsilon_i\}_{i\in I_0}$. We define $\varepsilon_i, \varepsilon_{iu}, \varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}$ $\varepsilon_{i3j}, \varepsilon_{i3} \triangleq \min_{j\in I_0\setminus i}\{\varepsilon_{i3j}\}, \varepsilon_{i4}, \varepsilon_{i5}, j \in I_0 \setminus \{i\}, i \in I_0$ as

$$0 < \varepsilon_i < \varepsilon_{iu} = \frac{1}{2}\min\{\varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3}, \varepsilon_{i4}, \varepsilon_{i5}\}, i \in I_0.$$

Note that β_i^{-1} is the set-valued preimage function.

In the proof we require that ε_{i3j} be selected such that $\mathscr{B}_i\left(\varepsilon_i\right) \cap \mathscr{B}_j\left(\varepsilon_j\right) = \emptyset \implies \beta_j(q) \geq \varepsilon_j, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right), \forall j \in I_0 \setminus \{i\}, \forall i \in I_0$. Since obstacles are pairwise disjoint, such an ε_{I_0} can always be selected. For properly defined β_i^{-1} level sets "near" obstacles we require that for all $i \in I_0$ there exists an $\varepsilon_{i4} > 0$ such that $\|\nabla\beta_i\| > 0, \forall q \in \overline{\mathscr{B}_i(\varepsilon_{i4})}$. Hereafter sets $\mathscr{F}_*$ are denoted omitting their arguments. Let $\mathscr{C}_f \triangleq \{q_c \in E^n | \nabla f = 0\}$ be the critical set of a function f.

III. PROBLEM DEFINITION

Let $x \in E^n$ be the state of a holonomic robot governed by the control law $\frac{dx}{dt}(t) = -\nabla_q\varphi\left(x(t)\right)$ where φ is a NF on $\mathscr{F}$ as defined in Eq.(1). The motion planning problem concerns finding a path for x from the initial condition $x(0)$ to the desired destination $q_d \in E^n$. NFs solve the motion planning problem in $\mathscr{F}$. As already noted, NFs exist for *any* world $\mathscr{F}$ and Eq.(1) is *one type* introduced and proved correct for sphere worlds in [7].

We are interested in proving that φ of the form of Eq.(1) is a NF for worlds of more complicated geometry. We do not focus on particular types of obstacles, but derive a curvature condition applicable in general.

IV. RELATIVE CURVATURE

The whole proof follows three steps. Firstly, *all* critical points $q_c \neq q_d$ are confined to obstacle neighborhoods. This can be done by a choice $k \geq \hat{k}_{\min}(\varepsilon_{I_0})$, where $\hat{k}_{\min}$ is a function of neighborhood "widths" ε_i. The second step is shrinking these widths, until all critical points become saddles. The third step is further shrinking of ε_i, in order to ensure that all saddles become non-degenerate.

A. Relative Curvature Function

Our target is to derive a geometric condition for the obstacles, in order for a KRNF to exist for any valid destination $q_d \in \mathscr{F}$. Some initial steps of the proof remain valid for non-spherical β_i, so the reader is referred to the case of spheres [7] and a brief description is provided here.

We want to work with $\hat{\varphi} \triangleq \frac{\gamma_d^k}{\beta}$, which is simpler than φ. To achieve this, we can map $\varphi = \sigma_k(\hat{\varphi})$ using the range diffeomorphism $\sigma_k(x) = (\frac{x}{x+1})^{\frac{1}{k}}$. Then, any good properties we prove about $\hat{\varphi}$, will also hold for φ, because σ_k conserves NF properties, by Prop.2.7 [7]. But $\hat{\varphi}$ can be diffeomorphically mapped to φ only within $\mathring{\mathscr{F}} \setminus q_d$. Therefore, it must be independently proved that $\partial\mathscr{F}$ is clear of critical points and q_d is a non-degenerate local minimum, which are Prop.3.3 and Prop.3.2 [7], respectively.

Working with $\hat{\varphi}$ within $\mathring{\mathscr{F}}$, if we select

$$k \geq \hat{k}_{\min}(\varepsilon_{I_0}) \triangleq \frac{1}{2}\max_{\mathscr{W}}\{\sqrt{\gamma_d}\}\sum_{i\in I_0}\frac{\max_{\mathscr{W}}\{\|\nabla\beta_i\|\}}{\varepsilon_i} \tag{2}$$

then by Prop.3.4 [7] no critical points $q_c \neq q_d$ remain in $\mathscr{F}_a$ "away" from obstacles and $q_c \in \mathscr{F}_n$, "near" obstacles.

Then we need to study the critical points of $\hat{\varphi}$. But we have shown that these can be "pushed" close to the obstacles, within $\mathscr{F}_n$. We will show that when a critical point q_c is close enough to the obstacle surface, it acquires certain properties from the surface. These are related to the eigenvalues of the Hessian matrix $D^2\hat{\varphi}$. In particular, the surface curvature is "inherited" by $D^2\hat{\varphi}(q_c)$.

To prove this dependence of eigenvalues of the Hessian $D^2\hat{\varphi}(q_c)$ on the surface principal curvatures, it will prove useful to separate the quadratic form associated with $D^2\hat{\varphi}(q_c)$, similarly to Prop.3.6 [7], which is generalized here. The Hessian matrix separation is as follows

Lemma 2 ([18]): At a critical point q_c, the Hessian quadratic form is the sum of two components

$$\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i \frac{\beta_i(q_c)\beta(q_c)}{\gamma_d(q_c)^{k-2}} = \nu_i(q_c, \hat{t}_i) + \beta_i(q_c)\zeta(q_c, \hat{t}_i)$$

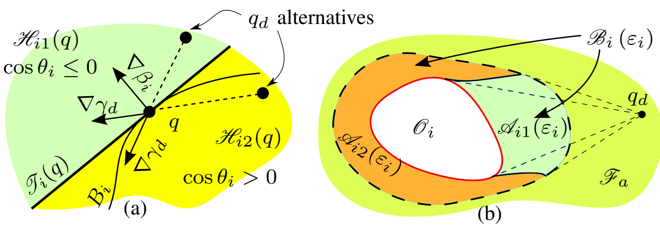


Fig. 2: Sign definite subsets for $\nabla\beta_i \cdot \nabla\gamma_d$.

where $\hat{t}_i \in \mathscr{T}_i(q)$, $\nu_i(q_c, \hat{t}_i)$ is defined later and $\zeta(q_c, \hat{t}_i) = \bar{\beta}_i(q_c)^{-1}\big(\frac{\nabla\bar{\beta}_i(q_c)\cdot\nabla\gamma_d(q_c)}{\|\nabla\gamma_d(q_c)\|^2} + \hat{t}_i^{\mathrm{T}}\big(\big(1-\frac{1}{k}\big)\frac{\nabla\bar{\beta}_i(q_c)\cdot\nabla\bar{\beta}_i(q_c)^{\mathrm{T}}}{\bar{\beta}_i(q_c)} - D^2\bar{\beta}_i(q_c)\big)\hat{t}_i\big)$.

An obstacle level set $B_i(q) \triangleq \beta_i^{-1}(\beta_i(q))$ has a tangent space $\mathscr{T}_i(q) \triangleq \{u \in T_q\mathscr{F} | u \cdot \nabla\beta_i(q) = 0\}$ and its orthogonal complement $\mathscr{R}_i(q) \triangleq \operatorname{span}\{\nabla\beta_i(q)\}$, Fig. 2. Then, the subsets of unit vectors are $U\mathscr{T}_i$ and $U\mathscr{R}_i = \{\hat{r}_i\} = \frac{\nabla\beta_i}{\|\nabla\beta_i\|}$.

Now the rationale is that the closer we "push" critical points to an obstacle, the smaller $\beta_i(q_c)$ can be made, canceling $\beta_i(q_c)\zeta(q_c, \hat{t}_i)$. Then we can make $\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\,\hat{t}_i$ to have the same sign as $\nu_i(q_c, \hat{t}_i)$. This indicates that $\nu_i(q_c, \hat{t}_i)$ is an important function to study for our problem.

Definition 3: Let $\hat{t}_i \in \mathscr{T}_i(q), q \in \mathscr{F}, i \in I_0$. The *relative curvature* function $\nu_i : UTB_i \to \mathbb{R}$ is defined as

$$\nu_i(q, \hat{t}_i) \triangleq \frac{\nabla\beta_i(q)\cdot\nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|^2}\hat{t}_i^{\mathrm{T}} D^2\gamma_d(q)\hat{t}_i - \hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i$$

As will be shown later, ν_i compares the curvature of level sets of the attractive function γ_d, with the curvature of level sets of the obstacle function β_i.

If $\gamma_d(q) = \|q - q_d\|^2$, then

$$\begin{aligned}\nu_i(q, \hat{t}_i) &= \nu_{i1}(q) + \nu_{i2}(q, \hat{t}_i)\\ &= \|\nabla\beta_i(q)\|\left(\nu_{i3}(q) + \nu_{i4}(q, \hat{t}_i)\right)\end{aligned} \tag{3}$$

where $\nu_{i1}(q) \triangleq 2\frac{\nabla\beta_i(q)\cdot\nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|^2}, \nu_{i2}(q, \hat{t}_i) \triangleq -\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i$, $\nu_{i3}(q) \triangleq \frac{\nu_{i1}(q)}{\|\nabla\beta_i(q)\|}, \nu_{i4}(q, \hat{t}_i) \triangleq \frac{\nu_{i2}(q, \hat{t}_i)}{\|\nabla\beta_i(q)\|}$.

If $\|\nabla\beta_i\| > 0$, then for any γ_d

$$\nu_i = \|\nabla\beta_i\|\left(\cos(\theta_i)\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d\hat{t}_i}{\|\nabla\gamma_d\|} - \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i}{\|\nabla\beta_i\|}\right) \tag{4}$$

where θ_i is the angle between $\nabla\gamma_d$ and $\nabla\beta_i$.

A critical point q_c cannot be a local minimum when $D^2\hat{\varphi}(q_c)$ has at least one negative eigenvalue. When a small enough ε_i is selected, the eigenvalues of $\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\hat{t}_i$ are determined by the sign of the relative curvature $\nu_i(q_c, \hat{t}_i)$. To ensure there exists a negative eigenvalue, there should exist at least one direction $\hat{t}_i$, such that $\nu_i(q, \hat{t}_i) < 0$ or

$$\cos(\theta_i)\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d\hat{t}_i}{\|\nabla\gamma_d\|} < \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i}{\|\nabla\beta_i\|}, \quad q \in \overline{\mathscr{B}_i(\varepsilon_{i4})}. \tag{5}$$

B. Critical point-free subsets

We now show that obstacle parts whose outward normal $\hat{r}_i$ "looks" towards the destination q_d do not obstruct us from getting there. Very close to $\partial\mathscr{O}_i$, $\nabla\beta_i$ dominates $\nabla\bar{\beta}_i$. Hence, in $\mathscr{B}_i(\varepsilon_i)$ the existence of critical points is dominated only by $\nabla\beta_i$ and $-\nabla\gamma_d$. At points where $\psi \triangleq \nabla\gamma_d \cdot \nabla\beta_i \le 0$, vectors $-\nabla\gamma_d, \nabla\beta_i$ cannot annihilate and $\nabla\hat{\varphi} \neq 0$. This happens in the "good" set $\mathscr{A}_1(\varepsilon_{I_0}) \triangleq \{q \in \bigcup_{i\in I_0}\mathscr{B}_i(\varepsilon_i) \,|\, \psi(q) \le 0\}$, whereas critical points may remain in the "bad" set $\mathscr{A}_2(\varepsilon_{I_0}) \triangleq \{q \in \bigcup_{i\in I_0}\mathscr{B}_i(\varepsilon_i) \,|\, 0 < \psi(q)\}$, illustrated in Fig. 2.

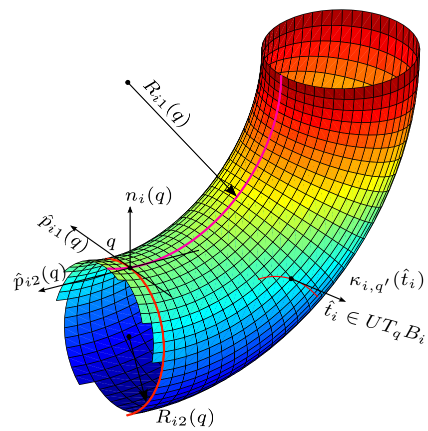


Fig. 3: Torus surface principal directions and radii of principal normal curvatures.

Proposition 4 ([18]): For a selected q_d, if $k > \hat{k}_{\min}(\varepsilon_{I_0})$ then no critical points arise in $\mathscr{A}_1(\varepsilon_{I_0})$.

This means that by setting $k \ge \hat{k}_{\min}(\varepsilon_{I_0})$ we confine critical points not just in $\bigcup_{i\in I_0}\mathscr{B}_i$, but in $\mathscr{A}_2(\varepsilon_{I_0}) \cap \bigcup_{i\in I_0}\mathscr{B}_i$. The proof in [18] is inspired by Proposition 3.7 of [7].

Lemma 5 ([18]): For $k \ge \hat{k}_{\min}(\varepsilon_{I_0})$ all critical points $q_c \neq q_d$ arise in $\mathscr{A}_2(\varepsilon_{I_0})$. Moreover, if $\gamma_d(q) = \|q - q_d\|^2$ then $0 < \nu_{i1}(q)$ for all $q \in \mathscr{A}_2(\varepsilon_{I_0})$.

Similarly, for each point, some destinations $q_d \in \mathring{\mathscr{F}} \setminus \{q\}$ cause $\nabla\beta_i(q) \cdot \nabla\gamma_d(q) \le 0$, as shown in Fig. 2, so we can define sets $\mathscr{H}_{i1}(q) \triangleq \{q_d \in \mathring{\mathscr{F}} \setminus \{q\} |\ \psi(q) \le 0\}$ and $\mathscr{H}_{i2}(q) \triangleq \{q_d \in \mathring{\mathscr{F}} \setminus \{q\} |\ 0 < \psi(q)\}$ to be used later.

V. SUFFICIENT CURVATURE CONDITION

A. Implicit Surface Differential Geometry

The following will prove useful in order to interpret $\nu_{i2}(q, \hat{t}_i)$ and $\nu_{i4}(q, \hat{t}_i)$. For an implicit (hyper)surface B_i, the *normal curvature* at point q in the tangent direction $\hat{t}_i \in UT_qB_i$ and *radius of normal curvature* are

$$\kappa_{i,q}(\hat{t}_i) = \hat{t}_i \cdot \mathrm{L}_q(\hat{t}_i) = \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i}{\|\nabla\beta_i(q)\|}, \quad R_{i,q}(\hat{t}_i) \triangleq \frac{1}{\kappa_{i,q}(\hat{t}_i)}$$

respectively, where $\mathrm{L}_q(\hat{t}_i)$ is the *Weingarten map* (or shape operator) [19], [20], [21] and $R_{i,q}(\hat{t}_i) = \pm\infty$ corresponds to flatness. A surface B_i is *convex* along $\hat{t}_i$ when $0 < \kappa_{i,q}(\hat{t}_i)$ and *non-convex* otherwise.

Definition 6 (Principal curvatures, principal directions): A (hyper)surface is characterized by its *principal curvatures* $\kappa_{ij}(q) \in \mathbb{R}$ in the associated tangent directions $\hat{p}_{ij}(q) \in UT_qB_i$, called the *principal directions* [19]. These are the eigenvalues and associated unit eigenvectors of the linear symmetric operator $\mathrm{L}_q(\hat{t}_i)$, where $i \in I_0, j \in \mathbb{N}^*_{\le n-1}$. Note that $\kappa_{ij}(q) = \kappa_{i,q}(\hat{p}_{ij}) = \frac{\hat{p}_{ij}^{\mathrm{T}} D^2\beta_i(q)\hat{p}_{ij}}{\|\nabla\beta_i(q)\|}$ and $R_{ij}(q) = R_{i,q}(\hat{p}_{ij}) = \frac{\|\nabla\beta_i(q)\|}{\hat{p}_{ij}^{\mathrm{T}} D^2\beta_i(q)\hat{p}_{ij}}$. These are shown in Fig. 3. The extremal principal curvatures bound $\kappa_{i,q}(\hat{t}_i)$, i.e $\min_j\{\kappa_{ij}(q)\} \le \kappa_{i,q}(\hat{t}_i) \le \max_j\{\kappa_{ij}(q)\}, \forall\hat{t}_i \in UT_qB_i$.

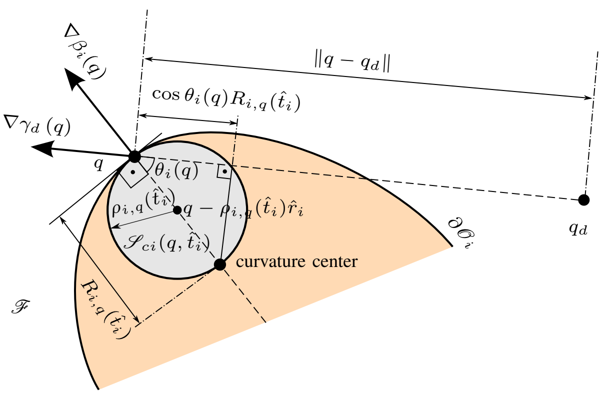
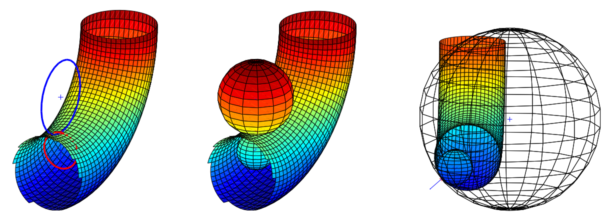


Fig. 4: Sufficient Curvature at q along $\hat{t}_i$.



(a) Curvature circles. (b) Sufficient and Non-convex. (c) Sufficient and Convex but not sufficient.

Fig. 5: Principal Curvature Balls.

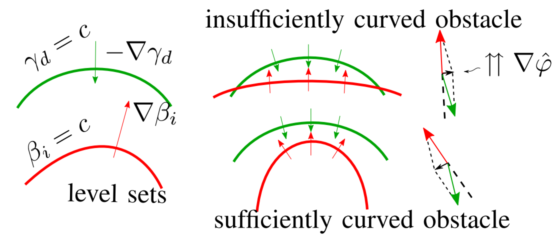


Fig. 6: Why relative curvature determines local minima.

B. Sufficient Curvature and its Geometry

The preceding development indicates that NF eigenvalues are related to principal curvatures κ_{ij} on the obstacle surface. Imagine that we are at a critical point q_c. Then, for each κ_{ij} for that point, we can draw an associated principal curvature ball $\mathscr{S}_{cij}$. We define a *curvature ball* with radius $\rho_{i,q} \triangleq \frac{1}{2}R_{i,q}$, touching B_i at q_0, as $\mathscr{S}_{ci}(q_0,\hat{t}_i) \triangleq \{q \in E^n \,|\, \|q-(q_0-\rho_{i,q}\hat{r}_i)\| \leq \rho_{i,q}\}$, as shown in Fig. 4.

Proposition 7: If $\gamma_d(q) = \|q-q_d\|^2$ and $\|\nabla\beta_i\| > 0$, then sufficient curvature along $\hat{t}_i$ is equivalent to $q_d \notin \mathscr{S}_{ci}(q,\hat{t}_i)$.

Proof: For all $q_d \in \mathscr{H}_{i2}(q)$ by Eq.(4) $\nu_i < 0 \iff \{R_{i,q}(\hat{t}_i) > 0 \wedge \cos(\theta_i)R_{i,q}(\hat{t}_i) < \|q-q_d\|\} \iff \{R_{i,q}(\hat{t}_i) > 0 \wedge q_d \notin \mathscr{S}_{ci}(q,\hat{t}_i)\}$, interestingly related to Meusnier's Theorem [22] and illustrated in Fig. 4. ■

If $\kappa_{ij}(q)$ is convex, $\mathscr{S}_{cij}(q)$ will touch $\partial\mathscr{O}_i$ from the inside at q. If non-convex, it will touch it from the outside. Now, if sufficiently curved, then the whole principal curvature ball $\mathscr{S}_{cij}(q)$ will be inside the obstacle $\mathscr{O}_i$. Otherwise, it will protrude from the "other side" of $\mathscr{O}_i$. Therefore, when completely inside $\mathscr{O}_i$, we cannot choose any $q_d \in \mathscr{S}_{cij}(q)$, because $\mathscr{S}_{cij}(q) \cap \mathring{\mathscr{F}} = \emptyset$. Examples are shown in Fig. 5.

This is the key to our understanding. When $\mathscr{S}_{cij}(q) \subseteq \overline{\mathscr{O}_i}$, then $\nu_i(q,\hat{p}_{ij}) < 0$ for every q_d. On the contrary, if $\mathscr{S}_{cij}(q)$ protrudes in free space, then there exist $q_d \in \mathring{\mathscr{F}}$ which we can pick inside it to make $0 \leq \nu_i(q,\hat{p}_{ij})$. Consequently, only sufficient principal curvatures κ_{ij} guarantee that we can make $D^2\hat{\varphi}$ negative definite along $\hat{p}_{ij}$.

Having proved that q can be a critical point only for $q_d \in \mathscr{H}_{i2}(q)$, the following divides tangents $\hat{t}_i$ at q accordingly.

Definition 8: A *direction* $\hat{t}_i \in T_q\partial\mathscr{O}_i$ is called

1) *Sufficiently curved* if $\nu_i(q,\hat{t}_i) < 0, \forall q_d \in \mathscr{H}_{i2}(q)$;
2) *Convex but not sufficiently curved* if $\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i > 0$ but $\exists q_d \in \mathscr{H}_{i2}(q):\ 0 \leq \nu_i(q,\hat{t}_i)$, see Fig. 7, Fig. 5c;
3) Non-convex if $\hat{t}_i^{\mathrm{T}} D^2\beta_i(q)\hat{t}_i \leq 0 \implies 0 < \nu_i(q,\hat{t}_i), \forall q_d \in \mathscr{H}_{i2}(q)$, see Fig. 5b.

Definition 9: A *world* $\mathscr{F}$ is called *everywhere sufficiently curved* if all its obstacles $\mathscr{O}_i$ are sufficiently curved at every boundary point, for every $\hat{t}_i$.

Sufficient curvature is meaningful only on $\partial\mathscr{O}_i$, because it involves all $q_d \in \mathring{\mathscr{F}}$. However, for a selected q_d, the sign of ν_i extends from $\partial\mathscr{O}_i$ to the level sets B_i in a sufficiently small neighborhood of $\partial\mathscr{O}_i$. This is due to C^2 continuity of β_i, which implies principal curvature continuity through level sets, as detailed in [18].

Therefore, for a selected q_d, sufficient κ_{ij} on $\partial\mathscr{O}_i$ ensures that there exists another level set sufficiently near $\partial\mathscr{O}_i$, with $\nu_i < 0$ for the same number of principal directions. Then, we can always "push" the critical points q_c to that level set and make them acquire the desired eigenvalue signs. Similarly, non-convex κ_{ij} determine the number of $\hat{p}_{ij}$ with $\nu_i \leq 0$ "near" $\partial\mathscr{O}_i$.

Fig. 6 intuitively illustrates that when a level set of β_i is locally *more* curved than a level set of γ_d, then $-\nabla\varphi \upuparrows \frac{\gamma_d}{k}\nabla\beta_i - \beta\nabla\gamma_d$ *cannot* form a local minimum, in case a critical point arises in that area.

VI. Sufficiently Curved Worlds

A. Geometry Controls NF Eigenvalues

We have previously shown that if $\gamma_d = \|q-q_d\|^2$, then $\nu_i(q,\hat{t}_i) = \|\nabla\beta_i(q)\|\left(\nu_{i3}(q) - \kappa_{i,q}(\hat{t}_i)\right)$. Since ν_{i3} is independent of $\hat{t}_i$, the dependence of ν_i on $\hat{t}_i$ stems only from $\kappa_{i,q}$. By the extremal properties of principal curvatures κ_{ij} in Def. 6 with respect to bounding curvature $\kappa_{i,q}$ on subspaces spanned by principal directions, it follows that

Lemma 10 ([18]): If $\gamma_d(q) = \|q-q_d\|^2$ and $\|\nabla\beta_i\| > 0$, then the span $\mathscr{P}_i$ of principal directions, each of which has negative (positive) ν_i, also has negative (positive) ν_i.

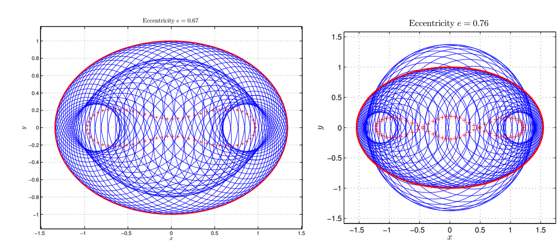


Fig. 7: Sufficiently curved vs convex but not sufficiently curved ellipse.

Let $\mathscr{P}_i^-(q) \triangleq \operatorname{span}\{P_i^-(q)\}$ be the span of the set P_i^- of principal directions $\hat{p}_{ij}$, indexed by I_i^-, with $\nu_i < 0$. Hereafter we set $\gamma_d = \|q - q_d\|^2$ and work in $\mathscr{B}_i(\varepsilon_{i4})$ to ensure $\|\nabla\beta_i(q)\| > 0$. Using the notions developed so far, it is now possible to generalize Prop. 3.6 [7]. The following connects principal relative curvature $\nu_i(q_c, \hat{p}_{ij}(q_c)), j \in I_i^-$ sign to NF Hessian quadratic form sign on span $\mathscr{P}_i$.

Proposition 11 ([18]): At a critical point q_c, the Hessian $D^2\hat{\varphi}$ can be made negative (positive) definite on the span $\mathscr{P}_i^-$ of principal directions $\hat{p}_{ij}$, each of which has negative (positive) ν_i, by setting $\varepsilon_i < \varepsilon_{i0}''$.

By Prop. 11 what happens with principal directions $\hat{p}_{ij}(q_c)$ at a critical point carries on to the sign definiteness of $D^2\hat{\varphi}$ on their spanned subspace. By "pushing" critical points close to $\partial\mathscr{O}_i$, we can *control* what happens with $\hat{p}_{ij}(q_c)$, Prop.31 [18]. This is achieved by setting $\varepsilon_i < \varepsilon_{i0}'$, Lemma 30 [18].

Proposition 12 ([18]): Every critical point $q_c \in \mathscr{B}_i(\varepsilon_i)$ has at least the number of negative eigenvalues as some boundary point $q \in \partial\mathscr{O}_i$ has *sufficiently curved* principal directions, provided $\varepsilon_i < \min\{\varepsilon_{i0}', \varepsilon_{i0}''\}$.

Proposition 13 ([18]): Every critical point has at least the number of positive eigenvalues as some obstacle boundary point has *non-convex* principal directions, provided $\varepsilon_i < \min\{\varepsilon_{i0}', \varepsilon_{i0}''\}$.

The following tells us that when everywhere one principal direction is sufficiently curved, we can "escape" in that direction. Degeneracy issues are treated later.

Lemma 14: If at every boundary point $q \in \partial\mathscr{O}_i$ there exists at least one sufficiently curved principal direction $\hat{p}_{ij}(q_c)$, then for every critical point $q_c \in \mathscr{B}_i(\varepsilon_i)$, the Hessian $D^2\hat{\varphi}(q_c)$ has at least one negative eigenvalue.

B. NF extension

First, we need to extend that part of Prop. 3.9 [7], which concerns "radial" positive definiteness of $D^2\hat{\varphi}(q_c)$.

Proposition 15 ([18]): The Hessian $D^2\hat{\varphi}(q_c)$ can be made positive definite in the "radial" direction $\hat{r}_i$. There exist $0 < \varepsilon_{i2}', \varepsilon_{i2}''$, such that the claim holds $\forall \varepsilon_i < \min\{\varepsilon_{i2}', \varepsilon_{i2}''\}$.

In an everywhere sufficiently curved world, every critical point $q_c \neq q_d$ will have $\nu_i < 0$ on all principal directions. Hence by Def. 9 and Prop. 12, $D^2\hat{\varphi}$ can be made negative definite on the tangent space of B_i, so q_c cannot be a local minimum. Moreover, since the radial direction can be made positive definite using Prop. 15, then $D^2\hat{\varphi}$ is sign definite on a direct sum decomposition of $T_q\mathscr{F}$. By Lemma 3.8 [7] q_c must be a non-degenerate saddle point.

VII. PARTIALLY NON-CONVEX WORLDS

Based on § VI, if all principal curvatures are sufficiently curved, then $D^2\hat{\varphi}$ can be made negative definite on the whole tangent space T_qB_i.

We will now relax this, by allowing some principal curvatures to be non-convex. A *partially non-convex world* has at least one sufficient principal curvature and the rest non-convex, at every point on $\partial\mathscr{O}_i$. Then, by the same arguments, $D^2\hat{\varphi}$ can be made positive definite on the subspace of T_qB_i which is spanned by these non-convex principal directions. The rest of T_qB_i will be negative definite as before. But this is not enough.

There remains the need to show that the span of those non-convex principal directions *and* the radial $\hat{r}_i$ is positive definite. Then, again $T_q\mathscr{F}$ decomposes into the direct sum of sign definite subspaces.

The following constitutes one of our main contributions, since it enables treatment of non-convex directions in conjunction with $\hat{r}_i$. Its proof relies again on showing that when β_i is small enough, the "residual" terms become negligible and the positive definiteness follows.

Proposition 16 ([18]): There exists an $\varepsilon_{i5} > 0$, such that for all $\varepsilon_i < \varepsilon_{i5}$ the Hessian $D^2\hat{\varphi}$ is positive definite on $\operatorname{span}\{P_i^+, \hat{r}_i\}$.

VIII. PARTIALLY SUFFICIENTLY CURVED WORLDS

Until now we have avoided convex principal curvatures κ_{ij} which are not sufficiently curved. These can cause degeneracy in Hessian $D^2\hat{\varphi}$. An example is the big curvature sphere in Fig. 5c. We are now going to allow one such κ_{ij}.

Any critical point with at least one negative eigenvalue is a saddle, irrespective of degeneracy [23]. Hence, requiring at least one sufficient curvature κ_{ij} at every point $q \in \partial\mathscr{O}_i$ ensures that all critical points $q_c \neq q_d$ can be made into (possibly degenerate) saddles. These motivate the following

Definition 17: We call *everywhere partially sufficiently curved* a world for which every obstacle boundary point has at least one sufficient principal curvature κ_{ij} and at most one convex but not sufficient κ_{ij}. The rest are non-convex.

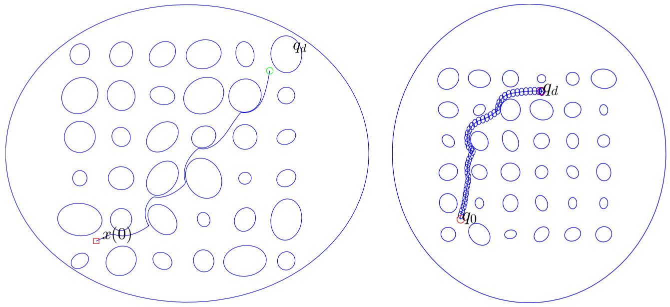
Relaxing the Morse property, we define an Extended Navigation Function (ENF). The the union of any critical points $q_c \neq q_d$ of an ENF must have a stable set of Lebesgue measure zero. An ENF has provably correct convergence by slightly adapting Prop. 2.4 [7] which is based on [24].

The following relies on limited degeneracy, which allows critical sets of at most one dimension and without branching, according to Thom's Splitting Lemma [25]. These are diffeomorphic to either a circle or a line segment. If a circle, then the Morse-Bott Lemma [26], [27] applies. If a line segment, then Morse-Bott theory applies to its interior and Thom's Splitting Lemma to the end-points.

Proposition 18 ([18]): If the Hessian $D^2\hat{\varphi}(q_c)$ has at least one negative, at least one positive and at most one zero eigenvalue, then only a measure zero set has the degenerate saddle point q_c in its positive limit set.

IX. SIMULATION RESULTS

This section presents some simulation results using KRNFs for environments of complicated geometry and topology. The worlds are known a priori and the tuning parameter k is selected to be constant. In Fig. 8a navigation of a point agent among ellipses is shown. The ellipses are 2d obstacles and need to be sufficiently curved because their tangent spaces are only 1-dimensional. This is equivalent to the requirement of limited eccentricity, Fig. 7. The KRNF is directly defined, without need for diffeomorphisms.



(a) Point agent. (b) Elliptic agent.

Fig. 8: Sufficiently curved worlds.

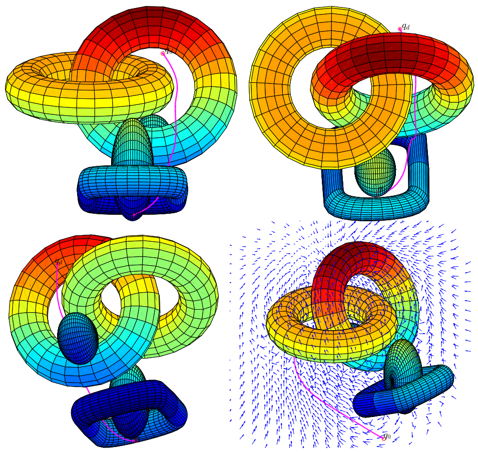


Fig. 9: NF in everywhere partially sufficiently curved world ($\mathscr{O}_0$ not shown, to make internal $\mathscr{O}_i$ visible).

A translating elliptic agent navigating in an elliptic 2d world is shown in Fig. 8b. Its C-space is sufficiently curved, because it is the Minkowski sum of sufficiently curved obstacles, whose radii of curvature add up [30]. To calculate $\nabla\beta$, we use the derivative of Rvachev conjunction [28], [29] of β_i on a set of agent boundary points. This is an approximation of the implicit Minkowski sum, which can be made as accurate as desired, by using more boundary points.

In Fig. 9 a point agent safely converges to q_d in an everywhere partially sufficiently curved world, illustrating how tori enable treatment of multiply connected obstacles, previously not representable by sphere worlds. A useful note is that as the C-space dimension increases, the KRNF method has an advantage, because more directions of "escape" become available and full non-convexity more rare.

X. Conclusions and Future Work

We have extended Koditschek-Rimon Navigation Functions firstly to sufficiently curved worlds, then to partially non-convex ones and finally altered the NF definition to allow limited degeneracy without affecting provably correct navigation properties, and proved KRNF extend to partially sufficiently curved worlds as well. Future work aims to explore tuning and extensions using the Morse-Kirwan Lemma [31]. Finally, the authors would like to thank the anonymous reviewers for their helpful suggestions.

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