Queens' College Cambridge

NST Workbook

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1. **A1** x^{-10}
2. **A2**
   (a) We have $x^2 - 1 = (x-1)(x+1)$.
   (b) We have $a^2 - 4ab + 4b^2 = (a+2b)(a-2b)$.
   (c) We have $x^3 - 1 = (x-1)(x^2+x+1)$.
3. **A3**
   (a) Given that $x^2 - 5x + 6 = (x-2)(x-3) = 0$ then it follows that the solutions are $x_{1,2} = 2, 3$.
   (b) Given that $x^2+2x = x(x+2) = 0$ then it follows that the solutions are $x_{1,2} = 0, -2$.
   (c) We have the quadratic equation $x^2 - x - 1 = 0$. Given that we cannot find any roots by inspection, let us applying the quadratic formula, which gives us the following solutions

$$x_{1,2} = \frac{-(-1) \pm \sqrt{1 - 4 \times 1 \times (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

   (d) We have the quartic equation $x^4 - 3x^2 + 2 = 0$. Using the substitution $u = x^2$, we have the quadratic equation $u^2 - 3u + 2 = 0$. Factorising this yields

$$u^2 - 3u + 2 = (u-2)(u-1) = 0.$$

   Hence the roots of the above equation are $u_{1,2} = 1, 2$. Since $u = x^2$, then we have the following solutions to the original equation

$$x_{1,\ldots,4} = \pm 1, \pm\sqrt{2}.$$

4. **A4**. Recall that the vertex from of univariate quadratic function is given by

$$f(x) = ax^2 + bx + c = a\left(x - \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$

   where $a, b, c \in \mathbb{R}$. Also note that $\forall x \in \mathbb{R}, x^2 \geq 0$, so it follows that

$$\left(x - \frac{b}{2a}\right)^2 \geq 0.$$

for all $x \in \mathbb{R}$. So the minima (or maxima) of f occurs when

$$\left(x - \frac{b}{2a}\right)^2 = 0 \iff x = \frac{b}{2a}.$$

And so any univariate has the turning point

$$\left(\frac{b}{2a}, c - \frac{b^2}{4a}\right).$$

(a) So writing the quadratic function in vertex form gives us

$$f(x) = x^2 - 2x + 6 = (x-1)^2 + 5.$$

Using the argument above, the minimum value of f is 5 when $x = 1$.

(b) Similarly

$$g(x) = x^4 + 2x^2 + 2 = (x^2+1)^2 + 1.$$

Hence the minimum value of g is 1 when $x = \pm 1$.

(c) Given the minimum of (a) occurs at $x = 1$ it follows that the shape of (a) in the domain $2 \leq x \leq 3$ is concave, thus the minimum value occurs at the boundary $x = 2$. Substituting this in gives us $f(2) = (2-1)^2 + 5 = 6$.

5. **A5**

(a) We have

$$\begin{aligned} & x^2 - 3x < 4 \\ \iff & x^2 - 3x - 4 < 0 \\ \iff & (x+1)(x-4) < 0. \end{aligned}$$

Hence $x^2 - 3 < 4 \iff x \in (-1, 4)$.

(b) We have

$$\begin{aligned} & y^3 < 2y^2 + 3y \\ \iff & y^3 - 2y^2 - 3y < 0 \\ \iff & y(y-3)(y+1) < 0. \end{aligned}$$

Hence $y^3 < 2y^2 + 3y \iff x \in (-\infty, -1) \cup (0, 3)$.

6. **A6**

(a) We note that $(x+4)$ is a factor of $f(x) = x^3 + 5x^2 - 2x - 24$ since $x = -4$ is a root of f, so by the factor theorem $(x+4)$ is a factor of f. So

$$f(x) = (x+4)g(x).$$

where g is some univariate polynomial of degree 2. So using polynomial division, we have

$$\begin{aligned} g(x) &= \frac{x^3 + 5x^2 - 2x - 24}{x+4} \\ &= \frac{x^2(x+4) + x^2 - 2x - 24}{x+4} \\ &= x^2 + \frac{x^2 - 2x - 24}{x+4} \\ &= x^2 + \frac{x(x+4) - 6x - 24}{x+4} \\ &= x^2 + x - 6\frac{x+4}{x+4} \\ &= x^2 + x - 6 \\ &= (x-2)(x+3). \end{aligned}$$

Hence the fully factorised form of f is

$$f(x) = (x+4)(x+3)(x-2).$$

(b) By inspection we note that $t = 1$ is a root of $f(t) = t^3 - 7t + 6$. So by the factor theorem we have

$$f(t) = (t-1)g(t),$$

where $g(t)$ is some univariate polynomial of degree 2. Intuitively we can see that g has the following form

$$g(t) = t^2 + at - 6.$$

Given that the t^2 coefficient of f is zero, it follows that $a - 1 = 0$, hence $a = 1$. Thus, we have

$$f(t) = (t-1)(t^2 + t - 6) = (t-1)(t+3)(t-2).$$

(c) Factorising the numerator and denominator, we get

$$\begin{aligned}\frac{x^3+x^2-2x}{x^3+2x^2-x-2} &= \frac{x(x-1)(x+2)}{(x-1)(x+1)(x+2)}\\ &= \frac{x}{x+1}.\end{aligned}$$

7. **A7**

(a) We have

$$\begin{aligned}\frac{2}{(x+1)(x-1)} &\equiv \frac{A}{x+1}+\frac{B}{x-1}\\ &\equiv \frac{(A+B)x+(B-A)}{(x+1)(x-1)}.\end{aligned}$$

Equating coefficients yields the following system for equations

$$\begin{aligned}A+B&=0\\ B-A&=2,\end{aligned}$$

which has the following solutions

$$\begin{aligned}A&=-1\\ B&=1.\end{aligned}$$

Substituting these into our original equivalence relation gives

$$\frac{2}{(x+1)(x-1)} = \frac{1}{x-1}-\frac{1}{x+1}.$$

(b) Similarly, we have

$$\begin{aligned}\frac{x+13}{(x+1)(x-2)(x+3)} &\equiv \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x+3}\\ &\equiv \frac{A(x-2)(x+3)+B(x+1)(x+3)+C(x+1)(x-2)}{(x+1)(x-2)(x+3)}\\ &\equiv \frac{(A+B+C)x^2+(A+4B-C)x+(3B-6A-2C)}{(x+1)(x-2)(x+3)}.\end{aligned}$$

Equating coefficients gives us the following system of equations

$$\begin{aligned}A + B + C &= 0\\ A + 4B - C &= 1\\ 3B - 6A - 2C &= 13.\end{aligned}$$

From equation (1) we have

$$-C = A + B.$$

Substituting this into equations (2) and (3) yields

$$\begin{aligned}2A + 5B &= 1\\ 5B - 4A &= 13.\end{aligned}$$

So the solution is $A = -2, B = 1, C = 1$. And so we finally have

$$\frac{x+13}{(x+1)(x-2)(x+3)} = \frac{-2}{x+1} + \frac{1}{x-2} + \frac{1}{x+3}.$$

(c) We have

$$\begin{aligned}\frac{4x+1}{(x+1)^2(x-2)} &\equiv \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\\ &\equiv \frac{A(x+1)^2 + B(x-2)(x+1) + C(x-2)}{(x-2)(x+1)^2}\\ &\equiv \frac{(A+B)x^2 + (2A - B + C)x + (A - 2B - 2C)}{(x-2)(x+1)^2}\end{aligned}$$

Equating coefficients produces the following system of equations

$$\begin{aligned}A + B &= 0\\ 2A - B + C &= 4\\ A - 2B - 2C &= 1,\end{aligned}$$

From equation (1) we get $A = -B$ hence

$$\begin{aligned}3A + C &= 4\\ 3A - 2C &= 1.\end{aligned}$$

So we have the solution $A = 1, B = -1, C = 1$. So the partial fraction decomposition is

$$\frac{4x+1}{(x+1)^2(x-2)} = \frac{1}{x-2} - \frac{1}{x+1} + \frac{1}{(x+1)^2}.$$

(d) We have

$$\frac{4x^2+x-2}{(x-1)(x^2+2)} \equiv \frac{A}{x-1}+\frac{Bx+C}{x^2+2}$$
$$\equiv \frac{(A+B)x^2+(C-B)x+(2A-C)}{(x-1)(x^2+2)}.$$

Equating coefficients gives us the following system of equations

$$A+B=4$$
$$C-B=1$$
$$2A-C=-2.$$

By inspection we see that the solution is $A=1, B=3, C=4$. And so the partial fraction decomposition is

$$\frac{4x^2+x-2}{(x-1)(x^2+2)} = \frac{1}{x-1}+\frac{3x+4}{x^2+2}.$$

8. **FC*** See paper.

9. **FC5**

(a) $x=-1/2$.

(b) Recall the definition of a logarithm. $x=k^{\log_k x}$. So if $\log_a b = c$, then it follows that for any base α we have

$$a^c = b$$
$$\iff \log_\alpha a^c = \log_\alpha b$$
$$\iff c\log_\alpha a = \log_\alpha b$$
$$\iff c = \frac{\log_\alpha b}{\log_\alpha a}.$$

As required.

(c) Changing to base e gives us

$$16\log_x 3 = 16\frac{\ln 3}{\ln x} = \frac{\ln x}{\ln 3} = \log_3 x$$
$$\iff (\ln x)^2 = (\ln 3^4)^2$$
$$\iff \ln x_{1,2} = \pm \ln 3^4$$
$$\iff x_{1,2} = e^{\pm \ln 3^4}$$

So the solutions are $x_{1,2} = 81, 1/81$.

10. **G1**

   (a) Note that given $AB = BC$ we can deduce that triangle ABC is an isosceles triangle, hence if $\angle A = \pi/3$ then it follows that $\angle B = \pi/3$. Given that

$$\angle A + \angle B + \angle C = \pi.$$

   Then $\angle C = \pi/3$. Thus ABC is in-fact a equilateral triangle, giving $AB = BC = CA = 1$.

   (b) Notice that ABC is an isosceles triangle with $AB = BC = 2$ and a base $AC = 3$. Using a method of dissection we produce two congruent right-angled triangles ABM and CBM where M is the midpoint of AC. Let only consider ABM (as the triangles are congruent). Since ABM is a right-angled triangle, we have

$$\cos \angle A = \frac{3}{4}.$$

   So $\angle A = \arccos(3/4)$. Since ABC is an isosceles triangle, then $\angle A = \angle B = \arccos(3/4)$. And given that

$$\angle A + \angle B + \angle C = \pi.$$

   Then $\angle C = \pi - 2\arccos(3/4)$.

11. **G2**

   (a) The length ℓ of a sector is given by $\ell = r\theta$, where r is the radius of the circle and θ is the sector angle (in radians). So when $r = 3$ and $\theta = \pi/3$ it follows that $\ell = \pi$.

   (b) The area A of a sector is given by $A = \frac{1}{2}r^2\theta$. So substituting $r = 3$ and $\theta = \pi/3$ gives us $A = 3/2\pi$.

12. **G3**. We have the lines (in Cartesian form)

$$x = y = z$$
$$x = y = 2z + 1.$$

Writing these lines in a vector form gives us

$$\begin{aligned}\vec{r}_1 &= \lambda \langle 1,1,1\rangle \\ \vec{r}_2 &= \langle 0,0,-1/2\rangle + \mu \langle 1,1,1/2\rangle .\end{aligned}$$

In order to find the angle between $\vec{r}_1$ and $\vec{r}_2$, let us consider their direction vectors $\vec{b}_1 = \langle 1,1,1\rangle$ and $\vec{b}_2 = \langle 1,1,1/2\rangle$. Recall that the dot product between to vectors $\vec{v}$ and $\vec{u}$ is $\vec{v}\cdot\vec{u} = \|\vec{v}\|\|\vec{u}\|\cos\theta$, where θ is the angle between the vectors. Rearranging for θ gives

$$\theta = \arccos\left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|\|\vec{u}\|}\right).$$

So we have

$$\theta = \arccos\left(\frac{5/2}{\sqrt{3}\sqrt{9/4}}\right) = \arccos\left(\frac{5\sqrt{3}}{9}\right).$$

In order to determine whether the lines intersect we need to find λ, μ such that

$$\lambda \langle 1,1,1\rangle = \langle 0,0,-1/2\rangle + \mu \langle 1,1,1/2\rangle .$$

which gives us the following system of equations

$$\begin{aligned}\lambda &= \mu \\ \lambda &= -1/2 + 1/2\mu,\end{aligned}$$

which has the solution $\lambda = \mu = -1$, hence the lines intersect when $\lambda = \mu = -1$.

13. **SS1**. Recall that the general term of an arithmetic progression with an initial term u_1 and a constant term difference d is given by $u_n = u_1 + (n-1)d$. And so given α is the 3rd term and β is the 9th term, we can then form the following system of equations

$$\begin{aligned}\alpha &= u_1 + 2d \\ \beta &= u_1 + 8d.\end{aligned}$$

Rearranging for d yields

$$d = \frac{\beta - \alpha}{6}.$$

Substituting this into equation (1) produces

$$u_1 = \frac{4\alpha - \beta}{3}.$$

Also recall that the sum of the first n terms of the arithmetic progression $u_1, u_2, \dots, u_n$ is

$$S_n = \sum_{r=1}^{n} u_r = \frac{1}{2}n(2u_1 + (n-1)d).$$

So for the first thirty terms we have

$$\begin{aligned} S_{30} &= \frac{1}{2}(30)\left(2\frac{4\alpha - \beta}{3} + 29\frac{\beta - \alpha}{6}\right) \\ &= \frac{5}{2}\left(25\beta - 13\alpha\right). \end{aligned}$$

14. **SS2**

(a) Applying the binomial theorem gives us

$$\begin{aligned} (1+x)^3 &= \sum_{r=0}^{3} \binom{3}{r} (1)^{3-r}(x)^r \\ &= 1 + 3x + 3x^2 + x^3. \end{aligned}$$

(b) Applying the binomial theorem gives us

$$\begin{aligned} (2+x)^4 &= \sum_{r=0}^{4} \binom{4}{r} (2)^{4-r}(x)^r \\ &= 16 + 32x + 24x^2 + 8x^3 + x^4. \end{aligned}$$

(c) We have

$$\begin{aligned} \left(2 + \frac{3}{x}\right)^5 &= \left(\frac{3+2x}{x}\right)^5 \\ &= \frac{1}{x^5}(3+2x)^5 \end{aligned}$$

Applying the binomial theorem yields

$$\begin{aligned}\frac{1}{x^5}(3+2x)^5 &= \frac{1}{x^5}\sum_{r=0}^{5}\binom{5}{r}(3)^{5-r}(2x)^r \\ &= \frac{1}{x^5}\left(243+810x+1080x^2+720x^3+240x^4+32x^5\right)\end{aligned}$$

15. **SS3**

(a) Recall that the sum of the first n terms of a arithmetic progression with general term $u_n = a+(n-1)d$ is

$$S_n = \sum_{r=1}^{n} u_r = \frac{1}{2}n(2a+(n-1)d).$$

Given the arithmetic progression $u_n = n$, it follows that $a=1, d=1$. So we have

$$\sum_{r=1}^{n} u_r = \frac{1}{2}n(2+n-1) = \frac{1}{2}n(n+1).$$

As required.

(b) Consider the arithmetic progression with general term $u_n = 2n+1$. So the sum of the first n terms of u_n is

$$S_n = \sum_{r=1}^{n}(2r+1) = 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = n(n+2).$$

Given u_n is a sequence of odd integers and $u_5 = 11$ and $u+49 = 99$. Then we have

$$\begin{aligned}S_{49} - S_5 &= \sum_{n=1}^{49}(2n+1) - \sum_{n=1}^{5}(2n+1) \\ &= 49(49+2) - 5(5+2) \\ &= 2469\end{aligned}$$

(c) We have

$$\sum_{n=1}^{5}(3n+2) = 3\sum_{n=1}^{5} n + 10 = \frac{3}{2}(5)(6) + 10 = 55.$$

(d) We have

$$\begin{aligned}\sum_{n=0}^{N}(an+b) &= a\sum_{n=1}^{N} n + (N+1)b \\ &= \frac{a}{2}N(N+1) + (N+1)b \\ &= (N+1)\left(\frac{aN}{2} + b\right)\end{aligned}$$

(e) Recall that the sum of the first n terms of a geometric progression with general term $u_n = ar^{n-1}$ is

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a\frac{1-r^n}{1-r} = a\frac{r^n-1}{r-1}.$$

We have

$$\sum_{n=0}^{10} 2^n = \sum_{n=1}^{11}(1)2^{n-1} = \frac{1-2^{11}}{1-2} = 2047.$$

(f) Similarly, we have

$$\begin{aligned}\sum_{n=9}^{N} ar^{2n} &= \sum_{n=1}^{N+1} ar^{2(n-1)} \\ &= a\sum_{n=1}^{N+1}(r^2)^{n-1} \\ &= a\frac{1-r^{2N+2}}{1-r^2}\end{aligned}$$

16. **SS4** Given that $u_{n+1} = ku_n$, then

$$u_n = k(k(k(\cdots ku_1))) = \underbrace{k \times k \times \cdots \times k}_{n-times} \times u_1 = k^n.$$

So we have the following cases as $n \to \infty$:

(a) $k > 1$. $\lim_{n\to\infty} u_n \to \infty$

(b) $k = 1$. $\lim_{n\to\infty} u_n = 1$.

(c) $0 < k < 1$. $\lim_{n\to\infty} u_n \to 0$

(d) $k = 0$. u_n is zero for all $n \neq 0$.

(e) $-1 < k < 0$. $\lim_{n\to\infty} u_n \to 0$, but oscillates about the line $x = 0$.

(f) $k = -1$. $\lim_{n\to\infty} u_n$ oscillates from 1 to -1.

(g) $k < -1$. $\lim_{n\to\infty} |u_n| \to \infty$, but oscillates about the line $x = 0$.

17. **SS5** Recall that the Maclaurin series for $(1+x)^n$ is

$$(1+x)^n = \sum_{r=0}^{\infty} \frac{n(n-1)\cdots(n-r+1)}{r!} x^r.$$

Which is valid for $|x| < 1$. Using this, we have:

(a)

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{8}x^3 + O(x^4).$$

This expansion is valid for $|x| < 1$.

(b)

$$\begin{aligned}(2+x)^{\frac{2}{5}} &= \sqrt[5]{4}\left(1 + \frac{x}{2}\right)^{\frac{2}{5}} \\ &= \sqrt[5]{4}\left(1 + \frac{1}{5}x - \frac{3}{100}x^2 + \frac{2}{125}x^3 + O(x^4)\right)\end{aligned}$$

This expansion is valid for $|x| < 2$

(c)

$$\begin{aligned}(1+2x)^{\frac{1}{2}}(2+x)^{-\frac{1}{3}} = 2^{-\frac{1}{3}} &\left(1 + x - \frac{1}{2}x^2 + x^3 + O(x^4)\right) \\ &\times \left(1 - \frac{1}{6}x + \frac{1}{18}x^2 - \frac{7}{162}x^3 + O(x^4)\right)\end{aligned}$$

This expansion is valid if and only if $|2x| < 1$ and $\left|\frac{x}{2}\right| < 1$, so the expansion is valid if and only if $|x| < \frac{1}{2}$.

18. **SS6** Using the approximations

$$\sin\theta \approx \theta - \frac{1}{6}\theta^3$$
$$\cos\theta \approx 1 - \frac{1}{2}\theta^2,$$

we have

$$\begin{aligned}\sin\left(\frac{\theta}{2}\right)\cos\theta + \sec 2\theta &\approx \left(\frac{\theta}{2} - \frac{1}{48}\theta^3\right)\left(1 - \frac{1}{2}\theta^2\right) + \frac{1}{1-2\theta^2} \\ &= \frac{\theta}{2}\left(1 - \frac{13}{24}\theta^2 + \frac{\theta^4}{48}\right) + \frac{1}{1-2\theta^2}.\end{aligned}$$

Recall the Maclaurin series for $1/(1-x)$ is

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Hence

$$\frac{1}{1-2\theta^2} = 1 + 2\theta^2 + 4\theta^4 + \cdots .$$

for all $\theta \in \mathbb{R}$. And so substituting this in yields

$$\begin{aligned}\sin\left(\frac{\theta}{2}\right)\cos\theta + \sec 2\theta &\approx \frac{\theta}{2} - \frac{13}{48}\theta^3 + 1 + 2\theta^2 \\ &= 1 + \frac{\theta}{2} + 2\theta^2 - \frac{13}{48}\theta^3\end{aligned}$$

19. **T1** We have

$$\begin{aligned} 2\sin^2\theta &= 1 \\ \iff \sin\theta &= \pm\frac{1}{\sqrt{2}}\end{aligned}$$

Using the CAST mnomic, we get the following general solution

$$\theta \in \left\{2n\pi \pm \frac{\pi}{4} : n \in \mathbb{Z}\right\} \cup \left\{(2n+1)\pi \pm \frac{\pi}{4} : n \in \mathbb{Z}\right\}.$$

Given we're only interested in solutions in the domain $[0, 2\pi]$, we then have

$$\theta_{1,\ldots,4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

20. **T2** We have

$$\begin{aligned}\frac{\cot^2 x + \sin^2 x}{\cos x + \operatorname{cosec} x} &\equiv \frac{\operatorname{cosec}^2 x - (1 - \sin^2 x)}{\cos x + \operatorname{cosec} x}\\ &\equiv \frac{\operatorname{cosec}^2 x - \cos^2 x}{\operatorname{cosec} x + \cos x}\\ &\equiv \frac{(\operatorname{cosec} x - \cos x)(\operatorname{cosec} x + \cos x)}{\operatorname{cosec} x + \cos x}\\ &\equiv \operatorname{cosec} x - \cos x.\end{aligned}$$

21. **T3** Recall that the compound angle formulae are

$$\begin{aligned}\sin(\phi \pm \psi) &= \sin\phi\cos\psi \pm \cos\phi\sin\psi\\ \cos(\phi \pm \psi) &= \cos\phi\cos\psi \mp \sin\phi\sin\psi.\end{aligned}$$

Applying these to the following questions gives:

(a)

$$\begin{aligned}\cos\pi/12 &= \cos(\pi/3 - \pi/4)\\ &= \cos\pi/3\cos\pi/4 + \sin\pi/3\sin\pi/4\\ &= \frac{1}{2}\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

(b)

$$\begin{aligned}\sin\pi/12 &= \sin(\pi/3 - \pi/4)\\ &= \sin\pi/3\cos\pi/4 - \cos\pi/3\sin\pi/4\\ &= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} - \frac{1}{2}\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

(c)

$$\begin{aligned}\cot \pi/12 &= \frac{\cos \pi/12}{\sin \pi/12} \\ &= \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} \\ &= 2+\sqrt{3}.\end{aligned}$$

22. **T4** If $t = \tan \frac{\theta}{2}$, then we have a right-angled triangle with opposite t, adjacent 1 and hypotenuse $\sqrt{1+t^2}$. So it follows that

$$\begin{aligned}\sin \frac{\theta}{2} &= \frac{t}{\sqrt{1+t^2}} \\ \cos \frac{\theta}{2} &= \frac{1}{\sqrt{1+t^2}}.\end{aligned}$$

Applying the double angle formulae yields

$$\begin{aligned}\cos\theta &= \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2} \\ \sin\theta &= 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}.\end{aligned}$$

And from definition of tan we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2t}{1-t^2}.$$

23. **T5** Recall that the compound angle formula for tan is

$$\tan(\phi \pm \psi) = \frac{\tan\phi \pm \tan\psi}{1 \mp \tan\phi\tan\psi}.$$

So applying this produces

$$\begin{aligned}\tan\left(\arctan\frac{1}{3} + \arctan\frac{1}{4}\right) &= \frac{\tan\arctan\frac{1}{3} + \tan\arctan\frac{1}{4}}{1 - \tan\arctan\frac{1}{3}\tan\arctan\frac{1}{4}} \\ &= \frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{3}\frac{1}{4}} \\ &= \frac{7}{11}.\end{aligned}$$

24. **T6** Let us consider any two angles ϕ, ψ. We wish to know what the product of two sines is. So applying the compound angle formula for cos we get

$$\begin{aligned}\cos(\phi-\psi)-\cos(\phi+\psi) &= (\cos\phi\cos\psi+\sin\phi\sin\psi)-(\cos\phi\cos\psi-\sin\phi\sin\psi)\\ &= 2\sin\phi\sin\psi.\end{aligned}$$

Substituting A, B and C in produces

$$\cos\left(\frac{B-C}{2}\right)-\cos\left(\frac{B+C}{2}\right)=2\sin\frac{B}{2}\sin\frac{C}{2}.$$

Now since A, B, C are angles of a triangle, it follows that

$$A+B+C=\pi.$$

So $B+C=\pi-A$ giving us

$$\cos\left(\frac{B-C}{2}\right)-\cos\left(\frac{\pi-A}{2}\right)=2\sin\frac{B}{2}\sin\frac{C}{2}.$$

Recall the identity

$$\cos\left(\frac{\pi}{2}-\theta\right)\equiv\sin\theta.$$

Thus

$$\cos\left(\frac{B-C}{2}\right)-\sin\frac{A}{2}=2\sin\frac{B}{2}\sin\frac{C}{2}.$$

As required.

25. **T7** We have

$$\sqrt{3}\sin\theta+\cos\theta\equiv A\sin(\theta+\alpha)=A\sin\theta\cos\alpha+A\cos\theta\sin\alpha.$$

Equating coefficients produces the following system of equations

$$\begin{aligned}\sqrt{3}&=A\cos\alpha\\ 1&=A\sin\alpha\end{aligned}$$

Squaring equations (1) and (2) and adding them together gives us

$$1+3=A^2(\cos^2\alpha+\sin^2\alpha)=A^2.$$

which implies $A = 2$. Now dividing equations (1) and (2)

$$\tan\alpha = \frac{1}{\sqrt{3}}.$$

Hence $\alpha = \pi/6$.

26. **T8** Recall the triple angle formulae are

$$\begin{aligned}\cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ \sin 3\theta &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

And so we have

$$\begin{aligned}\cos\theta + \cos 3\theta &= \sin\theta + \sin 3\theta \\ \iff \cos\theta + 4\cos^3\theta - 3\cos\theta &= \sin\theta + 3\sin\theta - 4\sin^3\theta \\ \iff 4\cos^3\theta - 2\cos\theta &= 4\sin\theta - 4\sin^3\theta \\ \iff \cos^3\theta(4 - 2\sec^2\theta) &= \cos^3\theta(4\tan\theta\sec^2\theta - 4\tan^3\theta) \\ \iff \cos^3\theta(2 - 2\tan^2\theta) &= \cos^3\theta(4\tan\theta) \\ \iff \cos^3(\tan^2\theta + 2\tan\theta - 1) = 0\end{aligned}$$

So we have a quadratic equation in terms of $\tan\theta$ or $\cos^3\theta = 0$. Solving the quadratic equation gives us

$$\tan\theta = -1 \pm \sqrt{2}.$$

and

$$\cos^3\theta = 0 \iff \cos\theta = 0.$$

So in the domain $[0, 2\pi]$, the solutions are

$$\theta \in \left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}, \frac{\pi}{2}, \frac{3\pi}{2}\right\}.$$

27. **V1**

(a) $\|\vec{A}\| = \sqrt{293}$, $\|\vec{B}\| = \sqrt{293}$, $\|\vec{C}\| = \sqrt{290}$ and $\|\vec{D}\| = 17$, hence $\|\vec{A}\| = \|\vec{B}\| > \|\vec{C}\| > \|\vec{D}$.

(b) Recall that the dot product between to vectors $\vec{v}$ and $\vec{u}$ is $\vec{v}\cdot\vec{u} = \|\vec{v}\|\|\vec{u}\|\cos\theta$, where θ is the angle between the vectors. Rearranging for θ gives

$$\theta = \arccos\left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|\|\vec{u}\|}\right).$$

Applying the above, we get

i. The dot product between $\vec{A}$ and $\vec{B}$ is $\vec{A}\cdot\vec{B} = -29$. So

$$\theta = \arccos\left(\frac{-29}{293}\right).$$

ii. The dot product between $\vec{B}$ and $\vec{C}$ is $\vec{B}\cdot\vec{C} = 1$. So

$$\theta = \arccos\left(\frac{2}{\sqrt{293}\sqrt{290}}\right).$$

28. **V2**

(a) Let us consider the vector $\overrightarrow{AB} = \vec{B} - \vec{A}$. So

$$\overrightarrow{AB} = \langle -12, 20, -10\rangle .$$

Given that the distance d between the points is equal to the magnitude of the vector $\overrightarrow{AB}$, then if follows that the distance d is

$$d = \sqrt{12^2 + 20^2 + 10^2} = 2\sqrt{161}.$$

29. **D1**

(a) Computing the first and second derivatives of y gives us

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$$

Hence we have a minima at $x = 0$ with no points of inflection.

(b) Computing the first and second derivatives of y gives us

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3(x^2-1) = 3(x+1)(x-1)\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 6x.\end{aligned}$$

Hence we have a point of inflection at $x = 0$ and a minima at $x = 1$ and a maximum at $x = -1$.

(c) Computing the first and second derivatives of y gives us

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2-6x+3 = 3(x^2-2x+1) = 3(x-1)^2\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 6(x-1).\end{aligned}$$

Hence we have a point of inflection at $x = 1$ and a stationary point at $x = 1$.

(d) Computing the first and second derivatives of y gives us

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2+3 = 3(x^2+1)\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 6x.\end{aligned}$$

So we have a point of inflection at $x = 0$.

30. **D2** Let $f(x) = x^2 + 1$, then by first principles we have

$$\begin{aligned}f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+\Delta x^2+1-x^2-1}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta x^2}{\Delta x}\\ &= \lim_{\Delta x\to 0} 2x+\Delta x\\ &= 2x.\end{aligned}$$

31. **D3**

(a)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\sin x^2.$$

(b)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = a^x \ln x.$$

(c) Note that
$$y = \ln(2x^a + 1) - \ln x^a = \ln(2x^a + 1) - a\ln x.$$
So we have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2ax^{a-1}}{2x^a + 1} - \frac{a}{x}.$$

(d) By definition of logarithms, we have
$$y = x^x = e^{\ln x^x} = e^{x\ln x}.$$
So
$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x\ln x} \times (\ln x + 1) = x^x(\ln x + 1).$$

(e) We have
$$\begin{aligned} y &= \arcsin x \\ \iff \sin y &= x \\ \iff \frac{\mathrm{d}}{\mathrm{d}x}\sin y &= \frac{\mathrm{d}}{\mathrm{d}x}x \\ \iff \cos y\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \iff \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos y} \end{aligned}$$
Given $\mathrm{rng}(\arcsin) = [-\pi/2, \pi/2]$ so it follows that $\cos y$ is positive for all $y \in [-\pi/2, \pi/2]$. Using the Pythagorean identity, we have
$$\cos y = \sqrt{1 - x^2}.$$
Hence
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 - x^2}.$$

32. **D4**

$$\begin{aligned} y + e^y &= x^3 + x + 1 \\ \iff \frac{\mathrm{d}}{\mathrm{d}x} y + e^y &= \frac{\mathrm{d}}{\mathrm{d}x} x^3 + x + 1 \\ \iff \frac{\mathrm{d}y}{\mathrm{d}x}(1 + e^y) &= 3x^2 + 1 \\ \iff \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2 + 1}{1 + e^y} \end{aligned}$$

33. **D5** We have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{(t-2) - (t+1)}{(t-2)^2} = -\frac{3}{(t-2)^2} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{2(t-3) - (2t+1)}{(t-3)^2} = -\frac{7}{(t-3)^2} \end{aligned}$$

Applying the chain rule yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{(t-2)^2} \frac{(t-3)^2}{7}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1} = \frac{12}{7}.$$

34. **I1**

(a) Let $x = \sqrt{2}\tan\theta$, then $\mathrm{d}x = \sqrt{2}\sec^2\theta\,\mathrm{d}\theta$. Hence

$$\begin{aligned} \int \frac{1}{2 + x^2}\,\mathrm{d}x &= \int \frac{1}{2 + 2\tan^2\theta}\sqrt{2}\sec^2\theta\,\mathrm{d}\theta \\ &= \frac{1}{\sqrt{2}} \int \frac{\sec^2\theta}{1 + \tan^2\theta}\,\mathrm{d}\theta \\ &= \frac{1}{\sqrt{2}} \int \mathrm{d}\theta \\ &= \frac{1}{\sqrt{2}}\theta + \kappa \end{aligned}$$

Thus

$$\int \frac{1}{2 + x^2}\,\mathrm{d}x = \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + \kappa.$$

(b) Let $x-1=2\sin\theta$, then $\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$ Thus

$$\begin{aligned}\int \frac{1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{4-(x-1)^2}}\,\mathrm{d}x \\ &= \int \frac{1}{\sqrt{4-4\sin^2\theta}} 2\cos\theta\,\mathrm{d}\theta \\ &= \frac{1}{2}\int \frac{\cos\theta}{\cos\theta}\,\mathrm{d}\theta \\ &= \frac{1}{2}\theta + \kappa\end{aligned}$$

Hence

$$\int \frac{1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x = \frac{1}{2}\arcsin\left(\frac{x-1}{2}\right) + \kappa.$$

(c) Let $u=\sqrt{1-x}$, then $\mathrm{d}u = -\frac{1}{2\sqrt{1-x}}$. So we have

$$\begin{aligned}I = \int \frac{1}{x\sqrt{1-x}}\,\mathrm{d}x &= -\int \frac{2\,\mathrm{d}x}{1-u^2} \\ &= \int \frac{2\,\mathrm{d}u}{(u+1)(u-1)}\end{aligned}$$

Notice that the integrand can be expressed as a partial fraction, hence

$$\begin{aligned}\frac{2}{(u+1)(u-1)} &\equiv \frac{A}{u+1} + \frac{B}{u-1} \\ &\equiv \frac{(A+B)u + (B-A)}{(u+1)(u-1)}\end{aligned}$$

Equating coefficients produces the following system of equations

$$\begin{aligned}A+B&=0\\ B-A&=2\end{aligned}$$

So by inspection we have the solution $A, B = -1, 1$. Substituting this back into our integrand we get

$$\begin{aligned} I &= \int \frac{1}{u-1}\,\mathrm{d}u - \int \frac{1}{u+1}\,\mathrm{d}u \\ &= \ln|u-1| - \ln|u+1| \\ &= \ln\left|\sqrt{1-x}-1\right| - \ln\left|\sqrt{1-x}+1\right| + \kappa \\ &= \ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| + \kappa \end{aligned}$$

(d) Recall the integration by parts states

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x\,.$$

Now let

$$u = \ln x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

Giving us

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \text{ and } v = x.$$

Applying the rule we state above yields

$$\int \ln x\,\mathrm{d}x = x\ln x - \int \frac{x}{x}\,\mathrm{d}x = x\ln x - x + \kappa.$$

35. **I2**

(a) We'll apply integration by parts again using

$$u = x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = e^{-x}.$$

then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } v = -e^{-x}.$$

So we have

$$\begin{aligned} \int_0^L xe^{-x}\,\mathrm{d}x &= \left[-xe^{-x}\right]_0^L + \int_0^L e^{-x}\,\mathrm{d}x \\ &= -Le^{-L} + \left[-e^{-x}\right]_0^L \\ &= -Le^{-L} + \left(-e^{-L}+1\right) \\ &= 1 - e^{-L}(L-1) \end{aligned}$$

Now let us consider the limiting value of the integral as $L \to \infty$. So we have

$$\lim_{L\to\infty} \int_0^L xe^{-x}\,\mathrm{d}x = \lim_{L\to\infty} 1 - \frac{L-1}{e^L}$$

On the right hand side, we have

$$1 - \frac{\infty}{\infty}.$$

Thus we must apply L'Hopital's rule. So

$$\begin{aligned}\lim_{L\to\infty} \frac{L-1}{e^L} &= \lim_{L\to\infty} \frac{1}{e^L} \\ &\to 0^+\end{aligned}$$

Therefore

$$\lim_{L\to\infty} \int_0^L xe^{-x}\,\mathrm{d}x = 1.$$

(b) Recall that the triple angle formula for $\sin 3\theta$ is

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Substituting this into the integrand yields

$$\begin{aligned}I = \int_0^{\pi/2} \sin 3\theta \cos\theta\,\mathrm{d}\theta &= \int_0^{\pi/2} (3\sin\theta - 4\sin^3\theta)\cos\theta\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} 3\sin\theta\cos\theta\,\mathrm{d}\theta - \int_0^{\pi/2} 4\sin^3\theta\cos\theta\,\mathrm{d}\theta\end{aligned}$$

Using the substitution $u = \sin\theta$ in both integrals gives us $\mathrm{d}u = \cos\theta\,\mathrm{d}\theta$ and

$$\begin{aligned}I &= \int_0^1 3u\,\mathrm{d}u - \int_0^1 4u^3\,\mathrm{d}u \\ &= \left[\frac{3}{2}u^2\right]_0^1 - \left[u^4\right]_0^1 \\ &= \frac{3}{2} - 1 = \frac{1}{2}\end{aligned}$$

(c) Let $u = x^3 + 3x + 2$, then $\mathrm{d}u = 3(x^2+1)\,\mathrm{d}u$. So

$$\begin{aligned}\int_0^1 \frac{x^2+1}{x^3+3x+2}\,\mathrm{d}x &= \frac{1}{3}\int_2^6 \frac{1}{u}\,\mathrm{d}u\\ &= \frac{1}{3}\left[\ln|u|\right]_2^6\\ &= \frac{1}{3}\ln 3.\end{aligned}$$

(d) Using the Weierstrass substitution $t = \tan\frac{\theta}{2}$ (and our answers from **T4**) which are

$$\begin{aligned}\cos\theta &= \frac{1-t^2}{1+t^2}\\ \sin\theta &= \frac{2t}{1+t^2}\end{aligned}$$

we then have

$$\begin{aligned}\mathrm{d}t &= \frac{1}{2}\sec^2\frac{\theta}{2}\,\mathrm{d}\theta\\ \iff \mathrm{d}t &= \frac{1}{2}(1+t^2)\,\mathrm{d}\theta\\ \iff \mathrm{d}\theta &= \frac{2\,\mathrm{d}t}{1+t^2}\end{aligned}$$

Hence

$$\begin{aligned}I &= \int_0^{\pi/2} \frac{1}{3+5\cos\theta}\,\mathrm{d}\theta\\ &= \int_0^1 \frac{1}{3+5\frac{1-t^2}{1+t^2}}\frac{2\,\mathrm{d}t}{1+t^2}\\ &= \int_0^1 \frac{1+t^2}{8-2t^2}\frac{2}{1+t^2}\,\mathrm{d}t\\ &= \int_0^1 \frac{\mathrm{d}t}{4-t^2}\\ &= \int_0^1 \frac{\mathrm{d}t}{(2-t)(2+t)}\end{aligned}$$

Notice that the integrand can be expressed as a partial fraction, thus

$$\begin{aligned}\frac{1}{(2-t)(2+t)} &\equiv \frac{A}{2-t}+\frac{B}{2+t}\\ &\equiv \frac{(A-B)t+2(A+B)}{(2-t)(2+t)}\end{aligned}$$

Equating coefficients produces the following system of equations

$$\begin{aligned}A-B&=0\\ 2(A+B)&=1\end{aligned}$$

So by inspection we have the solution $A=\frac{1}{4}$ and $B=-\frac{1}{4}$. Substituting this back into our integrand yields

$$\begin{aligned}I &= \frac{1}{4}\left\{\int_0^1 \frac{1}{2-t}\,\mathrm{d}t - \int_0^1 \frac{1}{2+t}\,\mathrm{d}t\right\}\\ &= \frac{1}{4}\left\{[-\ln|2-t|]_0^1 - [\ln|2+t|]_0^1\right\}\\ &= \frac{1}{4}\left\{(-\ln 1+\ln 2)-(\ln 3-\ln 2)\right\}\\ &= \frac{1}{4}\ln 3\end{aligned}$$

36. **DE1** We have

$$\begin{aligned}x\frac{\mathrm{d}y}{\mathrm{d}x}+1-y^2&=0\\ \iff x\frac{\mathrm{d}y}{\mathrm{d}x}&=y^2-1\\ \iff \frac{1}{y^2-1}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{1}{x}\end{aligned}$$

Integrating both sides with respect to x gives us

$$\begin{aligned}\int \frac{1}{y^2-1}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int \frac{\mathrm{d}x}{x}\\ \iff \int \frac{\mathrm{d}y}{y^2-1} &= \int \frac{\mathrm{d}x}{x}\end{aligned}$$

Notice that the integrand on the right hand side can be expressed as a partial fraction, hence

$$\begin{aligned}\frac{1}{(y+1)(y-1)} &\equiv \frac{A}{y+1} + \frac{B}{y-1} \\ &\equiv \frac{(A+B)y + (B-A)}{(y+1)(y-1)}\end{aligned}$$

Equating coefficient produces the following system of equations

$$\begin{aligned}A + B &= 0 \\ B - A &= 1\end{aligned}$$

By inspection, we have the solution $A = -1/2$ and $B = 1/2$. Substituting this back into our original integrand yields

$$\begin{aligned}\frac{1}{2}\left\{\int \frac{\mathrm{d}y}{y-1} - \int \frac{\mathrm{d}y}{y+1}\right\} &= \ln|x| + \kappa_1 \\ \iff \frac{1}{2}\left\{\ln|y-1| - \ln|y+1|\right\} + \kappa_2 &= \ln|x| + \kappa_1 \\ \iff \ln\left|\frac{y-1}{y+1}\right| &= 2\ln|x| + \kappa \\ \iff \frac{y-1}{y+1} &= \lambda x^2\end{aligned}$$

Given the initial value condition $(x, y) = (1, 0)$ we have

$$\frac{-1}{1} = \lambda \implies \lambda = -1.$$

So the particular solution is

$$\begin{aligned}\frac{y-1}{y+1} &= -x^2 \\ \iff y - 1 &= -x^2 y - x^2 \\ \iff y(1+x^2) &= 1 - x^2 \\ \iff y &= \frac{1-x^2}{1+x^2}\end{aligned}$$

37. **C1**

(a) Let us first express

$$z = \frac{1+i}{2-i}.$$

in the form $z = a + bi$, where $a, b \in \mathbb{R}$. Rationalising the denominator gives us

$$\begin{aligned} z &= \frac{1+i}{2-i}\frac{2+i}{2+i} \\ &= \frac{1+3i}{5}. \end{aligned}$$

And so

$$\begin{aligned} \Re(z) &= \frac{1}{5} \\ \Im(z) &= \frac{3}{5}. \end{aligned}$$

(b) Applying the quadratic formula gives us

$$\begin{aligned} z_{1,2} &= \frac{2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2} \\ &= 1 \pm i \end{aligned}$$

The modulus of each root is

$$|z_{1,2}| = \sqrt{1^2 + (\pm 1)^2} = \sqrt{2}.$$

The argument of z_1 is

$$\arg z_1 = \pi/4.$$

By symmetry argument, it follows that the argument of z_2 is

$$\arg z_2 = -\pi/4.$$

See paper for Argand diagram.

38. **C2**

(a) Recall that de Moivre's theorem states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

So let us consider consider $z = (\cos\theta + i\sin\theta)^5$, hence by de Moivre's theorem $\Re(z) = \cos 5\theta$. Applying the binomial theorem to z gives us

$$\begin{aligned} z = \cos 5\theta + i\sin 5\theta &= \sum_{r=0}^{5}\binom{5}{r}(\cos\theta)^{5-r}(i\sin\theta)^r \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta \\ &\quad + 5\cos\theta\sin^4\theta + i\sin^5\theta \\ &= \cos\theta(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta) \\ &\quad + i\sin\theta(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta) \end{aligned}$$

Hence

$$\Re(z) = \cos 5\theta = \cos\theta(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta).$$

(b) See paper

39. **VE1**

(a) The vectors $\vec{A}, \vec{B}, \vec{C}$ lie on the line on the line $\vec{r} = \vec{a} + \lambda\vec{b}$ if and only if $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, that is to say $\overrightarrow{BC} = \lambda\overrightarrow{AB}$, where $\lambda \in \mathbb{R}$. Note that

$$\overrightarrow{AB} = \vec{B} - \vec{A} = \langle 1, 1, -1\rangle.$$

and

$$\overrightarrow{BC} = \vec{C} - \vec{B} = -2\langle 1, 1, -1\rangle = -2\overrightarrow{AB}.$$

Hence $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel. Thus the equation of the line they all lie on must have the directional vector $\vec{b} = \langle 1, 1, -1\rangle$. So the equation of our line is

$$\vec{r} = \langle 1, 0, 1\rangle + \lambda\langle 1, 1, -1\rangle.$$

40. **M1**

(a)

$$\begin{aligned} A+B &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 7 & 6 \end{pmatrix}. \end{aligned}$$

(b)

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 10 & 5 \end{pmatrix}.$$

(c)

$$BA = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -5 & -8 \\ 10 & 16 \end{pmatrix}.$$

41. **M2** Let us consider the matrices

$$\begin{aligned} A &= \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 0 \\ q & r \end{pmatrix}, \end{aligned}$$

where $p, q, r \in \mathbb{R}$. So we have

$$\begin{aligned} AB &= \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ q & r \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

However,

$$\begin{aligned} BA &= \begin{pmatrix} 0 & 0 \\ q & r \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ pq & 0 \end{pmatrix} \end{aligned}$$

Why does this work (and why pick A and B)? It is because A and B are singular. If $AB = 0$, then it implies that at least one of the two matrices is singular.

Proof. Assume there exists two non-singular matrices A and B such that $AB = 0$. It follows that if they are non-singular, then the matrices A^{-1} and B^{-1} exist such that

$$A^{-1}AB = B$$
$$ABB^{-1} = A$$

But $AB = 0$, and since $M \times 0 = 0$, then it follows that $A = B = 0$. A contradiction! Thus proving the initial assumption that A, B are non-singular is false. (*I got bored...*) □

42. **M3** Let A be a 2×2 matrix such that

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Recall that if M is an $n \times n$ matrix and λ is some scalar, then

$$\det(\lambda M) = \lambda^n \det M.$$

and an enlargement by a scale factor λ on the matrix N is

$$N' = \lambda N.$$

Now if A is a transformation of a rotation R followed by an enlargement S of scale factor λ, then $A = SR = \lambda R$. We also note that R will be in the form

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Which by inspection has the determinant $\det R = 1$ (due to the Pythagorean identity). Given that

$$\det A = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 \times 1 + 1 \times 1 = 2.$$

Then

$$\det A = 2 = \lambda^2.$$

Hence $\lambda = \sqrt{2}$ (We ignore the negative solution as it's not relevant to this question). So

$$A = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \equiv \sqrt{2}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Equating elements of the matrices produces the following system of equations

$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\sin\theta = -\frac{1}{\sqrt{2}}$$

Giving us $\tan\theta = -1$. So the general solution is

$$\theta \in \left\{2n\pi - \frac{\pi}{4} : n \in \mathbb{Z}\right\} \cup \left\{(2n+1)\pi - \frac{\pi}{4} : n \in \mathbb{Z}\right\}.$$

For simplicity, we'll use the solution $\theta = 3\pi/4$. Hence

$$A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \cos\frac{3\pi}{4} & \sin\frac{3\pi}{4} \\ -\sin\frac{3\pi}{4} & \cos\frac{3\pi}{4} \end{pmatrix}.$$

43. **SE1**

(a)

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$

(b) Recall that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 \text{ and } \sum_{r=1}^{n} r = \frac{1}{2}n(n+1).$$

So using the above formulae and the linearity of the summation, we have

$$\begin{aligned} S_n = \sum_{r=1}^{n} r(r^2+2) &= \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r \\ &= \frac{1}{4}n^2(n+1)^2 + n(n+1) \\ &= n(n+1)\left(\frac{1}{4}n(n+1)+1\right) \end{aligned}$$

44. **SE2**

(a) Note that the general term of the summation can be expressed as a partial fraction, so

$$\begin{aligned}\frac{1}{r(r+1)} &\equiv \frac{A}{r} + \frac{B}{r+1}\\ &\equiv \frac{(A+B)r + A}{r(r+1)}\end{aligned}$$

Equating coefficients produces the following system of equations

$$\begin{aligned}A + B &= 0\\ A &= 1\end{aligned}$$

So the solution is $A = 1, B = -1$. Hence the summation S_n can now be expressed as

$$S_n = \sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1} \equiv \sum_{r=1}^{n} f(r) - f(r-1).$$

Note that the right hand side equivalence relation is the general form a telescopic series on which we can apply the method of difference. Hence we can apply the method of difference on S_n, giving us

$$\begin{aligned}S_n &= \frac{1}{1} - \frac{1}{2}\\ &+ \frac{1}{2} - \frac{1}{3}\\ &\vdots\\ &+ \frac{1}{n-1} - \frac{1}{n}\\ &+ \frac{1}{n} - \frac{1}{n+1}\end{aligned}$$

Cancelling terms produces

$$S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

45. **IN1** Given that $a_1 = 1$ and $a_{n+1} = 3a_n + 4$, then we have the sequence

$$1, 7, 25, 79, \ldots = 1, 3+4, 3^2 + 4(3+1), 3^3 + 4(3^2 + 3 + 1), \ldots.$$

By inspection, we see that

$$a_n = 3^{n-1} + 4\sum_{r=1}^{n-1} 3^{r-1}$$

Recall that

$$\sum_{k=1}^{n} ar^{k-1} = a\frac{1-r^n}{1-r} = a\frac{r^n - 1}{r-1}.$$

Hence

$$a_n = 3^{n-1} + 4\frac{3^{n-1}-1}{2} = 3^{n-1} + 2(3^{n-1} - 1) = 3^n - 2.$$

We will now prove that this is true for all $n \in \mathbb{Z}^+$ by induction on n.

Proof.
Base Case: When $n = 1$, we have $a_1 = 3^1 - 2 = 1 = a_1$. So the statement holds when $n = 1$.
Inductive Hypothesis: Suppose the statement holds for all integer values of n up to some integer k, $k \geq 1$.
Inductive Step: Now let us consider $n = k + 1$. So

$$\begin{aligned} a_{k+1} &= 3a_k + 4 \\ &= 3\left(3^k - 2\right) + 4 \\ &= 3^{k+1} - 6 + 4 \\ &= 3^{k+1} - 2. \end{aligned}$$

So, the statement holds for $n = k+1$. By the principle of mathematical induction, the statement holds for all $n \in \mathbb{Z}^+$. □

46. **IN2** Let

$$\Pi(n) = \int_0^\infty x^n e^{-x}\,\mathrm{d}x\,.$$

We wish to prove that

$$\Pi(n) = n!,$$

for all $n \in \mathbb{N}$. We will do so by induction on n.

Proof.
Base Case: When $n = 0$, we have

$$\begin{aligned}
\Pi(0) &= \lim_{L\to\infty} \int_0^L e^{-x}\,\mathrm{d}x \\
&= \lim_{L\to\infty} \left[-e^{-x}\right]_0^L \\
&= \lim_{L\to\infty} 1 - e^{-L} \\
&\to 1 = 0!
\end{aligned}$$

So the statement holds for $n = 0$.
Inductive Hypothesis: Suppose the statement holds for all integer values of n up to some integer k, $k \geq 0$.
Inductive Step: Now let us consider $n = k + 1$. So

$$\Pi(k+1) = \lim_{L\to\infty} \int_0^L x^{k+1}e^{-x}\,\mathrm{d}x$$

Recall that integration by parts states

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x\,.$$

So let $u = x^{k+1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^{-x}$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = (k+1)x^k$ and $v = -e^{-x}$. Hence

$$\begin{aligned}
\Pi(k+1) &= \lim_{L\to\infty} \left[-x^{k+1}e^{-x}\right]_0^L + (k+1)\int_0^L x^k e^{-x}\,\mathrm{d}x \\
&= \lim_{L\to\infty} -\frac{L^{k+1}}{e^L} + (k+1)\Pi(k)
\end{aligned}$$

Note that on the right hand side we have ∞/∞. So we must apply L'Hopital's rule $k+1$ times, giving us

$$\lim_{L\to\infty} \frac{L^{k+1}}{e^L} = \lim_{L\to\infty} \frac{(k+1)L^k}{e^L} = \cdots = \lim_{L\to\infty} \frac{(k+1)!}{e^L} \to 0.$$

Hence

$$\Pi(k+1) = (k+1)\Pi(k).$$

Applying our inductive hypothesis yields

$$\Pi(k+1) = (k+1)k! = (k+1)!.$$

So the statement holds for $n = k+1$. So by the principle of mathematical induction, the statement holds for all $n \in \mathbb{N}$. $\square$

47. **H1** Recall that the definitions of sinh and cosh are

$$\begin{aligned}\sinh x &= \frac{1}{2}(e^x - e^{-x}) \\ \cosh x &= \frac{1}{2}(e^x + e^{-x})\end{aligned}$$

So for $\cosh^2 x - \sinh^2 x = 1$, we have

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4}{4} \\ &= 1.\end{aligned}$$

As required.
For $\sinh 2x = 2\sinh x \cosh x$, we have

$$\begin{aligned}\sinh 2x &= \frac{e^{2x} - e^{-2x}}{2} \\ &= \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} \\ &= 2\frac{e^x - e^{-x}}{2}\frac{e^x + e^{-x}}{2} \\ &= 2\sinh x \cosh x.\end{aligned}$$

As required.

48. **H2** Recall that tanh is defined as

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Recall that the quotient rule states

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{u(x)}{v(x)} = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}.$$

So we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tanh x &= \frac{2e^{2x}(e^{2x}+1) - 2e^{2x}(e^{2x}-1)}{(e^{2x}+1)^2} \\
&= \frac{4e^{2x}}{e^{4x}+2e^{2x}+1} \\
&= \frac{4}{e^{2x}+2+e^{-2x}} \\
&= \frac{4}{(e^{x}+e^{-x})^2} \\
&= \frac{1}{\cosh^2 x} \\
&= \operatorname{sech}^2 x.
\end{aligned}$$

As required.