

Queens' College Cambridge

NST Workbook



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1. **A1**

- (a) We have $x^2 - 1 = (x - 1)(x + 1)$.
- (b) We have $a^2 - 4ab + 4b^2 = (a + 2b)(a - 2b)$.
- (c) We have $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

2. **A2**

- (a) Given that $x^2 - 5x + 6 = (x - 2)(x - 3) = 0$ then it follows that the solutions are $x_{1,2} = 2, 3$.
- (b) Given that $x^2 + 2x = x(x + 2) = 0$ then it follows that the solutions are $x_{1,2} = 0, -2$.
- (c) We have the quadratic equation $x^2 - x - 1 = 0$. Given that we cannot find any roots by inspection, let us apply the quadratic formula, which gives us the following solutions

$$x_{1,2} = \frac{-(-1) \pm \sqrt{1 - 4 \times 1 \times (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

- (d) We have the quartic equation $x^4 - 3x^2 + 2 = 0$. Using the substitution $u = x^2$, we have the quadratic equation $u^2 - 3u + 2 = 0$. Factorising this yields

$$u^2 - 3u + 2 = (u - 2)(u - 1) = 0.$$

Hence the roots of the above equation are $u_{1,2} = 1, 2$. Since $u = x^2$, then we have the following solutions to the original equation

$$x_{1,\dots,4} = \pm 1, \pm \sqrt{2}.$$

- 3. **A3.** Recall that the vertex form of univariate quadratic function is given by

$$f(x) = ax^2 + bx + c = a \left(x - \frac{b}{2a} \right)^2 + \left(c - \frac{b^2}{4a} \right).$$

where $a, b, c \in \mathbb{R}$. Also note that $\forall x \in \mathbb{R}, x^2 \geq 0$, so it follows that

$$\left(x - \frac{b}{2a} \right)^2 \geq 0.$$

for all $x \in \mathbb{R}$. So the minima (or maxima) of f occurs when

$$\left(x - \frac{b}{2a}\right)^2 = 0 \iff x = \frac{b}{2a}.$$

And so any univariate has the turing point

$$\left(\frac{b}{2a}, c - \frac{b^2}{4a}\right).$$

(a) So writing the quadratic function in vertex form gives us

$$f(x) = x^2 - 2x + 6 = (x - 1)^2 + 5.$$

Using the argument above, the minimum value of f is 5 when $x = 1$.

(b) Similarly

$$g(x) = x^4 + 2x^2 + 2 = (x^2 + 1)^2 + 1.$$

Hence the minimum value of g is 1 when $x = \pm 1$.

(c) Given the minimum of (a) occurs at $x = 1$ it follows that the shape of (a) in the domain $2 \leq x \leq 3$ is concave, thus the minimum value occurs at the boundary $x = 2$. Substituting this in gives us $f(2) = (2 - 1)^2 + 5 = 6$.

4. A4

(a) We have

$$\begin{aligned} x^2 - 3x &< 4 \\ \iff x^2 - 3x - 4 &< 0 \\ \iff (x + 1)(x - 4) &< 0. \end{aligned}$$

Hence $x^2 - 3 < 4 \iff x \in (-1, 4)$.

(b) We have

$$\begin{aligned} y^3 &< 2y^2 + 3y \\ \iff y^3 - 2y^2 - 3y &< 0 \\ \iff y(y - 3)(y + 1) &< 0. \end{aligned}$$

Hence $y^3 < 2y^2 + 3y \iff x \in (-\infty, -1) \cup (0, 3)$.

5. A5

- (a) We note that $(x + 4)$ is a factor of $f(x) = x^3 + 5x^2 - 2x - 24$ since $x = -4$ is a root of f , so by the factor theorem $(x + 4)$ is a factor of f . So

$$f(x) = (x + 4)g(x).$$

where g is some univariate polynomial of degree 2. So using polynomial division, we have

$$\begin{aligned} g(x) &= \frac{x^3 + 5x^2 - 2x - 24}{x + 4} \\ &= \frac{x^2(x + 4) + x^2 - 2x - 24}{x + 4} \\ &= x^2 + \frac{x^2 - 2x - 24}{x + 4} \\ &= x^2 + \frac{x(x + 4) - 6x - 24}{x + 4} \\ &= x^2 + x - 6\frac{x + 4}{x + 4} \\ &= x^2 + x - 6 \\ &= (x - 2)(x + 3). \end{aligned}$$

Hence the fully factorised form of f is

$$f(x) = (x + 4)(x + 3)(x - 2).$$

- (b) By inspection we note that $t = 1$ is a root of $f(t) = t^3 - 7t + 6$. So by the factor theorem we have

$$f(t) = (t - 1)g(t),$$

where $g(t)$ is some univariate polynomial of degree 2. Intuitively we can see that g has the following form

$$g(t) = t^2 + at - 6.$$

Given that the t^2 coefficient of f is zero, it follows that $a - 1 = 0$, hence $a = 1$. Thus, we have

$$f(t) = (t - 1)(t^2 + t - 6) = (t - 1)(t + 3)(t - 2).$$

(c) Factorising the numerator and denominator, we get

$$\begin{aligned}\frac{x^3 + x^2 - 2x}{x^3 + 2x^2 - x - 2} &= \frac{x(x-1)(x+2)}{(x-1)(x+1)(x+2)} \\ &= \frac{x}{x+1}.\end{aligned}$$

6. A6

(a) We have

$$\begin{aligned}\frac{2}{(x+1)(x-1)} &\equiv \frac{A}{x+1} + \frac{B}{x-1} \\ &\equiv \frac{(A+B)x + (B-A)}{(x+1)(x-1)}.\end{aligned}$$

Equating coefficients yields the following system for equations

$$\begin{aligned}A + B &= 0 \\ B - A &= 2,\end{aligned}$$

which has the following solutions

$$\begin{aligned}A &= -1 \\ B &= 1.\end{aligned}$$

Substituting these into our original equivalence relation gives

$$\frac{2}{(x+1)(x-1)} = \frac{1}{x-1} - \frac{1}{x+1}.$$

(b) Similarly, we have

$$\begin{aligned}\frac{x+13}{(x+1)(x-2)(x+3)} &\equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+3} \\ &\equiv \frac{A(x-2)(x+3) + B(x+1)(x+3) + C(x+1)(x-2)}{(x+1)(x-2)(x+3)} \\ &\equiv \frac{(A+B+C)x^2 + (A+4B-C)x + (3B-6A-2C)}{(x+1)(x-2)(x+3)}.\end{aligned}$$

Equating coefficients gives us the following system of equations

$$\begin{aligned}A + B + C &= 0 \\A + 4B - C &= 1 \\3B - 6A - 2C &= 13.\end{aligned}$$

From equation (1) we have

$$-C = A + B.$$

Substituting this into equations (2) and (3) yields

$$\begin{aligned}2A + 5B &= 1 \\5B - 4A &= 13.\end{aligned}$$

So the solution is $A = -2, B = 1, C = 1$. And so we finally have

$$\frac{x + 13}{(x + 1)(x - 2)(x + 3)} = \frac{-2}{x + 1} + \frac{1}{x - 2} + \frac{1}{x + 3}.$$

(c) We have

$$\begin{aligned}\frac{4x + 1}{(x + 1)^2(x - 2)} &\equiv \frac{A}{x - 2} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \\&\equiv \frac{A(x + 1)^2 + B(x - 2)(x + 1) + C(x - 2)}{(x - 2)(x + 1)^2} \\&\equiv \frac{(A + B)x^2 + (2A - B + C)x + (A - 2B - 2C)}{(x - 2)(x + 1)^2}\end{aligned}$$

Equating coefficients produces the following system of equations

$$\begin{aligned}A + B &= 0 \\2A - B + C &= 4 \\A - 2B - 2C &= 1,\end{aligned}$$

From equation (1) we get $A = -B$ hence

$$\begin{aligned}3A + C &= 4 \\3A - 2C &= 1.\end{aligned}$$

So we have the solution $A = 1, B = -1, C = 1$. So the partial fraction decomposition is

$$\frac{4x + 1}{(x + 1)^2(x - 2)} = \frac{1}{x - 2} - \frac{1}{x + 1} + \frac{1}{(x + 1)^2}.$$

(d) We have

$$\begin{aligned}\frac{4x^2 + x - 2}{(x-1)(x^2+2)} &\equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+2} \\ &\equiv \frac{(A+B)x^2 + (C-B)x + (2A-C)}{(x-1)(x^2+2)}.\end{aligned}$$

Equating coefficients gives us the following system of equations

$$\begin{aligned}A + B &= 4 \\ C - B &= 1 \\ 2A - C &= -2.\end{aligned}$$

By inspection we see that the solution is $A = 1, B = 3, C = 4$. And so the partial fraction decomposition is

$$\frac{4x^2 + x - 2}{(x-1)(x^2+2)} = \frac{1}{x-1} + \frac{3x+4}{x^2+2}.$$

7. **FC*** See paper.

8. **G1**

- (a) Note that given $AB = BC$ we can deduce that triangle ABC is an isosociles triangle, hence if $\angle A = \pi/3$ then it follows that $\angle B = \pi/3$. Given that

$$\angle A + \angle B + \angle C = \pi.$$

Then $\angle C = \pi/3$. Thus ABC is infact a equilateral triangle, giving $AB = BC = CA = 1$.

- (b) Notice that ABC is an isosociles triangle with $AB = BC = 2$ and a base $AC = 3$. Using a method of dissection we produce two congrugent right-angled triangles ABM and CBM where M is the midpoint of AC . Let only consider ABM (as the triangles are congrugent). Since ABM is a right-angled triangle, we have

$$\cos \angle A = \frac{3}{4}.$$

So $\angle A = \arccos(3/4)$. Since ABC is an isosociles triangle, then $\angle A = \angle B = \arccos(3/4)$. And given that

$$\angle A + \angle B + \angle C = \pi.$$

Then $\angle C = \pi - 2 \arccos(3/4)$.

9. G2

- (a) The length ℓ of a sector is given by $\ell = r\theta$, where r is the radius of the circle and θ is the sector angle (in radians). So when $r = 3$ and $\theta = \pi/3$ it follows that $\ell = \pi$.
- (b) The area A of a sector is given by $A = \frac{1}{2}r^2\theta$. So substituting $r = 3$ and $\theta = \pi/3$ gives us $A = 3/2\pi$.

10. **G3.** We have the lines (in cartesian form)

$$\begin{aligned} x &= y = z \\ x &= y = 2z + 1. \end{aligned}$$

Writing these lines in a vector form gives us

$$\begin{aligned} \vec{r}_1 &= \lambda \langle 1, 1, 1 \rangle \\ \vec{r}_2 &= \langle 0, 0, -1/2 \rangle + \mu \langle 1, 1, 1/2 \rangle. \end{aligned}$$

In order to find the angle between \vec{r}_1 and \vec{r}_2 , let us consider their direction vectors $\vec{b}_1 = \langle 1, 1, 1 \rangle$ and $\vec{b}_2 = \langle 1, 1, 1/2 \rangle$. Recall that the dot product between two vectors \vec{v} and \vec{u} is $\vec{v} \cdot \vec{u} = \|\vec{v}\| \|\vec{u}\| \cos \theta$, where θ is the angle between the vectors. Rearranging for θ gives

$$\theta = \arccos \left(\frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\| \|\vec{u}\|} \right).$$

So we have

$$\theta = \arccos \left(\frac{5/2}{\sqrt{3}\sqrt{9/4}} \right) = \arccos \left(\frac{5\sqrt{3}}{9} \right).$$

In order to determine whether the lines intersect we need to find λ, μ such that

$$\lambda \langle 1, 1, 1 \rangle = \langle 0, 0, -1/2 \rangle + \mu \langle 1, 1, 1/2 \rangle.$$

which gives us the following system of equations

$$\begin{aligned}\lambda &= \mu \\ \lambda &= -1/2 + 1/2\mu,\end{aligned}$$

which has the solution $\lambda = \mu = -1$, hence the lines intersect when $\lambda = \mu = -1$.

11. **SS1.** Recall that the general term of an arithmetic progression with an initial term u_1 and a constant term difference d is given by $u_n = u_1 + (n - 1)d$. And so given α is the 3rd term and β is the 9th term, we can then form the following system of equations

$$\begin{aligned}\alpha &= u_1 + 2d \\ \beta &= u_1 + 8d.\end{aligned}$$

Rearranging for d yields

$$d = \frac{\beta - \alpha}{6}.$$

Substituting this into equation (1) produces

$$u_1 = \frac{4\alpha - \beta}{3}.$$

Also recall that the sum of the first n terms of the arithmetic progression u_1, u_2, \dots, u_n is

$$S_n = \sum_{r=1}^n u_r = \frac{1}{2}n(2u_1 + (n - 1)d).$$

So for the first thirty terms we have

$$\begin{aligned}S_{30} &= \frac{1}{2}(30) \left(2\frac{4\alpha - \beta}{3} + 29\frac{\beta - \alpha}{6} \right) \\ &= \frac{5}{2}(25\beta - 13\alpha).\end{aligned}$$

12. **SS2** x^{-10} .

13. **SS3**

(a) Applying the binomial theorem gives us

$$\begin{aligned}(1+x)^3 &= \sum_{r=0}^3 \binom{3}{r} (1)^{3-r} (x)^r \\ &= 1 + 3x + 3x^2 + x^3.\end{aligned}$$

(b) Applying the binomial theorem gives us

$$\begin{aligned}(2+x)^4 &= \sum_{r=0}^4 \binom{4}{r} (2)^{4-r} (x)^r \\ &= 16 + 32x + 24x^2 + 8x^3 + x^4.\end{aligned}$$

(c) We have

$$\begin{aligned}\left(2 + \frac{3}{x}\right)^5 &= \left(\frac{3+2x}{x}\right)^5 \\ &= \frac{1}{x^5} (3+2x)^5\end{aligned}$$

Applying the binomial theorem yields

$$\begin{aligned}\frac{1}{x^5} (3+2x)^5 &= \frac{1}{x^5} \sum_{r=0}^5 \binom{5}{r} (3)^{5-r} (2x)^r \\ &= \frac{1}{x^5} (243 + 810x + 1080x^2 + 720x^3 + 240x^4 + 32x^5)\end{aligned}$$

14. SS4

- (a) $x = -1/2$.
- (b) Recall the definition of a logarithm. $x = k^{\log_k x}$. So if $\log_a b = c$, then it follows that for any base α we have

$$\begin{aligned}a^c &= b \\ \iff \log_\alpha a^c &= \log_\alpha b \\ \iff c \log_\alpha a &= \log_\alpha b \\ \iff c &= \frac{\log_\alpha b}{\log_\alpha a}.\end{aligned}$$

As required.

(c) Changing to base e gives us

$$\begin{aligned}
 16 \log_x 3 &= 16 \frac{\ln 3}{\ln x} = \frac{\ln x}{\ln 3} = \log_3 x \\
 \iff (\ln x)^2 &= (\ln 3^4)^2 \\
 \iff \ln x_{1,2} &= \pm \ln 3^4 \\
 \iff x_{1,2} &= e^{\pm \ln 3^4}
 \end{aligned}$$

So the solutions are $x_{1,2} = 81, 1/81$.

15. SS5

(a) Recall that the sum of the first n terms of an arithmetic progression with general term $u_n = a + (n-1)d$ is

$$S_n = \sum_{r=1}^n u_r = \frac{1}{2}n(2a + (n-1)d).$$

Given the arithmetic progression $u_n = n$, it follows that $a = 1, d = 1$. So we have

$$\sum_{r=1}^n u_r = \frac{1}{2}n(2 + n - 1) = \frac{1}{2}n(n+1).$$

As required.

(b) Consider the arithmetic progression with general term $u_n = 2n+1$. So the sum of the first n terms of u_n is

$$S_n = \sum_{r=1}^n (2r+1) = 2 \sum_{r=1}^n r + \sum_{r=1}^n 1 = n(n+2).$$

Given u_n is a sequence of odd integers and $u_5 = 11$ and $u_{49} = 99$. Then we have

$$\begin{aligned}
 S_{49} - S_5 &= \sum_{n=1}^{49} (2n+1) - \sum_{n=1}^5 (2n+1) \\
 &= 49(49+2) - 5(5+2) \\
 &= 2469
 \end{aligned}$$

(c) We have

$$\sum_{n=1}^5 (3n+2) = 3 \sum_{n=1}^5 n + 10 = \frac{3}{2}(5)(6) + 10 = 55.$$

(d) We have

$$\begin{aligned} \sum_{n=0}^N (an+b) &= a \sum_{n=1}^N n + (N+1)b \\ &= \frac{a}{2}N(N+1) + (N+1)b \\ &= (N+1) \left(\frac{aN}{2} + b \right) \end{aligned}$$

(e) Recall that the sum of the first n terms of a geometric progression with general term $u_n = ar^{n-1}$ is

$$S_n = \sum_{k=1}^n ar^{k-1} = a \frac{1-r^n}{1-r} = a \frac{r^n-1}{r-1}.$$

We have

$$\sum_{n=0}^{10} 2^n = \sum_{n=1}^{11} (1)2^{n-1} = \frac{1-2^{11}}{1-2} = 2047.$$

(f) Similarly, we have

$$\begin{aligned} \sum_{n=9}^N ar^{2n} &= \sum_{n=1}^{N+1} ar^{2(n-1)} \\ &= a \sum_{n=1}^{N+1} (r^2)^{n-1} \\ &= a \frac{1-r^{2N+2}}{1-r^2} \end{aligned}$$

16. **SS6** Given that $u_{n+1} = ku_n$, then

$$u_n = k(k(k(\cdots ku_1))) = \underbrace{k \times k \times \cdots \times k}_{n\text{-times}} \times u_1 = k^n.$$

So we have the following cases as $n \rightarrow \infty$:

- (a) $k > 1$. $\lim_{n \rightarrow \infty} u_n \rightarrow \infty$
- (b) $k = 1$. $\lim_{n \rightarrow \infty} u_n = 1$.
- (c) $0 < k < 1$. $\lim_{n \rightarrow \infty} u_n \rightarrow 0$
- (d) $k = 0$. u_n is zero for all $n \neq 0$.
- (e) $-1 < k < 0$. $\lim_{n \rightarrow \infty} u_n \rightarrow 0$, but oscillates about the line $x = 0$.
- (f) $k = -1$. $\lim_{n \rightarrow \infty} u_n$ oscillates from 1 to -1 .
- (g) $k < -1$. $\lim_{n \rightarrow \infty} |u_n| \rightarrow \infty$, but oscillates about the line $x = 0$.

17. **SS7** Recall that the Maclaurin series for $(1+x)^n$ is

$$(1+x)^n = \sum_{r=0}^{\infty} \frac{n(n-1)\cdots(n-r+1)}{r!} x^r.$$

Which is valid for $|x| < 1$. Using this, we have:

(a)

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{8}x^3 + O(x^4).$$

This expansion is valid for $|x| < 1$.

(b)

$$\begin{aligned} (2+x)^{\frac{2}{5}} &= \sqrt[5]{4} \left(1 + \frac{x}{2}\right)^{\frac{2}{5}} \\ &= \sqrt[5]{4} \left(1 + \frac{1}{5}x - \frac{3}{100}x^2 + \frac{2}{125}x^3 + O(x^4)\right) \end{aligned}$$

This expansion is valid for $|x| < 2$

(c)

$$\begin{aligned} (1+2x)^{\frac{1}{2}}(2+x)^{-\frac{1}{3}} &= 2^{-\frac{1}{3}} \left(1 + x - \frac{1}{2}x^2 + x^3 + O(x^4)\right) \\ &\quad \times \left(1 - \frac{1}{6}x + \frac{1}{18}x^2 - \frac{7}{162}x^3 + O(x^4)\right) \end{aligned}$$

This expansion is valid if and only if $|2x| < 1$ and $|\frac{x}{2}| < 1$, so the expansion is valid if and only if $|x| < \frac{1}{2}$.

18. **SS8** Using the approximations

$$\begin{aligned}\sin \theta &\approx \theta - \frac{1}{6}\theta^3 \\ \cos \theta &\approx 1 - \frac{1}{2}\theta^2,\end{aligned}$$

we have

$$\begin{aligned}\sin\left(\frac{\theta}{2}\right)\cos\theta + \sec 2\theta &\approx \left(\frac{\theta}{2} - \frac{1}{48}\theta^3\right)\left(1 - \frac{1}{2}\theta^2\right) + \frac{1}{1 - 2\theta^2} \\ &= \frac{\theta}{2}\left(1 - \frac{13}{24}\theta^2 + \frac{\theta^4}{48}\right) + \frac{1}{1 - 2\theta^2}.\end{aligned}$$

Recall the Maclaurin series for $1/(1-x)$ is

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Hence

$$\frac{1}{1-2\theta^2} = 1 + 2\theta^2 + 4\theta^4 + \dots$$

for all $\theta \in \mathbb{R}$. And so substituting this in yields

$$\begin{aligned}\sin\left(\frac{\theta}{2}\right)\cos\theta + \sec 2\theta &\approx \frac{\theta}{2} - \frac{13}{48}\theta^3 + 1 + 2\theta^2 \\ &= 1 + \frac{\theta}{2} + 2\theta^2 - \frac{13}{48}\theta^3\end{aligned}$$

19. **T1** We have

$$\begin{aligned}2\sin^2\theta &= 1 \\ \iff \sin\theta &= \pm\frac{1}{\sqrt{2}}\end{aligned}$$

Using the CAST mnemonic, we get the following general solution

$$\theta \in \left\{2n\pi \pm \frac{\pi}{4} : n \in \mathbb{Z}\right\} \cup \left\{(2n+1)\pi \pm \frac{\pi}{4} : n \in \mathbb{Z}\right\}.$$

Given we're only interested in solutions in the domain $[0, 2\pi]$, we then have

$$\theta_{1,\dots,4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

20. **T2** We have

$$\begin{aligned}
 \frac{\cot^2 x + \sin^2 x}{\cos x + \operatorname{cosec} x} &\equiv \frac{\operatorname{cosec}^2 x - (1 - \sin^2 x)}{\cos x + \operatorname{cosec} x} \\
 &\equiv \frac{\operatorname{cosec}^2 x - \cos^2 x}{\operatorname{cosec} x + \cos x} \\
 &\equiv \frac{(\operatorname{cosec} x - \cos x)(\operatorname{cosec} x + \cos x)}{\operatorname{cosec} x + \cos x} \\
 &\equiv \operatorname{cosec} x - \cos x.
 \end{aligned}$$

21. **T3** Recall that the compound angle formulae are

$$\begin{aligned}
 \sin(\phi \pm \psi) &= \sin \phi \cos \psi \pm \cos \phi \sin \psi \\
 \cos(\phi \pm \psi) &= \cos \phi \cos \psi \mp \sin \phi \sin \psi.
 \end{aligned}$$

Applying these to the following questions gives:

(a)

$$\begin{aligned}
 \cos \pi/12 &= \cos(\pi/3 - \pi/4) \\
 &= \cos \pi/3 \cos \pi/4 + \sin \pi/3 \sin \pi/4 \\
 &= \frac{1}{2} \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2}} \\
 &= \frac{\sqrt{6} + \sqrt{2}}{4}.
 \end{aligned}$$

(b)

$$\begin{aligned}
 \sin \pi/12 &= \sin(\pi/3 - \pi/4) \\
 &= \sin \pi/3 \cos \pi/4 - \cos \pi/3 \sin \pi/4 \\
 &= \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2}} - \frac{1}{2} \frac{1}{\sqrt{2}} \\
 &= \frac{\sqrt{6} - \sqrt{2}}{4}.
 \end{aligned}$$

(c)

$$\begin{aligned}
 \cot \pi/12 &= \frac{\cos \pi/12}{\sin \pi/12} \\
 &= \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} \\
 &= 2 + \sqrt{3}.
 \end{aligned}$$

22. **T4** If $t = \tan \frac{\theta}{2}$, then we have a right-angled triangle with opposite t , adjacent 1 and hypotenuse $\sqrt{1+t^2}$. So it follows that

$$\begin{aligned}
 \sin \frac{\theta}{2} &= \frac{t}{\sqrt{1+t^2}} \\
 \cos \frac{\theta}{2} &= \frac{1}{\sqrt{1+t^2}}.
 \end{aligned}$$

Applying the double angle formulae yields

$$\begin{aligned}
 \cos \theta &= \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2} \\
 \sin \theta &= 2 \frac{t}{\sqrt{1+t^2}} \frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}.
 \end{aligned}$$

And from definition of \tan we get

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{2t}{1-t^2}.$$

23. **T5** Recall that the compound angle formula for \tan is

$$\tan(\phi \pm \psi) = \frac{\tan \phi \pm \tan \psi}{1 \mp \tan \phi \tan \psi}.$$

So applying this produces

$$\begin{aligned}
 \tan \left(\arctan \frac{1}{3} + \arctan \frac{1}{4} \right) &= \frac{\tan \arctan \frac{1}{3} + \tan \arctan \frac{1}{4}}{1 - \tan \arctan \frac{1}{3} \tan \arctan \frac{1}{4}} \\
 &= \frac{\frac{1}{3} + \frac{1}{4}}{1 - \frac{1}{3} \frac{1}{4}} \\
 &= \frac{7}{11}.
 \end{aligned}$$

24. **T6** Let us consider any two angles ϕ, ψ . We wish to know what the product of two sines is. So applying the compound angle formula for \cos we get

$$\begin{aligned}\cos(\phi - \psi) - \cos(\phi + \psi) &= (\cos \phi \cos \psi + \sin \phi \sin \psi) - (\cos \phi \cos \psi - \sin \phi \sin \psi) \\ &= 2 \sin \phi \sin \psi.\end{aligned}$$

Substituting A, B and C in produces

$$\cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right) = 2 \sin \frac{B}{2} \sin \frac{C}{2}.$$

Now since A, B, C are angles of a triangle, it follows that

$$A + B + C = \pi.$$

So $B + C = \pi - A$ giving us

$$\cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{\pi-A}{2}\right) = 2 \sin \frac{B}{2} \sin \frac{C}{2}.$$

Recall the identity

$$\cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin \theta.$$

Thus

$$\cos\left(\frac{B-C}{2}\right) - \sin \frac{A}{2} = 2 \sin \frac{B}{2} \sin \frac{C}{2}.$$

As required.

25. **T7** We have

$$\sqrt{3} \sin \theta + \cos \theta \equiv A \sin(\theta + \alpha) = A \sin \theta \cos \alpha + A \cos \theta \sin \alpha.$$

Equating coefficients produces the following system of equations

$$\begin{aligned}\sqrt{3} &= A \cos \alpha \\ 1 &= A \sin \alpha\end{aligned}$$

Squaring equations (1) and (2) and adding them together gives us

$$1 + 3 = A^2(\cos^2 \alpha + \sin^2 \alpha) = A^2.$$

which implies $A = 2$. Now dividing equations (1) and (2)

$$\tan \alpha = \frac{1}{\sqrt{3}}.$$

Hence $\alpha = \pi/6$.

26. **T8** Recall the triple angle formulae are

$$\begin{aligned}\cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta \\ \sin 3\theta &= 3 \sin \theta - 4 \sin^3 \theta\end{aligned}$$

And so we have

$$\begin{aligned}\cos \theta + \cos 3\theta &= \sin \theta + \sin 3\theta \\ \iff \cos \theta + 4 \cos^3 \theta - 3 \cos \theta &= \sin \theta + 3 \sin \theta - 4 \sin^3 \theta \\ \iff 4 \cos^3 \theta - 2 \cos \theta &= 4 \sin \theta - 4 \sin^3 \theta \\ \iff \cos^3 \theta (4 - 2 \sec^2 \theta) &= \cos^3 \theta (4 \tan \theta \sec^2 \theta - 4 \tan^3 \theta) \\ \iff \cos^3 \theta (2 - 2 \tan^2 \theta) &= \cos^3 \theta (4 \tan \theta) \\ \iff \cos^3 (\tan^2 \theta + 2 \tan \theta - 1) &= 0\end{aligned}$$

So we have a quadratic equation in terms of $\tan \theta$ or $\cos^3 \theta = 0$. Solving the quadratic equation gives us

$$\tan \theta = -1 \pm \sqrt{2}.$$

and

$$\cos^3 \theta = 0 \iff \cos \theta = 0.$$

So in the domain $[0, 2\pi]$, the solutions are

$$\theta \in \left\{ \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}, \frac{\pi}{2}, \frac{3\pi}{2} \right\}.$$

27. **V1**

$$\begin{aligned}\text{(a) } \|\vec{A}\| &= \sqrt{293}, \|\vec{B}\| = \sqrt{293}, \|\vec{C}\| = \sqrt{290} \text{ and } \|\vec{D}\| = 17, \text{ hence} \\ \|\vec{A}\| &= \|\vec{B}\| > \|\vec{C}\| > \|\vec{D}\|.\end{aligned}$$

- (b) Recall that the dot product between two vectors \vec{v} and \vec{u} is $\vec{v} \cdot \vec{u} = \|\vec{v}\| \|\vec{u}\| \cos \theta$, where θ is the angle between the vectors. Rearranging for θ gives

$$\theta = \arccos \left(\frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\| \|\vec{u}\|} \right).$$

Applying the above, we get

- i. The dot product between \vec{A} and \vec{B} is $\vec{A} \cdot \vec{B} = -29$. So

$$\theta = \arccos \left(\frac{-29}{293} \right).$$

- ii. The dot product between \vec{B} and \vec{C} is $\vec{B} \cdot \vec{C} = 2$. So

$$\theta = \arccos \left(\frac{2}{\sqrt{293}\sqrt{290}} \right).$$

28. V2

- (a) The vectors $\vec{A}, \vec{B}, \vec{C}$ lie on the line on the line $\vec{r} = \vec{a} + \lambda \vec{b}$ if and only if \overrightarrow{AB} is parallel to \overrightarrow{BC} , that is to say $\overrightarrow{BC} = \lambda \overrightarrow{AB}$, where $\lambda \in \mathbb{R}$. Note that

$$\overrightarrow{AB} = \vec{B} - \vec{A} = \langle 1, 1, -1 \rangle.$$

and

$$\overrightarrow{BC} = \vec{C} - \vec{B} = -2 \langle 1, 1, -1 \rangle = -2 \overrightarrow{AB}.$$

Hence \overrightarrow{AB} and \overrightarrow{BC} are parallel. Thus the equation of the line they all lie on must have the directional vector $\vec{b} = \langle 1, 1, -1 \rangle$. So the equation of our line is

$$\vec{r} = \langle 1, 0, 1 \rangle + \lambda \langle 1, 1, -1 \rangle.$$

29. D1

- (a) Computing the first and second derivatives of y gives us

$$\begin{aligned} \frac{dy}{dx} &= 2x \\ \frac{d^2y}{dx^2} &= 2/ \end{aligned}$$

Hence we have a minima at $x = 0$ with no points of inflection.

- (b) Computing the first and second derivatives of y gives us

$$\begin{aligned}\frac{dy}{dx} &= 3(x^2 - 1) = 3(x + 1)(x - 1) \\ \frac{d^2y}{dx^2} &= 6x.\end{aligned}$$

Hence we have a point of inflection at $x = 0$ and a minima at $x = 1$ and a maximum at $x = -1$.

- (c) Computing the first and second derivatives of y gives us

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2 \\ \frac{d^2y}{dx^2} &= 6(x - 1).\end{aligned}$$

Hence we have a point of inflection at $x = 1$ and a stationary point at $x = 1$.

- (d) Computing the first and second derivatives of y gives us

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 + 3 = 3(x^2 + 1) \\ \frac{d^2y}{dx^2} &= 6x.\end{aligned}$$

So we have a point of inflection at $x = 0$.

30. **D2** Let $f(x) = x^2 + 1$, then by first principles we have

$$\begin{aligned}f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + \Delta x^2 + 1 - x^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 2x + \Delta x \\ &= 2x.\end{aligned}$$

31. **D3**

(a)

$$\frac{dy}{dx} = 2x \sin x^2.$$

(b)

$$\frac{dy}{dx} = a^x \ln x.$$

(c) Note that

$$y = \ln(2x^a + 1) - \ln x^a = \ln(2x^a + 1) - a \ln x.$$

So we have

$$\frac{dy}{dx} = \frac{2ax^{a-1}}{2x^a + 1} - \frac{a}{x}.$$

(d) By definition of logarithms, we have

$$y = x^x = e^{\ln x^x} = e^{x \ln x}.$$

So

$$\frac{dy}{dx} = e^{x \ln x} \times (\ln x + 1) = x^x (\ln x + 1).$$

(e) We have

$$\begin{aligned} y &= \arcsin x \\ \iff \sin y &= x \\ \iff \frac{d}{dx} \sin y &= \frac{d}{dx} x \\ \iff \cos y \frac{dy}{dx} &= 1 \\ \iff \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

Given $\text{rng}(\arcsin) = [-\pi/2, \pi/2]$ so it follows that $\cos y$ is positive for all $y \in [-\pi/2, \pi/2]$. Using the Pythagorean identity, we have

$$\cos y = \sqrt{1 - x^2}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{1 - x^2}.$$

32. **D4**

$$\begin{aligned}
 y + e^y &= x^3 + x + 1 \\
 \iff \frac{d}{dx}y + e^y &= \frac{d}{dx}x^3 + x + 1 \\
 \iff \frac{dy}{dx}(1 + e^y) &= 3x^2 + 1 \\
 \iff \frac{dy}{dx} &= \frac{3x^2 + 1}{1 + e^y}
 \end{aligned}$$

33. **D5** We have

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{(t-2) - (t+1)}{(t-2)^2} = -\frac{3}{(t-2)^2} \\
 \frac{dx}{dt} &= \frac{2(t-3) - (2t+1)}{(t-3)^2} = -\frac{7}{(t-3)^2}
 \end{aligned}$$

Applying the chain rule yields

$$\frac{dy}{dx} = \frac{3}{(t-2)^2} \frac{(t-3)^2}{7}.$$

Hence

$$\left. \frac{dy}{dx} \right|_{t=1} = \frac{12}{7}.$$

34. **I1**(a) Let $x = \sqrt{2} \tan \theta$, then $dx = \sqrt{2} \sec^2 \theta d\theta$. Hence

$$\begin{aligned}
 \int \frac{1}{2+x^2} dx &= \int \frac{1}{2+2\tan^2 \theta} \sqrt{2} \sec^2 \theta d\theta \\
 &= \frac{1}{\sqrt{2}} \int \frac{\sec^2 \theta}{1+\tan^2 \theta} d\theta \\
 &= \frac{1}{\sqrt{2}} \int d\theta \\
 &= \frac{1}{\sqrt{2}} \theta + \kappa
 \end{aligned}$$

Thus

$$\int \frac{1}{2+x^2} dx = \frac{1}{\sqrt{2}} \arctan \left(\frac{x}{\sqrt{2}} \right) + \kappa.$$

(b) Let $x - 1 = 2 \sin \theta$, then $dx = 2 \cos \theta d\theta$. Thus

$$\begin{aligned} \int \frac{1}{\sqrt{3 + 2x - x^2}} dx &= \int \frac{1}{\sqrt{4 - (x - 1)^2}} dx \\ &= \int \frac{1}{\sqrt{4 - 4 \sin^2 \theta}} 2 \cos \theta d\theta \\ &= \frac{1}{2} \int \frac{\cos \theta}{\cos \theta} d\theta \\ &= \frac{1}{2} \theta + \kappa \end{aligned}$$

Hence

$$\int \frac{1}{\sqrt{3 + 2x - x^2}} dx = \frac{1}{2} \arcsin \left(\frac{x - 1}{2} \right) + \kappa.$$

(c) Let $u = \sqrt{1 - x}$, then $du = -\frac{1}{2\sqrt{1-x}}$. So we have

$$\begin{aligned} I &= \int \frac{1}{x\sqrt{1-x}} dx = - \int \frac{2 dx}{1 - u^2} \\ &= \int \frac{2 du}{(u + 1)(u - 1)} \end{aligned}$$

Notice that the integrand can be expressed as a partial fraction, hence

$$\begin{aligned} \frac{2}{(u + 1)(u - 1)} &\equiv \frac{A}{u + 1} + \frac{B}{u - 1} \\ &\equiv \frac{(A + B)u + (B - A)}{(u + 1)(u - 1)} \end{aligned}$$

Equating coefficients produces the following system of equations

$$\begin{aligned} A + B &= 0 \\ B - A &= 2 \end{aligned}$$

So by inspection we have the solution $A, B = -1, 1$. Substituting

this back into our integrand we get

$$\begin{aligned}
 I &= \int \frac{1}{u-1} du - \int \frac{1}{u+1} du \\
 &= \ln|u-1| - \ln|u+1| \\
 &= \ln|\sqrt{1-x}-1| - \ln|\sqrt{1-x}+1| + \kappa \\
 &= \ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| + \kappa
 \end{aligned}$$

(d) Recall the integration by parts states

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

Now let

$$u = \ln x \text{ and } \frac{dv}{dx} = 1.$$

Giving us

$$\frac{du}{dx} = \frac{1}{x} \text{ and } v = x.$$

Applying the rule we state above yeilds

$$\int \ln x dx = x \ln x - \int \frac{x}{x} dx = x \ln x - x + \kappa.$$

35. I2

(a) We'll apply integration by parts again using

$$u = x \text{ and } \frac{dv}{dx} = e^{-x}.$$

then

$$\frac{du}{dx} = 1 \text{ and } v = -e^{-x}.$$

So we have

$$\begin{aligned}
 \int_0^L x e^{-x} dx &= [-x e^{-x}]_0^L + \int_0^L e^{-x} dx \\
 &= -L e^{-L} + [-e^{-x}]_0^L \\
 &= -L e^{-L} + (-e^{-L} + 1) \\
 &= 1 - e^{-L}(L + 1)
 \end{aligned}$$

Now let us consider the limiting value of the integral as $L \rightarrow \infty$.
So we have

$$\lim_{L \rightarrow \infty} \int_0^L x e^{-x} dx = \lim_{L \rightarrow \infty} 1 - \frac{L-1}{e^L}$$

On the right hand side, we have

$$1 - \frac{\infty}{\infty}.$$

Thus we must apply L'Hopital's rule. So

$$\lim_{L \rightarrow \infty} \frac{L-1}{e^L} = \lim_{L \rightarrow \infty} \frac{1}{e^L} \rightarrow 0^+$$

Therefore

$$\lim_{L \rightarrow \infty} \int_0^L x e^{-x} dx = 1.$$

(b) Recall that the triple angle formula for $\sin 3\theta$ is

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta.$$

Substituting this into the integrand yields

$$\begin{aligned} I &= \int_0^{\pi/2} \sin 3\theta \cos \theta d\theta = \int_0^{\pi/2} (3 \sin \theta - 4 \sin^3 \theta) \cos \theta d\theta \\ &= \int_0^{\pi/2} 3 \sin \theta \cos \theta d\theta - \int_0^{\pi/2} 4 \sin^3 \theta \cos \theta d\theta \end{aligned}$$

Using the substitution $u = \sin \theta$ in both integrals gives us
 $du = \cos \theta d\theta$ and

$$\begin{aligned} I &= \int_0^1 3u du - \int_0^1 4u^3 du \\ &= \left[\frac{3}{2} u^2 \right]_0^1 - \left[u^4 \right]_0^1 \\ &= \frac{3}{2} - 1 = \frac{1}{2} \end{aligned}$$

(c) Let $u = x^3 + 3x + 2$, then $du = 3(x^2 + 1) dx$. So

$$\begin{aligned} \int_0^1 \frac{x^2 + 1}{x^3 + 3x + 2} dx &= \frac{1}{3} \int_2^6 \frac{1}{u} du \\ &= \frac{1}{3} [\ln |u|]_2^6 \\ &= \frac{1}{3} \ln 3. \end{aligned}$$

(d) Using the Weierstrass substitution $t = \tan \frac{\theta}{2}$ (and our answers from **T4**) which are

$$\begin{aligned} \cos \theta &= \frac{1 - t^2}{1 + t^2} \\ \sin \theta &= \frac{2t}{1 + t^2} \end{aligned}$$

we then have

$$\begin{aligned} dt &= \frac{1}{2} \sec^2 \frac{\theta}{2} d\theta \\ \iff dt &= \frac{1}{2} (1 + t^2) d\theta \\ \iff d\theta &= \frac{2 dt}{1 + t^2} \end{aligned}$$

Hence

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{1}{3 + 5 \cos \theta} d\theta \\ &= \int_0^1 \frac{1}{3 + 5 \frac{1-t^2}{1+t^2}} \frac{2 dt}{1+t^2} \\ &= \int_0^1 \frac{1+t^2}{8-2t^2} \frac{2}{1+t^2} dt \\ &= \int_0^1 \frac{dt}{4-t^2} \\ &= \int_0^1 \frac{dt}{(2-t)(2+t)} \end{aligned}$$

Notice that the integrand can be expressed as a partial fraction, thus

$$\begin{aligned}\frac{1}{(2-t)(2+t)} &\equiv \frac{A}{2-t} + \frac{B}{2+t} \\ &\equiv \frac{(A-B)t + 2(A+B)}{(2-t)(2+t)}\end{aligned}$$

Equating coefficients produces the following system of equations

$$\begin{aligned}A - B &= 0 \\ 2(A + B) &= 1\end{aligned}$$

So by inspection we have the solution $A = \frac{1}{4}$ and $B = -\frac{1}{4}$. Substituting this back into our integrand yields

$$\begin{aligned}I &= \frac{1}{4} \left\{ \int_0^1 \frac{1}{2-t} dt - \int_0^1 \frac{1}{2+t} dt \right\} \\ &= \frac{1}{4} \{ [-\ln|2-t|]_0^1 - [\ln|2+t|]_0^1 \} \\ &= \frac{1}{4} \{ (-\ln 1 + \ln 2) - (\ln 3 - \ln 2) \} \\ &= \frac{1}{4} \ln 3\end{aligned}$$

36. **DE1** We have

$$\begin{aligned}x \frac{dy}{dx} + 1 - y^2 &= 0 \\ \iff x \frac{dy}{dx} &= y^2 - 1 \\ \iff \frac{1}{y^2 - 1} \frac{dy}{dx} &= \frac{1}{x}\end{aligned}$$

Integrating both sides with respect to x gives us

$$\begin{aligned}\int \frac{1}{y^2 - 1} \frac{dy}{dx} dx &= \int \frac{dx}{x} \\ \iff \int \frac{dy}{y^2 - 1} &= \int \frac{dx}{x}\end{aligned}$$

Notice that the integrand on the right hand side can be expressed as a partial fraction, hence

$$\begin{aligned}\frac{1}{(y+1)(y-1)} &\equiv \frac{A}{y+1} + \frac{B}{y-1} \\ &\equiv \frac{(A+B)y + (B-A)}{(y+1)(y-1)}\end{aligned}$$

Equating coefficient produces the following system of equations

$$A + B = 0$$

$$B - A = 1$$

By inspection, we have the solution $A = -1/2$ and $B = 1/2$. Substituting this back into our original integrand yields

$$\begin{aligned}\frac{1}{2} \left\{ \int \frac{dy}{y-1} - \int \frac{dy}{y+1} \right\} &= \ln|x| + \kappa_1 \\ \iff \frac{1}{2} \{ \ln|y-1| - \ln|y+1| \} + \kappa_2 &= \ln|x| + \kappa_1 \\ \iff \ln \left| \frac{y-1}{y+1} \right| &= 2 \ln|x| + \kappa \\ \iff \frac{y-1}{y+1} &= \lambda x^2\end{aligned}$$

Given the initial value condition $(x, y) = (1, 0)$ we have

$$\frac{-1}{1} = \lambda \implies \lambda = -1.$$

So the particular solution is

$$\begin{aligned}\frac{y-1}{y+1} &= -x^2 \\ \iff y-1 &= -x^2y - x^2 \\ \iff y(1+x^2) &= 1-x^2 \\ \iff y &= \frac{1-x^2}{1+x^2}\end{aligned}$$

(a) Let us first express

$$z = \frac{1+i}{2-i}.$$

in the form $z = a + bi$, where $a, b \in \mathbb{R}$. Rationalising the denominator gives us

$$\begin{aligned} z &= \frac{1+i}{2-i} \frac{2+i}{2+i} \\ &= \frac{1+3i}{5}. \end{aligned}$$

And so

$$\begin{aligned} \Re(z) &= \frac{1}{5} \\ \Im(z) &= \frac{3}{5}. \end{aligned}$$

(b) Applying the quadratic formula gives us

$$\begin{aligned} z_{1,2} &= \frac{2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2} \\ &= 1 \pm i \end{aligned}$$

The modulus of each root is

$$|z_{1,2}| = \sqrt{1^2 + (\pm 1)^2} = \sqrt{2}.$$

The argument of z_1 is

$$\arg z_1 = \pi/4.$$

By symmetry argument, it follows that the argument of z_2 is

$$\arg z_2 = -\pi/4.$$

See paper for Argand diagram.

38. **C2**

- (a) Recall that de Moivre's theorem states that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

So let us consider consider $z = (\cos \theta + i \sin \theta)^5$, hence by de Moivre's theorem $\Re(z) = \cos 5\theta$. Applying the binomial theorem to z gives us

$$\begin{aligned} z = \cos 5\theta + i \sin 5\theta &= \sum_{r=0}^5 \binom{5}{r} (\cos \theta)^{5-r} (i \sin \theta)^r \\ &= \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta \\ &\quad + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta \\ &= \cos \theta (\cos^4 \theta - 10 \cos^2 \theta \sin^2 \theta + 5 \sin^4 \theta) \\ &\quad + i \sin \theta (5 \cos^4 \theta - 10 \cos^2 \theta \sin^2 \theta + \sin^4 \theta) \end{aligned}$$

Hence

$$\Re(z) = \cos 5\theta = \cos \theta (\cos^4 \theta - 10 \cos^2 \theta \sin^2 \theta + 5 \sin^4 \theta).$$

- (b) See paper

39. M1

- (a)

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 7 & 6 \end{pmatrix}. \end{aligned}$$

- (b)

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 10 & 5 \end{pmatrix}.$$

- (c)

$$BA = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -5 & -8 \\ 10 & 16 \end{pmatrix}.$$

40. **M2** Let us consider the matrices

$$A = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ q & r \end{pmatrix},$$

where $p, q, r \in \mathbb{R}$. So we have

$$AB = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ q & r \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

However,

$$BA = \begin{pmatrix} 0 & 0 \\ q & r \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ pq & 0 \end{pmatrix}$$

Why does this work (and why pick A and B)? It is because A and B are singular. If $AB = 0$, then it implies that at least one of the two matrices is singular.

Proof. Assume there exists two non-singular matrices A and B such that $AB = 0$. It follows that if they are non-singular, then the matrices A^{-1} and B^{-1} exist such that

$$A^{-1}AB = B$$

$$ABB^{-1} = A$$

But $AB = 0$, and since $M \times 0 = 0$, then it follows that $A = B = 0$. A contradiction! Thus proving the initial assumption that A, B are non-singular is false. (*I got bored...*) \square

41. **M3** Let A be a 2×2 matrix such that

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Recall that if M is an $n \times n$ matrix and λ is some scalar, then

$$\det(\lambda M) = \lambda^n \det M.$$

and an enlargement by a scale factor λ on the matrix N is

$$N' = \lambda N.$$

Now if A is a transformation of a rotation R followed by an enlargement S of scale factor λ , then $A = SR = \lambda R$. We also note that R will be in the form

$$R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

Which by inspection has the determinant $\det R = 1$ (due to the Pythagorean identity). Given that

$$\det A = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 \times 1 + 1 \times 1 = 2.$$

Then

$$\det A = 2 = \lambda^2.$$

Hence $\lambda = \sqrt{2}$ (We ignore the negative solution as it's not relevant to this question). So

$$A = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \equiv \sqrt{2} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

Equating elements of the matrices produces the following system of equations

$$\begin{aligned} \cos \theta &= \frac{1}{\sqrt{2}} \\ \sin \theta &= -\frac{1}{\sqrt{2}} \end{aligned}$$

Giving us $\tan \theta = -1$. So the general solution is

$$\theta \in \left\{ 2n\pi - \frac{\pi}{4} : n \in \mathbb{Z} \right\} \cup \left\{ (2n+1)\pi - \frac{\pi}{4} : n \in \mathbb{Z} \right\}.$$

For simplicity, we'll use the solution $\theta = 3\pi/4$. Hence

$$A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \cos \frac{3\pi}{4} & \sin \frac{3\pi}{4} \\ -\sin \frac{3\pi}{4} & \cos \frac{3\pi}{4} \end{pmatrix}.$$

42. SE1

(a)

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1).$$

(b) Recall that

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2 \text{ and } \sum_{r=1}^n r = \frac{1}{2}n(n+1).$$

So using the above formulae and the linearity of the summation, we have

$$\begin{aligned} S_n &= \sum_{r=1}^n r(r^2 + 2) = \sum_{r=1}^n r^3 + 2 \sum_{r=1}^n r \\ &= \frac{1}{4}n^2(n+1)^2 + n(n+1) \\ &= n(n+1) \left(\frac{1}{4}n(n+1) + 1 \right) \end{aligned}$$

43. SE2

(a) Note that the general term of the summation can be expressed as a partial fraction, so

$$\begin{aligned} \frac{1}{r(r+1)} &\equiv \frac{A}{r} + \frac{B}{r+1} \\ &\equiv \frac{(A+B)r + A}{r(r+1)} \end{aligned}$$

Equating coefficients produces the following system of equations

$$\begin{aligned} A + B &= 0 \\ A &= 1 \end{aligned}$$

So the solution is $A = 1, B = -1$. Hence the summation S_n can now be expressed as

$$S_n = \sum_{r=1}^n \frac{1}{r} - \frac{1}{r+1} \equiv \sum_{r=1}^n f(r) - f(r-1).$$

Note that the right hand side equivalence relation is the general form a telescopic series on which we can apply the method of difference. Hence we can apply the method of difference on S_n , giving us

$$\begin{aligned} S_n &= \frac{1}{1} - \frac{1}{2} \\ &\quad + \frac{1}{2} - \frac{1}{3} \\ &\quad \vdots \\ &\quad + \frac{1}{n-1} - \frac{1}{n} \\ &\quad + \frac{1}{n} - \frac{1}{n+1} \end{aligned}$$

Cancelling terms produces

$$S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

44. **IN1** Given that $a_1 = 1$ and $a_{n+1} = 3a_n + 4$, then we have the sequence

$$1, 7, 25, 79, \dots = 1, 3 + 4, 3^2 + 4(3 + 1), 3^3 + 4(3^2 + 3 + 1), \dots$$

By inspection, we see that

$$a_n = 3^{n-1} + 4 \sum_{r=1}^{n-1} 3^{r-1}$$

Recall that

$$\sum_{k=1}^n ar^{k-1} = a \frac{1-r^n}{1-r} = a \frac{r^n-1}{r-1}.$$

Hence

$$a_n = 3^{n-1} + 4 \frac{3^{n-1} - 1}{2} = 3^{n-1} + 2(3^{n-1} - 1) = 3^n - 2.$$

We will now prove that this is true for all $n \in \mathbb{Z}^+$ by induction on n .

Proof.

Base Case: When $n = 1$, we have $a_1 = 3^1 - 2 = 1 = a_1$. So the statement holds when $n = 1$.

Inductive Hypothesis: Suppose the statement holds for all integer values of n up to some integer k , $k \geq 1$.

Inductive Step: Now let us consider $n = k + 1$. So

$$\begin{aligned} a_{k+1} &= 3a_k + 4 \\ &= 3(3^k - 2) + 4 \\ &= 3^{k+1} - 6 + 4 \\ &= 3^{k+1} - 2. \end{aligned}$$

So, the statement holds for $n = k + 1$. By the principle of mathematical induction, the statement holds for all $n \in \mathbb{Z}^+$. \square

45. **IN2** Let

$$\Pi(n) = \int_0^\infty x^n e^{-x} dx.$$

We wish to prove that

$$\Pi(n) = n!,$$

for all $n \in \mathbb{N}$. We will do so by induction on n .

Proof.

Base Case: When $n = 0$, we have

$$\begin{aligned} \Pi(0) &= \lim_{L \rightarrow \infty} \int_0^L e^{-x} dx \\ &= \lim_{L \rightarrow \infty} [-e^{-x}]_0^L \\ &= \lim_{L \rightarrow \infty} 1 - e^{-L} \\ &\rightarrow 1 = 0! \end{aligned}$$

So the statement holds for $n = 0$.

Inductive Hypothesis: Suppose the statement holds for all integer values of n up to some integer k , $k \geq 0$.

Inductive Step: Now let us consider $n = k + 1$. So

$$\Pi(k+1) = \lim_{L \rightarrow \infty} \int_0^L x^{k+1} e^{-x} dx$$

Recall that integration by parts states

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

So let $u = x^{k+1}$ and $\frac{dv}{dx} = e^{-x}$, then $\frac{du}{dx} = (k+1)x^k$ and $v = -e^{-x}$. Hence

$$\begin{aligned} \Pi(k+1) &= \lim_{L \rightarrow \infty} [-x^{k+1}e^{-x}]_0^L + (k+1) \int_0^L x^k e^{-x} dx \\ &= \lim_{L \rightarrow \infty} -\frac{L^{k+1}}{e^L} + (k+1)\Pi(k) \end{aligned}$$

Note that on the right hand side we have ∞/∞ . So we must apply L'Hopital's rule $k+1$ times, giving us

$$\lim_{L \rightarrow \infty} \frac{L^{k+1}}{e^L} = \lim_{L \rightarrow \infty} \frac{(k+1)L^k}{e^L} = \dots = \lim_{L \rightarrow \infty} \frac{(k+1)!}{e^L} \rightarrow 0.$$

Hence

$$\Pi(k+1) = (k+1)\Pi(k).$$

Applying our inductive hypothesis yields

$$\Pi(k+1) = (k+1)k! = (k+1)!.$$

So the statement holds for $n = k+1$. So by the principle of mathematical induction, the statement holds for all $n \in \mathbb{N}$. \square

46. **H1** Recall that the definitions of \sinh and \cosh are

$$\begin{aligned} \sinh x &= \frac{1}{2}(e^x - e^{-x}) \\ \cosh x &= \frac{1}{2}(e^x + e^{-x}) \end{aligned}$$

So for $\cosh^2 x - \sinh^2 x = 1$, we have

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4}{4} \\ &= 1. \end{aligned}$$

As required.

For $\sinh 2x = 2 \sinh x \cosh x$, we have

$$\begin{aligned}\sinh 2x &= \frac{e^{2x} - e^{-2x}}{2} \\ &= \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} \\ &= 2 \frac{e^x - e^{-x}}{2} \frac{e^x + e^{-x}}{2} \\ &= 2 \sinh x \cosh x.\end{aligned}$$

As required.

47. **H2** Recall that \tanh is defined as

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Recall that the quotient rule states

$$\frac{d}{dx} \frac{u(x)}{v(x)} = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}.$$

So we have

$$\begin{aligned}\frac{d}{dx} \tanh x &= \frac{2e^{2x}(e^{2x} + 1) - 2e^{2x}(e^{2x} - 1)}{(e^{2x} + 1)^2} \\ &= \frac{4e^{2x}}{e^{4x} + 2e^{2x} + 1} \\ &= \frac{4}{e^{2x} + 2 + e^{-2x}} \\ &= \frac{4}{(e^x + e^{-x})^2} \\ &= \frac{1}{\cosh^2 x} \\ &= \operatorname{sech}^2 x.\end{aligned}$$

As required.