Queens' College Cambridge

Further Graphics

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Contents

1 Differential Geometry

1.1 Curves

Definition 1.1.1. (**Parameterized Curve**) A parameterized curve C is a curve that is defined by the vector-valued function $\boldsymbol{\gamma} : \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}^n$, s.t

$$C = \{\boldsymbol{\gamma}(t) : t \in \mathcal{D}\}.$$

- A parameterized curve C with parameterization $\boldsymbol{\gamma}$ is in the class C^r if $\boldsymbol{\gamma}$ is r differentiable. We often write this as $\boldsymbol{\gamma} \in C^r(\mathcal{D}, \mathbb{R}^n)$.
- A curve $\boldsymbol{\gamma}$ is said to be smooth if $\boldsymbol{\gamma} \in C^\infty$. (*We will assume all parameterized curves to be smooth*).

Definition 1.1.2. (**Velocity Vector**) For a parameterized curve $\boldsymbol{\gamma} : \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}^n$, the *velocity-vector* at the point $t \in \mathcal{D}$ is defined as $\boldsymbol{\gamma}'(t)$.

- The speed at $t \in \mathcal{D}$ is $\|\gamma'(t)\|$.

1.1.1 Arc Length

- Let C be a parameterized curve with parameterization $\boldsymbol{\gamma} : \mathcal{D} \to \mathbb{R}^n$. The arc-length of γ at $t \in \mathcal{D}$ is

$$s(t) = \lim_{n\to\infty} \sum_{i=1}^{n} \|\boldsymbol{\gamma}'(t_i)\| \Delta t = \int_a^t \|\boldsymbol{\gamma}'(t)\| \, \mathrm{d}t,$$

where $s : \mathcal{D} \to \mathbb{R}$ is the *arc length function*. By the Fundamental theorem of Calculus, we have

$$s'(t) = \|\boldsymbol{\gamma}'(t)\| = \underbrace{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}_{\text{for } n=2}.$$

Definition 1.1.3. (**Tangent Vector**) Let $\boldsymbol{\gamma} : \mathcal{D} \to \mathbb{R}^n$ be a parametric curve. For a point $t \in \mathcal{D}$, we define the *unit tangent vector* $\hat{\mathbf{T}}(t)$ as

$$\hat{\mathbf{T}}(t) = \frac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|}.$$

- So we can write the velocity vector as $\boldsymbol{\gamma}'(t) = s'(t)\hat{\mathbf{T}}(t)$.

Definition 1.1.4. (**Normal Vector**) Let $\boldsymbol{\gamma} : \mathcal{D} \to \mathcal{R}^n$ be a parametric curve. For the point $t \in \mathcal{D}$, we define the *unit normal vector* $\hat{\mathbf{N}}(t)$ as

$$\hat{\mathbf{N}}(t) = \frac{\hat{\mathbf{T}}'(t)}{\|\hat{\mathbf{T}}'(t)\|}.$$

- We note that for all unit vectors $\mathbf{u}(t)$, $\mathbf{u}'(t) \cdot \mathbf{u}(t) = 0$, that is $\mathbf{u}'(t)$ is perpendicular to $\mathbf{u}(t)$.

1.1.2 Reparameterizations of Curves

Definition 1.1.5. (**Reparameterization**) Let $\boldsymbol{\gamma} : \mathcal{D} \to \mathbb{R}^n$ be a parameterized curve. A parameterized curve $\tilde{\boldsymbol{\gamma}} : \mathcal{I} \to \mathbb{R}^n$ is *reparameterization* of $\boldsymbol{\gamma}$ if there exists a smooth bijection $\varphi : \mathcal{I} \to \mathcal{D}$ (*The reparameterization map*) if

$$\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(\varphi(t)),$$

for all $t \in \mathcal{I}$. Note that since φ^{-1} is also smooth, then $\boldsymbol{\gamma}$ is a reparameterization of $\tilde{\boldsymbol{\gamma}}$:

$$\tilde{\boldsymbol{\gamma}}(\varphi^{-1}(t)) = \boldsymbol{\gamma}(\varphi(\varphi^{-1}(t))) = \boldsymbol{\gamma}(t),$$

for all $t \in \mathcal{D}$.

- A parameterized curve $\boldsymbol{\gamma}$ is said to be a *unit-speed* curve if $s'(t) = \|\boldsymbol{\gamma}'(t)\| = 1$ for all $t \in \mathcal{D}$.
- Often we reparameterize $\boldsymbol{\gamma}$ to be a unit-speed curve.

1.1.3 Curvature

- Let $\boldsymbol{\gamma} : \mathcal{D} \to \mathbb{R}^n$ be a parameterized curve. Recall that $\boldsymbol{\gamma}'(t) = s'(t)\hat{\mathbf{T}}(t)$. So

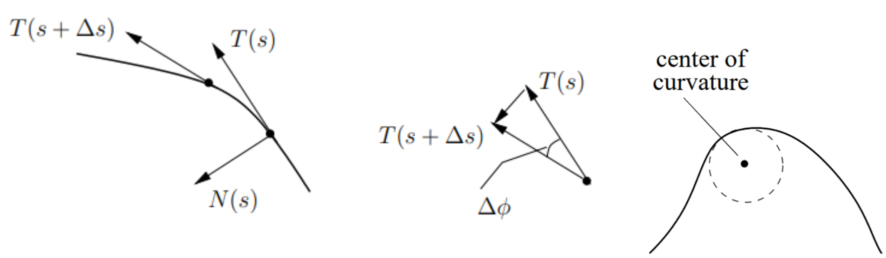
$$\boldsymbol{\gamma}''(t) = \underbrace{s''(t)\hat{\mathbf{T}}(t)}_{\text{Tagential Acceleration}} + \underbrace{s'(t)\hat{\mathbf{T}}'(t)}_{A}$$

  where A describes the rate of change of the tangent direction or how much the curve is "curving".

- Recall that the unit normal is given by $\hat{\mathbf{N}}'(t) = \hat{\mathbf{T}}'(t)/\|\hat{\mathbf{T}}'(t)\|$. Hence $\hat{\mathbf{T}}'(t) = \|\hat{\mathbf{T}}'(t)\|\hat{\mathbf{N}}'(t)$

Definition 1.1.6. (**Signed Curvature**) Let $\boldsymbol{\gamma} : \mathcal{D} \to \mathbb{R}^n$ be a parameterized curve. The *signed curvature* $\kappa_s(t)$ is defined by

$$\hat{\mathbf{T}}'(t) = s'(t)\kappa_s(t)\hat{\mathbf{N}}(t).$$



- $\kappa(t)$ is the rate of change in ϕ with respect to t, that is $\kappa(t) = \phi'(t)$, where ϕ is the *tangential angle*.

- The curvature is *signed* since ϕ may increase or decrease. (clockwise is positive).

- **Example**: The unit-speed parameterization of a circle of radius R centered at $\mathbf{c} = (x_c, y_c)$,

$$\gamma(t) = \left(x_c + R\cos\frac{t}{R}, y_c + R\sin\frac{t}{R}\right).$$

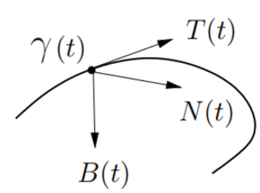
So we have

$$\gamma'(t) = \left(-\sin\frac{t}{R}.\cos\frac{t}{R}\right)$$
$$\gamma''(t) = -\frac{1}{R}\left(\cos\frac{t}{R}, \sin\frac{t}{R}\right) = -\frac{1}{R}\hat{\mathbf{N}}(t)$$

Hence $\kappa_S(t) = 1/R$.

The curvature a circle is extremely useful for visualizing curvature by trying to "fit" a circle to the curve.

Definition 1.1.7. (**Bi-Tangent Vector**) $\hat{\mathbf{B}}(t) = \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}(t)$, referred to as the *unit bi-tangent vector*.



- $\left\{\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}\right\}$ is an orthonormal basis with

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}, \quad \hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}}, \quad \hat{\mathbf{T}} = \hat{\mathbf{N}} \times \hat{\mathbf{B}}.$$

- Let us consider $\hat{\mathbf{N}}'(t)$ by taking the derivative of $\hat{\mathbf{B}}(t)$. So we have

$$\hat{\mathbf{B}}'(t) = \underbrace{\hat{\mathbf{T}}'(t) \times \hat{\mathbf{N}}(t)}_{\mathbf{0}} + \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}'(t) = \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}'(t).$$

Since $\hat{\mathbf{N}}(t)$ is perpendicular to $\hat{\mathbf{N}}'(t)$, then we can write $\hat{\mathbf{N}}'(t) = f(t)\hat{\mathbf{T}}(t) + g(t)\hat{\mathbf{B}}(t)$ for $f, g : \mathcal{I} \to \mathbb{R}$. Hence we deduce that

$$\hat{\mathbf{B}}'(t) = \hat{\mathbf{T}}(t) \times \left(f(t)\hat{\mathbf{T}}(t) + g(t)\hat{\mathbf{B}}(t)\right) = g(t)\hat{\mathbf{T}}(t) \times \hat{\mathbf{B}}(t) = -\underbrace{g(t)}_{A}\hat{\mathbf{N}}(t).$$

where A describes the rate of change of the bi-tangent direction (in the normal direction), or rate of rotation of the bi-tangent.

Definition 1.1.8. (**Torsion**) Let $\boldsymbol{\gamma} : \mathcal{D} \to \mathbb{R}^n$ be a parameterized curve. We define the *torsion function* $\tau : \mathcal{D} \to \mathbb{R}$ s.t

$$\hat{\mathbf{B}}'(t) = -s'(t)\tau(t)\hat{\mathbf{N}}(t).$$

- We have $\hat{\mathbf{T}}(t) \cdot \hat{\mathbf{N}}(t) = 0$ and $\hat{\mathbf{N}}(t) \cdot \hat{\mathbf{B}}(t) = 0$, taking the derivative wrt t yields

$$\begin{aligned} \hat{\mathbf{T}}'(t) \cdot \hat{\mathbf{N}}(t) + \hat{\mathbf{T}}(t) \cdot \hat{\mathbf{N}}'(t) &= 0 \\ \hat{\mathbf{N}}'(t) \cdot \hat{\mathbf{B}}(t) + \hat{\mathbf{N}}(t) \cdot \hat{\mathbf{B}}'(t) &= 0 \end{aligned}$$

  So for $\hat{\mathbf{N}}'(t) = f(t)\hat{\mathbf{T}}(t) + g(t)\hat{\mathbf{B}}(t)$, we have

$$\begin{aligned} f(t) &= \hat{\mathbf{N}}'(t) \cdot \hat{\mathbf{T}}(t) = -\hat{\mathbf{T}}'(t) \cdot \hat{\mathbf{N}}(t) = -s'(t)\kappa_s(t) \\ g(t) &= \hat{\mathbf{N}}'(t) \cdot \hat{\mathbf{B}}(t) = -\hat{\mathbf{N}}(t) \cdot \hat{\mathbf{B}}'(t) = s'(t)\tau(t) \end{aligned}$$

- The Frenet-Serret equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \hat{\mathbf{T}}(t) \\ \hat{\mathbf{N}}(t) \\ \hat{\mathbf{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & s'(t)\kappa_s(t) & 0 \\ -s'(t)\kappa_s(t) & 0 & s'(t)\tau(t) \\ 0 & -s'(t)\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{T}}(t) \\ \hat{\mathbf{N}}(t) \\ \hat{\mathbf{B}}(t) \end{bmatrix}$$

- **Example**: Curvature and torsion for $\gamma : \mathcal{D} \to \mathbb{R}^3$:
  - Curvature:

$$\begin{aligned} \boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t) &= \left(s'(t)\hat{\mathbf{T}}(t)\right) \times \left(s''(t)\hat{\mathbf{T}}(t) + (s'(t))^2\kappa_s(t)\hat{\mathbf{N}}(t)\right) \\ &= s'(t)s''(t)\underbrace{\left(\hat{\mathbf{T}}(t) \times \hat{\mathbf{T}}(t)\right)}_{\mathbf{0}} + (s'(t))^3\kappa_s(t)\hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}(t) \end{aligned}$$

$$\iff \kappa_s(t) = \frac{[\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)] \cdot \hat{\mathbf{B}}(t)}{\|\boldsymbol{\gamma}'(t)\|^3}$$

  - Torsion:

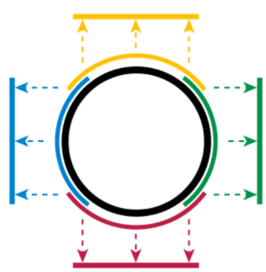
$$\begin{aligned} \boldsymbol{\gamma}^{(3)} &= s^{(3)}\hat{\mathbf{T}} + (3s''s'\kappa_s + (s')^2\kappa_s')\hat{\mathbf{N}} + (s')^3\kappa_s\left(-\kappa_s\hat{\mathbf{T}} + \tau\hat{\mathbf{B}}\right) \\ &= \left(s^{(3)} - (s')^3\kappa_s^2\right)\hat{\mathbf{T}} + (3s''s'\kappa_s + (s')^2\kappa_s')\hat{\mathbf{N}} + (s')^3\kappa_s\tau\hat{\mathbf{B}} \end{aligned}$$

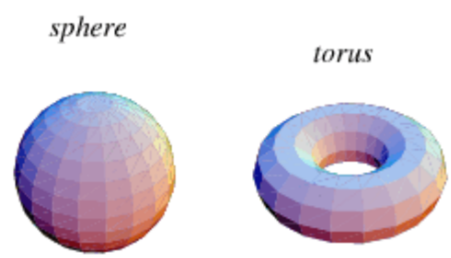
Taking the dot product of $\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''$ with $\boldsymbol{\gamma}^{(3)}$ yields

$$\begin{aligned}(\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)) \cdot \boldsymbol{\gamma}^{(3)}(t) &= (s'(t))^6 (\kappa_s(t))^2 \tau(t)\hat{\mathbf{B}}(t) \\ \iff \tau(t) &= \frac{(\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)) \cdot \boldsymbol{\gamma}^{(3)}(t)}{\left[(\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)) \cdot \hat{\mathbf{B}}(t)\right]^2}\end{aligned}$$

1.2 Manifolds

- A manifold is a topological space that "locally" resembles $\mathbb{R}^n$ e.g. The earth is a 2-manifold.
- Examples of 1-manifolds are line segments / any non-intersecting closed loop. Examples of 2-manifolds are any non-intersecting closed surfaces in $\mathbb{R}^3$ e.g. a sphere / torus.





Definition 1.2.1. (**Topology**) Let $\mathcal{M}$ be a set. A set $\mathcal{O} \subseteq \mathcal{P}(\mathcal{M})$ is called a **topology** if

1. $\emptyset \in \mathcal{O}, \mathcal{M} \in \mathcal{O}$
2. $U \in \mathcal{O}, V \in \mathcal{O} \implies U \cap V \in \mathcal{O}$
3. $(\forall \alpha \in \mathbb{N}. U_\alpha \in \mathcal{O}) \implies \bigcup_\alpha U_\alpha \in \mathcal{O}$

- $\mathcal{M}$ represents the surface (or topology) and $\mathcal{O}$ represents the neighborhoods.
- The tuple $(\mathcal{M}, \mathcal{O})$ is a topological space.

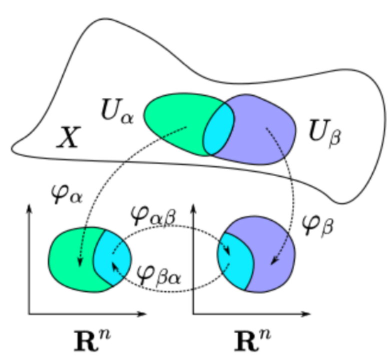
Definition 1.2.2. (**Manifold**) A n-dimensional topological manifold $\mathcal{M}$ is a topological space $(\mathcal{M}, \mathcal{O})$, such that for all points $\mathbf{p} \in \mathcal{M}$, there is an open neighbourhood $U \in \mathcal{O}$ of $\mathbf{p}$ and a homeomorphism $\varphi : U \to V$ where $V \subset \mathbb{R}^n$.

- A ϕ homeomorphism is a mapping that preserves topological features. (i.e. it's continuous and invertible). The inverse chart map is denoted as $\psi = \varphi^{-1}$.
- The mapping $\varphi : U \to V$ is the *coordinate system*, or **chart map**.
- U is the local coordinate neighbourhood.
- $\overrightarrow{\varphi}(U)$ are the local coordinates of $\mathbf{p}$.
- An **atlas** of $\mathcal{M}$ is a set

$$\mathcal{A} = \{\varphi_\alpha : \alpha \in \mathbb{N}\},$$

  such that

$$\mathcal{M} = \bigcup_\alpha U_\alpha.$$



1.2.1 Tangent Spaces

Definition 1.2.3. (**Differentiable Manifold**) A k-differentiable manifold $\mathcal{M}$ is a manifold with a k-differentiable atlas $\mathcal{A} = \{\phi : \alpha \in \mathbb{N}\}$, that is to say $\forall \varphi_\alpha \in \mathcal{A}. \varphi_\alpha \in C^k(U, V)$.

- A smooth manifold $\mathcal{M}$ is a ∞-differentiable manifold.

Definition 1.2.4. (**Differential**) Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. The **differential** $\mathrm{d}_{\mathbf{p}}f$ at $\mathbf{p} = (x_i) \in U$ is the linear map

$$\mathrm{d}_{\mathbf{p}}f : \mathbb{R}^n \to \mathbb{R}^m$$

defined by matrix

$$\mathrm{d}_{\mathbf{p}}f = \left(\frac{\partial f^i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_n} & \cdots & \frac{\partial f^m}{\partial x_n} \end{bmatrix}.$$

- The chain rule also follows, for two smooth maps $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m, g : V \to \mathbb{R}^p$. Let $\mathbf{p} \in U$ and $f(\mathbf{p}) \in V$, then

$$\mathrm{d}_{\mathbf{p}}(g \circ f) = \mathrm{d}_{f(\mathbf{p})}g \circ \mathrm{d}_{\mathbf{p}}f\,.$$

- Let $\mathcal{M}$ be a smooth n manifold embedded in $\mathbb{R}^m$. Let $\varphi : U \to V \subset \mathbb{R}^n$ be the chart map for the neighborhood U of $\mathbf{p}$. Assume that $\varphi(\mathbf{p}) = \mathbf{0}$. We may take it's differential of it's inverse: $\mathrm{d}_0\psi : \mathbb{R}^n \to \mathbb{R}^m$

Definition 1.2.5. (**Tangent Space**) Let $\mathcal{M}$ be a smooth n-dimensional manifold with $\mathbf{p} \in \mathcal{M}$. Then the **tangent space** of $\mathcal{M}$ at $\mathbf{p}$ is $\overrightarrow{d_{\mathbf{0}}\psi}(\mathbb{R}^n) \subset \mathbb{R}^m$ where $\varphi(\mathbf{p}) = \mathbf{0}$. Often denoted as $T_{\mathbf{p}}\mathcal{M}$

Theorem 1.2.1. $T_{\mathbf{p}}\mathcal{M}$ is independent of φ.

Proof. Suppose we have the following charts $\varphi : U \to V$ and $\tilde{\varphi} : \tilde{U} \to \tilde{V}$ (and inverses $\psi, \tilde{\psi}$ respectively). Taking intersections, we may assume that $U = \tilde{U}$ and $\psi(0) = \tilde{\psi}(0) = \mathbf{p}$.

Consider the **transition map** $\xi = \tilde{\varphi} \circ \psi : V \to \tilde{V}$. Similarly, we note that ξ^{-1} exists, $\xi^{-1} = \varphi \circ \tilde{\psi} : \tilde{V} \to V$. So we may write $\psi = \tilde{\psi} \circ \xi$.

By the chain rule, we have

$$\mathrm{d}_{\mathbf{0}}\psi = \mathrm{d}_{\xi(\mathbf{0})}\tilde{\psi} \circ \mathrm{d}_{\mathbf{0}}\xi \qquad \mathrm{d}_{\mathbf{0}}\tilde{\psi} = \mathrm{d}_{\xi(\mathbf{0})}\psi \circ \mathrm{d}_{\mathbf{0}}\xi^{-1}\,.$$

Note that $\xi(\mathbf{0}) = \mathbf{0}$, so $\mathrm{d}_{\mathbf{0}}\psi = \mathrm{d}_{\mathbf{0}}\tilde{\psi} \circ \mathrm{d}_{\mathbf{0}}\xi$. $\xi^{-1} \circ \xi = \mathrm{id}_V$, so applying the chain rule to id_V yields $\mathrm{id}_V = \mathrm{d}_{\mathbf{0}}\xi^{-1} \circ \mathrm{d}_{\mathbf{0}}\xi$. Hence $\overrightarrow{\mathrm{d}_{\mathbf{0}}\,\xi}(V \subseteq \mathbb{R}^n) = \mathbb{R}^n$. So

$$\overrightarrow{\mathrm{d}_{\mathbf{0}}\,\psi}(\mathbb{R}^n) = \overrightarrow{\mathrm{d}_{\mathbf{0}}\tilde{\psi}}\left(\overrightarrow{\mathrm{d}_{\mathbf{0}}\xi}(\mathbb{R}^n)\right) = \mathrm{d}_{\mathbf{0}}\tilde{\psi}(\mathbb{R}^n) = T_{\mathbf{p}}\mathcal{M}.$$

$\square$

- We now define a differential map between manifolds. Let $f : \mathcal{M} \to \mathcal{N}$ be a smooth map of manifolds where $\mathcal{M} \subseteq \mathbb{R}^k, \mathcal{N} \subseteq \mathbb{R}^\ell$.

  Let $\varphi : U \to V$ and $\tilde{\varphi} : \tilde{U} \to \tilde{V}$ where U and $\tilde{U}$ are local neighborhoods around $\mathbf{p} \in \mathcal{M}$ and $f(\mathbf{p}) \in \mathcal{N}$ respectively, where $V \subseteq \mathbb{R}^n$ and $\tilde{V} \subseteq \mathbb{R}^m$.

  We assume that $\varphi(\mathbf{p}) = \mathbf{0}$ and $\tilde{\varphi}(f(\mathbf{p})) = \mathbf{0}$, and $\overrightarrow{f}(U) \subseteq \tilde{U}$. We define the map $f' = \tilde{\varphi} \circ f \circ \varphi^{-1} : V \to \tilde{V}$.

Definition 1.2.6. (**Differential Map**) The **differential-map** of f at $\mathbf{p} \in \mathcal{M}$, denoted $\mathrm{d}_{\mathbf{p}} f$ is a linear map $\mathrm{d}_{\mathbf{p}} f : T_{\mathbf{p}}\mathcal{M} \to T_{f(\mathbf{p})}\mathcal{N}$, defined by

$$\mathrm{d}_{\mathbf{p}} f = \mathrm{d}_{\mathbf{0}}\tilde{\psi} \circ \mathrm{d}_{\mathbf{0}} f' \circ (\mathrm{d}_{\mathbf{0}}\psi)^{-1},$$

where $f' = \tilde{\varphi} \circ f \circ \psi : V \to \tilde{V}$.

$$\begin{tikzcd}
T_{\mathbf{p}}\mathcal{M} \arrow[r, "\mathrm{d}_{\mathbf{0}} f'"] & T_{f(\mathbf{p})}\mathcal{N} \\
\mathbb{R}^k \arrow[u, "\mathrm{d}_{\mathbf{0}}\psi"] \arrow[r, "\mathrm{d}_{\mathbf{0}}\, \tilde{\varphi} \circ f \circ \psi"'] & \mathbb{R}^\ell \arrow[u, "\mathrm{d}_{\mathbf{0}}\tilde{\psi}"']
\end{tikzcd}$$

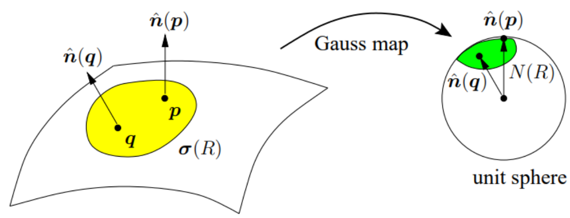
- **Notation**: Since a curve $\boldsymbol{\gamma} : \mathcal{D} \to R^n$ is a 1-manifold with chart $\boldsymbol{\gamma}^{-1}$, then we have $\boldsymbol{\gamma}'(t) = \mathrm{d}_t \gamma(1)$

Theorem 1.2.2. For a smooth n-manifold $\mathcal{M}$ with inverse chart $\psi(x_1, \ldots, x_n) : V \subseteq \mathbb{R}^n \to U$. For $\mathbf{p} \in U$ s.t $\psi(\mathbf{0}) = \mathbf{p}$, the basis of $T_{\mathbf{p}}\mathcal{M}$ is

$$\left\{ \left(\frac{\partial \psi}{\partial x_1}\right)_{\mathbf{0}}, \ldots, \left(\frac{\partial \psi}{\partial x_n}\right)_{\mathbf{0}} \right\}.$$

1.2.2 Gauss Map

Definition 1.2.7. (**Gauss Map**) The Gauss Map of a smooth n-manifold $\mathcal{M}$ with inverse chart $\psi : V \subseteq \mathbb{R}^n \to U$, denoted $\hat{\mathbf{N}} : U \to \mathcal{S}^3$ is a map s.t for all $\mathbf{p} \in U$, $\|\hat{\mathbf{N}}(\mathbf{p})\| = 1$ and $\forall \mathbf{v} \in T_{\mathbf{p}}\mathcal{M}. \mathbf{v} \cdot \hat{\mathbf{N}}(\mathbf{p}) = 0$.



- The Gauss Map is a map of the *normal*s for every point $\mathbf{p} \in \mathcal{M}$. $\hat{\mathbf{N}}(\mathbf{p}) \perp T_{\mathbf{p}}\mathcal{M}$.
- For a 2-manifold $\mathcal{M}$ embedded in $\mathbb{R}^3$ with inverse chart $\psi : V \subset \mathbb{R}^2 \to U$, s.t $\psi(\mathbf{0}) = \mathbf{p}$. The Gauss Map is

$$\hat{\mathbf{N}}(\mathbf{p}) = \frac{(\psi_u \times \psi_v)(\mathbf{0})}{\|(\psi_u \times \psi_v)(\mathbf{0})\|},$$

  where $\varphi(\mathbf{p}) = \mathbf{0}$.
- The differential $\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}} : T_{\mathbf{p}}\mathcal{M} \to T_{\hat{\mathbf{N}}(\mathbf{p})}\mathcal{S}^3$ is known as the *Shape Operator*. Note that $T_{\hat{\mathbf{N}}(\mathbf{p})}\mathcal{S}^3$ is a plane in $\mathbb{R}^3$ with normal $\hat{\mathbf{N}}(\mathbf{p})$, this is exactly $T_{\mathbf{p}}\mathcal{M}$. So $\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}} : T_{\mathbf{p}}\mathcal{M} \to T_{\mathbf{p}}\mathcal{M}$.
- Given a point $\mathbf{p}$ and a tangent $\hat{\mathbf{T}} \in T_{\mathbf{p}}\mathcal{M}$. $\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\hat{\mathbf{T}})$ gives the change in surface normal along the tangent line $\mathcal{L}(\mathbf{p}, \hat{\mathbf{T}}) = \mathbf{p} + \epsilon\hat{\mathbf{T}}$.
- Note that

$$\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\psi_u) = \hat{\mathbf{N}}_u$$
$$\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\psi_v) = \hat{\mathbf{N}}_v$$

  So $\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(x_1\psi_u + x_2\psi_v) = x_1\hat{\mathbf{N}}_u + x_2\hat{\mathbf{N}}_v$.

1.3 Fundamental Forms

Definition 1.3.1. (**First Fundamental Form**) Let $\mathcal{M}$ be a 2-manifold. The first fundamental form of $\mathcal{M}$ at the point $\mathbf{p} \in \mathcal{M}$ is the restriction of the dot product $\cdot$ to $T_{\mathbf{p}}\mathcal{M} \times T_{\mathbf{p}}\mathcal{M}$. That is

$$I_{\mathbf{p}}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y},$$

where $I_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \times T_{\mathbf{p}}\mathcal{M} \to \mathbb{R}$

- $I_{\mathbf{p}}$ is symmetric, bilinear and positive for $I_{\mathbf{p}}(\mathbf{x}) = I_{\mathbf{p}}(\mathbf{x}, \mathbf{x})$
- For all $\mathbf{x} \in T_{\mathbf{p}}\mathcal{M}$, $\mathbf{x} = x_1\psi_u(\mathbf{0}) + x_2\psi_v(\mathbf{0})$, where $\psi(0) = \mathbf{p}$. Hence

$$\begin{aligned} I_{\mathbf{p}}(\mathbf{u}, \mathbf{v}) &= (x_1\psi_u + x_2\psi_v) \cdot (y_1\psi_u + y_2\psi_v) \\ &= x_1 y_1 \|\psi_u\|^2 + (x_1 y_2 + x_2 y_1)(\psi_u \cdot \psi_v) + x_2 y_2 \|\psi_v\|^2 \\ &= \sum_{i=1}^{2} \sum_{j=1}^{2} x_i g_{ij} y_j \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{aligned}$$

where

$$(g_{ij}) = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \psi_u \cdot \psi_u & \psi_u \cdot \psi_v \\ \psi_v \cdot \psi_u & \psi_v \cdot \psi_v \end{bmatrix}.$$

- Consider the 2-manifold $\mathcal{M}$ with chart $\varphi : U \to V$. Now consider the curve $\gamma : \mathcal{D} \to \mathcal{M}$ where the curve γ lies within the neighborhood U, that is $\overrightarrow{\gamma}(\mathcal{D}) \subset U$. The arc length is given by

$$s(t) = \int_{t_0}^{t} \|\boldsymbol{\gamma}'(\tau)\| \,\mathrm{d}\tau = \int_{t_0}^{t} \sqrt{\boldsymbol{\gamma}'(\tau) \cdot \boldsymbol{\gamma}'(\tau)} \,\mathrm{d}\tau$$

Let $(u(t), v(t)) = (\varphi \circ \gamma)(t)$, so by the chain rule:

$$(u', v') = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi \circ \boldsymbol{\gamma})(t) = \left(\mathrm{d}_{\boldsymbol{\gamma}(t)}\varphi\right) \boldsymbol{\gamma}'(t).$$

Hence $\boldsymbol{\gamma}'(t) = (\mathrm{d}_{\boldsymbol{\gamma}(t)}\psi)(u', v') = u'\psi_u + v'\psi_v$. So we find that

$$s(t) = \int_{t_0}^{t} \sqrt{E\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^2 + 2F\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}v}{\mathrm{d}\tau}\right) + G\left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)^2} \,\mathrm{d}\tau = \int_{t_0}^{t} \sqrt{I_{\boldsymbol{\gamma}(\tau)}(u(\tau)', v(\tau)')} \,\mathrm{d}\tau\,.$$

We often write

$$(\mathrm{d}s)^2 = E(\mathrm{d}u)^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G(\mathrm{d}v)^2.$$

- Recall that area is given by

  $$A = \iint \|\psi_u \times \psi_v\| \,\mathrm{d}u\,\mathrm{d}v\,.$$

  and

  $$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2.$$

  Then $\|\psi_u \times \psi_v\|^2 = EG - F^2$. So we have

  $$\mathrm{d}A = \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = \sqrt{\det g}\,\mathrm{d}u\,\mathrm{d}v\,.$$

Definition 1.3.2. (**Second Fundamental Form**) Let $\mathcal{M}$ be a smooth 2-manifold and $\mathbf{p} \in \mathcal{M}$. We define the second fundamental form at the point $\mathbf{p}$ as

$$II_{\mathbf{p}}(\mathbf{x}) = -I_{\mathbf{p}}\left(\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\mathbf{x}), \mathbf{x}\right).$$

- The first fundamental form represents the length, angles (and area) of the geometry. The second fundamental form measures the change in the normal vector at $\mathbf{p}$ in the direction $\mathbf{x}$.

- We note that

  $$II_{\mathbf{p}}(\mathbf{x}) = \mathbf{x}^T \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \mathbf{x}.$$

  Let $\mathbf{x} = (x_1, x_2)$ in the basis of $T_{\mathbf{p}}\mathcal{M}$:

  $$\begin{aligned} II_{\mathbf{p}}(\mathbf{x}) &= -\mathbf{x}^T\,\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\mathbf{x}) \\ &= -\left(x_1\psi_u + x_2\psi_v\right) \cdot \left(x_1\hat{\mathbf{N}}_u + x_2\hat{\mathbf{N}}_v\right) \\ &= -\sum_{i=1}^{2}\sum_{j=1}^{2} x_i L_{ij} x_j \end{aligned}$$

  where $L_{ij} = -\hat{\mathbf{N}}_j \cdot \psi_i = \hat{\mathbf{N}} \cdot \psi_{ij}$ (by $\hat{\mathbf{N}} \cdot \psi_i = 0$ and the chain rule).

  $$(L_{ij}) = \begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{N}} \cdot \psi_{uu} & \hat{\mathbf{N}} \cdot \psi_{uv} \\ \hat{\mathbf{N}} \cdot \psi_{vu} & \hat{\mathbf{N}} \cdot \psi_{vv} \end{bmatrix}.$$

- Since $\operatorname{span}\{\psi_u, \psi_v\} = \operatorname{span}\left\{\hat{\mathbf{N}}_u, \hat{\mathbf{N}}_v\right\}$,

$$\hat{\mathbf{N}}_j = \sum_{i=1}^{n} a_j^i \psi_i$$

Then for $\mathbf{x} = (x_1, x_2)$ in the basis of $T_{\mathbf{p}}\mathcal{M}$:

$$\begin{aligned}
\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\mathbf{x}) &= x_1\hat{\mathbf{N}}_u + x_2\hat{\mathbf{N}}_v \\
&= \left(a_1^1 x_1 + a_2^1 x_2\right)\psi_u + \left(a_1^2 x_1 + a_2^2 x_2\right)\psi_v \\
&= \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}\mathbf{x}
\end{aligned}$$

However,

$$\begin{aligned}
-L_{ij} = \hat{\mathbf{N}}_i \cdot \psi_j = \left(\sum_{\ell=1}^{2} a_i^\ell \psi_\ell\right) \cdot \psi_j = \sum_{\ell=1}^{2} a_i^\ell g_{\ell j} \\
\iff -\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} \\
\iff \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} = -\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1}\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}
\end{aligned}$$

1.4 Normal, Principal, Mean and Gaussian Curvatures

1.4.1 Normal Curvature

Definition 1.4.1. (**Normal Curvature**) Let $\mathcal{M}$ be a smooth 2-manifold with $\mathbf{p} \in \mathcal{M}$, let U be a local neighborhood of $\mathbf{p}$ with chart $\varphi : U \to V \subset \mathbb{R}^2$. Let $\boldsymbol{\gamma} : \mathcal{D} \to \mathcal{M}$ be a smooth parameterized curve that lies in U, that is $\overrightarrow{\gamma}(\mathcal{D}) \subset U$. The *normal curvature* of $\mathcal{M}$ is

$$\kappa_n(t) = I_{\mathbf{p}}\left(\kappa_s(t)\hat{\mathbf{P}}(t), \hat{\mathbf{N}}(t)\right) = \frac{1}{s'(t)}\hat{\mathbf{T}}'(t) \cdot \hat{\mathbf{N}}(t),$$

where $\hat{\mathbf{P}}(t) = \frac{\hat{\mathbf{T}}'(t)}{s'(t)\kappa_s(t)}$, the *unit-normal* of the curve $\boldsymbol{\gamma}$ and $\hat{\mathbf{N}}$ is the gauss map.

Theorem 1.4.1. For $\boldsymbol{\gamma}(0) = \mathbf{p}$ and $\hat{\mathbf{T}}(0) = \hat{\mathbf{x}} \in T_{\mathbf{p}}\mathcal{M}$, then

$$\kappa_n(0) = II_{\mathbf{p}}(\hat{\mathbf{x}}).$$

Proof. Let $(u(t), v(t)) = \alpha(t) = (\varphi \circ \gamma)(t)$. The normal curve of $\mathcal{M}$ along $\boldsymbol{\gamma}$ is given by $\hat{\mathbf{N}}(t) = \hat{\mathbf{N}}(\boldsymbol{\alpha}(t))$. Since $\hat{\mathbf{T}}(t) \cdot \hat{\mathbf{N}}(t) = 0$, then by the chain rule

$$\hat{\mathbf{T}}'(t) \cdot \hat{\mathbf{N}} = -\hat{\mathbf{T}}(t) \cdot \hat{\mathbf{N}}'(t).$$

Hence

$$\kappa_n(t) = -\frac{1}{s'(t)}\hat{\mathbf{T}}(t) \cdot \hat{\mathbf{N}}'(t).$$

Note that $\hat{\mathbf{N}}'(t) = \hat{\mathbf{N}}_u u'(t) + \hat{\mathbf{N}}_v v'(t)$, so

$$\hat{\mathbf{N}}'(t) = \mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\alpha'(t)) = \mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(s'(t)\hat{\mathbf{T}}(t)).$$

Thus the normal curvature of $\mathcal{M}$ along $\boldsymbol{\gamma}$ at $\mathbf{p}$ is

$$\kappa_n = -\frac{1}{s'(0)}\hat{\mathbf{T}}(0) \cdot \mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(s'(0)\hat{\mathbf{x}}) = -\hat{\mathbf{x}} \cdot \mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\hat{\mathbf{x}}) = II_{\mathbf{p}}(\hat{\mathbf{x}}),$$

by the linearity of differential. □

1.4.2 Principal Curvature

Definition 1.4.2. (**Principal Curvatures**) Let $\mathcal{M}$ be a smooth 2-manifold, and let $\mathbf{p} \in \mathcal{M}$. The maximum and minimum normal curvatures κ_1 and κ_2 at $\mathbf{p}$ at the *principal curvatures* of $\mathcal{M}$ at $\mathbf{p}$. The corresponding *principal directions* are the tangents $\mathbf{e}_1$ and $\mathbf{e}_2$.

- By maximizing / minimizing $\kappa_n = II_{\mathbf{p}}(\mathbf{x})$ wrt $\mathbf{x} = (x_1, x_2)$ (using Lagrangians),
$$\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\mathbf{e}_i) = -\kappa_i \mathbf{e}_i.$$
- The principal directions are the eigenvectors of the second fundamental form.
- The principal directions form a vector basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ in the tangent plane $T_{\mathbf{p}}\mathcal{M}$.

Theorem 1.4.2. (**Euler's Formula**) Let $\hat{\mathbf{x}} \in T_{\mathbf{p}}\mathcal{M}$ be some arbitrary unit tangent vector in the tangent plane, then $\hat{\mathbf{x}} = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta$, where θ is the angle between $\hat{\mathbf{x}}$ and $\mathbf{e}_1$. So from the second-fundamental form, we have

$$\begin{aligned}
\kappa_n &= II_{\mathbf{p}}(\hat{\mathbf{x}}) \\
&= -\hat{\mathbf{x}} \cdot \mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}(\hat{\mathbf{x}}) \\
&= -\left(\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta\right) \cdot \mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}\left(\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta\right) \\
&= -\left(\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta\right) \cdot -\left(\mathbf{e}_1\kappa_1 \cos\theta + \mathbf{e}_2\kappa_2 \sin\theta\right) \\
&= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta
\end{aligned}$$

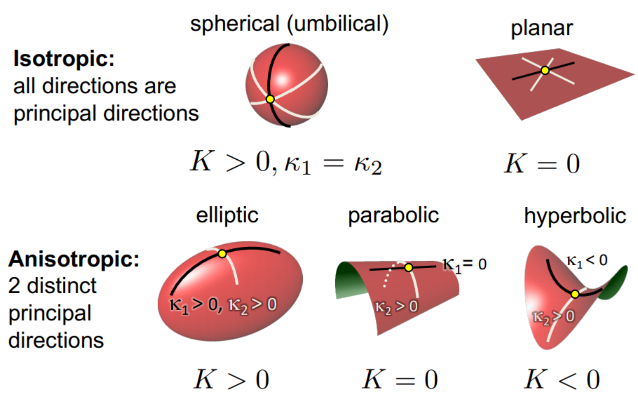
1.4.3 Gaussian and Mean Curvatures

Definition 1.4.3. (**Gaussian and Mean Curvatures**) Let κ_1 and κ_2 be the principal curvatures of a smooth 2-manifold $\mathcal{M}$ at $\mathbf{p}$. Define

1. The *Gaussian curvature* of $\mathcal{M}$ at $\mathbf{p}$ as $K = \kappa_1\kappa_2$
2. The *Mean curvature* of $\mathcal{M}$ at $\mathbf{p}$ as

$$H = \frac{\kappa_1 + \kappa_2}{2}.$$

- The Gaussian curvature is the determinant of $\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}$.
- We can classify manifolds using Gaussian curvature:



- The *mean curvature* is the negative of half of the trace of $\mathrm{d}_{\mathbf{p}}\hat{\mathbf{N}}$. (See linear algebra notes). Also

$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\,\mathrm{d}\theta\,.$$

1.5 Geodesic Curvature and Geodesics

- Let $\mathcal{M}$ be a smooth 2-manifold with chart $\psi : U \to V \subseteq \mathbb{R}^2$ where U is the neighborhood of $\mathbf{p} \in \mathcal{M}$ and $\boldsymbol{\gamma} : \mathcal{D} \to \mathcal{M}$ lie in $\overrightarrow{\gamma}(\mathcal{D}) \subseteq U$. We have the fernet frame $\left\{\hat{\mathbf{T}}, \hat{\mathbf{P}}, \hat{\mathbf{B}}\right\}$. Note that $\hat{\mathbf{P}} \neq \hat{\mathbf{N}}(\mathbf{p})$, the surface normal (Gauss-mapped).
- Note that $\left\{\hat{\mathbf{T}}, \hat{\mathbf{N}} \times \hat{\mathbf{T}}, \hat{\mathbf{N}}\right\}$ is an orthonormal basis.

Definition 1.5.1. (**Darboux Frame**) The *Darboux frame* of a smooth 2-manifold $\mathcal{M}$ is $\left\{\hat{\mathbf{T}}, \hat{\mathbf{U}}, \hat{\mathbf{N}}\right\}$ with

$$\hat{\mathbf{U}} = \hat{\mathbf{N}} \times \hat{\mathbf{T}},\ \ \hat{\mathbf{N}} = \hat{\mathbf{T}} \times \hat{\mathbf{U}}, \hat{\mathbf{T}} = \hat{\mathbf{U}} \times \hat{\mathbf{N}}.$$

1.5.1 Geodesic Curvature

- As with curves $\boldsymbol{\gamma}$, since $\hat{\mathbf{T}}' \perp \hat{\mathbf{T}}$, we have

$$\hat{\mathbf{T}}' = s'(t)\kappa_s(t)\hat{\mathbf{P}} = s'(t)\kappa_g(t)\hat{\mathbf{U}} + s'(t)\kappa_n(t)\hat{\mathbf{N}},$$

where κ_g is the *geodesic curvature* of $\boldsymbol{\gamma}$ at $\mathbf{p}$ on $\mathcal{M}$.

Definition 1.5.2. (**Geodesic Curvature**) Let $\mathcal{M}$ be a smooth 2-manifold with $\mathbf{p} \in \mathcal{M}$, with neighborhood U and chart $\varphi : U \to V \subseteq \mathbb{R}^2$. Let $\boldsymbol{\gamma} : \mathcal{D} \to \mathcal{M}$ be smooth curve in U. The *geodesic curvature* of $\mathcal{M}$ is

$$\kappa_g(t) = I_{\mathbf{p}}\left(\kappa_s(t)\hat{\mathbf{P}}(t), \hat{\mathbf{U}}(t)\right) = \frac{1}{s'(t)}\left[\hat{\mathbf{T}}'(t), \hat{\mathbf{N}}(t), \hat{\mathbf{T}}(t)\right].$$

- Note that

$$\kappa_s(t) = \pm\sqrt{\kappa_n(t)^2 + \kappa_g(t)^2}.$$

- As with the fernet frame, consider $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat{\mathbf{T}} & \hat{\mathbf{U}} & \hat{\mathbf{N}}\end{bmatrix}^T$. We have $\hat{\mathbf{T}}'(t) = s'(t)\left[\kappa_g(t)\hat{\mathbf{U}}(t) + \kappa_n(t)\hat{\mathbf{N}}(t)\right]$. We note that

$$\begin{aligned} \hat{\mathbf{T}}' \cdot \hat{\mathbf{T}} &= 0, & \hat{\mathbf{U}}' \cdot \hat{\mathbf{U}} &= 0, & \hat{\mathbf{N}}' \cdot \hat{\mathbf{N}} &= 0 \\ \hat{\mathbf{N}}' \cdot \hat{\mathbf{T}} &= -\hat{\mathbf{T}}' \cdot \hat{\mathbf{N}}, & \hat{\mathbf{N}}' \cdot \hat{\mathbf{U}} &= -\hat{\mathbf{U}}' \cdot \hat{\mathbf{N}}, & \hat{\mathbf{T}}' \cdot \hat{\mathbf{U}} &= -\hat{\mathbf{U}}' \cdot \hat{\mathbf{T}} \end{aligned}$$

hence

$$\hat{\mathbf{N}}' \cdot \hat{\mathbf{T}} = -s'\kappa_n \qquad \hat{\mathbf{U}}' \cdot \hat{\mathbf{T}} = -s'\kappa_g$$

Note that we require $\hat{\mathbf{U}}' \cdot \hat{\mathbf{N}}$.

Definition 1.5.3. (**Geodesic Torsion**) We define the geodesic torsion $\tau_g : \mathcal{D} \to \mathbb{R}$ s.t

$$\hat{\mathbf{U}}'(t) \cdot \hat{\mathbf{N}}(t) = s'(t)\tau_g(t).$$

- Intuitively, geodesic torsion is the rate of change $\hat{\mathbf{N}}'$ in the direction $\hat{\mathbf{U}}$.
- Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat{\mathbf{T}}(t) \\ \hat{\mathbf{U}}(t) \\ \hat{\mathbf{N}}(t)\end{bmatrix} = s'(t)\begin{bmatrix}0 & \kappa_g(t) & \kappa_n(t) \\ -\kappa_g(t) & 0 & \tau_g(t) \\ -\kappa_n(t) & -\tau_g(t) & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{T}}(t) \\ \hat{\mathbf{U}}(t) \\ \hat{\mathbf{N}}(t)\end{bmatrix}.$$

- These equations describe geometry of the curve $\boldsymbol{\gamma}$ and a 2-manifold $\mathcal{M}$, whereas the Fernet equations describe geometry of $\boldsymbol{\gamma}$.

1.5.2 Geodesics

Definition 1.5.4. (**Geodesic**) A geodesic is a curve $\boldsymbol{\gamma} : \mathcal{D} \to \mathcal{M}$ on the smooth 2-manifold $\mathcal{M}$ with geodesic curvature $\kappa_g(t) = 0$.

- Note that, for geodesic $\boldsymbol{\gamma}$, $\hat{\mathbf{T}}'(t) = s'(t)\kappa_s(t)\hat{\mathbf{P}}(t) = s'\kappa_n(t)\hat{\mathbf{N}}(t)$.

Theorem 1.5.1. Let $\boldsymbol{\gamma} : [0, 1] \to U$ be a curve on $\mathcal{M}$, where U is the neighborhood of $\mathbf{p} \in \mathcal{M}$. Suppose $\mathbf{a}, \mathbf{b} \in U$ s.t $\boldsymbol{\gamma}(0) = \mathbf{a}$ and $\boldsymbol{\gamma}(1) = \mathbf{b}$. $\boldsymbol{\gamma}$ minimizes the arc-length s if and only if $\boldsymbol{\gamma}$ is geodesic.

1.6 The Laplacian

Definition 1.6.1. (**The Laplacian**) The Laplacian of $f : \mathbb{R}^n \to \mathbb{R}$, denoted Δf, is defined as the divergence of the gradient: $\Delta f = \nabla^2 f = \nabla \cdot \nabla f$. Hence

$$\nabla f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}.$$

- The *Laplace-Beltrami* operator extends Δ to be defined on manifolds $\mathcal{M}$. Given a function $f : \mathcal{M} \to \mathbb{R}$,

$$\Delta_{\mathcal{M}} f = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} f.$$

Theorem 1.6.1. For a 2-manifold $\mathcal{M}$:

$$\Delta_{\mathcal{M}} \mathbf{x} = -2H\hat{\mathbf{N}},$$

where H is the mean curvature, where $\mathbf{x}$ is coordinate function for $\mathcal{M}$, that is $\mathbf{x} = (x, y, z)$. Hence $f = id_{\mathcal{M}}$.

1.6.1 Discrete Laplacian

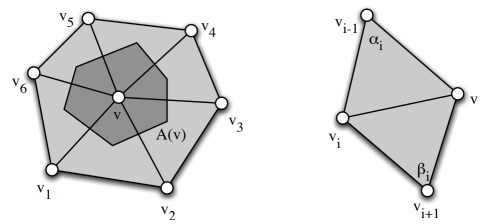
- **Problem**: Laplacian requires differentiable surface, however, triangle meshes are discontinuous.
- **Solution**: Compute differential property via spatial averages over neighborhood $\mathcal{N}(v)$, where $v \in V$ where T is a triangle mesh.
  - The *uniform discrete Laplacian*:

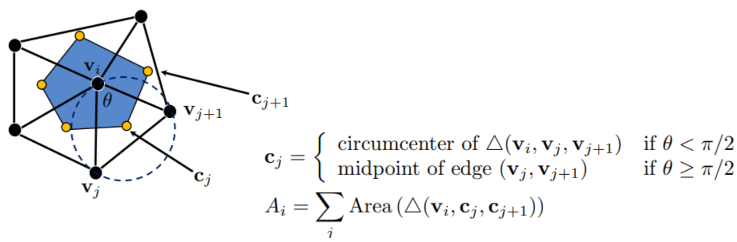
    $$L_u f = \Delta_{\mathcal{M}}^u f := \frac{1}{|\mathcal{N}(v)|} \sum_{v_i \in \mathcal{N}(v)} f(\mathbf{p}(v_i)) - f(\mathbf{p}(v)),$$

    where $\mathbf{p}(v_i)$ is the associated position vector to vertex $v_i \in V$.
  - The *cotangent discrete Laplacian*:

    $$L_{cot} f = \Delta_{\mathcal{M}}^{cot} f := \frac{1}{2A(v)} \sum_{v_i \in \mathcal{N}(v)} (\cot \alpha_i + \cot \beta_i) \left(f(\mathbf{p}(v_i)) - f(\mathbf{p}(v)) \right),$$

    where α_i and β_i are the angles and $A(v)$ is the the Voronoi area around vertex v, given by:





- The discrete Laplacian can be used to compute the mean curvature H:

$$\|H\| = \frac{\|\Delta_{\mathcal{M}}\mathbf{x}\|}{2},$$

giving us

$$\|H\| = \frac{1}{2}\|L_u\,\mathbf{x}\| = \frac{1}{2}\|L_{cot}\,\mathbf{x}\|$$

- For Gaussian curvature K, we have:

$$K(v) = \frac{1}{A(v)}\left(2\pi - \sum_{v_i \in \mathcal{N}(v)} \theta_i\right),$$

where θ_i are the angles of the incident triangles at v.

2 Geometry

2.1 Geometry Representations

2.1.1 Implicit Curves and Surfaces

Definition 2.1.1. (**Implicit Curve**) An implicit curve (surface) C (S) is a curve (surface) s.t for all points $\mathbf{x}$ on C, the implicit equation

$$f(\mathbf{x}) = 0,$$

holds, where $\mathbf{x} = (x, y)$ ($\mathbf{x} = (x, y, z)$) and f is the *implicit function* of the curve.

- Examples of implicit curves:
  - Circle with center $\mathbf{c} = (x_c, y_c)$ and radius r

    $$f(x, y) = (x - x_c)^2 + (y - y_c)^2 - r^2 = 0,$$

    or in vector notation

    $$\|\mathbf{x} - \mathbf{c}\|^2 - r = 0.$$

- The normal $\mathbf{n}$ at the point $\mathbf{p}$ to the implicit curve (or surface) $f(\mathbf{x}) = 0$, is given by

  $$\mathbf{n} = \nabla f(\mathbf{p}) = \left(\frac{\partial f(\mathbf{p})}{\partial x}, \frac{\partial f(\mathbf{p})}{\partial y}, \ldots, \frac{\partial f(\mathbf{p})}{\partial z}\right).$$

- The tangent line (or plane) to the implicit curve $f(\mathbf{x}) = 0$ (or line) at the point $\mathbf{p}$ is given by

  $$(\mathbf{r} - \mathbf{p}) \cdot \nabla f(\mathbf{p}) = 0.$$

- We can use a simple sign test to determine whether a given point $\mathbf{p}$ lines on, outside or inside the implicit curve (or surface) $f(\mathbf{x}) = 0$.
  - If $f(\mathbf{p}) > 0$, then the point lies outside of the implicit curve (or surface).
  - If $f(\mathbf{p}) = 0$, then the point lies on the implicit curve (or surface).
  - If $f(\mathbf{p}) < 0$, then the point lies inside the implicit curve (or surface).
- Advantages:
  - Easy to determine whether a point lies outside, on, or inside an implicit curve / surface.
  - Easy to combine implicit curve/surfaces by simply adding them.
- Disadvantages:
  - Implicit curves/surfaces can only be used in raytracing since we it's difficult to generate points on the surface without testing the entire domain $\mathbb{R}^2/\mathbb{R}^3$.
  - We can only represent a limited number of surfaces.
  - Does not lend itself to rasterization.

2.1.2 Parametric Curves and Surfaces

Definition 2.1.2. (**Parameterized Curve**) A parameterized curve C is a curve that is defined by the vector-valued function $\boldsymbol{\gamma} : \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}^n$, s.t

$$C = \{\boldsymbol{\gamma}(t) : t \in \mathcal{D}\}.$$

- See section ??
- In Graphics, sometimes a volumetric representation of a parametric surface is used (a scalar field) e.g. to represent densities, etc. Particularly useful in medical imagery.
- Advantages:
  - Easy to generate points on the curve / surface (by sampling the parameter domain $\mathcal{D}$).

  - Easy to combine curves (pointwise definitions of parametric curves).
  - Several important curves. e.g. Bezier Curves/Surfaces or B-spline.

- Disadvantages:
  - Difficult to determine whether a given point lies inside, on or outside the surface/curve, requires solving for parameters t (may not be possible).
  - Difficult to find a parametric representation of a given surface (reverse engineering).

2.1.3 Triangle Meshes

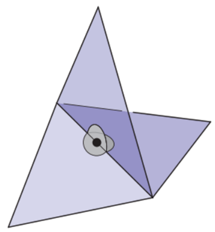
Definition 2.1.3. (**Triangle Mesh**) A triangle mesh $\mathcal{M}$ is a graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ is a set of vertices, and $E = \{e_1, \ldots, e_m\}$ is a set of edges where $e_i \in V^2$.

- The set of triangle faces $\mathcal{F} = \{f_1, \ldots, f_F\}$ where $f_i \in V^3$. This is the *topological component* of the mesh.
- The geometric component of $\mathcal{M}$ is the embedding of $\mathcal{M}$ in $\mathbb{R}^3$:

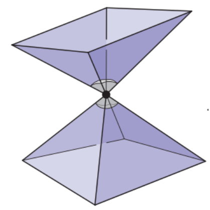
$$P = \{\mathbf{p}_1, \ldots, p_n\} \in \mathbb{R}^3.$$

  where $\mathbf{p}_i$ is the associated position vector to vertex $v_i \in V$.

- A mesh $\mathcal{M}$ is a 2-manifold iff it doesn't contain any non-manifold edges, non-manifold vertices or self-intersections:
  - A non-manifold edge has more than two incident triangles
  - A non-manifold vertex is a vertex that is incident to two fans of triangles



(a) Non-manifold edge



(b) Non-manifold vertex

2.1.4 Point Set Surfaces

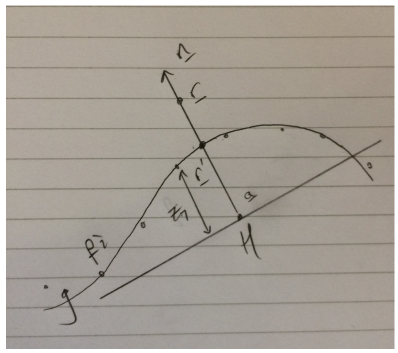
Definition 2.1.4. (**Point Set Surface**) A point set surface S is a surface defined by a sampled set of points $P = \{p_i\}$ (often acquired by a 3D scanning device).

- **Idea**: A given point set P defines a surface S. We can approximate S using the MLS (method of moving least squares) S_P.
- The idea of the MLS surface S_P is the projection procedure, which projects any points $\mathbf{r}$ in the neighborhood of the surface S onto the surface S_P. Hence a projection procedure F is such that $F(\mathbf{r}) = F(F(\mathbf{r}))$.
- **Projection Procedure**:
  1. Find a local reference domain (plane) H s.t that local weighted sum of square distances of $\mathbf{p}_i$ to H are minimized. e.g.
  $$H : (\mathbf{x} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0,$$
  then we minimize
  $$\sum_{i=1}^{N} \underbrace{((\mathbf{p}_i - \mathbf{a}) \cdot \hat{\mathbf{n}})^2}_{\text{sq dist}} \underbrace{\theta(\|\mathbf{p}_i - \mathbf{a}\|)}_{\text{weight}},$$
  where $\theta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a monotonically decreasing function.
  2. The local reference domain H for $\mathbf{r}$ is used to fit a polynomial g through the points in the neighborhood of $\mathbf{r}$. Similarly, a MLS approximation is used.
  $$\sum_{i=1}^{N} (g(x_i, y_i) - z_i)^2 \theta(\|\mathbf{p}_i - \mathbf{a}\|),$$

where (x_i, y_i, z_i) are the coordinates of $\mathbf{p}_i$ in the basis defined by H and $z_i = \hat{\mathbf{n}} \cdot (\mathbf{p}_i - \mathbf{a})$.

3. The projection of $\mathbf{r}$ onto S_p is then given by

$$\mathbf{r}' = MLS(\mathbf{r}) = \mathbf{a} + g(0,0)\mathbf{n}.$$



- Advantages:
  - Point set surfaces are robust to noise (due to MLS).
  - Can be converted to triangle meshes. Hence can be used for rasterization.
  - Easy to generate points on the surface
  - Easy to determine whether a point is lies inside, on or outside the surface due to implicit fitted curve g.
- Disadvantages:
  - Difficult to use for modelling tasks (since point set would have to be created manually).

2.2 Geometry Acquisition

- 2 sources of geometry: Modelling and Acquisition from the world.
- Optical Scanners:
  - Active scanners:

    * Active scanners emit some kind of radiation / light and detects it's reflection in order to determine the geometry of the object.
    * Examples: LIDAR: measures distance using timing calculations from the speed of light.
      Triangulation Laser: Uses camera and laser to determine distance between laser and object.
    * **Advantages**: Active scanners are better suited to large objects e.g. buildings.
    * **Disadvantages**: Due to the speed of light and timer delays, the accuracy of the distance measurements produced by active scanners can be low.
  - Passive Scanners:
    * Passive scanners rely on detecting reflected ambient radiation / light.
    * Examples: Cameras, multi-view stereo, a specialized camera (or 2 ordinary digital cameras).
    * **Advantages**: Very cheap (only requires a simple digital camera).
  - **Advantages**: Fast compared to active contact scanners such as touch probes.
  - **Disadvantages**: Only suited to reflective objects. (e.g. cannot scan glass objects).

- Contact Scanners:
  - Relies on contact with the object to determine the geometry of the object.
  - Examples: touch probe.
  - **Advantages**: Extremely precise.
  - **Disadvantages**: Extremely slow and not applicable to large objects (e.g. buildings).

2.2.1 Iterative Closest Point

Definition 2.2.1. (**Registration**) Registration is the process of transforming different point set surfaces (P_i) into a single coordinate system.

- Iterative Closest Point (ICP) is a common algorithm for minimizing the difference between two points set surface $P = \{\mathbf{p}_i\}$ and $Q = \{\mathbf{q}_i\}$. in $\mathbb{R}^\ell$.

- Define the distance between P and Q

$$d_g(P, Q) = \sum_{\mathbf{p}_i \in P} \sum_{\mathbf{q}_i \in Q} g(\mathbf{p}_i, \mathbf{q}_i),$$

  where g is a metric defined on the surfaces. e.g. $g(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|^2$ (Euclidean metric) and $\mathbf{p}_i$ is the "corresponding" point to $\mathbf{q}_i$. Determined using distance / curvature, etc.

- ICP determines rigid transformation $\mathbf{T}(\mathbf{x}) = \mathbf{R}(\mathbf{x}) + \mathbf{t}$ s.t $\min_{\mathbf{T}} d_{\|\cdot\|^2}(\mathbf{T}(P), Q)$.

- ICP algorithm:

  1. For each point $\mathbf{p}_i \in P$, determine the corresponding point $\mathbf{q}_i \in Q$ s.t

$$\mathbf{q}_i = \arg\min_{\mathbf{q} \in Q} \|\mathbf{p}_i - \mathbf{q}\|^2.$$

     Yields the *corresponding set* $C = \{(\mathbf{p}_i, \mathbf{q}_i) : 1 \leq i \leq n\}$. Define corresponding weights w_i s.t

$$w_i = 1 - \frac{\|\mathbf{p}_i - \mathbf{q}_i\|^2}{\max_i \|\mathbf{p}_i - \mathbf{q}_i\|^2}.$$

  2. Compute the centroids of P and Q:

$$\overline{\mathbf{p}} = \frac{1}{n} \sum_i \mathbf{p}_i \qquad \overline{\mathbf{q}} = \frac{1}{n} \sum_i \mathbf{q}_i$$

     and $\mathbf{p}'_i = \mathbf{p}_i - \overline{\mathbf{p}} \in P'$ and $\mathbf{q}'_i = \mathbf{q}_i - \overline{\mathbf{q}} \in Q'$. Note that $\sum_i \mathbf{p}'_i = \sum_i \mathbf{q}'_i = \mathbf{0}$.
  3. Define $\Lambda = \sum_i P_i^T Q_i$. Use SVD to compute the corresponding eigenvalues and eigenvectors of Λ: $\lambda_1 \geq \cdots \geq \lambda_4$ and $\mathbf{v}_1, \ldots, \mathbf{v}_4$. The optimal rigid transformation is given by

$$\mathbf{T}(\mathbf{x}) = \mathbf{R}(\mathbf{x}) + \mathbf{t} = L_q(\mathbf{x}) + L_q(\overline{\mathbf{p}}) - \overline{\mathbf{q}},$$

     where quaternion $q = \mathbf{v}_1$.

4. Apply $\mathbf{T}$ to P, yielding P'.
5. Compute the error:

$$E(\mathbf{T}) = \sum_{i=1}^{n} w_i \|\mathbf{T}\mathbf{p}_i - \mathbf{q}_i\|^2.$$

Iterate until $E(\mathbf{T}) \leq$ threshold.

Theorem 2.2.1. For point sets P and Q with corresponding points $C = \{(\mathbf{p}_i, \mathbf{q}_i) : 1 \leq i \leq n\}$. The rigid transformation $\mathbf{T}$ that minimizes $d_{\|\cdot\|^2}(\mathbf{T}(P), Q)$ is

$$\mathbf{T}(\mathbf{x}) = L_q(\mathbf{x}) + L_q(\overline{\mathbf{p}}) - \overline{\mathbf{q}},$$

where $q = \mathbf{v}_1$, the eigenvector of Λ with the maximum corresponding eigenvalue λ_1.

Proof. Let $\mathbf{T}(\mathbf{x}) = \mathbf{R}(\mathbf{x}) + \mathbf{t}$ be arbitrary rigid transformation. Note that

$$\begin{aligned} d_{\|\cdot\|^2}(\mathbf{T}(P), Q) &= \sum_{i=1}^{n} \|\mathbf{R}\left(\mathbf{p}'_i + \overline{\mathbf{p}}\right) + \mathbf{t} - (\mathbf{q}'_i + \overline{\mathbf{q}})\|^2 \\ &= \sum_{i=1}^{n} \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|^2 + 2\sum_{i=1}^{n} (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i) \cdot (\mathbf{R}\overline{\mathbf{p}} - \overline{\mathbf{q}} + \mathbf{t}) + n\|\mathbf{R}\overline{\mathbf{p}} - \overline{\mathbf{q}} + \mathbf{t}\|^2 \\ &= \sum_{i=1}^{n} \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|^2 + n\|\mathbf{R}\overline{\mathbf{p}} - \overline{\mathbf{q}} + \mathbf{t}\|^2 \end{aligned}$$

Hence

$$\min_{\mathbf{T}} d_{\|\cdot\|^2}(\mathbf{T}(P), Q) = \min_{\mathbf{R}} \sum_{i=1}^{n} \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|^2 + \min_{\mathbf{R},\mathbf{t}} n\|\mathbf{R}\overline{\mathbf{p}} - \overline{\mathbf{q}} + \mathbf{t}\|^2.$$

Observe that $\|\mathbf{R}\overline{\mathbf{p}} - \overline{\mathbf{q}} + \mathbf{t}\|^2 = 0$ iff $\mathbf{t} = \mathbf{R}\overline{\mathbf{p}} - \overline{\mathbf{q}}$. Hence $\mathbf{t} = \mathbf{R}\overline{\mathbf{p}} - \overline{\mathbf{q}}$ is the optimal translation. We wish to determine rotation $\mathbf{R}$ s.t $\sum_{i=1}^{n} \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|^2$ is minimized. Note that

$$\min_{\mathbf{R}} \sum_{i=1}^{n} \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|^2 = \sum_{i=1}^{n} \left(\|\mathbf{p}'_i\|^2 + \|\mathbf{q}'_i\|^2\right) - 2\max_{\mathbf{R}} \sum_{i=1}^{n} \mathbf{R}\mathbf{p}'_i \cdot \mathbf{q}'_i$$

Let us express $\mathbf{R}$ using the quaternion $q \in \mathbb{H}$. By definition of quaternion product, we have $L_q(\mathbf{p}_i') \cdot \mathbf{q}_i' = (q\mathbf{p}_i'q^*) \cdot \mathbf{q}_i' = (q\mathbf{p}_i') \cdot (\mathbf{q}_i'q)$. Hence

$$\begin{aligned}\sum_{i=1}^{n} \mathbf{R}\mathbf{p}_i' \cdot \mathbf{q}_i' &= \sum_{i=1}^{n} (q\mathbf{p}_i'q^*) \cdot \mathbf{q}_i' \\ &= \sum_{i=1}^{n} (q\mathbf{p}_i') \cdot (\mathbf{q}_i'q)\end{aligned}$$

Let us define

$$P_i = \begin{bmatrix} 0 & -p_{i1}' & -p_{i2}' & -p_{i3}' \\ p_{i1}' & 0 & p_{i3}' & -p_{i2}' \\ p_{i2}' & -p_{i3}' & 0 & p_{i1}' \\ p_{i3}' & p_{i2}' & -p_{i1}' & 0 \end{bmatrix} \qquad Q_i = \begin{bmatrix} 0 & -q_{i1}' & -q_{i2}' & -q_{i3}' \\ q_{i1}' & 0 & -q_{i3}' & q_{i2}' \\ q_{i2}' & q_{i3}' & 0 & -q_{i1}' \\ q_{i3}' & -q_{i2}' & q_{i1}' & 0 \end{bmatrix}$$

Then $P_i q = q\mathbf{p}_i'$ and $Q_i q = \mathbf{q}_i' q$. So

$$\begin{aligned}\sum_{i=1}^{n} \mathbf{R}\mathbf{p}_i' \cdot \mathbf{q}_i' &= \sum_{i=1}^{n} (P_i q) \cdot (Q_i q) \\ &= \sum_{i=1}^{n} q^T P_i^T Q_i q \\ &= q^T \left(\sum_{i=1}^{n} P_i^T Q_i \right) q\end{aligned}$$

Define $\Lambda = \sum_{i=1}^{n} P_i^T Q_i$. Note that $P_i^T Q_i$ is symmetric, hence Λ is symmetric. Let $\lambda_1 \geq \ldots \geq \lambda_4$ and $\mathbf{v}_1, \ldots, \mathbf{v}_4$ be the corresponding eigenvalues and eigenvectors of Λ. Since $\mathbf{v}_1, \ldots, \mathbf{v}_4$ form a basis, let $q = \sum_i \alpha_i \mathbf{v}_i$. Hence

$$q^T \Lambda q = \sum_i \lambda_i \alpha_i^2$$

So $q^T \Lambda q$ is maximized when $\alpha_1 = 1$ and $\alpha_i = 0, i \neq 1$.

□

- Other feature based version of ICP use curvature as a matric.

3 Animation

Definition 3.0.1. (**Animation**) Animation is a technique of using a sequence of images to create the illusion of movement (when shown in sequence).

- Animation has two main parts:
  - *Parameter definition*: Define a set of n parameters of the scene, denoted $\mathbf{q}$. $\mathbf{q}$ forms the state space of the scene.
  - *Parameter generation*: For each frame at time t, generate $\mathbf{q}(t)$ and render the scene.
- So the structure of an animation algorithm is:

```
define q(t) for the scene S
for (t ← 0; t < FRAMES; t++) {
    render(S, q(t))
}
```

- Animation techniques:
  - *Keyframing* is an animation technique where the parameters are interpolated through the states $(\mathbf{q}_1, \ldots, \mathbf{q}_T)$, called keyframes.

    Advantages: Expressive, Animator has complete control over state-space parameters.

    Disadvantages: Difficult to create convincing physical realism. Labor intensive defining keyframes.
  - *Physic-based* animation is a technique where a scene is simulated using **dynamics** (motion determined by mass and force).

    Advantages: Realism. Simulation is easy to implement.

    Disadvantages: Slow. No control of path.

- *Motion capture* is a technique where an actor has a number of markers attached to their body. Multiple cameras detect these markers, used to reconstruct their positions, forming the states for each frame.

  Advantages: Realistic character animation.
  Disadvantage: Noisy states. Marker reconstructions fail, requiring manual fixes. Accuracy limited by number of markers (e.g. faces).

- (forward) *Kinematics* describes the motion of points $\mathbf{p} = f(\mathbf{q})$ where $\mathbf{q}$ is a state vector, specifying translations, rotation, etc. Often used with keyframing.

3.1 Deformations

- Deformations:
  - *Free-Form*: consists of modifying the positions of vertices. The displacements are then interpolated to produce the modified geometry.
  - *Elastic deformations*: the surface is given elastic properties and is then simulated physically (using Hooke's law etc).
  - *Skeletal deformations*: See section ??.
  - *Cage-based (or structure-aware) deformations*: Define a cage C around the surface S, which is a crude approximation of the surface S.

    Each point $\mathbf{x} \in S$ is defined as a linear combination of cage points $(\mathbf{p}_i)$:

    $$\mathbf{x} = \sum_{i=1}^{|C|} w_i(\mathbf{x})\mathbf{p}_i.$$

    Modifying the positions $\mathbf{p}_i$ deforms S.

- Cutting and Fracturing surfaces: requires physical simulation. realistic effects, but is often complicated and slow.

3.1.1 Rigid-body Transformations

Definition 3.1.1. (**Rigid-body Transformation**) A rigid transformation is any transformation $\mathbf{T}$ such that

$$\mathbf{T}(\mathbf{x}) = \mathbf{R}(\mathbf{x}) + \mathbf{t},$$

where $\mathbf{R}$ is an orthogonal transformation and $\mathbf{t}$ is a translation.

- Hence a rigid transformation is a combination of a rotation / reflection and a translation.
- Rigid transformations are a subset of *affine transformations*:

  $$\mathbf{T}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \mathbf{t},$$

  where $\mathbf{A}$ is a linear transformation.
- Shape and size are preserved by rigid transformations.
- The manifold of rigid transformations of $\mathbb{R}^n$ is denoted $\mathbf{SE}(n)$.

3.2 Keyframing

Definition 3.2.1. (**Keyframing**) Keyframing is an animation technique where the parameters are interpolated through the states $(\mathbf{q}_1, \ldots, \mathbf{q}_T)$, called keyframes.

- The keyframes $(\mathbf{q}_1, \ldots, \mathbf{q}_T)$ are fitted to a state-space curve $\mathbf{q}(t) = \boldsymbol{\gamma}(t)$, the *animation curve*.
- Requirements of $\boldsymbol{\gamma}$:
  - C^1. Prevents sudden changes (violates animation principles) Typically a parametric curve is used. e.g. Catmull-Rom, Bezier, etc
  - Unit-speed parameterization. Prevents non-uniform interpolation.

3.2.1 Character Animation

- Articulating figures (characters) is often done via *rigging* and keyframing.

Definition 3.2.2. (**Rigging**) Rigging is a technique in which a manifold $\mathcal{M}$ is embedded onto a skeleton, which defines a set of *joints* or skeleton $j_1, \ldots, j_n$ (ordered into some hierarchy). Each point $\mathbf{p} \in \mathcal{M}$ is attached to the joints with weights $\mathbf{w} = (w_1, \ldots, w_n)$, s.t

$$w_i \geq 0 \qquad\qquad \sum_i w_i = 1$$

- When the rigid transformations $\mathbf{T}_1, \ldots, \mathbf{T}_n$ are applied to joints $j_1, \ldots, j_n$, the transformed point $\mathbf{p'}$ of $\mathbf{p}$ is

$$\mathbf{p}' = \left(\sum_{i=1}^{n} w_i \mathbf{T}_i\right) \mathbf{p}.$$

(See section ??)

- Weight properties:
  - Weights $\mathbf{w}$ should be smooth
  - Weights $\mathbf{w}$ should be *shape-aware*: they preserve certain geometry features e.g. aspect ratio.
- The skeleton is keyframed to animate the figure, where the state $\mathbf{q}_t$ consists of the rigid transformations $\mathbf{T}_1, \ldots, \mathbf{T}_n$ applied to joints $j_1, \ldots, j_n$ in keyframe t.

3.3 Quaternions

Definition 3.3.1. (**Quaternoin**) A quaternion $q \in \mathbb{H}$ is defined as as the sum of the scalar q_0 and the vector $\mathbf{q} = (q_1, q_2, q_3)$,

$$q = q_0 + \mathbf{q} = q_0 + q_1\hat{\mathbf{i}} + q_2\hat{\mathbf{j}} + q_3\hat{\mathbf{k}},$$

where

$$\hat{\mathbf{i}}^2 = \hat{\mathbf{j}}^2 = \hat{\mathbf{k}}^2 = \hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}} = -1$$

The *fundamental identity of quaternions.*

- Addition:

$$p + q = (p_0 + q_0) + (p_1 + q_1)\hat{\mathbf{i}} + (p_2 + q_2)\hat{\mathbf{j}} + (p_3 + q_3)\hat{\mathbf{k}}.$$

- Product:

$$\begin{aligned} pq &= (p_0 + p_1\hat{\mathbf{i}} + p_2\hat{\mathbf{j}} + p_3\hat{\mathbf{k}})(q_0 + q_1\hat{\mathbf{i}} + q_2\hat{\mathbf{j}} + q_3\hat{\mathbf{k}}) \\ &= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + p_0\left(q_1\hat{\mathbf{i}} + q_2\hat{\mathbf{j}} + q_3\hat{\mathbf{k}}\right) + q_0\left(p_1\hat{\mathbf{i}} + p_2\hat{\mathbf{j}} + p_3\hat{\mathbf{k}}\right) \\ &\qquad + (p_2q_3 - p_3q_2)\hat{\mathbf{i}} + (p_3q_1 - p_1q_3)\hat{\mathbf{j}} + (p_1q_2 - p_2q_1)\hat{\mathbf{k}} \\ &= p_0q_0 - \mathbf{p}\cdot\mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q} \end{aligned}$$

3.3.1 Conjugate, Norm and Inverse

Definition 3.3.2. (**Conjugate**) The conjugate of q, denoted q^*, is defined

$$q^* = q_0 - \mathbf{q} = q_0 - q_1\hat{\mathbf{i}} - q_2\hat{\mathbf{j}} - q_3\hat{\mathbf{k}}.$$

- Properties of the conjugate:

$$\begin{aligned} (q^*)^* &= q \\ q + q^* &= 2q_0 \\ q^*q &= qq^* \\ (pq)^* &= q^*p^* \end{aligned}$$

Definition 3.3.3. (**Norm**) The norm of a quaternion q, denoted $|q|$, is the scalar $|q| = \sqrt{q^*q}$

- Note that:

$$\begin{aligned} |pq|^2 &= (pq)(pq)^* = pqq^*p^* \\ &= p|q|^2p^* = |p|^2|q|^2 \end{aligned}$$

Definition 3.3.4. (**Inverse**) The inverse of a quaternion q is defined as

$$q^{-1} = \frac{q^*}{|q|^2}.$$

- $q^{-1}q = qq^{-1} = 1$
- For a unit quaternion $\hat{q}$, that is $|\hat{q}| = 1$, then $\hat{q}^{-1} = \hat{q}^*$.

3.3.2 Rotation Operator

- A vector $\mathbf{v} \in \mathbb{R}^3$ is a *pure quaternion* $q = 0 + \mathbf{v}$.

Definition 3.3.5. (**Quaternion Operator**) For a unit quaternion q, the quaternion operator of q, denoted $L_q : \mathbb{R}^3 \to \mathbb{R}^3$ applied to a vector $\mathbf{v} \in \mathbb{R}^3$ is

$$L_q(\mathbf{v}) = q\mathbf{v}q^* = (q_0^2 - \|\mathbf{q}\|^2)\mathbf{v} + 2(\mathbf{q} \cdot \mathbf{v})\mathbf{q} + 2q_0(\mathbf{q} \times \mathbf{v}).$$

- Properties:

$$\|L_q(\mathbf{v})\| = \|\mathbf{v}\|$$
$$L_q(\lambda\mathbf{q}) = \lambda\mathbf{q}$$
$$L_q(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2) = \lambda_1 L_q(\mathbf{v}_1) + \lambda_2 L_q(\mathbf{v}_2)$$

  Acts similar to rotation about $\mathbf{q}$

Theorem 3.3.1. For any unit quaternion

$$q = q_0 + \mathbf{q} = \cos\frac{\theta}{2} + \hat{\mathbf{u}}\sin\frac{\theta}{2},$$

the quaternion operator of q, L_q is a rotation through an angle of θ about $\hat{\mathbf{u}}$, the axis of rotation.

Proof. Let $\mathbf{v} \in \mathbb{R}^3$. Let $\mathbf{n}_q$ be perpendicular to $\mathbf{q}$, hence $\mathbf{v} = \underbrace{(\mathbf{v} \cdot \mathbf{q})\mathbf{q}}_{\mathbf{a}} + \underbrace{(\mathbf{v} \cdot \mathbf{n}_q)\mathbf{n}_q}_{\mathbf{b}}$.

We wish to show that:

(i) $\mathbf{a}$ is invariant under L_q

(ii) $\mathbf{b}$ is rotated about $\mathbf{q}$ through angle θ

We have (i). Consider (ii). We have

$$\begin{aligned} L_q(\mathbf{b}) &= (q_0^2 - \|\mathbf{q}\|^2)\mathbf{b} + 2(\mathbf{q} \cdot \mathbf{b}) + 2q_0\,(\mathbf{q} \times \mathbf{b}) \\ &= (q_0^2 - \|\mathbf{q}\|^2)\mathbf{b} + 2q_0\|\mathbf{q}\|(\hat{\mathbf{u}} \times \mathbf{b}) \end{aligned}$$

since $\hat{\mathbf{u}} = \mathbf{q}/\|\mathbf{q}\|$. Let $\mathbf{b}_\perp = \hat{\mathbf{u}} \times \mathbf{b}$. Note that $\|\mathbf{b}_\perp\| = \|\mathbf{b}\|\|\hat{\mathbf{u}}\| \sin \pi/2 = \|\mathbf{b}\|$. So we have

$$\begin{aligned} L_q(\mathbf{b}) &= (q_0^2 - \|\mathbf{q}\|^2)\mathbf{b} + 2q_0\|\mathbf{q}\|\mathbf{b}_\perp \\ &= \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}\right)\mathbf{b} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\mathbf{b}_\perp \\ &= \cos\theta\mathbf{b} + \sin\theta\mathbf{b}_\perp \end{aligned}$$

We have a plane defined by $\{\mathbf{b}, \mathbf{b}_\perp\}$, hence $\mathbf{b}$ is rotated by θ in the defined plane.

□

- Let p, q be unit quaternions. The composition $L_q \circ L_p$ is

$$\begin{aligned} (L_q \circ L_p)(\mathbf{v}) &= L_q(L_p(\mathbf{v})) \\ &= q(p\mathbf{v}p^*)q^* \\ &= (qp)\mathbf{v}(p^*q^*) \\ &= L_{qp}(\mathbf{v}) \end{aligned}$$

3.3.3 Power, Exponential and Logarithm

- A quaternion $q = q_0 + \mathbf{q}$ may be written as

  $$q = |q|(\cos\theta + \hat{\mathbf{u}}\sin\theta),$$

  where $\hat{\mathbf{u}} = \mathbf{q}/\|\mathbf{q}\|$ and $\theta = \arccos q_0/|q|$.

Definition 3.3.6. (**Exponential**) The definition of the quaternionic exponential is the converging series:

$$e^q = \sum_{k=0}^{\infty} \frac{q^k}{k!}.$$

Theorem 3.3.2. The exponential of a quaternion $q = q_0 + \mathbf{q}$ is

$$e^q = e^{q_0}\left(\cos\theta + \hat{\mathbf{u}}\sin\theta\right),$$

where $\mathbf{q} = \hat{\mathbf{u}}\theta$.

Proof. The definition of the quaternionic exponential is

$$e^q = \sum_{k=0}^{\infty} \frac{q^k}{k!}.$$

We note that $e^q = e^{q_0 + \hat{\mathbf{u}}\theta} = e^{q_0} e^{\hat{\mathbf{u}}\theta}$. We note that

$$\begin{aligned}
\hat{\mathbf{u}}^2 &= -\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = -1 \\
\hat{\mathbf{u}}^3 &= -\hat{\mathbf{u}} \\
\hat{\mathbf{u}}^4 &= 1 \\
&\vdots
\end{aligned}$$

So we have

$$\begin{aligned}
e^{\hat{\mathbf{u}}\theta} &= \sum_{k=0}^{\infty} \frac{(\hat{\mathbf{u}}\theta)^k}{k!} \\
&= 1 + \frac{\hat{\mathbf{u}}\theta}{1!} - \frac{\theta^2}{2!} - \frac{\hat{\mathbf{u}}\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\hat{\mathbf{u}}\theta^5}{5!} - \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + \hat{\mathbf{u}}\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + \hat{\mathbf{u}}\sin\theta.
\end{aligned}$$

Hence $e^q = e^{q_0}(\cos\theta + \hat{\mathbf{u}}\sin\theta)$. □

- We often define $e^{q_0} = |q|$.
- Hence the logarithm of q is

$$\begin{aligned}
\ln q &= \ln|q|(\cos\theta + \hat{\mathbf{u}}\sin\theta) \\
&= \ln|q| + \ln e^{\hat{\mathbf{u}}\theta} \\
&= \ln|q| + \hat{\mathbf{u}}\theta
\end{aligned}$$

  where $\theta = \arccos q_0/|q|$.

Definition 3.3.7. (**Power**) The ρth power of the quaternion q is defined as

$$q^\rho = |q|^\rho \left(e^{\hat{\mathbf{u}}\theta}\right)^\rho = |q|^\rho (\cos(\rho\theta) + \hat{\mathbf{u}}\sin(\rho\theta)),$$

where $\rho \in \mathbb{R}$.

- This follows from $q^\rho = e^{\rho \ln q}$.

3.4 Dual Quaternions

3.4.1 Dual Numbers

Definition 3.4.1. (**Dual Number**) A dual number $\overline{d} \in \mathbb{D}$ is defined to be

$$\overline{d} = a + \epsilon b,$$

where $a, b \in \mathbb{R}$ and ϵ is the *dual unit* with $\epsilon^2 = 0$.

- Addition:
$$\overline{d_1} + \overline{d_2} = (a_1 + a_2) + \epsilon(b_1 + b_2).$$

- Product:

$$\begin{aligned} \overline{d_1} \otimes \overline{d_2} &= a_1a_2 + \epsilon(a_1b_2 + b_1a_2) + \epsilon^2 b_1b_2 \\ &= a_1a_2 + \epsilon(a_1b_2 + b_1a_2) \end{aligned}$$

Definition 3.4.2. (**Inverse**) The inverse of a dual number $\overline{d}$ is defined as

$$\overline{d}^{-1} = \frac{1}{a}\left(1 - \epsilon\frac{b}{a}\right),$$

for $a \neq 0$.

- If $a = 0$, then $\overline{d} = \epsilon b$ has no inverse.
- Dual numbers form a ring (but not a field)
- For some function $f \in C^\infty$, $f(a + \epsilon b)$ is given by it's Taylor expansion:

$$\begin{aligned} f(a + \epsilon b) &= f(a) + \epsilon b\frac{f''(a)}{1!} + \epsilon^2 b^2\frac{f''(a)}{2!} + \cdots \\ &= f(a) + \epsilon b f'(a) \end{aligned}$$

3.4.2 Dual Quaternions

Definition 3.4.3. A dual quaternion σ is defined as

$$\sigma = p + \epsilon q,$$

where p, q are quaternions and ϵ is the dual unit.

- Dual number addition and product generalizes:

$$\begin{aligned}\sigma_1 + \sigma_2 &= (p_1 + p_2) + \epsilon(q_1 + q_2) \\ \sigma_1 \otimes \sigma_2 &= p_1 p_2 + \epsilon(p_1 q_2 + q_1 p_2)\end{aligned}$$

- The inverse of $\sigma = p + \epsilon q$ for $p \neq 0$ is

$$\sigma^{-1} = p^{-1}\left(1 - \epsilon q p^{-1}\right).$$

- Express σ in terms of a dual number $\overline{d} = p_0 + \epsilon q_0$ and a dual vector $\overline{\mathbf{d}} = \mathbf{p} + \epsilon \mathbf{q}$:

$$\sigma = \overline{d} + \overline{\mathbf{d}}.$$

  Hence

$$\begin{aligned}\sigma_1 \otimes \sigma_2 &= (\overline{d}_1 + \overline{\mathbf{d}}_1)(\overline{d}_2 + \overline{\mathbf{d}}_2) \\ &= \left(\overline{d_1} \otimes \overline{d_2} - \overline{\mathbf{d}}_1 \cdot \overline{\mathbf{d}}_2\right) + \overline{d}_1\overline{\mathbf{d}}_2 + \overline{d}_2\overline{\mathbf{d}}_1 + \overline{\mathbf{d}}_1 \times \overline{\mathbf{d}}_2\end{aligned}$$

Definition 3.4.4. (**Dual Conjugate**) The dual conjugate of σ, denoted $\sigma^\bullet$, is

$$\sigma^\bullet = p - \epsilon q.$$

- Properties of dual conjugate:

$$\begin{aligned}\sigma \otimes \sigma^\bullet &= (p + \epsilon q)(p - \epsilon q) = pp + \epsilon(qp - pq) \\ (\sigma_1 \otimes \sigma_2)^\bullet &= \sigma_1^\bullet \otimes \sigma_2^\bullet \\ (\sigma^\bullet)^\bullet &= \sigma\end{aligned}$$

- Rarely used apart from the *composite conjugate*.

Definition 3.4.5. (**Quaternion Conjugate**) The quaternion conjugate of σ, denoted σ^*, is

$$\sigma^* = p^* + \epsilon q^*,$$

where p^*, q^* are the conjugates of p, q.

- Properties of quaternion conjugate:

$$\begin{aligned}\sigma \otimes \sigma^* &= |p|^2 + 2\epsilon(p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) \\ (\sigma_1 \otimes \sigma_2)^* &= \sigma_2^* \otimes \sigma_1^* \\ (\sigma^*)^* &= \sigma\end{aligned}$$

Definition 3.4.6. (**Composite Conjugate**) The composite conjugate of σ, denoted σ°, is

$$\sigma^\circ = (\sigma^*)^\bullet = p^* - \epsilon q^*.$$

- Properties of composite conjugate:

$$\begin{aligned}\sigma \otimes \sigma^\circ &= pp^* + \epsilon(qp^* - pq^*) \\ (\sigma_1 \otimes \sigma_2)^\circ &= \sigma_2^\circ \otimes \sigma_1^\circ \\ (\sigma^\circ)^\circ &= \sigma\end{aligned}$$

Definition 3.4.7. (**Unit Dual Quaternion**) A dual quaternion $\sigma = p + \epsilon q$ is unit if $\sigma \times \sigma^* = 1$.

- Hence

$$\begin{aligned}|p|^2 &= 1 \\ p_0 q_0 + \mathbf{p} \cdot \mathbf{q} &= 0\end{aligned}$$

  So p is a unit quaternion and p and q are orthogonal.

3.4.3 Rigid Transformations

- Represent the rigid transformation $R(\mathbf{v}) + \mathbf{t}$ as a dual quaternion σ.
- A vector $\mathbf{v} \in \mathbb{R}^3$ is a pure dual quaternion, denoted $\mathfrak{D}(\mathbf{v}) = 1 + \epsilon \mathbf{v}$

Definition 3.4.8. (**Dual Quaternion Operator**) For a unit dual quaternion σ, the quaternion operator of σ, denoted $L_\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ applied to a vector $\mathbf{v} \in \mathbb{R}^3$ is

$$L_\sigma(\mathbf{v}) = \mathfrak{D}^{-1}(\sigma \otimes \mathfrak{D}(\mathbf{v}) \otimes \sigma^\circ).$$

Theorem 3.4.1. For any unit dual quaternion

$$\sigma = r + \frac{\epsilon}{2}\mathbf{t}r,$$

where $r = \cos\theta/2 + \hat{\mathbf{u}}\sin\theta/2$, a rotation about $\hat{\mathbf{u}}$ through θ. The dual quaternion operator σ, L_σ is a rotation through an angle of θ about $\hat{\mathbf{u}}$ followed by a translation $\mathbf{t}$.

Proof. Let $\mathbf{v} \in \mathbb{R}^3$ be arbitrary. We wish to show that $L_\sigma(\mathbf{v}) = R(\mathbf{v}) + \mathbf{t}$ where the linear map R corresponds to a rotation of θ about $\hat{\mathbf{u}}$. So we have

$$\begin{aligned}
L_\sigma(\mathbf{v}) &= \mathfrak{D}^{-1}\left[\left(r + \frac{\epsilon}{2}\mathbf{t}r\right) \otimes (1 + \epsilon\mathbf{v}) \otimes \left(r^* - \frac{\epsilon}{2}(\mathbf{t}r)^*\right)\right] \\
&= \mathfrak{D}^{-1}\left[\left(r + \epsilon\left\{\frac{1}{2}\mathbf{t}r + r\mathbf{v}\right\}\right) \otimes \left(r^* - \frac{\epsilon}{2}r^*\mathbf{t}^*\right)\right] \\
&= \mathfrak{D}^{-1}\left[rr^* + \epsilon\left(\frac{1}{2}\left(\mathbf{t}rr^* - rr^*\mathbf{t}^*\right) + r\mathbf{v}r^*\right)\right] \\
&= \mathfrak{D}^{-1}\left[1 + \epsilon\left(\frac{1}{2}\left(\mathbf{t} - \mathbf{t}^*\right) + L_r(\mathbf{v})\right)\right] \\
&= \mathfrak{D}^{-1}\left[1 + \epsilon\left(L_r(\mathbf{v}) + \mathbf{t}\right)\right] = L_r(\mathbf{v}) + \mathbf{t}
\end{aligned}$$

□

- The composite operator of $\sigma_1, \ldots, \sigma_n$ is given by

$$L_\sigma = L_{\sigma_n \otimes \cdots \otimes \sigma_1}.$$

3.4.4 Screw Axis

Definition 3.4.9. (**Screw Coordinates**) The screw coordinates of a line ℓ with direction $\hat{\mathbf{l}}$ and a point $\mathbf{p}$ on the line is given by $(\hat{\mathbf{l}}, \mathbf{m})$ where $\forall \mathbf{p} \in \ell . \mathbf{m} = \mathbf{p} \times \hat{\mathbf{l}}$, the *moment* of the line ℓ.

- $\|\mathbf{m}\|$ is the distance between the origin and ℓ, achieved by $\mathbf{p}_\perp$ s.t

$$\mathbf{p}_\perp = \mathbf{p} - (\hat{\mathbf{l}} \cdot \mathbf{p})\hat{\mathbf{l}} = \hat{\mathbf{l}} \times (\mathbf{p} \times \hat{\mathbf{l}}) = \hat{\mathbf{l}} \times \mathbf{m}.$$

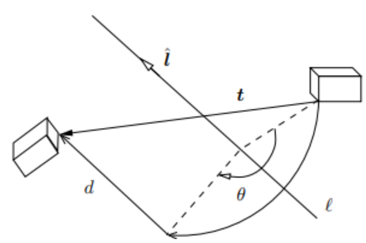
- The dual quaternion representation of ℓ is $l = \hat{\mathbf{l}} + \epsilon \mathbf{m}$.

Theorem 3.4.2. For the rigid transformation $\mathbf{T}(\mathbf{x}) = \mathbf{R}(\mathbf{x}) + \mathbf{t}$, where $\mathbf{R}$ is a rotation by θ about $\hat{\mathbf{u}}$ through the origin.

The screw motion on the screw axis $(\hat{\mathbf{l}}, \mathbf{m})$, which consists of a rotation by θ about $\hat{\mathbf{l}}$ followed by a translation $d\hat{\mathbf{l}}$, where d is the pitch, is given by

$$\begin{aligned}
\hat{\mathbf{l}} &= \hat{\mathbf{u}} \\
\mathbf{m} &= \frac{1}{2}\left(\mathbf{t} \times \hat{\mathbf{l}} - \hat{\mathbf{l}} \times (\hat{\mathbf{l}} \times \mathbf{t}) \cot \frac{\theta}{2}\right) \\
d &= \mathbf{t} \cdot \hat{\mathbf{l}}
\end{aligned}$$

Proof. Let $(\hat{\mathbf{l}}, \mathbf{m})$ be our screw axis. Let $\mathbf{p}$ be some arbitrary point on ℓ s.t $\mathbf{p} \cdot \hat{\mathbf{l}} = 0$. Hence $\mathbf{m} = \mathbf{p} \times \hat{\mathbf{l}}$.



Hence the coordinate system is translated to perform the screw motion:

$$\begin{aligned}
\mathbf{R}(\mathbf{x}) + \mathbf{t} &= \underbrace{\mathbf{R}(\mathbf{x} - \mathbf{p}) + \mathbf{p}}_{\text{coordinate shift}} + d\hat{\mathbf{l}} \\
&= \mathbf{R}(\mathbf{x}) + (\mathbf{I} - \mathbf{R})\mathbf{p} + d\hat{\mathbf{l}} \\
\iff \mathbf{t} &= (\mathbf{I} - \mathbf{R})\mathbf{p} + d\hat{\mathbf{l}}
\end{aligned}$$

Hence $\mathbf{p} = \mathbf{t} - d\hat{\mathbf{l}} + \mathbf{R}\mathbf{p}$, $d = \mathbf{t} \cdot \hat{\mathbf{l}}$ and $\hat{\mathbf{l}} = \hat{\mathbf{u}}$.

We wish to express $\mathbf{R}\mathbf{p}$ in terms of $\hat{\mathbf{l}}, \hat{\mathbf{t}}$. By Rodrigues formula, we have

$$\begin{aligned}
\mathbf{R}\mathbf{p} &= \mathbf{p} + \sin\theta \hat{\mathbf{l}} \times \mathbf{p} + (1 - \cos\theta)\hat{\mathbf{l}} \times (\hat{\mathbf{l}} \times \mathbf{p}) \\
&= \mathbf{p} + \sin\theta \hat{\mathbf{l}} \times \mathbf{p} + (1 - \cos\theta)\left[(\hat{\mathbf{l}} \cdot \mathbf{p})\hat{\mathbf{l}} - (\hat{\mathbf{l}} \cdot \hat{\mathbf{l}})\mathbf{p}\right] \\
&= \mathbf{p} + \sin\theta \hat{\mathbf{l}} \times \mathbf{p} - (1 - \cos\theta)\mathbf{p} \\
&= \sin\theta \hat{\mathbf{l}} \times \mathbf{p} + \cos\theta \mathbf{p}
\end{aligned}$$

Taking the cross product of $\hat{\mathbf{l}}$ and $\mathbf{t}$ yields:

$$\begin{aligned}\hat{\mathbf{l}} \times \mathbf{t} &= \hat{\mathbf{l}} \times [(\mathbf{I} - \mathbf{R})\mathbf{p}] \\ &= \hat{\mathbf{l}} \times \left[(1-\cos\theta)\mathbf{p} - \sin\theta\hat{\mathbf{l}} \times \hat{\mathbf{p}}\right] \\ &= (1-\cos\theta)\hat{\mathbf{l}} \times \mathbf{p} + \sin\theta\mathbf{p}\end{aligned}$$

Hence

$$\begin{aligned} (1-\cos\theta)^{-1}\hat{\mathbf{l}} \times \mathbf{t} &= \hat{\mathbf{l}} \times \mathbf{p} + \frac{\sin\theta}{1-\cos\theta}\mathbf{p} \\ \iff \frac{\sin\theta}{1-\cos\theta}\hat{\mathbf{l}} \times \mathbf{t} &= \sin\theta\hat{\mathbf{l}} \times \mathbf{p} + \frac{\sin^2\theta}{1-\cos\theta}\mathbf{p} \\ &= \sin\theta\hat{\mathbf{l}} \times \mathbf{p} + (1+\cos\theta)\mathbf{p}\end{aligned}$$

We note that $\cot\theta/2 = \sin\theta/(1-\cos\theta)$. So we have

$$\cot\frac{\theta}{2}\hat{\mathbf{l}} \times \mathbf{t} = \underbrace{\sin\theta\hat{\mathbf{l}} \times \mathbf{p} + \cos\theta\mathbf{p}}_{\mathbf{Rp}} + \mathbf{p} = \mathbf{Rp} + \mathbf{p}$$

So we have

$$\begin{aligned} \mathbf{p} &= \mathbf{t} - d\hat{\mathbf{l}} + \cot\frac{\theta}{2}\hat{\mathbf{l}} \times \mathbf{t} - \mathbf{p} \\ \iff \mathbf{p} &= \frac{1}{2}\left(\mathbf{t} - d\hat{\mathbf{l}} + \cot\frac{\theta}{2}\hat{\mathbf{l}} \times \mathbf{t}\right)\end{aligned}$$

Hence the moment $\mathbf{m}$ is given by

$$\mathbf{m} = \frac{1}{2}\left(\mathbf{t} \times \hat{\mathbf{l}} + \hat{\mathbf{l}} \times (\hat{\mathbf{l}} \times \mathbf{t})\cot\frac{\theta}{2}\right)$$

So we are done. □

Theorem 3.4.3. For any unit dual quaternion $\sigma = r + \frac{\epsilon}{2}\mathbf{t}r$, may be written as

$$\sigma = \cos\frac{\overline{\theta}}{2} + \sin\frac{\overline{\theta}}{2}l,$$

where $\overline{\theta} = \theta + \epsilon d$, $l = \hat{\mathbf{l}} + \epsilon\mathbf{m}$, and the screw motion parameters are the angle θ, and the pitch $d = \mathbf{t} \cdot \hat{\mathbf{l}}$.

Proof. By theorem ??, we have

$$\begin{aligned}
\mathbf{m} &= \frac{1}{2}\left(\mathbf{t}\times\hat{\mathbf{l}} + \hat{\mathbf{l}}\times(\hat{\mathbf{l}}\times\mathbf{t})\cot\frac{\theta}{2}\right)\\
\iff \sin\frac{\theta}{2}\mathbf{m} + \frac{d}{2}\cos\frac{\theta}{2}\hat{\mathbf{l}} &= \frac{1}{2}\left(\mathbf{t}\times\hat{\mathbf{l}}\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\mathbf{t}\right)
\end{aligned}$$

We also note that

$$\begin{aligned}
\mathbf{t}r &= \mathbf{t}\left(\cos\frac{\theta}{2} + \hat{\mathbf{l}}\sin\frac{\theta}{2}\right)\\
&= -\mathbf{t}\cdot\hat{\mathbf{l}}\sin\frac{\theta}{2} + \mathbf{t}\cos\frac{\theta}{2} + \mathbf{t}\times\hat{\mathbf{l}}\sin\frac{\theta}{2}\\
&= -d\sin\frac{\theta}{2} + \left(\mathbf{t}\cos\frac{\theta}{2} + \mathbf{t}\times\hat{\mathbf{l}}\sin\frac{\theta}{2}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\sigma = r + \frac{\epsilon}{2}\mathbf{t}r &= \left(\cos\frac{\theta}{2} + \hat{\mathbf{l}}\sin\frac{\theta}{2}\right) + \epsilon\left(-\frac{d}{2}\sin\frac{\theta}{2} + \frac{1}{2}\left(\mathbf{t}\cos\frac{\theta}{2} + \mathbf{t}\times\hat{\mathbf{l}}\sin\frac{\theta}{2}\right)\right)\\
&= \left(\cos\frac{\theta}{2} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}\right) + \left(\hat{\mathbf{l}}\sin\frac{\theta}{2} + \epsilon\left\{\sin\frac{\theta}{2}\mathbf{m} + \frac{d}{2}\cos\frac{\theta}{2}\hat{\mathbf{l}}\right\}\right)\\
&= \left(\cos\frac{\theta}{2} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}\right) + \left(\sin\frac{\theta}{2} + \epsilon\frac{d}{2}\cos\frac{\theta}{2}\right)\left(\hat{\mathbf{l}} + \epsilon\mathbf{m}\right)
\end{aligned}$$

By the Taylor series of sin and cos, we have

$$\begin{aligned}
\sin\frac{\theta+\epsilon d}{2} &= \sin\frac{\theta}{2} + \epsilon\frac{d}{2}\cos\frac{\theta}{2}\\
\cos\frac{\theta+\epsilon d}{2} &= \cos\frac{\theta}{2} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}
\end{aligned}$$

Hence

$$\sigma = \cos\frac{\overline{\theta}}{2} + \sin\frac{\overline{\theta}}{2}l,$$

where $\overline{\theta} = \theta + \epsilon d$ and $l = \hat{\mathbf{l}} + \epsilon\mathbf{m}$.

□

3.4.5 Power, Exponential and Logarithm

Definition 3.4.10. The definition of the dual quaternionic exponential is the converging series:

$$e^{\sigma} = \sum_{k=0}^{\infty} \frac{\sigma^k}{k!}.$$

Theorem 3.4.4. The exponential of $\sigma = \frac{\overline{\theta}}{2} l$ is

$$e^{\frac{\overline{\theta}}{2} l} = \cos \frac{\overline{\theta}}{2} + \sin \frac{\overline{\theta}}{2} l,$$

where $\overline{\theta} \in \mathbb{D}$ and $l \in \mathbb{DH}$.

- Hence the logarithm of σ is

$$\begin{aligned} \ln \sigma &= \ln \cos \frac{\overline{\theta}}{2} + \sin \frac{\overline{\theta}}{2} l \\ &= \frac{\overline{\theta}}{2} l \end{aligned}$$

Definition 3.4.11. (**Power**) The ρth power of the dual quaternion σ is defined as

$$\sigma^{\rho} = \cos \frac{\rho\overline{\theta}}{2} + \sin \frac{\rho\overline{\theta}}{2} l,$$

where $\rho \in \mathbb{R}$.

3.5 Linear Blending

- Linear blending or *skinning* is the interpolation of rigid transformations $\mathbf{T}_1, \dots, \mathbf{T}_n$ applied to joints $j_1, \dots, j_n$ to form a transformation $\mathbf{T}$ for point $\mathbf{p} \in \mathcal{M}$ with joint weights $\mathbf{w} = (w_1, \dots, w_n)$:

$$\mathbf{T} = \sum_{i=1}^{n} w_i \mathbf{T}_i.$$

- Simplest case: $\mathbf{T}(\tau) = (1-\tau)\mathbf{T}_1 + \tau\mathbf{T}_2$, interpolating between two rigid transformations $\mathbf{T}_1, \mathbf{T}_2$. with parameter $\tau \in [0, 1]$.

- Desirable properties:
  - *Constant speed*: The derivatives $\theta(\tau)$ and $\mathbf{t}(\tau)$ are constant, where $\theta(\tau)$ is the angle of rotation and $\mathbf{t}(\tau)$ is the translation for $\mathbf{T}(\tau)$.
  - *Shortest Path interpolation*: $\mathbf{T}(\tau)$ lies of the geodesic curve between $\mathbf{T}_1$ and $\mathbf{T}_2$ on the manifold of rigid transformations $\mathbf{SE}(n)$
  - *Coordinate system invariance*: Let $\mathbf{M}$ be some coordinate system transformation (change of basis), then

$$\mathbf{MT}(\tau)\mathbf{M}^{-1} = (1-\tau)(\mathbf{MT}_1\mathbf{M}^{-1}) + \tau(\mathbf{MT}_2\mathbf{M}^{-1}).$$

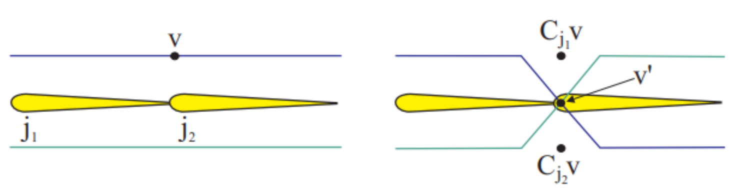
3.5.1 Homogenous Matrices

- Rigid transformations may be represented as matrices in their *homogenous form* (See IA notes).
- Properties:
  - *Coordinate invariance*: Suppose $\mathbf{M}$ is a matrix representing a change of basis. Joint transformation matrices in new basis are $\mathbf{MT}_i\mathbf{M}^{-1}$: Then

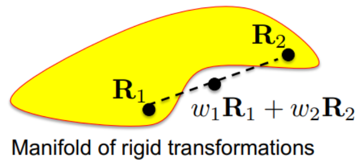
$$\left(\sum_{i=1}^{n} w_i \mathbf{MT}_i\mathbf{M}^{-1}\right) = \mathbf{M}\left(\sum_{i=1}^{n} w_i \mathbf{T}_i\right)\mathbf{M}^{-1}.$$

  - *"Candy-wrapper" artifacts.* The blended matrix $\sum_i w_i \mathbf{T}_i$ may no-longer be a rigid transformation, but an affine transformation which may contain scale / shear factors, since the set of orthonormal matrices isn't closed under addition.

    This results in "candy-wrapper" artifacts when one of the joints is rotated π radians about it's axis or if the blended matrix is singular (collapsing the transformed point)



  - *Shortest path interpolation.* Linear blending doesn't satisfy shortest path interpolation:



since the blended transformation $\sum_i w_i \mathbf{T}_i$ may not lie on the manifold of rigid transformations $\mathbf{SE}(n)$, hence cannot be on the geodesic curve (the shortest path).

3.5.2 Quaternion Blending

- Quaternion blending: split the rigid transformation $\mathbf{T}_1, \mathbf{T}_2$ into rotations $\mathbf{R}_1, \mathbf{R}_2$ and translations $\mathbf{t}_1, \mathbf{t}_2$.
- Linearly interpolate translations $\mathbf{t}_1, \mathbf{t}_2$:

$$\mathbf{t}(\tau) = (1-\tau)\mathbf{t}_1 + \tau\mathbf{t}_2,$$

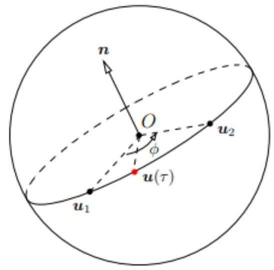
- Two interpolations of r_1, r_2:

$$r_i = \cos\frac{\theta_i}{2} + \hat{\mathbf{u}}_i \sin\frac{\theta_i}{2}.$$

  - *Naive Interpolation*: Let

$$r(\tau) = \cos\frac{\theta(\tau)}{2} + \hat{\mathbf{u}}(\tau)\sin\frac{\theta(\tau)}{2}.$$

    The angle of rotation is interpolated linearly: $\theta(\tau) = (1-\tau)\theta_1 + \tau\theta_2$. The vector $\hat{\mathbf{u}}(\tau)$ is interpolated by considering geodesic on the sphere $\mathcal{S}^3$, or *spherical linear interpolation.*



We note that $\phi = \arccos \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2$ and $\hat{\mathbf{n}} = \hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2 / \|\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2\|$. Hence $\hat{\mathbf{u}}(\tau)$ is determined from a rotation of $\hat{\mathbf{u}}_1$ about $\hat{\mathbf{n}}$ through the angle $\tau\phi$.

$$\begin{aligned}
\hat{\mathbf{u}}(\tau) &= \left(\cos\frac{\tau\phi}{2} + \hat{\mathbf{n}}\sin\frac{\tau\phi}{2}\right)\hat{\mathbf{u}}_1\left(\cos\frac{\tau\phi}{2} - \hat{\mathbf{n}}\sin\frac{\tau\phi}{2}\right) \\
&= \left(-\sin\frac{\tau\phi}{2}\hat{\mathbf{n}}\cdot\hat{\mathbf{u}}_1 + \cos\frac{\tau\phi}{2}\hat{\mathbf{u}}_1 + \sin\frac{\tau\phi}{2}\hat{\mathbf{n}}\times\hat{\mathbf{u}}_1\right)\left(\cos\frac{\tau\phi}{2} - \hat{\mathbf{n}}\sin\frac{\tau\phi}{2}\right) \\
&= -\left(\cos\frac{\tau\phi}{2}\hat{\mathbf{u}}_1 + \sin\frac{\tau\phi}{2}\hat{\mathbf{n}}\times\hat{\mathbf{u}}_1\right)\cdot\left(-\hat{\mathbf{n}}\sin\frac{\tau\phi}{2}\right) + \cos\frac{\tau\phi}{2}\left(\cos\frac{\tau\phi}{2}\hat{\mathbf{u}}_1 + \sin\frac{\tau\phi}{2}\hat{\mathbf{n}}\times\hat{\mathbf{u}}_1\right) \\
&\quad + \left(\cos\frac{\tau\phi}{2}\hat{\mathbf{u}}_1 + \sin\frac{\tau\phi}{2}\hat{\mathbf{n}}\times\hat{\mathbf{u}}_1\right)\times\left(-\hat{\mathbf{n}}\sin\frac{\tau\phi}{2}\right) \\
&= \cos\frac{\tau\phi}{2}\left(\cos\frac{\tau\phi}{2}\hat{\mathbf{u}}_1 + \sin\frac{\tau\phi}{2}\hat{\mathbf{n}}\times\hat{\mathbf{u}}_1\right) - \sin\frac{\tau\phi}{2}\left(\cos\frac{\tau\phi}{2}\hat{\mathbf{u}}_1\times\hat{\mathbf{n}} + \sin\frac{\tau\phi}{2}(\hat{\mathbf{n}}\times\hat{\mathbf{u}}_1)\times\hat{\mathbf{n}}\right)
\end{aligned}$$

We note that $\hat{\mathbf{u}}_1 \times \hat{\mathbf{n}} = \cos\phi\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2/\sin\phi$ and $(\hat{\mathbf{n}}\times\hat{\mathbf{u}}_1)\times\hat{\mathbf{n}} = \hat{\mathbf{u}}_1$ hence

$$\begin{aligned}
\hat{\mathbf{u}}(\tau) &= \left(\cos^2\frac{\tau\phi}{2} - \sin^2\frac{\tau\phi}{2}\right)\hat{\mathbf{u}}_1 - 2\sin\frac{\tau\phi}{2}\cos\frac{\tau\phi}{2}\hat{\mathbf{u}}\times\hat{\mathbf{n}} \\
&= \cos\tau\phi\hat{\mathbf{u}}_1 - \frac{\sin\tau\phi}{\sin\phi}(\cos\phi\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2) \\
&= \frac{\sin(1-\tau)\phi}{\sin\phi}\hat{\mathbf{u}}_1 + \frac{\sin\tau\phi}{\sin\phi}\hat{\mathbf{u}}_2
\end{aligned}$$

Hence $r(\tau)$ is given by

$$r(\tau) = \cos\frac{\theta(\tau)}{2} + \left(\frac{\sin(1-\tau)\phi}{\sin\phi}\hat{\mathbf{u}}_1 + \frac{\sin\tau\phi}{\sin\phi}\hat{\mathbf{u}}_2\right)\sin\frac{\theta(\tau)}{2}$$

- *Spherical Linear Interpolation*: The unit-quaternions form a unit 3-sphere Ω in $\mathbb{H}$. Suppose r_1 and r_2 have the angle ϕ between them on Ω.

  Then by spherical linear interpolation (see above),

  $$\text{Slerp}(\tau; r_1, r_2) = r(\tau) = \frac{\sin(1-\tau)\phi}{\sin\phi} r_1 + \frac{\sin\tau\phi}{\sin\phi} r_2.$$

  Note that r_2 and $-r_2$ represent the same rotation, known as *antipodality*:

  $$L_{-r_2}(\mathbf{v}) = (-r_2)\mathbf{v}(-r_2)^* = (-r_2)\mathbf{v}(-r_2^*) = r_2\mathbf{v}r_2^* = L_{r_2}(\mathbf{v}).$$

  However, $\text{Slerp}(\tau; r_1, r_2) \neq \text{Slerp}(\tau; r_1, -r_2)$. We determine the $\text{sgn}(r_2)$ s.t $r_1 \cdot (\text{sgn}(r_2)r_2) \geq 0$ (the angle between r_1 and $\text{sgn}(r_2)r_2$ is acute). This gives us *shortest path interpolation*.

  Slerp may be written as

  $$\text{Slerp}(\tau; r_1, r_2) = r_1(r_1^{-1}r_2)^\tau = (r_2 r_1^{-1})^\tau r_1.$$

  Since $r_2 = r_1(r_1^{-1}r_2)$. The term $r_1^{-1}r_2 = \cos\phi + \hat{\mathbf{v}}\sin\phi$. Hence linearly interpolating ϕ yields $(r_1^{-1}r_2)^\tau = \cos\tau\phi + \hat{\mathbf{v}}\sin\tau\phi$.

- Properties:
  - *Constant speed*: Both naive and slerp have constant speed interpolation.
  - *Shortest path interpolation*: Slerp has shortest path interpolation, since it performs spherical linear interpolation on $\Omega \subseteq \mathbb{H} \equiv \mathbf{SO}(3)$, the manifold of rotations.

    Provided the angle between r_1 and r_2 is acute, otherwise *antipodality* artifacts occur.
  - *Coordinate Dependence*: Interpolating rotations and translations in object space introduces a dependence on object space. Hence quaternion blending isn't coordinate independent.

- Slerp doesn't generalize to n quaternions, however, QLB (Quaternion Linear Blending) is used for n quaternions:

  $$\text{QLB}(\mathbf{w}, r_1, \ldots, r_n) = \sum_{i=1}^{n} w_i r_i.$$

  This is a sufficient approximation for small angles $\theta_1, \ldots, \theta_n$.

3.5.3 Dual Quaternion Blending

- Two interpolations:
  - *Screw Linear Interpolation* (ScLERP). A generalization of Slerp using dual quaternions:

$$\text{ScLERP}(\tau; \sigma_1, \sigma_2) = \sigma_1(\sigma_1^{-1}\sigma_2)^\tau.$$

As with Slerp, we have constant speed and shortest path. ScLERP is also coordinate invariant since

$$\begin{aligned}
\text{ScLERP}(\tau; \sigma\sigma_1, \sigma\sigma_2) &= (\sigma\sigma_1)\left((\sigma\sigma_1)^{-1}\sigma\sigma_2\right)^\tau \\
&= \sigma\sigma_1\left(\sigma^{-1}\sigma^{-1}\sigma\sigma_2\right)^\tau = \sigma\text{ScLERP}(\tau; \sigma_1, \sigma_2) \\
\text{ScLERP}(\tau; \sigma_1\sigma, \sigma_2\sigma) &= (\sigma_1\sigma)\left(\sigma^{-1}\sigma_1^{-1}\sigma_2\sigma\right)^\tau \\
&= (\sigma_1\sigma)\sigma^{-1}(\sigma_1^{-1}\sigma_2)^\tau\sigma = \text{ScLERP}(\tau; \sigma_1, \sigma_2)\sigma
\end{aligned}$$

  - Dual quaternion linear blending, denoted $\text{DLB}(\tau; \sigma_1, \sigma_2)$ is

$$\text{DLB}(\tau; \sigma_1, \sigma_2) = \frac{(1-\tau)\sigma_1 + \tau\sigma_2}{|(1-\tau)\sigma_1 + \tau\sigma_2|}.$$

DLB is coordinate invariant due to the distributivity of dual quaternions:

$$\begin{aligned}
\text{DLB}(\tau; \sigma\sigma_1\sigma', \sigma\sigma_2\sigma') &= \frac{(1-\tau)\sigma\sigma_1\sigma' + \tau\sigma\sigma_2\sigma'}{|(1-\tau)\sigma\sigma_1\sigma' + \tau\sigma\sigma_2\sigma'|} \\
&= \frac{\sigma\left((1-\tau)\sigma_1 + \tau\sigma_2\right)\sigma'}{|\sigma||(1-\tau)\sigma_1 + \tau\sigma_2||\sigma'|} \\
&= \sigma\text{DLB}(\tau; \sigma_1, \sigma_2)\sigma'
\end{aligned}$$

Note that

$$\begin{aligned}
\text{DLB}(\tau; \sigma_1, \sigma_2) &= \sigma_1\text{DLB}(\tau; 1, \sigma_1^{-1}\sigma_2) \\
&= \sigma_1\left(\frac{(1-\tau) + \tau\cos\frac{\overline{\phi}}{2} + l\tau\sin\frac{\overline{\phi}}{2}}{|1-\tau+\tau\sigma_1^{-1}\sigma_2|}\right) \\
&= \sigma_1\left(\cos\frac{\overline{\alpha}}{2} + l\sin\frac{\overline{\alpha}}{2}\right)
\end{aligned}$$

We also note that

$$\text{ScLERP}(\tau; \sigma_1, \sigma_2) = \sigma_1 \text{ScLERP}(\tau; 1; \sigma_1^{-1}\sigma_2) = \sigma_1 \left(\cos \frac{\tau\overline{\phi}}{2} + l \sin \frac{\tau\overline{\phi}}{2} \right).$$

So it follows from ScLERP that DLB has the property of shortest path interpolation.

DLB doesn't have the property of constant speed. (Although in practice, it approximately does).

- Properties:
  - *Constant speed*: ScLERP has constant speed interpolation. DLB approximately has constant speed interpolation.
  - *Shortest path interpolation*: Both ScLERP and DLB have shortest path interpolation. However, both suffer from *antipodality* artifacts.
  - *Coordinate Independence*: Both ScLERP and DLB have coordinate Independence.

4 Raytracing

4.1 Light

4.1.1 Radiometry

- Light consists of discrete packets of *energy*, called **photons** (or light *quanta*)
- The energy of a photon q is
$$q = hf = \frac{hc}{\lambda},$$
where f is frequency, λ is wavelength and h is *Planck's constant*.
- Q denotes the *radiant* energy emitted, reflected, transmitted or received.

Definition 4.1.1. (**Spectral Energy Distribution**) The spectral energy distribution Q_λ is the radiant energy Q per unit wavelength λ:

$$Q_\lambda = \frac{\partial Q}{\partial \lambda}.$$

- Hence $Q[\lambda_0, \lambda_1] = \int_{\lambda_0}^{\lambda_1} Q_\lambda \, \mathrm{d}\lambda$, denotes the radiant energy in the range $[\lambda_0, \lambda_1]$.

Definition 4.1.2. (**Radiant Power**) The radiant power, or *radiant flux*, is the radiant energy emitted, reflected, transmitted or received per unit time:

$$\Phi = \frac{\partial Q}{\partial t}.$$

The spectral power distribution Φ_λ is the radiant power (emitted, ..., received) per unit wavelength:

$$\Phi_\lambda = \frac{\partial Q_\lambda}{\partial t} = \frac{\partial \Phi}{\partial \lambda} = \frac{\partial^2 Q}{\partial \lambda \partial t}.$$

- Note that radiant power is given by $\Phi = \int \Phi_\lambda \, \mathrm{d}\lambda$.

Definition 4.1.3. (**Radiant Intensity**) The radiant intensity I is defined as the radiant flux (emitted, reflected, transmitted or received) per unit solid angle:

$$I = \frac{\partial \Phi}{\partial \Omega},$$

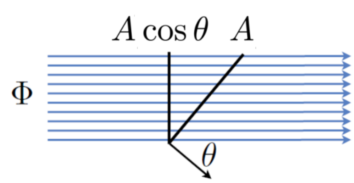
where Ω is the solid angle (see ??), and the *spectral radiant intensity* I_λ is

$$I_\lambda = \frac{\partial \Phi_\lambda}{\partial \Omega} = \frac{\partial I}{\partial \lambda} = \frac{\partial^3 Q}{\partial \lambda \partial \Omega \partial t}.$$

Definition 4.1.4. (**Radiance**) The radiance L is defined as the radiant intensity (emitted, ..., received) per unit projected area.

$$L = \frac{\partial I}{\partial A \cos\theta} = \frac{\partial^2 \Phi}{\partial \Omega \partial A \cos\theta},$$

where $A\cos\theta$ is the the projected area.



The spectral radiance L_λ is given by

$$L_\lambda = \frac{\partial L}{\partial \lambda} = \frac{\partial^2 I_\lambda}{\partial \Omega \partial A \cos\theta}.$$

- Radiance is invariant along the direction of wave propagation.
- Useful to distinguish between incident and emitted rays.
- *Irradiance* H is defined the radiant flux (received) per unit of area:

$$H = \frac{\partial \Phi}{\partial A} = \frac{\partial^2 Q}{\partial A \partial t}.$$

Similarly, for spectral irradiance H_λ is

$$H_\lambda = \frac{\partial \Phi_\lambda}{\partial A} = \frac{\partial H}{\partial \lambda} = \frac{\partial^3 Q}{\partial \lambda \partial A \partial t}.$$

- *Radiosity* E is defined as the radiant flux (emitted, reflected or transmitted) per unit of area:

$$E = \frac{\partial \Phi}{\partial A} = \frac{\partial^2 Q}{\partial A \partial t}$$
$$E_\lambda = \frac{\partial \Phi_\lambda}{\partial A} = \frac{\partial H}{\partial \lambda} = \frac{\partial^3 Q}{\partial \lambda \partial A \partial t}$$

Definition 4.1.5. (**Surface and Field Radiance**) The surface radiance L_s is defined as the radiance emitting from the surface:

$$L_s = \frac{\partial E}{\partial \Omega \cos \theta}.$$

Similarly, the field radiance L_f is defined as the radiance incident to the surface:

$$L_f = \frac{\partial H}{\partial \Omega \cos \theta}.$$

4.1.2 BRDF

- By conservation of energy:

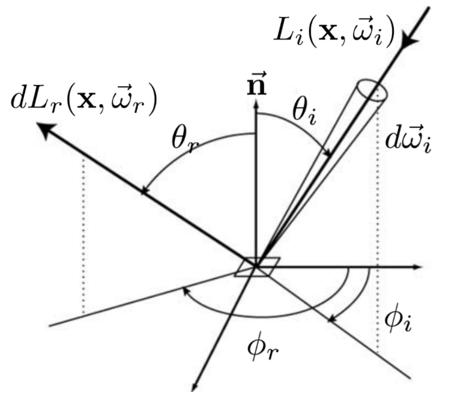
$$L_f \geq L_s - L_e = L_r + L_t,$$

  where L_r, L_t and L_e is the reflected, transmitted and emitted radiance.

- BRDFs are defined to determine L_r. (BTDFs are used to determine L_t)

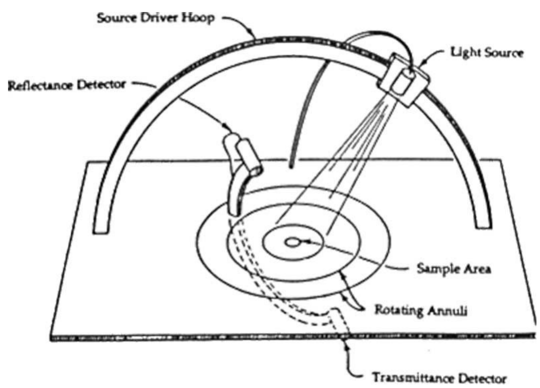
Definition 4.1.6. (**BRDF**) The bidirectional reflectance distribution function $f_r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)$ is the ratio of reflected radiance in direction $\boldsymbol{\omega}_r$ to the irradiance incident on the surface from direction $\boldsymbol{\omega}_i$:

$$f_r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r) = \frac{\mathrm{d}L_r(\boldsymbol{\omega}_r)}{\mathrm{d}H_i(\boldsymbol{\omega}_i)} = \frac{\mathrm{d}L_r(\boldsymbol{\omega}_r)}{L_f(\boldsymbol{\omega}_i) \cos \theta_i \, \mathrm{d}\boldsymbol{\omega}_i}.$$



- The BRDF is defined using differentials. Irradiating light other than $\mathrm{d}H_i(\boldsymbol{\omega}_i)$ (e.g. subsurface scattering / transmission) may affect $L_r(\boldsymbol{\omega}_r)$. Whereas $\mathrm{d}L_r(\boldsymbol{\omega}_r)$ is entirely dependent on $\mathrm{d}H_i(\boldsymbol{\omega}_i)$.
- In Graphics, incident and reflected wavelengths λ_i, λ_r are *ignored*.
- Properties:
  - Reciprocity: $f_r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r) = f_r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)$. BRDF should remain unchanged if reflected and incident directions are commuted.
  - Energy conservation:
  $$\int_\Omega f_r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r) \cos\theta_i \, \mathrm{d}\boldsymbol{\omega}_i \leq 1.$$
  - Range: $f_r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r) \in [0, \infty]$.
  - Units: $[1/sr]$.
- Examples:
  - Diffuse: $f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r) = \rho/\pi$, where π is the albedo.
  - Perfect Specular (mirror): $f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r) = \rho\delta(\boldsymbol{\omega}_i - \boldsymbol{\omega}_r)$
  - Specular: $f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r) = \rho(\mathbf{R}(\boldsymbol{\omega}_i) \cdot \boldsymbol{\omega}_r)^n$ where $\mathbf{R}(\cdot)$ is the reflection operator and n is the "roughness coefficient".
- Measuring BRDFs empirically:

1. For every directional pair $(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)$, measure the change in reflected radiance L_r, denoted δL_r for a given change in irradiance δH_i using luminance and illuminance meters respectively.
2. Compute
$$f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r) \leftarrow \frac{\delta L_r}{\delta H_i}.$$



4.1.3 The Rendering Equation

- **Observe**: BRDF can describe surface radiance L_r using L_f. With L_e, can approximate L_s (ignoring L_t).

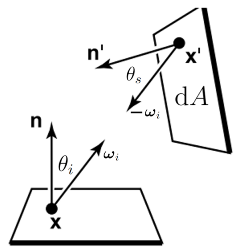
- The *reflection equation*:

$$\begin{aligned} f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r) &= \frac{\mathrm{d}L_r(\mathbf{x}, \boldsymbol{\omega}_r)}{L_f(\mathbf{x}, \boldsymbol{\omega}_i)\cos\theta_i\,\mathrm{d}\boldsymbol{\omega}_i} \\ \iff f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r)L_f(\mathbf{x}, \boldsymbol{\omega}_i)\cos\theta_i &= \frac{\mathrm{d}L_r(\mathbf{x}, \boldsymbol{\omega}_r)}{\mathrm{d}\boldsymbol{\omega}_i} \\ \iff L_r(\mathbf{x}, \boldsymbol{\omega}_r) &= \int_\Omega f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r)L_f(\mathbf{x}, \boldsymbol{\omega}_i)\cos\theta_i\,\mathrm{d}\boldsymbol{\omega}_i \end{aligned}$$

- The *rendering equation* (or transport equation) is

$$L_s(\mathbf{x}, \boldsymbol{\omega}_r) = L_e(\mathbf{x}, \boldsymbol{\omega}_r) + \int_\Omega f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r)L_f(\mathbf{x}, \boldsymbol{\omega}_i)\cos\theta_i\,\mathrm{d}\boldsymbol{\omega}_i,$$

- Recall that
$$\mathrm{d}\boldsymbol{\omega} = \frac{\hat{\mathbf{n}} \cdot \mathrm{d}\mathbf{A}}{r^2},$$
then for:



we have $\mathrm{d}\boldsymbol{\omega}_i = \dfrac{\cos\theta_s}{\|\mathbf{x} - \mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i)\|^2}\,\mathrm{d}A$. So
$$\begin{aligned} L_r(\mathbf{x}, \boldsymbol{\omega}_r) &= \int_{\text{visible } \mathbf{x}'} f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r) L_f(\mathbf{x}, \boldsymbol{\omega}_i) \frac{\cos\theta_s \cos\theta_i}{\|\mathbf{x} - \mathbf{x}'\|^2}\,\mathrm{d}A \\ &= \int_{\mathbf{x}'} f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r) L_f(\mathbf{x}, \boldsymbol{\omega}_i) V(\mathbf{x}, \mathbf{x}') \frac{\cos\theta_s \cos\theta_i}{\|\mathbf{x} - \mathbf{x}'\|^2}\,\mathrm{d}A \end{aligned}$$
where $\boldsymbol{\omega}_i = (\mathbf{x}' - \mathbf{x})/\|\mathbf{x}' - \mathbf{x}\|$ and *visibility* term:
$$V(\mathbf{x}, \mathbf{x}') = \begin{cases} 1 & \text{if } \mathbf{x}' \text{ is visible from } \mathbf{x} \\ 0 & \text{otherwise} \end{cases}$$

- The *Geometry term* is defined as:
$$G(\mathbf{x}, \mathbf{x}') = V(\mathbf{x}, \mathbf{x}') \frac{\cos\theta_s \cos\theta_i}{\|\mathbf{x} - \mathbf{x}'\|^2}.$$
  - $G(\mathbf{x}, \mathbf{x}') \propto P(\text{photon from } \mathbf{x}' \text{ hits } \mathbf{x})$.
  - As the patches face away from each other, $\cos\theta_s \cos\theta_i$ decreases.
  - As the patches move away from each other, $\|\mathbf{x} - \mathbf{x}'\|^2$ increases.

4.2 Global Illumination

Definition 4.2.1. (**Direct Illumination**) Direct illumination is a rendering technique that only considers contributions from light sources $\ell_1, \ldots, \ell_n$ that are visible from $\mathbf{x}$ on the surface S.

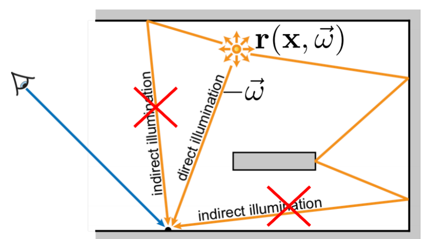
Definition 4.2.2. (**Global Illumination**) Global illumination is a rendering technique that considers *direct and indirect illumination*, cases in which photons are reflected by other surfaces that contribute to the field radiance of a given point $\mathbf{x}$ on the surface S.

- **Assumption**: Radiance remains constant along direction of propagation. So $L_f(\mathbf{x}, \boldsymbol{\omega}_i) = L_s(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i), -\boldsymbol{\omega}_i)$, where $\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i)$ is the point of intersection of the ray $\mathbf{r} = \mathbf{x} + \lambda\boldsymbol{\omega}_i$ with the scene $\mathcal{S}$.

- Rendering equation for direct illumination:

$$L_s(\mathbf{x}, \boldsymbol{\omega}_r) = L_e(\mathbf{x}, \boldsymbol{\omega}_r) + \int_\Omega f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r) L_e(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i), -\boldsymbol{\omega}_i)\, \mathrm{d}\boldsymbol{\omega}_i,$$

since (incident radiance) $L_f(\mathbf{x}, \boldsymbol{\omega}) = L_e(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i), -\boldsymbol{\omega}_i)$.



- For global illumination, we recursively calculate $L_s(\mathbf{x}, \boldsymbol{\omega})$ until we reach an emissive surface (or our recursion depth M).

4.2.1 Estimating the Rendering Equation

- Computing rendering equation:

$$L_s(\mathbf{x}, \boldsymbol{\omega}_r) = L_e(\mathbf{x}, \boldsymbol{\omega}_r) + \int_\Omega f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r) L_s(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i), -\boldsymbol{\omega}_i) \cos\theta_i \, \mathrm{d}\boldsymbol{\omega}_i,$$

requires approximation of $\int_\Omega$ using a Monte Carlo estimator (see Data Science).

Construct a dataset $\langle \boldsymbol{\omega}_i^j \rangle$ from the random sample $\langle \boldsymbol{\Omega}_i \rangle$ of size N, of incident directions, distributed over the domain Ω with pdf $f_{\boldsymbol{\Omega}}$

$$\langle L_s(\mathbf{x}, \boldsymbol{\omega}_r) \rangle^N = L_e(\mathbf{x}, \boldsymbol{\omega}_r) + \frac{1}{N} \sum_{j=1}^{N} \frac{f_r(\mathbf{x}, \boldsymbol{\omega}_i^j, \boldsymbol{\omega}_r) L_s(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i^j), -\boldsymbol{\omega}_i^j) \cos\theta_i}{f_{\boldsymbol{\Omega}}(\boldsymbol{\omega}_i^j)}.$$

Uniform Sampling

- Construct sample $\langle \mathbf{\Omega}_i \rangle$ of size N distributed uniformly over Ω:

$$f_{\mathbf{\Omega}}(\boldsymbol{\omega}) = \begin{cases} \frac{1}{2\pi} & \text{if } \boldsymbol{\omega} \in \Omega \\ 0 & \text{otherwise} \end{cases}.$$

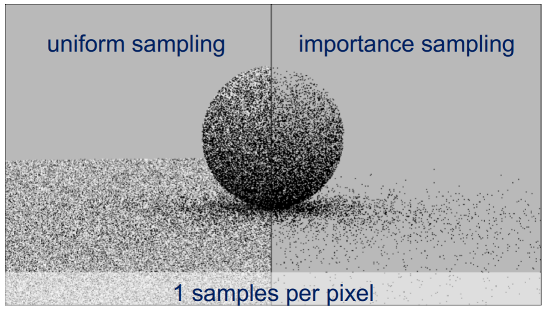
- Monte Carlo estimator:

$$\langle L_s(\mathbf{x}, \boldsymbol{\omega}_r) \rangle^N = L_e(\mathbf{x}, \boldsymbol{\omega}_r) + \frac{1}{2\pi N} \sum_{j=1}^{N} f_r(\mathbf{x}, \boldsymbol{\omega}_i^j, \boldsymbol{\omega}_r) L_s(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i^j), -\boldsymbol{\omega}_i^j) \cos\theta_i.$$

- **Problem**: Var $\left[\langle L_r(\mathbf{x}, \boldsymbol{\omega}_r) \rangle^N \right]$ converges with $O(\sqrt{N})$.
- **Solution**:
  - *Stratified Sampling.* Split Ω into N subdomains $\Omega_1, \ldots, \Omega_N$. Sample each subdomain. Converges with $O(N)$.
  - *Importance Sampling.*

Importance Sampling

- Select $f_{\mathbf{\Omega}}$ s.t Var $\left[\frac{f_r(\mathbf{x}, \boldsymbol{\omega}_i^j, \boldsymbol{\omega}_r) L_s(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i^j), -\boldsymbol{\omega}_i^j) \cos\theta_i}{f_{\mathbf{\Omega}}(\boldsymbol{\omega}_i^j)} \right]$ is minimized.
  Occurs when $f_{\mathbf{\Omega}}(\boldsymbol{\omega}_i^j) \propto f_r(\mathbf{x}, \boldsymbol{\omega}_i^j, \boldsymbol{\omega}_r) L_s(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i^j), -\boldsymbol{\omega}_i^j) \cos\theta_i$.
- Two choices for $f_{\mathbf{\Omega}}$:
  - The cosine term: $f_{\mathbf{\Omega}}(\boldsymbol{\omega}_i) \propto \cos\theta_i$,
  - BRDF: $f_{\mathbf{\Omega}}(\boldsymbol{\omega}_i) \propto f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r)$,
  - Emissive surfaces: $f_{\mathbf{\Omega}}(\boldsymbol{\omega}_i) \propto L_s(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i), -\boldsymbol{\omega}_i)$.
- Removes "shadow acne" (noise)



Definition 4.2.3. (**Ambient Occlusion**) The ambient occlusion $A_{\mathbf{x}}$ at the point $\mathbf{x}$ on the surface S with normal $\hat{\mathbf{n}}$ is defined as

$$A_{\mathbf{x}} = \frac{1}{\pi} \int_{\Omega} V(\mathbf{x}, \boldsymbol{\omega}) \cos\theta \, \mathrm{d}\boldsymbol{\omega},$$

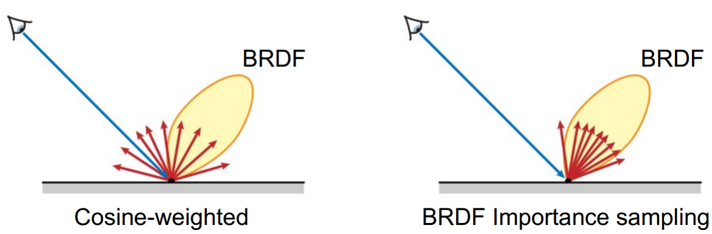
where V is the visible function and $\hat{\mathbf{n}} \cdot \boldsymbol{\omega} = \cos\theta$, where $\hat{\mathbf{n}}$ is the surface normal of S.

- **Cosine Term**: $f_{\mathbf{\Omega}}(\boldsymbol{\omega}_i) \propto \cos\theta_i$ for $A_{\mathbf{x}}$:

  $$\langle A_{\mathbf{x}} \rangle^N = \frac{1}{N} \sum_{j=1}^{N} V(\mathbf{x}, \boldsymbol{\omega}_i^j).$$

  Reduces noise for diffuse (or Lambertian) surfaces.

- **BRDF**: $f_{\mathbf{\Omega}}(\boldsymbol{\omega}_i) \propto f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_r)$:

  Reduces noise for specular surfaces.

- **Emissive**: $f_{\mathbf{\Omega}}(\boldsymbol{\omega}_i) \propto L_s(\mathbf{r}(\mathbf{x}, \boldsymbol{\omega}_i), -\boldsymbol{\omega}_i)$. Implemented by sampling the set of emissive surfaces. Necessary for point lights.

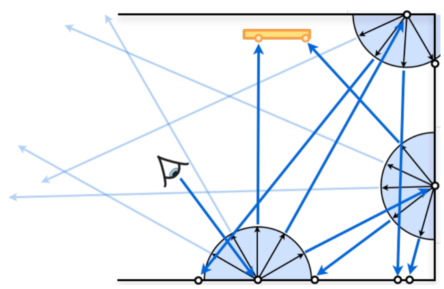
4.2.2 Recursive Ray Tracing and Path Tracing

- **Recursive Ray Tracing**: We have

```
float L_s(obj O, vec3 x, vec3 ω_r, int M) {
    if (M = 0 || is_emissive(O)) return L_e(x, ω_r)

    obtain dataset ⟨ω_i^j⟩ from random sample ⟨Ω_i⟩ of size N,
    with pdf f_Ω


    L ← 0
    for (j ← 0; j < N; j++) {
        O', x', n̂ ← scene.intersect(x, ω_i^j)
        if (O' = null) continue

        L ← L + O.f_r(x, ω_i^j, ω_r) L_s(O', x', -ω_i^j, M - 1)(n̂ · ω_i^j) / f_Ω(ω_i^j)
    }
    return L/N
}
```

$$L \leftarrow L + \frac{O.f_r(\mathbf{x}, \boldsymbol{\omega}_i^j, \boldsymbol{\omega}_r) L_s(O', \mathbf{x}', -\boldsymbol{\omega}_i^j, M-1)(\hat{\mathbf{n}} \cdot \boldsymbol{\omega}_i^j)}{f_{\boldsymbol{\Omega}}(\boldsymbol{\omega}_i^j)}$$

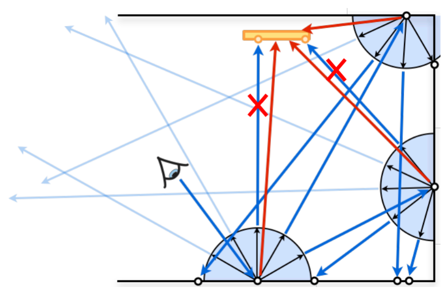
where M is the depth.



- **Accurate Direct lighting**:
  - Much of the contribution at $\mathbf{x}$ is from visible light sources $\ell_1, \ldots, \ell_n$.
  - Trace n *light rays* (called *shadow rays* in Lectures...) to ℓ_i.
  - Recursively trace N scatter rays (removing those that intersect with a light source ℓ_i)

We have:

```
float L_s(obj O, vec3 x, vec3 ω_r, int M) {
    ...
    L ← 0
    for (ℓ ∈ scene.lights) L ← L + L_e(ℓ.x, normalize(x − ℓ.x))

    for (j ← 0; j < N; j++) {
        ...
        if (O′ = null || is_emissive(O′)) continue
        ...
    }
    ...
}
```



- **Path Tracing** is recursive raytracing with $N = 1$. However, $\mathbf{f}_{\mathbf{\Omega}}$ is defined by the surface. e.g. Fensel approximation for dielectrics, Lambertian cosine for diffuse.

  As can add *accurate direct lighting* to path tracing: