Queens' College Cambridge

Mathematical Methods

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Contents

1 Vectors

Vectors may be thought of objects that have both *magnitude* and *direction*. They are used extensively in Physics and can generalise equations for all co-ordinate systems.

Definition 1.0.1. (**Vector**) A quantity that is specified by a magnitude and a direction in space is called a vector.

Some notation

- We represent the vector $\mathbf{a}$ by bold font.
- It's magnitude is written as $\|\mathbf{a}\|$
- The unit vector $\hat{\mathbf{a}}$ acts in the same direction as $\mathbf{a}$ but with unit (1) length.
- We may represent the vector from point A to point B by the notation $\overrightarrow{AB}$

We say a vector is **constant** if and only if they are not necessarily tied to a space. That is to say, two constant vectors if they have the same magnitude and direction. A **displacement** vector is a vector between two points.

A vector can also be written in terms of it's components. There are two commonly used notations for this, the *ordered set* notation and the *column* notation. For example, the vector $\mathbf{a}$ in $\mathbb{R}^n$ can be denoted as

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

1.0.1 Properties

Definition 1.0.2. (**Equality**) Two vectors are said to be **equal** if they have the same magnitude and direction.

Equivalently two vectors will be equal if their components are equal. So for two vectors,

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle \text{ and } \mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$$

are equal, if $a_i = b_i$ for all $a_i \in \mathbf{a}$ and $b_i \in \mathbf{b}$.

Definition 1.0.3. (**Magnitude**) The **magnitude** or length of the vector $\mathbf{a}$ is denoted by $\|\mathbf{a}\|$ (or less commonly, $|\mathbf{a}|$) is defined as the Euclidean length,

$$\|\mathbf{a}\| = \sqrt{\sum_{k=1}^{n} a_k^2}, \tag{1.1}$$

which is a consequence of the Pythagorean theorem. This also happens to be equal to the square root of the scalar product, as discussed in section ??, of the vector with itself:

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}. \tag{1.2}$$

A unit vector is any vector with a magnitude of one; which are normally used to simply indicate a direction. A vector of an arbitrary magnitude can be dividing by it's magnitude to create a unit vector. This is known as *normalizing* a vector. The unit vector of the vector $\mathbf{a}$, denoted $\hat{\mathbf{a}}$, is given by

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|}. \tag{1.3}$$

The zero vector is the vector with a magnitude of zero. Written out in components, the vector is $\langle 0, 0, \ldots, 0 \rangle$ and it is commonly denoted as $\mathbf{0}$. Unlike any other vector, it cannot be normalized (that is, there is no unit vector that is a multiple of the zero vector). We will later see that the zero vector is identity element with respect to the binary operation of vector addition.

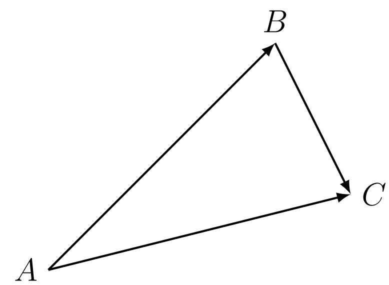
Definition 1.0.4. (**Orthogonal**) We say that two vectors $\mathbf{a}$ and $\mathbf{b}$ are **orthogonal** (perpendicular) if and only if

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

1.1 Algebra with vectors

1.1.1 Vector Addition

Definition 1.1.1. (**Addition**) We add vectors geometrically by placing them tip to tail



We see that $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$. We can also clearly see that vector addition is commutative and associative.

Definition 1.1.2. (**Additive Inverse Vector**) The **additive inverse vector $\mathbf{a}^*$** of a vector $\mathbf{a}$ under the binary operator of vector addition is defined by

$$\mathbf{a} + \mathbf{a}^* = \mathbf{0},$$

where $\mathbf{0}$ *is zero vector.*

It is obvious that inverse vectors have the same magnitude but opposite directions, and so $\mathbf{a}^* = -\mathbf{a}$.

Thus, subtracting by a vector is the same as adding the additive inverse vector, i.e. $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

Scalar Multiplication

Definition 1.1.3. (**Scalar Multiplication**) The **scalar multiplication** of the vector $\mathbf{a}$ by a scalar $\lambda \in \mathbb{R}$ is a vector $\lambda\mathbf{a}$ with magnitude $|\lambda|\|\mathbf{a}\|$ and is parallel to $\mathbf{a}$.

If $\lambda, \mu \in \mathbb{R}$, then

1. **Distributive Law**. For all $\mathbf{u}, \mathbf{v} \in V$

$$(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v} \text{ and } \lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}.$$

2. **Associative Law**. For all $\mathbf{v} \in V$

$$\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}.$$

3. Multiplication by $1, 0$ and -1. For all $\mathbf{v} \in V$

$$1\mathbf{v} = \mathbf{v}, 0\mathbf{v} = \mathbf{0} \text{ and } (-1)\mathbf{v} = -\mathbf{v}.$$

The combination of scalar multiplication and vector addition produces the concept of a linear combination of vectors.

Definition 1.1.4. (**Linear Combination**) *The vector* $\mathbf{c}$ *is described as a linear combination of the vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *if*

$$\mathbf{c} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i,$$

where $\lambda_i \in \mathbb{R}$ *for all* $1 \leq i \leq n$

1.1.2 Scalar Product

Definition 1.1.5. (**Scalar Product**) The **scalar product** $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} \cdot \mathbf{b}$, is defined as

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta \tag{1.4}$$

Some properties of the scalar product are:

- The scalar product is commutative,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}. \tag{1.5}$$

- The scalar product is distributive over vector addition,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}. \tag{1.6}$$

- Scalar multiplication is associative over the scalar product,

$$(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b}). \tag{1.7}$$

- The scalar product of a vector with itself is equal to it's magnitude squared,
$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2. \tag{1.8}$$

- The scalar product of a vector with the zero vector is equal to zero,
$$\mathbf{a} \cdot \mathbf{0} = 0, \tag{1.9}$$
from this we can deduce that if $\mathbf{a} \cdot \mathbf{a} = 0$, then $\mathbf{a} = \mathbf{0}$.

- We cannot take the dot product of three vectors consecutively since it is not a binary operator

- If $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a}, \mathbf{b}$ are non-zero vectors, then $\mathbf{a}$ and $\mathbf{b}$ must be orthogonal. i.e. $\theta = \pi/2$.

Theorem 1.1.1. The scalar product of two vectors $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$ is given by

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i \tag{1.10}$$

Proof. Let $B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the orthonormal basis such that

$$\mathbf{a}, \mathbf{b} \in \operatorname{span}(B),$$

then

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n) \cdot (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \cdots + b_n\mathbf{e}_n) \\ &= a_1\mathbf{e}_1 \cdot (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \cdots + b_n\mathbf{e}_n) + \cdots + a_n\mathbf{e}_n \cdot (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \cdots + b_n\mathbf{e}_n)\end{aligned}$$

Since $\mathbf{e}_i$ is orthogonal to $\mathbf{e}_j$ for all $i \neq j$, then $\mathbf{e}_i \cdot \mathbf{e}_j = 0$, and for all $1 \leq i \leq n$, $\mathbf{e}_i \cdot \mathbf{e}_i = 1$ then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i.$$

$\square$

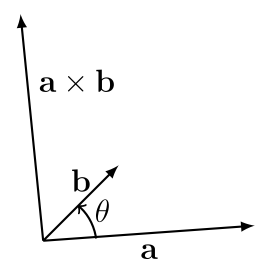
1.1.3 Vector Product

The vector product (also called the cross product) is a binary operation on the set of three dimensional vectors $\mathbb{R}^3$.

Definition 1.1.6. (**Vector Product**) The **vector product** $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, denoted $\mathbf{a} \times \mathbf{b}$, is a vector of magnitude $\|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$ in the direction perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ in the right-handed sense, so it is defined as

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta\, \hat{\mathbf{n}} \tag{1.11}$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.



Some Properties

1. The vector product is anti-commutative

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

   Consider a plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$ with an anticlockwise angle of θ between them, then by definition the direction of the vector product is given by $\sin\theta\hat{\mathbf{n}}$, whereas with a clockwise angle $-\theta$ then the direction is given by $-\sin\theta\hat{\mathbf{n}}$.

2. $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ implies the two vectors are parallel to each other, or one / both are the zero vector.

3. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

4. $\|\mathbf{a} \times \mathbf{b}\|$ gives the area of a parallelogram with sides given by $\mathbf{a}$ and $\mathbf{b}$

5. Distributive over vector addition $+$.

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

6. Not associative.

In terms of components, the vector product can be written as a determinant of a 3×3 matrix:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.12}$$

1.1.4 Scalar Triple Product

Definition 1.1.7. (**Scalar Triple Product**) The scalar triple product is

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

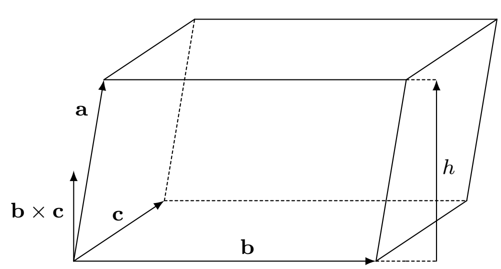
The scalar triple product is invariant under cyclic permutations, and changes sign under non-cyclic permutations. Hence

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}].$$

and

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}].$$

Consider the following Parallelpiped formed from the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.



The area of the base formed by $\mathbf{b}, \mathbf{c}$ is

$$\|\mathbf{b} \times \mathbf{c}\| = \|\mathbf{b}\| \|\mathbf{c}\| \sin\theta.$$

The height h corresponds to the magnitude of the projection of $\mathbf{a}$ onto the normalized vector of $\mathbf{b} \times \mathbf{c}$. Hence

$$h = \mathbf{a} \cdot \frac{\mathbf{b} \times \mathbf{c}}{\|\mathbf{b} \times \mathbf{c}\|}.$$

So

$$V_{parallelpiped} = [\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

In component form, the scalar triple product is

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

1.1.5 Vector Triple Product

Definition 1.1.8. (**Vector Triple Product**) The vector triple product is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Given $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, then consider $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, etc. Then

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \times \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_2c_2 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix} \\ &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \end{aligned}$$

Similarly, for $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, we have

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

Thus $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

1.2 Vector geometry

1.2.1 Lines and Planes

The vector equation of a line is

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}, \tag{1.13}$$

where $\mathbf{r}$ is the position vector of an arbitrary point on the line, $\mathbf{a}$ is the position vector of the fixed point A that lies on the line, $\mathbf{b}$ is a vector in the direction of the line and λ is a scalar multiplier. We can rearrange for

$$\mathbf{r} - \mathbf{a} = \lambda \mathbf{b}.$$

Taking the cross product of $\mathbf{b}$ on both sides yields and alternative form

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}.$$

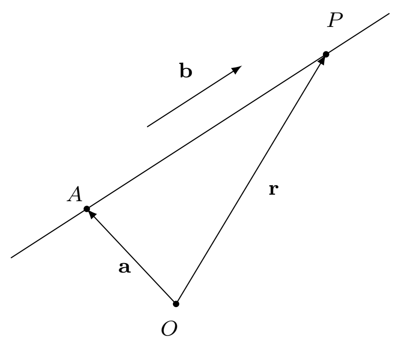


Figure 1.1: Vector equation of a line.

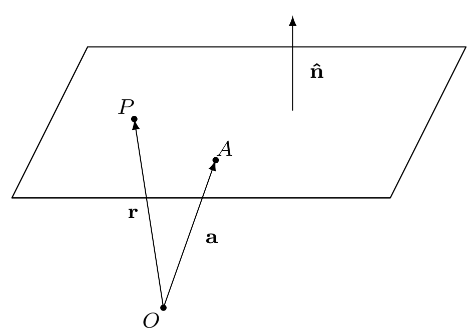


Figure 1.2: A plane.

Consider a plane that goes through a point A and that is orthogonal to a unit vector $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is said to normal to the plane. Let P be any point in the plane Π with position vector $\mathbf{r}$, then

$$\begin{aligned} \overline{AP} \cdot \hat{\mathbf{n}} &= 0 \\ \iff (\overline{AO} + \overline{OP}) \cdot \hat{\mathbf{n}} &= 0 \\ \iff (\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} &= 0 \end{aligned}$$

Let Q be a point on the plane Π such that $\overrightarrow{OQ}$ is parallel to $\hat{\mathbf{n}}$. Suppose that $\overrightarrow{OQ} = p\hat{\mathbf{n}}$, then d is the distance of the plane to O. Further, since Q is in the plane, it follows that

$$(p\hat{\mathbf{n}} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0 \implies \mathbf{a} \cdot \hat{\mathbf{n}} = p$$

and so we also have

$$\mathbf{r} \cdot \hat{\mathbf{n}} = p.$$

Using this, we may write the equation of Π in component form. Suppose that $\mathbf{r} = \langle r_1, r_2, \ldots, r_m \rangle$ and $\hat{\mathbf{n}} = \langle n_1, n_2, \ldots, n_m \rangle$, then by ?? we have

$$r_1 n_1 + r_2 n_2 + \cdots + r_m n_m = p.$$

The numbers $n_1, n_2, \ldots, n_m$ are called the *direction cosines*, since

$$n_1 = \hat{\mathbf{i}} \cdot \hat{\mathbf{n}} = \cos\theta.$$

where θ is the angle between the $\hat{\mathbf{i}}$ and $\hat{\mathbf{n}}$. Similarly n_2 is the cosine of the angle between $\hat{\mathbf{j}}$ and $\hat{\mathbf{n}}$, etc.

Suppose that $\mathbf{l}$ and $\mathbf{m}$ are two non-parallel vectors such that $\mathbf{l} \cdot \hat{\mathbf{n}} = 0$ and $\mathbf{m} \cdot \hat{\mathbf{n}} = 0$ (so both vectors are parallel to the plane), then any point $\mathbf{r}$ is given by

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{l} + \mu\mathbf{m},$$

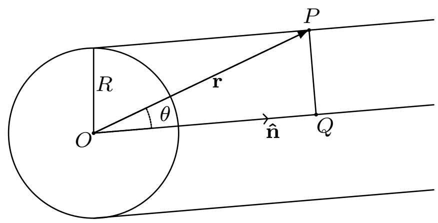
where $\lambda, \mu \in \mathbb{R}$.

1.2.2 Spheres, Cones and Cylinders

- **Spheres**. A sphere is a collection of points $\mathbf{r}$ in $\mathbb{R}^3$ that are equidistant from a center $\mathbf{c}$. Thus the magnitude of the vector $\overrightarrow{CR}$ is constant.

$$\|\mathbf{r} - \mathbf{c}\| = R.$$

- **Cylinder**. Let P be any point on the surface of a cylinder with center O and radius R with a direction vector $\hat{\mathbf{n}}$ through O , then from the figure below,

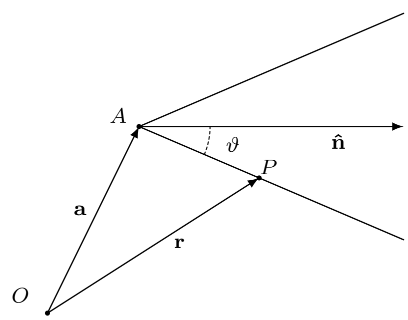


we see that $\|\overrightarrow{OQ}\| = \mathbf{r} \cdot \hat{\mathbf{n}}$, thus $\overrightarrow{OQ} = (\mathbf{r} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$. Note that $\|\overrightarrow{QP}\| = R$, and $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$. Hence

$$\|\mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}\| = R.$$

- **Cone**. A cone with vertex at the point A with position vector $\mathbf{a}$, with axis along the unit vector $\hat{\mathbf{n}}$ and making an angle ϑ to the axis vector $\hat{\mathbf{n}}$ is given by

$$(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = \|\mathbf{r} - \mathbf{a}\| \cos \vartheta.$$

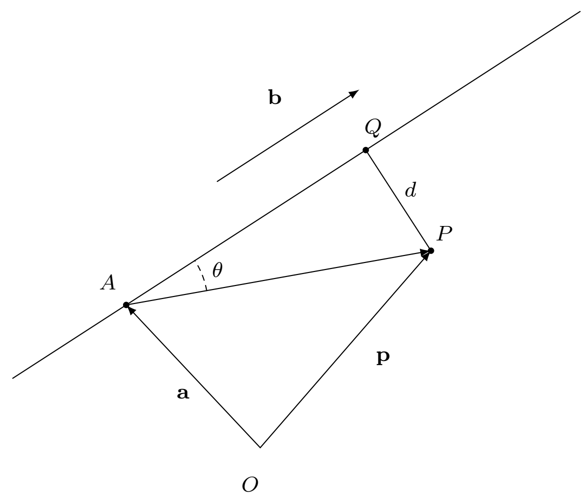


1.2.3 Distances

Shortest distance of a point from a line

The shortest distance between the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and the point P with position vector $\mathbf{p}$ is given by

$$d = \frac{\|(\mathbf{p} - \mathbf{a}) \times \mathbf{b}\|}{\|\mathbf{b}\|}.$$



Since APQ is a right angled triangle, it follows that $d = \|\mathbf{AP}\| \sin\theta$. Hence

$$d = \|(\mathbf{p} - \mathbf{a}) \times \hat{\mathbf{b}}\| = \frac{\|(\mathbf{p} - \mathbf{a}) \times \mathbf{b}\|}{\|\mathbf{b}\|}.$$

Alternatively, consider $\overrightarrow{PR}$ where R is some arbitrary point on the line. Thus

$$\overrightarrow{PR} = (\mathbf{a} - \mathbf{p}) + \lambda \mathbf{b}.$$

Notice that $\|\overrightarrow{PR}\| = d$ if and only if $\overrightarrow{PR} \cdot \mathbf{b} = 0$. So

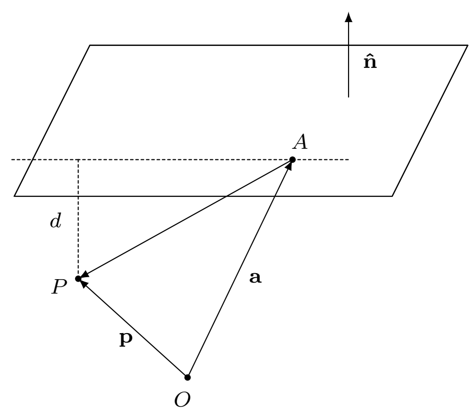
$$\begin{aligned} & [(\mathbf{a} - \mathbf{p}) + \lambda \mathbf{b}] \cdot \mathbf{b} = 0 \\ \iff & (\mathbf{a} - \mathbf{p}) \cdot \mathbf{b} + \lambda \|\mathbf{b}\|^2 = 0 \end{aligned}$$

Solving for λ and substituting back into $\|\overrightarrow{PR}\|$ will yield d.

Shortest distance of a point from a plane

The shortest distance between the plane $(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0$ and the point P with position vector $\mathbf{p}$ is given by

$$d = (\mathbf{p} - \mathbf{a}) \cdot \hat{\mathbf{n}}.$$



By inspection, the magnitude of the projection of $\overrightarrow{AP}$ on $\hat{\mathbf{n}}$ is equal to d. Hence

$$d = (\mathbf{p} - \mathbf{a}) \cdot \hat{\mathbf{n}}.$$

Shortest distance of a line from a line

Consider two lines L_1 and L_2 given by

$$\mathbf{r} = \mathbf{a}_1 + \lambda \hat{\mathbf{b}}_1 \text{ and } \mathbf{r} = \mathbf{a}_2 + \mu \hat{\mathbf{b}}_2,$$

respectively, for $\lambda, \mu \in \mathbb{R}$.

Let $\overrightarrow{XY}$ be a vector from L_1 to L_2. Suppose that $\|\overrightarrow{XY}\|$ is the shortest distance between the two lines. Then it follows that $\overrightarrow{XY}$ is orthogonal to both $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$. Thus the direction vector $\hat{\mathbf{u}}$ for $\overrightarrow{XY}$ is given by $\hat{\mathbf{u}} = \hat{\mathbf{b}}_1 \times \hat{\mathbf{b}}_2$, so $\overrightarrow{XY} = d\hat{\mathbf{u}}$ where d is the shortest distance between the lines.

For some λ and μ, we have

$$\overrightarrow{XY} = (\mathbf{a}_2 + \mu \hat{\mathbf{b}}_2) - (\mathbf{a}_1 + \lambda \hat{\mathbf{b}}_1) = (\mathbf{a}_2 - \mathbf{a}_1) + \mu \hat{\mathbf{b}}_2 - \lambda \hat{\mathbf{b}}_1.$$

Equating this to $d\hat{\mathbf{u}}$ and taking the dot product of $\hat{\mathbf{u}}$ on both sides yields

$$d = \left| (\mathbf{a}_2 - \mathbf{a}_1) \cdot (\hat{\mathbf{b}}_1 \times \hat{\mathbf{b}}_2) \right|.$$

2 Complex Numbers

In this section we will look at defining complex numbers, explore some of their elementary properties. Complex numbers occur in many branches of mathematics, and arise most directly from solving polynomial equations with non-real roots. Consider the quadratic equation

$$z^2 - 4z + 5 = 0.$$

This has two roots, z_1 and z_2, such that

$$(z - z_1)(z - z_2) = 0.$$

Using the quadratic formula we obtain

$$z_{1,2} = 2 \pm \frac{\sqrt{-4}}{2}.$$

Both roots contain the square root of a negative number. Recall the *fundamental theorem of algebra* states that a polynomial of degree n will have n solutions (included repeated roots), and in this case they are given above. The first part is called the *real part*, the second term is the *imaginary* part and the whole thing is called a *complex number*. The conventional representation of a complex number uses the variable z, with a real part a and an imaginary part b multiplied by the imaginary unit, $i = \sqrt{-1}$, i.e.

$$z = a + bi. \tag{2.1}$$

This is known as the *Cartesian form* of a complex number, there are other forms such as *modulus-argument* form (polar) and *exponential* form. The functions $\Re$ and $\Im$ gives the real and imaginary parts of a complex number, respectively, such that

$$\Re(z) = a \text{ and } \Im(z) = b. \tag{2.2}$$

The set of complex numbers $\mathbb{C}$ is defined as follows;

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}. \tag{2.3}$$

Definition 2.0.1. (**Complex Conjugate**) A complex conjugate of z, denoted by z^*, is found by changing the sign of the imaginary part of z. So if

$$z = a \pm bi \text{ then } z^* = a \mp bi. \tag{2.4}$$

The non-real roots of polynomials with entirely real coefficient come in **complex conjugate pairs**. That is, if z is a non-real root of the polynomial function f with real coefficients, then the complex conjugate of z (z^*) is a root of the polynomial as well. This can simply be rationalized by considering the sum and product of two complex conjugates,

$$\begin{aligned} z + z^* &= (a + bi) + (a - bi) \\ &= 2a \end{aligned}$$

and

$$\begin{aligned} zz^* &= (a + bi)(a - bi) \\ &= a^2 - b^2 i^2 \\ &= a^2 + b^2 \end{aligned}$$

and so the sum and product of two complex conjugates are entirely real, so $(z - z_1)(z - z_1^*)$ is entirely real, where z_1 is a non-real root of some arbitrary polynomial with entirely real coefficients. So the complex conjugate ensures the coefficients of the polynomial are real.

Complex numbers can be plotted in a plane Argand Diagram (a two dimensional space), with a horizontal real axis and a vertical imaginary axis, as shown below.

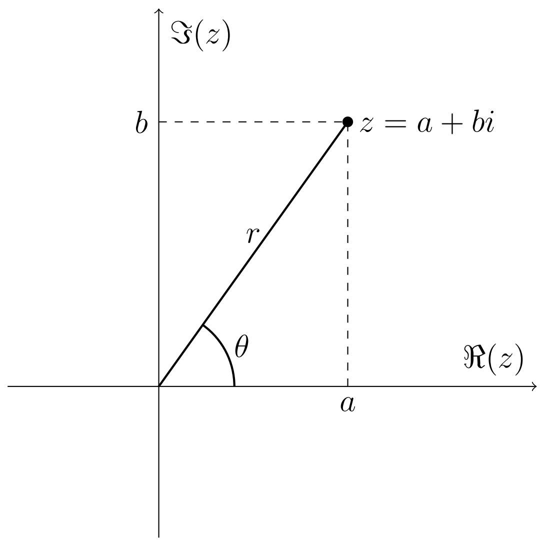


Figure 2.1: Representation of Cartesian and polar form of a complex number.

Notice how the complex conjugate of z corresponds to a reflection in the real axis.

2.1 Modulus and Argument

Definition 2.1.1. (**Modulus**) The **modulus** of a complex number is denoted as $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} \tag{2.5}$$

This corresponds (graphically) to the distance from the point z and the origin on an Argand diagram. Notice that $|z|^2 = zz^*$.

Definition 2.1.2. (**Argument**) The argument of a complex number is denoted by $\arg z$ and is defined as

$$\arg z = \arctan\left(\frac{b}{a}\right). \tag{2.6}$$

where $-\pi \leq \arg z \leq \pi$. This is the angle that a line segement (a half line) from the origin to z makes to the positive real axis on an Argand diagram.

Having defined the modulus and argument, we can now consider some of their interesting properties. Consider the product of two complex numbers z_1 and z_2. Writing them in polar form (see section ??), we find that

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\left(\cos\theta_1\sin\theta_2 + \sin\theta_2\cos\theta_2\right)\right] \\ &= r_1 r_2 \left[\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right], \end{aligned}$$

which implies the modulus of the product of two complex numbers is given by the product of their moduli, that is

$$|z_1 z_2| = |z_1||z_2|, \tag{2.7}$$

and the argument of the product of two complex numbers is given by the sum of their arguments,

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{2.8}$$

Now, let us consider the quotient of two complex numbers z_1 and z_2. Using a similar method to the one above, we find that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)],$$

giving us

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \tag{2.9}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{2.10}$$

2.2 Forms

There are three main forms of a complex number; Cartesian form, polar form and exponential form. We've already discussed Cartesian form in our introduction to complex numbers. Polar form, as the name suggests, is linked to the geometry of the Argand diagram representation in Figure ??. Exponential is a direct result of polar form and the expansion of the exponential function.

2.2.1 Polar Form

By considering the geometry of the Argand diagram representation in Figure ??, we can derive a polar form of complex numbers

$$z = r(\cos\theta + i\sin\theta), \tag{2.11}$$

where $r = |z|$ and $\theta = \arg\theta$.

2.2.2 Exponential Form

The Maclaurin expansion of the terms within the brackets of the polar form in equation ?? is

$$\begin{aligned}\cos\theta + i\sin\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \cdots\right)\\ &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots\end{aligned}$$

The final infinite series is the definition of $\exp(i\theta)$, and thus

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{2.12}$$

this is known as **Euler's formula**. It follows from this that

$$e^{i(n\theta)} = \cos n\theta + i\sin n\theta, \tag{2.13}$$

for all $n \in \mathbb{R}$. This is called **de Moivre's Theorem**. We notice that due to the periodic nature of trigonometric functions, we have

$$z = re^{i\theta} = re^{i(\theta+2n\pi)} \tag{2.14}$$

for all $n \in \mathbb{N}$. Multiplication and division of complex numbers is much more straightforward in this exponential form. Notice that since

$$\begin{aligned}e^{i\theta} &= \cos\theta + i\sin\theta\\ e^{-i\theta} &= \cos\theta - i\sin\theta,\end{aligned}$$

we can now express the trigonometric functions in terms of complex exponentials as follows:

$$\cos\theta = \Re(e^{i\theta}) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \tag{2.15}$$

$$\sin\theta = \Im(e^{i\theta}) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{2.16}$$

This allows Euler's formula to provide a powerful connection between analysis and trigonometry, for example complex exponentials can simplify trigonometry, because they are easier to manipulate than their sinusoidal components.

2.3 Loci

We can draw **loci** in an Argand diagram. These are a set of points that obey a given rule. Let us consider the locus of points that satisfy $|z| = r$ on an Argand diagram. From the definition of the modulus of a complex number $z = a + bi$, we find that $a^2 + b^2 = r^2$, hence sketch of the locus points that satisfy $|z| = r$ is a circle with a centre at the origin and a radius of r, as shown below.

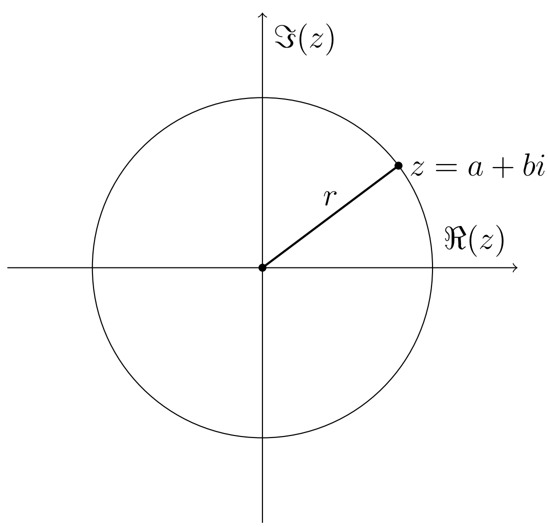


Figure 2.2: The locus of points satisfying $|z| = r$.

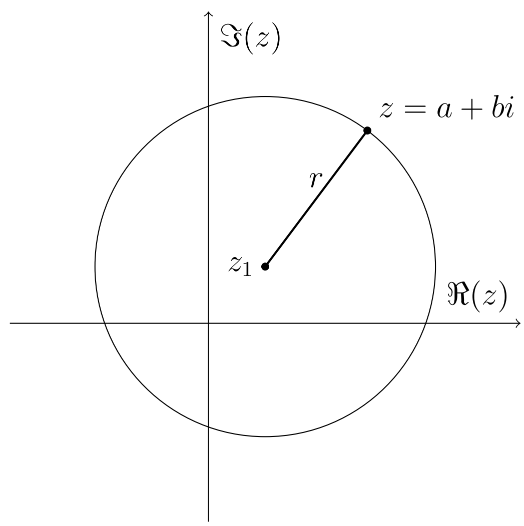


Figure 2.3: The locus of points satisfying $|z - z_1| = r$.

We can generalize this result, such that for any given complex number z_1, the locus of points satisfying $|z - z_1| = r$ will be a circle with a centre at z_1 and a radius of r.

Let us now consider the locus of points satisfying $\arg z = \theta$ on an Argand diagram. So we have $\theta = \arctan \frac{a}{b}$, thus the locus of points that satisfy $\arg z = \theta$ is a half line that makes an angle of θ with the positive real axis on an Argand diagram. Generally, the locus of points satisfying $\arg(z - z_1) = \theta$ is a **half line** from the point z_1 at an angle of θ to the positive real axis.

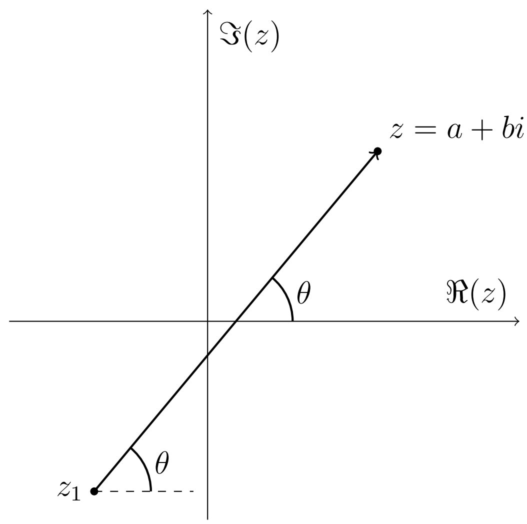


Figure 2.4: The locus of points satisfying $\arg(z - z_1) = \theta$.

Similarly, the locus of points satisfying $|z-z_1| = |z-z_2|$ is the perpendicular bisector of the line joining z_1 and z_2. This is because the locus includes all the points that are equidistant from the points z_1 and z_2.

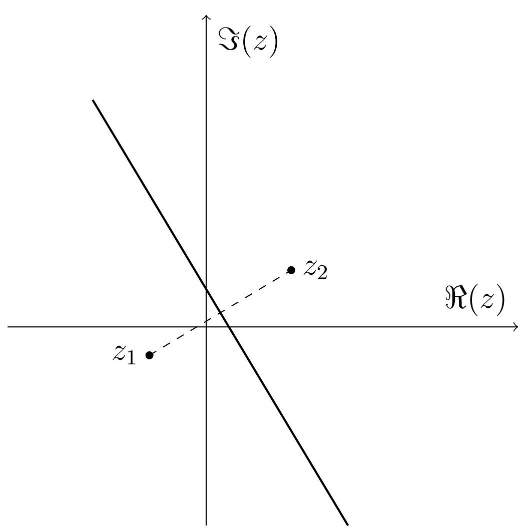


Figure 2.5: The locus of points satisfying $|z - z_1| = |z - z_2|$.

Note that due to the relationship between a Cartesian plane and an Argand diagram. The equation of line of the perpendicular bisector is given by

$$y - \frac{b_2 + b_1}{2} = \frac{a_1 - a_2}{b_2 - b_1}\left(x - \frac{a_2 + a_1}{2}\right), \tag{2.17}$$

where the points z_1 and z_2 are represented by the points (a_1, b_2) and (a_2, b_2) respectively.

2.4 Roots of Unity

The polynomial equation in the form $z^n = a + bi$ has n roots, as stated by the *fundamental theorem of algebra*. On an Argand diagram these are arranged at a regular angular intervals along a circle of a given radius. For example, the roots of the third degree polynomial $z^3 = 1$ are represented in Figure ??. In general, the roots of $z^n = a + bi = re^{i\theta}$ are given by

$$z_{1,2,\ldots,n} = r^{\frac{1}{n}} \exp\left(\frac{\theta + 2k\pi}{n} i\right), \tag{2.18}$$

for $k = 0, 1, 2, \ldots, n-1$. The complex solution for the equations in the form $z^n = 1$ is known as the $\mathbf{n^{th}}$ **roots of unity**, denoted by ω. From equation ??, we see that the roots of $z^n = re^{i\theta}$ are

$$z_{1,2,\ldots,n} = r^{\frac{1}{n}} \exp\left(\frac{i\theta}{n}\right) \exp\left(\frac{2k\pi}{n} i\right), \tag{2.19}$$

for $k = 0, 1, 2, \ldots, n-1$. We notice that $\exp\left(\frac{2k\pi}{n} i\right)$ for $k = 0, 1, 2, \ldots, n-1$ are the roots of the equation $z^n = 1$ and therefore are powers of the nth root of unity ω, that is.

$$\omega = \exp\left(\frac{2\pi i}{n}\right) \tag{2.20}$$

and

$$z_{1,2,\ldots,n} = r^{\frac{1}{n}} \exp\left(\frac{i\theta}{n}\right) \omega^k. \tag{2.21}$$

It follows from this, that the sum of the roots is a multiple of a geometric series with a common ratio of ω, that is

$$z_1 + z_2 + \ldots z_n = r^{\frac{1}{n}} \exp\left(\frac{i\theta}{n}\right) (\omega^0 + \omega^1 + \omega^2 + \ldots + \omega^{n-1}).$$

Applying the summation formula from equation ??, we find that

$$z_1 + z_2 + \dots z_n = r^{\frac{1}{n}} \exp\left(\frac{i\theta}{n}\right) \frac{1-\omega^n}{1-\omega},$$

however, since $\omega^n = 1$, it follows that

$$z_1 + z_2 + \dots z_n = 0 \tag{2.22}$$

Therefore, the sum of the roots of the equation $z^n = a + bi$ is zero for all $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.

2.5 Complex Logarithms and Powers

An arbitrary complex number z can be written in the form

$$z = re^{i(\theta+2n\pi)},$$

where $n \in \mathbb{Z}$. Thus, each point on the complex plane can be represented by a countably infinite number of equivalent expressions. For the real numbers, $\ln x$ is defined as the inverse function of $\exp x$, and both are one-to-one functions for real numbers. However, $\exp z$ is a many-to-one function in the complex domain, and thus it's inverse function would be one-to-many, unless the range were restricted. Thus there exist *two* functions that act as an inverse to $\exp z$.

Definition 2.5.1. The **multivalued** logarithm, $\operatorname{Ln} z$ of a complex number z is given by

$$\operatorname{Ln} z = \ln r + i(\theta + 2n\pi), \tag{2.23}$$

where $\ln r$ is the natural logarithm of the (real and positive) modulus of the complex number z, θ is the argument and n is any integer.

Definition 2.5.2. The **principle value** of $\operatorname{Ln} z$, denoted by $\ln z$, is given by

$$\ln z = \ln r + i\theta, \tag{2.24}$$

where $\ln r$ is the natural logarithm of the (real and positive) modulus of the complex number z and θ is the argument in the range $-\pi \leq \theta \leq \pi$.

Theorem 2.5.1. If z and t are both complex numbers, then the zth power of t is defined by

$$t^z = e^{z \operatorname{Ln} t}. \tag{2.25}$$

Notice that since $\operatorname{Ln} t$ is multivalued, then so is t^z.

Example 2.5.1. Simplify the expression $z = i^i$. We can rewrite this as $z = e^{i \operatorname{Ln} i}$. Since $i = \exp\left[\left(\frac{\pi}{2} + 2n\pi\right) i\right]$, then $\operatorname{Ln} i = \left(\frac{\pi}{2} + 2n\pi\right) i$. Completing the computation:

$$\begin{aligned} z &= e^{i\left(\frac{\pi}{2}+2n\pi\right)i} \\ &= e^{-\left(\frac{\pi}{2}+2n\pi\right)}. \end{aligned}$$

This result is surprising because it is an entirely real quantity!

3 Hyperbolic Functions

Hyperbolic function share similarities to the trigonometric functions but come from a more algebraic, rather than geometric, idea. In general, a function can be expressed as the sum of an event function and an odd function, where either of these two parts may just be zero. Let us define $e(x)$ as an even function of x, and $o(x)$ as an odd function of x. Now let us define two functions, f and g, as follows:

$$
\begin{aligned}
f(x) &= e(x) + o(x) \\
g(x) &= e(x) - o(x)
\end{aligned}
$$

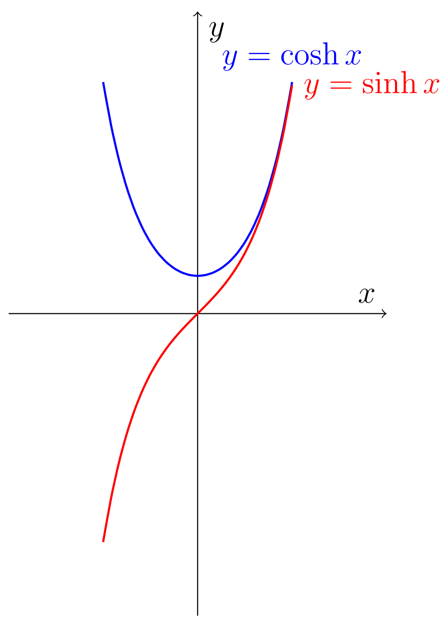
Suppose we want to find $e(x)$ and $o(x)$ given that we know $f(x)$ and $g(x)$. We can solve this pair of equations simultaneously to reveal that

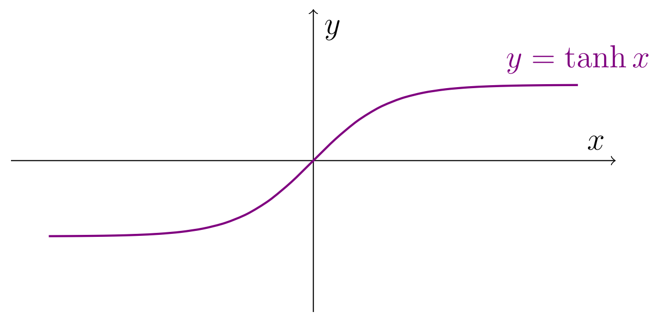
$$
\begin{aligned}
e(x) &= \frac{1}{2}(f(x) + g(x)) \\
o(x) &= \frac{1}{2}(f(x) - g(x))
\end{aligned}
$$

Setting $f(x) = e^x$ and $g(x) = e^{-x}$, we obtain the definitions of the two hyperbolic equivalents of $\cos x$ and $\sin x$:

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \tag{3.1}$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \tag{3.2}$$





The hyperbolic functions $\tanh x$, $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\coth x$ are defined by analogy to their trigonometric counterparts, such that

$$\tanh x = \frac{\sinh x}{\cosh x} \tag{3.3}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \tag{3.4}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} \tag{3.5}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} \tag{3.6}$$

3.1 Hyperbolic-Trigonometric Analogies

Recalling ?? and ?? but letting $\theta = ix$, we find that

$$\cos ix = \frac{1}{2}(e^x + e^{-x}) \tag{3.7}$$

$$\sin ix = \frac{1}{2}i(e^x - e^{-x}) \tag{3.8}$$

Hence, by the definitions given earlier:

$$\cosh x = \cos ix \tag{3.9}$$
$$i \sinh x = \sin ix \tag{3.10}$$
$$\cos x = \cosh ix \tag{3.11}$$
$$i \sin x = \sinh ix \tag{3.12}$$

Thus hyperbolic functions have identities similar to the trigonometric ones, with the difference begin that where $\sin^2 x$ appears in a trig identity, $-\sinh^2 x$ appears in the hyperbolic analogue. For the same reason, (since $i^2 = -1$), $\sin x \sin y$ coverts to $-\sinh x \sinh y$. This is known as **Osborn's rule**. So we have the so-called "hyperbolic-pythagorean identites":

$$\cosh^2 x - \sinh^2 x = 1 \tag{3.13}$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x \tag{3.14}$$
$$1 + \operatorname{cosech}^2 x = \coth^2 x \tag{3.15}$$

Similarly, we can apply Osborn's rule to the compound angle formulae, to find the "sum of argument formulae":

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{3.16}$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{3.17}$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \tag{3.18}$$

and finally, the "double argument formulae":

$$\sinh 2x = 2 \sinh x \cosh x = \frac{2 \tanh x}{1 - \tanh^2 x} \tag{3.19}$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1 \tag{3.20}$$
$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \tag{3.21}$$

3.2 Inverse of Hyperbolic Functions

If $y = \cosh x$, then there exists the inverse $x = \operatorname{arcosh} y$, however, we need to restrict the domain of $\cosh x$ to find such an inverse. Having done this we can find the inverse hyperbolic functions by algebraic methods.

Let us consider the expression for $\operatorname{arcosh} x$. First let $y = \operatorname{arcosh} x$, which implies that $x = \cosh y$. Using the definition of cosh, we obtain

$$2x = e^y + e^{-y}.$$

Next we multiply through by e^y and rearrange to

$$e^{2y} - 2xe^y + 1 = 0,$$

which is a quadratic equation in e^y. Using the quadratic formula, we find the roots

$$e^y_{1,2} = x \pm \sqrt{x^2 - 1}.$$

We define $\operatorname{arcosh} x$ by the principle root, so taking logarithms of both sides we find that

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right), \qquad \forall\, x \geq 1. \tag{3.22}$$

Similarly, it is possible to show that

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right), \qquad \forall\, x \in \mathbb{R} \tag{3.23}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), \qquad \forall\, |x| < 1. \tag{3.24}$$

4 Calculus

4.1 Differentiation

4.1.1 Limits and Continuity

This section is about looking at the limiting value taken by function $f(x)$ as a particular input x is approached.

A function is said to be continuous at a point if the limiting value of the function is the limiting value of the function is the same when approached from above as from below (i.e. there is no "jump" or discontinuity). A function is differentiable at a point if $f'(x)$ is continuous at that point. Thus understanding limits and continuity is a crucial concept in differential calculus.

Definition 4.1.1. (**Limit of a function**) If $\lim_{x\to a} f(x) = l$ then $\forall \varepsilon > 0\ \exists\, \delta \in (0, \varepsilon) \mid |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon$. This is read as "If $f(x)$ approaches l as x approaches a, then for any number ε, there must exist some number δ such that $|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$.

Definition 4.1.1 is sometimes referred to as the "epsilon – delta" $(\varepsilon - \delta)$ definition of a limit, and is commonly used to define a continuous function (definition 4.1.2).

Definition 4.1.2. (**Continuous function**) A function f is continuous at some point a of its domain, if the limit of $f(x)$, as x approaches a through the domain of f, exists and is equal to $f(a)$, formally this can be written as:

$$\lim_{x\to a} f(x) = f(a) \tag{4.1}$$

where $a \in \mathrm{dom}(f)$ and the limit exists.

Example 4.1.1. Let us test whether the function $f(x) = x\sin(1/x)$, where $x \neq 0\ \wedge\ x \in \mathbb{R}$, $f(0) = 0$ is continuous at $x = 0$.

Using the $\varepsilon - \delta$ definition of a limit, we must show that for any $\varepsilon > 0$ there exists a $\delta \in (0, \varepsilon)$ such that if $|x - 0| < \delta$ then $|f(x) - f(0)| < \varepsilon$.

To do so, we let $\varepsilon > 0$ be arbitrary. Now, suppose $|x - 0| = |x| < \delta$. Proceeding:

$$\begin{aligned}
|f(x) - f(0)| &= \left| x \sin\left(\frac{1}{x}\right) - 0 \right| \\
&= |x| \cdot \left| \sin\left(\frac{1}{x}\right) \right| \\
&\leq |x| \;\; (\because \max(\sin x) = 1 \; \forall x \in \mathbb{R}) \\
&< \delta \\
&< \varepsilon
\end{aligned}$$

Therefore for an arbitrary ε there exists a $\delta \in (0, \varepsilon)$ such that if $|x - 0| < \delta$ then $|f(x) - f(0)| < \varepsilon$, thus it has been shown that $f(x)$ is continuous at $x = 0$.

Evaluation of limits

The following observations may be useful in finding the limit of a function:

- A limit may be $\pm\infty$. For example, $\lim_{x \to 0} x^{-2} = \infty$
- A limit may take different values depending on whether it is approached from above or below. For example, consider the limit of $f(x) = 1/x$, as x approaches 0 from above and below:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = \infty$$

- It may be easier to evaluate a limit if the function under consideration is split into a sum, product or quotient. Then the following rules may be applied, provided the limit exists:

$$\begin{aligned}
\lim_{x \to a} [f(x) \pm g(x)] &= \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \\
\lim_{x \to a} [f(x) \cdot g(x)] &= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \\
\lim_{x \to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}
\end{aligned}$$

provided the numerator and numerator are not both equal to zero or infinity.

In order to differentiate a function, we must check whether the function is continuous, to do so we check whether the limit of the function exists. This will become apparent in the next section (4.1.2).

4.1.2 First Principles

If $f(x)$ is a function of x then a function for the gradient of f at the point x is given by:

$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{4.2}$$

This function $f'(x)$ is called the first *derivative* of $f(x)$. This notation is known as the prime notation. Higher derivatives can be found using a similar method, for example the second derivative:

$$f''(x) = \lim_{\Delta x \to 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}$$

And the n^{th} derivative:

$$f^{(n)}(x) = \lim_{\Delta x \to 0} \frac{f^{(n-1)}(x + \Delta x) - f^{(n-1)}(x)}{\Delta x} \tag{4.3}$$

4.1.3 The Chain Rule

Sometimes we encounter functions inside other functions (composition), e.g. $f(x) = (3 + x^2)^3$, where $u(x) = 3 + x^2$ and $f(x) = [u(x)]^3$. If Δf, Δu and Δx are small finite quantities, it follows that:

$$\frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta u} \frac{\Delta u}{\Delta x}.$$

Applying the limits to both sides and assuming the limit $\lim_{\Delta x \to 0}$ exists, we have

$$\begin{aligned} \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta u} \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \end{aligned}$$

We know, from the definition of a derivative, that u is differentiable if and only if u is continuous. Assuming u is differentiable at x, if follows that u is continuous at x, hence as Δx approaches 0, we have Δu approaches 0.

$$\begin{aligned}\frac{\mathrm{d}f(x)}{\mathrm{d}x} &= \lim_{\Delta u \to 0} \frac{\Delta f}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\ &= \frac{\mathrm{d}f(u(x))}{\mathrm{d}u} \frac{\mathrm{d}u(x)}{\mathrm{d}x}\end{aligned}$$

This constitutes the chain rule, which must be used when differentiating functional composition.

4.1.4 The Product Rule

Suppose it is necessary to differentiate a function composed by the product of two other functions, namely $f(x) = u(x)v(x)$.

$$\begin{aligned}f(x+\Delta x) - f(x) &= u(x+\Delta x)v(x+\Delta x) - u(x)v(x) \\ &= u(x+\Delta x)[v(x+\Delta x) - v(x)] + [u(x+\Delta x) - u(x)]v(x)\end{aligned}$$

From the definition of a derivative, we have:

$$\begin{aligned}\frac{\mathrm{d}f(x)}{\mathrm{d}x} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left\{ u(x+\Delta x) \left[\frac{v(x+\Delta x) - v(x)}{\Delta x} \right] + \left[\frac{u(x+\Delta x) - u(x)}{\Delta x} \right] v(x) \right\}\end{aligned}$$

In the limit $\lim_{\Delta x \to 0}$, the factors in the square brackets become the derivatives with respect to v and u respectively, and $u(x+\Delta x)$ simply becomes $u(x)$. Consequently we obtain:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[u(x)v(x)] = u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x} + \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)$$

This forms the so-called product rule.

4.1.5 The Quotient Rule

For differentiating a function made up of a quotient of two other functions, the product rule and chain rule could be combined. For example, if we have

$f(x) = \frac{u(x)}{v(x)}$, then we have

$$\begin{aligned} f'(x) &= \left(u(x) \cdot \frac{1}{v(x)}\right)' \\ &= u'(x)\frac{1}{v(x)} + u(x)\frac{-v'(x)}{[v(x)]^2} \qquad \textit{(By the product rule)} \\ &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \end{aligned}$$

This is known as the quotient rule.

4.2 Integration

4.2.1 First Principles

It is possible to draw rectangles under the graph of the curve $y = f(x)$ for some function f. The height of each rectangle would be $f(x)$. In the interval $[a, b]$, one can draw n individual rectangles of equal width $k = (b - a)/n$. The integral is defined as the limit of the sum of the areas of these rectangles as the number of rectangles n tends to ∞ (or, equally, as the width of the rectangles tend to 0). Thus the definite integral is defined as follows (provided the limits exist):

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{r=0}^{n-1} kf(a + kr), \tag{4.4}$$

where k is the width of each rectangle, i.e. $k = \frac{b-a}{n}$.

INSERT FIGURE

For a function that is continuous in the interval $[a, b]$, this corresponds to the area enclosed between the curve, the x-axis and the lines $x = a$ and $y = b$. A negative value indicates that the enclosed area lies below the x-axis.

The following is a list of self-evident properties of definite integrals:

$$\int_a^a f(x)\,\mathrm{d}x = 0, \tag{4.5}$$

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x, \tag{4.6}$$

$$\int_a^b [f(x)+g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x, \tag{4.7}$$

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x\,. \tag{4.8}$$

4.2.2 Fundamental Theorem of Calculus

We define the primitive $F(x)$ as the integral of a function f between a fixed lower limit a and a variable upper limit x,

$$F(x) = \int_a^x f(u)\,\mathrm{d}u \tag{4.9}$$

Clearly $F(x)$ is a continuous function in x, but how is it related to differentiation? If we consider the integral in ?? and using the elementary property in ??, we obtain

$$F(x+\Delta x) = \int_a^{x+\Delta x} f(u)\,\mathrm{d}u \tag{4.10}$$

$$= \int_a^x f(u)\,\mathrm{d}u + \int_x^{x+\Delta x} f(u)\,\mathrm{d}u \tag{4.11}$$

$$= F(x) + \int_x^{x+\Delta x} f(u)\,\mathrm{d}u\,. \tag{4.12}$$

Rearranging and dividing through by Δx yields

$$\frac{F(x+\Delta x)-F(x)}{\Delta x} = \frac{1}{\Delta x}\int_x^{x+\Delta x} f(u)\,\mathrm{d}u\,. \tag{4.13}$$

According to the mean value theorem for integration (see section ??), there exists a real number $c \in [x, x+\Delta x]$ such that

$$\int_x^{x+\Delta x} f(u)\,\mathrm{d}u = f(c)\Delta x \tag{4.14}$$

Substituting the above into equation ?? we get

$$\frac{F(x+\Delta x)-F(x)}{\Delta x}=f(c). \tag{4.15}$$

Letting Δx tend to 0 and using the definition from equation ?? we obtain

$$F'(x)=\lim_{\Delta x\to 0} f(c) \tag{4.16}$$

We use the squeeze theorem. The number c in the interval $[x, x+\Delta x]$, with $\lim_{\Delta x\to 0} x = x$ and $\lim_{\Delta x\to 0} x+\Delta x = x$. Therefore by the squeeze theorem,

$$\lim_{\Delta x\to 0} c = x \tag{4.17}$$

Substituting this into ?? gives

$$F'(x)=f(x). \tag{4.18}$$

This is known as the **Fundamental Theorem of Calculus**, which essentially states that "Integration is the reverse of differentiation".

4.2.3 Reduction formulae

One can use reduction formulae to find the answer to a complicated definite or indefinite integral by first evaluating a more elementary one and finding a recurrence relation to reach the more complicated integral.

For example consider the integral I_n such that

$$I_n=\int x^n e^{-x}\,\mathrm{d}x \tag{4.19}$$

Let us split I_n into two parts and use the method of *integration by parts* with the following substitutions $u=x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x}=e^{-x}$,

$$\begin{align} I_n &= -x^n e^{-x} + n\int x^{n-1}e^{-x}\,\mathrm{d}x \tag{4.20}\\ &= -x^n e^{-x} + nI_{n-1} \tag{4.21}\end{align}$$

Evaluating this integral in the limits of zero to infinity, gives rise to the gamma function Γ at $n+1$, such that

$$\begin{align} \Gamma(n+1) &= [-x^n e^{-x}]_0^\infty + n\Gamma(n) \tag{4.22}\\ &= n\Gamma(n). \tag{4.23}\end{align}$$

Therefore $\Gamma(2)=1\cdot\Gamma(1)=1$, $\Gamma(3)=2\cdot\Gamma(1)=2$, $\Gamma(4)=3\cdot 2\cdot 1$, etc ... So $\Gamma(n+1)=n!$. This gives a new definition for the factorial function which can be used to extend factorials to all real numbers.

4.2.4 Mean Value Theorem of Integration

The mean value theorem of integration states that for a function f that is continuous in the interval $[a, b]$ and differentiable in the interval (a, b), then there exists c in (a, b) such that

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a). \tag{4.24}$$

We say that $f(c)$ is the mean value m of the function f between the two limits a and b and can be calculated by evaluating the integral

$$m = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\,. \tag{4.25}$$

4.3 Taylor and Maclurin Series

Taylor's theorem provides a way of expression a function in terms of a power series in x, provided that it is continuous and differentiable in the range of interest.

Suppose we have a function f that we wish to expression as a power series in $x - a$ about the point $x = a$. We assume that $f(x)$ and all it's derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$ are continuous and exist in the range. Using the fundamental theorem of calculus (see section ??) we may write

$$\int_a^{a+h} f'(x)\,\mathrm{d}x = f(a+h) - f(a), \tag{4.26}$$

where a, $a + h$ are neighbouring values of x. We rearrange this equation to

$$f(a+h) = f(a) + \int_a^{a+h} f'(x)\,\mathrm{d}x \tag{4.27}$$

and subsequently replace $f'(x)$ with $f'(a)$ in the integrand to obtain

$$f(a+h) \approx f(a) + hf'(a).$$

This first order approximation is illustrated in the diagram below.

If we let $x = a + h$ (and thus $h = x - a$) we can write this approximation in terms of just a and x as

$$f(x) \approx f(a) + (x-a)f'(a)$$

Similarly, we find that

$$f'(x) \approx f'(a) + (x-a)f''(a)$$

$$f''(x) \approx f''(a) + (x-a)f'''(a)$$

and so on. If we now substitute our expression for $f'(x)$ into equation ?? we obtain

$$\begin{aligned} f(a+h) &\approx f(a) + \int_a^{a+h} f'(a) + (x-a)f''(a)\,\mathrm{d}x \\ &\approx f(a) + hf'(a) + \frac{h^2}{2}f''(a) \end{aligned}$$

which is a second order approximation of $f(a+h)$. Repeating this process, we find that the $(n-1)^{th}$ order approximation of $f(a+h)$ is

$$f(a+h) \approx f(a) + hf'(a) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a).$$

The error, or *remainder*, from approximating $f(a+h)$ by the $(n-1)^{th}$ order power series is the last term of the next order power series and is given by

$$R_n(h) = \frac{h^n}{n!}f^{(n)}(\zeta) \text{ or } R_n(x) = \frac{(x-a)^n}{n!}f^{(n)}(\zeta) \tag{4.28}$$

for some ζ that lies in the interval $[a, a+h]$ (or $[a, x]$). **Taylor's theorem** then states that we may write equality

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(\zeta) \tag{4.29}$$

If we let $x = a+h$ in equation ?? we obtain the more useful form

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(x-a)^n}{n!}f^{(n)}(\zeta) \tag{4.30}$$

We often remove the necessity for the remainder by using an infinite power series. However this adds another assumption about f, that is f can be expressed as a convergent infinite power series.

Definition 4.3.1. (**Taylor Expansion**) *The* ***Taylor expansion*** *of a function f about the point $x = a$ is given by*

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots = \sum_{k=0}^{\infty}\frac{(x-a)^k}{k!}f^{(k)}(a) \tag{4.31}$$

Definition 4.3.2. (**Maclaurin Series**) *The* ***Maclaurin series*** *of a function f is given by the Taylor expansion of f about the point $x = 0$, that is*

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots = \sum_{k=0}^{\infty}\frac{x^k}{k!}f^{(k)}(0) \tag{4.32}$$

So a Maclaurin series can be found for any function f that meets the following conditions

- f can be expressed as a convergent infinite power series.
- f is continuous and infinitely differentiable around the point $x = 0$.
- $f^{(n)}(0)$ is finite for all $n \in \mathbb{N}$.

Common Maclaurin Series

$$(1+x)^n = \sum_{k=0}^{\infty}\frac{n(n-1)\cdots(n+1-k)}{k!}x^k \quad \text{for } |x| < 1 \text{ and } n \in \mathbb{R} \tag{4.33}$$

$$\exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!} \quad \text{for all } x \in \mathbb{R} \tag{4.34}$$

$$\ln(1+x) = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^k}{k} \quad \text{for } x \in (-1,1] \tag{4.35}$$

$$\sin x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \quad \text{for all} x \in \mathbb{R} \tag{4.36}$$

$$\cos x = \sum_{k=0}^{\infty}(-1)^k\frac{x^2}{(2k)!} \quad \text{for all } x \in \mathbb{R} \tag{4.37}$$

5 Differential Equations

5.1 First-Order Equations

Definition 5.1.1. A first order ODE is an equation of the form

$$F\left(\frac{\mathrm{d}y}{\mathrm{d}x}, y, x\right) = 0,$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = G(y, x).$$

5.1.1 Separable Equations

A separable first-order ODE is one which the two variables appear in separate factors. That is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y).$$

We divide by $g(y)$ yielding

$$\frac{1}{g(y)}\frac{\mathrm{d}f}{\mathrm{d}x} = f(x).$$

The equation can now be integrated directly with respect to x

$$\int g(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int f(x)\,\mathrm{d}x\,.$$

The general solution is then

$$G(y) = F(x) + \kappa.$$

where $G(y)$ and $F(y)$ are the integrals of $g(y), f(x)$ wrt y and x respectively and κ is an arbitrary constant.

5.1.2 Initial Conditions

A general solution for a given ODE contains arbitrary constants of integration $\kappa_1, \ldots, \kappa_n$.

To specify a unique or *particular* solution for a first-order ODE we require an initial condition (or boundary condition) (x_0, y_0) to calculate the value of κ.

5.1.3 First-Order Linear Differential Equations

The general form of a first order linear ODE is

$$\mathcal{L}[y] = f(x).$$

where $\mathcal{L}$ is the *linear differential operator* in which y is the dependent variable and x is the dependent variable.

Theorem 5.1.1. The linear differential operator $\mathcal{L}$ is a linear transformation, that is to say

$$\mathcal{L}[\lambda y] = \lambda \mathcal{L}[y]$$
$$\mathcal{L}[y + z] = \mathcal{L}[y] + \mathcal{L}[z]$$

Proof. Follows from the linearity of the derivative. □

In the case of linear first-order ODEs, the linear differential operator is

$$\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x} + p(x)y.$$

If the term $f(x)$ is absent, the equation is said to be *homogeneous.* With the $f(x)$ it is *inhomogenous.*

Case: Homogenous

In the homogenous case

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -p(x)y.$$

We note that the ODE is separable. So we have

$$\begin{aligned}\int \frac{\mathrm{d}y}{y} &= -\int p(x)\,\mathrm{d}x \\ \iff \ln y &= -\int p(x)\,\mathrm{d}x \\ \iff y &= \exp\left[-\int p(x)\,\mathrm{d}x\right]\end{aligned}$$

Let $P(x)$ be the integral of $p(x)$, then the general solution is given by

$$y = e^{-P(x)+\kappa} = \lambda e^{-P(x)}.$$

where λ is the arbitrary multiplicative factor e^{κ}.

Case: Inhomogenous

In the inhomogenous case

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x).$$

Consider multiplying by $\mu(x)$, so we have

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)p(x)y = \mu(x)f(x).$$

By the chain rule, we note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)y\right] = \mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu'(x)y.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)y\right] = \mu(x)f(x) \iff \mu'(x) = \mu(x)p(x)$$

So we have the solution

$$y = \frac{1}{\mu(x)}\int \mu(x)f(x)\,\mathrm{d}x + \kappa$$

provided $\mu(x)$ exists and $\mu(x) \neq 0$.

Now consider $\mu(x)$. Note that

$$\begin{aligned} \frac{\mu'(x)}{\mu(x)} &= p(x) \\ \iff \frac{\mathrm{d}}{\mathrm{d}x} \ln|\mu(x)| &= p(x) \\ \iff \ln|\mu(x)| &= \int p(x)\,\mathrm{d}x \\ \iff \mu(x) &= \pm \exp\left[\int p(x)\,\mathrm{d}x\right] \end{aligned}$$

Since we only need a single solution for $\mu(x)$, we generally use $\mu(x) = \exp\left[\int p(x)\,\mathrm{d}x\right]$. We note that $\forall x. \exp x > 0$, hence $\mu(x) \neq 0$.

We often refer to $\mu(x)$ as the *integrating factor*. We also note that there are an infinite number of integrating factors due to multiplicative constants.

5.1.4 Solving Differential Equations with Substitutions

Homogenous Differential Equations

These differential equations aren't to be confused with linear first-order homogenous equations...

Definition 5.1.2. (**Homogenoous Differential Equations**) A homogenous differential equation is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right).$$

Note that the ODE is invariant to the scale transformation

$$\mathbf{M} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

We solve this ODE using the substitution $y = u(x)x$. This gives us

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[u(x)x\right] = f(u) \\ \iff \frac{\mathrm{d}u}{\mathrm{d}x}x + u(x) &= f(x) \\ \iff \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{f(u) - u}{x} \\ \iff \int \frac{\mathrm{d}u}{f(u) - u} &= \int \frac{\mathrm{d}x}{x} \\ &= \ln|x| + \kappa \end{aligned}$$

Bernoulli Differential Equations

Definition 5.1.3. (**Bernoulli Differential Equations**) A Bernoulli differential equation is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n,$$

where $n \in \mathbb{Z}$.

We solve this ODE using the substitution $z = y^{1-n}$. So we have

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} &= (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= (1-n)y^{-n}\left[q(x)y^n + p(x)y\right] \\ &= (1-n)\left[q(x) - p(x)z\right] \\ \iff \frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)p(x)z &= (1-n)q(x) \end{aligned}$$

giving us a first-order linear inhomogenous equation.

Transformed Linear Differential Equations

Definition 5.1.4. (**Transformed Linear Differential Equations**) A transformed linear differential equation is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\alpha x + \beta y + \gamma\right),$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

We solve this ODE using the substitution $z = \alpha x + \beta y + \gamma$. So we have

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= \alpha + \beta\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \alpha + \beta f(z) \\ \iff \int \frac{\mathrm{d}z}{\alpha + \beta f(z)} &= \int \mathrm{d}x \\ &= x + \kappa\end{aligned}$$

Transformed Exponential Differential Equations

Definition 5.1.5. (**Transformed Exponential Differential Equations**) A transformed exponential differential equation is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = yf\left(e^{\alpha x}y^{\beta}\right),$$

where $\alpha, \beta \in \mathbb{R}$.

Let us first re-arrange the equation into a more suitable form

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= yf\left(e^{\alpha x}y^{\beta}\right) \\ \iff \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= f\left(e^{\alpha x}e^{\beta \ln y}\right) \\ \iff \frac{\mathrm{d}(\ln y)}{\mathrm{d}x} &= f\left(e^{\alpha x + \beta \ln y}\right)\end{aligned}$$

We now use the substitution $z = \ln y$, giving us

$$\begin{aligned}\iff \frac{\mathrm{d}z}{\mathrm{d}x} &= f(e^{\alpha x + \beta z}) \\ &= g(\alpha x + \beta z)\end{aligned}$$

where $g(z) = f(e^z)$. Note that this is a special case of the transformed linear differential equation.

5.2 Second-Order Equations

Definition 5.2.1. (**Second-Order Differential Equation**) A second order ODE is an equation of the form

$$F\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \frac{\mathrm{d}y}{\mathrm{d}x}, y, x\right) = 0,$$

or

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = G\left(\frac{\mathrm{d}y}{\mathrm{d}x}, y, x\right).$$

A second-order **linear** ODE is

$$p(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + r(x)y = f(x).$$

As with first-order linear ODEs, if $f(x)$ is absent, the equation is said to be *homogenous* (and vice-versa). We often write a second-order linear ODE using the *linear differential operator* $\mathcal{L}$,

$$\mathcal{L}[y] = f(x),$$

where

$$\mathcal{L} = p(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x)\frac{\mathrm{d}}{\mathrm{d}x} + r(x).$$

5.2.1 Principle Of Superposition

Theorem 5.2.1. (**The Principle Of Superposition for Second-Order Linear Differential Equations**) If $\mathcal{L}[y] = 0$ is a homogeneous second-order linear differential equation and $y = y_1(x)$ and $y = y_2(x)$ are both solutions, then for all $c_1, c_2 \in \mathbb{R}$,

$$y = c_1 y_1(x) + c_2 y_2(x),$$

is also a solution.

Proof. Let us assume that $y = y_1(x)$ and $y = y_2(x)$ are both solutions to $\mathcal{L}[y] = 0$. Let $c_1, c_2 \in \mathbb{R}$ be arbitrary. Let us consider $y = c_1 y_1(x) + c_2 y_2(x)$, giving us

$$\begin{aligned}\mathcal{L}[y] &= \mathcal{L}\left[c_1 y_1(x) + c_2 y_2(x)\right] \\ &= \mathcal{L}\left[c_1 y_1(x)\right] + \mathcal{L}\left[c_2 y_2(x)\right] \\ &= c_1 \underbrace{\mathcal{L}\left[y_1(x)\right]}_{0} + c_2 \underbrace{\mathcal{L}[y_2(x)]}_{0} \\ &= 0\end{aligned}$$

Hence $y = c_1 y_1(c) + c_2 y_2(c)$ is a solution to $\mathcal{L}[y] = 0$. ∎

Recall that an inhomogenous second-order ODE is $\mathcal{L}[y] = f(x)$ where $f(x) \neq 0$. Let $\mathcal{L}[y] = 0$ be the *corresponding homogeneous ODE*.

Definition 5.2.2. The *complementary function* $y_c(x)$ of the inhomogenous second-order ODE $\mathcal{L}[y] = f(x)$ is the general solution to $\mathcal{L}[y] = 0$, the corresponding homogeneous ODE.

Theorem 5.2.2. For an inhomogenous second-order ODE $\mathcal{L}[y] = f(x)$. If $y = y_1(x)$ and $y = y_2(x)$ are solutions to $\mathcal{L}[y] = f(x)$, then $y = y_1(x) - y_2(x)$ is a solution to $\mathcal{L}[y] = 0$, the corresponding homogeneous ODE.

Proof. Let $\mathcal{L}[y] = f(x)$ be an arbitrary inhomogenous second-order ODE. Let us assume that $y = y_1(x)$ and $y = y_2(x)$ are solutions to $\mathcal{L}[y] = f(x)$. We note that

$$\begin{aligned}
\mathcal{L}[y_1(x) - y_2(x)] &= \mathcal{L}[y_1(x)] - \mathcal{L}[y_2(x)] \\
&= f(x) - f(x) \\
&= 0
\end{aligned}$$

Hence $\mathcal{L}[y_1(x) - y_2(x)] = 0$. So $y_1(x) - y_2(x)$ is a solution to $\mathcal{L}[y] = 0$. $\square$

So by the Principle of Superposition, if $y = z_1(x)$ and $y = z_2(x)$ are solutions to $\mathcal{L}[y] = 0$, then it follows that for all $c_1, c_2 \in \mathbb{R}$,

$$y_1(x) - y_2(x) = c_1 z_1(x) + c_2 z_2(x),$$

is a solution to $\mathcal{L}[y] = 0$, the corresponding homogeneous ODE.

Theorem 5.2.3. For an inhomogenous second-order ODE $\mathcal{L}[y] = f(x)$, $y = y_c(x) + y_p(x)$ is a general solution where $y_c(x)$ is the *complementary function* and $y_p(x)$ is a solution to $\mathcal{L}[y] = f(x)$, referred to as the *particular integral*.

5.2.2 Homogenous Second-Order Differential Equations w/ Constant Coefficients

The second-order linear differential operator with constant coefficients is given by

$$\mathcal{L} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + a\frac{\mathrm{d}}{\mathrm{d}x} + b,$$

where $a, b \in \mathbb{R}$. Hence a homogeneous second-order differential equation with constant coefficients is an equation of the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a\frac{\mathrm{d}y}{\mathrm{d}x} + by = 0.$$

Let us consider the general solution $y = e^{\lambda x}$. So we have

$$\begin{aligned}
\mathcal{L}[e^{\lambda x}] = \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0 \\
\iff e^{\lambda x}\left[\lambda^2 + a\lambda + b\right] = 0 \\
\iff e^{\lambda x} = 0 \vee \lambda^2 + a\lambda + b = 0
\end{aligned}$$

Since $\forall x \in \mathbb{R}. e^{\lambda x} > 0$, it follow that

$$\begin{aligned}
\iff \lambda^2 + a\lambda + b = 0 \\
\iff \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}
\end{aligned}$$

The equation $\lambda^2 + a\lambda + b = 0$ is referred to as the *characteristic equation* or *auxillary equation.*

The equation yields 3 distinct mutually exclusive cases:

1. $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$
2. $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 = \lambda_2$
3. $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_1 = \lambda_2^*$

Case 1: Real and Distinct Roots

Let us assume that $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$. So we have $y = e^{\lambda_1 x}$ and $y = e^{\lambda_2 x}$ which are both solutions to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a\frac{\mathrm{d}y}{\mathrm{d}x} + b = 0.$$

By the Principle of Superposition, the general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x},$$

where $c_1, c_2 \in \mathbb{R}$.

Case 2: Repeated Real Roots

Let us assume that $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 = \lambda_2 = \lambda$. We note from the characteristic equation, that $\lambda = -a/2$. Hence $y = e^{-ax/2}$ is a solution to $\mathcal{L}[y] = 0$.

We will now show that $y = xe^{\lambda x}$ is also a solution to $\mathcal{L}[y] = 0$. We note that

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \lambda e^{\lambda x}(1 + \lambda x) + \lambda e^{\lambda x} = \lambda e^{\lambda x}(2 + \lambda x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= e^{\lambda x} + \lambda x e^{\lambda x} = e^{\lambda x}(1 + \lambda x) \end{aligned}$$

substituting this yields

$$\begin{aligned} \mathcal{L}[xe^{\lambda x}] &= \lambda e^{\lambda x}(2 + \lambda x) + e^{\lambda x}(1 + \lambda x) + bxe^{\lambda x} \\ &= e^{\lambda x} \left[(2 + \lambda x) + x\left(\lambda^2 + a\lambda + b\right)\right] \end{aligned}$$

Since $\lambda = -a/2$ and $\lambda^2 + a\lambda + b = 0$, we have

$$\mathcal{L}[xe^{\lambda x}] = e^{\lambda x} \cdot 0 = 0,$$

as required.

Hence by the Principle of Superposition, for all $c_1, c_2 \in \mathbb{R}$,

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x} = (c_1 + c_2 x)e^{\lambda x},$$

is a solution to $\mathcal{L}[y] = 0$.

Case 3: Complex Roots

Let us assume that $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_1 = \lambda_2^*$. Let $\lambda_1 = \alpha + \beta i$, where $\alpha, \beta \in \mathbb{R}$. Hence $\lambda_2 = \alpha - \beta i$. So we have the solutions $y_1 = e^{(\alpha+\beta i)x}$ and $y_2 = (\alpha - \beta i)$. Recall that Euler's Formula states that $e^{i\theta} = \cos\theta + i\sin\theta$. So

$$\begin{aligned} y_1 &= e^{\alpha x} e^{(\beta x)i} \\ &= e^{\alpha x}\left[\cos(\beta x) + i\sin(\beta x)\right] \\ y_2 &= e^{\alpha x}\left[\cos(\beta x) - i\sin(\beta x)\right] \end{aligned}$$

Theorem 5.2.4. Let $\mathcal{L}[y] = 0$ be an arbitrary homogeneous differential equation. If $y = u(x) + v(x)i$ is a solution to $\mathcal{L}[y] = 0$, then $y_1 = u(x)$ and $y_2 = v(x)$ are solutions to $\mathcal{L}[y] = 0$.

Proof. Let us assume that $y = u(x) + v(x)i$ is a solution to $\mathcal{L}[y] = 0$. By the linearity of $\mathcal{L}$, we have

$$\mathcal{L}\left[u(x) + v(x)i\right] = \mathcal{L}[u(x)] + i\mathcal{L}[v(x)] = 0.$$

Since $\mathcal{L}\left[u(x) + v(x)i\right] = 0$, we have

$$\begin{aligned}
\Re\left\{\mathcal{L}\left[u(x) + v(x)i\right]\right\} &= 0 \\
\Im\left\{\mathcal{L}\left[u(x) + v(x)i\right]\right\} &= 0
\end{aligned}$$

Equating coefficients yields

$$\begin{aligned}
\mathcal{L}[u(x)] &= 0 \\
\mathcal{L}[v(x)] &= 0
\end{aligned}$$

Hence $u(x)$ and $v(x)$ are solutions to $\mathcal{L}[y] = 0$. □

Hence, from the theorem above, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are solutions to $\mathcal{L}[y] = 0$. So by the Principle of Superposition, for all $c_1, c_2 \in \mathbb{R}$,

$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right],$$

is a solution to $\mathcal{L}[y] = 0$.

6 Multivariate Calculus

6.1 Partial Derivatives

Definition 6.1.1. (**Partial Derivative**) For a function of n variables $f(x_1, \dots, x_n)$. The partial derivative with respect to x_i is

$$\begin{aligned}\frac{\partial f}{\partial x_i} = \left(\frac{\partial f}{\partial x_i}\right)_{x_1,\dots,x_{i-1},x_{i+1},\dots,x_n} &= \partial_{x_i} f = f_{x_i} \\ = \lim_{\Delta x_i \to 0} \frac{f(x_1, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, \dots, x_n)}{\Delta x_i}\end{aligned}$$

Hence the partial derivative f_{x_i} is the rate of change of f along the x_i axis, and is calculated by differentiating f with respect to x_i while holding the variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ constant.

Theorem 6.1.1. (**Commutativity of Partial Differentation**) For a function f of n variables. For all $1 \leq i, j \leq n$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Proof. Let $1 \leq i, j \leq n$ be arbitrary. So we have

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) \\
&= \frac{\partial}{\partial x_i}\left(\lim_{\Delta x_j \to 0} \frac{f(x_1, \ldots, x_j + \Delta x_j, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta x_j}\right) \\
&= \lim_{\Delta x_i \to 0, \Delta x_j \to 0} \frac{1}{\Delta x_i}\left(\frac{\overbrace{f(\ldots, x_i + \Delta x_i, \ldots, x_j + \Delta x_j, \ldots)}^{d} - \overbrace{f(\ldots, x_i + \Delta x_i, \ldots)}^{c}}{\Delta x_j}\right. \\
&\qquad \left. - \frac{\overbrace{f(\ldots, x_j + \Delta x_j, \ldots)}^{b} - \overbrace{f(\ldots)}^{a}}{\Delta x_j}\right) \\
&= \lim_{\Delta x_i \to 0, \Delta x_j \to 0} \frac{d - c - b + a}{\Delta x_i \Delta x_j}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_j \partial x_i} &= \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) \\
&= \frac{\partial}{\partial x_j}\left(\lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta x_i}\right) \\
&= \lim_{\Delta x_i \to 0, \Delta x_j \to 0} \frac{1}{\Delta x_j}\left(\frac{\overbrace{f(\ldots, x_i + \Delta x_i, \ldots, x_j + \Delta x_j, \ldots)}^{d} - \overbrace{f(\ldots, x_j + \Delta x_j, \ldots)}^{c}}{\Delta x_i}\right. \\
&\qquad \left. - \frac{\overbrace{f(\ldots, x_i + \Delta x_i, \ldots)}^{b} - \overbrace{f(\ldots)}^{a}}{\Delta x_i}\right) \\
&= \lim_{\Delta x_i \to 0, \Delta x_j \to 0} \frac{d - c - b + a}{\Delta x_i \Delta x_j}
\end{aligned}
$$

Hence

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i},$$

as required. □

This generalizes for n partial derivatives by induction on n.

6.1.1 Partial Integration

Recall that for a function f of n variables, the partial derivative f_{x_i} is calculated by differentiating f with respect to x_i, while holding the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ constant. Sometimes it's useful to "partial integrate", often denoted

$$I_i = \int f(x_1, \ldots, x_n)\,\mathrm{d}x_i,$$

and is computed by integrating f with respect to x_i, while holding variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ constant.

For example, suppose

$$\frac{\partial f}{\partial x} = g(x, y),$$

then it follows that

$$f = \int g(x, y)\,\mathrm{d}x = F(x, y) + h(y),$$

where $h(y)$ is an arbitrary function of y (replaces the constant of integration).

6.1.2 Differentials

Recall that Taylor's Theorem states that the taylor expansion of f about the point $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a).$$

We may write this as

$$f(a+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(a),$$

where $x = a + h$. Using a first order approximation of the series, we have

$$\delta f = f(a+h) - f(a) \approx f'(a)\delta x,$$

where $\delta x = h$. Provided f is differentiable, we have

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0.$$

The exact equality is denoted using *differentials*, that is

$$\mathrm{d}f = f'(x)\,\mathrm{d}x,$$

where $\mathrm{d}f = \lim_{h\to 0} \delta f$ and $\mathrm{d}x = \lim_{h\to 0} \delta x$.

For $f(x, y)$, we have the first order approximation

$$f(x+h, y+k) \approx f(x, y) + \alpha h + \beta k.$$

Let us denote the error of the approximation by

$$\epsilon(h, k) = f(x+h, y+k) - f(x, y) - \alpha h - \beta k.$$

If

$$\lim_{\sqrt{h^2+k^2}\to 0} \frac{\epsilon(h, k)}{\sqrt{h^2+k^2}} = 0,$$

then $f(x, y)$ is differentiable at the point (x, y). So let us assume that $f(x, y)$ is differentiable at (x, y). Then we note that

$$\begin{aligned}\lim_{h\to 0} \frac{\epsilon(h, 0)}{h} &= \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} - \alpha \\ &= \frac{\partial f}{\partial x} - \alpha\end{aligned}$$

So

$$\begin{aligned}\alpha &= \frac{\partial f}{\partial x} - \underbrace{\lim_{h\to 0} \frac{\epsilon(h, 0)}{h}}_{0} \\ &= \frac{\partial f}{\partial x}\end{aligned}$$

Similarly, we have

$$\beta = \frac{\partial f}{\partial y}.$$

Substituting this yields

$$\begin{aligned} f(x+h,y+k) &= f(x,y) + \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k + \epsilon(h,k) \\ \iff \delta f &= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \epsilon(\delta x, \delta y) \end{aligned}$$

Taking limits yields

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y\,.$$

6.2 Theorems

6.2.1 The Chain Rule

Consider the function $f(x,y)$, where x,y are functions of u,v. So we have

$$f(u,v) = f(x(u,v), y(u,v)) = f(x,y).$$

Let us consider the differentials of x,y and f. This yields

$$\begin{aligned} \mathrm{d}x &= \left(\frac{\partial x}{\partial u}\right)_v \mathrm{d}u + \left(\frac{\partial x}{\partial v}\right)_u \mathrm{d}v \\ \mathrm{d}y &= \left(\frac{\partial y}{\partial u}\right)_v \mathrm{d}u + \left(\frac{\partial y}{\partial v}\right)_u \mathrm{d}v \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}f &= \left(\frac{\partial f}{\partial x}\right)_y \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)_x \mathrm{d}y \\ &= \left(\frac{\partial f}{\partial u}\right)_v \mathrm{d}u + \left(\frac{\partial f}{\partial v}\right)_u \mathrm{d}v \end{aligned}$$

Substituting $\mathrm{d}x$ and $\mathrm{d}y$ gives us

$$\begin{aligned} \mathrm{d}f &= \left(\frac{\partial f}{\partial x}\right)_y \left[\left(\frac{\partial x}{\partial u}\right)_v \mathrm{d}u + \left(\frac{\partial x}{\partial v}\right)_u \mathrm{d}v\right] + \left(\frac{\partial f}{\partial y}\right)_x \left[\left(\frac{\partial y}{\partial u}\right)_v \mathrm{d}u + \left(\frac{\partial y}{\partial v}\right)_u \mathrm{d}v\right] \\ &= \left[\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v\right] \mathrm{d}u + \left[\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u\right] \mathrm{d}v \end{aligned}$$

Equating the coefficients of du and dv produces

$$\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$$
$$\left(\frac{\partial f}{\partial v}\right)_u = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

This is known as the Multivariate Chain Rule.

A common and special case of the Chain Rule is to consider when $x = u$, that is

$$f(x, y) = f(x, y(x, v)) = f(x, v).$$

Since $x = u$, we have

$$\left(\frac{\partial x}{\partial x}\right)_v = 1$$
$$\left(\frac{\partial x}{\partial v}\right)_x = 0$$

So we have

$$\left(\frac{\partial f}{\partial x}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_v$$
$$\left(\frac{\partial f}{\partial v}\right)_x = \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_x$$

6.2.2 The Reciprocity and Cyclic Relation

Let us consider an equation of the form $F(x, y, z) = 0$. Then it follows that we have $x(y, z)$, $y(x, z)$ and $z(x, y)$. So

$$\mathrm{d}x = \left(\frac{\partial x}{\partial y}\right)_z \mathrm{d}y + \left(\frac{\partial x}{\partial z}\right)_y \mathrm{d}z$$
$$\mathrm{d}y = \left(\frac{\partial y}{\partial x}\right)_z \mathrm{d}x + \left(\frac{\partial y}{\partial z}\right)_x \mathrm{d}z$$
$$\mathrm{d}x = \left(\frac{\partial z}{\partial x}\right)_y \mathrm{d}x + \left(\frac{\partial z}{\partial y}\right)_x \mathrm{d}y$$

Now let us assume that $\left(\frac{\partial y}{\partial x}\right)_z \neq 0$, then it follows that

$$\begin{aligned}
\left[1\bigg/\left(\frac{\partial y}{\partial x}\right)_z\right]\mathrm{d}y &= \mathrm{d}x + \left[\left(\frac{\partial y}{\partial z}\right)_x\bigg/\left(\frac{\partial y}{\partial x}\right)_z\right]\mathrm{d}z \\
\iff \mathrm{d}x &= \left[1\bigg/\left(\frac{\partial y}{\partial x}\right)_z\right]\mathrm{d}y - \left[\left(\frac{\partial y}{\partial z}\right)_x\bigg/\left(\frac{\partial y}{\partial x}\right)_z\right]\mathrm{d}z
\end{aligned}$$

By equating coefficients of the dy term, we note that

$$\left(\frac{\partial x}{\partial y}\right)_z = 1\bigg/\left(\frac{\partial y}{\partial x}\right)_z$$

This is known as the *Cyclic Relation*. Similarly, we note that

$$\begin{aligned}
\left(\frac{\partial x}{\partial z}\right)_y &= -\left[\left(\frac{\partial y}{\partial z}\right)_x\bigg/\left(\frac{\partial y}{\partial x}\right)_z\right] \\
&= -1\bigg/\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_z \\
\iff \left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial y}{\partial x}\right)_z\left(\frac{\partial z}{\partial y}\right)_x &= -1
\end{aligned}$$

6.3 Exact Differentials

Definition 6.3.1. (**Differential Form**) We say that $\omega(x, y)$ is the differential form of x and y if and only if

$$\omega(x, y) = P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y,$$

where $P(x, y)$ and $Q(x, y)$ are arbitrary. For convenience, we often refer to $\omega(x, y)$ as ω.

Definition 6.3.2. (**Exact Differential**) The differential ω is an **exact differential** if and only if there exists a function $f(x, y)$ such that

$$\omega = \mathrm{d}f = \left(\frac{\partial f}{\partial x}\right)_y \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)_x \mathrm{d}y\,.$$

So for the differential form ω of x, y, ω is an exact differential if

$$\left(\frac{\partial f}{\partial x}\right)_y = P(x,y) \qquad \left(\frac{\partial f}{\partial y}\right)_x = Q(x,y).$$

By the commutativity of the partial derivative, we also note that

$$\left(\frac{\partial Q}{\partial x}\right)_y = \frac{\partial^2 f}{\partial x \partial y} = \left(\frac{\partial P}{\partial y}\right)_x .$$

6.4 Stationary Points

Recall that the Taylor Expansion of a function f about a point $\mathbf{x} = \mathbf{a}$ is given by

$$f(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x} - \mathbf{a}) \cdot \nabla f(\mathbf{a}) + [(\mathbf{x} - \mathbf{a}) \cdot (\mathbf{H}(\mathbf{a})(\mathbf{x} - \mathbf{a}))] + \cdots ,$$

where $\mathbf{H}(\mathbf{x})$ is the **Hessian Matrix**, given by

$$\mathbf{H}(\mathbf{x}) = \begin{bmatrix} \nabla f_x(\mathbf{x}) \\ \nabla f_y(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

Applying the transformation $\mathbf{x} = \mathbf{a} + \Delta\mathbf{x}$ yields

$$\Delta f = f(\mathbf{a} + \Delta\mathbf{x}) - f(\mathbf{a}) = \Delta\mathbf{x} \cdot \nabla f(\mathbf{a}) + \Delta\mathbf{x}^T \mathbf{H}(\mathbf{a}) \Delta\mathbf{x} + \cdots .$$

We can classify the point $\mathbf{a}$ by considering Δf. If Δf is a minima, then it follows that $\Delta f > 0$ for all $\Delta\mathbf{x} \in \mathbb{R}^n$. Similarly, for a maxima, $\Delta f < 0$. Hence we have a stationary point at $\mathbf{x} = \mathbf{a}$ (either a maxima, minima or saddle-point) if and only if

$$\nabla f(\mathbf{a}) = \mathbf{0}.$$

We note that a saddle-point is a stationary point that is neither a maxima nor a minima. IMAGE HERE

Classification

- The point $\mathbf{a}$ is a minima of f if $\Delta\mathbf{x}^T \mathbf{H}(\mathbf{a}) \Delta\mathbf{x} > 0$, that is

  $$f_{xx} f_{yy} > f_{xy}^2,$$

  with $f_{xx}, f_{yy} > 0$.

- The point $\mathbf{a}$ is a maxima of f if $\Delta\mathbf{x}^T\mathbf{H}(\mathbf{a})\Delta\mathbf{x} < 0$, that is

  $$f_{xx}f_{yy} > f_{xy}^2,$$

  with $f_{xx}, f_{yy} < 0$.

- The point $\mathbf{a}$ is a saddle-point of f if $f_{xx}f_{yy} < f_{xy}^2$

7 Vector Calculus

8 Linear Algebra

8.1 Vector Spaces

Definition 8.1.1. (**Vector Space**) A vector space over a field K is a set V with two operations $+ : V \times V \to V$ and $\cdot : K \times V \to V$ such that the following axioms holds:

(i) **Addition**:

(A1) **Associativity**: $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V . (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(A2) **Identity**: $\exists \mathbf{0} \in V . \forall \mathbf{v} \in V . \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$.

(A3) **Inverses**: $\forall \mathbf{v} \in V . \exists (-\mathbf{v}) \in V . \mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$.

(A4) **Commutativity**: $\forall \mathbf{u}, \mathbf{v} \in V . \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(ii) **Multiplication**:

(M1) **Associativity**: $\forall \lambda, \mu \in K, \mathbf{v} \in V . (\lambda \mu) \cdot \mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$

(M2) **Identity**: $\exists 1 \in K . \forall \mathbf{v} \in V . 1 \cdot \mathbf{v} = \mathbf{v}$.

(iii) **Distributivity**:

(D1) $\forall \lambda \in K, \mathbf{u}, \mathbf{v} \in V . \lambda \cdot (\mathbf{u} + \mathbf{v}) = \lambda \cdot \mathbf{u} + \lambda \cdot \mathbf{v}$.

(D2) $\forall \mu, \lambda \in K, \mathbf{v} \in V . (\lambda + \mu) \cdot \mathbf{v} = \lambda \cdot \mathbf{v} + \mu \cdot \mathbf{v}$.

8.1.1 Properties of Vector Spaces

8.1.2 Linear Independence and Spanning Sets

Definition 8.1.2. (**Linear Combination**) Let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of n-vectors, and $\lambda_1, \ldots, \lambda_n \in K$. A linear combination of V is

$$\mathbf{x} = \sum_{k=1}^{n} \lambda_k \cdot \mathbf{v}_k = (\lambda_k \cdot \mathbf{v}_k).$$

Definition 8.1.3. (**Linear Span**) Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq V$ of a space V over K. The span of S is defined as

$$\text{span}(S) = \{(\lambda_k \cdot \mathbf{v}_k) : \lambda_k \in K\}.$$

Suppose $S \subseteq V$ and $\forall \mathbf{v} \in V. \mathbf{v} \in \text{span}(S)$, then it follows that

$$V = \text{span}(S).$$

Definition 8.1.4. (**Linearly Dependent**) Let V be a space over K. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are linearly dependent if

$$\exists \lambda_1, \ldots, \lambda_n \in K. \sum_{k=1}^{n} \lambda_k \cdot \mathbf{v}_k = \mathbf{0} \implies \exists \lambda_k . \lambda_k \neq 0.$$

Definition 8.1.5. (**Linearly Independent**) Let V be a space over K, The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are linearly independent if

$$\exists \lambda_1, \cdots, \lambda_n \in K. \sum_{k=1}^{n} \lambda_k \cdot \mathbf{v}_k = \mathbf{0} \implies \forall \lambda_k . \lambda_k = 0.$$

8.1.3 Dimension and Basis

Definition 8.1.6. (**Basis**) The set $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subseteq V$ form a basis of V over K if

(i) B is a spans V, that is $V = \text{span}(B)$

(ii) B is a linearly independent set.

Given a basis $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, then for all $\mathbf{v} \in V$,

$$\exists \lambda_k \in K. \mathbf{v} = (\lambda_k \cdot \mathbf{e}_k).$$

The n-tuple $(\lambda_1, \ldots, \lambda_n)$ are said to be the *coordinates* of $\mathbf{v}$ wrt the basis B.

Definition 8.1.7. (**Dimension**) The cardinality of the spanning set S of a finite-dimensional space V is the dimension of V, written as $\dim(V) = |S|$

8.1.4 Linear Maps

Definition 8.1.8. (**Mapping**) A mapping $\mathbf{A}$ is a total function $\mathbf{A} : V \to U$ where V and U are vector spaces.

The map is said to be **linear** if the following holds:

- $\forall \mathbf{u}, \mathbf{v} \in V. \mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u}) + \mathbf{A}(\mathbf{v})$
- $\forall \lambda \in K, \mathbf{v} \in V. \mathbf{A}(\lambda \cdot \mathbf{v}) = \lambda \cdot \mathbf{A}(\mathbf{x})$

Definition 8.1.9. (**Null Space**) The null space (or kernel) of a linear mapping $\mathbf{A}$ is the set of vectors null($\mathbf{A}$) s.t

$$\forall \mathbf{x} \in \text{null}(\mathbf{A}). \mathbf{A}(\mathbf{x}) = \mathbf{0}.$$

8.2 Matrices

Definition 8.2.1. (**Matrix**) A $m \times n$ (m rows by n columns) matrix $\mathbf{A}$ is a $m \times n$ rectangular array of scalars.

Notation

- We represent the matrix $\mathbf{A}$ by bold font (upper case)
- The entry in row i and column j is a_{ij} in $\mathbf{A}$.
- We may write

$$\mathbf{A} = (a_{ij}), \qquad (\mathbf{A})_{ij} = a_{ij}, \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

8.2.1 Algebra with Matrices

Definition 8.2.2. (**Addition**) Let $\mathbf{A}$ and $\mathbf{B}$ be two $m \times n$ matrices. Their sum $\mathbf{S} = \mathbf{A} + \mathbf{B}$ is a $m \times n$ matrix where

$$\mathbf{C} = (a_{ij} + b_{ij}).$$

Definition 8.2.3. (**Scalar Multiplication**) Let $\mathbf{A}$ be a $m \times n$ matrix and $\lambda \in K$ be an arbitrary scalar. The scalar multiple $\lambda \cdot \mathbf{A}$ is a $m \times n$ matrix where

$$\lambda \cdot \mathbf{A} = (\lambda \cdot a_{ij}).$$

Definition 8.2.4. (**Matrix Multiplication**) Let $\mathbf{A}$ be a $m \times p$ and $\mathbf{B}$ be a $p \times n$ matrix. Their product $\mathbf{A} \cdot \mathbf{B}$ is a $m \times n$ matrix $\mathbf{C}$ where

$$c_{ij} = \sum_{k=1}^{p} a_{ik} \cdot b_{kj}.$$

- It is essential the number of columns p of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$. Otherwise their product is undefined.

Theorem 8.2.1. The set of $m \times n$ matrices $M_{m,n}$ over the field K form a vector space.

$$M_{m,n} = \left\{ \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{m1} & \cdots & \lambda_{mn} \end{bmatrix} : \lambda_{ij} \in K \right\}.$$

Definition 8.2.5. (**Transpose**) The transpose of a $m \times n$ matrix $\mathbf{M}$ is the $n \times m$ matrix $\mathbf{M}^T$ such that

$$(\mathbf{M}^T)_{ij} = (\mathbf{M})_{ji}.$$

Theorem 8.2.2. For all matrices $\mathbf{A}, \mathbf{B}$,

(i) $(\mathbf{A}^T)^T = \mathbf{A}$

(ii) $(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T$

Definition 8.2.6. (**Symmetric Matrix**) A matrix $\mathbf{M}$ is said to be symmetric if and only if $\mathbf{M}^T = \mathbf{M}$.

Definition 8.2.7. (**Anti-Symmetric**) A matrix $\mathbf{M}$ is said to be anti-symmetric if and only if $\mathbf{M}^T = -\mathbf{M}$.

- Given a $m \times n$ matrix $\mathbf{A}$, we can construct it's symmetric and anti-symmetric matrices, denoted $\mathbf{S}$ and $\overline{\mathbf{S}}$ respectively where

$$\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$
$$\overline{\mathbf{S}} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$$

Hence $\mathbf{A} = \mathbf{S} + \overline{\mathbf{S}}$.

Definition 8.2.8. (**Diagonal Matrix**) A square matrix $\mathbf{A}$ is said to be diagonal if and only if

$$\forall i, j. i \neq j \implies a_{ij} = 0.$$

Definition 8.2.9. (**Zero Matrix**) The $m \times n$ zero matrix $\mathbf{0}$ has $0_{ij} = 0$.

Definition 8.2.10. (**Identity**) The $n \times n$ identity $\mathbf{I}$ matrix satisfies

$$i_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

where δ_{ij} is the **Kronecker delta**.

Definition 8.2.11. (**Orthogonal Matrix**) A square $n \times n$ matrix $\mathbf{A}$ is said to be orthogonal if and only if $\mathbf{A} \cdot \mathbf{A}^T = \mathbf{A}^T \mathbf{A} = \mathbf{I}$.

Definition 8.2.12. (**Complex Conjugate**) The complex conjugate of a $m \times n$ matrix $\mathbf{A}$ is a $m \times n$ matrix $\mathbf{A}^*$ where

$$(\mathbf{A}^*) = (a_{ij}^*).$$

Definition 8.2.13. (**Hermitian Conjugate**) The hermitian conjugate of a matrix $m \times n$ matrix $\mathbf{A}$ is a $n \times m$ matrix $\mathbf{A}^\dagger$ where

$$\mathbf{A}^\dagger = \left(\mathbf{A}^T\right)^* = (\mathbf{A}^*)^T = (a_{ji}^*).$$

Definition 8.2.14. (**Trace**) The trace of a $n \times n$ square matrix $\mathbf{A}$ is

$$\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}.$$

Theorem 8.2.3. For all matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, the trace satisfies

$$\text{trace}(\mathbf{A}_1 \cdot \mathbf{A}_2 \cdots \mathbf{A}_n) = \text{trace}(\pi(\mathbf{A}_1) \cdot \pi(\mathbf{A}_2) \cdots \pi(\mathbf{A}_n)),$$

where π is a cyclic permutation.

8.2.2 Scalar Product

- We may define the **scalar** product of two vectors $\mathbf{x}, \mathbf{y}$ using matrix operations.
- For real column vectors $\mathbf{x}, \mathbf{y}$

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i = \mathbf{x}^T \mathbf{y}.$$

- For complex column vectors $\mathbf{x}, \mathbf{y}$

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i^* y_i = \mathbf{x}^\dagger \mathbf{y}.$$

- Two column vectors are said to be orthogonal if and only if $\mathbf{x} \cdot \mathbf{y} = 0$
- An orthonormal basis $B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ satisfies

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}.$$

8.2.3 Relation to Linear Equations

- Matrices can be used to compactly write linear equations.
- Let $\mathbf{A}$ be a $m \times n$ matrix, $\mathbf{x}$ be a n-dimensional column vector and $\mathbf{b}$ be a m-dimensional column vector, then

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b},$$

is equivalent to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

8.3 Determinants

Definition 8.3.1. (**2×2 Determinant**) The determinant of a 2×2 matrix $\mathbf{A}$ is

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Similarly, for a 3×3 matrix, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

8.3.1 Minor and Cofactor Entries

Definition 8.3.2. (**Minor Entry**) Suppose $\mathbf{A}$ is a $n \times n$ matrix, the **minor entry** M_{ij} is the $(n-1) \times (n-1)$ submatrix of $\mathbf{A}$ after deleting the ith row and jth column from $\mathbf{A}$.

- The determinant $|M_{ij}|$ is the **minor** of the element a_{ij} of $\mathbf{A}$

Definition 8.3.3. (**Cofactor Entry**) The **cofactor entry** of a_{ij}, denoted C_{ij} is

$$C_{ij} = (-1)^{i+j}|M_{ij}|.$$

- Cofactor signs in the matrix space yields

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ \vdots & & & \vdots \end{bmatrix}.$$

Definition 8.3.4. (**Adjoint**) The adjoint matrix of $\mathbf{A}$ is

$$(\text{adj}\,\mathbf{A})_{ij} = C_{ji}.$$

8.3.2 Permutations and determinants

Definition 8.3.5. (**Permutation**) A permutation of S is a bijection $\sigma : S \to S$.

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}.$$

- Any permutation σ on $[1, n]$ is expressed as

$$\begin{bmatrix} \sigma(1) \\ \vdots \\ \sigma(n) \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 0 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{e}^T_{\sigma(1)} & \longrightarrow \\ \vdots & \\ \mathbf{e}^T_{\sigma(n)} & \longrightarrow \end{bmatrix}}_{M_\sigma} \begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix}.$$

$$\forall \mathbf{e}_{\sigma(i)}, \mathbf{e}_{\sigma(j)}. \mathbf{e}^T_{\sigma(i)} \mathbf{e}_{\sigma(j)} = \delta_{ij}.$$

- M_σ is orthogonal
- A permutation is even if an even number of transpositions (swaps) occur (vice versa).
- The sign of a permutation is

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}.$$

- Note that $\operatorname{sgn}(\sigma) = \det M_\sigma$

Definition 8.3.6. For a $n \times n$ matrix $\mathbf{A}$ is

$$\det \mathbf{A} = \sum_\sigma \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)},$$

where σ is a permutation of $[1, n]$

8.3.3 Cofactor Expansion Theorem

Theorem 8.3.1. (**Cofactor Exapnsion Theorem**) Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ matrix.

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{ij} C_{ij} \text{ for } 1 \leq i \leq n,$$

or

$$\det \mathbf{A} = \sum_{i=1}^{n} a_{ij} C_{ij} \text{ for } 1 \leq j \leq n,$$

referred to as cofactor expansions along a row or column respectively.

Proof. (sketch)

- Let $\mathbf{A} = (a_{ij})$. Let $\mathbf{A}'_{ij}$ be $\mathbf{A}$ where ith row is replaced with $\mathbf{e}_j$. So

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{ij} \det \mathbf{A}'_{ij}.$$

- Permuting row i of $\mathbf{A}'_{ij}$ to row 1 ($i-1$ swaps) is $\mathbf{A}''_{ij}$, so

$$\det \mathbf{A}''_{ij} = (-1)^{i-1} \det \mathbf{A}'_{ij}.$$

  and permuting column j of $\mathbf{A}''_{ij}$ to column 1 ($j-1$ swaps) is $\mathbf{A}^{(3)}_{ij}$, so

$$\det \mathbf{A}^{(3)}_{ij} = (-1)^{j-1} \det \mathbf{A}''_{ij} = (-1)^{i+j} \det \mathbf{A}'_{ij}.$$

- Since row 1 is $\mathbf{e}_1$,

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}^{(3)}_{ij}.$$

□

Corollary 8.3.1.1.

$$\forall 1 \leq i \leq n. \sum_{k=1}^{n} a_{ik} C_{jk} = \delta_{ij} \det \mathbf{A}$$

$$\forall 1 \leq j \leq n. \sum_{k=1}^{n} a_{kj} C_{ki} = \delta_{ji} \det \mathbf{A}$$

- A strategy for cofactor expansion is to select the row or column that contains the most zeros.

8.3.4 Properties of Determinants

1. Interchanging any two rows or columns of a matrix changes the sign of it's determinant.

   Proof. Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ matrix. Let $1 \leq r < s \leq n$. Let ρ be a permutation which transposes r and s. Hence ρ is odd, so $\operatorname{sgn}(\rho) = -1$. Let $\mathbf{A}' = (a'_{ij})$ be $\mathbf{A}$ w/ rows r and s transposed. By definition,

$$\begin{aligned}
\det \mathbf{A}' &= \sum_{\sigma} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a'_{i\sigma(i)} \right) \\
&= \sum_{\sigma} \left(\operatorname{sgn}(\rho) \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\rho(i)\sigma(i)} \right) \\
&= \operatorname{sgn}(\rho) \sum_{\sigma} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\rho(i)\sigma(i)} \right) \\
&= -\det \mathbf{A}
\end{aligned}$$

   $\square$

2. $|\mathbf{A}| = 0$ if any two rows or columns are the same.

   Proof. Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ matrix. Suppose $1 \leq r < s \leq n$ are duplicate rows. Swapping rows r and s yields the matrix $\mathbf{A}' = (a'_{ij})$. From (1), $\det \mathbf{A}' = -\det \mathbf{A}$. However, $\mathbf{A} = \mathbf{A}'$, hence

$$2 \det \mathbf{A}' = \det \mathbf{A} - \det \mathbf{A} = 0 \implies \det \mathbf{A}' = 0.$$

   $\square$

3. The matrix obtained by multiplying all the elements of any one row (or column) of $\mathbf{A}$ by $\lambda \in K$ has the determinant $\lambda|\mathbf{A}|$.

Proof. Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ matrix. Let $1 \leq k \leq n$. Let $\mathbf{A}' = (a'_{ij})$ be $\mathbf{A}$ w/ row k multiplied by scalar λ.

$$\begin{aligned}
\det \mathbf{A}' &= \sum_{\sigma} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a'_{i\sigma(i)} \right) \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots \lambda a_{k\sigma(k)} \dots a_{n\sigma(n)} \\
&= \lambda \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{k\sigma(k)} \dots a_{n\sigma(n)} \\
&= \lambda \det \mathbf{A}
\end{aligned}$$

□

4. Adding a multiple of one row (column) to another row (column) leaves the determinant unchanged. (Gaussian elimination).

Proof. Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ matrix. Suppose $1 \leq r \neq s \leq n$. Let

$$(\mathbf{A'})_{ij} = a'_{ij} = \begin{cases} a_{ij} & i \neq r \\ a_{rj} + \lambda a_{sj} & i = r \end{cases}.$$

$$\begin{aligned}
\det \mathbf{A}' &= \sum_{\sigma} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a'_{i\sigma(i)} \right) \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots (a_{r\sigma(r)} + \lambda a_{s\sigma(r)}) \dots a_{s\sigma(s)} \dots a_{n\sigma(n)} \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{r\sigma} \dots a_{n\sigma(n)} + \lambda \underbrace{\sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{s\sigma(r)} \dots a_{s\sigma(s)} \dots a_{n\sigma(n)}}_{\text{duplicate row } s} \\
&= \det \mathbf{A} + \lambda \cdot 0 = \det \mathbf{A}
\end{aligned}$$

□

5. $\mathbf{T}_n$ is a $n \times n$ (upper or lower) triangular matrix. $\det \mathbf{T}_n = \prod_{i=1}^{n} a_{ii}$.

Proof. (**Induction on** n)
Base Case. *trivial*
Inductive Step. We wish to show $\forall n \in \mathbb{Z}^+ . P(n) \implies P(n+1)$. Let $n \in \mathbb{Z}^+$ be arbitrary. Let us assume that $P(n)$ holds, that is

$$\det \mathbf{T}_n = \prod_{i=1}^{n} a_{ii}.$$

Let $\mathbf{T}_{n+1}$ be arbitrary. By cofactor expansion theorem (on $n+1$ row)

$$\begin{aligned}
\det \mathbf{T}_{n+1} &= \sum_{j=1}^{n} a_{(n+1)j} C_{(n+1)j} \\
&= a_{(n+1)(n+1)} C_{(n+1)(n+1)} & \forall k < n+1 . a_{(n+1)k} = 0 \\
&= a_{(n+1)(n+1)} (-1)^{n+1+n+1} \det \mathbf{T}_n \\
&= \prod_{i=1}^{n+1} a_{ii}
\end{aligned}$$

By the Principle of Mathematical Induction, the statement holds for all $n \in \mathbb{Z}^+$. □

6. $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$.

Proof. Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ matrices. Let $\mathbf{A}', \mathbf{B}'$ be the upper triangular forms of $\mathbf{A}, \mathbf{B}$. Note that the product of a upper triangular matrices $\mathbf{A}'\mathbf{B}'$ is an upper triangular matrix, with diagonal elements $a'_{ii} b'_{ii}$. So

$$\begin{aligned}
\det \mathbf{AB} &= \det \mathbf{A}'\mathbf{B}' \\
&= \prod_{i=1}^{n} a_{ii} b_{ii} \\
&= \prod_{i=1}^{n} a_{ii} \prod_{i=1}^{n} b_{ii} \\
&= (\det \mathbf{A}')(\det \mathbf{B}') \\
&= (\det \mathbf{A})(\det \mathbf{B})
\end{aligned}$$

□

7. $\det \mathbf{A}^T = \det \mathbf{A}$

Proof. Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ matrix. $\mathbf{A}^T = (a'_{ij}) = (a_{ji})$. So

$$\begin{aligned}\det \mathbf{A}^T &= \sum_{\sigma} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a'_{i\sigma(i)} \right) \\ &= \sum_{\sigma} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i)i} \right) \\ &= \det \mathbf{A}\end{aligned}$$

□

8.4 Inverses

Definition 8.4.1. (**Inverse**) The inverse of $\mathbf{A} = (a_{ij})$ of order $n \times n$ is $\mathbf{A}^{-1}$ of order $n \times n$ s.t

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}.$$

Theorem 8.4.1. If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}^{-1}$ is unique.

$$\forall \mathbf{A}_1^{-1}, \mathbf{A}_2^{-1}. \mathbf{A}_1^{-1} \wedge \mathbf{A}_2^{-1} \text{ are inverses of } \mathbf{A} \implies \mathbf{A}_1^{-1} = \mathbf{A}_2^{-1}.$$

Proof. Let $\mathbf{A}_1^{-1}, \mathbf{A}_2^{-1}$ be arbitrary. Let us assume that $\mathbf{A}_1^{-1}$ and $\mathbf{A}_2^{-1}$ are inverses of $\mathbf{A}$. That is $\mathbf{A}_1^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}_1^{-1} = \mathbf{I}$ and $\mathbf{A}_2^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}_2^{-1} = \mathbf{I}$. So

$$\begin{aligned} & \mathbf{A}_1^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}_2^{-1}\mathbf{A} \\ \iff & \mathbf{A}_1^{-1}(\mathbf{A}\mathbf{A}_1^{-1}) = \mathbf{A}_2^{-1}(\mathbf{A}\mathbf{A}_1^{-1}) \\ \iff & \mathbf{A}_1^{-1} = \mathbf{A}_2^{-1}\end{aligned}$$

□

8.4.1 Construction of Inverses

- By the Cofactor Expansion Theorem, for $n \times n$ $\mathbf{A} = (a_{ij})$

$$\begin{aligned}\forall 1 \leq i, j \leq n. \sum_{k=1}^{n} a_{ik} C_{jk} &= \delta_{ij} \det \mathbf{A} \\ \iff \mathbf{A}(\operatorname{adj} \mathbf{A}) &= \mathbf{I}(\det \mathbf{A}) \\ \iff \mathbf{A}\left(\frac{1}{\det \mathbf{A}} \operatorname{adj} \mathbf{A}\right) &= \mathbf{I}\end{aligned}$$

  Hence $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj} \mathbf{A}$.

- $\mathbf{A}$ is said to be singular if $\det \mathbf{A} = 0$.

- $\mathbf{A}^{-1}$ is undefined for singular matrices.

8.4.2 Solutions to Linear Equations

Cramer's Rule

- If $\mathbf{A}\mathbf{x} = \mathbf{y}$ and $\det \mathbf{A} \neq 0$, then

$$\begin{aligned}\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} &= \frac{1}{\det \mathbf{A}}(\operatorname{adj} \mathbf{A})\mathbf{y} \\ \iff x_i &= \frac{1}{\det \mathbf{A}} \sum_{k=1}^{n} C_{ki} y_k \\ &= \frac{\det \mathbf{A}_i}{\det \mathbf{A}} && \text{Cofactor expansion theorem}\end{aligned}$$

  where

$$(\mathbf{A}_i)_{lm} = \begin{cases} a_{lm} & \text{if } m \neq i \\ y_l & \text{otherwise} \end{cases}$$

Uniqueness of Solutions

- $\mathbf{A}\mathbf{x} = \mathbf{y}$ contains **redundant** equations iff the equations are **linearly dependent** (isomorphism between linear equation vector space and K^n).

- Equations can be inconsistent e.g.

$$\begin{aligned} x + y &= 2 \\ 2x + 2y &= 5 \end{aligned}$$

  Inconsistent if LHS are linearly dependent, but RHS doesn't correspond to scale factor λ.

$\mathbf{A}$ is a $m \times n$ matrix, $\mathbf{x}$ is $n \times 1$ and $\mathbf{y}$ is $m \times 1$. When solving $\mathbf{Ax} = \mathbf{y}$.

(C1) $m < n$. $\mathbf{Ax} = \mathbf{y}$ is underdetermined. If inconsistent no solutions, otherwise a **family** of solutions.

(C2) $m > n$. $\mathbf{Ax} = \mathbf{y}$. Two cases:

  (a) System is **inconsistent** $\implies$ **no** solution.

  (b) System contains redundant equations. Then discard them, reduce m to other cases.

(C3) $m = n$. Two cases

  (a) $\det \mathbf{A} \neq 0$. **unique** solution.

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}.$$

  rows / columns are **linearly independent**.

  (b) $\det \mathbf{A} = 0, \mathbf{y} = 0$.

$$\mathbf{A} = \begin{bmatrix} \mathbf{e}_1 & \longrightarrow \\ \vdots & \\ \mathbf{e}_n & \longrightarrow \end{bmatrix}.$$

  $B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is linearly dependent. $S \subseteq B$ such that S is linearly independent. S spans subspace $V \subseteq K^n$ w/ $\dim V = |S|$. System of equations is

$$\forall 1 \leq i \leq n . \mathbf{e}_i \cdot \mathbf{x} = 0.$$

  Special case: $n = 3$. Suppose $|S| = 2$ and $\mathbf{e}_1, \mathbf{e}_2$ are linearly independent. Construct $\mathbf{z} \notin V$ s.t.

$$\begin{aligned} \mathbf{Az} &= \mathbf{0} \\ \mathbf{z} &= \mathbf{e}_1 \times \mathbf{e}_2 \end{aligned}$$

So

$$\mathbf{x} = \lambda\mathbf{z} = \lambda(\mathbf{e}_1 \times \mathbf{e}_2).$$

(c) $\det \mathbf{A} = 0, \mathbf{y} \neq \mathbf{0}$.

$$\mathbf{A} = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ \downarrow & & \downarrow \end{bmatrix}.$$

$B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is linearly dependent. $S \subseteq B$ such that S is linearly independent. S spans subspace $V \subseteq K^n$ w/ $\dim V = |S|$.

$$\mathbf{y} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.$$

If $\mathbf{y} \notin V \implies$ no solution for $\mathbf{x}$.

Special case: $n = 3$. Suppose $|S| = 2$ and $\mathbf{e}_1, \mathbf{e}_2$ are linearly independent. So

$$\mathbf{y} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3(\mathbf{e}_1 \times \mathbf{e}_3).$$

If $x_3 \neq 0 \implies \mathbf{y} \notin V$. Hence $\mathbf{y} \in V$ iff

$$\mathbf{y} \cdot (\mathbf{e}_1 \times \mathbf{e}_2) = 0.$$

Find $\mathbf{x}_0$ s.t $\mathbf{A}\mathbf{x}_0 = \mathbf{y}$. Solution not unique since $\mathbf{A}\mathbf{z} = \mathbf{0}$ has solutions.

General solutions of form

$$\mathbf{x} = \mathbf{x}_0 + \sum_k \lambda_k \mathbf{z}_k.$$

8.5 Eigenvalues and Eigenvectors

Definition 8.5.1. (**Eigenvalues and Eigenvectors**) Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ matrix. If

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

then λ is an eigenvalue of $\mathbf{A}$ w/ a respective eigenvector $\mathbf{x} \neq \mathbf{0}$.

1. There are an infinite number of respective eigenvectors $\mu\mathbf{x}$ for an eigenvalue λ

$$\mathbf{A}(\mu\mathbf{x}) = \lambda(\mu\mathbf{x}).$$

2. We write
$$\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\ \iff (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= 0 \end{aligned}$$
Hence null space of $\det(\mathbf{A} - \lambda\mathbf{I}) \neq \emptyset \implies \det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
$$\det(\mathbf{A} - \lambda\mathbf{I}) \equiv P_{\mathbf{A}}(\lambda) = 0.$$
where $P_{\mathbf{A}}$ is the characteristic polynomial of $\mathbf{A}$.

3. $P_{\mathbf{A}}$ has degree $n \implies n$ eigenvalues for $\mathbf{A}$. (Inductive proof on n using Cofactor Expansion Theorem)

Theorem 8.5.1. Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ matrix.
$$\begin{aligned} \operatorname{trace}(\mathbf{A}) &= \sum_{i=1}^{n} \lambda_i \\ \det \mathbf{A} &= \prod_{i=1}^{n} \lambda_i \end{aligned}$$

Proof. Since $P_{\mathbf{A}}$ is a n-order polynomial and by Vieta's formula, we have
$$\begin{aligned} P_{\mathbf{A}}(\lambda) &= (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda) \\ &= (-1)^n \lambda^n + (-1)^{n-1} \left(\sum_{i=1}^{n} \lambda_i \right) \lambda^{n-1} + \cdots + \prod_{i=1}^{n} \lambda_i \end{aligned}$$
Let us consider the coeff of λ^{n-1}. By the definition, we have
$$P_{\mathbf{A}}(\lambda) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a'_{i\sigma(i)}$$
The λ^{n-1} coeff (and λ^n) is in the summation term occurs in the term w/ $\sigma(i) = i$, that is
$$(-1)^n \lambda^n + (-1)^{n-1} b_{n-1} \lambda^{n-1} + \cdots = (a_{11} - \lambda) \cdots (a_{nn} - \lambda).$$
By Vieta's formula,
$$\sum_{i=1}^{n} a_{ii} = -\frac{(-1)^{n-1} b_{n-1}}{(-1)^n} = b_{n-1} = \sum_{i=1}^{n} \lambda_i.$$

Hence, by definition,

$$\operatorname{trace} \mathbf{A} = \sum_{i=1}^{n} \lambda_i.$$

For $\det \mathbf{A} = \prod_{i=1}^{n} \lambda_i$, consider $\lambda = 0$, so we have

$$\det \mathbf{A} = \det(\mathbf{A} - 0\mathbf{I}) = P_{\mathbf{A}}(0) = \prod_{i=1}^{n} \lambda_i.$$

□

8.5.1 Hermitian Matrices

- $\mathbf{A}$ is said to be a **Hermitian matrix** iff $\mathbf{A}^\dagger = \mathbf{A}$. (A real symmetric matrix).

Theorem 8.5.2. Every hermitian matrix $\mathbf{A}$ has real eigenvalues.

Proof. Let $\mathbf{A}$ be an arbitrary hermitian matrix. By definition,

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda \mathbf{x} \\ \iff \mathbf{x}^\dagger \mathbf{A}\mathbf{x} &= \mathbf{x}^\dagger \lambda \mathbf{x} \\ &= \lambda(\mathbf{x}^\dagger \mathbf{x}) \end{aligned}$$

Note that $(\mathbf{x}^\dagger \mathbf{A}\mathbf{x})^\dagger = \mathbf{x}^\dagger \mathbf{A}^\dagger (\mathbf{x}^\dagger)^\dagger$. As $\mathbf{A}$ is hermitian and $(\mathbf{x}^\dagger)^\dagger = \mathbf{x}$. Then

$$(\mathbf{x}^\dagger \mathbf{A}\mathbf{x})^\dagger = \mathbf{x}^\dagger \mathbf{A}\mathbf{x}.$$

So $\mathbf{x}^\dagger \mathbf{A}\mathbf{x}$ is hermitian. Note that $\mathbf{x}^\dagger \mathbf{A}\mathbf{x} = \mu$ (a 1×1 matrix) and $\mathbf{x}^\dagger \mathbf{x} = 1$ (eigenvectors are normalized). Hence

$$\mu = \lambda.$$

However, $\mathbf{x}^\dagger \mathbf{A}\mathbf{x}$ is hermitian, so $\mu = \mu^*$ and $\mu^* = \lambda^*$, hence

$$\lambda = \lambda^*.$$

So $\lambda \in \mathbb{R}$. □

Theorem 8.5.3. Eigenvectors of hermitian matrices $\mathbf{A}$ are orthogonal.

Proof. Let $\mathbf{A}$ be an arbitrary hermitian matrix. By definition,

$$\begin{aligned} \mathbf{A}\mathbf{x}_1 &= \lambda_1 \mathbf{x}_1 \\ \iff \mathbf{x}_2^\dagger \mathbf{A}\mathbf{x}_1 &= \lambda_1 (\mathbf{x}_2^\dagger \mathbf{x}_1) \end{aligned}$$

and

$$\begin{aligned} \mathbf{A}\mathbf{x}_2 &= \lambda_2 \mathbf{x}_2 \\ \iff \mathbf{x}_2^\dagger \mathbf{A} &= \mathbf{x}_2^\dagger \lambda_2 \\ \iff \mathbf{x}_2^\dagger \mathbf{A}\mathbf{x}_1 &= \lambda_2 (\mathbf{x}_2^\dagger \mathbf{x}_1) \end{aligned}$$

Hence

$$\begin{aligned} \lambda_2 (\mathbf{x}_2^\dagger \mathbf{x}_1) &= \lambda_1 (\mathbf{x}_2^\dagger \mathbf{x}_1) \\ \iff (\lambda_2 - \lambda_1)(\mathbf{x}_2^\dagger \mathbf{x}_1) &= 0 \end{aligned}$$

Since $\lambda_2 \neq \lambda_1 \implies \mathbf{x}_2^\dagger \mathbf{x}_1 = 0$. □

- If repeated roots in $P_{\mathbf{A}}(\lambda)$ then construct the respective eigenvectors for the corresponding repeated eigenvalues using other eigenvectors. Note that in the general case eigenvectors necessarily must be linearly independent (non-zero orthogonal $\implies$ linear independence).
- Normalize $\mathbf{x}_\lambda$. The spectrum of eigenvectors forms an orthonormal basis.

8.5.2 Diagonalization of Hermitian Matrices

- Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ hermitian matrix w/ eigenvector/value pairs $(\lambda_1, \mathbf{x}_1), \ldots$
- Construct
$$\mathbf{S} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$
Note that
$$\mathbf{S}^T\mathbf{S} = \begin{bmatrix} \mathbf{x}_1 & \rightarrow \\ \vdots & \\ \mathbf{x}_n & \rightarrow \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \mathbf{I}.$$
So $\mathbf{S}$ is orthogonal $\implies \det \mathbf{S} = 1$.

- Hence

$$\mathbf{D} = \mathbf{S}^T\mathbf{A}\mathbf{S} = \begin{bmatrix} \mathbf{x}_1 & \rightarrow \\ \vdots & \\ \mathbf{x}_n & \rightarrow \end{bmatrix} \begin{bmatrix} \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_n\mathbf{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

  $\mathbf{D}$ has the same eigenvalues as $\mathbf{A}$ but the eigenvectors are the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, where $(\mathbf{e}_i)_k = \delta_{ik}$.

- So

$$\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^T.$$

  and

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{y} \\ \mathbf{S}\mathbf{D}\mathbf{S}^T\mathbf{x} &= \mathbf{y} \\ (\mathbf{D})(\mathbf{S}^t\mathbf{x}) &= (\mathbf{S}^T\mathbf{y}) \\ \mathbf{D}\mathbf{x}' &= \mathbf{y}' \end{aligned}$$

  where $\mathbf{x}' = \mathbf{S}^T\mathbf{x}$ is $\mathbf{x}$ in the basis $\mathbf{x}_1, \ldots, \mathbf{x}_n$

9 Fourier Series

9.1 Periodic Functions

Definition 9.1.1. (**Periodic Function**) A total function f is said to be periodic with period T if and only if

$$\forall t. f(t+T) = f(t).$$

- We note that if T is the period of f then nT, $n \in \mathbb{Z}$. Hence we define T to be the "least period".

9.2 Orthogonality

- Recall that the vectors $\mathbf{x}, \mathbf{y}$ are said to be orthogonal if $\mathbf{x} \cdot \mathbf{y} = 0$.
- Consider the function $f : [a, b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$. We may think of f as an "infinite" vector $\mathbf{x}$ s.t

  $$\mathbf{x} = \lim_{n \to \infty} \left(f(x_1), f(x_2), \ldots, f(x_n)\right).$$

  where $a = x_1 < x_2 < \cdots < x_n = b$. We define the scalar product of two functions $f, g : [a, b] \to \mathbb{R}$ as the scalar products of their "infinite" vectors $\mathbf{x}, \mathbf{y}$, that is

  $$\begin{aligned}
  \langle f, g \rangle = \mathbf{x} \cdot \mathbf{y} &= \lim_{n \to \infty} \left(f(x_1), f(x_2), \ldots, f(x_n)\right) \cdot \left(g(x_1), g(x_2), \ldots, g(x_n)\right) \\
  &= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) g(x_i) \\
  &\to \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) g(x_i) \, \mathrm{d}x \\
  &= \int_a^b f(x) g(x) \, \mathrm{d}x
  \end{aligned}$$

Definition 9.2.1. (**Inner Product of Functions**) The scalar product (or inner product in this context) of two functions f, g defined on $[a, b]$, denoted $\langle f, g\rangle$ is defined as

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x\,.$$

- Hence two functions $f, g : [a, b] \to \mathbb{C}$ are said to be *orthogonal* if and only if

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x = 0.$$

9.3 Systems of Functions

Definition 9.3.1. (**Orthogonal System of Functions**) A set of functions $\{f_1, f_2, \ldots\}$ defined on the interval $[a, b]$ is said to be an **orthogonal system of functions** if

$$\forall i \neq j.\, \langle f_i, f_j\rangle = 0.$$

- A system of functions $\{f_1, f_2, \ldots\}$ is said to be **orthonormal** if the system is orthogonal and $\forall i \geq 1.\, \langle f_i, f_i\rangle = 1$
- Hence if $\{f_1, f_2, \ldots\}$ is an orthogonal system of functions on $[a, b]$, then we can obtain an orthonormal system of functions on $[a, b]$ to be

$$\left\{\frac{f_1}{\|f_1\|}, \ldots, \frac{f_n}{\|f_n\|}\right\},$$

  where the **norm** is defined as $\|f\| = \sqrt{\langle f, f\rangle}$

Theorem 9.3.1. Let $\mathcal{S} = \{f_1, f_2, \ldots\}$ be an orthonormal system of functions on $[a, b]$ and let f be a function defined on $[a, b]$.

Let for all $i \in \{1, \ldots, n\}$, $c_i = \langle f, f_i\rangle$ and let $d_1, \ldots, d_n \in \mathbb{C}$ Define $f_\sim(x) = \sum_{k=1}^n c_k f_k(x)$ and $g(x) = \sum_{k=1}^n d_k f_k(x)$.

Then $\|f(x) - f_\sim(x)\| \leq \|f(x) - g(x)\|$. We say that $f_\sim$ is the **best approximation** of f with respect to $\mathcal{S}$.

Proof. Let us define $f_\sim$ and g as above. We now consider $\|f(x)-g(x)\|^2$. So we have

$$\begin{aligned}
\|f(x)-g(x)\|^2 &= \langle f(x)-g(x), f(x)-g(x)\rangle \\
&= \int_a^b (f(x)-g(x))\overline{(f(x)-g(x))}\,\mathrm{d}x \\
&= \int_a^b f(x)\overline{f(x)} - f(x)\overline{g(x)} - \overline{f(x)}g(x) + g(x)\overline{g(x)}\,\mathrm{d}x \\
&= \|f\|^2 - \langle f,g\rangle - \langle g,f\rangle + \|g\|^2
\end{aligned}$$

We note that

$$\begin{aligned}
\langle f,g\rangle = \left\langle f(x), \sum_{k=1}^n d_k f_k(x)\right\rangle &= \int_a^b f(x)\overline{\sum_{k=1}^n d_k f_k(x)}\,\mathrm{d}x \\
&= \sum_{k=1}^n \overline{d_k}\int_a^b f(x)\overline{f_k(x)}\,\mathrm{d}x \\
&= \sum_{k=1}^n \overline{d_k}c_k
\end{aligned}$$

By symmetry, we have

$$\langle g,f\rangle = \sum_{k=1}^n d_k\overline{c_k}.$$

We also note that

$$\begin{aligned}
\|g\|^2 = \left\langle \sum_{k=1}^n d_k f_k(x), \sum_{k=1}^n d_k f_k(x)\right\rangle &= \int_a^b \sum_{k=1}^n d_k f_k(x)\overline{\sum_{k=1}^n d_k f_k(x)}\,\mathrm{d}x \\
&= \sum_{k=1}^n\sum_{\ell=1}^n d_k\overline{d_\ell}\int_a^b f_k(x)\overline{f_\ell(x)}\,\mathrm{d}x \\
&= \sum_{k=1}^n |d_k|^2
\end{aligned}$$

Hence

$$\|f(x)-g(x)\|^2 = \|f\|^2 - \sum_{k=1}^n \overline{d_k}c_k - \sum_{k=1}^n d_k\overline{c_k} + \sum_{k=1}^n |d_k|^2.$$

We note that

$$\begin{aligned}
\sum_{k=1}^{n} |c_k - d_k|^2 &= \sum_{k=1}^{n} (c_k - d_k)\overline{(c_k - d_k)} \\
&= \sum_{k=1}^{n} c_k\overline{c_k} - c_k\overline{d_k} - d_k\overline{c_k} + d_k\overline{d_k} \\
&= \sum_{k=1}^{n} |c_k|^2 - \sum_{k=1}^{n} c_k\overline{d_k} - \sum_{k=1}^{n} \overline{c_k}d_k + \sum_{k=1}^{n} |d_k|^2
\end{aligned}$$

Substituting this in, yields

$$\|f(x) - g(x)\|^2 = \|f\|^2 + \sum_{k=1}^{n} |c_k - d_k|^2 - \sum_{k=1}^{n} |c_k|^2.$$

We note that $\|f(x) - g(x)\|^2$ is minimized when $d_k = c_k$ for all $k \in \{1, \ldots, n\}$, that is to say when $g = f_\sim$. Since $\| \cdot \| \geq 0$, then it follows that we have

$$\|f(x) - f_\sim(x)\| \leq \|f(x) - g(x)\|.$$

□

9.3.1 The Trigonometric System

Definition 9.3.2. (**Trigonometric System**) We define the trigonometric system $\mathcal{T}$ on the interval $[-L, L]$ as

$$f_0(x) = \frac{1}{\sqrt{2L}}, \qquad f_{2n-1} = \frac{1}{\sqrt{L}} \cos\frac{n\pi x}{L}, \qquad f_{2n}(x) = \frac{1}{\sqrt{L}} \sin\frac{n\pi x}{L}.$$

Theorem 9.3.2. (**The Trigonometric System is Orthnormal**) The trigonometric system $\mathcal{T}$ on the interval $[-L, L]$ is orthonormal.

Proof. We first proceed by showing $\mathcal{T}$ is orthonormal. First, recall the following trigonometric identities

$$\begin{aligned}
\cos\phi \pm \psi &= \cos\phi\cos\psi \mp \sin\phi\sin\psi \\
\sin\phi \pm \psi &= \sin\phi\cos\psi \pm \cos\phi\sin\psi
\end{aligned}$$

We have

$$\begin{aligned}\langle f_0, f_0\rangle = \int_{-L}^{L} \left(\frac{1}{\sqrt{2L}}\right)^2 \mathrm{d}x &= \frac{1}{2L}\int_{-L}^{L} \mathrm{d}x \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\langle f_{2m-1}, f_{2n-1}\rangle &= \int_{-L}^{L} \left(\frac{1}{\sqrt{L}}\right)^2 \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{1}{2L}\int_{-L}^{L} \cos\left[\frac{(m+n)\pi x}{L}\right] + \cos\left[\frac{(m-n)\pi x}{L}\right]\mathrm{d}x \\ &= \frac{1}{2L}\left\{\int_{-L}^{L} \cos\left[\frac{(m+n)\pi x}{L}\right]\mathrm{d}x + \int_{-L}^{L} \cos\left[\frac{(m-n)\pi x}{L}\right]\mathrm{d}x\right\} \\ &= \begin{cases} 1 & m = n \\ 0 & m \neq n\end{cases}\end{aligned}$$

And

$$\begin{aligned}\langle f_{2m-1}, f_{2n}\rangle &= \int_{-L}^{L} \left(\frac{1}{\sqrt{L}}\right)^2 \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{1}{2L}\int_{-L}^{L} \sin\left[\frac{(m+n)\pi x}{L}\right] + \sin\left[\frac{(m-n)\pi x}{L}\right]\mathrm{d}x \\ &= \frac{1}{2L}\left\{\int_{-L}^{L} \sin\left[\frac{(m+n)\pi x}{L}\right]\mathrm{d}x + \int_{-L}^{L} \sin\left[\frac{(m-n)\pi x}{L}\right]\mathrm{d}x\right\} \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\langle f_{2m}, f_{2n}\rangle &= \int_{-L}^{L} \left(\frac{1}{\sqrt{L}}\right)^2 \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{1}{2L}\int_{-L}^{L} \cos\left[\frac{(m-n)\pi x}{L}\right] - \cos\left[\frac{(m+n)\pi x}{L}\right]\mathrm{d}x \\ &= \begin{cases} 1 & m = n \\ 0 & m \neq n\end{cases}\end{aligned}$$

Thus showing that for all $m, n \in \mathbb{N}$, $m \neq n. \langle f_m, f_n\rangle = 0$ and for all $n \in \mathbb{N}. \langle f_n, f_n\rangle = 1$. Hence by definition, $\mathcal{T}$ is an orthonormal system of functions. □

9.4 Fourier Series

Definition 9.4.1. Let $\mathcal{S} = \{f_0, f_1, \ldots\}$ be an orthonormal system of functions on $I = [a, b]$ and let f be a function on I. The **Fourier Coefficients** of f are the numbers $c_k = \langle f, f_k \rangle$ and the **Fourier Series** of f with respect to $\mathcal{S}$ is

$$f(x) \sim \sum_{n=0}^{\infty} c_n f_n(x),$$

where $\sim$ is the equivalence relation between functions on I and their Fourier series with respect to $\mathcal{S}$.

- The canonical orthonormal system of functions is the trigonometric system $\mathcal{T}$. We call the Fourier series of f with respect to $\mathcal{T}$ the **Fourier series** of f.

- So we have

$$\begin{aligned}
f(x) &\sim \sum_{n=0}^{\infty} c_n f_n(x) \\
&\sim \frac{1}{\sqrt{2L}} \int_{-L}^{L} f(x)\,\mathrm{d}x \cdot \frac{1}{\sqrt{2L}} + \frac{1}{\sqrt{L}} \int_{-L}^{L} f(x) \cos\frac{\pi x}{L}\,\mathrm{d}x \cdot \frac{1}{\sqrt{L}} \cos\frac{\pi x}{L} \\
&\quad + \frac{1}{\sqrt{L}} \int_{-L}^{L} f(x) \sin\frac{\pi x}{L}\,\mathrm{d}x \cdot \frac{1}{\sqrt{L}} \sin\frac{\pi x}{L} + \cdots \\
&\sim \frac{1}{2L} \int_{-L}^{L} f(x) \cos\frac{0\pi x}{L}\,\mathrm{d}x + \sum_{n=1}^{\infty} \left(\frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,\mathrm{d}x \cdot \cos\frac{n\pi x}{L} \right. \\
&\quad \left. + \frac{1}{L} \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,\mathrm{d}x \cdot \sin\frac{n\pi x}{L} \right)
\end{aligned}$$

We define for $n \geq 0$,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,\mathrm{d}x,$$

and for $n \geq 1$,

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,\mathrm{d}x\,.$$

So

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

- If the Fourier series converges, then we write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

9.4.1 Parseval's Theorem

Theorem 9.4.1. (**Parseval's Theorem**) Let $\mathcal{S} = \{f_0, f_1, \ldots\}$ be an orthonormal system of functions on $[a, b]$, f be a function on $[a, b]$, with

$$f(x) \sim \sum_{n=0}^{\infty} c_n f_n(x).$$

If the Fourier series of f with respect to $\mathcal{S}$ is convergent, then

$$\|f\|^2 = \sum_{n=0}^{\infty} |c_n|^2.$$

Proof. Let f be as described. Let $f_\sim$ denote the Fourier series of f with respect to $\mathcal{S}$ finitely evaluated to n terms. By theorem ??, we have

$$\begin{aligned} \|f(x) - f_\sim(x)\|^2 &= \|f\|^2 + \sum_{k=0}^{n} |c_k - c_k|^2 - \sum_{k=0}^{n} |c_k|^2 \\ &= \|f\|^2 - \sum_{k=0}^{n} |c_k|^2 \end{aligned}$$

Let us assume that $f_\sim$ is convergent, that is to say

$$f(x) = \lim_{n\to\infty} f_\sim(x).$$

So it follows that

$$\begin{aligned} \|f\|^2 &= \lim_{n\to\infty} \underbrace{\|f(x) - f_\sim(x)\|^2}_{0} + \sum_{k=0}^{n} |c_k|^2 \\ &= \sum_{n=0}^{\infty} |c_n|^2 \end{aligned}$$

□

- We note that for a Fourier series of f with respect to $\mathcal{T}$, we have

$$
\begin{aligned}
\|f\|^2 &= \left(\frac{1}{\sqrt{2L}}\int_{-L}^{L} f(x)\,\mathrm{d}x\right)^2 + \sum_{n=1}^{\infty}\left[\left(\frac{1}{\sqrt{L}}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x\right)^2\right. \\
&\qquad \left. + \left(\frac{1}{\sqrt{L}}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x\right)^2\right] \\
&= \frac{1}{2L}(La_0)^2 + \frac{1}{L}\sum_{n=1}^{\infty}(La_n)^2 + (Lb_n)^2 \\
&= L\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]
\end{aligned}
$$

- We often write this as

$$
\frac{1}{2L}\|f\|^2 = \frac{1}{2L}\int_{-L}^{L}[f(x)]^2\,\mathrm{d}x = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).
$$

9.5 Convergences of Fourier Series

9.5.1 Gibbs Phenomenon

- If f is discontinuous at some point x_0 in the interval I, then the Fourier series of f is evaluated finitely, the series always overestimates at x_0 by approx 18 %.

IMAGE

Theorem 9.5.1. Let f be some function on the interval $I = [-L, L]$ with a discontinuous point $x_0 \in I$. The Fourier series of f, denoted $f_\sim$ satisfies

$$
f_\sim(x_0) = \frac{f(x_0^-) + f(x_0^+)}{2},
$$

where

$$
f(x_0^-) = \lim_{x \to x_0^-} f(x) \qquad\qquad f(x_0^+) = \lim_{x \to x_0^+} f(x)
$$

9.5.2 Differentiation and Integration of Fourier Series

- Let f be a function on $I = [-L, L]$ such that the Fourier series of f converges, that is to say

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

  Hence the derivative of f is given by

$$f'(x) \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} \left(b_n \cos \frac{n\pi x}{L} - a_n \sin \frac{n\pi x}{L} \right).$$

- We note that despite the Fourier series of f converging, the expression above for f' may not converge. (See sawtooth function). Since the Fourier coefficients of f' decline less rapidly with n than those of f.

- For the integral of f, we have

$$\int f(x)\,\mathrm{d}x \sim \frac{a_0 x}{2} + \sum_{n=1}^{\infty} \frac{L}{n\pi} \left(a_n \sin \frac{n\pi x}{L} - b_n \cos \frac{n\pi x}{L} \right) + \kappa.$$

- In this case, if the Fourier series of f converges, then $\int f(x)\,\mathrm{d}x$ converges, since the Fourier coefficients converge fast with n.

- Note that if $a_0 \neq 0$, then IMAGE