Discrete Mathematics
Exercise Sheet 6

Alistair O'Brien

February 19, 2023

Before Attempting the Problems

The primary goal of this course is to introduce to you *formal reasoning* and various (discrete) mathematical structures such as *sets, relations, and functions* along side some niche number and automata theory. It is vital that you master many of these topics for various other Tripos courses[1].

Please complete these exercises and submit your solutions 48 hours before our scheduled supervisions to `ajo41@cam.ac.uk` as a **PDF** attachment. Feel free to choose the format of your answers (handwritten, typed, LaTeX'd) – just ensure that they're (mostly) legible[2].

Attempt to complete at least 75% of the set exercises – if you get stuck simply make a note of it in your answer and move onto the next question. We will be able to discuss all the solutions during the supervisions.

These supervisions will focus on both examinable and *non-examinable material*, the latter being for pedigogical reasons. Each week I will provide a 'Material' section with notes/summaries of the lecture material . You are welcome to skip these sections, they are simply there for your benefit. However, some questions will (explicitly) refer to this material.

Some of the exercises are taken from Marcelo's exercise sheets which you can find here: https://www.cl.cam.ac.uk/teaching/2223/DiscMath/materials.html.

[1] In particular, Algorithms (IA), Semantics of Programming Languages (IB), Logic and Proof (IB), Computation Theory (IB), Complexity Theory (IB), Formal Models of Language (IB), Denotational Semantics (II), Types (II), Hoare Logic and Model Checking (II), and Category Theory (II)

[2] Don't worry too much about this though, I can't handwrite anything legible either

1 Material

Injections and Surjections

Definition 1.1. (**Injection**) A function $f : A \to B$ is said to be injective, or an injection, if and only if

$$\forall a_1, a_2 \in A. f(a_1) = f(a_2) \implies a_1 = a_2.$$

Properties

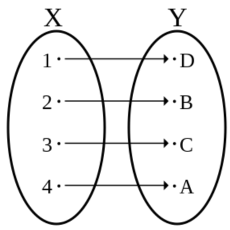
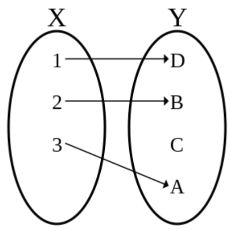
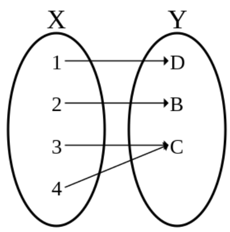
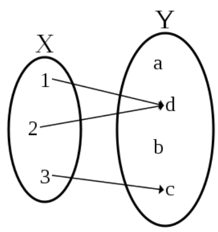
- For all injections $f : A \to B$ and $g : B \to C$, $g \circ f : A \to C$ is injective.
- For all functions $f : A \to B$ and $g : B \to C$,

  $$g \circ f \text{ is injective} \implies f \text{ is injective},$$

  but g is not necessarily injective.
- For all injections $f : A \to B$, there exists a left unique inverse $g : B \to A$ such that $g \circ f = id_A$.
- The set of injections from A to B is denoted $\text{Inj}(A, B)$.

Definition 1.2. (**Surjection**) A function $f : A \to B$ is said to be surjective, or an surjection, if and only if

$$\forall b \in B. \exists a \in A. f(a) = b.$$

<table>
<tr><th></th><th>Surjective</th><th>Non-surjective</th></tr>
<tr><td>Injective</td><td>
</td><td>
</td></tr>
<tr><td>Non-injective</td><td>
</td><td>
</td></tr>
</table>

Properties

- For all surjections $f : A \to B$ and $g : B \to C$, $g \circ f : A \to C$ is surjective.

- For all functions $f : A \to B$ and $g : B \to C$,

$$g \circ f \text{ is surjective} \implies g \text{ is surjective},$$

  but f is not necessarily surjective.

- For all surjections $f : A \to B$, there exists a unique right inverse $g : B \to A$ such that $f \circ g = id_B$.

- The set of surjections from A to B is denoted $\text{Sur}(A, B)$.

Bijections

Definition 1.3. A function $f : A \to B$ is said to be bijective, or a bijection, if f is injective and surjective.

Equivalently, a function $f : A \to B$ is a bijection if there exists a unique function $f^{-1} : B \to A$ such that

1. f^{-1} is a left inverse for f, that is

$$f^{-1} \circ f = id_A.$$

2. f^{-1} is a right inverse for f, that is

$$f \circ f = id_B.$$

Properties

- For all bijections $f : A \to B$ and $g : B \to C$, $g \circ f : A \to C$ is bijective.

- The set of bijections from A to B is denoted $\text{Bij}(A, B)$,

$$\text{Bij}(A, B) = \text{Inj}(A, B) \cap \text{Sur}(A, B).$$

  where

$$\left|\text{Bij}(A, B)\right| = \begin{cases} 0 & \text{if } |A| \neq |A| \\ n! & \text{if} |A| = |B| = n \end{cases}.$$

Definition 1.4. (**Isomorphism**) Two sets A and B are said to be isomorphic (and have the same cardinality) if there exists a bijection $\phi : A \to B$. That is to say

$$A \cong B \iff |A| = |B| \iff \exists \phi : A \to B. \phi \in \text{Bij}(A, B).$$

Images

Definition 1.5. Let $R \subseteq A \times B$ be a relation. The direct image of $X \subseteq A$ under R is the set $\overrightarrow{R}(X) \subseteq B$ defined as

$$\overrightarrow{R}(X) = \{b \in B : \exists x \in A. \, (x, b) \in R \wedge x \in X\} \, .$$

The inverse image of $Y \subseteq B$ under R is the set $\overleftarrow{R}(Y) \subseteq A$ defined as

$$\overleftarrow{R}(Y) = \{a \in A : \forall b \in B. \, (a, b) \in R \implies b \in Y\}$$

Enumerability, Countability and Other Theorems

Definition 1.6. (**Enumerable**) A set S is said to be enumerable if there exists a surjection $\mathbb{N} \to S$.

Definition 1.7. (**Countable**) A set S is said to be countable if it is either empty or enumerable.

Theorem 1.1. Every non-empty subset X of a enumerable set S is enumerable.

Definition 1.8. A set A is of less than or equal cardinality to a set B if there exists an injection $f : A \to B$. We write

$$A \lesssim B \iff |A| \leq |B|.$$

Since each injection has a left inverse (which is surjective), then if there exists a surjection $f : B \to A \implies |A| \leq |B|$.

2 Exercises

1. Let $f : A \to B$ be a injection. Show that (f, g) forms a *section-retraction* pair for some $g : B \to A$.

   Note: You cannot use the existence of left inverse for f in this question , you must define g.

2. Prove that $f : A \to B$ is surjective *if and only if*

   $$\forall C \in \mathrm{Set}, g, h : B \to C.\ g \circ f = h \circ f \implies g = h.$$

   Give and prove the analogous statement for injections.

3. Explain the intuition behind the definition of the relational images and inverse images.

   Note: No rigor required (feel free to use diagrams).

4. For a relation $R \subseteq A \times B$, show that

   (a) $\overrightarrow{R}(X) = \bigcup_{x \in X} \overrightarrow{R}(\{x\})$.

   (b) $\overleftarrow{R}(Y) = \{a \in A : \overrightarrow{R}(\{a\}) \subseteq Y\}$.

5. For all $X \subseteq A$ and injections $f : A \to B$, prove that

   $$X \cong \overrightarrow{f}(X)$$

6. Prove that for a surjective functions $f : A \to B$, the image $\overrightarrow{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ is surjective.

7. Show that for all functions $f : A \to B$, there exists a set X and a injection $i : X \to B$ and surjection $s : A \to X$ such that $f = i \circ s$.

8. Demonstrate that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable sets.

9. Let A be an infinite subset of $\mathbb{N}$. Show that $A \cong \mathbb{N}$.

10. Prove that:

    (a) For all countable sets A, B, the sets $A \times B$ and $A \uplus B$ are countable.

    (b) For a set A, there is no injection $\mathcal{P}(A) \to A$.