Discrete Mathematics
Exercise Sheet 4

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Before Attempting the Problems

The primary goal of this course is to introduce to you *formal reasoning* and various (discrete) mathematical structures such as *sets, relations, and functions* along side some niche number and automata theory. It is vital that you master many of these topics for various other Tripos courses[1].

Please complete these exercises and submit your solutions 48 hours before our scheduled supervisions to `ajo41@cam.ac.uk` as a **PDF** attachment. Feel free to choose the format of your answers (handwritten, typed, LaTeX'd) – just ensure that they're (mostly) legible[2].

Attempt to complete at least 75% of the set exercises – if you get stuck simply make a note of it in your answer and move onto the next question. We will be able to discuss all the solutions during the supervisions.

These supervisions will focus on both examinable and *non-examinable material*, the latter being for pedigogical reasons. Each week I will provide a 'Material' section with notes/summaries of the lecture material . You are welcome to skip these sections, they are simply there for your benefit. However, some questions will (explicitly) refer to this material.

Some of the exercises are taken from Marcelo's exercise sheets which you can find here: https://www.cl.cam.ac.uk/teaching/2223/DiscMath/materials.html.

[1]In particular, Algorithms (IA), Semantics of Programming Languages (IB), Logic and Proof (IB), Computation Theory (IB), Complexity Theory (IB), Formal Models of Language (IB), Denotational Semantics (II), Types (II), Hoare Logic and Model Checking (II), and Category Theory (II)

[2]Don't worry too much about this though, I can't handwrite anything legible either

1 Material

A *set* is a collection of distinct objects. This means that $\{1, 2, 3\}$ *is* set, but $\{1, 1, 2\}$ *is not* as the element 1 appears more than once. The second collection is called a *multiset*.

Sets are often constructed using a *set comprehension*, in which we define a set by means of a property.

Definition 1.1. (**Membership**) The *set membership symbol* $\in$ is used to say that an object is a member of a set.

Given a set A and a predicate $P(x)$ defined over the variable $x \in A$, we can use the following set-comprehension notation

$$S \triangleq \{x \in A : P(x)\},$$

to define the set S consisting of all elements $a \in A$ such that $P(a)$ holds.

Definition 1.2. (**Extensionality Axiom (Equality)**) We say two sets are *equal* if they have exactly the same members, that is

$$\forall A, B.\ A = B \iff (\forall x.x \in A \iff x \in B).$$

Definition 1.3. (**Empty Set**) The *empty set* is the set containing no elements, denoted $\emptyset$. The definable property of $\emptyset$ is $\forall x.x \notin \emptyset$.

Definition 1.4. (**Cardinality**) For a finite set S, the cardinality of a set is the number of members it contains, written as $|S|$.

1.1 Subsets

Definition 1.5. (**Subset**) For two sets A and B we say that A *is a subset of* B if each element of A is also an element of B.

$$A \subseteq B \iff \forall x.x \in A \implies x \in B.$$

If $A \subseteq B \land A \neq B$ then we write $A \subset B$ and we say A is a *proper subset* of B. The empty set $\emptyset$ is a subset of every set.

Definition 1.6. (**Powerset**) The *powerset* of the set S, denoted $\mathcal{P}(S)$, is defined as

$$\mathcal{P}(S) \triangleq \{X \in \text{Set} : X \subseteq S\}.$$

1.2 Set Operations

Definition 1.7. (**Intersection**) The *intersection* of two sets A and B is the collection of all objects that are in both sets. It is written $A \cap B$. We can write this using a set comprehension, such that

$$A \cap B \triangleq \{x : x \in A \wedge x \in B\}.$$

Definition 1.8. (**Disjoint sets**) If $A \cap B = \emptyset$, then we say that A and B are *disjoint*.

Definition 1.9. (**Union**) The *union* of two sets A and B is the collection of all objects that are in either set. It is written $A \cup B$, such that

$$A \cup B \triangleq \{x : x \in A \vee x \in B\}.$$

Definition 1.10. (**Compliment**) The *compliment* of a set S is the set of elements in the universal set U that are not in S. The compliment is written $\overline{S}$, such that

$$\overline{S} \triangleq \{x \in U : x \notin S\}.$$

Definition 1.11. (**Difference**) The *difference* of two sets A, B is the set of elements in A that are not in B. The difference is written $A \setminus B$, such that

$$A \setminus B \triangleq A \cap \overline{B} = \{x : x \in A \wedge x \notin B\}.$$

Definition 1.12. (**Cartesian product**) The *Cartesian product* between two sets A and B is the set of all possible ordered pairs with the first element from A and second element from B. The product is written $A \times B$, such that

$$A \times B \triangleq \{(x, y) : x \in A \wedge y \in B\}.$$

1.3 Pairs and Disjoint Unions

Definition 1.13. (**Pair**) The *pair* of elements a, b is the set $\{a, b\}$. Note that

$$\forall x. x \in \{a, b\} \iff x = a \vee x = b.$$

Definition 1.14. (**Singleton**) For all a, the pair $\{a, a\}$, denoted $\{a\}$, is said to be a *singleton*.

Definition 1.15. (**Ordered Pairs**) For all a, b, the *ordered pair* (a, b) is the set $\{\{a\}, \{a, b\}\}$

Definition 1.16. (**Tagging**) Let S be some set. The set of elements S tagged by labels ℓ is defined as

$$S_\ell = \{\ell\} \times S = \{(\ell, x) : x \in S\}.$$

Note that

$$\begin{aligned} &\forall y \in S_\ell. \exists! x \in S. y = (\ell, x) \\ &A_{\ell_1} = B_{\ell_2} \iff \ell_1 = \ell_2 \wedge A = B \end{aligned}$$

Definition 1.17. (**Disjoint Union**) The *disjoint union* $A \uplus B$ of the sets A, B is

$$A \uplus B = A_1 \cup B_2.$$

Hence

$$\forall x. x \in A \uplus B \iff (\exists! a \in A. x = (1, a)) \vee (\exists! b \in B. x = (2, b)).$$

1.4 Big Unions and Intersections

Definition 1.18. (**Big Union**) Let U be some universal set. For a collection of sets $\mathcal{F} \subseteq \mathcal{P}(U)$, the big union is defined as

$$\bigcup \mathcal{F} = \{x \in U : \exists X \in \mathcal{F} . x \in X\} \subseteq U.$$

Definition 1.19. (**Big Intersection**) Let U be some universal set. For a collection $\mathcal{F} \subseteq \mathcal{P}(U)$, the big intersection is defined as

$$\bigcap \mathcal{F} = \{x \in U : \forall X \in \mathcal{F} . x \in X\} \subseteq U.$$

2 Exercises

1. Prove that $\subseteq$ is a *partial order*.
2. Let $U \in \text{Set}$. For all $A, B \in \mathcal{P}(U)$, prove that:
   (i) $\overline{A} = B \iff (A \cup B = U \wedge A \cap B = \emptyset)$.
   (ii) $\overline{\overline{A}} = A$.
   (iii) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
3. Prove that for all sets $U \in \text{Set}$ and $A, B \in \mathcal{P}(U)$:
   (i) $\forall X \in \text{Set}.\ A \subseteq X \wedge B \subseteq X \iff A \cup B \subseteq X$
   (ii) $\forall X \in \text{Set}.\ X \subseteq A \wedge X \subseteq B \iff X \subseteq A \cap B$
4. Prove the following properties for families $\mathcal{F} \subseteq \mathcal{P}(A)$,
$$\forall U \subseteq \mathcal{P}(A).\ (\forall X \in \mathcal{F}.\ X \subseteq U) \iff \bigcup \mathcal{F} \subseteq U$$
$$\forall L \subseteq \mathcal{P}(A).\ (\forall X \in \mathcal{F}.\ L \subseteq X) \iff L \subseteq \bigcap \mathcal{F}$$
5. Let $A \in \text{Set}$ be an arbitrary set.
   (i) For family $\mathcal{F} \subseteq \mathcal{P}(A)$, define $\mathcal{U} \triangleq \{U \subseteq A : \forall X \in \mathcal{F}.X \subseteq U\}$. Prove that
$$\bigcup \mathcal{F} = \bigcap \mathcal{U}$$
   (ii) Analogously, define the family $\mathcal{L} \subseteq \mathcal{P}(A)$. Prove that
$$\bigcap \mathcal{L} = \bigcup \mathcal{F}$$
6. Let $U \in \text{Set}$ be a set. For all $A, B \in \mathcal{P}(U)$ prove that the following statements are equivalent:
   (i) $A \cup B = B$
   (ii) $A \subseteq B$
   (iii) $A \cap B = A$
   (iv) $\overline{B} \subseteq \overline{A}$
7. For sets $A, B, C, D \in \text{Set}$, prove or disprove the following statements:
   (i) $A \subseteq C \wedge B \subseteq D \implies A \times B \subseteq C \times D$
   (ii) $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$
   (iii) $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$
   (iv) $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
   (v) $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

8. For sets $A, B, C, D \in \mathrm{Set}$, prove or disprove the following statements:

   (i) $A \subseteq C \wedge B \subseteq D \implies A \uplus B \subseteq C \uplus D$

   (ii) $(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C)$

   (iii) $(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C$

   (iv) $(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C)$

   (v) $(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C$