Discrete Mathematics
Exercise Sheet 5

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Before Attempting the Problems

The primary goal of this course is to introduce to you *formal reasoning* and various (discrete) mathematical structures such as *sets, relations, and functions* along side some niche number and automata theory. It is vital that you master many of these topics for various other Tripos courses[1].

Please complete these exercises and submit your solutions 48 hours before our scheduled supervisions to `ajo41@cam.ac.uk` as a **PDF** attachment. Feel free to choose the format of your answers (handwritten, typed, LaTeX'd) – just ensure that they're (mostly) legible[2].

Attempt to complete at least 75% of the set exercises – if you get stuck simply make a note of it in your answer and move onto the next question. We will be able to discuss all the solutions during the supervisions.

These supervisions will focus on both examinable and *non-examinable material*, the latter being for pedigogical reasons. Each week I will provide a 'Material' section with notes/summaries of the lecture material . You are welcome to skip these sections, they are simply there for your benefit. However, some questions will (explicitly) refer to this material.

Some of the exercises are taken from Marcelo's exercise sheets which you can find here: https://www.cl.cam.ac.uk/teaching/2223/DiscMath/materials.html.

[1]In particular, Algorithms (IA), Semantics of Programming Languages (IB), Logic and Proof (IB), Computation Theory (IB), Complexity Theory (IB), Formal Models of Language (IB), Denotational Semantics (II), Types (II), Hoare Logic and Model Checking (II), and Category Theory (II)

[2]Don't worry too much about this though, I can't handwrite anything legible either

1 Exercises

1. Let $R \subseteq A \times A$ be an arbitrary relation.

   (a) The family of relations $\mathcal{T}_R$ containing the transitive supersets of R is defined as
   $$\mathcal{T}_R \triangleq \{S \subseteq A \times A : R \subseteq S \wedge S \text{ is transitive}\}.$$
   Prove that the set $\bigcap \mathcal{T}_R$ is the transitive closure of R.

   (b) Prove that the at least once iterated composition $R^{\circ +} = R \circ R^{\circ *}$ is the transitive closure of R.

   hint: The *transitive closure* of R is a relation $S \subseteq A \times A$ such that

   (i) S is transitive.

   (ii) $R \subseteq S$.

   (iii) S is *smallest*, that is:
   $$\forall Q \subseteq A \times A.\ Q \text{ is transitive} \wedge R \subseteq Q \implies S \subseteq Q.$$

2. Prove that the identity relation id_A is a partial function, and that the composition of partial functions is a partial function.

3. Let $\mathcal{F} \subseteq \mathrm{PFun}(A, B)$ be a non-empty family of partial functions from A to B.

   (a) Show that $\bigcap \mathcal{F}$ is a partial function.

   (b) Show that $\bigcup \mathcal{F}$ is not necessarily a partial function.

   (c) Prove that
   $$\forall h \in \mathrm{PFun}(A, B).\ (\forall f \in \mathcal{F}.\, f \subseteq h) \implies \bigcup \mathcal{F} \text{ is a partial function.}$$

4. Let $\chi : \mathcal{P}(U) \to \mathrm{Fun}(U, \{0, 1\})$ is a function mapping subsets $S \subseteq U$ to their *characteristic functions* $\chi_S : U \to \{0, 1\}$.

   (a) Prove that for all $x \in U$

      i. $\chi_{A \cup B}(x) = \chi_A(x) \vee \chi_B(x)$.

      ii. $\chi_{A \cap B}(x) = \chi_A(x) \wedge \chi_B(x)$.

      iii. $\chi_{\neg A}(x) = \neg \chi_A(x)$.

   (b) For what set operator ? does it hold that
   $$\chi_{A?B}(x) = (\chi_A(x) \oplus \chi_B(x)),$$
   for all $x \in U$, where $\oplus$ is the exclusive or operator.

5. Let A be a set with an element $a \in A$ and a function $f : A \to A$. We say that a relation $R \subseteq \mathbb{N} \times A$ is (a, f)-closed if

   (i) $(0, a) \in R$.

(ii) $\forall n \in \mathbb{N}, x \in A.\ (n, x) \in R \implies (n+1, f(x)) \in R$.

The relation $F \subseteq \mathbb{N} \times A$ is defined by

$$F \triangleq \bigcap \{R \subseteq \mathbb{N} \times A : R \text{ is } (a, f)\text{-closed}\}$$

(a) Prove that F is (a, f)-closed.

(b) Prove that F is total.

(c) Prove that F is a function $\mathbb{N} \longrightarrow A$.

(d) Show that for all $h : \mathbb{N} \longrightarrow A$, if $h(0) = a$ and $\forall n \in \mathbb{N}.\ h(n+1) = f(h(n))$ then $h = F$.

6. For $f : A \longrightarrow B$, prove that if there are $g, h : B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ then $g = h$.

   Conclude as a corollary that, whenever it exists, the inverse of a function is unique.

7. The functions $s : A \longrightarrow B$ and $r : B \longrightarrow A$ are a *section-retration* pair (s, r) if $r \circ s = \mathrm{id}_A$.

   A function $f : A \longrightarrow A$ is *idempotent* if $f \circ f = f$.

   (a) Let $f : C \longrightarrow D$ and $g : D \longrightarrow C$ be function such that $f \circ g \circ f = f$

      i. Prove or disprove whether $f \circ g$ is idempotent.

      ii. Prove or disprove whether $g \circ f$ is idempotent.

      iii. Define a function $g' : D \longrightarrow C$ that satisfies

      $$f \circ g' \circ f = f \qquad \text{and} \qquad g' \circ f \circ g' = g'.$$

   (b) If (s, r) is a section-retraction pair (on A, B), then $s \circ r : B \longrightarrow B$ is idempotent.

   (c) Prove that for every idempotent $f : B \longrightarrow B$, there exists a set A and a section-retraction pair (s, r) such that $s \circ r = f$.

8. Prove that the isomorphism relation $\cong$ between sets is an equivalence relation.

9. For an equivalence relation $E \subseteq A \times A$ on the set A, show that

   $$\forall a_1, a_2 \in A.\ [a_1]_E = [a_2]_E \iff (a_1, a_2) \in E,$$

   where $[a]_E \triangleq \{x \in A : (x, a) \in E\}$.