1. **[10 marks]** Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with unknown mean $\mu > 0$. Find a UMVUE for $e^{-\mu}$.

2. **[10 marks]** Let Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ variables with unknown μ and σ^2.

   (a) Find the MLE $\hat{\sigma}^2$ of σ^2.

   (b) Show that $\hat{\sigma}^2$ has a smaller MSE than the UMVUE $s^2 = \sum(X_i - \bar{X}_n)^2/(n-1)$.

3. **[5 marks]** The pdf of a Gamma(α, λ) distribution is

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}, \ x > 0$$

   where $\lambda > 0$, $\alpha > 0$ and $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

   If $X_1, \ldots, X_n$ are a random iid sample from this distribution, determine the minimal sufficient statistic for (α, λ).

4. **[10 marks]** Let $X_1, X_2, \ldots, X_5$ be a random sample of size 5 from a Poisson distribution with mean μ. Use the Neyman-Pearson Lemma to find an $\alpha = 0.05$ level test most powerful for testing $H_0: \ \mu = 1$ versus $H_1: \ \mu = 2$ and find the power of this test.

5. **[10 marks]** Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with unknown mean $\mu > 0$.

   (a) Show that any most powerful α-level test for $H_0: \ \mu = \mu_0$ versus $H_1: \ \mu = \mu_1$, $\mu_1 > \mu_0$ is based on the minimal sufficient statistic for μ.

   (b) Generalize this test to find a uniformly most powerful test for $H_0: \ \mu = \mu_0$ versus $H_1: \ \mu > \mu_0$.

   (c) Find an expression for the power and show that the test is unbiased.

1. **[10 marks]** Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with unknown mean $\mu > 0$. Find a UMVUE for $e^{-\mu}$.

The PMF of X_i is

$$P(X_i = x) = \frac{\mu^x e^{-\mu}}{x!}$$

Which can be written in exponential form, in joint case

$$P(X_i = x) = \prod_{i=1}^{n} \frac{1}{x_i!} \exp\left\{ \sum_{i=1}^{n} x_i \ln \mu - n\mu \right\}$$

Hence, we have $T = \sum_{i=1}^{n} X_i$ is sufficient.

Since $m = q = 1$ and $X_1, \ldots, X_n$ are from iid exponential family, T is complete.

Now, we need an unbiased estimator V for $e^{-\mu}$.

$$E(g(t)) = \sum_{t=0}^{\infty} g(t) P(T = t)$$

$$= \sum_{t=0}^{\infty} g(t) \frac{(n\mu)^t e^{-n\mu}}{t!} \overset{set}{=} e^{-\mu}$$

Dividing both sides by $e^{-n\mu}$ and using Taylor expansion,

$$\Rightarrow \sum_{t=0}^{\infty} g(t) \frac{(n\mu)^t}{t!} = e^{-\mu} e^{n\mu}$$

$$= e^{(n-1)\mu}$$

$$= \sum_{t=0}^{\infty} \frac{((n-1)\mu)^t}{t!}$$

Hence we need

$$g(t) n^t = (n-1)^t$$

$$\Rightarrow g(t) = \frac{(n-1)^t}{n^t} = \left(1 - \frac{1}{n}\right)^t = U$$

Applying Rao-Blackwell, $U = (1 - \frac{1}{n})^T$ is unbiased and $T = \sum_{i=1}^{n} X_i$ is complete and sufficient, then

$$E(U|T) = (1 - \tfrac{1}{n})^T$$

is the UMVUE for $g(\mu) = e^{-\mu}$.

2. **[10 marks]** Let Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ variables with unknown μ and σ^2.

(a) Find the MLE $\hat{\sigma}^2$ of σ^2.

(b) Show that $\hat{\sigma}^2$ has a smaller MSE than the UMVUE $s^2 = \sum(X_i - \bar{X}_n)^2/(n-1)$.

a) $$L(\mu, \sigma^2 | x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i)$$
$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \frac{(x_i-\mu)^2}{\sigma^2}\right\}$$
$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu)^2\right\}$$

$$\ell(\mu, \sigma^2 | x_1, \ldots, x_n) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - (2\sigma^2)^{-1} \sum_{i=1}^{n} (x_i-\mu)^2$$

$$\frac{\partial \ell(\mu, \sigma^2)}{\partial \sigma^2} = \frac{-n}{\sigma^2} + (\sigma^2)^{-2} \sum_{i=1}^{n} (x_i-\mu)^2 \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{n}{\cancel{\sigma^2}} = \frac{\sum_{i=1}^{n} (x_i-\mu)^2}{\cancel{\sigma^4}}$$

$$\Rightarrow n = \frac{\sum_{i=1}^{n} (x_i-\mu)^2}{\sigma^2}$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (x_i-\mu)^2}{n}$$

But since μ is unknown, we find its MLE estimator,

$$\frac{\partial l(\mu,\sigma^2)}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) \overset{set}{=} 0 \Rightarrow \hat{\mu} = \frac{\sum x_i}{n} = \bar{x}$$

Hence $\hat{\sigma}^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2/n$

b) $MSE(\hat{\sigma}^2) = Var(\hat{\sigma}^2) + Bias^2(\hat{\sigma}^2)$

We first find $Bias(\hat{\sigma}^2)$

$$E(\hat{\sigma}^2) = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]$$

$$= \frac{1}{n}E\left[\sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right]$$

$$= \frac{1}{n}E\left[\sum x_i^2 - n\bar{x}^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}E(x_i^2) - nE(\bar{x}^2)\right] \quad (*)$$

Since $\begin{cases} V(x_i) = E(x_i^2) - [E(x_i)]^2 \\ V(\bar{x}) = E(\bar{x}^2) - [E(\bar{x})]^2 \end{cases}$

Then $E(x_i^2) = V(x_i) + [E(x_i)]^2$
$= \sigma^2 + \mu^2$

And $E(\bar{x}^2) = V(\bar{x}) + [E(\bar{x})]^2$
$= \sigma^2/n + \mu^2$

Continue from $(*)$,

$$= \frac{1}{n}\left[\sum_{i=1}^{n}(\sigma^2+\mu^2) - n(\sigma^2/n + \mu^2)\right]$$

$$= \frac{1}{n}\left[n(\sigma^2+\mu^2) - \cancel{n}\left(\sigma^2/\cancel{n} + \mu^2\right)\right]$$

$$= \sigma^2 + \cancel{\mu^2} - \sigma^2/n - \cancel{\mu^2}$$

$$= \sigma^2(1 - 1/n)$$

So $\text{Bias}(\hat{\sigma}^2) = \sigma^2\left(\frac{n-1}{n}\right) - \sigma^2 = -\sigma^2/n$ ①

Next, we find $V(\hat{\sigma}^2)$

Recall if $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, then

$$\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2} = \frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$$

Also for χ^2 with $df = k$, $V(\chi^2_k) = 2k$

So, $V(\chi^2_{n-1}) = 2(n-1)$

Therefore,

$$V(\hat{\sigma}^2) = V\left(\frac{\sigma^2}{n} \cdot \frac{n\hat{\sigma}^2}{\sigma^2}\right) = \frac{\sigma^4}{n^2} V(\chi^2_{n-1})$$

$$= \frac{\sigma^4}{n^2} \cdot 2(n-1)$$ ②

Finally, using ① and ②,

$$MSE(\hat{\sigma}^2) = \left[-\frac{\sigma^2}{n}\right]^2 + \frac{\sigma^4}{n^2} \cdot 2(n-1)$$

$$= \boxed{\frac{\sigma^4}{n^2}(2n-1)}$$

Next, $MSE(s^2) = Var(s^2) + \cancel{Bias^2(s^2)}^{\to 0}$

We only need to find $Var(s^2)$ since s^2 is unbiased.

$$V(s^2) = E((s^2)^2) - (E(s^2))^2$$

So we need $E((s^2)^2)$

Since $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$, hence $s^2 = \frac{\sigma^2}{n-1} \chi^2_{n-1}$

$$\Rightarrow s^4 = \left(\frac{\sigma^2}{n-1}\right)^2 (\chi^2_{n-1})^2$$

$$\Rightarrow E(s^4) = \frac{\sigma^2}{(n-1)^2} E\left[(\chi^2_{n-1})^2\right]$$

where $E\left[(\chi^2_{n-1})^2\right] = V(\chi^2_{n-1}) + (E(\chi^2_{n-1}))^2$

$$= 2(n-1) + (n-1)^2$$

$$= (n+1)(n-1)$$

Hence,

$$E(s^4) = \frac{\sigma^4}{n-1}(n+1)$$

Again, $MSE(s^2) = V(s^2) = E((s^2)^2) - (E(s^2))^2$

$$= \frac{\sigma^4(n+1)}{n-1} - \sigma^4$$

$$= \boxed{\frac{2\sigma^4}{n-1}}$$

Lastly,

$$\Delta = MSE(s^2) - MSE(\hat{\sigma}^2)$$

$$= \frac{2\sigma^4}{n-1} - \frac{\sigma^4}{n^2}(2n-1)$$

$$= \frac{\sigma^4(3n-1)}{n^2(n-1)} > 0 \quad \text{since } n>1,\ \sigma^2>0$$

Hence, $MSE(\hat{\sigma}^2) < MSE(s^2)$

$\blacksquare$

3. [**5 marks**] The pdf of a Gamma(α, λ) distribution is

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}, \ x > 0$$

where $\lambda > 0$, $\alpha > 0$ and $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

If $X_1, \ldots, X_n$ are a random iid sample from this distribution, determine the minimal sufficient statistic for (α, λ).

Gamma distribution is a member of the exponential family. Hence its sufficient statistic is a minimum sufficient statistic.

Note, we can rewrite the PDF

$$f(x;\alpha,\lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0$$

$$= \exp\left[(\alpha-1)\ln(x) - \lambda x + \alpha \ln(\lambda) - \ln(\Gamma(\alpha))\right]$$

we can see $T(x) = (\ln(x), x)$

The likelihood is then,

$$L(x_1, \ldots, x_n; \alpha, \lambda) = \prod_{i=1}^{n} \frac{\lambda^\alpha x_i^{\alpha-1} e^{-\lambda x_i}}{\Gamma(\alpha)}$$

$$= \frac{\lambda^{n\alpha}}{[\Gamma(\alpha)]^n} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$= g(T(x_1,\dots,x_n);\alpha,\lambda)\cdot h(x_1,\dots,x_n)$$

We can similarly see, using the decomposition, that $T(x) = \left(\sum_{i=1}^{n}\ln(x_i), \sum_{i=1}^{n}x_i\right)$ is the (minimal) sufficient statistic.

ie. T preserves all information about α, λ

4. **[10 marks]** Let $X_1, X_2, \ldots, X_5$ be a random sample of size 5 from a Poisson distribution with mean μ. Use the Neyman-Pearson Lemma to find an $\alpha = 0.05$ level test most powerful for testing $H_0: \ \mu = 1$ versus $H_1: \ \mu = 2$ and find the power of this test.

$H_0: \mu = 1 \qquad H_1: \mu = 2$

$X_1, X_2, \ldots, X_5 \sim \text{Poisson}(\mu)$

The likelihood

$$L(\mu) = \prod_{i=1}^{5} \frac{e^{-\mu} \mu^{x_i}}{(x_i)!} = e^{-5\mu} \mu^{\sum_{i=1}^{5} x_i} \prod_{i=1}^{5} \frac{1}{(x_i)!}$$

Applying NP, the test statistic

$$T = \frac{L(\mu_1)}{L(\mu_0)} = \frac{L(\mu = 2)}{L(\mu = 1)}$$

$$= \frac{\cancel{e}^{-5(2)} \cdot 2^{\sum x_i} \cdot \cancel{\prod_{i=1}^{5} \frac{1}{(x_i)!}}}{\cancel{e}^{-5} \cdot 1^{\sum x_i} \cdot \cancel{\prod_{i=1}^{5} \frac{1}{(x_i)!}}}$$

$$= e^{-5} \cdot 2^{S} \qquad > C_\alpha$$

for some $C_\alpha > 0$ and $S = \sum x_i$. Furthermore,

$$\Rightarrow \quad S > k \quad \text{for some } k > 0$$

Under $H_0: \mu = 1$, $S = \sum_{i=1}^{5} x_i \sim \text{Poisson}(5)$.

So we are looking for

$$P(S \geq k \mid H_0) = \alpha = 0.05$$

$$\Leftrightarrow 1 - P(S < k \mid H_0) = 0.05$$

$$\Leftrightarrow P(S < k \mid H_0) = 0.95$$

Now we can find k for which the above expression holds.

```
> crit_vals <- c(1:10)
> cumulative_probs <- ppois(crit_vals, lambda = 5)
> data.frame(k = crit_vals, cumulative_probabilities = cumulative_probs)
    k cumulative_probabilities
1   1               0.04042768
2   2               0.12465202
3   3               0.26502592
4   4               0.44049329
5   5               0.61596065
6   6               0.76218346
7   7               0.86662833
8   8               0.93190637
9   9               0.96817194
10 10               0.98630473
```

$k=9$: $P(S \geq 9 \mid H_0) = 1 - P(S \leq 8 \mid H_0)$
$= 1 - 0.9319$
$= 0.0681$ not smaller than α

$k=10$: $P(S \geq 10 \mid H_0) = 1 - P(S \leq 9 \mid H_0)$
$= 1 - 0.9681$
$= 0.0319$

Hence, by NP lemma, the critical region formed by $S = \sum X_i \geq 10$ is the most powerful region for testing H_1 vs. H_0. The LRT rejects H_0 at $\alpha = 0.05$ if $S \geq 10$.

To find power of the test, we compute

$P(S \geq 10 \mid H_1 : \mu = 2)$ w/ $S \sim \text{Poisson}(10)$

$$\Rightarrow P(S \geq 10 \mid H_1 : \mu = 2) = 1 - P(S \leq 9 \mid H_1 : \mu = 2)$$

$$\overset{\text{by } R}{=} 1 - 0.458$$

$$= 0.542$$

Hence, power $= 0.542$.

5. **[10 marks]** Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with unknown mean $\mu > 0$.

(a) Show that any most powerful α-level test for $H_0: \ \mu = \mu_0$ versus $H_1: \ \mu = \mu_1$, $\mu_1 > \mu_0$ is based on the minimal sufficient statistic for μ.

(b) Generalize this test to find a uniformly most powerful test for $H_0: \ \mu = \mu_0$ versus $H_1: \ \mu > \mu_0$.

(c) Find an expression for the power and show that the test is unbiased.

a) Let $X_1, X_2, \ldots, X_n \sim \text{Poisson}(\mu)$ w/ $\mu > 0$

Let $S = \sum_{i=1}^{n} X_i$ be a statistic. From lecture notes, S is a sufficient statistic and hence minimal sufficient statistic since the Poisson distribution is from the exponential family.

To identify any most powerful α-level test for the stated simple hypothesis H_0 and H_1, we use the NP lemma. First, compute

$$T = \frac{L(\mu_1)}{L(\mu_0)}$$

$$= \frac{\prod_{i=1}^{n} e^{\mu_1} \mu_1^{x_i} / x_i!}{\prod_{i=1}^{n} e^{\mu_0} \mu_0^{x_i} / x_i!}$$

$$= \frac{e^{-n\mu_1}}{e^{-n\mu_0}} \frac{\mu_1^{S}}{\mu_0^{S}}$$

$$= e^{-n(\mu_0-\mu_1)} \left(\frac{\mu_1}{\mu_0}\right)^{S}$$

Note T is a monotonic decreasing function of S and all quantities except for S is fixed. The condition $T \geq C_\alpha$ is equivalent to $S \geq d_\alpha$ for some other constant d_α.

Recall: If, for all α, $0 < \alpha < 1$ there exist C_α s.t.

$$P(T \geq C_\alpha \mid H_0) = \alpha$$

then the critical region formed by $T \geq C_\alpha$ is the most powerful critical region for testing H_1 against H_0.

b) A test is uniformly most powerful (UMP) of level α for testing a simple hypothesis H_0 against a composite hypothesis H_1, if the test is most powerful of level α for testing H_0 verses H_1 for each simple hypothesis $H \in H_1$.

From a), we shown the most powerful test of level α for testing $H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1$ ($\mu_1 > \mu_0$) is rejecting H_0 if $S > d_\alpha$ where d_α is chosen to satisfy $P(S > d_\alpha | H_0) = \alpha$

So the RR formed by $S > d_\alpha$ doesn't depend on μ_1 from the simple hypothesis H_1's parameter space. In other words, the critical region we found is best (most powerful) for <u>any</u> $\mu_1 > \mu_0$. Hence it is the uniformly most powerful test.

Another method from discussing w/ friends

Alternatively, from a),

$$T = \frac{L(\mu_1)}{L(\mu_0)} \geq C_\alpha$$

$$\Rightarrow e^{-n\mu_1 + n\mu_0} \left(\frac{\mu_1}{\mu_0}\right)^S \geq C_\alpha$$

$$\Rightarrow S \geq k^*$$

where $k^* = \dfrac{\ln(C_\alpha) + n\mu_1 - n\mu_0}{\log(\mu_1/\mu_0)}$

Note the ratio of the likelihoods is big

ie. $\dfrac{L(\mu_1)}{L(\mu_0)} \geq C_\alpha$ iff $\sum x_i \geq k^*$

Hence, define $C = \{(x_1, \dots, x_n) : \sum x_i \geq k^* \}$ where k^* is chosen s.t. Type I error $\alpha = P(\sum x_i \geq k^* \mid H_0)$ when $\mu = \mu_0$.

c) To find power, we compute

$$P(S = \sum x_i \geq k^* \mid H_1)$$

with $\sum_{i=1}^{n} x_i \sim \text{Poisson}(n\mu)$

$$\Rightarrow P(S \geq k^* \mid H_1) = 1 - P(S < k^* \mid H_1)$$

$$= 1 - \sum_{i=0}^{k-1} \frac{e^{-n\mu}(n\mu)^i}{i!}$$

$$= 1 - e^{-n\mu} \sum_{i=0}^{k-1} \frac{(n\mu)^i}{i!}$$

which is the power for the test.

Now, to show if is unbiased, we show

$$\max_{\mu \in \mu_0} P_\mu(\text{Reject } H_0) \leq \min_{\mu \in \mu_1} P_\mu(\text{Reject } H_0)$$

This holds under $H_1: \mu > \mu_0$.

In general, larger μ leads to Poisson with rate $\lambda = \mu n$ larger. So that shifted the graph more to the right. Thus, for an increasing μ but the same k^*, the probability to the right of k^* increases.

By construction, we know under H_0

$$\max_{\mu \in \mu_0} P(\text{Reject } H_0) = \alpha$$

While under $H_1: \mu > \mu_0$, $P(S \geq k^* \mid H_1)$ (the rejection probability) is larger than α because the Poisson graph is shifted more to the right as μ increases.

Hence, we can conclude that the maximum rejection probability under H_0 is α, and for any $\mu > \mu_0$ under H_1 the rejection probability is greater than α. That is,

$$\max_{\mu \in \mu_0} P_\mu(\text{Reject } H_0) \leq \min_{\mu \in \mu_1} P_\mu(\text{Reject } H_0)$$

holds. Thus, test is unbiased.

Another method from discussing w/ friends

Recall $\text{Power}(\mu > \mu_0) = P(\sum x_i > d_\alpha \mid H_0)$

We know $P(\sum x_i \leq d_\alpha \mid H_0) = 1 - \alpha = \sum_{\sum x_i = 0}^{d_\alpha} \frac{(n\mu_0)^{\sum x_i} e^{-n\mu_0}}{(\sum x_i)!}$

Since $\sum x_i \sim \text{Poisson}(n\mu_0)$ under H_0

Then $d_\alpha = \min(t : F_{\text{poisson}(n\mu_0)}(t) \geq 1 - \alpha)$

$\text{Power}(\mu > \mu_0) = 1 - P(\sum x_i \leq d_\alpha \mid \mu > \mu_0)$

$= 1 - F_{\text{poisson}(n\mu)}(d_\alpha)$ where $\mu > \mu_0$

$= 1 - \sum_{\sum x_i = 0}^{d_\alpha} \frac{(n\mu)^{\sum x_i} e^{-n\mu}}{(\sum x_i)!}$ where $\mu > \mu_0$

Since $\mu > \mu_0$, $\sum_{\sum x_i = 0}^{d_\alpha} \frac{(n\mu)^{\sum x_i} e^{-n\mu}}{(\sum x_i)!} \leq 1 - \alpha$

So $\text{Power}(\mu > \mu_0) \geq \alpha$, hence

$$P_{\mu\in M_1}(\text{Reject } H_0) \geq P_{\mu\in M_0}(\text{Reject } H_0)$$

Thus, the UMP test in b) is unbiased $\boxed{\checkmark}$