

COMP 458/558  
Quantum Computing Algorithms

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# Chapter 1

## Phase I: Introduction and Background

### 1.1 Lecture 1: Overview of Quantum Computing Concepts

#### Definition 1.1.1: Quantum Computing

Quantum computing is a computational paradigm leveraging quantum mechanical principles such as superposition, entanglement, and interference to perform computations that can surpass the capabilities of classical systems for specific tasks.<sup>a</sup>

<sup>a</sup>Superposition allows quantum bits (qubits) to exist in multiple states simultaneously, and entanglement enables correlations between qubits even at a distance.

#### Historical Development of Quantum Computing

- **1980s-1990s:** Conception of quantum computing, with foundational ideas like the quantum Turing machine and quantum gates.
- **1990s-2000s:** Demonstration of key building blocks, such as quantum algorithms (e.g., Shor's and Grover's algorithms).
- **2016:** Emergence of quantum computing clouds, enabling access to quantum hardware via the internet.
- **2019:** First claims of **quantum advantage**, showcasing tasks where quantum computers outperform classical counterparts.
- **2024:** Increasing qubit counts and improvements in quantum error correction techniques.

#### Applications of Quantum Computing

Quantum computing offers speedup in areas such as:

1. **Quantum Simulation:** Applications in chemistry, physics, and materials science, such as simulating molecular energy levels and drug discovery.
2. **Security and Encryption:** Developing quantum-safe cryptographic protocols and random number generation.
3. **Search and Optimization:** Enhancing solutions for weather forecasting, financial modeling, traffic planning, and resource allocation.

#### Example 1.1.1 (Example: Quantum Speedup in Drug Discovery)

Drug discovery benefits from quantum simulation by enabling more accurate modeling of molecular interactions, which classical computers struggle to achieve efficiently.

## Classical vs. Quantum Computing Paradigms

- **Classical Computing:** Utilizes traditional processing units (CPU, GPU, FPGA) and executes deterministic computations.
- **Quantum Computing:** Employs quantum processing units (QPU) with probabilistic computation based on quantum states.

### Note:-

Note: Classical computing paradigms still dominate in tasks that require precision and deterministic results. Quantum computing excels in probabilistic or exponentially large state-space problems.

## 1.2 Lecture 2: Review of Linear Algebra Concepts

Linear algebra provides the foundation for manipulating quantum states, which are represented using vectors and matrices in a complex vector space.

### Definition 1.2.1: Vectors: Row and Column Vectors

A **vector** is an ordered list of numbers, which can be represented as either a row or column vector. The components of vectors in quantum computing belong to the field of complex numbers ( $\mathbb{C}$ ).

### Column Vectors

A column vector is a vertical arrangement of numbers:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad v_i \in \mathbb{C}.$$

### Row Vectors

A row vector is the complex conjugate transpose (adjoint) of a column vector:

$$\mathbf{v}^\dagger = [\overline{v_1} \quad \overline{v_2} \quad \dots \quad \overline{v_n}].$$

### Dirac Notation

In quantum computing, vectors are represented using **Dirac notation** (bra-ket notation):

- **Ket**  $|v\rangle$ : Represents a column vector.
- **Bra**  $\langle v|$ : Represents the adjoint (conjugate transpose) of the ket.
- Example:  $|v\rangle = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ ,  $\langle v| = [1-i \quad 2]$ .

### Definition 1.2.2: Euler's Formula

Euler's formula relates complex exponentials to trigonometric functions:

$$e^{i\omega} = \cos(\omega) + i \sin(\omega)$$

This is fundamental in representing quantum states and transformations.

### Definition 1.2.3: Inner Product

The **inner product** of two vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$  is defined as:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^\dagger \mathbf{w} = \sum_{i=1}^n \overline{v_i} w_i$$

which measures the overlap between two quantum states.

### Definition 1.2.4: Outer Product

The **outer product** of two vectors  $\mathbf{v} \in \mathbb{C}^m$  and  $\mathbf{w} \in \mathbb{C}^n$  produces an  $m \times n$  matrix:

$$\mathbf{v}\mathbf{w}^\dagger = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \begin{bmatrix} \overline{w_1} & \overline{w_2} & \dots & \overline{w_n} \end{bmatrix}$$

This operation is useful for constructing quantum operators.

### Definition 1.2.5: Tensor Product

The **tensor product** (or Kronecker product) allows us to describe multi-qubit systems. Given two vectors:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Their tensor product is:

$$\mathbf{v} \otimes \mathbf{w} = \begin{bmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{bmatrix}$$

The tensor product expands the state space, allowing representation of entangled states.

### Definition 1.2.6: Adjoint of a Matrix

The **adjoint** (or Hermitian conjugate) of a matrix  $A$  is obtained by taking the transpose and complex conjugate of each entry:

$$A^\dagger = \overline{A^T}$$

If  $A$  is:

$$A = \begin{bmatrix} 1 & i \\ 2 & 3 \end{bmatrix}$$

Then its adjoint is:

$$A^\dagger = \begin{bmatrix} 1 & 2 \\ -i & 3 \end{bmatrix}$$

### Definition 1.2.7: Unitary Matrix

A square matrix  $U$  is called **unitary** if its adjoint is equal to its inverse:

$$U^\dagger U = I$$

where  $I$  is the identity matrix. Unitary matrices preserve the norm of quantum states and represent reversible quantum operations. Example:

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad U^\dagger U = I$$

### Definition 1.2.8: Hermitian Matrix

A square matrix  $H$  is called **Hermitian** if it is equal to its adjoint:

$$H = H^\dagger$$

Hermitian matrices represent observable quantities in quantum mechanics and have real eigenvalues. Example:

$$H = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$$

Since  $H^\dagger = H$ , it is Hermitian.

### Definition 1.2.9: Eigenvalues and Eigenvectors

For a square matrix  $A \in \mathbb{C}^{n \times n}$ , a vector  $\mathbf{v} \neq \mathbf{0}$  is an **eigenvector** if:

$$A\mathbf{v} = \lambda\mathbf{v}$$

where  $\lambda \in \mathbb{C}$  is the **eigenvalue**. Eigenvalues provide insight into the structure of linear transformations.

#### Example 1.2.1 (Example: Eigenvalues)

For the matrix

$$A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

The characteristic equation is:

$$\det(A - \lambda I) = (1 - \lambda)^2 + 1 = 0$$

Solving gives eigenvalues  $\lambda = 1 \pm i$ .

### Question 1

Show that any unitary matrix preserves the inner product of two vectors.

**Solution:** Since a unitary matrix satisfies  $U^\dagger U = I$ , we have:

$$\langle U\mathbf{v}, U\mathbf{w} \rangle = \mathbf{v}^\dagger (U^\dagger U) \mathbf{w} = \mathbf{v}^\dagger \mathbf{w}$$

Thus, inner products are preserved.

## 1.3 Lecture 3: Quantum Bits and Quantum States

### Definition 1.3.1: Qubit

A **qubit** is the fundamental unit of quantum information. Unlike a classical bit, which is either 0 or 1, a qubit can exist in a **superposition** of states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \text{where } \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1$$

Key features of qubits include:

- **Superposition:** A qubit can exist simultaneously in multiple basis states.
- **Complex Amplitudes:** Coefficients  $\alpha$  and  $\beta$  are complex numbers carrying magnitude and phase information.
- **Interference:** Quantum states can interfere constructively or destructively.
- **Entanglement:** Qubits can be correlated in ways that classical bits cannot.

### Definition 1.3.2: Classical Computing Paradigms

Quantum computing introduces a fundamentally different computational model:

- **Deterministic Computing:** Uses discrete states (0 or 1) with predictable transitions.
- **Analog Computing:** Uses continuous values susceptible to noise accumulation.
- **Probabilistic Computing:** Represents probabilistic mixtures of states.
- **Quantum Computing:** Allows coherent superposition with complex amplitudes and quantum interference.

### Definition 1.3.3: Dirac Notation

Quantum states are represented using **Dirac notation** (bra-ket notation):

- **Ket:**  $|0\rangle, |1\rangle$  represent computational basis states
- Computational basis vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General state:  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

### Definition 1.3.4: Basis States

Common qubit bases include:

- **Computational Basis:**  $|0\rangle, |1\rangle$
- **Hadamard Basis:**

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- **Circular Polarization Basis:**

$$|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \quad |R\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

### Definition 1.3.5: Bloch Sphere

A geometric representation of a single qubit state:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

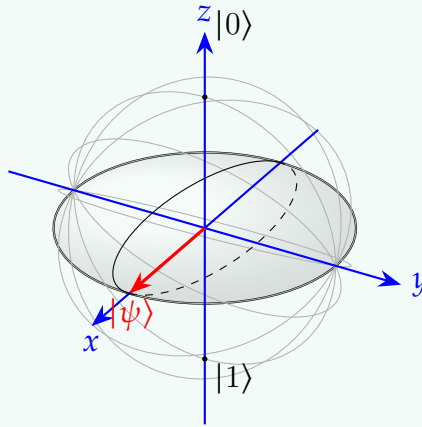
Where:

- $\theta \in [0, \pi]$  is the polar angle
- $\phi \in [0, 2\pi)$  is the azimuthal angle
- Cartesian coordinates:

$$x = \sin\theta \cos\phi, \quad y = \sin\theta \sin\phi, \quad z = \cos\theta$$

#### Example 1.3.1 (Example Bloch Sphere Representation)

For the state  $\theta = \frac{\pi}{2}, \phi = 0$ :



### Definition 1.3.6: Quantum Measurement

When a qubit is measured:

- The quantum state *collapses* to an eigenstate
- Measurement probability depends on squared amplitude
- Computational basis measurement probabilities:

$$P(0) = |\alpha|^2, \quad P(1) = |\beta|^2$$

- Post-measurement state:

$$|\psi_{\text{new}}\rangle = \frac{|b\rangle\langle b|\psi\rangle}{\sqrt{P(b)}}$$

#### Example 1.3.2 (Measurement Example)

For the state  $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$ :

- Probability of measuring  $|0\rangle$ :  $P(0) = \frac{1}{3}$



- Probability of measuring  $|1\rangle$ :  $P(1) = \frac{2}{3}$

### Question 2: Orthonormality Check

Verify the inner products of basis states:

$$\langle 0|1\rangle = 0$$

$$\langle 0|0\rangle = 1$$

$$\langle ++\rangle = 1$$

$$\langle +|-\rangle = 0$$

**Solution:** These relations hold due to the orthonormal nature of quantum basis states.

## 1.4 Lecture 4: Quantum Gates and Transformations

Quantum gates manipulate qubits through unitary transformations, preserving quantum information and enabling quantum computation. This section explores key quantum operations, their mathematical properties, and circuit representations.

### Definition 1.4.1: Qubit Superposition and Hilbert Space

A **qubit** exists in a complex vector space called a **Hilbert space**. The state of a qubit is given by:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \text{where } \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1.$$

The computational basis states are represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

### Definition 1.4.2: Measurement and Superposition Collapse

When a qubit is measured in the computational basis  $\{|0\rangle, |1\rangle\}$ , it collapses to one of the basis states with probability:

$$P(0) = |\alpha|^2, \quad P(1) = |\beta|^2.$$

The post-measurement state is:

$$|\psi_{\text{new}}\rangle = \frac{|b\rangle\langle b|\psi\rangle}{\sqrt{P(b)}}$$

where  $b \in \{0, 1\}$ . This formula captures the quantum measurement postulate and ensures proper normalization of the post-measurement state.

### Definition 1.4.3: Important Quantum States

Several quantum states are particularly important in quantum computing:

- **Computational Basis States:**  $|0\rangle$  and  $|1\rangle$

- **Plus/Minus States:**

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- **Complex Superposition States:**

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \quad |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

### Example 1.4.1 (Example: Equal Superposition State)

A qubit initially in state  $|0\rangle$  is transformed into an equal superposition using the Hadamard gate:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Measuring this state results in either  $|0\rangle$  or  $|1\rangle$  with equal probability  $P(0) = P(1) = \frac{1}{2}$ . Similarly, applying Hadamard to  $|1\rangle$ :

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

### Definition 1.4.4: Quantum Gates and Operations

Quantum gates are unitary matrices that transform qubits. A general qubit transformation is given by:

$$|\psi_{\text{final}}\rangle = U|\psi_{\text{initial}}\rangle$$

where  $U$  is a unitary matrix satisfying  $U^\dagger U = I$ . Key properties of quantum gates include:

- **Reversibility:** All quantum operations are reversible due to unitarity
- **Preservation of Norm:** The normalization condition  $|\alpha|^2 + |\beta|^2 = 1$  is preserved
- **Linearity:** Gates act linearly on superposition states

#### Definition 1.4.5: Rotation Gates

Rotation gates rotate a qubit state around the Bloch sphere:

- **Rotation about X-axis:**

$$R_X(\omega) = \begin{bmatrix} \cos \frac{\omega}{2} & -i \sin \frac{\omega}{2} \\ -i \sin \frac{\omega}{2} & \cos \frac{\omega}{2} \end{bmatrix}$$

Effect: Rotates state by angle  $\omega$  around X-axis

- **Rotation about Y-axis:**

$$R_Y(\omega) = \begin{bmatrix} \cos \frac{\omega}{2} & -\sin \frac{\omega}{2} \\ \sin \frac{\omega}{2} & \cos \frac{\omega}{2} \end{bmatrix}$$

Effect: Rotates state by angle  $\omega$  around Y-axis

- **Rotation about Z-axis:**

$$R_Z(\omega) = \begin{bmatrix} e^{-i\omega/2} & 0 \\ 0 & e^{i\omega/2} \end{bmatrix}$$

Effect: Adds a relative phase between  $|0\rangle$  and  $|1\rangle$  components

Special cases:

- $R_X(\pi) = iX$
- $R_Y(\pi) = iY$
- $R_Z(\pi) = iZ$

#### Definition 1.4.6: Pauli Matrices and Gates

The **Pauli matrices** define fundamental quantum operations:

- **Pauli-X (NOT Gate, Bit-Flip):**

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Effect:  $X|0\rangle = |1\rangle$ ,  $X|1\rangle = |0\rangle$

- **Pauli-Y (Combination of X and Z with phase):**

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Effect:  $Y|0\rangle = i|1\rangle$ ,  $Y|1\rangle = -i|0\rangle$

- **Pauli-Z (Phase-Flip Gate):**

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Effect:  $Z|0\rangle = |0\rangle$ ,  $Z|1\rangle = -|1\rangle$

Each of these matrices is both **Hermitian** ( $A = A^\dagger$ ) and **unitary** ( $A^\dagger A = I$ ).

Important relationships:

- $X^2 = Y^2 = Z^2 = I$
- $XY = iZ$ ,  $YZ = iX$ ,  $ZX = iY$
- $YX = -iZ$ ,  $ZY = -iX$ ,  $XZ = -iY$

#### Definition 1.4.7: Additional Important Gates

- **Hadamard Gate (H):**

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Creates superposition states:  $H|0\rangle = |+\rangle$ ,  $H|1\rangle = |-\rangle$

- **Phase Gate (S):**

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Adds a  $\pi/2$  phase to  $|1\rangle$

- **T Gate:**

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

Adds a  $\pi/4$  phase to  $|1\rangle$

#### Definition 1.4.8: Circuit Notation

Quantum circuits visually represent quantum operations. Each qubit is represented as a horizontal line, and gates are applied sequentially from left to right. Important circuit elements include:

- **Single-qubit gates:** Represented as boxes with gate symbols
- **Measurements:** Depicted with a meter symbol
- **Time flow:** Left to right in circuits (opposite of matrix multiplication order)
- **Initial states:** Usually started in  $|0\rangle$  unless specified otherwise

#### Example 1.4.2 (Example: Complex Circuit Analysis)

Consider the circuit applying the sequence  $HZH$  to  $|0\rangle$ :

$$|\psi_1\rangle = H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|\psi_2\rangle = Z|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|\psi_3\rangle = H|\psi_2\rangle = |1\rangle$$

This sequence performs a NOT operation on  $|0\rangle$  using only Hadamard and Phase-flip gates.

#### Definition 1.4.9: Measurement in Quantum Circuits

Measurement collapses a quantum state to a basis state with probabilities determined by the squared magnitudes of its coefficients. For a state  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ :

- Probability of measuring  $|0\rangle$ :  $P(0) = |\alpha|^2$
- Probability of measuring  $|1\rangle$ :  $P(1) = |\beta|^2$
- Post-measurement state is the measured basis state
- Multiple measurements of identically prepared states give statistical distributions

### Question 3: Exercise 1

Apply the sequence  $SXH$  to  $|0\rangle$  and calculate:

- The final state vector
- The probabilities of measuring  $|0\rangle$  and  $|1\rangle$
- The possible post-measurement states

### Question 4: Exercise 2

Show that the Hadamard gate is its own inverse by calculating  $H^2$ .

### Question 5: Exercise 3

Calculate the effect of applying  $R_Z(\pi/2)$  to the state  $|+\rangle$ .

**Solution:** Exercise 1 Solution:

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ XH|0\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ SXH|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \end{aligned}$$

Therefore:

- Final state:  $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$
- Measurement probabilities:  $P(0) = P(1) = \frac{1}{2}$
- Post-measurement states: Either  $|0\rangle$  or  $|1\rangle$  with equal probability

## 1.5 Lecture 5: Other Quantum Gates, Measurement, Multi-Qubit Systems

### Definition 1.5.1: Single-Qubit Gates

Quantum gates manipulate individual qubits. Key single-qubit gates include:

- **Hadamard Gate (H):**

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Creates superposition:  $H|0\rangle = |+\rangle, H|1\rangle = |-\rangle$

Properties:

- Self-inverse:  $H^2 = I$
- Maps computational basis to  $|\pm\rangle$  basis:

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

- **Phase Gate (S):**

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Adds a  $\pi/2$  phase to  $|1\rangle$ . Properties:

- Unitary but not Hermitian
- $S^2 = Z$
- Effect on  $|+\rangle$  :  $S|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

- **T Gate:**

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Adds a  $\pi/4$  phase to  $|1\rangle$ . Properties:

- $T^2 = S$
- $T^4 = Z$
- Often used in quantum error correction

- **General Phase Gate  $P(\theta)$ :**

$$P(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

Generalizes S and T gates:  $S = P(\pi/2), T = P(\pi/4)$

### Definition 1.5.2: Important Relations

$$Z = HXH$$

$$X = HZH$$

Demonstrates duality between X and Z gates via Hadamard transformation.

Proof:

$$HXH = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

### Definition 1.5.3: Measurement

Measurement collapses quantum states to basis states with probabilities determined by amplitudes.

- **Z-basis:** Standard computational basis ( $|0\rangle, |1\rangle$ )

- For state  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ :

$$P(0) = |\alpha|^2$$

$$P(1) = |\beta|^2$$

- **X-basis:** Hadamard basis ( $|+\rangle, |-\rangle$ )

- \* Measure in Z-basis after applying H gate
- \*  $P(+)=|\langle +|\psi\rangle|^2$
- \*  $P(-)=|\langle -|\psi\rangle|^2$

- **Y-basis:** Eigenstates of Y

- $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$
- $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

#### Definition 1.5.4: Multi-Qubit Systems

States for multiple qubits are represented as tensor products:

$$|\psi\rangle = \sum_{x=0}^{2^n-1} \alpha_x |x\rangle, \quad \sum |\alpha_x|^2 = 1$$

Properties of tensor products:

- Not commutative:  $(|0\rangle \otimes |1\rangle) \neq |1\rangle \otimes |0\rangle$
- Associative:  $((|a\rangle \otimes |b\rangle) \otimes |c\rangle = |a\rangle \otimes (|b\rangle \otimes |c\rangle))$
- Distributive:  $((\alpha|a\rangle + \beta|b\rangle) \otimes |c\rangle = \alpha(|a\rangle \otimes |c\rangle) + \beta(|b\rangle \otimes |c\rangle))$

#### Example 1.5.1 (Example: Two-Qubit System)

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

Bell state example:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

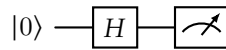
This is a maximally entangled state.

#### Definition 1.5.5: Circuit Diagrams

Quantum circuits visually represent quantum operations on qubits. They help to understand the sequence of quantum gates applied to quantum states. The conventions are:

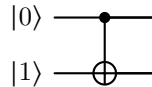
- Qubits are represented as horizontal lines (wires).
- Gates are shown as boxes or symbols on the wires.
- Time flows from left to right (the order of gate application).
- Measurement is depicted using a meter symbol.
- Classical information (post-measurement) is shown with double lines.

#### Single-Qubit Gate Example: Hadamard Gate



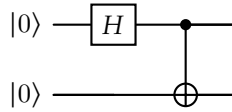
This circuit applies the Hadamard gate  $H$  to the state  $|0\rangle$ , creating the superposition state  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ , followed by measurement.

#### Multi-Qubit System Example: CNOT Gate



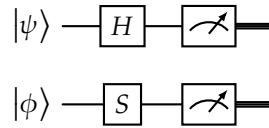
This represents a controlled-NOT (CNOT) gate where the first qubit is the control and the second is the target. If the control qubit is  $|1\rangle$ , the target qubit flips; otherwise, it remains unchanged.

## Bell State Preparation Circuit



This circuit prepares the Bell state  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ . The first qubit undergoes a Hadamard gate, creating a superposition, followed by a CNOT gate that entangles the qubits.

## Measurement in Different Bases



- The first qubit is measured in the X-basis by applying a Hadamard gate before measurement.
- The second qubit undergoes a phase shift with the  $S$  gate and is measured in the computational (Z) basis.

## Exercises

### Question 6: Exercise 1

Prove that the Hadamard gate is unitary and Hermitian.

### Question 7: Exercise 2

For  $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ , find measurement probabilities for  $|00\rangle$  and  $|11\rangle$ .

### Question 8: Exercise 3

Determine if  $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$  is unitary.

### Question 9: Exercise 4

If we apply  $H \otimes H$  to  $|00\rangle$ , what state do we get?

**Solution:** Exercise 1 Solution: To prove  $H$  is unitary and Hermitian:

$$H^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

$$HH = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus,  $H$  is both unitary ( $HH^\dagger = I$ ) and Hermitian ( $H = H^\dagger$ ).

**Solution:** Exercise 2 Solution: For  $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ :

$$P(00) = |\langle 00 | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

$$P(11) = |\langle 11 | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$



**Solution:** Exercise 3 Solution: To verify unitarity, compute  $UU^\dagger$ :

$$U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$
$$UU^\dagger = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore,  $U$  is unitary.

**Solution:** Exercise 4 Solution:

$$\begin{aligned} (H \otimes H)|00\rangle &= (H|0\rangle) \otimes (H|0\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \end{aligned}$$

This creates an equal superposition of all two-qubit basis states.

## Chapter 2

# Phase II: Fundamentals of Quantum Algorithms

## Chapter 3

# Phase III: Advanced Quantum Algorithms

## Chapter 4

# Phase IV: Special Topics in Quantum Computing

## Chapter 5

# Phase V: Concluding Lectures