MIT 18.01 Problem Set 7 Unofficial Solutions

Q1) (from PS6) The voltage V of house current is given by

$$V(t) = Csin(120\pi t)$$

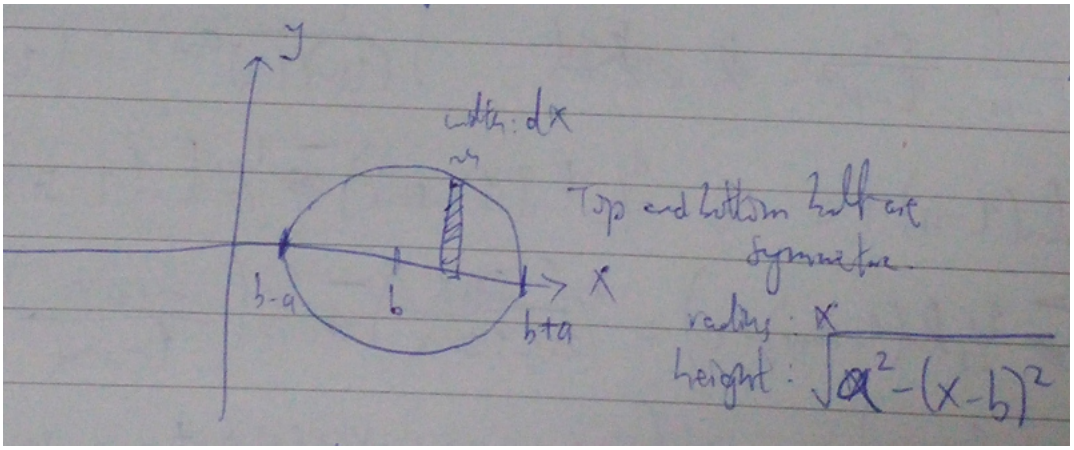
where t is time, in seconds and C is a constant amplitude. The square root of the average value of V^2 over one period of $V(t)$ (or cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant C. (The peak voltage delivered to the house is $\pm C$. The units of V^2 are square volts; when we take the square root again after averaging, the units become volts again.)

Average value of V^2 over 1 period of $V(t)$ is

$$\begin{aligned}
\frac{1}{60}\int_0^{\frac{1}{60}} C^2 sin^2(120\pi t)dt &= \frac{C^2}{60}\int_0^{\frac{1}{60}} sin^2(120\pi t)dt \\
&= \frac{C^2}{60}\int_0^{\frac{1}{60}} \frac{1-cos(240\pi t)}{2}dt \\
&= \frac{C^2}{120}\int_0^{\frac{1}{60}} 1-cos(240\pi t)dt \\
&= \frac{C^2}{120}(t - \frac{sin(240\pi t)}{240\pi})\Big]_0^{\frac{1}{60}} \\
&= \frac{C^2}{120}(\frac{1}{60} - \frac{sin(240\pi \cdot \frac{1}{60})}{240\pi}) \\
&= \frac{C^2}{120}(\frac{1}{60} - \frac{sin(4\pi)}{240\pi}) \\
&= \frac{C^2}{120}(\frac{1}{60}) \\
&= \frac{C^2}{7200}
\end{aligned}$$

Square root of average value of V^2 over 1 period of $V(t) = \sqrt{\frac{C^2}{7200}} = \frac{C}{\sqrt{3600*2}} = \frac{C}{60\sqrt{2}}$

Q2) The solid torus is the figure obtained by rotating the disk $(x-b)^2 + y^2 \leq a^2$ around the y-axis. Find its volume by the method of shells. (Hint: Substitute for $x - b$. As noted p. 229/11, the answer happens to be the area of the disk multiplied by the distance travelled by the center as it revolves.)



For a circle centered at $x = b, y = 0$ with radius a, the volume of the torus is:

$$2\int_{b-a}^{b+a} 2\pi x(\sqrt{a^2-(x-b)^2})dx = 4\pi\int_{b-a}^{b+a} x\sqrt{a^2-(x-b)^2}dx$$

Let $u = x - b$. Then $du = dx$. Also, $x = u + b$. Substitute those into the above:

$$\begin{aligned}
4\pi\int_{b-a}^{b+a} x\sqrt{a^2-(x-b)^2}dx &= 4\pi\int_{-a}^{a}(u+b)\sqrt{a^2-u^2}du \\
&= 4\pi(\int_{-a}^{a} u\sqrt{a^2-u^2}du + b\int_{-a}^{a}\sqrt{a^2-u^2}du) \\
&= 4\pi(-\frac{1}{2}\cdot\frac{(a^2-u^2)^{3/2}}{\frac{3}{2}}\Big]_{-a}^{a} + b\int_{-a}^{a}\sqrt{a^2-u^2}du) \\
&= 4\pi b\int_{-a}^{a}\sqrt{a^2-u^2}du \qquad \text{(area of semicircle of radius } a \text{ centered at origin)} \\
&= 4\pi b(\frac{1}{2}\pi a^2) \\
&= 2\pi^2 a^2 b
\end{aligned}$$