MIT 18.01 Problem Set 7 Unofficial Solutions

Q1) (from PS6) The voltage V of house current is given by

$$V(t) = Csin(120\pi t)$$

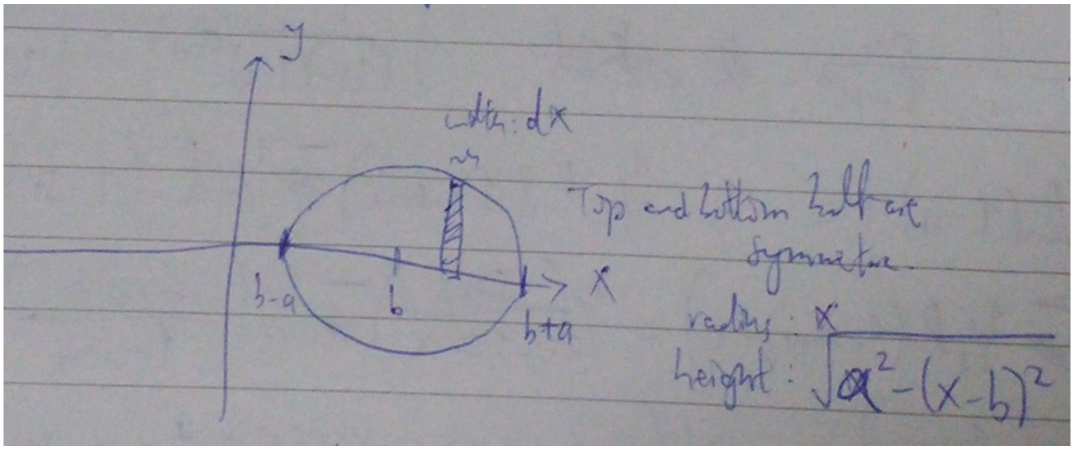
where t is time, in seconds and C is a constant amplitude. The square root of the average value of V^2 over one period of $V(t)$ (or cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant C. (The peak voltage delivered to the house is $\pm C$. The units of V^2 are square volts; when we take the square root again after averaging, the units become volts again.)

Average value of V^2 over 1 period of $V(t)$ is

$$\begin{aligned}
\frac{1}{60}\int_0^{\frac{1}{60}} C^2 sin^2(120\pi t)dt &= \frac{C^2}{60}\int_0^{\frac{1}{60}} sin^2(120\pi t)dt \\
&= \frac{C^2}{60}\int_0^{\frac{1}{60}} \frac{1-cos(240\pi t)}{2}dt \\
&= \frac{C^2}{120}\int_0^{\frac{1}{60}} 1-cos(240\pi t)dt \\
&= \frac{C^2}{120}(t-\frac{sin(240\pi t)}{240\pi})\Big]_0^{\frac{1}{60}} \\
&= \frac{C^2}{120}(\frac{1}{60}-\frac{sin(240\pi\cdot\frac{1}{60})}{240\pi}) \\
&= \frac{C^2}{120}(\frac{1}{60}-\frac{sin(4\pi)}{240\pi}) \\
&= \frac{C^2}{120}(\frac{1}{60}) \\
&= \frac{C^2}{7200}
\end{aligned}$$

Square root of average value of V^2 over 1 period of $V(t) = \sqrt{\frac{C^2}{7200}} = \frac{C}{\sqrt{3600*2}} = \frac{C}{60\sqrt{2}}$

Q2) The solid torus is the figure obtained by rotating the disk $(x-b)^2 + y^2 \leq a^2$ around the y-axis. Find its volume by the method of shells. (Hint: Substitute for $x - b$. As noted p. 229/11, the answer happens to be the area of the disk multiplied by the distance travelled by the center as it revolves.)



For a circle centered at $x = b, y = 0$ with radius a, the volume of the torus is:

$$2\int_{b-a}^{b+a} 2\pi x(\sqrt{a^2-(x-b)^2})dx = 4\pi\int_{b-a}^{b+a} x\sqrt{a^2-(x-b)^2}dx$$

Let $u = x - b$. Then $du = dx$. Also, $x = u + b$. Substitute those into the above:

$$\begin{aligned}
4\pi\int_{b-a}^{b+a} x\sqrt{a^2-(x-b)^2}dx &= 4\pi\int_{-a}^{a}(u+b)\sqrt{a^2-u^2}du \\
&= 4\pi(\int_{-a}^{a} u\sqrt{a^2-u^2}du + b\int_{-a}^{a}\sqrt{a^2-u^2}du) \\
&= 4\pi(-\frac{1}{2}\cdot\frac{(a^2-u^2)^{3/2}}{\frac{3}{2}}\Big]_{-a}^{a} + b\int_{-a}^{a}\sqrt{a^2-u^2}du) \\
&= 4\pi b\int_{-a}^{a}\sqrt{a^2-u^2}du \quad \text{(area of semicircle of radius } a \text{ centered at origin)} \\
&= 4\pi b(\frac{1}{2}\pi a^2) \\
&= 2\pi^2a^2b
\end{aligned}$$

Q3a) For any integer $n \geq 0$, use the substitution $tan^2x = sec^2x - 1$ to show that

$\int tan^{n+2}x\ dx = \frac{1}{n+1}tan^{n+1}x - \int tan^n x\ dx$

$$\begin{aligned}
\int tan^{n+2}x\ dx &= \int tan^2x\ tan^nx\ dx \\
&= \int (sec^2x-1)tan^nx\ dx \\
&= \int sec^2x\ tan^nx - tan^nx\ dx \\
&= \frac{tan^{n+1}x}{n+1} - \int tan^nx\ dx
\end{aligned}$$

Q3b) Deduce a formula for $\int tan^4 x\ dx$

$$
\begin{aligned}
\int tan^4 x\ dx &= \int tan^{2+2} x\ dx \\
&= \frac{1}{2+1} tan^{2+1} x - \int tan^2 x\ dx \\
&= \frac{1}{3} tan^3 x - \int tan^{0+2} x\ dx \\
&= \frac{1}{3} tan^3 x - (\frac{1}{0+1} tan^{0+1} x - \int tan^0 x\ dx \\
&= \frac{1}{3} tan^3 x - tan\ x + \int 1 dx \\
&= \frac{1}{3} tan^3 x - tan\ x + x + C
\end{aligned}
$$

Verify:

$$
\begin{aligned}
\frac{d}{dx}(\frac{1}{3} tan^3 x - tan\ x + x + C) &= tan^2 x\ sec^2 x - sec^2 x + 1 \\
&= tan^2 x(1 + tan^2 x) - (1 + tan^2 x) + 1 \\
&= tan^2 x + tan^4 x - 1 - tan^2 x + 1 \\
&= tan^4 x
\end{aligned}
$$

Q4a) Derive a formula for $\int sec\ x\ dx$ by writing $sec\ x = \frac{cos\ x}{1 - sin^2 x}$ (verify this), and then making a substitution for $sinx$ and using partial fractions. (Your final answer must be expressed in terms of x.)

$$
\begin{aligned}
\frac{cos\ x}{1 - sin^2 x} &= \frac{cos\ x}{cos^2 x} \\
&= \frac{1}{cos\ x} \\
&= sec\ x
\end{aligned}
$$

For $\int secx\ dx = \int \frac{cos\ x}{1 - sin^2 x} dx$, let $u = sinx$. Then $du = cos\ x\ dx$

$$
\begin{aligned}
\frac{cos\ x}{1 - sin^2 x} dx &= \int \frac{du}{1 - u^2} \\
&= \int \frac{1}{(1+u)(1-u)} du \\
&= \int \frac{1/2}{1+u} + \frac{1/2}{1-u} du \\
&= \frac{1}{2} ln(1+u) - \frac{1}{2} ln(1-u) + C \\
&= \frac{1}{2} ln(1 + sin\ x) - \frac{1}{2} ln(1 - sin\ x) + C
\end{aligned}
$$

Verify:

$$
\begin{aligned}
\frac{1}{2}ln(1+sin\ x)-\frac{1}{2}ln(1-sin\ x)+C &= \frac{1}{2}(\frac{cos\ x}{1+sin\ x})-\frac{1}{2}(\frac{-cos\ x}{1-sinx}) \\
&= \frac{1}{2}(\frac{cos\ x(1-sin\ x)+cos\ x(1+sin\ x)}{(1+sin\ x)(1-sin\ x)}) \\
&= \frac{1}{2}(\frac{cos\ x-sin\ x\ cos\ x+cos\ x+sin\ x\ cos\ x}{1-sin^2x}) \\
&= \frac{1}{2}(\frac{2cos\ x}{cos^2x}) \\
&= \frac{1}{cos\ x} \\
&= sec\ x
\end{aligned}
$$

Q4b) Convert the formula into the more familiar one by multiplying the fraction in the answer on both top and bottom by $1+sin\ x$. (Note that $(1/2)ln\ u = ln\sqrt{u}$

$$
\begin{aligned}
\frac{1}{2}ln(1+sin\ x)-\frac{1}{2}ln(1-sin\ x)+C &= \frac{1}{2}(ln(1+sin\ x)-ln(1-sin\ x)) \\
&= \frac{1}{2}ln(\frac{1+sin\ x}{1-sin\ x}) \\
&= \frac{1}{2}ln(\frac{1+sin\ x}{1-sin\ x}\cdot\frac{1+sin\ x}{1+sin\ x}) \\
&= \frac{1}{2}ln(\frac{sin^2x+2sin\ x+1}{1-sin^2x}) \\
&= \frac{1}{2}ln(\frac{sin^2x+2sin\ x+1}{cos^2x}) \\
&= \frac{1}{2}ln(\frac{sin^2x}{cos^2x}+\frac{2sin\ x}{cos^2x}+\frac{1}{cos^2x}) \\
&= \frac{1}{2}ln(tan^2x+2sec\ x\ tan\ x+sec^2x) \\
&= \frac{1}{2}ln((sec\ x+tan\ x)^2) \\
&= ln(\sqrt{(sec\ x+tan\ x)^2}) \\
&= ln(sec\ x+tan\ x)
\end{aligned}
$$