MIT 18.01 Problem Set 6 Unofficial Solutions

Q1) Do 7.4/12 and 13.

Skipped. We do not have the textbook.

Q2) The voltage V of the house current is given by

$$V(t) = Csin(120\pi t)$$

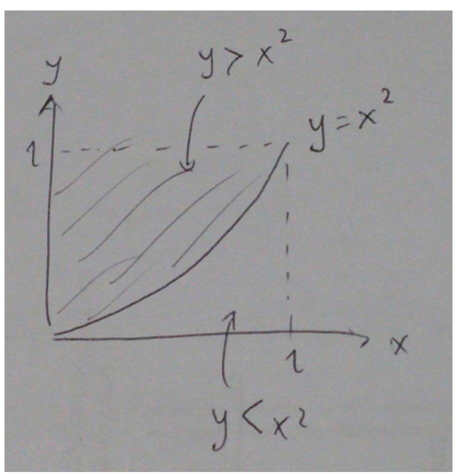
where t is time, in seconds and C is a constant amplitude. The square root of the average value of V^2 over one period of $V(t)$ (or cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant C. (The peak voltage delivered to the house is $\pm C$. The units of V^2 are square volts; when we take the square root again after averaging, the units become volts again.)

Every cycle of the *sin* function corresponds to 2π. This happens every $t = \frac{2\pi}{120\pi} = \frac{1}{60}$ seconds.
Since $V(t) = Csin(120\pi t)$, $V^2(t) = C^2 sin^2(120\pi t)$. The average value of V^2 over 1 cycle of $V(t)$ is:

$$\begin{aligned}
\frac{1}{\frac{1}{60} - 0} \int_0^{\frac{1}{60}} C^2 sin^2(120\pi t)dt &= 60C^2 \int_0^{\frac{1}{60}} sin^2(120\pi t)dt \\
&= 60C^2 (\frac{1}{2}t - \frac{1}{240\pi} sin(120\pi t)cos(120\pi t))\Big]_0^{1/60} \\
&= 60C^2 (\frac{1}{2} \cdot \frac{1}{60} - \frac{1}{240\pi} sin(2\pi)cos(2\pi) - (\frac{1}{2} \cdot 0 - \frac{1}{240\pi} sin(0)cos(0))) \\
&= 60C^2 \cdot \frac{1}{120} \\
&= \frac{C^2}{2}
\end{aligned}$$

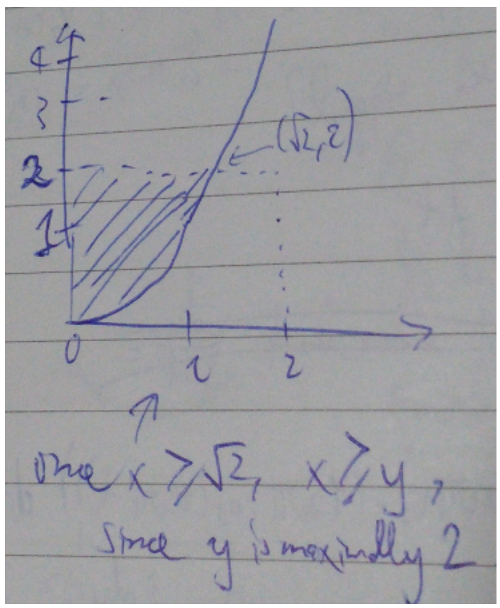
Then RMS $= \sqrt{\frac{C^2}{2}} = \frac{C}{\sqrt{2}} = \frac{\sqrt{2}C}{2}$

Q3a) What is the probability that $x^2 < y$ if (x, y) is chosen from the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ with probability equal to the area.



$$
\begin{aligned}
P(x^2 < y) &= \frac{\int_0^1 1dx - \int_0^1 x^2dx}{\int_0^1 1dx} \\
&= \frac{x\Big]_0^1 - \frac{x^3}{3}\Big]_0^1}{x\Big]_0^1} \\
&= \frac{1 - \frac{1}{3}}{1} \\
&= \frac{2}{3}
\end{aligned}
$$

Q3b) What is the probability that $x^2 < y$ if (x, y) is chosen from the square $0 \leq x \leq 2$, $0 \leq y \leq 2$ with probability **proportional** to the area. (Probability = Part/Whole).



For the line $y = x^2$ $y = 2$, $x = \sqrt{2}$. Once $x \geq \sqrt{2}$, $y > 2$. Hence $x \leq \sqrt{2}$.

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\begin{aligned}
P(x^2 < y) &= \frac{\int_0^2 2dx - \int_0^{\sqrt{2}} x^2 dx - \int_{\sqrt{2}}^2 2dx}{\int_0^2 2dx} \\
&= \frac{2x\Big]_0^2 - \frac{x^3}{3}\Big]_0^{\sqrt{2}} - 2x\Big]_{\sqrt{2}}^2}{2x\Big]_0^2} \\
&= \frac{4 - \frac{(\sqrt{2})^3}{3} - (4 - 2\sqrt{2})}{4} \\
&= \frac{4 - \frac{2\sqrt{2}}{3} - 4 + 2\sqrt{2}}{4} \\
&= \frac{2\sqrt{2} - \frac{2\sqrt{2}}{3}}{4} \\
&= \frac{\frac{6\sqrt{2} - 2\sqrt{2}}{3}}{4} \\
&= \frac{\frac{4\sqrt{2}}{3}}{4} \\
&= \frac{\sqrt{2}}{3}
\end{aligned}
$$

Q3c) Evaluate

$$W = \int_0^\infty e^{-at} dt = \lim_{N \to \infty} \int_0^N e^{-at} dt$$

This is known as an improper integral because it represents the area of an unbounded region.
We are using the letter W to signify "whole".
The probability that a radioactive particle will decay some time in the interval $0 \leq t \leq T$ is

$$P([0,T]) = \frac{\text{PART}}{\text{WHOLE}} = \frac{1}{W} \int_0^T e^{-at} dt$$

Note that $P([0,\infty)) = 1 = 100\%$

$$
\begin{aligned}
\int_0^\infty e^{-at} dt &= -\frac{1}{a} e^{-at}\Big]_0^\infty \\
&= -\frac{1}{a}(e^{-a\cdot\infty} - e^{-a\cdot 0}) \\
&= -\frac{1}{a}(\frac{1}{e^{a\cdot\infty}} - \frac{1}{e^{a\cdot 0}}) \\
&= -\frac{1}{a}(0 - \frac{1}{e^0}) \\
&= -\frac{1}{a}(-1) \\
&= \frac{1}{a}
\end{aligned}
$$