MIT 18.01 Problem Set 4 Unofficial Solutions

Q1a) Use the mean value property to show that if $f(0) = 0$ and $f'(x) \geq 0$, then $f(x) \geq 0$ for all $x \geq 0$.

For any $x > 0$, $f(x) = f(0) + f'(c)(x - 0)$, for some $0 < c < x$

Since $f'(x) \geq 0$ for all x, then for all $x > 0$, $f(x) \geq f(0)$.

Since $f(x) \geq f(0) \geq 0$ for all $x > 0$, then $f(x) \geq 0$ for all $x > 0$.

Since $f(0) = 0$ and $f(x) \geq 0$ for all $x > 0$, then $f(x) \geq 0$ for all $x \geq 0$.

Q1b) Deduce from part (a) that $ln(1 + x) \leq x$ for $x \geq 0$. Hint: Use $f(x) = x - ln(1 + x)$.

Let $f(x) = x - ln(1 + x)$

Then $f(0) = 0 - ln(1 + 0) = 0 - 0 = 0$

$f'(x) = 1 - \frac{1}{1+x}$

For all $x \geq 0$, $1 + x \geq 1$ and $\frac{1}{1+x} \leq 1$ and $1 - \frac{1}{1+x} \geq 0$

Since $f'(x) = 1 - \frac{1}{1+x} \geq 0$, therefore $f'(x) \geq 0$ for $x \geq 0$.

By 1(a), $f(x) = x - ln(1 + x) \geq 0$ for $x \geq 0$

Then $x \geq ln(1 + x)$ for $x \geq 0$

Q1c) Use the same method as in (b) to show $ln(1 + x) \geq x - x^2/2$ and $ln(1 + x) \leq x - x^2/2 + x^3/3$ for $x \geq 0$.

Let $f_1(x) = ln(1 + x) - x + \frac{x^2}{2}$

$f_1(0) = ln(1 + 0) - 0 + \frac{0^2}{2} = 0$

$f_1'(x) = \frac{1}{1+x} - 1 + x = \frac{1-(1+x)+x(1+x)}{1+x} = \frac{1-1-x+x+x^2}{1+x} = \frac{x^2}{1+x}$

Since $1 + x \geq 1$ for all $x \geq 0$ and $x^2 \geq 0$ for all $x \geq 0$, then $f_1'(x) = \frac{x^2}{1+x} \geq 0$ for all $x \geq 0$.

By 1(a), $f_1(x) \geq 0$ for all $x \geq 0$, which is equivalent to $ln(1 + x) - x - \frac{x^2}{2} \geq 0$ for all $x \geq 0$. Hence $ln(1 + x) \geq x - \frac{x^2}{2}$ for all $x \geq 0$.

Let $f_2(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ln(1 + x)$

$f_2(0) = 0 - \frac{0^2}{2} + \frac{0^3}{3} - ln(1 + 0) = 0$

$f_2'(x) = 1 - x + x^2 - \frac{1}{1+x} = \frac{1(1+x)-x(1+x)+x^2(1+x)-1}{1+x} = \frac{1+x-x-x^2+x^2+x^3}{1+x} = \frac{x^3}{1+x}$

Since $1 + x \geq 1$ for all $x \geq 0$ and $x^3 \geq 0$ for all $x \geq 0$, then $f_2'(x) = \frac{x^3}{1+x} \geq 0$ for all $x \geq 0$.

By 1(a), $f_2(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ln(1 + x) \geq 0$ for all $x \geq 0$. Hence $x - \frac{x^2}{2} + \frac{x^3}{3} \geq ln(1 + x)$ for all $x \geq 0$.

Q1d) Find the pattern in (b) and (c) and make a general conjecture.

General conjecture:

$ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^{2n+1}}{2n+1} = \sum\limits_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}$ for all $x \geq 0, n \geq 0$

$ln(1+x) \geq x - \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^{2n}}{2n} = \sum\limits_{k=1}^{2n} (-1)^{k+1} \frac{x^k}{k}$ for all $x \geq 0, n \geq 1$

Let $f_1(x) = (\sum\limits_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}) - ln(x+1)$ for any $n \geq 1$.

$f_1(0) = (\sum\limits_{k=1}^{2n+1} (-1)^{k+1} \frac{0^k}{k}) - ln(0+1) = 0$

$$\begin{aligned}
f_1'(x) &= (\sum_{k=1}^{2n+1} (-1)^{k+1} x^{k-1}) - \frac{1}{x+1} \\
&= \frac{(x+1)(\sum\limits_{k=1}^{2n+1} (-1)^{k+1} x^{k-1}) - 1}{x+1} \\
&= \frac{(\sum\limits_{k=1}^{2n+1} (-1)^{k+1} (x^k + x^{k-1})) - 1}{x+1} \\
&= \frac{(x^1 + x^0) - (x^2 + x^1) + ... + (-1)^{2n+1+1}(x^{2n+1} + x^{2n}) - 1}{x+1} \\
&= \frac{x^0 + x^{2n+1} - 1}{x+1} \\
&= \frac{x^{2n+1}}{x+1}
\end{aligned}$$

Since $x + 1 \geq 1$ for $x \geq 0$ and $x^{2n+1} \geq 0$ for $n \geq 0, x \geq 0$, then $f_1'(x) = \frac{x^{2n+1}}{x+1} \geq 0$ for all $x \geq 0$.

By 1(a), $f_1(x) = (\sum\limits_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}) - ln(x+1) \geq 0$ for all $x \geq 0, n \geq 0$. Hence $ln(x+1) \leq \sum\limits_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}$ for all $x \geq 0, n \geq 0$.

Let $f_2(x) = ln(x+1) - \sum\limits_{k=1}^{2n} (-1)^{k+1} \frac{x^k}{k}$ for any $n \geq 0$.

$f_2(0) = ln(0+1) - \sum\limits_{k=1}^{2n} (-1)^{k+1} \frac{0^k}{k} = 0$

$$
\begin{aligned}
f_2'(x) &= \frac{1}{x+1} - \sum_{k=1}^{2n}(-1)^{k+1}x^{k-1} \\
&= \frac{1-(x+1)\sum\limits_{k=1}^{2n} x^{k-1}}{x+1} \\
&= \frac{1-\sum\limits_{k=1}^{2n}(x^k+x^{k-1})}{x+1} \\
&= \frac{1-((x^1+x^0)-(x^2+x^1)+...+(-1)^{2n+1}(x^{2n}+x^{2n-1}))}{x+1} \\
&= \frac{1-(x^0-x^{2n})}{x+1} \\
&= \frac{x^{2n}}{x+1}
\end{aligned}
$$

For $x \geq 0$, $x+1 \geq 1$ and $x^{2n} \geq 0$ for $x \geq 0, n \geq 1$. Hence $f_2'(x) = \frac{x^{2n}}{x+1} \geq 0$ for $n \geq 1, x \geq 0$.

By 1(a), $f_2(x) = ln(x+1) - \sum\limits_{k=1}^{2n}(-1)^{k+1}x^{k-1} \geq 0$ for all $n \geq 1, x \geq 0$.

Then $ln(x+1) \geq \sum\limits_{k=1}^{2n}(-1)^{k+1}x^{k-1}$ for all $n \geq 1, x \geq 0$.

Q1e) Show that $ln(1+x) \leq x$ for $-1 < x \leq 0$. (Use the change of variable $u = -x$.)

Let $u = -x$. Then $x = -u$.

We want to show that $ln(1-u) \leq -u$ for $0 \leq u < 1$.

Let $f(u) = -u - ln(1-u)$

$f(0) = -0 - ln(1-0) = 0$

$f'(u) = -1 - \frac{-1}{1-u} = -1 + \frac{1}{1-u} = \frac{-1(1-u)+1}{1-u} = \frac{u-1+1}{1-u} = \frac{u}{1-u}$

Since $0 \leq u < 1$, then $1-u > 0$. Hence $f'(u) = \frac{u}{1-u} \geq 0$ for $0 \leq u < 1$

By 1(a), $f(u) = -u - ln(1-u) \geq 0$ for $0 \leq u < 1$.

Then $f(x) = -(-x) - ln(1-(-x)) = x - ln(1+x) \geq 0$ for $0 \leq -x < 1$ or $-1 < x \leq 0$.

Hence $ln(1+x) \leq x$ for $-1 < x \leq 0$

Q2a) Do 5.3/68

I do not have the textbook. Skipped.

Q2b) Show that both of the following integrals are correct, and explain.

$$\int tanx\ sec^2x\ dx = (1/2)tan^2x; \int tanx\ sec^2x\ dx = (1/2)sec^2x$$

Let $u = tan\ x$. We have

$$
\begin{aligned}
\int tanx\ sec^2 dx &= \int (sec^2 x)\ tanx\ dx \\
&= \int u'\ u\ du \\
&= \frac{1}{2}u^2 + c \\
&= \frac{1}{2}tan^2 x + c
\end{aligned}
$$

Now we want to prove that $\int tanx\ sec^2 x\ dx = (1/2)sec^2 x$. Let $u = sec\ x$. Then $\int u'\ u\ du = \frac{1}{2}u^2 + c = \frac{1}{2}sec^2 x + c$