

## MIT 18.01 Problem Set 7 Unofficial Solutions

**Q1)** (from PS6) The voltage  $V$  of house current is given by

$$V(t) = C \sin(120\pi t)$$

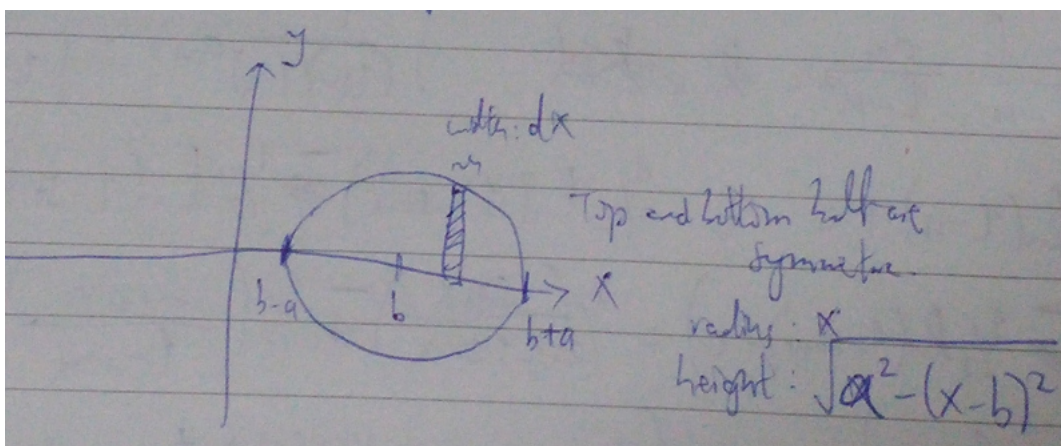
where  $t$  is time, in seconds and  $C$  is a constant amplitude. The square root of the average value of  $V^2$  over one period of  $V(t)$  (or cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant  $C$ . (The peak voltage delivered to the house is  $\pm C$ . The units of  $V^2$  are square volts; when we take the square root again after averaging, the units become volts again.)

Average value of  $V^2$  over 1 period of  $V(t)$  is

$$\begin{aligned} \frac{1}{60} \int_0^{\frac{1}{60}} C^2 \sin^2(120\pi t) dt &= \frac{C^2}{60} \int_0^{\frac{1}{60}} \sin^2(120\pi t) dt \\ &= \frac{C^2}{60} \int_0^{\frac{1}{60}} \frac{1 - \cos(240\pi t)}{2} dt \\ &= \frac{C^2}{120} \int_0^{\frac{1}{60}} 1 - \cos(240\pi t) dt \\ &= \frac{C^2}{120} \left( t - \frac{\sin(240\pi t)}{240\pi} \right) \Bigg|_0^{\frac{1}{60}} \\ &= \frac{C^2}{120} \left( \frac{1}{60} - \frac{\sin(240\pi \cdot \frac{1}{60})}{240\pi} \right) \\ &= \frac{C^2}{120} \left( \frac{1}{60} - \frac{\sin(4\pi)}{240\pi} \right) \\ &= \frac{C^2}{120} \left( \frac{1}{60} \right) \\ &= \frac{C^2}{7200} \end{aligned}$$

Square root of average value of  $V^2$  over 1 period of  $V(t) = \sqrt{\frac{C^2}{7200}} = \frac{C}{\sqrt{3600 \cdot 2}} = \frac{C}{60\sqrt{2}}$

**Q2)** The solid torus is the figure obtained by rotating the disk  $(x - b)^2 + y^2 \leq a^2$  around the  $y$ -axis. Find its volume by the method of shells. (Hint: Substitute for  $x - b$ . As noted p. 229/11, the answer happens to be the area of the disk multiplied by the distance travelled by the center as it revolves.)



For a circle centered at  $x = b, y = 0$  with radius  $a$ , the volume of the torus is:

$$2 \int_{b-a}^{b+a} 2\pi x (\sqrt{a^2 - (x-b)^2}) dx = 4\pi \int_{b-a}^{b+a} x \sqrt{a^2 - (x-b)^2} dx$$

Let  $u = x - b$ . Then  $du = dx$ . Also,  $x = u + b$ . Substitute those into the above:

$$\begin{aligned} 4\pi \int_{b-a}^{b+a} x \sqrt{a^2 - (x-b)^2} dx &= 4\pi \int_{-a}^a (u+b) \sqrt{a^2 - u^2} du \\ &= 4\pi \left( \int_{-a}^a u \sqrt{a^2 - u^2} du + b \int_{-a}^a \sqrt{a^2 - u^2} du \right) \\ &= 4\pi \left( -\frac{1}{2} \cdot \frac{(a^2 - u^2)^{3/2}}{\frac{3}{2}} \right) \Big|_{-a}^a + b \int_{-a}^a \sqrt{a^2 - u^2} du \\ &= 4\pi b \int_{-a}^a \sqrt{a^2 - u^2} du \quad (\text{area of semicircle of radius } a \text{ centered at origin}) \\ &= 4\pi b \left( \frac{1}{2} \pi a^2 \right) \\ &= 2\pi^2 a^2 b \end{aligned}$$

**Q3a)** For any integer  $n \geq 0$ , use the substitution  $\tan^2 x = \sec^2 x - 1$  to show that

$$\int \tan^{n+2} x \, dx = \frac{1}{n+1} \tan^{n+1} x - \int \tan^n x \, dx$$

$$\begin{aligned} \int \tan^{n+2} x \, dx &= \int \tan^2 x \tan^n x \, dx \\ &= \int (\sec^2 x - 1) \tan^n x \, dx \\ &= \int \sec^2 x \tan^n x - \tan^n x \, dx \\ &= \frac{\tan^{n+1} x}{n+1} - \int \tan^n x \, dx \end{aligned}$$

**Q3b)** Deduce a formula for  $\int \tan^4 x \, dx$

$$\begin{aligned}
 \int \tan^4 x \, dx &= \int \tan^{2+2} x \, dx \\
 &= \frac{1}{2+1} \tan^{2+1} x - \int \tan^2 x \, dx \\
 &= \frac{1}{3} \tan^3 x - \int \tan^{0+2} x \, dx \\
 &= \frac{1}{3} \tan^3 x - \left( \frac{1}{0+1} \tan^{0+1} x - \int \tan^0 x \, dx \right) \\
 &= \frac{1}{3} \tan^3 x - \tan x + \int 1 \, dx \\
 &= \frac{1}{3} \tan^3 x - \tan x + x + C
 \end{aligned}$$

Verify:

$$\begin{aligned}
 \frac{d}{dx} \left( \frac{1}{3} \tan^3 x - \tan x + x + C \right) &= \tan^2 x \sec^2 x - \sec^2 x + 1 \\
 &= \tan^2 x (1 + \tan^2 x) - (1 + \tan^2 x) + 1 \\
 &= \tan^2 x + \tan^4 x - 1 - \tan^2 x + 1 \\
 &= \tan^4 x
 \end{aligned}$$

**Q4a)** Derive a formula for  $\int \sec x \, dx$  by writing  $\sec x = \frac{\cos x}{1 - \sin^2 x}$  (verify this), and then making a substitution for  $\sin x$  and using partial fractions. (Your final answer must be expressed in terms of  $x$ .)

$$\begin{aligned}
 \frac{\cos x}{1 - \sin^2 x} &= \frac{\cos x}{\cos^2 x} \\
 &= \frac{1}{\cos x} \\
 &= \sec x
 \end{aligned}$$

For  $\int \sec x \, dx = \int \frac{\cos x}{1 - \sin^2 x} dx$ , let  $u = \sin x$ . Then  $du = \cos x \, dx$

$$\begin{aligned}
 \frac{\cos x}{1 - \sin^2 x} dx &= \int \frac{du}{1 - u^2} \\
 &= \int \frac{1}{(1+u)(1-u)} du \\
 &= \int \frac{1/2}{1+u} + \frac{1/2}{1-u} du \\
 &= \frac{1}{2} \ln(1+u) - \frac{1}{2} \ln(1-u) + C \\
 &= \frac{1}{2} \ln(1 + \sin x) - \frac{1}{2} \ln(1 - \sin x) + C
 \end{aligned}$$

Verify:

$$\begin{aligned}
 \frac{1}{2}\ln(1 + \sin x) - \frac{1}{2}\ln(1 - \sin x) + C &= \frac{1}{2}\left(\frac{\cos x}{1 + \sin x}\right) - \frac{1}{2}\left(\frac{-\cos x}{1 - \sin x}\right) \\
 &= \frac{1}{2}\left(\frac{\cos x(1 - \sin x) + \cos x(1 + \sin x)}{(1 + \sin x)(1 - \sin x)}\right) \\
 &= \frac{1}{2}\left(\frac{\cos x - \sin x \cos x + \cos x + \sin x \cos x}{1 - \sin^2 x}\right) \\
 &= \frac{1}{2}\left(\frac{2\cos x}{\cos^2 x}\right) \\
 &= \frac{1}{\cos x} \\
 &= \sec x
 \end{aligned}$$

**Q4b)** Convert the formula into the more familiar one by multiplying the fraction in the answer on both top and bottom by  $1 + \sin x$ . (Note that  $(1/2)\ln u = \ln\sqrt{u}$ )

$$\begin{aligned}
 \frac{1}{2}\ln(1 + \sin x) - \frac{1}{2}\ln(1 - \sin x) + C &= \frac{1}{2}(\ln(1 + \sin x) - \ln(1 - \sin x)) \\
 &= \frac{1}{2}\ln\left(\frac{1 + \sin x}{1 - \sin x}\right) \\
 &= \frac{1}{2}\ln\left(\frac{1 + \sin x}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x}\right) \\
 &= \frac{1}{2}\ln\left(\frac{\sin^2 x + 2\sin x + 1}{1 - \sin^2 x}\right) \\
 &= \frac{1}{2}\ln\left(\frac{\sin^2 x + 2\sin x + 1}{\cos^2 x}\right) \\
 &= \frac{1}{2}\ln\left(\frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{1}{\cos^2 x}\right) \\
 &= \frac{1}{2}\ln(\tan^2 x + 2\sec x \tan x + \sec^2 x) \\
 &= \frac{1}{2}\ln((\sec x + \tan x)^2) \\
 &= \ln(\sqrt{(\sec x + \tan x)^2}) \\
 &= \ln(\sec x + \tan x)
 \end{aligned}$$

**Q5)** Find the volume under the first hump of the function  $y = \cos x$  rotated around the  $y$ -axis by the method of shells.

$$\begin{aligned}
\text{Volume} &= \int_0^{\pi/2} 2\pi x \cos x \, dx \\
&= 2\pi \int_0^{\pi/2} x \cos x \, dx \\
&= 2\pi \left( x \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx \\
&= 2\pi \left( \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) - (-\cos x) \right]_0^{\pi/2} \\
&= 2\pi \left( \frac{\pi}{2} - (-\cos\left(\frac{\pi}{2}\right) - (-\cos 0)) \right) \\
&= 2\pi \left( \frac{\pi}{2} - (-0 - (-1)) \right) \\
&= 2\pi \left( \frac{\pi}{2} - (1) \right) \\
&= 2\pi \left( \frac{\pi}{2} - 1 \right)
\end{aligned}$$