MIT 18.01 Problem Set 6 Unofficial Solutions

Q1) Do 7.4/12 and 13.

Skipped. We do not have the textbook.

Q2) The voltage V of the house current is given by

$$V(t) = Csin(120\pi t)$$

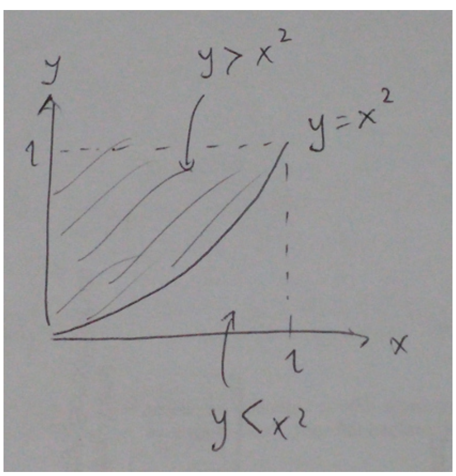
where t is time, in seconds and C is a constant amplitude. The square root of the average value of V^2 over one period of $V(t)$ (or cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant C. (The peak voltage delivered to the house is $\pm C$. The units of V^2 are square volts; when we take the square root again after averaging, the units become volts again.)

Every cycle of the *sin* function corresponds to 2π. This happens every $t = \frac{2\pi}{120\pi} = \frac{1}{60}$ seconds. Since $V(t) = Csin(120\pi t)$, $V^2(t) = C^2 sin^2(120\pi t)$. The average value of V^2 over 1 cycle of $V(t)$ is:

$$\begin{aligned}
\frac{1}{\frac{1}{60} - 0} \int_0^{\frac{1}{60}} C^2 sin^2(120\pi t)dt &= 60C^2 \int_0^{\frac{1}{60}} sin^2(120\pi t)dt \\
&= 60C^2 (\frac{1}{2}t - \frac{1}{240\pi} sin(120\pi t)cos(120\pi t))\Big]_0^{1/60} \\
&= 60C^2 (\frac{1}{2} \cdot \frac{1}{60} - \frac{1}{240\pi} sin(2\pi)cos(2\pi) - (\frac{1}{2} \cdot 0 - \frac{1}{240\pi} sin(0)cos(0))) \\
&= 60C^2 \cdot \frac{1}{120} \\
&= \frac{C^2}{2}
\end{aligned}$$

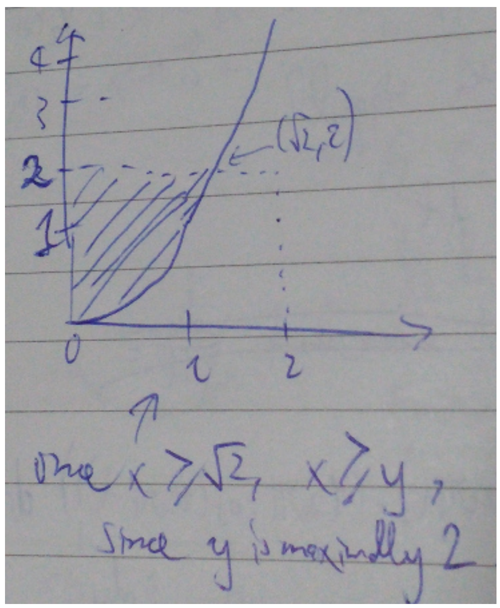
Then RMS $= \sqrt{\frac{C^2}{2}} = \frac{C}{\sqrt{2}} = \frac{\sqrt{2}C}{2}$

Q3a) What is the probability that $x^2 < y$ if (x, y) is chosen from the unit square $0 \le x \le 1$, $0 \le y \le 1$ with probability equal to the area.



$$
\begin{aligned}
P(x^2 < y) &= \frac{\int_0^1 1dx - \int_0^1 x^2 dx}{\int_0^1 1dx} \\
&= \frac{x\Big]_0^1 - \frac{x^3}{3}\Big]_0^1}{x\Big]_0^1} \\
&= \frac{1 - \frac{1}{3}}{1} \\
&= \frac{2}{3}
\end{aligned}
$$

Q3b) What is the probability that $x^2 < y$ if (x, y) is chosen from the square $0 \le x \le 2$, $0 \le y \le 2$ with probability **proportional** to the area. (Probability = Part/Whole).



For the line $y = x^2$ $y = 2$, $x = \sqrt{2}$. Once $x \ge \sqrt{2}$, $y > 2$. Hence $x \le \sqrt{2}$.

$$
\begin{aligned}
P(x^2<y) &= \frac{\int_0^2 2dx - \int_0^{\sqrt{2}} x^2 dx - \int_{\sqrt{2}}^2 2dx}{\int_0^2 2dx} \\
&= \frac{2x\Big]_0^2 - \frac{x^3}{3}\Big]_0^{\sqrt{2}} - 2x\Big]_{\sqrt{2}}^2}{2x\Big]_0^2} \\
&= \frac{4 - \frac{(\sqrt{2})^3}{3} - (4-2\sqrt{2})}{4} \\
&= \frac{4 - \frac{2\sqrt{2}}{3} - 4 + 2\sqrt{2}}{4} \\
&= \frac{2\sqrt{2} - \frac{2\sqrt{2}}{3}}{4} \\
&= \frac{\frac{6\sqrt{2}-2\sqrt{2}}{3}}{4} \\
&= \frac{\frac{4\sqrt{2}}{3}}{4} \\
&= \frac{\sqrt{2}}{3}
\end{aligned}
$$

Q3c) Evaluate

$$W = \int_0^\infty e^{-at}dt = \lim_{N\to\infty} \int_0^N e^{-at}dt$$

This is known as an improper integral because it represents the area of an unbounded region. We are using the letter W to signify "whole".
The probability that a radioactive particle will decay some time in the interval $0 \le t \le T$ is

$$P([0,T]) = \frac{\text{PART}}{\text{WHOLE}} = \frac{1}{W}\int_0^T e^{-at}dt$$

Note that $P([0,\infty)) = 1 = 100\%$

$$
\begin{aligned}
\int_0^\infty e^{-at}dt &= -\frac{1}{a}e^{-at}\Big]_0^\infty \\
&= -\frac{1}{a}(e^{-a\cdot\infty} - e^{-a\cdot 0}) \\
&= -\frac{1}{a}(\frac{1}{e^{a\cdot\infty}} - \frac{1}{e^{a\cdot 0}}) \\
&= -\frac{1}{a}(0 - \frac{1}{e^0}) \\
&= -\frac{1}{a}(-1) \\
&= \frac{1}{a}
\end{aligned}
$$

Q3d) The half-life is the time T for which $P([0,T]) = 1/2$. Find the value of a and W for which the half-life is $T = 1$. Suppose that a radioactive particle has a half-life of 1 second. What is the probability that it survives to time $t = 1$, but decays some time during the interval $1 \leq t \leq 2$? (Give an integral formula, and use a calculator to get an approximate numerical answer.)

We saw in part (c) that $W = \frac{1}{a}$. We want to find a and W such that $P([0,1]) = \frac{1}{2}$

$$\begin{aligned}
P([0,1]) &= \frac{\int_0^1 e^{-at}dt}{W} \\
&= a\int_0^1 e^{-at}dt \\
&= a(-\frac{1}{a}e^{-at}\Big]_0^1) \\
&= a \cdot (-\frac{1}{a})e^{-at}\Big]_0^1 \\
&= -1(e^{-a} - e^0) \\
&= -1(\frac{1}{e^a} - 1) \\
&= 1 - \frac{1}{e^a} \\
&= \frac{1}{2}
\end{aligned}$$

So $1 - \frac{1}{e^a} = \frac{1}{2}$. Then $\frac{1}{2} = \frac{1}{e^a}$ and $e^a = 2$ so $a = ln(2)$ and $W = \frac{1}{a} = \frac{1}{ln(2)}$

From $0 \leq t \leq 1$, the particle decays with probability $\frac{1}{2}$ and survives with probability $\frac{1}{2}$. These 2 cases are mutually exclusivge and from $1 \leq t \leq 2$, the particle decays with probability $\frac{1}{W}\int_1^2 e^{-at}dt$ regardless of what happens from $0 \leq t \leq 1$.

Hence probability of particule surviving to $t = 1$ and decaying some time in $1 \leq t \leq 2$ is

$$
\begin{aligned}
\frac{1}{2}\cdot\frac{1}{W}\int_1^2 e^{-at}dt &= \frac{1}{2} * a(-\frac{1}{a}e^{-at}\Big]_1^2) \\
&= -\frac{1}{2}(e^{-at}\Big]_1^2) \\
&= -\frac{1}{2}(e^{-2a} - e^{-a}) \\
&= \frac{1}{2}e^{-a} - \frac{1}{2}e^{-2a} \\
&= \frac{1}{2}e^{-ln(2)} - \frac{1}{2}e^{-2ln(2)} \\
&= \frac{1}{2}e^{ln(2^{-1})} - \frac{1}{2}e^{ln(2^{-2})} \\
&= \frac{1}{2}e^{ln(\frac{1}{2})} - \frac{1}{2}e^{ln(\frac{1}{4})} \\
&= \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{4} \\
&= \frac{1}{4} - \frac{1}{8} \\
&= \frac{1}{8}
\end{aligned}
$$