MIT 18.01 Problem Set 4 Unofficial Solutions

Q1a) Use the mean value property to show that if $f(0) = 0$ and $f'(x) \geq 0$, then $f(x) \geq 0$ for all $x \geq 0$.

For any $x > 0$, $f(x) = f(0) + f'(c)(x - 0)$, for some $0 < c < x$

Since $f'(x) \geq 0$ for all x, then for all $x > 0$, $f(x) \geq f(0)$.

Since $f(x) \geq f(0) \geq 0$ for all $x > 0$, then $f(x) \geq 0$ for all $x > 0$.

Since $f(0) = 0$ and $f(x) \geq 0$ for all $x > 0$, then $f(x) \geq 0$ for all $x \geq 0$.

Q1b) Deduce from part (a) that $ln(1 + x) \leq x$ for $x \geq 0$. Hint: Use $f(x) = x - ln(1 + x)$.

Let $f(x) = x - ln(1 + x)$

Then $f(0) = 0 - ln(1 + 0) = 0 - 0 = 0$

$f'(x) = 1 - \frac{1}{1+x}$

For all $x \geq 0$, $1 + x \geq 1$ and $\frac{1}{1+x} \leq 1$ and $1 - \frac{1}{1+x} \geq 0$

Since $f'(x) = 1 - \frac{1}{1+x} \geq 0$, therefore $f'(x) \geq 0$ for $x \geq 0$.

By 1(a), $f(x) = x - ln(1 + x) \geq 0$ for $x \geq 0$

Then $x \geq ln(1 + x)$ for $x \geq 0$

Q1c) Use the same method as in (b) to show $ln(1 + x) \geq x - x^2/2$ and $ln(1 + x) \leq x - x^2/2 + x^3/3$ for $x \geq 0$.

Let $f_1(x) = ln(1 + x) - x + \frac{x^2}{2}$

$f_1(0) = ln(1 + 0) - 0 + \frac{0^2}{2} = 0$

$f_1'(x) = \frac{1}{1+x} - 1 + x = \frac{1-(1+x)+x(1+x)}{1+x} = \frac{1-1-x+x+x^2}{1+x} = \frac{x^2}{1+x}$

Since $1 + x \geq 1$ for all $x \geq 0$ and $x^2 \geq 0$ for all $x \geq 0$, then $f_1'(x) = \frac{x^2}{1+x} \geq 0$ for all $x \geq 0$.

By 1(a), $f_1(x) \geq 0$ for all $x \geq 0$, which is equivalent to $ln(1 + x) - x - \frac{x^2}{2} \geq 0$ for all $x \geq 0$. Hence $ln(1 + x) \geq x - \frac{x^2}{2}$ for all $x \geq 0$.

Let $f_2(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ln(1 + x)$

$f_2(0) = 0 - \frac{0^2}{2} + \frac{0^3}{3} - ln(1 + 0) = 0$

$f_2'(x) = 1 - x + x^2 - \frac{1}{1+x} = \frac{1(1+x)-x(1+x)+x^2(1+x)-1}{1+x} = \frac{1+x-x-x^2+x^2+x^3}{1+x} = \frac{x^3}{1+x}$

Since $1 + x \geq 1$ for all $x \geq 0$ and $x^3 \geq 0$ for all $x \geq 0$, then $f_2'(x) = \frac{x^3}{1+x} \geq 0$ for all $x \geq 0$.

By 1(a), $f_2(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ln(1 + x) \geq 0$ for all $x \geq 0$. Hence $x - \frac{x^2}{2} + \frac{x^3}{3} \geq ln(1 + x)$ for all $x \geq 0$.

Q1d) Find the pattern in (b) and (c) and make a general conjecture.

General conjecture:

$ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^{2n+1}}{2n+1} = \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}$ for all $x \geq 0, n \geq 0$

$ln(1+x) \geq x - \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^{2n}}{2n} = \sum_{k=1}^{2n} (-1)^{k+1} \frac{x^k}{k}$ for all $x \geq 0, n \geq 1$

Let $f_1(x) = (\sum_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}) - ln(x+1)$ for any $n \geq 1$.

$f_1(0) = (\sum_{k=1}^{2n+1} (-1)^{k+1} \frac{0^k}{k}) - ln(0+1) = 0$

$$\begin{aligned}
f_1'(x) &= (\sum_{k=1}^{2n+1} (-1)^{k+1} x^{k-1}) - \frac{1}{x+1} \\
&= \frac{(x+1)(\sum_{k=1}^{2n+1} (-1)^{k+1} x^{k-1}) - 1}{x+1} \\
&= \frac{(\sum_{k=1}^{2n+1} (-1)^{k+1} (x^k + x^{k-1})) - 1}{x+1} \\
&= \frac{(x^1 + x^0) - (x^2 + x^1) + ... + (-1)^{2n+1+1}(x^{2n+1} + x^{2n}) - 1}{x+1} \\
&= \frac{x^0 + x^{2n+1} - 1}{x+1} \\
&= \frac{x^{2n+1}}{x+1}
\end{aligned}$$

Since $x+1 \geq 1$ for $x \geq 0$ and $x^{2n+1} \geq 0$ for $n \geq 0, x \geq 0$, then $f_1'(x) = \frac{x^{2n+1}}{x+1} \geq 0$ for all $x \geq 0$.

By 1(a), $f_1(x) = (\sum_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}) - ln(x+1) \geq 0$ for all $x \geq 0, n \geq 0$. Hence $ln(x+1) \leq \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}$ for all $x \geq 0, n \geq 0$.

Let $f_2(x) = ln(x+1) - \sum_{k=1}^{2n} (-1)^{k+1} \frac{x^k}{k}$ for any $n \geq 0$.

$f_2(0) = ln(0+1) - \sum_{k=1}^{2n} (-1)^{k+1} \frac{0^k}{k} = 0$

$$
\begin{aligned}
f_2'(x) &= \frac{1}{x+1} - \sum_{k=1}^{2n}(-1)^{k+1}x^{k-1} \\
&= \frac{1-(x+1)\sum_{k=1}^{2n} x^{k-1}}{x+1} \\
&= \frac{1-\sum_{k=1}^{2n}(x^k + x^{k-1})}{x+1} \\
&= \frac{1-((x^1+x^0)-(x^2+x^1)+...+(-1)^{2n+1}(x^{2n}+x^{2n-1}))}{x+1} \\
&= \frac{1-(x^0-x^{2n})}{x+1} \\
&= \frac{x^{2n}}{x+1}
\end{aligned}
$$

For $x \geq 0$, $x+1 \geq 1$ and $x^{2n} \geq 0$ for $x \geq 0, n \geq 1$. Hence $f_2'(x) = \frac{x^{2n}}{x+1} \geq 0$ for $n \geq 1, x \geq 0$.

By 1(a), $f_2(x) = ln(x+1) - \sum_{k=1}^{2n}(-1)^{k+1}x^{k-1} \geq 0$ for all $n \geq 1, x \geq 0$.

Then $ln(x+1) \geq \sum_{k=1}^{2n}(-1)^{k+1}x^{k-1}$ for all $n \geq 1, x \geq 0$.

Q1e) Show that $ln(1+x) \leq x$ for $-1 < x \leq 0$. (Use the change of variable $u = -x$.)

Let $u = -x$. Then $x = -u$.

We want to show that $ln(1-u) \leq -u$ for $0 \leq u < 1$.

Let $f(u) = -u - ln(1-u)$

$f(0) = -0 - ln(1-0) = 0$

$f'(u) = -1 - \frac{-1}{1-u} = -1 + \frac{1}{1-u} = \frac{-1(1-u)+1}{1-u} = \frac{u-1+1}{1-u} = \frac{u}{1-u}$

Since $0 \leq u < 1$, then $1-u > 0$. Hence $f'(u) = \frac{u}{1-u} \geq 0$ for $0 \leq u < 1$

By 1(a), $f(u) = -u - ln(1-u) \geq 0$ for $0 \leq u < 1$.

Then $f(x) = -(-x) - ln(1-(-x)) = x - ln(1+x) \geq 0$ for $0 \leq -x < 1$ or $-1 < x \leq 0$.

Hence $ln(1+x) \leq x$ for $-1 < x \leq 0$

Q2a) Do 5.3/68

I do not have the textbook. Skipped.

Q2b) Show that both of the following integrals are correct, and explain.

$$\int tanx\ sec^2x\ dx = (1/2)tan^2x; \int tanx\ sec^2x\ dx = (1/2)sec^2x$$

Let $u = tan\ x$. We have

$$
\begin{aligned}
\int tanx\ sec^2dx &= \int (sec^2x)\ tanx\ dx \\
&= \int u'\ u\ du \\
&= \frac{1}{2}u^2 + c \\
&= \frac{1}{2}tan^2x + c
\end{aligned}
$$

Now we want to prove that $\int tanx\ sec^2x\ dx = (1/2)sec^2x$. Let $u = sec\ x$. Then $\int u'\ u\ du = \frac{1}{2}u^2 + c = \frac{1}{2}sec^2x + c$

> **Q3)** (Lec 16, 6 pts: 3 + 3)
> a) Do 8.6/5 (answer in back of book)
> b) Do 8.6/6 (optional?)

I do not have the textbook. Skipped.

> **Q4)** (Lec 16, 7 pts: 2 + 3 + 2) Do 3F-5abc
>
> 3F-5 Air pressure satisfies the differential equation $dp/dh = -(.13)p$, where h is the altitude from sea level measured in kilometres.
> a) At sea level the pressure is $1kg/cm^2$. Solve the equation and find the pressure at the top of Mt. Everest (10 km).

$$
\begin{aligned}
\frac{dp}{dh} &= -(.13)p \\
\frac{1}{p}dp &= -.13dh \\
\int \frac{1}{p}dp &= \int -.13dh \\
ln\ p &= -.13h + c \\
p &= e^{-.13h+c} \\
p &= e^{-.13h} \cdot e^c \\
p &= Ae^{-.13h}
\end{aligned}
$$

where $A = e^c$. When $h = 0, p = 1$ (sea level pressure is 1). Then $p = 1 = Ae^{-.13(0)} = A$. Hence $p = Ae^{-.13h} = 1 \cdot e^{-.13h} = e^{-.13h}$

At the top of Mt Everest, $p = e^{-.13(10)} \approx 0.2725\ kg/cm^2$

> **Q4)** (Lec 16, 7 pts: 2 + 3 + 2) Do 3F-5abc
>
> 3F-5 b) Find the difference in pressure between the top and bottom of the Green building. (Pretend it's 100 meters tall starting at sea level.) Compute the numerical value using a calculator. Then use instead the linear approximation to e^x near $x = 0$ to estimate the percentage drop in pressure from the bottom to the top of the Green Building.

Difference $= e^{-.13(0.1)} - e^{-13(0)} = e^{-.013} - 1 \approx -0.012915865\ kg/cm^2$

Linear approximation to e^x near $x = 0$ is $1 + x$. But because we are dealing with $e^{-.13h}$, for $h \approx 0$, $e^{-.13h} \approx e^{-.13(0)} + -.13e^{-.13(0)}h = 1 - .13h$

Pressure at top - Pressure at bottom $\approx 1 - .13(0.1) - 1 \approx -0.013\ kg/cm^2$

Q4) (Lec 16, 7 pts: 2 + 3 + 2) Do 3F-5abc

3F-5 c) Use the linear approximation $\Delta p \approx p'(0)\Delta h$ and compute $p'(0)$ directly from the differential equation to find the drop in pressure from the bottom to top of the Green Building. Notice that this gives an answer without even knowing the solution to the differential equation. Compare with the approximation in part (b). What does the linear approximation $p'(0)\Delta h$ give for the pressure at the top of Mt. Everest?

$$\begin{aligned}
\Delta p &\approx p'(0)\Delta h \\
\frac{\Delta p}{\Delta h} &\approx p'(0) \\
\lim_{\Delta h \to 0} \frac{dp}{dh} &= \frac{dp}{dh} \\
\frac{\Delta p}{\Delta h} &\approx \frac{dp}{dh} = -.13p
\end{aligned}$$

We will use $p'(0) \approx \frac{\Delta p}{\Delta h} \approx -.13p$. When $h = 0, p = 1$. Hence $1 = p'(0) \approx -.13p = -.13(1) = -.13$, which implies that every kilometre increase in altitude causes pressure to decrease by $-.13\ kg/cm^2$.

Drop in pressure from bottom to top of green building $= -(-.13(0.1)) = 0.013\ kg/cm^2$ which was exactly the same as what we got using the linear approximation in part (b).

For pressure at top of Mt Everest, we have $\Delta h = 10$. Hence

$$\begin{aligned}
p(0) + p'(0)\Delta h &= 1 + -.13(10) \\
&= 1 - 1.3 \\
&= -0.3\ kg/cm^2
\end{aligned}$$

which is totally incorrect because $\Delta h = 10$ is much bigger than 0 and causes the approximation to be wildly inaccurate.

Q5) Calculate $\int_0^1 e^x dx$ using lower Riemann sums. (You will need to sum a geometric series to get a usable formula for the Riemann sum. To take the limit of Riemann sums, you will need to evaluate $\lim_{n\to\infty} n(e^{1/n} - 1)$, which can be done using the standard linear approximation to the exponential function.)

$$
\begin{aligned}
\int_0^1 e^x dx &= \lim_{n\to\infty} \sum_{k=0}^{n-1} e^{\frac{k}{n}} \frac{1}{n} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{\frac{k}{n}} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (e^{\frac{1}{n}})^k \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1-(e^{\frac{1}{n}})^n}{1-e^{\frac{1}{n}}} \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1-e}{1-e^{\frac{1}{n}}} \\
&= \lim_{n\to\infty} \frac{1-e}{n(1-e^{\frac{1}{n}})}
\end{aligned}
$$

The linear approximation to e^x near $x = 0$ is $e^0 + e^0 x = 1 + x$. Since $n \to \infty$, $\frac{1}{n} \to 0$ and $e^{\frac{1}{n}} \approx 1 + \frac{1}{n}$. Then

$$
\begin{aligned}
\int_0^1 e^x dx &= \lim_{n\to\infty} \frac{1-e}{n(1-e^{\frac{1}{n}})} \\
&\approx \lim_{n\to\infty} \frac{1-e}{n(1-(1+\frac{1}{n}))} \\
&= \lim_{n\to\infty} \frac{1-e}{n(-\frac{1}{n})} \\
&= \lim_{n\to\infty} \frac{1-e}{-1} \\
&= \lim_{n\to\infty} e-1 \\
&= e-1
\end{aligned}
$$