MIT 18.01 Problem Set 8A Unofficial Solutions

4E-2 Find the rectangular equation for $x = t + 1/t$ and $y = t - 1/t$ (compute x^2 and y^2).

$$\begin{aligned} x^2 &= t^2 + 2 + 1/t^2 \\ y^2 &= t^2 - 2 + 1/t^2 \\ y^2 &= x^2 - 4 \\ x^2 - y^2 &= 4 \end{aligned}$$

This is an equation of a hyperbola centred at the origin with width 2.

4E-3 Find the rectangular equation for $x = 1 + sin\ t, y = 4 + cos\ t$

$$\begin{aligned} x^2 &= 1 + 2sin\ t + sin^2\ t \\ y^2 &= 16 + 8sin\ t + cos^2\ t \\ x^2 + y^2 &= 17 + 2sin\ t + 8cos\ t + sin^2\ t + cos^2\ t \\ &= 18 + 2(sin\ t + 4cos\ t) \\ &= 2(1 + sin\ t) + 8(4 + cos\ t) - 16 \\ &= 2x + 8y - 16 \end{aligned}$$

Then

$$\begin{aligned} x^2 - 2x + y^2 - 8y + 16 = 0 \\ (x-1)^2 + (y-4)^2 - 1 = 0 \\ (x-1)^2 + (y-4)^2 = 1 \end{aligned}$$

This is an equation of a circle with radius 1 centred at $(1, 2)$.

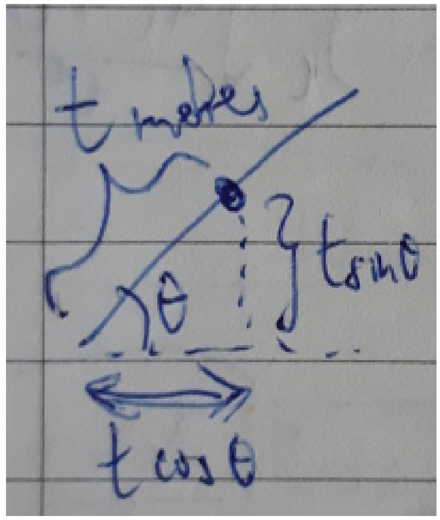
4E-8 At noon, a snail starts at the center of an open clock face. It creeps at a steady rate along the hour hand, reaching the end of the hand at 1:00 PM. The hour hand is 1 meter long. Write parametric equations for the position of the snail at time t, in some reasonable xy-coordinate system.

The average velocity of the snail is 1 metre / h. Let the centre of the clock be the origin.

Treat the positive x-axis as 0 radians. When the hour hand is 12pm, the angle with respect to the positive x-axis is $\pi/2$ radians. When the hour hand is at 1pm, the angle with respect to the positive x-axis is $\pi/3$ radians.

Hence, the angle traversed by the hour hand from 12pm to 1pm is $\pi/6$ radians.

$$\begin{aligned} x = tcos\ \theta = t\ cos(\frac{\pi}{2} - \frac{\pi}{6}t) \\ y = tsin\ \theta = t\ sin(\frac{\pi}{2} - \frac{\pi}{6}t) \end{aligned}$$



Verification:

At $t = 0$ (12pm), the snail should be at $(0, 0)$.

$$x = 0\ cos(\frac{\pi}{2} - \frac{\pi}{6} \cdot 0) = 0$$
$$y = 0\ sin(\frac{\pi}{2} - \frac{\pi}{6} \cdot 0) = 0$$

At $t = 1$ (1pm), the snail should be at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ (draw out a right angle triangle with hypothenus 1 and the angle being $\pi/3$ radians and you will see why these numbers):

$$x = 1\ cos(\frac{\pi}{2} - \frac{\pi}{6} \cdot 1) = cos(\frac{\pi}{3}) = \frac{1}{2}$$
$$y = 1\ sin(\frac{\pi}{2} - \frac{\pi}{6} \cdot 1) = sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

4F-1d Find the arclength of $y = (1/3)(2 + x^2)^{3/2},\ 1 \le x \le 2$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}(2x)(2 + x^2)^{1/2} = x(2 + x^2)^{1/2} \\
\sqrt{1 + (\frac{dy}{dx})^2} &= \sqrt{1 + (x(2 + x^2)^{1/2})^2} \\
&= \sqrt{1 + x^2(2 + x^2)} \\
&= \sqrt{1 + 2x^2 + x^4} \\
&= \sqrt{(x^2 + 1)^2} \\
&= x^2 + 1
\end{aligned}$$

$$ds = x^2 + 1\ dx$$

Arc length:

$$
\begin{aligned}
\int_1^2 x^2+1\,dx &= \frac{x^3}{3}+x\Big]_1^2 \\
&= \frac{2^3}{3}+2-(\frac{1}{3}+1) \\
&= \frac{8}{3}+2-\frac{1}{3}-1 \\
&= \frac{10}{3}
\end{aligned}
$$

4F-4 Find the length of the curve $x = t^2$, $y = t^3$ for $0 \leq t \leq 2$.

$$
\begin{aligned}
\frac{dx}{dt} &= 2t \\
\frac{dy}{dt} &= 3t^2 \\
ds &= \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2} \\
&= \sqrt{(2t)^2+(3t^2)^2} \\
&= \sqrt{4t^2+9t^4} \\
&= t\sqrt{4+9t^2} \\
\int_{s_0}^{s_1} ds &= \int_0^2 t\sqrt{4+9t^2}dt \\
&= \frac{2/3\cdot(4+9t^2)^{3/2}}{18}\Big]_0^2 \\
&= \frac{1}{27}(4+9t^2)^{3/2}\Big]_0^2 \\
&= \frac{1}{27}((4+9(2)^2)^{3/2}-(4+9(0)^2)^{3/2}) \\
&= \frac{1}{27}((4+36)^{3/2}-4^{3/2}) \\
&= \frac{1}{27}(40^{3/2}-8)
\end{aligned}
$$

4F-5 Find an integral for the length of the curve given parametrically in Exercise 4E-2 for $1 \leq t \leq 2$. Simplify the integrand as much as possible but do not evaluate.

What is given in 4E-2: $x = t + 1/t$, $y = t - 1/t$.

$$
\begin{aligned}
\frac{dx}{dt} &= 1 - \frac{1}{t^2} \\
\frac{dy}{dt} &= 1 + \frac{1}{t^2} \\
ds &= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \\
&= \sqrt{(1 - \frac{1}{t^2})^2 + (1 + \frac{1}{t^2})^2} dt \\
&= \sqrt{1 - \frac{2}{t^2} + \frac{1}{t^4} + 1 + \frac{2}{t^2} + \frac{1}{t^4}} dt \\
&= \sqrt{2 + \frac{2}{t^4}} dt \\
&= \sqrt{\frac{2t^4 + 2}{t^4}} dt \\
\int_{s_0}^{s_1} ds &= \int_1^2 \sqrt{\frac{2t^4 + 2}{t^4}} dt
\end{aligned}
$$

4F-8) Find the length of the curve $x = e^t cos\ t$, $y = e^t sin\ t$ for $0 \leq t \leq 10$.

$$
\begin{aligned}
x^2 + y^2 &= e^{2t} cos^2\ t + e^{2t} sin^2\ t = e^{2t} \\
\frac{dx}{dt} &= e^t cos\ t - e^t sin\ t \\
\frac{dy}{dt} &= e^t sin\ t + e^t cos\ t
\end{aligned}
$$

$$
\begin{aligned}
(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 &= e^{2t} cos^2\ t - 2e^{2t} cos\ t\ sin\ t + e^{2t} sin^2\ t + e^{2t} sin^2\ t + 2e^{2t} sin\ t\ cos\ t + e^{2t} cos^2\ t \\
&= e^{2t} cos^2\ t + e^{2t} sin^2\ t + e^{2t} sin^2\ t + e^{2t} cos^2\ t \\
&= 2e^{2t}
\end{aligned}
$$

$$
\begin{aligned}
ds &= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt \\
&= \sqrt{2e^{2t}}\ dt \\
&= \sqrt{2}\ e^t\ dt \\
\int_{s_0}^{s_1} ds &= \int_0^{10} \sqrt{2}\ e^t\ dt \\
&= \sqrt{2}\ e^t \Big]_0^{10} \\
&= \sqrt{2}\ (e^{10} - e^0) \\
&= \sqrt{2}\ (e^{10} - 1)
\end{aligned}
$$

4G-2 Find the area of the segment of $y = 1 - 2x$ in the first quadrant revolved around the x-axis.

$$\begin{aligned} ds &= \sqrt{1 + (\frac{dy}{dx})^2}\ dx \\ &= \sqrt{1 + (-2)^2}\ dx \\ &= \sqrt{5}\ dx \end{aligned}$$

Surface area:

$$\begin{aligned} \int_{s_0}^{s_1} 2\pi y\ ds &= \int_0^{1/2} 2\pi(1-2x)\sqrt{5}dx \\ &= 2\sqrt{5}\pi \int_0^{1/2} 1 - 2x\ dx \\ &= 2\sqrt{5}\pi(x - x^2)\Big]_0^{1/2} \\ &= 2\sqrt{5}\pi(\frac{1}{2} - \frac{1}{4})) \\ &= \frac{\sqrt{5}}{2}\pi \end{aligned}$$

4G-5 Find the area of $y = x^2$, $0 \leq x \leq 4$ revolved around the y-axis.

$$\begin{aligned} ds &= \sqrt{1 + (\frac{dy}{dx})^2}\ dx \\ &= \sqrt{1 + (\frac{1}{2}y^{-\frac{1}{2}})^2}\ dy \\ &= \sqrt{1 + \frac{1}{4y}}\ dy \end{aligned}$$

Surface area:

$$\begin{aligned} \int_{s_0}^{s_1} 2\pi x\ ds &= \int_0^{16} 2\pi x\sqrt{1 + \frac{1}{4y}}\ dy \\ &= 2\pi \int_0^{16} \sqrt{y}\sqrt{1 + \frac{1}{4y}}\ dy \\ &= 2\pi \int_0^{16} \sqrt{y + \frac{1}{4}}\ dy \\ &= 2\pi\frac{2}{3}(y + \frac{1}{4})^{3/2}\Big]_0^{16} \\ &= \frac{4\pi}{3}(y + \frac{1}{4})^{3/2}\Big]_0^{16} \\ &= \frac{4\pi}{3}((\frac{65}{4})^{3/2} - \frac{1}{8}) \end{aligned}$$

4H-1b Give the polar coordinates for the rectangular coordinate $(-2, 0)$

$r = 2$, $\theta = \pi$

Verify:

$$x = r\ cos\theta = 2cos\ \pi = 2(-1) = 2$$
$$y = r\ sin\theta = 2sin\ \pi = 0$$

4H-1f Give the polar coordinates for the rectangular coordinate $(0, -2)$

$r = 2$, $\theta = \frac{3\pi}{2}$

Verify:

$$x = r\ cos\theta = 2cos\ \frac{3\pi}{2} = 0$$
$$y = r\ sin\theta = 2sin\ \frac{3\pi}{2} = 2(-1) = -2$$

Alternatively, $r = 2$, $\theta = -\frac{\pi}{2}$

4H-1g Give the polar coordinates for the rectangular coordinate $(\sqrt{3}, -1)$

$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$

We know that $cos\ \theta = \frac{\sqrt{3}}{2}$ and $sin\ \theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}$

But in this case, we are in the 4th quadrant. So $\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ or equivalently, $\theta = -\frac{\pi}{6}$.

Verify:

$$x = r\ cos\theta = 2cos\ \frac{11\pi}{6} = 2cos\ \frac{\pi}{6} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$
$$y = r\ sin\theta = 2sin\ \frac{11\pi}{6} = -2sin\ \frac{\pi}{6} = -2 \cdot \frac{1}{2} = -1$$

4H-2a Find using two different methods the equation in polar coordinates for the circle of radius a with center at $(a, 0)$ on the x-axis, as follows:

(i) write its equation in rectangular coordinates, and then change it to polar coordinates (substitute $x = r\ cos\ \theta$ and $y = r\ sin\ \theta$, and then simplify).

(ii) treat it as a locus problem: let OQ be the diameter lying along the x-axis, and $P : (r, \theta)$ a point on the circle; use ΔOPQ and trigonometry to find the relation connecting r and θ.

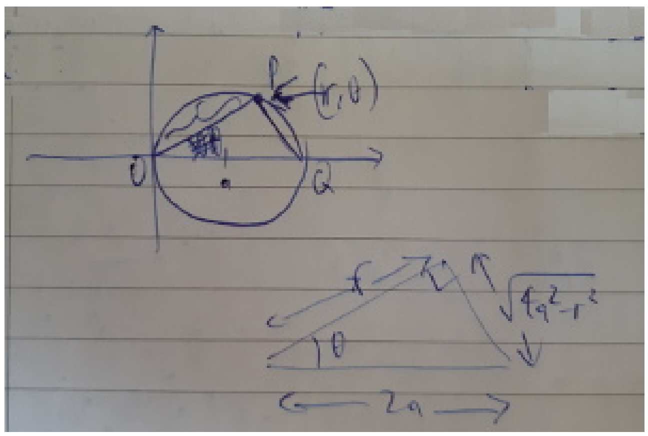
For part (i)

Equation of the circle is $(x-a)^2 + y^2 = a^2$.

Let $x = r\ cos\ \theta$, $y = r\ sin\ \theta$. Then

$$
\begin{aligned}
(r\ cos\ \theta - a)^2 + (r\ sin\ \theta)^2 &= a^2 \\
r^2\ cos^2\ \theta - 2ar\ cos\ \theta + a^2 + r^2\ sin^2\ \theta &= a^2 \\
r^2 - 2ar\ cos\ \theta &= 0 \\
r &= 2a\ cos\ \theta
\end{aligned}
$$

For part (ii)



$$
\begin{aligned}
OQ &= 2a \\
\angle OPQ &= \frac{\pi}{2} \\
\angle POQ &= \theta \\
\angle PQO &= \frac{\pi}{2} - \theta
\end{aligned}
$$

$$
\begin{aligned}
cos\ \theta &= \frac{r}{2a} \\
r &= 2a\ cos\ \theta
\end{aligned}
$$

4H-3f For $r = a\ cos(2\theta)$ (4-leaf rose)
(i) give the corresponding equation in rectangular coordinates;
(ii) draw the graph; indicate the direction of increasing θ

For part (i)

$$x = r\ cos\ \theta = a\ cos(2\theta)\ cos(\theta)$$
$$y = r\ sin\ \theta = a\ cos(2\theta)\ sin(\theta)$$

$$\begin{aligned}
x &= a\ cos(2\theta)\ cos(\theta) \\
&= a(cos^2\theta - sin^2\theta)\ cos\ \theta \\
&= a(cos^3\theta - cos\ \theta\ sin^2\theta) \\
&= a(cos^3\theta - cos\ \theta\ (1 - cos^2\theta)) \\
&= a(cos^3\theta - cos\ \theta\ \ + cos^3\theta) \\
&= a(2\ cos^3\theta - cos\ \theta)
\end{aligned}$$

$$\begin{aligned}
y &= a\ cos(2\theta)\ sin(\theta) \\
&= a\ (cos^2\theta - sin^2\theta)\ sin(\theta) \\
&= a\ (1 - sin^2\theta - sin^2\theta)\ sin(\theta) \\
&= a\ (1 - 2\ sin^2\theta)\ sin(\theta) \\
&= a\ (sin\ \theta - 2\ sin^3\theta)
\end{aligned}$$

$$\begin{aligned}
x^2 &= a^2(2\ cos^3\theta - cos\ \theta)^2 \\
&= a^2(4\ cos^6\theta - 4\ cos^4\theta + cos^2\theta)
\end{aligned}$$

$$\begin{aligned}
y^2 &= a^2\ (sin\ \theta - 2\ sin^3\theta)^2 \\
&= a^2\ (4\ sin^6\ \theta - 4\ sin^4\theta + sin^2\theta)
\end{aligned}$$

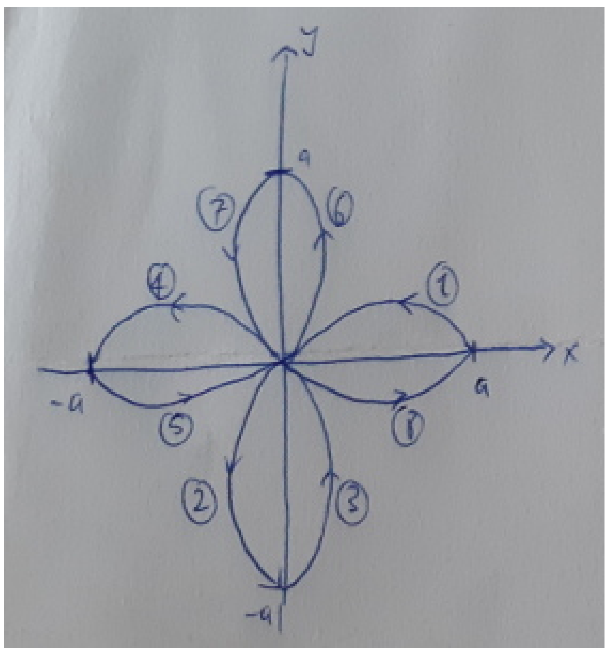
$$\begin{aligned}
r &= a\ cos(2\theta) = a(cos^2\theta - sin^2\theta) \\
r^3 &= ar^2\ cos(2\theta) \\
&= a(r^2 cos^2\theta - r^2 sin^2\theta) \\
&= a(x^2 - y^2)
\end{aligned}$$

Using polar coordinates, we know that $r = \sqrt{x^2 + y^2}$. Therefore $r^3 = (\sqrt{x^2 + y^2})^3$.
Equating $r^3 = a(x^2 - y^2)$ and $r^3 = (\sqrt{x^2 + y^2})^3$:

$$\begin{aligned}
a(x^2 - y^2) &= \sqrt{(x^2 + y^2)}^3 \\
(x^2 + y^2)^{3/2} &= a(x^2 - y^2)
\end{aligned}$$

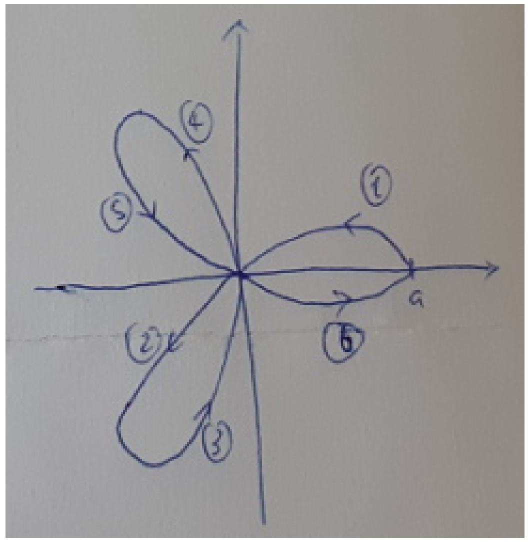
For part (ii)

θ	$r = a\ cos(2\theta)$
0	$a\ cos(0) = a$
$\pi/6$	$a\ cos(\pi/3) = a/2$
$\pi/4$	$a\ cos(\pi/2) = 0$
$\pi/3$	$a\ cos(2\pi/3) = -a/3$
$\pi/2$	$a\ cos(\pi) = -a$
$2\pi/3$	$a\ cos(4\pi/3) = -a/2$
$3\pi/4$	$a\ cos(3\pi/2) = 0$
$5\pi/6$	$a\ cos(5\pi/3) = a/2$
π	$a\ cos(2\pi) = a$
$7\pi/6$	$a\ cos(7\pi/3) = a/2$
$5\pi/4$	$a\ cos(5\pi/2) = 0$
$3\pi/2$	$a\ cos(3\pi) = -a$
$5\pi/3$	$a\ cos(10\pi/3) = -a/2$
$7\pi/4$	$a\ cos(7\pi/2) = 0$
$11\pi/6$	$a\ cos(11\pi/3) = a/2$
2π	$a\ cos(4\pi) = a$

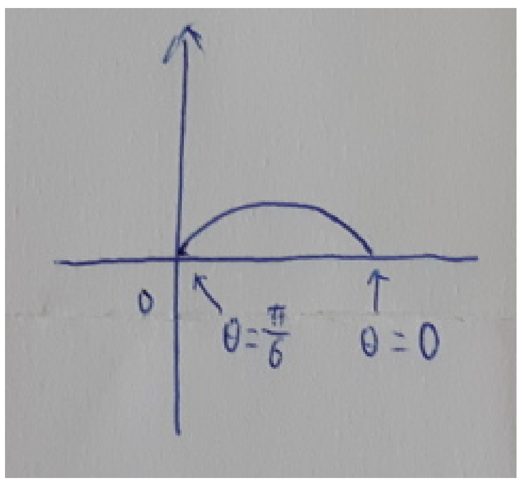


4I-2 Find the area of one leaf of a three-leaf rose $r = a\ cos(3\theta)$.

θ	$r = a\ cos(3\theta)$
0	$a\ cos(0) = a$
$\pi/12$	$a\ cos(\pi/4) = \sqrt{2}\ a/2$
$\pi/6$	$a\ cos(\pi/2) = 0$
$\pi/3$	$a\ cos(\pi) = -a$
$\pi/2$	$a\ cos(3\pi/2) = 0$
$2\pi/3$	$a\ cos(2\pi) = a$
$3\pi/4$	$a\ cos(9\pi/4) = \sqrt{2}\ a/2$
$5\pi/6$	$a\ cos(5\pi/2) = 0$
$11\pi/12$	$a\ cos(11\pi/4) = -\sqrt{2}\ a/2$
π	$a\ cos(3\pi) = -a$



Assume the petals are of equal area. We will find the area of the curve from $\theta = 0$ to $\theta = \pi/6$ and multiply by 2.



$$
\begin{aligned}
2\int_0^{\pi/6}\frac{1}{2}r^2 d\theta &= \int_0^{\pi/6} r^2 d\theta \\
&= \int_0^{\pi/6} a^2\ cos^2(3\theta)d\theta \\
&= a^2\int_0^{\pi/6} cos^2(3\theta)d\theta \\
&= a^2\int_0^{\pi/6} \frac{1+cos(6\theta)}{2}d\theta \\
&= \frac{a^2}{2}\int_0^{\pi/6} 1+cos(6\theta)d\theta \\
&= \frac{a^2}{2}(\theta+\frac{sin(6\theta)}{6})\Big]_0^{\pi/6} \\
&= \frac{a^2}{2}(\frac{\pi}{6}+\frac{sin(6\cdot\pi/6)}{6}) \\
&= \frac{a^2}{2}(\frac{\pi}{6}+\frac{sin(\pi)}{6}) \\
&= \frac{a^2\pi}{12}
\end{aligned}
$$

4I-3 Find the area of the region $0 \le r \le e^{3\theta}$ for $0 \le \theta \le \pi$

Skipped.

Q1a) Find the algebraic equation in x and y for the curve

$x = a\ cos^k t, y = a\ sin^k t$

Draw the portion of the curve $0 \le t \le \pi/2$ in the three cases $k = 1, k = 2, k = 3$.

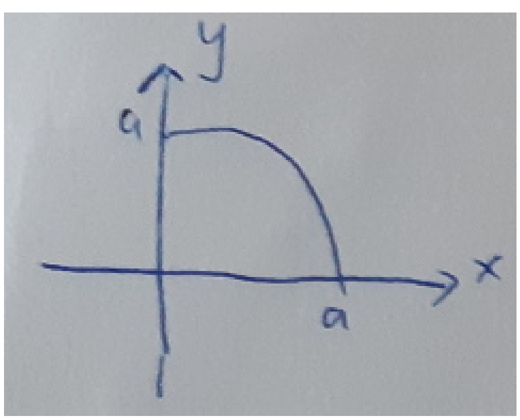
Raising both x and y to the power $2/k$, we get

$x^{2/k} = a^{2/k}\ cos^2 t$

$y^{2/k} = a^{2/k}\ sin^2 t$

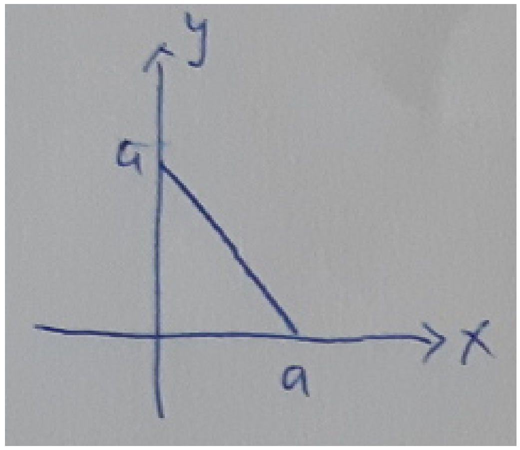
Summing them, we get $x^{2/k} + y^{2/k} = a^{2/k}cos^2 t + a^{2/k}sin^2 t = a^{2/k}$

For $k = 1$, this is $x^2 + y^2 = a^2$

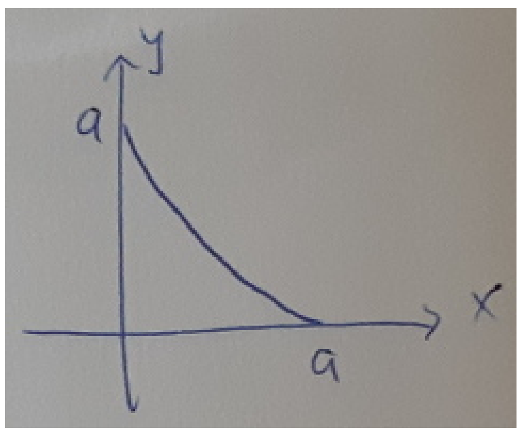


For $k = 2$, this is $x + y = a$ or equivalently, $y = a - x$.

Assuming $a > 0$, we get:



For $k = 3$, this is $x^{2/3} + y^{2/3} = a^{2/3}$



Q1b) Without calculation, find the arclength in the cases $k = 1$ and $k = 2$.

Frankly speaking, we need calculations for these.

Arc length for $k = 1$: $\frac{\pi a}{2}$

Arc length for $k = 2$: $\sqrt{2a^2}$

Q1c) Find a definite integral formula for the length of the curve for general k. Then evaluate the integral in the three cases $k = 1, k = 2$ and $k = 3$. (Your answer in the first two cases should match what you found in part (b), but the calculation takes more time.)

$$\begin{aligned}
x^{2/k} + y^{2/k} &= a^{2/k} \\
y^{2/k} &= a^{2/k} - x^{2/k} \\
y &= (a^{2/k} - x^{2/k})^{k/2} \\
\frac{dy}{dx} &= \frac{k}{2}(a^{2/k} - x^{2/k})^{k/2-1}(-\frac{2}{k}x^{(2/k)-1})
\end{aligned}$$

$$\begin{aligned}
ds &= \sqrt{1 + (dy/dx)^2}\ dx \\
&= \sqrt{1 + (\frac{k}{2}(a^{2/k} - x^{2/k})^{k/2-1}(-\frac{2}{k}x^{(2/k)-1}))^2}\ dx \\
&= \sqrt{1 + (\frac{k^2}{4}(a^{2/k} - x^{2/k})^{k-2}\ (\frac{4}{k^2}x^{2(2-k)/k})}\ dx \\
&= \sqrt{1 + (a^{2/k} - x^{2/k})^{k-2}\ (x^{2(2-k)/k})}\ dx
\end{aligned}$$

$$\int_{s_0}^{s_1} ds = \int_0^a \sqrt{1 + (a^{2/k} - x^{2/k})^{k-2}\ (x^{2(2-k)/k})}\ dx$$

For $k = 1$

$$\begin{aligned}
\int_0^a \sqrt{1 + (a^2 - x^2)^{-1}\ x^2}\ dx &= \int_0^a \sqrt{1 + \frac{x^2}{a^2 - x^2}}\ dx \\
&= \int_0^a \sqrt{\frac{a^2}{a^2 - x^2}}\ dx \\
&= a \int_0^a \sqrt{\frac{1}{a^2 - x^2}}\ dx
\end{aligned}$$

Let $x = a\ sin(u)$. Then $dx = a\ cos(u)\ du$. Substitute into above.

$$\begin{aligned}
a \int_0^{\pi/2} \sqrt{\frac{1}{a^2 - a^2 sin^2(u)}}\ a\ cos(u)\ du &= a^2 \int_0^{\pi/2} \sqrt{\frac{1}{a^2 cos^2(u)}}\ cos(u)\ du \\
&= a^2 \int_0^{\pi/2} \frac{cos(u)}{a\ cos(u)}\ du \\
&= a \int_0^{\pi/2} du \\
&= a\ u\Big]_0^{\pi/2} \\
&= a\pi/2
\end{aligned}$$

For $k = 2$

$$
\begin{aligned}
\int_0^a \sqrt{1+(a^{2/k}-x^{2/k})^{k-2}\ (x^{2(2-k)/k})}\ dx &= \int_0^a \sqrt{1+(a-x)^0\ x^{2(2-2)/2}}\ dx \\
&= \int_0^a \sqrt{1+1\cdot\ x^0}\ dx \\
&= \int_0^a \sqrt{2}\ dx \\
&= \sqrt{2}\ x\Big]_0^a \\
&= \sqrt{2}\ a
\end{aligned}
$$

For $k=3$

$$
\begin{aligned}
\int_0^a \sqrt{1+(a^{2/k}-x^{2/k})^{k-2}\ (x^{2(2-k)/k})}\ dx &= \int_0^a \sqrt{1+(a^{2/3}-x^{2/3})^{3-2}\ (x^{2(2-3)/3})}\ dx \\
&= \int_0^a \sqrt{1+(a^{2/3}-x^{2/3})\ (x^{-2/3})}\ dx \\
&= \int_0^a \sqrt{1+a^{2/3}x^{-2/3}-1}\ dx \\
&= \int_0^a \sqrt{a^{2/3}x^{-2/3}}\ dx \\
&= \int_0^a a^{1/3}x^{-1/3}\ dx \\
&= a^{1/3}\int_0^a x^{-1/3}\ dx \\
&= a^{1/3}\cdot\frac{3}{2}x^{2/3}\Big]_0^a \\
&= a^{1/3}\cdot\frac{3}{2}a^{2/3} \\
&= \frac{3}{2}a
\end{aligned}
$$