Chebyshev's Inequality

Concept

1. Chebyshev's inequality allows us to get an idea of probabilities of values lying near the mean even if we don't have a normal distribution. There are two forms:

$$P(|X-\mu| < k\sigma) = P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(|X-\mu| \geq r) \leq \frac{Var(X)}{r^2}.$$

The **Pareto** distribution is the PDF $f(x) = c/x^p$ for $x \geq 1$ and 0 otherwise. Then this is a PDF and $c = p-1$ if and only if $p > 1$. The mean exists and $\mu = \frac{p-1}{p-2}$ if and only if $p > 2$. Finally the variance exists and $\sigma^2 = \frac{p-1}{p-3} - (\frac{p-1}{p-2})^2$ if and only if $p > 3$.

Example

2. Let $f(x) = \frac{5}{x^6}$ for $x \geq 1$ and 0 otherwise. What bound does Chebyshev's inequality give for the probability $P(X \geq 2.5)$? For what value of a can we say $P(X \geq a) \leq 15\%$?

Problems

3. True False Chebyshev's inequality can tell us what the probability actually is.

4. True False For Chebyshev's inequality, the k must be an integer.

5. True False The Chebyshev's inequality also tells us $P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$.

6. True False Chebyshev's inequality can help us estimate $P(\mu - \sigma \leq X \leq \mu + \sigma)$.

7. True False We can use Chebyshev's inequality to prove the Law of Large Numbers.

8. Let $f(x)$ be $(2/3)x$ from $1 \leq x \leq 2$ and 0 everywhere else. Give a bound using Chebyshev's for $P(10/9 \leq X \leq 2)$.

9. Let $f(x)$ be the uniform distribution on $0 \leq x \leq 10$ and 0 everywhere else. Give a bound using Chebyshev's for $P(2 \leq X \leq 8)$. Calculate the actual probability. How do they compare?

10. Let $f(x) = e \cdot e^x$ for $x \leq -1$ and 0 otherwise. Give a bound using Chebyshev's for $P(-4 \leq X \leq 0)$.

11. Let $f(x)$ be $4/x^5$ for $x \geq 1$ and 0 everywhere else. Give a bound using Chebyshev's for $P(X \leq 3)$.

Bounding Probabilities

Simple intuition:

1. Draw the normal pdf. Highlight the portion of the pdf capturing $\{|X - \mu| \geq k\sigma\}$ for $k = 0.5, 1, 2, 5$, roughly.

2. What are some reasons Chebyshev may be lossy? What are some reasons it may be sharp?

Calculations:

1. Suppose X is now Poisson with parameter λ. What are μ and σ for this distribution?

   (a) Compute $\mathbb{P}[|X - \mu| \geq 2 \cdot \sigma]$.

   (b) Compute $\mathbb{P}[|X - \mu| > 2 \cdot \sigma]$.

   (c) Approximate $\mathbb{P}[|X - \mu| > 2 \cdot \sigma]$ using Chebyshev.

   (d) Approximate $\mathbb{P}[|X - \mu| \leq 0.5 \cdot \sigma]$ using Chebsyhev.

2. Suppose that X has Laplace distribution with mean 0, i.e. its pdf is

$$f(x) = \frac{1}{2} e^{-|x|}.$$

   Note that the variance of this distribution is 2.

   (a) Compute $\mathbb{P}[|X| > 4]$.

   (b) Compute $\mathbb{P}[|X| \geq 4]$.

   (c) Use Chebyshev to approximate $\mathbb{P}[|X| > 4]$.

Source: Rosen's *Discrete Mathematics and its Applications.*