Student Name:

MATH 10A – Fall 2018
Discussion 29 Solutions

1 Matrix Algebra

$\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -4 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix}$,

Find the following:

1. $A^T B$
   Soln: [16 7]

2. BA^T
   Soln: Cannot multiply these matrices

3. $B - 2C$
   Soln: $\mathbf{C} = \begin{bmatrix} 12 & -1 \\ 6 & -3 \\ -1 & 0 \end{bmatrix}$

2 Determinants

Find the determinants of the following:
$\mathbf{D} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\mathbf{E} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 1 & 3 & 3 \end{bmatrix}$

Is either matrix singular?

Soln:
$detD = 0$, $detE = 2$
Matrix D is singular (non-invertible). Notice that the rows are multiples of each other.

3 Inverse Matrices

True/False: A^T is the inverse of matrix A

Find the inverse of matrix $\mathbf{F} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Soln: False - clarify A^T vs A^{-1}
Soln:
detF = -1
$F^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

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4 Solving Linear Systems

1. Write the following as a system of equations, then solve for x using inverse matrices:

   $$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} c \\ 4 \end{bmatrix}$$

   Soln:
   $x = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c \\ 4 \end{bmatrix}$
   $x = \begin{bmatrix} 8 - 3c \\ -4 + 2c \end{bmatrix}$

2. Solve the same system of equations using Gaussian elimination.

   Soln:
   $\begin{bmatrix} 1 & 2 & c \\ 2 & 3 & 4 \end{bmatrix}$
   $\begin{bmatrix} 1 & 2 & c \\ 0 & -1 & 4 - 2c \end{bmatrix}$
   $-x_2 = 4 - 2c$
   $x_2 = -4 + 2c$
   Solve for $x_1 = 8 - 3c$

3. Challenge: Suppose that the voltage vector $\mathbf{v}_t$ of the heart changes from one beat to the next according to the equation:

   $$\mathbf{v}_{t+1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}_t$$

   An equilibrium is a value of the vector for which no change occurs (that is $\mathbf{v}_{t+1} = \mathbf{v}_t$). Find all possible equilibrium values $\mathbf{v}$.

   Soln:
   $\mathbf{v}_{t+1} = \mathbf{v}_t$
   $\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}$
   $\mathbf{v}$ - $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v} = 0$
   $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{v} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v} = 0$
   $(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})\mathbf{v} = 0$
   $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
   $detA = 0 - 0 = 0$
   A is singular. Therefore, by Theorem 5 the homogeneous system has infinitely many nontrivial solutions which must satisfy $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ where k is any real number.

5 Inverse Matrices (again!)

Find the inverse of the following matrix using gaussian elimination:

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 7 & 9 \\ 0 & 2 & 1 \end{bmatrix}$$

soln:

$$\begin{bmatrix} 1 & 0 & 3 & | & 1 & 0 & 0 \\ 2 & 7 & 9 & | & 0 & 1 & 0 \\ 0 & 2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & | & 1 & 0 & 0 \\ 0 & 7 & 3 & | & -2 & 1 & 0 \\ 0 & 2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 3/7 & | & -2/7 & 1/7 & 0 \\ 0 & 2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 3/7 & | & -2/7 & 1/7 & 0 \\ 0 & 0 & 1/7 & | & 4/7 & -2/7 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 3/7 & | & -2/7 & 1/7 & 0 \\ 0 & 0 & 1 & | & 4 & -2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -11 & 6 & -21 \\ 0 & 1 & 0 & | & -2 & 1 & -3 \\ 0 & 0 & 1 & | & 4 & -2 & 7 \end{bmatrix}$$

6 Three-Dimensional Geometry

1. Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.

   Soln:
   An equation for the sphere is $(x-2)^2 + (y+6)^2 + (z-4)^2 = 25$. The intersection of this sphere with the xy-plane is the set of points on the sphere whose z-coordinate is 0. Putting $z = 0$ into the equation, we have $(x-2)^2 + (y+6)^2 = 9$ which represents a circle with center (2, -6, 0) and radius 3.
   To find the intersection with the xz-plane we set y = 0: $(x-2)^2 + (z-4)^2 = -11$ no points satisfy this equation so the sphere does not intersect the xz plane.
   To find the intersection with the yz-plane we set x = 0: $(y+6)^2 + (z-4)^2 = 21$ a circle in the yz-plane with center (0, -6, 4) and radius $\sqrt{(}21)$

2. Write an inequality to describe the region between the yz-plane and the plane $x = 5$

Soln:
This describes all points whose x-coordinate is between 0 and 5, that is $0 < x < 5$

7 The Dot Product and Vector Projections

Review:
Vectors are matrices of what dimension?
Why can't we multiply vectors using our techniques for matrix algebra?
$a = [1, 2]$, $b = [-4, 1]$

1. Find the dot product of **a** and **b**

   soln:
   $1(-4) + 2(1) = -2$

2. Find the angle between **a** and **b**

   soln:
   $-2 = |a||b|cos\alpha$
   $-2 = \sqrt{(1^2 + 2^2)}\sqrt{(-4^2 + 1^2)}cos\alpha$
   $-2 = \sqrt{(5)(17)}cos\alpha$
   $-2/\sqrt{85} = cos\alpha$
   $\alpha = 102.5^o$

3. For what values of b are the vectors $[-6, b, 2]$ and $[b, b^2, b]$ orthogonal?

   soln:
   Orthogonal when $[-6, b, 2] \cdot [b, b^2, b] = 0$
   $(-6)(b) + (b)(b^2) + (2)(b) = 0$
   $b^3 - 4b = 0$
   $b(b + 2)(b - 2) = 0$
   $b = 0$ or $b = \pm 2$

8 Identifying Eigenvectors

Are the following vectors eigenvectors of the matrix? If so what are their corresponding eigenvalues?
$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

soln:
$\mathbf{Ax} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **v** NOT an eigenvector
$\mathbf{Av} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ **v** eigenvector with $\lambda = -1$

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9 Solving for Eigenvectors and Eigenvalues

Review: Do we need to use the characteristic equation to find eigenvalues of upper and lower triangular matrices? Why or why not?

soln:
No we do not. The values on the main diagonal are our eigenvalues.

Find the eigenvalues of the matrix: $\mathbf{A} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 2 & 1 \end{bmatrix}$

soln:
$det \begin{bmatrix} -1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 7 \\ 0 & 2 & 1-\lambda \end{bmatrix} = 0$

$(-1-\lambda)(1-\lambda)^2 - 14(1-\lambda) = (1-\lambda)[(1-\lambda)^2 - 14]$

$= (-1-\lambda)(1-\lambda+\sqrt{14})(1-\lambda-\sqrt{14})$

$\lambda = -1 \ , \ 1+\sqrt{14} \ , \ 1-\sqrt{14}$

10 Sequences

Find a direct formula for $a_{n+1} = 2a_n + 3a_{n1}$ for $n1$, $a_0 = 1$, $a_1 = 1$, by encoding the recurrence sequence with linear algebra

soln:
$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = 3, -1$

$\begin{bmatrix} 0-3 & 1 & | & 0 \\ 3 & 2-3 & | & 0 \end{bmatrix}$

$-3v_1 + v_2 = 0$

$v_2 = 3v_1$

$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 0+1 & 1 & | & 0 \\ 3 & 2+1 & | & 0 \end{bmatrix}$

$w_1 + w_2 = 0$

$w_2 = -w_1$

$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Intial values: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$1 = c_1 + c_2$, $1 = 3c_1 - c_2$

$c_1 = 1/2$, $c_2 = 1/2$

$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = (\frac{1}{2})(3)^n \begin{bmatrix} 1 \\ 3 \end{bmatrix} + (\frac{1}{2})(-1)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

11 Systems of Differential Equations

Solve the IVP $y'' - 2y' - 3y = 0$, $y(0) = 1$, $y'(0) = 1$ by encoding the DE with linear algebra.

soln:
$y_1 = y$
$y_2 = y'(= y_1)$
$y_1' = y'(= y_2)$
$y_2' = y''$

$y_1' = 0y_1 + 1y_2$
$y_2' = 3y_1 + 2y_2$

Use eigenvectors and eigenvalues from above.

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1(e^{3t}) \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2(e^{-1t}) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$c_1 = 1/2$, $c_2 = 1/2$

$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = (\frac{1}{2})(e^{3t}) \begin{bmatrix} 1 \\ 3 \end{bmatrix} + (\frac{1}{2})(e^{-t}) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$y(t) = \frac{1}{2}(e^{3t}) + \frac{1}{2}(e^{-t})$