MATH 10A with Prof. Stankova
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Gaussian Elimination

1 Gaussian Elimination to Solve Linear Systems

1.
$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 3 & 7 \end{array}\right) \xrightarrow{II-I/2, III-I/2} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 3/2 & 7/2 & 7 \end{array}\right) \xrightarrow{III-3II} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -1 & -2 \end{array}\right)$$

Thus we see that $-z = -2$ or $z = 2$. Then $1/2y + 3/2z = 3$ or $1/2y + 3 = 3$ so $y = 0$. Finally we have $2x + 3y - z = 0$ or $2x + 3(0) - 2 = 0$ so $2x = 2$ and $x = 1$. Thus, there is a unique solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

2. a) First, swap the third and first row so the top left entry is nonzero. Then,

$$\left(\begin{array}{ccc|c} 4 & 5 & 0 & 2 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 0 \end{array}\right) \xrightarrow{II-I/4} \left(\begin{array}{ccc|c} 4 & 5 & 0 & 2 \\ 0 & 3/4 & 1 & 1/2 \\ 0 & 1 & 3 & 0 \end{array}\right) \xrightarrow{III-4II/3} \left(\begin{array}{ccc|c} 4 & 5 & 0 & 2 \\ 0 & 3/4 & 1 & 1/2 \\ 0 & 0 & 5/3 & -2/3 \end{array}\right)$$

Thus we have that $5/3z = -2/3$ or $z = \frac{-2}{5}$. Then $3/4y + z = 1/2$ so $3/4y - 2/5 = 1/2$ and $y = \frac{4}{3} \cdot (1/2 + 2/5) = \frac{4}{3} \cdot \frac{9}{10} = \frac{6}{5}$. Finally $4x + 5y = 2$ so $4x + 5(6/5) = 4x + 6 = 2$ and $4x = -4$ so $x = -1$. So there is a unique solution and it is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 6/5 \\ -2/5 \end{pmatrix}$.

If we calculate the determinant of the original matrix, we get $0(2)(0) + 5(1)(3) + 4(1)(1) - 4(2)(3) - 5(1)(0) - 0(1)(1) = -5 \neq 0$ so it is invertible. This is what we expect because there is a unique solution.

b)

$$\left(\begin{array}{ccc|c} 2 & 1 & 8 & 1 \\ -1 & 1 & -1 & 0 \\ -2 & 5 & 4 & 1 \end{array}\right) \xrightarrow{II+I/2, III+I} \left(\begin{array}{ccc|c} 2 & 1 & 8 & 1 \\ 0 & 3/2 & 3 & 1/2 \\ 0 & 6 & 12 & 2 \end{array}\right) \xrightarrow{III-4II} \left(\begin{array}{ccc|c} 2 & 1 & 8 & 1 \\ 0 & 3/2 & 3 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The variable z can be whatever so we have $3/2y + 3z = 1/2$ or $y = 1/3 - 2z$. Then $2x + y + 8z = 1$ so $2x + 1/3 - 2z + 8z = 1$ and simplifying gives $x = 1/3 - 3z$. So there are an infinite number of solutions depending on z and they are of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 - 3z \\ 1/3 - 2z \\ z \end{pmatrix}$.

The determinant of the original matrix is $2 \cdot 1 \cdot 4 + 1 \cdot (-1)(-2) + 8(-1)(5) - 2(-1)(5) - 1(-1)(4) - 8(1)(-2) = 0$, so it is not invertible. This is what we expected because we have ∞ solutions.

c)

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 2 \\ 2 & 4 & -8 & 5 \\ -3 & -6 & 12 & -6 \end{array}\right) \xrightarrow{II-2\cdot I, II+3\cdot I} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Because the second row is (0 0 0| 1), so there are no solutions.

If we calculate the determinant of the original matrix, we get $1(4)(12)+2(-8)(-3)+(-4)(2)(-6)-(-4)(4)(-3)-(-8)(-6)(1)-(2)(2)(12)=0$ so it is not invertible. This is what we expect because there are no solutions.

2 Gaussian Elimination to find Matrix Inverse

1.

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{II-3I/2} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -1/2 & -3/2 & 1 \end{array}\right) \xrightarrow{I+6II} \left(\begin{array}{cc|cc} 2 & 0 & -8 & 6 \\ 0 & -1/2 & -3/2 & 1 \end{array}\right) \xrightarrow{I/2,-2II} \left(\begin{array}{cc|cc} 1 & 0 & -4 & 3 \\ 0 & 1 & 3 & -2 \end{array}\right)$$

Thus, $A^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$

This is the same answer we get using the other formula, $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

2. a)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -3 & 1 & 2 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{II+3I, III+2I} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 7 & -1 & 3 & 1 & 0 \\ 0 & 6 & -1 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{I-2II/7, III-6II/7} \left(\begin{array}{ccc|ccc} 1 & 0 & -5/7 & 1/7 & -2/7 & 0 \\ 0 & 7 & -1 & 3 & 1 & 0 \\ 0 & 0 & -1/7 & -4/7 & -6/7 & 1 \end{array}\right)$$

$$\xrightarrow{I-5III, II-7III} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 4 & -5 \\ 0 & 7 & 0 & 7 & 7 & -7 \\ 0 & 0 & -1/7 & -4/7 & -6/7 & 1 \end{array}\right)$$

$$\xrightarrow{II/7, III\cdot(-7)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 4 & -5 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right)$$

Thus $A^{-1} = \begin{pmatrix} 3 & 4 & -5 \\ 1 & 1 & -1 \\ 4 & 6 & -7 \end{pmatrix}$ and we multiply to check that $AA^{-1} = A^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

b)

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{Swap } I,II} \left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{III+I} \left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{\text{Swap } II,III} \left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{I-3II/2, I-III} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -1 & -1/2 & -3/2 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{I/2, II/2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/4 & -3/4 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

Thus $A^{-1} = \begin{pmatrix} -1/2 & -1/4 & -3/4 \\ 1/2 & 1/2 & \\ 1 & 0 & 0 \end{pmatrix}$ and we multiply to check that $AA^{-1} = A^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

3 Challenge

1. For what value of c will the following system of equations have at least one solution? Find all solutions of the resulting system.

$$\left|\begin{array}{rcrcrcl} x & + & 3y & - & z & = & 4 \\ 2x & - & y & + & 3z & = & 7 \\ & & 7y & - & 5z & = & c \end{array}\right.$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 2 & -1 & 3 & 7 \\ 0 & 7 & -5 & c \end{array}\right) \xrightarrow{II-2I} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -7 & 5 & -1 \\ 0 & 7 & -5 & c \end{array}\right) \xrightarrow{III+II} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -7 & 5 & -1 \\ 0 & 0 & 0 & c-1 \end{array}\right)$$

Now, from the last row we know that $c = 1$. If $c = 1$, then, we use back substitution to give the solution. Start with the second row. $-7y + 5z = -1$, so $y = \frac{-1}{7}(-1 - 5z) = \frac{1}{7} + \frac{5}{7}z$. The second row is $x + 3y - z = 4$, so $x = 4 - 3y + z = 4 - \frac{3}{7} - \frac{15}{7}z + z = \frac{25}{7} - \frac{8}{7}z$. The solutions are all of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{25}{7} - \frac{8}{7}z \\ \frac{1}{7} + \frac{5}{7}z \\ z \end{pmatrix}$.

2. Solve the following system of equations.

$$\left|\begin{array}{rcrcrcrcr} x & - & 3y & + & z & - & w & = & 7 \\ 2x & + & y & & & + & w & = & 0 \\ & & 3y & - & z & + & w & = & -6 \end{array}\right.$$

$$\left(\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 7 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -6 \end{array}\right) \xrightarrow{II-2I} \left(\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 7 \\ 0 & 7 & -2 & 3 & -14 \\ 0 & 3 & -1 & 1 & -6 \end{array}\right)$$

$$\xrightarrow{III-3II/7} \left(\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 7 \\ 0 & 7 & -2 & 3 & -14 \\ 0 & 0 & -1/7 & -2/7 & 0 \end{array}\right)$$

Solving the last row gives us $1/7z = -2/7w$ or $z = -2w$. Then $7y - 2z + 3w = -14 = 7y - 2(-2w) + 3w = 7y + 7w = -14$. Thus $y = -w - 2$. Finally $x - 3y + z - w = 7 = x - 3(-w-2) - 2w - w = x + 6 = 7$. Thus $x = 1$. Therefore, there are an infinite number of solutions depending on w of the form $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -w-2 \\ -2w \\ w \end{pmatrix}$.