MATH 10A with Prof. Stankova
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Intro to Ordinary Differential Equations

This document is a collective effort of multiple GSI's. Please thank them for their hard work. :)

I. Concepts

1. What is a differential equation? What are we solving for?

> It is an equation describing some relationship between a function y and its derivative(s) y' (or higher-ordered derivatives). We are solving for $y(t)$. That is, we are asking, which functions satisfy such a relationship?

2. What does it mean to be ordinary? What is autonomous? What is pure-time? What is first order? What is linear? What is homogeneous?

> - Ordinary: y is a function of one variable. In street maths, this means you cannot have any other variable other y and t, one is dependent, one is independent.
> - Autonomous: (Latin for self-referencing?). In street maths, it means no t anywhere in the equation (the derivative y' only depends on y.)
> - Pure-time: the opposite, only t, no y anywhere, except y', otherwise you wouldn't call it DE.
> - First-order: only first derivative of y.
> - Linear: (in y). The only functions of y are linear, that is no y^n where $n \neq 1$, no $cosy, e^y, arcsiny, ...$ Notice we don't care about what the function of t it is attached to.
> - Homogeneous: constant term in y is 0. The part that doesn't have any y is 0.

3. When solving an ODE, it is rather important to figure out what the type is because that often points us to the techniques required to solve it.

II. T/F

4. A solution to an ODE in $y(t)$ can be $t = 3$.

> This is FALSE! $t = 3$ is not a function of t in this context.

5. A solution to an ODE in $y(t)$ can be $y = 3$.

> This is TRUE! $y = 3$ is a function of t, the constant function in this context.

6. The method of integrating factor is applicable to solve both the exponential and logistic model.

This is TRUE! Notice both models are FOLODE.

- Exponential model: $y' = ky$
- Logistic model $y' = Ky(1 - y/K)$

7. To solve $ty' + yP(t) = Q(t)$, we find the integrating factor $I(t) = e^{\int P(t)dt}$.

This is FALSE! We must put the FOLODE in the form where the coefficient of y' is 1 first, before finding the integrating factor. That is to divide the whole equation by t before proceeding.

III. Creative Problems

8. Cook up an ODE where the $y(t) = 2t + cost - 5$ is a solution. Be creative, but correct! :-)

$y'(t) = 2 - sint$, or $ty' - y = 5 - tsint - cost$, or $y'' + y = 2t - 5$...

9. Can you give a different solution to the ODE you cooked up? PROVE that the solution you give is in fact a solution of the ODE you created.

For the first linear pure-time, any vertical shift will work, say $y(t) = 2t + cost + 5$.
For second DE, because it doesn't depend on $2t$, any scaled version of that should work, say $y(t) = 3t + cost - 5$.
For the third, this one depends on the relationship of second derivative, switching $cost$ and $sint$ (or any linear combination there of) should do, say $y(t) = 2t + sint - 5$.

IV. Algorithmic Problems

For all of the ODE below, label them by type (see question 2), and then proceed to solve if possible.

10. Determine if the differential equation is linear. Explain why or why not. You don't have to solve these.
    (a) $y' + \sin y = \tan t$;
    (b) $yy' + xy = x^2$;
    (c) $xy + \sqrt{x} = e^x y'$.

(a) It is not linear. Notice for linear differential equation every term involving y or its derivatives must be in linear form with respect to y. In this equation, $\sin y$ is not linear with respect to y, so it's not linear.

(b) It is not linear. In this equation yy' is not linear.

(c) It is linear. Every term involving y and every term involving y' is linear.

11. Solve the DE.
   (a) $y' = t + 5y$;
   (b) $xy' + y = \sqrt{x}$;
   (c) $y' + y = \sin(e^x)$;
   (d) $(1+t)\dfrac{du}{dt} + u = 1 + t$; $t > 0$.

> (a) Rewrite $y' = t + 5y$ as $y' - 5y = t$ to be in the form $y' + P(t)y = Q(t)$. Now we want to determine the intergrating factor $I(t)$: $I(t) = e^{\int P(t)dt} = e^{\int(-5)dt} = e^{-5t}$. Multiply $I(t)$ to both sides we get: $e^{-5t}y' - 5e^{-5t}y = te^{-5t}$. So $I(t)y = \int te^{-5t}dt = (-\frac{1}{5})(te^{-5t} - \int e^{-5t}dt) = -\frac{1}{5}te^{-5t} - \frac{1}{25}e^{-5t} + C$, so $y = -\frac{1}{5}t - \frac{1}{25} + Ce^{5t}$.
>
> (b) Notice that $xy' + y = (xy)'$, so $xy = \int \sqrt{x} = \frac{2}{3}x^{\frac{3}{2}} + C$. Hence $y = \frac{2}{3}\sqrt{x} + \frac{C}{x}$.
>
> (c) $I(x) = e^{\int 1dx} = e^x$, so multiply $I(t)$ to both sides we get: $e^x y' + e^x y = (e^x y)' = e^x \sin(e^x)$. So we have $e^x y = \int e^x \sin(e^x) = -\cos(e^x) + C$, so $y = \frac{-\cos(e^x)+C}{e^x}$.
>
> (d) Notice that $(1+t)\dfrac{du}{dt} + u = ((1+t)u)'$, so $(1+t)u = \int(1+t)dt = t + \frac{1}{2}t^2 + C$. Since in this equation the domain is determined to be $t > 0$, so $1 + t \neq 0$ and we can divide the equation by $1 + t$ to get $u = \frac{t+\frac{1}{2}t^2+C}{1+t}$.

12. Solve the IVP (Initial Value Problem).
   (a) $y' = x + y$, $y(0) = 2$;
   (b) $\dfrac{dv}{dt} - 2tv = 3t^2 e^{t^2}$, $v(0) = 5$;
   (c) $xy' = y + x^2 \sin x$, $y(\pi) = 0$.

> (a) Rewrite the equation: $y' - y = x$. We get $I(x) = e^{\int(-1)dx} = e^{-x}$, and hence $(e^{-x}y)' = xe^{-x}$. So $e^{-x}y = \int xe^{-x}dx = -(xe^{-x} - \int e^{-x}dx) = -e^{-x} - xe^{-x} + C$, which gives $y = -1 - x + Ce^x$. Since $y(0) = 2$, $-1 - 0 + Ce^0 = 2$, so $C = 3$. The solution would be: $y = -1 - x + 3e^x$.
>
> (b) $I(t) = e^{\int -2tdt} = e^{-t^2}$. Multiply both sides by $I(t)$: $e^{-t^2}v' - 2te^{-t^2}v = (e^{-t^2}v)' = 3t^2$. So we have $e^{-t^2}v = \int 3t^2 dt = t^3 + C$, which gives us the general solution: $v = e^{t^2}t^3 + Ce^{t^2}$. Since $v(0) = 5$, $C = 5$, we have a particular solution: $v = e^{t^2}t^3 + 5e^{t^2}$.
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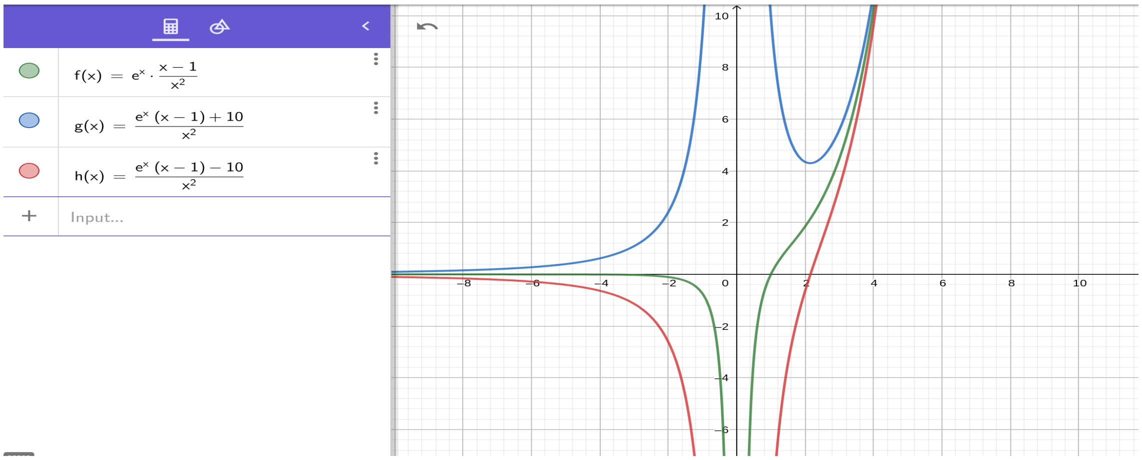
> (c) Rewrite the equation: $y' - \frac{y}{x} = x\sin(x)$, hence we have $I(x) = e^{\int(-\frac{1}{x})dx} = \frac{1}{x}$. So $(\frac{y}{x})' = \int \sin(x)dx = -\cos(x) + C$, which gives the general solution: $y = -x\cos(x) + Cx$. Since $y(\pi) = 0$, $-\pi\cos(\pi) + C\pi = -\pi + C\pi = 0, C = -1$, we have a particular solution: $y = -x\cos(x) - x$.

13. Solve the DE $xy' + 2y = e^x$.
   (*Optional*) Use computer software to graph several particular solutions.
   How does the solution change as C varies?

Rewrite this equation: $y' + \frac{2}{x}y = \frac{e^x}{x}$, then we have $I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = x^2$. Multiply $I(x)$ to both sides, we have $x^2y' + 2xy = xe^x$, so $x^2y = \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$. So $y = \frac{xe^x - e^x + C}{x^2}$.
Here is the graph for several possible C's.



14. (Challenge) Solve the second-order DE $xy'' + 2y' = 12x^2$.
(*Hint:* First substitute $u = y'$, which will lower the order of the DE, and solve for u. Finally, solve $y' = u$ for y, using the solution for u you just found.)

Follow the hint, let $u = y'$, then we can rewrite this equation into: $xu' + 2u = 12x^2$, or $u' + \frac{2}{x}u = 12x$. Now we follow the algorithm for solving first order linear DE: $I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = x^2$. And we have $(x^2u)' = 12x^3$, so $x^2u = \int 12x^3 dx = 3x^4 + C_0$, which gives us $u = 3x^2 + \frac{C_0}{x^2}$, which means $y = \int u dx = x^3 - \frac{C_0}{x} + C_1$ where C_0, C_1 are both constants.