Worksheet #24; Date: 11/13/2018
MATH 10A Methods of Mathematics
with Professor Stankova

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1 Eigenvalue and Eigenvector

1. Find eigenvalue of each matrix.

   (a) $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$

   (a) $\det\left(\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} - \lambda I\right) = \det\begin{bmatrix} 2-\lambda & 0 \\ 3 & -\lambda \end{bmatrix} = 0 \Rightarrow (2-\lambda)(-\lambda) - (0)(3) = 0 \Rightarrow \lambda(\lambda - 2) = 0$

   The eigenvalues are therefore $\lambda = 0$ and $\lambda = 2$.

   (b) $\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$

   (c) $\det\left(\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} - \lambda I\right) = \det\begin{bmatrix} 3-\lambda & -1 \\ 0 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (3-\lambda)(2-\lambda) = 0$

   The eigenvalues are therefore $\lambda = 2$ and $\lambda = 3$.

2. In general, the eigenvalues of an upper triangular matrix are given by the entries on the diagonal. The same is true for a lower triangular matrix. Verify this for 2 by 2 matrix.

   We determine the eigenvalues of 2×2 upper and lower triangular matrices U and L by calculating the characteristic polynomials

   $$\det\begin{bmatrix} u_{11}-\lambda & u_{12} \\ 0 & u_{22}-\lambda \end{bmatrix} = (u_{11}-\lambda)(u_{22}-\lambda) = 0 \Rightarrow \lambda = u_{11} \text{ and } \lambda = u_{22}$$

   $$\det\begin{bmatrix} l_{11}-\lambda & 0 \\ l_{21} & l_{22}-\lambda \end{bmatrix} = (l_{11}-\lambda)(l_{22}-\lambda) = 0 \Rightarrow \lambda = l_{11} \text{ and } \lambda = l_{22}$$

3. Suppose that an eigenvalue of matrix A is zero. Prove that A must therefore be singular.

20. If matrix A has an eigenvalue of zero then $\det(A - 0I) = 0 \;\Rightarrow\; \det(A) = 0 \;\Rightarrow\; A$ is singular.

5. Find the eigenvalues and eigenvectors of the matrix.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(a) Eigenvalues: $\det \begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) = -(1-\lambda)(1+\lambda) = 0$

The eigenvalues are therefore $\lambda = 1$ and $\lambda = -1$.

Eigenvectors: Starting with the eigenvalue $\lambda = 1$ we have

$$\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left\{ \begin{array}{r} 0 = 0 \\ -2v_2 = 0 \end{array} \right\}$$

The two equations are satisfied when $v_2 = 0$ and v_1 can be any value. Choosing $v_1 = 1$ gives the eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For the second eigenvalue $\lambda = -1$ we have

$$\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left\{ \begin{array}{r} 2v_1 = 0 \\ 0 = 0 \end{array} \right\}$$

The two equations are satisfied when $v_1 = 0$ and v_2 can be any value. Choosing $v_2 = 1$ gives the eigenvector $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix}$

Eigenvalues: $\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & -3-\lambda \end{bmatrix} = (1-\lambda)(-3-\lambda) - 6 = \lambda^2 + 2\lambda - 9 = 0 \quad \Rightarrow$

$$\lambda = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-9)}}{2(1)} = -1 \pm \frac{\sqrt{40}}{2} = -1 \pm \sqrt{10}$$

The eigenvalues are therefore $\lambda = -1 + \sqrt{10}$ and $\lambda = -1 - \sqrt{10}$.

Eigenvectors: Starting with the eigenvalue $\lambda = -1 + \sqrt{10}$ we have

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & -3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2-\sqrt{10} & 2 \\ 3 & -2-\sqrt{10} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{l} (2-\sqrt{10})\, v_1 + 2v_2 = 0 \\ 3v_1 - (2+\sqrt{10})\, v_2 = 0 \end{array} \right\}$$

[continued]

Both equations specify that $v_2 = -\left(\frac{2-\sqrt{10}}{2}\right) v_1$ (try multiplying the second equation by the appropriate radical conjugate if this is unclear). Choosing $v_1 = -2$ gives the eigenvector $\mathbf{v} = \begin{bmatrix} -2 \\ 2-\sqrt{10} \end{bmatrix}$.

For the second eigenvalue $\lambda = -1 - \sqrt{10}$ we have

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & -3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2+\sqrt{10} & 2 \\ 3 & -2+\sqrt{10} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{l} (2+\sqrt{10})\, v_1 + 2v_2 = 0 \\ 3v_1 - (2-\sqrt{10})\, v_2 = 0 \end{array} \right\}$$

Both equations specify that $v_2 = -\left(\frac{2+\sqrt{10}}{2}\right) v_1$ (try multiplying the second equation by the appropriate radical conjugate if this is unclear). Choosing $v_1 = -2$ gives the eigenvector $\mathbf{v} = \begin{bmatrix} -2 \\ 2+\sqrt{10} \end{bmatrix}$.

2 Fibonacci

1. given the vector to be $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ and $\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$ find the matrix for Fibonacci sequence.

   We know $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$ and $x_{n+1} = x_n + x_{n-1}$. Let A be a 2 by 2 matrix with $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ then $x_{n+1} = a_1 x_n + a_2 x_{n-1}$ and $x_n = a_3 x_n + a_4 x_{n-1}$. Therefore we have $a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 0$. The matrix A is then $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

2. Find the eigenvalues of the matrix you found above.

   $det \begin{bmatrix} 1-\lambda, & 1 \\ 1, & -\lambda \end{bmatrix} = 0$. then $\lambda^2 - \lambda - 1 = 0$. solve the equation we get:

$\lambda = \frac{1 \pm \sqrt{5}}{2}$

3. Suppose thatλ is an eigenvalue of A. Show that λ^2 is then an eigenvalue of A^2

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Rightarrow \quad A^2\mathbf{v} = \lambda A\mathbf{v} \quad \Rightarrow \quad A^2\mathbf{v} = \lambda\left(\lambda\mathbf{v}\right) \quad \Rightarrow \quad A^2\mathbf{v} = \lambda^2\mathbf{v}$$

Therefore, by Definition (2) λ^2 is an eigenvalue of the matrix A^2.

4. find the eigenvalues for generalized fibonacci sequences' matrix

   By 2 we know that $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
   Then $\begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$, $\begin{bmatrix} x_4 \\ x_3 \end{bmatrix} = A \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} == A^2 \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$. We can conclude that $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A^n \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$.
   Then by 3 we can see taht if we know the eigenvalue for A is $\lambda = \frac{1 \pm \sqrt{5}}{2}$, then the eigenvalue for A^n is $\lambda^n = (\frac{1 \pm \sqrt{5}}{2})^n$