I Inverses and Determinants

1. Determine if matrices A and B are inverses of one another.

   a) $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

   Yes.

   b) $A = \begin{bmatrix} 9 & 0 \\ 2 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} \frac{1}{9} & 0 \\ -\frac{2}{27} & \frac{1}{3} \end{bmatrix}$

   Yes.

2. Find the inverse of each matrix.

   a) $\begin{bmatrix} 9 & 2 \\ 7 & 5 \end{bmatrix}$

   $\frac{1}{45-14} \begin{bmatrix} 5 & -2 \\ -7 & 9 \end{bmatrix} = \begin{bmatrix} 5/31 & -2/31 \\ -7/31 & 9/31 \end{bmatrix}$

   b) $\begin{bmatrix} 3x & y \\ 2x & y \end{bmatrix}$

   If the determinant $3xy - 2xy = xy \neq 0$, then the inverse is $\frac{1}{xy} \begin{bmatrix} y & -y \\ -2x & 3x \end{bmatrix} = \begin{bmatrix} 1/x & -1/x \\ -2/y & 3/y \end{bmatrix}$

   Otherwise not invertible.

3. Find the determinant of each matrix. Determine whether they are nonsingular (i.e. invertible).

   a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (Use both Laplace expansion and the rule taught in class.)

   Laplace expansion: $1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 1(-3) - 2(-6) + 3(-3) = 0.$

   Rule in class: $1 \cdot 5 \cdot 9 + 3 \cdot 4 \cdot 8 + 2 \cdot 6 \cdot 7 - 3 \cdot 5 \cdot 7 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9 = 0.$

   Not invertible.

   b) $\begin{bmatrix} 1 & 0 & 5 \\ 4 & 2 & 6 \\ 3 & 2 & 1 \end{bmatrix}$

   By Laplace expansion, determinant is $1(2-12) - 0(4-18) + 5(8-6) = -10 + 10 = 0.$

   Not invertible.

4. Find all solutions to the system of equations or matrix. For systems of equations, first translate into the matrix form $A\mathbf{x} = \mathbf{b}$.

   a) $\begin{cases} 2x_1 = 10 \\ 3x_1 - x_2 = 14 \end{cases}$

   $A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$ $\quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \mathbf{b} \begin{bmatrix} 10 \\ 14 \end{bmatrix}$

   $\det A = -2$. A invertible. $A^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & 0 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 3/2 & -1 \end{bmatrix}$.

   Since $A^{-1}A\mathbf{x} = A^{-1}b$, we get $\mathbf{x} = \begin{bmatrix} 1/2 & 0 \\ 3/2 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

   Graphically, the two lines have exactly one intersection.

   b) $\begin{cases} 4x_1 + 6x_2 = 1 \\ 2x_1 + 3x_2 = 6 \end{cases}$

   $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ $\quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \mathbf{b} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

det $A = 0$. A is not invertible.
Multiplying the second equation by 2, we get $4x_1 + 6x_2 = 12$, which contradicts the first equation. Therefore the system has no solutions.
Graphically, the two lines are parallel.

c) $A = \begin{bmatrix} 1 & -1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ $\quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

det $A = 0$. A is not invertible. The system of equation we have is $\begin{cases} x_1 - x_2 = 2 \\ \frac{1}{2}x_1 - \frac{1}{2}x_2 = 1. \end{cases}$

The two equations are constant multiples of one another. We have infinitely many solutions.
Graphically, the two lines are parallel.

5. If A is nonsingular/invertible, show that $(A^{-1})^{-1} = A$ for any 2×2 matrix.
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then det $A = ad - bc \neq 0$.
So $A^{-1} = \begin{bmatrix} d/\det A & -b/\det A \\ -c/\det A & a/\det A \end{bmatrix}$.
Then $\det(A^{-1}) = \frac{1}{ad-bc}$. And $\frac{1}{\det(A^{-1})} = ad - bc = \det A$.
So $(A^{-1})^{-1} = \det A \begin{bmatrix} a/\det A & b/\det A \\ c/\det A & d/\det A \end{bmatrix} = A$.

6. (Challenge) Consider the following models for the population size $\mathbf{n}_t$ of an age-structured population with two age classes:

$$\mathbf{n}_{t+1} = \begin{bmatrix} b & 2 \\ \frac{1}{2} & 0 \end{bmatrix} \mathbf{n}_t$$

An equilibrium is a value of the vector for which no change occurs (that is, $\mathbf{n}_{t+1} = \mathbf{n}_t$).

a) Suppose $b \neq 0$. Find all possible equilibrium values.

b) Suppose $b = 0$. Find all possible equilibrium values.

We find $\hat{\mathbf{n}}$ such that $A\hat{\mathbf{n}} = \hat{\mathbf{n}}$, where $A = \begin{bmatrix} b & 2 \\ \frac{1}{2} & 0 \end{bmatrix}$. To find such $\hat{\mathbf{n}}$, we make the RHS a vector of constants.
We get $A\hat{\mathbf{n}} - I\hat{\mathbf{n}} = \mathbf{0}$. Then $\begin{bmatrix} b-1 & 2 \\ \frac{1}{2} & -1 \end{bmatrix} \hat{\mathbf{n}} = \mathbf{0}$.
Let the first matrix above be B. Then det $B = 1 - b - 1 = b$.

a) $b \neq 0$. Then B is invertible. $\hat{\mathbf{n}} = B^{-1}\mathbf{0} = \mathbf{0}$. We have one unique equilibria.

b) $b = 0$. Then we solve the system $\begin{cases} -x_1 + 2x_2 = 0 \\ \frac{1}{2}x_1 - x_2 = 0 \end{cases}$ and get $x_1 = 2x_2$. So $\mathbf{n} = \begin{bmatrix} 2k \\ k \end{bmatrix}$, where k is any real number. We have infinitely many equilibria.

7. True/False

a) If A, B, C are square matrices, then in general, it's not the case that $(AB)C = A(BC)$.
False.

b) Only square matrices are invertible.
True.

c) Suppose $A\mathbf{x} = \mathbf{b}$, where A is an $n \times n$ matrix, $\mathbf{x}$ is $n \times 1$ vector of unknowns, and $\mathbf{b}$ is a $n \times 1$ vector of constants. If det $A = 0$, the system of equations have either infinitely many solutions, or no solution.
True.