MATH 10A with Prof. Stankova
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Integrals and Fundamental Theorem of Calculus

1 Definite Integrals

1. (a) If f is a continuous function defined on $\mathbb{R}$, we know that for any real number, c,

$$\int_0^2 f(x)\mathrm{d}x = \int_0^c f(x)\mathrm{d}x + \int_c^2 f(x)\mathrm{d}x.$$

For $0 \leq c \leq 2$, we can interpret this geometrically. How can we interpret this equality geometrically for $c > 2$? What about for $c < 0$?

First, let's think about $c > 2$. Then, $\int_c^2 f(x)\mathrm{d}x = -\int_2^c f(x)\mathrm{d}x$. This means, we can think of $\int_c^2 f(x)\mathrm{d}x$ as the negative of the area under the graph of f from $x = 2$ to $x = c$. Then, the equation is saying that the area under the graph of f from $x = 0$ to $x = 2$ is the area under the graph of f from $x = 0$ to $x = c$ minus the area under the graph of f from $x = 2$ to $x = c$. This makes sense.

Now, think about $c < 0$. Then, we should think of $\int_0^c f(x)\mathrm{d}x$ as the area under the graph of f from $x = c$ to $x = 0$. The equation is then saying that the area under the graph of f from $x = 0$ to $x = 2$ is the area under the graph of f from $x = c$ to $x = 2$ minus the area under the graph of f from $x = c$ to $x = 0$. This also makes sense.

(b) Let $f(x) = \sqrt{\ln x}$. Evaluate

$$\int_1^4 f(x)\mathrm{d}x + \int_3^6 f(x)\mathrm{d}x + \int_6^1 f(x)\mathrm{d}x - \int_3^4 f(x)\mathrm{d}x.$$

Rewrite the third term as $-\int_1^6 f(x)\mathrm{d}x$. We can see geometrically that the integral is then 0. Algebraically,

$$\begin{aligned}
&\int_1^4 f(x)\mathrm{d}x + \int_3^6 f(x)\mathrm{d}x + \int_6^1 f(x)\mathrm{d}x - \int_3^4 f(x)\mathrm{d}x \\
&= \left(\int_1^3 f(x)\mathrm{d}x + \int_3^4 f(x)\mathrm{d}x\right) + \int_3^6 f(x)\mathrm{d}x - \int_1^6 f(x)\mathrm{d}x - \int_3^4 f(x)\mathrm{d}x \\
&= \int_1^3 f(x)\mathrm{d}x + \int_3^6 f(x)\mathrm{d}x - \int_1^6 f(x)\mathrm{d}x = 0.
\end{aligned}$$

2. There is a real number, c, such that $\int_{-2}^2 (c + \sqrt{4 - x^2}) = 0$. What is c?

Breakup the integral as $\int_{-2}^2 c\ \mathrm{d}x + \int_{-2}^2 \sqrt{4 - x^2}\ \mathrm{d}x$. The first term is $4c$. The second term, is the area under a semicircle of radius 2; hence, it is 2π. Thus, we want to find c such that $4c + 2\pi = 0$. $c = -\pi/2$.

2 Fundamental Theorem of Calculus

1. Using Fundamental Theorem of Calculus, evaluate

$$\int_{3\pi/2}^{4\pi} e^{\sin x} \cos x \, \mathrm{d}x.$$

The antiderivative of $e^{\sin x} \cos x$ is $e^{\sin x}$. Check that $e^{\sin(4\pi)} - e^{\sin(3\pi/2)} = e^0 - e^{-1} = 1 - \frac{1}{e}$.

2. g is a function such that g'' exists and is continuous at all real numbers. Given that $g(1) = 2$, $g(3) = -9$, $g'(1) = 0$, and $g'(3) = 4$, compute

$$\int_1^3 (g''(x) + g'(x))\mathrm{d}x.$$

Break up the integral as $\int_1^3 g''(x) \, \mathrm{d}x + \int_1^3 g'(x) \, \mathrm{d}x$. The antiderivative of g'' is g', so the first term is $g'(3) - g'(1) = 4$. The antiderivative of g' is g, so the second term is $g(3) - g(1) = -11$. The answer is $4 + (-11) = -7$.

3. (a) Let f be the function given by

$$f(x) = \int_3^x \frac{1}{t^3 + 1} \, \mathrm{d}t.$$

What is $f'(5)$? What about $f'(2)$?

By fundamental theorem of calculus, $f'(x) = \frac{1}{x^3+1}$. $f'(5) = \frac{1}{126}$ and $f'(2) = \frac{1}{9}$.

(b) Let g be the function given by

$$g(x) = \int_x^7 \tan t \, \mathrm{d}t$$

What is $g'(2)$? What about $g'(9)$?

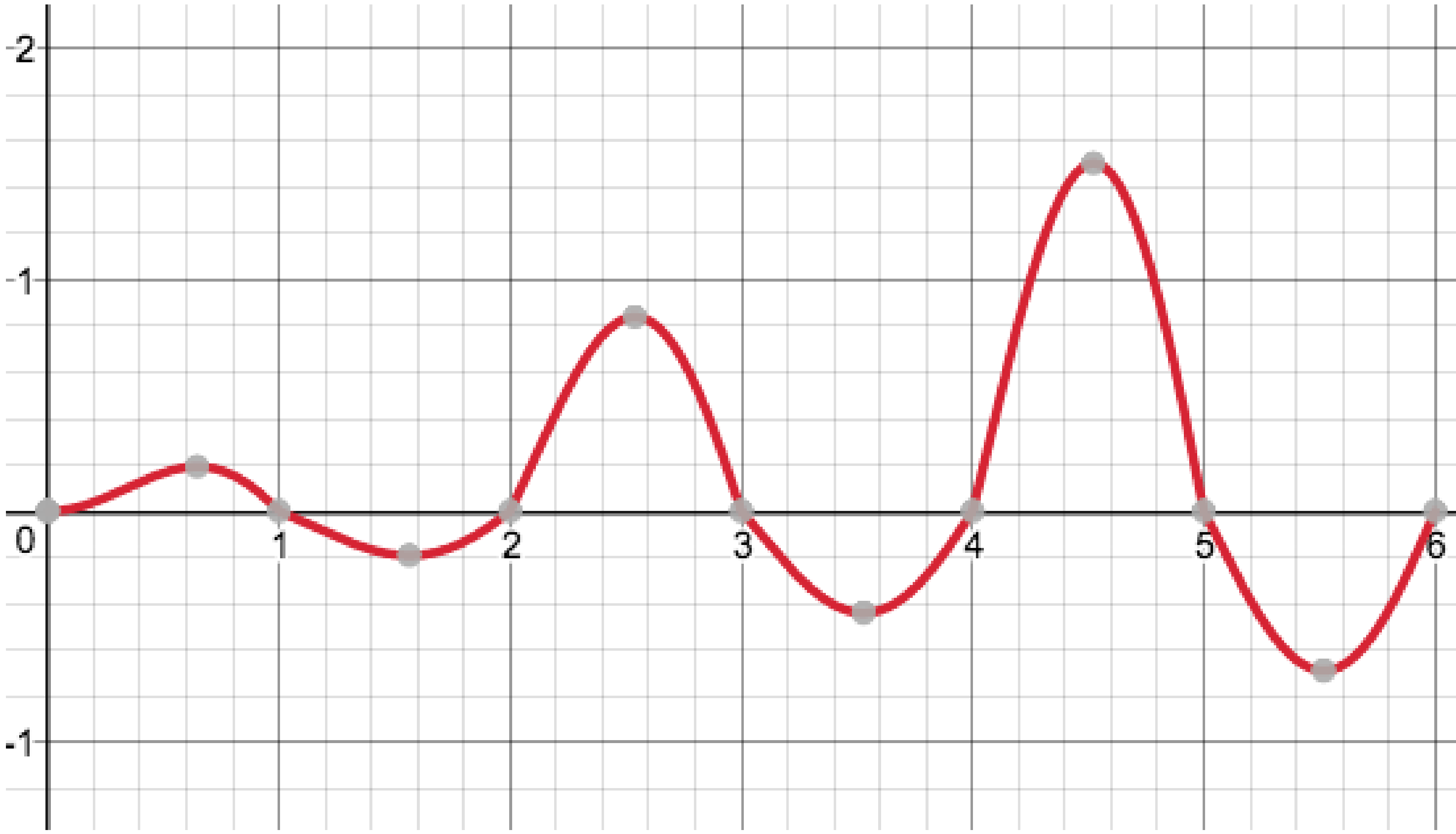
Note that $g(x) = -\int_7^x \tan t \mathrm{d}t$, so $g'(x) = -\tan x$. In particular, $g'(2) = -\tan 2$ and $g'(9) = -\tan 9$.

4. The antiderivative of $\frac{1}{x}$ is $\ln |x|$. Is the following reasoning correct?

$$\int_{-1}^{e} \frac{1}{x} \, \mathrm{d}x = \ln |x| |_{-1}^{e} = \ln |e| - \ln |-1| = 1.$$

No, $\frac{1}{x}$ is not continuous on the interval $[-1, e]$. The integral is not even defined.

5. Let f be the function with the following graph. For what value of $c \in [0, 6]$ is $\int_0^x f(t)\mathrm{d}t$ greatest?



The maximum possible value will be when $c = 5$. This is because the absolute area between 2 and 3 is more than the absolute area between 1 and 2 and the absolute area between 4 and 5 is more than the absolute area between 3 and 4.