Math 10B, Sprinh 2018 with Professor Stankova
Homework #35

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1 Problem Set

1. Show that the two given vector functions are solutions to the linear system of ODE's $\vec{y}' = A\vec{y}$ where A is the given matrix.

   (a) $\vec{y}(t) = \begin{pmatrix} 2e^{2t} \\ 5e^{2t} \end{pmatrix}$ and $\vec{z}(t) = \begin{pmatrix} e^{t} \\ 3e^{t} \end{pmatrix}$ where $A = \begin{pmatrix} 7 & -2 \\ 15 & -4 \end{pmatrix}$.

   Solution: $\vec{y}'(t) = \begin{pmatrix} 4e^{2t} \\ 10e^{2t} \end{pmatrix}$, and

$$\begin{aligned} A\vec{y} &= \begin{pmatrix} 7 \times 2e^{2t} - 2 \times 2e^{2t} \\ 15 \times 5e^{2t} - 4 \times 5e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 4e^{2t} \\ 10e^{2t} \end{pmatrix} \\ &= \vec{y}' \end{aligned}$$

   $\vec{z}'(t) = \begin{pmatrix} e^{t} \\ 3e^{t} \end{pmatrix}$, and

$$\begin{aligned} A\vec{z} &= \begin{pmatrix} 7e^{t} - 2 \times 3e^{t} \\ 15e^{t} - 4 \times 3e^{t} \end{pmatrix} \\ &= \begin{pmatrix} e^{t} \\ 3e^{t} \end{pmatrix} \\ &= \vec{z}' \end{aligned}$$

   (b) $\vec{y}(t) = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{z}(t) = e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where $A = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$.

   Solution: $\vec{y}'(t) = -3e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{-3t}\begin{pmatrix} -3 \\ -3 \end{pmatrix}$, and

$$\begin{aligned} A\vec{y} &= e^{-3t}\begin{pmatrix} -1-2 \\ -2-1 \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} -3 \\ -3 \end{pmatrix} \\ &= \vec{y}' \end{aligned}$$

$\vec{z}'(t) = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and

$$\begin{aligned} A\vec{z} &= e^t \begin{pmatrix} -1+2 \\ -2+1 \end{pmatrix} \\ &= e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \vec{z}' \end{aligned}$$

(c) (Challenge) $\vec{y}(t) = e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$ and $\vec{z}(t) = e^{-t} \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$ where $A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$.

Solution: You can derive $\vec{y}$ and $\vec{z}$ just like a usual product of functions:

$$\begin{aligned} \vec{y}'(t) &= -e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + e^{-t} \begin{pmatrix} -2\sin 2t \\ 2\cos 2t \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} -\cos 2t - 2\sin 2t \\ -\sin 2t + 2\cos 2t \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A\vec{y} &= e^{-t} \begin{pmatrix} -\cos 2t - 2\sin 2t \\ 2\cos 2t - \sin 2t \end{pmatrix} \\ &= \vec{y}' \end{aligned}$$

$$\begin{aligned} \vec{z}'(t) &= -e^{-t} \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + e^{-t} \begin{pmatrix} -2\cos 2t \\ -2\sin 2t \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} \sin 2t - 2\cos 2t \\ -\cos 2t - 2\sin 2t \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A\vec{z} &= e^{-t} \begin{pmatrix} \sin 2t - 2\cos 2t \\ -2\sin 2t - \cos 2t \end{pmatrix} \\ &= \vec{z}' \end{aligned}$$

2. In each case of Problem 1:

- Write down the eigenvalues and eigenvectors of A **without** reapplying the algorithm for finding them but just using what is already given. (Part (c) is still "Challenge" here.)

Solution: Based on the solution of the differential equation:

(a) $\lambda_1 = 2$ with corresponding eigenvector $\vec{v_1} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, and

$\lambda_2 = 1$ with corresponding eigenvector $\vec{v_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(b) $\lambda_1 = -3$ with corresponding eigenvector $\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and

$\lambda_2 = 1$ with corresponding eigenvector $\vec{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) $\lambda_1 = -1 - 2i$ with corresponding eigenvector $\vec{v_1} = \begin{pmatrix} 1 \\ i \end{pmatrix}$, and

$\lambda_2 = -1 + 2i$ with corresponding eigenvector $\vec{v_2} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

- Write down the **general** solution to the given system, **without** resolving the problem but just using what is already given.

Solution: We can take linear combinations of the solutions:

(a) $C_1 \begin{pmatrix} 2e^{2t} \\ 5e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^t \\ 3e^t \end{pmatrix}$

(b) $C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) $C_1 e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$

3. Find two different solutions of the system, and then write the general solution:

(a) $\vec{y}\,' = A\vec{y}$ where $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Solution: In each case, we want to find the eigenvalues and eigenvectors of the matrix A.

$$Det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda)^2 - 4$$
$$= (3-\lambda)(-1-\lambda)$$

Hence the eigenvalues of A are 3 and -1.

We must find non zero solution $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ to: $A\vec{v} = 3\vec{v}$, $A\vec{v} = \vec{v}$

We can rewrite the first one: $\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$.

In terms of equations: $-v_1 + v_2 = 0$.

So we can select $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For the second one, we can rewrite: $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$.

In terms of equations: $v_1 + v_2 = 0$.

So we can select $\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Hence the two particular solutions:

$\vec{y}(t) = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{z}(-t) = e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

The general solution is any linear combinations of these two solutions:

$C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, where C_1 and C_2 are any real numbers.

(b) $\vec{y}\,' = A\vec{y}$ where $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$.

Solution: Characteristic polynomial of the matrix:
$p(\lambda) = (-1-\lambda)^2 + 9$, The roots of which are $\lambda = -1 \pm 3i$

We look for a complex vector $z = (z_1, z_2)^T$ that is eigenvector corresponding to the eigenvalue $-1 - 3i$:

$$\begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -1 - 3i \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Which simplifies to

$$\begin{pmatrix} 3i & 3 \\ -3 & 3i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$z = (1, -i)^T$ is a solution, therefore

$$z(t) = e^{(-1-3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

is a complex values solution to our initial equation.
Taking the real and imaginary parts of z, we find:
$y^1(t) = e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$, $y^2(t) = e^{-t} \begin{pmatrix} -\sin 3t \\ -\cos 3t \end{pmatrix}$
And the general solution is:
$y(t) = C_1 e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -\sin 3t \\ -\cos 3t \end{pmatrix}$

(c) (Challenge) $\vec{y}' = A\vec{y}$ where $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$.

Solution: Characteristic polynomial:
$p(\lambda) = \lambda^2 - 4\lambda + 5$.
The roots are $\lambda = 2 \pm i$
Eigenvector for the eigenvalue $2 + i$:

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (2+i) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Which simplifies to:

$$\begin{pmatrix} -1-i & -1 \\ 2 & 1-i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$z = (1, -1-i)^T$ is a solution, therefore

$$z(t) = e^{(2+i)t} \begin{pmatrix} 1 \\ -1-i \end{pmatrix} = e^{2t}(\cos t + i\sin(t)) \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$$

is a complex values solution to our initial equation.
Taking the real and imaginary parts of z, we find:
$y^1(t) = e^{2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix}$, $y^2(t) = e^{2t} \begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix}$
And the general solution is:
$y(t) = C_1 e^{2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix}$

4. Solve the initial value problem:

(a) $\vec{y}\,'(t) = \begin{pmatrix} 1 & 2 \\ 6 & -3 \end{pmatrix} \vec{y}(t),\ \vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

Solution: Characteristic polynomial:
$p(\lambda) = \lambda^2 + 2\lambda - 15 = (\lambda + 5)(\lambda - 3).$
The roots are $\lambda_1 = -5$ and $\lambda_2 = 3$
Corresponding eigenvectos:
$v_1 = (1, -3)^T$, and $v_2 = (1, 1)^T$
Hence the two particular solutions:
$\vec{y_1}(t) = e^{-5t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\vec{y_2}(-t) = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
The general solution is any linear combinations of these two solutions:
$\vec{y}(t) = C_1 e^{-5t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where C_1 and C_2 are any real numbers.
The I.V. becomes:
$\begin{pmatrix} C_1 + C_2 \\ -3C_1 + C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$
So the final solution is:
$\vec{y}(t) = \frac{1}{2} e^{-5t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) (Challenge) $\vec{y}\,'(t) = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \vec{y}(t),\ \vec{y}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$

Solution: This is the same equation as 3.b. The general solution is:
$y(t) = C_1 e^{2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix}$
The I.V. becomes:
$\begin{pmatrix} C_1 \\ -C_1 - C_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$
So the final solution is:
$y(t) = 2e^{2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix} - 4e^{2t} \begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix}$

2 True / False

1. (Neither-nor or either?) $\vec{y}$ in $\vec{y}\,' = A\vec{y}$ where $A_{2\times 2}$ is the following object:
   - A function $y(t)$ whose derivative is $y'(t)$.
     False
   - A vector $\vec{y}$ whose derivative is $\vec{0}$
     False

- A vector function $\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ whose derivative is $\vec{y}' = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}$
  True

2. (Generalize correctly) If we end up with two different real eigenvalues λ_1 and λ_2 of $A_{2\times 2}$, then in analogy with what we have learned before, the two fundamental solutions to the system of DE's $\vec{y}' = A\vec{y}$ are $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$.
   False
   The educated guess will be:
   $\vec{y}(t) = e^{\lambda_1 t}\vec{v_1}$ and $\vec{z}(t) = e^{\lambda_2 t}\vec{v_2}$
   where $\vec{v_1}$ and $\vec{v_2}$ are eignevectors associated to the eigenvalues λ_1 and λ_2

3. (No double-counting!) The algorithm for solving an I.V.P. $\vec{y}' = A\vec{y}$, $y_1(0) = 8$,$y_2(0) = 64$, for some $A_{2\times 2}$ with two distinct real eigenvalues involves solving two quadratic equations (one for each $y_1(t)$ and $y_2(t)$), two linear systems to find the two eigenvectors, and one additional linear system to address the given I.V.
   False
   There is only one quadratic equation to solve to find the two eigenvalues.

4. (SOS) Euler's formula for $e^{(a+ib)t}$ rescues the solutions to the homogeneous linear DE systems whose auxiliary equation refuses to have any real roots.
   True
   We can then use the real and imaginary parts of the solution to be the real solutions to our initial equation.

5. (What is imagined may turn out to be real!) If $t \in \mathbb{R}$, the imaginary part of $5ti + 10$ is $5ti$.
   False
   The imaginary part is $5t$.

6. (With the speed of light) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the auxiliary (or characteristic) equation of A is $\lambda^2 - (a + d)\lambda + Det(A) = 0$, where $a + d$ is called the trace of A and is denoted by $tr(A)$. (Finally! Googling on my phone can help me with definitions and names!)
   True
   $Det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc$

7. (Short of a shortcut formula) We did not come up with a direct final formula for the solutions to the linear DE system $\vec{y}' = A\vec{y}$ because we still have find the eigenvectors of A by solving systems of linear equations.
   True

8. (Only for OH visitors who know what we are talking about!) The shortcut formula for solving a quadratic equation $ax^2 + bx + c = 0$ with integer coefficients a, b, c with which Zvezda has been showing off in class works efficiently roughly 50% of the time because b has an equal chance of being even (good!) or odd (bad!).

True

If b is even, then $b = 2k$ where $k \in \mathbb{N}$, then the roots are given by $x_{1,2} = \dfrac{-k \pm \sqrt{k^2 - ac}}{a}$ and if $a = 1$ this further simplifies to $x_{1,2} = -k \pm \sqrt{k^2 - c}$, which will win fair and square about 50% of the time in terms of speed over using the regular quadratic formula.