Math 10A, Fall 2018: Worksheet 26

1. Find a direct formula for $a_{n+1} = a_n + 2a_{n-1}$ for $n \geq 1$, $a_0 = 0, a_1 = 3$, by encoding the recurrence sequence with linear algebra. Follow the steps:

   (a) Let $\vec{v}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ for all $n \geq 0$. Write down $\vec{v}_0, \vec{v}_1, \vec{v}_2$, and $\vec{v}_3$.

   Solution. $\vec{v}_0 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 9 \\ 15 \end{pmatrix}$

   (b) Find a 2×2 matrix A such that $A\vec{v}_{n-1} = \vec{v}_n$ for all $n \geq 1$.
   Check your answer by verifying that $A\vec{v}_0 = \vec{v}_1$, $A\vec{v}_1 = \vec{v}_2$, and $A\vec{v}_2 = \vec{v}_3$.

   Solution. $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

   (c) Find the eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\vec{u}_1$ and $\vec{u}_2$ of A.

   Solution. Setting $\det(A - \lambda I) = 0$ gives us $\lambda^2 - \lambda - 2 = 0$, so $\lambda = 2$ or $\lambda = -1$. These are our two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. $A - 2I = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ solves the matrix equation $(A - 2I)\vec{x} = \vec{0}$, and $A + I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ solves the matrix equation $(A + I)\vec{x} = \vec{0}$. Therefore we can take our corresponding eigenvectors to be $\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

   (d) Write $\vec{v}_0$ as a linear combination of the eigenvectors $\vec{u}_1$ and $\vec{u}_2$; i.e., find constants c_1 and c_2 such that $\vec{v}_0 = c_1\vec{u}_1 + c_2\vec{u}_2$. (*Hint*: This boils down to a 2×2 system.)

   Solution. $\vec{v}_0 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{u}_1 - \vec{u}_2$ (so $c_1 = 1, c_2 = -1$).

   (e) Calculate $A\vec{u}_1$, $A^2\vec{u}_1$, $A^3\vec{u}_1$, and $A^n\vec{u}_1$ for any $n \geq 0$. Repeat for $\vec{u}_2$ instead of $\vec{u}_1$. (*Hint*: There are almost no calculations here. Use the property of eigenvectors.)

   Solution. $A\vec{u}_1 = 2\vec{u}_1$, $A^2\vec{u}_1 = 4\vec{u}_1$, $A^3\vec{u}_1 = 8\vec{u}_1$, $A^n\vec{u}_1 = 2^n\vec{u}_1$;
   $A\vec{u}_2 = -\vec{u}_2$, $A^2\vec{u}_2 = 1\vec{u}_2$, $A^3\vec{u}_2 = -\vec{u}_2$, $A^n\vec{u}_2 = (-1)^n\vec{u}_2$.

   (f) Find $A^n\vec{v}_0$ for any $n \geq 0$. (*Hint*: Use your lin. combo in (d) and $A^n(c_1\vec{w}_1 + c_2\vec{w}_2) = c_1A^n\vec{w}_1 + c_2A^n\vec{w}_2$; i.e., A^n splits linear combinations of vectors.)

   Solution. $A^n\vec{v}_0 = A^n\vec{u}_1 - A^n\vec{u}_2 = 2^n\vec{u}_1 - (-1)^n\vec{u}_2$

   (g) Noting that $\vec{v}_n = A^n\vec{v}_0$, extract from (f) the formula for the top component a_n.

Solution. The top components of $A^n\vec{v}_0$, $\vec{u}_1$, and $\vec{u}_2$ are a_n, 1, and 1 respectively, so $a_n = 2^n - (-1)^n$.

2. Solve the IVP $y'' - y' - 2y = 0$, $y(0) = 0, y'(0) = 3$ by encoding the DE with linear algebra. Follow the steps:

   (a) Set $y_1(t) = y(t)$ and $y_2(t) = y'(t)$. What are $y_1'(t)$ and $y_2'(t)$, writen only in terms of $y_1(t)$ and $y_2(t)$?

   Solution. $y_1'(t) = y'(t) = y_2(t)$;
   $y_2'(t) = y''(t) = 2y(t) + y'(t) = 2y_1(t) + y_2(t)$.

   (b) Write a homogeneous system of two linear 1st order DE's with unknowns $y_1(t)$ and $y_2(t)$ that corresponds to solving the given 2nd order DE. (*Hint*: The 2×2 matrix is the same as in Problem 1.) Encode also the initial condition via $y_1(t)$ and $y_2(t)$.

   Solution. $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ (this has the form $\vec{y}' = A\vec{y}$). The initial conditions are encoded as $\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

   (c) Solve the DE system, using the algorithm from class.

   Solution. We use the eigenvalues and eigenvectors found in Problem 1, which yield the general solution $\vec{y} = C_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

   (d) Which function from your result in (c) is the desired function $y(t)$?

   Solution. Plugging in the initial conditions gives $\begin{pmatrix} 0 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and solving for the coefficients gives us $C_1 = 1$ and $C_2 = -1$. So $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and the top row of the vector equation gives us $y(t) = e^{2t} - e^{-t}$.

   (e) Now apply the shortcut algorithm for finding solutions to 2nd order DE's to solve the given IVP. Compare with your result in (d). Which steps (a)-(d) did you skip, and which steps did you modify in your shortcut solution?

   Solution. The characteristic equation will be $r^2 - r - 2 = 0$, which has roots 2 and -1, so the general solution is $y = C_1 e^{2t} + C_2 e^{-t}$. Plugging in the initial conditions yields $C_1 = 1$ and $C_2 = -1$, so $y = e^{2t} - e^{-t}$ as in part (d). In the shortcut algorithm, steps (a) and (b) are skipped, while steps (c) and (d) are slightly modified.

3. (Challenge) Let A be the (same) 2×2 matrix that appears in Problems 1 and 2. Find A^{2018} and show all calculations to justify your answer.
(*Hint*: $A = EDE^{-1}$ for some diagonal 2×2 matrix D related to the eigenvalues of A and some 2×2 matrix E made somehow of the eigenvectors of A.)

Solution. $A = EDE^{-1}$ where $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. $E^{-1} = \frac{1}{-3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$.
Then $A^{2018} = (EDE^{-1})^{2018} = EDE^{-1}\, EDE^{-1}\, EDE^{-1}\, EDE^{-1} \cdots EDE^{-1} = ED^{2018}E^{-1}$.
So

$$\begin{aligned}
A^{2018} &= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2^{2018} & 0 \\ 0 & (-1)^{2018} \end{pmatrix} \frac{-1}{3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \\
&= -\frac{1}{3} \begin{pmatrix} 2^{2018} & 1 \\ 2^{2019} & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \\
&= -\frac{1}{3} \begin{pmatrix} -2^{2018} - 2 & -2^{2018} + 1 \\ -2^{2019} + 2 & -2^{2019} - 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{2^{2018}+2}{3} & \frac{2^{2018}-1}{3} \\ \frac{2^{2019}-2}{3} & \frac{2^{2019}+1}{3} \end{pmatrix}.
\end{aligned}$$