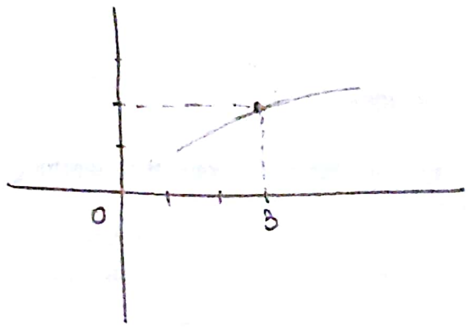
1. Graphing Functions

1. $f(3) = 2 \qquad f'(3) = 1/2$

$f'(x) > 0 \qquad f''(x) < 0 \qquad \forall x$

(Always increasing) (Always concave downward)

(a)



(b) One solution as it would only intersect the x-axis once. ($f'(x) > 0 \ \forall x$)

(c) $f'(2) = 1/3$ Not possible. as $f''(x) < 0$ thus the slope can only decrease as we increase x. (Note: $f'(3) = \frac{1}{2}$)

2. $f(x) = x^{2/3}(6-x)^{1/3}$

Domain : $\mathbb{R}$

Zeros of f : 0 and 6

$f'(x) = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}}$ $\qquad x \neq \{0, 6\} \qquad x = 4 \qquad$ CP $\rightarrow$ 0, 4 & 6

$f''(x) = \dfrac{-8}{x^{4/3}(6-x)^{5/3}}$ $\qquad x \neq 0 \quad x \neq 6 \qquad$ CP $\rightarrow$ 0, 6

Increasing interval. $(0, 4)$ $\qquad$ Decreasing interval $(-\infty, 0), (4, \infty)$

$f'(x) > 0$

$\dfrac{4-x}{x^{1/3}(6-x)^{2/3}} > 0$

Interval	$4-x$	$x^{1/3}$	$(6-x)^{2/3}$	$f'(x)$
$(-\infty, 0)$	+	−	+	−
$(0, 4)$	+	+	+	+
$(4, 6)$	−	+	+	−
$(6, \infty)$	−	+	+	−

x	$(-\infty,0)$	0	$(0,4)$	4	$(4,6)$	6	$(6,\infty)$
f	↘	∨	↗	⌢	↘		↘
f'	$-$	Local min	$+$	local max	$-$		$-$
f''	$-$		$-$		$-$	IP	$+$

First Derivative test

$- \to +$ at 0 local min

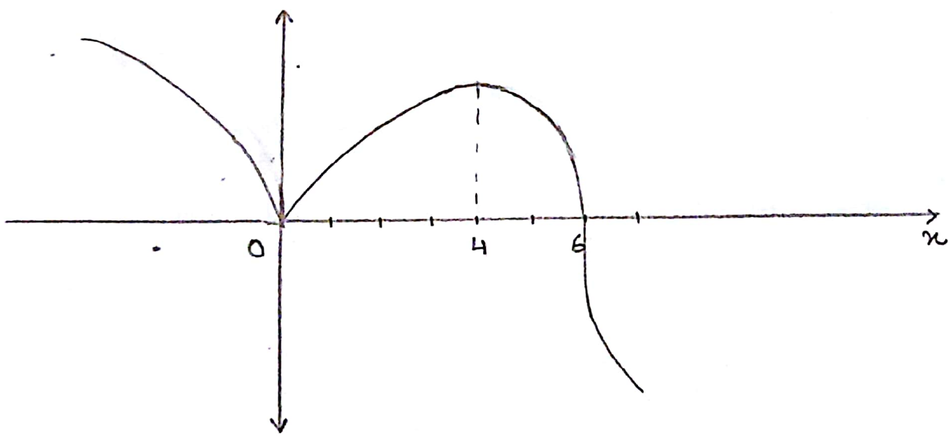
$+ \to -$ at 4 local max

$- \to -$ at 6 no minimum / maximum

(second derivative could be used at 4 but not at 0 or 6 as f'' DNE at 0, 6)

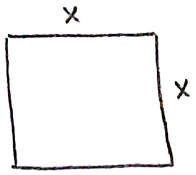
concavity changes from $- \to +$ at 6 $\Rightarrow$ 6 is the point of inflection.

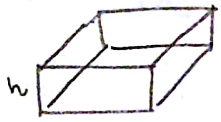
<u>Vertical Tangents</u> at 0 and 6 as $|f'(x)| \to \infty$ as $x \to 0$ and $x \to 6$



II. Optimization

1200 cm^2 Material.





Max Volume = Max $(x \cdot x \cdot h)$

Area of the box = $x \cdot x + 4(h \cdot x)$

$$1200 = x^2 + 4hx$$

$$h = \frac{1200 - x^2}{4x}$$

volume = $x \cdot x \cdot h = x^2 \cdot \left(\frac{1200 - x^2}{4x}\right)$

$$V(x) = 300x - \frac{x^3}{4}$$

$$V'(x) = 300 - \frac{3x^2}{4}$$

$$300 - \frac{3x^2}{4} = 0$$

$$\frac{3x^2}{4} = 300$$

$$x^2 = 400$$

$x = 20$ cm $\qquad h = \frac{1200 - 400}{4(20)} = \frac{80\cancel{0}}{8\cancel{0}} = 10$ cm

Largest possible volume $\rightarrow x \cdot x \cdot h = (20)^2 \cdot 10$ cm^3

$= 4000$ cm^3

III. L'Hospitals Rule

(a) $$\lim_{t\to 0} \frac{5^t - 3^t}{t} \underset{LH}{\overset{\frac{0}{0}}{=}} \lim_{t\to 0} \frac{(5^t - 3^t)'}{t'} = \lim_{t\to 0} 5^t \ln 5 - 3^t \ln 3 = 5^0 \ln 5 - 3^0 \ln 3 = \ln \frac{5}{3}$$

(b) $$\lim_{x\to\infty} x^3 e^{-x^2} \underset{LH}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{(x^3)'}{(e^{x^2})'} = \lim_{x\to\infty} \frac{3x^{\cancel{2}}}{e^{x^2} \cdot 2\cancel{x}}$$

since $x \to \infty$ and $e^{x^2} \to \infty$ as $x \to \infty$, the limit on the right hand side is also indeterminate. A second application of LH gives:

$$\lim_{x\to\infty} \frac{3x}{e^{x^2} \cdot 2} \underset{LH}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{(3x)'}{(e^{x^2} \cdot 2)'} = \lim_{x\to\infty} \frac{3}{2e^{x^2} \cdot 2x} = \lim_{x\to\infty} \frac{3}{4} \cdot \frac{\left(\frac{1}{e^{x^2}}\right)}{x}$$

$$= \frac{3}{4} \frac{\lim_{x\to\infty}\left(\frac{1}{e^{x^2}}\right)}{\lim_{x\to\infty} x}$$

$$= 0$$

(c) $$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x}}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = \frac{1}{\sqrt{1+\frac{1}{\infty}}} = 1.$$ (No need for LH)

(d) $$\lim_{x\to\infty} (x - \ln x)$$

$$= \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right) = \lim_{x\to\infty} x \cdot \lim_{x\to\infty}\left(1 - \overbrace{\frac{\ln x}{x}}^{\text{LH Rule}}\right)$$

$$= \lim_{x\to\infty} x \cdot \lim_{x\to\infty}\left(1 - \frac{(\ln x)'}{x'}\right)$$

$$= \lim_{x\to\infty} x \cdot \lim_{x\to\infty}\left(1 - \frac{1}{x}\right) = \infty \cdot 1 = \infty$$

2. $f(x) = xe^{-x} = \frac{x}{e^x}$ zeros of f $x=0$

- Vertical Asymptote is found where denominator $= 0$. Not possible here
- Horizontal Asymptote is found by calculating lim at $+\infty$ and $-\infty$.

$\lim_{x\to+\infty} \frac{x}{e^x} \overset{\frac{\infty}{\infty}}{\underset{LH}{=}} \lim_{x\to+\infty} \frac{(x)'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$ $\boxed{\text{HA at } y=0}$

$\lim_{x\to-\infty} xe^{-x} = \lim_{x\to-\infty} x \cdot \lim_{x\to-\infty} (e^{-x}) = (-\infty)\cdot\infty = -\infty$

$f' = (xe^{-x})' = e^{-x} + x \cdot e^{-x} \cdot (-1)$ CP $x=1$

$= e^{-x}(1-x)$

$f'' = (e^{-x}(1-x))' = -e^{-x}(1-x) + e^{-x}(-1)$ CP $x=2$.

$= e^{-x}(x-1-1)$

$= e^{-x}(x-2)$

Increasing / Decreasing Intervals.

$f'(x) > 0$

$(1-x)e^{-x} > 0$

$(1-x) > 0$ as e^{-x} always +ve

$1 > x$ $(-\infty, 1)$

$f'(x) < 0$

$1-x < 0$

$1 < x$

$(1, \infty)$

Concavity.

$f''(x) > 0$

$(x-2)e^{-x} > 0$

$x > 2$ concave upward

$f''(x) < 0$

$x-2 < 0$

$x < 2$

concave downward.

x	$(-\infty, 1)$	1	$(1, 2)$	2	$(2, \infty)$
f	↗ /	⌒	↘ \	↘	↘ \
f'	$+$	local max	$-$		$-$
f''	$-$		$-$	IP	$+$

HA at $y = 0$