Worksheet 5

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1. Consider the function $f(x) = x^2 + 5x + 3$.

   a. Compute the first derivative $f'(x)$.

   b. Compute the second derivative $f''(x)$.

   c. Compute the third derivative $f'''(x)$.

   a. To compute the first derivative we use the differentiation law which allows us to write the derivative of the sum as the sum of the derivative, so we simply compute the derivative of the terms one by one. This is $f'(x) = 2x + 5$. We have used the power rule, the constant rule, and the fact that the derivative of a constant is 0.

   b. Recall that in order to compute a second derivative of a function f, we just compute the ordinary derivative of the first derivative. This is
   $$\frac{d}{dx}(f'(x)) = \frac{d}{dx}(2x+5) = 2$$
   where we have used the sum DL, and the constant DL.

   c. Now to find the third derivative of the function, we take another derivative of the second derivative, but this is just 0 since the second derivative is constant.

2. Consider the function $f(x) = \sqrt{x}$.

   a. Compute the first derivative $f'(x)$.

   b. Compute the second derivative $f''(x)$.

   c. Compute the third derivative $f'''(x)$.

   a. If we write $f(x) = x^{1/2}$, we can use the power rule to calculate this derivative:
   $$f'(x) = \frac{1}{2\sqrt{x}}$$

b. Again, we just take the derivative of f' to get the second derivative. To do this, we can invoke the constant DL, and the power DL to get

$$\frac{d}{dx}\left(\frac{1}{2}x^{-1/2}\right) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4\sqrt{x^3}}$$

c. Lastly, we compute the derivative of the previous expression again:

$$\frac{d}{dx}\left(-\frac{1}{4}x^{-3/2}\right) = \frac{3}{8}x^{-5/2} = \frac{3}{8\sqrt{x^5}}$$

3. Use the chain rule to complete the following derivatives. Before differentiating anything, make sure to write explicitly the functions $g(x)$ and $h(x)$ are such that $f(x) = g \circ h(x)$.

   a. $f(x) = e^{x^2}$
   b. $f(x) = \cos(\sin(x))$
   c. $f(x) = \left(x^2 + x\right)^{1/3}$
   d. $f(x) = \sqrt{1 - x}$

   a. If we set $g(x) = e^x$, and $h(x) = x^2$, then $f = g \circ h$. Now the chain rule says that $f'(x) = g'(h(x))\, h'(x)$ so we first calculate

   $$g'(h(x)) = e^{x^2} \qquad\qquad h'(x) = 2x$$

   using the fact that $(e^x)' = e^x$ and the power rule. Now from the chain rule we can write:

   $$f'(x) = 2xe^{x^2}$$

   b. If we set $g(x) = \cos(x)$ and $h(x) = \sin(x)$ we have that $f = g \circ h$ as desired. Now we calculate

   $$g'(h(x)) = -\sin(\sin(x)) \qquad\qquad h'(x) = \cos(x)$$

   so from the chain rule our final answer is

   $$f'(x) = -\cos(x)\sin(\sin(x))$$

   c. If we set $g(x) = x^{1/3}$, and $h(x) = x^2 + x$, then we can first calculate

   $$g'(x) = \frac{1}{3x^{2/3}}$$

   which allows us to write

   $$g'(h(x)) = \frac{1}{3\left(x^2 + x\right)^{2/3}} \qquad\qquad h'(x) = 2x + 1$$

   Now by the chain rule we can write:

   $$f'(x) = \frac{2x + 1}{3\left(x^2 + x\right)^{2/3}}$$

d. If we set $g(x) = \sqrt{x}$, and $h(x) = 1 - x$, we can calculate

$$g'(h(x)) = \frac{1}{2\sqrt{1-x}} \qquad\qquad h'(x) = -1$$

so by the chain rule we can write

$$f'(x) = -\frac{1}{2\sqrt{1-x}}$$

4. Calculate the derivative of the function $f(x) = \ln\left(x\sqrt{x^2-1}\right)$.

First, we can write $f(x) = g \circ h(x)$ for $g(x) = \ln(x)$ and $h(x) = x\sqrt{x^2-1}$. Now we can use the product rule to calculate:

$$h'(x) = \sqrt{x^2-1} + x\left(\sqrt{x^2-1}\right)'$$

Now we need to use the chain rule to calculate:

$$\left(\sqrt{x^2-1}\right)' = \frac{2x}{2\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}$$

Now this allows us to write:

$$h'(x) = \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} = \frac{x^2-1+x^2}{\sqrt{x^2-1}} = \frac{2x^2-1}{\sqrt{x^2-1}}$$

Now we also calculate:

$$g'(h(x)) = \frac{1}{x\sqrt{x^2-1}}$$

since $(\ln x)' = 1/x$. Finally the chain rule allows us to write:

$$f'(x) = g'(h(x))\,h'(x) = \frac{2x^2-1}{x\left(x^2-1\right)}$$

5. Find the derivative of $f(x) = x^x$. [Hint: Use logarithmic differentiation!]

Recall the following is the list of steps for logarithmic differentiation:

- Take ln of both sides of the equation $y = f(x)$.
- Simplify the new equation using the laws of logarithms.
- Differentiate implicitly with respect to x.
- Solve the resulting equation for y'.

In this case, our equation is $y = x^x$. So this becomes

$$\ln y = \ln\left(x^x\right) = x \ln x$$

Now differentiating implicitly, we get

$$\begin{aligned}
\frac{d}{dx}\left(\ln y\right) &= \frac{d}{dx}\left(x \ln x\right) \\
\frac{y'}{y} &= \ln x + 1 \\
y' &= y\left(\ln x + 1\right)
\end{aligned}$$

but $y = x^x$, so

$$y' = x^x\left(\ln x + 1\right)$$