Worksheet 13

1 Discrete Probability Distributions

For each of the following problems, do the following things.

(i) Identify the name of the distribution of X.

(ii) Identify the range R_X of X.

(iii) Write a formula for the probability mass function $f(k)$.

(iv) Find $P(X = 4)$.

1. You decide to play the lottery over and over until you win some prize. Each time you play the lottery, the probability that you win a prize is 0.01. Let X be the number of times that you play the lottery and do not win any prize. Also find $P(X \geq 2)$.

Solution:
(i) geometric (ii) $R_X = \{0, 1, 2, ...\}$ (iii) $f(k) = 0.99^k \cdot 0.01$
(iv) $P(X = 4) = f(4) = 0.99^4 \cdot 0.01 \approx 0.009606$

$P(X \geq 2) = 1 - P(X < 2) = 1 - (P(X = 0) + P(X = 1)) = 1 - (f(0) + f(1)) = 1 - (0.99^0 \cdot 0.01 + 0.99^1 \cdot 0.01) = 0.9801$

2. You roll two fair 6-sided die 100 times. Let X be the number of times you roll a sum of 7.

Solution:
(i) Binomial (ii) $R_X = \{0, 1, 2, ..., 100\}$
(iii) The probability of rolling a sum of 7 on any given roll is $\frac{6}{36} = \frac{1}{6}$ since there are 6 outcomes that result in a sum of 7 (namely (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)) and there are $6^2 = 36$ total outcomes. Thus $f(k) = \binom{100}{k}(1/6)^k(5/6)^{100-k}$.
(iv) $P(X = 4) = f(4) = \binom{100}{4}(1/6)^4(5/6)^{96} \approx 7.5756 \cdot 10^{-5}$

3. Assume that the probability a child is a boy is 0.55 and that the sexes of children born into a family are independent. Let X be the number of boys in a family of 8 children.

Solution:
(i) Binomial (ii) $R_X = \{0, 1, 2, ..., 8\}$ (iii) $f(k) = \binom{8}{k}0.55^k 0.45^{8-k}$
(iv) $P(X = 4) = f(4) = \binom{8}{4}0.55^4 0.45^4 \approx 0.2627$

4. The undergraduate enrollment of a university is 5000 students, 200 of whom are math majors. You interview 50 undergraduates. Let X be the number of math majors you interview.

Solution:
(i) hypergeometric (ii) $R_X = \{0, 1, ..., 50\}$ (iii) $f(k) = \binom{200}{k}\binom{4800}{50-k}/\binom{5000}{50}$
(iv) $P(X = 4) = f(4) = \binom{200}{4}\binom{4800}{46}/\binom{5000}{50} \approx 0.09010$

5. Assume that the number of people who arrive in the emergency room at a hospital each night is a Poisson random variable with parameter $\lambda = 20$. Let X be the number of people who arrive in the emergency room tonight.

   Solution:
   (i) Poisson (ii) $R_X = \{0, 1, 2, ...\}$ (iii) $f(k) = \frac{20^k \cdot e^{-20}}{k!}$
   (iv) $P(X = 4) = f(4) = \frac{20^4 \cdot e^{-20}}{4!} \approx 1.3741 \cdot 10^{-5}$

2 Estimation via Poisson/Hypergeometric Distributions

1. (based on a true YouTube video; youtube.com/watch?v=MTmnVBJ9gCI) Matt goes to a statistics conference and wants to estimate the number of attendees. He randomly samples 64 people at the poster session and puts stickers on their name tags. Later on, he randomly samples 67 people at the closing plenary session, of whom 21 are stickered. What is Matt's estimate for the total number of attendees?

   Solution: $\lfloor 64 \cdot 67/21 \rfloor = 204$

2. I perform $n = 900$ independent Bernoulli trials with individual success probability $p = 0.01$.

   (a) Using the binomial P.M.F., what is the probability of exactly 5 successes?

   Solution:
   $\binom{900}{5} 0.01^5 \cdot 0.99^{895} \approx 0.06035$

   (b) Using the Poisson P.M.F., what is the probability of exactly 5 successes? (What is λ?)

   Solution:
   $\lambda = np = 900 \cdot 0.01 = 9$
   the probability of exactly 5 successes is $\frac{9^5 \cdot e^{-9}}{5!} \approx 0.06073$

   (c) In this particular example, was the Poisson distribution a good approximation to the binomial distribution?

   Solution:
   Yes! Our estimate of the probability of exactly 5 successes, which we computed using the Poisson P.M.F., was 0.06073. This is quite close to the true probability of exactly 5 successes, which we found to be 0.06035 using the binomial distribution.