Student Name:

MATH 10A with Prof. Stankova – Fall 2018
Discussion 22

Geometry of $\mathbb{R}^3$

1. What shapes do the equations $x^2 + y^2 + z^2 = 9$ and $x + y + z = 0$ describe? How would you describe the set of points satisfying both equations?

   $x^2 + y^2 + x^2 = 9$ is a sphere of radius 3, $x + y + z = 0$ is a plane through the origin. They intersect at an equator of the spehre (a circle).

2. If U and V are planes in $\mathbb{R}^3$, what forms can $U \cap V$ take? Construct explicit examples - for each possible form of $U \cap V$, find two planes whose intersection is if that form. What does this tell you about solutions to systems of 2 linear equations in 3 variables?

   $U \cap V$ can be a plane, a line, or empty (in particular $U \cap V$ cannot be a point). The system of equations $x + y + z = 1, x + y + z = 2$ has no solutions (the intersections of the planes is empty, i.e. the planes are parallel). The system of equations $x + y + z = 1, x = 0$ has solutions along a line in $\mathbb{R}^3$ (the line is the set of points $(0, y, 1-y)$). The system of equations $x + y + z = 1, 2x + 2y + 2z = 2$ has solutions on the plane $x + y + z = 1$.

   In general, a system of two equatiosn with 3 variables cannot have a single unique solution - it must either have infinitely many solutions or 0 solutions.

3. Describe the part of the plane $x + 2y + z = 4$ contained within the first octant.

   This is a triangle with vertices $(4, 0, 0), (0, 2, 0), (0, 0, 4)$.

4. (Challenge) Write down a system of inequalities describing a cone with base of radius 2 in the xy-plane centered at the origin and height 2 with top point $(0, 0, 2)$.

   The system is $z \geq 0$, $z \leq 2 - \sqrt{x^2 + y^2}$ (on the surface of the cone, z depends linearly on $\sqrt{x^2 + y^2}$).

Vectors and the Dot Product

1. What are the lengths of the vectors $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $u = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Compute $v \cdot v$ and $u \cdot u$ - what do you notice?

   $|v| = \sqrt{14}$, $|u| = \sqrt{17}$. $v \cdot v = 14, u \cdot u = 17$. In general, $u \cdot u = |u|^2$ for any vector u.

2. Find a vector that has the same direction as $\begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix}$ of length 2.

   Let u be the given vector. Then $|u| = 9$, so $|\frac{2}{9}u| = |\frac{2}{9}||u| = 2$, so $\frac{2}{9}u$ is the desired vector.

3. Consider the triangle in $\mathbb{R}^3$ with vertices $0, v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, u = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. What are the side lengths of the edges of this triangle? (Challenge) What is the area of this triangle?

The sides of the triangle are given by $u, v, u - v$. $|u| = \sqrt{14}, |v| = \sqrt{14}, |u - v| = 2\sqrt{2}$. To find the area, remember that the area of a triangle is $\frac{1}{2}bh$. Let u be the base, so $b = |u| = \sqrt{14}$.

To find the height, we need to find the altitude from the point v onto the side given by u. To do this, note the projection of v onto u is the vector in the deriction of u with magnitude $|v|\cos(\theta)$ where θ is the angle between u, v. This vector can be written as $\frac{u}{|u|}|v|\cos(\theta) = \frac{(u \cdot v)u}{|u|^2} = \frac{10}{14}u$. Then the altitude from the point v onto the side u is $v - \frac{10}{14}u$ and thus the height is the length of this altitude which is $\sqrt{(v - \frac{10}{14}u) \cdot (v - \frac{10}{14}u)} = \sqrt{14 - 100/14} = 4\sqrt{3/7}$.

Thus the area of the triangle is $\frac{1}{2}bh = 2\sqrt{6}$.

4. Determine whenther angles between the following pairs of vectors are acute, right, or obtuse: $v_1 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

   $v_1 \cdot v_2 = 0$, so v_1 and v_2 are orthogonal. $v_1 \cdot v_3 = -4$, so v_1, v_3 form an obtuse angle. $v_1 \cdot v_4 = 7$, so v_1, v_4 form an acute angle. $v_2 \cdot v_3 = -4$, so v_2, v_3 form an obtuse angle. $v_2 \cdot v_4 = 5$, so v_2, v_4 form an acute angle. $v_3 \cdot v_4 = -4$, so v_3, v_4 form an obtuse angle.

5. Find the equation of the plane that passes through the origin and is perpendicular to the vector $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$.

   The equation of the plane is $x - 2y + 5z = 0$.

6. (Challenge) Use the CBS inequality to prove the triangle inequality for vectors: $|u + v| \leq |u| + |v|$ for vectors u, v.

   Notice $|u+v|^2 = (u+v)\cdot(u+v) = v\cdot v + u\cdot u + 2(u\cdot v) = |v|^2 + |u|^2 + 2(u\cdot v) \leq |u|^2 + |v|^2 + 2|u||v| = (|u| + |v|)^2$ (we used CBS to state that $u \cdot v \leq |u||v|$). Thus $|u + v|^2 \leq (|u| + |v|)^2$, so $|u + v| \leq |u| + |v|$.