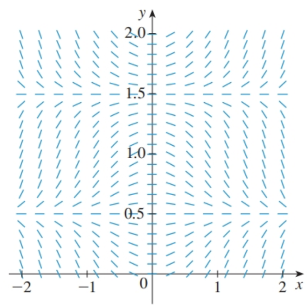
1 Direction Field

1. A direction field for the differential equation $y' = x\cos(\pi y)$ is shown



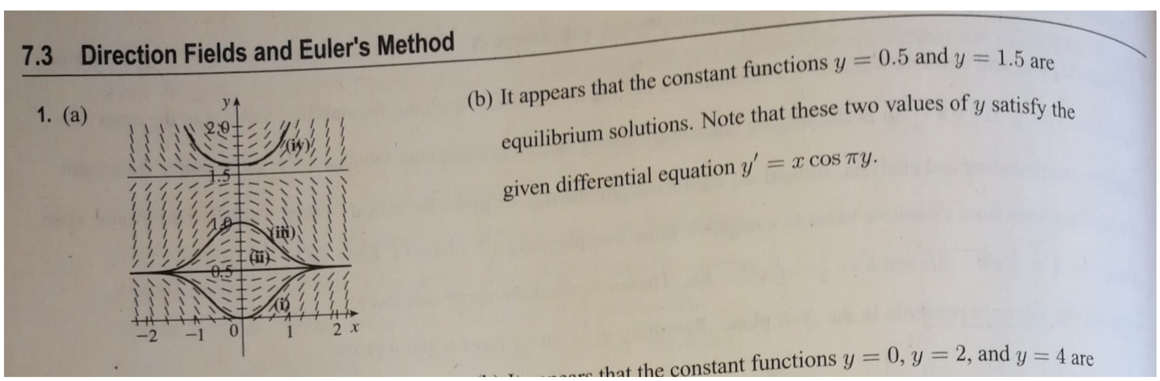
   (a) Sketch the graphs of the solutions that satisfy the given initial conditions.

      i. $y(0) = 0$

      ii. $y(0) = 0.5$

      iii. $y(0) = 1$

   (b) Find all the equilibrium solutions on the graph



7.3 Direction Fields and Euler's Method

1. (a)

(b) It appears that the constant functions $y = 0.5$ and $y = 1.5$ are equilibrium solutions. Note that these two values of y satisfy the given differential equation $y' = x \cos \pi y$.

that the constant functions $y = 0$, $y = 2$, and $y = 4$ are

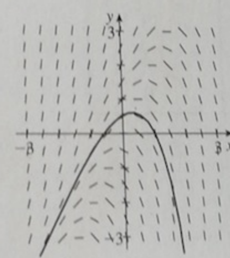
2. Sketch the direction field of the differential equation.Then use it to sketch a solution curve that passes through the given point.

   (a) $y' = y - 2x, \quad (1, 0)$

11.

x	y	$y' = y - 2x$
−2	−2	2
−2	2	6
2	2	−2
2	−2	−6

Note that $y' = 0$ for any point on the line $y = 2x$. The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through $(1, 0)$.

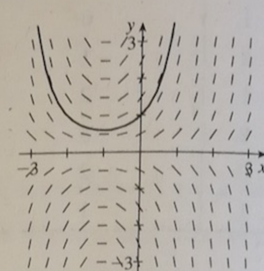


(b) $y' = y + xy, \qquad (0, 1)$

13.

x	y	$y' = y + xy$
0	± 2	± 2
1	± 2	± 4
−3	± 2	∓ 4

Note that $y' = y(x + 1) = 0$ for any point on $y = 0$ or on $x = -1$. The slopes are positive when the factors y and $x + 1$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(0, 1)$.



2 Euler method

3. (a) Use Euler's method with each of the following step sizes to estimate the value of $y(0.4)$, where y is the solution of the initial-value problem $y' = y, y(0) = 1$.

   i. $h = 0.4$

   ii. $h = 0.2$

   iii. $h = 0.1$

19. (a) $y' = F(x, y) = y$ and $y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$.

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \Rightarrow y_1 = 1 + 0.4 \cdot 1 = 1.4$. $x_1 = x_0 + h = 0 + 0.4 = 0.4$,

so $y_1 = y(0.4) = 1.4$.

(ii) $h = 0.2 \Rightarrow x_1 = 0.2$ and $x_2 = 0.4$, so we need to find y_2.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2$,

$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44$.

(iii) $h = 0.1 \Rightarrow x_4 = 0.4$, so we need to find y_4. $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1$,

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21$,

$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331$,

$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641$.

We see that the estimates are underestimates since

graph of $y = e^x$.

(b) We know that the exact solution of the initial-value problem in part (a) is $y = e^x$. The error in Euler's method is the difference between the exact value and the approximate value. Find the errors made in part (a) in using Euler's method to estimate the true value of $y(0.4)$, namely, $e^{0.4}$. What happens to the error each time the step size is halved? ($e^{0.4} \approx 1.49182$)

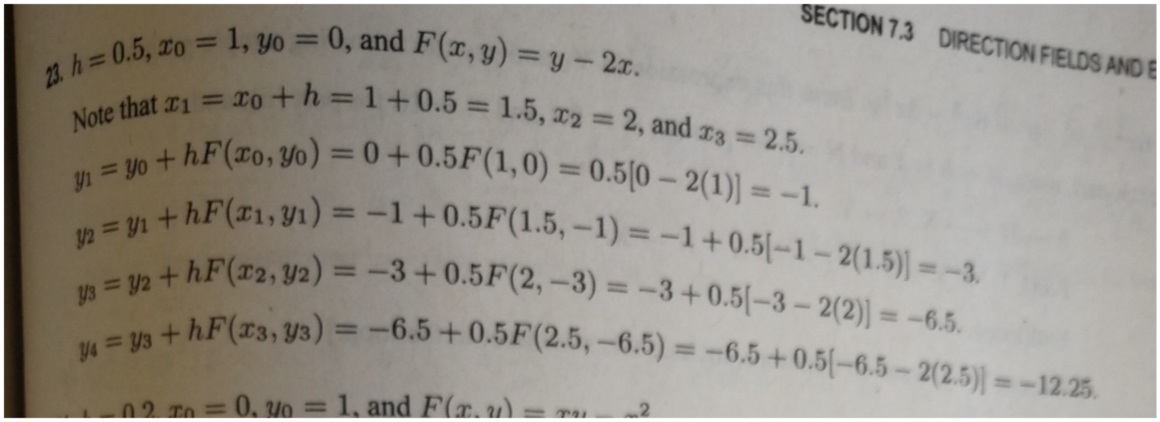
(c) (i) For $h = 0.4$: (exact value) – (approximate value) $= e^{0.4} - 1.4 \approx 0.0918$

(ii) For $h = 0.2$: (exact value) – (approximate value) $= e^{0.4} - 1.44 \approx 0.0518$

(iii) For $h = 0.1$: (exact value) – (approximate value) $= e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

4. Use Euler's method with step size 0.5 to compute the approximate y-values y_1, y_2, y_3, and y_4 of the solution of the initial-value problem $y' = y - 2x$, $y(1) = 0$.


SECTION 7.3 DIRECTION FIELDS AND E

23. $h = 0.5, x_0 = 1, y_0 = 0$, and $F(x, y) = y - 2x$.

Note that $x_1 = x_0 + h = 1 + 0.5 = 1.5$, $x_2 = 2$, and $x_3 = 2.5$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.5F(1, 0) = 0.5[0 - 2(1)] = -1$.

$y_2 = y_1 + hF(x_1, y_1) = -1 + 0.5F(1.5, -1) = -1 + 0.5[-1 - 2(1.5)] = -3$.

$y_3 = y_2 + hF(x_2, y_2) = -3 + 0.5F(2, -3) = -3 + 0.5[-3 - 2(2)] = -6.5$.

$y_4 = y_3 + hF(x_3, y_3) = -6.5 + 0.5F(2.5, -6.5) = -6.5 + 0.5[-6.5 - 2(2.5)] = -12.25$.

3 Extra Challenge

5. Find the general solution of the ODE $t^2y'' + 3ty' - 3y = 0$.(Hint: This linear, second-order ODE has variable coefficients and so the methods from class will not work. Instead, look for solutions of the form $y = t^r$)

$t^2y'' + 3ty' - 3y = 0$
$t^2\frac{d^2}{dt^2}(t^r) + 3t\frac{d}{dt}(t^r) - 3t^r = 0$
we can derive that$t^2\frac{d^2}{dt^2}(t^r) = (r-1)rt^{r-2}$ and $\frac{d}{dt}(t^r) = rt^{r-1}$
Then the equation become:
$r^2t^r + 2rt^r - 3t^r = 0$
$t^r(r^2 + 2r - 3) = 0$
$r^2 + 2r - 3 = 0$
$(r-1)(r+3) = 0$
$r_1 = 1, r_2 = -3$
$y(t) = \frac{C_1}{t^3} + C_1 t$