Minimal compactification of the affine plane with nef cannonical divisors / $\mathbb{C}$.

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1 Intro. & Main results

Definition

X : normal compact analy. surface

$\cup$

Γ : irreducible closed curve

① (X, Γ) : minimal (analy.) cpt'n of $\mathbb{C}^2$

$\iff X \setminus \Gamma \underset{\text{bihol}}{\sim} \mathbb{C}^2$

② (X, Γ) : minimal alg. cpt'n of $\mathbb{C}^2$

$\iff X \setminus \Gamma \simeq \mathbb{C}^2$ (alg. var.)

◎ (X, Γ) : min. cpt'n of $\mathbb{C}^2$

Fact [Morrow '73, Fujita '82]

$\pi : V \to X$: min. resol.

D : red. exc. div. of π

$C := \pi_*^{-1}(\Gamma)$

Then V : sm. proj. rat. surf. (s.t.) $V \setminus \mathrm{Supp}(C + D) \simeq \mathbb{C}^2$ (alg. var).

Example

$\mathbb{P}^2 \supseteq L$: proj. line. $\rightsquigarrow$ $(\mathbb{P}^2, L)$: min. alg. cpt'n of $\mathbb{C}^2$.

Theorem [Remmert - van de Ven '60]

(X, Γ) : min. cpt'n of $\mathbb{C}^2$ (s.t.) X : smooth.

Then $(X, \Gamma) = (\mathbb{P}^2, \text{line})$

Fact

$Cl(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[k_X]$ (k_X : a cannonical div. of X)

In particular, one of the following hold :

(i) $-k_X$: num. ample $\rightsquigarrow$ $\kappa = -\infty$

($\Leftrightarrow$ $(-k_X)^2 > 0$ & $(-k_X . B) > 0$ for $^\forall B$: closed curve on X)

(ii) k_X : num. triv. $\rightsquigarrow$ $\kappa = 0$

($\Leftrightarrow$ $k_X \equiv 0$)

(iii) k_X : num. ample $\rightsquigarrow$ $\kappa > 0$ ($\kappa = 1$ or 2)

Remark

V : sm. proj. rat. surf.

Then the Kodaira dim of V ($= \kappa(V)$) is $-\infty$.

Question

When does k_X become nef ?

$\underset{\text{def}}{\Leftrightarrow}$ $(k_X \cdot B) \geq 0$ for $^\forall B$: curve on X

$\underset{\text{iff}}{\Leftrightarrow}$ (ii) or (iii)

Theorem [Kojima - Takahashi '09]

(X, Γ) : min. cpt'n of $\mathbb{C}^2$. (s.t.) X has at most lc sing.

Then X : projective ($\rightsquigarrow$ X : alg.) & $-k_X$: num. ample.

Main result

(X, Γ) : min. cpt'n of $\mathbb{C}^2$.

k_X : nef $\Rightarrow$ X : non alg.

(i) alg.	non-alg	(ii) alg.	non-alg.	(iii) alg.	non-alg.	
	×	×	×	×	×	高々 lc sing
(∃)	(?)	×	(∃)	×	(∃)	lc より悪い

2. Proof of Main result

(X, Γ) : min cpt'n of $\mathbb{C}^2$. (s.t) X has sing's worse than lc sing's.

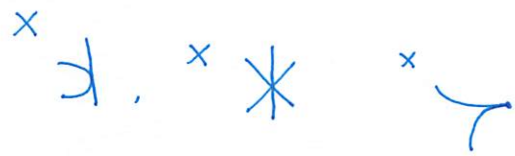
$\pi : V \longrightarrow X$: min. resol.

D : red. exc. div. of π, $C := \pi_*^{-1}(\Gamma)$.

Facts [Kojima '01], [Kojima - Takahashi '09]

① $\# \operatorname{Sing}(X) \leq 2$

② $C + D$: SNC - div. ←



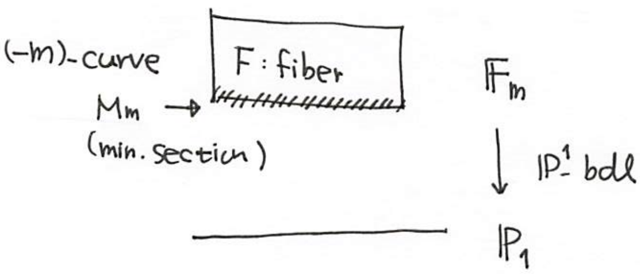
③ C : (-1) - curve

④ $\exists \nu : V \xrightarrow{\text{bir}} \mathbb{F}_m$ $(m \geq 2)$

(Hirzebruch surf)

$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

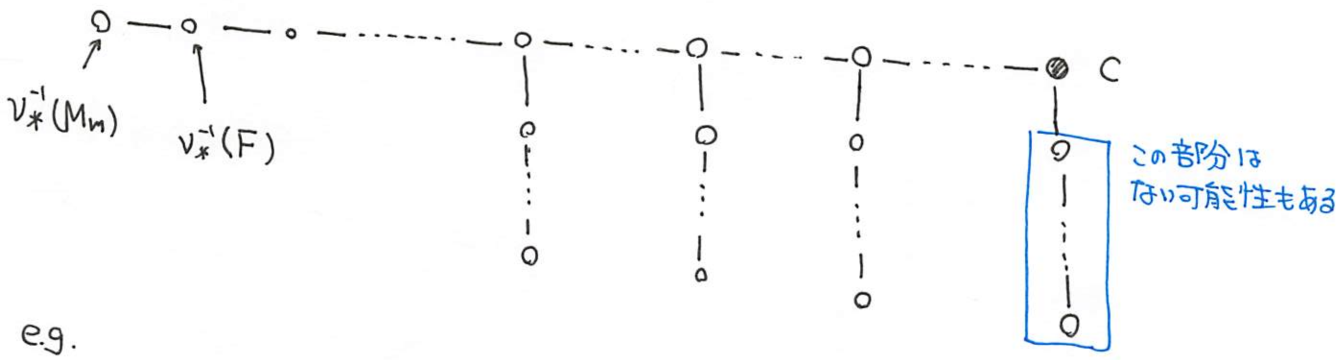
$\mathbb{F}_1 = \mathbb{P}^2$ の 1点 b.u.



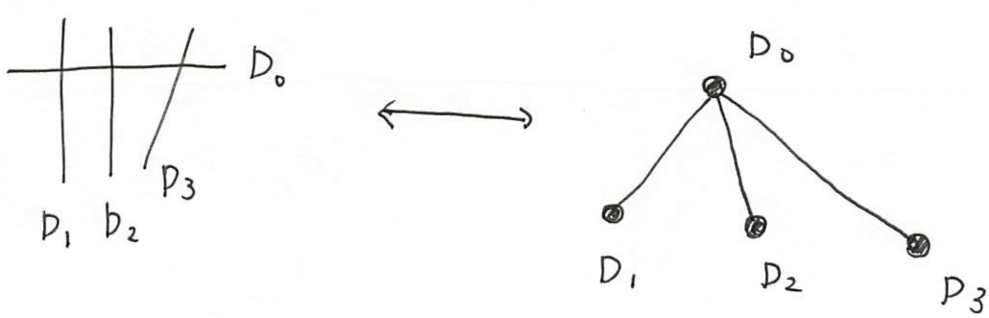
(s.t.) • $V \setminus \operatorname{Supp}(C + D) \simeq \mathbb{F}_m \setminus (M_m \cup F)$

• $\nu_*(C + D) = M_m + F$

• The dual graph $C + D$ is of the form



e.g.



Fact

$D = \sum_i D_i$: irr. decomp.

Then $\exists 1\ D^\# = \sum_i \alpha_i D_i$: effective $\mathbb{Q}$-div. on V $\rightsquigarrow$ $\alpha_i \in \mathbb{Q}_{\geq 0}$.

(s.t.) $(D_i \cdot K_V + D^\#) = 0$ for $\forall D_i$: irr. comp. of D. ------ ☆

Remark

・$\{\alpha_i\}_i$ は次の線型連立方程式の解：

$$\left((D_i \cdot D_j)\right)_{ij} \begin{pmatrix} \vdots \\ x_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ (D_i \cdot -K_V) \\ \vdots \end{pmatrix}$$

・$m_i := -(D_i)^2 \rightsquigarrow (D_i \cdot -K_V) = -(m_i - 2)$

2.1 Case : $K_X \equiv 0$ ($\overset{\text{def}}{\iff}$ $(K_X . D') = 0$ for $\forall D'$: divisor)

Claim : $D^\#$: $\mathbb{Z}$-div (i.e.) $\forall \alpha_i \in \mathbb{Z}_{\geq 0}$

Assume that this claim is true.

Note that $P_a(D^\#) := \frac{1}{2}(\underbrace{D^\# \cdot K_X + D^\#}_{= 0\ ☆}) + 1 = 1 > 0$

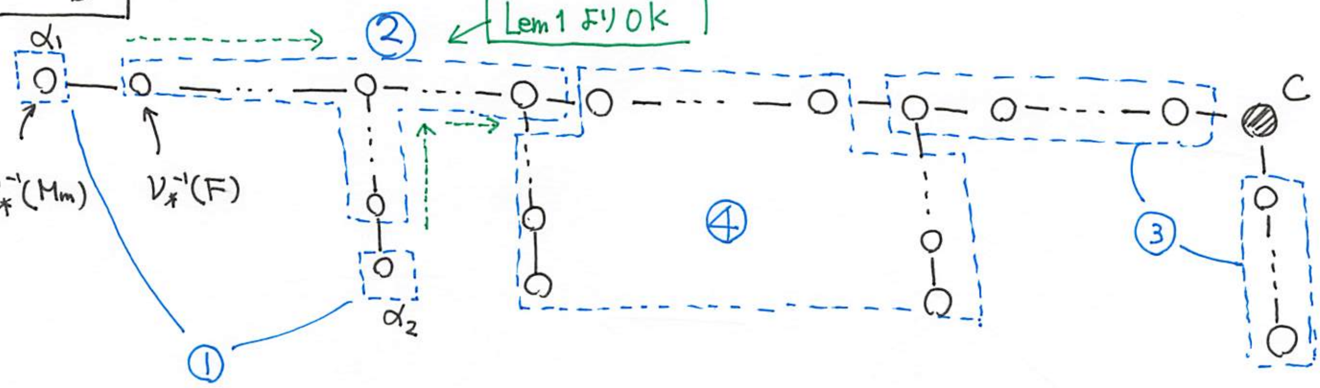
By [Artin '62], D can not be alg. contracted.

Dが代数的に contraction できるためには、

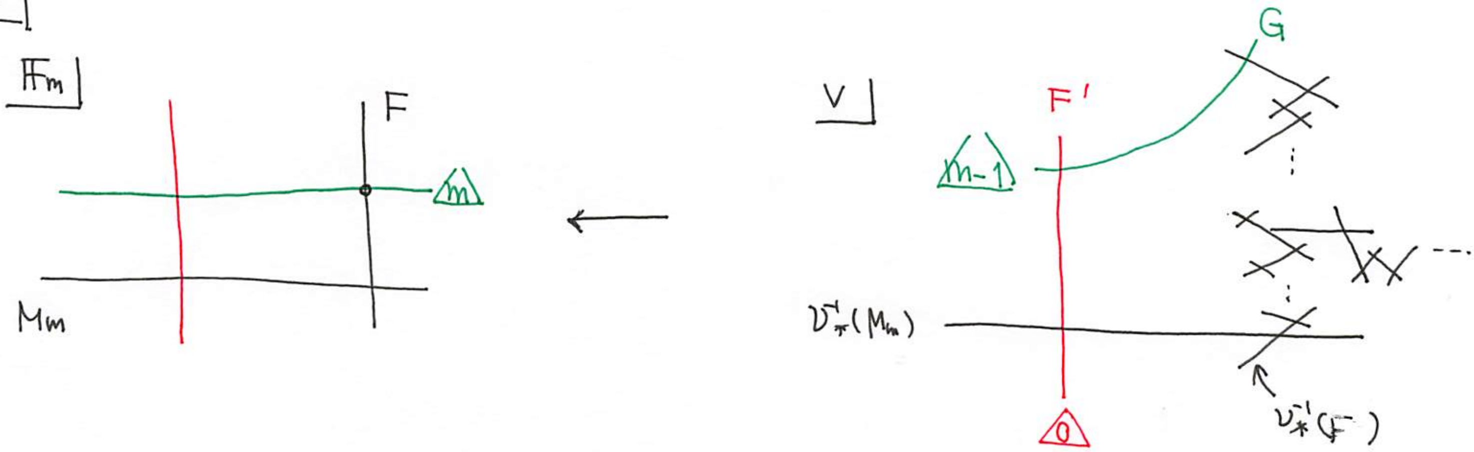
$\forall Z$: eff. $\mathbb{Z}$-div. with $\mathrm{supp}(Z) = \mathrm{Supp}(D)$ に対して $P_a(Z) \leq 0$ が必要条件.

$\rightsquigarrow$ X : non alg :

$C + D$



①



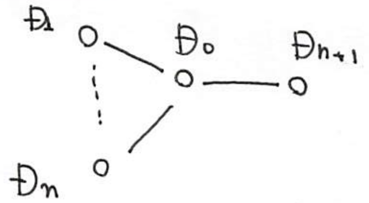
Note $K_V + D^\# \equiv \pi^*(K_X) \equiv 0$.

$(F' \cdot D^\#) = \sum_i \alpha_i (F' \cdot D_i)$
$= \alpha_1$

- $\alpha_1 = (F' \cdot D^\#) = (F' \cdot -K_V) = 2 \in \mathbb{Z}$.
- $\alpha_2 = (G \cdot D^\#) = (G \cdot -K_V) = m+1 \in \mathbb{Z}$.

②

Lemma 1



$$\left(\begin{array}{c} (D_0 \cdot D - D_0) = n+1 \\ n \in \mathbb{Z}_{\geq 0} \end{array} \right)$$

If $\alpha_0, \alpha_1, \cdots, \alpha_n \in \mathbb{Z}$, then $\alpha_{n+1} \in \mathbb{Z}$.

(proof)

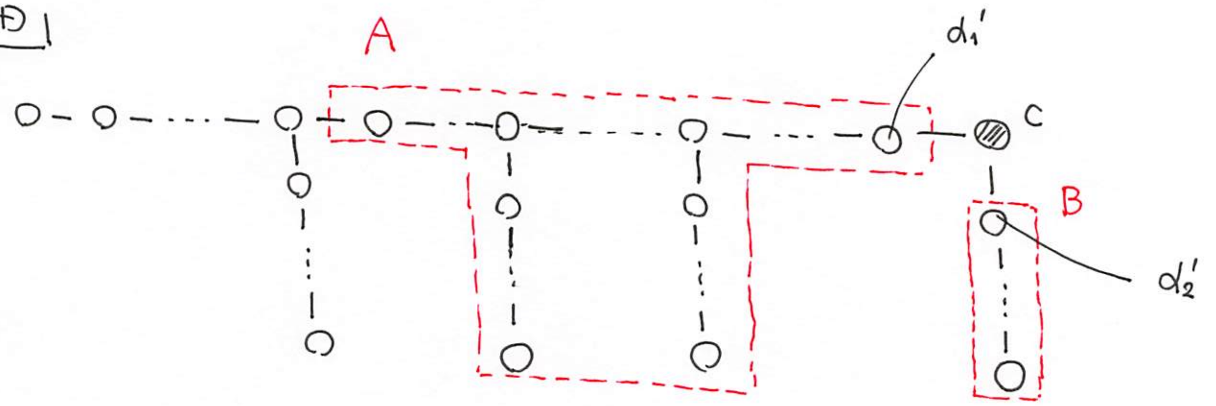
$m_0 := -(D_0)^2$.

Since $(D_0 \cdot K_V + D^\#) = 0$, $(m_0 - 2) - \alpha_0 m_0 + \alpha_1 + \cdots + \alpha_n + \alpha_{n+1} = 0$

$\therefore \alpha_{n+1} \in \mathbb{Z}$. □

③|

$C + D$ |



- $*$: weighted dual graph に対して

  $I(*)$: intersection matrix of $*$

  $d(*) := \det(-I(*)) \in \mathbb{Z}$.

Lemma 2

$\gcd(d(A), d(B)) = 1$.

(proof)

$$\det\begin{pmatrix} d(A) & -a & 0 \\ -1 & 1 & -1 \\ 0 & -b & d(B) \end{pmatrix} \qquad \det(1)$$

By [Fujita '82, §3], $d\left(\begin{smallmatrix} A - C \\ | \\ B \end{smallmatrix}\right) = d(E) = 1$

$\because$ $d(A)(d(B) - b) + d(B)\cdot(-a) = 1$. $\square$

- $d_1' + d_2' = (C \cdot D^{\#}) = (C \cdot -K_V) = 1$.

If $\# \mathrm{Sing}(X) = 2$, then it follows from ② & Cramer's formula that

$$d_1' \in \frac{1}{d(A)}\mathbb{Z}, \quad d_2' \in \frac{1}{d(B)}\mathbb{Z}.$$

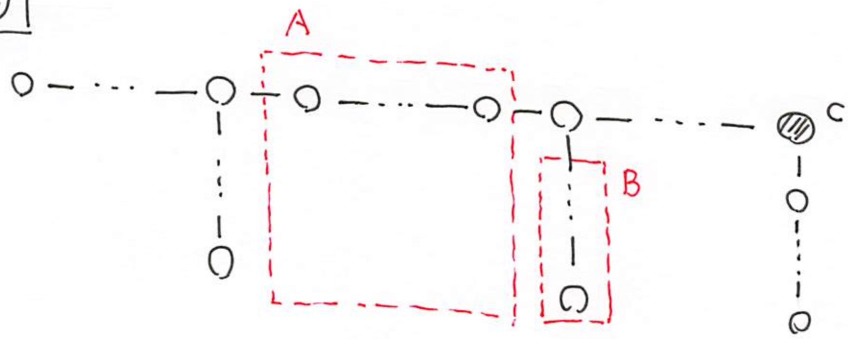
$$\rightsquigarrow d_1' \in \frac{1}{d(A)}\mathbb{Z} \cap \frac{1}{d(B)}\mathbb{Z} = \mathbb{Z}.$$

Simillary $d_2' \in \mathbb{Z}$.

($\pi(B)$ = a cyclic quotient sing pt. より $d_2' < 1 \rightsquigarrow d_1' = 1$, $d_2' = 0$.)

The rests can be showed by Lem 1.

④|



⇝ ③と同様の議論.

□.