A Simple Fire-Growth Model

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Forest fire control planning sometimes requires a mathematical model of how a forest fire grows with time. Such a formula is especially necessary in problems concerning the economics of fire detection and control. Although the model presented here is not completely original in concept, it is given in a simple but flexible algebraic form not previously used as far as the author is aware.

Assume that, after an initial short period of adjustment, the fire's linear rate of spread at each point on the perimeter remains constant. This rate will vary continuously from a maximum at the head to a minimum at the rear. For simplicity, select values of this linear rate of spread for the head, flanks, and rear of the fire, and assume a uniform fuel.

Next, assume that the fire's head burns a fan-shaped area that widens as the head advances; flank spread then proceeds from the sides of the fan. Furthermore, assume that the width of the fan is such that the fire's shape remains elliptical for any combination of head and flank rates.

Refer to Figure 1, and let the following symbols apply:

A — fire's area, an ellipse
a — long semiaxis of ellipse
b — short semiaxis of ellipse
v — linear rate of spread at head
u — linear rate of spread at flanks
w — linear rate of spread at rear
t — time since ignition

Then, according to the formula for the area of an ellipse,

$$A = \pi ab$$

But $a = (v + w)\,t/2$
and $b = 2\,ut/2 = ut$
Therefore

$$A = \frac{\pi}{2}(v + w)ut^2 \qquad (1)$$

This expression can be used if all required rates are known, or simplified if necessary. For example, if the fire advances at rate u at all points on its perimeter, then expression (1) reduces to

$$A = \pi u^2 t^2 \qquad (2)$$

the area of a circle of radius ut. Or, suppose w is negligible and u = v/4, then

$$A = \frac{\pi}{8} v^2 t^2 \qquad (3)$$

This is the area of an ellipse whose length is twice its width, and whose perimeter is about 1½ times that of a circle of equal area. This is the average fire shape found by Hornby (1936) in the Rocky Mountains and for which Pirsko (1961) made an alignment chart. Peet (1967) found the same 2 to 1 ratio for length and width of small fires in Western Australia.

The area supposedly burned by the head fire in Figure 1 is shown hatched. Strong winds will result in greater ratios of v to u; at the same time, it is reasonable to assume that the stronger the wind the less will be its directional variation and the narrower the angular width of the fan-shaped head-fire pattern. The width of the fire should thus be about the same near each end, preserving the approximate elliptical shape for all ratios of length to width. The length of a, the long semiaxis in Figure 1, is plainly half the sum of v and w, multiplied by time t. The short semiaxis b is not so plainly equal to ut, which is more exactly represented by the line c in the figure. The mathematical advantage of the elliptical shape, however, makes this approximation worthwhile for practical purposes.

It is worth noting that in expression (1) the area is proportional to the square of the time since ignition. The rate of area increase at time t will be given in terms of area per unit time by the first derivative:

$$\frac{dA}{dt} = \pi (v + w)\,u\,t \qquad (4)$$

This shows that the rate of area increase is not constant but increases in direct proportion to time. However, the acceleration rate at which the area increases is constant and is given by the second derivative with dimensions of area per (unit time)^2:

$$\frac{d^2A}{dt^2} = \pi (v + w)\,u \qquad (5)$$

In spite of its simplicity, expression (1) requires more information than is commonly known about fire behavior. Ideally, for a given fuel type the rates v, u, and w might be expressed as functions of a danger index or of specific burning conditions such as fuel moisture and wind.

A fire-growth model is not quite complete without mention of the perimeter. The perimeter of an

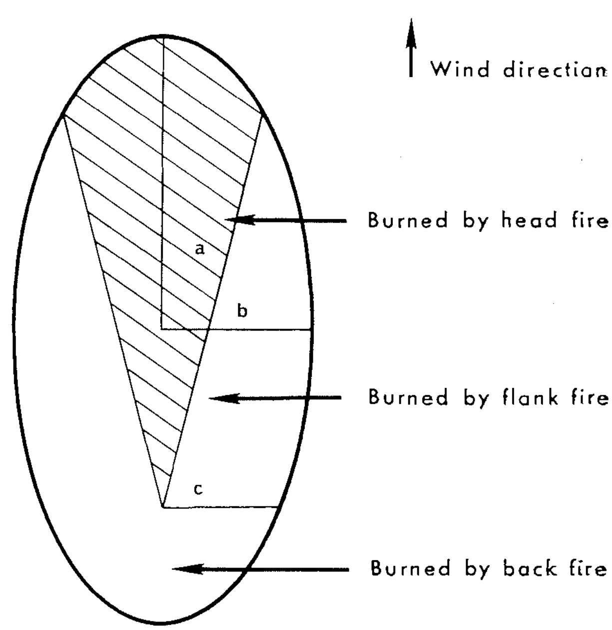

FIGURE 1. DIAGRAM OF SIMPLE FIRE GROWTH MODEL

ellipse in terms of its semiaxes is given by the rather awkward formula:

$$P = \pi (a + b) \left(1 + \frac{M^2}{4} + \frac{M^4}{64} + \frac{M^6}{256} + \ldots\right) \qquad (6)$$

where $M = \frac{a - b}{a + b}$

When a = 2b, for example, the series in M equals 1.03; it increases as the ellipse narrows, becoming 1.09 when a = 4b. For present purposes the terms in M^4 and so on can be omitted, and, with the spread rates substituted, the perimeter formula becomes

$$P = \pi t \left(\frac{v + w}{2} + u\right) \left(1 + \frac{M^2}{4}\right) \qquad (7)$$

The rate of perimeter increase with time is constant and equals

$$\frac{dP}{dt} = \pi \left(\frac{v + w}{2} + u\right) \left(1 + \frac{M^2}{4}\right) \qquad (8)$$

When a = b, the series in M equals 1, and expressions (6) and (7) reduce to the formula for the circumference of a circle, which is, of course, the shape with the greatest area for a given perimeter.

References

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The Harmonic Mean in Forest Mensuration

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"The harmonic mean basal area of a Bitterlich population is equal to the arithmetic mean basal area of the corresponding actual population."

Abstract

Under certain conditions, the harmonic mean rather than the arithmetic mean should be used to average measurements. Two examples which illustrate the proper use of the harmonic mean are the averaging of growth measurements in the form of number of rings in the last half-inch of radius, and the calculation of the arithmetic mean basal area of a forest stand from a point-sample tally.

Résumé

Dans certains problèmes, la moyenne harmonique est préférable à la moyenne arithmétique. L'auteur illustre le bon emploi de la moyenne harmonique au moyen de deux exemples:1° - pour calculer la moyenne de la mesure des accroissements exprimés par le nombre de cernes compris dans le demi-pouce extérieur des bâtonnets prélevés radialement, 2° - pour le calcul de la surface terrière moyenne d'un peuplement mesuré par la méthode de Bitterlich.

Perhaps the least well-known measure of the central tendency of a quantitative distribution is the harmonic mean. It is generally treated by authors of statistical texts as a statistic of limited value, which is true. Authors of mensuration texts seldom make reference to it at all, presumably because they feel that it has no application in forestry, which is not true. Though Husch (1963) describes the harmonic mean, he makes no attempt to demonstrate its use in forestry problems. The purpose of this paper is to call the attention of foresters to the existence of this measure of central tendency, to indicate the general type of problem in which it should be used, and to describe two specific applications where it can be of value.

The harmonic mean is an ancient term. In his **History of Greek Mathematics,** Sir Thomas Heath says "We are told that in Pythagoras's time there were three means, the arithmetic, the geometric, and the subcontrary, and that the name of the third (subcontrary) was changed by Archytas and Hippasus to 'harmonic'. A fragment of Archytas's work **On Music** actually defines the three: we have the arithmetic mean when, of three terms, the first exceeds the second by the same amount as the second exceeds the third; and the geometric mean when, of three terms, the first is to the second as the second is to the third; and the **'Subcontrary** which we call the **harmonic'** when the three terms are such that 'by whatever part of itself the first exceeds the second, the second exceeds the third by the same part of the third' "(Walker, 1949).

Problems involving the harmonic mean were "of great practical concern in earlier days when the water supply of a town depended upon public fountains and it was important to know whether the pipes through which the water flowed were large enough to deliver an adequate supply. As an illustration, we may quote a problem from a work called the **Lilavati** (= 'the beautiful') written in India in 1150 by Bhaskara: 'Say quickly, friend, in what portion of a day will four fountains being let loose together, fill a cistern if, when separately opened they would fill it in one day, half a day, the third part of a day, and the sixth part of a day respectively' " (Walker, 1949).

Probably one reason for the avoidance of the harmonic mean is the difficulty of describing it. According to Croxton and Cowden (1955), "The harmonic mean is the reciprocal of the arithmetic mean of the reciprocals of the values." The mathematical form is easier to comprehend:

$$\bar{X}_h = \frac{N}{\Sigma(1/X_i)},$$

where N represents the total number of values. When dealing with a frequency distribution, each reciprocal value of X_i must be multiplied by its ap-