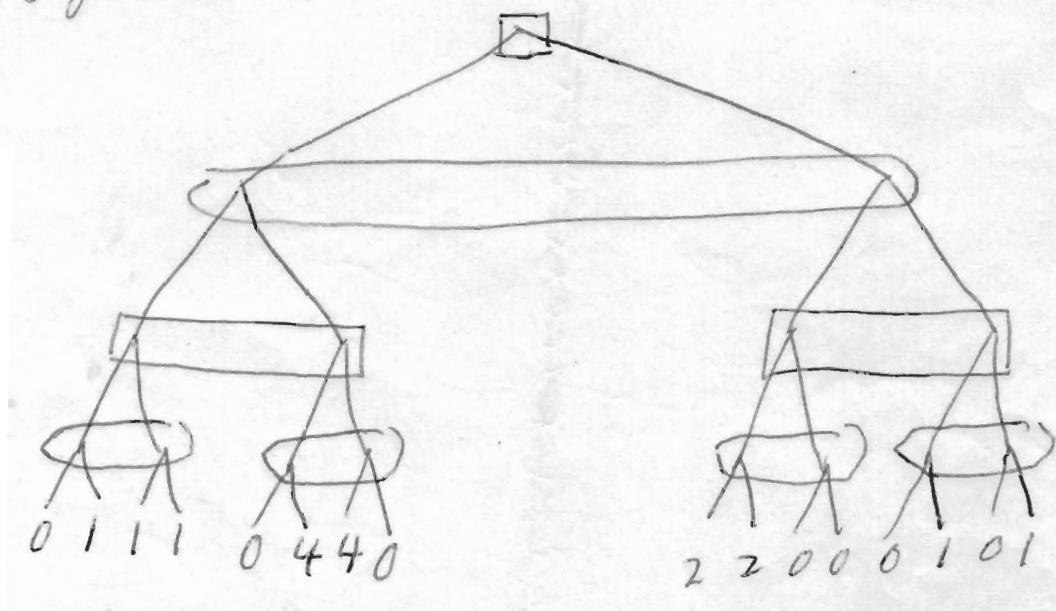
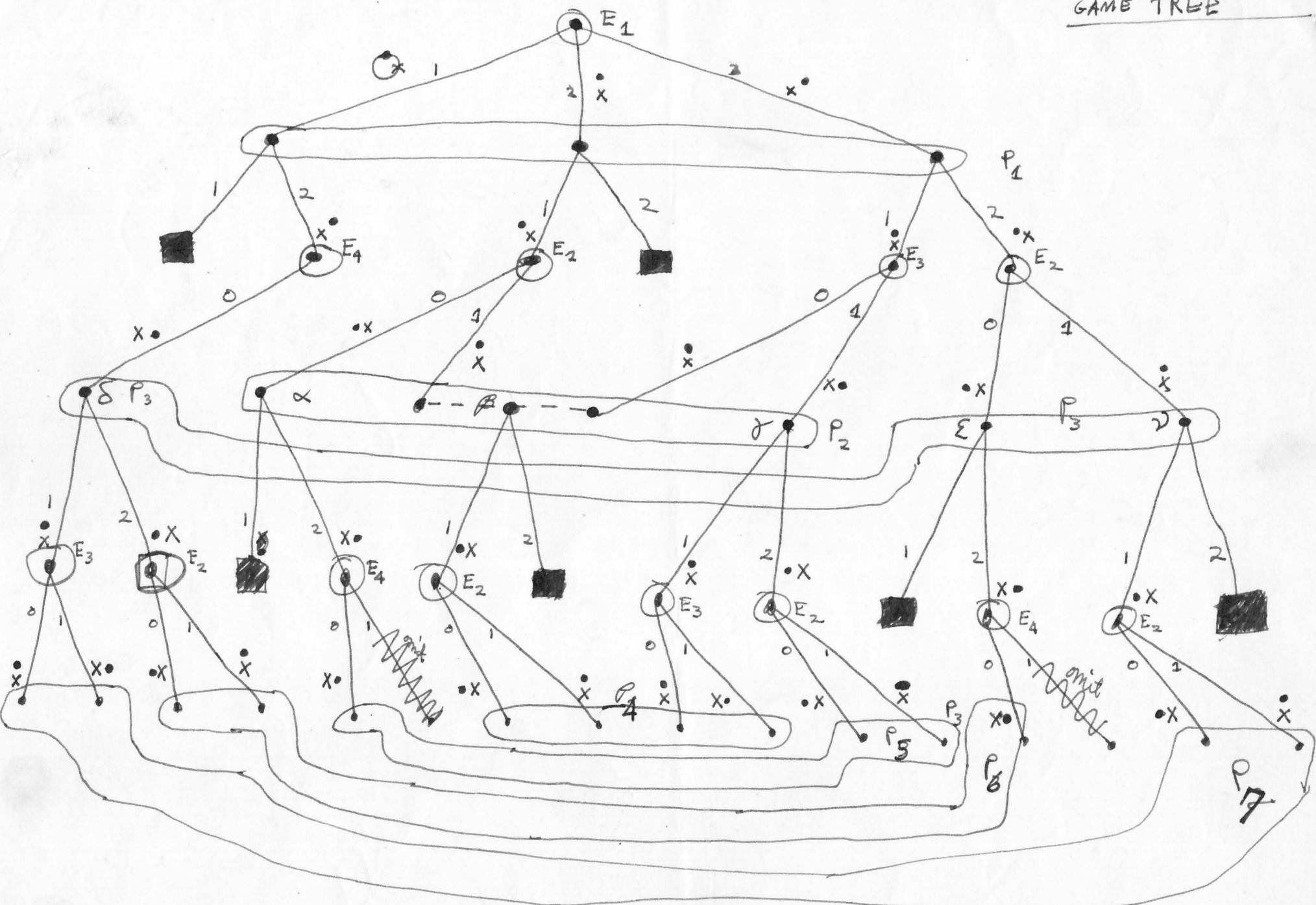


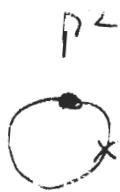
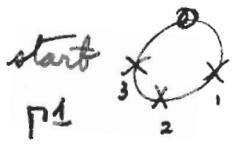
If you protest that anyway one player's relative game gives the right answer, try the (information-lag) game



GAME TREE



Eraders game:



$$\text{EV}_{P1} = \frac{1}{3} \left(1 + \frac{1}{1+V^4} \right)$$

$$V^2 = \frac{1}{2} \left(0 + \frac{1}{1+V^4} \right)$$

$$V^3 = \frac{1}{3} \left(0 + \frac{1}{1+V^2} \right)$$

Saddle

$$V^4 = \frac{1}{4} \left(0 + \frac{1}{1+V^2} \right)$$

(no choice for evader)

so for P^2 since $V^4 = 1+V^2$

$$V^3 = 1+V^2$$

$$V^4 = 1+V^2$$

$$\text{we get } V^2 \begin{pmatrix} 0 & 2+V^2 \\ 2+V^2 & 0 \end{pmatrix} = V^2$$

$$\Rightarrow \frac{(2+V^2)(1+V^2)}{3+2V^2} = V^2 \quad \Rightarrow 3 + 3V^2 + (V^2)^2 = 3V^2 + 2(V^2)^2$$

$$3 = (V^2)^2$$

$$\Rightarrow V^2 = \sqrt{3} = 1.732$$

so

$$V^3 = 1 + \sqrt{3} = 2.232$$

$$V^4 = 1 + \sqrt{2} = 2.414$$

$$\Rightarrow V^1 = \frac{10 + 6\sqrt{2}}{7} = 2.649$$

then P_1

1	$2+\sqrt{2}$
0	$2+\sqrt{2}$
$1+\sqrt{2}$	0
$2+\sqrt{2}$	$1+\sqrt{2}$

$$= \frac{(2+\sqrt{2})(2+\sqrt{2})}{3+\sqrt{2}} = \frac{4 + 4\sqrt{2} + 2}{3+\sqrt{2}} = \frac{6+4\sqrt{2}}{3+\sqrt{2}} \left(\frac{3-\sqrt{2}}{3-\sqrt{2}} \right) = \frac{18+6\sqrt{2}-8}{9-2} = \frac{10+6\sqrt{2}}{7}$$

decimally, value = 2.6490755

Value of Evader's relative game

$$\Gamma \begin{matrix} \alpha & \infty \\ \beta & +\Gamma_{\alpha, \beta} \\ \gamma & +\Gamma_{\beta, \gamma} \end{matrix} \begin{pmatrix} 0 & 1 + \Gamma_\delta \\ 1 + \Gamma_{\alpha, \beta} & 0 \\ 1 + \Gamma_{\beta, \gamma} & 1 + \Gamma_{\gamma, \alpha} \end{pmatrix}$$

Pursuer game,

where $\Gamma_{\alpha, \beta} \Rightarrow$ on next move evader restricted to play only strategies α, β

two consecutive move game

		1,1	1,2	2,1	2,2	
		0	0	$2 + \Gamma_{\alpha, \beta}$	$2 + \Gamma_{\alpha, \beta}$	
		1	$2 + \Gamma_\delta$	0	0	
β, β		$2 + \Gamma_{\alpha, \beta}$	1	0	0	
γ, α		X	X	1	$2 + \Gamma_\delta$	
γ, β		$2 + \Gamma_{\alpha, \beta}$	1	$2 + \Gamma_{\alpha, \beta}$	1	
γ, γ		$2 + \Gamma_{\alpha, \beta}$	$2 + \Gamma_{\alpha, \beta}$	X	X	

$$\begin{array}{c} \mathcal{S} \quad (1-\delta) \\ \text{---} \\ \mathcal{F} \quad \left(\begin{array}{cc} 0 & 1+V_f(\delta) \\ 1+V_{\alpha\beta}(\delta) & 0 \\ 1+V_{\beta\alpha}(\delta) & 1+V_{\alpha\beta}(\delta) \end{array} \right) \end{array} \quad \begin{array}{l} \text{assume} \\ V_x(\delta) \text{ etc exist} \end{array}$$

$$V_{\alpha\beta}(\delta) = \max \left(\delta(1+V_{\alpha\beta}(\delta)) + (1-\delta)(1+V_f(\delta)) \right)$$

$$\begin{aligned} V_f(\delta) &= \cancel{\delta(1-V_{\beta\alpha}(\delta))} \delta + (1-\delta)(1+V_{\alpha\beta}(\delta)) \\ &= \cancel{\delta} - \cancel{\delta V_{\beta\alpha}(\delta)} + \cancel{1} + V_{\alpha\beta}(\delta) - \cancel{\delta} - \cancel{\delta V_{\alpha\beta}(\delta)} \\ &= 1 - \delta[V_{\beta\alpha}(\delta) + V_{\alpha\beta}(\delta)] + V_{\alpha\beta}(\delta) \\ &= (1-\delta)V_{\alpha\beta}(\delta) + 1 - \delta V_{\beta\alpha}(\delta) \end{aligned}$$

$$V_{\beta\alpha}(\delta) = \max \left(\delta(1+V_{\alpha\beta}(\delta)), (1-\delta)V_{\alpha\beta}(\delta) + 1 - \delta V_{\beta\alpha}(\delta) \right)$$

Assuming stationary behavior strategies

①

for P_1	move 1	α
	move 2	$1-\alpha$

configurations



for P_2	move 0	C^1	C^2	C^3
	move 1	β	γ	1
		$1-\beta$	$1-\gamma$	0

would to move then
survive

fairly clear that in
configuration 2 beta is $\gamma =$

$$\beta\alpha + \gamma - \alpha - \beta +$$

$$= 1 - \alpha - \beta - 2\alpha$$

trans probability matrix:

STOP	C^1	C^2	C^3	
STOP	1	0	0	0
C^1	$\beta\alpha + (1-\alpha)(1-\beta)$	$\alpha(1-\beta)$	0	$\beta(1-\alpha)$
C^2	$\gamma(1-\alpha)$	$\gamma\alpha + (1-\gamma)(1-\alpha)$	$\alpha(1-\gamma)$	0
C^3	0	$1-\alpha$	α	0

if domination argument used ($\gamma = 0$) reduces to

S	C^1	C^2	C^3	
S	1	0	0	0
C^1	$\alpha\beta + (1-\alpha)(1-\beta)$	$\alpha(1-\beta)$	0	$\beta(1-\alpha)$
C^2	0	$1-\alpha$	α	0
C^3	0	$1-\alpha$	α	0

so now C^3
completely equivalent to C^2

and we get (omitting c_2)

$$\begin{matrix} & s & c^1 & c^3 \\ \begin{pmatrix} s \\ c^1 \\ c^3 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \alpha + (1-\alpha)(1-\beta) & \alpha(1-\beta) & \beta(1-\alpha) \\ 0 & 1-\alpha & \alpha \end{pmatrix} \end{matrix}$$

$$\text{Expectation} = \sum_{k=0}^m Q^k \rightarrow 1$$

$$\text{Expectation} = \sum_{k=1}^{\infty} Q^k$$

$$\text{but } = \frac{1}{(1-Q)} \Phi$$

$$(1-Q^{-1}) \sum Q^n = Q$$

$$\begin{aligned} (1-Q^{-1})(Q + Q^2 + Q^3 + \dots) \\ = Q - 1 + Q^{-2} - Q + Q^3 - Q^2 + \dots \\ = -1 \end{aligned}$$

$$(1-Q) \sum Q^n = Q$$

$$= (1-Q)(Q + Q^2 + Q^3 + \dots)$$

$$\begin{aligned} \sum Q^n = (1-Q)^{-1} Q \\ = Q - Q^2 + Q^2 - Q^3 + \dots \\ = Q \end{aligned}$$

(3)

so that

$$Q = \begin{pmatrix} \alpha(1-\beta) & \beta(1-\alpha) \\ 1-\alpha & \alpha \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\downarrow \quad \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$(1-Q) = \begin{pmatrix} 1-\alpha(1-\beta) & -\beta(1-\alpha) \\ -1+\alpha & 1-\alpha \end{pmatrix}$$

$$= [1-\alpha(1-\beta)][1-\alpha] - [-\beta(1-\alpha)-(1-\alpha)]$$

$$= 1-\alpha(1-\beta)-\alpha+\alpha^2(1-\beta)$$

$$-\beta+\alpha\beta-1+\alpha$$

$$= \cancel{1-\alpha+\alpha\beta} \cancel{-\alpha+\alpha^2} \cancel{-\alpha^2\beta} \cancel{-\beta+\alpha\beta} \cancel{-1+\alpha}$$

$$= \boxed{-\alpha+\alpha^2-\alpha^2\beta-\beta+2\alpha\beta}$$

$$\text{so } (1-Q)^{-1} = \frac{1}{-\alpha-\beta+\alpha^2-\alpha^2\beta+2\alpha\beta} \begin{pmatrix} 1-\alpha & \beta(1-\alpha) \\ 1-\alpha & 1-\alpha(1-\beta) \end{pmatrix}$$

$$\text{and } (1-Q)^{-1}Q$$

Divide by $\alpha^2 + 2\alpha\beta - \alpha - \beta - \alpha^2\beta$

$$\begin{aligned}
 &= \left[\begin{array}{l} ((1-\alpha)\alpha(1-\beta) + \beta(1-\alpha)(1-\alpha)) \\ ((1-\alpha)[\alpha(1-\beta) + \beta(1-\alpha)]) \\ = (1-\alpha)[\alpha - \alpha\beta + \beta - \alpha\beta] \end{array} \right] \left[\begin{array}{l} ((1-\alpha)\beta(1-\alpha) + \beta(1-\alpha)\alpha) \\ = (1-\alpha)[\beta(1-\alpha) + \beta\alpha] \\ = (1-\alpha)\beta \end{array} \right] \\
 &\quad \left[\begin{array}{l} ((1-\alpha)\alpha(1-\beta) + (1-\alpha)(1-\beta))(1-\alpha) \\ ((1-\alpha)[\alpha(1-\beta) + 1-\alpha(1-\beta)]) \end{array} \right] \left[\begin{array}{l} ((1-\alpha)\beta(1-\alpha) + \alpha)[1-\alpha(1-\beta)] \\ = (1-\alpha)[\beta(1-\alpha) + \alpha] \\ = (1-2\alpha+\alpha^2)\beta \\ \quad || \\ \beta - 2\alpha\beta + \alpha^2\beta + \alpha - \alpha^2 + \alpha^2\beta \end{array} \right] \\
 &= \frac{\alpha + \beta - 3\alpha\beta - \alpha^2 + 2\alpha^2\beta}{\alpha^2 + 2\alpha\beta - \alpha - \beta - \alpha^2\beta} \\
 &= \beta - 2\alpha\beta + \alpha - \alpha^2 + 2\alpha^2\beta
 \end{aligned}$$

$$\begin{aligned}
 &= \left[\begin{array}{l} [\alpha + \beta - 3\alpha\beta - \alpha^2 + 2\alpha^2\beta] [(1-\alpha)\beta] \\ [1-\alpha] [\beta - 2\alpha\beta + \alpha - \alpha^2 + 2\alpha^2\beta] \end{array} \right]
 \end{aligned}$$

$$\text{so } E_{C^2} = \frac{\alpha + \beta - 3\alpha\beta - \alpha^2 + 2\alpha^2\beta + \beta - \alpha\beta}{\alpha^2 + 2\alpha\beta - \alpha - \beta - \alpha^2\beta} = \frac{\alpha + 2\beta - 4\alpha\beta - \alpha^2 + 2\alpha^2\beta}{\alpha^2 + 2\alpha\beta - \alpha - \beta - \alpha^2\beta}$$

$$E_{C^2} = \frac{1 - \cancel{\alpha^2} + \beta - 2\alpha\beta + \cancel{\alpha} - \alpha^2 + 2\alpha^2\beta}{\alpha^2 + 2\alpha\beta - \alpha - \beta - \alpha^2\beta}$$

$$\frac{1 + \beta - 2\alpha\beta - \alpha^2 + 2\alpha^2\beta}{\alpha^2 + 2\alpha\beta - \alpha - \beta - \alpha^2\beta}$$



Evaders
Pursuer's choice
unknown to evader, so OK

Pursuer

$$\text{Evader } P^1: \begin{pmatrix} 1 & 2 \\ 0 & 1+P^3 \\ 1+P^1 & 0 \end{pmatrix}$$

Value $\sqrt{2}$

$$P^2: \begin{pmatrix} 1 & 2 \\ 0 & 1+P^2 \\ 1+P^2 & 1+P^1 \end{pmatrix}$$

note Val $P^2 > \text{Val } P^1$

Value $1+\sqrt{2}$

$$P^3: \begin{pmatrix} 1 & 2 \\ 0 & 1+P^2 \\ 1+P^2 & 1+P^1 \end{pmatrix}$$

Value $1+\sqrt{2}$

$$V^3 = 1 + V^1$$

$$V^2 = 1 + V^1$$

$$\text{so } P^1: \begin{pmatrix} 0 & 2+P^1 \\ 1+P^1 & 0 \end{pmatrix}$$

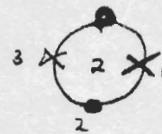
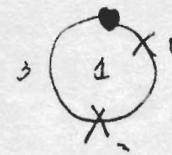
$$\begin{pmatrix} 0 & 2+x \\ 1+x & 0 \end{pmatrix}$$

$$\Rightarrow \frac{(2+x)(1+x)}{1+x+2+x} = x$$

$$\Rightarrow 2+3x+x^2 = 3x+x^2$$

$$x = -\sqrt{2}$$

Pursuer



Known
↓
 $P^3: 3*$

Pursuer's choice unknown when evader made last move so OK

$$P^4: \begin{matrix} \text{Value } \sqrt{2} \\ \text{Pursuer} \\ \begin{pmatrix} 1 & 2 \\ 0 & 1+P^3 \\ 1+P^1 & 0 \end{pmatrix} \end{matrix}$$

$$P^5: \begin{matrix} P^2 \text{ never occurs} \\ \begin{pmatrix} 1 & 2 \\ 0 & 1+P^5 \\ 1+P^4 & 1+P^5 \end{pmatrix} \end{matrix}$$

$$P^3: \begin{matrix} \begin{pmatrix} 1 & 2 \\ 1+P^4 & 1+P^2 \end{pmatrix} \\ 1+P^1 \\ \text{Value } 1+\sqrt{2} \end{matrix}$$

$$P^4: \begin{matrix} \begin{pmatrix} 2 \\ 3+P^5 \end{pmatrix} \\ 0 \\ \text{Value } 1+\sqrt{2} \rightarrow 4 \begin{pmatrix} 1+P^5 & 1+P^2 \\ 1+P^5 & 0 \end{pmatrix} \\ P^4 = 1+P^1 \end{matrix}$$

$$P^5: \begin{matrix} \begin{pmatrix} 1 & 2 \\ 0 & 1+P^5 \end{pmatrix} \\ 2 \\ \begin{pmatrix} 1+P^5 & 0 \\ 1+P^5 & 1+P^5 \end{pmatrix} \\ 1 \rightarrow 3 \end{matrix}$$

Q

Pursuer's game

$$\alpha \quad 1-\alpha$$

$$\beta^2 \begin{pmatrix} 0 & 1+\beta^4 \\ 1-\beta & 1+\beta^2 \end{pmatrix}$$


$$\alpha \quad 1-\alpha$$

$$\beta^3 \begin{pmatrix} 1+\beta^2 & 0 \\ 1+\beta^3 & 1+\beta^2 \end{pmatrix}$$


$$\alpha \quad 1-\alpha$$

$$\beta^4 \begin{pmatrix} 1+\beta^3 & 1+\beta^2 \\ 1+\beta^2 & 0 \end{pmatrix}$$


$$V_2 = \alpha(1-\beta)(1+V_2) + \beta(1-\alpha)(1+V_4)$$

$$V_3 = \alpha(1+V_3) + (1-\alpha)(1+V_2)$$

$$V_4 = \alpha(1+V_3) + (1-\alpha)(1+V_2)$$

$$= \cancel{\alpha} + \alpha V_3 + 1 - \cancel{\alpha} + V_2 - V_2 \cancel{\alpha}$$

$$= 1 + V_2 - \alpha(-V_3 + V_2)$$

so α and α
such that

$$\alpha(1+V^2) = (1-\alpha)(1+V^4)$$

$$\Rightarrow \alpha(1+V^2) + \alpha(1+V^4) = 1+V^4$$

$$\alpha(2+V^2+V^4) = 1+V^4$$

$$\Rightarrow \boxed{\alpha = \frac{1+V^4}{2+V^2+V^4}}$$

This game described by making pursuer play stationary strategy $(\alpha, 1-\alpha)$ on all rounds subsequent to first.
(must play same strategy in all 3 game elements of evader game, excluding initial move!)

(b)

$$V_2 = (\alpha - \alpha\beta)(1+V_2) + (\beta - \alpha\beta)(1 + 1 + V_2 - \alpha(V_3 - V_2))$$

$$= \cancel{\alpha} + \cancel{\alpha V_2} - \cancel{\alpha\beta} - \cancel{\alpha\beta V_2} + \cancel{2\beta} + \cancel{\beta V_2} + \cancel{\alpha\beta V_3} - \cancel{\alpha\beta V_2}$$

$$- 2\alpha\beta - \cancel{\alpha\beta V_2} + \cancel{\alpha^2\beta V_3} - \cancel{\alpha^2\beta V_2}$$

$$V_2 = \alpha + \alpha V_2 - 3\alpha\beta - 3\alpha\beta V_2 + 2\beta + \beta V_2 - \alpha\beta V_3$$

$$+ \alpha^2\beta V_3 - \alpha^2\beta V_2$$

but $V_3 - \alpha V_3 = \alpha + (1-\alpha)(1+\nu^2)$

$$V_3 = \frac{\alpha}{1-\alpha} + 1 + V_2$$

so that

$$V_2 = \cancel{\alpha} + \cancel{\alpha V_2} - \cancel{3\alpha\beta} - \cancel{3\alpha\beta V_2} + \cancel{2\beta} + \cancel{\beta V_2}$$

$$+ (-\alpha\beta + \alpha^2\beta) \left(\frac{\alpha}{1-\alpha} + 1 + V_2 \right) - \alpha^2\beta V_2$$

$$- (\alpha\beta)(1-\alpha)$$

$$\underline{-\alpha^2\beta} \quad \overset{11}{- \alpha\beta(1-\alpha)} - \cancel{\alpha\beta(1-\alpha)V_2} - \cancel{\alpha^2\beta V_2}$$

(C)

or

$$\left[\frac{1}{\alpha} - \alpha + 3\alpha\beta - \beta + \alpha\beta(1-\alpha) + \cancel{\alpha^2\beta} \right] V^2 =$$

$$\underline{\alpha} - \underline{3\alpha\beta} + \underline{2\beta} - \cancel{\alpha^2\beta} - \cancel{\alpha\beta} + \cancel{\alpha^2\beta}$$

||

$$\alpha - 4\alpha\beta + 2\beta$$

$$V_2 = \frac{\alpha - 4\alpha\beta + 2\beta}{1 - \alpha + 4\alpha\beta - \beta}$$

To be index off

$$\Rightarrow \frac{\alpha}{1-\alpha} = \frac{\alpha - 4\alpha + 2}{1 - \alpha + 4\alpha - 1}$$

$$\Rightarrow \frac{\alpha}{1-\alpha} = \frac{5\alpha}{3\alpha}$$

~~$$3\alpha^2 = 5\alpha + 2 - 5\alpha^2 - 2\alpha$$~~

$$8\alpha^2 = 3\alpha + 2$$

$$8\alpha^2 - 3\alpha - 2 = 0$$

$$\alpha = \frac{3 \pm \sqrt{9 + 64}}{16} = \frac{3 \pm \sqrt{73}}{16}$$

to that $V_2 = \frac{\alpha}{1-\alpha} = \frac{\frac{3+\sqrt{73}}{16}}{1 - \frac{3+\sqrt{73}}{16}} = \frac{3+\sqrt{73}}{13 - \sqrt{73}}$

$$\frac{(3+\sqrt{73})(3+\sqrt{73})}{(13-\sqrt{73})(13+\sqrt{73})} = \frac{39 + 16\sqrt{73} + 73}{169 - 73} = \frac{\frac{39}{73}}{\frac{96}{73}} = \frac{28 + 4\sqrt{73}}{24}$$

Param plays $\alpha = \frac{3+\sqrt{73}}{16}$

$$\Rightarrow V_2$$

$$= \frac{7+\sqrt{73}}{6}$$

but $V_3 = \frac{\alpha}{1-\alpha} + 1 + V_2$

$$= V_2 + 1 + V_2 = 2V_2 + 1$$

$$= \frac{7+\sqrt{73}}{3} + \frac{3}{3} = \frac{10+\sqrt{73}}{3} = V_3$$

$$V_4 = 1 + V_2 - \alpha(V_3 - V_2)$$

$$V_4 = V_3 = \frac{10+\sqrt{73}}{3}$$

$$= 1 + \frac{7+\sqrt{73}}{6} - \left(\frac{3+\sqrt{73}}{16} \right) \left(\frac{20+2\sqrt{73}}{6} - \frac{7+\sqrt{73}}{6} \right)$$

$$= 1 + \frac{7+\sqrt{73}}{6} - \left[\frac{39+16\sqrt{73}+73}{6 \cdot 16} - \frac{13+\sqrt{73}}{6} \right]$$

$$= 1 + \frac{7+\sqrt{73}}{6} + \frac{147}{6 \cdot 16} + \frac{16\sqrt{73}}{6 \cdot 16}$$

$$= 1 + \frac{7+\sqrt{73}}{3} = \frac{10+\sqrt{73}}{3} + \frac{1}{2}$$

(e)

for pursuer's game,

$$V_2 = \frac{7 + 2\sqrt{13}}{6} = \frac{15.544004}{6} = \underline{\underline{2.590667}}$$

$$V_3 = V_4 = \frac{10 + 2\sqrt{13}}{3} = \frac{18.544004}{3} = \underline{\underline{6.181335}}$$

so that

$$\begin{matrix} & \frac{13}{6} \\ P^1 & \left(\begin{array}{cc} 0 & \frac{26+2\sqrt{13}}{6} \\ \frac{26+2\sqrt{13}}{6} & \frac{13+2\sqrt{13}}{6} \end{array} \right) \end{matrix}$$

$$\frac{13 + 2\sqrt{13}}{3} = 1 + \frac{10 + 2\sqrt{13}}{3} = 7.$$

$$\text{Value} = \left(\frac{13 + 2\sqrt{13}}{3} \right)^2 \div \frac{39 + 2\sqrt{13}}{6} \text{ normalized by } \frac{39 + 2\sqrt{13}}{6}$$

$$= (7.181335)^2 \div \frac{13}{6} + 7.181335$$

$$= \frac{51.5716}{7.348002}$$

$$= \boxed{5.51686}$$

Value of pursuer's game

Value of evader's game was 2.6