

New Notation:

random variables X, Y, Z, \dots particular values x, y, z
 Distributions $P(X), P(Y), P(Z)$ or $P_i(x_i)$

General distribution $\underline{P_i(x) + P_i(x_i)}$

Random Variable X , continuous range x , discrete $\{\tilde{x}_i\}$

Probability distribution = $P(x) + \tilde{P}_i \leftarrow \text{Prob of } x_i$

$$\text{cond, } P, P_i \geq 0 \quad \sum P(x) + \sum_i P_i = 1$$

Information, two numbers $I_X = \int P(x) \ln P(x) dx$ (continuous info)

$$I_X = \sum_i P_i \ln P_i \quad (\text{discrete info})$$

Note both $I + \tilde{I}$ vanish if their ranges null.

Joint distribution $P(x, y), P(x, y_j), P(x_i, y), P(x_i, y_j)$
consists of 4 functions, hence 4 infos.

Conditional distribution $P_{\gamma}^x = \frac{P(x, y_j)}{P(x)}, \frac{P(x_i, y_j)}{P(x)}$
if x is fixed
 $= P(x_i | y_j), P(y_j | x)$
if x is $-x_j$

Development for continuous case only:

n, Variables $P_{XYZ\dots}$

$$P_{XYZ\dots} = P(x, y, z, \dots)$$

$$P_{XY} = \iiint P_{XYZ\dots} dz d\dots = \iiint P(x, y, z, \dots)$$

Conditional Prob $P_{XY}^Z = \frac{P(x, y, z)}{P(z)}$

Information: $I_{XYZ}^{uv} = \int P_{XYZ}^{uv} \ln P_{XYZ}^{uv} dx dy dz$

$$= \iiint \frac{P(x, y, z|u, v)}{P(u, v)} \ln \frac{P(x, y, z|u, v)}{P(u, v)} dx dy dz$$

Binary
Correlation: $C_{UV|XYZ}^{tw} = I_{UVXYZ}^{tw} - I_{UV}^{tw} - I_{XYZ}^{tw}$

Total Correlation mutual information of groups X, Y, and UV

$$\left\{ \begin{matrix} tw \\ X, Y, Z, UV \end{matrix} \right\} = I_{XYZUV}^{tw} - [I_X^{tw} + I_Y^{tw} + I_U^{tw}]$$

Identity $\left\{ \begin{matrix} tw \\ X, Y \end{matrix} \right\} = C_{X,Y}^{tw}$

bivariate cases, def:

joint distri of X with its self:

$$P(x, x) = \int P(x) \delta(x-x') = P(x) \delta(x-x)$$

$$\Rightarrow I_{xx} = \iint P(x, x') \ln P(x, x') dx dx'$$

$$= \iint P(x) \delta(x-x') \ln P(x) \delta(x-x') dx dx'$$

$$= \int P(x) \ln P(x) \delta(0) dx$$

$$= +\infty$$

discrete $I_{xx} = \sum_{ij} p_i S_{ij} \ln p_i S_{ij} = \sum_i p_i \ln p_i = I_x$

$$\Rightarrow C_{xx} = I_{xx} - I_x - I_x = 0$$

discrete: $\sum_{ij} p_{ij} \cdot p_{ii} S_{ij} \ln \frac{p_{ij} S_{ij}}{p_{ii} p_{jj}}$

attenuate

from conditional def: $P_X^{x'} = \frac{P(x) \delta(x-x')}{P(x)} = \delta(x-x')$

Record

$$C_{xx} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x-x') \ln \delta(x-x') dx dx'$$

$$I_X^{x'} = \int \delta(x-x') \ln \delta(x-x') dx$$

now; $P(u, v) = P(u, v) \delta(v-v')$

$$\Rightarrow I_{vv'} = \infty$$

$$C_{uvv'} = I_{vv'} - I_{vv} - I_{v'}$$

$$= \infty$$

unless $P(v) > 0$.

Degenerate Case:

continuous: $I_{UVV'} = +\infty$ (unless $P(V) = 0$)

in case $P(V) = \delta(V)$ $P(UVV') = P(U,V)\delta(V-V')$
 $= P(u)\delta(v)\delta(v-v')$

$$\Rightarrow \int P(u)\delta(v)\delta(v-v') \ln P(u)\delta(v)\delta(v-v') dv dv$$
$$= \int P(u)\delta(v)\delta(0) \ln P(u)\delta(v)\delta(0) du dv$$
$$= \int P(u)\delta(0)\delta(0) \ln P(u)\delta(0)\delta(0) du$$

$$P(x,y) = \delta(x)\delta(y)$$

$$\Rightarrow \cancel{\int dx}$$

$$C_{xy} = \int P(x,y) \ln \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)} dx = \underline{0}$$

really indeterminate, could define either way
as 0 or $+\infty$

discrete: $I_{XVV'} = P_{ijk} - P_{XVV'} = P_{ij} S_{jk}$

$$\Rightarrow I_{XVV'} = \sum_{ijk} P_{ij} S_{jk} \ln P_{ij} S_{jk}$$

$$= \sum_{ij} P_{ij} \ln P_{ij} = I_{XV}$$

$$i \in \mathcal{P}(X; X') \quad P_{ij} = P_i S_{ij}$$

$$\Rightarrow I_{XX'} = \sum_{ij} P_i S_{ij} \ln P_i S_{ij}$$

$$= \sum_i P_i \ln P_i = I_X$$

$$\Rightarrow C_{XX'} = I_{XX'} - I_X - I_X = -I_X !!$$

conditional method

~~$P_{X'_i} = S_{ij}$~~

$$I_{X'_i}^{X'_i} = \sum_j S_{ij} \ln S_{ij}$$

$$= 0 \quad P_i = P_j$$

from discrete relationships

$$\Rightarrow E_{X'} = 0$$

$$\Rightarrow C_{XX'} = 0 - I_X = -I_X$$

$$\text{finally } = \sum_{ij} \frac{P_{ij} \ln P_{ij}}{P_i P_j}$$

$$= \sum_{ij} P_i S_{ij} \ln \frac{P_i S_{ij}}{P_i P_j} = \sum_{ij} P_i \ln \frac{P_i}{P_i P_j}$$

$$= - \sum_i P_i \ln P_i + 1$$

Boggs: I_{UXX} :

$$P_{UXX} = P_{ij} \delta_{jk}$$

$$I_{UXX} = \sum_{ijk} P_{ij} \delta_{jk} \ln P_{ij} \delta_{jk}$$

$$= \sum_{ij} P_{ij} \ln P_{ij} = I_{UX}$$

Therefore: $\tilde{I}_{UXX} = \tilde{I}_{UX}$ discrete

$$\begin{aligned}\tilde{I}_{UX;X} &= \tilde{I}_{UXX} - \tilde{I}_{UX} - \tilde{I}_X = \\ &= \tilde{I}_{UX} - \tilde{I}_{UX} - \tilde{I}_X = \tilde{I}_X\end{aligned}$$

More generally $\tilde{I}_{UXXXVWW} = \tilde{I}_{UXVW}$
simply suppress doubled indices

Check of Continuous case:

$$P(uvw) = P(u,v)\delta(v-w)$$

$$\Rightarrow I_{uvw} = \int p(uv)\delta(v-w) \ln P(u,v) \delta(v-w) du dv$$
$$= \int p(u,v) \ln p(u,v) \delta(0) du dv$$

So

$$I_{uvv} = \infty$$

$$\tilde{I}_{uvw} = + I_{uv}$$

basic difference of continuous + discrete

$$\Rightarrow C_{uvv} = \infty \quad \tilde{C}_{uv;v} = - I_v$$

$$= \left\{ \begin{array}{c} t \\ uv; vwx, yz \end{array} \right\} = \infty \quad \left\{ \begin{array}{c} t \\ uv, vwx, yz \end{array} \right\}$$

Read !
these results

$$I_{uvwxyz} - I_{uv} - I_{vwx}$$

$$- I_{yz}$$

$$= I_{uvwxyz} - I_{uv} - I_{vwx}$$

$$- I_{yz}$$

$$= - I_{wx} - I_v = C_{v;wx}$$

$$\left\{ \begin{array}{c} t \\ uv, vwx, yz \end{array} \right\} - I_v - C_{v;wx}$$

Transformations

given $P(x, y)$ with $C_{x,y}$

what about $C_{z,y}$ where $z = f(x)$?
 $x = g(z)$

$$P(z, y) = P(g(z), y) Dg(z)$$

$$P(g(z)) = \int P(g(z), y) Dg(z) dy$$

$$C_{z,y} =$$

$$\int P(g(z), y) Dg(z) \ln \frac{P(g(z), y)}{P(g(z)) P(y)} dy dz$$

$$\omega = g(z)$$

$$= \int P(\omega, y) D\omega \ln \frac{P(\omega, y)}{P(\omega) P(y)} dy d\omega$$

$$= C_{x,y}$$

General Theorem:

$$C_{z,y} = C_{x,y}$$

if $z = f(x)$!

Prove more
explicitly!

Very important!
Generalization
of Correlation!

one more,

$$C_{0;XX}$$

$$= I_{0XX} - I_0 - I_{XX}$$

$$\text{in discrete} = I_{0X} - I_{0\phi} - I_X = \underline{\underline{C_{0;X}}}$$

In Continuous Case

$$P_{0XX'} = P(u, x) \delta(x-x')$$

$$C_{0;XX'} = P(x, x') = \delta(x-x') P(x)$$

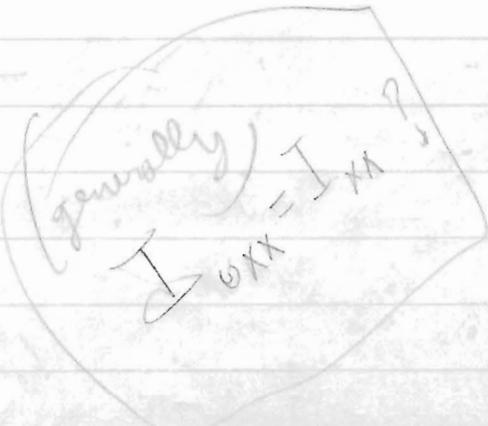
$$\iint P(u, x) \delta(x-x') \ln \frac{P(u, x) \delta(x-x')}{P(u) \delta(x-x') P(x)}$$

$$= \iiint P(u, x) \delta(x-x') \ln \frac{P(u, x)}{P(u) P(x)} dx dx' du$$

$$= \iint P(u, x) \ln \frac{P(u, x)}{P(u) P(x)} = C_{0;X}$$

So $\lim_{\text{record.}} C_{0;XX} = C_{0;X}$

both cont + discrete!



Work out carefully section on transformations

i.e. given $P(x, y, z)$

change

Transform to $P'(u, v, w)$

$$u = u(x, y, z)$$

$$v = v(x, y, z)$$

$$w = w(x, y, z)$$

what about $I' = I + \text{Exp} \ln I$

Should contain fact that if only one variable
changed \rightarrow Correlation unchanged
in manner which depends only on its value.

leads to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$P'(u, v, w) = P(x, y, z) J \left(\frac{x, y, z}{u, v, w} \right)$$

$$dudvdz = \left| J \left(\frac{x, y, z}{u, v, w} \right) \right| dv dw$$

$$\Rightarrow P'(u, v, w) dv dw = P(x, y, z) dxdydz$$