

PRINCETON UNIVERSITY  
Palmer Physical Laboratory

MINUTES OF ACCELERATOR MEETING HELD APRIL 15, 1953

Application of Accelerating Field

Dr. Shoemaker presented some comparative calculations made by himself and Dr. Mozley on the power requirements of acceleration using (a) a traveling wave system (WCE-1-53) and (b) a conventional system using ferrite.

He calculated the power requirements in the two cases as

$$(a) \quad P = \frac{f C_T V_T^2}{\pi^2 N^2 \cos^2 \theta}$$

f = frequency of R.F.

$$(b) \quad P = \frac{10^9 V_T^2}{2 \pi^2 f \mu l \ln \frac{r_1}{r_0} \cos^2 \theta}$$

V<sub>T</sub> = voltage gain per revolution.

where both expressions were for non-resonant systems and so represented the stored energy supplied and thrown away in each cycle (through class B tubes) (a) was calculated on the basis of a large number of drift tubes (and not a conducting wall tube) to apply the voltage and so included no inherent resistive dissipation. The terms mean: C<sub>T</sub> - total capacity, N - number of wavelengths per machine period, θ - phase of particles with respect to the r.f. wave, l - the total length of ferrite, r<sub>0</sub> r<sub>1</sub> the internal and external radii of cylindrical ferrite blocks.

If the following values were chosen:

rise time = 0.2 sec , injection energy = 4 Mev, final energy = 10 Bev,  $\frac{V_T}{\cos \theta} = 10^5$  volts, l = 10 meters, r<sub>0</sub> = 6 cms. to keep B<sub>max</sub> for the ferrite = 500 gauss,  $\ln \frac{r_1}{r_0} = 1$ , f<sub>max</sub> = 10 Mc/s , N = 10 μ<sub>max</sub> = 300,

$$\text{but } \int (u\delta^* + \delta u^* + \delta^* \delta) \ln \left[ (u^* u + u\delta^* + \delta u^* + \delta^* \delta) \left( \frac{u^* u}{u^* u} \right) \right]$$

$$= \int u^* u \left[ \frac{\delta^*}{u^*} + \frac{\delta}{u} + \frac{\delta^* \delta}{u^* u} \right] \left[ \ln \left( 1 + \frac{\delta^*}{u^*} + \frac{\delta}{u} + \frac{\delta^* \delta}{u^* u} \right) + \ln u^* u \right]$$

//

$$\int u^* u \frac{\delta^* \delta}{u^*}$$

$$\text{Let } \alpha = \frac{\delta}{u}$$

$$= \int u^* u [\alpha^* + \alpha + \alpha^* \alpha] \ln (1 + \alpha^* + \alpha + \alpha^* \alpha)$$

$$+ \int u^* u [\alpha^* + \alpha + \alpha^* \alpha] \ln u^* u$$

∴ from (2)

$$(1 + \alpha)^{-2}$$

$$\begin{aligned} \Delta I &= \int u^* u [1 + \alpha^* + \alpha + \alpha^* \alpha] \ln [1 + \alpha^* + \alpha + \alpha^* \alpha] \\ &\quad + \int u^* u [\alpha^* + \alpha + \alpha^* \alpha] \ln u^* u \end{aligned}$$

$$1 \quad \phi(k) = \alpha \int_{-\infty}^{\infty} e^{ikx} \psi(x) dx$$

for function  $u(\xi)$ ,

$$I_\xi = \int_{-\infty}^{\infty} u^* u(\xi) \ln u^* u(\xi) d\xi$$

$$\begin{aligned} \text{Let } u'(\xi) &= u(\xi) + \Delta u(\xi) \\ &= u + \delta \end{aligned} \quad u'^* = u^* + \delta^*$$

$$\text{then } I'_\xi = \int (u^* + \delta^*)(u + \delta) \ln (u^* + \delta^*)(u + \delta) d\xi$$

$$= \int (u^* u + u \delta^* + \delta u^* + \delta^* \delta) \ln (u^* u + \delta u^* + \delta^* u + \delta^* \delta) d\xi$$

$$= \int u^* u \ln (u^* u + \delta u^* + u \delta^* + \delta^* \delta) + \int (\delta u^* + u \delta^* + \delta^* \delta) \ln (u^* u + \dots)$$

$$\Delta I = I'_\xi - I_\xi = \int u^* u \ln \left[ \frac{u^* u + \delta u^* + u \delta^* + \delta^* \delta}{u^* u} \right] d\xi + \int (\delta u^* + u \delta^* + \delta^* \delta) \ln (u^* u + \dots)$$

$$= \int u^* u \ln \left( \frac{u^* u + \delta u^* + u \delta^* + \delta^* \delta}{u^* u} \right) d\xi$$

$$= \int u^* u \ln \left( 1 + \frac{\delta}{u} + \frac{\delta^*}{u^*} + \frac{\delta^* \delta}{u^* u} \right) d\xi + \int (\delta u^* + u \delta^* + \delta^* \delta) \ln \left( u^* u + u \delta^* + \delta^* u + \delta^* \delta \right)$$

4

$$\begin{aligned} &= \left[ n \left( 2 \ln \frac{4r_x^2}{2\pi} + \frac{x^2}{2r_x^2} - 2 \right) e^{-\frac{x^2}{8r_x^2}} \right] \\ &\quad + \frac{2}{r_x \sqrt{2\pi}} \left( n \frac{1}{2\pi r_x^2} - \frac{x^2}{r_x^2} \right) e^{-\frac{x^2}{2r_x^2}} \right] \pi(x) dx + \text{c.c.} \end{aligned}$$

working out:

(5)

$$\Delta P = \phi \omega^* + \omega \phi^* + \omega^* \omega$$

$$(\Delta P)^2 = \cancel{\phi^2 \omega^{*2} + \phi^* \omega^* \omega + \omega^* \omega^2} + \phi^*$$

$$\phi^2 \omega^{*2} + \omega^2 \phi^{*2} + (\omega^* \omega)^2$$

$$+ 2 \phi \phi^* \omega^* \omega + 2 \phi \omega^{*2} \omega + 2 \phi^* \omega^* \omega^2$$

To second term only:

$$\int \Delta P \ln \phi dP = \int [\phi \omega^* + \omega \phi^* + \omega^* \omega] \ln \phi dP$$
  
~~$$+ \frac{1}{2 \phi^* \phi} [\phi^2 \omega^{*2} + \omega^2 \phi^{*2} + (\omega^* \omega)^2 + 2 \phi \phi^* \omega^* \omega + 2 \phi \omega^{*2} \omega + 2 \phi^* \omega^* \omega^2]$$~~  
$$+ \frac{1}{2 \phi^* \phi} [\phi^2 \omega^{*2} + \omega^2 \phi^{*2} + (\omega^* \omega)^2 + 2 \phi \phi^* \omega^* \omega + 2 \phi \omega^{*2} \omega + 2 \phi^* \omega^* \omega^2] dP$$

Therefore, to second order:

$$\int \Delta P \ln \phi dP = \int [\phi \omega^* + \omega \phi^* + \omega^* \omega] \ln \phi dP$$
  
$$+ \int \frac{1}{2 \phi^* \phi} [\phi^2 \omega^{*2} + \omega^2 \phi^{*2} + 2 \phi \phi^* \omega^* \omega] dP$$

$$= \underbrace{\int \ln \phi \phi^* [\phi \omega^* + \omega \phi^*] dP}_{\text{but}} + \int [\ln \phi \phi^*] \omega^* \omega dP$$

~~$$\ln \phi + \frac{1}{2} \int \frac{\phi^2}{\phi} \omega^{*2} + \frac{\phi^2}{\phi} \omega^2 + 2 \omega^* \omega dP$$~~

(6)

$$\Delta \int \rho_{\ln P} dK = \underbrace{\int \ln \phi^* [\phi \omega^* + \omega \phi^*] dK}_{\text{first order } \omega} + \underbrace{\int \ln \phi^* [\omega^* \omega] dK + \frac{1}{2} \int \left( \frac{\phi^2}{\phi} \omega^{*2} + \frac{\phi^2}{\phi} \omega^2 + 2\omega^* \omega \right) dK}_{\text{second order } \omega}$$

Now, assume Gaussian i.e.  $\phi(x) = a e^{-bx^2}$  (real)

$$\phi(K) = c e^{-dK^2}$$

really only one free parameter  
say b  
then we determine

$$\Rightarrow \phi^* \phi = c^2 e^{-2dK^2}$$

$$\ln \phi^* \phi = \ln c^2 - 2dK^2$$

$$\ln \phi^* \phi = \ln a^2 - 2bx^2$$

$$\Rightarrow \Delta \int \rho_{\ln P} dK =$$

$$\int [ \ln c^2 - 2dK^2 ] [ x e^{-dK^2} (\omega^* + \omega) ] dK$$

first order

$$+ \int [ \ln c^2 - 2dK^2 ] [ \omega^* \omega ] dK + \frac{1}{2} \int_{-\infty}^{\infty} (\omega^* + \omega)^2 dK$$

but,  $\int \phi^* \phi = 1 \Rightarrow \int \phi^* \omega + \omega^* \phi + \omega^* \omega = 0$

now

Let

$\zeta(x)$  be variation of  $x$

$$\Rightarrow \omega(x) = n \int e^{ixz} \eta(z) dz$$

$$\omega^* = n \int e^{-izx} \eta^*(z) dz$$

$$\begin{aligned}
 & \int [\phi^* \phi + \phi^* \delta\phi + \phi \delta\phi^* + \delta\phi^* \delta\phi] \ln [\phi^* \phi + \phi^* \delta\phi + \phi \delta\phi^* + \delta\phi^* \delta\phi] dK \\
 & - \int \phi^* \phi \ln \phi^* \phi dK \\
 \Rightarrow & \int \phi^* \phi \ln \left[ \frac{\phi^* \phi + \phi^* \delta\phi + \phi \delta\phi^* + \delta\phi^* \delta\phi}{\phi^* \phi} \right] dK \\
 & + \int [\phi^* \delta\phi + \phi \delta\phi^* + \delta\phi^* \delta\phi] \ln [\phi^* \phi + \phi^* \delta\phi + \phi \delta\phi^* + \delta\phi^* \delta\phi] dK \\
 = & \left( 1 + \frac{\delta\phi}{\phi} + \frac{\delta\phi^*}{\phi^*} + \frac{\delta\phi^* \delta\phi}{\phi^* \phi} \right)
 \end{aligned}$$

$$\Delta I_K = \int \phi^* \phi \ln \left[ 1 + \frac{\Delta\phi}{\phi} + \frac{\Delta\phi^*}{\phi^*} + \frac{\Delta\phi^* \Delta\phi}{\phi^* \phi} \right] + \int [\phi^* \Delta\phi + \phi \Delta\phi^* + \Delta\phi^* \Delta\phi] \ln [\phi^* \phi + \phi^* \Delta\phi + \phi \Delta\phi^* + \Delta\phi^* \Delta\phi] dK$$

Proof that Variation vanishes:

1

We have ⑥ that to first order in  $\omega(k) = \Delta \phi(k) = n \int e^{ikx} \frac{\partial \phi(x)}{\partial x}$

$$\psi(x) = ce^{-dx^2} \quad \phi(k) = ce^{-dk^2}$$

$$\Delta I_R = \int [ln c^2 - 2dk^2] \left[ 2ce^{-dk^2} (\omega^* + \omega) \right] dk$$

$$\text{where } \omega = n \int e^{ikx} n(x) dx$$

∴

$$\Delta I_K = \int [ln c^2 - 2dk^2] \left[ 2ce^{-dk^2} \frac{n}{n \int e^{ikx} n(x) dx} \right] dk + \text{c.c.}$$

interchanging integ

$$\Rightarrow \Delta I_K = \iint [ln c^2 - 2dk^2] \left[ 2ce^{-dk^2} \frac{ne^{ikx}}{dk n(x)} \right] \cancel{dx} + \text{c.c.}$$

$$= \int f^+ \cancel{n(x) dx} + \cancel{f^-}$$

$$f = n \int e^{ikx} 2ce^{-dk^2 + ikx} dk$$

$$g = - \int 2dK^2 2c^2 e^{-dk^2 + ikx} dk + \text{c.c.}$$

Evaluation of  $f, g$

2

need

$$\int_{-\infty}^{\infty} e^{-dk^2 + i\kappa k} dk$$

$$\downarrow \quad a = \sqrt{d} \quad b = -ix$$

$$\therefore \int = \frac{\sqrt{\pi}}{d^{3/2}} e^{-\frac{x^2}{4d}} \quad b^2 = -x^2$$

$$\text{and } \int_{-\infty}^{\infty} k^2 e^{-dk^2 + i\kappa k} dk$$

$$a = \sqrt{d} \quad b = -ix$$

$$\int = \frac{\sqrt{\pi}}{d^{3/2}} \left( \frac{1}{2} - \frac{x^2}{4d} \right) e^{-\frac{x^2}{4d}}$$

$$\therefore f = 2mc \ln c^2 \frac{\sqrt{\pi}}{d} e^{-\frac{x^2}{4d}}$$

$$\begin{aligned} g &= -4dmc \frac{\sqrt{\pi}}{d^{3/2}} \left( \frac{1}{2} - \frac{x^2}{4d} \right) e^{-\frac{x^2}{4d}} \\ &= -4mc \frac{\sqrt{\pi}}{d} \left( \frac{1}{2} - \frac{x^2}{4d} \right) e^{-\frac{x^2}{4d}} \end{aligned}$$

$$\text{and } f+g = \frac{\sqrt{\pi}}{d} e^{-\frac{x^2}{4d}} mc \left( 2\ln c^2 + \frac{x^2}{d} - 2 \right)$$

so that

$$\Delta I_K = \int_{-\infty}^{\infty} mc \frac{\sqrt{\pi}}{d} \left( 2\ln c^2 + \frac{x^2}{d} - 2 \right) e^{-\frac{x^2}{4d}} n(x) dx + \text{C.C.}$$

first order

[3]

Now,

$$\Delta I_x = \int [ln a^2 - 2bx^2] [2ae^{-bx^2}] N(x) dx + C.C.$$

$$\therefore \Delta I_k + \Delta I_x$$

$$= \int_{-\infty}^{\infty} \left[ \left[ nc\sqrt{\frac{\pi}{d}} \left( 2\ln c^2 + \frac{x^2}{d} - 2 \right) e^{-\frac{x^2}{d}} \right] + \left[ (ln a^2 - 2bx^2) 2ae^{-bx^2} \right] \right] N(x) dx + C.C.$$

$$\text{but, } \psi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}} \quad \phi(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2\sigma_k^2}}$$

$$\text{i.e. } a = \frac{1}{\sigma_x \sqrt{2\pi}} \quad b = \frac{1}{2\sigma_x^2} \quad \rightarrow \Rightarrow \sigma_k = \frac{1}{2\sigma_x}$$

and  $\sigma_x \sigma_k = \frac{1}{2}$

$$c = \frac{1}{\sigma_k \sqrt{2\pi}} = \frac{2\sigma_k}{\sqrt{2\pi}}$$

$$d = \frac{1}{2\sigma_k^2} = \frac{1}{2} 4\sigma_x^2 = 2\sigma_x^2 \quad n =$$

$$\Delta I_k + \Delta I_x = \int_{-\infty}^{\infty} \frac{2\sigma_x}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2\sigma_k^2}} \left( 2 \ln \frac{4\sigma_x^2}{2\pi} + \frac{x^2}{2\sigma_x^2} - 2 \right) e^{-\frac{x^2}{8\sigma_x^2}}$$

$$+ \left[ \left( \ln \frac{1}{2\pi\sigma_x^2} - \frac{x^2}{\sigma_x^2} \right) \frac{2}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}} \right] N(x) dx + C.C.$$

$$\Delta(\rho \ln \rho) = (\bar{\rho} \ln \bar{\rho})(\Delta \rho) + \frac{1}{2\rho} (\Delta \rho)^2 - \frac{1}{6\rho^2} (\Delta \rho)^3 \quad \text{... (4)}$$

now, if  $\Delta \rho$  such that  $S(\bar{\rho} + \Delta \rho) = 1 = \int_{\bar{\rho}}^{\bar{\rho} + \Delta \rho} S \, d\rho = 1 + \int_{\Delta \rho} \Rightarrow \int_{\Delta \rho} = 0$

$$\begin{aligned} \therefore \int_{\bar{\rho}}^{\bar{\rho} + \Delta \rho} \Delta(\rho \ln \rho) &= \int_{\Delta \rho}^{\Delta \rho} + \int_{\bar{\rho}}^{\bar{\rho} + \Delta \rho} \bar{\rho} \ln \bar{\rho} + \int_{\Delta \rho}^{\Delta \rho} \left( \frac{1}{2\rho} (\Delta \rho)^2 - \int_{\Delta \rho}^{\Delta \rho} \frac{1}{6\rho^2} (\Delta \rho)^3 \right) \\ &= \int_{\Delta \rho}^{\Delta \rho} \bar{\rho} \ln \bar{\rho} + \int_{\Delta \rho}^{\Delta \rho} \frac{1}{2\rho} (\Delta \rho)^2 - \int_{\Delta \rho}^{\Delta \rho} \frac{1}{6\rho^2} (\Delta \rho)^3 \end{aligned}$$

Now, if  $\rho = \phi^* \phi$  and  $\phi = \phi + \Delta \phi$   
 $\Rightarrow \phi^* = \phi^* + (\Delta \phi)^*$

$$\Rightarrow (\phi^* \phi)' = \phi^* \phi + \phi(\Delta \phi)^* + \Delta \phi \phi^* + \Delta \phi(\Delta \phi)^*$$

$$\Rightarrow \Delta \phi^* \phi = \phi w^* + \omega \phi^* + \omega^* \omega = \Delta \rho$$

Let  $\omega = \Delta \phi$   
for short

$$\begin{aligned} \therefore \int_{\bar{\rho}}^{\bar{\rho} + \Delta \rho} \Delta(\rho \ln \rho) \, dt &= \int_{-\infty}^{\infty} [\phi w^* + \omega \phi^* + \omega^* \omega] \ln \phi^* \phi \, dt \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{2\phi^* \phi} [\phi w^* + \omega \phi^* + \omega^* \omega]^2 \, dt \\ &\quad - \int_{-\infty}^{\infty} \frac{1}{6(\phi^* \phi)^2} [\phi w^* + \omega \phi^* + \omega^* \omega]^3 \, dt \end{aligned}$$

(3)

Another method

$$f = x \ln x$$

$$\frac{\partial f}{\partial x} = \frac{x}{x} + \ln x = 1 + \ln x = f'$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{x} \cdot f'' \quad \frac{\partial^3 f}{\partial x^3} = -\frac{1}{x^2} = f'''$$

$$\text{Taylor series} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$\Rightarrow x \ln x = alna + (1+lna)(x-a) + \frac{1}{2a} (x-a)^2 + \frac{1}{2a^2} (x-a)^3$$

$$x = a + \Delta a$$

$$\therefore \text{and } \Delta(alna)$$

$$= x \ln x - alna$$

$$\Rightarrow \Delta alna = (1+lna)\Delta a + \frac{1}{2a} (\Delta a)^2 - \frac{1}{2a^2} (\Delta a)^3$$

$$e^{-\frac{bx^2}{2}} = e^{-\frac{b}{z^2}} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{\left(\frac{b}{z^2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{b^n}{z^{2n} n!}$$

$$\frac{-b(x-a)^2}{2}$$

$$\phi(p) = \int_{-\infty}^{\infty} e^{ipx} \psi(x) dx$$

$$I = \int_{-\infty}^{\infty} \phi^* \phi(p) \ln \phi^* \phi(p) dp + \int \psi^* \psi(x) \ln \psi^* \psi(x) dx$$

to find  $\psi(x)$ , subject to  $\int \psi^* \psi dx = 1$   
 for which I stationary.

$$\psi' = \psi + \varepsilon \eta$$

$$\phi(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{ikx} \psi(x) dx$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{ikx} \psi dx + \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{ikx} \varepsilon \eta(x) dx$$

$$= \phi(k) + \varepsilon \xi(k)$$

$$\phi^* \phi'(k) = \phi^* \phi(k) + 2\varepsilon (\phi^* \xi + \xi^* \phi) + \varepsilon^2 \xi'(k)$$

$$\xi(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{ikx} \eta(x) dx$$

$$\frac{d^2r}{dt^2} = -\frac{\partial V}{\partial r}; \quad \frac{d^2z}{dt^2} = -\frac{\partial V}{\partial z}$$

-2-

$$V = \pm \left\{ ar - bz + \frac{1}{2}c(r^2 - z^2) + drz + \frac{1}{3}e(r^3 - 3rz^2) - \frac{1}{3}f(3r^2z - z^3) \right. \\ \left. + \frac{1}{4}h(r^4 - 6r^2z^2 + z^4) - g(r^3z - rz^3) \right\}$$

up and a coder had been promised next week. The equation set up was:

$$\frac{d^2r}{dt^2} = \pm \left| a_0 + a_1 r + b_1 z + a_2 (r^2 - z^2) + b_2 (2rz) + a_3 (r^3 - 3rz^2) \right. \\ \left. + b_3 (z^3 - 3zr^2) \right|$$

where  $t$  is distance along the central equilibrium orbit and there was a similar equation for the  $z$  direction. The problem was to be coded so that each of the coefficients could be set constant, by-passed, or given small random deviations from an average value for the different sectors. Non-focusing guide sections could be accounted for separately. The integration was to be performed by the Runge-Kutta method. The maximum values of  $r$  and  $z$  were to be stored through a run and a run would be stopped if either  $r$  or  $z$  exceeded a predetermined value or after a predetermined number of revolutions.

The tentative program was:

- (1) To find closed orbits
- (2) To study resonant blow-ups due to variations in  $n$  and the effect of the non-linear terms on these.
- (3) To study driving terms.
- (4) To find the effect of small non-linear terms randomly distributed among the sectors.
- (5) To test the cumulative effect of the guide sectors, where  $n = 0$ .

No account was to be taken of fringing fields in this calculation. He agreed that it would not complicate the calculation very much to include fifth-order non-linear terms if desired as they could be fed in as a separate block.

$$\textcircled{2} \quad I'_T = \int_{-\infty}^{\infty} p(k) \ln p(k) dk + \int_{-\infty}^{\infty} p(x) \ln p(x) dx$$

$$I'_k = \int_{-\infty}^{\infty} \left[ \phi^* \phi(k) + 2\varepsilon (\phi_j^* + \zeta^* \phi) + \varepsilon^2 \zeta^2(k) \right] \ln \left[ \phi^* \phi(k) + 2\varepsilon (\phi_j^* + \zeta^* \phi) + \varepsilon^2 \zeta^2(k) \right] dk$$

$$\delta F(x) = F' \delta x + \frac{F''(\delta x)^2}{2!} \dots$$

$$\phi(k) = \int_{-\infty}^{\infty} e^{ikx} \psi(x) dx$$

$$f(x) = e^{-x} \psi(x)$$

$$\delta f(x) = f'(x)$$

$$\delta \phi(k) = e^{ikx} \psi'(x)$$

$$\boxed{\delta \phi(k) = \int_{-\infty}^{\infty} e^{ikx} \delta \psi(x) dx}$$

$$\delta \left[ \int \phi^* \phi \ln \phi^* \phi dk \right]$$

$$(\phi^* \delta \phi^*) (\phi + \delta \phi) \ln [\phi^* \phi]$$

$$\phi^* \phi + \phi \delta \phi^* + \phi^* \delta \phi + \delta \phi^* \delta \phi$$

$$I' = \int [\phi^* \phi + \phi \delta \phi^* + \phi^* \delta \phi + \delta \phi^* \delta \phi] \ln \text{dotta} dk$$

$$I' - I$$