

New Notation:

Original total ^{joint} prob distrib: $P_{\alpha \dots \beta \dots \delta, u \dots v}$

Unconditional distrib over any subset:

(1)

$$P_{\alpha \dots \beta \dots \delta} = \sum_{u \dots v} P_{\alpha \dots \beta \dots \delta, u \dots v}$$

(Partial)

total conditional

$$\text{Prob } P_{\alpha \dots \beta, u \dots v}^{\delta \dots \delta} = \text{Cond Prob of } \alpha \dots \beta, u \dots v \text{ given } \delta \dots \delta$$

(2)

$$P_{\alpha \dots \beta, u \dots v}^{\delta \dots \delta} = \frac{P_{\alpha \dots \beta, \delta \dots \delta, u \dots v}}{\sum_{\delta \dots \delta} P_{\alpha \dots \beta, \delta \dots \delta, u \dots v}} = \frac{P_{\alpha \dots \beta, \delta \dots \delta, u \dots v}}{P_{\delta \dots \delta}}$$

total
conditional

Partial Conditional

$$P_{\alpha \dots \beta}^{\delta \dots \delta} = \sum_{u \dots v} P_{\alpha \dots \beta, u \dots v}^{\delta \dots \delta} = \sum_{u \dots v} \frac{P_{\alpha \dots \beta, \delta \dots \delta, u \dots v}}{P_{\delta \dots \delta}}$$

(3)

$$= \frac{P_{\alpha \dots \beta, \delta \dots \delta}}{P_{\delta \dots \delta}}$$

Note, Expected Partial Conditional = $\sum_{\delta \dots \delta} P_{\delta \dots \delta} P_{\alpha \dots \beta}^{\delta \dots \delta} =$

$$1 = \sum_{\delta \dots \delta} P_{\delta \dots \delta} \frac{P_{\alpha \dots \beta, \delta \dots \delta}}{P_{\delta \dots \delta}} = \sum_{\delta \dots \delta} P_{\alpha \dots \beta, \delta \dots \delta} = P_{\alpha \dots \beta}$$

so, useful theorem:

(4) $\sum_{\delta \dots \delta} P_{\gamma \dots \delta} P_{\alpha \dots \beta}^{\gamma \dots \delta} = P_{\gamma \dots \beta}$

or more generally,

$$\sum_{\delta \dots \delta} P_{\gamma \dots \delta} P_{\alpha \dots \beta}^{\gamma \dots \delta \alpha \dots \nu} = \sum_{\delta \dots \delta} P_{\gamma \dots \delta} \frac{P_{\alpha \dots \beta \gamma \dots \nu}}{P_{\gamma \dots \delta \alpha \dots \nu}}$$

$$= \sum_{\delta \dots \delta} P_{\gamma \dots \delta}^{\alpha \dots \beta \gamma \dots \nu}$$

but $P_{\gamma \dots \beta}^{\alpha \dots \nu}$ ~~$= \frac{P_{\alpha \dots \beta \gamma \dots \nu}}{P_{\gamma \dots \nu}}$~~ $= \frac{\sum_{\delta \dots \delta} P_{\alpha \dots \beta \gamma \dots \nu}}{\sum_{\delta \dots \delta} P_{\gamma \dots \delta \alpha \dots \nu}}$

$$\frac{P_{\gamma \dots \delta}}{P_{\gamma \dots \delta \alpha \dots \nu}} =$$

$$P_{\alpha \dots \beta}^{\gamma \dots \delta} = \frac{P_{\gamma \dots \delta \alpha \dots \nu}}{P_{\gamma \dots \delta}}$$

$\Rightarrow P_{\gamma \dots \delta}^{\gamma \dots \delta} \Rightarrow P_{\gamma \dots \delta \alpha \dots \nu} = P_{\gamma \dots \delta} P_{\alpha \dots \nu}^{\gamma \dots \delta}$ (5)

consider $P_{\gamma \dots \delta} P_{\alpha \dots \beta u \dots v}$

$$= P_{\gamma \dots \delta} \frac{P_{\alpha \dots \beta, \gamma \dots \delta u \dots v}}{P_{\gamma \dots \delta u \dots v}}$$

but by ⑤ $P_{\gamma \dots \delta u \dots v} = P_{\gamma \dots \delta} P_{u \dots v}$

~~$$\therefore = P_{\gamma \dots \delta} \frac{P_{\alpha \dots \beta \gamma \dots \delta u \dots v}}{P_{\gamma \dots \delta} P_{u \dots v}}$$~~

$$= \frac{P_{\alpha \dots \beta \gamma \dots \delta u \dots v}}{P_{\gamma \dots \delta} P_{u \dots v}}$$

$$\sum_j P_{\gamma}^{\alpha} P_{\delta}^{\beta} = \sum_j \frac{P_{\alpha \gamma}}{P_{\gamma}} \frac{P_{\beta \delta}}{P_{\delta}} = \frac{1}{P_{\gamma}} \sum_j \frac{P_{\alpha \gamma} P_{\beta \delta}}{P_{\gamma} P_{\delta}}$$

$$\text{is } P_{\alpha \delta} = \sum_j \frac{P_{\alpha \gamma} P_{\beta \delta}}{P_{\gamma}} ?$$

$$P_{\alpha \delta} = \sum_j P_{\alpha \gamma} P_{\beta \delta}$$

try to prove
false

$$P_{\alpha \gamma} P_{\beta \delta} = \sum_j \frac{P_{\alpha \gamma} P_{\beta \delta}}{P_{\gamma}}$$

$$P_{\alpha} P_{\delta}^r = \frac{P_{\alpha} P_{\delta} \alpha}{P_{\delta}} = P_{\alpha}^r P_{\delta} \alpha$$

So lemma:

$$P_{\alpha} P_{\delta}^r = P_{\alpha}^r P_{\delta} \alpha$$

(an upper + lower
part can be
interchanged)

Now,

~~$P_{\delta} P_{\delta}^r = P_{\delta} \alpha$~~

$$P_{\delta} P_{\delta}^r = P_{\delta} \alpha \quad \text{since } P_{\delta}^r = \frac{P_{\delta} \alpha}{P_{\delta}}$$

$$\Rightarrow \sum_j P_{\delta} P_{\delta}^r = \sum_j P_{\delta} \alpha = P_{\delta}$$

also $P_{\delta} P_{\alpha}^{r\delta} = P_{\delta} \frac{P_{\delta} \alpha}{P_{\delta} \alpha}$

$$= P_{\delta} \frac{P_{\delta} \alpha}{P_{\delta} \alpha} = P_{\delta} \frac{P_{\delta} \alpha}{P_{\alpha}^r P_{\delta}^r} = \frac{P_{\delta} \alpha}{P_{\alpha}^r P_{\delta}^r}$$

but $\frac{P_{\delta} \alpha}{P_{\delta}} = P_{\delta}^r$

define Partial Conditional Information:

$$I_{(\alpha \dots \beta)}^{\gamma \dots \delta} = \sum_{\gamma \dots \delta} p_{\alpha \dots \beta}^{\gamma \dots \delta} \ln p_{\alpha \dots \beta}^{\gamma \dots \delta}$$

define

Partial

Correlation indices



Correlation index $C_{(\alpha \dots \beta, \gamma \dots \delta)} = I_{(\alpha \dots \beta, \gamma \dots \delta)} - I_{\alpha \dots \beta} - I_{\gamma \dots \delta}$

$$C_{\alpha \dots \beta, \gamma \dots \delta}$$

define conditional
correlation

$$C_{(\alpha \dots \beta, \gamma \dots \delta)}^{u \dots v} = I_{(\alpha \dots \beta, \gamma \dots \delta)}^{u \dots v} - I_{\alpha \dots \beta}^{u \dots v} - I_{\gamma \dots \delta}^{u \dots v}$$

(function of $u \dots v$)

try to show $\sum I_{\alpha \dots \beta}^{u \dots v} C_{\alpha \dots \beta, \gamma \dots \delta}^{u \dots v} = C_{\alpha \dots \beta, \gamma \dots \delta}$

$$\sum_{\gamma \dots \delta} p_{\gamma \dots \delta} I_{(\alpha \dots \beta)}^{\gamma \dots \delta} = \sum_{\gamma \dots \delta} p_{\gamma \dots \delta} \underbrace{p_{\alpha \dots \beta}}_{\alpha \dots \beta} \ln p_{\alpha \dots \beta}^{\gamma \dots \delta}$$

first) $I_{\alpha \dots \beta}^{\gamma \dots \delta} = \sum_{\alpha \dots \beta} p_{\alpha \dots \beta} \ln p_{\alpha \dots \beta}^{\gamma \dots \delta}$

$$= \sum_{\alpha \dots \beta} \frac{p_{\alpha \dots \beta \gamma \dots \delta}}{p_{\gamma \dots \delta}} \ln \left(\frac{p_{\alpha \dots \beta \gamma \dots \delta}}{p_{\gamma \dots \delta}} \right)$$

$$= \frac{1}{p_{\gamma \dots \delta}} \sum_{\alpha \dots \beta} p_{\alpha \dots \beta \gamma \dots \delta} \ln p_{\alpha \dots \beta \gamma \dots \delta}$$

$$- \frac{1}{p_{\gamma \dots \delta}} \sum_{\alpha \dots \beta} p_{\alpha \dots \beta \gamma \dots \delta} \ln p_{\gamma \dots \delta}$$

$$= \frac{1}{p_{\gamma \dots \delta}} \sum_{\alpha \dots \beta} p_{\alpha \dots \beta \gamma \dots \delta} \ln p_{\alpha \dots \beta \gamma \dots \delta} - \ln p_{\gamma \dots \delta}$$

$$\Rightarrow p_{\gamma \dots \delta} I_{(\alpha \dots \beta)}^{\gamma \dots \delta} = \sum_{\alpha \dots \beta} p_{\alpha \dots \beta \gamma \dots \delta} \ln p_{\alpha \dots \beta \gamma \dots \delta} - p_{\gamma \dots \delta} \ln p_{\gamma \dots \delta}$$

hence $\sum_{\gamma \dots \delta} p_{\gamma \dots \delta} I_{\alpha \dots \beta}^{\gamma \dots \delta} = \sum_{\alpha \dots \beta \gamma \dots \delta} p_{\alpha \dots \beta \gamma \dots \delta} \ln p_{\alpha \dots \beta \gamma \dots \delta} - \sum_{\gamma \dots \delta} p_{\gamma \dots \delta} \ln p_{\gamma \dots \delta}$

$$= I_{(\alpha \dots \beta \gamma \dots \delta)} - I_{(\gamma \dots \delta)}$$

(6)

so that

$$\sum_{\alpha\mu\nu} P_{\alpha\mu\nu} C_{\alpha\mu\beta\gamma\dots\delta}^{\alpha\mu\nu} = \sum_{\alpha\mu\nu} P_{\alpha\mu\nu} I_{\alpha\mu\beta\gamma\dots\delta}^{\alpha\mu\nu}$$

$$- \sum_{\alpha\mu\nu} P_{\alpha\mu\nu} I_{\alpha\mu\beta}^{\alpha\mu\nu} - \sum_{\alpha\mu\nu} P_{\alpha\mu\nu} I_{\beta\dots\delta}^{\alpha\mu\nu}$$

Previous
using theorem

$$= I_{\alpha\mu\beta\gamma\dots\delta\alpha\mu\nu} - I_{\alpha\mu\nu}$$

$$- [I_{\alpha\mu\beta\alpha\mu\nu} - I_{\alpha\mu\nu}]$$

$$- [I_{\beta\dots\delta\alpha\mu\nu} - I_{\alpha\mu\nu}]$$

$$= I_{\alpha\mu\beta\gamma\dots\delta\alpha\mu\nu} + I_{\alpha\mu\nu} - I_{\alpha\mu\beta\alpha\nu} - I_{\beta\dots\delta\alpha\mu\nu}$$

But, $C_{\alpha\mu\beta\gamma\dots\delta} = I_{\alpha\mu\beta\gamma\dots\delta} - I_{\alpha\mu\beta} - I_{\beta\dots\delta}$

Digression:

$$\text{Expected information change upon knowing } \beta = I_{\alpha}^{\beta} - I_{\alpha}$$

$$\Rightarrow \text{Expected change} = \sum_{\beta} P_{\beta} I_{\alpha}^{\beta} - I_{\alpha}$$

$$= I_{\alpha\beta} - I_{\beta} - I_{\alpha} \quad \text{OK.}$$

More generally,

inform change of $\alpha \dots \beta$ knowing $\gamma \dots \delta$

$$= I_{\alpha \dots \beta}^{\gamma \dots \delta} - I_{\alpha \dots \beta}$$

$$\text{Expected} = \sum_{\gamma \dots \delta} P_{\gamma \dots \delta} I_{\alpha \dots \beta}^{\gamma \dots \delta} - I_{\alpha \dots \beta}$$

$$= I_{\alpha \dots \beta \gamma \dots \delta} - I_{\gamma \dots \delta} - I_{\alpha \dots \beta} \quad \text{DEF}$$

(Proof of general change)

We must have $I_{\alpha \dots \beta \gamma \dots \delta} - I_{\gamma \dots \delta} = I_{\alpha \dots \beta}$?
Obviously false!

$$C_{\alpha\beta\gamma\delta} = I_{\alpha\beta\gamma\delta} - I_{\alpha\beta} - I_{\gamma\delta}$$

\Rightarrow

Back to formula

$$\sum_{\mu \dots \nu} P_{\mu \dots \nu} C_{\alpha \beta \gamma \delta; \mu \dots \nu}^{\alpha \beta \gamma \delta} = I_{\alpha \beta \gamma \delta \mu \nu} + I_{\mu \nu} - I_{\alpha \beta \mu \nu} - I_{\gamma \delta \mu \nu}$$

note $\cancel{I_{\alpha \beta \gamma \delta \mu \nu} - I_{\alpha \beta \mu \nu} - I_{\gamma \delta \mu \nu}} = C_{\alpha \beta \mu \nu; \gamma \delta}$

first) $C_{\alpha \beta \gamma \delta; \mu \nu} = I_{\alpha \beta \gamma \delta \mu \nu} - I_{\alpha \beta \gamma \delta} - I_{\mu \nu}$

$\Rightarrow \cancel{I_{\mu \nu}} = I$

fuck even further

$$= \cancel{C_{\alpha \beta \gamma \delta \mu \nu} + I_{\alpha \beta \gamma \delta}} = C_{\alpha \beta \gamma \delta \mu \nu} + I_{\alpha \beta \gamma \delta}$$

$$= I_{\alpha \beta \gamma \delta \mu \nu} - I_{\mu \nu}$$

$$- (I_{\alpha \beta \mu \nu} - I_{\mu \nu}) C_{\alpha \beta \mu \nu} + I_{\alpha \beta}$$

$$- (I_{\gamma \delta \mu \nu} - I_{\mu \nu}) C_{\gamma \delta \mu \nu} + I_{\gamma \delta}$$

$$= C_{\alpha \beta \gamma \delta \mu \nu} + I_{\alpha \beta \gamma \delta} - (C_{\alpha \beta \mu \nu} + I_{\alpha \beta}) - (C_{\gamma \delta \mu \nu} - I_{\gamma \delta})$$

$$= C_{\alpha \beta \gamma \delta \mu \nu} - C_{\alpha \beta \mu \nu} - C_{\gamma \delta \mu \nu} + I_{\alpha \beta \gamma \delta} - I_{\alpha \beta} - I_{\gamma \delta}$$

$$C_{\alpha \beta \gamma \delta}$$

so that

$$\sum_{u,v} P_{u,v} C_{\alpha \beta \gamma \delta}^{u,v} = C_{\alpha \beta \delta;uv} - C_{\alpha \beta;uv} - C_{\delta;uv} + C_{\alpha \beta \delta}$$

Now need to show that

$$C_{\alpha \beta \delta;uv} = C_{\alpha \beta;uv} + C_{\delta;uv}$$

! false!

$$\Rightarrow I_{\alpha \beta \delta;uv} - I_{\alpha \beta \delta} - I_{uv} = I_{\alpha \beta \delta;uv} - I_{\alpha \beta;uv} - I_{\delta;uv} + I_{\alpha \beta;uv} - I_{\alpha \beta} - I_{uv}$$



Let's build up slowly is $C_{\alpha \beta \delta} = C_{\alpha \beta; \delta}$?

$$C_{\alpha \beta \delta} = I_{\alpha \beta \delta} - I_{\alpha \beta} - I_{\beta \delta}$$

$$(i) I_{\alpha} + I_{\beta \delta} = I_{\alpha \beta} + I_{\delta} ?$$

$$I_{\alpha} + I_{\beta \delta} = \sum_{\alpha} P_{\alpha} \ln P_{\alpha} + \sum_{\beta \delta} P_{\beta \delta} \ln P_{\beta \delta}$$

$$= \sum_{\alpha \beta \delta} P_{\alpha \beta \delta} \ln P_{\alpha} + \sum_{\alpha \beta \delta} P_{\alpha \beta \delta} \ln P_{\beta \delta}$$

$$= \sum_{\alpha \beta \delta} P_{\alpha \beta \delta} \ln (P_{\alpha} P_{\beta \delta})$$

so, what about
 $P_{\alpha} P_{\beta \delta}$