

§ 5. Discussion of a spin measurement example.

We shall conclude this chapter with a discussion of an instructive example of Bohm¹. Bohm considers the measurement of the z component of the angular momentum of an atom, whose total angular momentum is $\frac{\hbar}{2}$, which is brought about by a Stern-Gerlach experiment.

¹ Bohm [1] pg 593

The measurement is brought about by passing an atomic beam through an inhomogeneous magnetic field which has the effect of giving the particle a momentum which is directed up or down depending upon whether the spin was up or down.

The measurement is treated as impulsive, so that during the time that the atom passes through the field the Hamiltonian is taken to be simply the interaction:

$$(5.1) \quad H_I = u(\vec{s} \cdot \vec{B}) \quad , \quad u = -\frac{e\hbar}{2mc}$$

where \vec{B} is the magnetic field and \vec{s} the spin operator for the atom. The particle is presumed to pass through a region of the field where the field is in the z direction, so that during the time of transit the field is approximately $B_z \approx B_0 + z B'_0$ ($B_0 = (B_z)_{z=0}$ and $B'_0 = (\frac{\partial B_z}{\partial z})_{z=0}$) and hence the interaction is approximately:

$$(5.2) \quad H'_I \approx u(B_0 + z B'_0) S_z$$

where S_z denotes the operator for the z component of the spin.

This ~~is~~ assumed that the ^{state of the} atom is, just prior to entry into the field, in a wave packet of the form:

$$(5.3) \quad \psi_0 = f_0(z) (c_+ v_+ + c_- v_-)$$

where v_+ and v_- are the spin functions for $s_z = 1$ and -1 respectively. Solving the Schrödinger equation for the ~~interaction~~ Hamiltonian (5.2) and initial condition (5.3) yields the state for a later time t :

$$(5.4) \quad \psi = f(z) [c_+ e^{-i\mu(\hbar_0 + 3\hbar'_0)t/\hbar} v_+ + c_- e^{+i\mu(\hbar_0 + 3\hbar'_0)t/\hbar} v_-]$$

²This time is
strictly speaking
not well-defined
but the results
do not depend
critically on its
value.

Therefore, if Δt is the time that it takes the atom to traverse the field,² each component of the wave packet has been multiplied by a phase factor $e^{\pm i\mu(\hbar_0 + 3\hbar'_0)\Delta t/\hbar}$, i.e. has ~~had~~ had its mean momentum changed in the z direction by an amount $\pm 3\hbar'_0 \mu \Delta t$, depending upon the spin direction. ~~Thus~~ Thus the initial wave packet is split into a superposition of two packets, one with mean momentum $+3\hbar'_0 \mu t$ and spin up, and the other with spin down and mean z -momentum $-3\hbar'_0 \mu t$. If the interaction (5.2) has therefore served to correlate the spin with the momentum in the z -direction. ~~Each~~ ^{two} of these packets ~~move~~ the resulting superposition now move in opposite z directions, so that after a short time they become widely separated (provided that the momentum changes $\pm 3\hbar'_0 \mu t$ are large compared to the momentum spread of the original packet), and the z coordinate is itself then correlated with the spin -- representing the "apparatus" coordinate in this case. The Stern-Gerlach apparatus therefore splits an incoming wave packet into a superposition of two diverging packets, corresponding to the two spin values.

We take this opportunity to caution against a certain viewpoint which can lead to difficulties. This is the idea that, after an apparatus has interacted with a system, in "actuality" one or another of the elements of the resultant superposition described by the composite state function has been realized to the exclusion of the rest, the existing one simply being unknown to an external observer (i.e. that instead of the superposition there is a genuine mixture). This position must be erroneous since there is always the possibility for the external observer to make use of interference properties between the elements of the superposition.

On the present example, for instance, it is in principle possible to deflect the two beams back toward one another with magnetic fields and recombine them in an ^{another} inhomogeneous field, which duplicates the first, in such a manner that the original spin state (before entering the ~~first~~ apparatus) is restored.³ This would not be possible if the original Stern-Gerlach apparatus performed the function of converting the original ~~beam~~ ^{wave packet} into a non-interfering mixture of packets for the two spin cases. Therefore the position that after the atom has passed through the ~~one~~ ^{other} inhomogeneous field it is "really" in one or the other beam with the corresponding spin, ~~even though~~ although we are ignorant of which one, is incorrect.

³ As pointed out by Bohm
Loc. cit. pg -

After two systems have interacted and become correlated it is true that marginal expectations for subsystem operators can be calculated correctly when the composite system is represented by a certain non-interfering mixture of states. Thus if the composite system state is $\Psi^{S_1 + S_2} = \sum_i d_i \phi_i^{S_2} N_i^{S_2}$ where the $\{N_i\}$ are orthogonal, then for purposes of calculating the expectations of operators on S_1 the state $\Psi^{S_1 + S_2}$ is equivalent to the non-interfering mixture of states $\phi_i^{S_2} N_i^{S_2}$ weighted by $p_i = d_i^* d_i$) and one can take the picture that one or another of the cases $\phi_i^{S_2} N_i^{S_2}$ has been realized to the exclusion of the rest, with probabilities p_i . However, this

⁴ Ψ

See III-61.

representation by ~~mixture~~ a mixture must be regarded as only a mathematical artifice which, although useful in many cases, is an incomplete description because it ignores phase relations

~~between all objects~~ between the separate elements which actually exist, and which become important in any interactions which involve more than just a subsystem.

In the present example the "composite system" is ~~assumed~~ made of the "subsystems" spin valve (object-system) and z-coordinate (apparatus), and the superposition of the two diverging wave packets is the state after interaction. It is only correct to regard this state as a mixture so long as ~~only~~ any contemplated future interactions or measurements will involve only the spin valve or only the z coordinate, but not both simultaneously. As we saw, ~~the~~ phase relations between the two packets

are present and become important when they are deflected back and recombined in another inhomogeneous field -- a process involving the spin values and z -coordinates simultaneously.

P It is therefore improper to attribute any less validity or "reality" to any element of a superposition than any other element, due to this ever-present possibility of obtaining interference effects between the elements. All elements of a superposition must be regarded as simultaneously existing.

At this time we should like to add a few remarks concerning the notion of transition probabilities in quantum mechanics. Often one considers a system, with Hamiltonian H and stationary states $\{\phi_i\}$, to be perturbed for a time by time-dependent addition to the Hamiltonian $H_I(t)$. Then under the action of the perturbed Hamiltonian $H' = H + H_I(t)$ the states $\{\phi_i\}$ are generally no longer stationary but change after time t into new states $\psi_i(t)$:

$$(5.5) \quad \phi_i \rightarrow \psi_i(t) = \sum_j (\phi_i, \psi_i(t)) \phi_j = \sum_j d_{ij}(t) \phi_j$$

~~the~~ which can be represented as a superposition of the old stationary states with time-dependent coefficients $d_{ij}(t)$.

If at time t a measurement with eigenstates ϕ_i is performed, such as an energy measurement (whose operator is the original H), then ~~the~~ according to the probabilistic interpretation the probability for finding the state ϕ_j , given that the state was originally ϕ_i , is $P_{ij}(t) = |d_{ij}(t)|^2$

~~and they are~~ \rightarrow The quantities $|d_{ij}(t)|^2$ are often referred to as transition probabilities. In this case, however, the name is a misnomer, since it carries the connotation that the original state ϕ_i is transformed into a mixture (of the ϕ_j weighted by $P_{ij}(t)$), and gives the erroneous impression that the quantum formalism ~~itself~~ implies the existence of "quantum jumps" (stochastic processes) independent of acts of observation. This is incorrect since there is still a pure state $\sum d_{ij}(t) \phi_j$ with phase relations between the ϕ_j , and expectations of operators other than the energy must be calculated from the superposition and not the mixture.

There is another case however, the one usually encountered in fact, where the transition probability concept is somewhat more justified. This is the case where the perturbation is due to interaction of the system (S_1) with another system S_2 , and not simply a time dependence of S_1 's Hamiltonian as in the case just considered. In this situation the interaction produces a composite system state, for which there are in general no independent subsystem states. ~~that~~ However, as we have seen, for purposes of calculating expectations of operators on S_1 alone we can regard S_1 as being represented by a certain mixture. According to this picture the states of subsystem S_1 are gradually converted into mixtures by the interaction with S_2 and the concept of transition probability makes some sense. Of course it must be remembered that this picture is only justified so long as further measurements on S_1 alone are contemplated, and any attempt to make a simultaneous determination in S_1 and S_2 involves the composite state where interference properties may be important.

An example is a hydrogen atom interacting with the electromagnetic field. After a time of interaction we can picture the atom as being in a mixture of its states, so long as we consider ~~any~~ future measurements on the atom^{only}. But actually ~~in~~ the state of the atom is dependent upon (correlated with) the state of the field, and some process involving both atom and field could conceivably depend on interference effects between the states of the ~~and~~ alleged mixture. With these restrictions, however, the concept of transition probability is quite useful and justified.