

# Selection of Max. Exp. Alternative.

(1)

Subj  
distrib  $P_i$  cost \$

Payoff  $\frac{1}{Q}$  or 0

$Q$  distributed randomly

$\mathbb{E} P_i (P_i, Q)$

$\mathbb{E} P_i$  Given that chooses  $i^{\text{th}}$  alt to bet on

$$= \frac{P_i}{Q} - 1 \quad (\text{Simpler, omit the Payment of } \$1)$$

$$\text{other} = \frac{P_i}{Q_i}$$

$Q_i$ 's index and uniformly distributed over  $[0, 1]$

assume chooses always  $\max_{\substack{\text{(after } Q_i \\ \text{announced)} \\ \text{of course}}} i \frac{P_i}{Q_i}$

$$\text{So Payoff of } (\{P_i\}, \{Q_i\}) = \max_i \frac{P_i}{Q_i}$$

$H(\vec{P}, \vec{Q})$

Now Wish to

compute  $\mathbb{E} P(H(\vec{P}, \vec{Q}))$  under assumption that  $Q_i$  uniform over  $[0, 1]$

So consider  $\text{Prob} \left\{ \frac{P_i}{Q_i} \in \{\xi + d\xi\} \text{ and } \frac{P_j}{Q_j} \in \{\text{all } j \neq i\} \right\}$

$$\text{Prob} \frac{P_i}{Q_i} \in \{\xi + d\xi\}$$

$$\Rightarrow Q_i \in \left[ \frac{P_i}{\xi}, \frac{P_i}{\xi + d\xi} \right]$$

$$\frac{d\left(\frac{P_i}{\xi}\right)}{d\xi} = -\frac{P_i}{\xi^2} d\xi$$

~~$$dQ_i = \frac{P_i}{\xi^2} d\xi$$~~

but  ~~$dQ_i$~~   $\text{Prob}(dQ_i \text{ ie } Q_i \in Q + dQ) = \frac{dQ}{1-\xi}$

$$\Rightarrow \text{Prob } P_i(\xi) = \frac{P_i}{\xi^2(1-\xi)} d\xi$$

note,  $Q$  from  $\xi$  to 1

$$\Rightarrow \frac{P_i}{Q_i} \text{ from } \frac{P_i}{\xi} \text{ to } P_i \quad (\text{ie } \xi \text{ from } P_i \text{ to } \frac{P_i}{Q_i})$$

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as check is  $\int_{P_i}^{P_i/\varepsilon} P_i(\xi) d\xi = 1$  ?

$$\begin{aligned} \int_{P_i}^{P_i/\varepsilon} \frac{P_i}{\xi^2(1-\varepsilon)} d\xi &= \frac{P_i}{1-\varepsilon} \left[ -\frac{1}{\xi} \right]_{P_i}^{P_i/\varepsilon} \\ &= \frac{P_i}{1-\varepsilon} \left( -\frac{\varepsilon}{P_i} + \frac{1}{P_i} \right) \\ &= \left( \frac{P_i}{1-\varepsilon} \right) \left( \frac{1-\varepsilon}{P_i} \right) = 1 \end{aligned}$$

so O.K

and established that  $P_i(\xi) d\xi = \frac{P_i}{\xi^2(1-\varepsilon)} d\xi = \text{Prob} \left\{ \frac{P_i}{Q_i} \in [\xi, \xi + d\xi] \right\}$

with range  $P_i \leq \xi \leq \frac{P_i}{\varepsilon}$

define  $R_j(\eta) = \text{Prob} \frac{P_i}{Q_j} \leq \eta$  (ie cumulative function)

$$n \geq \frac{p_j}{\varepsilon}$$

Then  $R_j(n) = \begin{cases} 1 & n \geq p_j \\ \int_{p_j}^n P_j(\xi) d\xi & n \geq p_j \text{ and } n \leq \frac{p_j}{\varepsilon} \\ 0 & n \leq p_j \end{cases}$

now  $\int_{p_j}^n P_j(\xi) d\xi = \int_{p_j}^n \frac{P_j}{\xi^2(1-\varepsilon)} d\xi$

$$= \frac{P_j}{(1-\varepsilon)} \left[ -\frac{1}{\xi} \right]_{p_j}^n = \frac{P_j}{(1-\varepsilon)} \left[ -\frac{1}{n} + \frac{1}{p_j} \right]$$

$$= \frac{P_j}{(1-\varepsilon)} \left[ \frac{n - p_j}{n p_j} \right]$$

$$= \frac{n - p_j}{n(1-\varepsilon)}$$

⑤

So we have that

$$R_j(n) = \begin{cases} 1 & \text{for } n \geq \frac{P_j}{\varepsilon} \\ \frac{n - P_j}{n(1-\varepsilon)} & \text{for } P_j \leq n \leq \frac{P_j}{\varepsilon} \\ 0 & \text{for } n \leq P_j \end{cases}$$

$= \left(\frac{1}{1-\varepsilon}\right) - \frac{P_j}{n(1-\varepsilon)}$

which is the Probability that  $\frac{P_j}{Q_j}$  shall be  $\leq n$

(note  $P_j$  fixed,  $Q_j$  uniform over  $[\varepsilon, 1]$ )

and Finally define  $S_K(u) = \text{Prob} \left\{ \frac{P_K}{Q_K} \in [u, u+du] \right\}$   
and  $\frac{P_j}{Q_j} \leq u \text{ all } j \neq K$

so that  $S_K(u) = P_K(u)du \times \prod_{j \neq K} R_j(u)$

Then define  $E_K = \int S_K(u) du$  = Expectation from the  $k^{th}$  alternative

and finally  $\underline{EXP} = \sum_K E_K$  is the total Expectation for the process

(and hope it looks like  $\sum_i P_i \ln P_i$ )

$$\text{so: } \underline{EXP} = \sum_K \int_0^\infty M P_K(u) \prod_{j \neq K} R_j(u) du$$

$$= \sum_K \int_0^\infty u \left[ \begin{array}{ll} \frac{P_K}{u^2(1-\varepsilon)}, & P_K \leq u \leq \frac{P_K}{\varepsilon} \\ 0, & \text{otherwise} \end{array} \right] \prod_{j \neq K} \left[ \begin{array}{ll} 1, & u \geq \frac{P_j}{\varepsilon} \\ \frac{u - P_j}{u(1-\varepsilon)}, & P_j \leq u \leq \frac{P_j}{\varepsilon} \\ 0, & u \leq P_j \end{array} \right] \times du$$

$$= \sum_K \int_0^\infty \left[ \begin{array}{ll} \frac{P_K}{u(1-\varepsilon)} & ; P_K \leq u \leq \frac{P_K}{\varepsilon} \\ 0 & ; \text{otherwise} \end{array} \right] \prod_{j \neq K} \left[ \begin{array}{ll} 1 & ; u \geq \frac{P_j}{\varepsilon} \\ \frac{1}{1-\varepsilon} - \frac{P_j}{u(1-\varepsilon)} & ; P_j \leq u \leq \frac{P_j}{\varepsilon} \\ 0 & ; u \leq P_j \end{array} \right]$$

What about  $\frac{P_K}{u} \prod_{j \neq K} \left(1 - \frac{P_j}{u}\right)$

i.e.  $\frac{P_K}{u(1-\varepsilon)} \prod_{j \neq K} \left(\frac{u - P_j}{u(1-\varepsilon)}\right)$

$$= \frac{1}{[u(1-\varepsilon)]^N} \cdot P_K \prod_{j \neq K} (u - P_j)$$

$$= \frac{1}{u(1-\varepsilon)^N} \frac{P_K}{u - P_K} \prod_{\text{all } j} (u - P_j)$$

now  $\sum_K = \frac{\prod (u - P_j)}{[u(1-\varepsilon)]^N} \sum_K \frac{P_K}{u - P_K}$

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$$\text{now } \sum_K \frac{P_K}{u - P_K} = \frac{\sum_K P_K \prod_{j \neq K} (u - p_j)}{\prod_K (u - P_K)}$$

back to where before

what is  $\int \frac{\prod (u - p_j)}{[u(1-\varepsilon)]^n} \sum_K \frac{P_K}{u - P_K} du$ ?

$$\ln(u - P_K) \Big|_{P_K}^{P_K/\varepsilon} = \ln\left(\frac{P_K}{\varepsilon} - P_K\right)$$

except for

would appear to give log function

$$\sum_K \frac{-\ln P_K}{P_K} \left(\frac{1}{\varepsilon} - 1\right)$$

would

i.e.  $\sum_K \int \frac{P_K}{u - P_K} dP \ln P$

try another method:

Absol Exp on  $i^{\text{th}}$  alt given one chooses it

$$= \int_{\varepsilon}^1 \frac{P_i}{Q_i} dQ = P_i \ln\left(\frac{1}{\varepsilon}\right)$$

Absolute

Prob that  $i^{\text{th}}$  alt will be chosen

$$= \int_0^\infty P_i(\xi) R_i(\xi) d\xi$$

(is this  $\propto \log P_i$ ? )  
(might be)

$$\log \frac{P_i}{Q_i} = \log P_i - \log Q_i$$

if distrib on  $Q$  is uniform

$$d \ln Q = \frac{1}{Q} dQ \Rightarrow dQ = Q d(\ln Q)$$

(P)

What is Prob  $\left[ \log P_i - \underbrace{\log Q_i}_{u} \right] \leq \{ \ ? \}$

$$\text{let } u_i = \log Q_i$$

$$\Rightarrow P(u) du = \frac{e^u}{(1-e)} du \quad du = \frac{dQ}{Q} = \frac{1}{e^u} dQ$$

$$e^u dQ = e^u du$$

$u$  from 0 to  $\ln \epsilon$

$$\text{check } \frac{1}{1-e} \int_0^{\ln \epsilon} e^u du$$

$$u = \ln Q \quad e^u = Q$$

$$du = \frac{dQ}{Q}$$

$$\Rightarrow dQ = e^u du$$

$$\text{Prob}\{dQ\} = dQ = \cancel{e^u du}$$

$$\text{Prob } du = \text{prob} \left[ \frac{dQ}{e^u} \right]$$

$$= \frac{1}{e^u} dQ$$

$$= e^{-u}$$

is  $Q$  uniform  $[\varepsilon, 1]$  what is  $\ln Q$  ?

$$\stackrel{u}{\text{if}} \quad Q = e^u$$

$$\text{Prob}\{\underline{dQ}\} = \frac{dQ}{1-\varepsilon}$$

$$du = \frac{1}{Q} dQ = e^{-u} dQ$$

$$\text{i.e. } \underline{dQ} = e^u du$$

so that  $P(u) du = \frac{1}{1-\varepsilon} e^u du$

$$Q[\varepsilon, 1]$$

$$\Rightarrow M[\ln \varepsilon, 0]$$

$$\int P(u) e^u du \Big|_0^{\ln \varepsilon}$$

neg

Correct  $P(u) du = \frac{1}{1-\varepsilon} e^u du$

$$\text{check} \int_{\ln \varepsilon}^0 \frac{1}{1-\varepsilon} e^u du = \left[ \frac{1}{1-\varepsilon} e^u \right]_{\ln \varepsilon}^0 = \frac{1}{1-\varepsilon} \left( 1 - e^{\ln \varepsilon} \right)$$

$$u_i = \ln Q_i$$

so  $P(u_i) = \frac{1}{1-\varepsilon} e^u du$  from  $\int_{\ln \varepsilon}^0$

Quesiton Given  $N$  random variables  $X_i$  uniformly over intervals  $[a_i, b_i]$  what is distib of  $\max X_i$ ? (12)

$$\text{so } \text{Prob} \left\{ \ln P_i - \ln Q_i \leq n \right\}$$

$$= \text{Prob} \left\{ \ln Q_i \geq \ln P_i - n \right\}$$

$$= \text{Prob} \left\{ U_i \geq \ln P_i - n \right\}$$

$$= \int_{\ln P_i - n}^0 \frac{1}{1-\varepsilon} e^u du$$

$$\text{if } \ln P_i - n \leq 0 \\ \Rightarrow \boxed{\ln P_i \leq n}$$

$$= \frac{1}{1-\varepsilon} \left[ 1 - e^{\ln P_i - n} \right]$$

$$\text{and } \ln P_i - n \geq 0$$

$$= \frac{1}{(1-\varepsilon)} \left[ 1 - P_i e^{-n} \right]$$

$$\Rightarrow \ln \left( \frac{P_i}{\varepsilon} \right) \geq n$$

$$\begin{aligned} \left( \text{Prob} \frac{P}{Q} \geq \right) &= \int_{\varepsilon}^{\frac{P}{Q}} \frac{P}{Q} dQ = P \ln \left( \frac{P}{\varepsilon Q} \right) \left( \frac{1}{1-\varepsilon} \right) \\ \Rightarrow \frac{P}{Q} &\geq Q \end{aligned}$$