

§ 3

Remarks on the choice of square amplitude measure:

While at first sight an artificial choice, and one which seems to give rise to the danger of begging the question, a little reflection shows that ~~this choice of measure~~ is not so arbitrary as it appears, but is the only reasonable choice for the purpose of making statistical deductions.

Let us consider the search for a general scheme for assigning a measure to the elements of a superposition of orthogonal states $\sum a_i \psi_i$. We require then a positive function m of the complex coefficients ~~of the states~~ of the elements of the superposition, so that $m(a_i)$ shall be the measure assigned to the element ψ_i . In order that this ~~the~~ general scheme shall be unambiguous we must ^{first} require that the states themselves always be normalized, so that we can distinguish the coefficients from the states ~~themselves~~. However, we can still only determine the coefficients ^{in distinction to the states,} up to an arbitrary phase factor, ^{and hence} ~~so that~~ the function m must be a function of the amplitudes of the coefficients alone, (i.e. $m(a_i) = m(\sqrt{a_i})$), in order to avoid ambiguities.

Problem

First there is an ambiguity about just what the coefficient is, called a phase factor. The requirement that the states are normalized, which we always assume, then determines this coefficient to an arbitrary phase factor. However since we can always associate the phase factor with the state, rather than with the coefficient if we choose, the function $m(a_i)$ must be a function of the amplitude of ϕ_i alone, i.e. $m(a_i) = m(\alpha_i \phi_i)$, so that our scheme shall be unambiguous.

If we now impose the additivity requirement that if we regard a subset of the superposition, say $\sum_{i=1}^n a_i \phi_i$, as a single element $\alpha \phi'$:

$$(3.1) \quad \alpha \phi' = \sum_{i=1}^n a_i \phi_i$$

then the measure assigned to ϕ' shall be the sum of the measures assigned to the ϕ_i (i from 1 to n):

$$(3.2) \quad m(\alpha) = \sum_i m(a_i)$$

thus we have already restricted the choice of m to the square amplitude alone. ($m(a_i) = a_i^* a_i$, apart from a multiplicative constant.)

To see this we note that the normality of ϕ' requires that $|\alpha| = \sqrt{\sum_{i=1}^n a_i^* a_i}$. From our remarks upon the dependence of m upon the amplitude alone, we replace the a_i by their amplitudes $u_i = |a_i|$.

(3.2) then requires that

$$(3.3) \quad m(\alpha) = m(\sqrt{\sum a_i^2} \alpha_i) = m(\sqrt{\sum u_i^2}) = \sum m(u_i) = \sum \underline{m}(\underline{\sqrt{u_i^2}})$$

Defining a new function ~~of the~~ $g(x)$:

$$(3.4) \quad g(x) = m(\sqrt{x})$$

~~(3.3) requires that we see that~~ we see that (3.3) requires that

$$(3.5) \quad g(\sum u_i^2) = \sum g(u_i^2)$$

so that g is restricted to be linear and necessarily has the form:

$$(3.6) \quad g(x) = cx \quad (c \text{ constant})$$

Therefore $\bullet g(x^2) = cx^2 = m(\sqrt{x^2}) = m(x)$

and we have deduced that m is restricted to the form

$$(3.7) \quad \cancel{m(\alpha_i)} \quad m(\alpha_i) = m(u_i) = cu_i^2 = c \alpha_i^2 \alpha_i$$

and we have ^{shown} ~~seen~~ that the only choice of measure consistent with our additivity requirement is the square amplitude measure, apart from an arbitrary multiplicative constant which may be fixed, if desired, by normalization requirements. (the requirement that the total measure be unity implies that this constant is 1.)

The situation here is fully analogous to that of classical statistical mechanics, where one puts a measure on trajectories of systems in the phase space by placing a measure on the phase space itself, and then making assertions which hold (either ergodicity, or quasi-ergodicity) for "almost all" trajectories, a notion which here also depends upon the choice of measure, which is taken to be the Lebesgue measure on the phase space.

One could, of course, ~~even though~~ contradict the statements of classical statistical mechanics by choosing a measure ~~such~~ for which only ~~of these~~ the exceptional trajectories had non-zero measure. Nevertheless the choice of Lebesgue measure on the phase space can be justified by the fact that it is the only choice ~~from which~~ for which "^{the} conservation of probability" (Liouville theorem) holds, and hence the only ~~possible~~ choice which makes possible ^{any} reasonable statistical deductions at all.

In our case we wish to make statements about "trajectories" of ~~ideal~~ observers. This case however, ^{for us} a trajectory is constantly branching (transforming from state to superposition) with each measurement, and a requirement analogous to the "conservation of probability" ^{in the classical case we demand} ~~is~~ that the measure assigned to a trajectory at one time shall equal the sum of the measures of its branches at a later time. This is precisely the additivity requirement which we imposed and which leads uniquely to the choice of the square amplitude measure. ~~Our procedure~~ therefore is quite as justified as that of classical statistical mechanics.

~~(give reference to Kacchini)~~

(i.e. ~~$\lim_{n \rightarrow \infty} (ae^{it})^n = a(e^{it})$~~ until it is ~~red~~)

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(such as ergodicity, quasi ergodicity, etc.).¹ This notion of "almost all" depends here also upon the choice of measure, which is in this case taken to be the Lebesgue measure on the phase space.

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To have a requirement analogous to the "conservation of probability" in the classical case, we demand that the measure assigned to a trajectory at one time shall equal the sum of the measures of its separate branches at a later time.