

Proof: for short, write $P(x_i, y_j) = P_{ij}$ $P(x_i) = P_i$ $P(y_j) = P_j$

Let $Q_{ij} = \frac{P_{ij}}{P_i P_j}$ so that $P_{ij} = Q_{ij} P_i P_j$

$$\begin{aligned} \text{then } \{X, Y\} &= \exp \left[\ln P_{ij} - \ln P_i - \ln P_j \right] \\ &= \exp \left[\ln \frac{P_{ij}}{P_i P_j} \right] = \exp [\ln Q_{ij}] \\ &= \sum_{ij} P_i P_j Q_{ij} \ln Q_{ij} \end{aligned}$$

but, making use of inequality $X \ln X \geq 1 - X$ except $x=1$
we have

$$\cancel{\sum_{ij} P_i P_j Q_{ij} \ln Q_{ij}} > \sum_{ij} P_i P_j (1 - Q_{ij}) \quad \text{unless } Q_{ij}=1$$

$$= \sum_{ij} P_i P_j Q_{ij} \ln Q_{ij} > \sum_{ij} P_i P_j - \sum_{ij} Q_{ij} P_i P_j = 0$$

unless for every i, j $P_i P_j = 0$ or $Q_{ij} = 0$ or 1

$$\Rightarrow P_{ij} = 0 \text{ or } Q_{ij} = 1$$

$$\Rightarrow P_{ij} = P_i P_j \quad \text{all } i, j$$

$\therefore \{X, Y\} > 0$ unless X, Y indep. QED