

Transformations :

$$\text{given } P(x, y, z)$$

$$\begin{aligned} \text{Let } & u = u(x, y, z) \\ & v = v(x, y, z) \\ & w = w(x, y, z) \end{aligned}$$

$$\text{Note } \int P(x, y, z) J_{uvw}^{x, y, z} du dv dw$$

$$= \int P(x, y, z) dxdydz = 1$$

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$$

$$\text{Then } P(u, v, w) = P(x, y, z) J_{uvw}^{x, y, z}$$

$$\Rightarrow I' = \int P(u, v, w) \ln P(u, v, w) du dv dw$$

$$= \int P(u, v, w) \ln P(x, y, z) J_{uvw}^{x, y, z} du dv dw$$

$$= \int P(x, y, z) J_{uvw}^{x, y, z} \ln P(x, y, z) du dv dw$$

$$+ \int P(u, v, w) \ln J_{uvw}^{x, y, z} du dv dw$$

$$P(x, y, z) J_{uvw}^{x, y, z} du dv dw + x dy dz$$

$$= \int P(x, y, z) \ln P(x, y, z) dx dy dz + \int P(x, y, z) \ln J_{uvw}^{x, y, z} dx dy dz$$

$$I' = I + \exp \left\{ \ln J_{uvw}^{x, y, z} \right\}$$

Basic info Transformation law!

Single Variable Change \Rightarrow

$$u = u(x) \quad J^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{du}{dx}$$

$$\Rightarrow J = \frac{dx}{du}$$

$$\Rightarrow P(u, y, z) = \frac{dx}{du} P(x, y, z)$$

$$\text{Proof: } \int P(u, y, z) du = \int \frac{dx}{du} P(x, y, z) du \\ = P(x, y)$$

$$\Rightarrow I_{uyz} = I_{xyz} + \exp \left\{ \ln \frac{dx}{du} \right\}$$

$$\text{Similarly } I_{uy} = I_{xy} + \exp \ln \frac{dx}{du}$$

$$I_u = I_x + \exp \ln \frac{dx}{du}$$

$$\Rightarrow C_{uyz} = I_{uyz} - I_u - I_{yz}$$

$$= I_{xyz} + \cancel{\exp \ln J} - I_x - \cancel{\exp \ln J} - I_{yz} \\ = C_{xyz} !$$

$$\text{conditional } I_{uyz}^t = I_{xyz}^t + \exp^t \left\{ \ln \frac{dx}{du} \right\}$$

$$\begin{aligned}
 \text{Also, } C_{VY,Z} &= I_{VYZ} - I_{VY} - I_Z \\
 &= I_{XYZ} + \cancel{Exploit} - I_{XY} - \cancel{Exploit} - I_Z \\
 &= C_{XY;Z}
 \end{aligned}$$

More Generally

$$\left\{ \begin{matrix} t \\ UV, WY, Z \end{matrix} \right\} = \cancel{\circlearrowleft}$$

$$I_{UVWYZ}^t - I_{UV}^t - I_{WY}^t - I_Z^t$$

$$= I_{XVWYZ}^t + \cancel{Exploit} - I_{XV}^t - \cancel{Exploit} - I_{WY}^t - I_Z^t$$

$$= \left\{ \begin{matrix} t \\ XV, WY, Z \end{matrix} \right\} = \{XV, WY, Z\}^t$$

i.e. in the { } symbols any random variable may be replaced by any other which is functionally dependent upon it.

In fact, any Group (within commas) can be so replaced.

General Invariance theorem!

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Proof of position correlation invariance:

Let joint distribution of particles 1, 2 be

$$P(\vec{x}_1, \vec{x}_2) = P(x_1, y_1, z_1, x_2, y_2, z_2)$$

$$\Rightarrow \{x_1, y_1, z_1\} = \{x_2, y_2, z_2\} = \{x_1 y_1 z_1, x_2 y_2 z_2\}$$

$$= \{u_1 v_1 w_1, u_2 v_2 w_2\} \text{ by general Invariance theorem!}$$

Some identities:

$$\text{Since } I_{uvwx} = I_{uv} + I_{wx} + \{uvwx\}$$

for w_x

$$\text{we have } \{g_{vwx}\} = I_{uvwx} - I_u - I_v - I_w - I_x$$

$$\text{fat } I_{uvwX} = I_{uvw} + I_X + \{_{uvw} \times \}$$

$$= I_{uv} + I_w + \{v, w\} + \{vw, x\} + I_x$$

$$= I_u + I_v + I_w + I_x + \{u, v\} + \{uv, w\} + \{uvw, x\}$$

$$\checkmark \text{ so that } \{u, v, w, x\} = \{u, v\} + \{uvw\} + \{uvw, x\}$$

$$\text{now } \{UV, WX, YZ\} = \{I_{UVWXYZ}, -I_{UV}, -I_{WX}, -I_{YZ}, -I_W, -I_X, \{W, X\}\}$$

$$\{uvw, x, yz\} = I_{uvwxyz} - I_{uv} - I_w - I_x - I_y$$

$$\text{∴ so, } \{uvw, x, yz\} = \{uv, wx, yz\} + \{w, x\}$$

Introduction

introduces

General synthesis rule

$$\textcircled{1} \quad \{\dots, uv, wx, \dots\} = \{\dots, uvwx, \dots\} + \{uv, wx\}$$

rule for removal of commas

gives:

General Commutation:

$$\{\dots, \cancel{uv}, \cancel{wx}, \cancel{yz}, \dots\}$$

$$\{\dots, \cancel{uvwxyz}\}$$

$$\{\dots, uv, wxyz, \dots\} = \{uvwxyz, \dots\} + \{uv, wxyz\}$$

$$= \cancel{\{...uvwx, yz\}} - \{uvwx, yz\} + \{uv, wxyz\}$$

$$\text{ie } \{\dots, uv, wxyz, \dots\} = \{\dots, uvwx, yz, \dots\} - \{uvwx, yz\} + \{uv, wxyz\}$$

\textcircled{2}

which means: $\{\dots, uv, wxyz, \dots\} - \{\dots, uvwx, yz, \dots\}$

$$\textcircled{3} \quad = \{uv, wxyz\} - \{uvwx, yz\}$$

Now, $\{v, wx\} - \{vw, x\}$

$$\begin{aligned} &= I_{vwx} - I_v - I_{wx} - I_{vwx} + I_{vw} + I_x \\ &= -I_v \end{aligned}$$

$$\begin{aligned} &= \cancel{\{v, wx\}} + \cancel{\{w, x\}} - \cancel{\{v, wx\}} - \cancel{\{v, x\}} \\ &= \cancel{\{w, x\}} - \cancel{\{v, x\}} \end{aligned}$$

Therefore, from 3 $\{\dots, uv, wxyz, \dots\} - \{\dots, uvwx, yz, \dots\}$

$$= \{uv, wxyz\} - \{uvwx, yz\}$$

wrong ^①

$$= \cancel{\{uv, wx, yz\}} - \{wx, yz\} - \cancel{\{uv, wx, yz\}} + \{uvwx\}$$

$$= \{uv, wx\} - \{wx, yz\}$$

Therefore General Commutation rule:

$$\begin{aligned} & \{ \dots, \overset{v}{u}, \overset{v}{w}, \overset{w}{x}, \overset{w}{y}, \overset{w}{z}, \dots \} - \{ \dots, \overset{v}{u}, \overset{v}{w}, \overset{w}{x}, \overset{w}{y}, \overset{w}{z}, \dots \} \\ = & \{uv, wxyz\} - \{uvwx, yz\} \\ \checkmark = & \{uv, wx\} - \{wx, yz\} \end{aligned}$$



To rules for bracket
mechanics
are the same
and commutation

Proof: $\{ \dots, \overset{v}{u}, \overset{v}{w}, \overset{w}{x}, \overset{w}{y}, \overset{w}{z}, \dots \} - \{ \dots, \overset{v}{u}, \overset{v}{w}, \overset{w}{x}, \overset{w}{y}, \overset{w}{z}, \dots \}$

$$\begin{aligned} \text{by ①} \\ = & \{ \dots, \overset{v}{u}, \overset{v}{w}, \overset{w}{x}, \overset{w}{y}, \overset{w}{z}, \dots \} - \{wx, yz\} - \{ \dots, \overset{v}{u}, \overset{v}{w}, \overset{w}{x}, \overset{w}{y}, \overset{w}{z}, \dots \} + \{uv, wx\} \\ = & \{uv, wx\} - \{wx, yz\} \end{aligned}$$

in fact, works with condition also

Another degenerate case: $\{x, y\}^{x'}$

$$P^x(x, y) = P^x(y) \delta(x - x')$$

$$P^x(y) \quad \text{cond into } I_y^x$$

$$\{ \dots, \overset{v}{w}, \overset{v}{x}, \overset{w}{y}, \overset{w}{z}, \dots \} - \{ \dots, \overset{v}{w}, \overset{v}{x}, \overset{w}{y}, \overset{w}{z}, \dots \}$$

$$= I_{\dots, wxzy\dots} - 1$$

Note $\{U, V, W, X\} = I_{UVWX} - I_U - I_{VW} - I_X$

$$= \int P(UVWX) \ln P(UVWX) dudvdwdx$$

$$- \int P(UVWX) \ln P(U) du dudvdwdx$$

$$- \int P(UVWX) \ln P(V,W) dudvdwdx$$

$$- \int P(UVWX) \ln P(X) dudvdwdx$$

$$= \boxed{\int P(UVWX) \ln \frac{P(UVWX)}{P(U)P(V,W)P(X)} dudvdwdx}$$

$$= \boxed{\exp \left\{ \ln \frac{P(UVWX)}{P(U)P(V,W)P(X)} \right\}}$$

$$\{U, V, W, X\}^{st} = \int P_{UVWX}^{st} \ln \frac{P_{UVWX}^{st}}{P_U^{st} P_{VW}^{st} P_X^{st}} dudvdwdx$$

$$= \boxed{\exp^{st} \left\{ \ln \frac{P_{UVWX}^{st}(UVWX)}{P_U^{st} P_{VW}^{st} P_X^{st}} \right\}}$$

Note: $\{U, V, W, X\}^{sw'} = \exp^{sw'} \left\{ \ln \frac{P_{UVWX}^{sw'}(UVWX)}{P_U^{sw'} P_{VW}^{sw'} P_X^{sw'}} \right\}$

but $P_{VW}^{sw'} = P(v) \delta(w-w)$

$$P_{UVWX}^{sw'} = P(UVWX) \delta(w-w')$$

Therefore

$$\{v, vw, x\}^{sw'} = \exp^{\text{sw}'} \left\{ \ln \frac{p^{sw'}(uvx) \delta(w-w')}{p^{sw'}(v) \delta(w-w) p^{sw'}(u) p^{sw'}(x)} \right\}$$
$$= \exp^{\text{sw}'} \left\{ \ln \frac{p^{sw'}(uvx)}{p^{sw'}(v) p^{sw'}(u) p^{sw'}(x)} \right\}$$
$$= \{v, v, x\}^{sw'}$$

Putting condition of variable
in bracket omits variable in bracket!

in case alone:

$$\{v, w, x\}^{sw'} = \exp^{\text{sw}'} \left\{ \ln \frac{p^{sw'}(vx) \delta(w-w)}{p^{sw'}(v) p^{sw'}(x) \delta(w-w')} \right\}$$
$$= \{v, x\}^{sw'}$$

Now, what about $\{v, w\}^{sw'} = \exp^{\text{sw}'} \left\{ \ln \frac{p^{sw'}(v) \delta(w-w)}{p^{sw'}(v) \delta(w-w')} \right\}$

$$= \textcircled{0}$$

Therefore $C_{v,w}^{w'} = \textcircled{0}$ define 1

so define $\{v\} = \textcircled{0}$ (product of only one number)

$$= \{v, \phi\} = I_{v\phi} - I_v - I_\phi$$

for empty set define $I_{\phi\phi} = 0$ also $I_{v\phi} = I_v$

so define $\{uv\} = \emptyset$ (ie no commas)

$$\text{then } \{u, vw, x\}^{sw'} = \{u, v, x\}^{sw'}$$

$$\text{also } \{u, vx, w\}^{sw'} = \{u, vx\}$$

$$\text{furthermore } \{u, w\}^{sw'} = \{u\}^{sw'} = \emptyset \text{ by def}$$

alternately $\{u, v, w, x\}^{sw}$

$$\{u, vw, x\}^{sw} = \{u, v, wx\} \neq \{v, w\} + \{w, x\}$$

$$= \{u, vx, w\}^{sw} + \{w, x\} + \{v, x\} + \{v, w\} + \{u, vx\}$$

$$= \{u, vx, w\}^{sw} + \{v, w\}^{sw} + \{v, x\}^{sw}$$

$$= \{uvx, w\}^{sw} + \{u, vx\}^{sw} + \{v, x\}^{sw} + \{v, w\}^{sw}$$

$$= \{uvx, w\}^{sw} + \{u, v, x\}^{sw} - \{v, w\}^{sw}$$

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$$= \{u, v, x\}^{sw} \quad \text{since independent}$$

easier proof!

$$\text{commutator } \{X, Yz\} = \{XY, z\}$$

$$= \{X, Yz\} - \{Y, Nz\}$$

cancel
non shift

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Also, as check of consistency of comma removal
and def of $\{v\} = 0$

$$\{U, Vz\} = \{UV\} + \{U, V\} \Rightarrow \{UV\} = 0$$

QED

functional transformations across a comma:

Consider $XY \rightarrow VW$

$$\{\dots, UVWZ, \dots\} = \{\dots, Uvw, Z, \dots\} + \{Uv, W\} - \{w, Z\}$$

$$= \{\dots, UX, YZ, \dots\} + \{U, VW\} - \{w, Z\}$$

$$+ \{V, W\} - \{U, V\}$$

$$= \{\dots, UX, YZ, \dots\} + \{Y, Z\} - \{UX, Y\} + \{U, XY\}$$

$$- \{W, Z\} + \{V, W\} - \{U, V\}$$

$$\rightarrow = \cancel{\{U, XY\}} + \{U, X\} + \cancel{\{U, YZ\}} - \{X, Y\}$$

$$= \{\dots, UX, YZ, \dots\} + \{Y, Z\} + \{U, X\} - \{X, Y\} - \{W, Z\} + \{V, W\}$$

$$- \{U, V\}$$