GSERM - Oslo 2019
Introduction to Survival Data

January 9, 2019 (afternoon session)

Survival Analysis

- Models for *time-to-event data*.
- Roots in biostats/epidemiology, plus engineering, sociology, economics.
- Examples...
  - Political careers, confirmation durations, position-taking, bill cosponsorship, campaign contributions, policy innovation/adoption, etc.
  - Cabinet/government durations, length of civil wars, coalition durability, etc.
  - War duration, peace duration, alliance longevity, length of trade agreements, etc.
  - Strike durations, work careers (including promotions, firings, etc.), criminal careers, marriage and child-bearing behavior, etc.

Characteristics of Time-To-Event Data

- *Discrete* events (i.e., not continuous),
- Take place over *time*,
- May not (or *never*) experience the event (i.e., possibility of *censoring*).

Survival Data Basics: Terminology

$$\begin{aligned}
Y_i &= \text{the duration until the event occurs,} \\
Z_i &= \text{the duration until the observation is ``censored''} \\
T_i &= \min\{Y_i, Z_i\}, \\
C_i &= \text{0 if observation } i \text{ is censored, 1 if it is not.}
\end{aligned}$$

Survival Data Basics: The Density

$$f(t) = \Pr(T_i = t)$$

density function, probability T_i takes particular value t
pdf

Issues:

- $T_i = t$ iff $T_i > t-1,\ t-2$, etc.
- $C_i = 0$ (censoring)

up to the point T_i = if it does not equal t before that (you need to be 21 years unmarried to get married at 22.

Survival Data Basics: Survivor Function

cdf

$$\Pr(T_i \leq t) \equiv F(t) = \int_0^t f(t)\,dt$$

$$\begin{aligned}\Pr(T_i \geq t) \equiv S(t) &= 1 - F(t) \\ &= 1 - \int_0^t f(t)\,dt\end{aligned}$$

Survival Data Basics: The Hazard

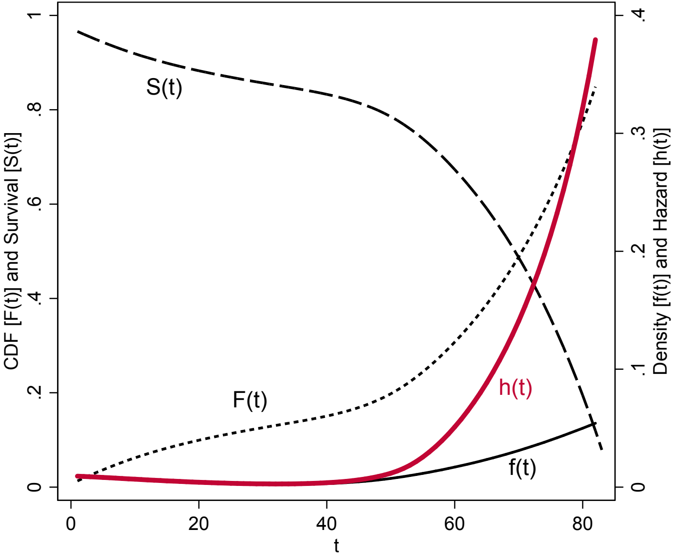
h is not a true probability, coz it can get bigger than 1 as f is bigger than S
=> correct term would be conditional risk.

$$\Pr(T_i = t | T_i \geq t) \equiv h(t) = \frac{f(t)}{S(t)}$$

$$= \frac{f(t)}{1 - \int_0^t f(t)\,dt}$$

given that I have not been married until t, what is the probability that i get married at t (conditional)

Example: Human Mortality



solid right axis, dashed left axis.

Some Useful Equivalencies

$$f(t) = \frac{-\partial S(t)}{\partial t}$$

negative derivative

Implies

$$\begin{aligned} h(t) &= \frac{\frac{-\partial S(t)}{\partial t}}{S(t)} \\ &= \frac{-\partial lnS(t)}{\partial t} \end{aligned}$$

hazard is negative derivative of logged survivor function

More Useful Things: Integrated Hazard

Define

$$H(t) = \int_0^t h(t)\,dt.$$

cumulative risk.

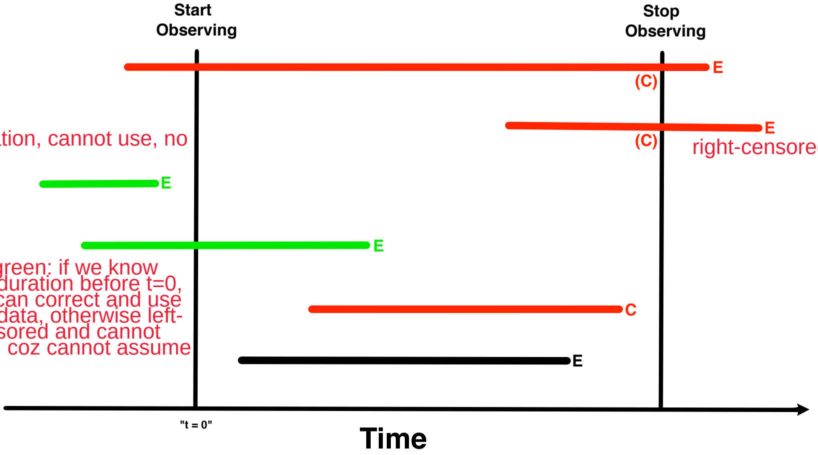
Implies

$$\begin{aligned} H(t) &= \int_0^t \frac{-\partial ln S(t)}{\partial t}\,dt \\ &= -ln[S(t)] \end{aligned}$$

and

$$S(t) = \exp[-H(t)]$$

Censoring and Truncation



truncation, cannot use, no data

right-censored.

for green: if we know the duration before t=0, we can correct and use the data, otherwise left-censored and cannot use, coz cannot assume t=0.

C=censored.
E=event

Censoring

(what is event, what is censoring "event", e.g. time to die, quit, get fired, etc. you need to define event of interest.)

- Defined by the researcher
- Conditionally independent of both T_i and $\mathbf{X}_i$
- Doesn't mean that the observation provides no information

censoring is conditionally independent sounds unreasonable at first.
but most covariates are typically conditionally dependent of X and you should control for it anyway.
so, if you are confident in the specified model, there is also no conditional dependence of T_i.

Estimating $S(t)$

Kaplan-Meier estimator.

no ties = no 2 observations at the exact same time t.

absorbing events=event occurred, cant happen again.

Assume N observations, *absorbing* events, and no ties. Then define

$$\begin{aligned} n_t &= \text{number of observations “at risk” for the event at } t \text{, and} \\ d_t &= \text{number of observations which experience the event} \\ & \quad \text{at time } t. \end{aligned}$$

Then

$$\widehat{S(t_k)} = \prod_{t \le t_k} \frac{n_t - d_t}{n_t}$$

descriptively we usually work with the survival function instead of hazard function.

proportion of at risk observations that had the event at timepoint t.

for each period where no event happens, the curve stays the same, otherwise it decreases a little bit, i.e less surviving happening.

censoring changes the risk set, but not the d_t; so censoring does not change survival function.

following no ties, d_t could be at most 1, but later we see d_t can also be bigger than 1.

Variance of $\widehat{S(t)}$

$$\mathrm{Var}[\widehat{S(t_k)}] = \left[\widehat{S(t_k)}\right]^2 \sum_{t \le t_k} \frac{d_t}{n_t(n_t - d_t)}$$

Note:

- $\mathrm{Var}[\widehat{S(t_k)}]$ is increasing in $S(t)$,
- is also increasing in d_t, but
- is decreasing in n_t.

Estimating $H(t)$

"Nelson-Aalen":

$$\widehat{H(t_k)} = \sum_{t \le t_k} \frac{d_t}{n_t}$$

it goes up by small amounts as d_t may be 1 and n_t large

...which gives an alternative estimator for the survival function equal to:

$$\begin{aligned}
\widehat{S(t_k)} &= \exp[-\widehat{H(t_k)}] \\
&= \exp\left[-\sum_{t \le t_k} \frac{d_t}{n_t}\right]
\end{aligned}$$

kaplan-meier and nelson-aalen are asympotically equivalent, but as estimates they may be slightly different in selected samples

Bivariate Hypothesis Testing

(similar to chi-squared test)

	Treatment	Placebo	Total
Event	d_{1t}	d_{0t}	d_t
No Event	$n_{1t} - d_{1t}$	$n_{0t} - d_{0t}$	$n_t - d_t$
Total	n_{1t}	n_{0t}	n_t

Log-Rank Test:

$$
\begin{aligned}
Q &= \frac{\left[\sum (d_{1t} - \frac{n_{1t} d_t}{n_t})\right]^2}{\left[\frac{n_{1t} n_{0t} d_t (n_t - d_t)}{n_t^2 (n_t - 1)}\right]} \\
&\sim \chi_1^2
\end{aligned}
$$

A Diversion: Survival Models and Counting Processes

Assume

- Event is *absorbing*,
- Y_i is duration to the event
- Z_i is duration to censoring
- Observe $T_i = \min(Y_i, Z_i)$, and
- C_i:
  - $C_i = 0$ if $T_i = Z_i$,
  - $C_i = 1$ if $T_i = Y_i$.
- $T_i \neq T_j \; \forall \; i \neq j$ (no "ties")

Three Key Variables

1. *Counting Process* Indicator:

$$N_i(t) = I(T_i \leq t, C_i = 1)$$

zero until event occured, then 1 for the rest.

2. *Risk* Indicator:

$$R_i(t) = I(T_i > t)$$

1 until event occuered, then 0 for the rest, overlaps with 1.

3. *Intensity Process*:

$$\lambda_i(t)\, dt = R_i(t) h(t)$$

hazard as long as you are at rest, if event occured, it goes to 0

With

$$\Lambda_i(t) = \int_0^t \lambda_i(t)\, dt$$

we can think of

$$N_i(t) = \Lambda_i(t) + M_i(t)$$

or

$$M_i(t) = N_i(t) - \Lambda_i(t).$$

kinda like "Residual = Observed - Expected"

it is a martingale process, so all the math can be used.

Martingales!

(memoryless process)

$$E(X_{t+s}|X_0, X_1, ...X_i, ...X_t) = X_t \; \forall \; s > 0$$

Data Structure and Organization: Non-Time-Varying

id	durat	censor	timein	timeout	X
1	4	0	30	34	0.12
2	2	1	12	14	0.19
3	5	1	5	10	0.09
...	...	...	...	...	...
N	10	1	21	31	0.22

Time-Varying Data

id	durat	censor	timein	timeout	X	Z
1	1	0	30	31	0.12	331
1	2	0	31	32	0.12	412
1	3	0	32	33	0.12	405
1	4	0	33	34	0.12	416
2	1	0	12	13	0.19	226
2	2	1	13	14	0.19	296
3	1	0	5	6	0.09	253
3	2	0	6	7	0.09	311
3	3	0	7	8	0.09	327
3	4	0	8	9	0.09	344
3	5	1	9	10	0.09	301
...	...	...	...	...	...	...

Analyzing Survival Data in R

survival object (non-time-varying):

```
library(survival)
NonTV<-read.csv(NonTVdata.csv)
NonTV.S<-Surv(NonTV$duration, NonTV$censor)
```

survival object (time-varying):

```
TV<-read.csv(TVdata.csv)
TV.S<-Surv(TV$starttime, TV$endtime, TV$censor)
```

An Example

OECD Cabinet survival [Strom (1985); King et al. (1990)],

$N = 314$ cabinets in 15 countries

Outcome: Duration of cabinet, in months

Covariates (all non-time varying):

- *Fractionalization*
- *Polarization*
- *Formation Attempts*
- **Investiture**
- *Numerical Status*
- *Post-Election*
- *Caretaker*

Also: Indicator for whether the cabinet ended within 12 months of the end of the "constitutional inter-election period" ($\rightarrow$ censored)

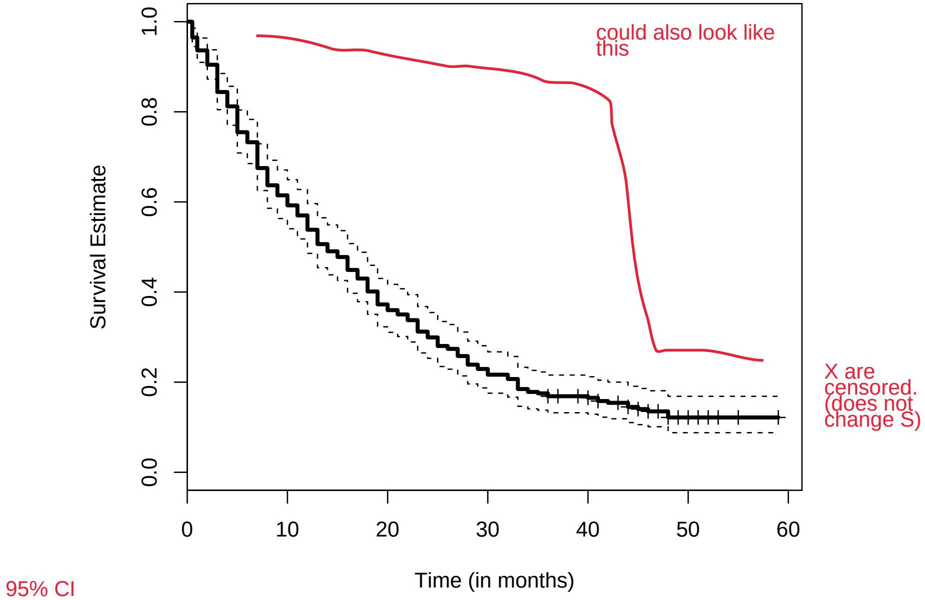
(coz cabinet may call early elections shortly before mandated end)

```
> head(KABL)
  id country durat ciep12 fract polar format invest numst2 eltime2 caretk2
1  1       1   0.5      1   656    11      3      1      0       1       0
2  2       1   3.0      1   656    11      2      1      1       0       0
3  3       1   7.0      1   656    11      5      1      1       0       0
4  4       1  20.0      1   656    11      2      1      1       0       0
5  5       1   6.0      1   656    11      3      1      1       0       0
6  6       1   7.0      1   634     6      4      1      1       1       0

> KABL.S<-Surv(KABL$durat,KABL$ciep12)

> KABL.S[1:50,]

 [1]  0.5   3.0   7.0  20.0   6.0   7.0   2.0  17.0  27.0  49.0+
[11]  4.0  29.0  49.0+  6.0  23.0  41.0+ 10.0  12.0   2.0  33.0
[21]  1.0  16.0   2.0   9.0   3.0   5.0   5.0   6.0  45.0+ 23.0
[31] 41.0   7.0  49.0+ 46.0   9.0  51.0+ 10.0  32.0  28.0   3.0
[41] 53.0+ 17.0  59.0+  9.0  52.0+  3.0  23.0  33.0   1.0  30.0
```

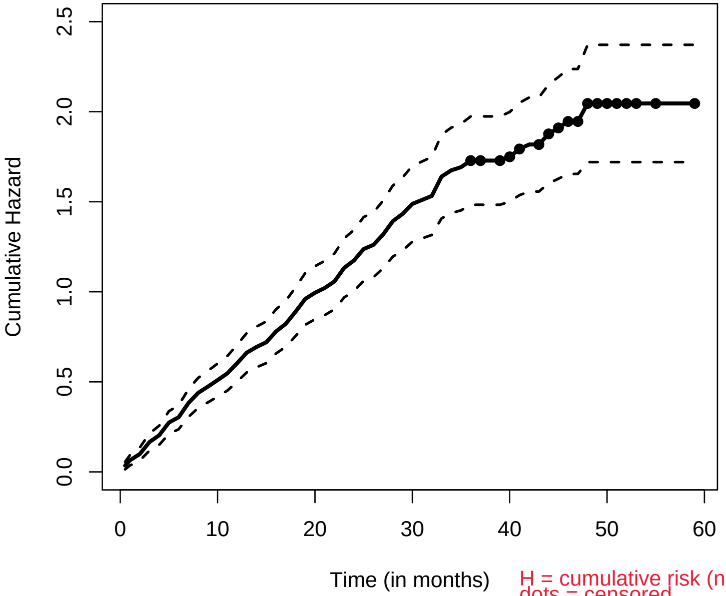
Example `survfit` Object

```
> KABL.fit<-survfit(KABL.S~1)

> str(KABL.fit)

List of 13
 $ n        : int 314
 $ time     : num [1:54] 0.5 1 2 3 4 5 6 7 8 9 ...
 $ n.risk   : num [1:54] 314 303 294 284 265 255 237 230 212 200 ...
 $ n.event  : num [1:54] 11 9 10 19 10 18 7 18 12 7 ...
 $ n.censor : num [1:54] 0 0 0 0 0 0 0 0 0 0 ...
 $ surv     : num [1:54] 0.965 0.936 0.904 0.844 0.812 ...
 $ type     : chr "right"
 $ std.err  : num [1:54] 0.0108 0.0147 0.0183 0.0243 0.0271 ...
 $ upper    : num [1:54] 0.986 0.964 0.938 0.885 0.856 ...
 $ lower    : num [1:54] 0.945 0.91 0.873 0.805 0.77 ...
 $ conf.type: chr "log"
 $ conf.int : num 0.95
 $ call     : language survfit(formula = KABL.S ~ 1)
 - attr(*, "class")= chr "survfit"
```

Plotting $\widehat{S(t)}$



Plotting $\widehat{H(t)}$



H = cumulative risk (not probability)
dots = censored.

Log-rank test:

```
> survdiff(KABL.S~invest,data=KABL,rho=0)

Call:
survdiff(formula = KABL.S ~ invest, data = KABL, rho = 0)

           N Observed Expected (O-E)^2/E (O-E)^2/V
invest=0 172      137    178.7      9.72      30.5
invest=1 142      134     92.3     18.81      30.5

 Chisq= 30.5  on 1 degrees of freedom, p= 3.26e-08
```

formal test if these two survival functions are statistically different.
you can also split the data and estimate two survival functions.

Comparing $\widehat{S(t)}$s