6520 Project

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```
set.seed(6520)

library(expm)
```

```
## Loading required package: Matrix
```

```
##
## Attaching package: 'expm'
```

```
## The following object is masked from 'package:Matrix':
##
##     expm
```

Simulate data for regression and classification

```
# simulate data: regression
n = 100 # sample size
p = 200 # number of predictors

# beta
k = round(0.05*p, 0) # number of nonzero coefficients
sd_beta = 0.01
nonzero_indexes = sample.int(n=p, size=k)
beta = rep(0, p)
beta[nonzero_indexes] = rnorm(n=k, mean=100, sd=sd_beta)
sum(which(beta !=0) != sort(nonzero_indexes)) # test that we made the right indexes nonzero
```

```
## [1] 0
```

```
beta = as.matrix(beta)

# x
X = matrix(rnorm(n=n*p, mean=0, sd=5), nrow=n)

# epsilon
E = matrix(rnorm(n=n, mean=0, sd=1), nrow=n)
```

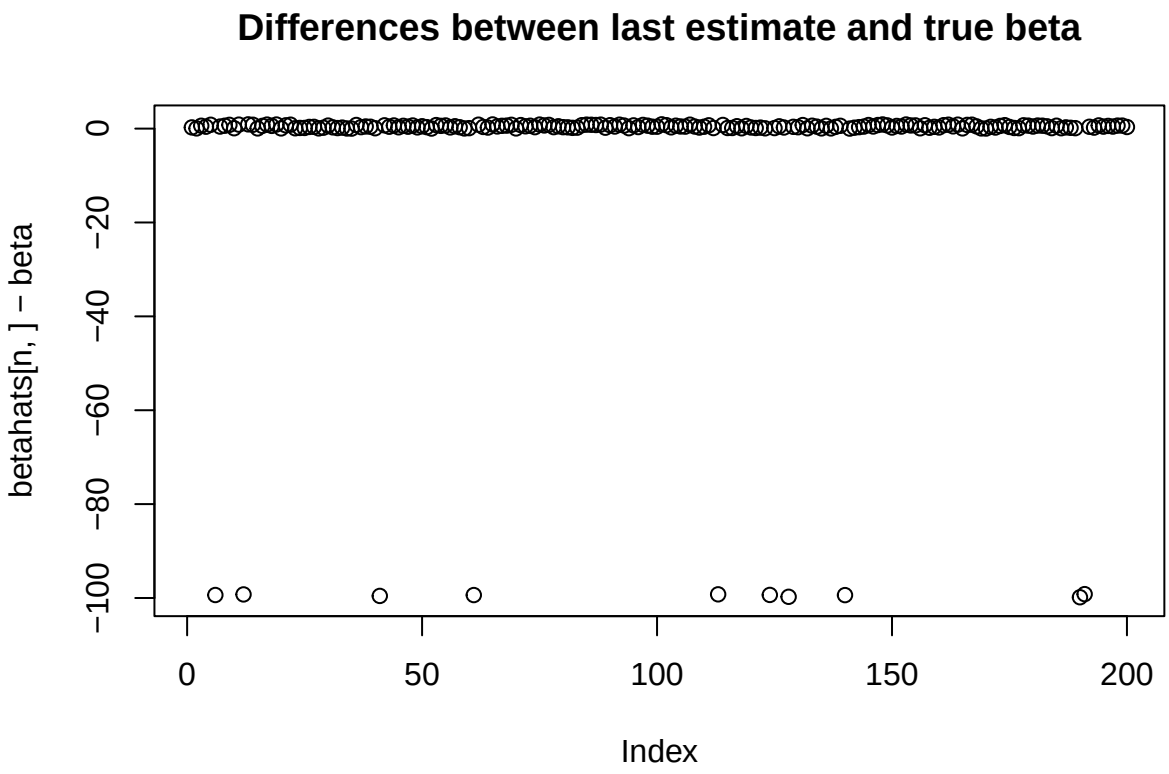
```r
# y

Y = X%*%beta + E

# note that in the online setting, each t^th row of X and Y is for time t
```

```r
# simulate data: classification
# X, beta same as above
probs = 1/(1+exp(-X%*%beta))
Y = rbinom(n=n, size=1, prob = probs) # Bernoulli
```

OGD

```r
# Online gradient descent for regression
my_OGD = function(X, Y, lr, beta_0) {
  n = nrow(X)
  p = ncol(X)
  betahats = matrix(nrow=n, ncol=p)
  betahats[1, ] = beta_0
  for (t in 1:(n-1)) {
    x_t = as.matrix(X[t, ])
    beta_t = as.matrix(betahats[t, ])
    y_t_hat = t(beta_t)%*%x_t
    Y_t = Y[t]
    # betahats[t+1, ] = beta_t - lr*as.numeric(y_t_hat-Y_t)*x_t # least means squares
    d_loss = 2*beta_t%*%t(x_t)%*%x_t - 2*x_t%*%Y_t
    betahats[t+1, ] = beta_t - lr*d_loss
  }
  return(betahats)
}
```

```r
# testing function
beta_0 = runif(p)
# beta_0 = rep(0, p)
betahats = my_OGD(X=X, Y=Y, lr=0.0000001, beta_0=beta_0)
plot(betahats[n, ] - beta, main="Differences between last estimate and true beta")
```



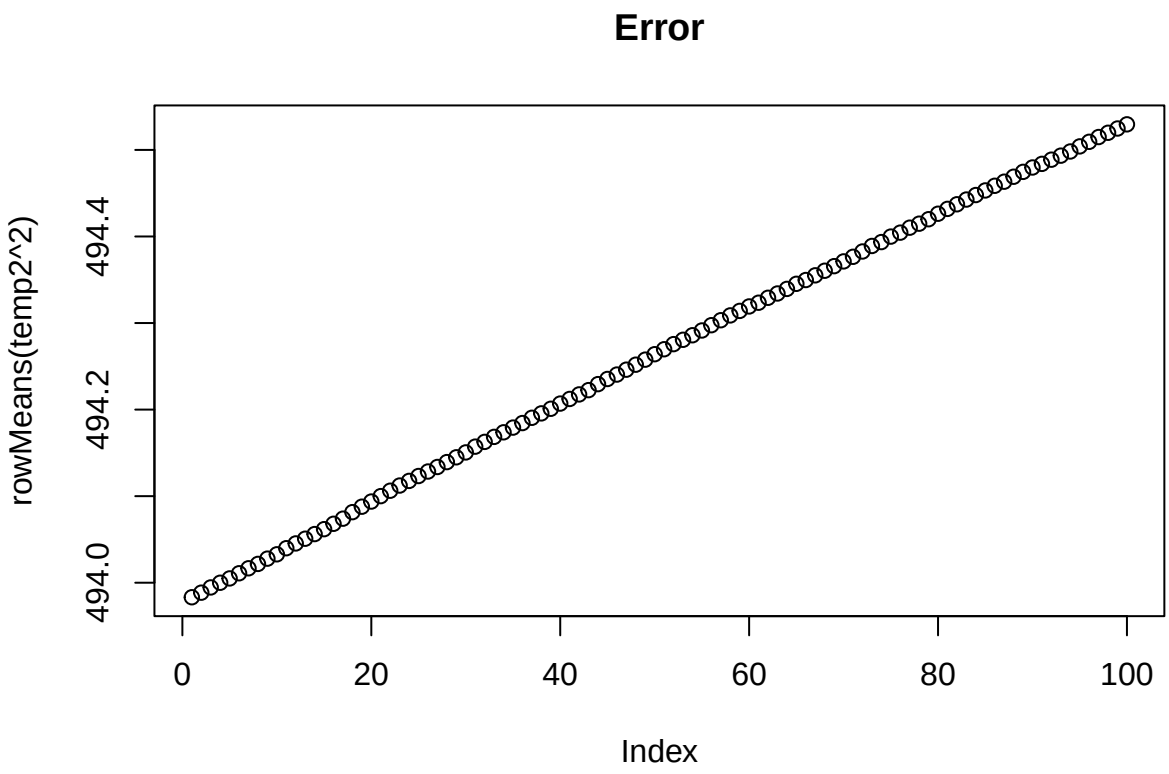

```
sum(abs(betahats[n, ] - beta) > 0.5*mean(beta[nonzero_indexes])) == k
```

```
## [1] TRUE
```

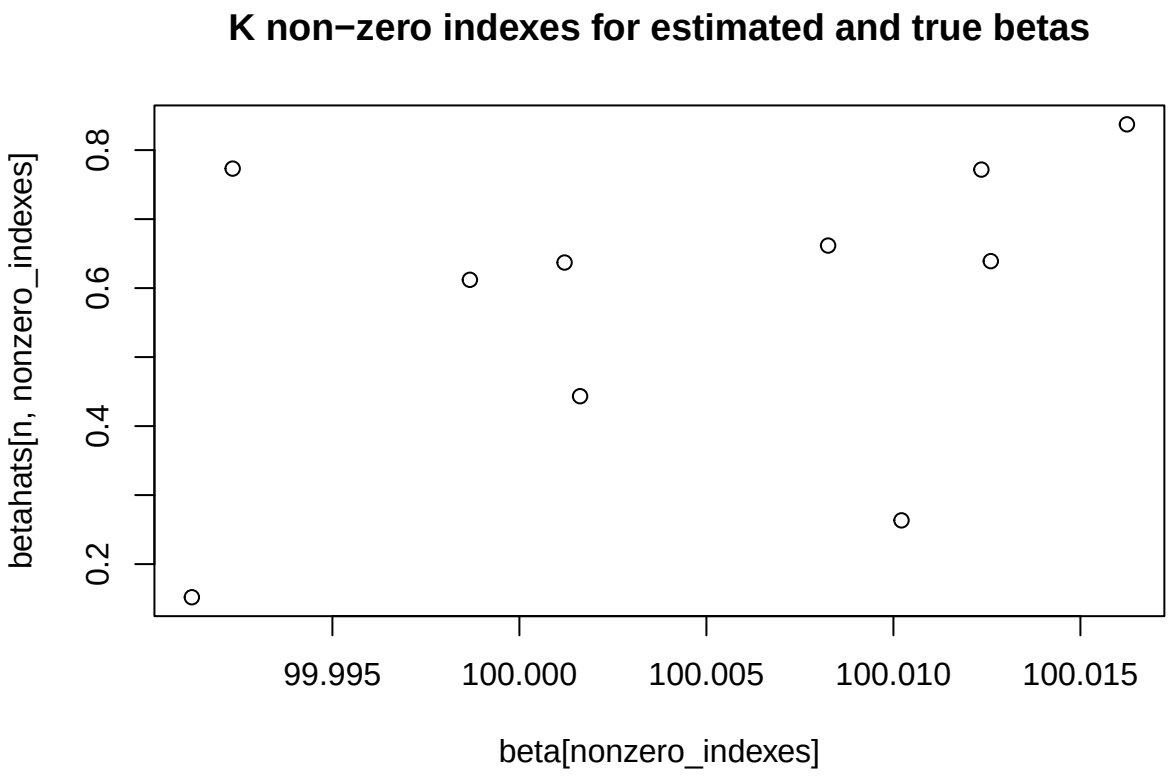
```
mean(betahats[n, nonzero_indexes]) - mean(betahats[n, -nonzero_indexes])
```

```
## [1] 0.1326939
```

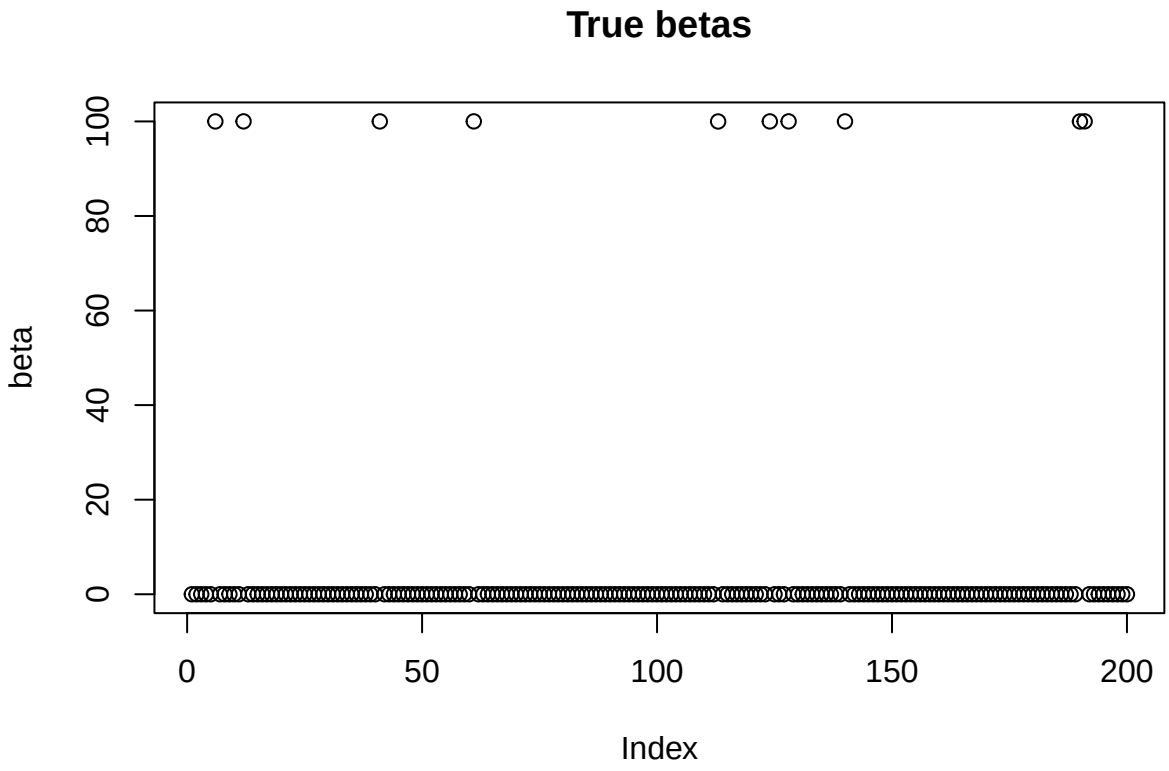
```
temp = matrix(rep(beta, n), nrow=n, ncol=p, byrow=T)
temp2 = betahats - temp
plot(rowMeans(temp2^2), main="Error")
```



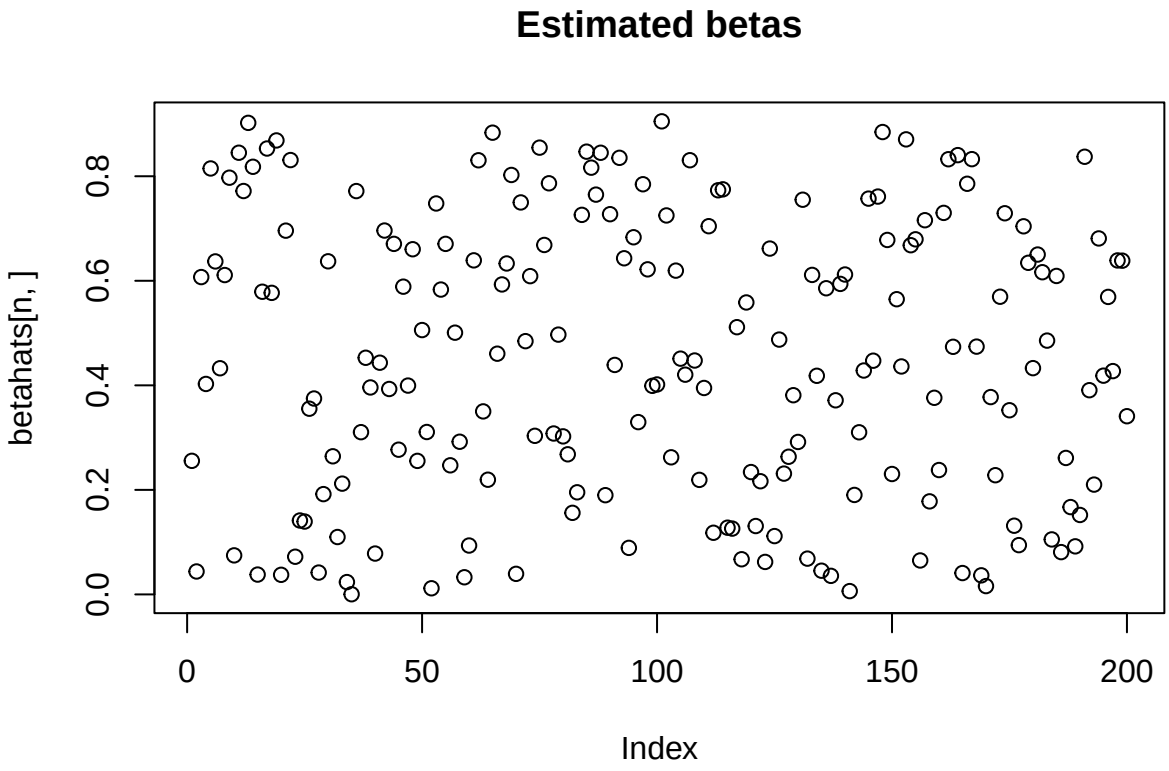

```
plot(beta[nonzero_indexes], betahats[n, nonzero_indexes], main="K non-zero indexes for estimated and tru
```




```
plot(beta, main="True betas")
```




```
plot(betahats[n, ], main="Estimated betas")
```



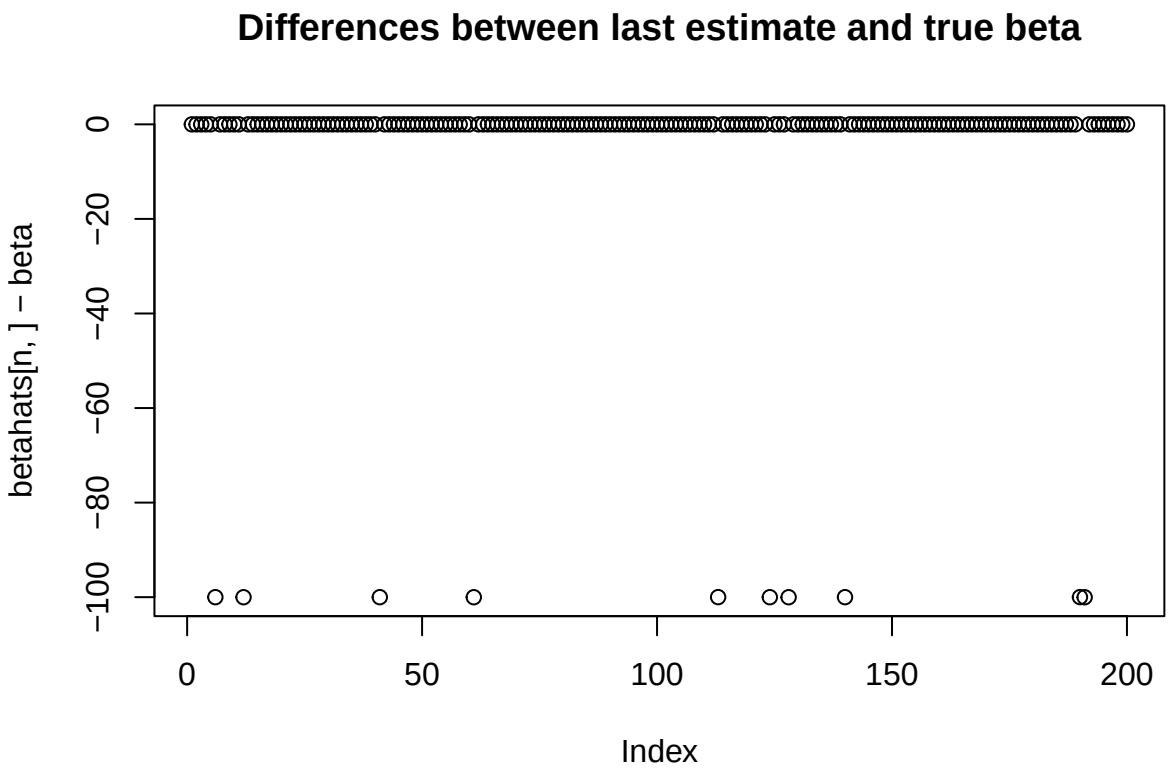

```
# function for online gradient descent (OGD)
# type: "classification" or "regresssion"
# X: rows are observations, columns are predictors
# Y: response variable
# learning rate (constant)
# beta_0: initialization for the estimate
# N: number of iterations for each data point (each row of X, Y)
# returns betahats for each data point and its N interations as a 3D array
# my_OGD = function(type, X, Y, lr, beta_0, N) {
#   if (type!='classification'&&type!='regression') {
#     stop("Argument 'type' must be 'classification' or 'regression'")
#   }
#
#   n = nrow(X)
#   betahats = array(data=rep(NA, n*p*N), dim = c(N, p, n)) # n arrays that are each Nxp arrays
#   betahats[1, , ] = beta_0 # initialize
#   if (type=='classification') {
#
#   } else {
#     # type is regression
#     for (k in 1:n) { # for each data point (each row of X, Y)
#       for (i in 1:(N-1)) { # for each iteration
#       d_loss = 2*as.matrix(betahats[i, , k])%*%as.matrix(t(X[k, ]))%*%as.matrix(X[k, ])-2*as.matrix(X
#       betahats[i+1, , k] = betahats[i, , k] - lr*d_loss
#       } # end for i
#     } # end for k
```

```
#     return(betahats)
#   } # end else for regression
# }
```

```
# use my_OGD on generated data
# betahats = my_OGD(type="regression", X=X, Y=Y, lr=0.00001, beta_0=rep(0, p), N=100)
```

```
# function for adaptive gradient descent (Adagrad)
# X: rows are observations, columns are predictors
# Y: response variable
# lr: global learning rate
# epsilon: noise for nonzero/invertibility
# beta_0: weight initialization
# full: boolean, uses full matrix for G if true, otherwise uses diagonal elements of G
my_adagrad = function(X, Y, lr, beta_0, full) {
  n = nrow(X)
  p = ncol(X)
  betahats = matrix(nrow=n, ncol=p)
  betahats[1, ] = beta_0
  g_vec = matrix(nrow=n, ncol=p) # save matrix for the gradients where each gradient g_t is the t^{th}
  G_t = matrix(data=rep(0, p^2), nrow=p, ncol=p) # matrix that is a cumulative sum

  for (t in 1:(n-1)) {
    x_t = as.matrix(X[t, ])
    beta_t = as.matrix(betahats[t, ])
    y_t_hat = t(beta_t)%*%x_t
    Y_t = Y[t]
    g_vec[t, ] = 2*beta_t%*%t(x_t)%*%x_t - 2*x_t%*%Y_t
    g_t = as.matrix(g_vec[t, ])
    G_t = G_t + g_t%*%t(g_t)
    diag_G_t = diag(diag(G_t), nrow=p, ncol=p)

    if (full) {
      # full
      betahats[t+1, ] = beta_t - lr*as.matrix(solve(sqrtm(G_t)))%*%g_t
    } else {
      # diagonal
      betahats[t+1, ] = beta_t - lr*as.matrix(diag(diag(diag_G_t^(-1/2)), nrow=p, ncol=p))%*%g_t
    }
  } # end for
  return(betahats)
}
```

```
beta_0 = rep(0, nrow(X))
betahats = my_adagrad(X=X, Y=Y, lr=0.0000001, beta_0=beta_0, full=F)

# plotting
plot(betahats[n, ] - beta, main="Differences between last estimate and true beta")
```



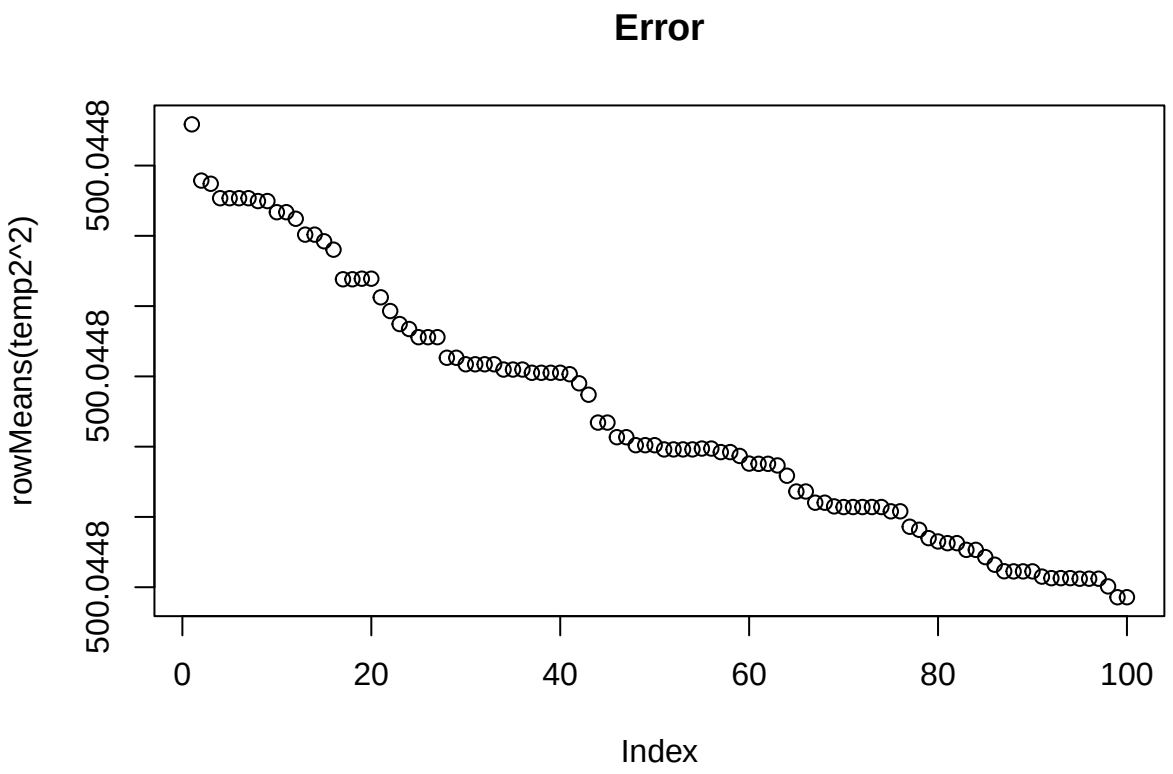

```
sum(abs(betahats[n, ] - beta) > 0.5*mean(beta[nonzero_indexes])) == k
```

```
## [1] TRUE
```

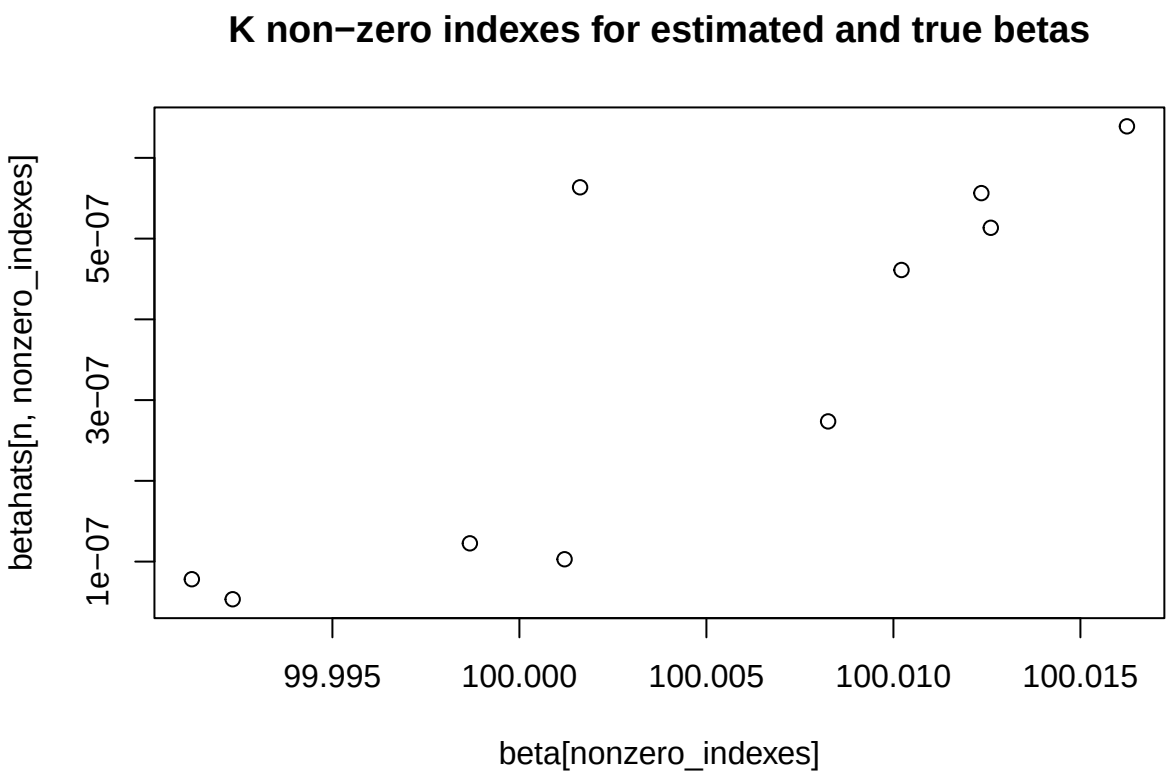
```
mean(betahats[n, nonzero_indexes]) - mean(betahats[n, -nonzero_indexes])
```

```
## [1] 3.370597e-07
```

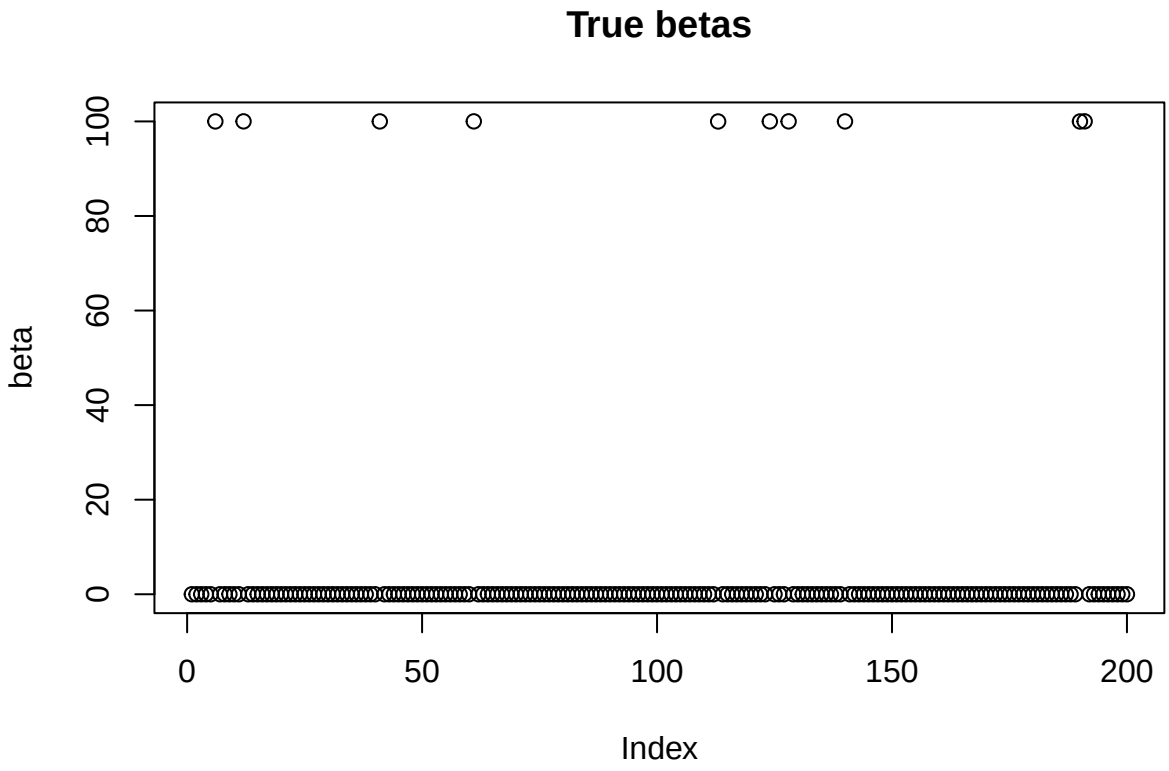
```
temp = matrix(rep(beta, n), nrow=n, ncol=p, byrow=T)
temp2 = betahats - temp
plot(rowMeans(temp2^2), main="Error")
```



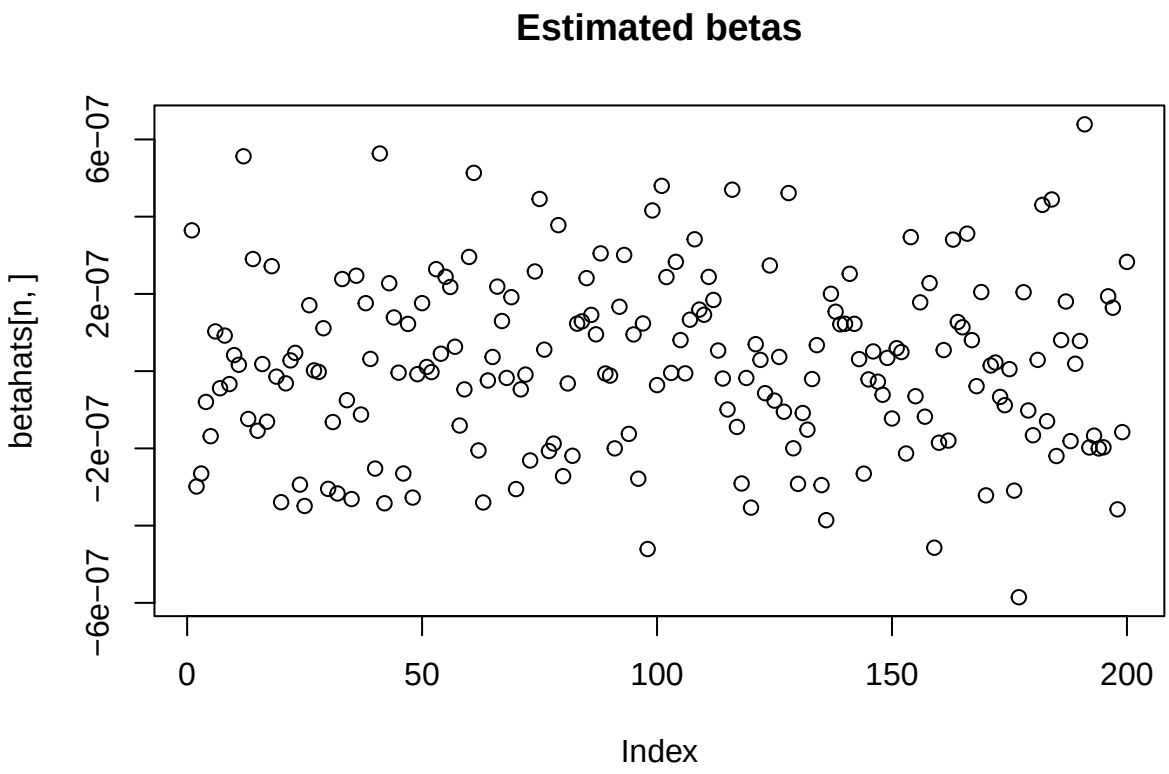

```
plot(beta[nonzero_indexes], betahats[n, nonzero_indexes], main="K non-zero indexes for estimated and tru
```




```
plot(beta, main="True betas")
```




```
plot(betahats[n, ], main="Estimated betas")
```

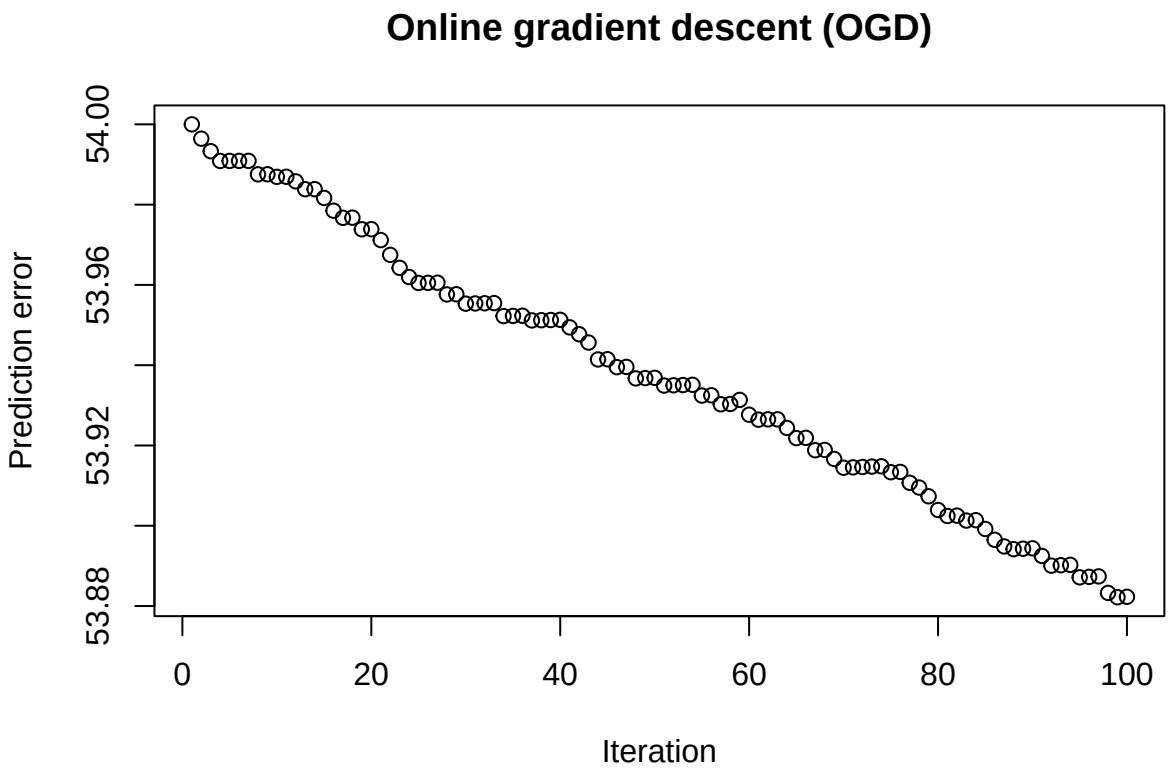


Analysis of $\hat{\beta}$'s

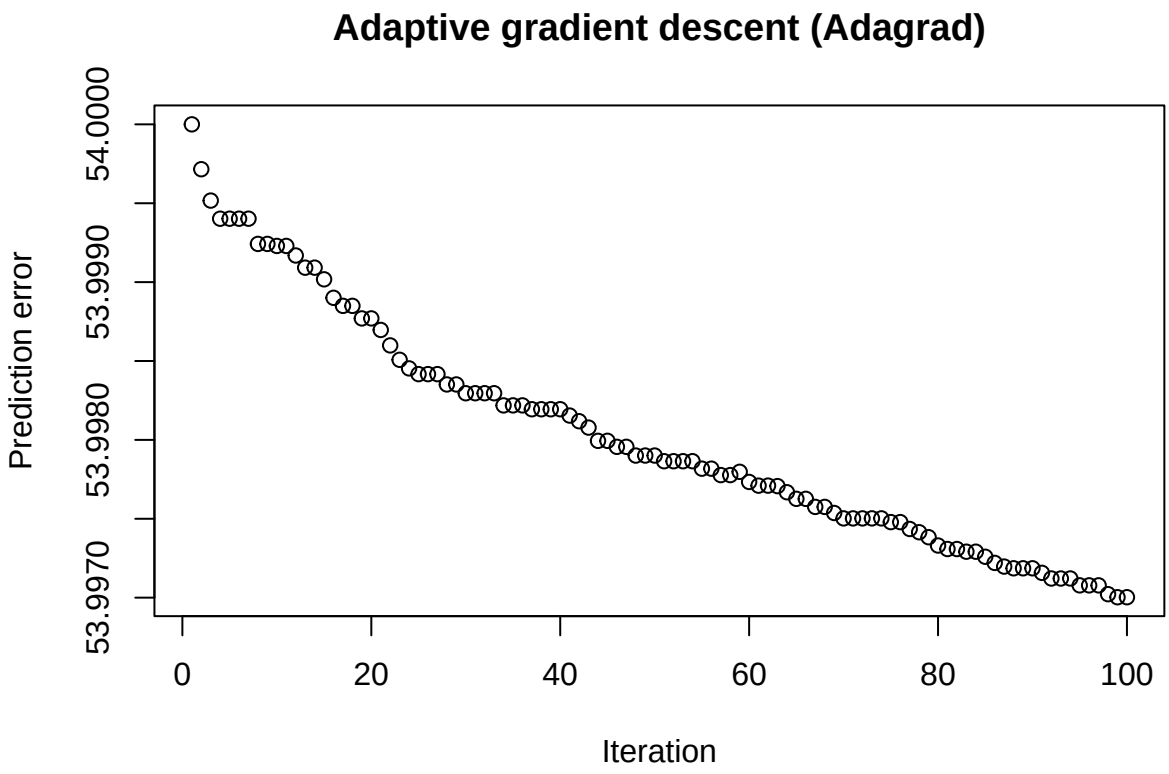
Plots + Prediction error vs iterations - Estimation error vs iterations - Betahats for each dimension, nonzero vs zero indexes - Comparison of different learning rates - Run time of full vs diagonal Adagrad - Run time of OGD, Adagrad, etc - Variance of betahats across iterations?

```
betahats_ogd = my_OGD(X=X, Y=Y, lr=0.0000001, beta_0=rep(0, p))
betahats_adagrad = my_adagrad(X=X, Y=Y, lr=0.0000001, beta_0=rep(0, p), full=F)
```

```
# plot prediction error vs iterations
pred_err = colSums((X%*%t(betahats_ogd) - matrix(rep(Y, n), nrow=n, ncol=n, byrow=F))^2) # row of the i
plot(pred_err, xlab="Iteration", ylab="Prediction error", main="Online gradient descent (OGD)")
```



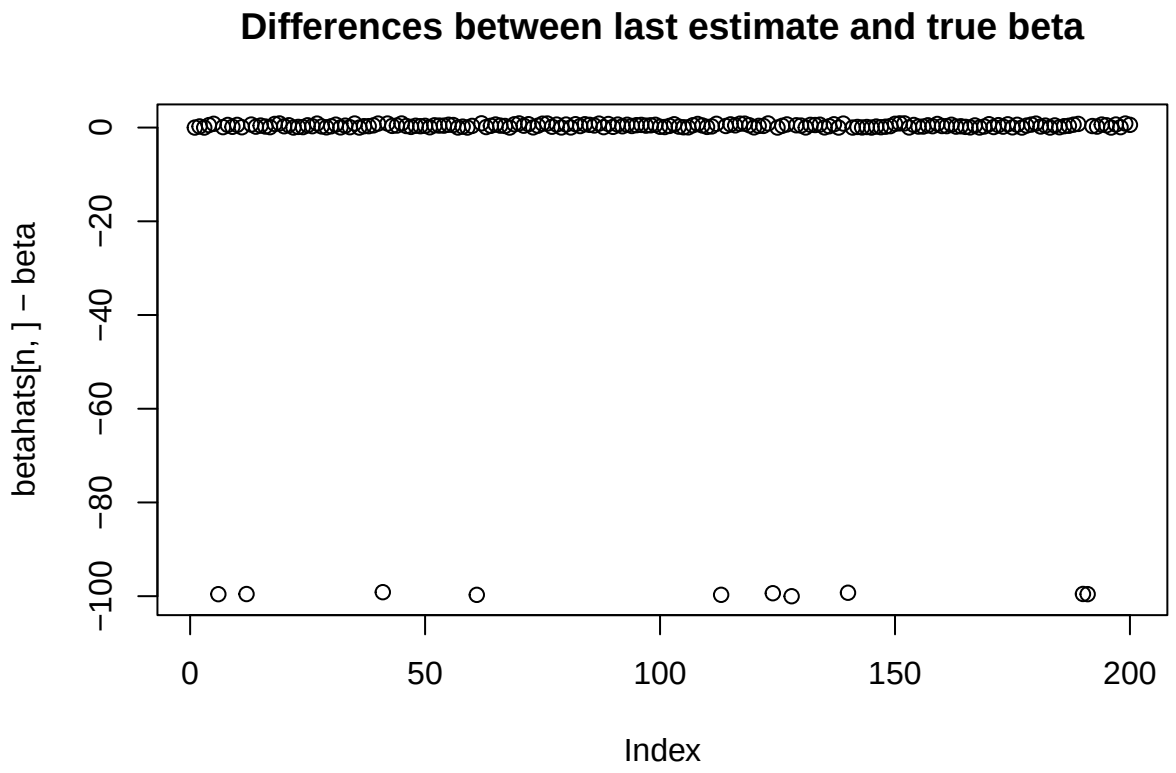

```
pred_err = colSums((X%*%t(betahats_adagrad) - matrix(rep(Y, n), nrow=n, ncol=n, byrow=F))^2) # row of t
plot(pred_err, xlab="Iteration", ylab="Prediction error", main="Adaptive gradient descent (Adagrad)")
```




```r
# function for adaptive moment estimation (Adam)
# X: rows are observations, columns are predictors
# Y: response variable
# lr: global learning rate (typical choice 0.001)
# beta_0: weight initialization
# epsilon: positive noise for nonzero/invertibility (typical choice 10^{-8})
# rho_1: 1st moment decay rate (typical choice 0.9)
# rho_2: 2nd moment decay rate (typical choice 0.999)
my_adam = function(X, Y, lr, beta_0, rho_1, rho_2, epsilon) {
  n = nrow(X)
  p = ncol(X)
  betahats = matrix(nrow=n, ncol=p)
  Ms = matrix(nrow=n, ncol=p) # 1st moment estimate
  Rs = matrix(nrow=n, ncol=p) # 2nd moment estimate
  Mhats = matrix(nrow=n, ncol=p) # 1st moment bias correction
  Rhats = matrix(nrow=n, ncol=p) # 2nd moment bias correction

  # initialize
  betahats[1, ] = beta_0
  Ms[1, ] = rep(0, p)
  Rs[1, ] = rep(0, p)

  for (t in 1:(n-1)) {
    x_t = as.matrix(X[t, ])
    beta_t = as.matrix(betahats[t, ])
    y_t_hat = t(beta_t)%*%x_t
```

```
    Y_t = Y[t]
    d_loss = 2*beta_t%*%t(x_t)%*%x_t - 2*x_t%*%Y_t
    Ms[t+1, ] = rho_1*as.matrix(Ms[t, ]) + (1-rho_1)*d_loss
    Rs[t+1, ] = rho_2*as.matrix(Rs[t, ]) + (1-rho_2)*d_loss^2
    Mhats[t+1, ] = Ms[t, ] / (1-rho_1^t)
    Rhats[t+1, ] = Rs[t, ] / (1-rho_2^t)

    betahats[t+1, ] = beta_t - lr*(Mhats[t+1, ]/(sqrt(Rhats[t+1, ]+epsilon))) # update
  } # end for
  return(betahats)
}
```

```
betahats = my_adam(X=X, Y=Y, lr=0.001, beta_0=runif(p), rho_1=0.9, rho_2=0.999, epsilon=10^(-8))

plot(betahats[n, ] - beta, main="Differences between last estimate and true beta")
```



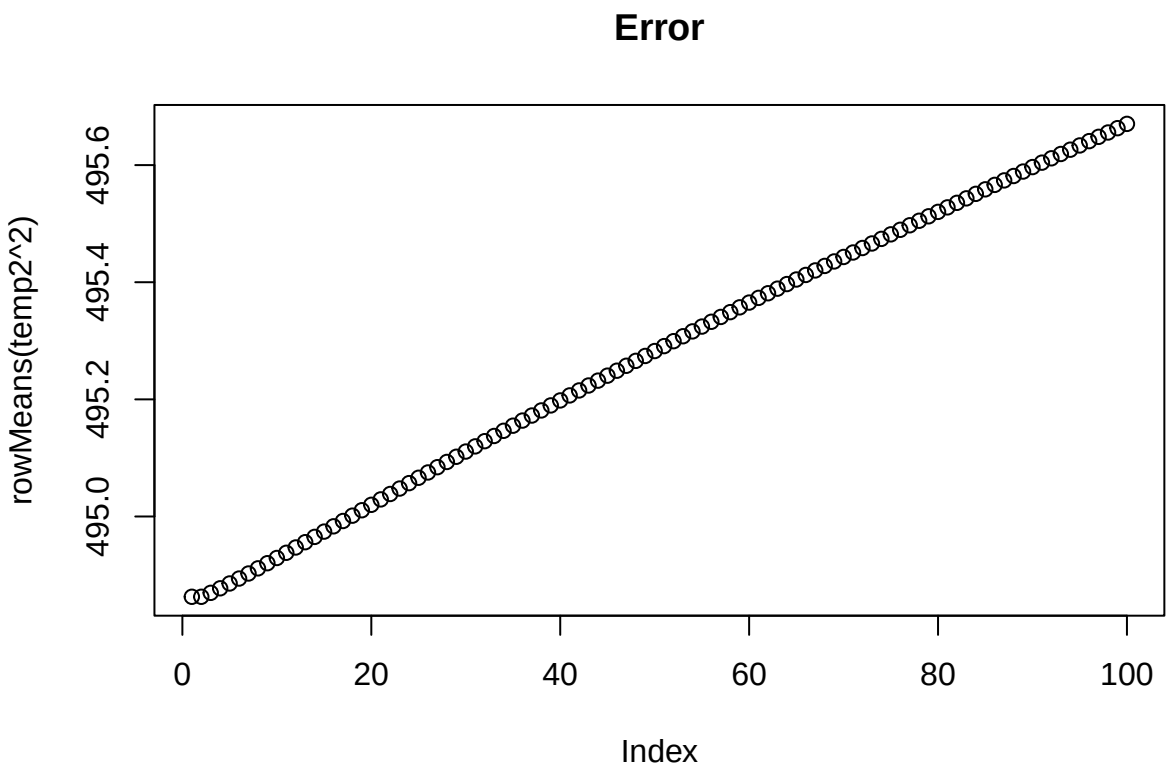

```
sum(abs(betahats[n, ] - beta) > 0.5*mean(beta[nonzero_indexes])) == k
```

```
## [1] TRUE
```

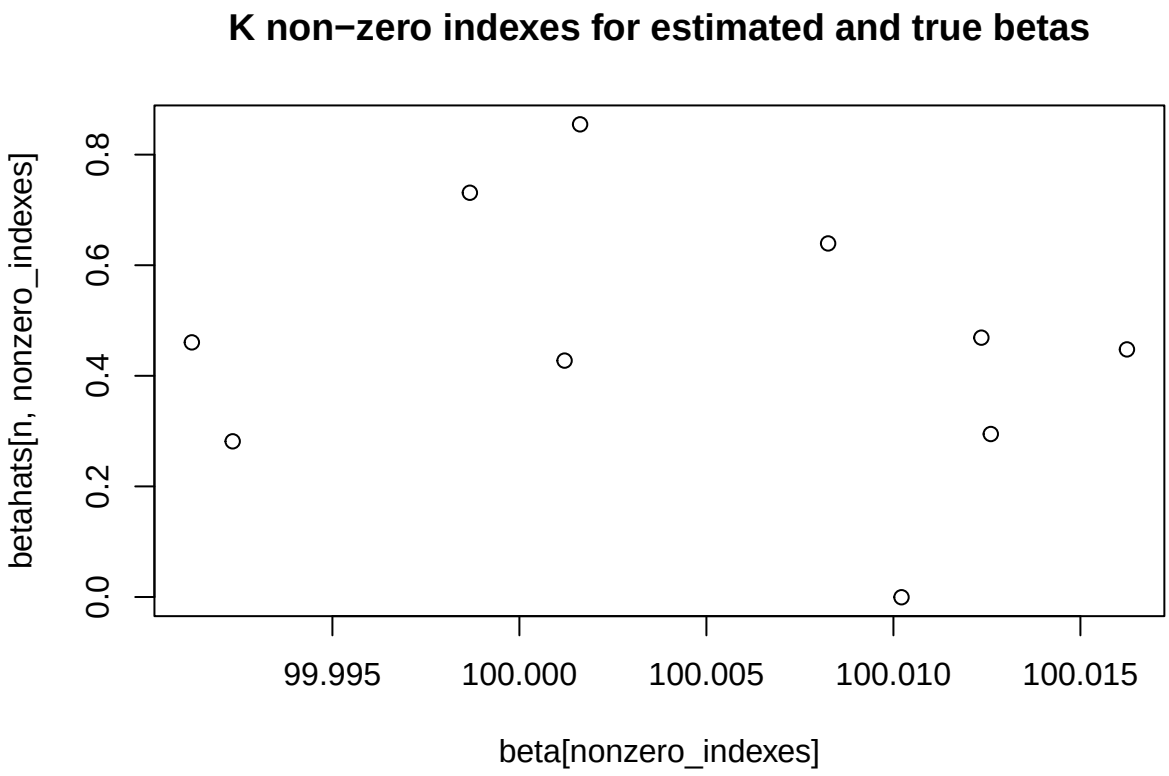
```
mean(betahats[n, nonzero_indexes]) - mean(betahats[n, -nonzero_indexes])
```

```
## [1] 0.06687797
```

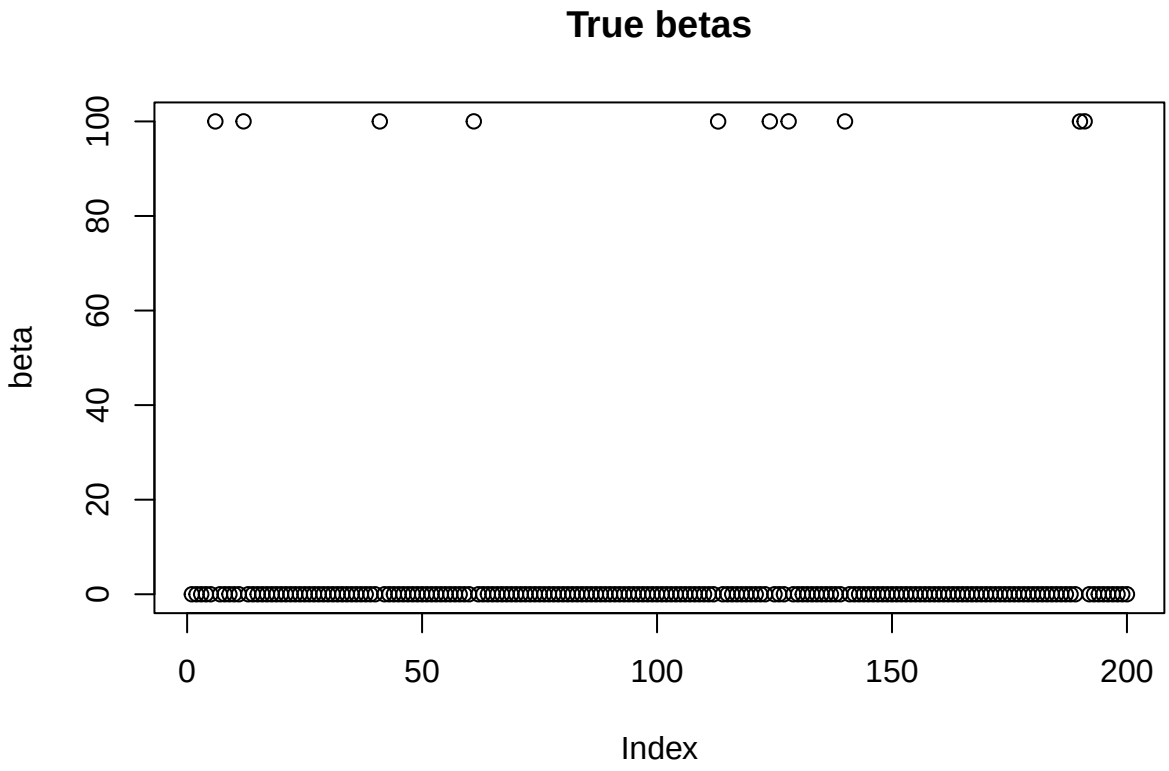
```
temp = matrix(rep(beta, n), nrow=n, ncol=p, byrow=T)
temp2 = betahats - temp
plot(rowMeans(temp2^2), main="Error")
```



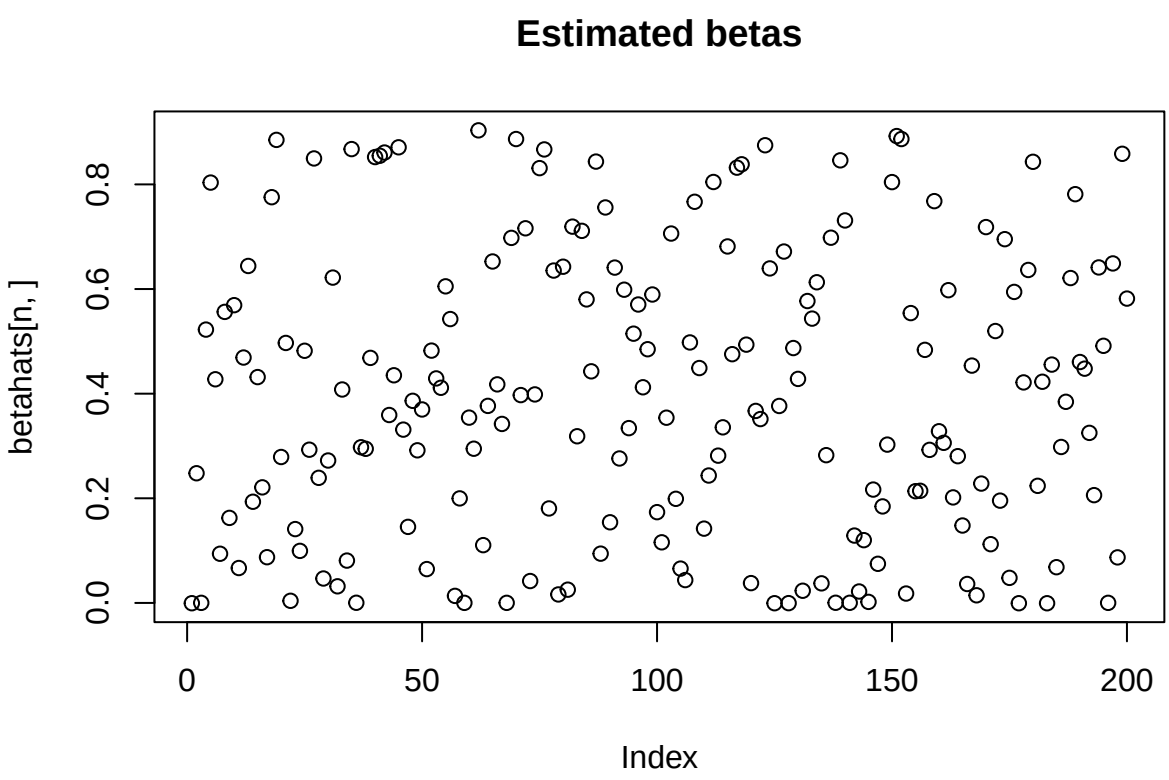

```
plot(beta[nonzero_indexes], betahats[n, nonzero_indexes], main="K non-zero indexes for estimated and tr
```




```
plot(beta, main="True betas")
```




```
plot(betahats[n, ], main="Estimated betas")
```


Estimated betas
betahats[n,]
0.0
0.2
0.4
0.6
0.8
0
50
100
150
200
Index

‘‘