Quiz

For $n \in \mathbb{N}$, let $f(n) = n^2 + 1001n + n^3$ and $g(n) = 10n^3$. Then $f(n)$ has the same asymptotic growth rate as $g(n)$ (meaning that $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}_+$).

true: $\lim\limits_{n\to\infty} \frac{n^2 + 1001n + 40n^3}{n^3} = \lim\limits_{n\to\infty} \Big(\underbrace{\frac{1}{n} + \frac{1001}{n^2}}_{\to 0} + 40\Big) = 40$

$n^4 \leq O\left(\frac{n^4}{\log(n)}\right)$

- True
- False

false: $\lim\limits_{n\to\infty} \frac{n^4}{\frac{n^4}{\log n}} = \lim\limits_{n\to\infty} n^4 \frac{\log n}{n^4} = \lim\limits_{n\to\infty} \log n = \infty$

$e^{3\ln(n)} \leq O(n^2)$

- True
- False

false: $e^{3\ln(n)} = e^{\ln(n^3)} = n^3 \not\leq O(n^2)$

Let $f(n) = 6n^2 + 5n + 10$. For each of the following definitions of $g(n)$, is $g(n) \leq O(f(n))$?

True	False	
		$g(n) = 123456789n - 200\sqrt{n}$
		$g(n) = 0.01n^2 \log(n)$
		$g(n) = 10n^3 + 5n + 1000$

true: $\lim\limits_{n\to\infty} \frac{1234567n - 200\sqrt{n}}{6n^2 + 5n + 10} = \lim\limits_{n\to\infty} \frac{123456789 - \overbrace{\frac{200}{\sqrt{n}}}^{\to 0}}{6n + 5 + \underbrace{\frac{10}{n}}_{\to 0}} = 0$

false: $\lim\limits_{n\to\infty} \frac{0.01n^2 \log n}{6n^2 + 5n + 10} = \lim\limits_{n\to\infty} \frac{0.01 \log n}{6 + \underbrace{\frac{5}{n} + \frac{10}{n^2}}_{\to 0}} = \infty$

false: $\lim\limits_{n\to\infty} \frac{10n^3 + 5n + 1000}{6n^2 + 5n + 10} = \lim\limits_{n\to\infty} \frac{10n + \overbrace{\frac{5}{n} + \frac{1000}{n^2}}^{\to 0}}{6 + \underbrace{\frac{5}{n} + \frac{10}{n^2}}_{\to 0}} = \infty$

Let $f : \mathbb{N} \to \mathbb{R}_+$ be some function for which we would like to prove $f(n) \geq n^2$ for every $n \geq 1$. Assume that you have proven that:

- $f(2) \geq 2^2$ → Base case with $k = 2$ instead of $k = 1$
- If $f(k) \geq k^2$ holds for an arbitrary positive integer k, then $f(k+1) \geq (k+1)^2$ holds.

Then, $f(n) \geq n^2$ holds for all positive integers $n \geq 1$. not for $n = 1$

false

Discussion Exercise Sheet 1

Exercise 1.1 *Sum of Cubes* **(1 point)**.

Prove by mathematical induction that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Wir iterieren über k

Base case: Sei $k = 1$. $\frac{1^2 \cdot (1+1)^2}{4} = \frac{4}{4} = 1 = 1^3$ ✓

I.H: Wir nehmen an, dass die Aussage für ein $k \geq 1$, $k \in \mathbb{N}$ gilt.

I.S: $\underbrace{1^3 + 2^3 + \ldots + k^3} + (k+1)^3 \overset{IH}{=} \underbrace{\frac{k^2(k+1)^2}{4}} + (k+1)^3$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4}$$

ausklammern

$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4}$$

binom. Formel

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2((k+1)+1)^2}{4} \quad \blacksquare$$

Exercise 1.2 *Sum of reciprocals of roots* **(1 point)**.

Consider the following claim:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq \sqrt{n}.$$

A student provides the following induction proof. Is it correct? If not, explain where the mistake is.

Base case: $n = 1$,

$$\frac{1}{\sqrt{1}} \leq 1, \text{ which is true.}$$

correct

Induction hypothesis: Assume the claim holds for $n = k$, i.e.

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \sqrt{k}.$$

correct

Erklären

Induction step: Then, starting from the claim we need to prove for $n = k+1$ and using logical equivalences:

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \sqrt{k+1} &\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \sqrt{k+1} - \frac{1}{\sqrt{k+1}} \\
&\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \frac{k+1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+1}} \\
&\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k+1}} \leq \frac{k}{\sqrt{k}} \leq \sqrt{k},
\end{aligned}$$

korrekte Termumformung

which is true, therefore the claim holds by the principle of mathematical induction.

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k+1}} &\;\cancel{\Leftarrow}\; \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k}} \\
&\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \sqrt{k}
\end{aligned}$$

Induktionsschritt in falsche Richtung: $k+1 \to k$

Exercise 1.4 *Proving Inequalities.*

(a) Prove the following inequality by mathematical induction

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n}\leq\frac{1}{\sqrt{3n+1}},\quad n\geq 1.$$

Base case: $\frac{1}{2} = \frac{1}{\sqrt{4}} = \frac{1}{\sqrt{3\cdot 1+1}}$ ✓

I.H: Wir nehmen an, dass die Aussage für ein $k\geq 1$ gilt.

I.S: $\underbrace{\frac{1}{2}\cdot\frac{3}{5}\cdot\ldots\cdot\frac{2k-1}{2k}}\cdot\frac{2k+1}{2k+2} \overset{\text{I.H.}}{\leq} \underbrace{\frac{1}{\sqrt{3k+1}}}\cdot\frac{2k+1}{2k+2}$

$\vdots$

$\leq \frac{1}{\sqrt{3(k+1)+1}}$

wollen wir zeigen

$$\frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2}\leq\frac{1}{\sqrt{3(k+1)+1}}$$

→ falls wir das zeigen können, ist der I.S. bewiesen!

$\Longleftrightarrow \frac{2k+1}{2k+2} \leq \frac{\sqrt{3k+1}}{\sqrt{3k+4}}$

Potenzgesetz

$\Longleftrightarrow \frac{2k+1}{2k+2} \leq \sqrt{\frac{3k+1}{3k+4}}$

beide Seiten quadrieren

$\Longleftrightarrow \frac{(2k+1)^2}{(2k+2)^2} \leq \frac{3k+1}{3k+4}$

$\Longleftrightarrow (2k+1)^2(3k+4) \leq (2k+2)^2(3k+1)$

$\Longleftrightarrow (4k^2+4k+1)(3k+4) \leq (4k^2+8k+4)(3k+1)$

$\Longleftrightarrow 12k^3+12k^2+3k+16k^2+16k+4 \leq 12k^3+24k^2+12k+4k^2+8k+4$

$\Longleftrightarrow \cancel{12k^3}+\cancel{28k^2}+19k+\cancel{4} \leq \cancel{12k^3}+\cancel{28k^2}+20k+\cancel{4}$

subtrahiere 19k

$\Longleftrightarrow 0\leq k$

Das heisst der I.S. ist gültig für alle $k\geq 0$. ∎

Exercise 1.3 *Asymptotic growth* **(1 point)**.

(b) $f(m) = \log(m^3)$ grows asymptotically slower than $g(m) = (\log m)^3$.

$$\lim_{m\to\infty} \frac{\log(m^3)}{\log(m)^3} = \lim_{m\to\infty} \frac{3\log(m)}{\log(m)^3} = \lim_{m\to\infty} \frac{3}{\log(m)^2} = 0$$

$\rightarrow$ True

(d)* If $f(m)$ grows asymptotically slower than $g(m)$, then $\log(f(m))$ grows asymptotically slower than $\log(g(m))$.

False. Counterexample: Let $f(m) = m$, $g(m) = m^2$

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{m}{m^2} = \lim_{m\to\infty} \frac{1}{m} = 0$$ $\rightarrow$ f grows slower than g

$$\lim_{m\to\infty} \frac{\log(f(m))}{\log(g(m))} = \lim_{m\to\infty} \frac{\log(m)}{\log(m^2)} = \lim_{m\to\infty} \frac{\log(m)}{2\log(m)} = \frac{1}{2}$$ $\rightarrow$ they have same growth

O- Notation

f und g sind Funktionen von N nach $\mathbb{R}^+$

Definition 1 (O-Notation). For $\underbrace{f : N \to \mathbb{R}^+}$, "es existiert eine Konstante C" "g ist höchstens um einen konstanten Faktor grösser als f"

$$O(f) := \{\underbrace{g : N \to \mathbb{R}^+} \mid \exists C > 0\, \underbrace{\forall n \in N}\, \overbrace{g(n) \leq C \cdot f(n)}\}$$

Definitionszeichen "bedingt dass" ↳ "sodass für alle Eingaben $n \in N$ gilt dass"

- O(f) ist die Menge aller Funktionen die asymptotisch höchstens so schnell wie f wachsen.
- Was f und g am Anfang (für kleine n) machen ist egal.
- Wir ignorieren konstante Faktoren
- Beispiel: $f(n) = n$, $g(n) = 3n$, dann gilt $g \in O(f)$ mit $C = 3$

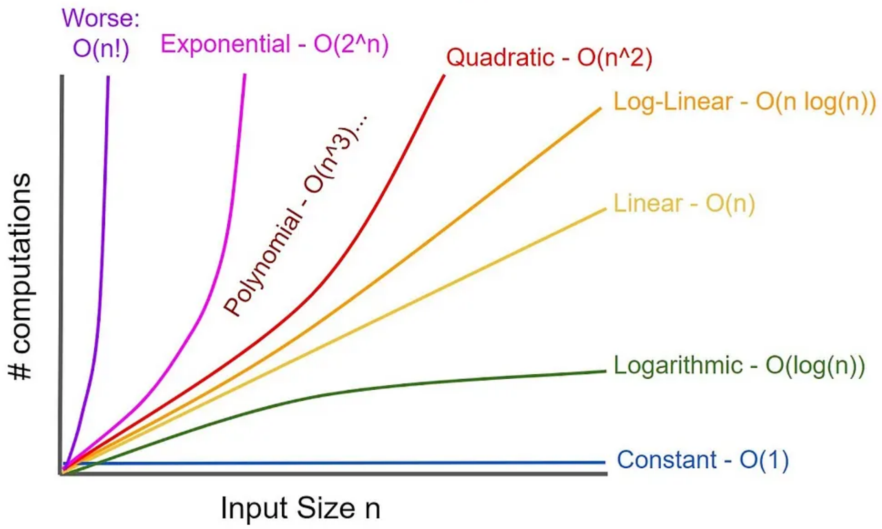
Theorem 1. *Let N be an infinite subset of $\mathbb{N}$ and $f : N \to \mathbb{R}^+$ and $g : N \to \mathbb{R}^+$.*

- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f \leq O(g)$ and $g \not\leq O(f)$.* g wächst schneller
- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C \in \mathbb{R}^+$, then $f \leq O(g)$ and $g \leq O(f)$.* beide wachsen gleich schnell
- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $f \not\leq O(g)$ and $g \leq O(f)$.* f wächst schneller

The following theorem can also be helpful when working with O-notation.

Theorem 2. *Let $f, g, h : N \to \mathbb{R}^+$. If $f \leq O(h)$ and $g \leq O(h)$, then*

1. *For every constant $c > 0$, $c \cdot f \leq O(h)$.* "wir ignorieren konstante Faktoren"
2. *$f + g \leq O(h)$.*



True or False: $\log n \leq O(n)$

$$\lim_{n\to\infty} \frac{\log n}{n} \overset{\text{L'Hôp.}}{=} \lim_{n\to\infty} \frac{1}{n} = 0$$

True or False: $2^n \leq O(n^5)$

$$\lim_{n\to\infty} \frac{2^n}{n^5} \overset{\text{L'Hôp}}{=} \lim_{n\to\infty} \frac{\ln(2)\, 2^n}{5n^4} \overset{\text{L'Hôp.}}{=} \lim_{n\to\infty} \frac{\ln(2)\ln(2)\, 2^n}{20n^3} \overset{\text{L'Hop.}}{=} \cdots = \lim_{n\to\infty} \frac{\ln(2)^5 \cdot 2^n}{120} = \infty$$

True or False: $n \log n \leq O(n^2)$

$$\lim_{n\to\infty} \frac{n \log n}{n^2} = \lim_{n\to\infty} \frac{\log n}{n} \overset{\text{L'Hôp}}{=} \lim_{n\to\infty} \frac{1}{n} = 0$$

True or False: $7n^4 + n \leq O(n^4)$

$$\lim_{n\to\infty} \frac{7n^4 + n}{n^4} = \lim_{n\to\infty} \frac{7n^4}{n^4} + \frac{n}{n^4} = \lim_{n\to\infty} 7 + \frac{1}{n^3} = 7$$

Order the following functions by their asymptotic growth rate as $n \to \infty$:

$$n!, \quad \sqrt{n}, \quad 2^n, \quad 1, \quad n^n, \quad \log n, \quad n, \quad n^{\frac{n}{2}}, \quad n^2$$

$$1 \leq \log n \leq \sqrt{n} \leq n \leq n^2 \leq 2^n \leq n^{\frac{n}{2}} \leq n! \leq n^n$$

Order the following functions:

$$n^{10}, \quad n \log n, \quad 3^n, \quad \sqrt{n}$$

$$\sqrt{n} \leq n \log n \leq n^{10} \leq 3^n$$

Order the following functions:

$$n^2 + \log n, \quad n + 100, \quad \sqrt{n} + \log^2 n, \quad n^3 + n$$

$$\sqrt{n} + \log^2 n \leq n + 100 \leq n^2 + \log n \leq n^3 + n$$

Order the following functions:

$$2^n + n^2, \quad n^4 + 3n, \quad n \log n + 5, \quad 3^n + n^3$$

$$n \log n + 5 \leq n^4 + 3n \leq 2^n + n^2 \leq 3^n + n^3$$

Order the following functions:

$$n^2 + 2^n, \quad n \log n + n^2, \quad n + \sqrt{n}, \quad 10^n + n^3$$

$$n + \sqrt{n} \leq n \log n + n^2 \leq n^2 + 2^n \leq 10^n + n^3$$