Quiz

$n^8 - 800n^7 - 10000n^6 + 200n^2 \leq O(0.25n^7)$

False

Let $f(n) = 4e^n$, for which of the following definitions of $g(n)$ is $f(n) \leq O(g(n))$?

True	False	
		$g(n) = n^4 + n^2$
		$g(n) = n^{100} + e^{100\ln(n)}$
		$g(n) = 2^{3n}$

False: $4e^n \not\leq O(n^4 + n^2)$

False: $e^{100 \ln n} = e^{\ln(n^{100})} = n^{100}$

$4e^n \not\leq O(2 \cdot n^{100})$

True: $2^{3n} = 8^n$, $4e^n \leq O(8^n)$ weil $8 > e$

You want to show using induction that a statement $A(n)$ holds for all n such that $n \geq 2$.

Which of the following combinations of *base case* and *induction step* would form a valid proof?

Wählen Sie eine oder mehrere Antworten:

- a. *Base case:* $A(2)$ holds.
  Induction step: $A(k) \implies A(k+1)$ for all integers $k \geq 2$
- b. *Base case:* $A(2), A(3)$ holds.
  Induction step: $A(k) \implies A(k+2)$ for all integers $k \geq 2$.
- c. *Base case:* $A(2)$ holds.
  Induction step: $A(k) \implies A(k+2)$ for all integers $k \geq 2$.

only proves A for even k

$\sum_{i=1}^{n} i^2 \leq O(n^2 \log(n))$?

False

$$\sum_{i=1}^{n} i^2 \geq \sum_{i=\frac{n}{2}}^{n} i^2 \geq \sum_{i=\frac{n}{2}}^{n} \left(\frac{n}{2}\right)^2 = \left(\frac{n}{2}\right)^3 = \frac{n^3}{8} \geq \Omega(n^3)$$

Consider the following recursive formula:

$T(1) = 0$

$T(n) = 2T(n/2) + 5n$

Assume n is of the form $n = 2^k$ for $k \in \{0, 1, 2, 3, \ldots\}$.

Is $T(n) \leq O(n)$?

$$T(n) = 2T\left(\frac{n}{2}\right) + 5n$$

$$= 2 \cdot \left(2T\left(\frac{n}{4}\right) + 5 \cdot \frac{n}{2}\right) + 5n$$

$$= 4T\left(\frac{n}{4}\right) + 2 \cdot 5n$$

$$= 4 \cdot \left(2 \cdot T\left(\frac{n}{8}\right) + 5\,\frac{n}{4}\right) + 2 \cdot 5n$$

$$= 8 \cdot T\left(\frac{n}{8}\right) + 3 \cdot 5n$$

$$\vdots$$

$$= \underbrace{n \cdot T(1)}_{=0} + \log_2(n) \cdot 5n \quad \nleq O(n)$$

Induktion

Exercise 2.1 *Induction.*

(a) Prove via mathematical induction that for all integers $n \geq 5$,

$$2^n > n^2 .$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

Wir iterieren über k

Base case: Sei $k = 5$. $2^5 = 32 > 25 = 5^2$. Base case gilt.

I.H: Wir nehmen an, dass die Aussage für ein $k \geq 5, k \in \mathbb{N}$ gilt,

also $2^k > k^2$

I.S: Wir wollen zeigen, dass die Aussage dann auch für $k+1$ gilt.

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&\overset{IH}{>} 2 \cdot k^2 \\
&= k^2 + k^2 \\
&\geq k^2 + 5k && \text{weil } k \geq 5 \\
&= k^2 + 2k + 3k \\
&> k^2 + 2k + 1 && \text{weil } k \geq 5 \\
&= (k+1)^2
\end{aligned}$$

$\blacksquare$

(b) Let x be any real number. Prove via mathematical induction that for every positive integer n, we have

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \,,$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \,.$$

Recall that the factorial of a positive integer n is defined as $n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$. We use a standard convention $0! = 1$, so $\binom{n}{0} = \binom{n}{n} = 1$ for every positive integer n.

In your solution, you should address the base case, the induction hypothesis and the induction step.

Hint: *You can use the following fact without proof: for every* $1 \le i \le n$,

$$\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i} .$$

Wir iterieren über k

Base case: Sei $k = 1$.

$$\sum_{i=0}^{1} \binom{1}{i} x^i = \binom{1}{0} x^0 + \binom{1}{1} x^1$$
$$= 1 \cdot 1 + 1 \cdot x$$
$$= 1 + x$$
$$= (1+x)^1$$

Base case gilt.

I.H: Wir nehmen an, dass die Aussage für ein $k \geq 1$, $k \in \mathbb{N}$ gilt,

also $\underbrace{(1+x)^k = \sum_{i=0}^{k} \binom{k}{i} x^i}$

I.S: Wir wollen zeigen, dass die Aussage dann auch für $k+1$ gilt.

$$(1+x)^{k+1} = (1+x)(1+x)^k$$
$$\overset{\text{I.H.}}{=} (1+x) \cdot \sum_{i=0}^{k} \binom{k}{i} x^i$$
ausmultiplizieren
$$= \sum_{i=0}^{k} \binom{k}{i} x^i + \sum_{i=0}^{k} \binom{k}{i} x^{i+1}$$
Index shifting
$$= \sum_{i=0}^{k} \binom{k}{i} x^i + \sum_{i=1}^{k+1} \binom{k}{i-1} x^i$$
Summanden raus ziehen

$$= \binom{k}{0}x^0 + \sum_{i=1}^{k}\binom{k}{i}x^i + \sum_{i=1}^{k}\binom{k}{i-1}x^i + \binom{k}{k}x^{k+1}$$

Distributiv Gesetz

$$= \binom{k}{0}x^0 + \sum_{i=1}^{k}\left(\binom{k}{i}+\binom{k}{i-1}\right)x^i + \binom{k}{k}x^{k+1}$$

Hint

$$= \binom{k+1}{0}x^0 + \sum_{i=1}^{k}\binom{k+1}{i}x^i + \binom{k+1}{k+1}x^{k+1}$$

Summe zusammenfassen

$$= \sum_{i=0}^{k+1}\binom{k+1}{i}x^i$$

$\blacksquare$

Exercise 2.2 *Fibonacci numbers* **(1 point)**.

There are a lot of neat properties of the Fibonacci numbers that can be proved by induction. Recall that the Fibonacci numbers are defined by $f_0 = 0$, $f_1 = 1$ and the recursion relation $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$. For example, $f_2 = 1$, $f_5 = 5$, $f_{10} = 55$, $f_{15} = 610$.

Prove that $f_n \geq \frac{1}{3} \cdot 1.5^n$ for $n \geq 1$.

Base case. Sei $k=1$. $f_1 = 1 \geq 0.5 = \frac{1}{3}\cdot 1.5^1$

Sei $k=2$. $f_2 = 1 \geq 0.75 = \frac{1}{3}\, 1.5^2$ stimmt für $k=1$ und $k=2$

I.H. Wir nehmen an, dass die Aussage für ein $k \in \mathbb{N}$ gilt und für $k+1$

Das heisst $f_k \geq \frac{1}{3}\cdot 1.5^k$ und $f_{k+1} \geq \frac{1}{3}\cdot 1.5^{k+1}$

I.S. Wir wollen zeigen, dass die Aussage dann auch für $k+2$ gilt.

(Idee: k und $k+1 \to k+2$)

$$f_{k+2} \overset{\text{def.}}{=} f_{k+1} + f_k$$
$$\overset{\text{IH}}{\geq} \frac{1}{3}\cdot 1.5^{k+1} + \frac{1}{3}\cdot 1.5^k$$

ausklammern

$$= \frac{1}{3}\cdot 1.5^k\,(1.5+1)$$
$$= \frac{1}{3}\cdot 1.5^k \cdot 2.5$$

$1.5^2 = 2.25 > 2.5$

$$> \frac{1}{3}\cdot 1.5^k \cdot 1.5^2$$

Potenzgesetze

$$= \frac{1}{3}\, 1.5^{k+2}$$

$\blacksquare$

Exercise 2.4 *Asymptotic growth of* $\sum_{i=1}^{n} \frac{1}{i}$ **(1 point)**.

The goal of this exercise is to show that the sum $\sum_{i=1}^{n} \frac{1}{i}$ behaves, up to constant factors, as $\log(n)$ when n is large. Formally, we will show $\sum_{i=1}^{n} \frac{1}{i} \leq O(\log n)$ and $\log n \leq O(\sum_{i=1}^{n} \frac{1}{i})$ as functions from $\mathbb{N}_{\geq 2}$ to $\mathbb{R}^{+}$.

For parts (a) to (c) we assume that $n = 2^k$ is a power of 2 for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will generalise the result to arbitrary $n \in \mathbb{N}$ in part (d). For $j \in \mathbb{N}$, define

$$S_j = \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i}.$$

(a) For any $j \in \mathbb{N}$, prove that $S_j \leq 1$.

Hint: Find a common upper bound for all terms in the sum and count the number of terms.

Welches ist der grösste Summand von S_j? $\rightarrow \frac{1}{2^{j-1}+1} \leq \underbrace{\frac{1}{2^{j-1}}}_{\text{erleichtert das Rechnen}}$

Wie viele Summanden hat S_j? $\rightarrow 2^{j-1}$ viele

$\Rightarrow S_j \leq 2^{j-1} \cdot \frac{1}{2^{j-1}} = 1$

(b) For any $j \in \mathbb{N}$, prove that $S_j \geq \frac{1}{2}$.

Welches ist der kleinste Summand von S_j? $\rightarrow \frac{1}{2^j}$

Wie viele Summanden hat S_j? $\rightarrow 2^{j-1}$ viele

$\Rightarrow S_j \geq 2^{j-1} \cdot \frac{1}{2^j} = \frac{2^{j-1}}{2^j} = 2^{-1} = \frac{1}{2}$

(c) For any $k \in \mathbb{N}_0$, prove the following two inequalities

$$\sum_{i=1}^{2^k} \frac{1}{i} \leq k+1$$

and

$$\sum_{i=1}^{2^k} \frac{1}{i} \geq \frac{k+1}{2}.$$

Hint: You can use that $\sum_{i=1}^{2^k} \frac{1}{i} = 1 + \sum_{j=1}^{k} S_j$. *Use this, together with parts (a) and (b), to prove the required inequalities.*

$$\sum_{i=1}^{2^k} \frac{1}{i} = 1 + \sum_{j=1}^{k} S_j \quad \Big\} \; S_j \leq 1 \quad \text{aus Teil a}$$
$$\leq 1 + \sum_{j=1}^{k} 1$$
$$= 1 + k$$

$$\sum_{i=1}^{2^k} \frac{1}{i} = 1 + \sum_{j=1}^{k} S_j \quad \Big\} \; S_j \leq 1 \quad \text{aus Teil b}$$
$$\geq 1 + \sum_{j=1}^{k} \frac{1}{2}$$
$$> \frac{1}{2} + \frac{k}{2}$$
$$= \frac{k+1}{2}$$

(d)* For arbitrary $n \in \mathbb{N}$, prove that

$$\sum_{i=1}^{n} \frac{1}{i} \leq \log_2(n) + 2$$

and

$$\sum_{i=1}^{n} \frac{1}{i} \geq \frac{\log_2 n}{2}.$$

Hint: *Use the result from part (c) for $k_1 = \lceil \log_2 n \rceil$ and $k_2 = \lfloor \log_2 n \rfloor$. Here, for any $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer that is at least x and $\lfloor x \rfloor$ is the largest integer that is at most x. For example, $\lceil 1.5 \rceil = 2$, $\lfloor 1.5 \rfloor = 1$ and $\lceil 3 \rceil = \lfloor 3 \rfloor = 3$. In particular, for any $x \in \mathbb{R}$, $x \leq \lceil x \rceil < x + 1$ and $x \geq \lfloor x \rfloor > x - 1$.*

$$\sum_{i=1}^{n} \frac{1}{i} \leq \sum_{i=1}^{2^{k_1}} \frac{1}{i}$$
$$\leq k_1 + 1$$
$$= \lceil \log_2 n \rceil + 1$$
$$\leq \log_2 n + 2$$

$$\sum_{i=1}^{n} \frac{1}{i} \geq \sum_{i=1}^{2^{k_2}} \frac{1}{i}$$
$$\geq \frac{k_2 + 1}{2}$$
$$= \frac{\lfloor \log_2 n \rfloor + 1}{2}$$
$$\geq \frac{(\log_2 n - 1) + 1}{2}$$
$$= \frac{\log_2 n}{2}$$

(a) Show that $\ln(n!) \leq O(n \ln n)$.

Hint: *You can use the fact that* $n! \leq n^n$ *for* $n \in \mathbb{N}_{\geq 2}$ *without proof.*

(b) Show that $n \ln n \leq O(\ln(n!))$.

Hint: *You can use the fact that* $\left(\frac{n}{2}\right)^{\frac{n}{2}} \leq n!$ *for* $n \in \mathbb{N}_{\geq 2}$ *without proof.*

merken für Prüfung!

a) $\ln(n!) \leq \ln(n^n)$

$= n \cdot \ln(n)$

$\leq O(n \ln n)$

b) $$\lim_{n\to\infty} \frac{n \ln n}{\ln(n!)} \leq \lim_{n\to\infty} \frac{n \ln n}{\ln\left(\frac{n}{2}^{\frac{n}{2}}\right)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n} \ln n}{\frac{\cancel{n}}{2} \ln\left(\frac{n}{2}\right)}$$

$$= \lim_{n\to\infty} \frac{2 \ln n}{(\ln n - \ln 2)}$$

$$= \lim_{n\to\infty} \frac{2 \ln n}{\ln n - \ln 2}$$

$$= \lim_{n\to\infty} \frac{2}{1 - \underbrace{\frac{\ln 2}{\ln n}}_{\to 0}}$$

$$= 2$$

Maximum Subarray Sum

Gegeben: Array der Länge n, Schreibweise $A[1..n]$ oder $(a_1, a_2, \ldots, a_n)$

Gesucht: Startindex i und Endindex j mit $1 \le i \le j \le n$

sodass $\sum_{k=i}^{j} a_k$ maximal ist

Teilproblem: Randmaximum R_i

R_i := Grösste Summe eines Subarrays, das in Index i endet

Array: $[1, 3, 2, -1, -4, 6, -20, 0, 2]$

Randmax: $[1, 4, 6, 5, 1, 7, -13, 0, 2]$

Lösung = $\max\limits_{1 \le i \le n} R_i$ oder 0 falls alle Array Elemente negativ sind

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MSS-INDUKTIV$(a_1, \ldots, a_n)$

```
randmax ← 0
maxS ← 0
Für i ← 1, ..., n:
    randmax ← randmax + a_i
    Wenn randmax > maxS:
        maxS ← randmax
    Wenn randmax < 0:
        randmax ← 0
Gib maxS zurück.
```

Laufzeit: $O(n)$

Θ und Ω Notation

$O(f) :=$ Menge aller Funktionen, die langsamer oder gleich schnell wachsen

$\Theta(f) :=$ Menge aller Funktionen, die gleich schnell wachsen

$\Omega(f) =$ Menge aller Funktionen, die gleich schnell oder schneller wachsen

Mathematisch:

$\Theta(f) := \{ g: \mathbb{N} \to \mathbb{R}^+ \mid f \leq O(g)$ und $g \leq O(f) \} = O(f) \cap \Omega(f)$

$\Omega(f) := \{ g: \mathbb{N} \to \mathbb{R}^+ \mid f \leq O(g) \}$

Beweismethode für Summen ☆

Show that $\sum_{i=1}^{n} i^3 = \Theta(n^4)$

Upper Bound: $\sum_{i=1}^{n} i^3 \leq \sum_{i=1}^{n} n^3 = n^4 \leq O(n^4)$

Lower Bound: $\sum_{i=1}^{n} i^3 \geq \sum_{i=\frac{n}{2}}^{n} i^3 \geq \sum_{i=\frac{n}{2}}^{n} \left(\frac{n}{2}\right)^3 = \left(\frac{n}{2}\right)^4 = \frac{1}{16} n^4 \geq \Omega(n^4)$

↳ wir nehmen nur die obere Hälfte der Summe

Show that $\sum_{i=1}^{n} \sqrt{i} = \Theta(n^{1.5})$

Upper Bound: $\sum_{i=1}^{n} \sqrt{i} \leq \sum_{i=1}^{n} \sqrt{n} = n \sqrt{n} = n^{1.5} \leq O(n^{1.5})$

Lower Bound: $\sum_{i=1}^{n} \sqrt{i} \geq \sum_{i=\frac{n}{2}}^{n} \sqrt{i} \geq \sum_{i=\frac{n}{2}}^{n} \sqrt{\frac{n}{2}} = \frac{n}{2} \cdot \sqrt{\frac{n}{2}} = \frac{n^{1.5}}{2\sqrt{2}} \geq \Omega(n^{1.5})$

Counting function calls

Algorithm 3

$i \leftarrow 0$
while $i \leq 3n$ **do**
  $f()$
  $i \leftarrow i + 1$

$j \leftarrow 1$
while $j \leq n + 5$ **do**
  $g()$
  $g()$
  $j \leftarrow j + 1$

$$\#calls = \sum_{i=0}^{3n} 1 + \sum_{j=1}^{n+5} 2$$

$$= 3n+1 + 2(n+5)$$

$$= 3n + 1 + 2n + 10$$

$$= 5n + 11$$

$$= \Theta(n)$$

Algorithm 4

$i \leftarrow 1$
while $i \leq n$ **do**
  $j \leftarrow 1$
  while $j \leq n$ **do**
    $f()$
    $j \leftarrow j + 1$
  $i \leftarrow i + 1$

$k \leftarrow 0$
while $k \leq 2n$ **do**
  $g()$
  $k \leftarrow k + 1$

$$\#calls = \sum_{i=1}^{n} \sum_{j=1}^{n} 1 + \sum_{k=0}^{2n} 1$$

$$= \sum_{i=1}^{n} n + 2n+1$$

$$= n^2 + 2n + 1$$

$$= \Theta(n^2)$$