First Exercise Session

Organisatorische Informationen.

- Übungsstunden am Montag 09.15, HG D 5.1
- Bonussystem
    - Mini-Quizzes (1P)
    - Wöchentliche Aufgaben (3P)
    - Peergrading (1P)
    - Code Expert (6P)
- Moodle
- n.ethz.ch/~jheger

Allgemeine Tips:

- Verstehen ist wichtiger als Bonuspunkte
- Vorlesungen am Anfang besuchen
- Übungsstunden sind wichtiger als Vorlesungen, aber kein Ersatz.
- Veschwendet nicht zu viel Zeit mit schwierigen Aufgaben
- ChatGPT kann sehr hilfreich sein je nach Einsatz
- Ihr müsst nicht 100% der Aufgaben lösen können
- Tablet ist nützlich aber nicht notwendig
- DiskMat wird schwierig
- EProg ist einfach in der ersten Hälfte (falls man schon programmieren kann)
- Jeder hat mal einen schlechten Tag
- ASVZ

Aufgaben

Exercise 0.1 *Induction.*

(a) Prove by mathematical induction that for any positive integer n,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

wir iterieren über k

Base Case: für $k=1$ gilt: $1 = \frac{1(1+1)}{2}$ was den base case beweist

I.H. Wir nehmen an dass die Aussage für ein beliebiges $k \in \mathbb{N}$ gilt.

I.S: Wir wollen zeigen, dass die Aussage dann auch für $k+1$ gilt:

$$1 + 2 + \ldots + k + (k+1) \overset{\text{IH.}}{=} \frac{k \cdot (k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2} \quad \blacksquare$$

(b) **(This subtask is from August 2019 exam).** Let $T : \mathbb{N} \to \mathbb{R}$ be a function that satisfies the following two conditions:

$$T(n) \geq 4 \cdot T(\tfrac{n}{2}) + 3n \qquad \text{whenever } n \text{ is divisible by 2;}$$
$$T(1) = 4.$$

Prove by mathematical induction that

$$T(n) \geq 6n^2 - 2n$$

holds whenever n is a power of 2, i.e., $n = 2^k$ with $k \in \mathbb{N}_0$. In your solution, you should address the base case, the induction hypothesis and the induction step.

Wir iterieren über k

Base Case: Sei $k=0$, dann ist $n = 2^0 = 1$

$T(1) = 4 \geq 6 \cdot 1^2 - 2 \cdot 1$ holds!

I.H Wir nehmen an, dass die Ungleichung für ein $k \in \mathbb{N}_0$ gilt.

I.S: Wir wollen zeigen, dass die Ungleichung dann auch für $k+1$ gilt.

Sei $n = 2^k \Rightarrow 2^{k+1} = 2n$

Sei $m = 2^k$

$$\begin{aligned}
T(2m) &\geq 4 \cdot T(m) + 3(2m) \\
&\overset{\text{I.H.}}{\geq} 4\left(6m^2 - 2m\right) + 6m \\
&= 4\, 6m^2 - 2m \\
&= 6\,(2m)^2 - 2m \\
&\geq 6 \cdot (2m)^2 - 4m \qquad \text{subtrahiere } 2m \\
&= 6 \cdot (2m)^2 - 2(2m)
\end{aligned}$$

$\blacksquare$

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Asymptotic Growth

When we estimate the number of elementary operations executed by algorithms, it is often useful to ignore smaller order terms, and instead focus on the asymptotic growth defined below. We denote by $\mathbb{R}^+$ the set of all (strictly) positive real numbers and by $\mathbb{R}_0^+$ the set of nonnegative real numbers.

Definition 1. Let $f, g : \mathbb{N} \to \mathbb{R}^+$ be two functions. We say that f *grows asymptotically faster than* g if $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$.

This definition is also valid for functions defined on $\mathbb{R}^+$ instead of $\mathbb{N}$. In general, $\lim\limits_{n\to\infty} \frac{g(n)}{f(n)}$ is the same as $\lim\limits_{x\to\infty} \frac{g(x)}{f(x)}$ if the second limit exists.

For all the following exercises, you can assume that $n \in \mathbb{N}_{\geq 10}$. We make this assumption so that all functions are well-defined and take values in $\mathbb{R}^+$.

Exercise 0.2 *Comparison of functions part 1.*

Show that

(a) $f(n) := n \log n$ grows asymptotically faster than $g(n) := n$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n}{n \cdot \log n} = \lim_{n\to\infty} \frac{1}{\log n} = 0$$

(b) $f(n) := n^3$ grows asymptotically faster than $g(n) := 10n^2 + 100n + 1000$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \frac{10n^2 + 100n + 1000}{n^3} = \lim_{n\to\infty} \left(\frac{10}{n} + \frac{100}{n^2} + \frac{1000}{n^3} \right)$$

$$= 0$$

(c) $f(n) := 3^n$ grows asymptotically faster than $g(n) := 2^n$.

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

The following theorem can be useful to compute some limits.

Theorem 1 (L'Hôpital's rule). *Assume that functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ and for all $x \in \mathbb{R}^+$, $g'(x) \neq 0$. If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = C \in \mathbb{R}_0^+$ or $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \infty$, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Exercise 0.3 *Comparison of functions part 2.*

Show that

(a) $f(n) := n^{1.01}$ grows asymptotically faster than $g(n) := n \ln n$.

$$\lim_{n\to\infty} \frac{n \cdot \ln n}{n^{1.01}} = \lim_{n\to\infty} \frac{\ln n}{n^{0.01}} \overset{\text{L'Hôp.}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{0.01 n^{-0.99}} = \lim_{n\to\infty} \frac{1}{0.01 n^{0.01}} = 0$$

(b) $f(n) := e^n$ grows asymptotically faster than $g(n) := n$.

$$\lim_{n\to\infty} \frac{n}{e^n} \overset{\text{L'Hôp.}}{=} \lim_{n\to\infty} \frac{1}{e^n} = 0$$

(c) $f(n) := e^n$ grows asymptotically faster than $g(n) := n^2$.

$$\lim_{n\to\infty} \frac{n^2}{e^n} \overset{\text{L'Hôp.}}{=} \lim_{n\to\infty} \frac{2n}{e^n} \overset{\text{L'Hôp}}{=} \lim_{n\to\infty} \frac{2}{e^n} = 0$$

(d)* $f(n) := 1.01^n$ grows asymptotically faster than $g(n) := n^{100}$.

$\ln(x^a) = a\ln(x)$

$x = e^{\ln(x)}$

$$\lim_{n\to\infty} \frac{n^{100}}{1.01^n} = \lim_{n\to\infty} \frac{e^{\ln(n^{100})}}{e^{\ln(1.01^n)}} = \lim_{n\to\infty} \frac{e^{100\ln(n)}}{e^{n\ln(1.01)}} = \lim_{n\to\infty} e^{\underbrace{100\ln(n) - n\ln(1.01)}_{\to -\infty}}$$
$$= 0$$

$$\lim_{n\to\infty} 100\ln(n) - n\ln(1.01) = \underbrace{\underbrace{\left(\lim_{n\to\infty} n\right)}_{\to\infty} \underbrace{\left(\underbrace{\lim_{n\to\infty} 100 \frac{\ln(n)}{n}}_{\to 0} - \underbrace{\underbrace{\ln(1.01)}_{>0}}_{<0}\right)}_{<0}} = -\infty$$

Andere Möglichkeit: 100 mal L'Hôpital oder Induktion

(e) $f(n) := \log_2 n$ grows asymptotically faster than $g(n) := \log_2 \log_2 n$.

$\log_2 x = \frac{\log_a x}{\log_a 2}$

Substitution: Sei $y := \log_2 n$

$$\lim_{n\to\infty} \frac{\log_2 \log_2 n}{\log_2 n} = \lim_{y\to\infty} \frac{\log_2 y}{y} = \lim_{y\to\infty} \frac{\ln(y)}{\ln(2) y} \overset{\text{L'Hôp}}{=} \lim_{y\to\infty} \frac{\frac{1}{y}}{\ln(2)} = 0$$

(f) $f(n) := 2^{\sqrt{\log_2 n}}$ grows asymptotically faster than $g(n) := \log_2^{100} n$.

$$\lim_{n\to\infty} \frac{\log_2^{100} n}{2^{\sqrt{\log_2 n}}} = \lim_{n\to\infty} \frac{2^{\log_2(\log_2^{100} n)}}{2^{\sqrt{\log_2 n}}} = \lim_{n\to\infty} \frac{2^{100 \log_2 \log_2 n}}{2^{\sqrt{\log_2 n}}} = \lim_{n\to\infty} 2^{\underbrace{100 \log_2 \log_2 n - \sqrt{\log_2 n}}_{\to -\infty}}$$

$$= 0$$

Substitution $y := \log_2 n$

$$\lim_{y\to\infty} 100 \log_2 y - \sqrt{y} = \underbrace{\left(\lim_{y\to\infty} -\sqrt{y}\right)}_{\to -\infty} \underbrace{\left(\lim_{y\to\infty} 1 - 100 \underbrace{\frac{\log_2 y}{\sqrt{y}}}_{\to 0}\right)}_{\to 1} = -\infty$$

$$\lim_{y\to\infty} \frac{\log_2 y}{\sqrt{y}} = \lim_{y\to\infty} \frac{\ln y}{\ln 2 \sqrt{y}} \overset{\text{L'Hôp}}{=} \lim_{y\to\infty} \frac{\frac{1}{y}}{\ln 2 \cdot \frac{1}{2\sqrt{y}}} = \lim_{y\to\infty} \frac{1}{\ln 2 \cdot \frac{\sqrt{y}}{2}} = 0$$

(g) $f(n) := n^{0.01}$ grows asymptotically faster than $g(n) := 2^{\sqrt{\log_2 n}}$.

$$\lim_{n\to\infty} \frac{2^{\sqrt{\log_2 n}}}{n^{0.01}} = \lim_{n\to\infty} \frac{2^{\sqrt{\log_2 n}}}{2^{\log_2(n^{0.01})}} = \lim_{n\to\infty} \frac{2^{\sqrt{\log_2 n}}}{2^{0.01 \log(n)}} = \lim_{n\to\infty} 2^{\underbrace{\sqrt{\log_2 n} - 0.01 \log_2 n}_{\to -\infty}}$$

$$= 0$$

Substitution: sei $y := \log_2 n$

$$\lim_{n\to\infty} \sqrt{\log_2 n} - 0.01 \log_2 n = \lim_{y\to\infty} \sqrt{y} - 0.01 y = \left(\lim_{y\to\infty} -0.01 y\right) \left(\lim_{y\to\infty} 1 - \frac{\sqrt{y}}{0.01 y}\right)$$

$$= \underbrace{\left(\lim_{y\to\infty} -0.01 y\right)}_{\to -\infty} \underbrace{\left(\lim_{y\to\infty} 1 - \underbrace{\frac{1}{0.01 \sqrt{y}}}_{\to 0}\right)}_{\to 1}$$

$$= -\infty$$

Exercise 0.4 *Simplifying expressions.*

Simplify the following expressions as much as possible without changing their asymptotic growth rates.

Concretely, for each expression $f(n)$ in the following list, find an expression $g(n)$ that is as simple as possible and that satisfies $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+$.

(a) $f(n) := 5n^3 + 40n^2 + 100$

$$g(n) = n^3, \quad \lim_{n\to\infty} \frac{5n^3+40n^2+100}{n^3} = \lim_{n\to\infty} \frac{5\cancel{n^3}}{\cancel{n^3}} + \underbrace{\frac{40\cancel{n^2}}{n^{\cancel{3}}}}_{\to 0} + \underbrace{\frac{100}{n^3}}_{\to 0} = 5$$

(b) $f(n) := 5n + \ln n + 2n^3 + \frac{1}{n}$

$$g(n) = n^3, \quad \lim_{n\to\infty} \frac{5n+\ln n+2n^3+\frac{1}{n}}{n^3} = \lim_{n\to\infty} \underbrace{\frac{5\cancel{n}}{n^{\cancel{3}2}}}_{\to 0} + \underbrace{\frac{\ln n}{n^3}}_{\to 0} + \frac{2\cancel{n^3}}{\cancel{n^3}} + \underbrace{\frac{\frac{1}{n}}{n^3}}_{\to 0} = 2$$

$$\text{weil } \lim_{n\to\infty} \frac{\ln n}{n^3} \overset{\text{L'Hôp}}{=} \lim_{n\to\infty} \frac{1}{3n^3} = 0$$

(c) $f(n) := n \ln n - 2n + 3n^2$

$$g(n) = n^2, \quad \lim_{n\to\infty} \frac{n\ln n - 2n + 3n^2}{n^2} = \lim_{n\to\infty} \underbrace{\frac{\cancel{n}\ln n}{n^{\cancel{2}}}}_{\to 0} - \underbrace{\frac{2\cancel{n}}{n^{\cancel{2}}}}_{\to 0} + \frac{3\cancel{n^2}}{\cancel{n^2}} = 3$$

$$\text{weil } \lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{L'Hôp}}{=} \lim_{n\to\infty} \frac{1}{n} = 0$$

(d) $f(n) := 23n + 4n \log_5 n^6 + 78\sqrt{n} - 9$

$$g(n) = n \ln n, \quad \lim_{n\to\infty} \underbrace{\frac{23\cancel{n}}{\cancel{n}\ln n}}_{\to 0} + \frac{4\cancel{n}\log_5 n^6}{\cancel{n}\ln n} + \underbrace{\frac{78\cancel{\sqrt{n}}}{\sqrt{n}\ln n}}_{\to 0} + \underbrace{\frac{9}{\ln n}}_{\to 0}$$

$$= \lim_{n\to\infty} \frac{4\cdot 6 \log_5 n}{\ln n} = \lim_{n\to\infty} \frac{24\cancel{\ln n}}{\ln 5 \cdot \cancel{\ln n}} = \frac{24}{\ln 5}$$

(e) $f(n) := \log_2 \sqrt{n^5} + \sqrt{\log_2 n^5}$

$$g(n) = \ln n, \quad \lim_{n\to\infty} \frac{\log_2 \sqrt{n^5} + \sqrt{\log_2 n^5}}{\ln n} = \lim_{n\to\infty} \frac{\frac{5}{2}\log_2 n}{\ln n} + \frac{\sqrt{5 \log_2 n}}{\ln n}$$

$$= \lim_{n\to\infty} \frac{\frac{5}{2}\cancel{\ln n}}{\ln 2 \cancel{\ln n}} + \underbrace{\frac{\sqrt{\frac{5}{\ln 2}} \cdot \cancel{\sqrt{\ln n}}}{\sqrt{\ln n}}}_{\to 0}$$

$$= \frac{5}{2 \ln 2}$$

(f)* $f(n) := 2n^3 + \underbrace{(\sqrt[4]{n})^{\log_5 \log_6 n}}_{n^{\frac{1}{4}\log_5 \log_6 n}} + \underbrace{(\sqrt[7]{n})^{\log_8 \log_9 n}}_{n^{\frac{1}{7}\log_8 \log_9 n}}$

$$\lim_{n\to\infty} \frac{n^{\frac{1}{7}\log_8 \log_9 n}}{n^{\frac{1}{4}\log_5 \log_6 n}} = \lim_{n\to\infty} n^{\frac{1}{7}\log_8 \log_9 n - \frac{1}{4}\log_5 \log_6 n} = 0$$

$$\frac{1}{7}\log_8 \log_9 n - \frac{1}{4}\log_5 \log_6 n \to -\infty$$

$\log_9 n < \log_6 n$

$\log_8 \log_9 n < \log_5 \log_6 n \qquad \Rightarrow g(n) = \sqrt[4]{n}^{\,\log_5 \log_6 n}$

Vorgehen beim Berechnen solcher Aufgaben:

- Könnte man ableiten? → L'Hôpital
- Gibt es eine vereinfachende Substitution?
- Kann ich log-/Potenzregeln anwenden?
- Hilft der $e^{\ln}$ oder $2^{\log_2}$ Trick?
- Ansonsten probieren, etwas zu vereinfachen/ausklammern ...