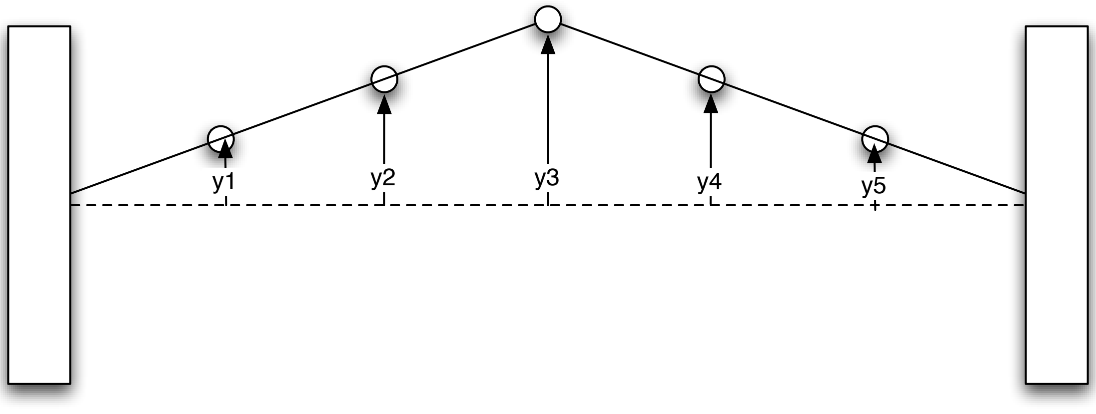

Figure 1: A set of 5 beads connected by elastic springs

Consider the simple model of an elastic string shown above. Here the string is modeled as n distinct point masses linked together by springs. The ends of the string are fixed and have zero displacement. We can model the vertical forces acting on each mass element as follows:

$$m\ddot{y}_i = k(y_{i-1} - y_i) + k(y_{i+1} - y_i) - c\dot{y}_i$$
$$\Rightarrow \ddot{y}_i = \frac{k}{m}(y_{i-1} - 2y_i + y_{i+1}) - \frac{c}{m}\dot{y}_i$$

Here the k captures the spring constant of the elastic links and the c captures the viscous drag/friction that each of the elements contends with.

We can aggregate the equations for all of the nodes into matrix form as shown below:

$$\frac{d}{dt}\begin{pmatrix}\dot{\mathbf{y}} \\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\frac{-c}{m}I & \frac{k}{m}T \\ I & 0\end{pmatrix}\begin{pmatrix}\dot{\mathbf{y}} \\ \mathbf{y}\end{pmatrix}$$

Where $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T$ and T is a tridiagonal matrix:

$$T = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

In your experiments you should simulate a string with $n = 20$ point masses, the spring constant $k = 5$ the damping coefficient $c = 0.01$ and the mass of each element $m = 1$.

Recall that the solution to a linear differential equation of the form $\frac{d}{dt}\mathbf{z}(t) = A\mathbf{z}(t)$ is given by. $\mathbf{z}(t) = e^{At}\mathbf{z}(0)$. This expression can be used to predict how the system will evolve over time.

Your task is to write a matlab script that can be used to visualize how the string will evolve if we pluck it in the center and let it evolve. You should create a vector, $\mathbf{z}(0)$ representing the initial state where the velocity of all n beads is 0 and the position is set so that the string is in a triangular shape as shown in the figure. The maximum displacement should be 1. Your simulation should then compute what the state of the string will be at each subsequent timestep and plot what it will look like using the equation $\mathbf{z}(t) = e^{At}\mathbf{z}(0)$. You can use the `expm` function which computes the exponential of a matrix. You can also look at the matlab script entitled plotExample which shows how you can go about producing a plot that changes over time. Your simulation should cover at least 10 seconds and the timestep between each simulation step should be 0.1 seconds.

Extra Credit (20 pts)

Note that we could extend what we have done here with strings to deal with two dimensional membranes like the surface of a trampoline. You can look at Matlab's `vibes` demo for inspiration. Here we just need to modify our equation to reflect the fact that each point mass is connected to four of its neighbors, north, south, east and west. Your job is to write a script similar to the one that you wrote in the first part that will simulate a trampoline composed of a 20×20 array of point masses with the same values of k, c and m as before. The initial state should be a plucked membrane where the center of the trampoline is indented, the 2D equivalent of the plucked string. The edges of the membrane are all fixed to zero as is the case with the string in Part 1. You can use the `mesh` function to plot the 2D membrane. Your script should also compute and print out the resonant frequencies of this system.