# Algorithm Theory, Tutorial 4

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# O(1) Priority Queue

- Assume we want to store $(key, data)$-pairs in a priority queue.
- The priorities (keys) are only from the set $\{1, \ldots, c\}$ and $c \in \mathbb{N}$ is constant.

Describe a priority queue that provides the operations `Insert`$(key, data)$, `Get-Min`, `Delete-Min`, and `Decrease-Key`$(pointer, newkey)$ all in constant time for the given scenario, and describe how these operations work on your data structure.

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duplicate keys are possible

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- `Delete-Min`: We iterate the Array starting from the beginning (in $\mathcal{O}(c) = \mathcal{O}(1)$), until we find a non-empty list at index $i$. We remove the first pair $(i, data)$ from that list and return it. $\mathcal{O}(1)$
- `Decrease-Key`$(pointer, newkey)$: Since we have a pointer to the $(key, data)$-pair in question, we can remove and change its key in $\mathcal{O}(1)$. Afterwards we reinsert it into the correct list also in $\mathcal{O}(1)$

# 1b

- State how fast Prim's algorithm to compute a minimum spanning tree is, under the assumption that edge weights are in the set $\{1, \ldots, c\}$ and $c \in \mathbb{N}$ is constant, using your implementation of a priority queue. Explain your answer.
- Prim's Algorithm now runs in $\mathcal{O}(|E| + |V|)$ using our implementation of the priority queue.
- The reason is that Prim's algorithm uses $\mathcal{O}(|E|)$ `Decrease-Key` operations and $\mathcal{O}(|V|)$ `Delete-Min`, `Get-Min` and `Insert` operations (see analysis in lecture slides).

# Exercise 2

We are given a maximum flow network $G = (V, E)$ with integer capacities together with a maximum $\boxed{\text{flow } \Phi.}$ Describe an algorithm with time complexity $O(|V| + |E|)$ to compute a new maximum flow for each of the following cases:

a) if the capacity of an arbitrary edge $(u, v) \in E$ increases by one unit.
b) if the capacity of an arbitrary edge $(u, v) \in E$ decreases by one unit.

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- Run one iteration of Ford-Fulkerson ($O(|E| + |V|)$)
- If we find an augmenting path, augment $\Phi$ by this path.

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- Search path from $s$ to $u$ and from $v$ to $t$ with positive flow. Reduce flow by one on each edge of those paths. (of course also reduce flow on $(u, v)$. (Reduces flow value by one)

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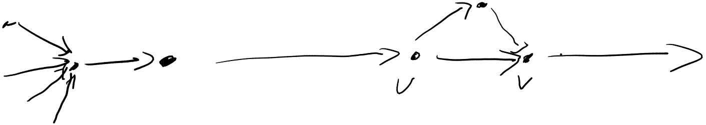
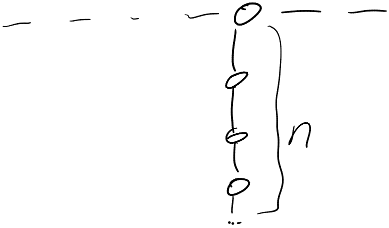
- Afterwards run FF to see if we can again increase the flow to its "original" size, if possible, augment it. $O(|E| + |V|)$



# Linear Chain in Fibonacci Heap

Show that for any positive integer $n$, there exists a sequence of Fibonacci Heap operations that can construct a Fibonacci Heap consisting of just one tree that is a linear chain of $n$ nodes. Provide the pseudocode of a recursive procedure to construct such a Fibonacci Heap, and show its correctness.

- Hint: Search for easy recursive solutions.
- Assume we can build a linear chain of lenght $n$ and extend it to $n + 1$.
- Recursion and Induction are basically the same then



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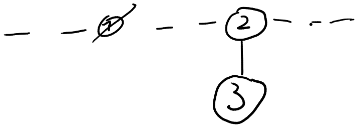

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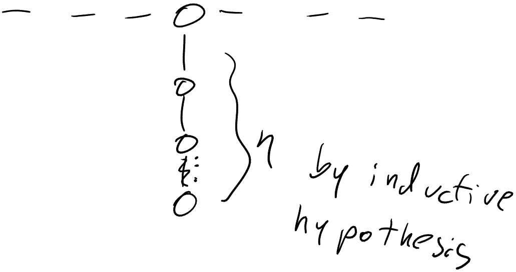
- $H$ is an empty Fib. Heap
- `H.insert(1)`
- `H.insert(2)`
- `H.insert(3)`
- `H.deleteMin()`
- We have successfully constructed a chain of length 1.

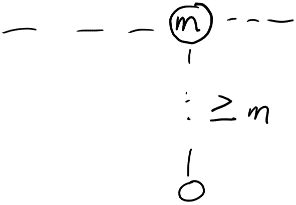
# Inductive Step, $n \mapsto n + 1$

- `H:=linChain(n)`



# Inductive Step, $n \mapsto n+1$

- `H:=linChain(n)`
- `m := H.getMin()`



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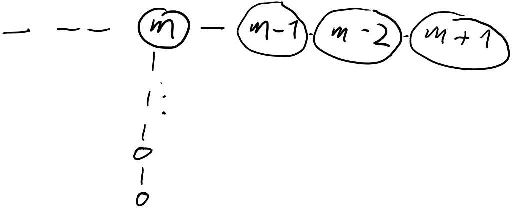

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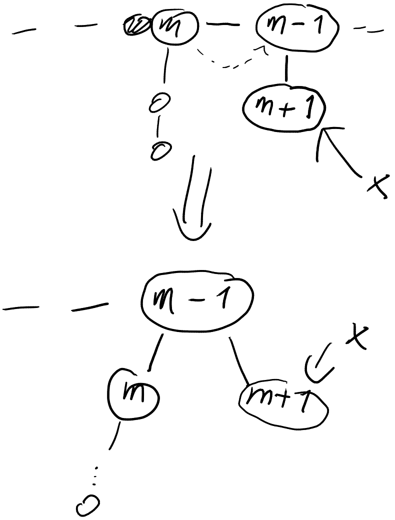
1. half of consolidate



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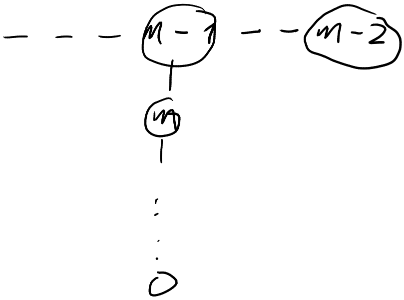
- `H.decreaseKey(x, m-2)`



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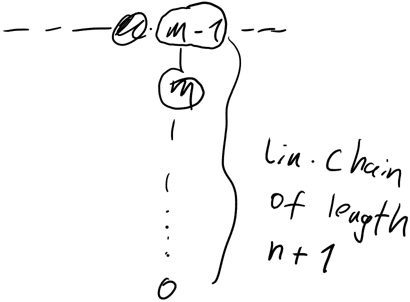

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- `H.decreaseKey(x, m-2)`
- `H.deleteMin()`
- We have successfully transformed a linear chain of length $n$ into a linear chain of length $n+1$.

**Algorithm 1** Chain-Construction($n$)

**if** $n = 1$ **then**
  Initialize-Heap($F$) ▷ assume $F$ is now globally known
  Insert($F$, 1) ▷ inserting a node with key 1 into $F$.
  Insert($F$, 2)
  Insert($F$, 3)
  Delete-Min($F$)
  **return**

Base case

Chain-Construction($n-1$)

Inductive Hypothesis
chain of length $n$

$min \leftarrow$ Get-Min($F$)
$x \leftarrow$Insert($F$, $min + 1$)
Insert($F$, $min - 1$, $\cancel{null}$)
Insert($F$, $min - 2$, $\cancel{null}$)
Delete-Min($F$)
Decrease-Key($x$, $min - 3$)
Delete-Min($F$)

Chain of length $n+1$

**return**

# Min cut with min number of edges

- This exercise will be considered as a bonus exercise, which earns points but does not count towards the threshold of exam admittance.
- Consider an undirected, weighted graph $G = (V, E)$ with integral edge weights. Among all cuts of $G$ with minimum weight you want to find a cut $(S, V \setminus S)$ with the smallest number of edges (i.e. edges with exactly one endpoint in $S$).
  a. Modify the weights of $G$ to create a new graph $G'$ in which any minimum cut in $G'$ is a minimum cut with the smallest number of edges in $G$.
  b. Prove that $G'$ has the property claimed in part (a).

# Solution

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- We define $G' = (V, E, w')$ with edge weights $w'(e) := |E| \cdot w(e) + 1$.

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- We have to prove two things:
- Every min cut of $G$ has less weight in $G'$ than every non-minimal cut of $G$ (1)
- Of two min-cuts in $G$ the one with fewer edges has less weight in $G'$. (2)

- Every min cut of $G$ has less weight in $G'$ than every non-minimal cut of $G$ (1)
- Let $M$ be a min cut in $G$ and $X$ a non-minimal cut in $G$
- Let $|M|, |X|$ be the number of edges of the two cuts and $w_G(M) < w_G(X)$ the weights of the two cuts in $G$ ($w_{G'}(\dots)$ in analogy).
- It holds that $w_G(M) \overset{+1}{<=} w_G(X)\ \cancel{+1}$ (because of the Integer weights)

$$W_G(X) = \sum_{e \in X} w_G(e)$$

$$
\begin{aligned}
W_{G'}(X) &= w_G(X) \cdot |E| + |X| \\
&\geq (w_G(M) + 1) \cdot |E| + |X| \\
&= w_G(M) \cdot |E| + \underbrace{|E| + |X|}_{> |M|} \\
&> w_G(M) \cdot |E| + |M| \\
&= W_{G'}(M)
\end{aligned}
$$

# Proof of Claim 2

- Of two min-cuts in $G$ the one with fewer edges has less weight in $G'$. (2)
- Let $M$ and $X$ be min cuts in $G$ ($w_G(M) = w_G(X)$) and let $M$ have fewer edges than $X$ ($|M| < |X|$).

$$
\begin{aligned}
W_{G'}(M) &= W_G(M) \cdot |E| + |M| \\
&= W_G(X) \cdot |E| + |M| \\
&< W_G(X) \cdot |E| + |X| \\
&= W_{G'}(X)
\end{aligned}
$$