Algorithm Theory, Tutorial 3

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General Hints

- Contact tutor (`johannes.kalmbach@gmail.com`) for questions concerning corrections etc.
- Contact forum (`daphne.informatik.uni-freiburg.de`) for everything else
- Suggestion: Submit in groups of two (better for understandable algorithms)
- Submit readable solutions (LaTeX as pdf, CLEAN handwriting ($+$ good scan if necessary))
- Spend enough time on exercise sheets and writeup (you and I have to understand your submission).

Algorithm Writeups

- Pseudocode, limit to important aspects
- Reader must be able to understand and implement it.
- E.g "Split Array A in two evenly-sized halves L and R"

Dynamic Programming in a nutshell

- Problem can be formulated recursively
- The number of subproblems / subresults that are actually needed/called is bounded (e.g. polynomial in input size)
- Use Memoization to only calculate each subresult once.
- Directly gives us an efficient algorithm
- **Do not forget the base cases for recursion**
- Alternatively: Bottom-up formulation (might be more efficient in practice (not asymptotically), but sometimes harder to reason about).

Knapsack Problem

- We are given n items that each have a weight $w_i > 0$ and a value $v_i > 0$.
- Additional we are given a maximum capacity W of our knapsack.
- What is the maximum value we can pack into our knapsack without exceeding the capacity?
- General case : $w_i, v_i \in \mathbb{R}$: **NP-COMPLETE**
- But we can do something in case we put additional constraints on w_i or v_i

Knapsack with integer **WEIGHTS**

- Example from Lecture, NOT from the exercise (many confused about this)
- $w_i \in \mathbb{N} \Rightarrow$ all valid combinations of items have a weight in $\{0, \ldots W\}$.
- Define helper function $OPT(i, w)$ to be the maximum value that we can achieve with a weight $\leq w$ and only the items $\{1, \ldots i\}$.
- $OPT(i, 0) = 0$, $OPT(0, k) = 0$ (no weight or no items allowed $\Rightarrow$ no value can be achieved.)
- $OPT(i, w) = 0$ if $w < 0$
- $OPT(i, w) = \max(OPT(i - 1, w), OPT(i - 1, w - w_i) + v_i)$
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-
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- We only need $n \cdot W$ different subresults to calculate $OPT(n, W)$, so using memoization gives us $\mathcal{O}(nW)$ algorithm.

Knapsack with integer **VALUES**

- Weights might be arbitrary real numbers now, algorithm from previous slide does not work efficiently.
- However we can do something similar with the values (Read exercises and hints carefully!)
- $v_i \in \mathbb{N} \Rightarrow$ all valid combinations of items have a value in $\{0, \ldots \sum_i v_i =: V\}$.
- Define helper function $OPT(i, v)$ to be the minimum weight that we can achieve with a value $\geq v$ and only the items $\{1, \ldots i\}$.
- $OPT(i, 0) = 0$, $OPT(0, v) = \infty$ if $v > 0$
- $OPT(i, v) = 0$ if $v < 0$
- $OPT(i, v) = \min\left(OPT(i - 1, v), OPT(i - 1, v - v_i) + w_i\right)$
-
- We only need $n \cdot V$ different subresults to calculate $OPT(n, V)$, so using memoization gives us $\mathcal{O}(nW)$ algorithm.

- Define helper function $OPT(i, v)$ to be the minimum weight that we can achieve with a value $\geq v$ and only the items $\{1, \ldots i\}$.
- Calculate $OPT(n, v)$ for all $v \in \{1, \ldots V\}$. The biggest v for which $OPT(n, v) \leq W$ is our maximum value.
- How to construct the actual **set** that achieves this value? (Was asked in the exercise)
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- $OPT(i, v) = \min\left(OPT(i - 1, v), OPT(i - 1, v - v_i) + w_i\right)$
- For each $OPT(i, v)$ we decide whether to include i or not, and which previous value we used. If we store these pointers / edges during computation, we can also retrieve the actual set.

Dynamic Programming

Conisder the following functions $f_i : \mathbb{N} \to \mathbb{N}$

$$f_1(n) = n - 1$$

$$f_2(n) = \begin{cases} \frac{n}{2} & \text{if 2 divides } n \\ n & \text{else} \end{cases}$$

$$f_3(n) = \begin{cases} \frac{n}{3} & \text{if 3 divides } n \\ n & \text{else} \end{cases}$$

"m divides n" means there is a $k \in \mathbb{N}$ with $k \cdot m = n$.

For a given $n \geq 1$, we want to find die minimal number of applications of the functions f_1, f_2, f_3 needed to reach 1. Formally: Find the minimal k for which there are $i_1, \ldots, i_k \in \{1, 2, 3\}$ with $f_{i_1}(f_{i_2}(\ldots (f_{i_k}(n)) \ldots) = 1$.

Consider the following algorithms:

```
def min_f(n: positive integer):
 if n == 1 : return 0
 if 3 divides n:
   return min_f(n / 3) + 1
 if 2 divides n:
   return min_f(n / 2) + 1
 return min_f(n-1) + 1
```

- What type of algorithm is this?
- What runtime has it?
- Does it solve the problem?

Dynamic Programming in O(n)

```
def helper(n: positive integer, memo: dictionary):
if n == 1 : return 0
if n in memo:
  return memo[n]

x:= helper(n-1)
if 3 divides n:
  x:= min(x, helper(n/3))
if 2 divides n:
    x:= min(x, helper(n/3))
memo[n] = x + 1
return x + 1
```

Faster approach

- One student found an even faster solution, based on the following idea:
- The optimal solution can never perform (-1) (-1) (/2), because (/2) (-1) would be faster.
- Similarly, (-1) (-1) (-1) (/3) is not optimal.
- Using this and a nice numerical property $\lfloor \frac{\lfloor \frac{n}{3} \rfloor}{2} \rfloor = \lfloor \frac{\lfloor \frac{n}{2} \rfloor}{3} \rfloor$ we can use this to implement an
- $\mathcal{O}((\log n)^2)$ algorithm (also based on DP)

- Please consider this topic in your exam preparations, they

Exercise 3

Suppose a sequence of *n* operations are performed on an (unknown) data structure in which the *i*-th operation costs *i* if *i* is an exact power of 2, and 1 otherwise.

Operation	1	2	3	4	5	6	7	8	9	...	15	16	17	...
Actual Cost	1	2	1	4	1	1	1	8	1	...	1	16	1	...

Use the **potential function** method to show that each operation has constant amortized cost.

Hint: *The number of consecutive operations that are not an exact power of 2 and are performed immediately before operation $(i + 1)$ is $i - 2^{\ell(i)}$ where $\ell(i) := \lfloor \log_2 i \rfloor$.*

Operation	1	2	3	4	5	6	7	8	9	...	15	16	17	...
Actual Cost	1	2	1	4	1	1	1	8	1	...	1	16	1	...

Operation	1	2	3	4	5	6	7	8	9	...	15	16	17	...
Actual Cost	1	2	1	4	1	1	1	8	1	...	1	16	1	...

Operation	1	2	3	4	5	6	7	8	9	. . .	15	16	17	. . .
Actual Cost	1	2	1	4	1	1	1	8	1	. . .	1	16	1	. . .

Operation	1	2	3	4	5	6	7	8	9	...	15	16	17	...
Actual Cost	1	2	1	4	1	1	1	8	1	...	1	16	1	...