Hi ! :-)

# Algorithm Theory, Tutorial 1

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# General Hints

- Contact tutor (johannes.kalmbach@gmail.com) for questions concerning corrections etc.
- Contact forum (daphne.informatik.uni-freiburg.de) for everything else
- Suggestion: Submit in groups of two (better for understandable algorithms)
- Submit readable solutions (LaTeX as pdf, CLEAN handwriting (+ good scan if necessary)) + NAME
- Spend enough time on exercise sheets and writeup (you and I have to understand your submission).

# Algorithm Writeups

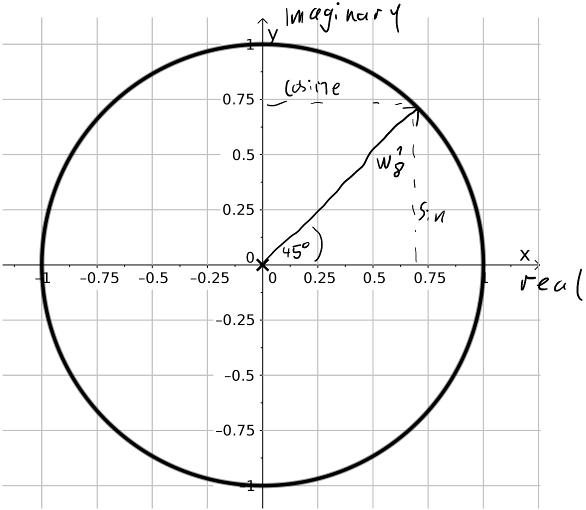
- Pseudocode, limit to important aspects
- Reader must be able to understand and implement it.
- E.g "Split Array $A$ in two evenly-sized halves $L$ and $R$"

# Exercise 1

Compute the convolution of the vectors $a = (5, 8, -2, 3)$ and $b = (-9, 4, -1)$ using the algorithm for polynomial multiplication from the lecture. Document all computation steps for evaluation, point-wise multiplication and interpolation.

- Basically: Calculate $p \cdot x$ where $p(x) = 3x^3 - 2x^2 + 8x^{1} + 5$ and $q(x) = -9x^2 + 4x - 1$
- Strategy: efficiently evaluate $p, q$ at the **8-th!!!** roots of unity using FFT.
- Perform pointwise multiplication
- Compute the inverse FFT to obtain result coefficients.

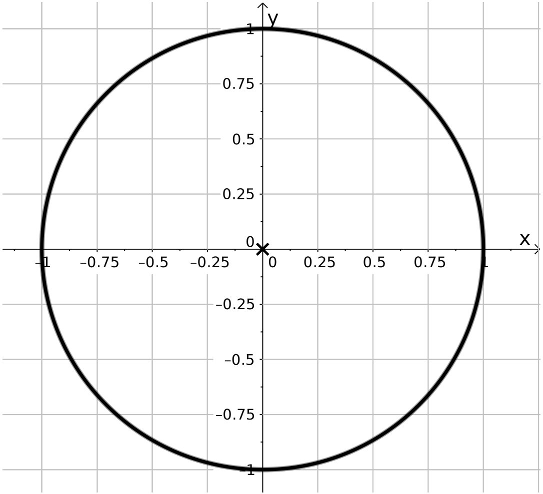
# Complex numbers and roots of unity



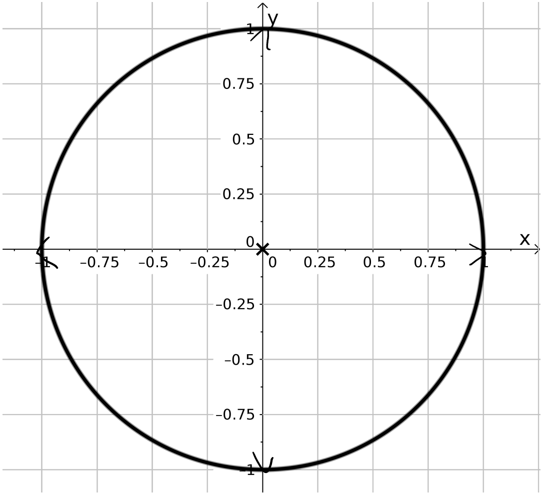

$$w_8^1 = \cos\left(\frac{\pi}{4}\right) + i \cdot \sin\left(\frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}} = \underline{\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}}$$
$$= \underline{e^{i\frac{\pi}{4}}}$$

$$x = \underbrace{e^{i\theta} + (3+4i)}_{\text{not really useful}}$$

# Complex numbers and roots of unity



# Complex numbers and roots of unity



- Evaluate $p(x) = 5 + 8x - 2x^2 + 3x^3$ at the 8-th roots of unity $(\omega_8^0 \dots \omega_8^7)$

$$\begin{aligned} p(x) &= 5 + 8x - 2x^2 + 3x^3 \\ &= \underbrace{(5 - 2x^2)}_{p_1(x^2)} + x \cdot \underbrace{(8 + 3x^2)}_{p_2(x^2)} \end{aligned}$$

- with $p_1(x) = 5 - 2x$, $p_2(x) = 8 + 3x$
- We only have to evaluate $p_1, p_2$ for $\{\omega_4^0 \dots \omega_4^3\} = \{1, i, -1, -i\}$

| | $x$ | 1 | i | -1 | -i |
|---|---|---|---|---|---|
| $p_1$ | $5 - 2x$ | 3 | 5 - 2i | 7 | 5 + 2i |
| $p_2$ | $8 + 3x$ | 11 | 8 + 3i | 5 | 8 - 3i |

- $p(x) = p_1(x^2) + x \cdot p_2(x^2)$

| $x$ | $\omega_4^0 = 1$ | $\omega_4^1 = i$ | $\omega_4^2 = -1$ | $\omega_4^3 = -i$ |
|---|---|---|---|---|
| $p_1(x)$ | 3 | 5 - 2i | 7 | 5 + 2i |
| $p_2(x)$ | 11 | 8 + 3i | 5 | 8 - 3i |

- $\underline{\omega_8^0 = 1}, (\omega_8^0)^2 = \omega_4^0 \Rightarrow p(\omega_8^0) = (3 + 11) = 14$
- $\omega_8^1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, (\omega_8^1)^2 = \omega_4^1 \Rightarrow p(\omega_8^1) = (5-2i)+(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)\cdot(8+3i) = \boxed{5+\frac{5}{\sqrt{2}} - 2i + \frac{11}{\sqrt{2}}i}$
- $\omega_8^2 = i, (\omega_8^2)^2 = \omega_4^2 \Rightarrow p(\omega_8^2) = 7 + 5i$
- $\omega_8^3 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, (\omega_8^1)^2 = \omega_4^3 \Rightarrow p(\omega_8^3) = 5 - \frac{5}{\sqrt{2}} + 2i + \frac{11}{\sqrt{2}}i$
- $\omega_8^4 = -1, p(\omega_8^4) = -8$
- $\omega_8^5 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, p(\omega_8^5) = 5 - \frac{5}{\sqrt{2}} - 2i - \frac{11}{\sqrt{2}}i$
- $\omega_8^6 = -i, p(\omega_8^6) = 7 - 5i$
- $\omega_8^7 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, p(\omega_8^7) = 5 + \frac{5}{\sqrt{2}} + 2i - \frac{11}{\sqrt{2}}i$

- Perform similar evaluation for q(x) (omitted here)
- then do pointwise multiplication

| x | $p(x)$ | $q(x)$ | $p(x) \cdot q(x)$ |
|---|---|---|---|
| $\omega_8^0$ | 14 | -6 | -84 |
| $\omega_8^1$ | $5 + \frac{5+11i}{\sqrt{2}} - 2i$ | $-9 - i + (2+2i)\sqrt{2}$ | $(-59+45i) - (3+46i)\sqrt{2}$ |
| $\omega_8^2$ | $7 + 5i$ | $-8 + 4i$ | $-76 - 12i$ |
| $\omega_8^3$ | $5 + \frac{-5+11i}{\sqrt{2}} + 2i$ | $-9 + i + (-2+2i)\sqrt{2}$ | $(-59 - 45i) + (3 - 46i)\sqrt{2})$ |
| $\omega_8^4$ | $-8$ | -14 | 112 |
| $\omega_8^5$ | $5 + o\frac{-5-11i}{\sqrt{2}} - 2i$ | $-9 - i + (-2-2i)\sqrt{2}$ | $(-59 + 45i) + (3 + 46i)\sqrt{2})$ |
| $\omega_8^6$ | $7 - 5i$ | $-8 - 4i$ | $-76 + 12i$ |
| $\omega_8^7$ | $5 + \frac{5-11i}{\sqrt{2}} + 2i$ | $-9 + i + (2-2i)\sqrt{2}$ | $(-59 - 45i) - (3 - 46i)\sqrt{2})$ |

$:= y_j$

- For the interpolation step we use the Inverse DFT from the lecture:
  - Given evaluations $y_k = p(\omega_n^k)$ for a polynomial with degree $\leq n-1$ we can compute the coefficients $a_k$ of the polynomial $p$ by

$$a_k = \frac{1}{n} \sum_{j=0}^{n-1} y_j \cdot (\omega_n^{-k})^j$$

  - This is equivalent to evaluating a polynome with the coefficients $y_k$ at the $n-th$ roots of unity (with proper rearrangement), so we can again use FFT.
  - Final result:

$$-3x^5 + 14x^4 - 43x^3 + 45x^2 - 52x - 45$$

python: $(-3{,}000 \ldots\ 1\ ,\ 3.7 \cdot 10^{-16}$

# Exercise 2, Greedy Scheduling

Error on sheet

Given $n$ jobs of lengths $t_1 \ldots, t_n$ with **one deadline** $d \geq 0$, we want to schedule these jobs such that the **average lateness** is minimized. That is, for each job $i$ we want to find a start and finishing time $0 \leq s(i) \leq f(i)$ with $f(i) - s(i) = t_i$ such that the intervals $[s(i), f(i)]$ are pairwise non-overlapping and the average over all $L(i) = \max\{0, f(i) - d\}$ is minimal (overlapping of start- and endpoints is allowed).

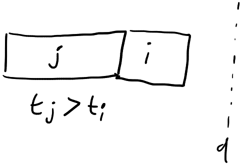
Describe a greedy algorithm for this problem and prove that it computes an optimal solution.

- ~~Maximizing~~ Minimizing the Average lateness is the same as ~~maximizing~~ minimizing the sum of lateness
- After the deadline, at each point in time our penalty gets bigger if we have more unfinished jobs.
- Thus we want as many jobs as possible to finish early.
- This gives us the idea of scheduling the shortest jobs first.
- THIS INTUITION IS NOT A PROOF

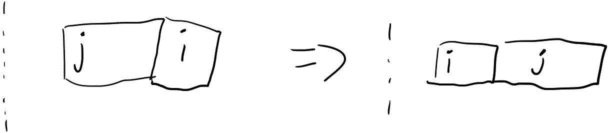
# Algorithm

without gaps

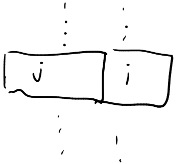
- Sort Jobs by ascending length, Schedule them non-overlapping in this order.
- For proof always use exchange argument (anything else is always fishy or wrong)
- Assume that $ALG$ is the greedy solution and $O$ is an optimal solution with inversions (wrt to the sorting by time).
- This means we have a longer job scheduled immediately before a shorter job in $OPT$.
- Need to show, that switching them does not increase (average/total) lateness.
- This implies that the greedy solution is optimal



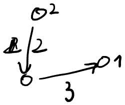

Switching makes no difference




lateness gets smaller



# Matroids



- We are given a directed weighted graph $G = (V, E)$
- $w : E \to \mathbb{R}^+$ defines (positive) weights of the edges
- $b : V \to N$ that defines some indegree bound for each node.
- Find subset $\boxed{E' \subseteq E}$ of maximum total weight such that every node $u \in V$ has indegree $\leq b(u)$ in graph $G' = (V, E')$.
- Show that the set of feasible solutions form a matroid and thus, this problem can be solved by using the greedy algorithm for matroids.

finite     subset of edges

- Need to show: $M := \{E' \subseteq E | \forall v \in V : indeg_{E'}(v) \leq b(u)\}$ is a matroid.
- Important: $M$ consists of sets of **Edges**.
- Need to show :
  - $\emptyset \in M$
  - $B \in M, A \subset B \Rightarrow A \in M$
  - $A \in M, B \in M, |A| < |B| \Rightarrow \exists x \in B \setminus A : A \cup \{x\} \in M$

# $\emptyset \in M$

$$E' = \emptyset \quad \Rightarrow \forall v : \mathrm{indeg}_{\emptyset} = 0 \leq b(v)$$

# $B \in M, A \subset B \Rightarrow A \in M$

$\forall v: \quad indeg_B(v) \leq b(v)$ $\qquad B \in M$

$indeg_A(v) \leq indeg_B(v)$ $\qquad A \subseteq B$

$\Rightarrow indeg_A(v) \leq b(v)$

# $A, B \in M, |A| < |B| \Rightarrow \exists x \in B \setminus A : A \cup \{x\} \in M$

$|A| < |B|$

$\Rightarrow \exists v : \text{indeg}_A < \text{indeg}_B \leq b(v)$

$\Rightarrow \exists (x, v) \in B \setminus A$

incoming at $v$

$\Rightarrow A \dot{\cup} \{(x,v)\} \in M$ (only $v$ has indegree $+1$ which is fine because of

other node's indegrees stay the same