Coupled problems: final project

January 19, 2022

1 Solve the heat equation using Dirichlet-Neumann iteration and adaptive SDIRK

After the space discretization step, we obtained a ODE of the form

$$\mathbf{u}_t = \mathcal{L}(t, \mathbf{u}). \tag{1}$$

Solving this using SDIRK time stepping requires finding the solution to an algebraic system in every stage. We have to solve

$$\mathcal{L}(\bar{t}, \bar{\mathbf{u}} + \alpha \mathbf{k}) = \mathbf{k} \tag{2}$$

for the vector $\mathbf{k}$.

Solving the same ODE using implicit Euler time stepping requires solving the system

$$\Delta t \mathcal{L}(\bar{t}, \mathbf{u}) = \mathbf{u} - \mathbf{u}^{old} \tag{3}$$

for the vector $\mathbf{u}$ on every time step. Set $\Delta t := \alpha$ and $\mathbf{u}^{old} := \bar{\mathbf{u}}$, then by substitution if $\mathbf{u}^*$ solves (3) then $\mathbf{k}^* := (\mathbf{u}^* - \bar{\mathbf{u}})/\alpha$ solves (2). Provided that we have a routine to solve the system (3) we can use it to solve (2). The Dirichlet-Neumann iteration

$$\mathbf{u} \approx DN_{\mathcal{L}}^{TOL,\Delta t}(\mathbf{u}^{old}) \tag{4}$$

we have used in the previous assignments is a solver for (3). Using the substitutions described above we obtain

$$\mathbf{k} \approx \frac{DN_{\mathcal{L}}^{TOL,\alpha}(\bar{\mathbf{u}}) - \bar{\mathbf{u}}}{\alpha} \tag{5}$$

a solver for (2). Note that the above expression is prone to cancellation errors if α is small. This problem can be mitigated by using the Shu-Osher representation[1] of the Runge-Kutta method, as in that formulation the system solved in every stage is the same as (3) and we do not have to compute $\mathbf{k}$.

1.1 Relaxation

The optimal relaxation parameter in the DN-iteration varies depending on the problem parameters α, λ and on the timestep size Δt. For small Δt the optimal relaxation parameter is $\theta^{big} := 1$, for large Δt the optimal relaxation parameters is approximately

$\theta^{small} := \lambda_N/(\lambda_D + \lambda_N)$ where λ_D and λ_N are the heat conductivity parameters for the Dirichlet and Neumann domains respectively. This was handled by choosing two time step sizes Δt^{small} and Δt^{big} and selecting appropriate relaxation parameters in the iteration using the (linear interpolation) rule

$$\theta(\Delta t) = \begin{cases} \theta^{big}, & \Delta t < \Delta t^{small} \\ \theta^{big}\zeta + \theta^{small}(1-\zeta), & \Delta t^{small} \leq \Delta t < \Delta t^{big} \\ \theta^{small}, & \text{otherwise} \end{cases} \tag{6}$$

where $\zeta = \frac{\Delta t - \Delta t^{small}}{\Delta t^{big} - \Delta t^{small}}$. After tuning Δt^{small} and Δt^{big} this worked well unless the time steps grew too large. A max timestep limit was enforced in the SDIRK adaptive time stepping, the limit was initialized to ∞ and then moved down if the DN iteration failed to converge for $\theta(\Delta t)$. The number of DN iterations required to reach the prescribed tolerance were usually 1-2 (except when the iteration diverged and we had to restart the time step). This setup allowed us to solve the room problem with λ_2 low and α_2 high to large t with a SDIRK relative tolerance of 10^{-5} without issues, see figure 4.

1.2 Adaptivity in the time stepper

Adaptivity was implemented as described in the lecture with the exception of adding a maximal time step to avoid using time steps so large that the DN iteration could not converge with the selected relaxation parameters.

1.3 Convergence test

The convergence test was conducted by computing a low tolerance reference solution and comparing it to various fixed time step solutions. The reference solution $\mathbf{u}_{ref,t_1}$ at the time t_1 was obtained using adaptive SDIRK. The error after n fixed size time steps of size $dt := t_1/n$ was computed as $\mathbf{e}_n = \mathbf{u}_{dt,t_1} - \mathbf{u}_{ref,t_1}$ and measured in the l_2 norm. As can be seen in figures 2 and 1, a convergence rate of approximately 2 was obtained both in the case of uniform $\lambda = \alpha = 1$ and in the case with different parameters in the different rooms.

References

[1] Luca Ferracina and Marc Spijker, *An extension and analysis of the shu-osher representation of runge-kutta methods*, Math. Comput. **74** (2005), 201–219.

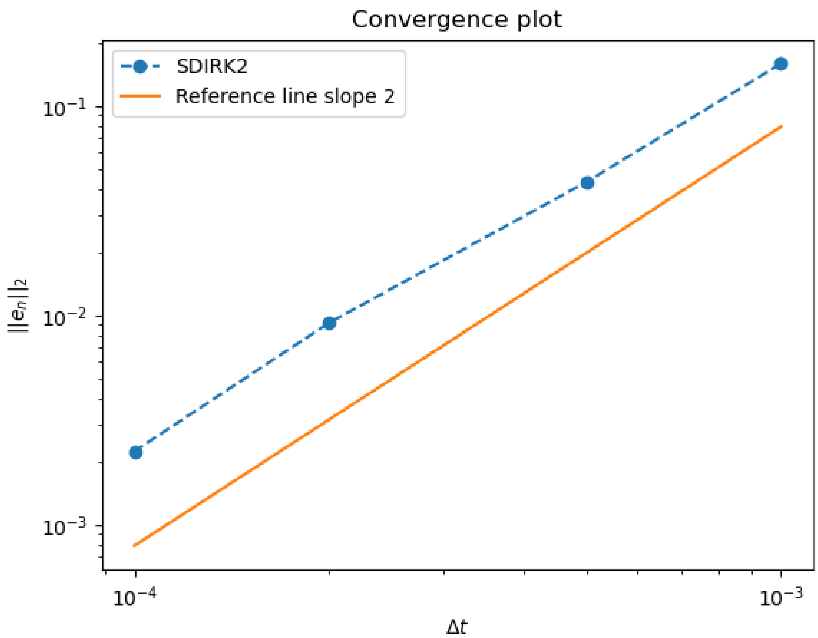

Figure 1: $\lambda = 1$ and $\alpha = 1$ in all rooms.

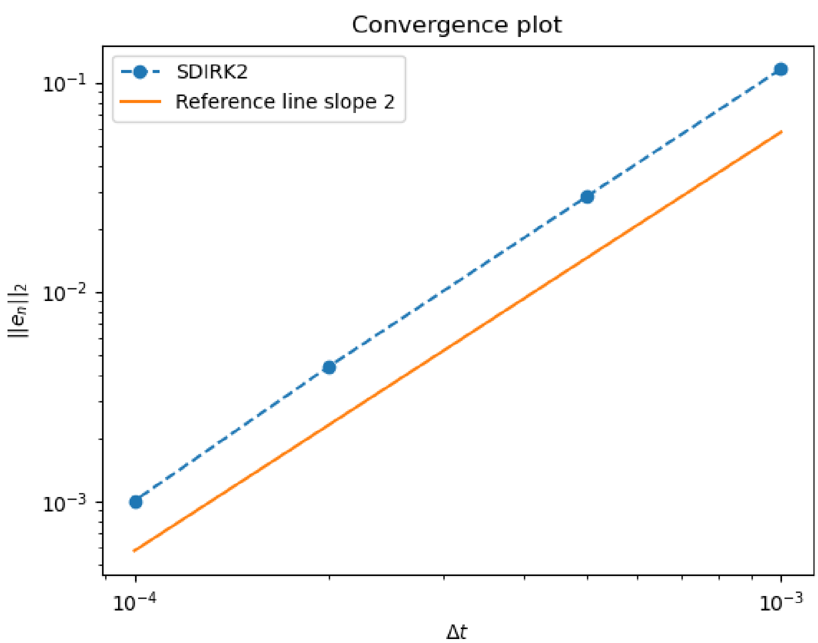

Figure 2: $\lambda = 0.0243$ and $\alpha = 1300$ in Γ_2.

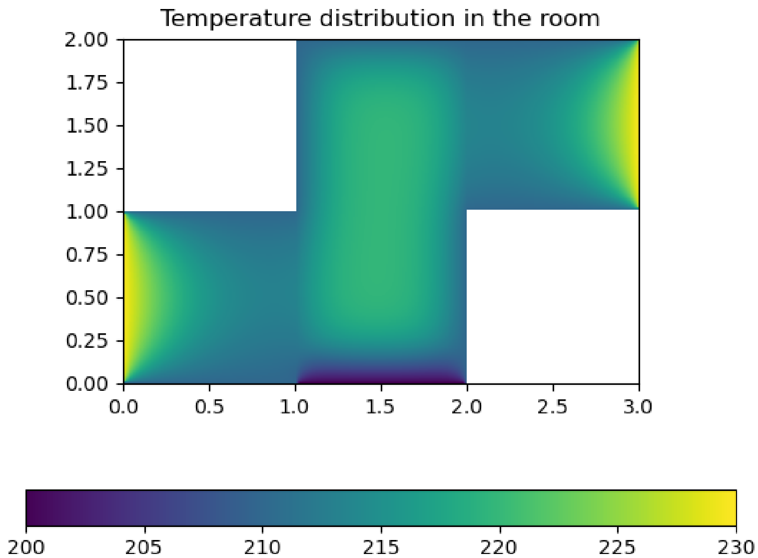

Figure 3: $T = 1000$, transient state. $\lambda = 0.0243$ and $\alpha = 1300$ in Γ_2.

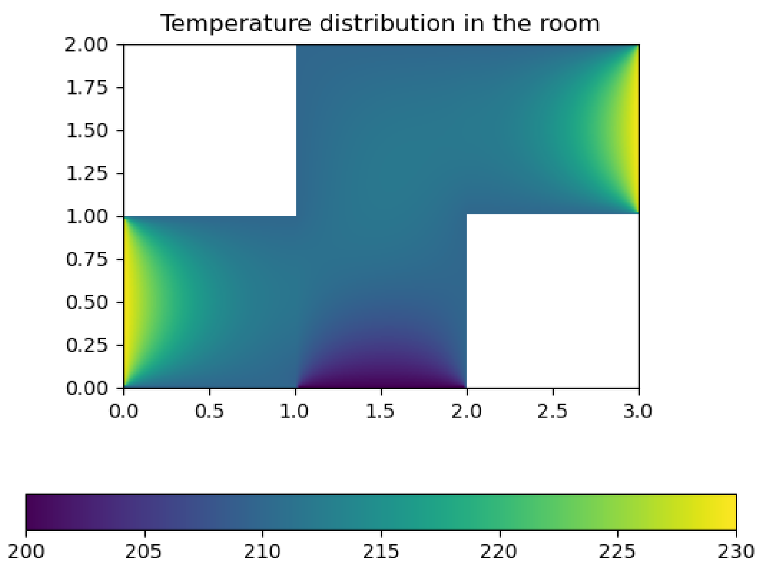

Figure 4: $T = 10000$, near steady state. $\lambda = 0.0243$ and $\alpha = 1300$ in Γ_2.