Image Sharpening Convolution using Toeplitz Matrix

Using First-Order Derivative

$$\frac{\partial f}{\partial x} = [f(x+1, y-1) + f(x+1, y+1) + 2f(x+1, y)] - [f(x-1, y-1) + f(x-1, y+1) + 2f(x-1, y)]$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -8 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$z_9 - z_5$

$z_8 - z_6$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$|\nabla f| = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]^{1/2} \approx \left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right|$$

a

z_1	z_2	z_3
z_4	z_5	z_6
z_7	z_8	z_9

b

−1	0
0	1

c

0	−1
1	0

d

−1	−2	−1
0	0	0
1	2	1

e

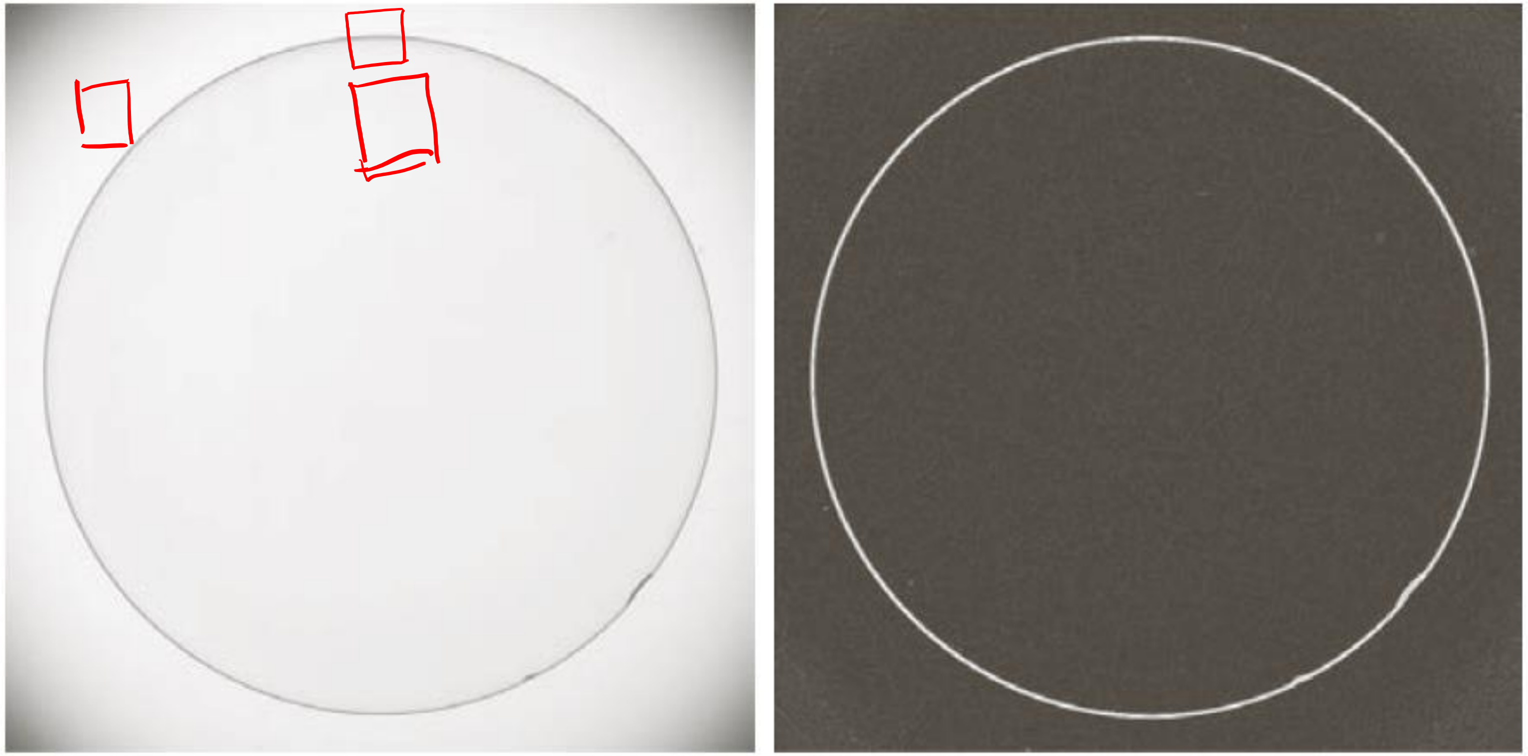
−1	0	1
−2	0	2
−1	0	1

FIGURE 3.41 A 3 × 3 region of an image (the zs are intensity values). (b)–(c) Roberts cross gradient operators. (d)–(e) Sobel operators. All the mask coefficients sum to zero, as expected of a derivative operator.

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \Bigg\} \text{ Prewitt edge operators}$$

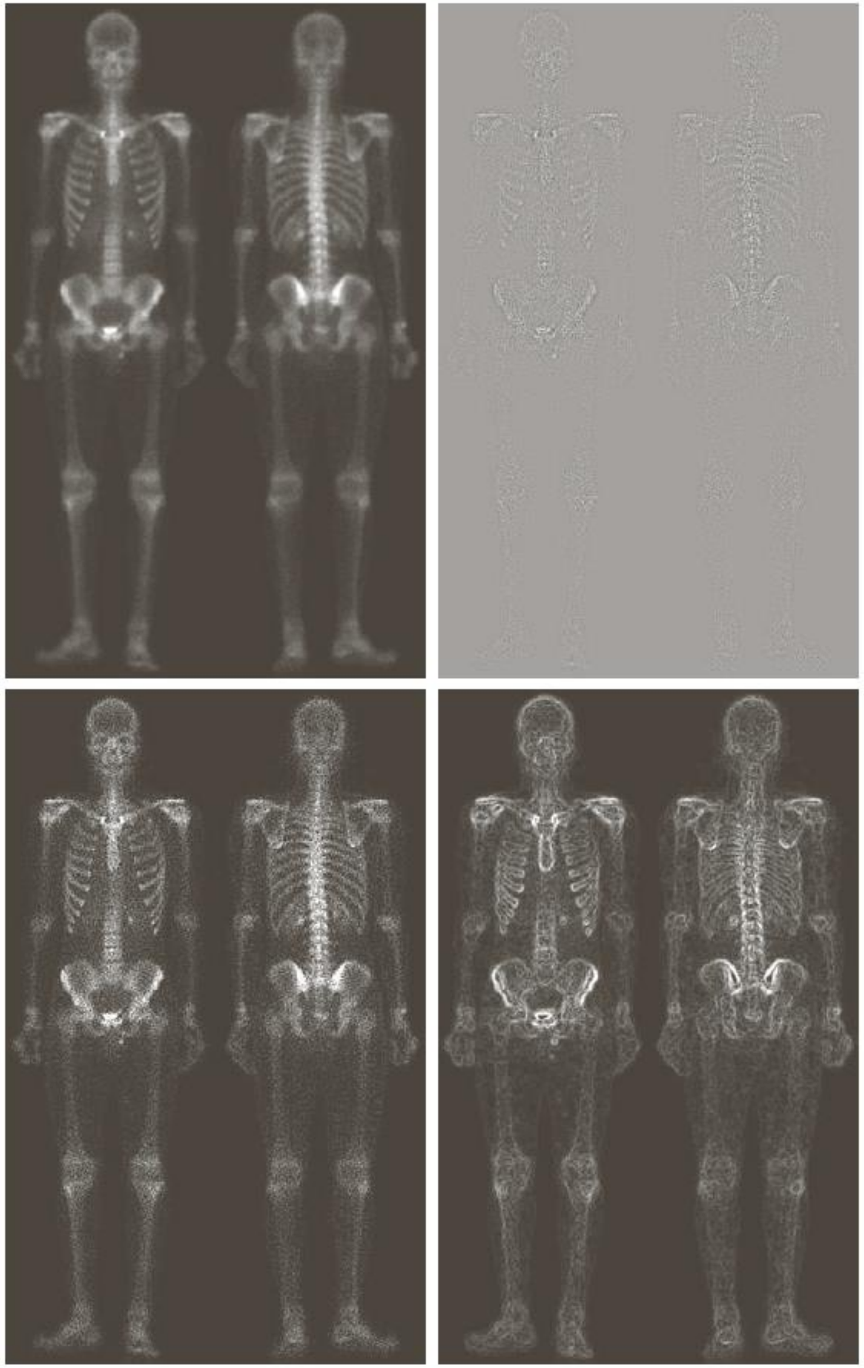
$$\begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \Bigg\} \text{ Sobel edge operators.}$$

Results



a b

FIGURE 3.42 (a) Optical image of contact lens (note defects on the boundary at 4 and 5 o'clock). (b) Sobel gradient. (Original image courtesy of Pete Sites, Perceptics Corporation.)



a b
c d

FIGURE 3.43
(a) Image of whole body bone scan.
(b) Laplacian of (a). (c) Sharpened image obtained by adding (a) and (b).
(d) Sobel gradient of (a).

e f
g h

FIGURE 3.43
(Continued)
(e) Sobel image smoothed with a 5×5 averaging filter. (f) Mask image formed by the product of (c) and (e).
(g) Sharpened image obtained by the sum of (a) and (f). (h) Final result obtained by applying a power-law transformation to (g). Compare (g) and (h) with (a). (Original image courtesy of G.E. Medical Systems.)

Toeplitz Matrices

- Elements with constant value along the main diagonal and sub-diagonals.
- $T(m,n) = t_{m-n}$
- Each row (column) is generated by a shift of the previous row (column).
  - The last element disappears
  - A new element appears

$$\mathbf{T} = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(N-1)} \\ t_1 & t_0 & t_{-1} & \ddots & \vdots \\ t_2 & \ddots & \ddots & \ddots & t_{-2} \\ \vdots & \ddots & \ddots & \ddots & t_{-1} \\ t_{N-1} & \cdots & t_2 & t_1 & t_0 \end{bmatrix}_{N\times N}$$

Convolution by matrix-vector operations

- 1-D linear convolution between two discrete signals may be expressed as the product of a Toeplitz matrix constructed by the elements of one of the signals and a vector constructed by the elements of the other signal.

1-D Linear Convolution

- 1-D linear convolution: $g[n] = f[n] \circledast h[n] = \sum f[k]h[n-k]$
- Without zero-padding, the length of $g[n]$ will be $N = N_1 + N_2 - 1$
- We create a Toeplitz matrix **H** from the elements of $h[n]$ (zero-padded if needed) with N rows (the length of the result) and N_1 columns (the length of the $f[n]$)
- The two signals may be interchanged.

1-D linear convolution using Toeplitz matrices

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

$$\mathbf{H}=\underbrace{\begin{bmatrix} \boxed{\begin{matrix}1\\-1\\0\\0\end{matrix}} & \begin{matrix}0\\1\\-1\\0\end{matrix} & \begin{matrix}0\\0\\1\\-1\end{matrix} \end{bmatrix}_{4\times 3}}_{\text{Length of } f[n]=3}$$

Length of the result =4

Zero-padded $h[n]$ in the first column

Notice that H is not circulant (e.g. a -1 appears in the second line which is not present in the first line.

Contd..

$\{1, 1, 0, -2\}$

$$f[n] = \{\underline{1}, 2, 2\}, \;\; h[n] = \{\underline{1}, -1\}, \;\; N_1 = 3, N_2 = 2$$

$$\mathbf{g} = \mathbf{Hf} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

$$g[n] = \{\underline{1}, 1, 0, -2\}$$

Block matrices

- A_{ij} are matrices.
- If the structure of **A**, with respect to its sub-matrices, is Toeplitz then matrix **A** is called block-Toeplitz.
- If each individual $\mathbf{A}_{ij}$ is also a Toeplitz matrix then A is called doubly block-Toeplitz.

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{bmatrix}$$

2-D Linear Convolution

- $g(m,n) = \sum_{k_1} \sum_{k_2} f(k_1,k_2)\, h(m-k_1, n-k_2)$

m

1	4	1
2	5	3

n

$f[m,n]$

M_1=2, N_1=3

m

1	1
1	-1

n

$h[m,n]$

M_2=2, N_2=2

- The result will be of size (M1 + M2 - 1)x(N1 + N2 - 1) = 3x4

2-D linear convolution using doubly block Toeplitz matrices

$$f[m,n] = \begin{array}{ccc} 1 & 4 & 1 \\ 2 & 5 & 3 \end{array} \qquad h[m,n] = \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}$$

(m vertical axis, n horizontal axis)

At first, $h[m,n]$ is zero-padded to 3×4 (the size of the result).

Then, for each row of $h[m,n]$, a Toeplitz matrix with 3 columns (the number of **columns** of $f[m,n]$) is constructed.

$$h[m,n] = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}$$

Contd..

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0
\end{array}
$$

m, n

$h[m,n]$

- For each row of $h[m,n]$, a Toeplitz matrix with 3 columns (the number of columns of $f[m,n]$) is constructed.

$$
\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}
\quad
\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\quad
\mathbf{H}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Contd..

$$f[m,n] = \begin{array}{ccc} 1 & 4 & 1 \\ 2 & 5 & 3 \end{array} \qquad h[m,n] = \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}$$

(axes: m vertical, n horizontal)

Using matrices $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{H}_3$ as elements, a doubly block Toeplitz matrix $\mathbf{H}$ is then constructed with 2 columns (the number of **rows** of $f[m,n]$).

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_3 \\ \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_3 & \mathbf{H}_2 \end{bmatrix}_{12\times 6}$$

Contd..

$$\begin{array}{c|ccc} m & & & \\ \uparrow & 1 & 4 & 1 \\ & 2 & 5 & 3 \\ \hline & & & \rightarrow n \end{array} \qquad \begin{array}{c|cc} m & & \\ \uparrow & 1 & 1 \\ & 1 & -1 \\ \hline & & \rightarrow n \end{array}$$

$$f\,[m,n] \qquad\qquad h\,[m,n]$$

We now construct a vector from the elements of $f\,[m,n]$.

$$\mathbf{f} = \begin{bmatrix} 2 \\ 5 \\ 3 \\ \hline 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (2 \quad 5 \quad 3)^T \\ \hdashline (1 \quad 4 \quad 1)^T \end{bmatrix}$$

Contd..

$$
\begin{array}{l|lll}
m & & & \\
 & 1 & 4 & 1 \\
 & 2 & 5 & 3 \\
\hline
 & & & n
\end{array}
\qquad\qquad
\begin{array}{l|ll}
m & & \\
 & 1 & 1 \\
 & 1 & -1 \\
\hline
 & & n
\end{array}
$$

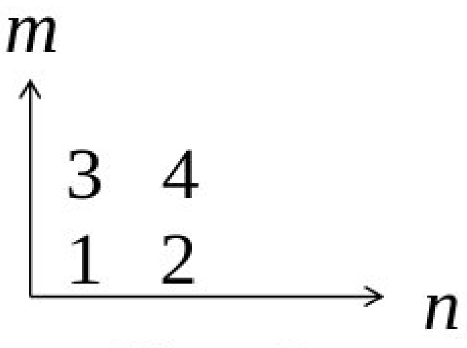
$$f\,[m,n] \qquad\qquad\qquad h\,[m,n]$$

$$
\mathbf{g} = \mathbf{H}\mathbf{f} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_3 \\ \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_3 & \mathbf{H}_2 \end{bmatrix} \left[\begin{array}{c} 2 \\ 5 \\ 3 \\ \hline 1 \\ 4 \\ 1 \end{array}\right]
$$

Contd..

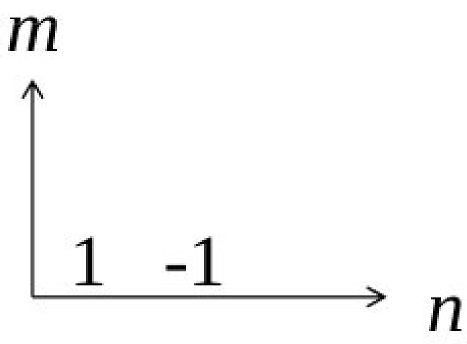
$$\mathbf{g} = \mathbf{Hf} = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} 2 \\ 5 \\ 3 \\ \hline 1 \\ 4 \\ 1 \end{array}\right] = \left[\begin{array}{r} 2 \\ 3 \\ -2 \\ 3 \\ \hdashline 3 \\ 10 \\ 5 \\ 2 \\ \hdashline 1 \\ 5 \\ 5 \\ 1 \end{array}\right] = \left[\begin{array}{c} (2 \quad 3 \quad -2 \quad 3)^T \\ \hdashline (3 \quad 10 \quad 5 \quad 2)^T \\ \hdashline (1 \quad 5 \quad 5 \quad 1)^T \end{array}\right]$$

Exercise (2-D Linear Convolution)



$f\,[m,n]$

$M_1=2,\ N_1=2$



$h\,[m,n]$

$M_2=1,\ N_2=2$

Bilateral Filtering

What is Bilateral filtering?

- It is a technique to smooth images while preserving edges.

(a) Input image (b) Output of the bilateral filter

[1] V. Aurich and J. Weule, “Non-linear gaussian filters performing edge preserving diffusion,” in Proceedings of the DAGM Symposium, pp. 538–545, 1995.
[2] C. Tomasi and R. Manduchi, “Bilateral filtering for gray and color images,” in Proceedings of the IEEE international Conference on Computer Vision, pp. 839–846, 1998.

Qualities of Bilateral filter

- Formulation is simple.
- Depends on less no. of parameters.
- Can be used non-iterative manners.
- Availability of numerical schemes makes the computation easier.

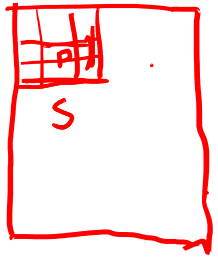
Image Smoothing with Gaussian Convolution

- Blurring is perhaps the simplest way to smooth an image.
- It can be done by convoluting the image with simple Gaussian kernel.

$$GC[I]_{\mathbf{p}} = \sum_{\mathbf{q} \in \mathcal{S}} G_\sigma(\|\mathbf{p} - \mathbf{q}\|)\, I_{\mathbf{q}},$$

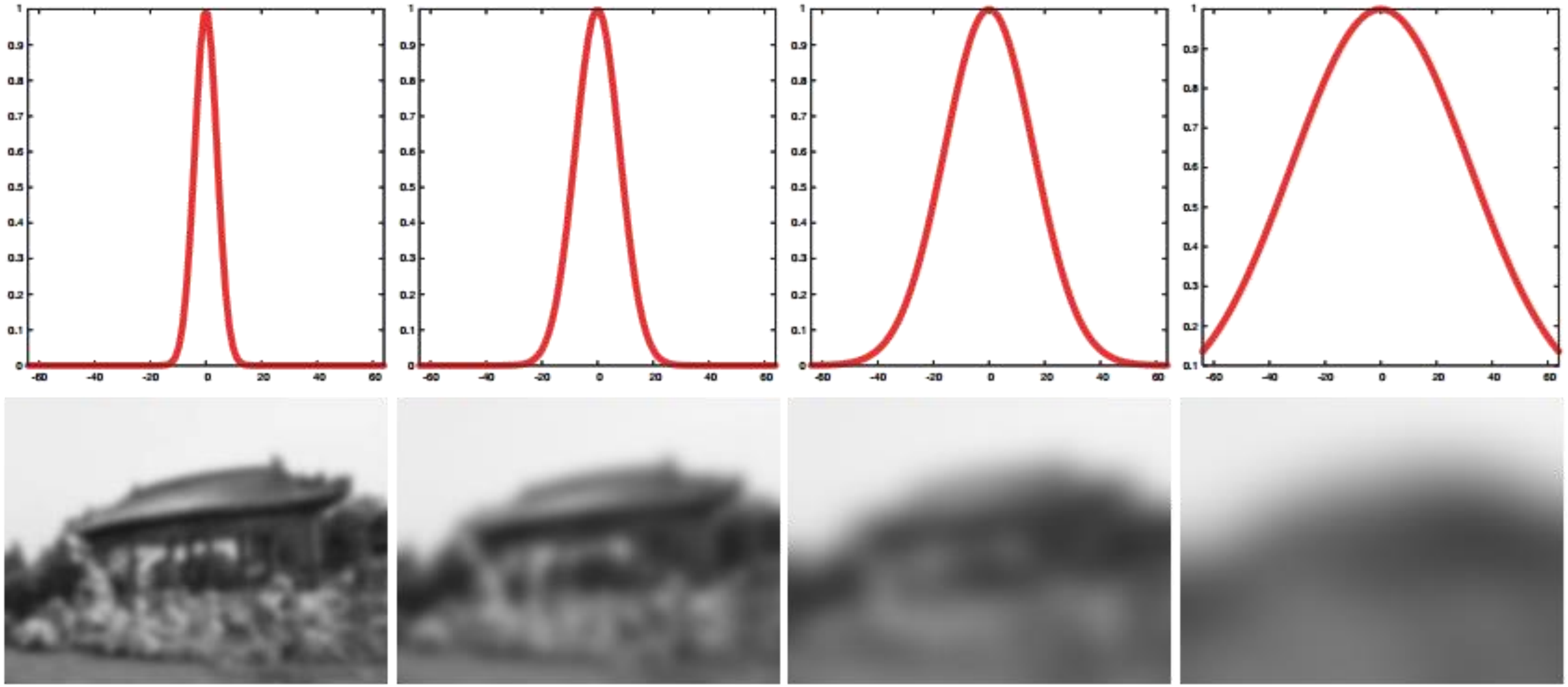
- where

$$G_\sigma(x) = \frac{1}{2\pi\,\sigma^2} \exp\left(-\frac{x^2}{2\,\sigma^2}\right)$$



1.5

An illustration



$\sigma = 4$ $\sigma = 8$ $\sigma = 16$ $\sigma = 32$

Problem with Gaussian smoothing

- It is independent of image content.
- Weight depends only on the spatial distance between the pixels.
- As a result it tends to smooth the edges, which is not desired.

Edge-preserving Filtering with the Bilateral Filter

- The key idea of the bilateral filter is that for a pixel to influence another pixel, it should not only occupy a nearby location but also have a similar value.

- Where

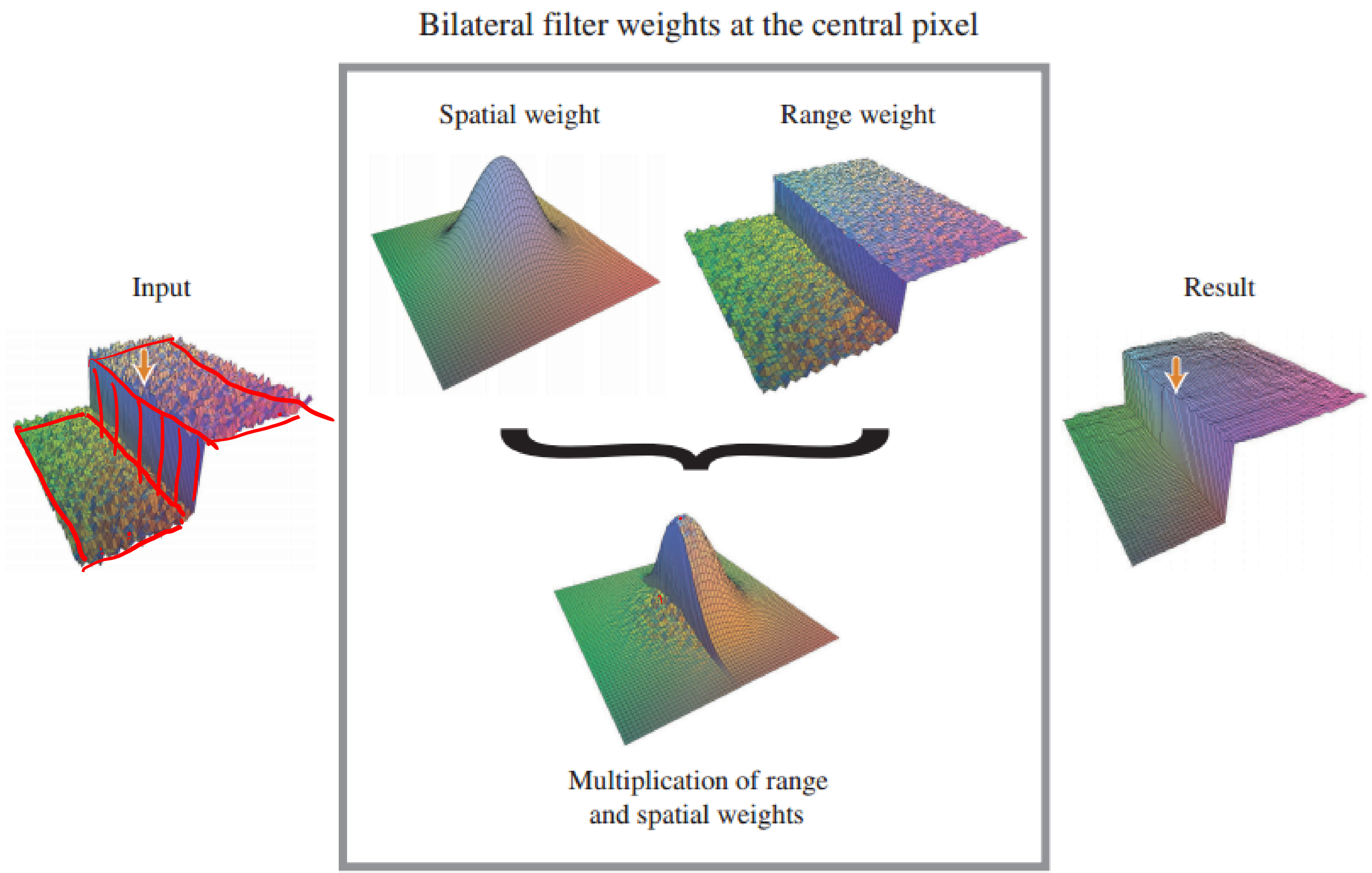
$$BF[I]_{\mathbf{p}} = \frac{1}{W_{\mathbf{p}}} \sum_{\mathbf{q} \in \mathcal{S}} G_{\sigma_s}(\|\mathbf{p} - \mathbf{q}\|)\, \underbrace{G_{\sigma_r}(|I_{\mathbf{p}} - I_{\mathbf{q}}|)}_{0.01}\, I_{\mathbf{q}},$$

$I_p \approx I_q$

1

$$W_{\mathbf{p}} = \sum_{\mathbf{q} \in \mathcal{S}} G_{\sigma_s}(\|\mathbf{p} - \mathbf{q}\|)\, G_{\sigma_r}(|I_{\mathbf{p}} - I_{\mathbf{q}}|).$$

$|I_p - I_q|$


Bilateral filter weights at the central pixel
Input
Spatial weight
Range weight
Multiplication of range
and spatial weights
Result

Parameters

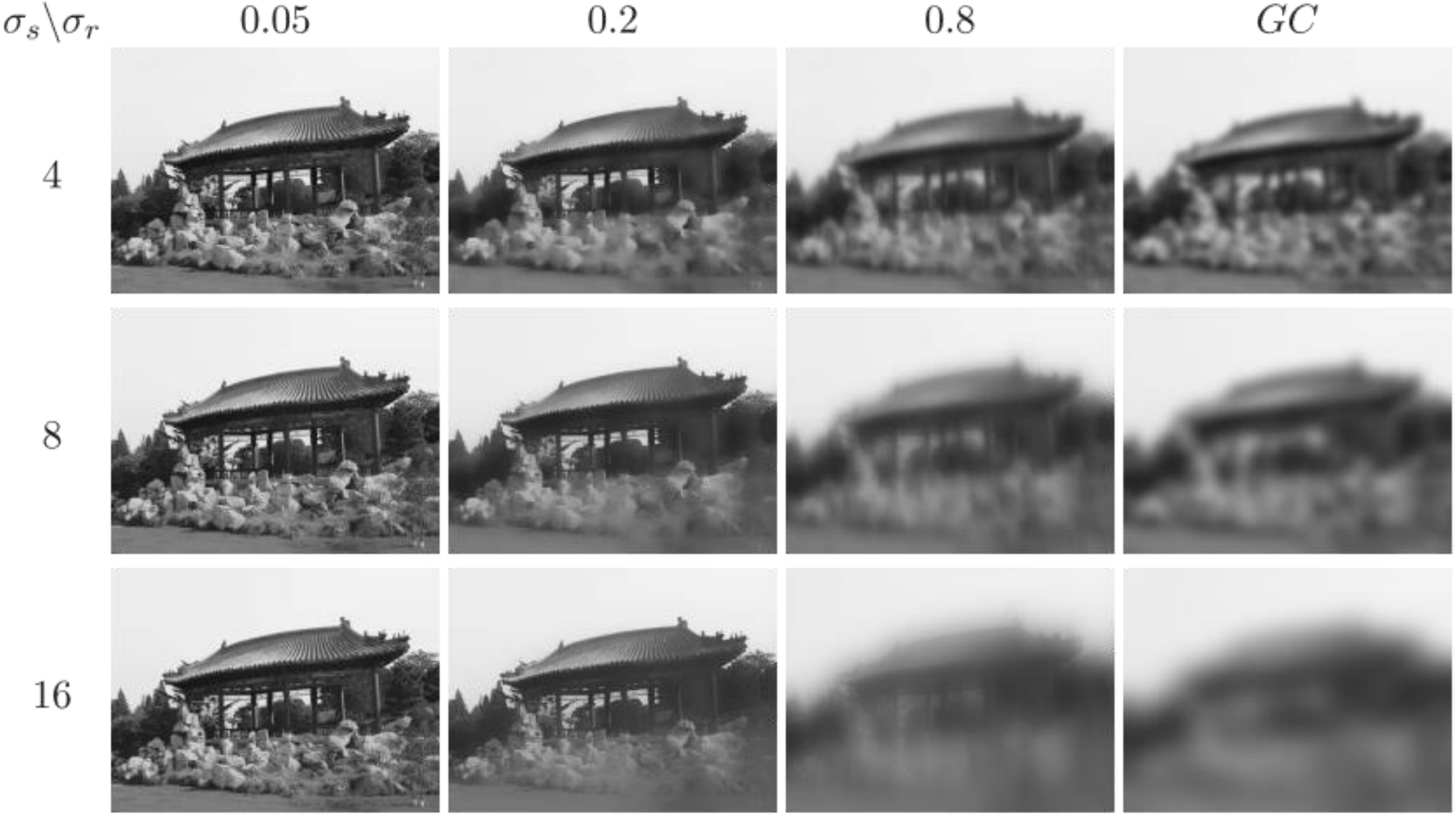
- It is controlled by two parameters: a) Range parameter, b) Spatial parameter.

a)

As the range parameter σ_r increases, the bilateral filter gradually approximates Gaussian convolution more closely because the range Gaussian G_{σ_r} widens and flattens, i.e., is nearly constant over the intensity interval of the image.

b)

Increasing the spatial parameter σ_s smooths larger features.



$\sigma_s \backslash \sigma_r$
0.05
0.2
0.8
GC
4
8
16

How to set parameters?

- Depends on the application.
- For instance:
- Space parameter: proportional to image size– e.g., 2% of image diagonal
- range parameter: proportional to edge amplitude– e.g., mean or median of image gradients
- independent of resolution and exposure