Lecture 10

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Linear Independence

Linear Independence and Linear Dependence

The set of vectors $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ in a vector space V is called *linearly independent* provided the only solution to the equation $c_1\mathbf{x}_1 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$ is the trivial solution $c_1 = \cdots = c_m = 0$. If the equation has a nontrivial solution or $\mathbf{0}$ is a linear combination of the vectors in S with coefficients that are not all zero, then the set S is called *linearly dependent*.

Remark

An infinite set S is called an linearly independent set if and only if every finite subset of S is linearly independent.

Examples

1. If $\mathbf{x}_1 = (2, -1, 0, 3), \mathbf{x}_2 = (1, 2, 5, -1)$ and $\mathbf{x}_3 = (7, -1, 5, 8)$, then the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly dependent set in $\mathbb{R}^4$, since $3\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0}$.
2. The polynomials $\mathbf{p}_1 = 1 - x, \mathbf{p}_2 = 5 + 3x - 2x^2, \mathbf{p}_3 = 1 + 3x - x^2$ are linearly dependent in $\mathbb{R}_2[x]$.
3. The polynomials $\mathbf{p}_1 = 1, \mathbf{p}_2 = x, \mathbf{p}_3 = x^2$ are linearly independent in $\mathbb{R}_2[x]$.
4. Let $\mathbf{A}$ be $m \times n$ matrix. Then what we can say about the linear independence or dependence of columns of $\mathbf{A}$?
5. Let S be the set of all solutions of $\mathbf{Ax} = \mathbf{0}$. Is S linearly independent? ?
6. Can there exists a finite subset $T \subset S$ which is linear independent?

Exercise

Determine whether the vectors $\mathbf{x}_1 = (1, -2, 3), \mathbf{x}_2 = (5, 6, -1), \mathbf{x}_3 = (3, 2, 1)$ forms a linearly dependent set or linearly independent set in $\mathbb{R}^3$.

Theorem

A set of vectors S with two or more vectors in a vector space V is

1. Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.
2. Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors.

Proof. Exercise

Remark

Let $S = \{v_1, \dots, v_p\}$. Then the set is linearly dependent if and only if there exists $1 \leq j \leq p$ such that $v_j = c_1 v_1 + \cdots + c_{j-1} v_{j-1}$ for some $c_1, \dots, c_{j-1} \in \mathbb{F}$.

Theorem

1. A finite set of vectors that contains a zero vector is linearly dependent.
2. A set with exactly two vectors is linearly independent if and only if neither vector is scalar multiple of the other.

Proof.

1. Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_r, \mathbf{0}\}$. Then S is linearly dependent since the equation $0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_r + 1(\mathbf{0}) = \mathbf{0}$ expresses $\mathbf{0}$ as a linear combination of the vectors in S with coefficients that are not all zero.
2. Exercise.

Question

What about the linear independence of the set $\{x\}$?

Theorem

Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ be a subset of $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

Proof. Let $\mathbf{x}_i = (x_{i1}, \ldots, x_{in})$ for all i. Consider the equation $c_1\mathbf{x}_1 + \cdots + c_r\mathbf{x}_r = \mathbf{0}$, i.e.,
$c_1(x_{11}, \ldots, x_{1n}) + c_2(x_{21}, \ldots, x_{2n}) + \cdots + c_r(x_{r1}, \ldots, x_{rn}) = (0, \ldots, 0)$.
Equate the corresponding components, we get

$$\begin{array}{rcl} c_1x_{11} + c_2x_{21} + \cdots + c_rx_{r1} & = & 0 \\ c_1x_{12} + c_2x_{22} + \cdots + c_rx_{r2} & = & 0 \\ & \vdots & \\ c_nx_{1n} + c_2x_{2n} + \cdots + c_rx_{rn} & = & 0 \end{array}$$

This is a homogeneous system in r variables and n equations. Since $r > n$, this system has infinite many solution. In particular, this system has a non-trivial solution. Hence S is Linearly dependent.

Corollary

Let S be a subset of $\mathbb{R}^n$. If S is linearly independent, then $|S| \leq n$.