Department of Mathematics
Indian Institute of Technology Jammu

CSD001P5M **Linear Algebra** **Tutorial: 03**

1. Show that the space of all 3×3 real (respectively complex) matrices is a vector space over $\mathbb{R}$ (respectively $\mathbb{C}$) with respect to the usual addition and scalar multiplication.

2. Let S:= The set of all 3×3 upper triangular matrices with real entries. Check whether S is a real vector space under usual addition and scalar multiplication of matrices.

3. In $\mathbb{R}$, consider the addition $x \oplus y := x + y - 1$ and $a \cdot x := a(x-1) + 1$. Show that $\mathbb{R}$ is a real vector space with respect to these operations with additive identity 1.

4. Which of the following are vector spaces:
   (a) $\mathbb{C}$ over $\mathbb{C}$ (b) $\mathbb{C}$ over $\mathbb{R}$ (c) $\mathbb{R}$ over $\mathbb{C}$ (d) $\mathbb{R}$ over $\mathbb{R}$

5. Which of the following are subspaces of $\mathbb{R}^3$:
   (a) $\{(x,y,z) | x+y=0\}$
   (b) $\{(x,y,z) | x^2+y^2=z\}$
   (c) $\{(x,y,z) | x^2+y^2=0\}$
   (d) $\{(x,y,z) | xy=1\}$
   (e) $\{(x,y,z) | x \geq 0\}$
   (f) $\{(x,y,z) | x+y=z\}$
   (g) $\{(x,y,z) | x=y^2\}$.

6. Which of the following are subspaces of $\mathbb{C}^3$ (over $\mathbb{C}$):
   (a) $\{(z_1,z_2,z_3) | z_1 \text{ is real}\}$
   (b) $\{(z_1,z_2,z_3) | z_1+z_2=10z_3\}$.

7. Find the conditions on real numbers a,b,c,d so that the set $\{(x,y,z) | ax+by+cz=d\}$ is a subspace of $\mathbb{R}^3$.

8. Which of the following are subspaces of $C[0,1] := \{f : [0,1] \longrightarrow \mathbb{R} : f \text{ is continuous}\}$?
   (a) $\left\{f \in C[0,1] : f\left(\frac{1}{2}\right) = 0\right\}$.
   (b) $\left\{f \in C[0,1] : f \text{ has a local maxima at } x = \frac{1}{2}\right\}$.

(c) $\left\{f \in C[0,1] : f \text{ has a local maxima or minima at } x = \frac{1}{2}\right\}$.

9. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and $\mathbb{W}_1, \ldots, \mathbb{W}_r$ subspaces of $\mathbb{V}$. Then which of the following is a subspace of $\mathbb{V}$:

   (a) $\mathbb{V} \setminus \mathbb{W}_1$

   (b) $\bigcup_{i=1}^{r} \mathbb{W}_i$

   (c) $\bigcap_{i=1}^{r} \mathbb{W}_i$

   (d) $\mathbb{W}_1 + \mathbb{W}_2 = \{w_1 + w_2 \ : \ w_1 \in \mathbb{W}_1, w_2 \in \mathbb{W}_2\}$

10. Express the 2×2 matrix $\begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix}$ as a linear combination of the matrices $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, $\begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$ and $\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$.

11. In the following W is a subspace of V? Base field is taken as $\mathbb{R}$ in all. Justify your answer.

    (a) $V := \mathbb{R}[x] =$ vector space of all polynomials with real coefficients, $W :=$ set of all polynomials with integer coefficients.

    (b) $V := \mathbb{R}^2$, $W := \{(x,y) \ : \ x + y \geq 0\}$.

    (c) $V := \mathbb{R}^2$, $W := \{(x,y) \ : \ x^2 + y^2 \geq 0\}$.

12. Consider the space V of all 2×2 matrices over $\mathbb{R}$. Which of the following sets of matrices A in V are subspaces of V? Justify (prove) your answers.

    (a) All upper triangular matrices $\left(\text{i.e. matrices of the form } \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right)$.

    (b) All A such that $AB = BA$ where B is some fixed matrix in V.

    (c) All A such that $BA = 0$ where B is some fixed matrix in V.

    (d) Would the above results hold for all $n \times n$ matrices where n is a general positive integer $(n \geq 2)$?

13. Given the following vectors in $\mathbb{R}^3 : \mathbf{u} = (1,3,5), \mathbf{v} = (1,4,6), \mathbf{w} = (2,-1,3)$ and $\mathbf{b} = (6,5,17)$.

    (a) Does $\mathbf{b} \in W = \operatorname{span}\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$?

    (b) If the answer to (a) is yes, express $\mathbf{b}$ as a linear combination of **u,v,w**.

14. Let U and W be two subspaces of the vector space V. We define $U + W = \{u + w \ : \ u \in U, w \in W\}$. Show that $U + W$ is a subspace of V, and moreover, $U + W$ is the smallest subspace of V which contains both U and W.