Department of Mathematics
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CSD001P5M **Linear Algebra** **Tutorial: 09**

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1. Let A be an $m \times n$ matrix. Suppose the columns of A are linearly independent. Show that there exist a matrix Q whose columns are orthogonal and an invertible upper triangular matrix R such that $A = QR$.

2. Let A be $m \times n$ matrix. Prove that

   a) $\mathbb{R}^n = \mathscr{R}(A) \oplus \mathscr{N}(A)$

   b) $\mathbb{R}^m = \mathscr{C}(A) \oplus \mathscr{N}(A^T)$.

3. Suppose $v_1, \ldots, v_m \in V$ Prove that $\{v_1, \ldots, v_m\}^\perp = (\text{span}(v_1, \ldots, v_m))^\perp$.

4. Let U is a finite dimensional subspace of V. Prove that $U^\perp = \{0\}$ if and only if $U = V$.

5. Suppose U is a subspace of V with basis $\{u_1, \ldots, u_m\}$ and $B = \{u_1, \ldots, u_m, w_1, \ldots, w_n\}$ is a basis of V. Apply the Gram-Schmidt procedure to the basis B. and it produce $\{e_1, \ldots, e_m, f_1, \ldots, f_n\}$. Prove that $\{e_1, \ldots, e_m\}$ is an orthonormal basis of U and $\{f_1, \ldots, f_n\}$ is an orthonormal basis of $U^\perp$.

6. Let U be a subspace of $\mathbb{R}^4$ defined by $\text{span}((1,2,3,-4),(-5,4,3,2))$. Find an orthonormal basis of U and an orthonormal basis of $U^\perp$.

7. Let U and W be subset of V. Then prove the following:

   (a) If $U \subset W$, then $W^\perp \subset U^\perp$.

   (b) $(U \cup W)^\perp = U^\perp \cap W^\perp$.

   (c) $U^\perp \cup W^\perp \subset (U \cap W)^\perp$.

8. Suppose U, W are subspaces of finite dimensional V. Then prove the following:

   a) $P_{U^\perp} = I - P_U$, where I is an identity map.

   b) $P_U P_W = 0$ if and only if $\langle u, w \rangle = 0$ for all $u \in U, w \in W$.

9. Let $U = \text{span}((1,1,0,0),(1,1,1,2))$. Find $u \in U$ such that $||u - (1,2,3,4)||$ is as small as possible.

10. Take your 5 favourite real inner product space. Find $U^\perp$ for some of their subspaces U.