Quadratic form

$Q(x) = x^T A x$, where A is symmetric mat.

change of variable $\quad x = Py$

$$Q(y) = (Py)^T A P y$$
$$= y\,(\underline{P^T A P})\,y$$

The principal Axis theorem

Let A be $n \times n$ symmetric matrix. Then there is an orthogonal change of variable $x = Py$, that transforms the quadratic form $x^T A x$ into a quadratic form $y^T D y$ with no mixed term.

$\longrightarrow$ The columns of P are called the Principal axis of the quadratic form.

Ex Let $A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$

$$Q(x) = x^T A x = [x_1 \quad x_2] \begin{bmatrix} 1 & -4 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Characteristic eq of A

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -4 \\ -4 & -5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-5-\lambda) - 16 = 0$$

$$\Rightarrow \lambda^2 + 4\lambda - 5 - 16 = 0$$

$$\Rightarrow \lambda^2 + 4\lambda - 21 = 0$$

$$\Rightarrow (\lambda + 7)(\lambda - 3) = 0$$

$\lambda = -7, 3$ are eigen values.

To find an eigenvector for $\lambda = -7$

$$A + 7I = \begin{vmatrix} 8 & -4 \\ -4 & 2 \end{vmatrix}$$

$$R_2 \to R_2 + \frac{1}{2} R_1$$

$$\sim \begin{vmatrix} 8 & -4 \\ 0 & 0 \end{vmatrix}$$

$$R_1 \to \frac{1}{8} R_1$$

$$\sim \begin{vmatrix} \boxed{1} & -1/2 \\ 0 & 0 \end{vmatrix}$$

$$x_1 - \frac{1}{2} x_2 = 0$$

eigenvector is

$$P_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \qquad x_1 = \frac{1}{2} x_2$$

normalize P_1 =

$$\frac{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}}{\sqrt{\frac{1}{4} + 1}} = \frac{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}}{\sqrt{5}/2} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

for $\lambda = 3$,

$$A - 3I = \begin{vmatrix} -2 & -4 \\ -4 & -8 \end{vmatrix}$$

$$R_2 \longrightarrow R_2 - 2R_1$$

$$\sim \begin{vmatrix} -2 & -4 \\ 0 & 0 \end{vmatrix}$$

$$R_1 \longrightarrow -\frac{1}{2} R_2$$

$$\sim \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}$$

eigen vector P_2 $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

normalize $P_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$$P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}$$

$$Q(y) = -7y_1^2 + 3y_2^2$$

$\langle x, y \rangle = x^T \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} y$ — Inner product

$\langle x, y \rangle = x^T \begin{bmatrix} 1 & -4 \\ -4 & -1 \end{bmatrix} y \longrightarrow$ not an inner product

(Think why?)

Classifying the quadratic forms.

A quadratic form $\varphi(x)$ is

a) Positive definite if $\varphi(x) > 0 \ \forall \ x \neq 0$

b) -ve definite if $\varphi(x) < 0 \ \forall \ x \neq 0$

c) Indefinite if $\varphi(x)$ assumes both value -ve as well as +ve.

d) positive semidefinite, if $\varphi(x) \geq 0 \ \forall \ x \neq 0$

(e) -ve semi definite if $\varphi(x) \leq 0 \ \forall \ x \neq 0$

ex
$$\begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

$$\varphi(x) = x^T A x$$

$$\varphi\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8 & -4-10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= -7 - 28 = -35$$

$$\varphi\begin{pmatrix}1\\2\end{pmatrix} < 0$$

$$\varphi\begin{pmatrix}-2\\1\end{pmatrix} = \begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}1 & -4\\-4 & -5\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}$$

$$= \begin{bmatrix}-2-4 & 8+5\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}$$

$$= 12 + 3 = 15 > 0$$

$$\varphi\begin{bmatrix}-2\\1\end{bmatrix} > 0$$

$\varphi(x)$ is an indefinite quadratic form.

---

ex $\quad A = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$

$$x = \begin{bmatrix}x_1 & x_2\end{bmatrix}$$

$$\varphi(x) = \begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$= 2x_1^2 - 2x_1x_2 + x_2^2$$

$$= x_1^2 + (x_1 - x_2)^2 > 0$$

$\varphi(x)$ is a +ve a definite form.

Let A be symmetric matrix.

and $x, y \in \mathbb{R}^n$, define $\langle x, y\rangle = x^T A y$.

Suppose $\langle \ \rangle$ is an inner product.

Then $\varphi(x) = x^T A x$ is +ve definite

Thm Let A be an $n \times n$ symmetric matrix.

Then $Q(x) = x^T A x$ is

a) Positive definite iff all the eigenvalues are +ve

b) -ve definite iff all the eigenvalues are -ve

c) +ve semidefinite iff all the eigenvalues are non-ve

d) -ve ______ iff all the eigenvalues are non +ve

e) Indefinite iff eigenvalues +ve as well as -ve.

---

ex $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Characteristic eq.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda) - 1 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3 \pm \sqrt{5}}{2} > 0$$

$\phi(x) = x^T A x$ is +ve definite.

Q8 $\quad \phi(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

Matrix of quadratic form is

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Characteristic eq $\quad |A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 2-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left((2-\lambda)(1-\lambda)-4\right)-2(2-2\lambda)=0$$

$$\Rightarrow (3-\lambda)(\lambda^2-3\lambda+\underbrace{2-4}_{-2})-4+4\lambda=0$$

$$\Rightarrow -\lambda^3+3\lambda^2+3\lambda^2-9\lambda+2\lambda+4\lambda-6-4=0$$

$$\Rightarrow -\lambda^3+6\lambda^2-3\lambda-10=0$$

$$\lambda^3-6\lambda^2+3\lambda+10=0$$

$\lambda=-1$ is an eigen value

$$\Rightarrow \lambda^2(\lambda+1)-7\lambda(\lambda+1)+10(\lambda+1)=0$$

$$\Rightarrow (\lambda^2-7\lambda+10)(\lambda+1)=0$$

$$\Rightarrow (\lambda-2)(\lambda-5)(\lambda+1)=0$$

Eigenvalues are $\lambda=-1, 2, 5$

$Q(x)$ is undefinite form.