Let V be a space and U be a subspace of V

Then $U^{\perp} = \{ v \in V : \langle u, v \rangle = 0 \ \forall \ u \in U \}$

If $U = V$, then

$$V^{\perp} = \{ v \in V : \langle u, v \rangle = 0 \ \forall \ u \in V \}$$

$$\Rightarrow \langle v, v \rangle = 0 \Rightarrow \|v\|^2 = 0 \Rightarrow v = 0$$

Diagonalization of symmetric matrix.

Def An $n \times n$ matrix A is said to be symmetric iff $A^T = A$.

Question Let A be a diagonalizable matrix. Then $\exists$ invertible matrix P and diagonal matrix D s.t. $P^{-1} A P = D$

Is it possible that P orthogonal?

Suppose P is orthogonal, then $P^{-1} = P^T$

$$\Rightarrow P^T A P = D \Rightarrow A = P D P^T$$

$$A^T = (P D P^T)^T$$

$$= (P^T)^T D^T P^T$$

$$= P D P^T$$

$$= A$$

$\Rightarrow$ A is symmetric.

$\longrightarrow$ If P is orthogonal, then A is symmt.

Ex $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\chi_A = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$

eigenvalues are $\underline{\lambda = \pm 1}$

eigen vector for $\lambda = 1$, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector corr. to $\lambda = 1$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vect —— $\lambda = -1$

$$\underline{\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle} = 1 \cdot 1 + (1)(-1)$$
$$= 1 - 1 = 0$$

We have seen that eigen values are real and eigen vectors are orthogonal.

Thm 1 Let A be an $n \times n$ symmetric matrix. Then eigen values of A are real.

Pf Let λ be an eigenvalue of A. We want to prove λ is real, i.e., $\bar{\lambda} = \lambda$

Let x be an eigen vector corr. to λ.

$$Ax = \lambda x$$

Take transpose $x^t A^t = \lambda x^t$

$$\Rightarrow x^t A = \lambda x^t$$

Take complex conj., $\overline{x^t \bar{A}} = \bar{\lambda}\, \overline{x^t}$

$\Rightarrow \overline{x^t} A = \bar{\lambda}\, \overline{x^t}$

post multiply by x $\Rightarrow \overline{x^t}\, \underline{Ax} = \bar{\lambda}\, \overline{x^t} x$

$\Rightarrow \lambda\, \overline{x^t} . x = \bar{\lambda}\, \overline{x^t} x$

$\Rightarrow (\lambda - \bar{\lambda}) \underbrace{(\overline{x^t} x)} = 0$

Since $(\overline{x^t} x \neq 0$, we get $\lambda - \bar{\lambda} = 0$

$\Rightarrow \bar{\lambda} = \lambda$

Hence λ is real

$\longrightarrow$ Since eigen values are real, we can see that x is also real.

<u>Thm</u> Let v_1, v_2 be eigenvectors of a symmetric matrix A corr. to eigenvalues λ_1, λ_2, resp., Then $v_1 \perp v_2$. $(\lambda_1 \neq \lambda_2)$.

<u>Pf</u> $A v_1 = \lambda_1 v_1$, $A v_2 = \lambda_2 v_2$,

WTP $v_1^t . v_2 = 0$

$$\langle A v_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle$$
$$= \lambda_1 \langle v_1, v_2 \rangle \quad \text{— (1)}$$

$$\langle A v_1, v_2 \rangle = \langle v_1, A^t v_2 \rangle$$
$$= \langle v_1, A v_2 \rangle$$
$$= \langle v_1, \lambda_2 v_2 \rangle$$

$$= \lambda_2 \langle v_1, v_2 \rangle \quad — (2)$$

from (1) & (2)

$$\lambda_1 \langle v_1, v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

$$\Rightarrow (\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0$$

Since $\lambda_1 \neq \lambda_2$, $\langle v_1, v_2 \rangle = 0$

$$\Rightarrow v_1 \perp v_2$$

$$\underline{v_1^t A v_2} = v_1^t \lambda_2 v_2 = \lambda_2 v_1^t v_2 \quad — (1)$$

$$v_1^t A v_2 = \underline{v_1^t A^t} v_2 = (Av_1)^t v_2$$
$$= \lambda_1 v_1^t v_2$$
$$= \lambda_1 v_1^t v_2 \quad — (2)$$

from (1) & (2) we get $\boxed{v_1^t v_2 = 0}$

Ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ eigen values are $1, 1, -1$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Thm Let A be real symmetric matrix. Then A is orthogonally diagonalizable.

Def A matrix A is said to be orthogonally

diagonalizable iff $\exists$ an orthogonal matrix P and
diagonal matrix D s.t $\boxed{P^T A P = D}$

Qust. Suppose A is symmetric matrix and P is an invertible matrix s.t $\boxed{P^T A P = D}$ → diagonal

Is it nec. that P is orthogonal.