Lecture 5

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Elementary Matrices

Theorem

A linear system of equations has either no solution, or a unique solution, or infinitely many solutions.

Corollary

Any homogeneous linear system of equations is either a trivial solution or infinitely many solutions.

Corollary

Let $\mathbf{Ax} = \mathbf{O}$ *be a linear system of* n *variables and* m *equations. If* $m < n$*, then system has infinitely many solutions.*

Proof. Exercise.

Remark

Suppose $\mathbf{Ax} = \mathbf{O}$ be a homogeneous system with number of variables equal to number of number of equations, say n. Further, suppose that $\mathbf{Ax} = \mathbf{O}$ has a trivial unique solution. The system does not have a free variable. Hence the number of non-zero rows of RREF of $\mathbf{A}$ is n, so $\mathbf{A}$ is row equivalent to I. On the other side suppose $\mathbf{A}$ is row equivalent to I, hence the system has no free variable. Thus system has only trivial solution.

Solutions of $\mathbf{Ax} = \mathbf{O}$ and $\mathbf{Ax} = \mathbf{b}$

Let $\mathbf{Ax} = \mathbf{O}$ and $\mathbf{Ax} = \mathbf{b}$ be systems of linear equations. Let S denote the set of solutions of the homogeneous system $\mathbf{Ax} = \mathbf{O}$. Let $\mathbf{x}_p$ be a particular solution of the system $\mathbf{Ax} = \mathbf{b}$. Then the set of all solutions of system $\mathbf{Ax} = \mathbf{b}$ is given by

$$\{\mathbf{x}_h + \mathbf{x}_p; \mathbf{x}_h \in S\}.$$

Suppose $\mathbf{x}_h \in S$ and $\mathbf{x}_p$ is a particular solution of $\mathbf{Ax} = \mathbf{b}$. Then $\mathbf{Ax}_h = \mathbf{O}$ and $\mathbf{Ax}_p = \mathbf{b}$ and hence $\mathbf{A}(\mathbf{x}_h + \mathbf{x}_p) = \mathbf{Ax}_h + \mathbf{Ax}_p = \mathbf{O} + \mathbf{b} = \mathbf{b}$. This implies that set of solutions of $\mathbf{Ax} = \mathbf{b}$ contains $\{\mathbf{x}_h + \mathbf{x}_p; \mathbf{x}_h \in S\}$. Other side is an exercise.

Elementary Row Matrices

Definition

A matrix is said to *elementary row matrix* if it is one of the following:

1. $E_i(c)$ is the matrix obtained from I_n by multiplying ith row by a non-zero scalar c
2. E_{ij} is obtained from I_n by interchanging ith rows and jth row.
3. $E_{ij}(c)$ is obtained from I_n by adding c time the jth row into ith row.

Effects. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $E \in \mathbb{R}^{m\times m}$. Then,

1. if $E = E_i(c)$, EA is the matrix obtained from A by multiplying ith row by a non-zero scalar c
2. if $E = E_{ij}$, then EA is obtained from A by interchanging ith rows and jth row.
3. if $E = E_{ij}(c)$, then EA is obtained from A by adding c time the jth row into ith row.

Proof. Exercise

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $E = E_{12}(3)$. Then $E = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and we get $E\mathbf{A} = \begin{bmatrix} 13 & 17 & 21 \\ 4 & 5 & 6 \end{bmatrix}$. Also note that, if we apply elementary row operation $R_1 \to R_1 + 3R_3$ on $\mathbf{A}$, we get $\begin{bmatrix} 13 & 17 & 21 \\ 4 & 5 & 6 \end{bmatrix}$.

Theorem

Let $\mathbf{A}$ be a matrix and E be an elementary matrix by performing some elementary row operation on I. Then $E\mathbf{A}$ is a matrix obtained from $\mathbf{A}$ by performing same elementary row operation.

Corollary

Elementary matrices are invertible.

Proof.

1. $E_i(c)E_i\left(\frac{1}{c}\right) = I = E_i\left(\frac{1}{c}\right)E_i(c)$, where $c \neq 0$.
2. $E_{ij}E_{ij} = I$.
3. $E_{ij}(c)E_{ij}(-c) = I = E_{ij}(-c)E_{ij}(c)$.

Theorem

Let $\mathbf{A}$ be an invertible matrix of size n and $\mathbf{b} \in \mathbb{R}^{1\times n}$. Then $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

Proof. Let $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Then $\mathbf{Ax} = \mathbf{A}(\mathbf{A}^{-1}\mathbf{b}) = (\mathbf{AA}^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$. Hence $\mathbf{A}^{-1}\mathbf{b}$ is a solution of $\mathbf{Ax} = \mathbf{b}$. Suppose $\mathbf{x}_1$ be the another solution of $\mathbf{Ax} = \mathbf{b}$. Then $\mathbf{Ax}_1 = \mathbf{b}$. Multiply both sides by $\mathbf{A}^{-1}$, we get $\mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{b}$. This gives $\mathbf{x}_1 = \mathbf{A}^{-1}\mathbf{b}$.

Theorem

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Then the following conditions are equivalent:

a) *$\mathbf{A}$ is an invertible matrix.*

b) *The matrix equation $A\mathbf{x} = \mathbf{O}$ has the unique solution $\mathbf{x} = \mathbf{O}$.*

c) *The reduced row-echelon form of $\mathbf{A}$ is I_n.*

Proof. (a) $\Rightarrow$ (b) Assume that $\mathbf{A}$ is an invertible matrix. Then there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = I_n = \mathbf{BA}$.

Now consider the matrix equation $\mathbf{Ax} = \mathbf{O}$. Then $\mathbf{B}(\mathbf{Ax}) = \mathbf{BO} = \mathbf{O}$ implies that $(\mathbf{BA})\mathbf{x} = \mathbf{O}$. Since $\mathbf{BA} = I_n$, it follows that $I_n\mathbf{x} = \mathbf{O}$, i.e., $\mathbf{x} = \mathbf{O}$.

(b)$\Rightarrow$ (c) Because the matrix equation $\mathbf{Ax} = \mathbf{O}$ has the unique solution $x_1 = 0, x_2 = 0, \ldots, x_n = 0$, there is no free variable. That is, all variables are leading variables. Since system has exactly n equations, the reduced system is

$$\begin{array}{rcl} x_1 \qquad\qquad & = & 0 \\ x_2 \qquad & = & 0 \\ \ddots \qquad & & \\ x_n & = & 0. \end{array}$$

Thus reduced row-echelon form of A is I_n.

(c) $\Rightarrow$ (a) We know that by a (finite) sequence of elementary row operations, the matrix A can be transformed into a reduced row-echelon matrix A'. Thus there are finitely many elementary matrices $E_1, E_2, \ldots, E_t$ such that

$$E_t E_{t-1} \cdots E_2 E_1 A = A' = I_n.$$

Thus $B = E_t E_{t-1} \cdots E_2 E_1$ is the inverse of A.