Relation b/w two basis.

$$\underbrace{B_1 = \{v_1, \dots v_n\}}_{V} \qquad \underbrace{B_2 = \{w_1, \dots, w_n\}}_{V}$$

$$w_1 = a_{11} v_1 + a_{21} v_2 + \dots + a_{n1} v_n$$

$$\vdots$$

$$w_n = a_{1n} v_1 + a_{2n} v_2 + \dots + a_{nn} v_n$$

$$[A]_{B_1}^{B_2} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ a_{31} & & & \\ \vdots & & & \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} \longrightarrow \text{Transition matrix}$$

$$v \in V, \quad \boxed{[v]_{B_2} = A[v]_{B_1}}$$

$$\begin{bmatrix} -k \\ k \\ k \\ k \end{bmatrix} \in \ker(T), \quad \text{where } T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix}$$

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

$$\left.\begin{aligned} a+b &= 0 \\ b-c &= 0 \\ a+d &= 0 \end{aligned}\right] \text{ find the values of } a, b, c, d$$

WT Show $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$ be a basis.

- S is L.I & span (S) $= \mathbb{R}^3$

T.S. S is L.I

$$a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^3 \qquad c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ $\quad c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 1 & 1 & b \\ 0 & 1 & -1 & c \end{array}\right] \longrightarrow \text{Convert to RREF}$$

$$\dim(\mathbb{R}^3) = 3 = |S|$$

$$T(x_1, x_2) \longrightarrow (x_1, 0, x_2, x_2^2)$$

— $T(\alpha x + \beta y) = T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$

$$= (\alpha x_1 + \beta y_1, 0\ \alpha x_2 + \beta y_2, (\alpha x_2 + \beta y_2)^2)$$

$$\alpha T(x) + \beta T(y) = (\alpha x_1 + \beta y_1, 0, \alpha x_2 + \beta y_2, \alpha x_2^2 + \beta y_2^2)$$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$$\Leftrightarrow (\alpha x_2 + \beta y_2)^2 = \alpha x_2^2 + \beta y_2^2$$

$$\Leftrightarrow \alpha^2 x_2^2 + \beta^2 y_2^2 + \alpha\beta x_2 y_2 = \alpha x_2^2 + \beta y_2^2$$

$x = (1, 2), \quad y = (1, 2) \quad T((1,2) + (1,2)) \neq T(1,2) + T(1,2)$

$\downarrow$ Not a L.T

$D : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_1[x], \quad D(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$

Let $a_0 + a_1 x + a_2 x, \ b_0 + b_1 x + b_2 x \in \mathbb{R}_2[x], \quad \alpha, \beta \in \mathbb{R}$

$$D\left(\alpha(a_0 + a_1 x + a_2 x^2) + \beta(b_0 + b_1 x + b_2 x^2)\right)$$

$$= D\left((\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + (\alpha a_2 + \beta b_2)x^2\right)$$

$$= (\alpha a_1 + \beta b_1) + 2(\alpha a_2 + \beta b_2)x$$

$$= \alpha(a_1 + 2a_2 x) + \beta(b_1 + 2b_2 x)$$

$$= \alpha D(a_0 + a_1 x + a_2 x^2) + \beta D(b_0 + b_1 x + b_2 x^2)$$

$\Rightarrow$ D is a L.T

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(x, y) = (|x|, |y|)$ Is it linear map?

$(1,1) = T(-1, -1) = T(1,1)$ $T(\alpha x) = \alpha T(x)$

$T(-1(1,1)) = -T(1,1) = (-1,-1)$ which is not true case

So this is not a L.T.

Thm Let $W \subseteq V$, where V is a finite dimensional

WTP $\dim(W) \leq \dim(V)$ and equality holds iff $W = V$

Pf Let B be a basis of W. Since B is L.I set in V

B can be extend to a basis of V, say B_1 (By Plus minus thm).

$$\dim(W) = |B| \leq |B_1| = \dim(V)$$

Suppose $\dim(W) = \dim(V)$

$|B| = \dim(V)$ and B is L.I

$\Rightarrow$ B is a basis of V

$W = \mathrm{Span}(B) = V$

$\Rightarrow \quad W = V.$

TP $\quad \dim(W_1 + W_2) = \dim(\underline{W_1}) + \dim(\underline{W_2}) - \dim(W_1 \cap W_2)$

Suppose

$\dim(W_1 \cap W_2) = r$ — Basis $\underline{B_1 = \{v_1, \dots, v_r\}}$

$\dim(W_1) = n$ — $B_2 = B_1 \cup \{u_1, \dots, u_{n-r}\}$

$\dim(W_2) = m$ $\quad \underline{B_3 = B_1 \cup \{w_1, \dots, w_{m-r}\}}$

$B = \{\underbrace{v_1, v_2, \dots, v_r, u_1, \dots, u_{n-r}}, w_1, \dots, w_{m-r}\}$

$n + m - r$

Claim B is a basis for $W_1 + W_2$

L.I

$c_1 v_1 + c_2 v_2 + \dots c_r v_r + d_1 u_1 + \dots + d_{n-r} u_{n-r} + \underbrace{\boxed{e_1 w_1 + e_2 w_2 + \dots + e_{m-r} w_{m-r}}}_{w} = 0$ — $(*)$

$w \in W_2$

$c_1 v_1 + c_2 v_2 + \dots c_r v_r + d_1 u_1 + \dots + d_{n-r} u_{n-r} = -w$

$\Rightarrow \quad w \in W_1$

$\Rightarrow \quad w \in \underline{W_1 \cap W_2}$

$w = f_1 v_1 + f_2 v_2 + \dots f_r v_r$

Substitute the value of w

$e_1 w_1 + e_2 w_2 + \dots + e_{n-r} w_{n-r} - f_1 v_1 - f_2 v_2 - \dots - f_r v_r = 0$

$\Rightarrow \quad e_1 = e_2 = \dots = e_{n-r} = f_1 = f_2 \dots = f_r = 0$

$\Rightarrow \quad w = 0$

from $(*)$ we get

$c_1 v_1 + c_2 v_2 + \dots + c_r v_r + d_1 u_1 + \dots d_{n-r} u_{n-r} = 0$

$\Rightarrow \quad c_1 = c_2 = \dots = c_r = d_1 = \dots = d_{n-r} = 0$

Hence B is L.I

Let $w \in W_1 + W_2$

$\Rightarrow \quad w = w_1 + w_2$

$$= \underline{\sum c_i v_i} + \sum d_i u_i + \underline{\sum e_i v_i} + \sum f_i w_i$$

$$= \sum (c_i + e_i) v_i + \sum d_i u_i + \sum f_i w_i$$

$$\in \text{Span}(B)$$

Tut I (15) $\quad rank(A+B) \leq rank(A) + rank(B)$

$rank(A) = \dim \text{ of } (col(A))$

$rank(B) = \dim(col(B))$

$rank(A+B) = \dim(col(A+B))$

WTP $\quad \dim \underline{col(A+B)} \leq \dim(\underline{col(A)}) + \dim(\underline{col(B)})$

Try to understand the relation b/w

$col(A+B)$ & $col(A)$, $col(B)$

Let $u \in col(A+B)$

Suppose $col(A) = \{u_1, \ldots, u_n\}$

$col(B) = \{v_1, \ldots, v_n\}$

$col(A+B) = \underline{\{u_1+v_1, \ldots, u_n+v_n\}}$

$\Rightarrow u = c_1(u_1+v_1) + c_2(u_2+v_2) + \ldots c_n(u_n+v_n)$

where $c_1, \ldots, c_n \in \mathbb{R}$.

$= \underline{(c_1 u_1 + c_2 u_2 + \ldots + c_n u_n)}$

$+ \underline{(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n)}$

$\in col(A) + col(B)$

$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$

$\Rightarrow \quad col(A+B) \subseteq col(A) + col(B)$

$\dim(col(A+B)) \leq \dim(col(A) + col(B))$

$= \dim(\underline{col(A)}) + (\dim(col(B)))$

$$- \dim(\mathrm{col}(A) \cap \mathrm{col}(B))$$

$$\Rightarrow \left[\mathrm{rank}(A+B) \le \mathrm{rank}\, A + \mathrm{rank}(B)\right] \underbrace{- \dim(\mathrm{col}(A) \cap \mathrm{col}(B))}$$

$$\le \mathrm{rank}(A) + \mathrm{rank}(B)$$

$S = \{x_1, x_2, x_3\}$ $\quad$ $\mathrm{span}(S) = \mathrm{span}(T)$

$T = \{y_1, y_2\}$ $\quad$ $T \subseteq \mathrm{span}(T)$

$$\Rightarrow \boxed{\begin{array}{l} T \subseteq \mathrm{span}\, S \\ S \subseteq \mathrm{span}\, T \end{array}}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\{(1,1,1,1),\ (1,2,1,2),\ (1,0,0,0),\ \underline{(0,1,0,0)},\ (0,0,1,0)\}$$

$$(1\ 2\ 1\ 2) - 2(1,1,1,1) = (-1, 0, -1, 0)$$

$$-(1\ 2\ 1\ 2) + 2(1,1,1,1) - (1,0,0,0)$$

$$= (0, 0, 1, 0)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$