Lecture 1

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Reference book:
Linear Algebra and its Applications (Third Edition) by David C. Lay

Grading Policy:

1. 2 Quizzes with weightage for each quiz is 15%
2. Mid semester exam weightage 30 %
3. End semester exam weightage 40 %

Matrices

Notation

- $\mathbb{N} = \{1, 2, 3, \dots\}$
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- $\mathbb{Q} =$ the set of rational numbers
- $\mathbb{R} =$ the set of real numbers
- $\mathbb{C} =$ the set of complex numbers

For $n \in \mathbb{N}$, we set $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}, i = 1, \dots, n\}$
For, $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we defined

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, \dots, x_n + y_n) \in \mathbb{R}^n \\ \alpha \mathbf{x} &= (\alpha x_1, \dots, \alpha x_n) \in \mathbb{R}^n \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + \cdots + x_n y_n \in \mathbb{R} \end{aligned}$$

Matrices

For $m, n \in \mathbb{N}$. An $m \times n$ *matrix* $\mathbf{A}$ with real (complex) entries is a rectangular array of real(complex) numbers arranged in m rows and n columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} = [a_{ij}],$$

where $a_{ij} \in \mathbb{R}(\mathbb{C})$ for $i = 1, \dots, m$ and $j = 1, \dots, n$.
Let $\mathbb{R}^{m\times n}$ $(\mathbb{C}^{m\times n})$ denote the set of all $m \times n$ matrices with real (complex) entries.
Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}] \in \mathbb{R}^{m\times n}$. Then we say that $\mathbf{A} = \mathbf{B}$ if and only if $a_{ij} = b_{ij}$ for all i, j.

An $n \times n$ matrix is called a *square matrix* of size n.

Let $\mathbf{A} = [a_{ij}]$ be a square matrix. Then

- elements a_{ii} of $\mathbf{A}$ are called *diagonal elements* of $\mathbf{A}$.
- $\mathbf{A}$ is called *symmetric* if $a_{ij} = a_{ji}$, e.g., $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$
- $\mathbf{A}$ is called *skew-symmetric* if $a_{ij} = -a_{ji}$, e.g., $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$

  Note that since for a skew-symmetric matrix $a_{ii} = -a_{ii}$ for all i, diagonal elements of a skew-symmetric matrix are 0.
- $\mathbf{A}$ is called *diagonal* if $a_{ij} = 0$ for $i \neq j$, e.g., $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\mathbf{A}$ is called *scalar* if $\mathbf{A}$ is diagonal and $a_{ii} = a_{jj}$ for all i, j, e.g., $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

- $\mathbf{A}$ is called *upper triangular* matrix if $a_{ij} = 0$ for all $i > j$, e.g.,
$$\begin{bmatrix} 0 & -1 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- $\mathbf{A}$ is called *lower triangular* matrix if $a_{ij} = 0$ for all $i < j$, e.g.,
$$\begin{bmatrix} 2 & 0 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Note that $\mathbf{A}$ is upper triangular as well as lower triangular if and only if $\mathbf{A}$ is diagonal.

- An element of $\mathbb{R}^{1\times n}$ is called a *row vector* of size n.
- An element of $\mathbb{R}^{n\times 1}$ is called a *column vector* of size n.

Operations on Matrices

Let $\mathbf{A} = [a_{ij}], \mathbf{B} = [b_{ij}] \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}$. Then

- the matrix $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}] \in \mathbb{R}^{m \times n}$ is called the *sum* of matrices $\mathbf{A}$ and $\mathbf{B}$.
- the matrix $\mathbf{A} - \mathbf{B} = [a_{ij} - b_{ij}] \in \mathbb{R}^{m \times n}$ is called the *difference* of matrices $\mathbf{A}$ and $\mathbf{B}$.
- the matrix $c\mathbf{A} = [ca_{ij}] \in \mathbb{R}^{m \times n}$ is called the *scalar multiple* of c and matrix $\mathbf{A}$.
- the matrix $\mathbf{A}^T = [a_{ji}] \in \mathbb{R}^{n \times m}$ is called the *transpose* of matrix $\mathbf{A}$.

Example

Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 5 \\ -4 & 3 & 0 \end{bmatrix}$. Then $\mathbf{A}^T = \begin{bmatrix} 2 & -4 \\ 0 & 3 \\ 5 & 0 \end{bmatrix}$

Note that $\mathbf{A}$ is upper triangular iff $\mathbf{A}^T$ is lower triangular.

Note that $\mathbf{A}$ is symmetric iff $\mathbf{A}^T = \mathbf{A}$, and skew symmetric iff $\mathbf{A}^T = -\mathbf{A}$.

Product of matrices

Let $\mathbf{A}$ be a matrix of size $m \times n$ and B is a matrices of order $n \times p$. Then the matrix $\mathbf{AB} = \left[\sum_{j=1}^{n} a_{ij}b_{jk}\right]$ is called the *product* of matrices $\mathbf{A}$ and $\mathbf{B}$. For example, let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 5 \\ -4 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$. Then

$$\mathbf{AB} = \begin{bmatrix} 2+0+15 & 4+0+5 & 6+0+20 \\ -4+6+0 & -8+0+0 & -12+3+0 \end{bmatrix} = \begin{bmatrix} 17 & 9 & 26 \\ 2 & -8 & -9 \end{bmatrix}.$$

Remark

- Product of two non-zero matrices can be zero.
- Let $\mathbf{A}$ and $\mathbf{B}$ matrices such that $\mathbf{AB}$ is defined. Then $\mathbf{BA}$ need not be defined.
- Suppose $\mathbf{AB}$ and $\mathbf{BA}$ be both defined. Then $\mathbf{AB}$ need not be equal to $\mathbf{BA}$.