Theorem Let A be $n \times n$ matrix. Then the following are equivalent

(1) A is diagonalizable

2. A has n linearly independent eigen vector.

Pf (1) $\Rightarrow$ (2) A is diagonalizable. $\exists$ a invertible matrix P and diagonal matrix D s.t $A = PDP^{-1}$

$$\Rightarrow AP = PD \quad \text{—— (1)}$$

Suppose $P = [x_1 \ x_2 \ - - \ x_n]$ $\longrightarrow$ Columns of P

$$AP = A[x_1 \ x_2 \ \cdots \ x_n]$$

$$= [Ax_1 \ Ax_2 \cdots \ Ax_n]$$

Let $D = \operatorname{diag}(d_1, \cdots, d_n)$

$$PD = [x_1 \ x_2 \cdots \ x_n] \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$$= [d_1x_1 \ d_2x_2 \ \cdots \ d_nx_n]$$

Using (1) we get $[Ax_1 \ Ax_2 \ \cdots \ Ax_n]$

$$= [d_1x_1 \ d_2x_2 \ \cdots \ d_nx_n]$$

$$\Rightarrow Ax_1 = d_1x_1, \ Ax_2 = d_2x_2, \cdots, Ax_n = d_nx_n$$

$\rightarrow$ $x_1, x_2, \ldots, x_n$ are eigenvector of A.

Since $x_1, x_2, \ldots x_n$ are columns of invertible matrix

$P,\ x_1, \ldots x_n$ are L.I

② ⇒ ① Suppose A has n L.I eigenvectors, say $x_1, \ldots, x_n$ corresponding to eigen value $\lambda_1, \ldots, \lambda_n$

$$Ax_i = \lambda_i x_i \quad \forall i$$

Take $P = [x_1 \ \ldots \ x_n]$. Since $x_1, \ldots, x_n$ are L.I, P is invertible.

WT P
$P^{-1}AP = D$
$AP = PD$

$$AP = A[x_1 \ldots \ x_n]$$

$$= [Ax_1 \ Ax_2 \ \ldots \ Ax_n]$$

$$= [\lambda_1 x_1 \ \lambda_2 x_2 \ldots \ \lambda_n x_n]$$

$$= [x_1 \ x_2 \ \ldots \ x_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ 0 & \vdots & \ddots & 0 \\ 0 & \cdots & & \lambda_n \end{bmatrix}$$

$$= PD, \quad \text{where } D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$AP = PD$$

$$\Rightarrow P^{-1}AP = D$$

$\Rightarrow A$ is diagonalizable.

Observation

→ A is diagonalizable iff A has n eigenvalues and A.M of each eigenvalue = G.M of each eigenval.

→ Suppose A is diagonalizable $\exists$ invertible matrix P s.t $P^{-1}AP = D$. Then the

columns eigenvectors of A

→ Diagonal elements of D are eigenvalues of A.

→ ith column of P is an eigenvector of A corr. to eigen value ith diagonal element of D

Ex: $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\lambda_1 = 0$, $\lambda_2 = 2$

For eigenvector $\lambda = 0$.

$$A - 0I = A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$R_2 \to R_1 + R_2$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$x_1 - x_2 = 0$

$x_1 - 1 = 0$

$x_1 = 1$

Eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$,

$$A - 2I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

Let $x_2 = 1$

$x_1 + 1 = 0$

$x_1 = -1$

Eigenvector $\begin{bmatrix} -1 \\ \end{bmatrix}$

$x_4 + \ldots$
$x_p = -1$

eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\boxed{P^{-1}AP = D}$ check.

$$\boxed{P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad P^{-1}AP = D. \quad \text{check}}$$

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ex $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, Find A^{1000}

$$P^{-1}AP = D \Rightarrow A = PDP^{-1}$$

$$A^2 = (PDP^{-1})^2 = (PDP^{-1})(PDP^{-1})$$
$$= PDID\,P^{-1}$$
$$= PD^2P^{-1}$$

$$A^n = PD^nP^{-1}$$
$$A^{n+1} = A^nA = (PD^nP^{-1})(PDP^{-1})$$
$$= PD^{n+1}P^{-1}$$

$\forall n$ $\quad A^n = PD^nP^{-1} \Rightarrow \quad P^{-1}A^nP = D^n$

$$A^{1000} = P\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}^{1000} P^{-1}$$

$$= P\begin{bmatrix} 2^{1000} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \quad P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

→ Suppose A is diagonalizable. Then A^n is Diagonalizable.

→ Find the eigenvalues and eigenvectors.

$\longrightarrow$ Find the eigenvalues and eigenvectors

$\rightarrow$ Suppose A^2 is diagonalizable, Then A need not be diagonalizable

Thm (Cayley-Hamilton Thm) Every square matrix satisfies its own characteristic polynomial.

i.e, Suppose $\mathrm{char}(A) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots a_n$,

Then $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots a_n I = 0$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ eigen values $1, 1$

AM — 2

G.M — 1

$$\mathrm{char}_A(x) = x^2 - 2x + 1$$

By C-H.T. $A^2 - 2A + I = 0$

$$\Rightarrow A^2 = 2A - I$$

$$A^3 = 2A^2 - A$$

$$= 4A - 2I - A$$

$$= 3A - 2I$$

$$A^4 = 3A^2 - 2A$$

$$= 6A - 3I - 2A$$

$$= 4A - 3I$$

$$A^n = nA - (n-1)I$$