Quadratic forms.

Def A quadratic form of $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose value at a vector x in $\mathbb{R}^n$ can be computed by an expression of the form $\boxed{Q(x) = x^T A x,}$ where A is $n \times n$ matrix.

Ex $Q(x) = \|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = x^T x = x^T I x$.

$\rightarrow$ $Q(x) = x^T A x$

$\Rightarrow$ $Q(x)^T = (x^T A x)^T = x^T A^T (x^T)^T$
$= x^T A^T x$

$Q(x)^T = Q(x)$

$Q(x) = x^T A^T x$

If A is not symmetric, we get two matrix A and A^T s.t $Q(x) = x^T A x$
$= x^T A^T x$

Remark For every quadratic form, $\exists$ a unique symmetric matrix A s.t $Q(x) = x^T A x$.
Such matrix is called the matrix of quadratic form $Q(x)$.

$\rightarrow$ Let $Q(x) = x^T B x$ be a quadratic form

$$Q(x) = x^T B^T x$$

$$2\,Q(x) = x^T B x + x^T B^T x$$

$$= x^T (B + B^T)\, x$$

$$Q(x) = x^T \left(\frac{B + B^T}{2}\right) x.$$

Let $A = \dfrac{B + B^T}{2}$, Then A is symme!

and $Q(x) = x^T A x$.

$\longrightarrow$ Suppose $\exists$ A_1 and A_2 two symmetric matrix s.t.

$$Q(x) = x^T A_1 x = x^T A_2 x \quad \forall x \in \mathbb{R}^n$$

Let $x = e_i$, $\quad \underline{e_i^T A_1 e_i} = e_i^T A_2 e_i$.

ith diagonal element of A_1

$=$ ith diagonal element of A_2

Let $x = e_i + e_j$ for $i \neq j$ $\quad \underline{(e_i + e_j)^T A_1 (e_i + e_j)} = (e_i + e_j)^T A_2 (e_i + e_j)$

Think about it (Exercise.)

This will prove that $A_1 = A_2$

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ex $\quad$ Let $\quad A = \begin{bmatrix} 1 & 5 \\ 2 & \end{bmatrix}$

$$\varphi(x) = x^T A x = [x_1 \quad x_2] \begin{bmatrix} 1 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \quad x_2] \begin{bmatrix} x_1 + 5x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

$$= x_1 (x_1 + 5x_2) + x_2 [3x_1 + 4x_2]$$

$$= x_1^2 + 5x_1 x_2 + 3 \cdot x_1 x_2 + 4 x_2^2$$

$$= \underline{x_1^2} + \underline{8} x_1 x_2 + \underline{4} x_2^2$$

$$= [x_1 \quad x_2] \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

↳ matrix of quadratic form of $Q(x)$

Find the expression for the quadratic form of matrix

① $\varphi(x) = [x_1 \quad x_2] \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2$

② $\varphi(x) = [x_1 \quad x_2] \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 - 4x_1 x_2 + 7x_2^2$

Find the matrix of quadratic form

① $Q(x) = x_1^2 + \underbrace{3x_1x_2}_{\downarrow} + \underline{4x_1x_3} + \underline{x_2^2}$

$\underline{(1,2)}$ $\underline{x_1x_2}$ $(1,3)$ — x_1x_3

x_2x_2

$$A = \begin{bmatrix} 1 & 3/2 & 2 \\ 3/2 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

② $Q(x) = x_1^2 + 3x_1x_2 + 4x_1x_2 + x_2^2$

$$A = \begin{bmatrix} 1 & 7/2 \\ 7/2 & 1 \end{bmatrix}$$

③ $Q(x) = 3x_1^2 - 3x_1x_2 + 4x_2x_3 - x_2^2 + 2x^3$

$$A = \begin{bmatrix} 3 & -3/2 & 0 \\ -3/2 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

Ex. $x = [1 \quad 4 \quad 1]$, $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

$$x^T A x = [1 \quad 4 \quad 1] \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$$x^T A x = 3x_1^2 + 4x_1x_2 + 2x_1x_3 + x_2^2 + 3x_3^2$$

$$\varphi(1,4,1) = 3(1)^2 + 4(1)(4) + 2(1)(1) + 4^2 + 3\cdot 1^2.$$

Change of the variables in quadratic form

Suppose $\{x_1, \dots, x_n\}$ is a given a set of variables and $\{y_1, \dots, y_n\}$ be a new set of variables.

$$y_i = \sum a_{ij} x_j$$

$$y_1 = a_{11}x_1 + a_{12}x_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We will get $Px = y$, where P is a $n \times n$ matrix.

Fact that P is an invertible matrix.

$$x = P^{-1}y$$

Let $Q(x) = x^T A x$ be the given quadratic form.

$$Q(y) = (Px)^T A (Px)$$

$$= x^T P^{-1} A \cdot P x$$

$$= x^T \underbrace{(P^T A P)}_{B} x$$

$$= x^T B x.$$

We get $B = P^T A P$

If P is an orthogonal matrix, then $P^{-1} = P^T$, and hence B and A are similar.

If A is orthogonally diagonalizable, then we can choose a matrix P s.t $P^T A P$ is a diagonal matrix.

Since A is symmetric matrix, we can choose always an orthogonal matrix P s.t $P^{-1} A P$ is a diagonal matrix. (Due to Spectral Theorem for symmetric matrix)

Principal axis thm.

Let A be an $n \times n$ symmetric matrix. Then $\exists$ an orthogonal change of variable $x = Py$ that transform the quadratic form $x^T A x$ into a quadratic form $y^T D y$ with no cross product, i.e, coefficient of $y_i y_j = 0$ $\forall\, i \neq j$.