$\Rightarrow$ 2021PCS1017.

$\Rightarrow$ AMBOJ MISHRA.

①. $\rightarrow$ eq.^{n} $\rightarrow$

$$2x_1 + x_2 + x_3 = 1$$
$$2x_1 + 2x_2 + ax_3 = -1$$
$$-2x_1 + x_2 + x_3 = b$$

$\Downarrow$

converting into matrix $\rightarrow$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & a \\ -2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ b \end{bmatrix} \Leftrightarrow \boxed{AX = b}$$

Augmented matrix $\rightarrow$ $[A \mid b]$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 2 & 2 & a & -1 \\ -2 & 1 & 1 & b \end{array}\right]$$

$\downarrow$ $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 + R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1 & a-1 & -2 \\ 0 & 2 & 2 & b+1 \end{array}\right] \xrightarrow{R_3 \rightarrow R_3 - 2*R_2} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1 & a-1 & -2 \\ 0 & 0 & 4-2a & b+5 \end{array}\right]$$

$\downarrow$ $R_1 \rightarrow \frac{1}{2} * R_1$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1/2 & 1/2 & 1/2 \\ 0 & \boxed{1} & a-1 & -2 \\ 0 & 0 & 4-2a & b+5 \end{array}\right]$$

{REF-form}.

ⓐ.

$\rightarrow$ for this system to not have any sol.^{n}, pivot element should exist in last col.^{n}. for this to happen $\rightarrow$

$\Rightarrow$ $\boxed{4-2a=0} \Rightarrow a=2$

& $b+5 \neq 0 \Rightarrow \boxed{b \neq (-5)}$ $\hookrightarrow$ for No. sol.^{n}: $\hookrightarrow \{a=2,\ b \neq (-5)\}$

(b) → for unique sol.^{n} to exist →

$$4-2a \neq 0$$

$$\boxed{a \neq 2}$$

⇓

calculating sol.^{n} →

$$(4-2a)\cdot x_3 = (b+5) \Rightarrow \boxed{x_3 = \frac{(b+5)}{(4-2a)}}$$

and →

$$x_2 + (a-1)\cdot x_3 = -2$$

$$x_2 = -2 - (a-1)\cdot\frac{(b+5)}{2(2-a)}$$

$$x_2 = -2 + \frac{(a-1)(b+5)}{2(a-2)}$$

$$x_2 = \frac{4a-8+ab+5a-b-5}{2(a-2)}$$

$$\boxed{x_2 = \frac{ab+9a-b-13}{2(a-2)}}$$

and →

$$x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = \frac{1}{2}$$

$$x_1 = \frac{1}{2} - \frac{1}{2}(x_2+x_3) = \frac{1}{2}\left\{1-x_2-x_3\right\}$$

$$x_1 = \frac{1}{2}\left\{1 - \frac{ab+9a-b-13}{2(a-2)} + \frac{(b+5)}{2(a-2)}\right\}$$

$$x_1 = \frac{1}{2}\left\{\frac{2a-4-ab-9a+b+13+b+5}{2(a-2)}\right\}$$

$$\boxed{x_1 = \frac{1}{2}\left\{\frac{-7a+2b+14-ab}{2(a-2)}\right\}}$$

unique sol.^{n} $\{x_1, x_2, x_3\}$ → as solved above

(c) → To have infinitely many sol.^{n} →

$4-2a=0 \Rightarrow \boxed{a=2}$

and → $b+5=0 \Rightarrow \boxed{b=(-5)}$

So →

$$[A \vdots b] \Rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 1/2 & 1/2 & 1/2 \\ 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let $\boxed{x_3=k}$;

So → $x_2 + x_3 = -2$

$\boxed{x_2 = (-2-k)}$

and → $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = \frac{1}{2}$

$x_1 = \frac{1}{2}\{1 - x_2 - x_3\} = \frac{1}{2}\{1+2+\cancel{k}-\cancel{k}\}$

$\boxed{x_1 = 3/2}$

So, sol.^{n} →

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -2-k \\ k \end{bmatrix} \Rightarrow \begin{bmatrix} 3/2 \\ -2 \\ 0 \end{bmatrix} + k \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} ; \forall k \in \mathbb{R}}$$

(d) → for homogeneous in above case →

$$A = \begin{bmatrix} \boxed{1} & 1/2 & 1/2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

let $\boxed{x_3 = Q}$

We'll cal. NULL(A) →

bcoz $Ax=0$; then $\boxed{x \in N(A)}$

$\Rightarrow x_2 + x_3 = 0$

$\boxed{x_2 = -Q}$

and → $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 0$

$x_1 - \cancel{\frac{Q}{2}} + \cancel{\frac{Q}{2}} = 0$

$x_1 = 0$.

So → sol.^{n} set → $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -Q \\ Q \end{bmatrix} \Rightarrow Q \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

for $Q = 1$ → $\Rightarrow \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

So → Basis of N(A) → $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ = {Answer}.

---

(2). → Que

$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & a \\ -2 & 1 & 3 \end{bmatrix}$; Since it is invertible so, all col^n's should have pivots.

$\Big\downarrow \begin{matrix} R_2 \to R_2 - 2 * R_1 \\ R_3 \to R_3 + 2 * R_1 \end{matrix}$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & a \\ 0 & -1 & 3 \end{bmatrix}$

$\Big\downarrow R_2 \to \frac{1}{3} * R_2$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & a/3 \\ 0 & -1 & 3 \end{bmatrix}$

$\Big\downarrow R_3 \to R_3 + R_2$

$\begin{bmatrix} \boxed{1} & -1 & 0 \\ 0 & \boxed{1} & a/3 \\ 0 & 0 & \boxed{3 + a/3} \end{bmatrix}$

So, even REf form should have pivots in all col.^{n} as well. So →

$3 + \frac{a}{3} \neq 0 \Rightarrow \frac{a}{3} \neq -3$

$\boxed{a \neq -9}$ = (Answer)

Now to find 3^{rd} col.^{m} of inverse $\rightarrow$

$$\underset{A}{\begin{bmatrix} \equiv \end{bmatrix}} \cdot \underset{A^{-1}}{\begin{bmatrix} & \vec{u_3} \end{bmatrix}} = \underset{I}{\begin{bmatrix} & \vec{e_3} \end{bmatrix}}$$

So $\rightarrow$ $A \cdot \vec{U_3} = \vec{e_3}$; let $\vec{u_3} = \{x_1, x_2, x_3\}$

So $\rightarrow$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & a \\ -2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

augmented mat $\rightarrow$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 1 & a & 0 \\ -2 & 1 & 3 & 1 \end{array}\right]$$

$\downarrow$ $R_2 \rightarrow R_2 - 2 \cdot R_1$
$R_3 \rightarrow R_3 + 2 \cdot R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & a & 0 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

$\downarrow$ $R_2 \rightarrow \frac{1}{3} * R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & a/3 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

$\downarrow$ $R_3 \rightarrow R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & a/3 & 0 \\ 0 & 0 & 3 + a/3 & 1 \end{array}\right]$$

so → $\left(3+\frac{a}{3}\right)\cdot x_3 = 1$

$$\boxed{x_3 = \frac{3}{a+9}}$$

② → $x_2 + \frac{a}{3}\cdot x_3 = 0$

$$x_2 = -\frac{a}{\cancel{3}} * \frac{\cancel{3}}{a+9} \Rightarrow \boxed{x_2 = \left(-\frac{a}{a+9}\right)}$$

① → $x_1 - x_2 = 0$

$$\boxed{x_1 = x_2 = \frac{-a}{a+9}}$$

so → 3rd col. of A^{-1} ⇒ $\begin{bmatrix} \left(\frac{-a}{a+9}\right) \\ \left(\frac{-a}{a+9}\right) \\ \left(\frac{3}{a+9}\right) \end{bmatrix}$

Ques ⑥ → $A = a\cdot\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$ ⇒ Symmetric matrix.

$$[A-\lambda I] = \begin{bmatrix} a-\lambda & -2a & 2a \\ -2a & 4a-\lambda & -4a \\ 2a & -4a & 4a-\lambda \end{bmatrix}$$

$\det(A-\lambda I) = 0$

$$(a-\lambda)\left\{(4a-\lambda)^2 - 16a^2\right\} + 2a\cdot\left\{-\cancel{8a^2} + 2a\lambda + \cancel{8a^2}\right\}$$
$$+ 2a\cdot\left\{\cancel{8a^2} - \cancel{8a^2} + 2a\lambda\right\} = 0$$

$$(a-\lambda)\cdot\left\{\cancel{16a^2} + \lambda^2 - 8a\lambda - \cancel{16a^2}\right\} + 4a^2\lambda + 4a^2\lambda = 0$$

$$(a-\lambda)\cdot(\lambda^2-8a\lambda)+8a^2\lambda=0$$

$$a\lambda^2-\cancel{8a^2\lambda}-\lambda^3+8a\lambda^2+\cancel{8a^2\lambda}=0$$

$$\lambda^3-9a\lambda^2=0$$

$$\boxed{\lambda^2(\lambda-9a)=0}$$

$$\boxed{\lambda=0,0,9a}$$

E.V. corresponding to $\lambda=9a$ $\rightarrow$

$$(A-\lambda I)\cdot x=\begin{bmatrix}-8a & -2a & 2a\\ -2a & -5a & -4a\\ 2a & -4a & -5a\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Downarrow R_3\rightarrow R_3+R_2$$

$$\Rightarrow\begin{bmatrix}-8a & -2a & 2a\\ -2a & -5a & -4a\\ 0 & -9a & -9a\end{bmatrix}$$

$$\Downarrow R_1\rightarrow \tfrac{1}{4}*R_1$$

$$\begin{bmatrix}-2a & -a/2 & a/2\\ -2a & -5a & -4a\\ 0 & -9a & -9a\end{bmatrix}$$

$$\Downarrow R_2\rightarrow R_2-R_1$$

$$\begin{bmatrix}-2a & -a/2 & +a/2\\ 0 & -9a/2 & -9a/2\\ 0 & -9a & -9a\end{bmatrix}$$

$$\Downarrow R_3\rightarrow R_3-2*R_2$$

$$\begin{bmatrix}-2a & -a/2 & +a/2\\ 0 & -9a/2 & -9a/2\\ 0 & 0 & 0\end{bmatrix}$$

$\rightarrow$ let $x_3=K$:

$$\left(-\frac{9a}{2}\right)x_2 - \left(\frac{9a}{2}\right)\cdot x_3 = 0$$

$$-\left(\cancel{\frac{9a}{2}}\right)\cdot x_2 = x_3\cdot\left(\cancel{\frac{9a}{2}}\right)$$

$$\boxed{x_2 = (-k)}$$

and →

$$(-2a)\cdot x_1 - \frac{a}{2}\times x_2 + \frac{a}{2}\times x_3 = 0$$

$$-2x_1 + \frac{k}{2} + \frac{k}{2} = 0$$

$$\boxed{x_1 = k/2}$$

So → E.V. for $k=1$

⇓

$$\begin{bmatrix} 1/2 \\ -1 \\ 1 \end{bmatrix}$$

So, Normal E.V. ⇒

$$\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

and →

$$\cancel{(-2a)\cdot x_1 - \frac{a}{2}\times -k - \frac{a}{2}\times k = 0}$$

$$\cancel{x_1 = 0};$$

So → ~~E.V. ⇒~~ $\cancel{\begin{bmatrix} 0 \\ -k \\ k \end{bmatrix} \Rightarrow k}$ ~~So, for~~ $\cancel{k=1}$ → $\cancel{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$

~~orthonormal~~ → $\cancel{\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$

for $\lambda = 0$ → $\boxed{(A-\lambda I)\cdot X = 0}$

$$\begin{bmatrix} a & -2a & 2a \\ -2a & 4a & -4a \\ 2a & -4a & 4a \end{bmatrix}\cdot\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

⇓ $R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} a & -2a & 2a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

↳ let $x_2 = P$ & $x_3 = Q$;

so → $ax_1 + (-2a)\cdot x_2 + (2a)\cdot x_3 = 0$

$x_1 - 2P + 2Q = 0$

$\boxed{x_1 = 2P - 2Q}$

for $(P,Q) = (1,0) \rightarrow$ sol$^n \rightarrow \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$ Normal E.V. $\Rightarrow \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

& for $(P,Q) = (0,1) \rightarrow$ sol$^n \rightarrow \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ Normal E.V. $\Rightarrow \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$

$\rightarrow$ Both of them are not orthogonal to each other

So, applying gram-schmidt $\rightarrow$

$$u_1 = \frac{v_1}{\|v_1\|} \Rightarrow u_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$u_2 = v_2 - \langle v_2, u_1 \rangle . u_1 \Rightarrow \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} + \frac{4}{5} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -\frac{2}{5\sqrt{5}} \\ \frac{4}{5\sqrt{5}} \\ 1/\sqrt{5} \end{bmatrix}$$

So $\rightarrow$

Answer $\rightarrow$ next page.

So → $P = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/5\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 4/5\sqrt{5} \\ 2/3 & 0 & 1/\sqrt{5} \end{bmatrix}$

Que (4) → (1) → $\boxed{\text{True}}$ Since A is matrix of quadratic form, we can say that A is symmetric. ⇒
so, using spectral decomposition →

$A = PDP^T$

we can write this as →

$\boxed{AP = PD}$

Since P is orthogonal.

(2) → $\boxed{\text{false}}$. R.H.S. → $(A-B).\cancel{(B-A)}(A+B)$

$\Rightarrow A^2 + A.B - B.A - B^2$

$\Rightarrow (A^2 - B^2) + \underbrace{(AB - BA)}_{\text{Not always equal to 0.}}$

$\neq$ L.H.S.

~~(3) → Not false → True → b/c (ATA) = (AAT)~~

(3). → $\boxed{\text{True.}}$. bcoz. A^TA is a symm. matrix as $\boxed{(A^TA)^T = A^TA}$
and as we have studied during SVD, that all the eigen values of (A^TA) will be positive. similarly here as well. So → {+ve definite}

(4). $\boxed{\cancel{\text{True}}}$ $\boxed{\text{false}}$ $\Rightarrow$ U is echelon form & there are only k zero rows. $(m-k)$ diff. rows.

$$\text{Dim.(Row space)} = (m-k)$$

$$\text{Dim.(Null space)} = \cancel{\text{Dim}}\ m - \cancel{\text{Dim}}(\text{Rank})$$

$$\Rightarrow m-(m-k)$$

$$\Rightarrow k.$$

(5). $\boxed{\text{True.}}$ $\rightarrow$ Since $\text{Dim}(B) = \text{Dim}(B')$

& $c_1(v_1+v_2) + c_2(v_2+v_3) + c_3.(v_3+v_4) + c_4(v_4+v_1)$

Lin Comb. $\Rightarrow$

$\Rightarrow v_1.(c_1+c_4) + v_2(c_1+c_2) + v_3(c_3+c_2) + v_4(c_4+c_3)$

So $\rightarrow$ $\text{span}(B) = \text{span}(B')$

Thus, basis.

(6) $\rightarrow$ $\boxed{\text{True}}$. Diagonalization is only possible when $\boxed{AM. = GM.}$.

(7). $\rightarrow$ $\boxed{\text{True}}$. using pythagoras theorem, ~~the~~ just one diff. that dir.^{n} of vector $\vec{v}$ is reversed.

So $\rightarrow$ $\|u+(-v)\|^2 = \|u\|^2 + \|-v\|^2$; if $u \perp (-v)$ (or) $u \perp v$

(8). $\boxed{\text{false.}}$ $\rightarrow$ Only true if it is square matrix which means transforming from same dimensional space to same.

Que (5) →

→ (a). To find the least squared sol.^{n};

we need to minimize ~~distance~~ $\|b-Ax\|$ distance

So → if it is a consistent system then →
directly we can cal.^{c} particular sol.^{n}

O.W. →

$$\text{minimize } \{ \|b-Ax\| ;\ x \in \mathbb{R}^n \}.$$

suppose sol.^{n} is $\hat{x}$. So →

$$\text{min}^m = \|b - A\hat{x}\| \ ; \ \text{where } A\hat{x} = \text{proj}_{\text{col}(A)}(b)$$

To find x → for $AX = b$

if we write $b = b_1 + b_2$; such that
$b_1 \in \text{col.}(A)$
$b_2 \in N(A^T)$

so, we can say, we want $\boxed{A\hat{X} = b_1}$

Now → $b - b_1 = b_2$

So → $b - A\hat{x} = b_2 \in N(A^T)$

$$A^T(b - A\hat{x}) = A^T b_2 = 0$$

So → $\boxed{A^T b = A^T A\hat{x}}$; this is Normal eq.^{n}, using this, we can find $\hat{x}$.

(b) → $\boxed{\text{True}}$. Least squared sol^n will be unique bcoz A^TA will be an invertible matrix.

Que (3) → (a). let $\vec{v} = (v_1, v_2, v_3, \dots, v_n)$

$\vec{u} = (u_1, u_2, u_3, \dots, u_n)$

Now → $c_1\vec{u} + c_2\vec{v} = (c_1u_1 + c_2v_1, \; c_1u_2 + c_2v_2, \dots, \; c_1u_n + c_2v_n)$

So → $T(c_1\vec{u} + c_2\vec{v}) = (0, \; c_1u_1 + c_2v_1, \; c_1u_2 + c_2v_2, \dots, \; ac_1u_{n-1} + ac_2v_{n-1})$

$\Rightarrow (0, c_1u_1, c_1u_2, \dots, ac_1u_{n-1}) + (0, c_2v_1, c_2v_2, \dots, ac_2v_{n-1})$

$\Rightarrow c_1 \cdot (0, u_1, u_2, \dots, au_{n-1}) + c_2 \cdot (0, v_1, v_2, \dots, av_{n-1})$

$\Rightarrow c_1 \cdot T(\vec{u}) + c_2 \cdot T(\vec{v})$

So → $$\boxed{T(c_1\vec{u} + c_2\vec{v}) = c_1 \cdot T(\vec{u}) + c_2 \cdot T(\vec{v})}$$

So, T is a linear transformation

(b). → logic → in each transformation, one element from left is getting '0'. and values being shifted to right.

so → point → if we keep on applying ~~transf~~ linear transformation on the result, the after a certain point, all the values in the ~~vals~~ vector will be 0.

So $\rightarrow$ $T(\vec{v}) = \vec{u}$

$T(\vec{u}) = \vec{x}$

⋮

after some point result $\Rightarrow$ zero vector $\Rightarrow (\vec{0})$

$$\boxed{T^m(\vec{v}) = (\vec{0})}$$

we can say that $\boxed{A^m = 0}$

$(T^m \cdot A)$

Now $\rightarrow$ $\boxed{m = \text{Rank}(A)}$; ~~we~~ bcoz it'll take these many steps to make $\boxed{A^m = 0}$.

$\boxed{\text{logic-wise}}$ $\rightarrow$

Que (8) $\rightarrow$ A, B are symmetric $\Rightarrow$ $\boxed{A^T = A}$; $\boxed{B^T = B}$;

Since we know that eigen values of $(A+B)$ will be $\rightarrow$ (eigen value of A) + (eigen value of B)

bcoz $\rightarrow$

$$AX = \lambda_1 X$$

& $$BX = \lambda_2 X$$

$$\boxed{(A+B) \cdot X = (\lambda_1 + \lambda_2) \cdot X}$$; So, eigen values of $(A+B)$ will also be $(-ve)$.