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Def Let $(V, \langle\ \rangle)$ be a inner product space. Let v be a vector in V. Then length (Norm) of a vector v, denoted $\|v\|$, $\|v\| = \sqrt{\langle v, v\rangle}$

Ex 1 $V = \mathbb{R}^n$, $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}.$$

If $n=2$, $v = \langle 1, 1\rangle$

$$\|v\| = \sqrt{1^2+1^2} = \sqrt{2}$$

2. $V = \mathbb{C}^n$, $\mathbb{F} = \mathbb{C}$, $\langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1\cdot\bar{x}_1 + x_2\cdot\bar{x}_2 + \cdots x_n\cdot\bar{x}_n}$$
$$= \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

$x = \alpha + i\beta$
$\bar{x} = \alpha - i\beta$
$x\bar{x} = \alpha^2 + \beta^2$
$= |x|^2$

3. $V = \mathbb{R}^2$, $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$$\langle x, y\rangle = [x_1\ x_2]\, A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\langle x, x\rangle \geq 0$ & $= 0 \Leftrightarrow x = 0$ $x \in V$
$\langle \alpha x + \beta y, z\rangle$
$= \alpha\langle x, z\rangle$
$+ \beta\langle y, z\rangle$
$\alpha, \beta \in \mathbb{F}$

$$\|x\| = \sqrt{[x_1\ x_2]\, A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$
$$= \sqrt{[x_1\ x_2] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$
$$= \sqrt{2x_1^2 - 2x_1x_2 + x_2^2}$$

Let $x = (1, 1)$.
Then $\|x\| = \sqrt{2 - 2 + 1} = \sqrt{1} = 1$

Observation 1 $\|v\| = 0 \Longleftrightarrow v = 0$

$\|v\| = 0 \iff \sqrt{\langle v, v\rangle} = 0 \iff \langle v, v\rangle = 0 \iff v = 0$

2. Let $\alpha \in F$, $v \in V$. $\|\alpha v\| = |\alpha|\,\|v\|$

$$\|\alpha v\| = \sqrt{\langle \alpha v, \alpha v\rangle} = \sqrt{\alpha \langle v, \alpha v\rangle} = \sqrt{\alpha \overline{\langle \alpha v, v\rangle}}$$

$$= \sqrt{\alpha \cdot \overline{\alpha} \langle v, v\rangle} = \sqrt{|\alpha|^2 \langle v, v\rangle}$$

$$= |\alpha| \sqrt{\langle v, v\rangle}$$

$$= |\alpha|\,\|v\|.$$

Def Let V be inner product space and $u, v \in V$. Then we say that u is orthogonal to v if $\langle u, v\rangle = 0$, denoted as $u \perp v$.

Ex 1. $V = \mathbb{R}^n$, $\langle x, y\rangle = \sum x_i y_i$

Let $\{e_1, \dots, e_n\}$ be a standard basis of V. Then $\langle e_i, e_j\rangle = 0$ $\forall\, i \neq j$. Hence e_i is orthogonal to e_j for all $i \neq j$.

Def A set of vectors $\{v_1, \dots, v_n\}$ is called mutually orthogonal if $v_i \perp v_j$ $\forall\, i \neq j$.

In the above example, $\{e_1, \dots, e_n\}$ is mutually orthogonal set.

Ex $V = \mathbb{R}^2$, $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $\langle x, y\rangle = [x_1 \ x_2]\, A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\langle e_1, e_2\rangle = [1 \ 0] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [2 \ -1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$$

e_1 is not orthogonal to e_2.

Discussion

Let $\{v_1, v_2\}$ be L.D. $\exists\ \alpha \in \mathbb{F}$,

s.t. $v_1 = \alpha v_2$, $\alpha \neq 0$

$$\langle v_1, v_2\rangle = \langle \alpha v_2, v_2\rangle = \alpha \underbrace{\langle v_2, v_2\rangle}_{\neq 0}$$

$\Rightarrow$ If $v_1 \perp v_2$, then $\{v_1, v_2\}$ is L.I.

Fact 2 Let $\{v_1, v_2, \ldots, v_n\}$ be a mutually orthogonal set of non-zero vectors. Then $\{v_1, v_2, \ldots, v_n\}$ is L.I.

Pf Let $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$, for some $c_i \in \mathbb{F}$

$$\langle c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, v_i\rangle = \underbrace{\langle 0, v_i\rangle}_{= 0}$$

$$\Rightarrow c_1\langle v_1, v_i\rangle + c_2\langle v_2, v_i\rangle + \cdots + c_i\langle v_i, v_i\rangle + \cdots + c_n\underbrace{\langle v_n, v_i\rangle}_{=0}$$

$$c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_i\|v_i\|^2 + \cdots + c_n \cdot 0 = 0$$

$$\Rightarrow c_i = 0$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

$\Rightarrow \{v_1, v_2, \ldots, v_n\}$ is L.I.

$\rightarrow$ The distance b/w two vectors u and v is defined by $\|u - v\|$.

Ex. 1. 0 is orthogonal to every vector in V.

2. 0 is the only vector in V which is orthogonal to itself.