Lecture 16

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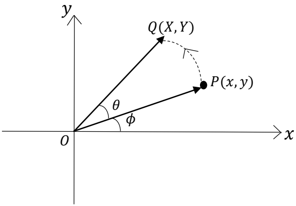
Linear Transformation

Matrix of rotation

Let $P = \begin{bmatrix} x \\ y \end{bmatrix}$ and $Q = \begin{bmatrix} X \\ Y \end{bmatrix}$ be the point obtained from P by rotation (anticlockwise direction) of an angle θ about origin. Then show that
$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$.

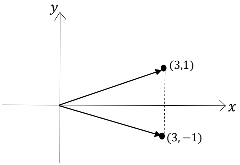


Let the line through O and P makes an angle ϕ with x-axis (as shown in figure). Now set $OP = OQ = r$. Then $X = r\cos(\theta + \phi)$ and $Y = r\sin(\theta + \phi)$. Using the fact that $x = r\cos\phi$, $y = r\sin\phi$ and formulas for $\cos(\theta + \phi)$, $\sin(\theta + \phi)$, the result follows. Thus

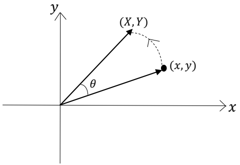
$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Linear transformation

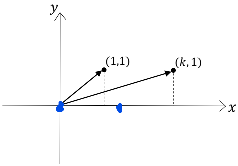
Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then A reflects vectors along the x-axis. For example $\mathbf{A} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$



Let $\mathbf{B} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Then $\mathbf{B}$ rotates a vector counterclockwise by an angle θ.



Let $\mathbf{A} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$. Then **A** compresses or expand a vector in x-direction by a factor k. vectors along the x-axis



$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} xk \\ y \end{bmatrix}$$

Note that under these transformations, lines get mapped to lines. More generally linear combinations get mapped to linear combinations, that is, $\mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2$.

Definition

Let V and W be two vector spaces over $\mathbb{F}$. Then a function $T : V \to W$ is called a *linear transformation* if it satisfy the following conditions:

$$T(c_1v + c_2w) = c_1T(v) + c_2T(w) \text{ for all } v, w \in V \text{ and } c_1, c_2 \in \mathbb{F}.$$

If $W = V$, then we say that T is a *linear operator* on V

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$. Then T is a linear transformation.

Examples

Which of the following are linear transformations?

1. Let $\mathbf{A}$ be an $m \times n$ matrix. Define $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$.
2. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $g(x_1, x_2, x_3) = (x_1, x_2, 0)$.
3. Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $g(x_1, x_2, x_3) = (x_1, x_2, 4)$.
4. Let $h : \mathbb{R}^2 \to \mathbb{R}^4$ be defined as $h(x_1, x_2) = (x_1, 0, x_2, x_1^2)$.
5. Let $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by $T(x_1, x_2, x_3, \ldots) = (x_1 + x_2, x_2 + x_3, \ldots)$.
6. Let $D : \mathcal{P}_2 \to \mathcal{P}_1$ be defined as $D(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$.

When want to prove..this is not LT give counter example

Just check whether linear combination concept is followed or not...

Remark

Translation transformation is not a linear transformation.

Theorem

Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Zero Vector stays at same poistion

Proof. Since $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0})$ and T is a linear transformation, we know that $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$. Adding inverse $-T(\mathbf{0})$ on both sides gives $T(\mathbf{0}) = \mathbf{0}$.

Exercise

Show that a linear transformation $T : V \to W$ maps a subspace of V to a subspace of W.

Proof this by taking Linear combination - As LT will always obey it..then images of v...that are in W...will do same

Definition

Let $T : V \to W$ be a linear transformation between vector spaces V and W. Then $\ker(T) = \mathcal{N}(T) = \{\mathbf{v} : T(\mathbf{v}) = \mathbf{0}\}$ is called *null space* (kernel) of T and $\mathcal{R}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$ is called the *range space* of T. Note that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ be subspaces of V and W, respectively.

Let V be a vector space with basis $B = \{v_1, \ldots, v_n\}$ and W be another vector space. Then to define a linear transformation from V to W it is enough define on B.

Theorem

Let V and W be vector spaces over $\mathbb{F}$, and $\{v_1, \ldots, v_n\}$ be a basis for V. Let $T : V \to W$ be a linear transformation. Then for any vector $v \in V$, there exist $c_1, \ldots, c_n \in \mathbb{F}$ such that $T(v) = c_1 T(v_1) + \cdots + c_n T(v_n)$.

v = c1v1 + c2v2 + ...+CnVn

Note that in the above set up $\boxed{\mathcal{R}(T) = \text{span}\{T(v_1), \ldots, T(v_n)\}}$.

Examples

Find the range space and null space of the following linear transformations.

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation given by
$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \left(\begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix}\right).$$

Nullspace of T = vector like
(k (-1 , 1 , 1 , 1))

Check....

2. Let $\mathbf{A}$ be an $m \times n$ matrix. Define $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(x) = \mathbf{A}x$.
3. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $g(x_1, x_2, x_3) = (x_1, x_2, 0)$.
4. Let $D : \mathbb{R}_2[x] \to \mathbb{R}_1[x]$ be defined as $D(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$.

Theorem (Rank-Nullity Theorem)

Let V and W be finite dimensional vector spaces. If $T : V \to W$ is a linear transformation, then $\dim V = \dim(\mathcal{R}(T)) + \dim(\mathcal{N}(T))$.

Exercise

Define a linear transformation $T : \mathbb{R}_4[x] \to \mathbb{R}_2[x]$ given by $T(p(x)) = p''(x)$. Verify rank nullity theorem.