Lecture 14

Rajiv Kumar
rajiv.kumar@iitjammu.ac.in

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Subspaces of $\mathbb{R}^n$

Some subspaces of $\mathbb{R}^n$

Let $\mathbf{A} = [a_{ij}]$ be $m \times n$ matrix, $R_1, \ldots, R_m$ be rows of $\mathbf{A}$ and $C_1, \ldots, C_n$ be columns of $\mathbf{A}$. Then a subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$, i.e., $\operatorname{span}(R_1, \ldots, R_m)$ is called the *row space* of $\mathbf{A}$ which is denoted as $\mathcal{R}(\mathbf{A})$, and the subspa~~Rank and Nullity~~ce of $\mathbb{R}^m$ spanned by columns of $\mathbf{A}$, i.e., $\operatorname{span}(C_1, \ldots, C_n)$ is called the *column space* of $\mathbf{A}$ which is denoted as $\mathcal{C}(\mathbf{A})$. The solution space of the homogeneous system of equations $\mathbf{Ax} = \mathbf{0}$ is called the *null space* of $\mathbf{A}$ which is denoted as $\mathcal{N}(\mathbf{A})$.

Example

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}$. Then

- Row space of $\mathbf{A}$, $\mathcal{R}(\mathbf{A}) = \operatorname{span}\{(1,3,4), (2,6,8)\}$.
- The column space of $\mathbf{A}$, $\mathcal{C}(\mathbf{A}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix} \right\}$.
- The null space of $\mathbf{A}$ is the set of all solution of $\mathbf{Ax} = \mathbf{0}$.

Theorem

Elementary row operations do not change the null space of a matrix, i.e., for $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$, if $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, then $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{B})$.

https://math.stackexchange.com/questions/108041/linear-algebra-preserving-the-null-space

Theorem

Elementary row operations do not change the row space of a matrix, i.e., for $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$, if $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, then $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{B})$.

Note that $\mathcal{C}(\mathbf{A})$ need not be equal to $\mathcal{C}(\mathbf{B})$.

Operations defined on vectorspace
Closed under addition and multiplication by scaler

Example

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$.

- $\mathcal{R}(\mathbf{A}) = \operatorname{span}\{(1,3,4),(2,6,8)\} = \operatorname{span}\{(1,3,4)\} = \mathcal{R}(\mathbf{B})$.
- $\mathcal{N}(\mathbf{A}) = \operatorname{span}\{(x_1,x_2,x_3) : x_1 + 3x_2 + 4x_3 = 0 = 2x_1 + 6x_2 + 8x_3\} = \operatorname{span}\{(x_1,x_2,x_3) : x_1 + 3x_2 + 4x_3 = 0 = 0x_1 + 0x_2 + 0x_3\} = \mathcal{N}(\mathbf{B})$.
- $\mathcal{C}(\mathbf{A}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix} \right\}$
- $\mathcal{C}(\mathbf{B}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Theorem

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$. Then $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \mathcal{C}(\mathbf{A})$.

x1C1+ x2C2 +...= b   Refer D.P. 2   Thm1.

Theorem

Let $\mathbf{A}$ be an $m \times n$ matrix. Then $\mathbf{Ax} = \mathbf{0}$ has a non-trivial solution if and only if columns of $\mathbf{A}$ are linearly dependent.

Thm 2.

Theorem

Thm3.

Let $\mathbf{A}$ and $\mathbf{B}$ be row equivalent matrices. Then a given set of column vectors in $\mathbf{A}$ is linear dependent if and only if the corresponding columns of $\mathbf{B}$ are linear dependent.

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a subset of linear dependent columns of $\mathbf{A}$. Then there exist scalars not all zero such that

$$c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r = \mathbf{0}. \tag{1}$$

After applying certain row operations columns $\mathbf{v}_1, \ldots, \mathbf{v}_r$ changes to $\mathbf{v}'_1, \ldots, \mathbf{v}'_r$. These column vectors satisfies the relation

$$c_1\mathbf{v}'_1 + \cdots + c_r\mathbf{v}'_r = \mathbf{0} \tag{2}$$

with same coefficients. Thus $\mathbf{v}'_1, \ldots, \mathbf{v}'_r$ are also linearly dependent.

Theorem

Let $\mathbf{A}$ and $\mathbf{B}$ be row equivalent matrices. Then a given set of column vectors of $\mathbf{A}$ forms a basis for $\mathcal{C}(\mathbf{A})$ if and only if the corresponding column vectors of $\mathbf{B}$ forms a basis of $\mathcal{C}(\mathbf{B})$.

E.g. A -> RREF(A)