Lecture 18

Rajiv Kumar
rajiv.kumar@iitjammu.ac.in

October 25, 2021

Matrix of Linear Transforms

Coordinates

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis for the vector space V. Let $\mathbf{v}$ be a vector in V, and let $c_1, \ldots, c_n$ be the unique scalars such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. Then $c_1, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ relative to B. In this case we write $[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and refer to the vector $[\mathbf{v}]_B$ as the coordinate vector (matrix) of $\mathbf{v}$ relative to B.

Exercise

Let $V = \mathbb{R}_2[x]$ and $B = \{1, x-1, (x-1)^2\}$ be its basis. Then find the coordinates of $p(x) = 2x^2 - 2x + 1$ with respect to B.

Matrix of Linear Transformation

Matrices have played an important role in our study of linear algebra. In this section we establish the connection between matrices and linear transformations. To illustrate the idea, recall that given any $m \times n$ matrix $\mathbf{A}$, we can define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$. For example, if $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation, and B be the standard basis of $\mathbb{R}^3$ with $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then

$$T(\mathbf{v}) = v_1 T(\mathbf{e}_1) + v_2 T(\mathbf{e}_2) + v_3 T(\mathbf{e}_3) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Thus $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$, where $\mathbf{A} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

Let V and W be finite dimensional vector spaces with ordered bases $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. Since $T(\mathbf{v}_i) \in W$, there exists unique scalars $a_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$ such that

$$
\begin{aligned}
T(\mathbf{v}_1) &= a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m \\
T(\mathbf{v}_2) &= a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m \\
&\vdots \\
T(\mathbf{v}_n) &= a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m.
\end{aligned}
$$

Then the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} [T(\mathbf{v}_1)]_{B'} & [T(\mathbf{v}_2)]_{B'} & \cdots & [T(\mathbf{v}_n)]_{B'} \end{bmatrix}$$ is

called matrix of linear transformation T relative to B and B', denoted by $[T]_B^{B'}$. In case if $T : V \to V$ is a linear transformation (operator) and B is a fixed basis for V, then we write $[T]_B$ for $[T]_B^B$

Exercise

Let V and W be finite dimensional vector spaces with ordered bases $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ respectively, and let $T : V \to W$ be a linear transformation. Then the coordinates of $T(\mathbf{v})$ relative to B are given by $[T(\mathbf{v})]_{B'} = [T]_B^{B'} [\mathbf{v}]_B$.

Exercise

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\mathbf{v}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$ and let

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \text{ and } B' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

be ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$. Then

1. find the matrix $[T]_B^{B'}$
2. for $\mathbf{v} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, verify $[T(\mathbf{v})]_{B'} = [T]_B^{B'}[\mathbf{v}]_B$.

Answer. $[T]_B^{B'} = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$.