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CSD001P5M | **Linear Algebra** | **Tutorial: 07**

1. Let A be an $n \times n$ matrix and α be a scalar. Find the eigenvalues of $A - \alpha I$ in terms of eigenvalues of A. Further show that A and $A - \alpha I$ have the same eigenvectors.

2. Let A be an $n \times n$ matrix. Show that:

   a) If A is projection ($A^2 = A$), then eigenvalues of A are either 0 or 1.

   b) If A is nilpotent ($A^m = 0$ for some $m \geq 1$), then all eigenvalues of A are 0.

3. In the above question, check whether the matrix is diagonalizable or not.

4. Find the minimal and characteristic polynomial of the following matrices

   (a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 1 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix}$

5. Construct a basis of $\mathbb{R}^3$ consisting of eigenvectors of the following matrices

   (a) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

6. Using the characteristic equation, find the A^{100} for the matrices given in previous question.

7. Let λ be the only eigenvalue of A. Then prove that A is diagonalizable if and only if $A = \lambda I$.

8. Using the Cayley-Hamilton theorem find the inverse of the following matrices, if exists.

   a $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

   b $\begin{bmatrix} 7 & -5 & 15 \\ 6 & -4 & 15 \\ 0 & 0 & 1 \end{bmatrix}$.

9. Let V be an inner product space and $u, v \in V$. Then show that $||u+v|| \leq ||u|| + ||v||$

10. Let V be inner product space. Then prove that for $v \in V$, $\langle v, v \rangle \in \mathbb{R}$

11. Show that the norm of a vector in a vector space V has the following three properties
    (a) $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.
    (b) $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in \mathbb{R}$ and $v \in V$.
    (c) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.
    Note that $\| \cdot \| = < \cdot, \cdot >^{1/2}$, where $< \cdot, \cdot >$ is an inner product defined on V.

12. Let $\mathbb{R}_3(x)$ be a vector space of all polynomials of degree at most 3. Then check that $\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx$ is an inner product on $\mathbb{R}_3(x)$.

13. Let R^n be an inner product space of row vectors and A be $n \times n$ matrix. For $x, y \in \mathbb{R}^n$, define $\langle x, y \rangle = xy^t$. Then show that $\langle xA, y \rangle = \langle x, yA^t \rangle$.

14. Let $z_1, \ldots, z_n$ be a set of complex numbers. The using Cauchy-Schwartz inequality prove that $(z_1 + z_2 + \cdots + z_n)^2 \leq n(z_1^2 + \cdots + z_n^2)$.

15. Let V be a rear inner product space and $u, v \in V$. Then show that

$$\langle u + v, u - v \rangle = ||u||^2 - ||v||^2.$$

16. Let V be a rear inner product space and $u, v \in V$. Find the condition that on u and v such that $(u + v) \perp (u - v)$.