Thm <u>Cauchy - Schwartz Inequality</u>

Let V be an inner product space and $u, v \in V$. Then
$|\langle u, v\rangle| \leq \|u\|\|v\|$ and equality holds iff
$\{u, v\}$ is L.D.

Pf $v=0$, Then $|\langle u,v\rangle| = \|u\|\|v\|$

If $v \neq 0$. $\exists$ $w \perp v$ st $u = w + \frac{\langle u, v\rangle}{\|v\|^2} v.$

$$\|u\|^2 = \left\langle w + \frac{\langle u,v\rangle}{\|v\|^2} v, \; w + \frac{\langle u,v\rangle}{\|v\|^2} v \right\rangle$$

$$= \left\langle w, w + \frac{\langle u,v\rangle}{\|v\|^2} v \right\rangle + \frac{\langle u,v\rangle}{\|v\|^2} \left\langle v, w + \frac{\langle u,v\rangle}{\|v\|^2} v \right\rangle$$

$$= \overline{\left\langle w + \frac{\langle u,v\rangle}{\|v\|^2} v, w \right\rangle} + \frac{\langle u,v\rangle}{\|v\|^2} \overline{\left\langle w + \frac{\langle u,v\rangle}{\|v\|^2} v, v \right\rangle}$$

$$= \overline{\langle w, w\rangle} + \overline{\frac{\langle u,v\rangle}{\|v\|^2}} \overline{\langle v, w\rangle} + \frac{\langle u,v\rangle}{\|v\|^2} \overline{\langle w, v\rangle}$$

$$+ \frac{\langle u,v\rangle}{\|v\|^2} \cdot \frac{\overline{\langle u,v\rangle}}{\|v\|^2} \langle v, v\rangle$$

$$= \|w\|^2 + 0 + 0 + \frac{|\langle u,v\rangle|^2}{\|v\|^2} \qquad z\bar{z} = |z|^2$$

$$\|u\|^2 = \|w\|^2 + \frac{|\langle u,v\rangle|^2}{\|v\|^2} \leq \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

and equality hold
iff $\|w\| = 0$

$\|u\|^2\|v\|^2 \leq |\langle u,v\rangle|^2$ and equality holds iff $\|w\| = 0$

iff $w = 0$

$\Rightarrow |\langle u, v\rangle| \le \|u\| \|v\|$ and equality holds iff $w=0$

iff $\{u, v\}$ is L.D.

Q Let $\{v_1, \dots, v_r\}$ be an orthonormal set. Let $v \in \text{Span}\{v_1, \dots, v_r\}$. Then $\exists$ $c_1, \dots, c_r$, s.t $v = c_1 v_1 + c_2 v_2 + \dots + c_r v_r$.

What we can say about c_i's in terms of v, v_i's?

Ans $c_i = \langle v, v_i\rangle$

$$\langle v, v_i\rangle = \langle c_1 v_1 + c_2 v_2 + \dots + c_i v_i + \dots c_r v_r, v_i\rangle$$

$$= c_1 \underline{\langle v_1, v_i\rangle} + c_2 \langle v_2, v_i\rangle + \dots + c_i \langle v_i, v_i\rangle + \dots + c_r \langle v_r, v_i\rangle$$

$$= c_1 \cdot 0 + c_2 \cdot 0 + \dots + c_r \underline{\langle v_i, v_i\rangle} + \dots c_r \cdot 0$$

$$= c_i \cdot \underline{\|v_i\|}^2 = c_i \cdot 1 = c_i$$

$\Rightarrow c_i = \langle v, v_i\rangle$.

$\{(1,2,3), (1,3,2), (1,1,1)\}$

$(x, y, z) = c_1 (1,2,3) + c_2 (1,3,2), c_3 (1,1,1)$.

Def A set $B \subseteq V$ is called an orthonormal basis if it is basis for V and an orthonormal set.

Ex. 1. $V = \mathbb{R}^n$, Then $\{e_1, \dots, e_n\}$ be an orthonormal basis.

2. $V = \mathbb{R}^2$, $\left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \right\}$ is an orthonormal basis.

Observation: An orthonormal set B is a basis of V iff $|B| = \dim(V)$

Question. Let V be an inner product space. Then does there exists an orthonormal basis for V.

$V = \mathbb{R}^2$, $\langle x, y \rangle = x^T A y$, $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Answer — Yes.

Gram - Schmidt procedure.

Suppose $\{v_1, \dots, v_m\}$ is a L.I set of vectors in an innerproduct space. Let $u_1 = \frac{v_1}{\|v_1\|}$, and for $j = 2, \dots, m$,

$$u_j = \frac{v_j - \langle v_j, u_1 \rangle u_1 - \langle v_j, u_2 \rangle u_2 - \dots - \langle v_j, u_{j-1} \rangle u_{j-1}}{\| v_j - \langle v_j, u_1 \rangle u_1 - \dots - \langle v_j, u_{j-1} \rangle u_{j-1} \|}$$

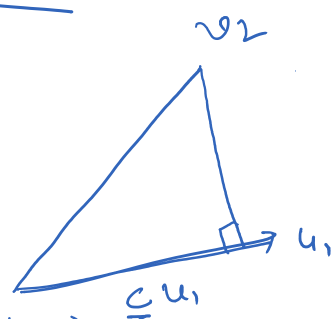
Then $\{u_1, \dots, u_m\}$ is an orthonormal set s.t

$$\operatorname{span}\{v_1, \dots, v_j\} = \operatorname{span}\{u_1, \dots, u_j\} \quad \forall \ 1 \le j \le m$$

Ex. $V = \mathbb{R}^3$, $B = \{(1, 2, 3), (1, 3, 2), (1, 1, 1)\}$.

$\langle x, y \rangle = \sum x_i y_i$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = \frac{(1,2,3)}{\sqrt{14}}$$



$$u_2 = \frac{v_2 - \langle u_1, v_2\rangle u_1}{\| v_2 - \langle u_1, v_2 \rangle u_1 \|}$$

$$= \frac{(1,3,2) - \frac{\langle (1,2,3),(1,3,2)\rangle}{\sqrt{14}} \cdot \frac{(1,2,3)}{\sqrt{14}}}{\left\| (1,3,2) - \frac{1+6+6}{14}(1,2,3) \right\|}$$

$\langle u_1, v_2 \rangle \cdot u_1$?

$$= \frac{\frac{1}{14}(14-13,\ 42-26,\ 28-39)}{\left\| \frac{1}{14}, \frac{16}{14}, \frac{-11}{14} \right\|}$$

$$u_3 = \frac{v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2 \rangle u_2}{\| v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2 \|}$$

Check that $\{u_1, u_2, u_3\}$ is an orthonormal set

and $\text{Span}\{u_1\} = \text{Span}\{v_1\}$

$\text{Span}\{u_1, u_2\} = \text{Span}\{v_1, v_2\}$

$\text{Span}\{u_1, u_2, u_3\} = \text{Span}\{v_1, v_2, v_3\}$