① Eigen Value: let A be $n\times n$ matrix. Then $\lambda \in \mathbb{R}$ is a eigen value of A if $\exists$ non zero vector $x \in \mathbb{R}^n$ s.t $Ax = \lambda x$.

② Characteristic polynomial: $\det(A - \lambda I)$ is called the characteristic polynomial.

3. The zeros of characteristic polynomial are eigen values of A.

→ let A be a matrix with characteristic poly
$p_A(\lambda) = (\lambda - a_1)^{r_1}(\lambda - a_2)^{r_2} \cdots (\lambda - a_k)^{r_k}$ (Algebraic)

→ $Ax = \lambda x \Rightarrow Ax - \lambda x = 0$
$\Rightarrow Ax - \lambda I x = 0$
$\Rightarrow (A - \lambda I)x = 0$
$\Rightarrow x \in N(A - \lambda I)$

let $b_\lambda = \dim(N(A - \lambda I)) \longrightarrow$ Geometrical

What is connection b/w r_λ and b_λ

Def: let λ be an eigenvalue of a linear operator T on a finite dimensional V.S V and $A = [T]_B$ for some basis V.

* Then the dimension of the eigenspace $A_\lambda = N(A - \lambda I)$ is called the geometric

multiplicity of λ, denoted as $\dim(A_\lambda)$

* The algebraic multiplicity of λ is the multiplicity of λ as a root of $\det(A-\lambda I)=0$.

Ex. $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Char eq. $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 1 = 0$$

$$1+\lambda^2-2\lambda-1=0$$

$$\lambda^2-2\lambda=0$$

$$\lambda(\lambda-2)=0, \quad \lambda=0, \ \lambda=2$$

algebraic multiplicity of $\lambda=0$, $=1$
$\lambda=2$, 1

For $\lambda=0$, $A-0I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$R_2 \to R_1+R_2$

$\sim \begin{bmatrix} \boxed{1} & -1 \\ 0 & 0 \end{bmatrix}$

Pivotal column

Non pivotal co.

free variable -1

$\Rightarrow \dim N(A-0I) = 1$

In this case find the eigen vector.

G.M of $0 = 1$

Illy ——— $2 = 1$ (check.

Ex 2 $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$|A-\lambda I| = 0$ $\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$-(1-\lambda)(1+\lambda) + 1 = 0$$
$$-1 + \lambda^2 + 1 = 0$$
$$\lambda^2 = 0$$

eigen value, 0, 0

A.M of $0 = 2$

G.M $A - 0I = A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

free variable -1

G.M of $0 = 1 <$ A.M of 0.

Thm The Geometric multiplicity of an eigenvalue does not exceed its Algebraic multiplicity.

Pf Let λ be an eigenvalue of A with geometric multiplicity r. This implies that

$$\dim(\underbrace{N(A-\lambda I)}) = r$$

$$\Rightarrow \boxed{\dim(A_\lambda) = r}$$

Let $\{x_1, x_2, \dots, x_r\}$ be a basis of A_λ.

Let $\{\underbrace{x_1, x_2, \ldots, x_r}, \underbrace{x_{r+1}, \ldots, x_n}\}$ be a basis of $\mathbb{R}^n$.

Let $P = [\underbrace{x_1\ x_2\ \ldots\ x_r}\ x_{r+1}\ \ldots\ x_n]$ and

$B = P^{-1}AP$.

$$\begin{aligned} Be_i &= (P^{-1}AP)e_i, \qquad i = \underbrace{1, \ldots, r}. \\ &= (P^{-1}A)(\underbrace{Pe_i}) \\ &= P^{-1}Ax_i \\ &= P^{-1}\lambda x_i \\ &= \lambda P^{-1}x_i \\ &= \lambda e_i \end{aligned}$$

$$\boxed{\begin{aligned} Ax &= A_1 \\ Ae_1 &= A_1 \\ Ae_i &= A_i \end{aligned}}$$

$Pe_i = x_i$.
$\Rightarrow P^{-1}x_i = e_i$

$$B = \begin{bmatrix} \lambda & & 0 & a_{1\,r+1} & \cdots & a_{1n} \\ & \lambda & & a_{2\,r+1} & & a_{2n} \\ 0 & & \ddots & \vdots & & \vdots \\ & & \lambda & \vdots & & \vdots \\ & 0 & & \underline{a_{n\,r+1}} & & \underline{a_{nn}} \end{bmatrix}$$

$$|B - xI| = \begin{vmatrix} \lambda - x & 0 & & a_{1\,r+1} & \cdots & a_{1n} \\ & \lambda - x & & & & \vdots \\ 0 & & \ddots & & & \vdots \\ & & \lambda - x & \vdots & & \vdots \\ 0 & 0 & \cdots\ 0 & a_{n\,r+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= (\lambda - x)^r \underbrace{\begin{vmatrix} a_{r+1,\,r+1} - x & \cdots & a_{r+1\,n} \\ & & \\ a_{n\,r+1} & \cdots & a_{nn} - x \end{vmatrix}}_{C}$$

$$= (\lambda - x)^{r} \det(C)$$

$$\det(B - xI) = (\lambda - x)^{r} \det(C)$$

Now $$B = P^{-1} A P$$

$$B - xI = P^{-1} A P - xI$$
$$= P^{-1} A P - x P^{-1} I P$$
$$= P^{-1} A P - P^{-1} x I P$$
$$= P^{-1} (A P - x I P)$$
$$= P^{-1} (A - x I) P$$

$\Rightarrow$ $A - xI$ is similar to $B - xI$.

A is similar to B if $\exists$ invertible P s.t $A = P^{-1} B P$

Determinant of two similar matrices is same. (Exercise)

Hint: $\det(AB) = \det A \cdot \det B$

$$\operatorname{Det}(A - xI) = \operatorname{Det}(B - xI)$$
$$= (\lambda - x)^{r} \det(C)$$

$\Rightarrow$ A.M of $\lambda \geq r =$ G.M.

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* Two similar matrices have same characteristic eqn.

$\rightarrow$ Two similar matrices have same eigenvalue.

Diagonalizable A $n \times n$ matrix over $\mathbb{R}$ is called diagonalizable if it similar to a diagonal matrix i.e., $\exists$ invertible matrix P and diagonal matrix D s.t $A = P^{-1}DP$.

Thm If A is $n \times n$ matrix over $\mathbb{R}$, then the following are equivalent

1. A is diagonalizable
2. A has n linearly independent eigenvectors.

Consequence (*) Suppose $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of A. A.M of $\lambda_i = 1$ $\forall i$ is 1. Since G.M of any eigenvalue can not be 0. Hence G.M of $\lambda_i = 1$ $\forall i$

Exercise. Fact Let λ_1, λ_2 be two distinct eigenvalue of A and x_1 and x_2 be corresponding eigenvector of A. Then $\{x_1, x_2\}$ is L.I

Using the fact, we can say that in (*) A has n L.I eigenvector. A is Diagonalizable

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \longrightarrow$ Diagonalizable.

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \longrightarrow$ Not Diagonalizable

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$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix} \rightarrow$$ Find that matrix is diagonalizable or not.