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**Def$^n$** If $A$ is $m \times n$ matrix and $b \in \mathbb{R}^n$, a least square solution of $Ax = b$ is an $\hat{x} \in \mathbb{R}^n$ s.t

$$\| b - A\hat{x} \| \leq \| b - Ax \| \quad \forall\, x \in \mathbb{R}^n$$

**Remark** 1. Suppose $Ax = b$ has a solution. Then

$$\min \{ \| b - Ax \| : x \in \mathbb{R}^n \} = 0$$

Hence in this case, a least square solution is set of solutions of $Ax = b$

2. If $Ax = b$ is inconsistent.

Then $\min \{ \| b - \underline{Ax} \| : x \in \mathbb{R}^n \}$

$$= \| b - A\hat{x} \|, \text{ where } A\hat{x} = \text{proj}_{\text{col} A}(b)$$

Hence we need to find $x$, st $A x = \underline{\text{proj}_{\text{Col}(A)}(b)}$

(3) $\mathbb{R}^m = \mathcal{C}(A) \oplus \mathcal{C}(A)^{\perp}$

$$= \mathcal{C}(A) \oplus \mathcal{N}(A^t)$$

4. If $Ax = b$ is inconsistent, then

$b = b_1 + b_2$, where $b_1 \in \text{Col}(A)$

$b_2 \in \mathcal{N}(A^T)$

(5) We need $\hat{x}$ s.t $\underline{A\hat{x} = b_1}$

(6) $b - b_1 = b_2 \in \mathcal{N}(A^T)$

$\Rightarrow \quad b - A\hat{x} = b_2 \in N(A^T)$

$\Rightarrow \quad A^T(b - A\hat{x}) = A^T(b_2)$

$\Rightarrow \quad A^T b - A^T A\hat{x} = 0$

$\Rightarrow \quad \boxed{A^T A\hat{x} = A^T b.} \longrightarrow$ Normal eq.

<u>Thm</u> The set of least square solution of $Ax=b$ coincide with the non-empty set of solutions of normal eq $A^T A x = A^T b$

<u>Ex</u> For
$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix},$$

find the set of least square solution.

multiply both sides by $\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Consider

$$[A \,\vdots\, b] = \left[\begin{array}{cc|c} 17 & 1 & 19 \\ 1 & 5 & 11 \end{array}\right]$$

$$R_1 \longleftrightarrow R_2$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 5 & 19 \\ 17 & 1 & 11 \end{array}\right]$$

$$R_2 \rightarrow R_2 - 17 R_1$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 5 & 19 \\ 0 & -84 & 11 - 17 \times 19 \end{array}\right]$$

Find the value of $x_1, x_2$

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Question: Let $A$ and $B$ be two matrices.

Can you say $\operatorname{rank}(AB) \le \min\{\operatorname{rk}(A), \operatorname{rk}(B)\}$

Suppose $B = A^T$.

Fact: $\boxed{\operatorname{rank}(AA^T) = \operatorname{rank}(A)}$

Thm: The matrix $A^TA$ is invertible iff the columns of $A$ are L.I.

Pf: Let $A_{m \times n}$. Then $A^TA$ is a matrix of size $n \times n$.

Assume $A^TA$ is invertible, $\operatorname{rank}(A^TA) = n$.

Using the fact $\underline{\text{rank}(A^T A) = \text{rk}(A)}$,

we get $\text{rk}(A) = n$.

$\Rightarrow$ columns of $A$ are L.I.

Suppose columns of $A$ are L.I., Then $\text{rk}(A) = n$.

Hence $\text{rk}(A^T A) = n$

$\Rightarrow A^T A$ is invertible.

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In this situation, we get a system

$A x = b$
$\underline{A}$ invertible matrix.

$$A^T A x = A^T b$$

$$\Rightarrow \boxed{x = (A^T A)^{-1} A^T b}$$

$$A x = \underline{A(A^T A)^{-1} A^T} b$$

check that $\left(A(A^T A)^{-1} A^T\right)^2 \overset{?}{=} A(A^T A)^{-1} A^T$

$\hookrightarrow$ projection matrix.

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$\underline{\underline{\text{Pf of the fact.}}}$

$\text{rk}(A^T A) = \text{rk}(A)$

$\text{rk}(A^T A) \leq \text{rk}(A)$ ✓

① $A x = 0 \Rightarrow A^T A x = 0$

$N(A) \subseteq N(A^T A)$

2. Let $x \in N(A^T A)$,

$\underline{A^T A x = 0}$.

$\underline{A x = 0}$
want to prove.

$$A^T A x = 0$$

want to prove

$$\Rightarrow x^T A^T A x = 0$$

$$\Rightarrow (Ax)^T A x = 0$$

$$\Rightarrow Ax \perp Ax$$

$$\Rightarrow Ax = 0$$

$$\text{So } N(A^T A) \subseteq N(A)$$

$$\text{So } N(A^T A) = N(A)$$

$$\text{rank}(A) + n(A) = \#\ \text{colums}(A) \quad -(1)$$

$$\text{rank}(A^T A) + n(A^T A) = \#\ \text{columns}(A) \quad -(2)$$

(1) − (2)

$$\text{rank}(A) - \text{rank}(A^T A) + n(A) - n(A^T A)$$

$$= \#\ \text{cols}(A) - \#\ \text{cols}(A^T A)$$

$$\Rightarrow \text{rank}(A) = \text{rank}(A^T A)$$

Remark The system $Ax = b$ has a unique least sq sol. iff the columns of $A$ are L.I.