Quadratic forms

Let A be a symmetric matrix, and $\varphi(x) = x^T A x$ be the corresponding quadratic form.

Then define $m = \min\{x^T A x : \|x\| = 1\}$

$M = \max\{x^T A x : \underline{\|x\| = 1}\}$

Question what is the value of m & M?

Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$\varphi(x) = 4x_1^2 + 2x_2^2 + 5x_3^2$$

Find the value of m & M.

$\|x\| = 1 \Rightarrow x_1^2 + x_2^2 + x_3^2 = 1$

$$4x_1^2 \le 5x_1^2$$
$$2x_2^2 \le 5x_2^2$$

$$\begin{aligned} \varphi(x) &= 4x_1^2 + 2x_2^2 + 5x_3^2 \\ &\le 5x_1^2 + 5x_2^2 + 5x_3^2 \\ &= 5(x_1^2 + x_2^2 + x_3^2) \\ &= 5\|x\|^2 \end{aligned}$$

$\Rightarrow \quad \varphi(x) \leq 5\, \|x\|^2$

If $\|x\| = 1$, $\quad \varphi(x) \leq 5$

If $x = (0,0,1)$, then $\varphi(x) = 4 \cdot 0^2 + 2 \cdot 0^2 + 5 \cdot 1^2$

$= 5$

$$\boxed{M = 5 \qquad x = e_3}$$

Find the value of $m = 2$ (Prove it.)

If $x = (0, 1, 0)$, then $\varphi(x) = 2$.

$$\boxed{m = 2, \qquad x = e_2}$$

Suppose $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ what is M & m
$\quad M \downarrow x \qquad m \downarrow y$

In previous example

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad M = 5 \qquad m = 2$$

$M = 5$: largest eigenvalue; e_3: corresponding eigenvector

$m = 2$: least eigen value; e_2

Thm Let A be a symmetric matrix. Then M is the greatest eigenvalue of A and m is the least eigenvalue of A. The value of

$x^T A x$ is M when x is a unit eigenvector of A, Hly the value $x^T A x$ is m when x is a unit eigenvector corr. to m of A.

<u>Pf</u> Let A be a symmetric matrix.

Let P be an orthogonal matrix s.t. $P^T A P = D$, diagonal elements of D are eigenvalues of A.

Now maximize $\underline{x^T A x}$ subject to $\underline{\|x\| = 1}$

Let $x = Py$

$$\|x\| = \|Py\| = \sqrt{(Py)^T Py} = \sqrt{y^T \underbrace{P^T P}_{} y}$$

$$= \sqrt{y^T I y} = \sqrt{y^T y} = \|y\|$$

$\|x\| = 1$ iff and only if $\|y\| = 1$

$$\varphi(x) = x^T A x = (Py)^T A Py$$
$$= y^T P^T A P y$$
$$= y^T D y = \varphi(y)$$

The theorem is reduce to

maximize $\underline{y^T D y}$ subject to $\|y\| = 1$

M = maximum eigenvalue of D
= maximum eigenvalue of A

The value M can be obtained at e_i if ith diagonal element of D is M

$x = Py = Pe_i$ = ith column of P
= unit eigenvector corresponding to eigenvalue M.

$\boxed{\text{Illy Prove for } m \quad (\text{Exercise})}$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Maximise $Q(x) = 4x_1^2 + 2x_2^2 + 5x_3^2$
subject to $\|x\| = 1$
$e_3^T x = 0$

In this case the maximum value is 4 at e_1

$e_3^T x = 0 \Rightarrow (0,0,1)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow x_3 = 0$

$\|x\| = 1$

$x_1^2 + x_2^2 + x_3^2 = 1$

$\boxed{x_1^2 + x_2^2 = 1, \quad x_3 = 0}$

Thm Let A be a symmetric matrix and λ be the largest eigenvalue of A and u be a corresponding eigen vector. Then the maximum value of $x^T A x$ subject to $||x|| = 1$, $u^T \cdot x = 0$ is the second largest eigenvalue of A. The value attained at correspond unit eigen vector.

(Proof is easy) Exercise

Ques. Let A be a symmetric matr.

Maximize $x^T A x$

subject to $||x|| = 1$

$u_1^T x = 0 \rightarrow u_1$ unit eigen vec cor to largest.

$u_2^T x = 0$, u_2 - eigenvec to corr. to 2nd largest eigen

Let us see an example

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad e_1^T x = 0$$

In this way, we can find all the eigenvalues in a decreasing order.

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ex Let $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

Then the l.t maps unit sphere to an ellipse. ($\mathbb{R}^2$)

Find a unit vector x s.t $\|Ax\|$ is largest.

Want to maximize $\|Ax\|$ subject to $\|x\|=1$

$$\|Ax\|^2 = (Ax)^T Ax$$

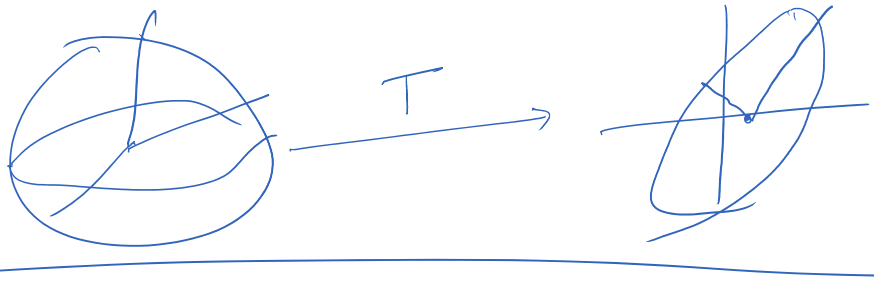
$$= x^T A^T A x$$

$$= x^T B x, \quad \text{where } B = A^T A$$

$\rightarrow$ Fact B is symmetric

The largest value of $\|Ax\|^2$ is the largest eigenvalue of $B = A^T A$

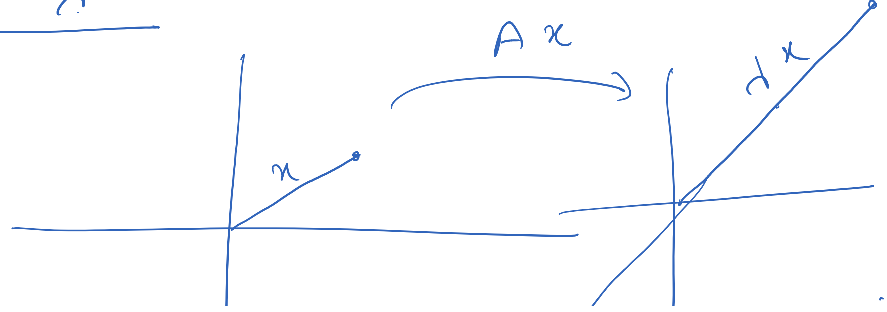
corresponding to unit eigenvector.

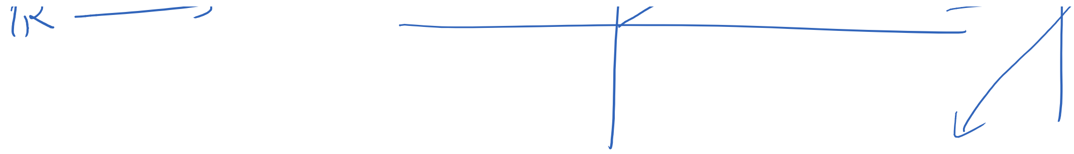


$Ax = \lambda x$

$x \longrightarrow Ax$

$\mathbb{R}^n \longrightarrow \mathbb{R}^n$





$$\begin{bmatrix} 1 & -1 \\ -1 & + \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$