Orthogonal Projection

Let U be finite dimensional vector space.

Then $V = U \oplus U^{\perp}$. Let P_U be a map from V to V,

by $P_U(v) = u$, where $v = u + w$, $u \in U$, $w \in U^{\perp}$ $\forall v \in V$.

Properties of P_U

① P_U is well defined

because if $V = U \oplus U^{\perp}$

$\forall v$, $P_U v \in V$

$\forall v$; if $P_U v = v_1$

& $P_U v = v_2$

$v_1 = v_2$

We know $V = U \oplus U^{\perp}$. Let $v = u_1 + w_1$, ① where $u_1 \in U$, $w_1 \in U^{\perp}$

Suppose $v = u_2 + w_2$ ②, $u_2 \in U$, $w_2 \in U^{\perp}$

We need to prove $u_1 = u_2$, $w_1 = w_2$

① − ② $\quad u_1 - u_2 + w_1 - w_2 = 0$

$\Rightarrow u_1 - u_2 = -w_1 + w_2$

But $-w_1 + w_2 \in U^{\perp}$ and $u_1 - u_2 \in U$

$\Rightarrow u_1 - u_2 \in U^{\perp}$

$\Rightarrow u_1 - u_2 \in U \cap U^{\perp} = \{0\}$

$\Rightarrow u_1 = u_2$

IIIly $w_1 = w_2$

$\Rightarrow$ for every vector v, $\exists$ unique vector u s.t $P_U v = u$.

$\underline{\underline{2}}$ P_U is a Linear Transformation

Let $\underline{v_1}, v_2 \in V$, $\alpha, \beta \in \mathbb{R}$

$\exists$ $u_1, u_2 \in U$, and $w_1, w_2 \in U^\perp$

st $v_1 = u_1 + w_1$, $v_2 = u_2 + w_2$.

Then $P_U(v_1) = u_1$, $P_U(v_2) = u_2$

$$\alpha v_1 + \beta v_2 = \alpha(u_1 + w_1) + \beta(u_2 + w_2)$$

$$= \underbrace{(\alpha u_1 + \beta u_2)}_{\in U} + \underbrace{(\alpha w_1 + \beta w_2)}_{\in U^\perp}$$

$$P_U(\alpha v_1 + \beta v_2) = \alpha u_1 + \beta u_2$$

$$= \alpha P_U(v_1) + \beta P_U(v_2)$$

$\Rightarrow$ P_U is a L.T

$\underline{\underline{3}}$ Let $u \in U$. Then $P_U(u) = u$

Because $u = \underbrace{u}_{\in U} + \underbrace{0}_{\in U^\perp}$

4. Let $w \in U^\perp$. Then $P_U(w) = 0$ (why?)

$$w = \underbrace{0}_{\in U} + \underbrace{w}_{\in U^\perp}$$

5 Range(P_U)?, is $U \subseteq$ Range(P_U)?

What about $\text{Range}(P_U) \subseteq U$? Yes.

Yes.

$\text{Range } P_U = U$ (because of 3 & Defin)

6. Kernel of $P_U := U^\perp$ (Use 4) and Rank nullity.

7 Let $v \in V$. Then $v - P_U(v) \in U^\perp$

8 $P_U \circ P_U = P_U$

Let $v \in V$, Then $\exists$ $u \in U$, and $w \in U^\perp$

$$v = u + w$$

$$P_U(v) = u$$

$$(P_U \circ P_U)(v) = P_U(\underline{P_U(v)})$$

$$= P_U(u) = u$$

$$= P_U(v) \quad \forall v \in V$$

9 $\|P_U(v)\| \leq \|v\|$

Let $v = u + w$, $u \in U$, $w \in U^\perp$

$$\|P_U(v)\|^2 = \|u\|^2$$

$$\leq \|u\|^2 + \|w\|^2 \quad \text{— (3)}$$

Since $u \perp w$, using Pythagoras thm.

$$\|u\|^2 + \|w\|^2 = \|u+w\|^2 = \|v\|^2 \quad \text{— (4)}$$

From (3) and (4), $\|P_U(v)\|^2 \leq \|v\|^2$

$$\Rightarrow \|P_U(v)\| \leq \|v\|$$

(10) Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis for U. Then $P_U(v) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$

Pf Let $v \in V$. Then $v = u + w$, $u \in U$, $w \in W^\perp$

Let $u = c_1 e_1 + \cdots + c_m e_m$

Hence $v = c_1 e_1 + c_2 e_2 + \cdots + c_m e_m + w$

$$P_U(v) = c_1 e_1 + \cdots + c_m e_m$$

$$\langle v, e_1\rangle = \langle c_1 e_1 + c_2 e_2 + \cdots + c_m e_m + w, e_1\rangle$$

By using linearity pr. $= c_1 \underbrace{\langle e_1, e_1\rangle}_{1} + \underline{c_2\langle e_2, e_1\rangle} + \cdots + c_m\langle e_m, e_1\rangle + \underline{\langle w, e_1\rangle}$

$$= c_1 + 0 + \cdots + 0 + 0$$

$$\Rightarrow c_1 = \langle v, e_1\rangle$$

111ly $c_i = \langle v, e_i \rangle$ (Verify)

Hence $P_U(v) = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \cdots + \langle v, e_m \rangle e_m$

(11) $P_{U^\perp} = I - P_U$

$\forall\, v \in V,$ $P_{U^\perp}(v) = \underline{v} - P_U(v)$

$= I(v) - P_U(v)$

$= (I - P_U)(v)$

where I is an identity map

(12) $P_U \circ P_{U^\perp} = 0$ (Verify)

Question: Let A be an orthogonal matrix and $b \in \mathbb{R}^n$. Consider the system

$Ax = b$. What can you say about x?

(Think about it)

Fact 1. $b \in$ Column space A

2. Columns of A forms an orthonormal basis of $\mathbb{R}^n$; say $\{e_1, \ldots, e_n\}$

$b \in \text{Span}\{e_1, \ldots, e_n\}$

$b = \langle b, e_1 \rangle e_1 + \langle b, e_2 \rangle e_2 + \cdots \langle b, e_n \rangle e_n$

$$b = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \underbrace{\begin{bmatrix} \langle b, e_1 \rangle \\ \langle b, e_2 \rangle \\ \vdots \\ \langle b, e_n \rangle \end{bmatrix}}_{\text{solution}}$$

Ex $\quad A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\langle b, e_1 \rangle = \left(1 \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$$

$$\langle b, e_2 \rangle = \left(\frac{1}{\sqrt{2}} \cdot 1 - \frac{1}{\sqrt{2}} \cdot 2\right) = -\frac{1}{\sqrt{2}}$$

solution is $\begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

Thm Let U be a finite dimensional subspace of V. and $v \in V$, $u \in U$. Then

$\|v - P_U(v)\| \le \|v - u\|$ and equality holds iff $P_U(v) = u$.

Pf $\quad \|v - P_U(v)\|^2 \le \|v - P_U(v)\|^2 + \underline{\|P_U(v) - u\|^2}$ (5)

Since $(v - P_U(v)) \perp U$ and

$P_U(v) - u \in U$, By Pythagoras

thm

$$\|v - P_U(v)\|^2 + \|P_U(v) - u\|^2$$

$$= \|v - P_U(v) + P_U(v) - u\|^2 \quad -(6)$$

From (5) and (6), we get.

$$\|v - P_U(v)\|^2 \leq \|v - u\|^2$$

$$\Rightarrow \|v - P_U(v)\| \leq \|v - u\|$$

In eq^n (5) the equality hold iff

$$\|P_U(v) - u\|^2 = 0 \Rightarrow P_U(v) - u = 0$$

$$\Rightarrow P_U(v) = u.$$

Def^n. Let $Ax = b$ be a system linear eq.

Then a least square solution of $Ax = b$ is an $\hat{x}$

s.t. $\|A\hat{x} - b\| \leq \|Ax - b\| \ \forall x \in \mathbb{R}^n.$

Discussion $b \in \mathbb{R}^m = N(A^T) \oplus C(A)$

$$\Rightarrow b \in N(A^T) \oplus C(A)$$

$$\Rightarrow b = b_1 + b_2 \qquad P(b) = b_2$$

$$\Rightarrow \quad b = \underbrace{b_1}_{\in \mathcal{N}(A^T)} + \underbrace{b_2}_{\in \mathcal{C}(A)} \qquad \Big| \; P_{\mathcal{C}(A)}(b) = b_2$$

Let $\hat{x} \in \mathbb{R}^n$, Then $A\hat{x} \in \mathcal{C}(A)$.

By using previous result,

$$\| b - \underbrace{\mathrm{Proj}_{\mathcal{C}(A)}(b)} \| \le \| b - u \| \qquad u \in \mathcal{C}(A)$$

$$\Rightarrow \| b - \underline{b_2} \| \le \| b - u \|$$

$$\Rightarrow \| b - A\hat{x} \| \le \| b - Ax \|,$$

where $\hat{x}$ is solution of $Ax = b_2$

Let $x \in \mathcal{N}(A)$, Then $Ax = 0$

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x} = 0$$

$$\Rightarrow \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$\vdots$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = 0$$

$$\Rightarrow \quad \langle A_i, X \rangle = a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = 0$$

$$\Rightarrow \quad A_i \perp X$$

$$\Rightarrow \quad R(A) \perp N(A)$$

In fact, $N(A)^{\perp} = R(A)$.

$$\mathbb{R}^n = N(A) \oplus \underbrace{N(A)^{\perp}}$$

$$= N(A) \oplus R(A)$$

Illy. $\mathbb{R}^m = N(A^T) \oplus \mathcal{C}''(A)$