Department of Mathematics
Indian Institute of Technology Jammu

CSD001P5M **Linear Algebra** **Tutorial: 06**

---

1. Construct a matrix A with the required property, or explain why you can't:

   (a) Column space of A contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, row space of A contains $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\5\end{bmatrix}$

   (b) Column space of A has basis $\begin{bmatrix}1\\2\\3\end{bmatrix}$, null space of A has basis $\begin{bmatrix}3\\2\\1\end{bmatrix}$.

   (c) Row space of A = column space of A, null space of $A \neq$ null space of A^T.

   (d) Column space of A contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$, but not $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

   (e) Dimension of null space of $A = 1+$ dimension of null space of A^T.

   (f) Null space of A^T contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space of A contains $\begin{bmatrix}3\\1\end{bmatrix}$.

   (g) Null space of $A =$ null space of A^T.

2. (a) Find the coordinates of the vectors $v_1 = (2,3,4)$ and $v_2 = (1,-1,2)$ with respect to the ordered basis $\beta = \{(1,1,1),(1,2,3),(1,3,6)\}$.
   (Note: the vectors have been written as 3-tuples, but should be regarded as column vectors.)

   (b) If $[v]_\beta = \begin{bmatrix}2\\3\\2\end{bmatrix}_\beta$, find $[v]_S$ where S is the standard basis for $\mathbb{R}^3$.

3. Find the matrix relative to the standard basis of the linear operator T on $\mathbb{R}^3$ given by:

$$T(x_1,x_2,x_3) = (x_1+x_3, x_1+2x_2+x_3, -x_1+x_2).$$

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $T(x_1,x_2,x_3) = (x_1+x_2, 2x_3-x_1)$.

   (a) Find the matrix of T with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

(b) Verify that $\beta = \{(1,0,-1),(1,1,1),(1,0,0)\}$ is a basis for $\mathbb{R}^3$.

(c) Now, determine the matrix of T with respect to the ordered bases β and $\beta' = \{(0,1),(1,0)\}$ for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

(d) Find the null space of matrix of T.

(e) Deduce the kernel of T.

5. Let V be an n-dimensional space and let T be a linear operator on V such that Range $(T) =$ Kernel (T). Show that n must be even. Give an example of such an operator.
(Note: a linear operator T on V is a linear transformation $T : V \to V$, i.e. the co-domain is the same as the domain.)

6. (a) Find the matrix relative to the standard basis of the linear operator T on $\mathbb{R}^3$ given by:
$$T(x_1,x_2,x_3) = (x_1+x_3, x_1+2x_2+x_3, -x_1+x_2).$$
(b) Find the matrix of the same linear operator T relative to the ordered basis $\beta = \{(1,1,1),(1,2,3),(1,3,6)\}$.

7. Prove that there does not exist a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^2$ such that
$$\text{Ker } T = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

8. Let $V = \mathbb{R}^{2\times 2} =$ vector space of 2×2 matrices with real entries, and consider the function $U : V \to V$ given by $U(A) = A + A^T$, for all $A \in V$, where A^T indicates the transpose of A.

(a) Show that U is a linear operator.

(b) Determine the matrix of U with regard to any suitable ordered basis β of V.

(c) Determine a basis for Ker U and determine a basis for Range U.

(d) Determine the dimension of $\mathrm{Sym}_n(\mathbb{R})$, the space of symmetric $n\times n$ matrices with real entries. Briefly explain your answer.

9. Show that a linear transformation $T : V \to W$, where V and W are finite-dimensional with $\dim V = \dim W$, is injective if and only if it is surjective.

10. Give an example of a vector space V, and two linear transformations $T,U : V \to V$, such that T is surjective but not injective, and U is injective but not surjective. (More advanced: this problem should be tried last.)

11. Let $V = \mathbb{R}^2$, and consider the ordered bases $\alpha = \{u_1,u_2\}$ and $\beta = \{v_1,v_2\}$, where the vectors are as given below. (NB: regard all vectors as column vectors in V)
$$\mathbf{u_1} = (3,1), \qquad \mathbf{u_2} = (\mathbf{11},\mathbf{4}), \qquad \mathbf{v_1} = (\mathbf{3},\mathbf{2}), \qquad \mathbf{v_2} = (\mathbf{7},\mathbf{5})$$

(a) Find the change of basis matrix I_{α}^{β}.

(b) Hence find $[v]_{\beta}$ given that $[v]_{\alpha} = (10, 20)$.

(c) Is there some way to check your answer to (b)? Explain your method and use it to check your answer.

12. Find the eigenvalues and corresponding eigenvectors for the matrix A given below. Is A diagonalizable ? Justify your answer in at most one sentence.

$$\begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$$

13. For each matrix find all eigenvalues and a basis of each eigenspace. Which matrix can be diagonalized and why? If yes, indicate the diagonal matrix D and the invertible matrix P such that $A = PDP^{-1}$. [Hint: $\lambda = 4$ is an eigenvalue.]

(a) $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

14. Let λ be an eigenvalue of A and $f(x)$ be a polynomial. Then show that $f(\lambda)$ is an eigen value of $f(A)$.

15. A matrix 7×7 matrix A has three eigenvalues. One eigenspace is 2-dimensional and one of the others is 3-dimensional. Is it possible for A to be not diagonalizable? Justify your answer.

16. Suppose A is an $n \times n$ square matrix and Rank $(A) = k$. Show that A can have at most $(k+1)$ distinct eigenvalues.

17. (a) If A is row-equivalent to the identity matrix, then A must be diagonalizable. Is this statement TRUE or FALSE?

    (b) Justify your answer to (a). Give a proof if TRUE or a concrete counter-example if FALSE. In the second case, you should verify that your counter-example is row-equivalent to identity matrix but not diagonalizable.

18. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Show that $\det(A) = \lambda_1 \cdots \lambda_n$ and tr $A = \lambda_1 + \cdots + \lambda_n$. Further show that A is invertible if and only if its all eigenvalues are non-zero.