Lecture 9

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Linear Span

Linear Span of Vectors

Definition

Let V be a vector space and let $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ be a set of vectors in V. Then the *linear span* of S, denoted by span(S), is the set

$$\{c_1\mathbf{x}_{i_1} + \cdots + c_m\mathbf{x}_{i_m} : c_1, \ldots, c_m \in \mathbb{R}\}.$$

Example

Let $V = \mathbb{R}^n$ and $S = \{e_1, \ldots, e_n\}$, where e_i denotes the element of $\mathbb{R}^n$ ith component is 1 and all other are zero. Then any vector of $\mathbb{R}^n$ can be written as a linear combination of vectors of S.

Proposition

If $S = \{\mathbf{x}_1, \mathbf{x}_2 \ldots\}$ be a set of vectors in V, then span(S) is the smallest subspace of V containing S.

Proof. Clearly, $S \subseteq$ span(S). Let $\mathbf{x}, \mathbf{y} \in$ span(S) and c, d are scalars. Then there exist scalars $c_1, \ldots, c_n, d_1, \ldots, d_n$, and $\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_n} \in S$ such that $\mathbf{x} = c_1\mathbf{x}_{i_1} + \cdots + c_n\mathbf{x}_{i_n}$ and $\mathbf{y} = d_1\mathbf{x}_{i_1} + \cdots + d_n\mathbf{x}_{i_n}$

$$\begin{aligned} c\mathbf{x} + d\mathbf{y} &= c(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n) + d(d_1\mathbf{x}_1 + \cdots + d_n\mathbf{x}_n) \\ &= (cc_1 + dd_1)\mathbf{x}_1 + \cdots + (cc_n + dd_n)\mathbf{x}_n \in \text{span}(S). \end{aligned}$$

Hence span(S) is a subspace of V. Now, let W be any other subspace of V containing S. Then $\mathbf{x}_i \in W$ for all i. Since W is a subspace of V, all linear combinations of $\mathbf{x}_i$'s belong to W, and hence span$(S) \subseteq W$.

Example

1. The linear span of a single nonzero vector in $\mathbb{R}^3$ is a line passing through origin.
2. The linear span of two vectors $(1, 1, 0)$ and $(0, 0, 1)$ is a plane in $\mathbb{R}^3$ passing through origin.
3. Show that linear span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^3$.
4. Show that linear span $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 5 \end{bmatrix} \right\} \neq \mathbb{R}^3$.

Example

Let S be a subset of the vector space $\mathbb{R}^3$ defined by $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$. Show that $\mathbf{x} = \begin{bmatrix} -4 \\ 4 \\ 6 \end{bmatrix}$ is in the span(S).

Solution. To determine if $\mathbf{v}$ is in the span(S), we consider the equation

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 6 \end{bmatrix}.$$

Solving these equations we get, $c_1 = -2, c_2 = 1, c_3 = -1$.