Lecture 12,13

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October 8, 2021

Dimension

Definition

The *dimension* of a vector space V, denoted by $\dim V$, is the number of vectors in any basis of V.

Example

$\dim \mathbb{R}^n = n$, $\dim \mathbb{R}^{2\times 2} = 4$, $\dim \mathbb{R}^{m\times n} = mn$, $\dim \mathbb{R}_n[x] = n+1$.

Definition

A non-zero vector space V is called *finite dimensional* vector space if it contains a finite set of vectors that forms a basis of V. Otherwise it is called *infinite dimensional* vector space.

Example

1. $\mathbb{R}^n, \mathbb{R}^{m\times n}, \mathbb{R}_n[x]$ are finite dimensional.
2. Let $\mathbb{R}(x)$ denotes the set of all polynomials with real coefficients. Then $\mathbb{R}[x]$ is a vector space under usual addition and scalar multiplication of polynomials and $\{1, x, x^2, \ldots\}$ is a basis of $\mathbb{R}[x]$. Hence $\mathbb{R}[x]$ is an infinite dimensional vector space.

Plus-Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

1. If S is a linearly independent set, and $\mathbf{v} \in V$ such that $\mathbf{v} \notin \operatorname{span}(S)$, then the set $S \cup \{\mathbf{v}\}$ is still linearly independent.
2. If $\mathbf{v} \in S$ and $\mathbf{v} \in \operatorname{span}(S \setminus \{\mathbf{v}\})$, then $S \setminus \{\mathbf{v}\}$ spans the same space i.e. $\operatorname{span}(S) = \operatorname{span}(S \setminus \{\mathbf{v}\})$.

Proof. Exercise.

Theorem B

Let V be an n-dimensional vector space and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$. Then S is a basis for V if either S spans V or S is linearly independent.

Proof.

a). Let $\mathrm{span}(S) = V$. Suppose that S is linearly dependent. Then there exist a vector $\mathbf{v}_j \in S$ such that $\mathbf{v} \in \mathrm{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}\}$, i.e., $\mathbf{v}_j = c_1\mathbf{v}_1 + \cdots + c_{r-1}\mathbf{v}_{j-1}$. Then by Plus-Minus Theorem, $\mathrm{span}(S) = \mathrm{span}(S \setminus \{\mathbf{v_j}\})$. By continuing this process, we get a set $T \subset S$ such that T is linearly independent and $\mathrm{span}(S) = \mathrm{span}(T)$. Note that $|T| < n$ gives a contradiction to the fact that $\dim V = n$.

b). Let S be a linearly independent set and $\mathbf{v} \in V$. We claim that $\mathbf{v} \in \mathrm{span}(S)$. On the contrary, assume that $\mathbf{v} \notin \mathrm{span}(S)$. By Plus-minus theorem, we get $S \cup \{v\}$ is a linearly independent set. This is a contradiction to the fact that the number of elements in any linearly independent set is less or equal to dimension of the vector space.

Theorem C

Let S be a finite set of vectors in a finite dimensional vector space V.

1. If S spans V but not a basis, then S can be reduced to a basis for V by removing appropriate vectors from S.
2. If S is linearly independent set that is not a basis, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Proof. Exercise.

Theorem

Every subspace W of a finite dimensional vector space V is again finite dimensional and $\dim W \leq \dim V$. Moreover, if $\dim W = \dim V$, then $V = W$.

Sum of two Spaces

Definition

Let V be a vector space over $\mathbb{F}$ and W_1, W_2 subspaces of V. Then $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$ is called the sum of W_1 and $W2$.

Remark

- $W_1 + W_2$ is again a subspace of V.
- As $0 \in W_2$, W_1 be a subspace of $W_1 + W_2$. Similarly, W_2 be a subspace of $W_1 + W_2$.
- $W_1 + W_2$ is the smallest subspace which contains $W_1 \cup W_2$.
- If V is finite dimensional, then so $W_1 + W_2$.

Theorem

Let V be a finite dimensional vector space over $\mathbb{F}$, and W_1, W_2 be subspaces of V. Then
$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$*.*

Proof. Since V is finite dimensional, $W_1, W_2, W_1 \cap W_2$ are finite dimensional. Assume that $\dim(W_1) = m$, $\dim(W_2) = n$ and $\dim(W_1 \cap W_2) = r$. Note that $r \leq m$ and $r \leq n$. Let $B = \{v_1, \ldots, v_r\}$ be a basis for $W_1 \cap W_2$. Since B is linearly independent set in W_1, B can we extend to a basis for W_1, say $B_1 = \{v_1, \ldots, v_r, u_1, \ldots, u_{m-r}\}$. Similarly, $B_2 = \{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$ is a basis for W_2. We claim that $B_3 = \{v_1, \ldots, v_r, u_1, \ldots, u_{m-r}, w_1, \ldots, w_{n-r}\}$ be a basis for $W_1 + W_2$. Note that $|B_3| = m + n - r$. Hence to prove the theorem it is enough to prove the claim.
Proof of claim: Exercise

Consequence

Corollary

Let P_1 and P_2 be two planes passing through origin in $\mathbb{R}^3$. Then $\dim(P_1 \cap P_2) \geq 1$.

Proof. Since $\dim(P_1) = 2 = \dim(P_2)$ and $\dim(\mathbb{R}^3) = 3$, $2 \leq \dim(P_1 + P_2) \leq 3$. By the above theorem,

$$\dim(P_1 + P_2) = \dim(P_1) + \dim(P_2) - \dim(P_1 \cap P_2),$$

and hence $3 \geq 2 + 2 - \dim(P_1 \cap P_2)$. This gives that $\dim(P_1 \cap P_2) \geq 1$.

Definition

Let V be a vector space over $\mathbb{F}$ and W_1, W_2 subspaces of V. If $W_1 \cap W_2 = (0)$, then $W_1 + W_2$ is called the *direct sum* of W_1 and W_2 and it is denoted as $W_1 \oplus W_2$.

Note that since $W_1 \cap W_2 = (0)$, we have $\dim(W_1 \oplus W_2) = \dim(W_1) + \dim(W_2)$

Example

Let $V = \mathbb{R}^2$, $W_1 = \{(x, 0) : x \in \mathbb{R}\}$ and $W_2 = \{(0, y) : y \in \mathbb{R}\}$. Then $\mathbb{R}^2 = W_1 \oplus W_2$

Exercise

Let l_1 and l_2 be two distinct lines passing through origin in $\mathbb{R}^2$. Then show that $\mathbb{R}^2 = l_1 \oplus l_2$.