Lecture 6

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Vector Spaces

Theorem

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$*. Then the following conditions are equivalent:*

a) $\mathbf{A}$ *is an invertible matrix.*

b) *The matrix equation* $A\mathbf{x} = \mathbf{O}$ *has the unique solution* $\mathbf{x} = \mathbf{O}$*.*

c) *The reduced row-echelon form of* $\mathbf{A}$ *is* I_n*.*

An Application. Above Theorem can be used to find the inverse of an invertible matrix. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then by a sequence of elementary row operations, $\mathbf{A}$ can be transformed into the row-echelon form A'. Since $\mathbf{A}$ is an invertible matrix, $A' = I_n$. Thus $E_t E_{t-1} \cdots E_1 A = I_n$ which implies that $B = E_t E_{t-1} \cdots E_1 I_n$ is the inverse of $\mathbf{A}$. Applying elementary row operations on $n \times 2n$ matrix $\left[\begin{array}{c|c} A & I_n \end{array}\right]$, it can be transformed into $\left[\begin{array}{c|c} I_n & B \end{array}\right]$. Then B is the inverse of A.

Example

Compute the inverse of matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$.

Consider the matrix $[\mathbf{A}|I]$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & \frac{1}{2} & 1 \end{array}\right].$$

$$\xrightarrow{R_2 \to \frac{2}{5}R_2} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} \end{array}\right] \xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} \end{array}\right].$$

the inverse of the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ is $\begin{bmatrix} \dfrac{2}{5} & -\dfrac{1}{5} \\ -\dfrac{1}{5} & \dfrac{2}{5} \end{bmatrix}$.

Definition

An n-tuple of real numbers is an ordered list $(x_1, x_2, \ldots, x_n)$ of real numbers. For example $(1, 2, 4)$ is a 3-tuple.

Definition

The n-dimensional space $\mathbb{R}^n$ is the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The elements of $\mathbb{R}^n$ are called n-dimensional vectors or simply vectors, we often use boldface letters to denote vectors. The ith entry of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called the ith coordinate or ith component. For consistency with matrix operations, we denote the vector x by $(x_1, x_2, \ldots, x_n)$ or $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

The addition of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined to be the matrix addition of $n \times 1$ matrices

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \tag{1}$$

and multiplication of a vector $\mathbf{x} \in \mathbb{R}^n$ by a scalar $\lambda \in \mathbb{R}$ to be the multiplication of a matrix by scalar

$$\lambda \mathbf{x} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} \tag{2}$$

Then from properties of matrix addition and multiplication of a matrix by a scalar, we see that

1. Addition of vectors is commutative: $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$.
2. Addition of vectors is associative: $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$ for all $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^n$.
3. The vector $\mathbf{0}\in\mathbb{R}^n$ with all its entries 0 is called zero vector and $\mathbf{x}+\mathbf{0}=\mathbf{x}=\mathbf{0}+\mathbf{x}$ for each vector $\mathbf{x}\in\mathbb{R}^n$.
4. For each vector $\mathbf{x}\in\mathbb{R}^n$, there is vector $-\mathbf{x}\in\mathbb{R}^n$, called inverse of vector $\mathbf{x}$, such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}=(-\mathbf{x})+\mathbf{x}$.
5. For all vectors $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$, we have $\lambda(\mathbf{x}+\mathbf{y})=\lambda\mathbf{x}+\lambda\mathbf{y}$.
6. For each vector $\mathbf{x}\in\mathbb{R}^n$ and $\lambda,\mu\in\mathbb{R}$, we have $(\lambda+\mu)\mathbf{x}=\lambda\mathbf{x}+\mu\mathbf{x}$.
7. For each vector $\mathbf{x}\in\mathbb{R}^n$ and $\lambda,\mu\in\mathbb{R}$, we have $\lambda(\mu\mathbf{x})=(\lambda\mu)\mathbf{x}$.
8. For $1\in\mathbb{R}$, we have $1\mathbf{x}=\mathbf{x}$; for each vector $\mathbf{x}\in\mathbb{R}^n$.

The set $\mathbb{R}^n$ together with vector addition, scalar multiplication and satisfying the properties $(\mathbf{1}) - (\mathbf{8})$ given above is called a *vector space over* $\mathbb{R}$ (or *real vector space*).

We can extend the concept of a vector by using the basic properties of vectors in $\mathbb{R}^n$ as axioms, which if satisfied by a set of objects, then we say those objects are vectors.

Let $\mathbb{V}$ be a set of objects and $\mathbb{F}$ be a field. Then we say that $\mathbb{V}$ is a *vector space* over $\mathbb{F}$ if it satisfy the following:

1. Closure of addition: $\mathbf{x}+\mathbf{y} \in \mathbb{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$,
2. Commutativity of addition: $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$,
3. Associativity of addition: $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{V}$,
4. There exists $\mathbf{0} \in \mathbb{V}$ (*zero vector*) such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for all $\mathbf{x} \in \mathbb{V}$,
5. For $\mathbf{x} \in \mathbb{V}$, there exists $-\mathbf{x} \in \mathbb{V}$ (*inverse* of $\mathbf{x}$) such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$,
6. Closure of scalar multiplication: $\alpha \mathbf{x} \in \mathbb{V}$ for all $\mathbf{x} \in \mathbb{V}$ and $\alpha \in \mathbb{F}$,
7. For all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $\lambda \in \mathbb{F}$, $\lambda(\mathbf{x}+\mathbf{y})=\lambda \mathbf{x}+\lambda \mathbf{y}$,
8. For all $\mathbf{x} \in \mathbb{V}$ and $\lambda, \mu \in \mathbb{F}$, $(\lambda+\mu) \mathbf{x}=\lambda \mathbf{x}+\mu \mathbf{x}$,
9. For all $\mathbf{x} \in \mathbb{V}$ and $\lambda, \mu \in \mathbb{F}$, $\lambda(\mu \mathbf{x})=(\lambda \mu) \mathbf{x}$,
10. For $1 \in \mathbb{F}$, we have $1 \mathbf{x}=\mathbf{x}$; for all $\mathbf{x} \in \mathbb{V}$.

If $\mathbb{F}=\mathbb{R}$, then we say that $\mathbb{V}$ is a real vector space. If $\mathbb{F}=\mathbb{C}$, then we say that $\mathbb{V}$ is a complex vector space.

Examples

- $\mathbb{R}^n$ is a real vector space with usual addition of vectors and scalar multiplication by a real number.
- **Real vector space of infinite sequences of real numbers**. Let $\mathbb{V} = \{\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots) : x_i \in \mathbb{R}\}$. Define addition and scalar multiplication as follows. For $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n, \ldots)$

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n, \ldots), \\ k\boldsymbol{x} &= (kx_1, kx_2, \ldots, kx_n, \ldots). \end{aligned}$$

We denote this vector space by $\mathbb{R}^\infty$.

Vector Spaces of Matrices

- Let $\mathbb{V}$ be a set of 2×2 matrices with real entries. Then V is a vector space with addition

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} x_{11} + y_{11} & x_{12} + y_{12} \\ x_{21} + y_{21} & x_{22} + y_{22} \end{bmatrix}.$$

  and scalar (real) multiplications as
  $k\mathbf{x} = v \begin{bmatrix} kx_{11} & kx_{12} \\ kx_{21} & kx_{22} \end{bmatrix}$. Check all axioms.
- The vector space of all $m \times n$ matrices with real entries.

Non-example

A set that is not a vector space. Let $\mathbb{V} = \mathbb{R}^2$. Define addition and scalar multiplication as

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

and

$$k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ 0 \end{bmatrix}.$$

Check that $\mathbb{V}$ is not a vector space.

K=1