Ex Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$u_1 = \frac{x_1}{\|x_1\|} = \frac{(1, 0, 1)^t}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$w_2 = x_2 - \langle x_2, u_1 \rangle u_1$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \underline{\frac{1}{\sqrt{2}} \underbrace{(1\cdot 1 + 0\cdot 1 + 1\cdot 0)}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$u_2 = \frac{w_2}{\|w_2\|}, \quad \frac{\begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}^t}{\sqrt{\frac{1}{4} + 1 + \frac{1}{4}}} = \sqrt{\frac{2}{3}} \left(\frac{1}{2} \quad 1 \quad -\frac{1}{2} \right)^t$$

$$= \frac{1}{2} \cdot \sqrt{\frac{2}{3}} (1, 2, -1)^t$$

$$= \frac{1}{\sqrt{6}} (1, 2, -1)^t$$

$$w_3 = x_3 - \underline{\langle x_3, u_1 \rangle} u_1 - \underline{\langle x_3, u_2 \rangle} u_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \underline{\frac{1}{\sqrt{2}}(1)} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \underline{\frac{1}{\sqrt{6}}(1)} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{6} \\ 1 - 0 - 1/3 \\ \end{bmatrix}$$

$$\begin{pmatrix} \\ 1 \end{pmatrix} \quad 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{}{6}\begin{pmatrix} \\ \end{pmatrix} = \begin{bmatrix} 1-0-\frac{1}{3} \\ 1-\frac{1}{2}+\frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$= \frac{2}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{\cancel{\frac{2}{3}}(-1,1,1)^t}{\cancel{\frac{2}{3}}\sqrt{3}} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

R,

$$x_1 = \sqrt{2}\, u_1$$

$$u_1 = \frac{1}{\sqrt{2}}\underline{(1,0,1)}^t = \frac{1}{\sqrt{2}} x$$

$$x = \sqrt{2}\, u_1$$

$$u_2 = \frac{x_2 - \langle x_2, u_1\rangle u_1}{\| x_2 - \langle x_2, u_1\rangle u_1 \|}$$

$$x_2 = \underline{\| x_2 - \langle x_2, u_1\rangle u_1 \|}\, u_2 + \langle x_2\, u_1\rangle\, u_1$$

$$x_2 = \|w_2\|\, u_2 + \langle \underline{x_2, u_1} \rangle\, u_1$$

$$x_2 = \sqrt{\frac{3}{2}}\, u_2 + \underline{\frac{1}{\sqrt{2}}}\, u_1$$

$$x_3 = \|w_3\|\, u_3 + \langle \underline{x_3, u_1} \rangle\, u_1 + \langle \underline{x_3, u_2} \rangle\, u_2$$

$$= \frac{2}{\sqrt{3}} u_3 + \frac{1}{\sqrt{2}} u_1 + \frac{1}{\sqrt{6}} u_2$$

$$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

You can verify that $\boxed{QR = A}$

Let A be an invertible matrix of size $n \times n$ and $b \in \mathbb{R}^n$. Then we know that $Ax = b$ has a solution.

By the QR-decomposition $\exists$ an orthogonal matrix Q and upper triangular matrix R s.t.

$$A = QR$$

$$QRx = b$$

pre multiply both side by Q^T

$$\Rightarrow \quad Rx = Q^T b$$

↳ upper triangular matrix with all the diagonals are +ve

Using the Gauss elimination (or by back substitution) we will get a solution of the given problem.

<u>Fact</u> Let A be an $m \times n$ matrix with all the columns are L.I. Then $\exists$ a matrix Q whose columns forms

an orthonormal basis of col. $(A)^{m\times n}$ and an upper triangular invertible matrix R s.t. $QR = A$.

Let $Ax = b$ be a system.

using QR Decomposit

$\Rightarrow QRx = b$

Suppose A is $n\times n$

$$Q^T Q R x = Q^T b$$

$$Rx = Q^T b$$

$$\Rightarrow x = \boxed{R^{-1} Q^T b}$$

Question Is $R^{-1}Q^T b$ gives solution to $Ax = b$

Let $Ax = b$, $A = QR$

$$Ax = QRx = QRR^{-1}Q^T b$$

$$= QQ^T b$$

Answer $R^{-1}Q^T b$ is the solution of $Ax = b$ iff $QQ^T = b$.

Question Does $R^{-1}Q^T b$ give the least square solution to $Ax = b$.

To Answer the above question we need to understand $QQ^T b$

$Q = [u_1 \ldots u_n]$, where $\{u_1, \ldots u_n\}$ is an orthonormal set (forms an orthonormal basis col(A))

$$Q^T = \begin{bmatrix} u_1^t \\ u_2^t \\ \vdots \\ u_n^t \end{bmatrix}$$

$$QQ^Tb = [u_1 \ \cdots \ u_n] \underbrace{\begin{bmatrix} u_1^t \\ u_2^t \\ \vdots \\ u_n^t \end{bmatrix}}_{n\times m} b_{m\times 1}$$

$$= [u_1 \ \cdots \ u_n] \begin{bmatrix} u_1^t b \\ u_2^t b \\ \vdots \\ u_n^t b \end{bmatrix}$$

$$= [u_1 \ \cdots \ u_n] \begin{bmatrix} \langle b, u_1 \rangle \\ \langle b, u_2 \rangle \\ \vdots \\ \langle b, u_n \rangle \end{bmatrix}$$

$$= \langle b, u_1 \rangle u_1 + \langle b, u_2 \rangle u_2 + \cdots + \langle b, u_n \rangle u_n$$

$$= \operatorname{Proj}_{col(A)}(b)$$

$\Rightarrow$ $R^{-1}Q^Tb$ gives the least square solution to $Ax = b$

Let A be $m \times n$ matrix of rank r. Does there exist a matrix Q whose columns are orthonormal and a matrix R st $A = QR$.

$$[x_1 \ x_2 \ \cdots \ x_n] = Q_{m\times r} \, R_{r\times n}$$

What can you

what can you about R

(1) Does there exist such Q and R

(2) What we can say about R.