Department of Mathematics
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CSD001P5M **Linear Algebra** **Tutorial: 05**

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1. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of vector space V over $\mathbb{F}$ and $0 \neq \alpha \in \mathbb{F}$. Consider $T_1 = \{v_1 + \alpha v_2, v_2, \ldots, v_n\}$, $T_2 = \{\alpha v_1, v_2, \ldots, v_n\}$. Then prove that $\text{span}(S) = \text{span}(T_1) = \text{span}(T_2)$.

2. V is a vector space with $\dim(V) = n$. W_1 and W_2 are subspaces of V such that $\dim(W_1) = \dim(W_2) = n-1$ and $W_1 \cap W_2 = \{0\}$. Find n?

3. If U and W are subspaces of the vector space V, then $V = U \oplus W$ (i.e, every element $v \in V$ is uniquely expressible as $v = u + w$, where $u \in U$ and $w \in W$) if and only if $V = U + W$, and $U \cap W = \{0\}$.

4. Given the vector space $\mathbb{R}^3$, let W_1 be the set of vectors of the form $(x, y, 0)$ and let W_2 be the set of vectors of the form $(0, a, b)$.

   (a) Show that W_1 and W_2 are subspaces of $\mathbb{R}^3$.

   (b) Find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

   (c) Find two distinct subspaces U_1 and U_2 of $\mathbb{R}^3$ such that $\mathbb{R}^3 = W_1 \oplus U_1 = W_1 \oplus U_2$. Justify your answer.

5. Suppose that X, Y and Z are subspaces of V. Then prove that $X + Y + Z$ is a direct sum if and only if $X \cap Y = \{0\}$ and $Z \cap (X + Y) = \{0\}$.

6. Extend the following sets to a basis of $\mathbb{R}^4$:

   a) $\{(1,0,1,0), (0,-1,1,0)\} \subset \mathbb{R}^4$

   b) $\{(1,1,1,1), (1,2,1,2)\} \subset \mathbb{R}^4$.

7. Extend the set $\{(3,-1,2)\}$ to two different bases for $\mathbb{R}^3$.

8. (a) Let $U = \{f(t) \in \mathbb{R}_4[t] \mid f(6) = 0\}$. Find a basis of U.

   (b) Extend the basis in part (a) to a basis of $\mathbb{R}_4[t]$.

   (c) Find a subspace W of $\mathbb{R}_4[t]$ such that $\mathbb{R}_4[t] = U \oplus W$.

9. (a) Let $U = \{f(t) \in \mathbb{R}_4[t] \mid \int_{-1}^{1} f(t)dt = 0\}$. Find a basis of U.

   (b) Extend the basis in part (a) to a basis of $\mathbb{R}_4[t]$.

(c) Find a subspace W of $\mathbb{R}_4[t]$ such that $\mathbb{R}_4[t] = U \oplus W$.

10. Let $V = \mathbb{F}^{2\times 2}$ be the vector space of all 2×2 matrices over the field $\mathbb{F}$. Let W_1 be the set of matrices of the form
$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$
and let W_2 be the set of matrices of the form
$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$
   (a) Prove that V has dimension 4 by exhibiting a basis for V which has four elements.
   (b) Prove that W_1 and W_2 are subspaces of V.
   (c) Find dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

11. Prove that $\{v_1, v_2, v_3, v_4\}$ basis for V if and only if $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$ also a basis for V.

12. Given the matrix A below:
$$A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 5 \\ 13 & 39 & 17 \end{bmatrix}$$
   (a) Find a basis for each of the spaces Nul A, Col A and Row A.
   (b) Find a basis for Row A consisting of rows of the given matrix A. This should be different from the one given in part (a).
   (c) Is A invertible? Justify your answer.

13. Given the matrix A and B below:
$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$
   (a) Find a basis for the row space of A and a basis for the row space of B. You must show your calculations.
   (b) Let $U = \text{Span}\,\{(1,2,-1,3),(2,4,-1,2),(3,6,3,-7)\}$ and
   let $W = \text{Span}\,\{1,2,-4,11),(2,4,-5,14)\}$. Is $U = W$? Justify your answer.

14. Given any $m\times n$ matrix A, show that $\text{rank}(A) \le \min\{m,n\}$. Give a non-trivial example in which equality is achieved, and a non-trivial example in which strict inequality holds.

15. Given any two $m\times n$ matrices A and B, prove that $\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$. Give a non- trivial example in which equality is achieved, and a non-trivial example in which strict inequality holds.