Lecture 7

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September 21, 2021

Vector Spaces

Non-example

- A set that is not a vector space. Let $\mathbb{V} = \mathbb{R}^2$. Define addition and scalar multiplication as

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

  and

$$k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ 0 \end{bmatrix}.$$

  Check that $\mathbb{V}$ is not a vector space.

- Let $\mathbb{R}'_n[x]$ = set of polynomials of degree n with 0 polynomial. Then $\mathbb{R}'_n[x]$ is not a vector space.

More Example

- Let $V = \mathbb{R}^{>0}$ with operations

$$u + v = uv$$

and

$$ku = u^k,$$

where $k \in \mathbb{R}$. Then V is a real vector space.

- Let V be set of all functions from $\mathbb{R}$ to $\mathbb{R}$ with the following operations: for $f, g \in V$ and $\alpha \in \mathbb{R}$

$$(f+g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x)$$

- Let V be set of all bijective functions from $\mathbb{R}$ to $\mathbb{R}$ with operation defined above. Is V a vector a space?
- Let $V = \{a\}$ be set containing one object. Then define operations such that V is a real vector space.

Examples

- Let V be line in $\mathbb{R}^2$ with usual addition and multiplication. Is V a vector space?
- Let V be plane in $\mathbb{R}^3$ with usual addition and multiplication. Is V a vector space?
- Let V be line in $\mathbb{R}^2$ passing through origin with usual addition and multiplication. Is V a vector space?
- Let V be plane in $\mathbb{R}^3$ passing through origin with usual addition and multiplication. Is V a vector space?
- Let $\mathbf{A}$ be a matrix and S is the set of solution of equation $\mathbf{Ax} = \mathbf{O}$. Is S a vector space over $\mathbb{R}$?

Theorem

Let V be a real (complex) vector space and $\mathbf{x} \in V, c \in \mathbb{F}$. Then

1. $0 \odot \mathbf{x} = \mathbf{0}$
2. $c \odot \mathbf{0} = \mathbf{0}$
3. $(-1) \odot \mathbf{x} = -\mathbf{x}$
4. *If $c \odot \mathbf{x} = \mathbf{0}$, then either $c = 0$ or $\mathbf{x} = \mathbf{0}$.*

Proof.

1. By axiom 8 in definition of vector space ($(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$ for all $\lambda, \mu \in \mathbb{F}$ and $\mathbf{x} \in \mathbb{V}$), we have $0 \odot \mathbf{x} = (0 + 0) \odot \mathbf{x} = (0 \odot \mathbf{x}) \oplus (0 \odot \mathbf{x})$. Adding the inverse $-(0 \odot \mathbf{x})$ on both sides

$$\begin{aligned} \mathbf{0} &= ((0 \odot \mathbf{x}) \oplus -(0 \odot \mathbf{x})) \oplus (0 \odot \mathbf{x}) \\ &= \mathbf{0} \oplus (0 \odot \mathbf{x}) = (0 \odot \mathbf{x}). \end{aligned}$$

2. By axiom 7 in definition of vector space ($\lambda(\mathbf{x}+\mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$ for all $\lambda \in \mathbb{F}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{V}$), we have

$$c \odot \mathbf{0} = c \odot (\mathbf{0} \oplus \mathbf{0}) = (c \odot \mathbf{0}) \oplus (c \odot \mathbf{0}).$$

   Adding the inverse $-(c \odot \mathbf{0})$ gives the desired result.

3. Exercise.

4. Let $c \odot \mathbf{x} = \mathbf{0}$. If $c = 0$, then the conclusion holds. Suppose $c \neq 0$. Then multiply both sides of $c \odot \mathbf{x} = \mathbf{0}$ by $\frac{1}{c}$ and apply part 2 of this theorem to obtain $\frac{1}{c} \odot (c \odot \mathbf{x}) = \frac{1}{c} \odot \mathbf{0} = \mathbf{0}$. By axiom 9, we get $1 \odot \mathbf{x} = \mathbf{0}$ and apply axiom 10 to obtain $\mathbf{x} = \mathbf{0}$.