What is Spectral theorem

Let A be a symmetric matrix. Then $\exists$ orthogonal matrix P, diagonal matrix D s.t.

$$\boxed{A = PDP^T}$$

Question Can we decompose $A = PDQ$ where P, Q are orthogonal matrix?

To Answer the above question, we need to understand Singular Value decomposition (SVD)

Let A be an $m \times n$ matrix. Then A^TA is a symmetric matrix of size $n \times n$.

$\Rightarrow$ all the eigenvalues of A^TA are real say $\lambda_1, \lambda_2, \ldots, \lambda_n$, and $\{v_1, \ldots, v_n\}$ be corresponding orthonormal eigenvectors of A^TA.

Let $Q(x) = x^T A^T A x$ be a quadratic form.

$= (Ax)^T Ax$

$$= (Ax)^T Ax$$
$$= \|Ax\|^2 \geq 0$$
$\Rightarrow q(x)$ is +ve semidefinite.
$\Rightarrow$ all the eigenvalues of A^TA are non-ve

$$\|Av_i\|^2 = (Av_i)^T Av_i$$
$$= v_i^T \underline{A^TAv_i}$$
$$= v_i^T (\lambda_i v_i)$$
$$= \lambda_i \cdot \underline{v_i^T v_i} = \lambda_i \cdot \underline{\|v_i\|^2}$$
$$= \lambda_i \cdot 1 = \lambda_i$$

$$\Rightarrow \boxed{\|Av_i\| = \sqrt{\lambda_i}}$$

Def Singular value: singular values of A are the square root of the eigenvalues of A^TA, denoted by $\sigma_1, \ldots, \sigma_n$ and arrange in the descending order $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$

and $\sigma_i = \sqrt{\lambda_i}$ $\forall\, i = 1, \ldots, n$.

Remark: The singular values of A are the length of $Av_1, \ldots, Av_n$, where $\{v_1, \ldots, v_n\}$ are orthonormal eigenvectors of A.

Ex

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 16+64 & 44+56 & 56-16 \\ 44+56 & 121+49 & 154-14 \\ 56-\cancel{7}6 & 154-14 & 196+4 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Eigenvalues of $A^T A$

$$\begin{vmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (80-\lambda)\left((170-\lambda)(200-\lambda) - 19600\right)$$

$$-100\left(20000 - 100\lambda - 5600\right)$$

$+ 40(14000 - 6800 + 40\lambda) = 0$

You can solve it.

$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

eigenvalues $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(4-\lambda) - 4 = 0$

$\Rightarrow \cancel{4} - 5\lambda + \lambda^2 - \cancel{4} = 0$

$\lambda = 0, 5$

eigenvector for $\lambda = 5$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

normalized eigenvector $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

eigenvector for $\lambda = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

eigenvect $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

normalized $\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$

$$A v_1 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}} + \frac{2 \cdot 2}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{5}} = \underline{\sqrt{5}}$$

$A v_2 = 0$

$$\begin{bmatrix} \sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{5} & 2\sqrt{5} \end{bmatrix} = \sqrt{5} \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$= \sqrt{5} \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$= \sqrt{5}\,[1 \quad 2]$

Thm Suppose $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors $A^T A$ corresponding. So that the eigenvalues $A^T A$ are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and suppose that $A^T A$ has r non zero eigenvalues.

Then show that $\{Av_1, \ldots, Av_r\}$ forms an orthogonal basis of Col space of A and $\operatorname{rank}(A) = r$

Pf Fix $i \neq j$. Then $v_i \perp v_j$

$\Rightarrow v_i^t \cdot v_j = 0$ ✓

Want to prove $Av_i \perp Av_j$

$\Rightarrow (Av_i)^T A v_j = 0$

$$(Av_i)^T (Av_j) = \left(v_i^T A^T A v_j\right)$$

$$= v_i^T \cdot \lambda_j v_j = \lambda_j \underline{v_i^T v_j} = 0$$

$\Rightarrow \{Av_1, \ldots, Av_n\}$ is an orthogonal set.

Since $\|Av_i\| = \sqrt{\lambda_i} = \sigma_i$ (From the Previous result) and $\alpha_i \neq 0 \ \forall\ 1 \le i \le r$, $\boxed{A v_i \neq 0 \ \forall\ 1 \le i \le r}$ — (*)

$\Rightarrow$ $\{Av_1, \ldots, Av_r\}$ is an orthogonal set and hence $\{Av_1, \ldots, Av_r\}$ is L.I set.

Let $y \in \text{Col}(A)$

$\Rightarrow$ $y = Ax$

Since $x \in \mathbb{R}^n$ and $\{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$

$$x = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$$

$$y = Ax = c_1 Av_1 + c_2 Av_2 + \cdots + c_n Av_n$$

$$= c_1 Av_1 + c_2 Av_2 + \cdots c_r Av_r + c_{r+1} Av_{r+1} + \cdots + c_n Av_n$$

$\shortparallel$ 0

(using (*))

$$y = c_1 Av_1 + c_2 Av_2 + \cdots c_r Av_r$$

$\Rightarrow$ $y \in \text{Span}(\underline{Av_1, \ldots, Av_r})$

$\Rightarrow$ $\{Av_1, \ldots, Av_r\}$ is a orthogonal basis of $\text{Col}(A)$

and $\text{rank}(A) = \dim C(A) = r$