Lecture 2

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Matrices and System of Linear Equations

Properties of product

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be matrices and $\alpha \in \mathbb{R}$. Then one can easily see that $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$, $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ and $(\alpha\mathbf{A})\mathbf{B} = \alpha(\mathbf{AB}) = \mathbf{A}(\alpha\mathbf{B})$ provided sums and products are well defined.

Theorem

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$ and $\mathbf{C} \in \mathbb{R}^{p\times q}$. Then $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

Proof. Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{ij}]$ and $\mathbf{C} = [c_{ij}]$. Suppose $(\mathbf{AB})\mathbf{C} = [\alpha_{ij}]$ and $\mathbf{A}(\mathbf{BC}) = [\beta_{ij}]$. Now,

$$\alpha_{ij} = \sum_{l=1}^{p}\left(\sum_{k=1}^{n} a_{ik}b_{kl}\right)c_{lj} = \sum_{k=1}^{n} a_{ik}\left(\sum_{l=1}^{p} b_{kl}c_{lj}\right) = \beta_{ij} \forall i, j.$$

Theorem

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$. Then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Matrix multiplication revisited

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} = \mathbb{R}^{n\times p}$. Let $\mathbf{a}_i = [a_{i1}, \ldots, a_{in}]$ be the ith row of $\mathbf{A}$ and $\mathbf{b}_j = \begin{bmatrix} b_{1j}, \\ \vdots \\ b_{mj} \end{bmatrix}$ jth column of $\mathbf{B}$. Then $\mathbf{AB} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \ldots & \mathbf{Ab}_n \end{bmatrix}$

and

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1\mathbf{B} \\ \mathbf{a}_2\mathbf{B} \\ \vdots \\ \mathbf{a}_m\mathbf{B} \end{bmatrix}$$

Summary: jth column of $\mathbf{AB} = \mathbf{A}[j$th column vector of $\mathbf{B}]$, and
ith row of $\mathbf{AB} = [i$th row vector of $\mathbf{A}]\mathbf{B}$

Product of matrices as a linear combination

Let $\mathbf{A}_1, \ldots, \mathbf{A}_r \in \mathbb{R}^{m \times n}$ and $c_1, \ldots, c_r \in \mathbb{R}$. Then an expression is of the form $c_1\mathbf{A}_1 + \cdots + c_r\mathbf{A}_r$ is called a *linear combination* of $\mathbf{A}_1, \ldots, \mathbf{A}_r$ and $c_1, \ldots, c_r$.

Example

$$1\begin{bmatrix} 2 \\ -4 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 5 \\ -4 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Theorem

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is $n \times 1$. Then $\mathbf{AB}$ can be written as linear combination of columns of $\mathbf{A}$ in which scalars are the entries of $\mathbf{B}$.

Thus, columns of product of matrices $\mathbf{AB}$ can be seen as linear combination of columns of $\mathbf{A}$ with coefficients are the entries of B.

Example

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 \\ 0 & -1 & 3 \\ 2 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 \\ 8 & -4 & 26 \end{bmatrix}$$

. Also note that

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Inverse of a square matrix

Let $\mathbf{A}$ be a square matrix of size n. Then $\mathbf{A}$ is called *invertible* if there exists a square matrix $\mathbf{B}$ of size n such that $\mathbf{AB} = I = \mathbf{BA}$, and B is called an *inverse* of $\mathbf{A}$, where I is the identity matrix.

Example

Let $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$. Then $\mathbf{A}$ is invertible, since for matrix $\mathbf{B} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/3 \end{bmatrix}$ we have $\mathbf{AB} = I = \mathbf{BA}$.

Now. let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Let $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\mathbf{AB} = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} \neq I$ for all $a, b, c, d \in \mathbb{R}$. Hence A is not invertible.

Properties of Inverse

Proposition

Let $\mathbf{A}$ be an invertible matrix. Then $\mathbf{A}$ has a unique inverse.

Proof. Suppose $\mathbf{A}$ has two inverse, say $\mathbf{B}$ and $\mathbf{C}$. Now

$$\mathbf{B} = I\mathbf{B} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{C}I = \mathbf{C}.$$

Hence $\mathbf{A}$ has a unique inverse.

If $\mathbf{A}$ is invertible, then inverse of A is denoted by $\mathbf{A}^{-1}$.

Proposition

Let $\mathbf{A}$ be a square matrix. Then $\mathbf{A}$ is invertible if and only $\mathbf{A}^T$ is invertible. Further, $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$.

Proposition

Let $\mathbf{A}$ and $\mathbf{B}$ be invertible matrices. Then $\mathbf{AB}$ also invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Linear equation

- A line in xy-coordinate system can be represented by an equation $ax + by = c$, where $a, b, c \in \mathbb{R}$ and both a, b are not zero.
- A plane in xyz-coordinate system can be represented by an equation $ax + by + cz = d$, where $a, b, c, d \in \mathbb{R}$ and a, b, c all are not zero.

Note that equation of line is a linear equation in two variable, and equation of plane is a linear equation in three variable. More generally, a *linear equation* in n-variable is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_i \in \mathbb{R}$ for all i, $b \in \mathbb{R}$ and atleast one of a_i should be nonzero. In the above equation, if $b = 0$, then we say that *homogeneous linear equation* in n variables.

Examples.

- $2\sqrt{x} + 3y - 4z = 5$ is not a linear equation.
- $\sin(x) = y$ is not a linear equation.
- $x^2 + y = 0$ is not a linear equation.
- $2x_1 + 3x_4 + 5x_3 = 5$ is linear equation but not homogeneous.
- $x = y$ is homogeneous linear equation.

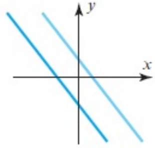
A finite set of linear equations is known as *linear system of equation*. A general system of linear equations of n-unknowns $x_1, \ldots, x_n$ can be written

$$\begin{array}{ccccc} a_{11}x_1 + & a_{12}x_2 & + \cdots + & a_{1n}x_n = & b_1 \\ a_{21}x_1 + & a_{22}x_2 & + \cdots + & a_{2n}x_n = & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 & + \cdots + & a_{mn}x_n = & b_m \end{array} \tag{1}$$

If $b_i = 0$ for all i, then the above system is known as *system of homogeneous linear equations.*

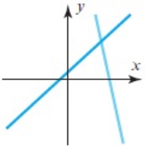
Solution of Linear system of equation

- A solution to the system of linear equation (1) in n-unknown is a sequence of numbers $s_1, s_2, \ldots, s_n$ such that substitution of $x_i = s_i$ for all i satisfies (1). This solution can be written as $(s_1, s_2, \ldots, s_n)$.
- Solution in two variable:



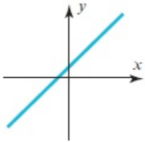
No solution.

$$x + y = 3$$
$$2x + 2y = 8$$



One solution.

$$x + y = 3$$
$$2x + y = 8$$

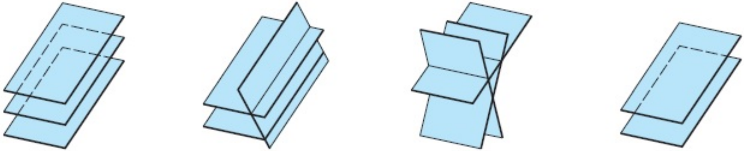


Infinite solutions.

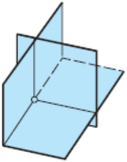
$$x + y = 3$$
$$2x + 2y = 6$$

Solution in Three variables

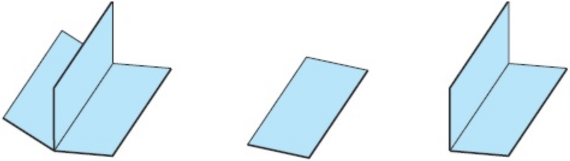
- No solution:



- One solution:



- Infinite solutions:



Question

How to solve a linear system in n-variables?

- Write system of linear equation (1) in matrix form $\mathbf{A}\mathbf{x} = \mathbf{b}$,where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

- The matrix $[A|b]$ is called *augmented* matrix associated to linear system (1).
- We need to define some operations on matrix to solve the linear system.

Two Variable

$$a_1X + b_1Y = c_1 \tag{2}$$

$$a_2X + b_2Y = c_2 \tag{3}$$

How to find the solution of above system?
Assume $a_1 \neq 0$. Multiply (2) equation by a_2/a_1 and subtract from (3).
We get

$$0X + \frac{a_1b_2 - a_2b_1}{a_1}Y = \frac{a_1c_2 - a_2c_1}{a_1}$$

We get a new system of equation

$$a_1X + b_1Y = c_1 \tag{4}$$

$$0X + \frac{a_1b_2 - a_2b_1}{a_1}Y = \frac{a_1c_2 - a_2c_1}{a_1} \tag{5}$$

The augmented matrix associated to original system is

$$\begin{bmatrix} a_1 & b_1 & : c_1 \\ a_2 & b_2 & : c_2 \end{bmatrix}$$

The augmented matrix associated to a new system is

$$\begin{bmatrix} a_1 & b_1 & : & c_1 \\ 0 & \frac{a_1 b_2 - a_2 b_1}{a_1} & : & \frac{a_1 c_2 - a_2 c_1}{a_1} \end{bmatrix}$$

Note that we can obtain the second matrix from the first matrix if we multiply the first row of first matrix by a_2/a_1 and subtract from 2nd row.

Elementary Row Operations

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. We can transformed the matrix $\mathbf{A}$ using the following process:

- Interchange ith and jth rows of A
- Multiply ith row of $\mathbf{A}$ by non-zero scalar
- Add a scalar multiple of ith row of $\mathbf{A}$ to jth row of $\mathbf{A}$.

These three operations are known as *elementary row operations*.

Example

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -3 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -9 & -4 \end{bmatrix}$$

Reduced Row Echelon Form

- The first non-zero entry of a non-zero row is called the *pivot*.
- A matrix $\mathbf{A}$ is said to be in *row reduced echelon form* if it satisfy the following:
  - The non-zero rows of $\mathbf{A}$ precede the zero rows of $\mathbf{A}$.
  - Suppose $\mathbf{A}$ has r non-zero rows and the pivot of ith row is in k_ith column. Then $k_1 < k_2 < \cdots < k_r$.
  - All pivots of $\mathbf{A}$ are equal to 1.
  - All the entries above pivots are zero.

Examples

- $$\begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{bmatrix}$$ Not in row echelon form.
- $$\begin{bmatrix} 0 & \boxed{1} & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Not in row echelon form
- $$\begin{bmatrix} 0 & \boxed{1} & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Row Reduced echelon form