29 November 2021 11:59

Def. Let U be a subset of V. Then the orthogonal complement of U, denoted by $U^\perp$, and defined as

$$U^\perp = \{ v \in V : \underline{\langle v, u \rangle} = 0 \ \forall u \in U \}$$

$$= \{ v \in V : v \perp u \ \forall u \in U \}$$

Ex $\mathbb{R}^2$ for fix α, $\quad U_\alpha = \{ (a, \alpha a) : a_1 \in \mathbb{R}, \alpha_1 \neq 0 \in \mathbb{R} \}$

$$U^\perp = \{ v : \langle v, u \rangle = 0 \}$$

$$= \{ (b_1, b_2) : (\langle b_1 b_2, a, \alpha a) = 0 \}$$

$$= \{ (b_1, b_2) : \underline{b_1 a + \alpha b_2 a = 0} \}$$

$$= \{ b(-\alpha, 1) : b \in \mathbb{R} \}$$

$U^\perp = \{(-\alpha, 1)$

$U^\perp = \{(-\alpha, 1) b \ \forall b \in \mathbb{R} \}$

Remark ① Let $U \subseteq V$ and $W = \operatorname{span} U$. Suppose B is basis of W. Then $B^\perp = W^\perp = U^\perp$ (Exer)

② Let $v, w \in U^\perp$. Then $\langle v, u \rangle = 0$

$\langle w, u \rangle = 0 \ \forall u \in U$

For $\alpha, \beta \in \mathbb{R}$, $\langle \alpha v + \beta w, u \rangle$

$$= \alpha \underline{\langle v, u \rangle} + \beta \underline{\langle w, u \rangle}$$

$$= \alpha \cdot 0 + \beta \cdot 0 = 0 \quad \forall u \in U$$

$$\Rightarrow \alpha v + \beta w \in U^\perp$$

Hence $U^\perp$ is subspace of V provided $U^\perp \neq \phi$

Since $\langle 0, u\rangle = 0 \ \forall\ u \in U$, $0 \in U^\perp$

$\Rightarrow U^\perp$ is a subspace of V.

③ $\{0\}^\perp = V$ (Exercise)

④ $V^\perp = \{v : \langle v, u\rangle = 0 \ \forall\ u \in V\}$

let $v \in V^\perp$, $\langle v, u\rangle = 0 \ \forall\ u \in V$

? In particular, $\langle v, v\rangle = 0 \xrightarrow{\quad} v = 0$ using ① cond^n.

$\|v\|^2 = 0$

$\Rightarrow \|v\| = 0$

$\Rightarrow v = 0$

$\Rightarrow V^\perp = \{0\}$

⑤ Let $U \subseteq V$. Then $U \cap U^\perp = \{0\}$

(Exercise)

⑥ Let $U \subseteq W$. Then $W^\perp \subseteq U^\perp$.

$W^\perp = \{v \in V : v \perp w \ \forall\ w \in W\}$

Let $v \in W^\perp \Rightarrow v \perp w \quad \forall\ w \in W$

In particular $v \perp w \quad \forall\ w \in U$

$\Rightarrow v \in U^\perp$

Hence $W^\perp \subseteq U^\perp$.

Ex ① Let A be an $m \times n$ matrix. Then

$$R(A) \perp N(A)$$

$R(A)$ = space spanned by rows of A.

$N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$ $\qquad A_{m \times n}$

Let $R_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^n$ be an ith row of A.

$$Ax = 0 \Rightarrow a_{i1}x_1 + \cdots a_{in}x_n = 0$$

$$\Rightarrow \langle R_i, \underline{x} \rangle = 0 \quad \forall \quad i = 1, \ldots, m$$

$\Rightarrow$ x is orthogonal to R_i $\forall$ i

$\Rightarrow$ x is orthogonal to row space of A.

Is it true $(R(A))^\perp = N(A)$

Ans Yes.

② $\mathcal{C}(A)^\perp = N(A^t)$

$(\mathcal{C}(A))^\perp = (R(A^t))^\perp = N(A^t)$

$\boxed{\text{Only true If the inner Product is Standard}}$

Ques. H Let U be a subset of V, where V is finite dimension inner product space. What is your guess of $\dim(U^\perp)$

2. Let U be a subspace of $\mathbb{R}^n$ what is

$\dim(U^{\perp}) = n - \dim(U)$ (from Rank - nullity Thm.)

How to find orthogonal complement of $U \subseteq \mathbb{R}^n$ with standard inner product.

① write the basis for U, say $\{u_1, \ldots, u_r\}$.

② Let $A = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$, where u_i's are written as Row vector.

③ find the null space A.

Ex Let $U = \{(\alpha, \alpha+\beta, \beta) : \alpha, \beta \in \mathbb{R}\}$

A basis for U is $\{(1,1,0), (0,1,1)\}$

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \end{bmatrix}$

$N(A) = \{(\alpha, -\alpha, \alpha) : \alpha \in \mathbb{R}\}$

Basis $\{(1,-1,1)\}$ for $N(A) = U^{\perp}$

$\Rightarrow U^{\perp} = \{(\alpha, -\alpha, \alpha) : \alpha \in \mathbb{R}\}$

Question $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

$\dim(U + U^{\perp}) = \dim(U) + \dim(U^{\perp})$

If this is true. $\boxed{V = U \oplus U^{\perp}}$ $U \cap U^{\perp} = \{0\}$

$\Rightarrow \quad \dim(U^{\perp}) = \dim(V) - \dim(U)$