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Ex Let $V = \mathbb{R}^4$ with standard inner product.

Convert $B = \{ \underset{v_1}{(1,-1,1,1)}, \underset{v_2}{(1,0,1,0)}, \underset{v_3}{(0,1,0,1)} \}$ L.I into Orthonormal set.

$$u_1 = \frac{(1,-1,1,1)}{\sqrt{1^2+(-1)^2+1^2+1^2}} = \frac{1}{2}(1,-1,1,1)$$

$$u_2 = \frac{w_2}{\|w_2\|}, \quad w_2 = v_2 - \langle v_2, u_1 \rangle u_1$$

$$= (1,0,1,0) - \frac{1}{2}(1.1 + 1.1)\frac{1}{2}(1,-1,1,1)$$

$$= (1,0,1,0) - \frac{1}{2}(1,-1,1,1)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}(1,1,1,-1)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{\frac{1}{2}(1,1,1,-1)}{\frac{1}{2}\sqrt{1^2+1^2+1^2+(-1)^2}} = \frac{1}{2}(1,1,1,-1)$$

$$u_3 = \frac{w_3}{\|w_3\|}, \quad w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$$

$$= (0,1,0,1) - 0 - 0$$

$$= (0,1,0,1)$$

$$u_3 = \frac{(0,1,0,1)}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}(0,1,0,1)$$

Ex 2. Let $V = \mathbb{R}_2[x]$, with inner product
$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$$

Find an orthonormal basis for $\mathbb{R}_2[x]$

Sol Let $\{1, x, x^2\}$ be a basis of $\mathbb{R}_2[x]$.

$$u_1 = \frac{v_1}{\|v_1\|}, \quad \|1\| = \sqrt{\int_{-1}^{1} 1\cdot 1\, dx} = \sqrt{x\Big|_{-1}^{1}} = \sqrt{1+1} = \sqrt{2}$$

$$u_1 = \frac{1}{\sqrt{2}}$$

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1$$

$$= x - \left(\int_{-1}^{1} x\cdot 1\, dx\right)\cdot 1$$

$$= x$$

$$\|w_2\|^2 = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$\Rightarrow \|w_2\| = \sqrt{\frac{2}{3}}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}\, x$$

Exercise: Find the third vector which is orthogonal to u_1 and u_2

Remark 1 Let V be a finite dimensional inner product space. Then V has orthonormal basis.

2. Let $B \subseteq V$ of orthonormal vectors in V. Then B can be extend to an orthonormal basis of V.

Let $B = \{u_1, \ldots, u_r\}$ be an orthonormal set.

$\Rightarrow$ B is L.I. Hence B can be extend to a a basis $B' = \{u_1, \ldots, u_r, f_1, \ldots, f_m\}$. Apply G.S. method, we will get $\{u_1, \ldots, u_r, g_1, \ldots, g_m\}$ a orthonormal basis.

(3) Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of V. For every $v \in V$, we have

$$v = \langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + \cdots + \langle v, v_n\rangle v_n$$

(4) Let $V = \mathbb{R}^n$ with standard inner product.

Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of V.

Think v_i as Row vector.

Let $A = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = B^T$

Let $B = \{e_1, \ldots, e_n\}$ be a standard basis

$v_1 = \alpha_{11} e_1 + \alpha_{21} e_2 + \cdots \alpha_{n1} e_n$, $v_1 = (\alpha_{11}, \alpha_{21}, \ldots \alpha_{n1})$

$\vdots$ $\quad v_2 = (\alpha_{12}\ \alpha_{22} \ldots \alpha_{n2})$

$v_i = \alpha_{1i} e_1 + \alpha_{2i} e_2 + \cdots \alpha_{ni} e_n$. $\quad v_i = (\alpha_{1i}\ \alpha_{2i} \ldots \alpha_{ni})$

$$\begin{bmatrix} [v_1]_B & [v_2]_B & \cdots & [v_n]_B \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & & \alpha_{nn} \end{bmatrix} = B$$

$$B^T B = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \vdots & & & \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & & \cdots & \alpha_{nn} \end{bmatrix}$$

$$\begin{bmatrix} \|v_1\|^2 & \langle v_1, v_2\rangle & \cdots & \langle v_1, v_n\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \|v_1\|^2 & \langle v_1, v_2\rangle & \cdots & \langle v_1, v_n\rangle \\ \langle v_2, v_1\rangle & \|v_2\|^2 & \cdots & \langle v_2, v_n\rangle \\ \vdots & & & \\ \langle v_n, v_1\rangle & \cdots & \cdots & \|v_n\|^2 \end{bmatrix}$$

As we know that $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal set

$$B^T B = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & \ddots & \\ \cdots & & & 1 \end{bmatrix} = I$$

$\Rightarrow \quad B^T B = I \quad , \quad A A^T = I$ $\boxed{\text{Think How to prove } A^T A = I}$

<u>Def</u> A square matrix A is said to be orthogonal matrix if $A A^T = I = A^T A$.

<u>Ex</u> of an orthonormal matrix

① $\begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix}$, $\begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}$

2. I.

<u>Question</u> Is an orthogonal matrix is invertible?

<u>Ans</u> Yes, $A^{-1} = A^T$

<u>Question?</u> Are there infinitely many or finite number of orthogonal matrix?

Answer <u>Yes</u>

We can construct infinitely many orthogonal matrix using Gram - Schmidt procedure.