Lecture 11

Rajiv Kumar
rajiv.kumar@iitjammu.ac.in

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Basis

Basis of a vector space

A subset B of a vector space V is a *basis* for V provided

1. B is linearly independent set of vectors in V
2. span$(B) = V$.

Examples

- The set $B = \left\{ \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.
- The set $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Example

1. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for the vector space $\mathbb{R}^{2\times 2}$.
2. Let W be the subspace of $\mathbb{R}^{2\times 2}$ of matrices with trace equal to 0, and let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$. Then S is a basis for W.
3. Let $\mathbb{R}_n(x)$ be the vector space of all polynomials of degree less than equals to n. Then $\{1, x, \ldots, x^n\}$ is a basis for $\mathbb{R}_n(x)$.
4. $B = \{x + 1, x - 1, x^2\}$ is a basis for $\mathbb{R}_2(x)$.

Theorem

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V and $0 \neq c \in \mathbb{F}$. Then $B_c = \{c\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis.

Proof. Let $\mathbf{v} \in V$. Since B is a basis, there exist scalars $c_1, \ldots, c_n$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. Note that $c \neq 0$. Thus, we can write

$$\mathbf{v} = \frac{c_1}{c}(c\mathbf{v}_1) + \cdots + c_n\mathbf{v}_n.$$

Hence $\mathbf{v}$ is a linear combination of the vectors in B_c. Therefore span$(B_c) = V$. To show that B_c is linearly independent, consider the equation $c_1(c\mathbf{v}_1) + \cdots + c_n\mathbf{v}_n = \mathbf{0}$. By axiom 9 of vector space we can write

$$(c_1c)\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Since B is a linearly independent set, only solution to above equation is trivial equation $cc_1 = 0, c_2 = 0, \ldots, c_n = 0$. Now since $c \neq 0$, we have $c_1 = 0$. Therefore, B_c is linearly independent.

Theorem A

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V and $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an independent set of V. Then $m \leq n$.

Proof. Suppose that $m > n$. Observe that, every vector in T can be written as a linear combination of the vectors from B. That is,

$$\begin{aligned}
\mathbf{u}_1 &= \lambda_{11}\mathbf{v}_1 + \lambda_{12}\mathbf{v}_2 + \cdots + \lambda_{1n}\mathbf{v}_n \\
\mathbf{u}_2 &= \lambda_{21}\mathbf{v}_1 + \lambda_{22}\mathbf{v}_2 + \cdots + \lambda_{2n}\mathbf{v}_n \\
&\vdots \\
\mathbf{u}_m &= \lambda_{m1}\mathbf{v}_1 + \lambda_{m2}\mathbf{v}_2 + \cdots + \lambda_{mn}\mathbf{v}_n
\end{aligned}$$

Now consider the equation $c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m = \mathbf{0}$. Substituting the value of $\mathbf{u}_i$'s from above and collecting like terms, we get

$$\begin{aligned}
&(c_1\lambda_{11} + c_2\lambda_{21} + \cdots + c_m\lambda_{m1})\mathbf{v}_1 \\
&+ (c_1\lambda_{12} + c_2\lambda_{22} + \cdots + c_m\lambda_{m2})\mathbf{v}_2 \\
&\qquad\vdots \\
&+(c_1\lambda_{1n} + c_2\lambda_{2n} + \cdots + c_m\lambda_{mn})\mathbf{v}_n \quad = \quad \mathbf{0}.
\end{aligned}$$

Since B is a basis, it is linearly independent, hence

$$\begin{aligned}
(c_1\lambda_{11} + c_2\lambda_{21} + \cdots + c_m\lambda_{m1}) &= 0 \\
(c_1\lambda_{12} + c_2\lambda_{22} + \cdots + c_m\lambda_{m2}) &= 0 \\
&\vdots \\
(c_1\lambda_{1n} + c_2\lambda_{2n} + \cdots + c_m\lambda_{mn}) &= 0.
\end{aligned}$$

Since $m > n$, the linear system has a nontrivial solution and hence T is linearly dependent, a contradiction. Hence $m \leq n$.

Corollary

Let V be a vector space with basis containing n elements. Then any two bases of V contain n elements.