Thm: suppose U is a finite dimensional v.s of V.

Thm $V = U \oplus U^{\perp}$

Pf We know that $U \cap U^{\perp} = \{0\}$. Hence it is enough to prove that $V = U + U^{\perp}$

Since U and $U^{\perp}$ are subspaces of V,

$U + U^{\perp} \subseteq V$.

We need to prove $\underline{V} \subseteq \underline{U + U^{\perp}}$

Let $v \in V$. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of U.

Let $u = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \cdots - \langle v, e_m \rangle e_m$

and $w = v - u \Rightarrow v = u + w$

Clearly $u \in U$, $w \in U^{\perp}$

For $i = 1, \ldots, m$,

$$\langle w, e_i \rangle = \langle v - u, e_i \rangle$$

$$= \langle v - \langle v, e_1 \rangle e_1, \ldots - \langle v, e_m \rangle e_m, e_i \rangle$$

$$= \langle v, e_i \rangle - \langle v, e_1 \rangle \underline{\langle e_1, e_i \rangle}$$

$$- - \langle v, e_i \rangle \underline{\langle e_i, e_i \rangle}$$

$$\cdots - \langle v, e_m \rangle \langle e_m, e_i \rangle$$

$$= \langle v, e_i \rangle - \langle v, e_i \rangle \cdot 1 = 0$$

$\Rightarrow \quad w \perp U \Rightarrow \quad w \in U^{\perp}$

which proves the result.

Cor. $\underline{1}$ $\quad \dim U^{\perp} = \dim(V) - \dim(U)$

2. Let A be an $m \times n$ matrix. Then

$$\mathbb{R}^n = \mathcal{R}(A) \oplus \mathcal{N}(A)$$

$$\mathbb{R}^m = \mathcal{C}(A) \oplus \mathcal{N}(A^T)$$

Let $Ax = b$ be system of linear eq.

The given is inconsistent iff $b \notin \mathcal{C}(A)$

But $b \in \mathbb{R}^m$. By previous thm.

$$b = b_{\mathcal{C}(A)} + b_{\mathcal{N}(A^T)}$$

The system is inconsistent iff $b_{\mathcal{N}(A^T)} \neq 0$

$$\begin{aligned} Ax &= b \\ &= b_{\mathcal{C}(A)} + b_{\mathcal{N}(A^T)} \end{aligned}$$

Multiply A^T on both sides

$$\begin{aligned} A^T A x &= A^T b \\ &= A^T b_{\mathcal{C}(A)} + \underbrace{A^T b_{\mathcal{N}(A^T)}} \end{aligned}$$

$\Rightarrow \boxed{A^T A x = A^T b_{\mathcal{C}(A)}}$ This system

$$\Rightarrow \boxed{A^T A x = A^T b_{C(A)}}$$ This system is consistent

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Q Let V be a finite dimensional inner product space and U be a subspace of V.

How to find $U^\perp$?

Ex Let $V = \mathbb{R}^2$, $\langle x, y \rangle = [x_1 \;\; x_2] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Let $U = \{(\alpha, \alpha) : \alpha \in \mathbb{R}\}$.

Find $U^\perp$.

Let $B = \{\underline{(1, 1),}\ (1, -1)\}$ be a basis of V.

$$\|(1,1)\|^2 = [1 \;\; 1] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$$

$$w_2 = (1, -1) - \langle (1, -1), (1, 1) \rangle (1, 1)$$

$$= (1, -1) - [1 \;\; -1] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} (1, 1)$$

$$= (1, -1) - [3 \;\; -2] \begin{bmatrix} 1 \\ -1 \end{bmatrix} (1, 1)$$

$$= (1, -1) - 5(1, 1) = (-4, -6)$$

$$\|w_2\|^2 = [-4 \;\; -6] \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} -4 \end{bmatrix}$$

$$\|w_2\|^2 = \begin{bmatrix} -4 & -6 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 \end{bmatrix} \begin{bmatrix} -4 \\ -6 \end{bmatrix} = 20$$

Orthonormal basis $= \{ (1,1), \frac{1}{\sqrt{20}}(-4,-6) \}$

$U^\perp = \{(1,1)\}^\perp = \{ \frac{1}{\sqrt{20}}(-4,-6)\alpha : \alpha \in \mathbb{R} \}$

Gram-Schmidt procedure

1. write the basis for U
2. Extend it to a basis for V
3. Apply Gram-Schmidt
4. write the basis for $U^\perp$

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another way / write a basis for U.
Apply the definition of orthogonal complement.

Thm Let $U \subseteq V$ (finite dim). Then $(U^\perp)^\perp = U$

Pf Let $u \in U$, $\langle u, v \rangle = 0 \ \forall \ v \in U^{\perp}$

$\Rightarrow u \in (U^{\perp})^{\perp}$

$\Rightarrow U \subseteq (U^{\perp})^{\perp}$

Now we need to prove

$(U^{\perp})^{\perp} \subseteq U$

Let $v \in (U^{\perp})^{\perp}$, $v \in V$.

$\Rightarrow v = u + w$, $u \in U$, $w \in U^{\perp}$

Since $u \in U$ and $U \subseteq (U^{\perp})^{\perp}$

$\Rightarrow u \in (U^{\perp})^{\perp}$

$\Rightarrow v - u \in (U^{\perp})^{\perp}$

$\Rightarrow w \in (U^{\perp})^{\perp}$

$\Rightarrow w \in \cap \ (U^{\perp})^{\perp} \cap U^{\perp}$

$\Rightarrow w = 0$

$\Rightarrow u - v = 0$

$\Rightarrow v = u$

$\Rightarrow v \in U$

and hence $(U^{\perp})^{\perp} \subseteq U$.

Orthogonal projection.

Let V be a finite dimensional v.s.

and $U \subseteq V$. Then $v \in V$, $v = u + w$,

$u \in U$, $w \in U^\perp$, Then a map

$P_U : V \to V$ defined as follows.

$P_U(v) = u$, is called the orthogonal

Projection of V on U.

Remark P_U is well defined.

Example Let V be a vector space and

$x \in V$, Take $U = \text{Span}\{x\}$.

Then $P_U(v) = \dfrac{\langle v, x \rangle x}{\|x\|^2}$

How we find for vector v, u and w

St $v = u + w$, and $u \perp w$