Lecture 17

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Linear Transformation

Definition

Let V and W be vector spaces.

- Then a mapping $T : V \to W$ is called *one to one (injective)* linear transformation, if T maps distinct vectors of V into distinct vector of W.
- Then a mapping $T : V \to W$ is called *onto (surjective)* linear transformation, if every vector in W is the image of at least one vector in V.

Theorem

If $T : V \to W$ is a linear transform, then T is one-one if and only if $\mathcal{N}(T) = \{\mathbf{0}\}$

This implies no other vector in V apart from 0 vector....maps to zero vector in W.

Proof. Assume that $\mathcal{N}(T) = \{\mathbf{0}\}$. If $u, v \in V$ such that $T(u) = T(v)$, then $T(u) - T(v) = \mathbf{0}$. Using linearity property of T, we get $T(u - v) = \mathbf{0}$. Since $\mathcal{N}(T) = \{\mathbf{0}\}$, $u - v = \mathbf{0}$, and hence $u = v$. Assume T is one-one.(Exercise)Health and medical

Let V and W be vector spaces.

- Let $T : V \to W$ be a one to one linear transformation. Then a mapping $T^{-1} : \mathcal{R}(T) \to V$ defined by $T^{-1}(\mathbf{w}) = \mathbf{v}$ if and only if $T(\mathbf{v}) = \mathbf{w}$ is called the *inverse* of T. If T is onto, then T^{-1} is defined on all of W.
- Then a linear transformation $T : V \to W$ that is both one to one and onto is called an isomorphism. In this case the vector spaces V and W are said to be isomorphic.

Doubt Note that T^{-1} is also a linear transformation.(Exercise.)

Proof ..by taking linear combination

Finite dimensional vector spaces

Theorem

Let V be an n-dimensional real vector space. Then V is isomorphic to $\mathbb{R}^n$.

Proof. Since V is n-dimensional, there exists a basis $\{v_1, \dots, v_n\}$ of V. If $v \in V$, there exist $c_1, \dots, c_n$ such that $v = c_1 v_1 + \cdots + c_n v_n$. Now we define a map $T : V \to \mathbb{R}^n$ as follows: $T(v) = (c_1, \dots, c_n)$.

linear T is linear(Exercise)

One-one We need to prove that $\mathcal{N}(T) = \{\mathbf{0}\}$. Suppose $T(v) = (0, \dots, 0)$. Then $v = 0v_1 + \cdots + 0v_n$. Hence $\mathcal{N}(T) = \{\mathbf{0}\}$.

Onto Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. Then take $v = c_1 v_1 + \cdots + c_n v_n$, and by definition, $T(v) = c$.

Coordinates

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis for the vector space V. Let $\mathbf{v}$ be a vector in V, and let $c_1, \ldots, c_n$ be the unique scalars such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. Then $c_1, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ relative to B. In this case we write $[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and refer to the vector $[\mathbf{v}]_B$ as the coordinate vector (matrix) of $\mathbf{v}$ relative to B.

Exercise

Let $V = \mathbb{R}_2[x]$ and $B = \{1, x-1, (x-1)^2\}$ be its basis. Then find the coordinates of $p(x) = 2x^2 - 2x + 1$ with respect to B.