Singular value decomposition

A matrix $m \times n$ is near to diagonal matrix is of the form

$$\Sigma = \begin{bmatrix} D_{r \times r} & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}$$

rank

$\downarrow m-r$

$\downarrow n-r$

where D is a diagonal matrix with all the diagonal elements are non zero.

Thm Let A be an $m \times n$ matrix with rank r
$\exists$ an $m \times n$ matrix Σ near to diagonal with all the entries are non-ve, i.e. all the entries of D are +ve. and an $m \times m$ orthogonal matrix U and $n \times n$ orthogonal matrix V s.t

$$A = \underset{m\times m}{U} \, \underset{m \times n}{\Sigma} \, \underset{n \times n}{V}$$

$U, V \rightarrow$ orthogonal

$\Sigma \rightarrow$ near to diagonal.

Further the entries of D are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$, where σ_i are singular value of A and rank$(A) = r$.

Remark ① Σ is unique
② U and V need not be unique.

Pf To find the singular values of A, we need to find eigenvalues of $A^T A$

Let $\underbrace{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r}\geq \underbrace{\cdots \geq \lambda_n} \geq 0$ be eigenvalues of $A^T A$ and $\{v_1, \ldots, v_n\}$ be corresponding orthonormal eigenvector of $A^T A$

$A v_i = 0$ iff $i > r$

$\|A v_i\| = \sqrt{\lambda_i} = \sigma_i \quad \forall\, 1 \leq i \leq r$

We know that $\{A v_1, \ldots, A v_r\}$ is an orthogonal set.

Let $u_i = \dfrac{A v_i}{\|A v_i\|} = \dfrac{1}{\sigma_i} A v_i \quad \forall\ i = 1, \ldots, r,$

Now $\{u_1, \ldots, u_r\}$ is an orthonormal set.

Extend $\{u_1, \ldots, u_r\}$ to an orthonormal basis of $\mathbb{R}^m$, say $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_m\}$

Let $U = [u_1 \ldots u_r\ u_{r+1} \ldots u_m]$

$V = [v_1 \ldots v_r\ v_{r+1} \ldots v_n]$

By construction U and V are orthonormal.

$$AV = [Av_1 \quad Av_2 \ \cdots \ Av_r \quad Av_{r+1} \ \cdots \ Av_n]$$

$$= [Av_1 \quad Av_2 \ \cdots \ Av_r \quad 0 \ \cdots \ 0] \longrightarrow (1)$$

$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \sigma_2 & & 0 \\ & & \ddots \ \sigma_r & \\ \hline & 0 & & 0 \end{array}\right]$$

$$U\Sigma = [u_1 \quad u_2 \ \cdots \ u_r \quad u_{r+1} \ \cdots \ u_m]\,\Sigma$$

$$= \left[\frac{Av_1}{\sigma_1} \quad \frac{Av_2}{\sigma_2} \ \cdots \ \frac{Av_r}{\sigma_r} \quad u_{r+1} \ \cdots \ u_m\right] \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots \ \sigma_r & \\ 0 & & 0 \end{bmatrix}$$

$$= \left[\sigma_1 \frac{Av_1}{\sigma_1} \quad \sigma_2 \frac{Av}{\sigma_2} \ \cdots \ \sigma_r \frac{Av_r}{\sigma_r} \quad 0 \ \cdots \ 0\right]$$

$$= [Av_1 \quad Av_2 \ \cdots \ Av_r \quad 0 \ \cdots \ 0]$$

$$= AV \quad (\text{using } (1)$$

$$U\Sigma = AV$$

By Post multiplying by V^T, we get

$$\boxed{A = U\Sigma V^T} \longrightarrow \text{Singular Value Decompo-}$$

Ex Let $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

Verify the eigenvalue of $A^T A$, are $\underline{\underline{18, 0}}$

eigenvector of $A^T A$ for $\lambda = 18$

$$\begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

vector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

orthonormal eigen vect $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

eigenvector for $\lambda = 0$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

orthonormal $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Singular values $= 3\sqrt{2}$

$$\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 3}$$

<u>To find</u>

$$A v_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$$

$$\frac{A v_1}{3\sqrt{2}} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Now we want to extend $\left\{ \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\}$ to an orthonormal basis of $\mathbb{R}^3$.

Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be another element.

Then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \perp \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

$\Rightarrow$ $\boxed{x - 2y + 2z = 0}$

$$\begin{bmatrix} 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—} \ (2)$$

The basis for solution set of (2) is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Apply the Gram-Schmidt procedure

$\{u_2, u_3\} \longrightarrow$ Exercise

$$U = \begin{bmatrix} 1/3 & & \\ -2/3 & u_2 & u_3 \\ 2/3 & & \end{bmatrix}$$

$$A = U \Sigma V$$