Lecture 8

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Sub Spaces

Subspaces

Definition

Let $(V, +, \cdot)$ be a real or complex vector space. A non-empty subset W of a vector space V is called a *subspace* of V if W is a vector space under the addition and scalar multiplication defined on V

Theorem

Let W be a non-empty subset of a vector space. Then W is a subspace if and only if the following conditions are satisfied.

a) *If $\mathbf{x}, \mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.*
b) *If $\mathbf{x} \in W$ and $\alpha \in \mathbb{F}$, then $\alpha\mathbf{x} \in W$.*

Remark

1. Let $\mathbf{v} \in W$. Then $0 \cdot \mathbf{v} = \mathbf{0} \in W$.
2. If W is a subspace of V, then W is a vector space with same operations as in V.

Examples.

1. The xy-plane in $\mathbb{R}^3$ given by $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$ is a vector space with same standard componentwise operations defined in $\mathbb{R}^3$.
2. Let $W = \left\{ \begin{bmatrix} a \\ a+1 \end{bmatrix} : a \in \mathbb{R} \right\}$ be subset of vector space $\mathbb{R}^2$ with standard componentwise addition and scalar multiplication. Check whether W is a subspace of V.
3. Let $M_{2\times 2}(\mathbb{R})$ be a vector space of 2×2 matrices with standard matrix addition and scalar multiplication, and let W be a subset of all 2×2 matrices with trace 0. Then W is a subspace of $M_{2\times 2}(\mathbb{R})$.
4. Let $\mathbf{A}$ be a matrix of size $m \times n$. Then set of solutions of $\mathbf{Ax} = \mathbf{O}$ with the operations defined on $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Theorem

Let W_1, W_2 be subspaces of a vector space V. Then $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. Let $W_1 \cup W_2$ is a subspace of V. Suppose $W_1 \not\subseteq W_2$. Then there exists $\mathbf{x} \in W_1$ such that $\mathbf{x} \notin W_2$. We claim that $W_2 \subseteq W_1$. Let $\mathbf{y} \in W_2$. Then $\mathbf{x} + \mathbf{y} \in W_1 \cup W_2$. This implies $\mathbf{x} + \mathbf{y} \in W_1$ or $\mathbf{x} + \mathbf{y} \in W_2$. If $\mathbf{x} + \mathbf{y} \in W_2$, then $\mathbf{x} = (\mathbf{x} + \mathbf{y}) + (-1)(\mathbf{y}) \in W_2$, a contradiction, and hence $\mathbf{x} + \mathbf{y} \in W_1$. Therefore, $\mathbf{y} = (\mathbf{x} + \mathbf{y}) + (-1)\mathbf{x} \in W_1$.

Remark

Union of two distinct straight lines passing through origin in $\mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$.

Exercise

The intersection of any two subspaces of a vector space V is always a subspace of V.

Linear Combination

Consider an ordered set $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ of vectors in a real (complex) vector space V. Then a vector of form

$$\mathbf{y} = (c_1 \cdot \mathbf{x}_{i_1}) + (c_2 \cdot \mathbf{x}_{i_2}) + \cdots + (c_m \cdot \mathbf{x}_{i_m}) \in V,$$

where $c_1, c_2, \ldots, c_m \in \mathbb{F}$, is called a *linear combination* of vectors in S.

Example

Let $V = \mathbb{R}^2$ be a real vector space with usual addition and scalar multiplication. Then $\begin{bmatrix} 4 \\ 10 \end{bmatrix}$ is a linear combination of vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Linear Span of Vectors

Definition

Let V be a vector space and let $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ be a set of vectors in V. Then the *linear span* of S, denoted by span(S), is the set

$$\{c_1\mathbf{x}_{i_1} + \cdots + c_m\mathbf{x}_{i_m} : c_1, \ldots, c_m \in \mathbb{R}\}.$$

Example

Let $V = \mathbb{R}^n$ and $S = \{e_1, \ldots, e_n\}$, where e_i denotes the element of $\mathbb{R}^n$ ith component is 1 and all other are zero. Then any vector of $\mathbb{R}^n$ can be written as a linear combination of vectors of S.