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Thm (Gram - Schmidt procedure) Let V be an inner product space and $\{v_1, \ldots, v_n\}$ be an L.I set in V. Let $u_1 = \frac{v_1}{\|v_1\|}$ and for $2 \le j \le n$,

$$u_j = \frac{v_j - \langle v_j, u_1\rangle u_1 - \langle v_j, u_2\rangle u_2 - \cdots - \langle v_j, u_{j-1}\rangle u_{j-1}}{\| v_j - \langle v_j, u_1\rangle u_1 - \langle v_j, u_2\rangle u_2 - \cdots - \langle v_j, u_{j-1}\rangle u_{j-1}\|}$$

Then $\{u_1, \ldots, u_n\}$ is an orthonormal set and

$$\operatorname{Span}\{v_1, \ldots, v_j\} = \operatorname{Span}\{u_1, \ldots, u_j\} \quad \forall\ 1 \le j \le n.$$

Pf We use induction on n. Suppose $n=1$

$\{v_1\}$ is a L.I set. , $u_1 = \frac{v_1}{\|v_1\|}$

Since $\{v_1\}$ is L.I, $v_1 \neq 0$ and hence $\|v_1\| \neq 0$

Verify $\|u_1\| = 1$, Since u_1 is a multiple v_1, and hence $\operatorname{Span}\{v_1\} = \operatorname{Span}\{u_1\}$

Assume that statement is true for $n-1$

(i) $\{u_1, \ldots, u_{n-1}\}$ is orthonormal set; i.e.,

(i) $\{u_1, \ldots, u_{n-1}\}$ is orthonormal set; i.e.,

$\langle u_i, u_j \rangle = 0$ for $i \neq j$, and

$\|u_i\| = 1$

(ii) $\text{Span}\{v_1, \ldots, v_j\} = \text{Span}\{u_1, \ldots, u_j\}$

$\forall \; 1 \leq j \leq n-1$

We want to prove the statement for n.

First aim $\{u_1, \ldots, u_n\}$ is an orthonormal set.

$$\langle u_n, u_i \rangle = \Big\langle u_n, \frac{\overbrace{u_n - \langle u_n, u_1\rangle u_1 - \cdots - \langle u_n, u_1\rangle u_{n-1}}^{w}}{\| w \|} \Big\rangle$$

$$= \quad \langle u_n, u_1 \rangle u_1 - \cdots - \langle u_n, u_{n-1}\rangle u_{n-1}$$

$$= \frac{1}{\|w\|} \{ \langle u_n, u_1 \rangle u - \langle u_n, u_{n-1}\rangle u_{n-1} \} \quad u_n$$

$$= \frac{1}{\|w\|} \langle u_n, u_n \rangle - \langle u_n, u_1\rangle \langle u_1, u_n \rangle - \cdots - \langle u_n, u_{n-1}\rangle \langle u_{n-1}, u_n \rangle$$

=

for $1 \leq i \leq n-1$ $\langle u_n, u_i \rangle$

$$= \langle \overbrace{v_n - \langle v_n, u_1\rangle u_1 - \langle v_n, u_2\rangle u_2}^{w}$$

$$\frac{\quad\cdots - \langle v_n, u_{n-1}\rangle u_{n-1}, u_i\rangle}{\|w\|}$$

$$= \frac{1}{\|w\|}\Big[\langle v_n, u_i\rangle - \langle v_n, u_1\rangle \underbrace{\langle u_1, u_i\rangle}_{0} - \langle v_n, u_2\rangle \underbrace{\langle u_2, u_i\rangle}_{0} \cdots \langle v_n, u_i\rangle \underbrace{\langle u_i, u_i\rangle}_{1} + \cdots - \langle v_n, u_{n-1}\rangle \langle u_{n-1}, u_i\rangle\Big]$$

$$= \frac{1}{\|w\|}\left[\langle v_n, u_i\rangle - 0 - 0 \cdots \langle v_n, u_i\rangle \cdot 1 \cdots 0\right]$$

$$= 0$$

$\Rightarrow \quad u_n \perp u_i \quad \forall \quad 1 \leq i \leq n-1$

From the definition of u_n, it is clear that $\|u_n\| = 1$

We want to prove

$$\text{Span}\{v_1, \ldots, v_j\} = \text{Span}\{u_1, \ldots, u_j\} \ \forall j.$$

From the hypothesis, we get

$$\text{Span}\{v_1, \ldots, v_j\} = \text{Span}\{u_1, \ldots, u_j\} \quad \forall \quad j \leq n-1$$

We need to prove $\text{Span}\{v_1,\dots,v_n\}$
$= \text{Span}\{u_1,\dots,u_n\}$.

Now $u_n = v_n - \underbrace{\langle v_n, u_1\rangle u_1 - \cdots - \langle v_n, u_{n-1}\rangle u_{n-1}}$

$v_n - u_n \in \text{Span}\{u_1,\dots,u_{n-1}\}$
$= \text{Span}\{v_1,\dots,v_{n-1}\}$

$\Rightarrow \quad u_n \in \text{Span}\{v_1,\dots,v_n\}$

$\Rightarrow \quad \text{Span}\{u_1,\dots,u_n\} \subseteq \text{Span}\{v_1,\dots,v_n\}$

IlLy $\quad \text{Span}\{v_1,\dots,v_n\} \subseteq \text{Span}\{u_1,\dots,u_n\}$
which proves the result.

Cor. Let V be a finite dimensional inner product space. Then $\exists$ a orthonormal basis for V.

Ex $\quad V = \mathbb{R}^2, \quad \langle x, y\rangle = [x_1 \ \ x_2]\begin{bmatrix} 2 & -1 \\ -1 & 1\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix}$

$\langle e_1, e_2\rangle = [1 \ \ 0]\begin{bmatrix} 2 & -1 \\ -1 & 1\end{bmatrix}\begin{bmatrix} 0 \\ 1\end{bmatrix}$

$= [2 \ \ -1]\begin{bmatrix} 0 \\ 1\end{bmatrix} = -1$

In this example e_1 is not orthogonal to e_2.

Let $B = \{e_1, e_2\}$.

$$\|e_1\| = \sqrt{[1, 0]\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \sqrt{[2 \ -1]\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$= \sqrt{2}$$

$$u_1 = \frac{e_1}{\sqrt{2}}$$

$$w_2 = e_2 - \langle e_2, u_1 \rangle u_1$$

$$= (0,1) - [0 \quad 1]\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{(1, 0)}{2}$$

$$= (0, 1) - [-1 \quad 1]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{(\cancel{1}, 0)}{2}$$

$$= (0, 1) + \frac{(1, 0)}{2} = \left(\frac{1}{2}, 1\right)$$

$$\|w_2\| = \left[\frac{1}{2}, 1\right]\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$= \sqrt{[0 \quad 1/2]\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(\frac{1}{2}, 1\right)}{\frac{1}{\sqrt{2}}} = \left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$$

$\{u_1, u_2\}$ is an orthonormal set

w.r.t $\langle x, y \rangle = (x_1, x_2]\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$V = \mathbb{R}^2$, $B = \{(1, 0), (1, 1)\}$

$V = \mathbb{R}^2$, $B = \{(1,0),(1,1)\}$

Apply the Gram Schmidt, you will get

$\{(1,0), (0,1)\}$

But $\{(1,1),(1,0)\}$, Then you will get.

$$\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}.$$

Let $B = \{(1,1),(a,b)\}$ s.t $b \neq a$.

Apply the Gram Schmidt method.

Orthonormal set $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}$.

upto sign

Please verify.