# Lecture 3

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# Gauss Elimination

# Reduced Row Echelon Form

- The first non-zero entry of a non-zero row is called the *pivot*.
- A matrix $\mathbf{A}$ is said to be in *row echelon form*(REF) if it satisfy the following:
  - The non-zero rows of $\mathbf{A}$ precede the zero rows of $\mathbf{A}$.
  - Suppose $\mathbf{A}$ has $r$ non-zero rows and the pivot of $i$th row is in $k_i$th column. Then $k_1 < k_2 < \cdots < k_r$.
  - All pivots of $\mathbf{A}$ are equal to $1$.
- A column containing a pivot called a *pivotal* column.
- Further, suppose that pivot is the only non zero entry of a pivotal column of $\mathbf{A}$. Then we say that $\mathbf{A}$ is said to be in *reduced row echelon form*(RREF).

- $$\begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{bmatrix}$$ Not in row echelon form.
- $$\begin{bmatrix} 0 & \boxed{1} & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Not in row echelon form
- $$\begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Row echelon form
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- $$\begin{bmatrix} 0 & \boxed{1} & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Reduced row echelon form

# Elementary Row Operations

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. We can transformed the matrix $\mathbf{A}$ using the following process:

- Interchange $i$th and $j$th rows of $A$, i.e., $R_i \leftrightarrow R_j$
- Multiply $i$th row of $\mathbf{A}$ by non-zero scalar, i.e., $R_i \to \alpha R_i$, where $\alpha \neq 0 \in \mathbb{R}$
- Add a scalar multiple of $i$th row of $\mathbf{A}$ to $j$th row of $\mathbf{A}$, i.e., $R_j \to R_i + \alpha R_j$, where $\alpha \in \mathbb{R}$.

These three operations are known as *elementary row operations*.

## Example

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -3 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -9 & -4 \end{bmatrix}$$

# Convert to Reduced Row Echelon form

1. Find the left most non-zero column of $\mathbf{A}$.
2. Interchange rows, if necessary to make the non zero entry to the top of column found in step 1.
3. By multiplying a nonzero scalar make the top nonzero entry found above to $1$.
4. Add scalar multiple of top rows to below rows so that all entries below pivot becomes zero.
5. Now, cover the top row and repeat the steps to the remaining submatrix until the matrix is in row echelon form.
6. Beginning with the last nonzero row and working upward, add suitable multiple of each row to the rows above to make zeros above the pivot.

# Example

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & 0 \end{bmatrix} \xrightarrow{R_3 \to -\frac{1}{3} R_3}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 \to -\frac{1}{2} R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \text{(REF)}$$

$$\xrightarrow[\phantom{x}]{\substack{R_2 \to R_2 - 2R_3 \\ R_1 \to R_1 - R_3}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Row Equivalent Matrices

## Definition

Matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be *row equivalent* if either can be obtained from the other by a sequence of elementary row operations.

## Theorem

*If the augmented matrices of two system of linear equations are row equivalent, then both system have the same set of solution.*

## Theorem

*Any matrix is row equivalent to row echelon form matrix.*