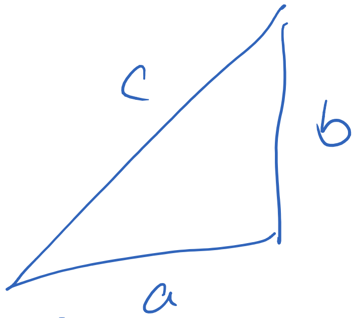
Fact ① Pythagorean Theorem

Let u and v are orthogonal. Then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$



$$c^2 = a^2 + b^2$$

Pf Since u and v are orthogonal,

$\langle u, v\rangle = 0.$

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$

$$= \langle u, u+v\rangle + \langle v, u+v\rangle$$

$$= \overline{\langle u+v, u\rangle} + \overline{\langle u+v, v\rangle}$$

$$= \overline{\langle u, u\rangle} + \overline{\langle v, u\rangle} + \overline{\langle u, v\rangle} + \overline{\langle v, v\rangle}$$

$$= \langle u, u\rangle + 0 + 0 + \langle v, v\rangle$$

$$= \|u\|^2 + \|v\|^2$$

② An orthogonal Decomposition.

Let u, v be two vectors in V, $v \neq 0$

Can we write $u = w + cv$, where $w \perp v$?

Ans Take $c = \dfrac{\langle u, v\rangle}{\|v\|^2}$

$$w = u - \frac{\langle u, v\rangle}{\|v\|^2} v.$$

Check that $u = w + cv.$

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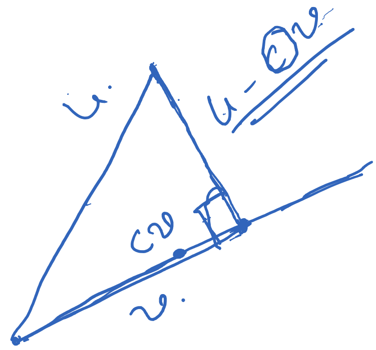
we need to check that $w \perp v.$

$$\langle w, v\rangle = \left\langle u - \frac{\langle u,v\rangle v}{\|v\|^2}, v\right\rangle$$

$$= \langle u, v\rangle - \frac{\langle u,v\rangle}{\cancel{\|v\|^2}} \cancel{\langle v, v\rangle}$$

$$= 0$$

Hence $w \perp v.$



③ Cauchy Schwartz inequality.

Let u, v be vectors in an inner product space V. Then $|\langle u, v\rangle| \le \|u\|\,\|v\|$.
and equality holds iff $\{u, v\}$ is L.D.

Pf

If $v = 0$, then, $\langle u, v\rangle = 0$

$0 \le \|u\|\, 0$ ∴ The Result holds.

Suppose $v \ne 0$. Then using ② we get

$u = w + cv$, where $c = \frac{\langle u,v\rangle}{\|v\|^2}$

Proof we will discuss tomorrow

$$|\langle u, v\rangle| \le \|u\|\,\|v\|$$

$$\Rightarrow \frac{|\langle u, v\rangle|}{\|u\|\|v\|} \le 1$$

$$\Rightarrow -1 \le \frac{\langle u, v\rangle}{\|u\|\|v\|} \le 1$$

If V is a real inner product space

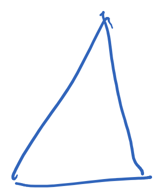
Def Let V be an inner product space over $\mathbb{R}$. Let $u, v \in V$. Then $\exists$ real no. $0 \le \theta \le \pi$ s.t. $\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}$. The real number θ is called the angle b/w u and v.

This is happening because $\cos\theta : [0, \pi] \to [-1, 1]$ is a surjective map

Remark If $\theta = 90$, $\cos\theta = 0$. Hence Two vectors are orthogonal iff $\langle u, v\rangle = 0$

(4) **Triangle inequality** $\forall\, u, v \in V$.

$$\|u+v\| \le \|u\| + \|v\|$$



(5) **parallelogram law** Let V be real vector space.

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

6 **Def^n** <u>Orthonormal set</u> :- Let V be inner product space. Then an <u>orthogonal set</u> $\{v_1, \ldots, v_n\}$ is called orthonormal set if $\|v_i\| = 1 \quad \forall i$

Ex Let $V = \mathbb{R}^n$, $\langle x, y\rangle = \sum x_i y_i$

(i) $\{e_1, e_2, \ldots, e_r\}$, $r \le n$.

Check that $\|e_i\| = 1$, and $e_i \perp e_j \quad \forall i, j$

(ii) $n=2$, $\left\{ \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$

$$\left\langle \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle = \frac{1}{2} - \frac{1}{2} = 0$$

vectors are orthogonal

$$\left\| \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\| = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Ily $\left\| \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\| = 1$

Hence the given set is orthonormal.

Ex $n=3$ $\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) \right\}$

Verify that the above set is orthonormal set.

→ Question Can be transformed every orthogonal set to an orthonormal set. ?

$\|\alpha v\| = |\alpha| \|v\|$

→ Why orthonormal sets are important.?

→ Orthonormal set are L.I

Let $\{v_1, \dots, v_n\}$ be orthonormal set. let

$v \in \text{Span}\{v_1, \dots, v_n\}$,

$\exists\ c_1, c_2, \dots, c_n,$ s.t

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

What can you say about c_i ?