Lecture 4

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Gauss Elimination

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Let $\mathbf{Ax} = \mathbf{b}$ be the given system and let $\mathbf{A}'$ be the REF of the augmented matrix $[\mathbf{A}|\mathbf{b}]$.

- Variables corresponding to pivotal columns of $\mathbf{A}'$ are known as *leading* variables.
- Variables corresponding to non-pivotal columns of $\mathbf{A}'$ are known as *free* variables.
- Identify free variables and assign arbitrary values to them.
- By back substitution, find the solution of the problem.
- Gauss-Jordan elimination: Convert the augmented matrix $[\mathbf{A}|\mathbf{b}]$ to RREF instead of REF

Example

Let $[\mathbf{A} : \mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 1 & : & 5 \\ 2 & 3 & 2 & 4 & : & 6 \\ 3 & 4 & 3 & 5 & : & 11 \end{bmatrix}$ .One can check that REF of augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 & : & 5 \\ 0 & 1 & 0 & 2 & : & -4 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$.

Now, 1st and 2nd columns are pivotal columns. So x_1 and x_2 are leading variables, and x_3, x_4 are free variable. Now we assign $x_3 = s$ and $x_4 = t$, where $s, t \in \mathbb{R}$. Now we get system of equations is

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 5 \\ 0x_1 + x_2 + 0x_3 + 2x_4 &= -4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

After substituting the values of x_3 and x_4 in 2nd equation, we get $x_2 + 0s + 2t = -4 \implies x_2 = -4 - 2t$.

Now substitute the value of x_2, x_3 and x_4 in first equation, we get $x_1 - 4 - 2t + s + t = 5 \implies x_1 = 9 - s + t$. Thus, we get $\{(9 - s + t, -4 - 2t, s, t) : s, t \in \mathbb{R}\}$.

Definition

If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called a *general solution* of the system.

In the above example, solution $\{(9 - s + t, -4 - 2t, s, t) : s, t \in \mathbb{R}\}$ is a general solution of the system.

Gauss-Jordan Elimination

In order to use Gauss-Jordan elimination method, note that RREF of augmented matrix is $\begin{bmatrix} 1 & 0 & 1 & -1 & : & 9 \\ 0 & 1 & 0 & 2 & : & -4 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$.

From the above matrix, we get that x_3 and x_4 are free variables. We assign $x_3 = s$ and $x_4 = t$, where $s, t \in \mathbb{R}$. The system of equations is

$$\begin{aligned} x_1 + 0x_2 + x_3 - x_4 &= 9 \\ 0x_1 + x_2 + 0x_3 + 2x_4 &= -4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

The first equation gives that $x_1 = 9 - s + t$ and 2nd equation gives that $x_2 = -4 - 2t$. So we get the same solution set using Gauss-Jordan elimination methods.

Theorem

A linear system of equations has either no solution, or a unique solution, or infinitely many solutions.

Proof. Suppose $\mathbf{Ax} = \mathbf{b}$ be the given linear system of equation. Suppose the system is reduced row equivalent to

$$\begin{array}{ccccccccc}
x_{j1} & + & & \cdots & & + & \sum_{k=r+1}^{n} c_{1k} x_{j_k} & = & d_1 \\
& & x_{j2} & + & \cdots & + & \sum_{k=r+1}^{n} c_{2k} x_{j_k} & = & d_2 \\
& & & \ddots & & & \vdots & & \vdots \\
& & & & x_{jr} & + & \sum_{k=r+1}^{n} c_{rk} x_{j_k} & = & d_r \\
& & & & & & 0 & = & d_{r+1} \\
& & & & & & & \vdots & \\
& & & & & & 0 & = & d_m,
\end{array}$$

where the summation is runs over free variable.

Now, if $d_j \neq 0$ for some $r+1 \leq j \leq m$, then we have an equation

$$0x_1 + 0x_2 + \cdots + 0x_n = d_j.$$

Since non of the values of $x_i's$ satisfy the above equation, this equation does not have a solution. Hence given system does not have a solution. On the other hand suppose that $d_j = 0$ for all $r+1 \leq j \leq n$. Suppose $r < n$. In the system, we have free variable and we can assign arbitrary value to those variables. Hence this has infinitely many solution. If $r = n$, then number of leading variables is equal to number of variables. Hence we do not have any free variable and we have $(d_1, \ldots, d_n)$ be the unique solution of given system.

Corollary

Any homogeneous linear system of equations is either a trivial solution or infinitely many solutions.

Proof follows from the fact that $(0, \ldots, 0)$ is always a solution to homogeneous linear system of equation.

Summary

Let $\mathbf{Ax} = b$ be linear system of equation and $\mathbf{A}'$ be the REF of the augmented matrix $[\mathbf{A}|b]$. Then we the following:

- The system has no solution if and only if last column of $\mathbf{A}'$ is pivotal column.
- If the last column of $\mathbf{A}'$ is not pivotal column, then system has a solution and we have the following:
  - The system has a unique solution if all the columns are pivotal columns (absence of free variables).
  - The system has a infinite solution if there exists a column which is not pivotal column(presence of free variables).