Suppose the $\chi_A(x) = (x-2)^2(x-1)$ of A. Then the possibility for minimal polynomial is $(x-2)(x-1)$ or $(x-2)^2(x-1)$

Thm Let $v_1, \dots, v_n$ be eigen vector of a matrix A corresponding to eigen value $\lambda_1, \dots, \lambda_n$, with $\lambda_i \neq \lambda_j$ $\forall\, i \neq j$.

Then $\{v_1, \dots, v_n\}$ is L.I.

Pf We use the induction. For $n=1$

$\{v_1\}$ is L.I, as $v_1 \neq 0$.

Let us assume that $\{v_1, \dots, v_r\}$ is L.I. We want to prove $\{v_1, \dots, v_r, v_{r+1}\}$ is L.I.

Let

$$c_1 v_1 + c_2 v_2 + \dots + c_r v_r + c_{r+1} v_{r+1} = 0 \quad \text{— (1)}$$

$$A(c_1 v_1 + c_2 v_2 + \dots + c_{r+1} v_{r+1}) = A0$$

$$c_1 A v_1 + c_2 A v_2 + \dots + c_{r+1} A v_{r+1} = 0$$

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_{r+1} \lambda_{r+1} v_{r+1} = 0 \quad \text{— (2)}$$

multiply (1) by λ_{r+1} and subtract from (2)

$$c_1(\lambda_1 - \lambda_{r+1}) v_1 + c_2(\lambda_2 - \lambda_{r+1}) v_2 + c_3(\lambda_3 - \lambda_{r+1}) + \dots + c_r(\lambda_r - \lambda_{r+1}) v_r = 0$$

Since $\{v_1, \dots, v_r\}$ is L.I;

$$c_1(d_1 - d_{r+1}) = 0,\ c_2(d_2 - d_{r+1}) = 0, \dots, c_r(d_r - d_{r+1}) = 0$$

Since $d_i - d_{r+1} \neq 0$, $\quad c_1 = 0 = c_2 = c_3 = \dots = c_r = 0$

Substitute in (1)

$$c_{r+1} v_{r+1} = 0, \Rightarrow \quad c_{r+1} = 0$$

as, $v_{r+1} \neq 0$.

Hence $\{v_1, \dots, v_r\}$ is L.I set.

Inner product spaces.

Def Let V be a finite dimensional vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Then a map $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{F}$ is called inner product if it satisfy the following conditions.

(i) $\langle x, x \rangle \geq 0$ equality holds iff $x = 0$ $\forall x \in V$.

(ii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, $\alpha, \beta \in \mathbb{F}$, $x, y, z \in V$.

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\forall x, y \in V$, and $\overline{\langle x, y \rangle}$ is a complex conjugate of $\langle x, y \rangle$

Ex Let $V = \mathbb{R}^n$, Define $\langle\ ,\ \rangle : V \times V \rightarrow \mathbb{R}$ by

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

(i) $\langle x, x \rangle = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$

$$x_1^2 + x_2^2 + \dots + x_n^2 = 0 \Rightarrow x_1 = x_2 = \dots = x_n = 0$$

(ii) Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $z = (z_1, \dots, z_n)$
$\alpha, \beta \in \mathbb{R}$.

$$\alpha x+\beta y = (\alpha x_1+\beta y_1, \alpha x_2+\beta y_2, \dots \alpha x_n+\beta y_n)$$

$$\langle \alpha x+\beta y, z\rangle = (\alpha x_1+\beta y_1)z_1 + (\alpha x_2+\beta y_2)z_2 + \dots + (\alpha x_n+\beta y_n)z_n$$

$$= \alpha x_1 z_1 + \beta y_1 z_1 + \alpha x_2 z_2 + \beta y_2 z_2 + \dots + \alpha x_n z_n + \beta y_n z_n$$

$$= \alpha(x_1 z_1 + x_2 z_2 + \dots + x_n z_n) + \beta(y_1 z_1 + y_2 z_2 + \dots + y_n z_n)$$

$$= \alpha\langle x, z\rangle + \beta\langle y, z\rangle$$

(iii) $\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$$= y_1 x_1 + y_2 x_2 + \dots + y_n x_n = \langle y, x\rangle = \overline{\langle y, x\rangle}$$

$\Rightarrow$ $\langle\, ,\, \rangle$ is an inner product.

Def Let V be vector space with inner product $\langle\ \rangle$. Then $(V, \langle\, ,\, \rangle)$ is called the inner product space (IPS).

In the last example, $(\mathbb{R}^n, \langle\, ,\, \rangle)$ is IPS.

Ex $V = \mathbb{C}^n$, $\mathbb{F} = \mathbb{C}$

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$

$$\langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + \dots + x_n\bar{y}_n$$

Verify that $(\mathbb{C}^n, \langle\, ,\, \rangle)$ is an IPS.

Ex Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$\langle x, y\rangle = [x_1, x_2]\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\langle x, x\rangle = [x_1\ x_2]\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1\ x_2]\begin{bmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$$

$$\langle x, x\rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$$

$$= 2x_1^2 - x_1 x_2 - x_1 x_2 + x_2^2$$

$$= 2x_1^2 - 2x_1 x_2 + x_2^2$$

$$= x_1^2 + x_1^2 - 2x_1 x_2 + x_2^2$$

$$= x_1^2 + (x_1 - x_2)^2 \geq 0$$

$\langle x, x\rangle = 0$ iff $x = 0$

$x_1^2 + (x_1 - x_2)^2 = 0 \Rightarrow x_1 = 0$ and $x_1 - x_2 = 0$

$x_2 = 0 \Rightarrow x = 0$

Verify the remaining two conditions.

Ex (i) Let $V = \mathbb{R}^2$, $\langle x, y\rangle = x_1 y_1$

You can check that it satisfy the 2nd and 3 condition.

but $\langle x, x\rangle = 0 \not\Rightarrow x = 0$

As $\langle (0,1), (0,1)\rangle = 0$

(ii) $V = \mathbb{R}^2$, $\langle x, y\rangle = x_1^2 + x_2^2 + y_1^2 + y_2^2$

This fails the 2^{nd} condition..

III $V = \mathbb{R}^2$, $\langle x, y\rangle = x_1 y_1^3 + x_2 y_2^3$

This fails the III condition

Verify

Ex Let V be an IPS with inner product $\langle\ \rangle$. over F. Then prove that $\langle x, \alpha y + \beta z\rangle$

$$= \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle$$

$\forall\ x, y, z \in V, \quad \alpha, \beta \in \mathbb{F}$