Suppose A is invertible. Find the inverse of a matrix using Cayley Hamilton.

Let A be a matrix and $p(x)$ be characteristic polynomial then constant term of $p(x)$ is equal to the determinant of a matrix upto a sign. (Exercise)

Let $f(x)$ be a polynomial. Then the constant of the polynomial $f(0)$.

$$p(x) = \text{Char}(A) = \det(A - xI)$$

$$p(0) = \det(A - 0I) = \det(A)$$

constant of char poly of a matrix A is non zero iff $\det(A) \neq 0 \Leftrightarrow$ A is invertible

Let $\det(A) \neq 0$ and $p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_0 I$.

By Cayley Hamilton

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots a_1 A + a_0 I = 0$$

$$\Rightarrow A(A^{n-1} + a_{n-1}A^{n-2} + a_{n-3}A^{n-3} + \cdots a_1 I) = -a_0 I$$

$$\Rightarrow A\left(\frac{-1}{a_0}A^{n-1} + \frac{a_{n-1}}{-a_0}A^{n-2} + \frac{a_{n-3}}{-a_0}A^{n-3} + \cdots \frac{a_1}{-a_0}I\right) = I$$

we get $AB = I$,

where $B = \underbrace{-\frac{1}{a_0}A^{n-1} - \frac{a_{n-1}}{a_0}A^{n-2} - \cdots \frac{a_1}{a_0}I}_{}$

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$\Rightarrow$ $\qquad A^{-1}$ ||

Ex $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Char poly(A) = $\det \begin{bmatrix} 1-x & 1 & 1 \\ 1 & -x & 1 \\ 1 & 1 & -x \end{bmatrix}$

$$= (1-x)(x^2-1) - 1(-x-1) + 1(1+x)$$

$$= x^2 - \cancel{1} - x^3 + x + x + \cancel{1} + 1 + x$$

$$= -x^3 + x^2 + 3x + 1$$

By Cayley Hamilton $\quad -A^3 + A^2 + 3A + I = 0$

$$I = A^3 - A^2 - 3A$$

multiply both side by A^{-1} $\quad A^{-1} = A^2 - A - 3I$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$A A^{-1} = I$

Minimal polynomial / A monic polynomial $p(x)$ is

called a minimal polynomial of matrix A if $p(A)=0$ and $q(A) \neq 0$ $\forall$ $q(x)$ with $\deg(q(x)) < \deg(p(x))$.
denoted as $m_A(x)$.

→ Since the degree of char polynomial is n for a matrix of size n, the degree of minimal polynomial is $\leq n$.

→ $\chi_A(x) \longrightarrow$ characteristic polynomial of A.

→ $m_A(x) \mid \chi_A(x)$

→ $m_A(x)$ and $\chi_A(x)$ polynomial have same root.

→ Let A and B be similar matrices. Then $m_A(x) = m_B(x)$

$A = I_{n\times n}$, $\chi_I(x) = (1-x)^n = \det|I - xI|$

$m_I(x) = x - 1$

→ Let A and B are similar and $m_A(x)$ and $m_B(x)$ be minimal polynomial of A and B resp.

Since A and B are similar, $\exists$ invertible matrix P s.t. $B = PAP^{-1}$

Let $m_A(x) = x^n + a_1 x^{n-1} + \dots a_n$

$\Rightarrow A^n + a_1 A^{n-1} + \dots a_n I = 0$

Pre multiply by P and Post P^{-1}.

$P(A^n + a_1 A^{n-1} + \dots + a_n I)P^{-1} = 0$

$$\Rightarrow \underline{(P A^n P^{-1})} + a_1 (P A^{n-1} P^{-1}) + \cdots + a_n \underbrace{P I P^{-1}}_{I} = 0$$

$$\Rightarrow B^n + a_1 B^{n-1} + \cdots \quad a_n I = 0$$

$\Rightarrow$ B^n satisfy $\underline{m_A(x)}$

But minimal polynomial of B^n is $m_B(x)$

$$\Rightarrow \quad m_B(x) \mid m_A(x)$$

Illy $\quad m_A(x) \mid m_B(x)$

$$\Rightarrow \boxed{m_A(x) = m_B(x)}$$

$\longrightarrow$

Let A be diagonalizable matrix. Then

A is similar diagonal matrix say D. then $m_A(x) = m_D(x)$

$\longrightarrow$ $A = \begin{bmatrix} 2 & & 0 \\ 0 & 2 & \\ & & 1 \end{bmatrix}$, $m_A(x) = \underline{(x-2)}(x-1)$

$$\underline{(A-2I)}(A-I) = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\cancel{\chi_A(x)} = (2-x)^2 (1-x)$$

$\longrightarrow$ Let $D = \operatorname{diag}(\underbrace{d_1, d_1, \ldots d_1}_{r_1}, \underbrace{d_2 \ldots d_2}_{r_2}, \ldots, \underbrace{d_k \ldots d_k}_{r_k})$

Thm $m_D(x) = (x-d_1)(x-d_2)\cdots(x-d_k)$

$\chi_D(x) = (x-d_1)^{r_1}(x-d_2)^{r_2}\cdots(x-d_k)^{r_k}$

→ $\boxed{\text{A matrix is diagonalizable iff } m_A(x) \text{ has distinct roots and splits into linear factor.}}$