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CSD001P5M **Linear Algebra** **Tutorial: 04**

1. Let S be a nonempty set of vectors in a vector space V.

   (a) If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V such that $\mathbf{v} \notin \text{Span}(S)$, then the set $S \cup \{\mathbf{v}\}$ is also linearly independent.

   (b) If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of the other vectors in S, then $S \setminus \{\mathbf{v}\}$ spans the same space i.e. $\text{Span}(S) = \text{Span}(S \setminus \{\mathbf{v}\})$.

2. Let S be a finite set of vectors in a finite dimensional vector space V.

   (a) If S spans V but not a basis, then S can be reduced to a basis for V by removing appropriate vectors from S.

   (b) If S is linearly independent set that is not a basis, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

3. Every subspace W of a finite dimensional vector space V is again finite dimensional and $\dim W \leq \dim V$. Moreover, if $\dim W = \dim V$, then $V = W$.

4. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V over field $\mathbb{F}$. Then for $\alpha \in \mathbb{F}$, $B_{ij}(\alpha) = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_i + \alpha \mathbf{v}_j, \ldots, \mathbf{v}_n\}$ also a basis for V.

5. Show that following sets form a basis for corresponding vector spaces:(without showing spanning of the set)

   (a) $\{(3,7),(5,5)\}$ forms a basis for $\mathbb{R}^2$

   (b) $\{(2,0,-1),(4,0,7),(-1,-1,4)\}$ forms a basis for $\mathbb{R}^3$

6. Determine whether the following sets of vectors are linearly independent or not. If linearly independent, then check whether it is a basis or not

   (a) Consider $\mathbb{C}$ as a vector space over $\mathbb{C}$ and let $S = \{1, i\}$.

   (b) Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$ and let $S = \{1, i\}$.

   (c) $S = \{1+2x+x^2, 2+x+4x^2, 3+3x+5x^2\}$ of $\mathbb{R}_2[x]$).

   (d) $S = \{(1,2,6),(-1,3,4),(-1,-4,-2),(2,3,4)\}$ of $\mathbb{R}^3$.

   (e) $S = \{u+v, v+w, w+u\}$ in a vector space V given that $\{u,v,w\}$ is basis for V.

7. Is the set $W = \{f(x) \in \mathbb{R}_4[x] : f(-1) = f(1) = 0\}$ a subspace of $\mathbb{R}_4[x]$? If yes, determine a basis and dimension of W.

8. Let V be the vector space of all real sequences and let,

$$W := \{< a_n > \in V \ : \text{ only finitely many of the terms in } < a_n > \text{ are non-zero}\}.$$

Show that W is a subspace of V. Is W finite-dimensional ? Justify your answer.

9. For $k = 0, 1, \ldots, n$, let

$$p_k(t) := t^k + t^{k+1} + \cdots + t^n.$$

Then prove that $\{p_0(t), p_1(t), \ldots, p_n(t)\}$ is a basis of $\mathbb{R}_n[t]$.

10. For a fixed $t_0 \in \mathbb{R}$, determine the dimension of the subspace of $\mathbb{R}_n[t]$ defined by

$$\{f \in \mathbb{R}_n[t] : f(t_0) = 0\}.$$

11. Let S be a basis of a vector space V over $\mathbb{F}$. Given $0 \neq v \in V$, show that there exist unique $\{v_1, \ldots, v_m\} \subseteq S$ and unique $\{\alpha_1, \ldots, \alpha_m\} \subseteq \mathbb{F} - \{0\}$ such that $v = \alpha_1 v_1 + \cdots + \alpha_m v_m$.

12. Let V be a vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let W_1 be the subset of even functions, $f(-x) = f(x)$ and W_2 be the subset of odd functions, $f(-x) = -f(x)$.

    (a) Prove that W_1 and W_2 are subspaces of V.

    (b) Prove that $W_1 + W_2 = V$.

    (c) $W_1 \cap W_2 = \{0\}$.

13. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ where the v_i's are vectors in $\mathbb{R}^3$ given below:

$$v_1 = (1,2,3), \quad v_2 = (2,5,7), \quad v_3 = (10,24,34), \quad v_4 = (.1,.5,.6), \quad v_5 = (3,7,11).$$

Let $W :=$ Span S.

    (a) Reduce S to a basis for W. You must explain your method briefly and show your calculations.

    (b) Is W all of $\mathbb{R}^3$? Justify your answer (YES or NO) in at most one sentence.