Lecture 15

Rajiv Kumar
rajiv.kumar@iitjammu.ac.in

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Rank and Nullity

Basis of the column space of A

Observe that if $\mathbf{A}_E = \begin{bmatrix} 0 & 0 & 1 & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a reduced row-echelon matrix, then the first non-zero column of $\mathbf{A}$ is $\boldsymbol{e}_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^t$ which is a pivotal column, say, at j_1 position. Now if $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r$ are first r pivotal columns of $\mathbf{A}$ occurring at $j_1, j_2, \ldots, j_r$ positions, then the next column of $\mathbf{A}$ at $(j_r + 1)^{th}$ position is either the pivotal column $\boldsymbol{e}_{r+1}$ or is a linear combination of the preceding pivotal columns $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r$. Clearly, pivotal columns forms a basis of $\mathcal{C}(\mathbf{A}_E)$. The columns of $\mathbf{A}$ corresponding to the pivotal columns of $\mathbf{A}_E$ are called the basic columns. The basic columns of $\mathbf{A}$ form a basis for $\mathcal{C}(\mathbf{A})$.

Finding basis of a span of a set in $\mathbb{R}^m$

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of vectors in $\mathbb{R}^m$. To find a basis of $\mathrm{span}(S)$, consider a $m \times n$ matrix $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$. To find a basis of $\mathrm{span}(S)$, convert the matrix $\mathbf{A}$ into the reduced row echelon form $\mathbf{A}_E$. Now, vector corresponding to basic columns of $\mathbf{A}_E$ forms a basis for $\mathrm{span}(S)$

Here C(A) = Span(S)..

Definition

Given an $m \times n$ matrix $\mathbf{A}$

- the rank of $\mathbf{A}$ is the dimension of the column space of $\mathbf{A}$: rank$(\mathbf{A}) = \dim \mathcal{C}(\mathbf{A})$.
- the nullity of $\mathbf{A}$ is the dimension of null space of $\mathbf{A}$: nullity$(\mathbf{A}) = \dim \mathcal{N}(\mathbf{A})$.

Example

Find a basis for $\operatorname{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}3\\3\\4\end{bmatrix}\right)$.

Let $\mathbf{A} = \begin{bmatrix}1 & 2 & 2 & 3\\2 & 4 & 1 & 3\\3 & 6 & 1 & 4\end{bmatrix}$. Then $\mathbf{A}_E = \begin{bmatrix}1 & 2 & 0 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$ with first and third columns as pivotal columns. Thus a basis of column space of A is $\{\begin{bmatrix}1 & 2 & 3\end{bmatrix}^t, \begin{bmatrix}2 & 1 & 1\end{bmatrix}^t\}$. Hence a basis for $\operatorname{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}3\\3\\4\end{bmatrix}\right)$ is $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$. Now, we can see that rank$(\mathbf{A}) = 2$ and nullity$(\mathbf{A}) = 2$.

Basis of the null space of $\mathbf{A}$

To find a basis of the null space of $\mathbf{A}$, solve the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$. Let rank $\mathbf{A} = r$. Then we have the following observations:

- Since elementary row operations are invertible, the solution set of $\mathbf{A}\mathbf{x} = \mathbf{0}$ equals to the solution set of $\mathbf{A}_E\mathbf{x} = \mathbf{0}$.
- The unknown variables corresponding the positions of basic columns are called basic variables and other variables are called free variables.
- There are exactly r basic variables and $n - r$ free variables.
- Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}$ denote the solutions obtained by sequentially setting each free variable equal to 1 and other free variables equal to zero.
- The set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ forms a basis of null space of $\mathbf{A}$.
- rank$(\mathbf{A})$+ nullity$(\mathbf{A})$= number of columns of $(\mathbf{A})$.

Theorem (rank nullity theorem)

Let $\mathbf{A}$ be an $m \times n$ matrix. Then rank$(\mathbf{A})$ + nullity$(\mathbf{A}) = n$.

Example

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix}$. Then $\mathbf{A}_E = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus the original homogeneous system is equivalent to the following reduced homogeneous system: $x_1 + 2x_2 + x_4 = 0,\ x_3 + x_4 = 0$, where x_2 and x_4 are free variables. By taking $x_2 = 1$ and $x_4 = 0$, we get a solution $\mathbf{x}_1 = \begin{bmatrix} -2 & 1 & 0 & 0 \end{bmatrix}^t$. Similarly, by taking $x_2 = 0$ and $x_4 = 1$, we get a solution $\mathbf{x}_2 = \begin{bmatrix} -1 & 0 & -1 & 1 \end{bmatrix}^t$. Thus a basis of null space of $\mathbf{A}$ is $\{\begin{bmatrix} -2 & 1 & 0 & 0 \end{bmatrix}^t, \begin{bmatrix} -1 & 0 & -1 & 1 \end{bmatrix}^t\}$ and a basis of column space of $\mathbf{A}$ is $\{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^t, \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}^t\}$.