Lecture 20

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Eigen Values and Eigen Vectors

Eigenvalues and Eigenvectors

Definition

- Let $T : V \to V$ be a linear transformation. Then an *eigenvector* of T is a nonzero vector $\mathbf{v}$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$ for some scalar λ. The scalar $\lambda \in \mathbb{F}$ is called *eigenvalue* of T.
- For any $n \times n$ matrix A over $\mathbb{F}$, a nonzero vector $X \in \mathbb{F}^n$ is called a eigenvector of A, if $AX = \lambda X$ for some scalar $\lambda \in \mathbb{F}$. The subspace $A_\lambda = \{X \in \mathbb{F}^n : AX = \lambda X\}$ is called the eigenspace of A corresponding to λ. Clearly, $A_\lambda = N(A - \lambda I)$.

Remark

- Let $T : V \to V$ be a linear transformation, where V is a vector space (real or complex) and B be a basis of V. If $A = [T]_B$, then λ is an eigenvalue of A if and only if λ is an eigenvalue of T. Because $T(\mathbf{v}) = \lambda\mathbf{v}$ implies that $[T(\mathbf{v})]_B = \lambda[\mathbf{v}]_B$. Therefore, $[T]_B[\mathbf{v}]_B = \lambda[\mathbf{v}]_B$, and hence $AX = \lambda X$ for $X = [\mathbf{v}]_B$. Conversely, if λ is an eigenvalue of A then $AX = \lambda X$ for some nonzero $X \in \mathbb{F}^n$, this implies $[T]_B[\mathbf{v}]_B = [T(\mathbf{v})]_B = \lambda[\mathbf{v}]_B$ for some nonzero vector $\mathbf{v} \in V$. This gives $T(\mathbf{v}) = \lambda\mathbf{v}$.
- For any $n \times n$ matrix A over $\mathbb{F}$, a scalar $\lambda \in \mathbb{F}$ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$. Suppose $\lambda \in \mathbb{F}$. Then λ is an eigenvalue of A if and only if there exists a nonzero vector $X \in \mathbb{F}^n$ such that $AX = \lambda X$ or equivalently $(A - \lambda I)X = 0$. Since X is nonzero this is equivalent to $\det(A - \lambda I) = 0$.

Definition

For any $n \times n$ matrix A over $\mathbb{F}$, the polynomial $\det(A - xI)$ is called the *characteristic polynomial* of A. The equation $\det(A - xI) = 0$ is called the *characteristic equation* of A. Thus the eigen values of A are the solutions of $\det(A - xI) = 0$ that lie in $\mathbb{F}$.

Example Consider the linear operator T on $\mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 12x_2 \\ x_1 - 5x_2 \end{bmatrix}$. Find eigenvalues and eigenvectors of T.

Solution. The matrix of T relative to the standard basis of $\mathbb{R}^2$ is given by $\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$. To determine the eigenvalues we solve $\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & -12 \\ 1 & -5 - \lambda \end{bmatrix} = 0$. Thus $\lambda_1 = -1$ and $\lambda_2 = -2$ are the eigenvalues. To find the eigenvectors, we need to find the null space of $A - \lambda_1 I$ and $A - \lambda_2 I$. Check that for $\lambda_1 = -1$,

$N((A - \lambda I)) = \operatorname{span}\left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$. Thus $X_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigenvector.

Similarly, for $\lambda_2 = -2$, check that $X_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector.

Exercise

- Find the eigenvalues and eigenvectors of $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$.
- Find the eigenvalues of $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$ find a basis for each of the corresponding eigenspaces.

Answer. $\lambda = 1, 1, 2, 3$ and $A_{\lambda_1} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$,

$A_{\lambda_2} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$, $A_{\lambda_3} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$.