Thm (Spectral theorem for symmetric matrix)

Let A be $n \times n$ real symmetric matrix. Then we have the following.

(1) A has n real eigenvalues counting with mult.

2. $A \cdot M(\lambda) = G \cdot M(\lambda)$ for all eigenvalue of A.

3. Let A_λ and A_μ be the eigenspace of A corr. to eigen value λ and μ, resp. Then $A_\lambda \perp A_\mu$

4. A is orthogonaly diagonalisable.

Pf Let λ be an eigen value of A and u be an eigen vector of A corr to λ.

$Au = \lambda u$, since, $\frac{u}{\|u\|}$ is also an eigenvector of A corr. to λ, we can assume that $\|u\| = 1$

Let $\{u, x_2, \ldots, x_n\}$ be an orthonormal basis of $\mathbb{R}^n$

Consider $P = [u \; x_2 \; \cdot \; \cdot \; x_n]$. Since $\|u\|=1$, $\|x_i\| = 1$ and $\{u, x_2, \ldots, x_n\}$ is ortho, P is orthonormal matrix.

$\Rightarrow \quad P^{-1} = P^T$.

To prove (A1) we use induction on n.

For $n=1$, the result trivial holds..

Let us assume the result will hold for $n-1$

Consider $P^T A P$

$$= \begin{bmatrix} u^t \\ x_2^t \\ \vdots \\ x_n^t \end{bmatrix} \underline{A \, [u \; x_2 \; \cdots \; x_n]}$$

$$= \begin{bmatrix} u^t \\ x_2^t \\ \vdots \\ x_n^t \end{bmatrix} \begin{bmatrix} Au & Ax_2 & \cdots & Ax_n \end{bmatrix} = \begin{bmatrix} u^t \\ x_2^t \\ \vdots \\ x_n^t \end{bmatrix} \begin{bmatrix} \lambda u & Ax_2 & \cdots & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda u^t u & u^t A x_2 & \cdots & u^t A x_n \\ x_2^t \lambda u & x_2^t A x_2 & x_2^t A x_3 \cdots & x_2^t A x_n \\ x_3^t \lambda u & \vdots & & \\ \vdots & & & \\ x_n^t \lambda u & x_n^t A x_2 & x_n^t A x_3 \cdots & x_n^t A x_n \end{bmatrix}$$

For some $i = 2, \ldots, n$.

$$u^t A x_i = u^t A^t x_i = (Au)^t x_i = (\lambda u^t) x_i$$

$$= \lambda u^t x_i = 0$$

Since A is symmetric, $P^T A P$ is also symmetric.

$$\Rightarrow \quad P^T A P = \begin{bmatrix} \lambda & 0 \cdots 0 \\ 0 & \\ \vdots & B \\ 0 & \end{bmatrix},$$

where B is $n-1 \times n-1$ symmetric matrix.

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By induction hypothesis $\exists$ an orthogonal matrix Q and diagonal matrix D s.t

$$Q^T B Q = D$$

Let $Q'' = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ 0 & Q \\ \vdots & \\ 0 & \end{bmatrix},$

Verify Q' is an orthogonal matrix.

$$Q'^T \underbrace{P^T A P}\, Q = \left[Q'^T \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} Q' \right] \quad (\text{Check})$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix}$$

Hence A is orthogonally diagonalizable.

(2) follows from (4).

Spectral Decomposition of a symmetric mat.

Let A be a symmetric matrix. Then $\exists$ $\{u_1, \ldots, u_n\}$ orthogonal eigenvector of A corr to eigen value $\lambda_1, \ldots, \lambda_n$.

$$\begin{bmatrix} u_1^t \\ \vdots \\ u_n^t \end{bmatrix} A \, [u_1 \cdots u_n] = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$$\Rightarrow A = [u_1 \cdots u_n] \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} [u_1^t \cdots u_n^t]$$

$$= d_1 \underline{u_1 u_1^t} + d_2 u_2 u_2^t + \cdots + d_n u_n u_n^t$$

this is known as spectral decomp. of a symmetric matrix.

<u>Remark</u>: $u_i u_i^t$ is matrix of rank 1.

QR - Decomposition.

<u>Thm</u> Let A be non-singular matrix of order n. Then there exists unique Q and R such that $A = QR$, where Q is an orthogonal matrix, and R is a upper triangular matrix with all the diagonal elements are +ve.

<u>Pf</u> Since A is non-singular matrix, columns of A are L.I. (equivalently, columns of A form a basis of $\mathbb{R}^n$.)

Using Gram-Schmidt procedure on the set of columns of A, say $\{x_1, \ldots, x_n\}$, we get an orthonormal basis of $\mathbb{R}^n$ say $\{u_1, \ldots, u_n\}$.

S.t

$$\text{Span}\{x_1, \ldots, x_j\} = \text{Span}\{u_1, \ldots, u_j\}$$
$$\forall j = 1, \ldots, n.$$

Let $Q = [u_1, \ldots, u_n]$. Then Q is orthogonal matrix.

Now we know that

$$x_j \in \text{Span}\{u_1, \ldots, u_j\}$$

$$x_j = a_{1j} u_1 + a_{2j} u_2 + \cdots a_{jj} u_j \quad \forall j.$$

Let

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & a_{23} & & a_{2n} \\ & & a_{33} & & \vdots \\ & & & & a_{nn} \end{bmatrix}.$$

Now

$$QR = \left[Q\begin{bmatrix} a_{11} \\ 0 \\ 0 \\ \vdots \end{bmatrix} \; Q\begin{bmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \; \cdots \; Q\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} \right]$$

$$= \begin{bmatrix} \underline{a_{11} u_1} & a_{12} u_1 + a_{22} u_2 & \cdots \end{bmatrix}$$

$$= [x_1 \; x_2 \; \cdots \; x_n] = A$$

$\underline{QR = A}$

... element of R are

We will get the diagonal element of R are +ve, by changing the appropriate sign of u_i

Let $Q_1 R_1 = Q_2 R_2 = A$

$$\Rightarrow \quad Q_2^t Q_1 R_1 = R_2$$

$$\Rightarrow \quad \underline{Q_2^t Q_1} = \underbrace{R_2 R_1^{-1}}$$

check $R_2 R_1^{-1}$ is upper triangular.

Since Q_2^t and Q_1 are orthogonal, $Q_2^t Q_1$ is orthogonal.

On the other hand, $R_2 R_1^{-1}$ is upper triangular.

$\Rightarrow$ $R_2 R_1^{-1}$ should be a diagonal matrix.

$$\Rightarrow \quad R_2 R_1^{-1} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$$\Rightarrow \quad Q_2^t Q_1 = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$$\Rightarrow \quad Q_1 = Q_2 \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

Suppose $Q_2 = [y_1 \ldots y_n]$

$$Q_1 = [d_1 y_1 \quad d_2 y_2 \quad \cdots \quad d_n y_n]$$

$$\Rightarrow \|d_i y_i\| = 1 \qquad \forall i$$

$$\Rightarrow |d_i| \underbrace{\|y_i\|}_{} = 1 \Rightarrow \boxed{|d_i| = 1}$$

$$\Rightarrow d_i = \pm 1$$

But $d_i > 0$, as d_i is a product of ith diagonal element of product of upper triangular matrix with all diagonals are +ve.

$$Q_2^t Q_1 = R_2 R_1^{-1} = I$$

$$\Rightarrow Q_1 = Q_2, \qquad R_1 = R_2$$