Computer Science & Engineering
Probability and Statistics
Tutorial-1

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1. **(a)** There are 10 distinguishable telegrams $\{t_1, t_2, t_3, ..., t_{10}\}$ to be distributed between 2 indistinguishable messengers $\{m_1, m_2\}$ where $m_1 m_2$ is same as $m_2 m_1$. Thus, for each telegram there are 2 choices i.e. either it may be taken by m_1 or m_2.

$$2 \times 2 \times 2 \times \cdots \times 2(10\ times) = 2^{10} \text{ ways}$$

.

(b) $\{t_1, t_2, ..., t_{10}\} \longrightarrow \{m_1, m_2\} \longrightarrow \{p_1, p_2, ..., p_{10}\}$

The each messenger has p choices and each telegram has m choices.

$$\text{Thus, the mapping can be in } p^{m^t} = 10^{2^{10}} \text{ ways}$$

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(c) If the telegrams are identical then it means that there will be many mappings which are same. So, this will reduce the mappings. In such case, we can apply stick partition approach in order to distribute between two messengers. Consider a stick in between all the identical telegrams, leaving it to permutate in $(m-1+t)!$. Since, the telegrams are identical and there is only one identical stick. The total permutation can be divided into $t!$ and $1!$.

$$\text{Total ways } = \frac{(2-1+10)!}{1! \cdot 10!} = 11$$

2. **(a)** There are $d = 4$ places where any digit from $S = \{2, 3, 5, 7\}$ can sit any number of times. The sum of 2,3,5 and 7 will occur exactly $(d-1)$ times and each place will have $|S| = 4$ choices. Every place will have a value incremented by a factor of 10 i.e. $(10^0 + 10^1 + 10^2 + 10^3) = 1111$. After multiplying all the factors we have

$$(2+3+5+7) \times |S|^{(d-1)} \times 1111 = 1,208,768$$

(b) If the digits are not repeated then there will be $(d-1)!$ choices on $(d-1)$ places.

$$(2+3+5+7) \times 3! \times 1111 = 113,322$$

3. Let $n = 1$, for atleast one digit case, d should start from 1 to n. Here, $n = 1$. Thus, it can have 4 options {2,4,6,8} but not {0}. Consider $n = 2$, there are two cases i.e. $d = 1$ and $d = 2$. For $f(d=1) + f(d=2)$, ${}^4C_1 \times^{10} C_1 +^4 C_1 \times^5 C_1$.

$$\begin{array}{rc} f(d=1) = & 4 \\ f(d=2) = & 4 \times 10 + 4 \times 5 \\ f(d=3) = & 4 \times 10^2 + 4 \times 5 \times 10 + 4 \times 5^2 \\ f(d=4) = & 4 \times 10^3 + 4 \times 5 \times 10^2 + 4 \times 5^2 \times 10 + 4 \times 5^3 \\ \cdots & \cdots \\ f(d=n) = & f(n-1) \times 10 + f(1) \times 5^{n-1} \end{array}$$

4. **(a)** Consider the problem where we need to count the number of bit sequence of size n that do not have two consecutive 0's. If a bit sequence ends with 1 then we consider $n-1$ bit binary sequence having no two consecutive 0's i.e. B_{n-1}. But if the bit sequence ends with 10 then we take $n-2$ bit sequence having no two consecutive 0's i.e. B_{n-2}. Moreover, we will not consider 00 or 01 ending bits since there may be a chance that $n-2$ bit sequence get two consecutive 0's. In total, there are two cases to be considered.

$$B_n = B_{n-1} + B_{n-2} \text{ with } B_1 = 2, B_2 = 3$$

$$B(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{1}$$

by solving linear homogeneous recurrence relation.

(b) Atleast = Total subsets-(no two consecutive sequence)

$$t = 2^n - B(n)$$

5. Number of ways to choose 10 elements subset is given by ${}^{100}C_{10}$. Consider a set $odd = \{1, 3, 5, ...99\}$. If we choose 10 elements from *odd* in ${}^{50}C_{10}$. Then, there is no chance to get a subset have neighbors. Similarly, in case of $even = \{2, 4, 6, ..., 100\}$ i.e. ${}^{50}C_{10}$. So, the subsets of S formed such that each subset contains at least a pair of neighbours is given by ${}^{100}C_{10} - 2 \times^{50} C_{10}$. It resembles a situation of bipartite graph in which there are two disjoint sets.

6. In such problem, we can denote '$\longrightarrow$' as left to right horizontal move and '$\uparrow$' up vertical moves. The number of moves required to reach B from A is suppose to have any $m+n$ moves in any order. So, we can permutate the moves in $(m+n)!$

ways where '$\longrightarrow$' and '$\uparrow$' moves are identical. Thus, the number of path possible is given by

$$\text{paths} = \frac{(m+n)!}{m! \cdot n!}$$

7.

8.

9. Given, $f : S_1 \longrightarrow S_2$ and $|S_1| = m, |S_2| = n$. Consider both the sets are sorted in 1 ways. Then, the functions possible to map n outputs from m inputs is given by ${}^nC_m$.

10. This problem can be thought of binary bits representing numbers of the set $S = \{1, 2, 3, ..., n\}$. For e.g. 0101 represents $\{2, 4\}$ which is disjoint from 1010 representing $\{1, 3\}$ such that there is a edge between 0101 and 1010.

$$\sum v \cdot d = \sum_{i=1}^{n} \binom{n}{i} \cdot (2^{n-i} - 1)$$

The number of edges e is given by

$$e = \frac{\sum v \cdot d}{2}$$

11. **(a)** The number of edges of a n-sided polygon is n. The total number of edges possible in a n-sided polygon is $\binom{n}{2}$. Then, the number of diagonal is given by

$$d = \binom{n}{2} - n = \frac{n \cdot (n-3)}{2}$$

(b) Consider n sided polygon having vertices $A_i : i \in [0, n-1]$. The diagonals from vertex A_0 contributes 0 intersections where as A_1 contributes $(1+2+3+\cdots+n-3)$ intersections. The other vertices A_2 to A_{n-3} contributes $\sum_{j=1}^{i}(n-2-i)$ for vertex A_i. Thus, total number of intersection is given by

$$\sum_{i=1}^{n-3} i + \sum_{i=1}^{n-4} i \cdot (n-2-i)$$

12. From Eq. 1, the number of ways to select distinct numbers such that no consecutive integers exist in a set of k numbers is given by $B(k)$. Thus, required number of ways is $B(k)$.

13. This problem is similar to that one of distributing objects to distinguishable boxes such that a every friend gets atleast one postcard. The problem can be converted into the following equation

$$f_1 + f_2 + f_3 + \cdots + f_n = a_1 + a_2 + \cdots + a_i + \cdots + a_k$$

where $f_1, f_2, ..., f_n \geq 1$. So, the stick partition method the number of ways to distribute are

$$w = \frac{((n-1) - n + \sum_{k=1}^{i} a_k)!}{(n-1)! \cdot \prod_{k=1}^{i} a_k!}$$

14.

15.

16. Consider a $2n$-bit sequence where first k is having as many 0's as that of 1's. So, the number of bit sequences possible that satisfy the condition $n(0) \geq n(1)$ for k-bit is

$$\binom{k}{\lceil \frac{k}{2} \rceil} + \binom{k}{\lceil \frac{k}{2} \rceil + 1} + \cdots + \binom{k}{\lceil k \rceil}$$

The other $2n - k$ bits can produce 2^{2n-k} along with the k-bit sequence. The function $f(n, k)$ defines the required number of sequences.

$$f(n, k) = 2^{2n-k} \cdot \sum_{t=\lceil \frac{k}{2} \rceil}^{k} \binom{k}{t}$$

17. **(a)** The following combinatorial proof can be proved using Vandermonde's identity. The identity is given by

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k} \tag{2}$$

Putting $r = n$ in Eq. 2.

$$\binom{m+n}{n} = \sum_{k=0}^{n} \binom{m}{n-k} \binom{n}{k} \tag{3}$$

Since, $\binom{n}{k} = \binom{n}{n-k}$. The Eq. 3 becomes

$$\binom{m+n}{n} = \sum_{k=0}^{n} \binom{m}{n-k} \binom{n}{n-k} \tag{4}$$

$$= \binom{m}{n}\binom{n}{n} + \cdots + \binom{m}{k}\binom{n}{k} + \cdots + \binom{m}{0}\binom{n}{0} \tag{5}$$

Hence, proved.

(b) Putting $m = n$ in Eq. 5

$$\binom{n+n}{n} = \binom{n}{n}\binom{n}{n} + \cdots + \binom{n}{k}\binom{n}{k} + \cdots + \binom{n}{0}\binom{n}{0} \tag{6}$$

$$\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}^2 \tag{7}$$

Hence, proved.

(c) The problem is similar to find number of ways of choosing $r+1$ number of 1's from $n+1$ bit binary sequence in $\binom{n+1}{r+1}$ ways. This can be done by fixing last 1 at k^{th} bit in the sequence. Thus, we can choose r number of 1's from the previous $(k-1)$ positions where $1 \leq k \leq n+1$. This is given by

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{p=(k-1)=r}^{(n+1)-1} \binom{p}{r} = \binom{n+1}{r+1} \tag{8}$$

18. **(a)** The steps of the proof are given below.

$$\frac{2^n - 1}{n} = \frac{\binom{n-1}{0}}{1} + \frac{\binom{n-1}{1}}{2} + \cdots + \frac{\binom{n-1}{n-1}}{n} \tag{9}$$

$$\frac{2^n - 1}{n} = \frac{(n-1)!}{1 \cdot 0! \cdot (n-1)!} + \frac{(n-1)!}{2 \cdot 1! \cdot (n-2)!} + \frac{(n-1)!}{3 \cdot 2! \cdot (n-3)!} + \cdots \tag{10}$$

$$2^n - 1 = \frac{n}{1} + \frac{n(n-1)}{2 \cdot 1} + \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} + \cdots + \frac{n(n-1)\cdots 2 \cdot 1}{1 \cdot 2 \cdot \cdots \cdot n} \tag{11}$$

$$2^n - 1 = \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} \tag{12}$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} \tag{13}$$

Since, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ is valid by Binomial theorem when $x = 1$ and $y = 1$ for $p = (x+y)^n$. Thus, completes our proof.

(b) Consider the equation

$$\sum_{k=m}^{n} \binom{k}{m}\binom{m}{k} = 2^{n-m}\binom{n}{m} \tag{14}$$

$$\sum_{k=m}^{n} \frac{(n-m)!}{(k-m)!(n-k)!} = 2^{n-m} \because \text{ both sides divided by } \binom{n}{m} \tag{15}$$

$$\sum_{\alpha=0}^{p} \frac{p!}{\alpha!\beta!} = 2^p \tag{16}$$

where, $p = n-m$, $k-m = \alpha$ and $n-k = \beta$. Thus, $\alpha + \beta = p$. Therefore, the Eq. 16 can be written as

$$\sum_{\alpha=0}^{p} \binom{p}{\alpha} = 2^p$$

Hence, proved.

19. The right hand side suggests the number of ways of choosing r balls from a set of m red balls and n white balls in $\binom{m+n}{r}$ ways. It can be done by choosing p balls from set of red balls and choosing $r-p$ balls from the set of white balls or vice versa. Here, p is the number of balls chosen from either of the set.

$$\therefore \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0} \tag{17}$$
$$= \sum_{p=0}^{r} \binom{m}{p}\binom{n}{r-p} \tag{18}$$

for $n \geq r \geq 1$ and $m \geq r \geq 1$. This identity is also known as Vandermonde's Identity.