CSL003P1M : Probability and Statistics
QuestionSet - 01: Combinatorics

September 10, 2021

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1. (a) There are 10 telegrams and 2 messenger boys. In how many different ways can the telegrams be distributed to the messenger boys if the telegrams are distinguishable?

   (b) In how many different ways can the telegrams be distributed to the messenger boys and then delivered to 10 different people if the telegrams are distinguishable?

   (c) Solve (a) under the assumption that telegrams are indistinguishable.

2. (a) Find the sum of all 4-digit numbers that can be obtained by using the digits $2, 3, 5$ and 7?

   (b) Find the sum of all 4-digit numbers that can be obtained by using the digits $2, 3, 5$ and 7 and no digit is repeated?

3. How many n-digit (the most significant digit must be non-zero) numbers are possible which contain at least one digit from the set $\{0, 2, 4, 6, 8\}$?

4. (a) How many subsets of $\{1, 2, \cdots, n\}$ have no two successive numbers?

   (b) How many subsets of $\{1, 2, \cdots, n\}$ have at least two successive numbers?

5. Let $S = \{1, 2, 3, \cdots, 100\}$. How many 10-element subsets of S can be formed so that each subset contains at least a pair of neighbours?

6. Consider an $m \times n$ grid shown below.

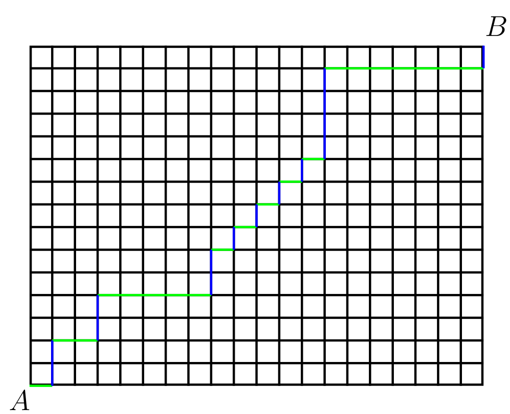

Figure 1: An $m \times n$ Grid.

How many paths are possible from A to B if any path consists of only (a) left to right horizontal and (b) up vertical moves? (See green and blue moves in the grid).

7. Prove that the number of partitions of the number n into (a sum of) no more than r terms is equal to the number of partitions of n into any number of terms, each at most r.
   Example: Let $n = 4$ and $r = 3$. Then partitions of 4 into no more than 3 terms are
   (a) 4
   (b) $3 + 1$
   (c) $2 + 2$
   (d) $2 + 1 + 1$

   and, partitions of 4 into any number of terms, each at most r are
   (a) $3 + 1$
   (b) $2 + 2$
   (c) $2 + 1 + 1$
   (d) $1 + 1 + 1 + 1$

   In both cases, the number of partitions is 4.

8. Prove that the number of partitions of n into (any number of) distinct terms is equal to the number of partitions of n into odd terms.

   Example: Let $n = 4$ and $r = 3$. Then partitions of 4 into any number of distinct terms are
   (a) 4
   (b) $3 + 1$

   and, partitions of 4 into odd terms are
   (a) $3 + 1$
   (b) $1 + 1 + 1 + 1$

   In both cases, the number of partitions is 2.

9. Let $S_1 = \{1, 2, \cdots, m\}$ and $S_2 = \{1, 2, , \cdots, n\}$. Let $f : S_1 \to S_2$ be a function. How many f's are possible which are strictly increasing?

10. How many two non-empty disjoint subsets A and B can be selected from $\{1, 2, 3, \cdots, n\}$?

11. (a) Find the number of diagonals of a convex n-gon.
    (b) Suppose no three diagonals pass through one point in a convex n-gon. Find the number of intersection points of all diagonals in that convex n-gon.

12. In how many ways can k distinct numbers be chosen from the set $\{1, 2, \cdots, n\}$ such that there does not exist any consecutive integers?

13. There are k kinds of the postcards, but only in a limited number of each, there being a_i copies of the i^{th} one. What is the number of possible ways of sending all of them to n friends? (We may send more than one copy of the same postcard to the same person. We many send different kinds of the postcard to the same person.)

14. Let $f : \{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$. How many f's are possible which are monotonically (not strictly) increasing?

15. How many monotone increasing functions f of $\{1, 2, \cdots, n\}$ into itself satisfy the condition $f(x) \leq x$ for every $1 \leq x \leq n$?

16. What is the number of sequences of n 0's and n 1's such that there are at least as many 0's as 1's among the first k digits for each $1 \leq k \leq 2n$?

17. Given combinatorial proofs for the following:

(a) $\binom{m}{0}\binom{n}{0}+\binom{m}{1}\binom{n}{1}+\cdots+\binom{m}{r}\binom{n}{r}+\cdots+\binom{m}{n}\binom{n}{n}=\binom{m+n}{n}$ for integers $m \geq n \geq 1$.

(b) Using (a), give a one-line proof that $\binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{r}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}$ for $n \geq 1$.

(c) $\binom{r}{r}+\binom{r+1}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1}$ for any positive integer $n \geq r \geq 1$.

18. Give algebraic proofs for the following

(a) $\frac{\binom{n-1}{0}}{1}+\frac{\binom{n-1}{1}}{2}+\frac{\binom{n-1}{2}}{3}+\cdots+\frac{\binom{n-1}{n-1}}{n}=\frac{2^n-1}{n}$ for $n \geq 2$.

(b) $\sum_{k=m}^{n}\binom{k}{m}\binom{n}{k}=\binom{n}{m}2^{n-m}$, for $n \geq m \geq 1$.

19. Give both combinatorial and algebraic proofs for the following:

$$\binom{n}{0}\binom{m}{r}+\binom{n}{1}\binom{m}{r-1}+\cdots+\binom{n}{r}\binom{m}{0}=\binom{n+m}{r}$$

for integers $n \geq r \geq 1$ and $m \geq r \geq 1$.

20.

Theorem 1. *If A and B are two finite sets, then*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This is known as "The Principle of Inclusion-Exclusion (PIE)" .

(a) Prove using PIE, if A, B and C are finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(b) Prove using PIE, if $A_1, A_2, \cdots, A_n$ are finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n}|A_i| - \sum_{i,j}|A_i \cap A_j| + \sum_{i,j,k}|A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n|.$$

21. (a) Among the permutations of $\{1, 2, \cdots, n\}$, there are some called derangements, in which none of the n integers appears in its natural place. Thus, $(i_1, i_2, \cdots, i_n)$ is a derangement if $i_1 \neq 1$, $i_2 \neq 2$, $\cdots$, and $i_n \neq n$. Let D_n be the number of derangements of $\{1, 2, \cdots, n\}$. Prove that

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right].$$

(b) Let n books be distributed to n students. Suppose that the books are returned and distributed to the students again later on. In how many ways can the books be distributed so that no student will get the same book twice?

22. (a) Give a combinatorial proof of the following relation: $D_n - nD_{n-1} = (-1)^n$ for $n \geq 2$ (for D_n, see previous question).

(b) Prove the following identity:

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^k = \begin{cases} 0 & \text{if } 0 \leq k < n, \\ (-1)^n n! & \text{if } k = n. \end{cases}$$