CSL003P1M : Probability and Statistics
Lecture 07 (Independent Events)

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September 20, 2021

Independent Events

Two events E and F are independent if

$$P(EF) = P(E)P(F).$$

(Why?)

- When we say E is independent of F (or F is independent of E), we mean that the occurrence of F (or E) does not change the chances of occurrence of E (or F).
- So, $P(E|F) = P(E)$ (or, $P(F|E) = P(F)$).
- Since
$$P(E) = P(E|F) = \frac{P(EF)}{P(F)}.$$
- Thus, $P(EF) = P(E)P(F)$.

Independent Events

Two events E and F are independent if

$$P(EF) = P(E)P(F).$$

- Now,

$$P(F|E) = \frac{P(EF)}{P(E)}.$$

- Thus $P(EF) = P(F|E)P(E)$.
- If E is independent of F, then $P(EF) = P(E)P(F)$.
- Hence, from the above two equations, $P(F|E) = P(F)$.
- Therefore, whenever the event E is independent of F, the event F is also independent of E.

Dependent Events

If two events are not independent, they are dependent events.

Exercise-1

A card is selected at random from an ordinary deck of 52 playing cards. Let E be the event that the selected card is an ace and F be the event that it is a spade. Are E and F independent?

Solution:

- $P(E) = \dfrac{4}{52} = 1/13.$
- $P(F) = \dfrac{13}{52} = 1/4.$
- $P(EF) = \dfrac{1}{52} = P(E)P(F).$
- Therefore, E and F are independent.

Exercise-2

Two coins are flipped, and all 4 outcomes are assumed to be equally likely. Let E be the event that the first coin lands on heads and F be the event that the second lands on tails. Are E and F independent?

Solution:

- $S = \{(H, H), (H, T), (T, H), (T, T)\}$.
- $E = \{(H, H), (H, T)\}$. $P(E) = 2/4 = 1/2$.
- $F = \{(H, T), (T, T)\}$. $P(F) = 2/4 = 1/2$.
- $EF = \{(H, T)\}$. $P(EF) = 1/4$.
- Thus, $P(EF) = P(E)P(F)$ and so E and F are independent.

Exercise-3(a)

Suppose that we toss 2 fair dice. Let E denote the event that the sum of the dice is 6 and F denote the event that the first die equals 4. Are E and F independent?

Solution:

- $|S| = 36$.
- $E = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$. $P(E) = 5/36$.
- $F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$.
  $P(F) = 6/36 = 1/6$.
- $EF = \{(4,2)\}$. $P(EF) = 1/36$.
- $P(EF) \neq P(E)P(F)$. Thus, E and F are not independent.

Exercise-3(b)

Suppose that we toss 2 fair dice. Let E denote the event that the sum of the dice is 7 and F denote the event that the first die equals 4. Are E and F independent?

Solution:

- $|S| = 36$.
- $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$.
  $P(E) = 6/36 = 1/6$.
- $F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$.
  $P(F) = 6/36 = 1/6$.
- $EF = \{(4,3)\}$. $P(EF) = 1/36$.
- $P(EF) = P(E)P(F)$. Thus, E and F are independent.

Exercise-4

Let S be the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ in the plane. Consider the uniform probability space on the square, and let A be the event

$$\{(x, y) : 0 \leq x \leq 1/2, 0 \leq y \leq 1\}$$

and B be the event

$$\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1/4\}.$$

Are A and B independent events?

Solution:

- Area corresponding to the sample space S is 1.
- Area corresponding to event A is $1/2$.
- Area corresponding to event B is $1/4$.
- $P(A) = 1/2$, $P(B) = 1/4$.

Exercise-4

Solution (contd...):

- $$A \cap B = \{(x, y) : 0 \leq x \leq 1/2, 0 \leq y \leq 1/4\}.$$

  is a subrectangle of the square S having area $1/8$.
- Therefore, $P(A \cap B) = 1/8 = P(A)P(B)$.
- Thus, A and B are independent events.

Independent Events

If E and F are independent, then so are E and $\bar{F}$.

Proof:

- $E = EF \cup E\bar{F}$.
- Since, $F\bar{F} = \phi$, hence $EF \cap E\bar{F} = \phi$.
- So,

$$P(E) = P(EF) + P(E\bar{F}).$$

- Since E and F are independent, hence

$$\begin{array}{rlcl} & P(E) & = & P(E)P(F) + P(E\bar{F}) \\ \Leftrightarrow & P(E)(1 - P(F)) & = & P(E\bar{F}) \\ \Leftrightarrow & P(E)P(\bar{F}) & = & P(E\bar{F}). \end{array}$$

- Thus, E and $\bar{F}$ are independent.

Independent Events

Let there be three events A, B and C. Take $S = \{1, 2, 3, 4\}$ and $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = 1/4$. Let $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{1, 4\}$.

1. Are A and B independent?
2. Are B and C independent?
3. Are A and C independent?
4. Are A, B and C mutually independent i.e.

$$P(A \cap B \cap C) = P(A)P(B)P(C)?$$

Solution:

- $P(A) = P(B) = P(C) = 1/2$.
- $P(A \cap B) = P(\{1\}) = 1/4 = P(A)P(B)$. So, A and B are independent. Similarly, B and C are independent and A and C are independent.

Independent Events

- $P(ABC) = P(A)P(B|A)P(C|AB) = P(A)P(B)P(C|AB)$.
- If $P(C|AB) = P(C)$, then we can say A, B and C are mutually independent.
- $P(C) = 1/2$ whereas $P(C|AB) = 1$.
- Thus, $P(ABC) \neq P(A)P(B)P(C)$.
- Though A, B and C are pairwise independent, they are not mutually independent.

Mutually Independent Events

Three events A, B and C are mutually independent events if

$$\begin{aligned} P(ABC) &= P(A)P(B)P(C) \\ P(AB) &= P(A)P(B) \\ P(AC) &= P(A)P(C) \\ P(BC) &= P(B)P(C) \end{aligned}$$

Mutually Independent Events

The events $E_1, E_2, \ldots, E_n$ are said to be mutually independent (or independent) if, for every subset $E_{1'}, E_{2'}, \ldots, E_{r'}$, $r \leq n$, of these events,

$$P(E_{1'}E_{2'} \cdots E_{r'}) = P(E_{1'})P(E_{2'}) \cdots P(E_{r'}).$$

Finally, we define an infinite set of events to be mutually independent (or independent) if every finite subset of those events is independent.

Exercise-5

A sequence of n independent trials is to be performed. Each trial results in a success with probability p and a failure probability $1 - p$. What is the probability that

1. at least 1 success occurs in the first n trials;
2. exactly k success occur in the first n trials;
3. all trials result in successes?

- Let E_i denote the event of a failure on the i^{th} trial. Thus, $P(E_i) = (1 - p)$ for $1 \leq i \leq n$.
- Now,

$$P(E_1 E_2 \cdots E_n) = P(E_1)P(E_2) \cdots P(E_n) = (1 - p)^n$$

(1) Hence, the answer is $1 - (1 - p)^n$.

Exercise-5

Now, we solve part (2).

- For any fixed k successful events and $n-k$ unsuccessful events, the probability is $p^k(1-p)^{n-k}$.
- There are $\binom{n}{k}$ ways to choose k events out of n events, thus the desired probability is

$$\binom{n}{k} p^k (1-p)^{n-k}.$$

Now, we solve part (3).

- We are interested in the event $\bar{E}_1 \bar{E}_2 \cdots \bar{E}_n$.
- $$P(\bar{E}_1 \bar{E}_2 \cdots \bar{E}_n) = P(\bar{E}_1) P(\bar{E}_2) \cdots P(\bar{E}_n) = p^n.$$

Exercise-6

Suppose a machine produces bolts, 10% of which are defective. Find the probability that a box of 3 bolts contains at most one defective bolt.

Solution:

- Let $p = 0.1$ be the probability that a randomly chosen bolt is defective.
- Let E_0 be the event that there is no defective bolt in a box of 3 bolts.
- Let E_1 be the event that there is one defective bolt in a box of 3 bolts.
- We are interested in $P(E_0 \cup E_1) = P(E_0) + P(E_1)$ since $E_0 E_1 = \phi$ (or E_0 and E_1 are mutually exclusive.)

Solution (contd...):

- Now,

$$P(E_0) = (1-p)^3, \quad P(E_1) = \binom{3}{1} p(1-p)^2.$$

- Thus, the desired probability is

$$(1-p)^3 + 3p(1-p)^2.$$

- By putting the value of $p = 0.1$, we get

$$(0.9)^3 + 3(0.9)^2(0.1) = 0.972.$$

Thank You