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Probability
Problem Set - 08

Q1

sol Let the random variable $X_i = 1$ when i^{th} throw is one otherwise $X_i = 0$.

Let the random variable $Y_i = 1$ when i^{th} throw is six otherwise $Y_i = 0$

and $i = 1, 2, \ldots\ldots, n$

Then,

$X = X_1 + X_2 + X_3 + \ldots\ldots + X_n$

$Y = Y_1 + Y_2 + Y_3 + \ldots\ldots + Y_n$

We need to find $Cov(X, Y)$

So,

$$Cov(X, Y) = n\, Cov(X_i, Y_j)$$

as

* $Cov(X_i, Y_j) = 0$ when $i \neq j$
* Each term $Cov(X_i\, Y_i)$ is the same

So,

$$Cov(X_1, Y_1) = E(X_1 Y_1) - E(X_1)\, E(Y_1)$$

$$= 0 - \frac{1}{6} \times \frac{1}{6}$$

$$= \frac{-1}{36}$$

So,

$$Cov(X, Y) = \frac{-n}{36}$$

Q2

Sol Let S_n be random variable which denotes the sum of selected number

We want $E[S_n]$ and $Var[S_n]$

Let X_i be the selected number where $i = 1, 2, \ldots n$

So,

$$S_n = X_1 + X_2 + X_3 + \ldots + X_n$$

$$S_n = \sum_{i=1}^{n} X_i$$

Because X_i is equally likely to be any of the values from 1 to N,

it follows that

$$E[X_i] = \sum_{i=1}^{N} i\left(\frac{1}{N}\right) = \frac{\cancel{N}(N+1)}{2\cancel{N}} = \frac{(N+1)}{2}$$

Thus,

$$E[S_n] = E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \sum_{i=1}^{n} E[X_i]$$

$$= \cancel{\sum_{i=1}^{n} \frac{(N+1)}{2}}$$

$$= \frac{n(N+1)}{2}$$

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$$Var(S_n) = var\left[\sum_{i=1}^{n} X_i\right] = Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} cov(X_i, X_i) + \sum_{i=1}^{n}\sum_{j\neq i} Cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} var(X_i) + 2\sum_{i=1}^{n}\sum_{j<i} cov(X_i, X_j)$$

After this see book's question 114 (Pg 28) solution : Pg 162

Q3

$$P\{X=k\} = \frac{\lambda^k}{(e^\lambda - 1)k!} \quad ; k = 1, 2, \ldots\ldots$$

So,

$$E(X) = \sum_{k=1}^{\infty} \frac{k \cdot \lambda^k}{(e^\lambda - 1)k!}$$

$$= \frac{1}{(e^\lambda - 1)} \sum_{k=1}^{\infty} \frac{k\, \lambda^k}{k!}$$

$$= \frac{1}{(e^\lambda - 1)} \sum_{k=1}^{\infty} \frac{\cancel{k}\, \lambda\, \lambda^{k-1}}{\cancel{k}(k-1)!}$$

$$= \frac{\lambda}{(e^\lambda - 1)} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \frac{\lambda}{(e^\lambda - 1)} e^\lambda$$

$$E(X) = \frac{\lambda}{(1 - e^{-\lambda})}$$

Also we know that

$$E(X^2) = E(X^2 - X + X) = E(X(X-1) + X)$$

$$= E(X(X-1)) + E(X).$$

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Now

$$E(X(X-1)) = \sum_{k=2}^{\infty} k(k-1)\frac{\lambda^k}{(e^{\lambda}-1)k!} = \cancel{\frac{\lambda}{(e^{\lambda}-1)}\sum_{k=2}^{\infty}\frac{k(k-1)\lambda^2\lambda^{k-2}}{k(k-1)(k-2)!}}$$

$$= \frac{1}{(e^{\lambda}-1)}\sum_{k=2}^{\infty}\frac{\cancel{k(k-1)}\lambda^2\lambda^{k-2}}{\cancel{k(k-1)}(k-2)!}$$

$$= \frac{\lambda^2}{(e^{\lambda}-1)}\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!} = \frac{\lambda^2 e^{\lambda}}{(e^{\lambda}-1)}$$

$$= \frac{\lambda^2}{(1-e^{-\lambda})}$$

Now.

$$Var(X) = E(X(X-1)) + E(X) - (E(X))^2$$

$$= \frac{\lambda^2}{(1-e^{-\lambda})} + \frac{\lambda}{(1-e^{-\lambda})} - \frac{\lambda^2}{(1-e^{-\lambda})^2}$$

$$= \frac{\lambda^2+\lambda}{(1-e^{-\lambda})} - \frac{\lambda^2}{(1-e^{-\lambda})^2}$$

$$= \frac{\lambda(1-\lambda e^{-\lambda}-e^{-\lambda})}{(1-e^{-\lambda})^2}$$

Q4) n_1 = number of white balls.

n_2 = number of black balls.

X = Random variable which denotes the number of ball drawn.

Here getting a black ball is success (p) Hence It follows geometric distribution.

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- we want to find $E[x]$ hence expectation of geometric ~~distribution~~ random variable with probability p is $\frac{1}{p}$.

p = probability of getting black ball

$$P = \frac{{}^{n_1}C_{i-1} \times {}^{n_2}C_1}{{}^{n_1+n_2}C_i}$$

So, $$E(x) = \frac{1}{p} = \frac{{}^{n_1+n_2}C_i}{{}^{n_1}C_{i-1}\ {}^{n_2}C_{i-2}}$$

Q5

Sol Let X be the random ~~variable~~ variable which denotes that the largest selected number is m

So,
$$P(X=m) = \frac{m^n}{N^n}$$

we want to find $E(X)$

So,

(a) $$E(x) = \sum_{i=1}^{N} i\, P(x)$$

$$= N \cdot \frac{m^n}{N^n}$$

$$= \frac{m^n}{N^{n-1}}$$

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Q7 X be the random variable denotes the number of matches

$$\cancel{P(X=i) = \frac{{}^nC_i}{n!}}$$

So

$$\cancel{E(X) = \sum_{k=1}^{n} k \frac{{}^nC_i}{n!}}$$

let X_i the random variable that i^{th} is at it's correct place.

So

$$X = X_1 + X_2 + X_3 \dots X_n$$

$$X = \sum_{i=1}^{n} X_i$$

Now

$$E(X_i) = \sum_{i=1}^{n} i P(X_i)$$

$$P(X_i) = \frac{1}{n}$$

So,

$$E(X_i) = \sum_{i=1}^{n} \frac{1}{n} = \frac{\cancel{n}}{\cancel{n}} = 1$$

Now

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} E(X_i)$$

$$= n \times 1$$

$$= n$$