CSL003P1M : Probability and Statistics
Lecture 25 (Joint Moment Generating Functions)

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An Important Property of MGF

The moment generating function uniquely determines the distribution. That is, if $M_X(t)$ exists and is finite in some region about $t = 0$, then the distribution of X is uniquely determined.

For instance, if

$$M_X(t) = \left(\frac{1}{2}\right)^{10} (e^t + 1)^{10}$$

then it follows that X is a binomial random variable with parameters 10 and $\frac{1}{2}$. Note that the mgf of binomial random variable X is

$$M_X(t) = (pe^t + 1 - p)^n.$$

Exercise-1

Suppose that the moment generating function of a random variable X is given by $M(t) = e^{3(e^t - 1)}$. What is $P\{X = 0\}$?

The mgf of the Poisson random variable X with parameter λ is

$$M_X(t) = e^{-\lambda} e^{\lambda e^t}$$

It is clear from $M_X(t)$ that X is a Poisson random variable with parameter $\lambda = 3$. Thus,

$$P\{X = 0\} = e^{-3} \frac{3^0}{0!} = e^{-3}.$$

Exercise-2

Prove that the sum of independent random variables equals the product of the individual moment generating functions.

- Suppose that X and Y are independent and have moment generating functions $M_X(t)$ and $M_Y(t)$ respectively.
- Then $M_{X+Y}(t)$, the moment generating function of $X + Y$ is given by

$$
\begin{aligned}
M_{X+Y}(t) &= E[e^{t(X+Y)}] \\
&= E[e^{tX}e^{tY}] \\
&= E[e^{tX}]E[e^{tY}] \quad \text{(since } X \text{ and } Y \text{ are independent)} \\
&= M_X(t)M_Y(t)
\end{aligned}
$$

MGF of the sum of a random number of random variables

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables, and let N be a nonnegative integer. We want to compute the moment generating function of

$$Y = \sum_{i=1}^{N} X_i$$

It follows easily from the previous result that

$$M_Y(t) = [M_{X_1}(t)]^N$$

Joint Moment Generating Functions

For any n random variables $X_1, \ldots, X_n$, the joint moment generating function, $M(t_1, \ldots, t_n)$, is defined, for all real values of $t_1, \ldots, t_n$ by

$$M(t_1, \ldots, t_n) = E[e^{t_1 X_1 + \cdots + t_n X_n}]$$

The individual moment generating functions can be obtained from $M(t_1, \ldots, t_n)$ by letting all but one of the t_j's be 0. That is,

$$M_{X_i}(t) = E[e^{tX_i}] = M(0, \ldots, 0, t, 0, \ldots, 0)$$

where t is in the ith place.

Joint Moment Generating Functions

- It can be proven that the joint moment generating function $M(t_1, t_2, \ldots, t_n)$ uniquely determines the joint distribution of $X_1, \ldots, X_n$.
- This result can then be used to prove that the n random variables $X_1, \ldots, X_n$ are independent if and only if

$$M(t_1, \ldots, t_n) = M_{X_1}(t_1) \cdots M_{X_n}(t_n)$$

Proof: If the n random variables are independent, then

$$\begin{aligned} M(t_1, \ldots, t_n) &= E[e^{(t_1 X_1 + \cdots + t_n X_n)}] \\ &= E[e^{t_1 X_1} \cdots e^{t_n X_n}] \\ &= E[e^{t_1 X_1}] \cdots E[e^{t_n X_n}] \quad \text{(by independence)} \\ &= M_{X_1}(t_1) \cdots M_{X_n}(t_n) \end{aligned}$$

For the other direction, try yourself. (Hint: use the fact that the joint moment generating function uniquely determines the joint distribution.)

Exercise-3

If X and Y are independent binomial random variables with parameters (n, p) and (m, p), respectively, what is the distribution of $X + Y$?

The moment generating function of $X + Y$ is given by

$$\begin{aligned} M_{X+Y}(t) = M_X(t)M_Y(t) &= (pe^t + 1 - p)^n(pe^t + 1 - p)^m \\ &= (pe^t + 1 - p)^{m+n} \end{aligned}$$

However, $(pe^t + 1 - p)^{m+n}$ is the moment generating function of a binomial random variable having parameters $m + n$ and p. Thus, this must be the distribution of $X + Y$.

Exercise-3

If X and Y are independent Poisson random variables with parameters λ_1 and λ_2, what is the distribution of $X+Y$?

The moment generating function of $X+Y$ is given by

$$\begin{aligned} M_{X+Y}(t) &= M_X(t)M_Y(t) \\ &= e^{-\lambda_1}e^{\lambda_1 e^t}e^{-\lambda_2}e^{\lambda_2 e^t} \\ &= e^{-(\lambda_1+\lambda_2)}e^{(\lambda_1+\lambda_2)e^t} \end{aligned}$$

However, $e^{-(\lambda_1+\lambda_2)}e^{(\lambda_1+\lambda_2)e^t}$ is the moment generating function of a Poisson random variable having parameter $\lambda_1+\lambda_2$. Thus, this must be the distribution of $X+Y$.

Exercise-4

Let X be a negative binomial random variable with parameters r and p. Find the mgf of X given that the mgf of a geometric random variable with parameter p is

$$\frac{pe^t}{1-(1-p)e^t}$$

Solution:

- For $1 \leq i \leq r$, let X_i be independent geometric random variables each with parameter p.
- Then

$$X = X_1 + X_2 + \cdots + X_r$$

- Thus,

$$M_X(t) = [M_{X_1}(t)]^r = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r$$

Thank You