CSL003P1M : Probability and Statistics
QuestionSet - 09: Moment Generating Functions and Inequalities

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1. Let X be uniformly distributed on (a, b). Find $M_X(t)$.
2. Express the moment generating function of $Y = a + bX$ in terms of $M_X(t)$ (here a and b are constants).
3. Let X be a continuous random variable having the density $f_X(x) = (1/2)e^{-|x|}$, $-\infty < x < \infty$.
   (a) Find $M_X(t)$.
   (b) Use $M_X(t)$ to find a formula for $E[X^{2n}]$ and $E[X^{2n+1}]$.
4. Let $X_1, \ldots, X_n$ be independent, identically distributed random variables such that $M_{X_1}(t)$ is finite for all t. Use moment generating functions to show that
$$E[(X_1 + \cdots + X_n)^3] = nE[X_1^3] = 3n(n-1)E[X_1^2]E[X_1] + n(n-1)(n-2)(E[X_1])^3$$
5. Let X have a gamma distribution with parameters α and λ. Use the previous result to show that
$$P\left\{X \geq \frac{2\alpha}{\lambda}\right\} \leq \left(\frac{2}{e}\right)^{\alpha}.$$
6. If $g(x) \geq 0$ for every x and $g(x) \geq c$ for $x \in (\alpha, \beta)$, then
$$P\{X \in (\alpha, \beta)\} \leq c^{-1}E[g(x)]$$
7. (Continuation). Show that for every constant $t > 0$
$$P\{X > t\} \leq \frac{1}{(t+c)^2}E[(X+c)^2]$$
8. If X_1 and X_2 are independent and identically distributed random variables then for every $t > 0$
$$P\{|X_1 - X_2| > t\} \leq 2P\left\{|X_1| > \frac{1}{2}t\right\}$$
9. If $g(x) \geq 0$ and even, i.e., $g(x) = g(-x)$ and in addition $g(x)$ is non-decreasing for $x > 0$, show that for every $c > 0$
$$P\{|X| \geq c\} \leq \frac{E[g(x)]}{g(c)}$$
10. (Continuation). If $g(x)$ of the previous exercise satisfies $|g(x)| \leq M < \infty$, then
$$P\{|X| \geq c\} \geq \frac{E[g(X)] - g(c)}{M}.$$
11. (Continuation). Let g be same as in the previous exercise and $P\{|X| \leq M\} = 1$, then
$$P\{|X| \geq c\} \geq \frac{E[g(X)] - g(c)}{g(M)}$$