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Ques ① → X, Y & Z are independent geometric R.V.

Now → taking a random instance →

suppose → $X = x, Y = y, Z = k$.

So → $\boxed{P(X+Y \leq Z) = \sum_{\forall x, y, k} P(x+y \leq k)}$

→ This is behaving like compositively joint prob. distribution of R.V. (X+Y).

$$\Downarrow$$

$$\boxed{P(X+Y \leqslant Z) = \sum_{\forall k} P(X+Y \leqslant k)}$$

So → we can say that →

$$P(X+Y \leqslant Z) = \frac{1 + P(X+Y=Z)}{2} = \frac{1 + \sum_{\forall k} P(X+Y=k)}{2}$$

Now → $P(X+Y=Z) \Rightarrow \sum_{\forall K} P(X+Y=k)$

⇒ this can be written as →

$$\Rightarrow \sum_{K=1}^{x+y} P_X(X=i) \cdot P_Y(Y=k-i) \; ; \; \left\{ \begin{array}{l} \text{Since } X \,\&\, Y \\ \text{are indepen.} \end{array} \right\}$$

$$\Rightarrow \sum_{K=1}^{x+y} \{(1-p)^{i-1} \cdot p\} \cdot \{(1-p)^{k-i-1} \cdot p\}$$

$$\Rightarrow \sum_{K=1}^{X+Y} \{ p^2 \cdot (1-p)^{k-2} \}$$

or

in general ⇒ $\boxed{P(X+Y=Z) = p^2 \cdot (1-p)^{X+Y-2}}$

So $\rightarrow$ $$\boxed{P(X+Y\leq Z) = \frac{1+p^2(1-p)^{x+y-2}}{2}}$$

(2). X = R.V. denoting no. of black balls in sample.

So $\rightarrow$ $$\boxed{X = x_1 + x_2 + x_3 + \cdots + x_r}$$

where x_i are R.V., such that $\rightarrow$

$$x_i = \begin{cases} 1 ; & \text{if ith ball taken is black.} \\ 0 ; & \text{~~oth~~ otherwise.} \end{cases}$$

So $\rightarrow$ for every x_i $\rightarrow$ $P(X_i = 1) = \left(\frac{b}{n}\right) = (P)$

where $\boxed{n = (b+g)}$ $\quad P(X_i = 0) = \left(\frac{g}{n}\right) = (1-P)$

So $\rightarrow$ $E[X_i] = 1.P + 0.(1-P) = P \Rightarrow \boxed{E[X_i] = P}$

Now $\rightarrow$ we know that:

$$E[X] = E[\sum_i X_i] = \sum_i E[X_i]$$

So $\rightarrow$ (since every x_i, distribution is identical) $\rightarrow$ $$\boxed{E[X] = r.P = \left(\frac{r.b}{b+g}\right)}$$

Now $\rightarrow$ $$Var[X] = Var.[\sum_{i=1}^{r} x_i]$$

$$\Rightarrow \sum_{i=1}^{r} Var[x_i] + 2\sum_{1\leq i<j\leq r} Cov[x_i, x_j]$$

$$\Rightarrow \underbrace{\sum_{i=1}^{r} \left\{E[x_i^2] - (E[x_i])^2\right\}}_{\text{easy}} + \underbrace{2\sum_{1\le i,j\le r} Cov.[x_i, x_j]} \quad \text{— (1)}$$

$$Cov.(x_i, x_j) = E[x_i x_j] - E[x_i]\cdot E[x_j]$$

$$\Rightarrow \sum x_i \cdot x_j \cdot P(x_i x_j) - E[x_i]\cdot E[x_j]$$

$$\Rightarrow \frac{b\cdot(b-1)/\cancel{2}}{n(n-1)/\cancel{2}} - \left(\frac{b}{n}\right)^2$$

$$\Rightarrow \frac{b(b-1)}{n(n-1)} - \left(\frac{b}{n}\right)^2$$

$$\boxed{Cov.(x_i x_j) \Rightarrow \frac{b(b-n)}{n^2(n-1)}}$$

putting this in eq. (1) $\rightarrow$

$$\Rightarrow \sum_{i=1}^{r} \left\{E[x_i] - (E[x_i])^2\right\} + \cancel{2}\cdot\frac{r(r-1)}{\cancel{2}} \cdot \frac{b(b-n)}{n^2(n-1)}$$

$$\Rightarrow \frac{br(n-b)}{n^2} + r(r-1)\cdot\frac{b(b-n)}{n^2(n-1)}$$

$$\boxed{Var[x] \Rightarrow \frac{rbg}{n^2}\cdot\left(1 - \frac{(r-1)}{(n-1)}\right)}$$

③ → $n_R = 3$ red balls.

$n_b = 2$ black balls.

sample $(k) = 2$.

Let RV. → U → No. of red balls. selected.

V → " " black " "

So →

$$\rho(U,V) = \frac{Cov(U,V)}{\sqrt{Var(U) \cdot Var(V)}}$$

$$Cov(U,V) = E(UV) - E(U) \cdot E(V)$$

probability fn. calculations → making combinations of red & black balls & calculating prob. →

$$P(0,2) \Rightarrow \frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} = \left(\frac{1}{10}\right)$$

$$P(1,1) \Rightarrow \frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = \left(\frac{6}{10}\right)$$

$$P(2,0) \Rightarrow \frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} = \left(\frac{3}{10}\right)$$

$$E(UV) = \sum_{\forall x,y} xy \cdot P(x,y)$$

$$\Rightarrow 0 + 1 * 1 * \frac{6}{10} + 0$$

$$\boxed{E(UV) = \frac{3}{5}}$$

Now, since U & V are hypergeometric distributions. So,

Directly putting expectations →

$$E[U] = k \cdot \frac{n_R}{n_R + n_b} \Rightarrow \left(\frac{6}{5}\right)$$

& → $$E[V] = \frac{k \cdot n_b}{n_b + n_R} \Rightarrow \left(\frac{4}{5}\right)$$

$$\text{Cov.}(X,Y) = \frac{3}{5} - \frac{24}{25} \Rightarrow \left(\frac{-9}{25}\right)$$

Now → Since we already know that →

$$\text{Var}(X) = k \cdot p \cdot (1-p) \cdot \left(1 - \frac{k-1}{n-1}\right)$$

if X is hypergeometric)

$$\text{Var}(U) = 2 \times \frac{3}{5} \times \frac{2}{5} \times \left(1 - \frac{2-1}{5-1}\right)$$

$$\Rightarrow 2 \times \frac{3}{5} \times \frac{2}{5} \times \frac{3}{4} \Rightarrow \frac{36}{25}$$

$$\text{Var}(V) \Rightarrow 2 \times \frac{2}{5} \times \frac{3}{5} \times \left(1 - \frac{2-1}{5-1}\right) \Rightarrow \frac{36}{25}$$

Thus →

$$\rho(U,V) = \frac{\frac{-9}{25}}{\sqrt{\frac{36}{25} \cdot \frac{36}{25}}} \Rightarrow \frac{-9}{\cancel{25}} \times \frac{\cancel{25}}{36} \Rightarrow \frac{-1}{4}$$

(4) $\rightarrow$ R.V. $\rightarrow$ $x_1, x_2, x_3, \ldots, x_n$

Since $\rightarrow$ MGf $\rightarrow$

$$M[t] = E[e^{tx}] = \sum_{\forall x} e^{tx} \cdot P(x)$$

& Joint MGf is defined as $\rightarrow$

$$M_{x+y}(t) = M_x(t) \cdot M_y(t)$$

Now $\rightarrow$

$$MGf(X_1 + X_2 + \cdots + X_n) =$$

$$M_{X_1}(t) * M_{X_2}(t) * \cdots * M_{X_n}(t)$$

So $\rightarrow$

$$M_X(t) = [M_{X_1}(t)]^n$$

But $\rightarrow$ $M_{X_1}(t) = \sum_x e^{tx_1} P(x)$

Now $\rightarrow$

L.H.S $\rightarrow$ $= E[(X_1 + \cdots X_n)^3]$

$\Rightarrow E[Y^3] \longrightarrow$ it is 3^{rd} moment.