CSL003P1M : Probability and Statistics
Lecture 06 (Bayes' Theorem)

Sumit Kumar Pandey

September 15, 2021

Bayes' Theorem

Bayes' Theorem

Suppose $A_1, A_2, \cdots, A_n$ are mutually disjoint events with union S, the sample space. Let B be the event such that $P(B) > 0$ and suppose $P(B|A_k)$ and $P(A_k)$ are specified for $1 \leq k \leq n$. Then,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)}.$$

Proof:

- $$P(A_i|B)P(B) = P(A_iB) = P(BA_i) = P(B|A_i)P(A_i).$$

  So,

- $$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}. \tag{1}$$

Bayes' Theorem

Now, we calculate $P(B)$.

- Note that,

$$B = S \cap B = (A_1 \cup A_2 \cup \cdots \cup A_n) \cap B.$$

- So,

$$B = (A_1 B) \cup (A_2 B) \cup \cdots \cup (A_n B).$$

- Since, A_i's are mutually exclusive, therefore BA_i's are also mutually exclusive for $1 \leq i \leq n$.
- So, from the result R_2

$$P(B) = \sum_{k=1}^{n} P(BA_k) = \sum_{k=1}^{n} P(A_k) P(B|A_k).$$

- Putting the value of $P(B)$ in equation (1), we get

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)}$$

Exercise-1

Suppose that the population of a certain city is 40% male and 60% female. Suppose also that 50% of the males and 30% of the females smoke. Find the probability that a smoker is male.

Solution:

- Let A_1 denote the event that a person selected is a male.
- Let A_2 denote the event that a person selected is a female.
- Let B denote the event that a person selected smokes.
- Then,

$$P(A_1) = 0.4, \;\; P(A_2) = 0.6, \;\; P(B|A_1) = 0.5, \;\; P(B|A_2) = 0.3.$$

- We are interested in $P(A_1|B)$.

Exercise-1

- Note that $A_1 \cup A_2 = S$.
- Thus, we can apply Bayes' Theorem.
- So,

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{\sum_{k=1}^{2} P(A_k)P(B|A_k)} = \frac{0.4 \times 0.5}{0.4 \times 0.5 + 0.6 \times 0.3} = \frac{0.2}{0.38}$$

- Therefore,

$$P(A_1|B) = \frac{0.2}{0.38} \approx 0.53.$$

Exercise-2

An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident-prone. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

- Let A_1 be the event that a person is accident-prone.
- Let A_2 be the event that a person is not accident-prone.
- Let B be the event that a person will have an accident within an year.
- We are interested in $P(B)$.

Exercise-2

An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident-prone. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

- Then, $A_1 \cup A_2 = S$, $A_1 \cap A_2 = \phi$ and

$$P(A_1) = 0.3,\ \ P(A_2) = 0.7, P(B|A_1) = 0.4,\ \ P(B|A_2) = 0.2.$$

Exercise-2

We have $A_1 \cup A_2 = S$, $A_1 \cap A_2 = \phi$ and

$$P(A_1) = 0.3,\ P(A_2) = 0.7, P(B|A_1) = 0.4,\ P(B|A_2) = 0.2.$$

Now, we find $P(B)$.

- Note that

$$B = SB = (A_1 \cup A_2)B = (A_1B) \cup (A_2B)$$

- Since $A_1A_2 = \phi$, hence

$$P(B) = P(A_1B) + P(A_2B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2)$$

- Therefore,

$$P(B) = 0.3 \times 0.4 + 0.7 \times 0.2 = 0.12 + 0.14 = 0.26.$$

Exercise-2 (Extension)

An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident-prone. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

Exercise-2 (Extension)

We have

$$P(A_1) = 0.3, P(B|A_1) = 0.4, \ P(B) = 0.26.$$

We want to find $P(A_1|B)$.

- We can apply Bayes' theorem. We get,

$$P(A_1|B)P(B) = P(B|A_1)P(A_1) = 0.4 \times 0.3 = 0.12.$$

- So,

$$P(A_1|B) = \frac{0.12}{P(B)} = \frac{0.12}{0.26} = \frac{6}{13} \approx 0.46.$$

Exercise-3

In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $1 - p$ be the probability that the student guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he or she answered it correctly?

- Let K_1 be the event that a student knows the answer.
- Let K_2 be the event that a student does not know the answer.
- Let C be the event that a student's guess is correct.
- We are interested in $P(K_1|C)$.

Exercise-3

We have $K_1 \cup K_2 = S$, $K_1 \cap K_2 = \phi$ and

$$P(K_1) = p, \; P(K_2) = 1 - p, P(C|K_1) = 1, P(C|K_2) = 1/m.$$

Now, we find $P(K_1|C)$. We will apply Bayes' theorem.

- $$\begin{aligned} P(K_1|C) &= \frac{P(K_1C)}{P(C)} \\ &= \frac{P(C|K_1)P(K_1)}{P(C|K_1)P(K_1) + P(C|K_2)P(K_2)} \\ &= \frac{p}{p + (1/m)(1-p)} \\ &= \frac{mp}{1 + (m-1)p} \end{aligned}$$

Exercise-4

Suppose there are three chests each having two drawers. The first chest has a gold coin in each drawer, the second chest has a gold coin in one drawer and a silver coin in the other drawer, and the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer opened. If the drawer contains a gold coin, what is the probability that the other drawer also contains a gold coin?

- Let C_i be the event that the i chest for $i = 1, 2, 3$ was chosen randomly. Then $P(C_i) = 1/3$.
- Let D be the event that the chosen drawer has a gold coin in it. Then,

$$P(D|C_1) = 1,\ P(D|C_2) = 1/2,\ P(D|C_3) = 0.$$

- We are interested in $P(C_1|D)$.

$\mathit{Exercise}$-4

We have $C_1 \cup C_2 \cup C_3 = S$ and $C_i \cap C_j = \phi$ for $i \neq j$.

$$P(C_i) = 1/3, P(D|C_1) = 1, P(D|C_2) = 1/2, P(D|C_3) = 0.$$

We want to find $P(C_1|D)$.

- We can apply Bayes' theorem. We get

$$\begin{aligned} P(C_1|D) &= \frac{P(D|C_1)P(C_1)}{P(D|C_1)P(C_1) + P(D|C_2)P(C_2) + P(D|C_3)P(C_3)} \\ &= \frac{1 \times (1/3)}{1 \times (1/3) + (1/2) \times (1/3) + 0 \times (1/3)} \\ &= \frac{2}{3}. \end{aligned}$$

Thank You