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→ 2021PCS1017

Ques(1) → let X → is a R.V. which denotes the total number of matches.

$$X = X_1 + X_2 + \cdots + X_n$$

where X_i is also a R.V. which denotes, if ith card is in it's position (or) not. So →

$$X_i \Rightarrow \begin{cases} 1 ; & \text{if } i\text{th card is at correct place} \\ 0 ; & \text{otherwise.} \end{cases}$$

So →

$$\boxed{X = \sum_{i=1}^{n} X_i} \Rightarrow E(X) = E\left(\sum_{i=1}^{n} X_i\right)$$

$$\boxed{E(X) = \sum_{i=1}^{n} E(X_i)} \text{ —— } (1).$$

But we know that →

$$\cancel{E(X_i) = \sum_{i=1}^{n} i \cdot P(X_i)}$$

amongst all the cards, every card is equally likely to be at it's correct place. So →

$$E[X_i] = P\{X_i = 1\} = \left(\frac{1}{n}\right)$$

so, from eq.^{n} (1) →

$$E[X] = \sum_{i=1}^{n} \left(\frac{1}{n}\right) \Rightarrow \frac{\cancel{n}}{\cancel{n}} = 1 \Rightarrow \boxed{E[X] = 1}$$

{Answer}.

Now for variance →

$$\boxed{Var(X) = E[X^2] - (E[X])^2}$$

→ we already know this part

So $\rightarrow$ $$E[X^2] = E\left[\left(\sum_{i=1}^{n} X_i\right)\cdot\left(\sum_{j=1}^{n} X_j\right)\right]$$

$$\Rightarrow \sum_{i=1}^{n} E[X_i^2] + \sum_{i=1}^{n}\sum_{j\neq i} E[X_i X_j]$$

$$\Rightarrow \underbrace{\sum_{i=1}^{n} E[X_i]}_{\text{we know this}} + \underbrace{\sum_{i=1}^{n}\sum_{j\neq i} E[X_i X_j]}\ ; \quad \left\{\text{Since we know that } \boxed{X_i^2 = X_i}\right. \quad \text{---} (2).$$

Since $\rightarrow$
$$X_i X_j = \begin{cases} 1 ; & \text{if } X_i = 1 \text{ \& } X_j = 1 \\ 0 ; & \text{OW.} \end{cases}$$

$$E[X_i X_j] = P\{X_i = 1, X_j = 1\}$$

$$\boxed{E[X_i X_j] \Rightarrow \left(\frac{1}{n}\right)^2}$$

from eq.^{n} (2) $\rightarrow$

$$E[X^2] = E[X] + \sum_{i=1}^{n}\sum_{j\neq i} E[X_i X_j]$$

$$\Rightarrow 1 + \cancel{n}(n-1) \times \frac{1}{n^{\cancel{2}}} \Rightarrow \left(1 + \frac{n-1}{n}\right)$$

Hence $\rightarrow$ $$Var(X) = E[X^2] - (E[X])^2$$

$$\Rightarrow \cancel{1} + \left(\frac{n-1}{n}\right) - \cancel{1} \Rightarrow \left(\frac{n-1}{n}\right)$$

So $\rightarrow$ $$\boxed{Var(X) = \left(1 - \frac{1}{n}\right)}$$

⇒ Que. 5 →

$$f(x,Y) = \frac{2}{(1+x+Y)^3} \quad ; \boxed{x>0,\ Y>0}$$

(a). Now, since we know that:

$$F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\cdot dy\cdot dx$$

So →

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{2}{(1+P+Q)^3}\cdot dQ\cdot dP$$

$$= 2\cdot\int_{0}^{x}\left\{\int_{0}^{Y} \frac{1}{(1+P+Q)^3}\cdot dQ\right\} dP$$; [limits have changed bcoz of condn given]

$$\Rightarrow 2\cdot\int_{0}^{x} \frac{1}{2}\cdot\left\{\frac{1}{(1+P)^2} - \frac{1}{(1+P+y)^2}\right\}\cdot dP$$

$$\Rightarrow \int_{0}^{x} \frac{1}{(1+P)^2}\cdot dP - \int_{0}^{x}\frac{1}{(1+P+y)^2}\cdot dP$$

$$F(x,y) = \left(1-\frac{1}{1+x}\right) - \left(\frac{1}{1+y} - \frac{1}{1+x+y}\right)$$

$$\boxed{F(x,y) = \left\{1 - \frac{1}{1+x} + \frac{1}{1+x+y} - \frac{1}{1+y}\right\}}$$

(b) $\rightarrow$ $f_X(x) = \int_{-\infty}^{+\infty} f(x,y) \cdot dy$

$$\Rightarrow \int_0^{\infty} \frac{2}{(1+x+y)^3} \cdot dy$$; [limit is changed bcoz of condn.]

$$\Rightarrow 2 * \frac{1}{2} \cdot \left[\frac{-1}{(1+x+y)^2} \right]_0^{\infty}$$

$$\Rightarrow \left[\frac{1}{(1+x+y)^2} \right]_{\infty}^{0} \Rightarrow \frac{1}{(1+x)^2} - 0$$

$$\boxed{f_X(x) = \frac{1}{(1+x)^2}} \quad \text{—— (2).}$$

(c) $\rightarrow$ $f_Y(y/X=x) = \dfrac{f(x,y)}{f_X(x)}$

$$= \frac{\frac{2}{(1+x+y)^3}}{\frac{1}{(1+x)^2}} \Rightarrow \boxed{\frac{2(1+x)^2}{(1+x+y)^3}}$$

By putting answer from part (b).

Que 7. →

$$f(x_1, x_2, \ldots x_n/\theta) = \frac{e^{-x^2/2\theta}}{\sqrt{2\pi\theta}}$$

we can also write it as →

$$\boxed{f(x/\theta) \Rightarrow \frac{1}{\sqrt{\theta}\cdot\sqrt{2\pi}} \cdot e^{-(x-0)^2/2\cdot(\sqrt{\theta})^2}}$$

⇓

My observation is that, this is following normal distribution with mean = 0 & variance = θ

Now, we know that →

cond.^{n} for unbiased estimator → we should get →

$\boxed{E[\hat{\theta}] = \theta}$; then it'll be unbiased.

Now first of all cal. $E[x^2]$ ⇒

$$\boxed{E[x^2] = \int_{-\infty}^{+\infty} x^2 \cdot f(x/\theta) \cdot dx}$$

; ⇒ we may solve this, but we already know that →

in case of Normal distribution →

$E[x] = $ mean

and $\boxed{E[x^2] = (\text{mean})^2 + (\text{variance})}$

So →

Hence $E[x^2] = (0)^2 + \theta \Rightarrow \boxed{E[x^2] = \theta}$

Now →

we are given $\rightarrow$ $$\hat{\theta} = \frac{\sum_{i=1}^{n} X_i^2}{n}$$

so $\rightarrow$ $$E[\hat{\theta}] = \frac{E\left[\sum_{i=1}^{n} X_i^2\right]}{n}$$

$$\Rightarrow \frac{\sum_{i=1}^{n} E[X_i^2]}{n} \Rightarrow \frac{\sum_{i=1}^{n} \theta}{n}$$

$$\Rightarrow \frac{\theta \cdot \sum_{i=1}^{n} 1}{n}$$

$$\Rightarrow \frac{\theta \cdot \cancel{n}}{\cancel{n}} \Rightarrow \theta$$

so $\rightarrow$ $$\boxed{E[\hat{\theta}] = \theta}$$

$\rightarrow$ Hence, it is unbiased.

$\Rightarrow$ Que $\rightarrow$ 4. we are given $\rightarrow$

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & ; x \geq 0 \\ 0 & , \text{o.w.} \end{cases}$$

(a). for $\boxed{Y = X^2}$ $\rightarrow$

$$F_Y(y) = P\{Y \leq y\} \quad ; \text{ for all values } \geq 0$$

$$= P\{X^2 \leq y\}$$

$$\Rightarrow P\{-\sqrt{y} \leq X \leq \sqrt{y}\}$$

$\Rightarrow F_X(\sqrt{y}) - F_X(-\sqrt{y})$

So $\rightarrow$

$F_Y(y) = \cancel{(1-e^{-\lambda\sqrt{y}}) - (1-e^{-\lambda\sqrt{y}})}$

$\Rightarrow (1-e^{-\lambda\sqrt{y}}) - 0$; {Bcoz CDF of all (-ve) no. is 0, in exponential distribution}.

$$\boxed{F_Y(y) = 1-e^{-\lambda\sqrt{y}}}$$

Now, After differentiation $\rightarrow$ $f_Y(y) = -1 \times \frac{-\lambda}{2\sqrt{y}} \cdot e^{-\lambda\sqrt{y}}$

$$\Rightarrow \boxed{f_Y(y) = \frac{\lambda}{2\sqrt{y}} \cdot e^{-\lambda\sqrt{y}}}$$

(b) $\rightarrow$ for $\boxed{Y=|X|}$ $\rightarrow$ $F_Y(y) = P\{Y \leq y\}$

$\Rightarrow P\{|X| \leq y\}$

$\Rightarrow P\{-y \leq x \leq y\}$

$\Rightarrow F_X(y) - F_X(-y)$

$= (1-e^{-\lambda y}) - 0$ {bcoz CDF of all -ve no. in exponential dist. is 0}.

$$\boxed{F_Y(y) = (1-e^{-\lambda y})}$$

Differentiating both sides $\rightarrow$

$f_Y(y) = -1 \times -\lambda \cdot e^{-\lambda y}$

$$\boxed{f_Y(y) \Rightarrow \lambda \cdot e^{-\lambda y}}$$

Que. (8) → We are given →

$$U = X + Y \quad \text{—} (1)$$

$$V = X/Y \quad \text{—} (2)$$

So →

$$J = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} \Rightarrow \frac{-x}{y^2} - \frac{1}{y} = \left(\frac{-xy - y^2}{y^2} \right)$$

After solving eq.^n (1) & (2) → we'll get ⇒ $\boxed{x = \frac{UV}{V+1}}$ & $\boxed{y = \frac{U}{V+1}}$

Now →

$$f_{U,V}(U,V) = f_{x,y}\left(\frac{UV}{V+1}, \frac{U}{V+1}\right) \cdot \frac{y^2}{-xy - y^2}$$

Since X, & Y are independent. So →

$$f(x,y) = f(x) \cdot f(y)$$

$$\Rightarrow \theta . e^{-\theta x} . \theta . e^{-\theta y}$$

$$\Rightarrow \theta . e^{-\theta(x+y)}$$

So →

$$f_{U,V}(U,V) = \theta . e^{-\theta\left(\frac{U + UV}{V+1}\right)} \cdot \frac{y^2}{-xy - y^2}$$

$$= \theta . e^{-\theta . U} \cdot \frac{y^2}{-xy - y^2}$$

Now, putting values of x & $y \rightarrow$

$$f_{U,V}(U,V) = \theta . e^{-\theta U} . \left(\frac{U}{V+1}\right)^2 . \frac{(-)1}{\frac{U^2V + U^2}{(V+1)^2}}$$

$$\Rightarrow \theta . e^{-\theta U} . \left(\frac{U}{\cancel{V+1}}\right)^2 - \frac{\cancel{(V+1)^2}}{\cancel{U^2}.(V+1)}$$

$$\boxed{f_{U,V}(U,V) \Rightarrow \theta . e^{-\theta U} . \frac{-1}{(V+1)}}$$

Que 8 $\rightarrow$ Since confidence interval is given $\rightarrow$

Now $\rightarrow$ $100.(1-\alpha) = 90$

$$\alpha = 1 - \frac{90}{100}$$

$$\boxed{\alpha = 0.1} \; : \; \boxed{\alpha/2 = 0.05}$$

and $\rightarrow$ $n = 12$

& $m = 14$

$$\boxed{n+m = 26}$$

So, D.O.f (degree of freedom) = $n+m-2$

$\Rightarrow 24$

$t_{score} \Rightarrow 2.797$.

Now, we know that →

$$S_p = \frac{(n-1)\cdot S_1^2 + (m-1)\cdot S_2^2}{(n+m-2)}$$

$$\Rightarrow \frac{11 \times S_1^2 + 13 * S_2^2}{24}$$

for (S_1) →

$$S_1^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1} = 49.36$$

$$S_2^2 = \sum_{i=1}^{n} \frac{(Y_i - \bar{Y})^2}{m-1} = 33.52$$

⇒ Now →

$$S_p = \frac{11 * 49.36 + 13 * 33.52}{24}$$

$$\Rightarrow \frac{549.96 + 435.76}{24}$$

$$\boxed{S_p \Rightarrow 41.07}$$

Now, calculating the confidence intervals →

$$\Rightarrow \left(\bar{x} - \bar{y} - t_{\alpha/2, n+m-2}\, S_p \sqrt{\frac{1}{n} + \frac{1}{m}},\ \bar{x} - \bar{y} + t_{\alpha/2, n+m-2}\, S_p \sqrt{\frac{1}{n} + \frac{1}{m}}\right)$$

$$\Rightarrow \left(143 - 135.7 - 2.797 * 41.07 * \sqrt{\frac{1}{12} + \frac{1}{14}},\ 143 - 135.7 + 2.797 * 41.07 * \sqrt{\frac{1}{12} + \frac{1}{\sqrt{14}}}\right)$$

$\Rightarrow \left(143 - 135.7 - 45.18 \,,\ 143 - 135.7 + 45.18 \right)$

$$\boxed{\Rightarrow (-37.88 \,,\ 52.48)}$$

$\rightarrow$ Required confidence interval

Que. 2 $\rightarrow$ Given $\rightarrow$ $f_X(x) = \frac{1}{2} \cdot e^{-|x|}$; $-\infty < x < \infty$

(or).
$$\Rightarrow \begin{cases} \frac{1}{2} e^{x} \,, & -\infty < x < 0 \\ \frac{1}{2} e^{-x} \,; & 0 \leq x < \infty \end{cases}$$

(a). $\rightarrow$ Now $\rightarrow$

$$M_X(t) = \int_{-\infty}^{0} e^{tx} \cdot \frac{1}{2} e^{x} \cdot dx + \int_{0}^{\infty} e^{tx} \cdot \frac{1}{2} \cdot e^{-x} \cdot dx$$

$$\Rightarrow \frac{1}{2} \cdot \int_{-\infty}^{0} e^{(t+1)\cdot x} \cdot dx + \frac{1}{2} \int_{0}^{\infty} e^{(t-1)\cdot x} \cdot dx$$

$$\Rightarrow \frac{1}{2} \cdot \left[\frac{e^{(t+1)\cdot x}}{(t+1)} \right]_{-\infty}^{0} + \frac{1}{2} \cdot \left[\frac{e^{(t-1)\cdot x}}{(t-1)} \right]_{0}^{\infty}$$

$$\Rightarrow \frac{1}{2} \left[\frac{1}{(t+1)} - 0 \right] + \frac{1}{2} \left[0 - \frac{1}{(t-1)} \right]$$

$$\Rightarrow \frac{1}{2} \left[\frac{1}{t+1} - \frac{1}{(t-1)} \right] \Rightarrow \frac{1}{2} * \left[\frac{\cancel{t} - 1 - \cancel{t} - 1}{t^2 - 1} \right]$$

$$\Rightarrow \frac{-1}{(t^2 - 1)}$$

So $\rightarrow$ $$\boxed{M_X(t) = \frac{-1}{(t^2 - 1)}}$$

(b) Now, since

$$M_x(t) = \frac{-1}{(t^2-1)} \Rightarrow \left(\frac{1}{1-t^2}\right)$$

Now

$$M_x'(t) = +\frac{2t}{(1-t^2)^2};$$

So $\boxed{M_x'(0) = 0} \Rightarrow E(x) = 0$

Now Similarily cal.^{c} double derivative

$$M''(t) = \frac{2(1-t^2)^2 - 2(1-t^2)\cdot 2t}{(1-t^2)^4} * 2t$$

$$M''(t) \Rightarrow \frac{2(1+3t^2)}{(1-t^2)^3}$$

$$\boxed{M''(0) = 2 \Rightarrow ({}^2P_1)}$$

Now $$M'''(t) = \frac{6t(1-t^2)^3 - \left(3(1-t^2)^2 \cdot (-2t)(1+3t^2)\right)}{(1-t^2)^6}$$

$$\boxed{M'''(0) = 0}$$

Similarily we can cal^c further, that values will be in pattern.

So $M[x^{2n}] = ({}^{2n}P_n)$

$M[x^{2n+1}] = 0$

③ → Gamma Distribution →

$$f(\lambda, x) = \frac{\lambda \cdot e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma \alpha}$$

Now, we have to calc. →

$$P(X \geq 2\alpha/\lambda)$$

$$\Rightarrow 1 - P(X < (2\alpha/\lambda))$$

$$\Rightarrow 1 - \left\{ \cancel{\int_{-\infty}^{0} f(\lambda, x) \cdot dx} + \int_{0}^{2\alpha/\lambda} f(\lambda, x) \cdot dx \right\}$$

$$\Rightarrow 1 - \int_{0}^{2\alpha/\lambda} \frac{\lambda \cdot e^{-\lambda x} \cdot (\lambda x)^{\alpha - 1}}{\Gamma \alpha} \cdot dx$$

$$\Rightarrow 1 - \frac{\lambda}{\Gamma \alpha} \cdot \int_{0}^{2\alpha/\lambda} e^{-\lambda x} \cdot (\lambda x)^{\alpha - 1} \cdot dx$$

solve further to get answer, Sorry (No time)