CSL003P1M : Probability and Statistics
Lecture 17 (Conditional Distributions)

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October 12, 2021

Recall that,

For any two events E and F, the conditional probability of E given F is defined, provided that $P(F) > 0$, by

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Hence, if X and Y are discrete random variables, it is natural to define the conditional probability mass function of X given that $Y = y$, by

$$
\begin{aligned}
p_{X|Y(x|y)} &= P\{X = x|Y = y\} \\
&= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\
&= \frac{p(x, y)}{p_Y(y)}
\end{aligned}
$$

for all values of y such that $p_Y(y) > 0$.

Conditional Distribution

Similarly, the conditional probability distribution function of X given that $Y = y$ is defined, for all y such that $p_Y(y) > 0$, by

$$\begin{aligned} F_{X|Y}(x|y) &= P\{X \leq x|Y = y\} \\ &= \sum_{a \leq x} p_{X|Y}(a|y) \end{aligned}$$

In other words, the definitions are exactly the same as in the unconditional case, except that everything is now conditional on the event that $Y = y$.

Conditional Distribution

If X is independent of Y, then the conditional mass function and the distribution function are the same as the respective unconditional ones. This follows because if X is independent of Y, then

$$\begin{aligned} p_{X|Y}(x|y) &= P\{X = x|Y = y\} \\ &= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\ &= \frac{P\{X = x\}P\{Y = y\}}{P\{y = y\}} \\ &= P\{X = x\} \end{aligned}$$

Exercise-1

Suppose that $p(x, y)$, the joint probability mass function of X and Y, is given by

$$p(0,0) = 0.4,\ \ p(0,1) = 0.2,\ \ p(1,0) = 0.1,\ \ p(1,1) = 0.3$$

Calculate the conditional probability mass function of X given that $Y = 1$.

- Note that

$$p_Y(1) = \sum_x p(x,1) = p(0,1) + p(1,1) = 0.5.$$

- Hence,

$$p_{X|Y}(0|1) = \frac{p(0,1)}{p_Y(1)} = \frac{2}{5} \ \text{ and } \ p_{X|Y}(1|1) = \frac{p(1,1)}{p_Y(1)} = \frac{3}{5}.$$

Exercise-2

If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X given that $X + Y = n$.

Solution:

- We want to calculate $P\{X = k | X + Y = n\}$.
- $$\begin{aligned} P\{X = k | X + Y = n\} &= \frac{P\{X = k, X + Y = n\}}{P\{X + Y = n\}} \\ &= \frac{P\{X = k, Y = n - k\}}{P\{X + Y = n\}} \\ &= \frac{P\{X = k\}P\{Y = n - k\}}{P\{X + Y = n\}} \end{aligned}$$

Exercise-2

$$
\begin{aligned}
\frac{P\{X=k\}P\{Y=n-k\}}{P\{X+Y=n\}} &= \frac{\dfrac{e^{-\lambda_1}\lambda_1^k}{k!}\,\dfrac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}}{\left[\dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}\right]} \\
&= \frac{n!}{(n-k)!k!}\,\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}
\end{aligned}
$$

The conditional distribution of X given that $X+Y=n$ is the binomial distribution with parameters n and $\lambda_1/(\lambda_1+\lambda_2)$.

Exercise-3

Let X and Y be independent random variables each geometrically distributed with parameter p. Find

$$P\{Y = y|X + Y = z\} \text{ for } y = 1, 2, \dots, z - 1.$$

Solution:

- We know that $X + Y$ follows negative binomial distribution with parameters $(2, p)$.
- Thus, $P\{X + Y = z\} = (z - 1)p^2(1 - p)^{z-2}$.
- Now,

$$\begin{aligned} P\{Y = y|X + Y = z\} &= \frac{P\{Y = y, X + Y = z\}}{P\{X + Y = z\}} \\ &= \frac{P\{X = z - y, Y = y\}}{P\{X + Y = z\}} \end{aligned}$$

Exercise-3

$$
\begin{aligned}
\frac{P\{X = z-y, Y = y\}}{P\{X+Y=z\}} &= \frac{P\{X = z-y\}P\{Y=y\}}{P\{X+Y=z\}} \\
&= \frac{p(1-p)^{z-y-1}p(1-p)^{y-1}}{(z-1)p^2(1-p)^{z-2}} \\
&= \frac{1}{z-1}.
\end{aligned}
$$

Exercise-4

Let X and Y be independent random variables each follows negative binomial distribution with parameters (n_1, p) and (n_2, p). Find

$$P\{X = j | X + Y = k\}.$$

Solution:

- We know $X + Y$ follows negative binomial distribution with parameters $(n_1 + n_2, p)$.
- Thus, $P\{X + Y = k\} = \binom{k-1}{n_1 + n_2 - 1} p^{n_1+n_2}(1-p)^{k-n_1-n_2}$.
- Now,

$$\begin{aligned} P\{X = j | X + Y = k\} &= \frac{P\{X = j, X + Y = k\}}{P\{X + Y = k\}} \\ &= \frac{P\{X = j, Y = k - j\}}{P\{X + Y = k\}} \end{aligned}$$

Let $q = 1 - p$.

$$\begin{aligned}
\frac{P\{X = j, Y = k - j\}}{P\{X + Y = k\}} &= \frac{P\{X = j\}P\{Y = k - j\}}{P\{X + Y = k\}} \\
&= \frac{\binom{j-1}{n_1 - 1} p^{n_1} q^{j - n_1} \binom{k - j - 1}{n_2 - 1} p^{n_2} q^{k - j - n_2}}{\binom{k-1}{n_1 + n_2 - 1} p^{n_1 + n_2} q^{k - n_1 - n_2}} \\
&= \frac{\binom{j-1}{n_1 - 1}\binom{k - j - 1}{n_2 - 1}}{\binom{k-1}{n_1 + n_2 - 1}}
\end{aligned}$$

Thank You