CSL003P1M : Probability and Statistics
Lecture 28 (Some Problems on Inequalities)

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Markov's and Chebyshev's Inequality

Markov's Inequality

If X is a random variable that takes only nonnegative values, then for any value $a > 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

Chebyshev's Inequality

If X is a random variable with finite mean μ and variance σ^2, then, for any value $k > 0$,

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

One-Sided Chebyshev Inequality

If X is a random variable with mean 0 and finite variance σ^2, then, for any $a > 0$,

$$P\{X \geq a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Corollary

If $E[X] = \mu$ and $Var(X) = \sigma^2$, then, for $a > 0$,

$$P\{X \geq \mu + a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$
$$P\{X \leq \mu - a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Chernoff Bound

Chernoff Bound

$$P\{X \geq a\} \leq e^{-ta}M(t) \qquad \text{for all } t > 0$$
$$P\{X \leq a\} \leq e^{-ta}M(t) \qquad \text{for all } t < 0$$

Jensen's Inequality

If $f(x)$ is a convex function, then

$$E[f(X)] \geq f(E[X])$$

provided that the expectations exist and are finite.

Exercise-1

If the number of items produced in a factory during a week is a random variable with mean 100 and variance 400, compute an upper bound on the probability that this week's production will be at least 120.

Solution:

- From Markov's inequality

$$P\{X \geq 120\} \leq \frac{E[X]}{120} = \frac{100}{120} = \frac{5}{6}$$

- From one-sided Chebyshev inequality

$$P\{X \geq 120\} = P\{X - 100 \geq 20\} \leq \frac{400}{400 + 20^2} = \frac{1}{2}$$

Exercise-2

Consider a gambler who is equally likely to either win or lose 1 unit on every play, independently of his past results. That is, if X_i is the gambler's winnings on the ith play, then the X_i are independent and

$$P\{X_i = 1\} = P\{X_i = -1\} = \frac{1}{2}$$

Let $S_n = \sum_{i=1}^{n} X_i$ denote the gambler's winning after n plays. Use the Chernoff bound to find $P\{S_n \geq a\}$.

Solution: From Chernoff bounds, we obtain

$$P\{X \geq a\} \leq e^{-ta} M(t) \quad \text{for all } t > 0$$

Let's calculate $M(t)$ and then we obtain the best bound on $P\{X \geq a\}$ by using the t that minimizes $e^{-ta} M(t)$.

Exercise-2

$$E[e^{tX}] = \frac{e^t + e^{-t}}{2}$$

Now, using the McLaurin expansions of e^t and e^{-t}, we see that

$$\begin{aligned}
e^t + e^{-t} &= \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) + \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \\
&= 2\left\{1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right\} \\
&= 2\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \\
&\leq 2\sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!} \qquad \text{since } (2n)! \geq n!2^n \\
&= 2e^{t^2/2}
\end{aligned}$$

Therefore, $E[e^{tX}] \leq e^{t^2/2}$

Exercise-2

Since the moment generating function of the sum of independent random variables is the product of their moment generating functions, we have

$$E[e^{tS_n}] = (E[e^{tX}])^n \leq e^{nt^2/2}$$

So, using the Chernoff bound, we obtain

$$P\{S_n \geq a\} \leq e^{-ta} e^{nt^2/2} = e^{nt^2/2 - ta} \qquad t > 0$$

The value $nt^2/2 - ta$ is minimum at $t = a/n$. Supposing that $a > 0$ and letting $t = a/n$, we obtain

$$P\{S_n \geq a\} \leq e^{-a^2/2n} \qquad a > 0$$

For example,

$$P\{S_{10} \geq 6\} \leq e^{-36/20} \approx 0.1653$$

whereas the exact probability is

$$\begin{aligned} P\{S_{10} \geq 6\} &= P\{\text{gambler wins at least 8 of the first 10 games}\} \\ &= \frac{\binom{10}{8} + \binom{10}{9} + \binom{10}{10}}{2^{10}} = \frac{56}{1024} \approx 0.0547 \end{aligned}$$

Exercise-3

A set of 200 people consisting of 100 men and 100 women is randomly divided into 100 pairs of 2 each. Give an upper bound to the probability that at most 30 of these pairs will consist of a man and a woman.

Solution: Number the men arbitrarily from 1 to 100, and for $i = 1, 2, \ldots, 100$, let

$$X_i = \begin{cases} 1 & \text{if man } i \text{ is paired with a woman} \\ 0 & \text{otherwise} \end{cases}$$

Then X, the number of man-woman pairs, can be expressed as

$$X = \sum_{i=1}^{100} X_i$$

Because a man is equally likely to be paired with any of the other 199 people, of which 100 are women, we have

$$E[X_i] = P\{X_i = 1\} = \frac{100}{199}$$

So,

$$E[X] = \sum_{i=1}^{100} E[X_i] = (100)\frac{100}{199} \approx 50.25$$

$$\begin{aligned} Var(X_i) &= E[X_i^2] - E[X_i]^2 \\ &= E[X_i] - E[X_i]^2 \qquad \text{since } X_i^2 = X_i \\ &= \frac{100}{199} - \left(\frac{100}{199}\right)^2 = \frac{100}{199}\frac{99}{199} \end{aligned}$$

Exercise-3

Now, for $i \neq j$

$$X_i X_j = \begin{cases} 1 & \text{when } X_i = 1, X_j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} E[X_i X_j] &= 0 \cdot P\{X_i X_j = 0\} + 1 \cdot P\{X_i X_j = 1\} \\ &= P\{X_i = 1, X_j = 1\} \\ &= P\{X_i = 1\} P\{X_j = 1 | X_i = 1\} = \frac{100}{199} \frac{99}{197} \end{aligned}$$

For $i \neq j$,

$$\begin{aligned} Cov(X_i, X_j) &= E[X_i X_j] - E[X_i] E[X_j] \\ &= \frac{100}{199} \frac{99}{197} - \left(\frac{100}{199} \right)^2 \end{aligned}$$

Exercise-3

So,

$$
\begin{aligned}
Var(X) &= \sum_{i=1}^{100} Var(X_i) + 2\sum_{i<j}\sum Cov(X_i, X_j) \\
&= 100\frac{100}{199}\frac{99}{199} + 2\binom{100}{2}\left[\frac{100}{199}\frac{99}{197} - \left(\frac{100}{199}\right)^2\right] \approx 25.126
\end{aligned}
$$

Thus, from Chebyshev inequality

$$P\{X \leq 30\} \leq P\{|X - 50.25| \geq 20.25\} \leq \frac{25.126}{(20.25)^2} \approx 0.061$$

And, from one-sided Chebyshev inequality

$$
\begin{aligned}
P\{X \leq 30\} &= P\{X \leq 50.25 - 20.25\} \\
&\leq \frac{25.126}{25.126 + (20.25)^2} \\
&\approx 0.058
\end{aligned}
$$

Exercise-4

Let X be a Poisson random variable with parameter λ. Which one of the following is/are true?

1. $P\left\{X \leq \frac{\lambda}{2}\right\} \leq \frac{4}{\lambda}$.
2. $P\{X \geq 2\lambda\} \leq \frac{1}{\lambda}$.

Solution: Use Chebyshev's inequality in both cases:

- $$P\left\{X \leq \frac{\lambda}{2}\right\} \leq P\left\{|X - \lambda| \geq \frac{\lambda}{2}\right\} \leq \frac{\lambda}{\lambda^2/4} = \frac{4}{\lambda}$$
- $$P\{X \geq 2\lambda\} \leq P\{|X - \lambda| \geq \lambda\} \leq \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

Thank You