CSL003P1M : Probability and Statistics
Lecture 24 (Moment Generating Function)

Sumit Kumar Pandey

November 09, 2021

Moment Generating Function

The moment generating function (mgf) $M(t)$ of the random variable X is defined for all real values of t by

$$M(t) = E[e^{tX}] = \sum_x e^{tx} p(x)$$

We call $M(t)$ the moment generating function because all of the moments of X can be obtained successively differentiating $M(t)$ and then evaluating the result at $t = 0$.

Moment Generating Function

$$
\begin{aligned}
M'(t) &= \frac{d}{dt} E[e^{tX}] \\
&= E\left[\frac{d}{dt}(e^{tX})\right] \\
&= E[Xe^{tX}]
\end{aligned}
$$

Hence, the above equation evaluated at $t = 0$ yields

$$M'(0) = E[X]$$

Moment Generating Function

Similarly,

$$
\begin{aligned}
M''(t) &= \frac{d}{dt}M'(t) \\
&= \frac{d}{dt}E[Xe^{tX}] \\
&= E\left[\frac{d}{dt}(Xe^{tX})\right] \\
&= E[X^2e^{tX}]
\end{aligned}
$$

Thus,

$$M''(0) = E[X^2]$$

In general, the nth derivative of $M(t)$ is given by

$$M^n(t) = E[X^n e^{tX}] \quad n \geq 1$$

implying that

$$M^n(0) = E[X^n] \quad n \geq 1$$

Exercise-1

If X is a binomial random variable with parameters n and p, then find $E[X]$ and $Var(X)$ using mgf.

$$
\begin{aligned}
M(t) &= E[e^{tX}] \\
&= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} \\
&= (pe^t + 1 - p)^n
\end{aligned}
$$

After differentiation, we obtain

$$M'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

Thus,

$$E[X] = M'(0) = np.$$

After differentiating again, we obtain

$$M''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1}pe^t$$

So,

$$E[X^2] = M''(0) = n(n-1)p^2 + np$$

Thus,

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= np(1-p). \end{aligned}$$

Exercise-2

If X is a Poisson random variable with parameter λ, then find $E[X]$ and $Var(X)$ using mgf.

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^t} \end{aligned}$$

After differentiation, we obtain

$$M'(t) = \lambda e^t e^{-\lambda} e^{\lambda e^t}$$

Thus,

$$E[X] = M'(0) = \lambda.$$

After differentiating again, we obtain

$$M''(t) = \{(\lambda e^t)^2 + \lambda e^t\}(e^{-\lambda} e^{\lambda e^t})$$

So,

$$E[X^2] = M''(0) = \lambda^2 + \lambda$$

Thus,

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= \lambda \end{aligned}$$

Exercise-3

If X is a geometric random variable with parameter p, then find $E[X]$ and $Var(X)$ using mgf.

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1}p \\ &= pe^t \sum_{k=1}^{\infty} e^{t(k-1)}(1-p)^{k-1} \\ &= pe^t \sum_{k=1}^{\infty} \{e^t(1-p)\}^{k-1} \\ &= \frac{pe^t}{1-(1-p)e^t} \end{aligned}$$

Exercise-3

After differentiation, we obtain

$$M'(t) = \frac{pe^t}{\{1-(1-p)e^t\}^2}$$

Thus,

$$E[X] = M'(0) = \frac{1}{p}$$

After differentiating again, we obtain

$$M''(t) = \frac{pe^t\{1+(1-p)e^t\}}{\{1-(1-p)e^t\}^3}$$

Thus,

$$E[X^2] = M''(0) = \frac{2-p}{p^2}$$

So, $Var(X) = E[X^2] - (E[X])^2 = \dfrac{1-p}{p^2}.$

Exercise-4

If X is a negative binomial random variable with parameters r and p, then find $E[X]$ and $Var(X)$ using mgf.

$$
\begin{aligned}
M(t) &= E[e^{tX}] \\
&= \sum_{k=r}^{\infty} e^{tk} \binom{k-1}{r-1} (1-p)^{k-r} p^r \\
&= (pe^t)^r \sum_{k=r}^{\infty} \binom{k-1}{r-1} \{e^t(1-p)\}^{k-r} \\
&= \left[\frac{pe^t}{1-(1-p)e^t}\right]^r
\end{aligned}
$$

using the identity for positive integer r,

$$(1-x)^{-r} = \sum_{l=0}^{\infty} \binom{r+l-1}{l} x^l$$

Exercise-4

Try!!

$$E[X] = M'(0) = \frac{r}{p}$$

and,

$$Var(X) = E[X^2] - (E[X])^2 = \frac{r(1-p)}{p^2}$$

Thank You