# *CSL003P1M : Probability and Statistics Lecture 30 (Expectation and Variance of Continuous Distributions)*

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# *Expectation of a Function of a Continuous Random Variable*

If $X$ is a continuous random variable with probability density function $f(x)$, then, for any real-valued function $g$,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Sorry ... proof not provided.

## *Exercise*-1

The density function of $X$ is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[e^X]$.

Solution:

$$E[e^x] = \int_0^1 e^x dx = e - 1$$

# *Exercise*-2

A stick of length 1 is split at a point $U$ that is uniformly distributed over $(0,1)$. Determine the expected length of the piece that contains the point $p$, $0 \leq p \leq 1$.

Solution: Let $L_p(U)$ denote the length of the substick that contains the point $p$, and note that

$$L_p(U) = \begin{cases} 1 - U & U < p \\ U & U > p \end{cases}$$

So,

$$\begin{aligned}
E[L_p(U)] &= \int_0^1 L_p(u)du \\
&= \int_0^p (1-u)du + \int_p^1 udu \\
&= \frac{1}{2} - \frac{(1-p)^2}{2} + \frac{1}{2} - \frac{p^2}{2} \\
&= \frac{1}{2} + p(1-p)
\end{aligned}$$

## *Exercise*-3

Suppose that if you are $s$ minutes early for an appointment, then you incur the cost $cs$, and if you are $s$ minutes late, then you incur the cost $ks$. Suppose also that the travel time from where you presently are to the location of your appointment is a continuous random variable having probability density function $f$. Determine the time at which you should depart if you want to minimize your expected cost.

Solution: Let $X$ denote the travel time. If you leave $t$ minutes before your appointment, then your cost - call it $C_t(X)$ - is given by

$$C_t(X) = \begin{cases} c(t - X) & \text{if } X \leq t \\ k(X - t) & \text{if } X \geq t \end{cases}$$

Therefore,

$$\begin{aligned}
E[C_t(X)] &= \int_0^\infty C_t(x) f(x) dx \\
&= \int_0^t c(t-x) f(x) dx + \int_t^\infty k(x-t) f(x) dx \\
&= ct \int_0^t f(x) dx - c \int_0^t x f(x) dx + \\
&\quad k \int_t^\infty x f(x) dx - kt \int_t^\infty f(x) dx
\end{aligned}$$

The value of $t$ that minimizes $E[C_t(X)]$ can now be obtained after differentiating $E[C_t(X)]$ with respect to $t$. So,

$$\begin{aligned}
\frac{d}{dt} E[C_t(X)] &= \frac{d}{dt}\left( ct \int_0^t f(x) dx \right) - \frac{d}{dt}\left( c \int_0^t x f(x) dx \right) + \\
&\quad \frac{d}{dt}\left( k \int_t^\infty x f(x) dx \right) - \frac{d}{dt}\left( kt \int_t^\infty f(x) dx \right)
\end{aligned}$$

# *Exercise*-3

$$\begin{aligned}\frac{d}{dt}\left(ct\int_0^t f(x)dx\right) &= ct\frac{d}{dt}\left(\int_0^t f(x)dx\right)+c\left(\int_0^t f(x)dx\right)\frac{d}{dt}(dt)\\ &= ctf(t)+cF(t)\end{aligned}$$

$$\frac{d}{dt}\left(c\int_0^t xf(x)dx\right)=c\frac{d}{dt}\left(\int_0^t g(x)dx\right)=cg(t)=ctf(t)$$

$$\begin{aligned}\frac{d}{dt}\left(k\int_t^\infty xf(x)dx\right) &= k\frac{d}{dt}\left(E[X]-\int_{-\infty}^t g(x)dx\right)=-cg(t)\\ &= -ktf(t)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left(kt\int_t^{\infty} f(x)dx\right) &= kt\frac{d}{dt}\left(\int_t^{\infty} f(x)dx\right) + k\left(\int_t^{\infty} f(x)dx\right)t' \\ &= kt\frac{d}{dt}\left(1-\int_{-\infty}^{t} f(x)dx\right) + \\ &\quad k\left(1-\int_{-\infty}^{t} f(x)dx\right)t' \\ &= -ktf(t) + k[1-F(t)]\end{aligned}$$

Equating

$$\frac{d}{dt}E[C_t(X)] = 0$$

we get,

$$F(t) = \frac{k}{k+c}$$

# *A Property of an Expectation*

***Proposition***

If $a$ and $b$ are constants, then

$$E[aX + b] = aE[X] + b$$

Proof: (Try yourself!!)

# *Variance of a Continuous Random Variable*

### *Variance*

The variance of a continuous random variable is defined exactly as it is for a discrete random variable, namely, if $X$ is a random variable with expected value $\mu$, then the variance of $X$ is defined by

$$Var(X) = E[(X - \mu)^2]$$

### *The Alternative Formula*

$$Var(X) = E[X^2] - (E[X])^2$$

# *Exercise*-4

Find $Var(X)$ when the density function of $X$ is

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution: We first compute $E[X^2]$.

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_0^1 2x^3 dx \\ &= \frac{1}{2} \end{aligned}$$

Since, $E[X] = 2/3$. Thus,

$$Var(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

# *A Property of a Variance*

**Proposition**

If $a$ and $b$ are constants, then

$$Var(aX + b) = a^2 Var(X)$$

Proof: (Try yourself!!)

Thank You