CSL003P1M : Probability and Statistics Lecture 32 (Some Problems on Some Standard Continuous Distributions)

Sumit Kumar Pandey

November 29, 2021

Exercise-1

An expert witness in a paternity suit testifies that the length (in days) of human gestation is approximately normally distributed with parameters $\mu = 270$ and $\sigma^2 = 100$. The defendant in the suit is able to prove that he was out of the country during a period that began 290 days before the birth of the child and ended 240 days before the birth. If the defendant was, in fact, the father of the child, what is the probability that the mother could have had the very long or very short gestation indicated by the testimony?

Solution: Let X denote the length of the gestation, and assume that the defendant is the father. Then the probability that the birth could occur within the indicated period is

$$P\{X > 290 \text{ or } X < 240\} = P\{X > 290\} + P\{X < 240\}$$

Exercise-1

$$
\begin{aligned}
P\{X > 290 \text{ or } X < 240\} &= P\{X > 290\} + P\{X < 240\} \\
&= P\left\{\frac{X-270}{10} > 2\right\} + P\left\{\frac{X-270}{10} < -3\right\} \\
&= 1 - \Phi(2) + 1 - \Phi(3) \\
&\approx 0.0241
\end{aligned}
$$

Exercise-2

Let X be uniformly distributed over (α, β). Find (a) $E[X]$ and (b) $Var(X)$.

Solution: We find $E[X]$.

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha} dx \\
&= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} \\
&= \frac{\beta + \alpha}{2}
\end{aligned}
$$

Exercise-2

Let X be uniformly distributed over (α, β). Find (a) $E[X]$ and (b) $Var(X)$.

Solution: We find $Var(X)$

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{\alpha}^{\beta} \frac{x^2}{\beta-\alpha}dx \\
&= \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\beta^2+\beta\alpha+\alpha^2}{3}
\end{aligned}
$$

So,

$$
\begin{aligned}
Var(X) &= E[X^2]-(E[X])^2 \\
&= \frac{\beta^2+\beta\alpha+\alpha^2}{3} - \frac{(\beta+\alpha)^2}{4} \\
&= \frac{(\beta-\alpha)^2}{12}
\end{aligned}
$$

Exercise-3

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait

1. more than 10 minutes;
2. between 10 and 20 minutes.

Solution: Let X denote the length of the call made by the person in the booth. Then

1. $$P\{X > 10\} = 1 - F(10) = e^{-1} \approx 0.368.$$
2. $$P\{10 < X < 20\} = F(20) - F(10) = e^{-1} - e^{-2} \approx 0.233$$

Exercise-4

Find $E[X]$ and $Var(X)$ when X is a normal random variable with parameters μ and σ^2.

Solution: Let $Z = (X - \mu)/\sigma$. We know that Z is also a normal random variable with parameters 0 and 1. First we find $E[Z]$ and $Var(Z)$ and then $E[X]$ and $Var(X)$. We have

$$
\begin{aligned}
E[Z] &= \int_{-\infty}^{\infty} x f_Z(x) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} dx \\
&= -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Big|_{-\infty}^{\infty} \\
&= 0
\end{aligned}
$$

$\mathit{Exercise}$-4

Since $E[Z] = 0$, hence $Var(Z) = E[Z^2]$. Now we calculate $E[Z^2]$.

$$E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx$$

Apply integration by parts with $f(x) = x$ and $g(x) = xe^{-x^2/2}$. We obtain

$$\begin{aligned}
Var(Z) &= \frac{1}{\sqrt{2\pi}} \left\{ \left(x \int xe^{-x^2/2} dx \right) \Big|_{-\infty}^{\infty} - \right. \\
&\qquad \left. \int_{-\infty}^{\infty} \left(\frac{dx}{dx} \int xe^{-x^2/2} dx \right) dx \right\} \\
&= \frac{1}{\sqrt{2\pi}} \left\{ (-xe^{-x^2/2})|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx \right\} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx \\
&= 1
\end{aligned}$$

Exercise-4

Since $Z = (X - \mu)/\sigma$, hence

$$0 = E[Z] = \frac{1}{\sigma}E[X] - \frac{\mu}{\sigma}$$

Or,

$$E[X] = \mu.$$

Similarly,

$$1 = Var(Z) = \frac{1}{\sigma^2}Var(X)$$

Thus,

$$Var(X) = \sigma^2.$$

Exercise-5

Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at $7, 7:15, 7:30, 7:45$, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and $7:30$, find the probability that he waits

1. less than 5 minutes for a bus;
2. more than 10 minutes for a bus.

Solution: Let X denote the number of minutes past 7 that the passenger arrives at the stop. Since X is a uniform random variable over the interval $(0, 30)$, it follows that the passenger will have to wait less than 5 minutes if (and only if) he arrives between $7:10$ and $7:15$ or between $7:25$ and $7:30$.

Hence, the desired probability is

1. $$\begin{aligned} P\{10 < X < 15\} + P\{25 < X < 30\} &= \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx \\ &= \frac{1}{3} \end{aligned}$$

Similarly, he would have to wait more than 10 minutes if he arrives between 7 and 7 : 05 or between 7 : 15 and 7 : 20, so the desired probability is

2. $$P\{0 < X < 5\} + P\{15 < X < 20\} = \frac{1}{3}$$

Let X be an exponential random variable with parameter λ. Calculate (a) $E[X]$ and (b) $Var(X)$.

Solution: We first calculate $E[X^n]$ for $n > 0$.

$$E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} dx$$

Integrating by parts with $f(x) = x^n$ and $g(x) = \lambda e^{-\lambda x}$ yields

$$\begin{aligned} E[X^n] &= -x^n e^{-\lambda x}|_0^\infty + \int_0^\infty e^{-\lambda x} n x^{n-1} dx \\ &= 0 + \frac{n}{\lambda} \int_0^\infty \lambda e^{-\lambda x} x^{n-1} dx \\ &= \frac{n}{\lambda} E[X^{n-1}] \end{aligned}$$

$\mathit{Exercise}$-6

Letting $n = 1$ and then $n = 2$, we get

$$E[X] = \frac{1}{\lambda}$$

and

$$E[X^2] = \frac{2}{\lambda}E[X] = \frac{2}{\lambda^2}$$

So,

$$Var(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Thank You