CSL003P1M : Probability and Statistics
Lecture 05 (Conditional Probability)-II

Sumit Kumar Pandey

September 14, 2021

The Multiplication Rule

$$P(E_1E_2\cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$$

Proof:

- We prove it by applying induction on the number of events, k.
- Base case: The result is true for $k = 2$ as we know $P(EF) = P(E)P(F|E)$.
- Induction hypothesis: Assume that the result is true for $k = n - 1$, i.e.

$$P(E_1E_2\cdots E_{n-1}) = P(E_1)P(E_2|E_1)\cdots P(E_{n-1}|E_1\cdots E_{n-2})$$

- Inductive step: We prove it for $k = n$.

$$P(E_1E_2\cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$$

Proof (for $k = n$):

- Let $E = E_1E_2\cdots E_{n-1}$ and $F = E_n$. Then,

$$\begin{aligned} P(E_1E_2\cdots E_n) &= P(EF) = P(E)P(F|E) \\ &= P(E_1E_2\cdots E_{n-1})P(E_n|E_1E_2\cdots E_{n-1}). \end{aligned}$$

- From our induction hypothesis,

$$P(E_1E_2\cdots E_{n-1}) = P(E_1)P(E_2|E_1)\cdots P(E_{n-1}|E_1\cdots E_{n-2}).$$

- Therefore,

$$P(E_1E_2\cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1}).$$

Exercise-1

The Matching Problem

Suppose that each of N men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. The probability that none of the men selects his own hat is

$$P_N = \sum_{i=0}^{N} \frac{(-1)^i}{i!}$$

What is the probability that exactly k of the N men select his hat correctly?

$\mathit{Exercise}$-1

Solution:

- Choose k people out of N. It can be done in $\binom{N}{k}$ ways.
- Let E denote the event that the chosen k people choose their hats correctly.
- Let F denote the event that none of the remaining $N-k$ people chooses his/her hat correctly.
- Then we are interested in finding $P(EF) = P(E)P(F|E)$.
- Let $G_i, i = 1, \ldots, k$ be the event that the i^{th} member of the set has a match, then

$$
\begin{aligned}
P(E) &= P(G_1 G_2 \cdots G_k) \\
&= P(G_1)P(G_2|G_1)P(G_3|G_1 G_2) \cdots P(G_k|G_1 \cdots G_{k-1}) \\
&= \frac{1}{N}\frac{1}{N-1}\frac{1}{N-2}\cdots\frac{1}{N-k+1} \\
&= \frac{(N-k)!}{N!}
\end{aligned}
$$

$\mathit{Exercise}$-1

- $P(F|E) = P_{N-k} = \sum_{i=0}^{N-k} \frac{(-1)^i}{i!}$
- Therefore,

$$P(EF) = P(E)P(F|E) = \frac{(N-k)!}{N!} P_{N-k}.$$

- Since k out of N people can be chosen in $\binom{N}{k}$ ways, thus the desired probability is

$$\binom{N}{k} P(EF) = \binom{N}{k} \frac{(N-k)!}{N!} P_{N-k} = \frac{P_{N-k}}{k!}$$

- If N is large,

$$\frac{P_{N-k}}{k!} \approx \frac{e^{-1}}{k!}.$$

Exercise-2

An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

Solution:

- First, we calculate the number of favourable cases.
- The 4 aces can be divided into 4 piles in 4! ways.
- The remaining 48 cards can be divided into 4 equal piles in

$$\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12} \text{ ways.}$$

- The total number of favourable cases -

$$F = 4!\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}.$$

Exercise-2

An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

Solution (contd...):

- The total number of cases -

$$T = \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}.$$

- So, the probability is

$$\frac{F}{T} = \frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49} \approx 0.105.$$

Exercise-2

Alternate solution:

An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

Solution:

- Define events E_i, $i = 1, 2, 3, 4$ as follows:
  - $E_1 = \{$the ace of spades is in any one of the piles$\}$.
  - $E_2 =$ $\{$the ace of spades and the ace of hearts are in different piles$\}$.
  - $E_3 =$ $\{$the ace of spades, hearts and diamonds are all in different piles$\}$.
  - $E_4 = \{$all 4 aces are in different piles$\}$.
- We are interested in

$$P(E_1E_2E_3E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3).$$

Exercise-2

- It is clear that

$$P(E_1) = 1.$$

- Now,

$$P(E_2|E_1) = \frac{\binom{50}{12}}{\binom{51}{12}} = \frac{39}{51}.$$

- Similarly,

$$P(E_3|E_1E_2) = \frac{\binom{49}{24}}{\binom{50}{24}} = \frac{26}{50}.$$

- And,

$$P(E_4|E_1E_2E_3) = \frac{\binom{48}{36}}{\binom{49}{36}} = \frac{13}{49}.$$

- So,

$$\begin{aligned} P(E_1E_2E_3E_4) &= P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3) \\ &= \tfrac{39\cdot 26\cdot 13}{51\cdot 50\cdot 49} \approx 0.105. \end{aligned}$$

Exercise-3

Three players P_1, P_2 and P_3 throw a die in that order (Start with P_1, then P_2, then P_3, again P_1 and so on ...). The first one to get 1 on the face of the die will be the winner. Find the probability of winning of P_1, P_2 and P_3.

Solution:

- Let A_i be the event that player P_i wins $(i = 1, 2, 3)$.
- Let B_j be the event that 1 does not appear at the j^{th} throw.
- Then, $A_2 \subseteq B_1$ and $A_3 \subseteq B_1 B_2$; so $A_2 = A_2 B_1$ and $A_3 = A_3 B_1 B_2$.

Exercise-3

- Let $P(A_i) = p_i$ $(i = 1, 2, 3)$.
- So,

$$p_2 = P(A_2) = P(A_2B_1) = P(B_1)P(A_2|B_1) = \frac{5}{6}p_1,$$

beacuse if P_1 does not win the game on the first throw then P_2 plays "first" and hence $P(A_2|B_1) = P(A_1) = p_1$.

- Similarly,

$$p_3 = P(A_3) = P(A_3B_1B_2) = P(B_1B_2)P(A_3|B_1B_2) = \frac{25}{36}p_1.$$

- Because $p_1 + p_2 + p_3 = 1$, we get

$$p_1 = \frac{36}{91},\ p_2 = \frac{30}{91},\ p_3 = \frac{25}{91}.$$

Thank You