CSL003P1M : Probability and Statistics
QuestionSet - 08: Expectation, Variance and Covariance of Random Variables

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1. Find the covariance of the number of ones and sixes in n throws of a die.
2. Let n numbers be selected from the N numbers $1, 2, \ldots, N$. Let S_n be the random variable which denotes the sum of selected numbers. Find $E[S_n]$ and $Var(S_n)$.
3. Consider the random variable X which has the following probability function:

$$P\{X = k\} = \frac{\lambda^k}{(e^\lambda - 1)k!}, \qquad k = 1, 2, \ldots$$

   Find $E[X]$ and $Var(X)$.

4. Suppose a box contains n_1 white balls and n_2 black balls. A ball is drawn successively (without replacement) until a black ball is drawn. Let X be the random variable which denotes the number of balls drawn. Find $E[X]$.
5. In PVR, there are $2n + 1$ seats in a row. n females and $n + 1$ males sit in a row at random. Find the expected number of male and female pairs who sit alternatively.
6. There are N balls numbered $1, 2, \cdots, N$. "n" balls are selected at random with replacement. Let X be the random variable which denotes that the largest selected number is m? (a) Find $E[X]$. (b) Assume $n \ll N$, find the simplified expression of $E[X]$.
7. A deck of n numbered cards $(1, 2, \ldots, n)$ is arranged randomly. Let X be the random variable which denotes the number of matches (cards in their natural place). Find $E[X]$ and $Var(X)$.
8. Suppose that a bag consists of b black and g green balls. A random sample of size r is taken without replacement. Let X be the random variable which denotes the number of black balls in the sample. Find $E[X]$ and $Var(X)$ using the formula for expectation and variance of sum of random variables.
9. In a sequence of Bernoulli trials, (a) let X be the length of the run (of either successes or failures) started by the first trial. Find $E[X]$ and $Var(X)$. (b) Let Y be the length of the second run. Find $E[Y]$ and $Var(Y)$.
10. Let X and Y be two random variables both of which assume only two values each. Suppose $Cov(X, Y) = 0$. Are X and Y independent?
11. Let (X, Y) be random variables whose joint distribution is trinomial distribution. Find $E(X)$, $Var(X)$ and $Cov(X, Y)$ by direct computation by representating X and Y as sums of n variables each and using the methods.

12. Suppose n balls are distributed at random into r boxes. Let $X_i = 1$ if box i is empty and let $X_i = 0$ otherwise. Let S_r denote the number of empty boxes defined by $S_r = X_1 + X_2 + \cdots + X_r$. (a) For $i \neq j$, compute $E[X_i X_j]$. (b) Compute $Var(S_r)$.

13. Let X_1, X_2 and X_3 be independent random variables having finite positive variances σ_1^2, σ_2^2 and σ_3^2 respectively. Find the correlation between $X_1 - X_2$ and $X_2 + X_3$.

14. Suppose X and Y are two random variables such that $\rho(X, Y) = 1/2$, $Var(X) = 1$ and $Var(Y) = 2$. Compute $Var(X - 2Y)$.

15. A box has 3 red balls and 2 black balls. A random sample of size 2 is drawn without replacement. Let U be the number of red balls selected and let V be the number of black balls selected. Compute $\rho(U, V)$.