CSL003P1M : Probability and Statistics
Lecture 20 (Some Properties of Expectation)

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Exercise-1

If $P\{a \leq X \leq b\} = 1$, then $a \leq E[X] \leq b$.

Proof:

- $$\begin{aligned} E[X] &= \sum_{x:p(x)>0} xp(x) \\ &\geq \sum_{x:p(x)>0} ap(x) \\ &= a \sum_{x:p(x)>0} p(x) \\ &= a. \end{aligned}$$
- Similarly, we can show that $E[X] \leq b$.
- Thus, $a \leq E[X] \leq b$.

Exercise-2

Suppose that for random variables X and Y,

$$X \geq Y.$$

That is, for any outcome of the probability experiment, the value of the random variable X is greater than or equal to the value of the random variable Y. Then,

$$E[X] \geq E[Y].$$

Proof:

- Let $Z = X - Y$. Thus $Z \geq 0$ or $P\{Z \geq 0\} = 1$.
- From the previous result, $E[Z] \geq 0$, or $E[X - Y] \geq 0$, or $E[X] - E[Y] \geq 0$.
- Thus, $E[X] \geq E[Y]$.

Exercise-3

Let X and Y be two random variables having finite expectation. Then,

$$|E[X]| \leq E[|X|].$$

Proof:

- $$|E[X]| = \left| \sum_{x:p(x)>0} xp(x) \right| \leq \sum_{x:p(x)>0} |xp(x)|$$

  because $|a+b| \leq |a| + |b|$.
- Now, since $p(x) > 0$, therefore

  $$\sum_{x:p(x)>0} |xp(x)| = \sum_{x:p(x)>0} |x|p(x) = E[|X|].$$

- Thus, $|E[X]| \leq E[|X|]$.

Exercise-4

Let X be a random variable such that for some constant M, $P\{|X| \leq M\} = 1$. Then $|E[X]| \leq M$.

Proof:

- $$E[|X|] = \sum_{x:p(x)>0} |x|p(x) \leq \sum_{x:p(x)>0} Mp(x)$$
  because $P\{|X| \leq M\} = 1$.
- Now,
  $$\sum_{x:p(x)>0} Mp(x) = M \sum_{x:p(x)>0} p(x) = M.$$
- Thus, $E[|X|] \leq M$.
- From the previous result, $|E[X]| \leq E[|X|]$.
- So, $|E[X]| \leq M$.

Exercise-5

Let X and Y be two independent random variables. Then,

$$E[XY] = (E[X])(E[Y]).$$

Solution:

- Since X and Y are independent,

$$p(x, y) = p_X(x)p_Y(y).$$

- Therefore,

$$\begin{aligned} E[XY] &= \sum_{x,y} xyp_X(x)p_Y(y) \\ &= \left(\sum_x xp_X(x)\right)\left(\sum_y yp_Y(y)\right) \\ &= E[X]E[Y]. \end{aligned}$$

Exercise-6

Find the expected value of a Negative Binomial random variable with parameters (r, p).

Solution:

- If X denotes the number of trials needed to amass a total of r successes, then X is a negative binomial random variable that can be represented by

$$X = X_1 + X_2 + \cdots + X_r$$

  where X_1 is the number of trials required to obtain the first success, X_2 the number of additional trials until the second success is obtained, X_3 the number of additional trials until the third success is obtained, and so on.
- That is, X_i represents the number of additional trials required after the $(i-1)$st success until a total of i successes is amassed.

Exercise-6

- Each of the random variable X_i is a geometric random variable with parameter p.
- We know that $E[X_i] = 1/p$ for $i = 1, 2, \ldots, r$.
- Thus,

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_r] = \frac{r}{p}.$$

Exercise-7

Find the mean of a Hypergeometric random variable with parameters (n, N, m). In other words, if n balls are randomly selected from an urn containing N balls of which m are white, find the expected number of white balls selected.

Solution:

- Let X denote the number of white balls selected, and represent X as

$$X = X_1 + X_2 + \cdots + X_m$$

where

$$X_i = \begin{cases} 1 & \text{if the } i\text{th white ball is selected} \\ 0 & \text{otherwise} \end{cases}$$

Exercise-7

$$
\begin{aligned}
E[X_i] &= P\{X_i = 1\} \\
&= P\{i\text{th ball is selected}\} \\
&= \frac{\binom{1}{1}\binom{N-1}{n-1}}{\binom{N}{n}} \\
&= \frac{n}{N}.
\end{aligned}
$$

Thus,

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_m] = \frac{mn}{N}.$$

Alternative Method:

- Let

$$X = Y_1 + Y_2 + \cdots + Y_n$$

where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{th ball selected is white} \\ 0 & \text{otherwise} \end{cases}$$

- Since the ith ball selected is equally likely to be any of the N balls, it follows that

$$E[Y_i] = \frac{m}{N}.$$

- So,

$$E[X] = E[Y_1] + E[Y_2] + \cdots + E[Y_n] = \frac{mn}{N}.$$

Thank You