CSL003P1M : Probability and Statistics Lecture 34 (Joint Distribution Of A Function Of Continuous Random Variables)

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Exercise-1

Let X be uniformly distributed over $(0,1)$. Find the density of the random variable Y, defined by $Y = X^n$.

Solution: For $0 \leq y \leq 1$,

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} \\ &= P\{X^n \leq y\} \\ &= P\{X \leq y^{1/n}\} \\ &= F_X(y^{1/n}) \\ &= y^{1/n} \end{aligned}$$

So,

$$f_Y(y) = \begin{cases} \frac{1}{n} y^{1/n-1} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise-2

If X is a continuous random variable with probability density f_X, then find the distribution of $Y = X^2$.

Solution: For $y \geq 0$,

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} \\ &= P\{X^2 \leq y\} \\ &= P\{-\sqrt{y} \leq X \leq \sqrt{y}\} \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

Differentiation yields

$$f_Y(y) = \frac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})] \quad y \geq 0$$

Exercise-3

If X has a probability density f_X, then has a density function of $Y = |X|$.

Solution: For $y \geq 0$,

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} \\ &= P\{|X| \leq y\} \\ &= P\{-y \leq X \leq y\} \\ &= F_X(y) - F_X(-y) \end{aligned}$$

Differentiation yields

$$f_Y(y) = f_X(y) + f_X(-y) \quad y \geq 0$$

Joint Probability Distribution of Functions of Random Variables

- Let X_1 and X_2 be jointly continuous random variable with joint probability density function f_{X_1,X_2}.
- It is sometimes necessary to obtain the joint distribution of the random variables Y_1 and Y_2, which arise as functions of X_1 and X_2.
- Specifically, suppose that $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ for some functions g_1 and g_2.

Joint Probability Distribution of Functions of Random Variables

Assumptions:

1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2, with solutions given by, say, $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.
2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the 2×2 determinant

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

at all points (x_1, x_2).

Under the previous two assumptions, it can be shown that the random variables Y_1 and Y_2 are jointly continuous with joint density function given by

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J(x_1, x_2)|^{-1}$$

where $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.

Exercise-4

Let X_1 and X_2 be jointly continuous random variables with probability density function f_{X_1,X_2}. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of f_{X_1,X_2}.

Let $g_1(x_1, x_2) = x_1 + x_2$ and $g_2(x_1, x_2) = x_1 - x_2$. Then

$$J(x_1, x_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

Now, since the equations $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ have $x_1 = (y_1 + y_2)/2$, $x_2 = (y_1 - y_2)/2$ as their solution, it follows that the desired density is

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2} f_{X_1,X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right)$$

Exercise-5

If X and Y are independent gamma random variables with parameters (α, λ) and (β, λ), respectively, compute the joint density of $U = X + Y$ and $V = X/(X + Y)$.

Solution: The joint density of X and Y is given by

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)} \\ &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda(x+y)} x^{\alpha-1} y^{\beta-1} \end{aligned}$$

Now, if $g_1(x, y) = x + y$, $g_2(x, y) = x/(x + y)$, then

$$\frac{\partial g_1}{\partial x} = \frac{\partial g_1}{\partial y} = 1 \quad \frac{\partial g_2}{\partial x} = \frac{y}{(x+y)^2} \quad \frac{\partial g_2}{\partial y} = -\frac{x}{(x+y)^2}$$

So,

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix} = -\frac{1}{x+y}$$

Finally, as $u = x + y$, $v = x/(x + y)$, so $x = uv$, $y = u(1 - v)$.

Therefore,

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}[uv, u(1-v)]u \\ &= \left(\frac{\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right)\left(\frac{v^{\alpha-1}(1-v)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right) \end{aligned}$$

Hence $X+Y$ and $X/(X+Y)$ are independent, with $X+Y$ having a gamma distribution with parameters $(\alpha+\beta, \lambda)$ and $X/(X+Y)$ having a beta distribution with parameters (α, β).

Thank You