CSL003P1M : Probability and Statistics
Lecture 21 (Some Problems on Expectation)

Sumit Kumar Pandey

October 26, 2021

Exercise-1

Matching Problem

Suppose that N people throw their hats into the center of a room. The hats are mixed up, and each person randomly selects one. Find the expected number of people that select their own hat.

Solution:

- Let X denote the number of matches.
- We can compute $E[X]$ by expressing X in the form

$$X = X_1 + X_2 + \cdots + X_N$$

where

$$X_i = \begin{cases} 1 & \text{if the } i\text{th person selects his own hat} \\ 0 & \text{otherwise} \end{cases}$$

Exercise-1

- Since, for each i, the ith person is equally likely to select any on the N hats,

$$E[X_i] = P\{X_i = 1\} = \frac{1}{N}.$$

- Thus,

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_N] = \left(\frac{1}{N}\right) N = 1.$$

Exercise-2

Coupon-Collecting Problem

Suppose that there are N different types of coupons, and each time one obtains a coupon, it is equally likely to be any one of the N types. Find the expected number of coupons one need amass before obtaining a complete set of at least one of each type.

Solution:

- Let X denote the number of coupons collected before a complete set is attained.
- Let

  $$X = X_0 + X_1 + \cdots + X_{N-1}$$

  where X_i, $i = 0, 1, \ldots, N-1$ to be the number of additional coupons that need to be obtained after i distinct types have been collected in order to obtain another distinct types.

Exercise-2

- When i distinct types of coupons have already been collected, a new coupon obtained will be of a distinct type with probability $(N-i)/N$. Therefore,

$$P\{X_i = k\} = \frac{N-i}{N}\left(\frac{i}{N}\right)^{k-1} \quad k \geq 1.$$

- Or, in other words, X_i is a geometric random variable with parameter $(N-i)/N$. Hence,

$$E[X_i] = \frac{N}{N-i}.$$

- Thus,

$$\begin{aligned} E[X] &= 1 + \frac{N}{N-1} + \frac{N}{N-2} + \cdots + \frac{N}{1} \\ &= N\left[1 + \cdots + \frac{1}{N-1} + \frac{1}{N}\right]. \end{aligned}$$

Exercise-3

Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability p, compute the expected number of ducks that escape unhurt when a flock of size 10 flies overhead.

- Let X_i equals 1 if the ith duck escapes unhurt and 0 otherwise, for $i = 1, 2, \ldots, 10$.
- The expected number of ducks to escape can be expressed as

$$E[X_1 + \cdots + X_{10}] = E[X_1] + \cdots + E[X_{10}].$$

Exercise-3

- To compute $E[X_i] = P\{X_i = 1\}$, we note that each of the hunters will, independently, hit the ith duck with probability $p/10$, so
$$P\{X_i = 1\} = \left(1 - \frac{p}{10}\right)^{10}.$$
- Hence,
$$E[X] = 10\left(1 - \frac{p}{10}\right)^{10}.$$

Expected number of runs

Suppose that a sequence of n 1's and m 0's is randomly permuted so that each of the $(n+m)!/(n!m!)$ possible arrangements is equally likely. Any consecutive string of 1's is said to constitute a run of 1's $-$ for instance $-$, if $n = 6, m = 4$, and the ordering is $1, 1, 1, 0, 1, 1, 0, 0, 1, 0$, then there are 3 runs of 1's $-$. Find the mean number of such runs.

Solution:

- Let

$$X_i = \begin{cases} 1 & \text{if a run of 1's start at the } i\text{th position} \\ 0 & \text{otherwise} \end{cases}$$

Exercise-4

- Let X denote the number of runs of 1. Then,

$$X = \sum_{i=1}^{n+m} X_i.$$

- So,

$$E[X] = \sum_{i=1}^{n+m} E[X_i]$$

- Now,

$$\begin{aligned} E[X_1] &= P\{\text{“1” in position 1}\} \\ &= \frac{n}{n+m} \end{aligned}$$

- And for $1 < i \leq n+m$,

$$\begin{aligned} E[X_i] &= P\{\text{“0” in position } i-1, \text{ “1” in position } i\} \\ &= \left(\frac{m}{n+m}\right)\left(\frac{n}{n+m-1}\right) \end{aligned}$$

Exercise-4

- Hence,

$$E[X] = \frac{n}{n+m} + (n+m-1)\frac{nm}{(n+m)(n+m-1)}.$$

- Similarly, let Y denote the number of runs of 0. Then,

$$E[X] = \frac{m}{n+m} + \frac{nm}{n+m}.$$

- So, the expected number of runs of either type is

$$E[X+Y] = 1 + \frac{2nm}{n+m}.$$

Thank You