CSL003P1M : Probability and Statistics
Lecture 04 (Conditional Probability)-I

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Exercise-1

Suppose that we toss 2 dice, and suppose that each of the 36 possible outcomes is equally likely to occur and hence has the probability $\frac{1}{36}$. Supose further that the first die is a 3. Then what is the probability that sum of 2 dice equals eight.

Solution:

- Let A be the event that the sum of the 2 dice equals 8. Then

$$A = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$$

- Let B be the event that the first die is a 3. Then,

$$B = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}.$$

- Let C be the event that the sum of die is 8 given the first die is a 3. Then

$$C = \{(3,5)\} = A \cap B.$$

Exercise-1

Suppose that we toss 2 dice, and suppose that each of the 36 possible outcomes is equally likely to occur and hence has the probability $\frac{1}{36}$. Supose further that the first die is a 3. Then what is the probability that sum of 2 dice equals eight.

Solution (contd...):

- So, the conditional probability that the sum of a die is 8 given the first die is a 3 is equal to

$$\frac{P(C)}{P(B)} = \frac{P(A \cap B)}{P(B)} = \frac{1}{6}.$$

If $P(F) > 0$, then

$$P(E|F) = \frac{P(EF)}{P(F)}.$$

where $P(E|F)$ is the "conditional probability that E occurs given that F has occurred".

Since we know that F has occurred, it follows that F becomes our new, or reduced, sample space; hence the probability that the event EF occurs will equal the probability of EF relative to the probability of F.

Exercise-2

A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $x/2$ for all $0 \leq x \leq 1$. Then, given that the student is still working after 0.75 hour, what is the conditional probability that the full hour is used?

Solution:

- Let E be the event that the student has used full hour.
- Let F be the event that the student has not finished (or in other words, still working) in 0.75 hour.
- Thus, we are interested in $P(E|F)$ which is

$$P(E|F) = \frac{P(EF)}{P(F)}$$

.

- Now, we calculate $P(F)$ and $P(EF)$.

Exercise-2

A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $x/2$ for all $0 \leq x \leq 1$. Then, given that the student is still working after 0.75 hour, what is the conditional probability that the full hour is used?

Solution (contd...):

- Note that E be the event that the student has used full hour, or in other words, the student could not complete in less than one hour.
- The probability that the student completes in less than x hour is $x/2$ (given) for all $0 \leq x \leq 1$. Thus, the probability that the student completes in less than an hour (here $x = 1$) is $1/2 = 0.5$.

Exercise-2

A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $x/2$ for all $0 \leq x \leq 1$. Then, given that the student is still working after 0.75 hour, what is the conditional probability that the full hour is used?

Solution (contd...):

- Therefore, the probability could not complete in less than an hour (or has used full hour) is

$$P(E) = 1 - 0.5 = 0.5.$$

- Now, we calculate $P(F)$.

Exercise-2

A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $x/2$ for all $0 \leq x \leq 1$. Then, given that the student is still working after 0.75 hour, what is the conditional probability that the full hour is used?

Solution (contd...):

- The probability that the student has finished in less than 0.75 hour is $0.75/2$.
- So, the probability that the student has not finished in 0.75 hour (or still working after 0.75 hour) is

$$P(F) = 1 - 0.75/2 = 1 - 0.375 = 0.625.$$

- Now, we need to calculate $P(EF)$.

Exercise-2

A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $x/2$ for all $0 \leq x \leq 1$. Then, given that the student is still working after 0.75 hour, what is the conditional probability that the full hour is used?

Solution (contd...):

- Note that $EF = E \cap F = E$.
- The reason is that if the student has utilised full one hour, he/she must be working after 0.75 hour. So, we can say that if the event E has occured, then F definitely has occurred. Therefore, $E \subseteq F$.
- Thus,

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(E)}{P(F)} = \frac{0.5}{0.625} = 0.8.$$

Exercise-3

Suppose that the population of a certain city is 40% male and 60% female. Suppose also that 50% of the males and 30% of the females smoke. Find the probability that a smoker is male.

Solution:

- Let M denote the event that a person selected is a male.
- Let F denote the event that a person selected is a female.
- Let S denote the event that a person selected smokes.
- Let N denote the event that a person selected does not smoke.
- Then,

$$P(M) = 0.4,\ P(F) = 0.6,\ P(S|M) = 0.5,\ P(S|F) = 0.3.$$

- We are interested in $P(M|S)$.

Exercise-3

- Note that,

$$P(M|S)P(S) = P(MS) = P(SM) = P(S|M)P(M).$$

- Thus, $P(M|S)P(S) = 0.5 \times 0.4 = 0.2$.
- So, if we find $P(S)$, we are done.
- Now, we can write

$$S = (S \cap M) \cup (S \cap F).$$

- Since $(S \cap M)$ and $(S \cap F)$ are disjoint,

$$\begin{aligned} P(S) &= P(SM) + P(SF) \\ &= P(S|M)P(M) + P(S|F)P(F) \\ &= 0.5 \times 0.4 + 0.3 \times 0.6 \\ &= 0.2 + 0.18 = 0.38. \end{aligned}$$

- Thus, $P(M|S) = \frac{0.2}{0.38} \approx 0.53$.

Exercise-4

Polya's urn scheme

Suppose an urn has r red balls and b blue balls. A ball is drawn and its color noted. Then it together with $c > 0$ balls of the same color as the drawn ball are added to the urn. The procedure is repeated $n - 1$ additional times so that the total number of drawings made from the urn is n. Calculate the probability that

1. the first drawn ball is red.
2. the first drawn ball is blue.
3. the second drawn ball is red.
4. the second drawn ball is blue.

Exercise-4

- Let R_j, $1 \leq j \leq n$, denote the event that the j^{th} ball drawn is red.
- Let B_j, $1 \leq j \leq n$, denote the event that the j^{th} ball drawn is blue.
- Of course for each j, R_j and B_j are disjoint.

Exercise-4

- $$P(R_1) = \frac{r}{b+r}, \quad P(B_1) = \frac{b}{b+r}.$$
- Now we calculate $P(R_2)$ and $P(B_2)$.
- $$\begin{aligned} P(R_2) &= P(R_1 \cap R_2) + P(B_1 \cap R_2) \\ &= P(R_1)P(R_2|R_1) + P(B_1)P(R_2|B_1) \end{aligned}$$
- Now, we calculate $P(R_2|R_1)$ and $P(R_2|B_1)$.
- $$P(R_2|R_1) = \frac{r+c}{b+r+c}, \ P(R_2|B_1) = \frac{r}{b+r+c}.$$
- So,

$$\begin{aligned} P(R_2) &= P(R_1)P(R_2|R_1) + P(B_1)P(R_2|B_1) \\ &= \left(\frac{r}{b+r}\right)\left(\frac{r+c}{b+r+c}\right) + \left(\frac{b}{b+r}\right)\left(\frac{r}{b+r+c}\right) \\ &= \frac{r}{b+r}. \end{aligned}$$

Exercise-4

- So,

$$P(R_1) = P(R_2) = \frac{r}{b+r}.$$

- Now,

$$P(B_2) = 1 - P(R_2) = 1 - \frac{r}{b+r} = \frac{b}{b+r}.$$

- Therefore,

$$P(B_1) = P(B_2) = \frac{b}{b+r}.$$

Thank You