CSL003P1M : Probability and Statistics Lecture 14 (Jointly Distributed Random Variables)

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Joint Distribution Functions

- So far, we have concerned ourselves only with probability distributions for single random variables.
- However, we are often interested in probability statements concerning two or more random variables.

Joint Cumulative Probability Distribution Function

For any two random variables X and Y, the joint cumulative probability distribution function of X and Y is

$$F(a, b) = P\{X \leq a, Y \leq b\} \quad -\infty < a, b < \infty$$

Joint Distribution Functions

The distribution of X can be obtained from the joint distribution of X and Y as follows:

$$\begin{aligned} F_X(a) &= P\{X \leq a\} \\ &= P\{X \leq a, Y < \infty\} \\ &= P\left(\lim_{b\to\infty}\{X \leq a, Y \leq b\}\right) \\ &= \lim_{b\to\infty} P\{X \leq a, Y \leq b\} \\ &= \lim_{b\to\infty} F(a, b) \\ &= F(a, \infty). \end{aligned}$$

Similarly,

$$\begin{aligned} F_Y(b) &= P\{Y \leq b\} \\ &= \lim_{a\to\infty} F(a, b) \\ &= F(\infty, b). \end{aligned}$$

Marginal Distributions of X and Y

Marginal Distributions of X and Y

The distribution functions F_X and F_Y are sometimes referred to as the marginal distributions of X and Y.

Exercise-1

Prove that

$$P\{X > a, Y > b\} = 1 - F_X(a) - F_Y(b) + F(a, b).$$

Solution;

- $$\begin{aligned} &P\{X > a, Y > b\} = 1 - P(\overline{\{X > a, Y > b\}}) \\ &= 1 - P(\overline{\{X > a\}}) \cup \overline{\{Y > b\}}) \\ &= 1 - P(\{X \leq a\} \cup \{Y \leq b\}) \\ &= 1 - [P\{X \leq a\} + P\{Y \leq b\} - P\{X \leq a, Y \leq b\}] \\ &= 1 - F_X(a) - F_Y(b) + F(a, b) \end{aligned}$$

Prove that

$$P\{a_1 < X \le a_2, b_1 < Y \le b_2\}$$
$$= F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1)$$

whenever $a_1 < a_2, b_1 < b_2$.

Solution:

- $$P\{a_1 < X \le a_2, Y \le b_2\}$$
  $$= P\{X \le a_2, Y \le b_2\} - P\{X \le a_1, Y \le b_2\}$$
  $$= F(a_2, b_2) - F(a_1, b_2).$$
- Similarly,

$$P\{a_1 < X \le a_2, Y \le b_1\} = F(a_2, b_1) - F(a_1, b_1).$$

- Thus,

$$
\begin{aligned}
&P\{a_1 < X \le a_2, b_1 < Y \le b_2\} \\
&= P\{a_1 < X \le a_2, Y \le b_2\} - P\{a_1 < X \le a_2, Y \le b_1\} \\
&= F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1).
\end{aligned}
$$

Joint Probability Mass Function

The joint probability mass function of X and Y is

$$p(x, y) = P\{X = x, Y = y\}$$

The probability mass function of X can be obtained from $p(x, y)$ by

$$p_X(x) = P\{X = x\} = \sum_{y:p(x,y)>0} p(x, y)$$

Similarly,

$$p_Y(y) = P\{Y = y\} = \sum_{x:p(x,y)>0} p(x, y)$$

Exercise-3

Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote, respectively, the number of red and white balls chosen, then find the joint probability mass function of X and Y, $p(i,j) = P\{X = i, Y = j\}$.

$$p(0,0) = \binom{5}{3} \Big/ \binom{12}{3} = \frac{10}{220}$$

$$p(0,1) = \binom{4}{1}\binom{5}{2} \Big/ \binom{12}{3} = \frac{40}{220}$$

$$p(0,2) = \binom{4}{2}\binom{5}{1} \Big/ \binom{12}{3} = \frac{30}{220}$$

$$p(0,3) = \binom{4}{3} \Big/ \binom{12}{3} = \frac{4}{220}$$

$$p(1,0) = \binom{3}{1}\binom{5}{2}\Big/\binom{12}{3} = \frac{30}{220}$$
$$p(1,1) = \binom{3}{1}\binom{4}{1}\binom{5}{1}\Big/\binom{12}{3} = \frac{60}{220}$$
$$p(1,2) = \binom{3}{1}\binom{4}{2}\Big/\binom{12}{3} = \frac{18}{220}$$
$$p(2,0) = \binom{3}{2}\binom{5}{1}\Big/\binom{12}{3} = \frac{15}{220}$$
$$p(2,1) = \binom{3}{2}\binom{4}{1}\Big/\binom{12}{3} = \frac{12}{220}$$
$$p(3,0) = \binom{3}{3}\Big/\binom{12}{3} = \frac{1}{220}$$

These probabilities can most easily be expressed in tabular form.

Exercise-3

$i \backslash j$	0	1	2	3	$P\{X = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
$P\{Y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Exercise-4

Suppose that 15 percent of the families in a certain community have no children, 20 percent have 1 child, 35 percent have 2 children, and 30 percent have 3. Suppose further that in each family each child is equally likely (independently) to be a boy or a girl. Let B and G be the random variables which denote the number of boys and the number of girls respectively in a family. If a family is chosen at random from this community, find the joint probability mass function.

$$
\begin{aligned}
P\{B=0, G=0\} &= P\{\text{no children}\} = 0.15 \\
P\{B=0, G=1\} &= P\{\text{1 girl and total of 1 child}\} \\
&= P\{\text{1 child}\}P\{\text{1 girl}|\text{1 child}\} = (0.2)\left(\frac{1}{2}\right) \\
P\{B=0, G=2\} &= P\{\text{2 girls and total of 2 children}\} \\
&= P\{\text{2 children}\}P\{\text{2 girls}|\text{ 2 children}\} \\
&= (0.35)\left(\frac{1}{2}\right)^2
\end{aligned}
$$

$i \backslash j$	0	1	2	3	$P\{B = i\}$
0	0.15	0.10	0.0875	0.0375	0.3750
1	0.10	0.175	0.1125	0	0.3875
2	0.0875	0.1125	0	0	0.2000
3	0.0375	0	0	0	0.0375
$P\{G = j\}$	0.3750	0.3875	0.2000	0.0375	

We can also define joint probability distribution for n random variables in exactly the same manner as we did for $n = 2$.

Joint Cumulative Probability Distribution

The joint cumulative probability distribution $F(a_1, a_2, \ldots, a_n)$ of the n random variables $X_1, X_2, \ldots, X_n$ is defined by

$$F(a_1, a_2, \ldots, a_n) = P\{X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n\}$$

The Multinomial Distribution

Let there be a sequence of n independent and identical experiments. Suppose that each experiment can result in any one of r possible outcomes, with respective probabilities $p_1, p_2, \ldots, p_r$, such that $\sum_{i=1}^{r} p_i = 1$. Let X_i be a random variable which denotes the number of the outcome number i. Then

$$P\{X_1 = n_1, X_2 = n_2, \ldots, X_r = n_r\} = \frac{n!}{n_1! n_2! \cdots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

where $\sum_{i=1}^{r} n_i = n$.

Exercise-5

Suppose that a fair die is rolled 9 times. Find the probability that 1 appears three times, 2 and 3 twice each, 4 and 5 once each, and 6 not at all.

Solution:

- Let X_i be a random variable which denotes the number of the outcome number i.
- Let p_i be the probability that the number i occurs. Then, $p_i = 1/6$ for $i = 1, 2, \ldots, 6$.
- Now,

$$
\begin{aligned}
&P\{X_1 = 3, X_2 = 2, X_3 = 2, X_4 = 1, X_5 = 1, X_6 = 0\} \\
&= \frac{9!}{3!2!2!1!1!0!}\left(\frac{1}{6}\right)^3\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^0 \\
&= \frac{9!}{3!2!2!}\left(\frac{1}{6}\right)^9.
\end{aligned}
$$

Thank You