CSL003P1M : Probability and Statistics
Lecture 08 (Discrete Random Variables)

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Introduction

Frequently, when an experiment is performed, we are interested mainly in some functions of the outcome as opposed to the actual outcome itself.

- For instance, in tossing dice, we are often interested in the sum of the two dice and are not really concerned about the separate values in each die. That is, we may be interested in knowing that the sum is 7.
- For instance, in flipping a coin, we may be interested in the total number of heads that occur and not care at all about the actual head-tail sequence that results.

Random variables

These quantities of interest, or more formally, these real-valued functions defined on the sample space, are known as **random variables**.

Discrete Random Variable

Let S be the sample space. A discrete real-valued random variable X is a function X with domain S and range a finite or countably infinite subset $\{x_1, x_2, \ldots\}$ of the real number $\mathbb{R}$ such that $\{\omega : X(\omega) = x_i\}$ is an event for all i.

Because the value of a random variable is determined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable.

Example-1

Suppose that our experiment consists of tossing 3 fair coins. If we let Y denote the number of heads that appear, then Y is a random variable taking on one of the values $0, 1, 2$ and 3 with respective probabilities

$$P\{Y = 0\} = P(T, T, T) = \frac{1}{8}$$
$$P\{Y = 1\} = P\{(T, T, H), (T, H, T), (H, T, T)\} = \frac{3}{8}$$
$$P\{Y = 2\} = P\{(T, H, H), (H, T, H), (H, H, T)\} = \frac{3}{8}$$
$$P\{Y = 3\} = P\{(H, H, H)\} = \frac{1}{8}$$

It is easy to check that

$$1 = P\left(\bigcup_{i=0}^{3}\{Y = i\}\right) = \sum_{i=0}^{3} P\{Y = i\}.$$

Exercise-1

Three balls are to be randomly selected without replacement from an urn containing 20 balls numbered 1 through 20.If we bet that at least one of the balls that are drawn has a number as large as or larger than 17. What is the probability that we win the bet?

Solution:

- Let X denote the largest number selected.
- Then X is a random variable taking one of the values $3, 4, \ldots, 20$.
- Furthermore, if we suppose that each of the $\binom{20}{3}$ possible selections are equally likely to occur, then

$$P\{X = i\} = \frac{\binom{i-1}{2}}{\binom{20}{3}}, i = 3, \ldots, 20.$$

Exercise-1

We are interested in

$$P\{X \geq 17\} = P\{X = 17\} + P\{X = 18\} + P\{X = 19\} + P\{X = 20\}.$$

- $$P\{X = 20\} = 0.150, \quad P\{X = 19\} = 0.134.$$
  $$P\{X = 18\} = 0.119, \quad P\{X = 17\} = 0.105.$$
- $P[X \geq 17] = 0.150 + 0.134 + 0.119 + 0.105 = 0.508.$

Example-2

Independent trials consisting of the flipping of a coin having probability p of coming up heads are continually performed until either a head occurs or a total of n flips is made. If we let X denote the number of times the coin is flipped, then X is a random variable taking on one of the values $1, 2, 3, \ldots, n$ with respective probabilities

$$
\begin{aligned}
&P\{X = 1\} = P\{H\} = p \\
&P\{X = 2\} = P\{(T, H)\} = (1 - p)p \\
&P\{X = 3\} = P\{(T, T, H)\} = (1 - p)^2 p \\
&\vdots \\
&P\{X = n - 1\} = P\{(\underbrace{T, T, \ldots, T,}_{n-2} H)\} = (1 - p)^{n-2} p \\
&P(X = n) = P\{(\underbrace{T, T, \ldots, T,}_{n-1} T), (\underbrace{T, T, \ldots, T,}_{n-1} H)\} = (1 - p)^{n-1}.
\end{aligned}
$$

Example-2

As a check, note that

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{n}\{X=i\}\right) &= \sum_{i=1}^{n} P\{X=i\} \\
&= \sum_{i=1}^{n-1} p(1-p)^{i-1} + (1-p)^{n-1} \\
&= p\left[\frac{1-(1-p)^{n-1}}{1-(1-p)}\right] + (1-p)^{n-1} \\
&= 1-(1-p)^{n-1} + (1-p)^{n-1} \\
&= 1
\end{aligned}
$$

Example-3

Three balls are randomly chosen from an urn containing 3 white, 3 red, and 5 black balls. Suppose that we win INR 1 for each white ball selected and lose INR 1 for each red ball selected. If we let X denote our total winnings from the experiment, then X is a random variable taking on the possible values $0, \pm 1, \pm 2, \pm 3$ with respective probabilities

$$P\{X = 0\} = \frac{\binom{5}{3} + \binom{3}{1}\binom{3}{1}\binom{5}{1}}{\binom{11}{3}} = \frac{55}{165}$$

$$P\{X = 1\} = P\{X = -1\} = \frac{\binom{3}{1}\binom{5}{2} + \binom{3}{2}\binom{3}{1}}{\binom{11}{3}} = \frac{39}{165}$$

Example-3

$$P\{X=2\}=P\{X=-2\}=\frac{\binom{3}{2}\binom{5}{1}}{\binom{11}{3}}=\frac{15}{165}$$

$$P\{X=3\}=P\{X=-3\}=\frac{\binom{3}{3}}{\binom{11}{3}}=\frac{1}{165}$$

Now we check

$$\sum_{i=0}^{3} P\{X=i\}+\sum_{i=1}^{3} P\{X=-i\}=\frac{55+2\cdot 39+2\cdot 15+2\cdot 1}{165}=1.$$

The probability that we win money is given by $\sum_{i=1}^{3} P\{X=i\}=55/165=1/3$.

Exercise-2

Consider n independent repetitions of the simple success-failure experiment with success probability p. Let X_n denote the number of successes in the n trials. Then find $P\{X_n = k\}$.

Solution:

- $$P\{X_n = k\} = \binom{n}{k} p^k (1-p)^{n-k}.$$
- Now,
$$P\left(\bigcup_{i=0}^{n} \{X_n = i\}\right) = \sum_{i=0}^{n} P\{X_n = i\}.$$

Consider n independent repetitions of the simple success-failure experiment with success probability p. Let X_n denote the number of successes in the n trials. Then find $P\{X_n = k\}$.

Solution (contd...):

- Note that,

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

- Put $x = p$ and $y = 1 - p$. We obtain

$$\sum_{i=0}^{n} \binom{n}{i} p^i (1 - p)^{n-i} = 1.$$

Exercise-3

Suppose that independent trials, each having probability $p, 0 < p < 1$, of being a success are performed until a total of r successes is accumulated. If we let X_r equals the number of trials required, then find $P\{X_r = n\}$.

- $$P\{X_r = n\} = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \quad n = r, r+1, \ldots$$
- Check,
$$P\left(\bigcup_{i=r}^{\infty}\{X_r = i\}\right)?$$

Thank You