CSL003P1M : Probability and Statistics
Lecture 02 (Axioms of Probability-I)

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How to Define Probability?

One way of defining the probability of an event is in terms of its relative frequency. Such a definition usually goes as follows:

- We suppose that an experiment, whose sample space is S, is repeatedly performed under exactly the same conditions.
- For each event E of the sample space S, we define $n(E)$ to be the number of times in the first n repetitions of the experiment that the event E occurs.
- Then $P(E)$, the probability of the event E, is defined as

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}.$$

- Thus, $P(E)$ is defined as the limiting frequency of E.

How to Define Probability?

Drawback:

- How do we know that $n(E)/n$ will converge to some constant limiting value that will be the same for each possible sequence of repetitions of the experiment?

Assumption:

- The convergence of $n(E)/n$ to a constat limiting value is an assumption, or an axiom, of the system.
  - However, to assume that $n(E)/n$ will necessarily converge to some constant value seems to be an extraordinary complicated assumption.
  - For, although we might indeed hope that such a constant limiting frequency exists, it does not at all seem to be a priori evident that this need be the case.

How to Define Probability?

- It is thus more reasonable to assume a set of simpler and more self-evident axioms about probability and then attempt to prove that such a constant limiting frequency does in some sense exist.

This approach is the modern axiomatic approach to probability theory.

Sample Space

- Consider an experiment whose outcome is not predictable by certainty.
- But suppose that the set of all possible outcomes is known.

Sample Space

This set of all possible outcomes of an experiment is known as the sample space of the experiment.

Examples

1. If the outcome of an experiment consists in the determination of a newborn child, then the sample space S is

$$S = \{\text{girl}, \ \text{boy}\}.$$

2. If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions $1, 2, 3, 4, 5, 6$ and 7, then the sample space S is

$$S = \{\text{all } 7! \text{ permutations of } (1, 2, 3, 4, 5, 6, 7)\}.$$

Examples (contd...)

3. If the experiment consists of flipping two coins, then the sample space S is

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

where H and T stand for "Head" and "Tail" respectively.

4. If the experiment consists of measuring (in hours) the lifetime of a transistor, then the sample space S consists of all nonnegative real numbers; that is

$$S = \{x : 0 \leq x < \infty\}.$$

Event

Event

Any subset E of the sample space is known as an event.

Examples

1. In the previous example 1, the sample space was $S = \{\text{girl}, \ \text{boy}\}$.
   - If $E_1 = \{\text{girl}\}$, then the event is that the child is a girl.
   - If $E_2 = \{\text{boy}\}$, then the event is that the child is a boy.
2. In the previous example 2, the sample space was

$$S = \{\text{all } 7! \text{ permutations of } (1, 2, 3, 4, 5, 6, 7)\}.$$

   If $E = \{\text{all outcomes in } S \text{ starting with a } 3\}$, then E is the event that horse 3 wins the race.

Examples (contd...)

1. In the previous example 3, the sample space was $S = \{(H,H),(H,T),(T,H),(T,T)\}$.
   - If $E = \{(H,H),(H,T)\}$, then E is the event that a head appears on the first coin.
2. In the previous example 4, the sample space was

$$S = \{x : 0 \leq x < \infty\}.$$

   If $E = \{x : 0 \leq x \leq 5\}$, then E is the event that the transistor does not last longer than 5 hours.

Union of Events

Union of Events

For any two events E and F of a sample space S, we define the new event $E \cup F$ to consist of all outcomes that are either in E or in F or in both E and F. That is, event E will occur if either E or F occurs.

Examples

1. In the previous example 1, if $E = \{\text{girl}\}$ and $F = \{\text{boy}\}$, then
$$E \cup F = \{\text{girl}, \text{boy}\}.$$
2. In the previous example 3, if $E = \{(H, H), (H, T)\}$ and $F = \{(T, H)\}$, then
$$E \cup F = \{(H, H), (H, T), (T, H)\}.$$
Thus, $E \cup F$ would occur if a head appeared on the either coin.

Intersection of Events

Intersection of Events

For any two events E and F, we may also define the new event EF (or $E \cap F$), called the intersection of E and F, to consist of all outcomes that are both in E and F. That is the event EF will occur if both E and F occur.

Examples

1. If $E = \{(H,H),(H,T),(T,H)\}$ is the event that at least one head occurs and $F = \{(H,T),(T,H),(T,T)\}$ is the event that at least one tail occurs, then

$$EF = \{(H,T),(T,H)\}$$

is the event that exactly 1 head and 1 tail occur.

2. If $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ is the event that the sum of dice is 7 and $F = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ is the event that the sum is 6, then the event EF does not contain any outcomes and hence could not occur. We shall refer to this event as null event and we denote it as

$$EF = \phi.$$

That is, $EF = \phi$, then E and F are said to be mutually exclusive.

Complement of an Event

Complement of an Event

For any event E, we define the event $\bar{E}$, referred to as the complement of E, to consist of all the outcomes in the sample space S that are not in E. That is $\bar{E}$ occurs if and only if E does not occur.

Examples

1. If $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$, then $\bar{E}$ will occur when the sum of the dice does not equal 7.
2. Note that $\bar{S} = \phi$.

Containment of an Event

Containment of an Event

For two events E and F, if all the outcomes in E are also in F, then we say that E is contained in F, or E is a subset of F, and write $E \subseteq F$, or equivalently, $F \supseteq E$, which we say F is a superset of E. Thus, if $E \subseteq F$, then the occurrence of E implies the occurrence of F.

If $E \subseteq F$ and $F \subseteq E$, then $E = F$.

Some Laws

1. Commutative Laws::
   - $E \cup F = F \cup E$,
   - $EF = FE$.
2. Associative Laws:
   - $(E \cup F) \cup G = E \cup (F \cup G)$,
   - $(EF)G = E(FG)$.
3. Distributive Laws:
   - $(E \cup F)G = EG \cup FG$,
   - $EF \cup G = (E \cup G)(F \cup G)$.

DeMorgan's Laws

1. $$\overline{\left(\bigcup_{i=1}^{n} E_i\right)} = \bigcap_{i=1}^{n} \bar{E}_i$$

2. $$\overline{\left(\bigcap_{i=1}^{n} E_i\right)} = \bigcup_{i=1}^{n} \bar{E}_i$$

Proof: (Exrecise!!)

Axioms of Probability

Consider an experiment whose sample space is S. For **each event** E of the sample space S, we assume that a number $P(E)$ is defined and satisfies the following three axioms:

1. Axiom 1:
$$0 \leq P(E) \leq 1.$$
2. Axiom 2:
$$P(S) = 1.$$
3. Axiom 3:
For any sequence of mutually exclusive events $E_1, E_2, \cdots$
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Exercises

1. $P(\phi) = ?$
2. For any finite sequence of mutually exclusive events $E_1, E_2, \ldots, E_n$,

$$P\left(\bigcup_{i=1}^{n} E_i\right) \stackrel{?}{=} \sum_{i=1}^{n} P(E_i)$$

Answer:

1. $P(\phi) = 0$.
2. For any finite sequence of mutually exclusive events $E_1, E_2, \ldots, E_n$,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

1. $P(\phi) = 0$.

Proof:

- Consdier $E_1 = S$ and $E_i = \phi$ for $i \geq 2$. Then from Axiom 3,

$$P(S) = \sum_{i=1}^{\infty} E_i = P(S) + \sum_{i=2}^{\infty} P(\phi)$$

which implies $P(\phi) = 0$.

Exercises

1. For any finite sequence of mutually exclusive events $E_1, E_2, \ldots, E_n$,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Proof:

- Consider $E_i = \phi$ for $i \geq n+1$.
- Then,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{\infty} E_i\right) \text{ because } E_i = \phi \text{ for } i \geq n+1.$$

So,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) \text{ from Axiom 3.}$$

1. $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i).$

Proof (contd...):

- And,

$$\sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{n} P(E_i) + \sum_{i=n+1}^{\infty} P(E_i) = \sum_{i=1}^{n} P(E_i) + 0$$

because $P(E_i) = P(\phi) = 0$ for $i \geq n+1$.

Thank You