CSL003P1M : Probability and Statistics Lecture 09 (Discrete Probability Distributions)

Sumit Kumar Pandey

September 22, 2021

Discrete Random Variables

Discrete Random Variables

A random variable that can take on at most a countable number of possible values is said to be discrete. For a discrete random variable X, we define the **probability mass function p(a)** of X by

$$p(a) = P\{X = a\}$$

The probability mas function $p(a)$ is positive for at most a countable number of values of a. That is, if X assume one of the values $x_1, x_2. \ldots,$ then

$$\begin{aligned} p(x_i) &\geq 0 \quad \text{for } i = 1, 2, \ldots \\ p(x) &= 0 \quad \text{ for all other values of } x \end{aligned}$$

Since X must take on one of the values x_i, we have $\sum_{i=1}^{\infty} p(x_i) = 1$.

Bernoulli Random Variable

Bernuolli Random Variable

Suppose that a trial, or an experiment, whose outcome can be classified as either a **success** or a **failure** is performed. If we let $X = 1$ when the outcome is a success and $X = 0$ when it is a failure, then the probability mass function of X is given by

$$\begin{aligned} p(0) &= P\{X = 0\} = 1 - p \\ p(1) &= P\{X = 1\} = p. \end{aligned}$$

where $p, 0 \leq p \leq 1$, is the probability that the trial is a success.

A random variable X is said to be **Bernoulli random variable** with the parameter p if its probability mass function is given by the above equation.

Binomial Distribution

Binomial Distribution

Consider n independent repetitions of the simple success-failure experiment with success probability p. Let X denote the number of successes in the n trials. Then,

$$p(i) = P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, \ldots, n.$$

X is said to be a **Binomial random variable** with parameters (n, p).

Bernoulli random variable is a Binomial random variable with parameters $(1, p)$.

Suppose X is a Binomial random variable with parameters (n, p) when n is large and p is small enough so that np is of moderate size. Let

$$\lambda = np.$$

Then,

$$\begin{aligned} P\{X = i\} &= \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i} \\ &= \frac{n!}{(n-i)!i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} \\ &= \frac{n(n-1)\cdots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i} \end{aligned}$$

Noe, we calculate

$$P\{X = i\} = \frac{n(n-1)\cdots(n-i+1)}{n^i}\frac{\lambda^i}{i!}\frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}.$$

Now, for large n and moderate λ,

- $$\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda},$$
- $$\frac{n(n-1)\cdots(n-i+1)}{n^i} \approx 1,$$
- $$\left(1-\frac{\lambda}{n}\right)^i \approx 1,$$

So, for large n and moderate λ,

$$P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}$$

Poisson Distribution

Suppose n is large and p is small enough to make $\lambda = np$ moderate. Let there be n independent trials, each of which results in a success with probabiliy p. If we let X equals the number of successes, then

$$p(i) = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

A random variable X whose probability mass function is given by the above equation is said to be a **Poisson random variable** with parameter λ.

Poisson Distribution

- $$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1.$$

Negative Binomial Distribution

Suppose that independent trials, each having probability $p, 0 < p < 1$, of being a success are performed until a total of r successes is accumulated. If we let X equals the number of trials required, then

$$p(i) = P\{X = i\} = \binom{i-1}{r-1} p^r (1-p)^{i-r}, \quad i = r, r+1, \ldots$$

Any random variable X whose probability mass function is given by the above equation is said to be **Negative Binomial random variable** with parameters (r, p).

Why this name - Negative Binomial?

Geometric Distribution

Suppose that independent trials, each having probability $p, 0 < p < 1$, of being a success are performed until a success is achieved. If we let X equals the number of trials required, then

$$p(i) = P\{X = i\} = (1-p)^{i-1}p, \quad i = 1, 2, \ldots$$

Any random variable X whose probability mass function is given by the above equation is said to be **Geometric random variable** with parameter p.

Geometric random variable is a Negative Binomial random variable with parameters $(1, p)$.

- $$\sum_{i=1}^{\infty} p(i) = \sum_{i=1}^{\infty} (1-p)^{i-1} p = p \sum_{i=1}^{\infty} (1-p)^{i-1}.$$
- As $p < 1$, therefore

$$p \sum_{i=1}^{\infty} (1-p)^{i-1} = \frac{p}{1-(1-p)} = 1.$$

Thank You