CSL003P1M : Probability and Statistics
Lecture 41 (Testing of Hypothesis-I)

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Introduction

- A statistical hypothesis is usually a statement about a set of parameters of a population distribution.
- It is called hypothesis because it is not known whether or not it is true.
- A primary problem is to develop a procedure for determining whether or not the values of a random sample from this population are **consistent** with the hypothesis.
- If the random sample is deemed to be consistent with the hypothesis under considertaion, we say that the hypothesis has been "accepted"; otherwise we say that it has been "rejected".

- Note that in **accepting a given hypothesis**, we are not actually claiming that it is true but rather **data appear to be consistent with it**.
  - For instance, in the case of a normal $(\theta, 1)$ population, if a resulting sample of size 10 has an average value of 1.25, then although such a result cannot be regarded as being evidence in favor of the hypothesis "$\theta < 1$", it is not inconsistent with this hypothesis, which would thus be accepted.
  - On the other hand, if the sample of size 10 has an average value of 3, then even though a sample value that large is possible when $\theta < 1$, it is so unlikely that it seems inconsistent with this hypothesis, which would thus be rejected.

Null Hypothesis

- Consider a population having distribution D_θ, where θ is unknown, and suppose we want to test a hypothesis about θ. We shall denote this hypothesis by H_0 and call it the **null hypothesis**.
  - For example, if D_θ is a normal distribution function with mean θ and variance equal to 1, then two possible null hypotheses about θ are

$$\text{(a)}H_0 : \theta = 1$$
$$\text{(b)}H_0 : \theta \leq 1$$

  Thus the first of these hypotheses states that population is normal with mean 1 and variance 1, whereas the second states that it is normal with variance 1 and mean less than or equal to 1.

- Note that the null hypthesis in (a), when true, completely specifies the population distribution; whereas the null hypothesis in (b) does not.

Simple and Composite Hypothesis

Simple Hypothesis

A hypothesis that, when true, completely specifies the population distribution is called a simple hypothesis.

Composite Hypothesis

A hypothesis that, when true, does not completely specify the population distribution is called a composite hypothesis.

Critical Region

- Suppose now that in order to test a specific null hypothesis H_0, a population sample of size n - say $X_1, \ldots, X_n$ - is to be observed.
- Based on these values, we must decide whether or not to accept H_0.
- A test for H_0 can be specified by defining a region C in n-dimensional space with the provision that the hypothesis is to be rejected if the random sample $X_1, \ldots, X_n$ turns out to lie in C and accepted otherwise.
- The region C is called the **critical region**.
- In other words, the statistical test determined by the critical region C is the one that

$$\begin{array}{l} \text{accepts } H_0 \text{ if } (X_1, X_2, \ldots, X_n) \notin C \\ \text{rejects } H_0 \text{ if } (X_1, X_2, \ldots, X_n) \in C \end{array}$$

Example-1

A common test of the hypothesis that θ, the mean of a normal population with variance 1, is equal to 1 has a critical region given by

$$C = \left\{ (X_1, \ldots, X_n) : \left| \frac{\sum_{i=1}^{n} X_i}{n} - 1 \right| > \frac{1.96}{\sqrt{n}} \right\}$$

Thus, this test calls for rejection of the null hypothesis that $\theta = 1$ when the sample average differs from 1 by more than 1.96 divided by the square root of the sample size.

Type-I and Type-II Error

Type-I Error

Type-I error is said to result if the test incorrectly calls for rejecting H_0 when it is indeed correct.

Type-II Error

Type-II error is said to result if the test incorrectly calls for accepting H_0 when it is indeed incorrect.

Level of the Significance of the Test

- The objective of a statistical test of H_0 is not to explicitly determine whether or not H_0 is true but rather to determine if its validity is consistent with the resultant data.
- Hence, with this objective it seems reasonable that H_0 should only be rejected if the resultant data are very unlikely when H_0 is true.
- The classical way of accomplishing is by specifying a value α which is called the **level of significance of the test**.

The level of significance of the test

It is required that the test have the property that the probability of a type-I error can never be greater than α. The value α is called the level of significance of the test.

The commonly chosen values of α are being $0.1, 0.05, 0.005$.

Exercise-1

It is known that if a signal of value μ is sent from location A, then the value received at location B is normally distributed with mean μ and standard deviation 2. That is, the random noise added to the signal is an $N(0,4)$ random variable. There is reason for the people at location B to suspect that the signal value $\mu = 8$ will be sent today. Test this hypothesis if the same signal value is independently sent five times and the average value received at location B is $\bar{X} = 9.5$.

Solution: Let

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

will have a standard normal distribution. Now,

$$P\left\{|Z| > \frac{|\bar{X} - \mu|\sqrt{n}}{\sigma}\right\} = \alpha$$

Exercise-1

We compute the test statistic taking $\bar{X} = 9.5$ and $\mu_0 = 8$.

$$\frac{\sqrt{n}|\bar{X} - \mu_0|}{\sigma} = \frac{\sqrt{5}(1.5)}{2} = 1.68$$

Let

$$P\{Z > z_{\alpha/2}\} = \alpha/2$$

For $\alpha = 0.05$, we have $z_{0.025} = 1.96$, or, in other words

$$P\{Z > z_{0.025} = 1.96\} = 0.025$$

Since $1.68 < 1.96$, the data are not inconsistent with the null hypothesis.

Exercise-1

If we would have chosen a test that had a 10 percent chance of rejecting H_0 when H_0 was true, then the null hypothesis would have been rejected. Because for $\alpha = 0.1$,

$$P\{Z > z_{0.05} = 1.645\} = 0.05$$

and $1.68 > 1.645$.

p-Value of the Test

- For any observed value of the test statistic, call it ν, the test calls for rejection of the null hypothesis if the probability that the test statistic would be as large as ν when H_0 is true is less than or equal to the significance level α.
- From this, it follows that we an determine whether or not to accept the null hypothesis by computing, first, the value of the test statistic and, second, the probability that a unit normal would (in absolute value) exceed that quantity.
- **This probability - called the p-value of the test - gives the critical significance level in the sense that H_0 will be accepted if the significance level α is less than the p-value and rejected if it is greater than or equal**.

Exercise-2

In the previous exercise, suppose that the average of the 5 values received is $\bar{X} = 8.5$. Find the p-value.

Solution: In the previous exercise, $\mu_0 = 8$.

$$\frac{\sqrt{n}|\bar{X} - \mu_0|}{\sigma} = \frac{\sqrt{5}}{4} \approx 0.559$$

Since

$$P\{|Z| > 0.559\} = 2P\{Z > 0.559\} = 2 \times 0.288 = 0.576$$

it follows that the p-value is 0.576 and thus the null hypothesis H_0 that the signal sent has value 8 would be accepted at any significance level $\alpha < 0.576$.

Exercise-3

In the previous example, suppose that the average of the 5 values received is $\bar{X} = 11.5$. Find the p-value.

Solution: In the previous exercise, $\mu_0 = 8$.

$$\frac{\sqrt{n}|\bar{X} - \mu_0|}{\sigma} = 1.75\sqrt{5} \approx 3.913$$

Since

$$P\{|Z| > 3.913\} = 2P\{Z > 3.913\} \approx 0.00005$$

For such a small p-value, the hypothesis that the value 8 was sent is rejected.

One-Sided Tests

- In testing the null hypothesis that $\mu = \mu_0$, we have chosen a test that calls for rejection when $\bar{X}$ is far from μ_0. That is, a very small value of $\bar{X}$ or a very large value appears to make it unlikely that μ (which $\bar{X}$ is estimating) could equal μ_0.
- However, what happens when the alternative hypothesis to $H_0 : \mu = \mu_0$ is $H_1 : \mu > \mu_0$.
- Clearly, in this latter case we would not want to reject H_0 when $\bar{X}$ is small (since a small $\bar{X}$ is more likely when H_0 is true than when H_1 is true).

- Thus, in testing

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu > \mu_0$$

we should reject H_0 when $\bar{X}$, the point estimate of μ_0, is much greater than μ_0. That is the critical region should be of the following form:

$$C = \{(X_1, \ldots, X_n) : \bar{X} - \mu_0 > c\}$$

- Since the probability of rejection should equal α when H_0 is true (that is, when $\mu = \mu_0$) we require that c be such that

$$P\{\bar{X} - \mu_0 > c\} = \alpha.$$

Exercise-4

In the previous exercise, we know in advance that the signal value is at least as large as 8. What can be concluded in this case?

Solution: To see if the data are consistent with the hypothesis that the mean is 8, we test

$$H_0 : \mu = 8$$

against the one-sided alternative

$$H_1 : \mu > 8$$

The value of the test statistic is $\sqrt{n}(\bar{X} - \mu_0)/\sigma = \sqrt{5}(9.5 - 8)/2 = 1.68$, and the p-value is the probability that a standard normal would exceed 1.68, namely,

$$p\text{-value } = 1 - \Phi(1.68) = 0.0465$$

Eercise-5

All cigarettes presently on the market have an average nicotine content of at least 1.6 mg per cigarette. A firm that produces cigarettes claims that it has discovered a new way to cure tobacco leaves that will result in the average nicotine content of a cigarette being less than 1.6 mg. To test this claim, a sample of 20 of the firm's cigarettes were analyzed. If it is known that the standard deviation of a cigarette's nicotine content is 0.8 mg, what conclusions can be drawn, at the 5 percent level of significance, if the average nicotine content of the 20 cigarettes is 1.54?

Solution: We should test

$$H_0 : \mu \geq 1.6 \quad \text{versus} \quad H_1 : \mu < 1.6$$

Now, the value of the test statistic is

$$\sqrt{n}(\bar{X} - \mu_0)/\sigma = \sqrt{20}(1.54 - 1.6)/8 = -0.336$$

and so the p-value is given by

$$p\text{-value} = P\{Z < -0.336\} = 0.368$$

Since this value is greater than 0.05, the foregoing data do not enable us to reject, at the 5 percent level of significance.

Thank You