CSL003P1M : Probability and Statistics Lecture 33 (Some More Continuous Distributions)

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Exercise-1

If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$, find (a) $P\{2 < X < 5\}$; (b) $P\{X > 0\}$; $P\{|X - 3| > 6\}$.

Solution: (a)

$$\begin{aligned} P\{2 < X < 5\} &= P\left\{\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right\} \\ &= P\left\{\frac{-1}{3} < Z < \frac{2}{3}\right\} \\ &= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) \\ &= \Phi\left(\frac{2}{3}\right) - \left[1 - \Phi\left(\frac{1}{3}\right)\right] \\ &\approx 0.3779 \end{aligned}$$

Solution: (b)

$$\begin{aligned} P\{X > 0\} &= P\left\{\frac{X-3}{3} > \frac{0-3}{3}\right\} \\ &= P\{Z > -1\} \\ &= 1 - \Phi(-1) \\ &= \Phi(1) \\ &\approx 0.8413 \end{aligned}$$

Exercise-1

Solution: (c)

$$
\begin{aligned}
P\{|X-3|>6\} &= P\{X>9\}+P\{X<-3\} \\
&= P\left\{\frac{X-3}{3}>\frac{9-3}{3}\right\}+P\left\{\frac{X-3}{3}<\frac{-3-3}{3}\right\} \\
&= P\{Z>2\}+P\{Z<-2\} \\
&= 1-\Phi(2)+\Phi(-2) \\
&= 2[1-\Phi(2)] \\
&\approx 0.0456
\end{aligned}
$$

Gamma, Cauchy and Beta Distribution

The Gamma Distribution

A random variable is said to have a gamma distribution with parameters $(\alpha, \lambda), \lambda > 0, \alpha > 0$, if its density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\Gamma(\alpha)$, called the **gamma function**, is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$$

Integration of $\Gamma(\alpha)$ by parts yields

$$\begin{aligned}\Gamma(\alpha) &= -e^{-y}y^{\alpha-1}\Big|_0^\infty + \int_0^\infty e^{-y}(\alpha-1)y^{\alpha-2}dy \\ &= (\alpha-1)\int_0^\infty e^{-y}y^{\alpha-2}dy \\ &= (\alpha-1)\Gamma(\alpha-1)\end{aligned}$$

For integral values of α, say $\alpha = n$, we obtain,

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\cdots 3\cdot 2\cdot \Gamma(1)$$

and

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1.$$

Thus, for integral values of n,

$$\Gamma(n) = (n-1)!$$

The Gamma Distribution

- When $\alpha = n$, the gamma distribution with parameters (α, λ) often arises, in practice as the distribution of the amount of time one has to wait until a total of n events has occurred.
- More specifically, if events are occurring randomly, then it turns out that the amount of time one has to wait until a total of n events has occurred will be a gamma random variable with parameters (n, λ).

We will prove this fact now.

The Gamma Distribution

Let T_n denote the time at which the nth event occurs, and note that T_n is less than or equal to t if and only if the number of events that occurred by time t is at least n. That is, with $N(t)$ equal to the number of events in $[0, t]$.

$$\begin{aligned} P\{T_n \leq t\} &= P\{N(t) \geq n\} \\ &= \sum_{j=n}^{\infty} P\{N(t) = j\} \\ &= \sum_{j=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \end{aligned}$$

where the final identity follows because the number of events in $[0, t]$ has a Poisson distribution with parameter λt.

Differentiation will yield the density function of T_n:

$$\begin{aligned} f(t) &= \sum_{j=n}^{\infty} \frac{e^{-\lambda t} j(\lambda t)^{j-1}\lambda}{j!} - \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!} \\ &= \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^{j-1}}{(j-1)!} - \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!} \\ &= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$

χ^2 *(chi-squared) distribution*

The gamma distribution with $\lambda = 1/2$ and $\alpha = n/2$, n a positive integer, is called the χ_n^2 distribution with n degrees of freedom.

The Cauchy Distribution

A random variable is said to have a Cauchy distribution with parameter θ, $-\infty < \theta < \infty$, if its density is given by

$$f(x) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2} \quad -\infty < x < \infty$$

A random variable is said to have a beta distribution if its density is given by

$$f(x) = \begin{cases} \dfrac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

The Beta Distribution

- The beta distribution can be used to model a random phenomenon whose set of possible values is some finite interval $[c, d]$ - which, by letting c denote the origin and taking $d - c$ as a unit measurement, can be transformed into the interval $[0, 1]$.
- When $a = b$, the beta density is symmetric about $1/2$, giving more and more weight to regions about $1/2$ as the common value a increases.
- When $b > a$, the density is skewed to the left and it is skewed to the right when $a > b$.

The relationship

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

can be shown to exist between

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

and the gamma function.

Thank You