CSL003P1M : Probability and Statistics Lecture 35 (Joint Distribution Of A Function Of Continuous Random Variables)

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December 01, 2021

Joint Density Function of Continuous Random Variables

Suppose that the joint density function of the n random variables $X_1, X_2, \ldots, X_n$ is given and we want to compute the joint density function of $Y_1, Y_2, \ldots, Y_n$, where

$$Y_1 = g_1(X_1, \ldots, X_n),\ Y_2 = g_2(X_1, \ldots, X_n), \ldots, Y_n = g_n(X_1, \ldots, X_n)$$

Assumptions:

1. The functions g_i have continuous partial derivatives.
2. The equations $y_1 = g_1(x_1, \ldots, x_n)$, $y_2 = g_2(x_1, \ldots, x_n)$, $\ldots$, $y_n = g_n(x_1, \ldots, x_n)$ have a unique solution, say, $x_1 = h_1(y_1, \ldots, y_n)$, $x_2 = h_2(y_1, \ldots, y_n)$, $\ldots$, $x_n = h_n(y_1, \ldots, y_n)$.
3. $J(x_1, x_2, \ldots, x_n) \neq 0$.

Joint Density Function of Continuous Random Variables

$$J(x_1, \ldots, x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix} \neq 0$$

Under these assumptions, the joint density function of the random variables Y_i is given by

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)|J(x_1, \ldots, x_n)|^{-1}$$

where $x_i = h_i(y_1, \ldots, y_n)$, $i = 1, 2, \ldots, n$.

Exercise-1

Let X_1, X_2 and X_3 be independent standard normal random variables. If $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_1 - X_2$, and $Y_3 = X_1 - X_3$, compute the joint density function of Y_1, Y_2, Y_3.

Solution: Let $y_1 = g_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $y_2 = g_2(x_1, x_2, x_3) = x_1 - x_2$, $y_3 = g_3(x_1, x_2, x_3) = x_1 - x_3$, the Jacobian of these transformations is given by

$$J = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 3$$

Moreover,

$$x_1 = \frac{y_1 + y_2 + y_3}{3}, \ x_2 = \frac{y_1 - 2y_2 + y_3}{3}, \ x_3 = \frac{y_1 + y_2 - 2y_3}{3}$$

Exercise-1

So,

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3)$$
$$= \frac{1}{3} f_{X_1,X_2,X_3}\left(\frac{y_1 + y_2 + y_3}{3}, \frac{y_1 - 2y_2 + y_3}{3}, \frac{y_1 + y_2 - 2y_3}{3}\right)$$

Since,

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}} e^{-\sum_{i=1}^{3} x_i^2/2}$$

hence,

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \frac{1}{3(2\pi)^{3/2}} e^{-Q(y_1,y_2,y_3)/2}$$

where

$$Q(y_1, y_2, y_3) = \left(\tfrac{y_1+y_2+y_3}{3}\right)^2 + \left(\tfrac{y_1-2y_2+y_3}{3}\right)^2 + \left(\tfrac{y_1+y_2-2y_3}{3}\right)^2$$
$$= \tfrac{1}{3}y_1^2 + \tfrac{2}{3}y_2^2 - \tfrac{2}{3}y_2y_3$$

Exercise-2

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables with parameter λ. Let

$$Y_i = X_1 + X_2 + \cdots + X_i \quad i = 1, \ldots, n$$

1. Find the joint density function of $Y_1, \ldots, Y_n$.
2. Use the result of part (1) to find the density of Y_n.

Solution: (1)

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \quad 0 < x_i < \infty, i = 1, \ldots, n$$

Exercise-2

The Jacobian is

$$J = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{vmatrix} = 1$$

Moreover, we obtain

$$X_1 = Y_1, X_2 = Y_2 - Y_1, \ldots, X_i = Y_i - Y_{i-1}, \ldots, X_n = Y_n - Y_{n-1}$$

Now, the joint density function of Y_i is given by

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)|J(x_1, \ldots, x_n)|^{-1}$$

Exercise-2

$$\begin{aligned}
f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) &= f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)|J(x_1,\ldots,x_n)|^{-1} \\
&= f_{X_1,\ldots,X_n}(y_1, y_2 - y_1, \ldots, y_n - y_{n-1}) \\
&= \lambda^n \exp\left\{-\lambda\left[y_1 + \sum_{i=2}^{n}(y_i - y_{i-1})\right]\right\} \\
&= \lambda^n e^{-\lambda y_n} \quad 0 < y_1, 0 < y_i - y_{i-1}, i = 2, \ldots, n \\
&= \lambda^n e^{-\lambda y_n} \quad 0 < y_1 < y_2 < \cdots < y_n
\end{aligned}$$

Solution (2): Try!!

Miscellaneous

Jointly Continuous

Joint Probability Density Function

We say that X and Y are jointly continuous if there exists a function $f(x, y)$, defined for all real x and y, having the property that, for every set C of pairs of real numbers (that is, C is a set in the two-dimensional plane),

$$P\{(X, Y) \in C\} = \int\int_{(x,y)\in C} f(x, y)dxdy$$

The function $f(x, y)$ is called the **joint probability density function** of X and Y.

If A and B are any sets of real numbers, then, by defining $C = \{(x, y) : x \in A, y \in B\}$, we obtain

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x, y)dxdy$$

Joint Distribution Function

Joint Distribution Function

$$
\begin{aligned}
F(a,b) &= P\{X \in (-\infty, a], Y \in (-\infty, b]\} \\
&= \int_{-\infty}^{b}\int_{-\infty}^{b} f(x,y)\,dxdy
\end{aligned}
$$

Joint Density Function

Joint Density Function

$$f(a,b) = \frac{\partial^2}{\partial a \partial b} F(a,b)$$

where the partial derivatives are defined.

Alternatively,

$$\begin{aligned} P\{a < X < a + da, b < Y < b + db\} &= \int_b^{d+db} \int_a^{a+da} f(x,y)dxdy \\ &\approx f(a,b)dadb \end{aligned}$$

when da and db are small and $f(x,y)$ is continuous at a, b.

Marginal Density Function

Let X and Y be jointly continuous and furthermore they are individually continuous. Their probability density function can be obtained as follows:

$$\begin{aligned} P\{X \in A\} &= P\{X \in A, Y \in (-\infty, \infty)\} \\ &= \int_A \int_{-\infty}^{\infty} f(x, y) dy dx \\ &= \int_A f_X(x) dx \end{aligned}$$

where

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

is thus the probability density function of X. Similarly, the probability density function of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Exercise-3

The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Compute (a) $P\{X > 1, Y < 1\}$, (b) $P\{X < Y\}$ and (c) $P\{X < a\}$.

Solution: (a)

$$\begin{aligned} P\{X > 1, Y < 1\} &= \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dxdy \\ &= \int_0^1 2e^{-2y}\left(-e^{-x}|_1^\infty\right)dy \\ &= e^{-1}\int_0^1 2e^{-2y}\,dy = e^{-1}(1 - e^{-2}) \end{aligned}$$

(b)

$$\begin{aligned}
P\{X < Y\} &= \iint_{(x,y):x<y} 2e^{-x}e^{-2y}\,dxdy \\
&= \int_0^{\infty}\int_0^{y} 2e^{-x}e^{-2y}\,dxdy \\
&= \int_0^{\infty} 2e^{-2y}(1-e^{-y})dy \\
&= \int_0^{\infty} 2e^{-2y}dy - \int_0^{\infty} 2e^{-3y}dy \\
&= 1 - \frac{2}{3} = \frac{1}{3}
\end{aligned}$$

(c)

$$\begin{aligned}
P\{X < a\} &= \int_0^{a}\int_0^{\infty} 2e^{-2y}e^{-x}\,dydx \\
&= \int_0^{a} e^{-x}dx = 1 - e^{-a}
\end{aligned}$$

Independent Random Variables

Independent Random Variables

The random variables X and Y are said to be independent if, for any two sets of real numbers A and B,

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}$$

In other words, X and Y are independent if, for all A and B, the events $E_A = \{X \in A\}$ and $F_B = \{Y \in B\}$ are independent.

It can be shown that the above equation holds if and only if

$$P\{X \leq a, Y \leq b\} = P\{X \leq a\}P\{Y \leq b\}$$

Independent Random Variables

In terms of the joint distribution function of X and Y, X and Y are independent if

$$F(a, b) = F_X(a)F_Y(b) \quad \text{for all } a, b$$

In the jointly continuous case, the condition of independence is equivalent to

$$f(x, y) = f_X(x)f_Y(y) \quad \text{for all } x, y$$

If X and Y have a joint probability density function $f(x, y)$, then the conditional probability density function of X given that $Y = y$ is defined, for all values of y such that $f_Y(y) > 0$, by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

Conditional Distributions

1. If X and Y are jointly continuous, then, for any set A,

$$P\{X \in A|Y = y\} = \int_A f_{X|Y}(x|y)dx$$

2. In particular, by letting $A = (-\infty, a]$, we can define the conditional cumulative distribution function of X given that $Y = y$ by

$$F_{X|Y}(a|y) = P\{X \leq a|Y = y\} = \int_{-\infty}^{a} f_{X|Y}(x|y)dx$$

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the conditional density of X given that $Y = y$, where $0 < y < 1$.

Solution: For $0 < x < 1$, $0 < y < 1$, we have

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)dx} \end{aligned}$$

Exercise-4

$$
\begin{aligned}
\frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)dx} &= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)dx} \\
&= \frac{x(2-x-y)}{\frac{2}{3}-\frac{y}{2}} \\
&= \frac{6x(2-x-y)}{4-3y}
\end{aligned}
$$

Thank You