Probability

Question Set 1

(2) (a) With repeation   digit (2, 3, 5, 7)

$\overset{4}{\square} * \overset{4}{\square} * \overset{4}{\square} * \overset{4}{\square} = 4^4 = 256$ ← No. of arrangement

Total possible numbers = 256

Now ~~each no.~~ all four digits occurs equal no. of times at each of the position (like ones, tens, hundreds, thousands).

Each digit occurs $= \frac{256}{4} = 64$ times in each of the pos^n.

Sum at one's position ⇒ $64 * (2+3+5+7) * 1 = 1088$

at 10's pos^n ⇒ $64 * (2+3+5+7) * 10 = 10880$

at 100's pos^n ⇒ $64 * (2+3+5+7) \times 100 = 108800$

at 1000's pos^n ⇒ $64 * (2+3+5+7) * 1000 = 1088000$

So sum $= 1088 + 10880 + 108800 + 1088000$

$= 1088 [1 + 10 + 100 + 1000]$

$= 1088 \times 1111$

$= 1208768$ Ans

(b) Without repeation ⇒

$\overset{4}{\square} * \overset{3}{\square} * \overset{2}{\square} * \overset{1}{\square} = 24$ No. of arrangements $= {}^4P_4 = 4! = 24$

Each digit occurs $= \frac{24}{4} = 6$ times

Sum at one's pos^n ⇒ $6 * (2+3+5+7) * 1 = 102$

10's pos^n ⇒ $6 * (2+3+5+7) \times 10 = 1020$

100's pos^n ⇒ $6 * (2+3+5+7) * 100 = 10200$

1000's pos^n ⇒ $6 * (2+3+5+7) * 1000 = 102000$

So Sum $= 102 [1111]$

$= 113322$ Ans

2 Q)

(b) Find the sum of all 4-digit numbers that can be obtained by using the digits 2, 3, 5 and 7. (without repeating)

Sol:>

Total no of digit is = 4 ie 2, 3, 5 and 7.

No of 4 digit no. that can be formed = ${}^4P_4 = 4! = 24$

No. of times each digit will appear = $\frac{24}{4} = 6$

Sum of the digit at unit place = $6 \times (2+3+5+7)$
$= 6 \times 17 = 102$

Simillary is the case in ~~the~~ ten's hundre's and thousands place.

$\therefore$ The sum is = $102 + 102 \times 10 + 102 \times 100 + 102 \times 1000$
$= \underline{113,322}$

(a) Find the sum of all 4-digit numbers that can be obtained by using the digits 2, 3, 5 and 7 and ~~no~~ digit can ~~be~~ repeated?

Total no of digit is = 4

No of 4 digit that can be formed = 4^4 = ~~24~~
$= 256$

No of times each no will appear = $\frac{256}{4}$ = ~~80~~ $\underline{64}$

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Sum of all digits at unit place = $\cancel{6}4(2+3+5+7)$
$= \cancel{1513}\ 1088$

Similary for tens, hundred and thousand places

The total sum is $= \cancel{1513 + 1513\times10 + 1513\times100 + 1513\times1000}$
$= \cancel{1680942}$
$= 1088 + 1088\times10 + 1088\times100 + 1088\times1000$
$= 1208768$

3) How many n-digit (the most significant digit must be non-zero) number are possible which contains at least one digit from the set (0,2,4,6,8)?

(1) If repeatation is not allowed
total no of digits is = 5

So no. of 4 digits maximum can be formed = 5 digit number.

(i) So no of 5 digit no can be formed = $\cancel{5!}$
$\underline{4}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1} = 4\times4!$
$= 96$ ways

(2) For 4 digit no. can be formed =
$\underline{4}\times\underline{4}\times\underline{3}\times\underline{2} = 4\times{}^4P_3 = 96$ ways.

(3) For 3 digit no. can be formed =
$\underline{4}\times\underline{4}\times\underline{3} = 4\times{}^4P_2 = 48$ way

(4) For 2 digit no can be formed =
$\underline{4}\times\underline{4} = 4\times{}^4P_1 \Rightarrow 16$ ways.

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(8) For 1 digit no can be formed $\Rightarrow$ ~~5~~ 4

Total digits no. are = $96 + 96 + 48 + 16 + 4$

$= \underline{\underline{260}}$

or 261 If we con 1 digs as 0 al.

$\cancel{*}$ (wrong)

If (no. can be repeated.

case 1) 5 digit no $\cancel{5} \times \underline{5} \times \underline{5} \times \underline{5} \times \underline{5} \Rightarrow \cancel{5} \times 5^4 = 2500$

(ii) 4 digit no $\cancel{5} = \underline{4} \times \underline{5} \times \underline{5} \times \underline{5} \Rightarrow 4 \times 5^3 = 500$

(iii) 3 digit no. $\underline{4} \times \underline{5} \times \underline{5} \Rightarrow 4 \times 5^2 = 100$

(iv) 2 digit no. $4 \times 5 \Rightarrow 45$

(v) 1 digit no. $\cancel{5}$ 4

Total no of ways of all possible digits $\Rightarrow$

$2500 + 500 + 100 + 45 + 4 = 3150$

If we count 0 as digit in 1 digit no then $\underline{\underline{3151}}$

Some way n digits can be formed if repeated

~~5~~ (

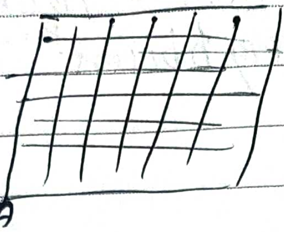
$4(5^{n-1} + 5^{n-2} \ldots \cancel{5^0})$

$4 \times 1 \dfrac{(5^{n-1} - 1)}{5-1} \Rightarrow \dfrac{4 \times (5^{n-1} - 1)}{4}$

$\Rightarrow \underline{\underline{5^{n-1} - 1}}$

5) Consider an mxn grid shown below

B



A

How many path are possible from A to B if any path consists of only (a) left to right horizontal and (b) up vertical moves?
i see gree and blue move in the grid.

Sing our traveler is limited to move from A to B position.

Let horizontal move from left to right move indicated by Green colour as "G"
and vertical move indicated by blue colour as "B"

Let assume a possible path from A to B as shown in the figure its direction will be.
As show to mxn Grid is of 15X20 matrix
GBBGG BBGGGGGBBGBGBGBGBGBBBB
GGGGGGGB

Total moves = 35
G = 20 B = 15

So total path possible from A to B is

$$= \frac{35!}{20!\,15!}$$

For m×n Grid

The possible path from A to B will be

$$= \frac{(m+n)!}{m!\,n!}$$

7) Prove that the number of partitions of the number n into (a sum of) no more then r terms is equal to the number of partition of n into any number of terms, each at most r.

It is of the form of Generic function
so
lets ass $n=4$
then partition of 4 will be

4 ~~no of 1's = 3~~

3+1

2+2 ~~no of 2's = 3~~

2+1+1

1+~~1~~+1+1 ~~no of 3's = 1~~

no. of partition of 4 is 5

lets denote the no of partition of n be P_n

let look it as generic function

so

no of 1's $= (1+x^1+x^2+\ldots) = \frac{1}{1-x}$

no of 2's $= (1+x^2+x^4+\ldots) = \frac{1}{1-x^2}$

n of 3's $= (1+x^3+x^6+\ldots) = \frac{1}{1-x^3}$

⋮

no of n's $= (1+x^n+x^{2n}+x^{4n}\ldots) = \dfrac{1}{1+x^n}$

So the generic function for partition of n is

$$(1+x+x^2+\ldots)(1+x^2+x^4\ldots)\ldots\ldots(1+x^n+x^{2n}\ldots)$$

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\ldots\left(\frac{1}{1-x^n}\right)$$

$$\sum_{i\geq 0}^{\infty}(x^3)\,P_i(n) = (x^3)\left(\prod_{i\geq 1}^{\infty}\frac{1}{1-x^i}\right)$$

10) How many two non-empty disjoint subsets A and B can be selected from $\{1, 2, 3 \dots n\}$?

A and B are two subsets of set $\{1, 2, 3 \dots n\}$

By condition, subset A and B to be ~~select~~ Disjoint so,

$A \cap B = \phi$ where A and B are non-empty set

~~We can differ or classify it into 3 cases.~~

Let the original set be X.

So we can classify the set X into three pieces.

(i) those in A (ii) those in B (iii) and those in neither of them.

We are mainly focused on the case (i) and (ii)

Suppose $|A| = K$ where $K \geq 1 < n$

So we can choose B of the for $|B| = n-K$ where $1 < (n-K) < n$

So, the possible ways to choose A with $|A| = K$ is $\binom{n}{K}$

and the possible way to choose B with $|B| = n-K$ is 2^{n-K}

So total no of possibilities is

$$\frac{1}{2} \times \sum_{K=1}^{n-1} \binom{n}{K} 2^{n-K}$$

$\frac{1}{2}$ is for un-ordered subsets.

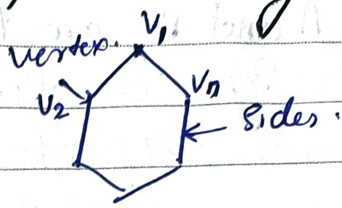
(11).

(a) Find the no of diagonals of convex n-gon.

Sol =)

Polygon = n sides

No of vertex = n



from n points we can form lines

$= {}^nC_2$

Diagonals of the polygon is formed, if two non-consecutive vertices of polygon are joined.

lets take a vertex V_1 and trace the diagonls from it so we can say that it closent form a diagonal with V_2 and V_n with is its neighbour vertex.

Hence it can form diagons with remaining $(n-3)$ vertex which is $(n-3)$ diagonals.

Hence n vertex can form "$n*(n-3)$" diagonals

but each diagonal has been counted twice, as it was counted as first diagonal when each of the vertexes at its end was considerd, but actually we have one diagon.

Hence no of diagonals of a convex polygon on n-side is $= \dfrac{n(n-3)}{2}$

Mathematically:-

Total diagonals = Total no of lines - no. of sides.

$$= {}^{n}C_2 - n$$

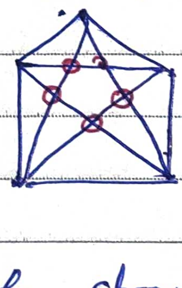
$$= \frac{n \times (n-1)}{2} - n$$

$$= \frac{n^2 - n - 2n}{2}$$

$$= \frac{n(n-3)}{2}$$

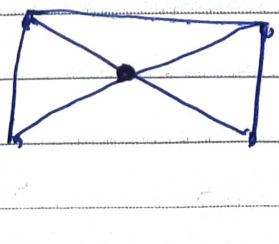
(b) Suppose no three diagonals pass through one point in a convex n-gon. Find the number of intersection points of all diagonals in the convex n-gon.

Lets assume $n = 5$ which is pentagon, if we ~~take~~ draw diagonals of pentagon then diagonals will be 5 and intersection points are also 5



If we take 4 vertex and try to draw the diagonal of it then its diagonals will be 2 and intersection point will be 1.

i.e



So for n-side polygon where no three diagonal intersect we choose ~~4 vertex from n~~ set of 4 vertices from

i.e $\binom{n}{4} = {}^{n}C_4$

sets of 4 vertices of the n-side polygon uniquely determines a pair of intersecting diagonals.

Q(ii) (a) Dearrangement ~~*~~ is a permutations of elements of a set, in ~~a set~~ such a manner that no element appear in its original position

Ex:

$\{1,2,3\} \xrightarrow{\text{dearg.}} \{3,1,2\}, \{2,3,1\}, \boxed{\{3,2,1\}}$

This is not dearrangement

- Dearrangement of n elements = D_N
  - its complement is atleast one element in its original position

D_N = Total Permutations without any restrictions − No. of permutation in which atleast one element is in its original position

no. of elements in

$= n! - n(A_1 \cup A_2 \cup \dots \cup A_n)$

$\{A_i \rightarrow$ Set of perm. in which i^{th} element is in its original position

$$n(A_1 \cup A_2 \dots \cup A_n) = \sum_{^nC_1} n(A_i) - \sum_{^nC_2} n(A_i \cap A_j) + \sum_{^nC_3} n(A_i \cap A_j \cap A_k)$$

$$- \dots + \dots + (-1)^{n-1} n(A_1 \cap A_2 \dots \cap A_n)$$

meaning

→ i^{th} elements is in its original position → so remaining element $(n-1)$ can be arrange in $(n-1)!$ ways

→ $\sum_{^nC_2} n(A_i \cap A_j) \xrightarrow{\text{meaning}}$ i^{th} & j^{th} elements in its original posn.

then remaining $(n-2) \longrightarrow (n-2)!$

$$n(A_1 \cup A_2 \cup \cdots \cup A_n) = {}^nC_1 * (n-1)! - {}^nC_2 (n-2)! + {}^nC_3 (n-3)!$$

$$+ \cdots\cdots + (-1)^n \, {}^nC_n * 1$$

So now

$$D_n = n! - \left[{}^nC_1 * (n-1)! - {}^nC_2 * (n-2)! + {}^nC_3 (n-3)! + \cdots + (-1)^n \, {}^nC_n * 1 \right]$$

$$= n! - \left[\frac{n!}{(n-1)!\,1!} * (n-1)! - \frac{n!}{(n-2)!\,2!} * (n-2)! + \frac{n!}{(n-3)!\,3!} * (n-3)! + \cdots + (-1)^n \frac{n!}{(n-n)!\,n!} * 1 \right]$$

$$= n! - \left[\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!} \right]$$

$$D_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right]$$

Q21

(b) ⇒ The first time the books are distributed = $n!$ ways Since no student gets the same book that he got first time and the second time the books are distributed in D_n ways.

Therefore, the total no. of ways

$$= n! * D_n$$

$$= n! * n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right]$$

$$= (n!)^2 \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right]$$

Q. 22 (a)

Proof $\Rightarrow$ $D_n - nD_{n-1} = (-1)^n$, for $n \geq 2$

Sol^n $\Rightarrow$

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right] \quad -(1)$$

$$D_{n-1} = (n-1)!\left[1 - \frac{1}{1!} + \cdots + \frac{(-1)^{n-1}}{n!}\right]$$

$$nD_{n-1} = n*(n-1)!\left[1 - \frac{1}{1!} + \cdots - \frac{(-1)^n}{n!}\right]$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots - \frac{(-1)^n}{n!}\right] \quad (2)$$

Now eq(1) - (2)

$$D_n - nD_{n-1} = n!\left[\frac{(-1)^n}{n!} + \frac{(-1)^n}{n!}\right]$$

$$D_n - nD_{n-1} = 2*(-1)^n \text{ से } (-1)^n$$

~~Now~~

~~$n = 1 \Rightarrow$ $D_1 - D_0 = 2$~~

$$\Rightarrow D_n - nD_{n-1} = 2*(-1)^n$$

from eq(1) $D_1 = 0$, & $D_2 = 1$

for D_3

$\Rightarrow$ $D_3 - 3D_2 = \cancel{2*(-1)^3}$

$D_3 = 3D_2$

$= 3*1$