CSL003P1M : Probability and Statistics
QuestionSet - 10: Continuous Distributions

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1. Verify that each of the following functions f is a probability density function and sketch the graph in MATLAB.
   (a) $f(x) = 1 - |1 - x|, \quad 0 < x < 2$
   (b) $f(x) = \dfrac{1}{\pi} \dfrac{\beta}{\beta^2 + (x - a)^2}, \quad -\infty < x < \infty$
   (c) $f(x) = \dfrac{1}{2\sigma} e^{-(|x-\mu|)/\sigma}, \quad -\infty < x < \infty$
   (d) $f(x) = \frac{1}{4} x e^{-x/2}, \quad 0 < x < \infty$

2. The amount of bread (in hundreds of kilos) that a bakery sells in a day is a random variable with density

$$f(x) = \begin{cases} cx & \text{for } 0 \le x < 3, \\ c(6 - x) & \text{for } 3 \le x < 6, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of c which makes f a probability density function.
   (b) What is the probability that the number of kiols of bread that will be sold in a day is, (a) more than 300 kilos? (b) between 150 and 450 kilos?
   (c) Denote by A and B the events in (a) and (b), respectively. Are A and B independent events?

3. Suppose that the duration in minutes of long-distance telephone conversations follows an exponential density function,

$$f(x) = \frac{1}{5} e^{-x/5}, \quad x > 0$$

   Find the probability that the duration of a conversation:
   (a) will exceed 5 minutes;
   (b) will be between 5 and 6 minutes;
   (c) will be less than 3 minutes;
   (d) will be less than 6 minutes given that it was greater than 3 minutes.

4. The height of men is normally distributed with mean $\mu = 167$ cms and standard deviation $\sigma = 3$ cms.
   (a) What is the percentage of the population of men that have height, (a) greater than 167 cms, (b) greater than 170 cms, (c) between 161 cms and 173 cms?
   (b) In a random sample of four men, what is the probability that:
      i. all will have height greater than 170 cms;
      ii. two will have height smaller than the mean (and two bigger than the mean)?

5. A machine produces bolts the length of which (in centimeters) obeys a normal probability law with mean 5 and standard deviation $\sigma = 0.2$. A bolt is called defective if its length falls outside the interval $(4.8, 5.2)$.

   (a) What is the proportion of defective bolts that this machine produces?

   (b) What is the probability that among ten bolts none will be defective?

6. If X is a continuous random variable with cumulative distribution function F and density function f, show that the random variable $Y = X^2$ is also continuous and express its cumulative distribution function and density in terms of F and f.

7. Find the density of $Y = X^2$ when X has:

   (a) the normal distribution $N(\mu, \sigma^2)$;

   (b) the Laplace distribution (see Wikipedia)

   (c) the Cauchy distribution

8. Solve the previous two exercises with $Y = |X|$.

9. If the $\log X$ is normally distributed then X is said to have a lognormal distribution. Find its density.

10. Let $f(x)$ denote the density function of the random variable X. Suppose that X has a symmetric distribution about a, that is, $f(x + a) = f(a - x)$ for every x. Show that the mean $E[X]$ equals a, provided it exists.

11. Show that for a continuous random variable X with density function f and cumulative distribution function F

$$\mu = E[X] = \int_0^\infty [1 - F(x)]dx - \int_{-\infty}^0 F(x)dx.$$

12. Let X be a random variable with distribution function

$$F(x) = \begin{cases} 1 - 0.8e^{-x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0 \end{cases}$$

    Calculate $E[X]$.

13. Verify whether the following function

$$F(x, y) = \begin{cases} 0 & \text{for } x + y < 1, \\ 1 & \text{for } x + y \geq 1 \end{cases}$$

    is a joint distribution function?

14. X and Y have the joint density

$$f(x, y) = cx^{n_1-1}(y - x)^{n_2-1}e^{-y}, \quad 0 < x < y < \infty$$

    Find (a) the constant c, (b) the marginal distribution of X and Y.

15. A bivariate normal distribution has density

$$f(x, y) = c\exp[-x^2 + xy - y^2].$$

    Find the constant c and the moment of order 2.

16. Given that the density of X and Y is

$$f(x, y) = \frac{2}{(1 + x + y)^3}, \quad x > 0, y > 0,$$

    find (a) $F(x, y)$, (b) $f_X(x)$, (c) $f_Y(y|X = x)$.

17. Find the density function $f(x,y)$ of the uniform distribution in the circle $x^2+y^2 \leq 1$. Find the marginal distribution of X and Y. Are the variables X and Y independent?

18. The joint density of the variables X, Y, Z is

$$f(x,y,z) = 8xyz, \quad 0 < x,y,z < 1$$

Find $P[X < Y < Z]$.

19. For each of the following densities $f(x,y)$, find $F(x,y)$, $F_X(x)$, $F_Y(y)$, $f_X(x)$, $f_Y(y)$, $f_X(x|Y=y)$, $f_Y(y|X=x)$.

    (a) $f(x,y) = 4xy, \quad 0 < x,y < 1,$

    (b) $f(x,y) = \frac{1}{8}(x^2 - y^2)e^{-x}, \quad 0 < x < \infty,\ |y| < x.$

20. Suppose that X and Y are independent random variables with the same exponential density

$$f(x) = \theta e^{-\theta x}, \quad x > 0.$$

Are $X+Y$ and X/Y independent? Justify your answer.

21. If X_1 and X_2 are independent and uniform on the interval $(0,1)$, find the densities of (a) $X_1 + X_2$, (b) $X_1 - X_2$, (c) $|X_1 - X_2|$, (d) X_1/X_2.

22. Two friends A and B agree to meet between 12 (noon) and 1 PM at a restaurant. Supposing that they arrive at random between 12 and 1 PM independently of each other and the lunch lasts 30 minutes, what is the probability that they meet in the restaurant? Let T be the instant of their meeting. What is the conditional distribution of T, (a) given that they meet, (b) given that they meet and A arrives first?

23. Let X and Y be independent with densities

$$f_X(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, \quad |x| < 1, \qquad f_Y(y) = \frac{y}{\sigma^2}e^{-y/2\sigma^2}.$$

Show that XY is $N(0, \sigma^2)$.

24. Given that $Var(X) = 9$, find the number n of observations (the sample size) required in order that with probability less than 5% the mean of the sample differs from the unknown mean μ of X or more, (a) than 5% of the standard deviation of X, (b) than 5% of μ given that $\mu > 5$. Compare the answers obtained by using Chebyshev's inequality and the CLT.

25. In a poll designed to estimate the percentage p of men who support a certain bill, how many men should be questioned in order that, with probability at least 95% the percentage of the sample differs from p

    (a) less than 1%

    (b) less than 5%

given that (a) $p < 30\%$ and (b) p is completely unknown.

26. Each of the 300 workers of a factory takes his lunch in one of three competing restaurants. How many seats should each restaurant have so that, on average, at most one in 20 customers will remain unseated?