CSL003P1M : Probability and Statistics
Lecture 03 (Axioms of Probability-II)

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Axioms of Probability

Consider an experiment whose sample space is S. For **each event** E of the sample space S, we assume that a number $P(E)$ is defined and satisfies the following three axioms:

1. Axiom 1:
$$0 \leq P(E) \leq 1.$$
2. Axiom 2:
$$P(S) = 1.$$
3. Axiom 3:
For any sequence of mutually exclusive events $E_1, E_2, \cdots$
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Some Results

R1: $P(\phi) = 0$.

R2: For any finite sequence of mutually exclusive events $E_1, E_2, \ldots, E_n$,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Examples

1. Let the experiment consists of tossing a coin. Assume that the a head is as likely to appear as a tail, then we would have

$$P(H) = P(T) = \frac{1}{2}$$

On the other hand, if the coins were biased and we felt that a head were twice as likely to appear as a tail, then we would have

$$P(H) = \frac{2}{3}, \quad P(T) = \frac{1}{3}$$

Examples

2. If a die is rolled and we suppose that all six sides are equally likely to appear, then we would have
   $P[1] = P[2] = P[3] = P[4] = P[5] = P[6] = \frac{1}{6}$.

   From the result $R2$, it would thus follow that the probability of rolling an even number would equal

   $$P(\{2,4,6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{2}$$

Exercise - 1

$$P(\bar{E}) = 1 - P(E)$$

Proof:

- $E \cup \bar{E} = S$.
- $E \cap \bar{E} = \phi$, thus E and $\bar{E}$ are mutually exclusive events.
- Therefore, from Axioms 2 and the result $R2$,

$$1 = P(S) = P(E \cup \bar{E}) = P(E) + P(\bar{E}).$$

- Hence, $P(\bar{E}) = 1 - P(E)$.

Exercise-2

If $E \subseteq F$, then $P(E) \leq P(F)$.

Proof:

- Since $E \subseteq F$, it follows that we can express F as

$$F = E \cup (\bar{E} \cap F)$$

- Because E and $\bar{E} \cap F$ are mutually exclusive, we obtain from the result $R2$,

$$P(F) = P(E) + P(\bar{E}F)$$

whcih proves the result, since $P(\bar{E}F) \geq 0$.

$\mathit{Exercise}$ - 3

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Proof:

- $P(E \cup F) = P(E \cup \bar{E}F) = P(E) + P(\bar{E}F)$.
- Since $F = EF \cup \bar{E}F$, we obtain

$$P(F) = P(EF) + P(\bar{E}F),$$

therefore, $P(\bar{E}F) = P(F) - P(EF)$.

- Putting the value of $P(\bar{E}F)$ in the first equation, we get

$$P(E \cup F) = P(E) + P(F) - P(EF).$$

Exercise - 4

Alice is taking two books along one her holiday vacation. With probability 0.5, she will like the first book; with probability 0.4, she will like the second book; and with probability 0.3, she wll like both books. What is the probability that she likes neither book?

Solution:

- Let B_i denote the event that Alice likes book i where $i = 1, 2$.
- Then we are interested in $P(\bar{B}_1 \bar{B}_2)$.
- First, we calculate

$$P(B_1 \cup B_2) = P(B_1) + P(B_2) - P(B_1 B_2) = 0.5 + 0.4 - 0.3 = 0.6.$$

- So, by DeMorgan's law

$$P(\bar{B}_1 \bar{B}_2) = P(\overline{B_1 \cup B_2}) = 1 - P(B_1 \cup B_2) = 1 - 0.6 = 0.4.$$

Exercises

Prove that (Hint: Use Mathematical Induction)

1. $P(E \cup F \cup G) =$
   $P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$
2. Inclusion-Exclusion Identity -

$$\begin{aligned} P(E_1 \cup E_2 \cup \cdots \cup E_n) &= \sum_{i=1}^{n} P(E_i) - \sum_{i_i < i_2} P(E_{i_1} E_{i_2}) + \cdots \\ &+ (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r}) \\ &+ \cdots + (-1)^{n+1} P(E_1 E_2 \cdots E_n) \end{aligned}$$

The summation $\sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r})$ is taken over all of the $\binom{n}{r}$ possible subsets of size r of the set $\{1, 2, \cdots, n\}$.

Spaces Having Equally Likely Outcomes

In many experiments, it is natural to assume that all outcomes in the sample space are equally likely to occur. That is consider, an experiment whose sample space is a finite set, say $S = \{1, 2, \ldots, N\}$. Then it is often natural to assume that

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{N\})$$

which implies from Axiom 2 and the result $R2$, that

$$P(i) = \frac{1}{N}, \quad i = 1, 2, \cdots, N$$

From the above equation, it follows from the result $R2$ that, for any event E,

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{F}{T}.$$

Examples

If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

Solution:

- $S = \{(1,1),(1,2),\ldots,(1,6),\ldots,(2,1),(2,2),\ldots,(2,6),\ldots,(6,6)\}$.
- $|S| = 36$.
- $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$.
- $|E| = 6$.
- If we assume all 36 outcomes are **equally likely**, then

$$P(E) = \frac{6}{36} = \frac{1}{6}.$$

Examples

If 2 balls are "randomly drawn" from a bowl containing 2 white and 1 black balls. What is the probability that one of the balls is white and one is black?

Solution:

- Denote the balls by W_1, W_2, B_1. (Pause for a moment and think ... can we do it?)
- $S = \{(W_1, W_2), (W_1, B_1), (W_2, W_1), (W_2, B_1), (B_1, W_1), (B_1, W_2)\}$. So, $|S| = 3 \times 2 = 6$. (Wait for a moment ... here we considered that the ordering is important ... which means (a, b) and (b, a) are different.)
- $E = \{(W_1, B_1), (B_1, W_1), (W_2, B_1), (B_1, W_2)\}$. So, $|E| = 2 \times 2 = 4$.

Examples

If 2 balls are "randomly drawn" from a bowl containing 2 white and 1 black balls. What is the probability that one of the balls is white and one is black?

Solution (contd):

- Since balls are chosen randomly, we assume that all outcomes are **equally likely**.
- Thus,

$$P(E) = \frac{4}{6} = \frac{2}{3}.$$

But what if the ordering is not important?

Examples

If 2 balls are "randomly drawn" from a bowl containing 2 white and 1 black balls. What is the probability that one of the balls is white and one is black?

Solution:

- Denote the balls by W_1, W_2, B_1. (Pause for a moment and think ... can we do it?)
- $S = \{(W_1, W_2), (W_1, B_1), (W_2, B_1)\}$. So, $|S| = (3 \times 2)/2 = 6/2 = 3$. (Here we considered that the ordering is not important ... which means (a, b) and (b, a) are same.)
- $E = \{(W_1, B_1), (W_2, B_1)\}$. So, $|E| = (2 \times 2)/2 = 2$.

Examples

If 2 balls are "randomly drawn" from a bowl containing 2 white and 1 black balls. What is the probability that one of the balls is white and one is black?

Solution (contd):

- Since balls are chosen randomly, we assume that all outcomes are **equally likely**.
- Thus,

$$P(E) = \frac{2}{3}.$$

Thank You