CSL003P1M : Probability and Statistics
Lecture 29 (Continuous Distributions)

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Introduction

- We have considered discrete random variables whose set of possible values is either finite or countably infinite.
- However, there also exist random variable whose set of possible values is uncountable.
- Some examples - (a) a train arrives at a specified stop, (b) the lifetime of a transistor etc.

Continuous Random Variables

Continuous Random Variables

We say that X a continuous random variable if there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that, for any set B of real numbers,

$$P\{X \in B\} = \int_B f(x)d(x)$$

The function f is called the **probability density function** of the random variable X.

f must satisfy

$$1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx$$

All probability statements about X can be answered in terms of f. For instance,

Let $B = [a, b]$, we obtain

$$P\{a \leq X \leq b\} = \int_a^b f(x)dx$$

If we let $a = b$,

$$P\{X = a\} = \int_a^b f(x)dx = 0$$

Continuous Random Vaiable

Cumulative Distribution Function

$$P\{X < a\} = P\{X \leq a\} = F(a) = \int_{-\infty}^{a} f(x)dx$$

Exercise-1

Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

1. What is the value of C?
2. Find $P\{X > 1\}$.

Solution: (1) Since f is a probability density function, we must have $\int_{-\infty}^{\infty} f(x)dx = 1$, implying that

$$C \int_0^2 (4x - 2x^2)dx = 1$$

$$C\left[2x^2 - \frac{2x^3}{3}\right]_{x=0}^{x=2} = 1$$

Thus,

$$C = 3/8.$$

(2)

$$\begin{aligned} P\{X > 1\} &= \int_1^\infty f(x)dx \\ &= \frac{3}{8}\int_1^2 (4x - 2x^2)dx \\ &= \frac{1}{2} \end{aligned}$$

Exercise-2

The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

What is the probability that

1. a computer will function between 50 and 150 hours before breaking down?
2. it will function for fewer than 100 hours?

Solution: Since

$$1 = \int_{-\infty}^{\infty} f(x)dx = \lambda \int_{0}^{\infty} e^{-x/100} dx$$

we obtain

$$1 = -\lambda(100)e^{-x/100}|_0^{\infty} = 100\lambda \Rightarrow \lambda = \frac{1}{100}.$$

(1) The probability that a computer will function between 50 and 150 hours before breaking down is

$$\begin{aligned} P\{50 < X < 150\} &= \int_{50}^{150} \frac{1}{100} e^{-x/100} dx = -e^{-x/100}|_{50}^{150} \\ &= e^{-1/2} - e^{-3/2} \approx 0.384 \end{aligned}$$

(2)

$$P\{X < 100\} = \int_0^{100} \frac{1}{100} e^{-x/100} dx = -e^{-x/100}|_0^{100} = 1 - e^{-1} \approx 0.633$$

Exercise-3

The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by

$$f(x) = \begin{cases} 0 & x \leq 100 \\ \dfrac{100}{x^2} & x > 100 \end{cases}$$

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume that the events E_i, $i = 1, 2, 3, 4, 5$, that the ith such tube will have to be replaced within this time are independent.

Exercise-3

$$\begin{aligned} P(E_i) &= \int_0^{150} f(x)dx \\ &= 100\int_{100}^{150} x^{-2}dx \\ &= \frac{1}{3} \end{aligned}$$

Since E_i's are independent, the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation is

$$\binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{80}{243}$$

Cumulative Distribution Function and Probability Density Function

The relationship between the cumulative distribution F and the probability density f is expressed by

$$F(a) = P\{X \in (-\infty, a]\} = \int_{-\infty}^{a} f(x)dx$$

Cumulative Distribution Function and Probability Density Function

Differentiating both sides of the preceeding equation yields

$$\frac{d}{da}F(a) = f(a)$$

That is, the density is the derivative of the cumulative distribution function. A somewhat more intuitive interpretation of the density function may be obtained as follows:

$$P\left\{a - \frac{\epsilon}{2} \le X \le a + \frac{\epsilon}{2}\right\} = \int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} f(x)dx \approx \epsilon f(a)$$

Exercise-4

If X is continuous with distribution function F_X and density function f_X, find the density function of $Y = 2X$.

Solution: First approach

$$\begin{aligned} F_Y(a) &= P\{Y \leq a\} \\ &= P\{2X \leq a\} \\ &= P\{X \leq a/2\} \\ &= F_X(a/2) \end{aligned}$$

Differentiation gives

$$f_Y(a) = \frac{1}{2} f_X(a/2)$$

Second approach:

$$
\begin{aligned}
\epsilon f_Y(a) &= P\{a - \frac{\epsilon}{2} \leq Y \leq a + \frac{\epsilon}{2}\} \\
&= P\{a - \frac{\epsilon}{2} \leq 2X \leq a + \frac{\epsilon}{2}\} \\
&= P\left\{\frac{a}{2} - \frac{\epsilon}{4} \leq X \leq \frac{a}{2} + \frac{\epsilon}{4}\right\} \\
&\approx \frac{\epsilon}{2} f_X(a/2)
\end{aligned}
$$

Dividing through by ϵ gives the same results as before.

Expectation of Continuous Random Variables

We defined the expected value of a discrete random variable X by

$$E[X] = \sum_x xP\{X = x\}$$

If X is a continuous random variable having probability density function $f(x)$, then, because

$$f(x)dx \approx P\{x \leq X \leq x + dx\} \qquad \text{for } dx \text{ small}$$

it is easy to see that the analogous definition is to define the expected value of X by

Expected Value of a Continuous Random Variable X

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

Exercise-5

Find $E[X]$ when the density function of X is

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^1 2x^2 dx \\ &= \frac{2}{3} \end{aligned}$$

Exercise-6

The density function of X is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[e^X]$.

Solution: Let $Y = e^X$. We first determine F_Y, the probability density function of Y. Now, for $1 \leq x \leq e$,

$$\begin{aligned} F_Y(x) &= P\{Y \leq x\} \\ &= P\{e^X \leq x\} \\ &= P\{X \leq \log(x)\} \\ &= \int_0^{\log(x)} f(t)dt \\ &= \log(x) \end{aligned}$$

Exercise-6

By differentiating $F_Y(x)$, we get

$$f_Y(x) = \frac{1}{x} \quad 1 \leq x \leq e$$

Hence,

$$\begin{aligned} E[e^X] = E[Y] &= \int_{-\infty}^{\infty} x f_Y(x) dx \\ &= \int_1^e dx \\ &= e - 1 \end{aligned}$$

Thank You