Q8

Total no. of balls $\Rightarrow n = b + g$

$X \rightarrow$ random variable (No. of black ball in sample of r)

Sample size $= r$

$X_i \rightarrow$ Sample of black balls.

$$X_i = \begin{cases} 1 & \text{if black ball is present} \\ 0 & \text{other wise} \end{cases}$$

For every black ball,

$P(X_i = 1) = \dfrac{b}{n} = p$ (Let)

$P(X_i = 0) = 1 - p$

$E[X_i] = 1 \cdot p + 0 \cdot (1-p) = p$

$E[X] = E[\sum_i X_i] = \sum_i E[X_i]$

$\therefore E[X] = p + p + \cdots p$ (r times)

$$\boxed{E[X] = rp = \frac{rb}{n} = \frac{rb}{b+g}}$$

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$$Var(X_i) = E(X_i^2) - (E(X_i))^2$$

Since $X_i = 0$ or $1 \Rightarrow X_i^2 = X_i$

$$\therefore E[X_i^2] = E[X_i]$$

$$Var[X_i] = p - p^2 = \frac{b(n-b)}{n^2}$$

$$Var[X] = Var\left(\sum_i^r X_i\right)$$

$$= \sum_{i=1}^{r} Var(X_i) + 2\sum_{1 \le i < j \le r} Cov(X_i, X_j)$$

(r terms) ($^{r}C_2$ terms)

$$Cov(X_i, X_j) = E[X_i X_j] - E(X_i)E(X_j)$$

$$X_i X_j = \begin{cases} 1 & \text{if } X_i = X_j = 1; \\ 0 & \text{otherwise} \end{cases}$$

$$P(X_i X_j = 1) = P(X_i = X_j = 1) = \frac{b(b-1)}{n(n-1)}$$

$$E(X_i X_j) = \frac{b(b-1)}{n(n-1)}$$

$$Cov(X_i X_j) = \frac{b(b-1)}{n(n-1)} - \left(\frac{b}{n}\right)^2$$

$$= \frac{b(b-n)}{n^2(n-1)}$$

$$\text{Var}[X] = \frac{r\,b(n-b)}{n^2} +$$

$$\frac{2r(r-1)}{2}\,\frac{b(b-n)}{n^2(n-1)}$$

$$= \cancel{\frac{r\,b\,g}{(b+g)^2}\left(1-\frac{r-1}{b+g-1}\right)}$$

$$= \frac{r\,b(n-b)}{n^2}\left(1-\frac{(r-1)}{(n-1)}\right)$$

$$\text{r}[x] \quad \boxed{= \frac{r\,b\,g}{(b+g)^2}\left(1-\frac{(r-1)}{(b+g-1)}\right)}$$

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Q9

Given X be length of first run of either success or faliur.

Let X_1 be length of first run of success
X_2 be length of first run of faliure

$X = X_1 + X_2$

$E[X] = E[X_1] + E[X_2]$

$P(X_1 = k) = p^k q$

$P(X_2 = k) = p q^k$

(where p is probability of success
q is probability of faliure)

This takes the form of Geometric Distribution but with $X_* = X_1 + 1$

$E[X_1] = \frac{1}{q} - 1 = \frac{p}{q}$

$E[X_2] = \frac{1}{p} - 1 \quad \frac{q}{p}$

$$\boxed{E[X] = \frac{p}{q} + \frac{q}{p}}$$

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$Var(X) = Var(X_1) + Var(X_2) + 2Cov(X,Y)$

$Var(X_1) = E(X_1^2) - (E(X_1))^2$ — ①

$E[X_1^2] = \sum_{i=1}^{\infty} i^2 q^{i-1} p = \sum_{i=1}^{\infty} (i-1+1)^2 q^{i-1} p$

$= \sum_{i=1}^{\infty} (i-1)^2 q^{i-1} p + \sum_{i=1}^{\infty} 2(i-1) q^{i-1} p$

$+ \sum_{i=1}^{\infty} q^{i-1} p$

$= q\,E[X^2] + 2q\,E[X] + 1.$

$= q\,E[X^2] + 2\left(\frac{q}{p}\right) + 1$

$E[X^2] = \frac{(q+1)}{p^2}$

From ①

$Var[X_1] = \frac{q}{p^2}$

$Var[X_2] = \frac{p}{q^2}$

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$$\text{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y]$$

$$\text{Cov}(X_1, X_2) = \underset{\downarrow \, 0}{E[X_1 X_2]} - \frac{p}{q} \times \frac{q}{p}$$

$$= -1$$

$$\boxed{\text{Var}(X) = \frac{p}{q^2} + \frac{q}{p^2} - 2}$$

Now let Y_1, Y_2 be length of second ~~of~~ rth run for success & failure

$$P(Y = k) = \binom{r+k-1}{k} p^r q^k, \quad k = 0, 1 \ldots$$

$$E[Y] = \sum_{k=0}^{\infty} k \binom{r+k-1}{k} p^r (q)^k$$

$$= \sum_{k=1}^{\infty} \frac{(r+k-1)!}{(k-1)!\,(r-1)!} p^r (q)^k$$

$$= \sum_{k=1}^{\infty} \frac{r(q)}{p} \binom{r+k-1}{k-1} p^{r+1} q^{k-1}$$

$$= \frac{rq}{p}$$

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Simlarly $Var(Y) = \frac{rq}{p^2}$

for $r = 2$

$E[Y] = E[Y_1 + Y_2] = E[Y_1] + E[Y_2]$

$$\boxed{E[Y] = \frac{2q}{p} + \frac{2p}{q}}$$

$Var(Y) = Var(Y_1) + Var(Y_2) - 2Cov(Y_1, Y_2)$

~~Var(Y)~~

$Cov(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2]$

0

$= -4$

$$\boxed{Var(Y) = \left(\frac{2p}{q^2} + \frac{2q}{p^2} - 8\right)}$$

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Q 10

If X & Y are independent then $Cov(X, Y) = 0$

But converse is not true

Consider an example :—

Let X, Y be random variable that

$$X = \begin{cases} -1 \text{ or } 1 \quad \text{with } p = 0.5 \end{cases}$$

$$Y = \begin{cases} 0 & \text{if } X = -1 \\ -1 \text{ or } +1 & \text{if } X = 1 \end{cases} \quad \text{with } p = 0.5$$

Clearly Y & X are dependent.

$$E[XY] = (-1).0 + 1.1 + (1.(-1))$$
$$= 0$$

$$E[X] = E[Y] = 0.5 \times 1 + 0.5 \times (-1)$$
$$= 0$$

$$\therefore Cov(X, Y) = E[XY] - E[X]E[\cancel{X}]$$
$$= 0$$

Hence proof

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$\boxed{Q13}$

Given X_1, X_2, X_3 be random variable with independent value.

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}}$$

i) $Cov(X_1 - X_2, X_2 + X_3) =$

$= E[(X_1 - X_2)(X_2 + X_3)] - E(X_1 - X_2) E(X_2 + X_3)$

(using $Cov(A, B) = E[AB] - E[A] E[B]$)

2) $= E[X_1X_2 + X_1X_3 - X_2^2 - X_2X_3] - (E[X_1] - E[X_2])(E[X_2] + E[X_3])$

$= \cancel{E[X_1X_2]} + \cancel{E[X_1X_3]} - E[X_2^2] - \cancel{E[X_2X_3]} - [\cancel{E[X_1]E[X_2]} + \cancel{E[X_1]E[X_3]} - E[X_2]E[X_2] - \cancel{E[X_2][E X_3]}]$

$\left(\begin{matrix} E[X_i] E[X_j] = E[X_i X_j] \\ i \neq j \end{matrix} \right)$

$= E[X_2]^2 - E[X_2^2]$

$= -Var[X_2]$

$$\mathrm{Var}(X_1 - X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$$

(Since X_1 & X_2 are independent)

$$\mathrm{Var}(X_2 + X_3) = \mathrm{Var}(X_2) + \mathrm{Var}(X_3)$$

$$\rho = \frac{-\mathrm{Var}(X_2)}{\sqrt{(\mathrm{Var}(X_2) + \mathrm{Var}(X_3))(\mathrm{Var}\,X_2 + \mathrm{Var}\,X_1)}}$$

$$\cancel{\rho = \frac{-\mathrm{Var}(X_2)}{\sqrt{\ }}}$$

$$\rho = \frac{-\sigma_2^2}{\sqrt{(\sigma_2^2 + \sigma_3^2)(\sigma_2^2 + \sigma_1^2)}}$$

$$\boxed{\rho = \frac{-\sigma_2^2}{\sqrt{\sigma_2^4 + (\sigma_2\sigma_1)^2 + (\sigma_2\sigma_3)^2 + (\sigma_1\sigma_3)^2}}}$$

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$\boxed{Q\ 17}$

$Var(X-2Y) = ?$

Given
$$\boxed{\begin{aligned} \rho(X,Y) &= \frac{1}{2} \\ Var(X) &= 1 \\ Var(Y) &= 2 \end{aligned}}$$

$$Var(X-2Y) = Var(X) + Var(2Y) - 2\,Cov(X, 2Y)$$

$$= Var(X) + 2^2\,Var(Y) - 2*2\,Cov(X,Y)$$

$$= Var(X) + 4\,Var(Y) - 4\,Cov(X,Y)$$

$[\ Var(aX+b) = a^2\,Var(X)\ ]$

$[\ Cov(aX,Y) = a\,Cov(X,Y)\ ]$

$$Cov(X,Y) = \sqrt{Var(X)\,Var(Y)} * \rho(X,Y)$$

$$= \sqrt{1\cdot 2} * \frac{1}{2}$$

$$= \frac{1}{\sqrt{2}}$$

$$Var(X-2Y) = 1 + 2\times4 - 4\left(\frac{1}{\sqrt{2}}\right)$$

$$\boxed{= 9 - 2\sqrt{2}}$$

Q 15

Given $n_R = 3$ $n_b = 2$

Sample Size $k = 2$

$U \rightarrow$ Random variable denoting no. of red ball selected

$V \rightarrow$ Random variable denoting no. of black ball selected

$$\rho(U,V) = \frac{Cov(U,V)}{\sqrt{var(U)\,var(V)}}$$

$$Cov(U,V) = E(UV) - E(U)E(V)$$

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$f(x, y) =$ probability functi

Red balls   black balls.

$$f(0,2) = \frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} = \frac{1}{10}$$

$$f(1,1) = \frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{6}{10}$$

$$f(2,0) = \frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} = \frac{3}{10}$$

$$E(UV) = \sum xy\, f(x,y)$$

$$= 0 + 1 \times 1 \times \frac{6}{10} + 0$$

$$\boxed{E(UV) = \frac{3}{5}}$$

$\frac{\cancel{12}}{5} \times \frac{3}{5}$

Given distribution is hypergeometric.

$\therefore E[U] = k \frac{n_r}{n_r + n_b}$

$= 2 \frac{3}{5} = \frac{6}{5}$

$E[V] = \frac{k\, n_b}{n_r + n_b} = \frac{2 \times 2}{5} = \frac{4}{5}$

$Cov(X, Y) = \frac{3}{5} - \frac{4}{5} \times \frac{6}{5}$

$$\boxed{Cov(X, Y) = -\frac{9}{25}}$$

~~Var[U]~~

For hypergeometric

$Var(X) = Kp(1-p)\left(1 - \frac{(k-1)}{(n-1)}\right)$

$Var(U) = 2 \times \frac{3}{5} \times \frac{2}{5} \left(1 - \frac{(2-1)}{(5-1)}\right)$

$$\boxed{Var(U) = \frac{\cancel{12}}{\cancel{8}} \cdot \frac{36}{25}}$$

$$Var(V) = \cancel{16}\, 2 \times \frac{2}{5} \times \frac{3}{5} \left(1 - \frac{(2-1)}{(5-1)} \right)$$

$$\boxed{Var(V) = \frac{36}{25}}$$

$$\rho(U, V) = \frac{\frac{-9}{25}}{\sqrt{\frac{36}{25} \times \frac{36}{25}}}$$

$$= \frac{\cancel{25}}{\cancel{36}\,4} \times \frac{-\cancel{9}}{\cancel{25}}$$

$$\boxed{\rho(U, V) = -\frac{1}{4}}$$