CSL003P1M : Probability and Statistics Lecture 18 (Some Problems on Random Variables)

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Exercise-1

Let X be a Poisson random variable with parameter λ. Find the distribution of X^2.

Solution:

- $X^2 = k^2$ if and only if $X = k$ for $k = 0, 1, 2, \ldots$.
- Thus,

$$P\{X^2 = k^2\} = P\{X = k\} = \frac{e^{-\lambda}\lambda^k}{k!}.$$

  and, 0 otherwise.
- Thus,

$$P\{X^2 = l\} = \begin{cases} \dfrac{e^{-\lambda}\lambda^{\sqrt{l}}}{(\sqrt{l})!} & \text{if } l \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose a box has 12 balls labeled $1, 2, \ldots, 12$. Two independent repetitions are made of the experiment of selecting a ball at random from the box with replacement. Let Z denotes the larger of the two numbers on the balls selected. Find the distribution of Z.

Solution:

- Let X be the random variable which denotes the ball number in the first draw.
- Let Y be the random variable which denotes the ball number in the second draw.
- Note that X and Y are independent.
- We are interested in finding the distribution of $Z = \max(X, Y)$.
- Or, we are interested in $P\{Z = z\}$ for $z = 1, 2, \ldots, 12$.

Exercise-2

- Note that $\{Z = z\}$ when
  - $\{X = z, Y = z\}$, or
  - disjoint unions of $\{X = z, Y = z - r\}$ for $r = 1, 2, \ldots, z - 1$, or
  - disjoint unions of $\{X = z - r, Y = z\}$ for $r = 1, 2, \ldots, z - 1$.
- Observe that all three events discussed are mutually exclusive.
- $P\{X = z, Y = z\} = P\{X = z\}P\{Y = z\} = \left(\frac{1}{12}\right)^2$.
- $$\begin{aligned} \sum_{r=1}^{z-1} P\{X = z, Y = z - r\} &= \sum_{r=1}^{z-1} P\{X = z\}P\{Y = z - r\} \\ &= (z - 1)\left(\frac{1}{12}\right)^2 \end{aligned}$$
- Similarly, $\sum_{r=1}^{z-1} P\{X = z - r, Y = z\} = (z - 1)\left(\frac{1}{12}\right)^2$.

Exercise-2

- Thus,

$$P\{Z=z\} = [1+2(z-1)]\left(\frac{1}{12}\right)^2 = (2z-1)\left(\frac{1}{12}\right)^2$$

z	$P\{Z=z\}$	z	$P\{Z=z\}$
1	$1/144$	7	$13/144$
2	$3/144$	8	$15/144$
3	$5/144$	9	$17/144$
4	$7/144$	10	$19/144$
5	$9/144$	11	$21/144$
6	$11/144$	12	$23/144$

Exercise-3

Let X and Y be independent random variables each geometrically distributed with parameter p. Find $P\{\min(X,Y)=X\}$.

Solution:

- Note that the event $\min(X,Y)=X$ is same as $Y \geq X$.
- Thus,

$$\begin{aligned} P\{\min(X,Y)=X\} &= P\{Y \geq X\} \\ &= \sum_{x=1}^{\infty} P\{X=x, Y \geq x\} \\ &= \sum_{x=1}^{\infty} P\{X=x\}P\{Y \geq x\} \\ &= \sum_{x=1}^{\infty} pq^{x-1}(pq^{x-1}+pq^{x}+pq^{x+1}+\ldots) \end{aligned}$$

$$
\begin{aligned}
p^2\sum_{x=1}^{\infty}(q^2)^{x-1}(1+q+q^2+\ldots) &= p^2\sum_{x=1}^{\infty}(q^2)^{x-1}\frac{1}{1-q} \\
&= p^2\sum_{x=1}^{\infty}(q^2)^{x-1}\frac{1}{p} \\
&= p\sum_{x=1}^{\infty}(q^2)^{x-1} \\
&= p(1+q^2+q^4+\ldots) \\
&= p\left(\frac{1}{1-q^2}\right) \\
&= p\left(\frac{1}{(1-q)(1+q)}\right) \\
&= \frac{1}{1+q}
\end{aligned}
$$

Exercise-4

Let X and Y be independent random variables each geometrically distributed with parameter p. Find the distribution of the random variable $\min(X, Y)$.

Solution:

- Let $Z = \min(X, Y)$ be the random variable.
- We are interested in $P\{Z = z\}$ for $z = 1, 2, \ldots$
- Note that $\{Z = z\}$ when
  - $\{X = z, Y = z\}$, or
  - disjoint unions of $\{X = z, Y = z + r\}$ for $r = 1, 2, \ldots$, or
  - disjoint unions of $\{X = z + r, Y = z\}$ for $r = 1, 2, \ldots$.
- Observe that all three events discussed are mutually exclusive.

So,

$$P\{Z=z\} = \sum_{r=1}^{\infty} P\{X=z, Y=z+r\} + \sum_{r=1}^{\infty} P\{X=z+r, Y=z\} + P\{X=z, Y=z\}$$

Since, X and Y are independent, thus

$$P\{Z=z\} = \sum_{r=1}^{\infty} P\{X=z\}P\{Y=z+r\} + \sum_{r=1}^{\infty} P\{X=z+r\}P\{Y=z\} + P\{X=z\}P\{Y=z\}$$

Moreover,

$$\sum_{r=1}^{\infty} P\{X=z\}P\{Y=z+r\} = \sum_{r=1}^{\infty} P\{X=z+r\}P\{Y=z\}$$

Therefore,

$$\begin{aligned}
P\{Z=z\} &= 2\sum_{r=1}^{\infty} P\{X=z\}P\{Y=z+r\} + P\{X=z\}P\{Y=z\} \\
&= P\{X=z\}\left(2\sum_{r=1}^{\infty} P\{Y=z+r\} + P\{Y=z\}\right) \\
&= pq^{z-1}[2pq^{z-1}(q+q^2+\ldots) + pq^{z-1}] \\
&= p^2(q^2)^{z-1}[2q(1+q+q^2+\ldots)+1] \\
&= p^2(q^2)^{z-1}\left[\frac{2q}{1-q}+1\right] \\
&= p^2(q^2)^{z-1}\left[\frac{2q}{p}+1\right] \\
&= p(q^2)^{z-1}(2q+p) \\
&= p(q^2)^{z-1}[2(1-p)+p] \\
&= p(2-p)[(1-p)^2]^{z-1}
\end{aligned}$$

Exercise-4

$$P\{Z = z\} = p(2 - p)[(1 - p)^2]^{z-1}.$$

- Let $p' = p(2 - p)$. Then $1 - p' = (1 - p)^2$.
- Moreover, $0 \leq p' \leq 1$.
- Thus, for $l = 0, 1, 2, \ldots$

$$P\{Z = z\} = p'(1 - p')^{z-1}.$$

Hence $Z = \min(X, Y)$ is a geometric random variable with parameter $p' = p(2 - p)$.

Thank You