CSL003P1M : Probability and Statistics
Lecture 26 (Inequalities-I)

Sumit Kumar Pandey

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Cauchy-Schwarz Inequality

Cauchy-Schwarz Inequality

Let X and Y have finite second moments. Then

$$(E[XY])^2 \leq (E[X^2])(E[Y^2])$$

Furthermore, equality holds if and only if either $P\{Y = 0\} = 1$ or $P\{X = aY\} = 1$ for some constant a.

Proof:

- If $P\{Y = 0\} = 1$, then $P\{XY = 0\} = 1$ which implies $E[XY] = 0$ and $E[Y^2] = 0$. Thus the Cauchy-Schwarz inequality actually holds with equality.
- If $P\{X = aY\} = 1$, then both sides will be equal to $(a^2E[Y^2])^2$.

Cauchy-Schwarz Inequality

Now, we prove that the inequality always holds. From the discussion, we can assume that $P\{Y = 0\} < 1$ and hence $E[Y^2] > 0$. Observe that,

$$0 \leq E[(X - \lambda Y)^2] = \lambda^2 E[Y^2] - 2\lambda E[XY] + E[X^2].$$

This is a quadratic function of λ. The minimum value of this function is achieved at $\lambda = a = (E[XY])(E[Y^2])^{-1}$. So, putting this value of λ, we obtain

$$0 \leq E[(X - aY)^2] = E[X^2] - \frac{(E[XY])^2}{E[Y^2]}$$

where equality occurs when $E[(X - aY)^2] = 0$, or $P\{(X - aY) = 0\} = 1$. Thus,

$$(E[XY])^2 \leq (E[X^2])(E[Y^2])$$

Correlation Coefficient

Correlation Coefficient

Let X and Y be random variables. The correlation coefficient $\rho(X, Y)$ is defined as

$$\rho = \rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

Show that $|\rho| \leq 1$.

Proof: (Hint: Apply Cauchy-Schwarz inequality)
Let $Z_1 = X - E[X]$ and $Z_2 = Y - E[Y]$. Then, from Cauchy-Schwarz inequality

$$\begin{array}{rrcl} & (E[Z_1 Z_2])^2 & \leq & E[Z_1^2]E[Z_2^2] \\ \Leftrightarrow & E[(X - E[X])(Y - E[Y])]^2 & \leq & E[(X - E[X])^2]E[(Y - E[Y])^2] \\ \Leftrightarrow & (Cov(X, Y))^2 & \leq & Var(X)Var(Y) \end{array}$$

Correlation Coefficient

$$(Cov(X, Y))^2 \leq Var(X)Var(Y)$$

So,

$$(\rho(X, Y))^2 = \frac{Cov(X, Y)^2}{Var(X)Var(Y)} \leq 1$$

Thus, we obtain $|\rho(X, Y)| \leq 1$. And,

$$|\rho(X, Y)| = 1$$

if and only if $P\{X = aY\} = 1$ for some constant a.

Markov's Inequality

Markov's Inequality

If X is a random variable that takes only nonnegative values, then for any value $a > 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

Proof: For $a > 0$, let

$$I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{otherwise} \end{cases}$$

and note that, since $X \geq 0$,

$$I \leq \frac{X}{a}$$

Markov's Inequality

Taking expectations of the preceeding inequality, we get

$$E[I] \leq \frac{E[X]}{a}$$

which, because $E[I] = P\{X \geq a\}$, proves the result.

Chebyshev's Inequality

Chebyshev's Inequality

If X is a random variable with finite mean μ and variance σ^2, then, for any value $k > 0$,

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

Proof: Since $(X - \mu)^2$ is a nonnegative random variable, we can apply Markov's inequality (with $a = k^2$) to obtain

$$P\{(X - \mu)^2 \geq k^2\} \leq \frac{E[(X - \mu)^2]}{k^2}$$

But since $(X - \mu)^2 \geq k^2$ if and only if $|X - \mu| \geq k$. The above equation is equivalent to

$$P\{|X - \mu| \geq k\} \leq \frac{E[(X - \mu)^2]}{k^2} = \frac{\sigma^2}{k^2}.$$

Thank You