CSL003P1M : Probability and Statistics
Lecture 15 (Independent Random Variables)

Sumit Kumar Pandey

October 06, 2021

Independent Random Variables

Independent Random Variables

The random variables X and Y are said to be independent if, for any two sets of real numbers A and B,

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}.$$

In other words, X and Y are independent if, for all A and B, the events $E_A = \{X \in A\}$ and $F_B = \{Y \in B\}$ are independent.

Independent Random Variables

It can be shown by using the three axioms of probability that for any two sets of real numbers A and B

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}.$$

if and only if, for all a, b,

$$P\{X \leq a, Y \leq b\} = P\{X \leq a\}P\{Y \leq b\}.$$

Hence, in terms of the joint distribution function of X and Y, X and Y are independent if

$$F(a, b) = F_X(a)F_Y(b) \quad \text{for all } a, b.$$

Independent Random Variables

When X and Y are discrete random variables, the condition of independence is equivalent to

$$p(x, y) = p_X(x)p_Y(y) \quad \text{for all } x, y.$$

Solution: Assume the condition of independence holds, i.e. for any two sets of real numbers A and B

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}.$$

Then,

- Take $A = \{x\}$ and $B = \{y\}$.
- We get

$$p(x, y) = P\{X = x, Y = y\} = P\{X = x\}P\{Y = y\} = p_X(x)p_Y(y).$$

Conversely, assume

$$p(x,y) = p_X(x)p_Y(y) \quad \text{for all } x, y.$$

Then,

$$\begin{aligned} P\{X \in A, Y \in B\} &= \sum_{y \in B} \sum_{x \in A} p(x,y) \\ &= \sum_{y \in B} \sum_{x \in A} p_X(x)p_Y(y) \\ &= \sum_{y \in B} p_Y(y) \sum_{x \in A} p_X(x) \\ &= P\{Y \in B\}P\{X \in A\}. \end{aligned}$$

Exercise-1

The random variables X and Y are independent random variables if and only if whenever $a \leq b$ and $c \leq d$, then

$$P\{a < X \leq b, c < Y \leq d\} = P\{a < X \leq b\}P\{c < Y \leq d\}.$$

Solution: Assume that X and Y are independent random variables. Then,

- Put $A = (a, b]$ and $B = (c, d]$.

$Exercise$-1

The random variables X and Y are independent random variables if and only if whenever $a \leq b$ and $c \leq d$, then

$$P\{a < X \leq b, c < Y \leq d\} = P\{a < X \leq b\}P\{c < Y \leq d\}.$$

Alternate Solution: Assume X and Y are independent. Then,

- $F(x, y) = F_X(x)F_Y(y)$ for all x, y.
- Consider $F(b, d) - F(a, d) - F(b, c) + F(a, c)$ which is

$$= P\{a < X \leq b, c < Y \leq d\}.$$

- Now, since X and Y are independent,
  $F(b, d) - F(a, d) - F(b, c) + F(a, c)$

$$\begin{aligned} &= F_X(b)F_Y(d) - F_X(a)F_Y(d) - F_X(b)F_Y(c) + F_X(a)F_Y(c) \\ &= (F_X(b) - F_X(a))(F_Y(d) - F_Y(c)) \\ &= P\{a < X \leq b\}P\{c < Y \leq d\}. \end{aligned}$$

Loosely speaking, X and Y are independent if knowing the value of one does not change the distribution of the other.

Dependent Random Variables

Random variables that are not independent are said to be dependent.

Exercise-2

Suppose that $n + m$ independent trials having a common probability of success p are performed. If X is the number of successes in the first n trials, and Y is the number of successes in the final m trials, then X and Y are independent, since knowing the number of successes in the first n trials does not affect the distribution of the number of successes in the final m trials (by the auumption of independent trials). Find

$$P\{X = x, Y = y\}.$$

Solution: Let $q = 1 - p$.

$$\begin{aligned} P\{X = x, Y = y\} &= \binom{n}{x} p^x q^{n-x} \binom{m}{y} p^y q^{m-y} \\ &= P\{X = x\}P\{Y = y\}. \end{aligned}$$

where $0 \leq x \leq n$ and $0 \leq y \leq m$.

Exercise-3

Suppose that $n+m$ independent trials having a common probability of success p are performed. If X is the number of successes in the first n trials, and Z is the number of successes in the $n+m$ trials, will X and Z be independent?

Solution: No. (Why?)

- $$P\{X=x\}=\binom{n}{x}p^x(1-p)^{n-x}, \quad 0\le x\le n.$$
- $$P\{Z=z\}=\binom{n+m}{z}p^z(1-p)^{n+m-z}, \qquad 0\le z\le n+m.$$
- But,

$$P\{X=1,Z=0\}=0\neq P\{X=1\}P\{Z=0\}.$$

Exercise-4

Suppose that the number of people who enter a post office on a given day is a Poisson random variable with parameter λ. Show that if each person who enters the post office is a male with probability p and a female with probability $1-p$, then the number of males and females entering the post office are independent Poisson random variables with respective parameters λp and $\lambda(1-p)$.

: Hint: (Combination of Poisson and Binomial distribution.)

- Let X and Y be random variables, respectively, which denote the number of males and females who enter the post office.
- Find $P\{X=i, Y=j\}$, $P\{X=i\}$ and $P\{Y=j\}$. .
- Check $P\{X=i, Y=j\} \stackrel{?}{=} P\{X=i\}P\{Y=j\}$.

Solution:

- Apply $P(E) = P(E|F)P(F) + P(E|\bar{F})P(\bar{F})$.
-

$$
\begin{aligned}
&P\{X = i, Y = j\} = \\
&P\{X = i, Y = j | X + Y = i + j\} P\{X + Y = i + j\} + \\
&P\{X = i, Y = j | X + Y \neq i + j\} P\{X + Y \neq i + j\}.
\end{aligned}
$$

- Note that $P\{X = i, Y = j | X + Y \neq i + j\} = 0$.
- Thus,

$$
\begin{aligned}
&P\{X = i, Y = j\} = \\
&P\{X = i, Y = j | X + Y = i + j\} P\{X + Y = i + j\}.
\end{aligned}
$$

Exercise-4

- From the question, $X + Y$ follows Poisson distribution with parameter λ.
- Therefore,

$$P\{X + Y = i + j\} = e^{-\lambda}\frac{\lambda^{i+j}}{(i+j)!}, \quad i + j = 0, 1, \ldots$$

- Given that $i + j$ people enter the post office, it follows that exactly i of them will be male (and thus j of them female) follows Binomial distribution with parameters $i + j$ and p (from the question).
- So,

$$P\{X = i, Y = j | X + Y = i + j\} = \binom{i+j}{i} p^i (1-p)^j.$$

- Therefore,

$$
\begin{aligned}
P\{X=i, Y=j\} &= \binom{i+j}{i} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!} \\
&= e^{-\lambda} \frac{(\lambda p)^i}{i!j!} [\lambda(1-p)]^j \\
&= e^{-\lambda p} \frac{(\lambda p)^i}{i!} e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^j}{j!}
\end{aligned}
$$

- Hence,

$$
P\{X=i\} = e^{-\lambda p} \frac{(\lambda p)^i}{i!} \sum_{j=0}^{\infty} e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^j}{j!} = e^{-\lambda p} \frac{(\lambda p)^i}{i!}.
$$

- Similarly,

$$
P\{Y=j\} = e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^j}{j!}
$$

Thank You