# Problem Set - 7

Date ___/___/___

Page ______

1\. i.) $P\{X=Y\} = P\bigcup_{K=1}^{\infty}\{X=Y=K\} = \sum_{K=1}^{\infty} P(X=Y=K)$

$= \sum_{K=1}^{\infty} P((X=K)\cap(Y=K)) \qquad = \sum_{K=1}^{\infty} P\{X=K\}\cdot P\{Y=K\}$

Since, events X and Y are disjoint and X and Y are independent variables.

Using the pmf (probability mass function) of the Geometric distribution, we get:

$$P\{X=Y\} = \sum_{K=1}^{\infty} p(1-p)^{K-1}\cdot p(1-p)^{K-1}$$

$$= p^2 \sum_{K=0}^{\infty} (1-p)^{2k}$$

Using Geometric Series sum, we get:

$$P\{X=Y\} = \frac{p^2}{1-(1-p)^2} = \frac{p^2}{\cancel{1}-(\cancel{1}+p^2-2p)}$$

$$= \frac{p^{\cancel{2}}}{\cancel{p}(2-p)}$$

$$= \frac{p}{(2-p)}$$

ii.) $P\{X \geq 2Y\} = P\{X=2Y\} + P\{X>2Y\}$ ——— (1)

and for $P\{X>2Y\}$ and $P\{X<2Y\}$

Since,

$P\{X=2Y\} + P\{X>2Y\} + P\{X<2Y\} = 1$ ——— (2)

for $P\{X=2Y\} = \sum_{K=1}^{\infty} P((X=K)\cap(2Y=K))$

$(1-p) = q$

$\frac{1}{}$

$\cancel{K}-1+\frac{K}{2}-1 = \frac{3K}{2}$

Date / /

Page

Since X and Y are independent variables.

So,

$$P(X = 2Y) = \sum_{K=1}^{\infty} P(X = 2K) \cdot P(2Y = K)$$

$$= \sum_{K=1}^{\infty} p \cdot (1-p)^{(2K-1)} \cdot p(1-p)^{K-1}$$

$$= p^2 \sum_{K=0}^{\infty} (1-p)^{3K}$$

Using Geometric Series sum, we get:

$$P(X = 2Y) = p^2 * \frac{1}{1-(1-p)^3}$$

$$= \frac{p^2}{\cancel{1} - \{\cancel{1} - 3p + 3p^2 - p^3\}}$$

$$= \frac{p^{\cancel{2}}}{p\{3 - 3p + p^2\}} = \frac{p}{3-3p+p^2} \quad \text{--- (3)}$$

Now from Eq$^n$ (2), we get:

$$P\{X > 2Y\} = P\{X < 2Y\} \quad \text{[By Symmetry]}$$

Thus, $P\{X > 2Y\} = \frac{1 - P\{X = 2Y\}}{2}$

Therefore, from (1)

$$P\{X \geq 2Y\} = P\{X = 2Y\} + P\{X > 2Y\}$$

$$= P\{X = 2Y\} + \frac{1 - P\{X = 2Y\}}{2}$$

$$= \frac{P\{X = 2Y\} + 1}{2}$$

Date / /
Page

Using eq$^n$ (3), we get :

$$P\{X \geq 2Y\} = \frac{\frac{p}{3-3p+p^2} + 1}{2}$$

$$= \frac{p + 3 - 3p + p^2}{6 - 6p + 2p^2}$$

$$= \frac{3 - 2p + p^2}{6 - 6p + 2p^2} .$$

iii) $P\{X + Y \leq Z\}$

Date ___/___/___
Page ______

2. A/q. X and Y are independent geometric random variables with parameter p.

Let $U = \min(X, Y)$ and $V = X - Y$.

Possible cases will be $X > Y$, $X < Y$ and $X = Y$.

**Case I:** $X > Y$

Then, $U = Y$ and $V = X - Y > 0$

Then,

$$P(U=u, V=v) = P(Y=u, X=v+u) \{\text{where, } v>0\}$$

$$\boxed{P(Y, X) = \sum_{V, U=0}^{\infty} P(Y) \cdot P(X) = \sum_{V, U=0}^{\infty} p(1-p)^{u-1} \cdot p(1-p)^{u+v-1} = p^2(1-p)^{2u+v-2}}$$

**Case II:** $X < Y$

Then, $U = X$ and $V = X - Y < 0$

Then,

$$P(U=u, V=v) = P(X=u, Y=u+v) \{\text{where, } v<0\}$$

$$\boxed{P(X, Y) = \sum_{V, U=0}^{\infty} P(X) \cdot P(Y) = \sum_{U, V=0}^{\infty} p(1-p)^{u-1} \cdot p(1-p)^{u+v-1} = p^2(1-p)^{2u+v-2}}$$

**Case III:** $X = Y$

Then, $U = X = Y$ and $V = X - Y = 0$.

Then,

$$P(U=u, V=0) = P(X=Y=u, V=0)$$

$$\boxed{P(X, Y) = \sum_{U=0}^{\infty} p(1-p)^{u-1} p(1-p)^{u-1} = p^2(1-p)^{2u-2}}$$

Now, In joint probability mass function (pmf), we get:

$$P(U=u, V=v) = \begin{cases} p^2(1-p)^{2u-2} & ; v=0, u=1,2,3,\ldots \\ p^2(1-p)^{2u+v-2} & ; v>0, u=1,2,3,\ldots \\ p^2(1-p)^{2u+v-2} & ; v<0, u=1,2,3,\ldots \end{cases}$$

Hence, U and V are also independent variables.

Pg 240 Have to find.

Date / /
Page

4. A/q. $X_1, X_2, \ldots, X_K$ are distributed according to multinomial distribution. Then, their probabilities of each $(X_1, X_2, \ldots)$

$n$ = # no. variable $X_i$
$n_i$ = for each $i \in \{1, 2, \ldots, k\}$
$n_i$ is count of each type of $X_i$ is present.

$$P(X) = P_1^{n_1} P_2^{n_2} P_3^{n_3} \ldots P_{K-1}^{n_{K-1}}$$

$$= \frac{n!}{n_1! \, n_2! \ldots n_r!} P_1^{n_1} P_2^{n_2} P_3^{n_3} \ldots P_r^{n_r} \quad (1)$$

for $n_i > 0$, $i = 1, 2, \ldots, K$

$n_1 + n_2 + \ldots + n_{K-1} + n_K \leq n$. We have!

$$P[X_K = n_K \mid X_1 = n_1, X_2 = n_2, \ldots, X_{K-1} = n_{K-1}]$$

$$= \frac{n!}{n_1! \, n_2! \ldots n_{K-1}!} P_1^{n_1} P_2^{n_2} \ldots P_{K-1}^{K-1} \quad (2)$$

Since, Given, parameter of are $n - (n_2 + \ldots + n_{K-1})$ and $\frac{P_1}{P_1 + P_K} \Rightarrow P_1 + P_K > 0 \Rightarrow P_1 > 0$ & $P_K > 0$

from the eq$^n$ (1) we get a multinomial distribution {i.e. joint distribution whose pmf is eq$^n$ (1)}

Now, if $r = 1$ ~~case~~ then it will turn into Binomial dist.

$\therefore$ Binomial Distribution of $X_1$

$$P[X_1 = n_1 \mid X_2 = n_2, \ldots, X_{K-1} = n_{K-1}]$$

where, $N \subset \{1, 2, \ldots, r\}$

$$= \frac{\binom{NP_2}{n_2}\binom{NP_K}{n_K} \Big/ \binom{N}{n}}{\binom{NP_{K-1}}{n_1} \cdots \binom{NP_{K-1}}{n_{K-1}} \binom{N(P_{K+1} + P_K)}{n - x_K}}$$

$$= \frac{NP_K}{n_K}$$

Date __/__/__
Page ____

7. A/q. We have:

| | Acceptable | Defective |
|---|---|---|
| ① Inspected | $pp'$ | $p'q$ |
| ② Undiscovered | $pq'$ | $qq'$ |

where, $q = (1-p)$ and $q' = (1-p')$

and,

$N \equiv$ No. of items passing inspection desk (both ① & ②) before 1st defective is found.

$K \equiv$ No. of undiscovered defectives.

Let, 1st defective item is found at $(n+1)$th trial, and 'n' items are either acceptable or undiscovered.

$\therefore P\{\text{Acceptable} \cup \text{Inspected}\}^c = P\{\text{Defective, Inspected}\}^c$
$= 1 - p'q$

Here, $N$ is the waiting time for first defective found. which follow. geometric distribution.

$$P\{N=n\} = \{1-p'q\}^n p'q \quad \text{——— (i)}$$

And, $K$ is undiscovered defectives.

$$\therefore P\{K=k \mid N=n\} = \binom{n}{k}(qq')^k (p)^{n-k} \quad \text{——— (ii)}$$

Hence

For joint Distribution of $(N, K)$, we have:

$$P(n, K) = \binom{n}{k} p^{n-k} q^k \times qp' \times (q')^k$$

$$= \binom{n}{k} p^{n-k} \times (qq')^k \times qp'.$$

Date / /
Page

For marginal distribution,
---- of N.

~~$f(n, k)$~~

$$P(N=n_i) = \sum_{n} P(N=n_i, K=k_i) \quad \{\text{where}, i=1,2,\ldots r\}$$

~~Marginal distribution~~

$$= \left\{\binom{n_1}{k_1} p^{n_1-k_1} \times (qq')^{k_1} \times qp'\right\} + \left\{\binom{n_2}{k_2} p^{n_2-k_2} * (qq')^{k_2} * qp'\right\}$$

$$+ \ldots\ldots + \left\{\binom{n_r}{k_r} p^{n_r-k_r} * (qq')^{k_r} * qp'\right\}$$

Date___/___/___
Page______

6. Let, p be the probability of coin landing heads.
and q be the probability " " tails
i.e. $q = (1-p)$

Since, ~~we have~~ a coin is biased.
Then, let consider we have coin having $p = 0.6$
If we toss the coin twice and coins faces are different
the probability will be either
$P(HH) = P(H) \cdot P(H) = 0.36$