CSL003P1M : Probability and Statistics Lecture 16 (Sums Of Independent Random Variables)

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If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, find the distribution of $X + Y$.

Solution:

- Note that the event $\{X + Y = n\}$ can be written as the union of the disjoint events $\{X = k, Y = n - k\}$, $0 \leq k \leq n$, we have

$$\begin{aligned} P\{X + Y = n\} &= \sum_{k=0}^{n} P\{X = k, Y = n - k\} \\ &= \sum_{k=0}^{n} P\{X = k\}P\{Y = n - k\} \\ &= \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} \end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} &= e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{\lambda_1^k\lambda_2^{n-k}}{k!(n-k)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n
\end{aligned}$$

Thus $X + Y$ follows Poisson distribution with parameter $\lambda_1 + \lambda_2$.

If X and Y are independent Binomial random variables with respective parameters (n, p) and (m, p), find the distribution of $X + Y$.

Solution:

- Note that the event $\{X + Y = k\}$ can be written as the union of the disjoint events $\{X = i, Y = k - i\}$, $0 \leq i \leq n$, we have

$$\begin{aligned} P\{X + Y = k\} &= \sum_{i=0}^{n} P\{X = i, Y = k - i\} \\ &= \sum_{i=0}^{n} P\{X = i\} P\{Y = k - i\} \end{aligned}$$

Sum of Independent Binomial Random Variables

Thus,

$$P\{X+Y=k\} = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \binom{m}{k-i} p^{k-i} q^{m-k+i}$$

where $q = 1-p$ and where $\binom{r}{j} = 0$ when $j < 0$. Now,

$$\begin{aligned} P\{X+Y=k\} &= p^k q^{n+m-k} \sum_{i=0}^{n} \binom{n}{i} \binom{m}{k-i} \\ &= \binom{n+m}{k} p^k q^{n+m-k} \end{aligned}$$

Thus, $X+Y$ follows Binomial distribution with parameters $(n+m, p)$.

Sum of Independent Geometric Random Variables

If X_1 and X_2 are independent geometric random variables with parameter p, find the distribution of $X_1 + X_2$.

Solution:

- Note that the event $\{X_1 + X_2 = k\}$ can be written as the union of the disjoint events $\{X_1 = j, X_2 = k - j\}$, $1 \leq j \leq k - 1$, we have

$$\begin{aligned} P\{X_1 + X_2 = k\} &= \sum_{j=1}^{k-1} P\{X_1 = j, X_2 = k - j\} \\ &= \sum_{j=1}^{k-1} P\{X_1 = j\} P\{X_2 = k - j\} \\ &= \sum_{j=1}^{k-1} pq^{j-1} pq^{k-j-1} \end{aligned}$$

$$
\begin{aligned}
\sum_{j=1}^{k-1} pq^{j-1}pq^{k-j-1} &= p^2q^{k-2}\sum_{j=1}^{k-1} 1 \\
&= p^2q^{k-2}(k-1) \\
&= \binom{k-1}{2-1} p^2q^{k-2}.
\end{aligned}
$$

Thus, $X_1 + X_2$ follows Negative Binomial distribution with parameters $(2, p)$.

Sum of Independent Geometric Random Variables

If $X_1, X_2, \dots, X_n$ are independent geometric random variables with parameter p, find the distribution of $X_1 + X_2 + \cdots + X_n$.

Solution: $X_1 + X_2 + \cdots + X_n$ follows Negative Binomial Distribution with parameters (n, p). (Hint: Use Mathematical Induction)

- Let $S_n = X_1 + X_2 + \cdots + X_n$.
- Base Case: When $n = 1$, S_1 follows geometric distributon with parameter p which is same as negative binomial distribution with parameters $(1, p)$.
- Induction hypothesis: We assume it's true for $n = k$, i.e. $S_k = X_1 + X_2 + \cdots + X_k$ follows negative binomial distribution with parameters (k, p).

Sum of Independent Geometric Random Variables

- Inductive Step: Let $n = k + 1$.
- $S_{k+1} = S_k + X_{k+1}$.
- Note that the event $\{S_k + X_{k+1} = l\}$ can be written as the union of the disjoint events $\{S_k = j, X_{k+1} = l - j\}$, $k \leq j \leq l - 1$, we have

$$\begin{aligned} P\{S_k + X_{k+1} = l\} &= \sum_{j=k}^{l-1} P\{S_k = j, X_{k+1} = l - j\} \\ &= \sum_{j=k}^{l-1} P\{S_k = j\} P\{X_{k+1} = l - j\} \\ &= \sum_{j=k}^{l-1} \binom{j-1}{k-1} p^k q^{j-k} p q^{l-j-1} \end{aligned}$$

$$\begin{aligned}
\sum_{j=k}^{l-1} \binom{j-1}{k-1} p^k q^{j-k} p q^{l-j-1} &= p^{k+1} q^{l-(k+1)} \sum_{j=k}^{l-1} \binom{j-1}{k-1} \\
&= \binom{l-1}{k} p^{k+1} q^{l-(k+1)} \\
&= \binom{l-1}{k+1-1} p^{k+1} q^{l-(k+1)}
\end{aligned}$$

because

$$\binom{k-1}{k-1} = \binom{k}{k} \text{ and } \binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$$

Thus, $S_{k+1} = X_1 + X_2 + \cdots + X_{k+1}$ follows Negative Binomial distribution with parameters $(k+1, p)$.

If X_1 and X_2 are independent geometric random variables with parameters p_1 and p_2 respectively, find the distribution of $X_1 + X_2$.

Solution:

- Note that the event $\{X_1 + X_2 = k\}$ can be written as the union of the disjoint events $\{X_1 = j, X_2 = k - j\}$, $1 \le j \le k - 1$, we have

$$
\begin{aligned}
P\{X_1 + X_2 = k\} &= \sum_{j=1}^{k-1} P\{X_1 = j, X_2 = k - j\} \\
&= \sum_{j=1}^{k-1} P\{X_1 = j\} P\{X_2 = k - j\} \\
&= \sum_{j=1}^{k-1} p_1 q_1^{j-1} p_2 q_2^{k-j-1}
\end{aligned}
$$

Sum of Independent Geometric Random Variables

$$
\begin{aligned}
\sum_{j=1}^{k-1} p_1 q_1^{j-1} p_2 q_2^{k-j-1} &= p_1 p_2 q_2^{k-2} \sum_{j=1}^{k-1} (q_1/q_2)^{j-1} \\
&= p_1 p_2 q_2^{k-2} \left(\frac{1-(q_1/q_2)^{k-1}}{1-(q_1/q_2)} \right) \\
&= \frac{p_1 p_2 q_2^{k-1}}{q_2-q_1} - \frac{p_1 p_2 q_1^{k-1}}{q_2-q_1} \\
&= p_2 q_2^{k-1} \frac{p_1}{p_1-p_2} + p_1 q_1^{k-1} \frac{p_2}{p_2-p_1}
\end{aligned}
$$

Sum of Independent Geometric Random Variables

If $X_1, X_2, \ldots, X_n$ are independent geometric random variables with parameters $p_1, p_2, \ldots, p_n$ respectively, find the distribution of $X_1 + X_2 + \cdots + X_n$.

Exercise!!

Thank You