CSL003P1M : Probability and Statistics Lecture 13 (Expectation and Variance of Some Standard Discrete Random Variable)

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Expectation of a Binomial Random Variable

Let X be a binomal random variable with parameters n and p. Find $E[X]$.

Solution:

- $$E[X] = \sum_{i=0}^{n} iP\{X = i\} = \sum_{i=1}^{n} iP\{X = i\} = \sum_{i=1}^{n} i\binom{n}{i}p^i(1-p)^{n-i}$$
- Since
$$i\binom{n}{i} = n\binom{n-1}{i-1},$$
- Therefore,
$$\sum_{i=1}^{n} i\binom{n}{i}p^i(1-p)^{n-i} = np\left(\sum_{i=1}^{n}\binom{n-1}{i-1}p^{i-1}(1-p)^{n-i}\right)$$

Expectation of a Binomial Random Variable

- Take $i-1=j$. Then,

$$\sum_{i=1}^{n}\binom{n-1}{i-1}p^{i-1}(1-p)^{n-i}=\sum_{j=0}^{n-1}\binom{n-1}{j}p^{j}(1-p)^{n-j-1}$$

- Note that,

$$\sum_{j=0}^{n-1}\binom{n-1}{j}p^{j}(1-p)^{n-j-1}=1.$$

- Thus,

$$E[X]=np.$$

Let X be a binomal random variable with parameters n and p. Find $Var[X]$.

Solution:

- $$E[X^2] = \sum_{i=0}^{n} i^2 P\{X = i\} = \sum_{i=1}^{n} i^2 \binom{n}{i} p^i (1-p)^{n-i}$$
- Since
$$i\binom{n}{i} = n\binom{n-1}{i-1},$$
- Therefore,

$$\sum_{i=1}^{n} i^2 \binom{n}{i} p^i (1-p)^{n-i} = np\left(\sum_{i=1}^{n} i \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}\right)$$

Variance of a Binomial Random Variable

- Take $i - 1 = j$. Then,

$$\sum_{i=1}^{n} i \binom{n-1}{i-1} p^{i-1}(1-p)^{n-i} = \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} p^{j}(1-p)^{n-j-1}$$
$$= E[Y+1]$$

  where Y is a binomial random variable with parameters $n-1$ and p.

- Thus,

$$E[X^2] = npE[Y+1] = np(E[Y]+1) = np\{(n-1)p+1\}.$$

- So,

$$Var(X) = E[X^2] - (E[X])^2 = np\{(n-1)p+1\} - (np)^2 = np(1-p).$$

kth Moment of a Binomial Random Variable

Let X be a binomal random variable with parameters n and p. Find $E[X^k]$.

Solution:

- $$E[X^k] = \sum_{i=0}^{n} i^k P\{X = i\} = \sum_{i=1}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$
- Since
$$i\binom{n}{i} = n\binom{n-1}{i-1},$$
- Therefore,

$$\sum_{i=1}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i} = np\left(\sum_{i=1}^{n} i^{k-1}\binom{n-1}{i-1} p^{i-1}(1-p)^{n-i}\right)$$

kth Moment of a Binomial Random Variable

- Take $i - 1 = j$. Then,

$$\sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1}(1-p)^{n-i}$$
$$= \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^{j}(1-p)^{n-j-1} = E[(Y+1)^{k-1}]$$

  where Y is a binomial random variable with parameters $n-1$ and p.

- Thus,

$$E[X^k] = npE[(Y+1)^{k-1}].$$

- Setting $k = 1$, we get $E[X] = npE[1] = np$.
- Setting $k = 2$, we get $E[X^2] = npE[Y+1]$.

Expectation of a Poisson Random Variable

Let X be a Poisson random variable with parameter λ. Find $E[X]$.

Solution:

- $$\begin{aligned} E[X] &= \sum_{i=0}^{\infty} i e^{-\lambda} \frac{\lambda^i}{i!} \\ &= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \frac{\lambda^{i-1}}{(i-1)!} \\ &= \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} && \text{by letting } j = i-1 \\ &= \lambda && \text{since} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{\lambda} \end{aligned}$$

Variance of a Poisson Random Variable

Let X be a Poisson random variable with parameter λ. Find $Var[X]$.

Solution:

- $$\begin{aligned} E[X^2] &= \sum_{i=0}^{\infty} i^2 e^{-\lambda}\frac{\lambda^i}{i!} \\ &= \lambda \sum_{i=1}^{\infty} i e^{-\lambda}\frac{\lambda^{i-1}}{(i-1)!} \\ &= \lambda \sum_{j=0}^{\infty} (j+1) e^{-\lambda}\frac{\lambda^j}{j!} \qquad \text{by letting } j = i-1 \\ &= \lambda E[X+1] = \lambda(E[X]+1) = \lambda(\lambda+1). \end{aligned}$$
- So, $Var(X) = E[X^2] - (E[X])^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$.

kth Moment of a Poisson Random Variable

Let X be a Poisson random variable with parameter λ. Find $E[X^k]$.

Solution:

- $$\begin{aligned} E[X^k] &= \sum_{i=0}^{\infty} i^k e^{-\lambda} \frac{\lambda^i}{i!} \\ &= \lambda \sum_{i=1}^{\infty} i^{k-1} e^{-\lambda} \frac{\lambda^{i-1}}{(i-1)!} \\ &= \lambda \sum_{j=0}^{\infty} (j+1)^{k-1} e^{-\lambda} \frac{\lambda^j}{j!} \qquad \text{by letting } j = i-1 \\ &= \lambda E[(X+1)^{k-1}] \end{aligned}$$
- Setting $k = 1$, we get $E[X] = \lambda E[1] = \lambda$.
- Setting $k = 2$, we get $E[X^2] = \lambda E[X+1]$.

Expectation of a Geometric Random Variable

Let X be a geometric random variable with parameter p. Find $E[X]$.

Solution: Let $1 - p = q$. So, $p + q = 1$.

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} iq^{i-1}p = \sum_{i=1}^{\infty}(i-1+1)q^{i-1}p \\
&= \sum_{i=1}^{\infty}(i-1)q^{i-1}p + \sum_{i=1}^{\infty} q^{i-1}p \\
&= q\sum_{j=0}^{\infty} jq^{j-1}p + 1 = q\sum_{j=1}^{\infty} jq^{j-1}p + 1 \\
&= qE[X] + 1
\end{aligned}
$$

Thus, $(1-q)E[X] = 1$, or $E[X] = 1/p$.

Let X be a geometric random variable with parameter p. Find $Var[X]$.

Solution: Let $1-p=q$. So, $p+q=1$.

$$\begin{aligned}
E[X^2] &= \sum_{i=1}^{\infty} i^2 q^{i-1} p = \sum_{i=1}^{\infty} (i-1+1)^2 q^{i-1} p \\
&= \sum_{i=1}^{\infty} (i-1)^2 q^{i-1} p + \sum_{i=1}^{\infty} 2(i-1) q^{i-1} p + \sum_{i=1}^{\infty} q^{i-1} p \\
&= q \sum_{j=0}^{\infty} j^2 q^{j-1} p + 2q \sum_{j=0}^{\infty} j q^{j-1} p + 1 \\
&= qE[X^2] + 2qE[X] + 1 = qE[X^2] + 2\left(\frac{q}{p}\right) + 1
\end{aligned}$$

Variance of a Geometric Random Variable

Thus, we have,

$$E[X^2] = qE[X^2] + 2\left(\frac{q}{p}\right) + 1$$

Or, $E[X^2] = \dfrac{(q+1)}{p^2}$.
Now, we calculate $Var(X)$.

- $$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= \frac{(q+1)}{p^2} - \frac{1}{p^2} \\ &= \frac{q}{p^2}. \end{aligned}$$

kth Moment of a Geometric Random Variable

Let X be a geometric random variable with parameter p. Prove that

$$E[X^k] = \frac{1}{p}E[(Y-1)^{k-1}]$$

where Y is a negative binomial random variable with parameters $(2, p)$.

Hint: Prove that

Let X be a negative binomial random variable with parameter (r, p). Then

$$E[X^k] = \frac{r}{p}E[(Y-1)^{k-1}]$$

where Y is a negative binomial random variable with parameters $(r+1, p)$.

Thank You