CSL003P1M : Probability and Statistics
Lecture 27 (Inequalities-II)

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Markov's and Chebyshev's Inequality

Markov's Inequality

If X is a random variable that takes only nonnegative values, then for any value $a > 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

Chebyshev's Inequality

If X is a random variable with finite mean μ and variance σ^2, then, for any value $k > 0$,

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

Exercise-1

Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

1. What can be said about the probability that this week's production will exceed 75?
2. If the variance of a week's production is known to equal 25, then what can be said about the probability that the week's production will be between 40 and 60?

Solution: Let X be the number of items that will be produced in a week.

1. By Markov's inequality

$$P\{X > 75\} \leq \frac{E[X]}{75} = \frac{50}{75} = \frac{2}{3}$$

2. By Chebyshev's inequality,

$$P\{|X - 50| \geq 10\} \leq \frac{\sigma^2}{10^2} = \frac{1}{4}$$

Hence,

$$P\{|X - 50| < 10\} \geq 1 - \frac{1}{4} = \frac{3}{4}$$

so the probability that this week's production will be between 40 and 60 is at least 0.75.

Chebyshev's Inequality

If $Var(X) = 0$, then

$$P\{X = E[X]\} = 1$$

In other words, the only random variables having variances equal to 0 are those which are constant with probability 1.

Proof: By Chebyshev's inequality, we have, for any $n \geq 1$,

$$P\left\{|X - \mu| > \frac{1}{n}\right\} = 0$$

In discrete case, we can find a positive integer $n \geq 1$ such that $P\{X = x\} = 0$ for all $x \in [\mu - 1/n, \mu + 1/n]$ except μ.

One-Sided Chebyshev Inequality

If X is a random variable with mean 0 and finite variance σ^2, then, for any $a > 0$,

$$P\{X \geq a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Proof: Let $b > 0$ and note that

$$X \geq a \text{ is equivalent to } X + b \geq a + b$$

Hence,

$$\begin{aligned} P\{X \geq a\} &= P\{X + b \geq a + b\} \\ &\leq P\{(X + b)^2 \geq (a + b)^2\} \end{aligned}$$

where the inequality is obtained by noting that since $a + b > 0$, $X + b \geq a + b$ implies that $(X + b)^2 \geq (a + b)^2$.

Upon applying Markov's inequality, the preceding yields that

$$P\{X \geq a\} \leq \frac{E[(X+b)^2]}{(a+b)^2} = \frac{\sigma^2 + b^2}{(a+b)^2}.$$

Letting $b = \sigma^2/a$. This value of b minimizes $(\sigma^2 + b^2)/(a+b)^2$. And thus the result.

One-Sided Chebyshev Inequality

Corollary

If $E[X] = \mu$ and $Var(X) = \sigma^2$, then, for $a > 0$,

$$P\{X \geq \mu + a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$
$$P\{X \leq \mu - a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Proof:

- X has mean μ and variance σ^2.
- Both $X - \mu$ and $\mu - X$ have mean 0 and variance σ^2.
- Hence, it follows from one-sided Chebyshev inequality that, for $a > 0$

$$P\{X - \mu \geq a\} \leq \frac{\sigma^2}{\sigma^2 + a^2} \text{ and } P\{\mu - X \geq a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Chernoff Bound

Chernoff Bound

$$
\begin{array}{ll}
P\{X \geq a\} \leq e^{-ta}M(t) & \text{for all } t > 0 \\
P\{X \leq a\} \leq e^{-ta}M(t) & \text{for all } t < 0
\end{array}
$$

Proof: For $t > 0$

$$
\begin{aligned}
P\{X \geq a\} &= P\{e^{tX} \geq e^{ta}\} \\
&\leq E[e^{tX}]e^{-ta} \quad \text{by Markov's inequality} \\
&= M(t)e^{-ta}
\end{aligned}
$$

Similarly, for $t < 0$,

$$
\begin{aligned}
P\{X \leq a\} &= P\{e^{tX} \geq e^{ta}\} \\
&\leq E[e^{tX}]e^{-ta} \quad \text{by Markov's inequality} \\
&= M(t)e^{-ta}
\end{aligned}
$$

Thank You