CSL003P1M : Probability and Statistics Lecture 31 (Some Standard Continuous Distributions-I)

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Uniform Random Variable

Uniform Random Variable

A random variable is said to be uniformly distributed over the interval $(0, 1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

f is a probability density function, or pdf in short, because

1. $f(x) \geq 0$ and
2. $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 dx = 1.$

$$P\{a \leq X \leq b\} = \int_a^b f(x)dx = b - a, \quad 0 < a < b < 1.$$

Uniform Random Variable

In general,

Uniform Random Variable

We say that X is a uniform random variable on the interval (α, β) if the pdf of X is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

Since $F(a) = \int_{-\infty}^{a} f(x)dx$, it follows that the distribution function of a uniform random variable on the interval (α, β) is given by

$$F(a) = \begin{cases} 0 & a \leq \alpha \\ \dfrac{a - \alpha}{\beta - \alpha} & \alpha < a < \beta \\ 1 & a \geq \beta \end{cases}$$

Normal Random Variable

Normal Random Variable

We say that X is a normal random variable, or simply that X is normally distributed, with parameters μ and σ^2 if the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty$$

f is a pdf because

1. $f(x) \geq 0$ and
2. $\int_{-\infty}^{\infty} f(x)dx = 1$. (Prove it!!!)

Prove that if X is normally distributed with parameters μ and σ^2, then $Y = aX + b$ is normally distributed with paramaeters $a\mu + b$ and $a^2\sigma^2$ given $a \neq 0$.

Proof: Suppose $a > 0$. Let F_Y denote the cumulative distribution function of Y. Then

$$\begin{aligned} F_Y(x) &= P\{Y \leq x\} \\ &= P\{aX + b \leq x\} \\ &= P\left\{X \leq \frac{x-b}{a}\right\} \\ &= F_X\left(\frac{x-b}{a}\right) \end{aligned}$$

So,

$$F_Y(x) = F_X\left(\frac{x-b}{a}\right)$$

After differentiation, the density function of Y is then

$$\begin{aligned}
f_Y(x) &= \frac{1}{a} f_X\left(\frac{x-b}{a}\right) \\
&= \frac{1}{\sqrt{2\pi}a\sigma}\exp\left\{-\left(\frac{x-b}{a}-\mu\right)^2/2\sigma^2\right\} \\
&= \frac{1}{\sqrt{2\pi}a\sigma}\exp\left\{-(x-b-a\mu)^2/2(a\sigma)^2\right\} \\
&= \frac{1}{\sqrt{2\pi}(a\sigma)}\exp\left[-\{x-(a\mu+b)\}^2/2(a\sigma)^2\right]
\end{aligned}$$

which shows that Y is normal with parameters $a\mu + b$ and $a^2\sigma^2$.

Normal Random Variable

An important implication of the preceding result is that if X is normally distributed with parameters μ and σ^2, then $Z = (X - \mu)/\sigma$ is normally distributed with parameters 0 and 1.

Standard or Unit Normal Random Variable

Such a random variable is said to be standard or unit normal random variable.

It is customary to denote the cumulative distribution function of a standard normal random variable by $\Phi(x)$. That is,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

Observe that

$$\Phi(-x) = 1 - \Phi(x) \qquad -\infty < x < \infty$$

One can show that if Z is a standard normal random variable, then

$$P\{Z \leq -x\} = P\{Z > x\} \qquad -\infty < x < \infty$$

Since $Z = (X - \mu)/\sigma$ is a standard normal random variable whenever X is normally distributed with parameters μ and σ^2, it follows that the distribution function of X can be expressed as

$$F_X(a) = P\{X \leq a\} = P\left\{\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right\} = \Phi\left(\frac{a-\mu}{\sigma}\right)$$

The Normal Approximation to the Binomial Distribution

An important result in probability theory known as the DeMoivre-Laplace limit theorem states that when n is large, a binomial random variable with parameters n and p will have approximately the same distribution as a normal random variable with the same mean and variance as the binomial.

- This result was originally proved for the special case of $p = 1/2$ by DeMoivre in 1733.
- It was then extended to general p by Laplace in 1812.

The Normal Approximation to the Binomial Distribution

The DeMoivre-Laplace Limit Theorem

If S_n denotes the number of successes that occur when n independent trials, each resulting in a success with probability p, are performed, then, for any $a < b$,

$$P\left\{a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right\} \to \Phi(b) - \Phi(a)$$

as $n \to \infty$.

The Normal Approximation to the Binomial Distribution

Now, we have two possible approximations to binomial probabilities:

1. The Poisson approximation, which is good when n is large and p is small and
2. The normal approximation, which can be shown to be quite good when $np(1-p)$ is large [The normal approximation will, in general, be quite good for values of n satisfying $np(1-p) \geq 10$.]

Exponential Random Variable

Exponential Random Variable

A continuous random variable whose probability density function is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is said to be an exponential random variable with parameter λ.

Exponential Random Variable

In practice, the exponential distribution often arises as the distribution of the amount of time until some specific event occurs. For instance,

1. The amount of time until an earthquake occurs, or
2. until a new war breaks out, or
3. until a telephone call you receive turns out to be a wrong number.

Exponential Random Variable

The cumulative distribution function $F(a)$ of an exponential random variable is given by

$$
\begin{aligned}
F(a) &= P\{X \leq a\} \\
&= \int_0^a \lambda e^{-\lambda x} dx \\
&= -e^{-\lambda x}|_0^a \\
&= 1 - e^{-\lambda a}
\end{aligned}
$$

Memoryless Random Variable

We say that a nonnegative random variable X is memoryless if

$$P\{X > s+t | X > t\} = P\{X > s\} \quad \text{for all } s, t \geq 0$$

The above equation is equivalent to

$$\frac{P\{X > s+t, X > t\}}{P\{X > t\}} = P\{X > s\}$$

or,

$$P\{X > s+t\} = P\{X > s\}P\{X > t\}$$

Prove that exponentially distributed random variables are memoryless.

Thank You