CSL003P1M : Probability and Statistics Lecture 10 (Some Problems on Discrete Probability Distributions)

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Exercise-1

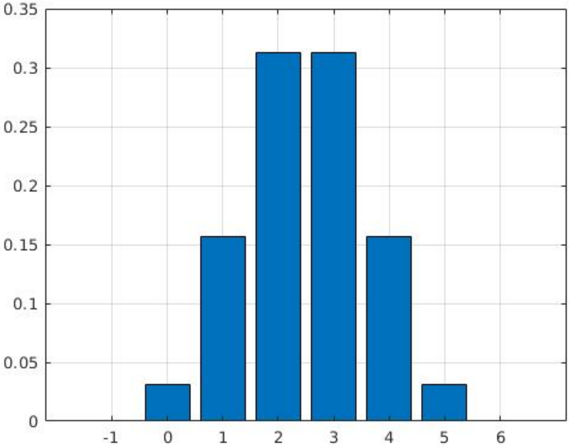
Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Solution:

- Let X denote the number of heads, then X is a binomial random variable with parameters $(n = 5, p = 1/2)$. Hence,

$$\begin{aligned}
P\{X=0\} &= \binom{5}{0}\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^5 &= 1/32 \\
P\{X=1\} &= \binom{5}{1}\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^4 &= 5/32 \\
P\{X=2\} &= \binom{5}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^3 &= 10/32 \\
P\{X=3\} &= \binom{5}{3}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^2 &= 10/32 \\
P\{X=4\} &= \binom{5}{4}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^1 &= 5/32 \\
P\{X=5\} &= \binom{5}{5}\left(\tfrac{1}{2}\right)^5\left(\tfrac{1}{2}\right)^0 &= 1/32
\end{aligned}$$

Exercise-1



$$p(i), \;\; i = 0, 1, 2, 3, 4, 5$$

Exercise-2

A book of 500 pages contains 500 misprints. Find out the probability that a given page contains at least three misprints.

Solution:

- $\lambda = 500/500 = 1$. It means that, on an average, there is one error per page.
- Let X be a random variable which denotes the number of misprints in a given page. Observe that, X is a Poisson random variable with the parameter $\lambda = 1$.
- Then, we are interested in $P\{X \geq 3\}$ which is equal to $1 - P\{X \leq 2\} = 1 - (P\{X = 0\} + P\{X = 1\} + P\{X = 2\})$ which is equal to $1 - e^{-\lambda}(1 + \lambda + \lambda^2/2) = 1 - 5e^{-1}/2 = 0.0803$.

Exercise-3

At all times, a pipe-smoking mathematician carries 2 matchboxes - 1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match, he is equally likely to take it from either pocket. Consider the moment when the mathematician first discovers that one of his matchboxes is empty. If it is assumed that both matchboxes initially contained N matches, what is the probability that there are exactly k matches, $k = 0, 1, \ldots, N$, in the other box?

Solution: (Hint: Apply Negative Binomial Distribution)

- Let E_r (similarly E_l)denote the event that the mathematician first discovers that the right-hand (left-hand) matchbox is empty and there are k matches in the left-hand (right-hand) box at the time.
- Now, this event will occur if and only if $(N+1)^{\text{th}}$ choice of the right-hand matchbox is made at the $(N+1+N-k)^{\text{th}}$ trial.

Exercise-3

- Whenever a mathematician chooses a right matchbox, we consider it a success.
- Let X be a random variable which denotes the number of trials required until we get $(N+1)$ successes, i.e. the mathematician chooses the right-hand matchbox $N+1$ times.
- Observe that X is a Negative Binomial random variable with parameters $r = N+1$ and $p = 1/2$.
- Then, we are interested in $P\{X = 2N-k+1\}$ which is equal to

$$\binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k+1} = P(E_r).$$

- So, the desired probability is

$$P(E_r \cup E_l) = P(E_r) + P(E_l) = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k}.$$

Exercise-4

A book of n pages contains on the average λ misprints per page. Find out the probability that at least one page will contain more than k misprints.

Solution:

- Let X_i be a random variable which denotes the number of misprints in page i. Observe that X_i's are independent Poisson random variables with the same parameter λ.
- We first calculate the probability that there is no page that contains more than k misprints.
- Let the i^{th} page does not contain more than k misprints. Then, we are interested in $P\{X_i \leq k\}$ which is equal to

$$\sum_{j=0}^{k} e^{-\lambda} \frac{\lambda^j}{j!} = e^{-\lambda} \sum_{j=0}^{k} \frac{\lambda^j}{j!}$$

Exercise-4

- So the probability that no page contains more than k misprints is

$$\prod_{i=1}^{n} P\{X_i \leq k\} = \left(e^{-\lambda} \sum_{j=0}^{k} \frac{\lambda^j}{j!}\right)^n$$

- Therefore, the probability that at least one page contains more than k misprints is

$$1 - \left(e^{-\lambda} \sum_{j=0}^{k} \frac{\lambda^j}{j!}\right)^n.$$

Exercise-5

Suppose that the probability of an insect laying r eggs follow Poisson distribution with parameter λ. Assume that the probability of an egg developing is p. Assuming mutual independence of the eggs, find out the probability of a total of k survivors.

Solution: (Hint: Combination of Poisson and Binomial Distribution)

- Let X be the random variable which denotes the number of eggs that an insect lay. From the question, X is a Poisson random variable with parameter λ.
- Then,

$$P\{X = r\} = e^{-\lambda}\frac{\lambda^r}{r!}.$$

Exercise-5

- Let Y_r be the random variable which denotes the number of eggs which survive. Observe that Y_r is a binomial random variable with parameter (r, p).
- Then,

$$P\{Y_r = k\} = \binom{r}{k} p^k (1-p)^{r-k}.$$

- Since X and Y_r are independent, hence

$$\begin{aligned} P\{X = r \wedge Y_r = k\} &= P\{X = r\}P\{Y_r = k\} \\ &= \binom{r}{k} e^{-\lambda} \frac{\lambda^r}{r!} p^k (1-p)^{r-k}. \end{aligned}$$

- Therefore, the probability that there will be k survivors is

$$\sum_{r=k}^{\infty} P\{X = r\}P\{Y_r = k\} = \sum_{r=k}^{\infty} \binom{r}{k} e^{-\lambda} \frac{\lambda^r}{r!} p^k (1-p)^{r-k}$$

Exercise-5

- After some calculations, it can be shown that

$$\sum_{r=k}^{\infty} \binom{r}{k} e^{-\lambda} \frac{\lambda^r}{r!} p^k (1-p)^{r-k} = (e^{-\lambda p}) \frac{(\lambda p)^k}{k!}.$$

- If Z is a random variable which denotes the number of survivors, then Z is a Poisson random variable with the parameter λp.

Thank You