CSL003P1M : Probability and Statistics
Lecture 19 (Expectations of Sums of Discrete Random Variables)

Sumit Kumar Pandey

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Recall

Expectation of a Random Variable

If X is a discrete random variable having a probability mass function $p(x)$, then the *expectation*, or the *expected value*, of X, denoted by $E[X]$, is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

In words, the expected value of X is the weighted average of the possible values that X can take on, each value being weighted by the probability that X assumes it.

See Lecture Slides 11.

Recall

Expectation of a Function of a Random Variable

If X is a random variable that takes on one of the values x_i, $i \geq 1$, with respective probabilities $p(x_i)$, then, for any real-valued function g,

$$E[g(X)] = \sum_i g(x_i)p(x_i)$$

Corollary

If a and b are constants, then

$$E[aX + b] = aE[X] + b.$$

See Lecture Slides 12.

Recall

nth Moment of X

The quantity $E[X^n]$, $n \geq 1$ is called the nth moment of X. Note that,

$$E[X^n] = \sum_{x:p(x)>0} x^n p(x).$$

See Lecture Slides 12.

Recall

Variance of a Random Variable

If X is a random variable with mean μ, then the variance of X, denoted by $Var(X)$, is defined by

$$Var(X) = E[(X - \mu)^2]$$

Variance of a Random Variable

If X is a random variable with mean μ, then the variance of X, denoted by $Var(X)$, is

$$Var(X) = E[X^2] - \mu^2 = E[X^2] - (E[X])^2$$

See Lecture Slides 12.

Recall

Corollary

Prove that for any constants a and b, if X is a discrete random variable, then

$$Var(aX + b) = a^2 Var(X).$$

Standard Deviation of a Random Variable

The square root of the $Var(X)$ is called the *standard deviation* of X, and we denote it by $SD(X)$. That is,

$$SD(X) = \sqrt{Var(X)}.$$

See Lecture Slides 12.

Expected Value of Sums of Random Variables

- For a random variable X, let $X(s)$ denote the value of X when $s \in S$ is the outcome of the experiment.
- If X and Y are both random variables, then so is their sum. That is $Z = X + Y$. Moreover, $Z(s) = X(s) + Y(s)$.
- Let $p(s) = P(\{s\})$ be the probability that s is the outcome of the experiment.
- We can write any event A as the finite or countable infinite union of the mutually exclusive events $\{s\}$, $s \in A$, it follows from the axioms of probability

$$P(A) = \sum_{s \in A} p(s).$$

When $A = S$, the preceding equation gives

$$1 = \sum_{s \in S} p(s).$$

Expected Value of Sums of Random Variables

Proposition

$$E[X] = \sum_{s \in S} X(s)p(s)$$

Proof:

- Suppose that the distinct values of X are $x_i, i \geq 1$.
- For each i, let S_i be the event that X is equal to x_i.
- That is $S_i = \{s : X(s) = x_i\}$.
- Then,

$$\begin{aligned} E[X] &= \sum_i x_i P\{X = x_i\} \\ &= \sum_i x_i P\{S_i\} \\ &= \sum_i x_i \sum_{s \in S_i} p(s) \end{aligned}$$

Expected Value of Sums of Random Variables

$$
\begin{aligned}
\sum_i x_i \sum_{s\in S_i} p(s) &= \sum_i \sum_{s\in S_i} x_i p(s) \\
&= \sum_i \sum_{s\in S_i} X(s)p(s) \\
&= \sum_{s\in S_i} X(s)p(s)
\end{aligned}
$$

Suppose that two independent flips of a coin that comes up heads with probability p are made, and let X denote the number of heads obtained. Find $E[X]$.

Solution:

- $$\begin{aligned} P\{X=0\} &= P(T,T) = (1-p)^2, \\ P\{X=1\} &= P(H,T) + P(T,H) = 2p(1-p), \\ P\{X=2\} &= P(H,H) = p^2 \end{aligned}$$

- It follows from the definition of expected value that

$$E[X] = 0 \cdot (1-p)^2 + 1 \cdot 2p(1-p) + 2 \cdot p^2 = 2p.$$

- Let's calculate alternatively

Exercise-1

$$\begin{aligned} E[X] &= X(H,H)p^2 + X(H,T)p(1-p) + X(T,H)(1-p)p \\ &\quad + X(T,T)(1-p)^2 \\ &= 2p^2 + p(1-p) + (1-p)p + 0(1-p)^2 \\ &= 2p. \end{aligned}$$

Expected Value of Sums of Random Variables

Corollary

For random variables $X_1, X_2, \ldots, X_n$,

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

- Let $Z = \sum_{i=1}^{n} X_i$. Then,

$$\begin{aligned}
E[Z] &= \sum_{s \in S} Z(s)p(s) \\
&= \sum_{s \in S} (X_1(s) + X_2(s) + \cdots + X_n(s))p(s) \\
&= \sum_{s \in S} X_1(s)p(s) + \sum_{s \in S} X_2(s)p(s) + \cdots + \sum_{s \in S} X_n(s)p(s) \\
&= E[X_1] + E[X_2] + \cdots + E[X_n]
\end{aligned}$$

Exercise-2

Find the expected value of the sum obtained when n fair dice are rolled.

Solution:

- Let X be the sum.
- We will compute $E[X]$ by using the representation

$$X = \sum_{i=1}^{n} X_i$$

where X_i is the upturned value on die i.

Exercise-2

- Because X_i is equally likely to be any of the values from 1 to 6, it follows that

$$E[X_i] = \sum_{i=1}^{6} i(1/6) = 21/6 = 7/2.$$

- Thus,

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = 7n/2.$$

Exercise-3

Find the expected total number of successes that result from n trials when trial i is a success with probability p_i, $i = 1, \ldots, n$.

- Let

$$X_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & \text{if trial } i \text{ is a failure} \end{cases}$$

- Then, $E[X_i] = 0 \cdot (1 - p_i) + 1 \cdot p_i = p_i$.
- Consider

$$X = \sum_{i=1}^{n} X_i.$$

- Thus,

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p_i.$$

Some Pathologies

- The result $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$ does not require that X_i be independent.
- Exercise-3 includes as a special case the expected value of binomial random variable, which assumes independent trials, and thus has mean np.
- Exercise-3 gives the expected value of a hypergeometric random variable also. (Exercise!!!)

Exercise-4

Derive an expression for the variance of the number of successful trials in the previous exercise, and apply it to obtain the variance of a random variable with parameters n and p, and of a hypergeometric variable equal to the number of white balls chosen when n balls are randomly chosen from an urn containing N balls of which m are white.

Solution:

- Let X be the number of successful trials.
- Let

$$X = \sum_{i=1}^{n} X_i.$$

Exercise-4

Then, we have

$$\begin{aligned} E[X^2] &= E\left[\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{j=1}^{n} X_j\right)\right] \\ &= E\left[\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n}\sum_{j\neq i} X_i X_j\right] \\ &= \sum_{i=1}^{n} E[X_i^2] + \sum_{i=1}^{n}\sum_{j\neq i} E[X_i X_j] \\ &= \sum_{i=1}^{n} p_i + \sum_{i=1}^{n}\sum_{j\neq i} E[X_i X_j] \end{aligned}$$

where the final equation used that $X_i^2 = X_i$ because X_i takes only two values 0 and 1.

Now we calculate $\sum_{i=1}^{n} \sum_{j \neq i} E[X_i X_j]$.

Because the possible values of both X_i and X_j are 0 or 1, it follows that

$$X_i X_j = \begin{cases} 1 & \text{if } X_i = 1, X_j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$E[X_i X_j] = P\{X_i = 1, X_j = 1\}.$$

Exercise-4

If X is Binomial, then

- For $i \neq j$, X_i and X_j are independent with success probability p. Therefore,

$$E[X_iX_j] = p^2, \quad i \neq j$$

- So for a Binomial random varaible X,

$$E[X^2] = \sum_{i=1}^{n} p_i + \sum_{i=1}^{n}\sum_{j\neq i} E[X_iX_j] = np + n(n-1)p^2$$

- Hence,

$$Var(X) = E[X^2] - (E[X])^2 = np + n(n-1)p^2 - n^2p^2 = np(1-p).$$

Exercise-4

If X is Hypergeometric, then
Exercise!!!

Thank You