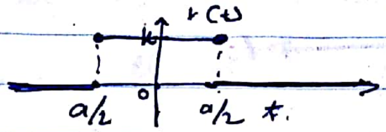
(1) Convolution and Fourier Transform

(a)

$$r(t) = \begin{cases} K, & |t| \le a/2. \\ 0, & |t| > a/2. \end{cases}$$



Fourier Transformation of function $f(t)$ is:

$$\hat{f}(\omega) = F\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt.$$

(provided the integral converges).

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} r(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{-a/2} r(t)\, e^{-j\omega t}\, dt + \int_{-a/2}^{a/2} r(t)\, e^{-j\omega t}\, dt + \int_{a/2}^{\infty} r(t)\, e^{-j\omega t}\, dt.$$

$$= 0 + \int_{-a/2}^{a/2} r(t)\, e^{-j\omega t}\, dt + 0.$$

$$= k \int_{-a/2}^{a/2} e^{-j\omega t}\, dt = -k.\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-a/2}^{a/2}.$$

$$= -k.\frac{1}{j\omega}\left[e^{-j\omega\frac{a}{2}} - e^{j\omega\frac{a}{2}}\right].$$

$$= -k.\frac{1}{j\omega}\left[\cos\left(-\frac{\omega a}{2}\right) + j\sin\left(-\frac{\omega a}{2}\right) - \cos\left(\frac{\omega a}{2}\right) - j\sin\left(\frac{\omega a}{2}\right)\right]$$

$$= -k.\frac{1}{j\omega}\left[\cos\left(\frac{\omega a}{2}\right) - \cos\left(\frac{\omega a}{2}\right) - j\sin\left(\frac{\omega a}{2}\right) - j\sin\left(\frac{\omega a}{2}\right)\right]$$

$$= -k.\frac{1}{j\omega}\left[-2j\sin\left(\frac{\omega a}{2}\right)\right] = k\frac{2}{\omega}\sin\left(\frac{\omega a}{2}\right)$$

$$= ka.\frac{\sin\left(\frac{\omega a}{2}\right)}{\left(\frac{\omega a}{2}\right)} = k\,a\,\mathrm{sinc}\left(\frac{\omega a}{2}\right) \quad \text{(Ans)}$$

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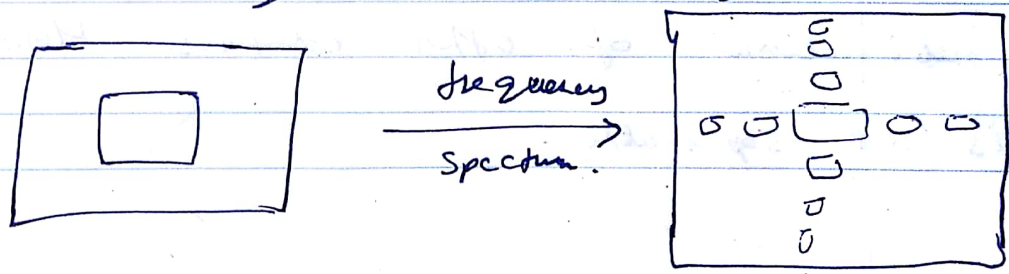
b In the frequency spectrum the sine function can be observed in both the x and y axes, so the original function has been rectangular.

From the figure, since the oscillations are same in both the x and y axis, we can conclude that the image must be square.



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6 Consider the Dirac delta impulse :

$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\boxed{\int_{-\infty}^{+\infty} \delta(t)\, dt = \int_{-\infty}^{+\infty} \delta(t)\, dt = 1}$$

We can see that $\int_{-\infty}^{\infty} f(t)\, \delta(\tau - t)\, dt = f(\tau)$.

$$\delta(t) * f(t) = f(t)$$

Proof :

$$FT[\delta(t) * f(t)] = FT[\delta(t)]\, FT[f(t)]$$

$$= 1 \cdot F(\omega).$$

or, $FT^{-1}\left[FT[\delta(t) * f(t)]\right] = FT^{-1}[F(\omega)]$

or, $\delta(t) * f(t) = f(t)$.

Hence $\delta(t)$ is the identity function for convolution.

Fourier transformation of $\delta(t)$ is 1.

Convolution of any function by $\delta(t)$ keeps the function unchanged.

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d

$$A = \begin{pmatrix} 3 & 1 & -9 & -2 & 0 \\ 5 & 2 & 2 & 3 & -1 \\ 9 & 4 & -9 & -8 & 1 \\ 2 & 10 & -20 & -20 & 0 \\ 4 & 8 & 4 & -6 & 0 \end{pmatrix}$$

$R_3/2$, $R_4/2$

$$A = \begin{pmatrix} 3 & 1 & -9 & -2 & 0 \\ 5 & 2 & 2 & 3 & -1 \\ 9 & 4 & -9 & -8 & 1 \\ 1 & 5 & -10 & -10 & 0 \\ 2 & 4 & 2 & -3 & 0 \end{pmatrix}$$

$R_3 - R_2$

$$A = \begin{pmatrix} 3 & 1 & -9 & -2 & 0 \\ 5 & 2 & 2 & 3 & -1 \\ 9 & 4 & -9 & -8 & 1 \\ -8 & 1 & -1 & -2 & 0 \\ 2 & 4 & 2 & -3 & 0 \end{pmatrix}$$

$C_1 + C_3$

$$A = \begin{pmatrix} 3 & -1 & -9 & -2 & 0 \\ 5 & 5 & 2 & 3 & -1 \\ 9 & -4 & -9 & -8 & 1 \\ -8 & -1 & -1 & -2 & 0 \\ 2 & 1 & 2 & -3 & 0 \end{pmatrix}$$

$C_2 = C_2 + C_1$

$$A = \begin{pmatrix} 3 & -1 & -6 & -2 & 0 \\ 5 & 5 & 7 & 3 & -1 \\ 9 & -4 & 0 & -8 & 1 \\ -8 & -1 & -7 & -2 & 0 \\ 2 & 1 & 4 & -3 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -1 & -6 & -2 & 0 \\ -3 & 4 & 0 & 1 & -1 \\ 9 & -4 & 0 & -8 & 1 \\ -8 & -1 & -7 & -2 & 0 \\ 2 & 1 & 4 & -3 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -1 & -6 & -2 & 0 \\ 6 & 0 & 0 & -7 & 0 \\ 9 & -4 & 0 & -8 & 1 \\ -8 & -1 & -7 & -2 & 0 \\ 2 & 1 & 4 & -3 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} [illegible] & -1 & -6 & -2 & 0 \\ -1 & 0 & 0 & -7 & 0 \\ 1 & -4 & 0 & -8 & 1 \\ -6 & -1 & -7 & -2 & 0 \\ -1 & 1 & 4 & -3 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & -6 & -2 & 0 \\ -1 & 0 & 0 & -7 & 0 \\ -3 & -4 & 0 & -8 & 1 \\ -5 & -1 & -7 & -5 & 0 \\ 0 & 1 & 4 & -3 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & -6 & -2 & 0 \\ -1 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -5 & 0 & -3 & -3 & 0 \\ 0 & 1 & 4 & -3 & 0 \end{pmatrix}$$

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$$A = \begin{pmatrix} 0 & -1 & -6 & -2 & 0 \\ -1 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -5 & 0 & -3 & -3 & 0 \\ 0 & 1 & 4 & -3 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & -6 & -8 & 0 \\ -1 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -5 & 0 & -3 & 0 & 0 \\ 0 & 1 & 4 & -7 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & -2 & -1 & 0 \\ -1 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -5 & 0 & -3 & 0 & 0 \\ 0 & 1 & 4 & -7 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & -2 & -1 & 0 \\ -1 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -4 & 0 & -1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ \\ 1+ \\ 1+ \\ \\ \end{array}$$

Rank of $A > 1$

Hence matrix is not linearly separable into outer products.

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e

$$A = \begin{pmatrix} -21 & 6 & 3 & 12 & 9 \\ 7 & -2 & -1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 35 & -10 & -5 & -20 & -15 \\ -14 & 4 & 2 & 8 & 6 \end{pmatrix}$$

$R_2 = 3 \cdot R_1$

$$A = \begin{pmatrix} -21 & 6 & 3 & 12 & 9 \\ 21 & -6 & -3 & -12 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 35 & -10 & -5 & -20 & -15 \\ -14 & 4 & 2 & 8 & 6 \end{pmatrix}$$

$R_2 = 3R_1$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 21 & -6 & -3 & -12 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 7 & -2 & -1 & -4 & -3 \\ -7 & 2 & 1 & 4 & 3 \end{pmatrix}$$

$R_1 = R_1 + R_2$

$R_3 = R_3 / 5$

$R_4 = R_4 / 2$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 7 & -2 & -1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -7 & 2 & 1 & 4 & 3 \end{pmatrix}$$

$R_1 = R_1 / 3$

$R_4 = R_4 + R_3$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -7 & 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_4 = R_4 + R_1$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[swap rows]

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~~A~~

Rank of $A = 1$.

Hence matrix is separable.

$$\begin{pmatrix} -21 & 6 & 3 & 12 & 9 \\ 7 & -2 & -1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 35 & -10 & -5 & -20 & -15 \\ -14 & 4 & 2 & 8 & 6 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} \otimes (b_1 \ b_2 \ b_3 \ b_4 \ b_5)$$

$$= \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 & a_1 b_4 & a_1 b_5 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 & a_2 b_4 & a_2 b_5 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 & a_3 b_4 & a_3 b_5 \\ a_4 b_1 & a_4 b_2 & a_4 b_3 & a_4 b_4 & a_4 b_5 \\ a_5 b_1 & a_5 b_2 & a_5 b_3 & a_5 b_4 & a_5 b_5 \end{pmatrix}$$

Solving linear equations

$a_3 = 0, \quad b_1 \neq 0.$

$a_1 b_1 = -21$ | $a_4 b_1 = 35$

$a_2 b_1 = 7$ | $a_5 b_1 = -14$

$\frac{a_2}{a_4} = \frac{1}{5}$ $\quad \frac{a_1}{a_2} = -3$ $\quad \frac{a_4}{a_5} = -\frac{5}{2}$

$a_4 = 5a_2$ — ①

$a_1 = -3a_2$ — ②

$2a_4 = -5a_5$ — ③

$a_3 = 0$ — ④

$a_1 b_2 = 6$

$a_1 b_3 = 3$

$\frac{b_2}{b_3} = 2.$

$a_2 = -\frac{1}{3}a_1$

$a_4 = 5a_2 = -\frac{5}{3}a_1$

$3a_4 = 5(a_2 - a_5)$

$\frac{a_2}{a_5} = -\frac{1}{2}.$

$3a_4 = 5(a_2 + 2a_2)$

$2a_2 = -a_5$ — ⑤

$3a_4 = 15a_2$

$a_4 = 5a_2$

$a_5 = -2a_2 = -2 \cdot -\frac{1}{3}a_1 = \frac{2}{3}a_1$

$$\begin{array}{c} a_1 \\ -\frac{1}{3}a_1 \\ 0 \\ -\frac{5}{3}a_1 \\ \frac{2}{3}a_1 \end{array}$$

$$a_1 \begin{pmatrix} 1 \\ -\frac{1}{3} \\ 0 \\ -\frac{5}{3} \\ \frac{2}{3} \end{pmatrix}$$

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$$\frac{b_1}{b_2} = -\frac{21}{6} = -\frac{7}{2}.$$

$$2b_1 = -7b_2 \quad -① . \qquad \underline{b_2 = -\frac{2}{7} b_1}$$

$$\frac{b_1}{b_3} = -\frac{21}{3} = -7$$

$$b_1 = -7 b_3 \quad -② . \qquad \underline{b_3 = -\frac{1}{7} b_1}$$

$$\frac{b_1}{b_4} = -\frac{14}{8} = -\frac{7}{4}.$$

$$b_1 = -\frac{7}{4} b_4 \Rightarrow \underline{b_4 = -\frac{4}{7} b_1}$$

$$\frac{b_5}{b_1} = -\frac{63}{147} = -\frac{3}{7}.$$

$$\underline{b_5 = -\frac{3}{7} b_1}$$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix} = b_1 \begin{pmatrix} 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{4}{7} & -\frac{3}{7} \end{pmatrix}$$

$$a \otimes b = \begin{pmatrix} 1 \\ -\frac{1}{3} \\ 0 \\ -\frac{5}{3} \\ \frac{2}{3} \end{pmatrix} \otimes \begin{pmatrix} 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{4}{7} & -\frac{3}{7} \end{pmatrix}$$

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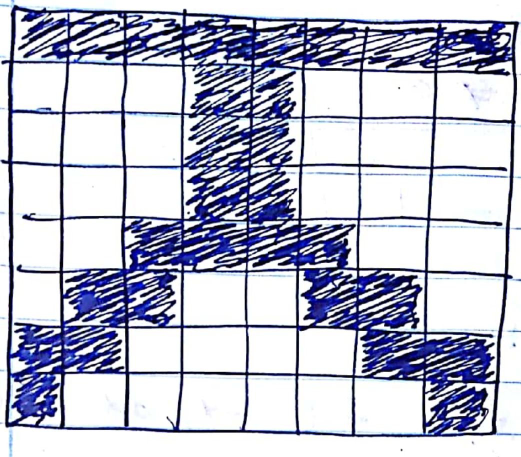
$$A = \begin{pmatrix} 21 \\ 7 \\ 0 \\ 35 \\ -14 \end{pmatrix} \otimes \begin{pmatrix} 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{4}{7} & -\frac{3}{7} \end{pmatrix}$$

Linear separation of the matrix into outer products. (Ans)

1 d

Initialization of the image :



$\xrightarrow{\text{Initialize}}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \infty & \infty & \infty & 0 & 0 & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 & 0 & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 & 0 & \infty & \infty & \infty \\ \infty & \infty & 0 & 0 & 0 & 0 & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty & 0 & 0 & \infty \\ 0 & 0 & \infty & \infty & \infty & \infty & 0 & 0 \\ 0 & \infty & \infty & \infty & \infty & \infty & \infty & 0 \end{pmatrix}$$

Forward pass (distant) :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 1 & 2 & 2 \\ 3 & 3 & 3 & 0 & 0 & 1 & 2 & 3 \\ 4 & 4 & 0 & 0 & 0 & 0 & 1 & 2 \\ 5 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Backward Pass :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 & 0 & 1 & 2 & 2 \\ 3 & 2 & 1 & 0 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

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