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Three-Body Systems with Square-Well Potentials in $L = 0$ States

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Abstract. The angular part of the Faddeev equations is solved analytically for *s*-states in case of two-body square-well potentials. The results are, still analytically, generalized to arbitrary short-range potentials for both small and large distances. We consider systems with three identical bosons, three non-identical particles, and two identical spin-$\frac{1}{2}$ fermions plus a third particle with arbitrary spin. The angular wave functions are in general linear combinations of trigonometric and exponential functions. The Efimov conditions are obtained at large distances. General properties and applications to short-range potentials are discussed. Gaussian potentials are used for illustrations. The results are useful for numerical calculations, where, for example, large distances can be treated analytically and matched to the numerical solutions at smaller distances. The saving in computational efforts could be substantial.

1 Introduction

A new method to study the quantum mechanical three-body problem for short-range binary potentials was recently formulated [1]. The method solves the coordinate-space Faddeev equations in two steps. First the angular (and most difficult) parts of the three-component wave functions are calculated by elaborate use of the analytical knowledge of the large-distance behaviour. Then the coupled set of effective radial equations is solved numerically [2]. The method may be considered as a further development of an approach used to study the properties of H^- [3]. The approach exploits the adiabatic basis, complete in the angular space, and defined as a function of the hyperradius [3–5]. The basis size necessary for convergence is normally very small for the ground state and the lowest-lying excited states, where only small angular momenta are involved.

The first significant advantage of the method is the precise treatment of large distances where the correct asymptotic behaviour of the wave function is incorporated. Intuitive interpretation of the physics is rather direct. Another advantage is the parallel treatment of bound and continuum states, what in combination with the long-range treatment opens the interesting possibility of approaching the celebrated three-body

Coulomb problem in the continuum [6–9]. The advantages are seen in the analytical and numerical treatment of extreme cases like the so-called Efimov states [1, 10], which has been suggested and looked for in both molecules [11–14] and nuclei [15]. These states occur when at least two of the two-body subsystems simultaneously have an *s*-state at zero energy arising from short-range interactions. The resulting infinitely many bound three-body states of 0^+ nature are extremely extended in space and extremely weakly bound. The long-range Coulomb interaction destroys the effect and finite angular momenta substantially hinder or prevent it.

The three-body problem has been extensively studied and we refer here to some recent surveys dedicated to various physical systems, i.e. baryons [16], Borromean systems [17, 18], molecular systems [19, 20] and Coulomb-interacting particles [21, 22]. These methods are used in studies of the properties of a variety of different systems of interest in physics, see, for example, refs. [23–26]. Some of the methods solve the Schrödinger equation directly, but the Faddeev equations are more suitable for a description of all the subtle correlations. This is especially evident in systems where two- and three-body asymptotic behaviours are mixed [2]. Borromean systems, characterized by genuine three-body asymptotics only, are relatively easy to handle, whereas weakly bound systems, where both two- and three-particle correlations are essential, require a very careful treatment of the large distances.

Analytic results are rare in quantum mechanics where the Schrödinger equation should be solved. The Faddeev equations further complicate analytic analyses. However, so far at least one exception exists for identical and non-identical spinless particles interacting via two-body harmonic-oscillator potentials [27]. This potential is infinite at large distance and scattering states therefore cannot be studied. Also the behaviour for short-range potentials is excluded in general. Simplifying mathematical assumptions may allow analytical solutions to the Faddeev equations. An example of this type is the schematic one-dimensional model with delta functions as binary interactions [28, 29]. Instead of complete analytical solutions it could be an advantage to investigate analytically the asymptotic large-distance behaviour of the adiabatic eigenvalues and the coupling terms between them. This has been done in several cases based on the Schrödinger equation [3, 30, 31].

The large-distance coupling for short-range binary potentials takes place only between *s*-states in different relative two-body subsystems [2]. Furthermore, low-lying bound three-body states often contain large components of such *s*-states. Different treatments are usually needed when long-range interactions like the Coulomb potential are involved and we shall in the following only consider short-range interactions. We shall furthermore neglect the coupling to higher angular momenta and entirely confine ourselves to relative *s*-states. The purpose of this paper is to classify the lowest eigenvalues obtained for such restricted cases and furthermore to describe how the solutions for small as well as large distances can be obtained analytically for the angular part of the Faddeev equations. These solutions turn out to be exact for square-well potentials where we also can find exact solutions at intermediate distances. Thus we shall derive a semianalytic *s*-state square-well solution to the Faddeev equations.

The solutions describe the Efimov anomaly in details, e.g., how the infinitely many states continuously appear and disappear as a function of the parameters of the potential. The advantages of the method are especially related to the analytical treatment of large distances which must be handled with particular care for loosely bound quantum

systems in low angular-momentum states [18, 32]. Also the connection between two- and three-body large-distance behaviours can be explored. Using the analytical results for small and large distances in numerical calculations for arbitrary potentials improve both precision and computational speed and thereby enable investigations of otherwise inaccessible problems like the Efimov effect. For each particular application one must carefully investigate the necessary basis including the components of finite angular momenta. The results for *s*-states in the present paper are in general insufficient for practical applications. The use is essentially in the insight and in the asymptotic behaviour, which combined allow improvements of the analytical and numerical methods.

The binary interactions in the present paper are square-well potentials and therefore finite at the origin and zero outside their radii. We shall treat both the cases of attraction and repulsion. Such potentials exhibit in general the characteristic properties of the strong interaction and especially of the mean field potential for nucleons. They are, consequently, well suited for investigations of nuclear-physics problems. Accurate and realistic applications require inclusion of angular momenta larger than zero. Infinite potentials at the origin and long-range potentials like the Coulomb potential are not directly treated although extensions including these cases seem to be possible.

The potentials used in the following are especially appropriate in investigations of loosely bound three-body quantum systems. The weaker the binding energy the more extended is the wave function for low angular momenta and in particular for *s*-states. This implies that the wave function may be situated outside the ranges of the binary potentials. It is then independent of the details of the interactions and essentially only depending on scattering length, binding energy, and angular momentum. Examples are halo nuclei [17, 18, 26], but a larger class of physical systems accessible to our treatment is the Borromean systems, i.e. bound three-body systems where all binary subsystems are unbound. One particularly important system is ^{11}Li where the subsystems (^{10}Li and the dineutron) are unbound. The treatment as a three-body system has been very successful [33]. The details of the interaction are unimportant and 98% of the wave function consists of *s*-waves. Although ^{11}Li is the most prominent example many other light nuclei can be treated in the same way.

In this paper we outline in Sect. 2 the general theoretical framework and in Sect. 3 we solve the angular eigenvalue problem for *s*-states for a system of three identical bosons and spin-independent interactions. In Sect. 4 we generalize to systems of three different particles and in Sect. 5 we consider a system with two identical spin-$\frac{1}{2}$ particles plus a third particle. In Sect. 6 we give numerical illustrations and indicate qualitatively how to generalize the results. Tedious mathematical derivations are collected in the Appendices. Finally we give a summary and the conclusions in Sect. 7.

2 Theoretical Framework

In this section we shall briefly sketch the method which will be presented by use of a previously established notation, see refs. [2, 34]. The intrinsic Hamiltonian of the three-body system is given by

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m_i} - \frac{P^2}{2M} + \sum_{i>j=1}^{3} V_{ij} \,, \tag{1}$$

where m_i, $\mathbf{r}_i$, and $\mathbf{p}_i$ are mass, coordinate, and momentum of the i-th particle, V_{ij} are the two-body potentials, P and M are, respectively, the total momentum and the total mass of the system. We shall use the (three sets of) hyperspherical coordinates which consist of one radial coordinate ρ (hyperradius) and five generalized angles Ω_i, where $i = 1, 2, 3$. The precise definitions are given in Appendix A. One of these sets of hyperspherical coordinates is in principle sufficient for a complete description. The volume element is given by $\rho^5\, d\Omega\, d\rho$ where $d\Omega = \sin^2\alpha\, \cos^2\alpha\, d\alpha\, d\Omega_x\, d\Omega_y$.

2.1 General Procedure

The total wave function Ψ of the three-body system is written as a sum of three components $\psi^{(i)}$ which in turn for each ρ are expanded in a complete set of generalized angular functions

$$\Psi = \sum_{i=1}^{3} \psi^{(i)} = \frac{1}{\rho^{5/2}} \sum_{i,n} f_n(\rho) \Phi_n^{(i)}(\rho, \Omega_i) \,, \tag{2}$$

where the radial expansion coefficients $f_n(\rho)$ are component independent and $\rho^{-5/2}$ is the phase-space factor. These wave functions satisfy the three Faddeev equations [34]

$$(T - E)\psi^{(i)} + V_{jk}(\psi^{(i)} + \psi^{(j)} + \psi^{(k)}) = 0 \,, \tag{3}$$

where E is the total energy, T is the kinetic-energy operator and $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. The Faddeev equations may have non-trivial spurious solutions, where each component is non-vanishing while the sum corresponding to the Schrödinger wave function is identically equal to zero. The components of such solutions are eigenfunctions of the kinetic-energy operator with eigenvalues equal to the total energy.

After insertion of $\psi^{(i)}$ from Eq. (2) into Eq. (3) we collect the angular-dependent part and write the corresponding Faddeev equations for fixed ρ as

$$\frac{\hbar^2}{2m}\frac{1}{\rho^2}\hat{\Lambda}^2 \Phi_n^{(i)} + V_{jk}(\Phi_n^{(i)} + \Phi_n^{(j)} + \Phi_n^{(k)}) = \frac{\hbar^2}{2m}\frac{1}{\rho^2}\lambda_n(\rho)\Phi_n^{(i)} \,, \tag{4}$$

where $\hat{\Lambda}^2$ is the ρ-independent part of the kinetic-energy operator defined by

$$T \equiv T_\rho + \frac{\hbar^2}{2m}\frac{1}{\rho^2}\hat{\Lambda}^2, \qquad T_\rho = -\frac{\hbar^2}{2m}\left(\rho^{-5/2}\frac{\partial^2}{\partial\rho^2}\rho^{5/2} - \frac{1}{\rho^2}\frac{15}{4}\right) . \tag{5}$$

Eq. (4) may be considered as a generalization of the definition of the adiabatic basis introduced previously in the Schrödinger formulation [3–5]. This equation has already been used in several applications [1, 15, 34]. The procedure is now for each ρ to solve the eigenvalue problem in Eq. (4) for the five-dimensional angular part of the Faddeev operator.

Explicitly the generalized angular momentum operator $\hat{\Lambda}^2$ is given by

$$\hat{\Lambda}^2 = -\frac{1}{\sin\alpha\cos\alpha}\frac{\partial^2}{\partial\alpha^2}\sin\alpha\cos\alpha + \frac{\hat{l}_x^2}{\sin^2\alpha} + \frac{\hat{l}_y^2}{\cos^2\alpha} - 4, \tag{6}$$

in terms of an arbitrary α-coordinate and the angular-momentum operators $\hat{l}_x^2$ and $\hat{l}_y^2$ related to the Jacobi coordinates. The spurious states are characterized by angular eigenvalues equal to those of the angular kinetic-energy operator $\hat{\Lambda}^2$, i.e. $K(K+4)$, $K = 0, 1, 2, \ldots$.

Insertion of $\psi^{(i)}$ defined in Eq. (2) into Eq. (3) then leads to the coupled set of "radial" differential equations

$$\left(-\frac{d^2}{d\rho^2}-\frac{2mE}{\hbar^2}+\frac{1}{\rho^2}\left(\lambda_n(\rho)+\frac{15}{4}\right)\right)f_n=\sum_{n'}\left(2P_{nn'}\frac{d}{d\rho}+Q_{nn'}\right)f_{n'}\ , \tag{7}$$

with the functions P and Q defined by

$$P_{nn'}(\rho)\equiv\sum_{i,j=1}^{3}\int d\Omega\ \Phi_n^{(i)*}(\rho,\Omega)\frac{\partial}{\partial\rho}\Phi_{n'}^{(j)}(\rho,\Omega), \tag{8}$$

$$Q_{nn'}(\rho)\equiv\sum_{i,j=1}^{3}\int d\Omega\ \Phi_n^{(i)*}(\rho,\Omega)\frac{\partial^2}{\partial\rho^2}\Phi_{n'}^{(j)}(\rho,\Omega)\ . \tag{9}$$

The diagonal part of the P-matrix vanishes, i.e. $P_{nn}=0$, since the wave functions $\sum_{i=1}^{3}\Phi_n^{(i)}(\rho,\Omega)$ for each ρ are normalized in the angular space. Then we see immediately that the vanishing derivative of the normalization integral with respect to ρ is equal to P_{nn} for real wave functions, what is always possible for Hermitean Hamiltonians, see, for example, ref. [30].

2.2 Angular Eigenvalue Equation

It is convenient to show explicitly the spin dependence of the wave function $\Phi_n^{(i)}(\rho,\Omega_i)$ in Eq. (2), see, e.g., ref. [34]. Let us consider s-waves only and assume that $\mathbf{s}$ is the intermediate spin obtained by coupling of the spins $\mathbf{s}_j$ and $\mathbf{s}_k$ of particles j and k. The spin $\mathbf{s}$ is afterwards coupled to $\mathbf{s}_i$ to give the total spin S of the system. The total angular wave function for the i-th channel then factorizes into the spatial part $\phi_{n,s}^{(i)}(\rho,\Omega_i)$ and the spin-dependent part $\chi_{n,s}^{(i)}$, i.e.

$$\Phi_n^{(i)}(\rho,\Omega_i)=\frac{1}{\sin(2\alpha_i)}\sum_s\phi_{n,s}^{(i)}(\rho,\Omega_i)\chi_{n,s}^{(i)}\ , \tag{10}$$

where we explicitly extracted the phase-space factor $\sin(2\alpha_i)$. Both ϕ and χ may depend on the intermediate coupling.

The Faddeev components in Eq. (3) must be expressed in one Jacobi coordinate set. For the s-waves the wave functions $\phi_{n,s}^{(k)}$, which only depend on α_k and ρ, are first expressed in terms of the i-th set of hyperspherical coordinates and subsequently integrated over the four angular variables describing the directions of $\mathbf{x}_k$ and $\mathbf{y}_k$. This projection of one s-wave component in the k-th Jacobi coordinate system on the s-waves in the i-th Jacobi coordinate system can formally be expressed by the operator R_{ik} defined by [35]

$$R_{ik}\left[\frac{\phi_{n,s}^{(k)}}{\sin(2\alpha_k)}\right]\equiv\frac{1}{\sin(2\varphi_j)}\frac{1}{\sin(2\alpha_i)}\int_{|\varphi_j-\alpha_i|}^{\pi/2-|\pi/2-\varphi_j-\alpha_i|}\phi_{n,s}^{(k)}(\rho,\alpha_k)\ d\alpha_k\ , \tag{11}$$

where the angle φ_j is given by the masses as

$$\tan\varphi_j=\sqrt{\frac{m_j(m_1+m_2+m_3)}{m_im_k}}\ . \tag{12}$$

It is closely related to the transformation angle φ_{ik} used in Appendix A and defined

by

$$\varphi_{ik} = (-1)^p \varphi_j = \arctan\left((-1)^p \sqrt{\frac{m_j(m_1+m_2+m_3)}{m_i m_k}} \right) , \tag{13}$$

where $(-1)^p$ is the parity of the permutation $\{i,k,j\}$.

Substituting Eq. (10) into Eq. (4) we obtain, after multiplication from the left with $\chi^{(i)}_{n,s}$, the angular eigenvalue equation

$$\left(-\frac{\partial^2 \phi^{(i)}_{n,s}(\rho,\alpha_i)}{\partial \alpha_i^2} + (\rho^2 \langle \chi^{(i)}_{n,s}|v_i(\rho\sin\alpha_i)|\chi^{(i)}_{n,s}\rangle - \tilde{\lambda}_n(\rho))\phi^{(i)}_{n,s}(\rho,\alpha_i) + \rho^2 \sin(2\alpha_i) \right.$$
$$\left. \times \sum_{s's''} \langle \chi^{(i)}_{n,s}|v_i(\rho\sin\alpha_i)|\chi^{(i)}_{n,s''}\rangle \left(C^{ij}_{s''s'} R_{ij}\left[\frac{\phi^{(j)}_{n,s'}}{\sin(2\alpha_j)}\right] + C^{ik}_{s''s'} R_{ik}\left[\frac{\phi^{(k)}_{n,s'}}{\sin(2\alpha_k)}\right] \right) \right) = 0, \tag{14}$$

where $v_i(x) = V_{jk}(x/a_{jk})2m/\hbar^2$ with a_{jk} defined in Appendix A, $\tilde{\lambda}_n(\rho) = \lambda_n(\rho) + 4$, and the coefficients $C^{ik}_{ss'}$ expressing the overlap of the spin functions are given by

$$C^{ik}_{ss'} = \langle \chi^{(i)}_{n,s}|\chi^{(k)}_{n,s'}\rangle \, . \tag{15}$$

These matrix elements are diagonal for $i = k$, i.e. $C^{ii}_{ss'} = \delta_{ss'}$, and symmetric, i.e. $C^{ik}_{ss'} = C^{ki}_{s's}$.

If the potential is diagonal in spin space we obtain the much simpler set of equations

$$\left(-\frac{\partial^2 \phi^{(i)}_{n,s}(\rho,\alpha_i)}{\partial \alpha_i^2} + (\rho^2 \langle \chi^{(i)}_{n,s}|v_i(\rho\sin\alpha_i)|\chi^{(i)}_{n,s}\rangle - \tilde{\lambda}_n(\rho))\phi^{(i)}_{n,s}(\rho,\alpha_i) + \rho^2 \sin(2\alpha_i) \right.$$
$$\left. \times \langle \chi^{(i)}_{n,s}|v_i(\rho\sin\alpha_i)|\chi^{(i)}_{n,s}\rangle \sum_{s'} \left(C^{ij}_{ss'} R_{ij}\left[\frac{\phi^{(j)}_{n,s'}}{\sin(2\alpha_j)}\right] + C^{ik}_{ss'} R_{ik}\left[\frac{\phi^{(k)}_{n,s'}}{\sin(2\alpha_k)}\right] \right) \right) = 0. \tag{16}$$

Eqs. (14) and (16) constitute sets of equations obtained for $i = 1, 2, 3$ and all possible (i-dependent) values of s. As usual the values of $\{i,j,k\}$ must here be a permutation of $\{1,2,3\}$.

2.3 Spin-Independent Interactions

If the interactions are independent of spin, each Faddeev component must factorize into a spin part and a spatial part. Furthermore, the spin can be factorized out of Eq. (4) and the structure of the Faddeev equations then remains unchanged for the spatial parts alone. The spin-dependent wave function must then be the same for all three Faddeev components and the spatial parts of the wave function in Eq. (10) can at most differ by a normalization constant, i.e.

$$\phi^{(i)}_{n,s}(\rho,\Omega) \equiv b^{(i)}_s \phi^{(i)}_n(\rho,\Omega) \, , \tag{17}$$

$$\sum_s b^{(i)}_s \chi^{(i)}_{n,s} = \sum_s b^{(k)}_s \chi^{(k)}_{n,s} \, , \qquad i,k = 1,2,3. \tag{18}$$

Multiplication of Eq. (18) by $\chi^{(i)}_{n,s}$ from the left then gives

$$b^{(i)}_s = \sum_{s'} C^{ik}_{ss'} b^{(k)}_{s'} \, , \qquad i,k = 1,2,3. \tag{19}$$

These equations are not independent, since

$$\sum_{s'} C^{ik}_{ss'} b^{(k)}_{s'} = \sum_{s'} C^{il}_{ss'} b^{(l)}_{s'}, \tag{20}$$

for all values of s, k, and l. This is easily seen by use of the closure relation of Eq. (15), i.e.

$$C^{ik}_{ss'} = \sum_{s''} C^{il}_{ss''} C^{lk}_{s''s'} , \qquad l = 1, 2, 3 . \tag{21}$$

Thus Eq. (19) is valid for all k, if it only holds for one of the values of k. If $b^{(3)}_s$ is arbitrarily chosen and $b^{(1)}_s$ and $b^{(2)}_s$ calculated ($i = 1, 2$ and $k = 3$) from Eq. (19), we can see by using Eqs. (21) and (20) that Eq. (19) is valid for all other values of i and k. Thus any choice of $b^{(i)}_s$ for one value of i provides the same spin-independent solution of the Faddeev equations.

In the symmetric case when the three particles furthermore have equal masses and spatial interactions, the wave functions $\Phi^{(i)}_n$ corresponding to the different Faddeev components are independent of i. Then Eq. (10) implies that

$$\sum_s \phi^{(i)}_{n,s}(\rho, \Omega)\chi^{(i)}_{n,s} = \sum_s \phi^{(k)}_{n,s}(\rho, \Omega)\chi^{(k)}_{n,s} , \tag{22}$$

which by use of Eq. (17) directly leads to Eq. (18).

Inserting Eq. (17) into Eq. (16) we obtain by use of Eq. (19) the relation

$$\left(-\frac{\partial^2 \phi_n(\rho, \alpha_i)}{\partial \alpha_i^2} + (\rho^2 v_i(\rho \sin\alpha_i) - \tilde{\lambda}_n(\rho))\phi_n(\rho, \alpha_i) \right.$$
$$\left. + \rho^2 \sin(2\alpha_i) v_i(\rho \sin\alpha_i) \left(R_{ij}\left[\frac{\phi_n}{\sin(2\alpha_j)}\right] + R_{ik}\left[\frac{\phi_n}{\sin(2\alpha_k)}\right] \right) \right) = 0 , \tag{23}$$

which determines $\phi_n \equiv \phi^{(i)}_n$ for $i = 1, 2, 3$.

3 Three Identical Bosons with Spin-Independent Interactions

In the symmetric case with spin-independent interactions the remaining single Faddeev equation (23) can, by use of Eqs. (11) and (12), explicitly be written as

$$-\frac{\partial^2 \phi(\rho, \alpha)}{\partial \alpha^2} + (\rho^2 v(\rho \sin\alpha) - \tilde{\lambda}(\rho))\phi(\rho, \alpha)$$
$$+ \frac{4}{\sqrt{3}} \rho^2 v(\rho \sin\alpha) \int_{|\pi/3-\alpha|}^{\pi/2-|\pi/6-\alpha|} \phi(\rho, \alpha')\, d\alpha' = 0 , \tag{24}$$

where we omitted the label "n". We shall only consider short-range potentials and often further restrict ourselves to square wells. It is most convenient first to solve exactly for the schematic square-well potential and afterwards to generalize as far as possible. The two-body potential is then a step function

$$V(r) = -V_0 \Theta(r < R_0) \tag{25}$$

and the reduced potential in Eq. (24) is therefore another step function

$$v(x) = -v_0 \Theta(x < X), \tag{26}$$

where $v_0 = 2mV_0/\hbar^2$ and $X = R_0/\sqrt{2}$, if the normalization mass is equal to the mass of

one of the particles. The angular Faddeev equation (24) is now solved analytically in different intervals corresponding to increasing values of ρ. The decisive quantity for the square-well potential is $\rho \sin\alpha$, which determines whether v is finite or vanishes. When $\rho \leq X$ we have $v = -v_0$ for all values of $\alpha \in [0, \pi/2]$.

3.1 Short-Distance Behaviour: $0 \leq \rho \leq R_0/\sqrt{2}$

The potential is now constant for all α, and Eq. (24) simplifies to

$$-\frac{\partial^2\phi(\rho,\alpha)}{\partial\alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi(\rho,\alpha) = \frac{4}{\sqrt{3}}\rho^2 v_0 \int_{|\pi/3-\alpha|}^{\pi/2-|\pi/6-\alpha|} \phi(\rho,\alpha')\, d\alpha'. \tag{27}$$

The differential equation, without the right-hand side, has vanishing solutions at $\alpha = 0$ of the form $\sin(k\alpha)$. The integration of this function on the right-hand side of Eq. (27) is then performed in the three different α-regions, $\alpha < \pi/6$, $\pi/6 < \alpha < \pi/3$, $\pi/3 < \alpha$, and we observe that this functional form only remains if k is an even integer, i.e. $k = 2n$, where n is an integer. A solution to Eq. (27),

$$\phi(\rho,\alpha) \propto \sin(2n\alpha)\ , \tag{28}$$

is then by insertion found to correspond to the values of $\tilde{\lambda}_n$ given by

$$\tilde{\lambda}_n = 4n^2 - \rho^2 v_0 \left(1 - \frac{2}{n}(-)^n \frac{2}{\sqrt{3}}\sin(n\pi/3)\right). \tag{29}$$

This wave function is furthermore also the solution for all potentials at $\rho = 0$, where the corresponding eigenvalue $\tilde{\lambda}_n = 4n^2$ is then obtained by demanding a vanishing wave function at $\alpha = \pi/2$. Since the potential only enters in combination with ρ^2, a perturbative solution to second order in ρ^2 is obtained for potentials that are finite at the origin by using $v(x) \approx v(0) = -v_0$. Thus Eqs. (29) and (28) are also solutions for any potential in first-order perturbation theory provided the depth of the square-well potential is replaced by the value $v(0)$ of the potential at $\rho = 0$.

The solution for $n = 2$ is independent of ρ and corresponds to the trivial solution (identically zero) to the Schrödinger equation. For $n > 2$ the solution is only an approximation, since the higher angular momenta (neglected here) contribute to these states.

3.2 Intermediate Distances: $\pi/3 \leq \alpha_0 \leq \pi/2, R_0/\sqrt{2} \leq \rho \leq R_0\sqrt{2/3}$

The potential now vanishes for $\alpha_0 \leq \alpha \leq \pi/2$ (region II), where

$$\alpha_0 = \arcsin\left(R_0/\rho\sqrt{2}\right), \tag{30}$$

and it remains constant and finite for $0 \leq \alpha \leq \alpha_0$. The solution to Eq. (24) in region II (vanishing potential) is therefore

$$\phi_{\rm II} = A_{\rm II}\sin\left((\alpha-\pi/2)\sqrt{\tilde{\lambda}}\right) = \frac{1}{2i}A_{\rm II}\left(\exp\left[i(\alpha-\pi/2)\sqrt{\tilde{\lambda}}\right] - \exp\left[-i(\alpha-\pi/2)\sqrt{\tilde{\lambda}}\right]\right), \tag{31}$$

where we already explicitly selected the solution vanishing at $\alpha = \pi/2$.

To proceed further we divide the α-space into subregions:

$$\begin{aligned} A: &\quad 0 \le \alpha \le \alpha_0 - \frac{\pi}{3}, \\ B: &\quad \alpha_0 - \frac{\pi}{3} \le \alpha \le \frac{2\pi}{3} - \alpha_0, \\ C: &\quad \frac{2\pi}{3} - \alpha_0 \le \alpha \le \alpha_0, \\ \mathrm{II}: &\quad \alpha_0 \le \alpha \le \pi/2. \end{aligned} \tag{32}$$

These regions are marked in Fig. 1, which shows the integration limits of α' as function of α for the integral in Eq. (24). The coupling between the different regions can then be seen and expressed by the set of equations

$$-\frac{\partial^2 \phi_A(\rho,\alpha)}{\partial \alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi_A(\rho,\alpha) = \frac{4}{\sqrt{3}}\rho^2 v_0 \int_{|\pi/3-\alpha|}^{\pi/3+\alpha} \phi_C(\rho,\alpha')\, d\alpha', \tag{33}$$

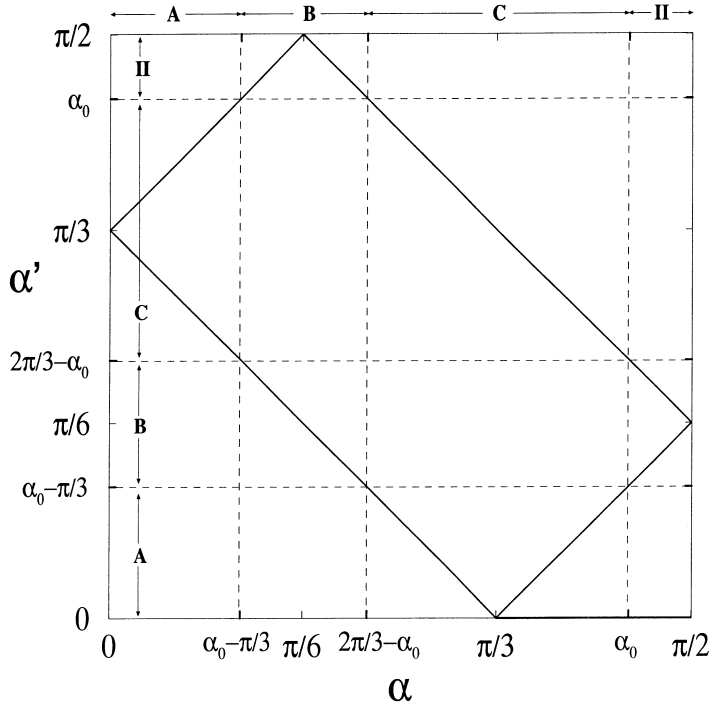

Fig. 1. Representation of the integration area in Eq. (24) by the tilted rectangle (solid lines). The regions *A*, *B*, *C*, and II (see Eq. (32)) are marked on the figure. The size of α_0 corresponds to intermediate values of ρ: $R_0/\sqrt{2} \le \rho \le R_0\sqrt{2/3}$. The thick solid line on the horizontal axis indicates the range of α_0-values

$$-\frac{\partial^2\phi_C(\rho,\alpha)}{\partial\alpha^2}-(\rho^2v_0+\tilde{\lambda}(\rho))\phi_C(\rho,\alpha)=\frac{4}{\sqrt{3}}\rho^2v_0\Bigg(\int_{|\pi/3-\alpha|}^{\alpha_0-\pi/3}\phi_A(\rho,\alpha')\,d\alpha'+\int_{2\pi/3-\alpha_0}^{2\pi/3-\alpha}\phi_C(\rho,\alpha')\,d\alpha'+\int_{\alpha_0-\pi/3}^{2\pi/3-\alpha_0}\phi_B(\rho,\alpha')\,d\alpha'\Bigg), \tag{34}$$

$$-\frac{\partial^2\phi_B(\rho,\alpha)}{\partial\alpha^2}-(\rho^2v_0+\tilde{\lambda}(\rho))\phi_B(\rho,\alpha)=\frac{4}{\sqrt{3}}\rho^2v_0\Bigg(\int_{|\pi/3-\alpha|}^{2\pi/3-\alpha_0}\phi_B(\rho,\alpha')\,d\alpha'+\int_{\alpha_0}^{\pi/2-|\pi/6-\alpha|}\phi_{\mathrm{II}}(\rho,\alpha')\,d\alpha'+\int_{2\pi/3-\alpha_0}^{\alpha_0}\phi_C(\rho,\alpha')\,d\alpha'\Bigg), \tag{35}$$

where the solutions are labeled according to the subregion. Only ϕ_A and ϕ_C are directly coupled since ϕ_B enters into Eq. (34) as an integral over a constant interval. Analogously, only ϕ_B and ϕ_{II} are directly coupled.

Integrating the ϕ_{II}-term in Eq. (35) we obtain exponentially increasing and decreasing functions, $\exp(\pm i\alpha\sqrt{\tilde{\lambda}})$, which by further integration and differential derivation still remain of the same functional form. Thus such functions matching ϕ_{II} are necessary in the solution, but in addition other exponentials are also possible. By insertion we then find that the wave function

$$\phi_B=B_+^{\mathrm{II}}\exp\left[i\alpha\sqrt{\tilde{\lambda}}\right]+B_-^{\mathrm{II}}\exp\left[-i\alpha\sqrt{\tilde{\lambda}}\right]+\sum_{k=1}^{3}\left(B_+^{(k)}\exp\left[\alpha\kappa_B^{(k)}\right]+B_-^{(k)}\exp\left[-\alpha\kappa_B^{(k)}\right]\right) \tag{36}$$

is a solution to Eq. (35) when the B-coefficients and $\kappa_B^{(k)}$ are related as shown in Appendix B. We have unique solutions for B_+^{II} and B_-^{II}, except for the pathological case of $\tilde{\lambda}=16/3$, and three different solutions for $\kappa_B^{(k)}$ with three corresponding constraints between $B_+^{(k)}$ and $B_-^{(k)}$. This is the explanation for the three terms of the same form in Eq. (36). In addition, there is a link to the C-region providing one constraint between the integrated ϕ_C-function and the $B_\pm^{(k)}$-values, see Eqs. (B.11) and (B.12).

In the coupled A- and C-regions we must also look for exponential functions as solutions. By insertion into Eqs. (33) and (34) we then find that the wave functions

$$\phi_A=\sum_{k=0}^{3}A^{(k)}\left(\exp\left[\alpha\kappa_{AC}^{(k)}\right]-\exp\left[-\alpha\kappa_{AC}^{(k)}\right]\right), \tag{37}$$

$$\phi_C=\sum_{k=0}^{3}\left(C_+^{(k)}\exp\left[\alpha\kappa_{AC}^{(k)}\right]+C_-^{(k)}\exp\left[-\alpha\kappa_{AC}^{(k)}\right]\right) \tag{38}$$

are indeed solutions provided the A- and C-coefficients and the $\kappa_{AC}^{(k)}$-values are related as

shown in Appendix C. In Eq. (37) we have already imposed the constraint that $\phi_A(\alpha = 0) = 0$, which related the coefficients in front of the two exponentials. They are different in the other regions. We obtain four different solutions for $\kappa_{AC}^{(k)}$, as indicated by the four k-values, and the two $C^{(k)}$-coefficients are uniquely determined by $A^{(k)}$. In addition, there is a link to the B-region providing one constraint between the integrated ϕ_B-function and the $A^{(k)}$-values, Eqs. (C.9) and (C.10).

The solutions are now explicitly written down in the four regions named A, B, C, and II. We have found wave functions containing the 8 parameters $A_{\rm II}$, $B_-^{(k)}$, $A^{(0)}$, and $A^{(k)}$ for $k = 1, 2, 3$. Two linear constraints exist between them as seen from Eqs. (B.11) and (B.12) in Appendix B and Eqs. (C.9) and (C.10) in Appendix C. The matching conditions at the three boundaries between the regions (see Fig. 1) then provide additional 6 linear constraints on the remaining 6 free parameters. This leads, as usual, to the quantization condition for the eigenvalue $\tilde{\lambda}$.

3.3 Intermediate Distances: $\pi/6 \le \alpha_0 \le \pi/3,\ R_0\sqrt{2/3} \le \rho \le R_0\sqrt{2}$

The potential again vanishes for $\alpha_0 \le \alpha \le \pi/2$, where α_0 is given in Eq. (30). Subregion C has now been absorbed into region II and subregions A and B are, as shown in Fig. 2, now defined by

$$A: \quad 0 \le \alpha \le \frac{\pi}{3} - \alpha_0,$$

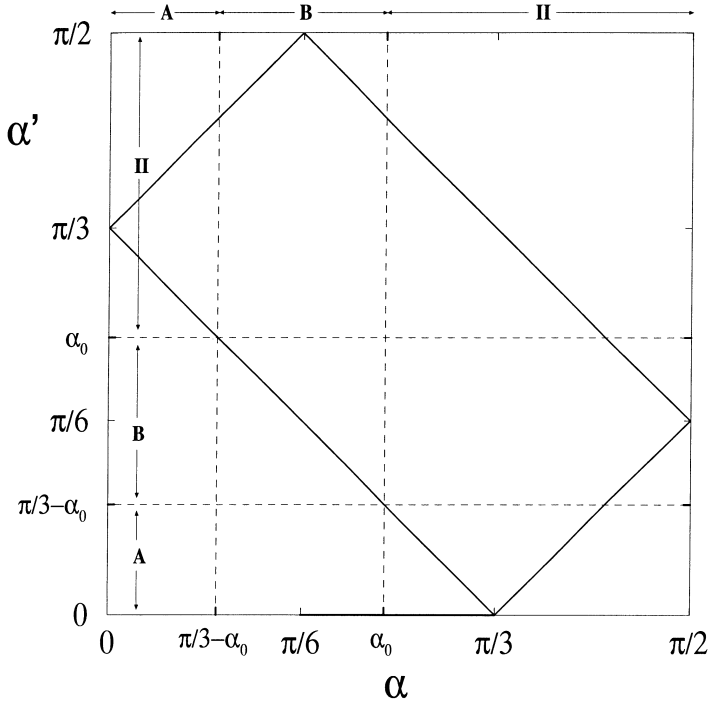

Fig. 2. The same as in Fig. 1 for α_0 corresponding to intermediate values of ρ: $R_0\sqrt{2/3} \le \rho \le R_0\sqrt{2}$. The regions marked on the figure are defined in Eq. (39). The thick solid line on the horizontal axis indicates the range of α_0-values

$$B: \quad \frac{\pi}{3} - \alpha_0 \leq \alpha \leq \alpha_0,$$
$$\text{II}: \quad \alpha_0 \leq \alpha \leq \pi/2. \tag{39}$$

The coupled set of equations is now:

$$-\frac{\partial^2 \phi_A(\rho,\alpha)}{\partial \alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi_A(\rho,\alpha) = \frac{4}{\sqrt{3}}\rho^2 v_0 \int_{\pi/3-\alpha}^{\pi/3+\alpha} \phi_{\text{II}}(\rho,\alpha')\, d\alpha', \tag{40}$$

$$-\frac{\partial^2 \phi_B(\rho,\alpha)}{\partial \alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi_B(\rho,\alpha) = \frac{4}{\sqrt{3}}\rho^2 v_0 \left(\int_{\pi/3-\alpha}^{\alpha_0} \phi_B(\rho,\alpha')\, d\alpha' + \int_{\alpha_0}^{\pi/2-|\pi/6-\alpha|} \phi_{\text{II}}(\rho,\alpha')\, d\alpha' \right). \tag{41}$$

The couplings are much simpler and essentially only ϕ_B enters into an integro-differential equation. Since ϕ_{II} is still given by the expression (31), a solution to Eq. (40) would have to be proportional to $\sin(\alpha\sqrt{\tilde{\lambda}})$. The solution, where the right-hand side vanishes, is analogously proportional to $\sin(\alpha\kappa)$, where

$$\kappa \equiv \sqrt{\rho^2 v_0 + \tilde{\lambda}(\rho)}\,. \tag{42}$$

By insertion we then obtain the complete solution to Eq. (40) as

$$\begin{aligned}\phi_A &= A_f \left(\exp\left[i\alpha\sqrt{\tilde{\lambda}} \right] - \exp\left[-i\alpha\sqrt{\tilde{\lambda}} \right] \right) + \frac{1}{2i} A_h (e^{i\alpha\kappa} - e^{-i\alpha\kappa}) \\ &= 2iA_f \sin\left(\alpha\sqrt{\tilde{\lambda}} \right) + A_h \sin(\alpha\kappa),\end{aligned} \tag{43}$$

where the coefficient A_h is arbitrary and A_f is given by

$$A_f = \frac{2A_{\text{II}}}{\sqrt{3\tilde{\lambda}}} \left(-\exp\left[i\alpha\sqrt{\tilde{\lambda}}\ \pi/6 \right] + \exp\left[-i\alpha\sqrt{\tilde{\lambda}}\ \pi/6 \right] \right) = -\frac{4iA_{\text{II}}}{\sqrt{3\tilde{\lambda}}} \sin\left(\alpha\sqrt{\tilde{\lambda}}\ \pi/6 \right). \tag{44}$$

The wave function in the B-region is a solution to Eq. (41), which apart from a constant term arising from ϕ_C and the upper limit of the integral of ϕ_B, is identical to Eq. (35). The solution is therefore given by Eq. (36), which by insertion into Eq. (41) in this case too leads to the expressions for the coefficients given in Eqs. (B.1)–(B.4) in Appendix B. The only differences are the constants in Eq. (B.5), which is now changed into

$$\begin{aligned}&\frac{A_{\text{II}}}{2\sqrt{\tilde{\lambda}}} \left(\exp\left[i\sqrt{\tilde{\lambda}}(\alpha_0 - \pi/2) \right] + \exp\left[-i\sqrt{\tilde{\lambda}}(\alpha_0 - \pi/2) \right] \right) \\ &\quad + \frac{1}{i\sqrt{\tilde{\lambda}}} \left(B_+^{\text{II}} \exp\left[i\sqrt{\tilde{\lambda}}\alpha_0 \right] - B_-^{\text{II}} \exp\left[-i\sqrt{\tilde{\lambda}}\alpha_0 \right] \right) \\ &\quad + \sum_{k=1}^{3} \left[\frac{1}{\kappa_B^k} \left(B_+^{(k)} \exp\left[\kappa_B^{(k)}\alpha_0 \right] - B_-^{(k)} \exp\left[-\kappa_B^{(k)}\alpha_0 \right] \right) \right] = 0.\end{aligned} \tag{45}$$

With the expressions in Eqs. (B.6), (B.7), and (B.10) for the coefficients we can rewrite

the constraint from Eq. (45) as

$$\sum_{k=1}^{3}\left[\frac{B_{-}^{(k)}}{\kappa_B^{(k)}}\left(\pm i\exp\left[\kappa_B^{(k)}(\pi/3-\alpha_0)\right]-\exp\left[-\kappa_B^{(k)}(2\pi/3-\alpha_0)\right]\right)\right]$$
$$+\frac{A_{\mathrm{II}}}{2\sqrt{\tilde{\lambda}}}\left[\left(\exp\left[i\sqrt{\tilde{\lambda}}(\pi/2-\alpha_0)\right]+\exp\left[-i\sqrt{\tilde{\lambda}}(\pi/2-\alpha_0)\right]\right)\right.$$
$$+\frac{1}{1-(16/3\tilde{\lambda})}\ \frac{4}{i\sqrt{3\tilde{\lambda}}}\left(\exp\left[i\sqrt{\tilde{\lambda}}(\alpha_0-\pi/6)\right]\left(1-\frac{4}{i\sqrt{3\tilde{\lambda}}}\right)\right.$$
$$\left.\left.-\exp\left[-i\sqrt{\tilde{\lambda}}(\alpha_0-\pi/6)\right]\left(1+\frac{4}{i\sqrt{3\tilde{\lambda}}}\right)\right)\right]=0. \tag{46}$$

We are now left with 5 parameters, i.e. A_{II}, A_h, and $B_{-}^{(k)}$ for $k = 1, 2, 3$. One of these is determined by the constraint in Eq. (46). The matching conditions at the two boundaries between the regions (see Fig. 2) provide additional 4 linear constraints on the remaining 4 free parameters. Therefore we obtain again a quantization condition for the eigenvalue $\tilde{\lambda}$.

3.4 Large-Distance Behaviour: $0 \le \alpha_0 \le \pi/6\ ,\ R_0\sqrt{2} \le \rho \le \infty$

The potential vanishes as usual for $\alpha_0 \le \alpha \le \pi/2$, where α_0 again is given in Eq. (30). Subregion B has now also been absorbed into region II and subregion A is, as shown in Fig. 3, now defined by

$$\begin{aligned} A:&\quad 0 \le \alpha \le \alpha_0,\\ \mathrm{II}:&\quad \alpha_0 \le \alpha \le \pi/2. \end{aligned} \tag{47}$$

The equations now reduce to Eq. (40), which couples ϕ_A and ϕ_{II}. The latter is again given by the expression (31) and consequently ϕ_A is given by Eq. (43) with one arbitrary coefficient and the other coefficient expressed in Eq. (44).

The matching conditions at α_0 provide 2 linear constraints on the remaining 2 free parameters and we obtain again a quantization condition for the eigenvalue $\tilde{\lambda}$. An explicit expression is most conveniently derived by rewriting the solution in Eq. (43) as

$$\phi_A = \frac{8A_{\mathrm{II}}}{\sqrt{3\tilde{\lambda}}}\sin(\sqrt{\tilde{\lambda}}\ \pi/6)\sin(\alpha\sqrt{\tilde{\lambda}}) + A_h\sin(\alpha\kappa)\ . \tag{48}$$

Matching the wave functions in Eqs. (31) and (48) and their first derivatives at $\alpha = \alpha_0$ gives immediately

$$\frac{8A_{\mathrm{II}}}{\sqrt{3\tilde{\lambda}}}\sin(\sqrt{\tilde{\lambda}}\ \pi/6)\sin(\alpha_0\sqrt{\tilde{\lambda}}) + A_h\sin(\alpha_0\kappa) = A_{\mathrm{II}}\sin\left((\alpha_0-\pi/2)\sqrt{\tilde{\lambda}}\right), \tag{49}$$

$$\frac{8A_{\mathrm{II}}}{\sqrt{3}}\sin(\sqrt{\tilde{\lambda}}\ \pi/6)\cos(\alpha_0\sqrt{\tilde{\lambda}}) + A_h\kappa\cos(\alpha_0\kappa) = A_{\mathrm{II}}\sqrt{\tilde{\lambda}}\cos\left((\alpha_0-\pi/2)\sqrt{\tilde{\lambda}}\right). \tag{50}$$

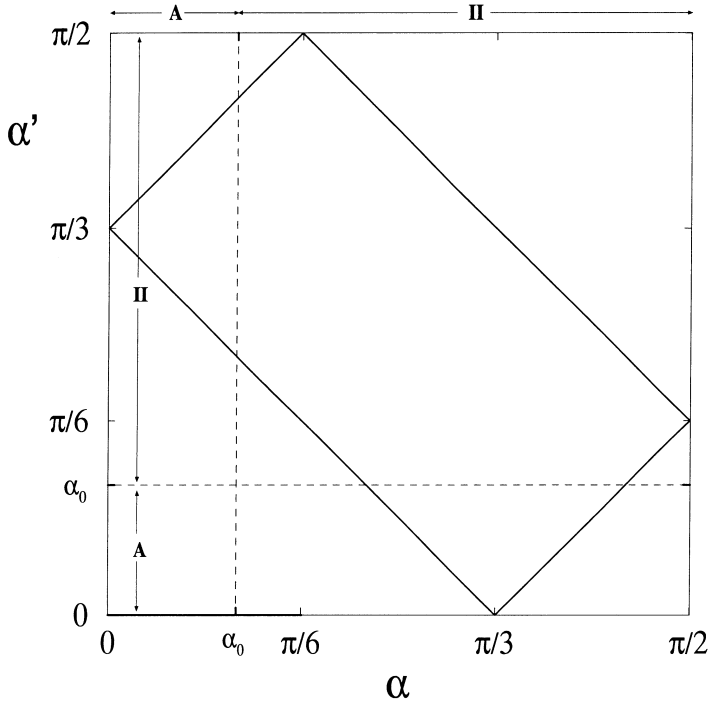

Fig. 3. The same as in Fig. 1 for α_0 corresponding to large values of ρ: $R_0\sqrt{2} \leq \rho \leq \infty$. The regions marked on the figure are defined in Eq. (47). The thick solid line on the horizontal axis indicates the range of α_0-values

The determinant D corresponding to this linear set of equations for $A_{\rm II}$ and A_h is

$$D = \frac{\kappa}{\sqrt{\tilde{\lambda}}}\cos(\alpha_0\kappa)\left[\frac{8}{\sqrt{3}}\sin(\sqrt{\tilde{\lambda}}\,\pi/6)\sin(\alpha_0\sqrt{\tilde{\lambda}}) - \sqrt{\tilde{\lambda}}\sin\Big((\alpha_0-\pi/2)\sqrt{\tilde{\lambda}}\Big)\right]$$
$$- \sin(\alpha_0\kappa)\left[\frac{8}{\sqrt{3}}\sin(\sqrt{\tilde{\lambda}}\,\pi/6)\cos(\alpha_0\sqrt{\tilde{\lambda}}) - \sqrt{\tilde{\lambda}}\cos\Big((\alpha_0-\pi/2)\sqrt{\tilde{\lambda}}\Big)\right]. \quad (51)$$

The non-trivial solutions are obtained for $D = 0$, which therefore is the eigenvalue equation for $\tilde{\lambda}$.

We can simplify even further in the limit of very large distances. We must then distinguish between the cases of different behaviours of $\tilde{\lambda}$. First we assume that $\tilde{\lambda}$ remains finite when ρ increases towards infinity. With these assumptions we obtain

$$\alpha_0 \approx \frac{R_0}{\rho\sqrt{2}}, \qquad \kappa \approx \rho\sqrt{v_0}, \qquad \kappa\alpha_0 \approx R_0\sqrt{v_0/2}\,, \quad (52)$$

where we used Eqs. (42) and (30). The eigenvalue equation $D = 0$ obtained from Eq. (51) by expansion to the lowest orders in $1/\rho$ then becomes

$$\frac{8}{\sqrt{3}}\sin(\sqrt{\tilde{\lambda}}\,\pi/6) - \sqrt{\tilde{\lambda}}\cos(\sqrt{\tilde{\lambda}}\,\pi/2) = \frac{\rho\sqrt{2}}{a_{\rm scat}}\sin(\sqrt{\tilde{\lambda}}\,\pi/2)\,, \quad (53)$$

where we introduced the two-body scattering length $a_{\rm scat}$ of the square-well potential

$$\frac{1}{a_{\rm scat}} = \frac{\sqrt{v_0/2}}{-R_0\sqrt{v_0/2} + \tan(R_0\sqrt{v_0/2})} \ . \tag{54}$$

The factor of 2 appearing here is due to the mass (not the reduced mass) in the definition of v_0. If ρ approaches infinity, Eq. (53) can be valid only if $\tilde{\lambda} \to 4n^2$, where n must be an integer. This is the same spectrum as obtained in Eq. (29) for $\rho = 0$. Expanding to the next order in $1/\rho$ gives

$$\tilde{\lambda}_n = 4n^2\left(1 - \frac{12a_{\rm scat}}{\pi\rho\sqrt{2}}\right) . \tag{55}$$

If the scattering length is infinitely large, Eq. (53) reduces to the equation derived by Efimov [10], which instead has the solution $\tilde{\lambda} = -1.01251$ leading to infinitely many bound states obtained from Eq. (7). Clearly, if necessary, these expressions can be extended to any order of correction in $1/\rho$.

We shall now consider the cases when $\tilde{\lambda}$ diverges as $\rho \to \infty$. If also $\tilde{\lambda}/\rho^2$ diverges, the eigenvalue equation $D = 0$ obtained from Eq. (51) does not have any solutions. If $\tilde{\lambda}/\rho^2$ approaches a constant, we define ϵ_x in the large-distance limit by $\tilde{\lambda} \to 2m\epsilon_x\rho^2/\hbar^2 \equiv \bar{\epsilon}_x\rho^2$. Using Eqs. (42) and (30) we then obtain at large ρ instead of Eq. (52) the result

$$\begin{aligned} &\sqrt{\tilde{\lambda}} \approx -i\rho\sqrt{-\bar{\epsilon}_x}, \qquad \kappa \approx \rho\sqrt{v_0 + \bar{\epsilon}_x}, \qquad \alpha_0\sqrt{\tilde{\lambda}} \approx -iR_0\sqrt{-\bar{\epsilon}_x/2}, \\ &\frac{\kappa}{\sqrt{\tilde{\lambda}}} \approx i\sqrt{-(v_0 + \bar{\epsilon}_x)/\bar{\epsilon}_x}, \qquad \kappa\alpha_0 \approx R_0\sqrt{(v_0 + \bar{\epsilon}_x)/2} \ , \end{aligned} \tag{56}$$

where we assumed that ϵ_x is negative since the eigenvalue equation $D = 0$ derived from Eq. (51) otherwise has no solutions for large ρ. Expansion of the equation $D = 0$ to the lowest orders in $1/\rho$ now (instead of Eq. (53)) leads to

$$\tan\left(R_0\sqrt{(v_0 + \bar{\epsilon}_x)/2}\right) \approx -\sqrt{\frac{v_0 + \bar{\epsilon}_x}{-\bar{\epsilon}_x}} \ , \tag{57}$$

which is the eigenvalue equation for the two-body subsystem with an energy corresponding to $\bar{\epsilon}_x$. If two-body bound states exist, ϵ_x must be one of the energies. In other words, for each bound two-body state one $\tilde{\lambda}$-value diverges parabolically according to Eq. (56).

For general short-range potentials that vanish at distances larger than a given finite radius R_0 we obtain again the solutions described in this subsection. The Efimov conditions, in particular, therefore remain the same for such potentials.

3.5 Large-Distance Behaviour of $P_{nn'}$ and $Q_{nn'}$

According to Eqs. (8) and (9) we must find the derivative of the Schrödinger wave function, $\Phi_n \equiv \sum_{i=1}^{3} \phi_n^{(i)}$, with respect to ρ. In the present symmetric case the three Faddeev components are equal. We choose to work in the first Jacobi coordinate system and must then express the other two components in this set of coordinates. The transformation is given in Eq. (11) and the angular wave function is

$$\Phi \propto \phi_n(\alpha_1) + 2R_{12}\left[\frac{\phi_n}{\sin(2\alpha_2)}\right] = \phi_n(\alpha_1) + \frac{4}{\sqrt{3}}\int_{|\pi/3-\alpha_1|}^{\pi/2-|\pi/6-\alpha_1|} \phi_n(\rho, \alpha_2)\ d\alpha_2 \ , \tag{58}$$

where the factor 2 arises since the R_{13}-operation gives the same result as that of R_{12}. The wave function $\phi_n(\alpha_1)$ is explicitly shown in Eqs. (31) and (48) with A_h obtained from Eq. (49) as

$$N(\sqrt{\tilde{\lambda}},\kappa,\alpha_0) \equiv \frac{A_h}{A_{\rm II}} = \frac{1}{\sin(\alpha_0\kappa)}\left[\frac{8}{\sqrt{3\tilde{\lambda}}}\sin(\sqrt{\tilde{\lambda}}\,\pi/6)\sin(\alpha_0\sqrt{\tilde{\lambda}}) - \sin\Big((\alpha_0-\pi/2)\sqrt{\tilde{\lambda}}\Big)\right]. \tag{59}$$

It is then clear that Φ_n only depends on ρ through $\sqrt{\tilde{\lambda}_n}$, κ_n and α_0. Thus

$$\frac{\partial}{\partial\rho}\Phi_n = \frac{\partial\sqrt{\tilde{\lambda}_n}}{\partial\rho}\frac{\partial\Phi_n}{\partial\sqrt{\tilde{\lambda}_n}} + \frac{\partial\kappa_n}{\partial\rho}\frac{\partial\Phi_n}{\partial\kappa_n} + \frac{\partial\alpha_0}{\partial\rho}\frac{\partial\Phi_n}{\partial\alpha_0}\ . \tag{60}$$

We have $P_{nn}=0$ and for $n\neq n'$ we obtain immediately $P_{nn'}$ from Eq. (8), i.e.

$$P_{nn'} \propto \frac{\partial\alpha_0}{\partial\rho}\int_0^{\pi/2} d\alpha_1\ \Phi_n^*\frac{\partial}{\partial\alpha_0}\Phi_{n'}\ , \tag{61}$$

since the orthogonality of the different Φ_n implies that

$$\frac{\partial\kappa_n'}{\partial\rho}\int_0^{\pi/2} d\alpha_1\ \Phi_n^*\frac{\partial}{\partial\kappa_n'}\Phi_{n'} = \frac{\partial\kappa_n'}{\partial\rho}\frac{\partial}{\partial\kappa_n'}\int_0^{\pi/2} d\alpha_1\ \Phi_n^*\Phi_{n'} = 0, \tag{62}$$

and analogously for $\sqrt{\tilde{\lambda}_n'}$. Furthermore, the derivative of the overall normalization factor $A_{\rm II}$, which also depends on α_0, does not contribute to Eq. (61) again due to the orthogonality of the wave functions. The large-distance behaviour of $\partial\alpha_0/\partial\rho$ is seen from the definition in Eq. (30) to be $\propto 1/\rho^2$.

We now divide the integration interval into the four parts appropriate for the combination of the transformation between Jacobi coordinates contained in Eq. (58) and the different wave functions given in the two intervals of Eq. (47), i.e.

$$I_1=[0,\alpha_0], \qquad I_2=[\alpha_0,\pi/3-\alpha_0], \qquad I_3=[\pi/3-\alpha_0,\pi/3+\alpha_0]\ ,$$
$$I_4=[\pi/3+\alpha_0,\pi/2]\ . \tag{63}$$

In the intervals I_2 and I_4 the integration variable α_1 is larger than α_0 (region II) and therefore $\phi_n(\alpha_1)$ in Eq. (58) is given by Eq. (31). In these intervals also the integration in Eq. (58) involves only the functions $\phi_n(\alpha_2)$ in region II, see Fig. 3. In the intervals I_2 and I_4 the wave function in Eq. (61), apart from $A_{\rm II}$, is then independent of α_0 (see Eq. (31)), and the contribution to $P_{nn'}$ is therefore zero.

In the interval I_3 the integration variable α_1 is also larger than α_0; again the function ϕ_{α_1} in Eq. (58) is given by Eq. (31) and it is therefore independent of α_0. However, in this case, the limits of the integration in Eq. (58) are such that in the integration interval the function $\phi_n(\alpha_2)$ involves both the solution in region II given by Eq. (31) and the solution in region A given by Eq. (48), see Fig. 3. Therefore the α_0-dependence of the wave function Φ (again apart from $A_{\rm II}$) arises entirely from the A_h-term of the two transformed Faddeev components. The size of the interval I_3 vanishes with $\alpha_0\propto 1/\rho$ and the derivative of Φ is proportional to $\kappa\alpha_0^2\,\partial N/\partial\alpha_0$, which vanishes as $\alpha_0\propto 1/\rho$. The contribution to $P_{nn'}$ from the third interval is then vanishing at least as fast as $1/\rho^4$.

The size of the first interval I_1 vanishes as $\alpha_0\propto 1/\rho$. The α_0-dependence of the wave function arises (apart from $A_{\rm II}$) entirely from the A_h-term in the first Faddeev

component. The derivative of N is proportional to $\kappa\alpha_0$, which remains finite together with the total wave function. Therefore, $P_{nn'}$ in Eq. (61) vanishes as $1/\rho^3$ for all n, n'. The leading-order term comes entirely from I_1, where the transformed Faddeev components combined with the derivative of the A_h-term of the first component contribute to the order $1/\rho^3$.

For the non-diagonal $Q_{nn'}$, in analogy to Eq. (61), we obtain

$$Q_{nn'} \propto \left(\frac{\partial\alpha_0}{\partial\rho}\right)^2 \int_0^{\pi/2} d\alpha_1 \; \Phi_n^* \frac{\partial^2}{\partial\alpha_0^2} \Phi_{n'} + \frac{\partial^2\alpha_0}{\partial\rho^2} \int_0^{\pi/2} d\alpha_1 \; \Phi_n^* \frac{\partial}{\partial\alpha_0} \Phi_{n'}, \tag{64}$$

where both terms therefore approach zero at least as $1/\rho^4$ (the first term due to $(\partial\alpha_0/\partial\rho)^2 \propto 1/\rho^4$ and the last term due to $\partial^2\alpha_0/\partial\rho^2 \propto 1/\rho^3$, and the integral found in Eq. (61) vanishing as $1/\rho$).

The diagonal term Q_{nn} can be written as a sum of six terms corresponding to Eq. (64) and the analogous terms where α_0 is replaced by $\sqrt{\tilde{\lambda}_n}$ and κ_n. The three terms with first-order derivatives of the wave function all vanish due to the normalization of Φ_n. Since $(\partial\alpha_0/\partial\rho)^2 \propto 1/\rho^4$, we are left with the leading terms

$$Q_{nn} \propto \left(\frac{\partial\kappa_n}{\partial\rho}\right)^2 \int_0^{\pi/2} d\alpha_1 \; \Phi_n^* \frac{\partial^2}{\partial\kappa_n^2} \; \Phi_n + \left(\frac{\partial\sqrt{\tilde{\lambda}_n}}{\partial\rho}\right)^2 \int_0^{\pi/2} d\alpha_1 \; \Phi_n^* \frac{\partial^2}{\partial(\sqrt{\tilde{\lambda}_n})^2} \Phi_n, \tag{65}$$

which for $n \neq n'$ would vanish due to orthogonality. The factor $(\partial\kappa_n/\partial\rho)^2$ approaches a constant when $\rho \to \infty$ and so does $(\partial\sqrt{\tilde{\lambda}_n}/\partial\rho)^2$ if $\sqrt{\tilde{\lambda}_n}$ corresponds to a bound two-body state; otherwise $(\partial\sqrt{\tilde{\lambda}_n}/\partial\rho)^2 \propto 1/\rho^4$.

We again make an appropriate division into four intervals $I_1, \; I_2, \; I_3, \; I_4$. Let us first study the case without two-body bound states, where the leading-order term arises from the κ-derivation, since then $(\partial\sqrt{\tilde{\lambda}_n}/\partial\rho)^2 \propto 1/\rho^4$. The second and fourth intervals contain no κ-dependence and consequently the contribution vanishes. The third interval contributes at most to the order $1/\rho^4$, i.e. $\alpha_0 \propto 1/\rho$ from the interval size, $\alpha_0^2 \propto 1/\rho^2$ from the second derivative of the A_h-term of the two transformed Faddeev components, and finally α_0 from the interval size of the transformation integral. In the first interval we have $\alpha_0 \propto 1/\rho$ from the size of the interval and $\alpha_0^2 \propto 1/\rho^2$ from the second derivative of the A_h-term in the first Faddeev component. The leading term of the total wave function arises from the two transformed Faddeev components and it approaches a finite constant for $\rho \to \infty$.

The κ_n-dependence of the normalization constant A_{II} did not contribute to $P_{nn'}$ due to the orthogonality of the wave functions. However, for Q_{nn} we must consider this dependence. If we define $\Phi_n \equiv A_{\mathrm{II}}\tilde{\Phi}_n$, it is easy to show that

$$\frac{\partial A_{\mathrm{II}}}{\partial\kappa_n} = -|A_{\mathrm{II}}|^3 \int_0^{\pi/2} d\alpha_1 \tilde{\Phi}_n^* \frac{\partial}{\partial\kappa_n} \tilde{\Phi}_n \; . \tag{66}$$

Going through the different intervals we find again vanishing contributions from I_2 and I_4, a contribution proportional to $1/\rho^3$ from I_3, and the leading-order term $\propto 1/\rho^2$ from I_1 arising from the two transformed Faddeev components combined with the derivative of the first Faddeev component. We therefore have $\partial A_{\mathrm{II}}/\partial\kappa_n \propto 1/\rho^2$. Furthermore, since

$$\int_0^{\pi/2} d\alpha_1 \; \Phi_n^* \frac{\partial^2}{\partial\kappa_n^2} \Phi_n = 2A_{\mathrm{II}} \frac{\partial A_{\mathrm{II}}}{\partial\kappa_n} \int_0^{\pi/2} d\alpha_1 \tilde{\Phi}_n^* \frac{\partial}{\partial\kappa_n} \tilde{\Phi}_n +$$

$$+ A_{\rm II} \frac{\partial^2 A_{\rm II}}{\partial \kappa_n^2} \int_0^{\pi/2} d\alpha_1 \, \tilde{\Phi}_n^* \tilde{\Phi}_n + |A_{\rm II}|^2 \int_0^{\pi/2} d\alpha_1 \, \tilde{\Phi}_n^* \frac{\partial^2}{\partial \kappa_n^2} \tilde{\Phi}_n, \tag{67}$$

the terms with derivatives of $A_{\rm II}$ approach zero at least as fast as $1/\rho^3$ and so does the last term as discussed above. Thus Q_{nn} approaches zero at least as fast as $1/\rho^3$.

In the case where $\tilde{\lambda}_n$ corresponds to a bound two-body state, we get the same results for the same reasons as described above. However, then $(\partial\sqrt{\tilde{\lambda}_n}/\partial\rho)^2$ approaches a constant for $\rho \rightarrow \infty$ and the second term in Eq. (65) must also be considered. Although possible this is a rather elaborate procedure due to the many terms containing $\tilde{\lambda}_n$. We shall instead give arguments based directly on the equation of motion.

At large distances the Faddeev components decouple as, for example, seen from Eq. (24), where the potential in the last term vanishes unless α approaches zero at least as fast as $1/\rho$. However, if $\alpha \rightarrow 0$ the size of the integration interval vanishes together with the integral itself. Thus the angular wave function is in this limit determined by an ordinary Schrödinger equation, i.e.

$$\left[-\frac{d^2}{dz^2} - \frac{\tilde{\lambda}}{\rho^2} + v(z) \right] \phi(z) = 0, \tag{68}$$

where we introduced the new variable $z = \rho \sin\alpha \approx \rho\alpha$. The boundary conditions $\phi(z = \rightarrow 0) = \phi(z \rightarrow \infty) = 0$ make Eq. (68) equivalent to the Schrödinger equation for one of the (three identical) two-body subsystems, i.e.

$$\left[-\frac{d^2}{dz^2} + v(z) \right] u(z) = \frac{2m\epsilon_x}{\hbar^2} u(z), \tag{69}$$

where $z = r\sqrt{m/\mu}$, r is the relative coordinate, ϵ_x is the energy, and μ the reduced mass of the two-body system.

From Eqs. (68) and (69) we obtain

$$\tilde{\lambda} = \lambda + 4 = \frac{2m\epsilon_x}{\hbar^2}\rho^2 + \int dz \; u(z) \left[z\frac{du}{dz} + z^2 \frac{d^2u}{dz^2} \right] \tag{70}$$

and $\phi(z) = \sqrt{\rho} u(z)$. Introducing $\phi(z)$ in the Q_{nn}-equation we get

$$Q_{nn} = -\frac{1}{4\rho^2} + \frac{1}{\rho^2} \int dz \; u(z) \left[z\frac{du}{dz} + z^2 \frac{d^2u}{dz^2} \right]. \tag{71}$$

The integrals in Eqs. (70) and (71) cancel in the combination $(\lambda + 15/4)/\rho^2 - Q_{nn}$ found in the coupled set of radial equations, see Eq. (7). The result, $2m\epsilon_x/\hbar^2$, restores the proper two-body radial asymptotics, which describe the motion of one of the particles relative to the bound system of the other two particles.

We can conclude that the coupled set of radial equations (7) decouple in the limit of large ρ, since both coupling terms $P_{nn'}$ and $Q_{nn'}$ approach zero at least as fast as $1/\rho^3$, i.e. faster than the leading centrifugal-barrier term $1/\rho^2$. This behaviour at large distances is a general result valid for all short-range potentials.

4 Three Non-Identical Particles with Spin-Independent Interactions

In the asymmetric case with spin-independent interactions the three Faddeev equations

(16) can, by use of Eqs. (11), (12), (17), and (19), explicitly be written as

$$-\frac{\partial^2\phi^{(i)}(\rho,\alpha_i)}{\partial\alpha_i^2}+(\rho^2 v_i(\rho\sin\alpha_i)-\tilde{\lambda}(\rho))\phi^{(i)}(\rho,\alpha_i)$$
$$+\rho^2 v_i(\rho\sin\alpha_i)\left(\frac{1}{\sin(2\varphi_j)}\int_{|\varphi_j-\alpha_i|}^{\pi/2-|\pi/2-\varphi_j-\alpha_i|}\phi^{(k)}(\rho,\alpha_k)\;d\alpha_k\right.$$
$$\left.+\frac{1}{\sin(2\varphi_k)}\int_{|\varphi_k-\alpha_i|}^{\pi/2-|\pi/2-\varphi_k-\alpha_i|}\phi^{(j)}(\rho,\alpha_j)\;d\alpha_j\right)=0, \tag{72}$$

for $i = 1, 2, 3$ (we have again omitted the label *"n"*). We shall follow the same procedure as in the symmetric case and mostly use square-well potentials, i.e.

$$V_i(r) = -V_0^{(i)}\Theta(r < R_i), \tag{73}$$

and the reduced potentials in Eq. (72) are therefore the other step functions

$$v_i(x) = -v_0^{(i)}\Theta(x < X_i), \tag{74}$$

where $v_0^{(i)} = 2mV_0^{(i)}/\hbar^2$ and $X_i = R_i\mu_{jk}$. The angular Faddeev equations (72) are now solved analytically in different intervals corresponding to increasing values of ρ. The decisive quantity for each of the square-well potentials is $\rho\sin\alpha_i$, which determines whether $v_i(x)$ is finite or vanishes. If $\rho \le X_i$ we have $v_i(x) = -v_0^{(i)}$ for all values of $\alpha_i \in [0, \pi/2]$.

Without loss of generality we can assume that $X_1 \le X_2 \le X_3$ implying $\alpha_0^{(1)} \le \alpha_0^{(2)} \le \alpha_0^{(3)}$, where in analogy to Eq. (30) we define

$$\alpha_0^{(i)} \equiv \arcsin(X_i/\rho)\ . \tag{75}$$

In this section we shall confine ourselves to small or large distances. The intermediate distances can be solved in analogy to the symmetric case by division into ρ-intervals. The related eigenfunctions are combinations of simple functions, but they are rather tedious to write down in their full length and they are also less interesting.

4.1 Short-Distance Behaviour: $0 \le \rho \le X_1$

The potentials are now all constants for all α-values, and Eq. (72) is an inhomogeneous differential equation for $\phi^{(i)}$, where the inhomogeneous part contains $\phi^{(j)}$ and $\phi^{(k)}$. It can therefore be solved by adding one inhomogeneous solution

$$\phi^{(i)}(\rho,\alpha_i) = a_i\sin(2n\alpha_i) \tag{76}$$

to the complete set of homogeneous solutions

$$\phi^{(i)}(\rho,\alpha_i) = b_i\sin(\alpha_i\kappa_i), \qquad \kappa_i = \sqrt{v_0^{(i)}\rho^2+\tilde{\lambda}(\rho)}\ , \tag{77}$$

where we already imposed the boundary condition $\phi^{(i)}(\rho,\alpha_i = 0) = 0$.

The inhomogeneous components in Eq. (76) must all be proportional to $\sin(2n\alpha)$, since the integrated values in Eq. (72) otherwise do not return the same function as required by this equation. The homogeneous components in Eq. (77) can only be solutions if $v_0^{(1)} = v_0^{(2)} = v_0^{(3)}$ and $\varphi_1 = \varphi_2 = \varphi_3 = \pi/3$, which is the symmetric case discussed in Sect. 3. Any asymmetry therefore leads to $b_1 = b_2 = b_3 = 0$. A set of

solutions is therefore obtained, if and only if the three equations

$$\rho^2(v_0^{(i)} - \epsilon)a_i = \rho^2 v_0^{(i)}(a_k d_j + a_j d_k) \tag{78}$$

are fulfilled for $\{i,j,k\}$ being the three permutations of $\{1,2,3\}$. Here we have defined

$$\tilde{\lambda} \equiv 4n^2 - \rho^2\epsilon, \qquad d_i \equiv -\frac{\sin(2n\varphi_i)}{n\sin(2\varphi_i)} \ . \tag{79}$$

In the limiting case of $\rho = 0$ we immediately find the three times degenerate solution $\tilde{\lambda} = 4n^2$.

Non-trivial solutions for $\rho \neq 0$ only occur if the corresponding determinant vanishes, i.e.

$$\begin{vmatrix} \epsilon - v_0^{(1)} & v_0^{(1)} d_3 & v_0^{(1)} d_2 \\ v_0^{(2)} d_3 & \epsilon - v_0^{(2)} & v_0^{(2)} d_1 \\ v_0^{(3)} d_2 & v_0^{(3)} d_1 & \epsilon - v_0^{(3)} \end{vmatrix} = 0. \tag{80}$$

This determines the possible solutions $\tilde{\lambda}(\rho)$.

In the symmetric case ($v_0^{(1)} = v_0^{(2)} = v_0^{(3)} \equiv v_0$, $d_1 = d_2 = d_3 \equiv d_0$) we obtain the two solutions

$$\tilde{\lambda}(\rho) = 4n^2 - \rho^2 v_0(1 + d_0), \tag{81}$$

$$\tilde{\lambda}(\rho) = 4n^2 - \rho^2 v_0(1 - 2d_0), \tag{82}$$

where the first one is two times degenerate and the last one corresponds to the symmetric solution, as seen from Eq. (78).

In the general case we define

$$\begin{aligned} S_0 &\equiv v_0^{(1)} v_0^{(2)} v_0^{(3)}(2d_1 d_2 d_3 + d_1^2 + d_2^2 + d_3^2 - 1), \\ S_1 &\equiv v_0^{(1)} v_0^{(2)}(1 - d_3^2) + v_0^{(1)} v_0^{(3)}(1 - d_2^2) + v_0^{(2)} v_0^{(3)}(1 - d_1^2), \\ S_2 &\equiv -v_0^{(1)} - v_0^{(2)} - v_0^{(3)} \end{aligned} \tag{83}$$

and rewrite Eq. (80) as

$$\epsilon^3 + S_2\epsilon^2 + S_1\epsilon + S_0 = 0 \ , \tag{84}$$

which is independent of ρ. Thus the short-distance behaviour of $\tilde{\lambda}(\rho)$ is quadratic in ρ and given by Eq. (79).

For $n = 1$ we have $d_i = -1$, $S_0 = S_1 = 0$ and consequently the solutions $\epsilon = -S_2$ and the doubly degenerate spurious solution $\epsilon = 0$. This is also seen from Eq. (81) with $n = 1$. The corresponding wave function is determined by Eq. (76) with $a_i \propto v_0^{(i)}$ as seen from Eq. (78).

For $n = 2$ we have

$$d_i = -\cos(2\varphi_i) = \frac{\tan^2\varphi_i - 1}{\tan^2\varphi_i + 1} \tag{85}$$

and $S_0 = 0$. Consequently we find the spurious solution $\epsilon = 0$ and the two solutions

$$\epsilon = \frac{1}{2}\left(-S_2 \pm \sqrt{S_2^2 - 4S_1}\right) \ , \tag{86}$$

where ϵ is real, since $S_2^2 \geq 4S_1$ for all masses and all values of the strength parameters $v_0^{(i)}$. The related wave functions are found from Eq. (78).

For arbitrary n-values exceeding 2 we have three real solutions to Eq. (84) as discussed in Appendix D. For $n \to \infty$, we have $d_i \to 0$.

As in the symmetric case, here too the potentials enter only in combination with ρ^2, and therefore a perturbative solution to first order in ρ^2 is obtained for arbitrary potentials by using $v_i(x) \approx v_i(0) = -v_0^{(i)}$. Thus the solutions in this subsection are also solutions for any potential in first-order perturbation theory provided the depth of the square-well potential is replaced by the value $v_i(0)$ of the potential at $\rho = 0$.

4.2 Large-Distance Behaviour

We define in this connection large distances to mean that

$$\alpha_0^{(1)} \leq \alpha_0^{(2)} \leq \alpha_0^{(3)} \leq |\varphi_1 - \alpha_0^{(1)}| \leq |\varphi_2 - \alpha_0^{(2)}| \leq |\varphi_3 - \alpha_0^{(3)}| \tag{87}$$

for all sets 1, 2, 3; this implies that $\alpha_0^{(i)} \leq \frac{1}{2}\,\mathrm{Min}(\varphi_i)$ for all sets $i = 1, 2, 3$. The potentials all vanish for $\alpha_i \geq \alpha_0^{(i)}$ and the solutions to Eq. (72) are therefore the same as in Eq. (31), i.e.

$$\phi^{(i)}(\alpha_i) = a_i \sin\left((\alpha_i - \pi/2)\sqrt{\tilde{\lambda}}\right) . \tag{88}$$

In the other region where $\alpha_i \leq \alpha_0^{(i)}$ we also obtain decoupled solutions, i.e.

$$\phi^{(i)}(\alpha_i) = b_i \sin(\alpha_i \kappa_i) + c_i \sin\left(\alpha_i \sqrt{\tilde{\lambda}}\right) , \tag{89}$$

where the first term is the solution to the homogeneous equation. The constants b_i are therefore completely free, since the functions to be integrated in Eq. (72) only involve the functions in Eq. (88), i.e. values of $\alpha_i \geq \alpha_0^{(i)}$, $i = 1, 2, 3$.

The connection between a_i and c_i is now found by insertion of Eqs. (88) and (89) into Eq. (72). The result is

$$\bar{c}_i = \frac{2F^3}{\sqrt{\tilde{\lambda}}}(\bar{a}_j + \bar{a}_k) , \tag{90}$$

where

$$\bar{a}_i = a_i/f_i, \qquad \bar{c}_i = -c_i\, f_i, \qquad f_i = \frac{\sin\left((\varphi_i - \pi/2)\sqrt{\tilde{\lambda}}\right)}{\sin(2\varphi_i)}, \qquad F = (\, f_1\, f_2\, f_3)^{1/3}. \tag{91}$$

This determines c_i from given values of a_i.

The matching conditions at $\alpha_0^{(1)}, \alpha_0^{(2)}$, and $\alpha_0^{(3)}$ provide 6 linear constraints for the parameters a_i and b_i, where c_i is found from Eqs. (90)–(91). They have non-trivial solutions only if the corresponding determinant vanishes. In Appendix E it is shown that this condition determining the eigenvalues $\tilde{\lambda}$ can be formulated as

$$D = B_1 B_2 B_3 + 2A_1 A_2 A_3 - B_1 A_2 A_3 - A_1 A_2 B_3 - A_1 B_2 A_3 = 0 , \tag{92}$$

where A_i and B_i are defined in Appendix E.

In the symmetric case ($B_1 = B_2 = B_3 = B_0$, $A_1 = A_2 = A_3 = A_0, f_1 = f_2 = f_3$), we have $D = (A_0 - B_0)^2(2A_0 + B_0)$, where the solutions to $D = 0$ correspond to those of Eqs. (81) and (82).

Eq. (92) simplifies in the limit of very large distances, where

$$\alpha_0^{(i)} \approx \frac{X_i}{\rho}, \qquad \kappa_i \approx \rho\sqrt{v_0^{(i)}}, \qquad \kappa_i\alpha_0^{(i)} \approx X_i\sqrt{v_0^{(i)}}\,, \tag{93}$$

where we have assumed that $\tilde{\lambda}$ remains finite. Expansion to lowest order in $1/\rho$ then gives, see Eqs. (E.4) and (E.7),

$$A_i \approx -2F\mu_{jk}\sqrt{v_0^{(i)}}a_{\rm scat}^{(i)}\cos(X_i\sqrt{v_0^{(i)}}), \tag{94}$$

$$B_i \approx -\frac{f_i^2}{F^2}\cos(X_i\sqrt{v_0^{(i)}})\sqrt{v_0^{(i)}}\left(\rho\sin\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right) + a_{\rm scat}^{(i)}\mu_{jk}\sqrt{\tilde{\lambda}}\cos\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\right), \tag{95}$$

where the scattering length for the two-body system in analogy with Eq. (54) is given by

$$\frac{1}{a_{\rm scat}^{(i)}} = \frac{\mu_{jk}\sqrt{v_0^{(i)}}}{-X_i\sqrt{v_0^{(i)}} + \tan\left(X_i\sqrt{v_0^{(i)}}\right)}\,. \tag{96}$$

The eigenvalues for $\rho \to \infty$ therefore approach solutions to $\sin((\pi/2)\sqrt{\tilde{\lambda}}) = 0$, i.e. the hyperspherical spectrum of $\tilde{\lambda} = 4n^2$. The equation to the next order in $1/\rho$ is instead

$$\tan\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right) = -\frac{\sqrt{\tilde{\lambda}}}{\rho}\sum_{i=1}^{3} a_{\rm scat}^{(i)}\mu_{jk}\,, \tag{97}$$

which immediately generalizes the symmetric case in Eq. (55) to

$$\tilde{\lambda} \approx 4n^2\left(1 - \frac{4}{\pi\rho}\sum_{i=1}^{3} a_{\rm scat}^{(i)}\mu_{jk}\right). \tag{98}$$

The different pathological cases of extremely large scattering lengths are very different. Without loss of generality we can here assume that $|a_{\rm scat}^{(1)}| \le |a_{\rm scat}^{(2)}| \le |a_{\rm scat}^{(3)}|$ and consider various cases. Using Eqs. (92), (94), and (95) we obtain for large values of ρ that in case $|a_{\rm scat}^{(2)}| \ll \rho \ll |a_{\rm scat}^{(3)}|$ the angular eigenvalues are still given by $\sin((\pi/2)\sqrt{\tilde{\lambda}}) = 0$; in case $|a_{\rm scat}^{(1)}| \ll \rho \ll |a_{\rm scat}^{(2)}|$ they are instead given by

$$\pm\sqrt{\tilde{\lambda}}\cos\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\sin(2\varphi_1) = 2\sin\left((\varphi_1 - \pi/2)\sqrt{\tilde{\lambda}}\right); \tag{99}$$

finally, if $\rho \ll |a_{\rm scat}^{(1)}|$, we find

$$\left(\frac{\sqrt{\tilde{\lambda}}\cos\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)}{2F}\right)^3 - \left(\frac{\sqrt{\tilde{\lambda}}\cos\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)}{2F}\right)\frac{(f_1^2 + f_2^2 + f_3^2)}{F^2} + 2 = 0\,. \tag{100}$$

The last equation has the symmetric solution $\sqrt{\tilde{\lambda}}\cos((\pi/2)\sqrt{\tilde{\lambda}}) = -4F$, if all the quantities are independent of i. These results reflect the fact that the Efimov anomaly with the resulting infinitely many bound three-body states obtained from Eq. (7) is only present in the last two cases where at least two of the scattering lengths are infinitely large. Clearly we are able to extend these expressions to any order of correction in $1/\rho$.

If $\tilde{\lambda}/\rho^2$ diverges for $\rho \to \infty$, the eigenvalue equation $D = 0$ has no solution as in the symmetric case. If $\tilde{\lambda}/\rho^2$ approaches a finite constant, we can again define ϵ_x in the large-

distance limit by $\tilde{\lambda} \to 2m\epsilon_x\rho^2/\hbar^2 \equiv \bar{\epsilon}_x\rho^2$. Eq. (92) can at most have solutions for negative $\bar{\epsilon}_x$. At large ρ we have approximately

$$\sqrt{\tilde{\lambda}} \approx -i\rho\sqrt{-\bar{\epsilon}_x}, \qquad \kappa_i \approx \rho\sqrt{v_0^{(i)} + \bar{\epsilon}_x}, \qquad \alpha_0^{(i)}\sqrt{\tilde{\lambda}} \approx -iX_i\sqrt{-\bar{\epsilon}_x/2},$$
$$\frac{\kappa_i}{\sqrt{\tilde{\lambda}}} \approx i\sqrt{-(v_0^{(i)} + \bar{\epsilon}_x)/\bar{\epsilon}_x}, \qquad \kappa_i\alpha_0^{(i)} \approx X_i\sqrt{v_0^{(i)} + \bar{\epsilon}_x}. \tag{101}$$

The dominating term in the determinant in Eq. (92) is $B_1B_2B_3$, which diverges as ρ^3. To leading order in ρ we therefore obtain the eigenvalue solutions from $B_i = 0$. This is equivalent (see Eqs. (E.4) and (E.7)) to $d_{ii} = 0$ which, to leading order, is immediately seen to be identical to Eq. (57) for the i-th two-body subsystem. Thus we obtain the general result that there is a one-to-one correspondence between the total number of two-body bound states in all the subsystems and the parabolically diverging $\tilde{\lambda}$-values at large ρ.

The behaviour of $P_{nn'}$ and $Q_{nn'}$ at large distances is qualitatively the same as for three identical particles. This result, as well as the various solutions described in this subsection, is valid in general for all short-range potentials. In particular, the Efimov conditions remain the same for such potentials.

5 Two Identical Spin-$\frac{1}{2}$ Particles

We consider a system of two identical spin-$\frac{1}{2}$ particles (labelled "f" and referred to as fermions) and a third particle (labelled by "c" and called the core) of spin s_c. The total angular momentum J can then take the values $J = s_c, s_c \pm 1$, since we only consider vanishing orbital angular momentum. The spin wave functions $\chi_s^{(i)}$ from Eq. (10) are $\chi_0^{(1)}$, $\chi_{s_c\pm1/2}^{(2)}$, and $\chi_{s_c\pm1/2}^{(3)}$, which are defined as in Subsect. 2.2. We have labelled the core as particle 1 and the two fermions as particles 2 and 3. The spin state $\chi_1^{(1)}$ is not possible due to the required antisymmetrization of the wave function under exchange of the fermions. Since we include only s-waves, the two fermions cannot be coupled to spin 1. The total angular momentum is therefore confined to be $J = s_c$.

The form of the Faddeev equations is determined by the spin dependence of the interactions. The fermion-fermion interaction, $V_{ff}(r)$, is only effective in relative s-states and no spin dependence is needed. For the fermion-core interaction we take the spin-dependence to be of the form

$$V_2(r) = V_3(r) = (1 + \gamma_s\mathbf{s}_c \cdot \mathbf{s}_f)V_{fc}(r). \tag{102}$$

This potential is diagonal in spin space, and the diagonal matrix elements for $i = 2, 3$ are:

$$\langle\chi_{s_c+1/2}^{(i)}|V_i(r)|\chi_{s_c+1/2}^{(i)}\rangle \equiv V_{fc}^+(r) = \left(1 + \gamma_s\frac{s_c}{2}\right)V_{fc}(r), \tag{103}$$

$$\langle\chi_{s_c-1/2}^{(i)}|V_i(r)|\chi_{s_c-1/2}^{(i)}\rangle \equiv V_{fc}^-(r) = \left(1 - \frac{\gamma_s(s_c+1)}{2}\right)V_{fc}(r). \tag{104}$$

The spin-overlaps defined in Eq. (15) are now found to be

$$C_{0,s_c-1/2}^{12} = C_{0,s_c-1/2}^{13} = -\sqrt{\frac{s_c}{2s_c+1}},$$

$$C^{12}_{0,s_c+1/2} = -C^{13}_{0,s_c+1/2} = \sqrt{\frac{s_c+1}{2s_c+1}},$$

$$C^{23}_{s_c-1/2,s_c-1/2} = C^{23}_{s_c+1/2,s_c+1/2} = -\frac{1}{2s_c+1}\ ,$$

$$C^{23}_{s_c-1/2,s_c+1/2} = -C^{23}_{s_c+1/2,s_c-1/2} = \frac{\sqrt{4s_c(s_c+1)}}{2s_c+1}. \tag{105}$$

The Pauli principle requires that the solutions are antisymmetric under a simultaneous interchange of all the coordinates of the two fermions labelled 2 and 3. This means that exchange of α_2 and α_3 and exchange of the order of the couplings in the spin functions must give a change of sign of the total wave function. Imposing this constraint the components of the three-body wave function in Eq. (10) must be related as

$$\phi^{(3)}_{s_c-1/2} = \phi^{(2)}_{s_c-1/2}, \qquad \phi^{(3)}_{s_c+1/2} = -\phi^{(2)}_{s_c+1/2}\ . \tag{106}$$

The corresponding Faddeev equations obtained from Eq. (16) by integrating out the spin degrees of freedom are then given by

$$\begin{aligned}
&\left[-\frac{\partial}{\partial\alpha_1^2} - (\tilde{\lambda}(\rho) - \rho^2 v_{ff}(\rho\sin\alpha_1))\right]\phi^{(1)}_0(\rho,\alpha_1) \\
&\quad = -\rho^2 v_{ff}(\rho\sin\alpha_1)\left[\frac{1}{\sin(2\varphi)}\left(C^{12}_{0,s_c-1/2}\int_{|\varphi-\alpha_1|}^{\pi/2-|\pi/2-\varphi-\alpha_1|} d\alpha_2\ \phi^{(2)}_{s_c-1/2}(\rho,\alpha_2)\right.\right. \\
&\qquad \left. + C^{12}_{0,s_c+1/2}\int_{|\varphi-\alpha_1|}^{\pi/2-|\pi/2-\varphi-\alpha_1|} d\alpha_2\ \phi^{(2)}_{s_c+1/2}(\rho,\alpha_2)\right) \\
&\qquad + \frac{1}{\sin(2\varphi)}\left(C^{12}_{0,s_c-1/2}\int_{|\varphi-\alpha_1|}^{\pi/2-|\pi/2-\varphi-\alpha_1|} d\alpha_3\ \phi^{(3)}_{s_c-1/2}(\rho,\alpha_3)\right. \\
&\qquad \left.\left. - C^{12}_{0,s_c+1/2}\int_{|\varphi-\alpha_1|}^{\pi/2-|\pi/2-\varphi-\alpha_1|} d\alpha_3\ \phi^{(3)}_{s_c+1/2}(\rho,\alpha_3)\right)\right],
\end{aligned} \tag{107}$$

$$\begin{aligned}
&\left[-\frac{\partial}{\partial\alpha_2^2} - (\tilde{\lambda}(\rho) - \rho^2 v^{\mp}_{fc}(\rho\sin\alpha_2))\right]\phi^{(2)}_{s_c\mp1/2}(\rho,\alpha_2) \\
&\quad = -\rho^2 v^{\mp}_{fc}(\rho\sin\alpha_2)\left[C^{12}_{0,s_c\mp1/2}\frac{1}{\sin(2\varphi)}\int_{|\varphi-\alpha_2|}^{\pi/2-|\pi/2-\varphi-\alpha_2|} d\alpha_1\ \phi^{(1)}_0(\rho,\alpha_1)\right. \\
&\qquad + \frac{1}{\sin(2\tilde{\varphi})}\left(C^{23}_{s_c\mp1/2,s_c-1/2}\int_{|\tilde{\varphi}-\alpha_2|}^{\pi/2-|\pi/2-\tilde{\varphi}-\alpha_2|} d\alpha_3\ \phi^{(3)}_{s_c-1/2}(\rho,\alpha_3)\right. \\
&\qquad \left.\left. + C^{23}_{s_c\mp1/2,s_c+1/2}\int_{|\tilde{\varphi}-\alpha_2|}^{\pi/2-|\pi/2-\tilde{\varphi}-\alpha_2|} d\alpha_3\ \phi^{(3)}_{s_c+1/2}(\rho,\alpha_3)\right)\right],
\end{aligned} \tag{108}$$

where the reduced potentials, $v_{ff} = 2mV_{ff}/\hbar^2, v^{\mp}_{fc} = 2mV^{\mp}_{fc}/\hbar^2$, are defined as in the previous sections. The two equations with the second derivative of α_3, analogous to those of Eq. (108), turn out to be identical to Eq. (108) due to the constraints in Eq. (106). We have then three independent Faddeev equations.

The angles φ and $\tilde{\varphi}$ obtained from Eqs. (12) and (13) are given explicitly by

$$\varphi = \arctan\left(\frac{M+2m}{M}\right)^{1/2}, \tag{109}$$

$$\tilde{\varphi} = \arctan\left(\frac{M(M+2m)}{m^2}\right)^{1/2}, \tag{110}$$

where $M = m_1$ is the mass of the core and $m = m_2 = m_3$ the mass of each of the two fermions.

We shall again use the square-well potentials for the radial shapes of the two-body potentials between both fermion and core and between the two fermions, i.e.

$$V_{ff}(r) = -V_0^{(ff)}\Theta(r < R_{ff}), \tag{111}$$

$$V_{fc}(r) = -V_0^{(fc)}\Theta(r < R_{fc}). \tag{112}$$

The related reduced potentials are then defined as

$$v_{ff}(x) = -v_0^{(ff)}\Theta(x < X_{ff}), \tag{113}$$

$$v_{fc}(x) = -v_0^{(fc)}\Theta(x < X_{fc}), \tag{114}$$

and the depth parameters for the two fermion-core relative spin states are

$$v_0^{(fc+)} = v_0^{(fc)}\left(1 + \gamma_s\frac{s_c}{2}\right), \tag{115}$$

$$v_0^{(fc-)} = v_0^{(fc)}\left(1 - \frac{\gamma_s(s_c+1)}{2}\right), \tag{116}$$

where $X_{fc} = R_2\mu_{13}$ and $X_{ff} = R_1\mu_{23}$ are defined as in Sect. 4.

In this section we shall also restrict ourselves to small or large distances and omit the lengthy but straightforward calculations and expressions for the less interesting intermediate distances.

5.1 Short-Distance Behaviour: $0 \le \rho \le \mathrm{Min}(X_{ff}, X_{fc})$

The potentials are again constants for all α-values and, as in Subsect. 4.1, Eqs. (107) and (108) have the general solutions

$$\phi_s^{(i)}(\rho, \alpha_i) = a_{s,i}\sin(2n\alpha_i)\,, \tag{117}$$

where the constants $a_{0,1}$, $a_{s_c\pm1/2,2}$, and $a_{s_c\pm1/2,3}$ are found by substitution of Eq. (114) into Eqs. (107) and (108). We obtain the following system of linear equations for

$$\begin{aligned}(\epsilon - v_0^{(ff)})a_{0,1} = & -v_0^{(ff)}\big[dC^{12}_{0,s_c-1/2}(a_{s_c-1/2,2} + a_{s_c-1/2,3}) \\ & + dC^{12}_{0,s_c+1/2}(a_{s_c+1/2,2} - a_{s_c+1/2,3})\big],\end{aligned} \tag{118}$$

$$\begin{aligned}(\epsilon - v_0^{(fc\mp)})a_{s_c\mp1/2,2} = & -v_0^{(fc\mp)}\big[dC^{12}_{0,s_c\mp1/2}a_{0,1} + \tilde{d}(C^{23}_{s_c\mp1/2,s_c-1/2}a_{s_c-1/2,3} \\ & + C^{23}_{s_c\mp1/2,s_c+1/2}a_{s_c+1/2,3})\big],\end{aligned} \tag{119}$$

where in analogy to Eq. (79) we have defined

$$\tilde{\lambda}(\rho) = 4n^2 - \rho^2\epsilon, \qquad d = -\frac{1}{n}\frac{\sin(2n\varphi)}{\sin(2\varphi)}, \qquad \tilde{d} = -\frac{1}{n}\frac{\sin(2n\tilde{\varphi})}{\sin(2\tilde{\varphi})}. \tag{120}$$

The antisymmetry expressed in Eq. (106) relates the unknown constants by

$$a_{s_c-1/2,3} = a_{s_c-1/2,2}, \qquad a_{s_c+1/2,3} = -a_{s_c+1/2,2}, \tag{121}$$

and by use of Eqs. (118) and (119) reduces the number of equations and the number of unknown constants to three.

The solutions for ϵ or $\tilde{\lambda}(\rho)$ are now obtained by demanding non-vanishing solutions for the constants $a_{s,i}$. This amounts effectively to three linear equations, and the corresponding determinant must vanish. Thus, ϵ is obtained from

$$\begin{vmatrix} \epsilon - v_0^{(ff)} & 2v_0^{(ff)} d C^{12}_{0,s_c-1/2} & 2v_0^{(ff)} d C^{12}_{0,s_c+1/2} \\ v_0^{(fc-)} d C^{12}_{0,s_c-1/2} & \epsilon - v_0^{(fc-)}(1 - \tilde{d} C^{23}_{s_c-1/2,s_c-1/2}) & -v_0^{(fc-)} \tilde{d} C^{23}_{s_c-1/2,s_c+1/2} \\ v_0^{(fc+)} d C^{12}_{0,s_c+1/2} & -v_0^{(fc+)} \tilde{d} C^{23}_{s_c-1/2,s_c+1/2} & \epsilon - v_0^{(fc+)}(1 + \tilde{d} C^{23}_{s_c-1/2,s_c-1/2}) \end{vmatrix} = 0\ . \tag{122}$$

This means that there are at most three λ-solutions for each n; sometimes less than three, since some of them can be the trivial spurious solutions characterized by $\epsilon = 0$.

If the determinant in Eq. (122) vanishes for $\epsilon = 0$, we find that these spurious solutions must satisfy one or both of the following two conditions:

$$n - \frac{\sin(4n\varphi)}{\sin(4\varphi)} = 0\ , \tag{123}$$

$$n^2 + \frac{\sin(4n\varphi)}{\sin(4\varphi)} n - 2\frac{\sin^2(2n\varphi)}{\sin^2(2\varphi)} = 0. \tag{124}$$

For $n = 1$ ($d = \tilde{d} = -1$) these two equations are both satisfied, implying that two of the solutions are spurious, and only one antisymmetric solution exists. Solving the determinant in Eq. (122) the antisymmetric solution is found to be

$$\tilde{\lambda}(\rho) = \lambda(\rho) + 4 = 4 - \rho^2\left(v_0^{(ff)} + v_0^{(fc-)}\frac{2s_c}{2s_c+1} + v_0^{(fc+)}\frac{2s_c+2}{2s_c+1}\right), \tag{125}$$

which reduces to Eq. (78) for identical and spin-independent potentials.

For $n = 2$, where $2d^2 = 1 - \tilde{d}$, only the condition in Eq. (124) is satisfied, and two non-spurious antisymmetric solutions appear. They are the solutions of the second-order equation

$$\begin{aligned} &\epsilon^2 - \epsilon\left(v_0^{(ff)} + v_0^{(fc+)} + v_0^{(fc-)} + \frac{\tilde{d}}{2s_c+1}(v_0^{(fc-)} - v_0^{(fc+)})\right) \\ &\quad + (1+\tilde{d})\left(v_0^{(ff)} v_0^{(fc-)}\frac{s_c+1}{2s_c+1} + v_0^{(ff)} v_0^{(fc+)}\frac{s_c}{2s_c+1} + v_0^{(fc+)} v_0^{(fc-)}(1-\tilde{d})\right) = 0, \end{aligned} \tag{126}$$

which combined with Eq. (120) results in two solutions for $\tilde{\lambda}(\rho)$.

For $n \geq 3$ none of the conditions (123) and (124) is satisfied, and three non-spurious antisymmetric solutions are found. For very big values of n, both d and $\tilde{d}$ approach zero, and the three solutions of the determinant converge towards $\epsilon = v_0^{(ff)}$, $\epsilon = v_0^{(fc-)}$, and $\epsilon = v_0^{(fc+)}$.

5.2 Large-Distance Behaviour

As in Eqs. (30) and (75), we define

$$\alpha_0^{(ff)} = \arcsin(X_{ff}/\rho), \tag{127}$$

$$\alpha_0^{(fc)} = \arcsin(X_{fc}/\rho), \tag{128}$$

such that the potential is non-vanishing only when the corresponding $\alpha \leq \alpha_0$.

Since $\alpha_0^{(ff)}$ and $\alpha_0^{(fc)}$ both approach zero for increasing ρ, we can define large distances by

$$\alpha_0^{(ff)} \leq \alpha_0^{(fc)} \leq |\varphi - \alpha_0^{(ff)}| \leq |\varphi - \alpha_0^{(fc)}|, \tag{129}$$

$$\alpha_0^{(ff)} \leq \alpha_0^{(fc)} \leq |\tilde{\varphi} - \alpha_0^{(ff)}| \leq |\tilde{\varphi} - \alpha_0^{(fc)}|, \tag{130}$$

where we have assumed that $X_{ff} < X_{fc}$.

In Fig. 4 we plot $\alpha_{fc} = \alpha_2$ versus $\alpha_{ff} = \alpha_1$, where only the region $\alpha_{ff} < \alpha_{fc}$ is relevant for our calculation. In the plot we can distinguish four different zones:

$$\text{II} : \alpha_{fc} > \alpha_0^{(fc)} \text{ and } \alpha_{ff} > \alpha_0^{(ff)} \Longrightarrow V_{fc}(\rho \sin\alpha_{fc}) = 0 \text{ and } V_{ff}(\rho \sin\alpha_{ff}) = 0,$$

$$B : \alpha_{fc} > \alpha_0^{(fc)} \text{ and } \alpha_{ff} < \alpha_0^{(ff)} \Longrightarrow V_{fc}(\rho \sin\alpha_{fc}) = 0 \text{ and } V_{ff}(\rho \sin\alpha_{ff}) \neq 0,$$

$$C : \alpha_{fc} < \alpha_0^{(fc)} \text{ and } \alpha_{ff} > \alpha_0^{(ff)} \Longrightarrow V_{fc}(\rho \sin\alpha_{fc}) \neq 0 \text{ and } V_{ff}(\rho \sin\alpha_{ff}) = 0,$$

$$D : \alpha_{fc} < \alpha_0^{(fc)} \text{ and } \alpha_{ff} < \alpha_0^{(ff)} \Longrightarrow V_{fc}(\rho \sin\alpha_{fc}) \neq 0 \text{ and } V_{ff}(\rho \sin\alpha_{ff}) \neq 0.$$

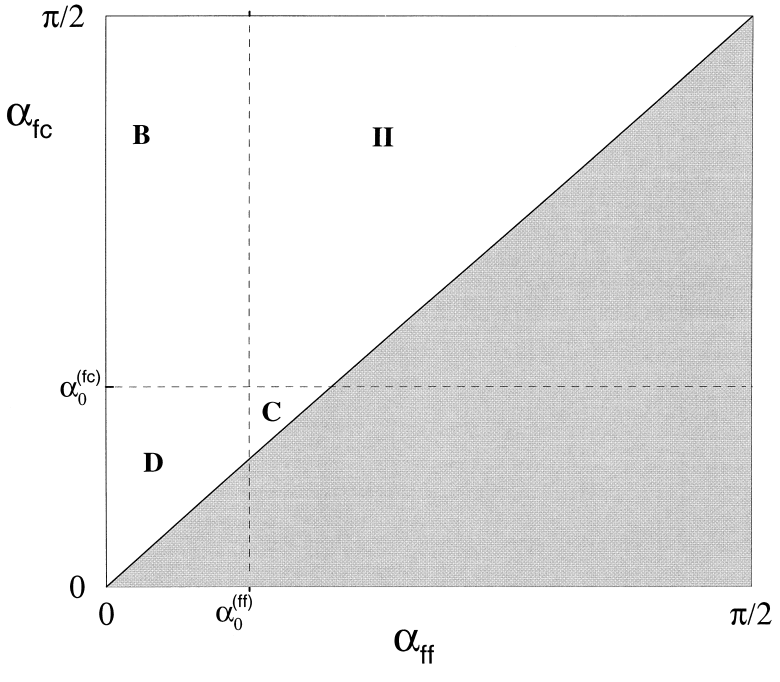

Fig. 4. The regions II, *B*, *C*, and *D* as defined in Subsect. 5.2 in $(\alpha_{fc}, \alpha_{ff})$-space arising for large distances in case of two identical spin-$\frac{1}{2}$ particles. The shaded area does not enter into the present computation

Since $\alpha_0^{(ff)}$ and $\alpha_0^{(fc)}$ are close to zero, the regions B, C, and D are very small. Due to the inequalities in Eqs. (129) and (130), the integrations appearing in the Faddeev equations involve only the functions in the region where the potentials vanish.

The wave functions must vanish at $\alpha = 0$ and $\alpha = \pi/2$, and the solutions to Eqs. (107) and (108) in region II, where all potentials are zero, are therefore of the form

$$\phi_s^{(i)}(\alpha_i) = b_{s,i} \sin\left((\alpha_i - \pi/2)\sqrt{\tilde{\lambda}}\right) . \tag{131}$$

In the other regions (B, C, and D) the same form of the wave function is a solution if the corresponding potentials vanish, i.e. for $\alpha_i > \alpha_0^{(i)}$. If $\alpha_i < \alpha_0^{(i)}$ in these regions, we have solutions of the form

$$\phi_s^{(i)}(\alpha_i) = c_{s,i} \sin(\alpha_i \kappa_s) + d_{s,i} \sin\left(\alpha_i \sqrt{\tilde{\lambda}}\right), \tag{132}$$

where the first term is the solution to the homogeneous equation and consequently the different κ_s-values are given by

$$\kappa_0 = \sqrt{v_0^{(ff)}\rho^2 + \tilde{\lambda}(\rho)}, \qquad \kappa_{s_c \pm 1/2} = \sqrt{v_0^{(fc\pm)}\rho^2 + \tilde{\lambda}(\rho)} . \tag{133}$$

The constants $c_{s,i}$ are therefore completely free, since the functions to be integrated in Eqs. (107) and (108) only involve the functions in Eq. (131), i.e. values of $\alpha_i > \alpha_0^{(i)}$ for all three components (see Fig. 4). The constants $b_{s,i}$ and $d_{s,i}$ are related linearly through the Faddeev equations.

The detailed solutions in the different regions are given in Appendix F. The eigenvalue equation again takes the form of a vanishing determinant where the matrix elements are given in Eq. (F.13).

In the limit of very large distances we can for finite $\tilde{\lambda}$-values use an expansion to lowest order in $1/\rho$ as given in Appendix F, where also the case of diverging $\tilde{\lambda}$-values is considered. The linear set of equations (F.12) then reduces to

$$A_1\left(\sqrt{\tilde{\lambda}}\cos\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\frac{a_{\rm scat}^{(ff)}}{\rho}\mu_{23} + \sin\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\right) = a_0\sqrt{\tilde{\lambda}}\frac{a_{\rm scat}^{(ff)}}{\rho}\mu_{23}, \tag{134}$$

$$A_2\left(\sqrt{\tilde{\lambda}}\cos\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\frac{a_{\rm scat}^{(fc-)}}{\rho}\mu_{12} + \sin\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\right) = a_{s_c-1/2}\sqrt{\tilde{\lambda}}\frac{a_{\rm scat}^{(fc-)}}{\rho}\mu_{12}, \tag{135}$$

$$A_3\left(\sqrt{\tilde{\lambda}}\cos\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\frac{a_{\rm scat}^{(fc+)}}{\rho}\mu_{12} + \sin\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\right) = a_{s_c+1/2}\sqrt{\tilde{\lambda}}\frac{a_{\rm scat}^{(fc+)}}{\rho}\mu_{12}, \tag{136}$$

where μ_{12} and μ_{23} are constants defined in Appendix A, and $a_0, a_{s_c\pm1/2}$ depend linearly on the coefficients A_i as defined in Appendix F. The eigenvalues for $\rho \to \infty$ for finite scattering lengths therefore approach solutions to $\sin((\pi/2)\sqrt{\tilde{\lambda}}) = 0$, i.e. the hyperspherical spectrum of $\tilde{\lambda} = 4n^2$.

The solutions to Eqs. (134)–(136) to the next order in $1/\rho$ only exist if the corresponding determinant is zero, what then defines the behaviour of $\tilde{\lambda}(\rho)$ for large values of ρ. We have previously assumed that $\alpha_0^{(ff)} < \alpha_0^{(fc)}$ and both quantities are close to zero in the large-distance limit. If we further assume that $\alpha_0^{(ff)} = 0$ or equivalently $X_{ff} = 0$, only the regions II and C survive in Fig. 4. Then $A_1 = 0$, $|a_{\rm scat}^{(ff)}|/\rho \approx 0$ (for finite

$a_{\rm scat}^{(ff)}$ and large ρ) and the fermion-fermion interaction completely disappears. The determinant is reduced to a 2-by-2 determinant that gives the following equation for $\tilde{\lambda}(\rho)$

$$\frac{\rho^2}{a_{\rm scat}^{(fc+)} a_{\rm scat}^{(fc-)}} \sin^2\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right) + \frac{1}{2}\sqrt{\tilde{\lambda}}\sin(\pi\sqrt{\tilde{\lambda}})\mu_{12}\left(\frac{\rho}{a_{\rm scat}^{(fc+)}} + \frac{\rho}{a_{\rm scat}^{(fc-)}}\right)$$
$$+ \frac{2}{2s_c+1}\left(\frac{\rho}{a_{\rm scat}^{(fc-)}} - \frac{\rho}{a_{\rm scat}^{(fc+)}}\right)\tilde{f}\sin\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)$$
$$+ \tilde{\lambda}\cos^2\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\mu_{12}^2 - 4\mu_{12}^2 f^2 = 0\,, \tag{137}$$

where f and $\tilde{f}$ are defined in Eq. (F.5).

In the limit where both scattering lengths are large compared to ρ, i.e. $|a_{\rm scat}^{(fc\pm)}| \gg \rho$, we arrive again at Eq. (99). The occurrence conditions and some properties of the Efimov states can be derived from Eq. (137). The details are discussed in ref. [34].

The behaviour of $P_{nn'}$ and $Q_{nn'}$ at large distances is qualitatively the same as for three spinless particles. The results are again, along with the various solutions described in this subsection, valid in general for all short-range potentials. In particular, also the Efimov conditions remain the same for such potentials.

6 Numerical Illustrations and Generalizations

The analytical solutions in the previous sections are derived for square-well potentials. The results are similar for other short-range potentials. Illustrations by use of smoother Gaussian potentials are therefore appropriate, and we shall first show numerically calculated angular eigenvalue spectra for different symmetries of the three-body system. Then we shall give a survey of the analytical results for different distances and show how the results can be used for arbitrary short-range potentials. The simplest case of intrinsic spins of the particles is also considered in the previous sections. Similarly, other cases with more complicated spin structures can be worked out. It is straightforward but may easily result in rather extended formulae. Here we shall indicate how to proceed in the case of two different spin-$\frac{1}{2}$ particles.

6.1 Numerical Illustrations

The angular eigenvalues, still for *s*-states only, in the symmetric case of three identical bosons are shown in Fig. 5 as functions of ρ/b for a Gaussian potential. The parameters of the potential result in a two-body bound state. The lowest eigenvalue consequently bends over and diverges parabolically as described in Eq. (56). The higher-lying levels then come down and the hyperspherical spectrum seen at $\rho = 0$ is approached at large distances. The values of $\lambda = 4n^2 - 4$ at $\rho = 0$ are 0, 32, 60 and the spurious state starting from $\lambda = 12$, corresponding to $n = 2$ in Eq. (29), is omitted. Due to the symmetry requirement only one state appears for each value of n.

In Fig. 6 we show the angular eigenvalues as functions of ρ/b, again for *s*-states only, in the asymmetric case of three different spinless particles interacting via Gaussian potentials. Only one of the potentials has a bound two-body state, and consequently the

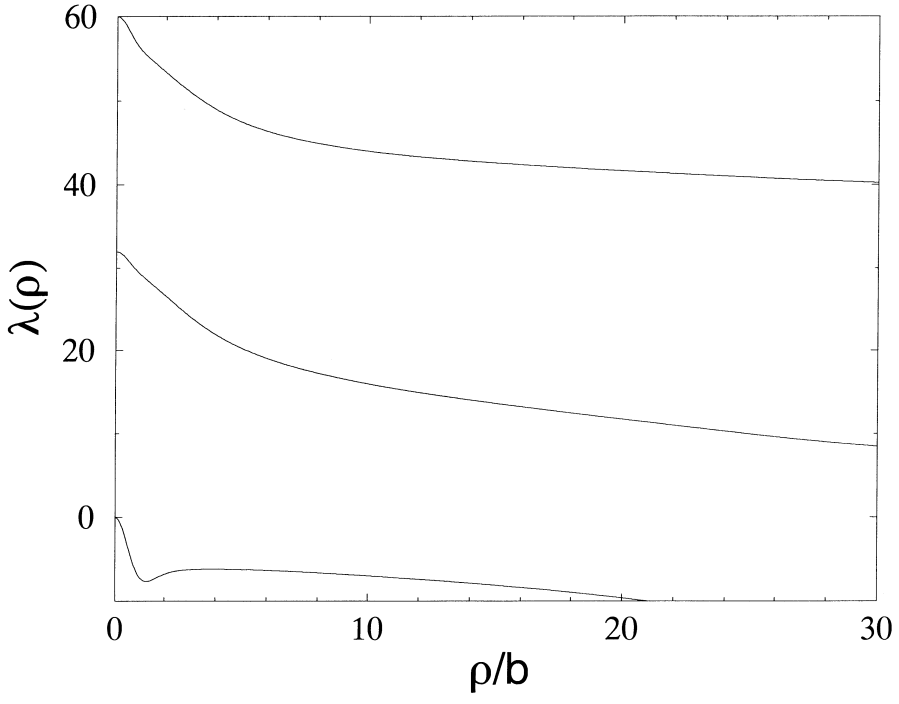

Fig. 5. The lowest angular eigenvalues $\lambda = \tilde{\lambda} - 4$, in case of all orbital angular momenta zero, as functions of ρ/b for three identical bosons interacting via a Gaussian potential, $\bar{V}_0\hbar^2/mb^2 \exp(-(r/b)^2)$, with m being the mass of the bosons. The actual value of $\bar{V}_0 = -3.08$ corresponds to a bound two-body state

lowest eigenvalue bends over and diverges parabolically as described in Eq. (101). Again the higher-lying levels come down and the spectrum at $\rho = 0$ emerges also in this case at large distances. The spurious level corresponding to $n = 2$ is again omitted, but now two other levels appear at $\lambda = 12$ corresponding to (spatially) non-symmetric configurations. Also, for $n = 3$, at $\lambda = 32$ two more levels corresponding to non-symmetric states appear in addition to the totally symmetric state. For all values of $n \geq 3$ the structures of one symmetric and two asymmetric states remain unchanged.

In Fig. 7 we show analogous numerical results for two identical spin-$\frac{1}{2}$ fermions plus one third particle (core) of spin $s_c = \frac{3}{2}$. Now one additional degree of freedom appears. It is related to the two different fermion-core spin couplings. However, simultaneously the two fermions are restricted to totally antisymmetric two-body states. This clearly removes some of the otherwise possible states. Since the hyperspherical spectrum is always both the starting point at $\rho = 0$ and the asymptotic limit for large ρ, the combined result is a spectrum very similar to that of Fig. 6 corresponding to three different particles without spin degrees of freedom. If the core had been spinless, the fermion-core spin coupling would have been unique and one state less would have appeared for every value of $n \geq 2$.

6.2 Survey and Generalization of the Solutions

Different analytical *s*-state solutions to the Faddeev equations for square-well potentials are obtained depending on the region in the two-dimensional (ρ, α) coordinate space. A

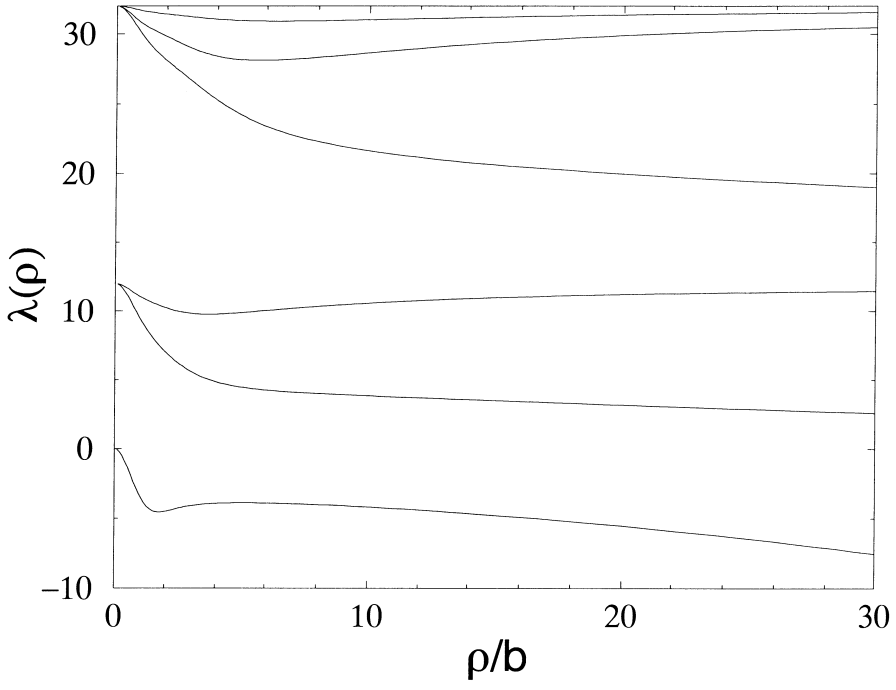

Fig. 6. The lowest angular eigenvalues $\lambda = \tilde{\lambda} - 4$, in case of all orbital angular momenta zero, as functions of ρ/b for three different spinless particles interacting via Gaussian potentials, $\bar{V}_0^{(i)}\hbar^2/m_i b^2 \exp(-(r/b)^2)$. The actual values $\bar{V}_0^{(i)} = -1.28, -1.54, -3.08$ and $m_i = m_1, 2m_1, 3m_1$ for $i = 1, 2, 3$, respectively, correspond to a bound two-body state in the subsystem labelled $i = 1$ and no bound states in the other subsystems

survey of the symmetric case is shown in Fig. 8, where the regions labelled I, *A*, *B*, *C* and II refer to the division in the previous subsections. (The more general asymmetric case requires more divisions at intermediate distances. However, the picture basically remains unchanged.) The simplest cases of small and large distances (I and II) actually constitute a very large fraction of the total coordinate space. These solutions are also interesting and useful for general short-range potentials, where approximate solutions can be obtained from specific perturbative treatments.

At large distances, where α_0 and κ can be expanded to first order in $1/\rho$, we may obtain considerable further simplifications. In this limit the eigenvalue equation can be expressed entirely in terms of the scattering length of the potential. In Fig. 9 we show the lowest angular eigenvalue for a square-well potential as a function of ρ. We compare with the approximate solution at very large distances given in Eq. (55). The $1/\rho$ asymptotic convergence towards the limit is clearly seen. This approximation is not accurate enough for the interesting shorter distances.

A numerical procedure to treat the long distance behaviour for general short-range potentials now suggests itself. First we construct the equivalent square-well potential with the same scattering length and effective range as the potential in question. The solutions are then obtained in α-space as described in this paper. They can be expected to be very accurate solutions outside the range of the potentials. At smaller distances direct numerical methods must be used. If a high accuracy at very large distance is

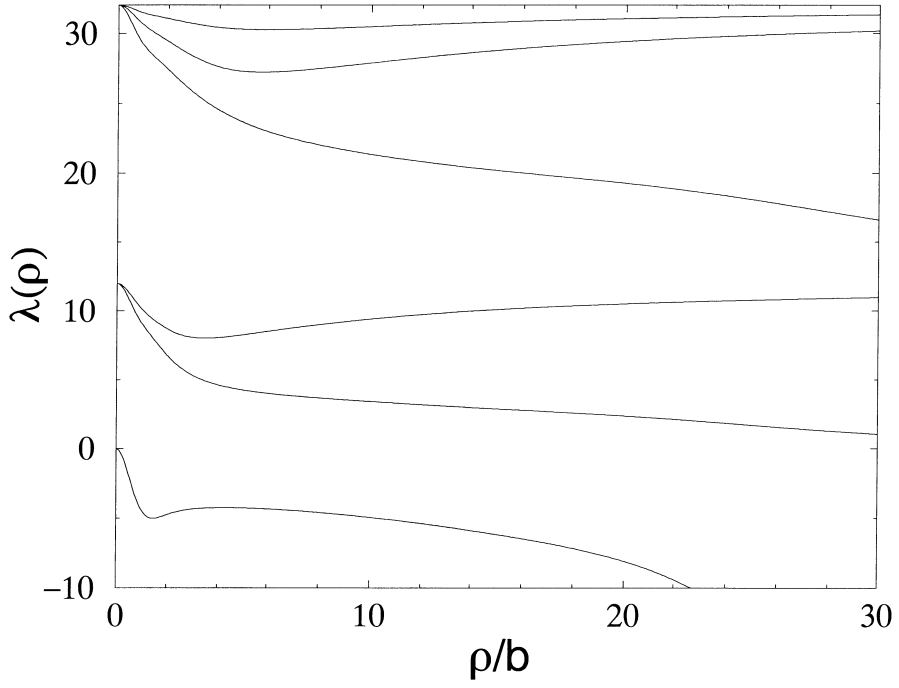

Fig. 7. The lowest angular eigenvalues $\lambda = \tilde{\lambda} - 4$, in case of all orbital angular momenta zero, as functions of ρ/b for a system of two identical spin-$\frac{1}{2}$ fermions plus a third particle of spin $s_c = \frac{3}{2}$ interacting via Gaussian potentials, $\bar{V}_0^{(i)}\hbar^2/m_i b^2 \exp(-(r/b)^2)$. The actual values $\bar{V}_0^{(i)} = -6.16, -0.99, -1.09$ and $m_i = 2m_2, m_2, m_2$ for $i = 1, 2, 3$, respectively, correspond to the fermion-fermion interaction, the fermion-core interaction for relative spin of 2, and the fermion-core interaction for relative spin of 1

needed, as for the Efimov effect, this or similar implementation of analytical knowledge is probably unavoidable. Whenever angular momenta higher than zero contribute (what is normally the case), the large-distance behaviour is nevertheless governed by *s*-states and the procedure can still be useful. The gain in accuracy and speed might be substantial.

An example is shown in Fig. 9 where we used an attractive Gaussian potential of range b and strength $-2.19\hbar^2/mb^2$ with a scattering length, $a_{\rm scat} = 5b$, and an effective range, $R_{\rm eff} = 1.64b$, with b being the range of the potential. The corresponding radius and depth of the square-well potential are then $R_0 = 1.47b$ and $V_0 = -0.93\hbar^2/mb^2$, respectively.

The curves for the two potentials are indistinguishable at large distances down to $\rho = R_0\sqrt{2} \approx 2b$. Then the square-well result is lowest until $\rho \approx b$, where the curves cross and the Gaussian result stays as the lowest all the way down to zero. The largest deviation is less than about 0.75 compared to the minimum value of about -6. This relatively small region, $0.3 \leq \rho/b \leq 2$, at intermediate distances, corresponds to the surface region for the two-body potential. The perturbative short-distance solution for the Gaussian potential is accurate up to about $\rho \approx 0.3b$. The large-distance behaviour describing the approach towards the asymptotic limit is only a reasonable approximation at very large distances.

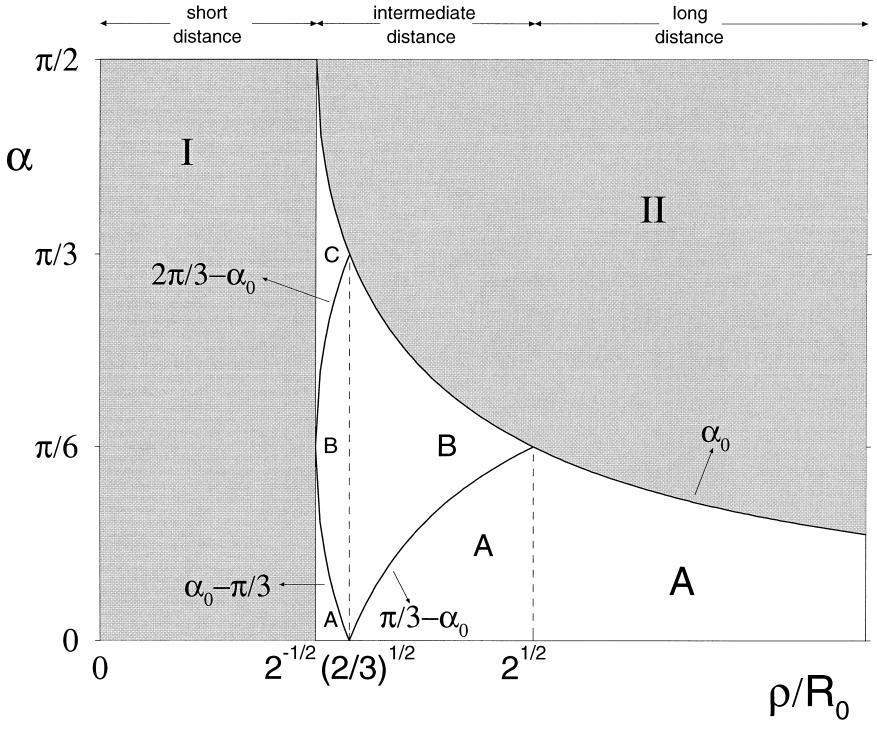

Fig. 8. The different regions in α-space as functions of ρ/R_0 for a square-well potential of radius R_0 in case of three identical bosons. The shaded areas indicate the small- and large-distance regions I and II, where the wave functions are particularly simple

The exact square-well solution is a good approximation almost quantitatively. The two energies would be close due to the similarities and the fact that a larger potential at smaller distance is compensated by a smaller potential at larger distance. The square well tends to confine the wave function in a somewhat narrower region around the minimum.

The (smaller) differences between the two potentials occur in an essential region, where a substantial part of the attractive pocket of the effective three-body potential is contained. High accuracy for an arbitrary potential therefore requires a treatment better than that corresponding to the square-well solution. In practice the most efficient procedure is numerical integration starting with the small-distance perturbative solution. Reaching large distances corresponding to $\rho \leq R_0\sqrt{2}$ the exact s-state square-well solution, which then is particularly simple, can be used to a very high accuracy.

6.3 Two Different Spin-$\frac{1}{2}$ Particles

We now consider a system consisting of two non-identical spin-$\frac{1}{2}$ particles and a third particle of arbitrary spin s_c. The procedure is then the same as in Sect. 5, but now the spin state $\chi_1^{(1)}$ is allowed and the antisymmetrization constraint Eq. (106) is not required. The three Faddeev equations can then be written as a system of six equations, where the angles φ_1, φ_2, and φ_3 all are different. Since the two fermions can couple to spin 1, the total spin of the three-body system can be $J = s_c, s_c \pm 1$. In the case when $J = s_c \pm 1$,

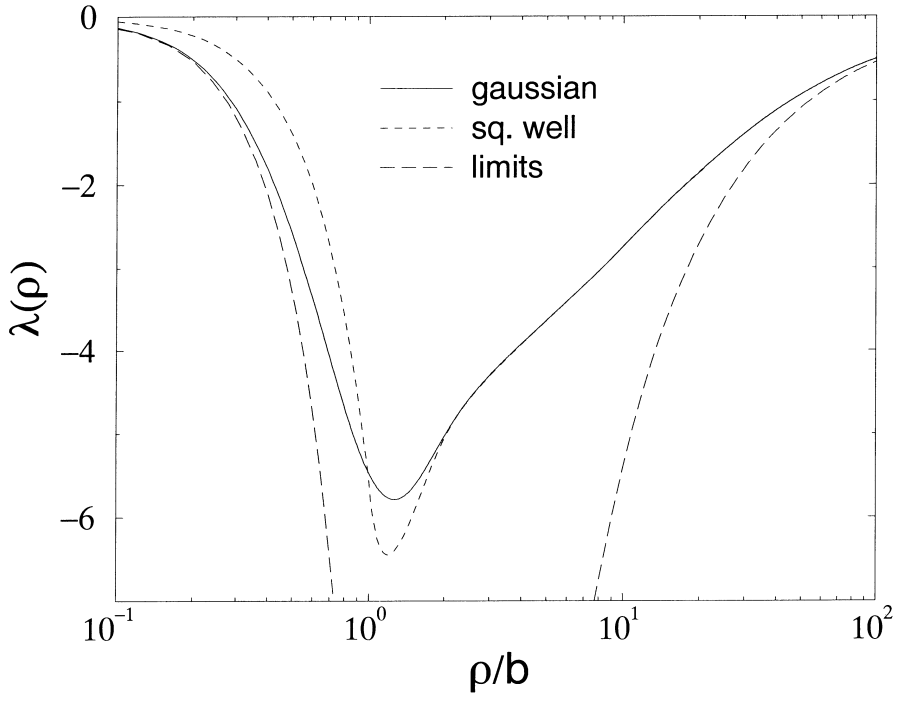

Fig. 9. The lowest angular eigenvalue $\lambda = \tilde{\lambda} - 4$ as functions of ρ/b for a system of three identical bosons. The interactions correspond to a Gaussian potential (solid curve), $\bar{V_0}\hbar^2/mb^2 \exp(-(r/b)^2)$, $\bar{V_0} = -2.19$, and a square-well potential (short-dashed curve), radius $R_0 = 1.47b$ and depth $-0.93\hbar^2/mb^2$, with the same scattering length $a_{\rm scat} = 5b$, and effective range $R_{\rm eff} = 1.64b$. The perturbative solution for the Gaussian potential at small distance (Eq. (29)) and the limiting behaviour at large distance (Eq. (55)) are also shown (long-dashed curves)

the spin states $\chi_0^{(1)}$, $\chi^{(2)}_{s_c \mp 1/2}$, and $\chi^{(3)}_{s_c \mp 1/2}$ are not possible and the total wave function is symmetric under exchange of the two fermions, and the number of Faddeev equations is again reduced to three.

At short distances the general solutions are given by Eq. (117). They give a 6 by 6 determinant when $J = s_c$ and a 3 by 3 determinant when $J = s_c \pm 1$. Making these determinants equal to zero and solving the equation for $\epsilon = 0$ one can extract the spurious solutions and the non-spurious ones for $\epsilon \neq 0$.

At large distances the procedure is also analogous to the one presented in Subsect. 5.2. The only difference is that now we have $\alpha_0^{(f_1 f_2)}$, $\alpha_0^{(f_1 c)}$, and $\alpha_0^{(f_2 c)}$, and the number of regions we have to consider (similar to regions II, *A*, *B*, and *C* in Subsect. 5.2) is much larger. Technically the problem is more cumbersome, but conceptually it is identical.

7 Summary and Conclusions

The method exploited in the present paper to solve the Faddeev equations in coordinate space is especially useful in the description of the long-distance behaviour. The reason is the access to the asymptotic analytic properties made possible by the two-step sequential procedure. The generalized angular equations are first solved for each average distance (hyperradius) between the three particles. The periodic behaviour or

the finite intervals characteristic for angular variables provide discrete quantized solutions that are used subsequently as a complete basis set in an expansion of the total wave function. The expansion coefficients (as well as the basis) are functions of the radial coordinate. They are in the next step determined together with the total energy from the coupled set of radial equations.

The method is based on the expansion on the adiabatic complete basis which is most efficient for the ground state and the lowest-lying excited states (including continuum states) with relatively small angular momenta. We have in this paper first formulated the general procedure for *s*-states with the inclusion of intrinsic spins for each of the three particles. For spin-independent interactions we recover the previous general *s*-state Faddeev equations.

We study the symmetric case described by three equal Faddeev components. We assume spin-independent interactions and arrive at one integro-differential angular Faddeev equation for *s*-states. This equation is solved analytically for a square-well potential. The angular wave functions are especially simple at small (one sine function) and large distances (one or two sine functions). These solutions are, with an appropriate interpretation, more general than their origin as square-well solutions seems to suggest. At small distance, they are the first-order perturbative solutions for an arbitrary potential with a value at the centre equal to the square-well depth. The corresponding eigenvalues are explicitly given at small distance where they vary parabolically with distance starting from the hyperspherical spectrum.

At intermediate distances, still for the square-well, the solutions are linear combinations of up to four sine functions of varying arguments. The eigenvalues are solutions to cubic equations, and a simple transcendental equation involving trigonometric functions arises at large distances. At sufficiently large distances where the general short-range potential has vanished the solution obviously is identical to the square-well solutions outside its radius. We show explicitly that the coupled set of radial equations decouple at large distances.

The length scale is defined by the radius of the square-well potential and the large-distance solution is valid when the average radius exceeds $R_0\sqrt{2}$. Expansion of the eigenvalue equation to lowest order in the inverse distance brings additional simplification, but introduces also the two-body scattering length as another length parameter. The Efimov condition leading to infinitely many bound three-body states is then obtained and seen to be independent of the short-range potential.

The asymmetric case, still for *s*-states and spin-independent interactions, is described by the three Faddeev equations. We can again solve the angular part analytically for square-well potentials acting betweeen each pair of particles. Each Faddeev component of the wave functions is at small and large distances of the same form as in the symmetric case. These solutions are again appropriately interpreted solutions to general potentials either in perturbation theory or asymptotically at large distances. The corresponding eigenvalues are in this case obtained either as solutions to a cubic equation or to a transcendental equation involving trigonometric functions. At intermediate distances the solutions can still be found analytically to be combinations of exponentials and trigonometric functions. However, the number of terms is now substantial and we refrain from writing down these solutions.

Expanding the eigenvalue equation to first order in the inverse distance again brings additional simplifications and introduces the three (different) scattering lengths. The

Efimov conditions leading to infinitely many bound three-body states is then seen to exist if at least two scattering lengths are infinitely large. Asymptotic large-distance behaviours of the effective radial potentials are derived to first order in the inverse distance.

The case of two identical spin-$\frac{1}{2}$ fermions and one additional core particle of arbitrary spin can of course also be described by three Faddeev equations. We still include only *s*-states and assume a spin-spin type of interaction between fermion and core particles. Elimination of the spin degrees of freedom results in three angular Faddeev equations where the reduction from six to three equations arises due to the requirement of an antisymmetric wave function. Again these equations are solved for radial square-well potentials both for small and large distances. Expansion to first order in the inverse distance expresses the angular eigenvalue equation in terms of three (different) scattering lengths. The Efimov conditions are described and discussed.

We finally considered qualitatively the case of two different spin-$\frac{1}{2}$ particles interacting mutually and with a third particle. The original three Faddeev equations result now, still only including *s*-states, in six coupled angular equations after elimination of the spin degrees of freedom. Small and large-distance behaviours can be studied analogously for radial square-well potentials. The systems considered here containing spin-$\frac{1}{2}$ particles are chosen as the simplest examples of three interacting particles with an intrinsic structure showing up in the form of spin degrees of freedom. Other similar examples can be worked out in the same way.

The angular solutions obtained with square-well potentials have properties characteristic for short-range potentials. To calculate the total solution we must in addition solve the coupled set of radial differential equations. This is a rather simple numerical problem. The coupled set of radial equations decouple at large distances. Only *s*-waves are included in the discussion, but at large distances they are often the most interesting components in the wave functions. For bound states only *s*-waves extend far beyond the radii of the short-range potentials. They decouple from higher angular-momentum components at large distances and often constitute the dominating part of the total wave function.

The square-well solutions are intrinsically interesting. Furthermore, they approach at large distance the solutions for arbitrary short-range potentials provided the scattering lengths and effective ranges are identical. The general solutions are rather accurately reproduced down to distances close to the radius of the square-well potentials. Approximate solutions for general potentials can therefore be found by adjusting depth and radius of a square-well potential to reproduce the scattering length and effective range of the potential in question. The large-distance solutions, as defined in this paper, are then correct down to $\sqrt{2}$ times the radius of the square-well potential, i.e. at much smaller distances than given by the large-distance expansion for general potentials. These results can therefore probably be used to improve both speed and accuracy of the numerical computations.

In this work we have considered short-range potentials and confined the solutions to *s*-states only. They decouple for many cases, at least in a qualitative description from other angular momentum components and often in addition even quantitatively at large distances. The derivations might be extended to include higher angular momenta. In any case the couplings between different *l*-values would result in many additional terms in the equations. The analytical expressions would by far be more lengthy than the present ones and therefore impractical.

On the other hand, *s*-states alone provide the qualitative and sometimes even the quantitative understanding of some very loosely bound quantum systems. Then the wave function is situated outside the range of the potentials and the details of the potentials are unimportant. Only low-energy scattering properties like scattering length and effective range are needed in the description. Efimov states, nuclear halos and Borromean systems in general are very good examples of applications.

In conclusion, we have discussed a method and derived *s*-state square-well solutions providing basic insight into the Faddeev equations in coordinate space. Practical numerical improvements are suggested for arbitrary short-range potentials. The present work may be considered as a detailed description and a continuation of the procedure suggested in ref. [1]. We actually solve analytically the angular part of the *s*-state Faddeev equations for square-well potentials. We generalize to asymmetric systems with intrinsic spins. We derive a series of detailed properties of the solutions and focus especially on small and large distances where they are particularly useful.

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Appendix A. Coordinates

We consider a system of three particles with masses m_i and coordinates $\mathbf{r}_i$. The Jacobi coordinates are defined as [2, 17]:

$$\mathbf{x}_i = \mu_{jk}\mathbf{r}_{jk}, \qquad \mathbf{y}_i = \mu_{(jk)i}\mathbf{r}_{(jk)i},$$
$$\mu_{jk} = \left(\frac{1}{m}\,\frac{m_j m_k}{m_j + m_k}\right)^{1/2}, \qquad \mu_{(jk)i} = \left(\frac{1}{m}\,\frac{(m_j + m_k)m_i}{m_1 + m_2 + m_3}\right)^{1/2},$$
$$\mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k, \qquad \mathbf{r}_{(jk)i} = \frac{m_j\mathbf{r}_j + m_k\mathbf{r}_k}{m_j + m_k} - \mathbf{r}_i, \tag{A.1}$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$ and μ^2 are the reduced masses of the subsystems in units of an arbitrary normalization m.

The hyperspherical variables are introduced as

$$\rho, \quad \mathbf{n}_{x_i} = \mathbf{x}_i/|\mathbf{x}_i|, \quad \mathbf{n}_{y_i} = \mathbf{y}_i/|\mathbf{y}_i|, \quad \alpha_i, \tag{A.2}$$

where α_i is from the interval $[0, \pi/2]$, and

$$\rho^2 = \mathbf{x}_i^2 + \mathbf{y}_i^2, \qquad |\mathbf{x}_i| = \rho\sin\alpha_i, \qquad |\mathbf{y}_i| = \rho\cos\alpha_i\ . \tag{A.3}$$

We omit the indices where we need not emphasize the particular set of Jacobi coordinates. Note that ρ is independent of what set is used.

The relation between three different sets of Jacobi coordinates is given by

$$\mathbf{x}_k = \mathbf{x}_i\cos\varphi_{ik} + \mathbf{y}_i\sin\varphi_{ik}, \qquad \mathbf{y}_k = -\mathbf{x}_i\sin\varphi_{ik} + \mathbf{y}_i\cos\varphi_{ik}, \tag{A.4}$$

where the transformation angle φ_{ik} is given by the masses as

$$\varphi_{ik} = \arctan\left((-1)^p\sqrt{\frac{m_j(m_1 + m_2 + m_3)}{m_k m_i}}\right) \tag{A.5}$$

with $(-1)^p$ the parity of the permutation $\{i, k, j\}$.

Appendix B. *B*-Region Solution at Intermediate Distances

If the wave function in Eq. (36) is a solution to Eq. (35), we must have that

$$B_+ \exp\left[i\sqrt{\tilde{\lambda}}\pi/3\right] + \frac{4}{i\sqrt{3\tilde{\lambda}}} B_-^{\mathrm{II}} = \frac{2A_{\mathrm{II}}}{\sqrt{3\tilde{\lambda}}} \exp\left[i\sqrt{\tilde{\lambda}}\pi/6\right], \tag{B.1}$$

$$\frac{4}{i\sqrt{3\tilde{\lambda}}} B_+^{\mathrm{II}} \exp\left[i\sqrt{\tilde{\lambda}}\pi/3\right] - B_-^{\mathrm{II}} = -\frac{2A_{\mathrm{II}}}{\sqrt{3\tilde{\lambda}}} \exp\left[i\sqrt{\tilde{\lambda}}\pi/6\right], \tag{B.2}$$

$$((\kappa_B^{(k)})^2 + v_0\rho^2 + \tilde{\lambda}) B_+^{(k)} \exp\left[\kappa_B^{(k)}\pi/3\right] + \frac{4}{\sqrt{3}} \frac{v_0\rho^2}{\kappa_B^{(k)}} B_-^{(k)} = 0, \tag{B.3}$$

$$\frac{4}{\sqrt{3}} \frac{v_0\rho^2}{\kappa_B^{(k)}} B_+^{(k)} \exp\left[\kappa_B^{(k)}\pi/3\right] - ((\kappa_B^{(k)})^2 + v_0\rho^2 + \tilde{\lambda}) B_-^{(k)} = 0 \tag{B.4}$$

for each of the values of $k = 1, 2, 3$ and furthermore

$$\begin{aligned} &\frac{A_{\mathrm{II}}}{2\sqrt{\tilde{\lambda}}} \left(\exp\left[i\sqrt{\tilde{\lambda}}(\alpha_0 - \pi/2)\right] + \exp\left[-i\sqrt{\tilde{\lambda}}(\alpha_0 - \pi/2)\right) \right] \\ &\quad + \frac{1}{i\sqrt{\tilde{\lambda}}} \left(B_+^{\mathrm{II}} \exp\left[i\sqrt{\tilde{\lambda}}(2\pi/3 - \alpha_0)\right] - B_-^{\mathrm{II}} \exp\left[-i\sqrt{\tilde{\lambda}}(2\pi/3 - \alpha_0)\right] \right. \\ &\quad + \sum_{k=1}^{3} \left[\frac{1}{\kappa_B^{(k)}} \left(B_+^{(k)} \exp\left[\kappa_B^{(k)}(2\pi/3 - \alpha_0)\right] - B_-^{(k)} \exp\left[-\kappa_B^{(k)}(2\pi/3 - \alpha_0)\right] \right) \right] \\ &\quad + \int_{2\pi/3-\alpha_0}^{\alpha_0} \phi_C(\rho, \alpha')\, d\alpha' = 0. \end{aligned} \tag{B.5}$$

Eqs. (B.1) and (B.2) have for $\lambda \neq \frac{16}{3}$ the unique solution

$$B_+^{\mathrm{II}} \exp\left[i\sqrt{\tilde{\lambda}}\pi/3\right] = \frac{1 - \dfrac{4}{i\sqrt{3\tilde{\lambda}}}}{1 - \dfrac{16}{3\tilde{\lambda}}} \frac{2A_{\mathrm{II}}}{\sqrt{3\tilde{\lambda}}} \exp\left[i\sqrt{\tilde{\lambda}}\pi/6\right], \tag{B.6}$$

$$B_-^{\mathrm{II}} = \frac{1 + \dfrac{4}{i\sqrt{3\tilde{\lambda}}}}{1 - \dfrac{16}{3\tilde{\lambda}}} \frac{2A_{\mathrm{II}}}{\sqrt{3\tilde{\lambda}}} \exp\left[i\sqrt{\tilde{\lambda}}\pi/6\right]. \tag{B.7}$$

For $\tilde{\lambda} = \frac{16}{3}$ the two equations determining $B_\pm^{\mathrm{II}}$ are identical and infinitely many sets of coefficients exist. The constraint, in addition to Eq. (B.5), on $B_\pm^{\mathrm{II}}$ is then

$$B_+^{\mathrm{II}} \exp\left[i\sqrt{\tilde{\lambda}}\pi/3\right] + \frac{4}{i\sqrt{3\tilde{\lambda}}} B_-^{\mathrm{II}} = \frac{2A_{\mathrm{II}}}{\sqrt{3\tilde{\lambda}}} \exp\left[i\sqrt{\tilde{\lambda}}\pi/6\right]. \tag{B.8}$$

Eqs. (B.3) and (B.4) have non-trivial solutions only if

$$(\kappa_B^{(k)})^2 + v_0\rho^2 + \tilde{\lambda} = \pm \frac{4i}{\sqrt{3}} \frac{v_0\rho^2}{\kappa_B^{(k)}}, \tag{B.9}$$

which for each of the possible signs is a cubic equation with the three complex solutions $\kappa_B^{(k)}$ for $k = 1, 2, 3$. The corresponding coefficients are then related by

$$\pm i B_-^{(k)} = B_+^{(k)} \exp\left[\kappa_B^{(k)}\pi/3\right], \qquad k = 1, 2, 3. \tag{B.10}$$

The solutions corresponding to the different signs are related by an interchange of the $B_+^{(k)}$ and $B_-^{(k)}$ terms in Eq. (36).

With the expressions in Eqs. (B.6), (B.7), and (B.10) for the coefficients we can rewrite the link to the *C*-region from Eq. (B.5) as

$$\int_{2\pi/3-\alpha_0}^{\alpha_0} \phi_C(\rho, \alpha')\, d\alpha' = -\sum_{k=1}^{3} \left[\frac{B_-^{(k)}}{\kappa_B^{(k)}} \left(\pm i \exp\left[\kappa_B^{(k)}(\pi/3 - \alpha_0)\right] - \exp\left[-\kappa_B^{(k)}(2\pi/3 - \alpha_0)\right] \right) \right]$$

$$- \frac{A_{\mathrm{II}}}{2\sqrt{\tilde{\lambda}}} \frac{1}{1 - \dfrac{16}{3\tilde{\lambda}}} \left(\exp\left[i\sqrt{\tilde{\lambda}}(\pi/2 - \alpha_0) \right] \left(1 + \frac{4}{i\sqrt{3\tilde{\lambda}}} \right) \right.$$

$$\left. + \exp\left[-i\sqrt{\tilde{\lambda}}(\pi/2 - \alpha_0) \right] \left(1 - \frac{4}{i\sqrt{3\tilde{\lambda}}} \right) \right). \tag{B.11}$$

Using the wave function ϕ_C explicitly together with the expressions for the C-coefficients from Appendix C, we calculate the integral in Eq. (B.5) and arrive at

$$\int_{2\pi/3-\alpha_0}^{\alpha_0} \phi_C(\rho, \alpha')\, d\alpha' = \sum_{k=1}^{3} \left[\frac{\mp i A^{(k)} \sqrt{3}}{\kappa_{AC}^{(k)}} \left(\exp\left[\kappa_{AC}^{(k)}(\alpha_0 - \pi/3) \right] - \exp\left[-\kappa_{AC}^{(k)}(\alpha_0 - \pi/3) \right] \right) \right], \tag{B.12}$$

where the summation excluded $k = 0$, since this contribution vanishes. Combined with Eq. (B.11) this gives one constraint between the coefficients A_{II}, $B_{-}^{(k)}$, and $A^{(k)}$ for $k = 1, 2, 3$.

Appendix C. *A*- and *C*-Region Solutions at Intermediate Distances

If the wave functions in Eqs. (37) and (38) are solutions to Eqs. (33) and (34), we must have that

$$((\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda})A^{(k)} + \frac{4}{\sqrt{3}} \frac{v_0\rho^2}{\kappa_{AC}^{(k)}} \left(C_{+}^{(k)} \exp\left[\kappa_{AC}^{(k)}\pi/3 \right] + C_{-}^{(k)} \exp\left[-\kappa_{AC}^{(k)}\pi/3 \right] \right) = 0, \tag{C.1}$$

$$((\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda})C_{+}^{(k)} \exp\left[\kappa_{AC}^{(k)}\pi/3 \right] - \frac{4}{\sqrt{3}} \frac{v_0\rho^2}{\kappa_{AC}^{(k)}} \left(A^{(k)} + C_{-}^{(k)} \exp\left[-\kappa_{AC}^{(k)}\pi/3 \right] \right) = 0, \tag{C.2}$$

$$((\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda})C_{-}^{(k)} \exp\left[-\kappa_{AC}^{(k)}\pi/3 \right] - \frac{4}{\sqrt{3}} \frac{v_0\rho^2}{\kappa_{AC}^{(k)}} \left(A^{(k)} + C_{+}^{(k)} \exp\left[\kappa_{AC}^{(k)}\pi/3 \right] \right) = 0, \tag{C.3}$$

for each of the values of $k = 0, 1, 2, 3$ and furthermore

$$\sum_{k=0}^{3} \left[\frac{A^{(k)}}{\kappa_{AC}^{(k)}} \left(\exp\left[\kappa_{AC}^{(k)}(\alpha_0 - \pi/3) \right] + \exp\left[-\kappa_{AC}^{(k)}(\alpha_0 - \pi/3) \right] \right) \right.$$

$$\left. - \frac{1}{\kappa_{AC}^{(k)}} \left(C_{+}^{(k)} \exp\left[\kappa_{AC}^{(k)}(2\pi/3 - \alpha_0) \right] - C_{-}^{(k)} \exp\left[-\kappa_{AC}^{(k)}(2\pi/3 - \alpha_0) \right] \right) \right] + \int_{\alpha_0 - \pi/3}^{2\pi/3 - \alpha_0} \phi_B(\rho, \alpha')\, d\alpha' = 0. \tag{C.4}$$

Eqs. (C.1), (C.2), and (C.3) have non-trivial solutions only if the determinant vanishes for this linear system of equations. This is equivalent to either

$$(\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda} = 0 \tag{C.5}$$

or instead, when Eq. (C.5) is false, that

$$(\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda} = \pm i \frac{4v_0\rho^2}{\kappa_{AC}^{(k)}}. \tag{C.6}$$

The first of these equations (C.5) is a second-order equation with two solutions for $\kappa_{AC}^{(k)}$. They only differ by a sign in κ and they therefore correspond to the same wave function and count as one solution, here labelled by $k = 0$. The corresponding coefficients are then related by

$$C_{+}^{(0)} \exp\left[\kappa_{AC}^{(0)}\pi/3 \right] = -C_{-}^{(0)} \exp\left[-\kappa_{AC}^{(0)}\pi/3 \right] = A^{(0)}. \tag{C.7}$$

The expression in Eq. (C.6) is for each of the possible signs a cubic equation with three complex solutions $\kappa_{AC}^{(k)}$, here labelled by $k = 1, 2, 3$. The corresponding coefficients are given by

$$C_{+}^{(k)} \exp\left[\kappa_{AC}^{(k)}\pi/3 \right] = \frac{A^{(k)}}{2}(1 \mp i\sqrt{3}), \qquad C_{-}^{(k)} \exp\left[-\kappa_{AC}^{(k)}\pi/3 \right] = -\frac{A^{(k)}}{2}(1 \pm i\sqrt{3}) \tag{C.8}$$

for $k = 1, 2, 3$. The solutions corresponding to the different signs are related by an interchange of the $C_+^{(k)}$ and $C_-^{(k)}$ terms in Eq. (38).

With the expressions in Eqs. (C.7) and (C.8) for the coefficients we can rewrite the link to the B-region from Eq. (C.4) as

$$\int_{\alpha_0-\pi/3}^{2\pi/3-\alpha_0} \phi_B(\rho,\alpha')\, d\alpha' = -\sum_{k=1}^{3}\left[\frac{3A^{(k)}}{2\kappa_{AC}^{(k)}}\Big((1\mp i/\sqrt{3})\exp\left[\kappa_{AC}^{(k)}(\alpha_0-\pi/3)\right] + (1\pm i/\sqrt{3})\exp\left[-\kappa_{AC}^{(k)}(\alpha_0-\pi/3)\right]\Big)\right]. \tag{C.9}$$

Here the summation does not include $k = 0$, since this contribution vanishes. Using the wave function ϕ_B explicitly together with the expressions for the B-coefficients from Appendix B, we calculate the integral in Eq. (C.9) and arrive at

$$\begin{aligned}\int_{\alpha_0-\pi/3}^{2\pi/3-\alpha_0} \phi_B(\rho,\alpha')\, d\alpha' = &-\sum_{k=1}^{3}\left(\frac{B_-^{(k)}(1\pm i)}{\kappa_B^{(k)}}\left(\exp\left[\kappa_B^{(k)}(\alpha_0-2\pi/3)\right]-\exp\left[-\kappa_B^{(k)}(\alpha_0-\pi/3)\right]\right)\right)\\ &+\Big(\exp\Big[i\sqrt{\tilde{\lambda}}(\alpha_0-2\pi/3)\Big]\\ &-\exp\Big[-i\sqrt{\tilde{\lambda}}(\alpha_0-\pi/3)\Big]\Big)\frac{4iA_{\rm II}}{\tilde{\lambda}\sqrt{3}}\;\frac{1}{1-\dfrac{16}{3\tilde{\lambda}}}\exp\Big[i\sqrt{\tilde{\lambda}}\,\pi/6\Big],\end{aligned} \tag{C.10}$$

which combined with Eq. (C.9) gives one constraint between the coefficients $A_{\rm II}$, $B_-^{(k)}$, and $A^{(k)}$ for $k = 1, 2, 3$.

Appendix D. Properties of the Asymmetric Solutions

We have not found a general rigorous proof for the claim that three real solutions exist for $n \geq 3$. However, the numerical computations all unanimously support such a theorem, and several very different limiting cases will be discussed below. The intermediate cases are probably similar and may possibly be proved with some effort.

When n increases, d_i^2 approaches zero and

$$S_0 \to -v_0^{(1)}v_0^{(2)}v_0^{(3)}, \qquad S_1 \to v_0^{(1)}v_0^{(2)} + v_0^{(1)}v_0^{(3)} + v_0^{(2)}v_0^{(3)}, \qquad S_2 \to -v_0^{(1)} - v_0^{(2)} - v_0^{(3)} \tag{D.1}$$

with the solution $\epsilon = v_0^{(i)}$ to Eq. (84).

In general the cubic equation (84) has three real solutions if

$$\left(\tfrac{1}{3}S_1 - \tfrac{1}{9}S_2^2\right)^3 + \left(-\tfrac{1}{6}S_1S_2 + \tfrac{1}{2}S_0 + \tfrac{1}{27}S_2^3\right)^2 \leq 0\,, \tag{D.2}$$

which can be rewritten as

$$\tfrac{1}{27}S_1^3 - \tfrac{1}{108}S_1^2S_2^2 + \tfrac{1}{4}S_0^2 + \tfrac{1}{27}S_0S_2^3 - \tfrac{1}{6}S_0S_1S_2 \leq 0. \tag{D.3}$$

If one of the potentials, for example $v_0^{(3)}$, is very small or vanishes, we get in this limit

$$S_0 \to 0, \qquad S_1 \to v_0^{(1)}v_0^{(2)}(1-d_3^2), \qquad S_2 \to -v_0^{(1)} - v_0^{(2)} \tag{D.4}$$

and the condition in Eq. (D.3) becomes

$$\tfrac{1}{27}S_1^3 - \tfrac{1}{108}S_1^2S_2^2 = -\tfrac{1}{27}\left(\tfrac{1}{4}(v_0^{(1)} - v_0^{(2)})^2 + v_0^{(1)}v_0^{(2)}d_3^2\right) \leq -\tfrac{1}{27}(v_0^{(1)})^2(1-d_3^2) \leq 0. \tag{D.5}$$

If all $v_0^{(i)}$ are equal, the condition in Eq. (D.3) reduces to

$$\tfrac{1}{27}x^3 - \tfrac{1}{12}x^2 + \tfrac{1}{4}y^2 - y - \tfrac{1}{2}xy \leq 0\,, \tag{D.6}$$

where the overall factor $(v_0^{(i)})^6$ has been removed and the new variables x and y are defined as

$$x = 3 - d_1^2 - d_2^2 - d_3^2, \qquad y = 2d_1d_2d_3 + d_1^2 + d_2^2 + d_3^2 - 1. \tag{D.7}$$

Direct computation of the quantities recasts Eq. (D.6) into

$$-\tfrac{1}{27}(d_1^2 + d_2^2 + d_3^2)^3 + (d_1 d_2 d_3)^2 \leq 0\ , \tag{D.8}$$

which is fulfilled for any set of d_i.

Appendix E. Eigenvalue Equation at Large Distances for Three Different Particles

The matching conditions at $\alpha_0^{(1)}, \alpha_0^{(2)}$, and $\alpha_0^{(3)}$ at large distances for the asymmetric case provide the eigenvalue equation. This is found by equating the functions in Eqs. (88) and (89) and their first derivatives, i.e.

$$\left[a_i \sin\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) - c_i \sin\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\right] = b_i \sin(\alpha_0^{(i)}\kappa_i), \tag{E.1}$$

$$\left[a_i \cos\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) - c_i \cos\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\right]\sqrt{\tilde{\lambda}} = b_i\kappa_i \cos(\alpha_0^{(i)}\kappa_i), \tag{E.2}$$

and then eliminating b_i resulting in

$$\begin{aligned}&\left[a_i \sin\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) - c_i \sin\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\right]\kappa_i \cos(\alpha_0^{(i)}\kappa_i)\\ &\quad = \left[a_i \cos\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) - c_i \cos\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\right]\sqrt{\tilde{\lambda}} \sin(\alpha_0^{(i)}\kappa_i).\end{aligned} \tag{E.3}$$

Furthermore eliminating c_i by use of Eqs. (90)–(91) then provides 3 linear equations in a_i. They have non-trivial solutions only if the corresponding determinant, $D = \det\{d_{ik}\}$ vanishes. The matrix elements are

$$d_{ii} = \kappa_i \sin\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right)\cos(\alpha_0^{(i)}\kappa_i) - \sqrt{\tilde{\lambda}}\cos\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right)\sin(\alpha_0^{(i)}\kappa_i), \tag{E.4}$$

$$d_{ik} = \frac{A_i\, f_j}{F}, \qquad \text{for} \quad i \neq k, \tag{E.5}$$

where f_j is defined in Eq. (91) and

$$A_i = \frac{2F}{\sqrt{\tilde{\lambda}}}\left[\kappa_i \sin\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\cos(\alpha_0^{(i)}\kappa_i) - \sqrt{\tilde{\lambda}}\cos\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\sin(\alpha_0^{(i)}\kappa_i)\right]. \tag{E.6}$$

By further defining

$$B_i = \frac{d_{ii}\, f_i^2}{F^2} \tag{E.7}$$

the determinant can then be written as

$$D = B_1B_2B_3 + 2A_1A_2A_3 - B_1A_2A_3 - A_1A_2B_3 - A_1B_2A_3. \tag{E.8}$$

The eigenvalues $\tilde{\lambda}$ are then determined by $D = 0$.

Appendix F. Solutions to the Case of Two Identical Spin-$\frac{1}{2}$ Particles

The solutions to the Faddeev equations (107) and (108) are first found independently for each of the four regions of Fig. 4. We shall use the notation $\alpha_{ff} = \alpha_1$ and $\alpha_{fc} = \alpha_2$. In region II, where all potentials are zero, we have

$$\phi_s^{(i)}(\alpha_i) = A_i \sin\left((\alpha_i - \pi/2)\sqrt{\tilde{\lambda}}\right)\ , \tag{F.1}$$

where $(i, s) = (1, 0), (2, s_c - \frac{1}{2}), (3, s_c + \frac{1}{2})$, and Eq. (106) relates these to the remaining spin components of $\phi^{(3)}$ and $\phi^{(2)}$.

In region B of Fig. 4, where only one of the potentials is identically zero, we have instead

$$\phi_0^{(1)}(\alpha_{ff}) = b_1 \sin(\alpha_{ff}\kappa_0) + a_0 \sin(\alpha_{ff}\sqrt{\tilde{\lambda}}), \tag{F.2}$$

$$\phi_s^{(i)}(\alpha_{fc}) = b_i \sin\Big((\alpha_{fc} - \pi/2)\sqrt{\tilde{\lambda}}\Big), \tag{F.3}$$

where $(i,s) = (2, s_c - \frac{1}{2}), (3, s_c + \frac{1}{2})$, $\kappa_0 = \sqrt{v_0^{(ff)}\rho^2 + \tilde{\lambda}(\rho)}$ and

$$a_0 = -\frac{4f}{\sqrt{\tilde{\lambda}}}(C^{12}_{0,s_c-1/2}A_2 - C^{12}_{0,s_c+1/2}A_3); \tag{F.4}$$

here, in analogy to Eq. (91), we have defined

$$f = \frac{\sin\Big((\varphi - \pi/2)\sqrt{\tilde{\lambda}}\Big)}{\sin(2\varphi)}, \qquad \tilde{f} = \frac{\sin\Big((\tilde{\varphi} - \pi/2)\sqrt{\tilde{\lambda}}\Big)}{\sin(2\tilde{\varphi})}\ . \tag{F.5}$$

In region C of Fig. 4, where the other potential is identically zero, we have

$$\phi_0^{(1)}(\alpha_{ff}) = c_1 \sin\Big((\alpha_{ff} - \pi/2)\sqrt{\tilde{\lambda}}\Big), \tag{F.6}$$

$$\phi_s^{(i)}(\alpha_{fc}) = c_i \sin(\alpha_{fc}\kappa_s) + a_s \sin(\alpha_{fc}\sqrt{\tilde{\lambda}}), \tag{F.7}$$

where $(i,s) = (2, s_c - \frac{1}{2}), (3, s_c + \frac{1}{2})$, $\kappa_{s_c\pm1/2} = \sqrt{v_0^{(fc\pm)}\rho^2 + \tilde{\lambda}(\rho)}$ and

$$a_{s_c-1/2} = -\frac{2}{\sqrt{\tilde{\lambda}}}\big[C^{12}_{0,s_c-1/2}A_1 f + (C^{23}_{s_c-1/2,s_c-1/2}A_2 + C^{23}_{s_c-1/2,s_c+1/2}A_3)\tilde{f}\big], \tag{F.8}$$

$$a_{s_c+1/2} = \frac{2}{\sqrt{\tilde{\lambda}}}\big[C^{12}_{0,s_c+1/2}A_1 f + (-C^{23}_{s_c-1/2,s_c+1/2}A_2 + C^{23}_{s_c+1/2,s_c+1/2}A_3)\tilde{f}\big]. \tag{F.9}$$

In region D of Fig. 4, where both potentials are finite, we have

$$\phi_s^{(i)}(\alpha_i) = d_i \sin(\alpha_i\kappa_s) + a_s \sin(\alpha_i\sqrt{\tilde{\lambda}})\ , \tag{F.10}$$

where $(i,s) = (1,0), (2, s_c - \frac{1}{2}), (3, s_c + \frac{1}{2})$.

Imposing continuity of the functions and their first derivatives at the boundaries between the different regions of Fig. 4, the constants must be related by

$$c_1 = A_1, \qquad b_2 = A_2, \qquad b_3 = A_3, \qquad d_1 = b_1, \qquad d_2 = c_2, \qquad d_3 = c_3, \tag{F.11}$$

and the remaining six constants obey the following set of six linear equations:

$$\begin{aligned}
A_1 \sin\Big(\Big(\alpha_0^{(ff)} - \frac{\pi}{2}\Big)\sqrt{\tilde{\lambda}}\Big) &= b_1 \sin(\alpha_0^{(ff)}\kappa_0) + a_0 \sin(\alpha_0^{(ff)}\sqrt{\tilde{\lambda}}),\\
A_1\sqrt{\tilde{\lambda}} \cos\Big(\Big(\alpha_0^{(ff)} - \frac{\pi}{2}\Big)\sqrt{\tilde{\lambda}}\Big) &= b_1\kappa_0 \cos(\alpha_0^{(ff)}\kappa_0) + a_0\sqrt{\tilde{\lambda}} \cos(\alpha_0^{(ff)}\sqrt{\tilde{\lambda}}),\\
A_2 \sin\Big(\Big(\alpha_0^{(fc)} - \frac{\pi}{2}\Big)\sqrt{\tilde{\lambda}}\Big) &= c_2 \sin(\alpha_0^{(fc)}\kappa_{s_c-1/2}) + a_{s_c-1/2} \sin(\alpha_0^{(fc)}\sqrt{\tilde{\lambda}}),\\
A_2\sqrt{\tilde{\lambda}} \cos\Big(\Big(\alpha_0^{(fc)} - \frac{\pi}{2}\Big)\sqrt{\tilde{\lambda}}\Big) &= c_2\kappa_{s_c-1/2} \cos(\alpha_0^{(fc)}\kappa_{s_c-1/2}) + a_{s_c-1/2}\sqrt{\tilde{\lambda}} \cos(\alpha_0^{(fc)}\sqrt{\tilde{\lambda}}),\\
A_3 \sin\Big(\Big(\alpha_0^{(fc)} - \frac{\pi}{2}\Big)\sqrt{\tilde{\lambda}}\Big) &= c_3 \sin(\alpha_0^{(fc)}\kappa_{s_c+1/2}) + a_{s_c+1/2} \sin(\alpha_0^{(fc)}\sqrt{\tilde{\lambda}}),\\
A_3\sqrt{\tilde{\lambda}} \cos\Big(\Big(\alpha_0^{(fc)} - \frac{\pi}{2}\Big)\sqrt{\tilde{\lambda}}\Big) &= c_3\kappa_{s_c+1/2} \cos(\alpha_0^{(fc)}\kappa_{s_c+1/2}) + a_{s_c+1/2}\sqrt{\tilde{\lambda}} \cos(\alpha_0^{(fc)}\sqrt{\tilde{\lambda}}).
\end{aligned} \tag{F.12}$$

Since a_s depend only on A_i we can easily eliminate b_1, c_2, c_3 from the equations (F.12). This leaves 3 homogeneous linear equations in A_1, A_2, A_3 with a corresponding determinant $D = \det\{d_{ik}\}$. The matrix elements are given by

$$\begin{aligned}
d_{22} = D_2 + F_{23}C^{23}_{s_c-1/2,s_c-1/2}, \qquad d_{33} = D_3 - F_{32}C^{23}_{s_c+1/2,s_c+1/2},\\
d_{11} = D_1, \qquad d_{ik} = F_{ik}C^{ik}_{s_i,s_k},
\end{aligned} \tag{F.13}$$

where $s_1 = 0, s_2 = s_c - \frac{1}{2}, s_3 = s_c + \frac{1}{2}$, the spin overlap functions are defined in Eqs. (15), and

$$D_i = \kappa_s \sin\Big(\Big(\alpha_0^{(i)} - \frac{\pi}{2}\Big)\sqrt{\tilde{\lambda}}\Big) \cos(\alpha_0^{(i)}\kappa_s) - \sqrt{\tilde{\lambda}} \cos\Big(\Big(\alpha_0^{(i)} - \frac{\pi}{2}\Big)\sqrt{\tilde{\lambda}}\Big) \sin(\alpha_0^{(i)}\kappa_s), \tag{F.14}$$

$$F_{ik} = \left[\kappa_s \sin\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\cos(\alpha_0^{(i)}\kappa_s) - \sqrt{\tilde{\lambda}}\cos\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\sin(\alpha_0^{(i)}\kappa_s)\right]\frac{2(1+\delta_{i,1})}{\sqrt{\tilde{\lambda}}}f_j, \tag{F.15}$$

with f_j defined in Eq. (91).

The values of $\tilde{\lambda}$ are as usual determined from $D = 0$.

Simplified expressions can be obtained in the limit of very large distances where

$$F_{ik} \approx -2(1+\delta_{i,1})\mu_{jk}\sqrt{v_0^{(i)}}a_{\text{scat}}^{(i)}\cos(X_i\sqrt{v_0^{(i)}})\,f_j, \tag{F.16}$$

$$D_i \approx -\cos(X_i\sqrt{v_0^{(i)}})\sqrt{v_0^{(i)}}\left(\rho\sin\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right) + a_{\text{scat}}^{(i)}\mu_{jk}\sqrt{\tilde{\lambda}}\cos\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\right). \tag{F.17}$$

Here $i = 1, 2, 3$ correspond to $ff, fc-, fc+$, respectively. Then $v_0^{(i)}$ is defined in Eqs. (115) and (116) and the scattering lengths, $a_{\text{scat}}^{(ff)}$, $a_{\text{scat}}^{(fc+)}$, $a_{\text{scat}}^{(fc-)}$, for the two-body systems are defined in analogy to Eq. (96) in terms of radii, potentials, and reduced masses.

If $\tilde{\lambda}/\rho^2$ remains finite, the dominating terms in the determinant are D_i. Thus D is approximately diagonal with the matrix elements D_i given in Eq. (F.14). The eigenvalues in the limit of large ρ then again correspond to the two-body bound states obtained by using Eq. (F.14) and solving $D_i = 0$.

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