PROJECT EULER PROBLEM 142

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We want to find three positive integers $x > y > z > 0$ with minimal sum, for which the positive differences and sums are all perfect squares:

$$x + z = k^2 \tag{1}$$
$$x - z = l^2 \tag{2}$$
$$y + z = m^2 \tag{3}$$
$$y - z = n^2 \tag{4}$$
$$x + y = q^2 \tag{5}$$
$$x - y = r^2 \tag{6}$$

Note that $m > n$ because $z > 0$. From these equations we get:

$$x + z = y + z + x - y \Rightarrow k^2 = m^2 + r^2 \tag{7}$$
$$x - z = y - z + x - y \Rightarrow l^2 = n^2 + r^2 \tag{8}$$

Geometric interpretation for this is that (m, r, k) and (n, r, l) are pythagorean triplets. And from (5) we get an additional condition:

$$x + y = x - z + y + z \Rightarrow n^2 + r^2 + m^2 = q^2 \tag{9}$$

From the equations (1)-(6) we can solve x, y, and z:

$$2x = m^2 + n^2 + 2r^2 \tag{10}$$
$$2y = m^2 + n^2 \tag{11}$$
$$2z = m^2 - n^2 \tag{12}$$

So in order for x, y, z be integer, both m and n must have same parity. So we can search for pythagorean triples sharing a leg and see if they can be combined to satisfy (9) and the parity requirement.

Listing 1. Problem 142 Solution

```
from collections import Counter, defaultdict
from itertools import combinations
from util import pythag, square

c, P = Counter(), defaultdict(list)

for i,j,k in pythag(999, False):
    c[i] += 1
    c[j] += 1
    P[i].append(j)
    P[j].append(i)

def gen():
    for r,num in c.most_common():
        if num<2: break
        for n,m in combinations(sorted(P[r]), 2):
            if not (n+m)&1 and square(n*n+m*m+r*r):
                yield (m*m+n*n+2*r*r)//2,
                    (m*m+n*n)//2, (m*m-n*n)//2

print(sum(min(gen(), key=sum)))
```