FACULDADE DE
CIÊNCIAS E TECNOLOGIA
DEPARTAMENTO DE INFORMÁTICA

The Relative Power of Synchronization Operations

lectures 11 & 12 (2025-04-14)

Master in Computer Science and Engineering

— Concurrency and Parallelism / 2024-25 —

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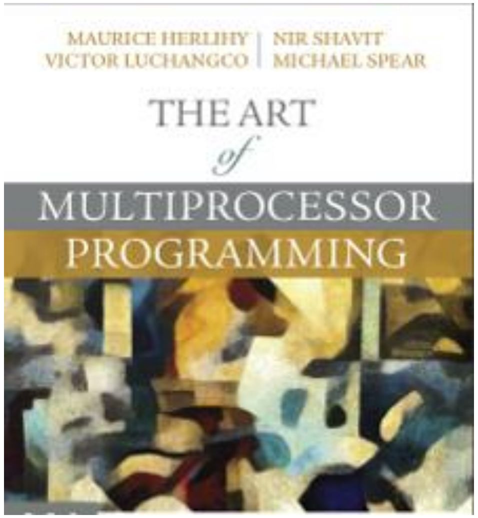
Based in the companion slides from “The Art of Multiprocessor Programming”

Outline

- Concurrent Objects
  - Correctness
  - Sequential Objects
  - Quiescent and Sequential Consistency
  - Linearizability
  - Progress Conditions

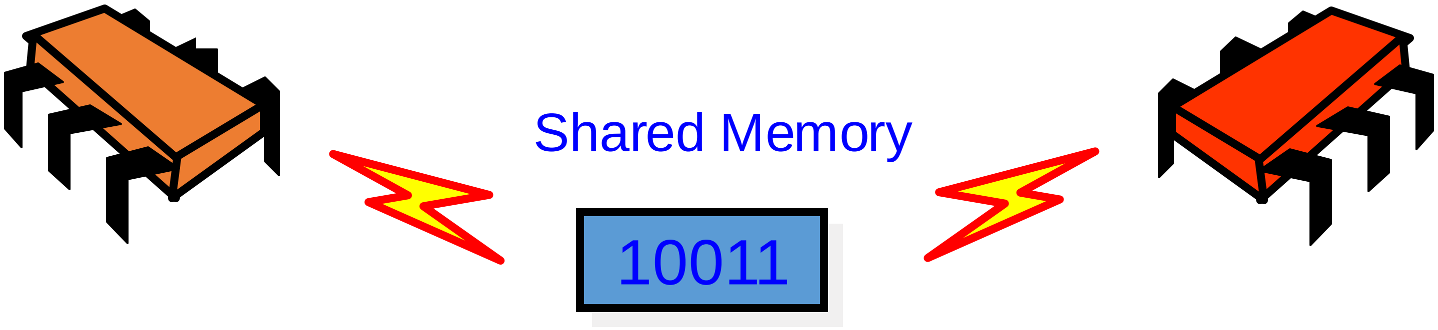


  - Bibliography:
    - **Chapters 5** of book

      Herlihy M., Shavit N., Luchangco V., Spear M.;
      The Art of Multiprocessor Programming;
      Morgan Kaufmann (2020); ISBN: 978-0-12-415950-1

Last Lecture

Shared-Memory Computability?

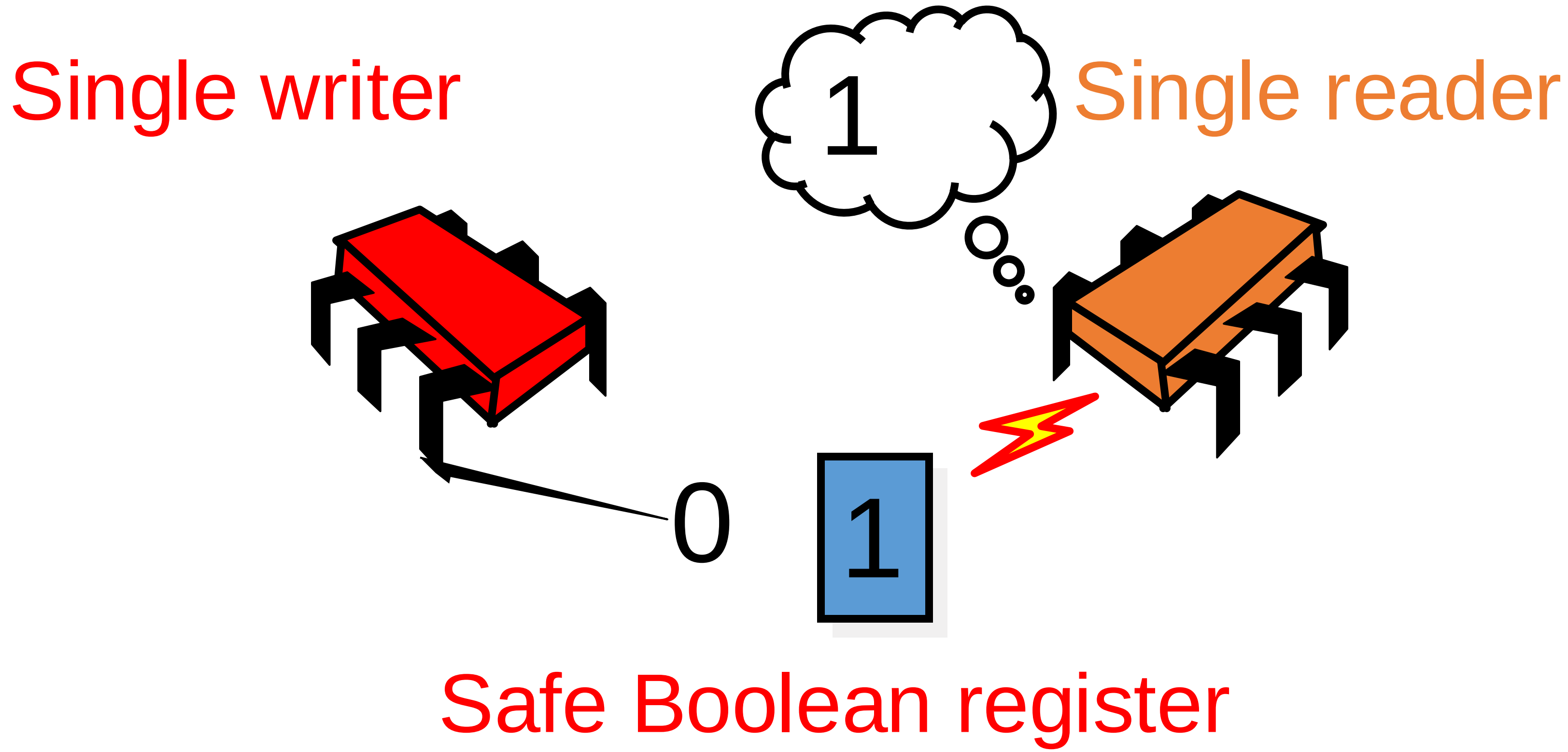


- Mathematical model of concurrent computation
- What is (and is not) concurrently computable
- Efficiency (mostly) irrelevant

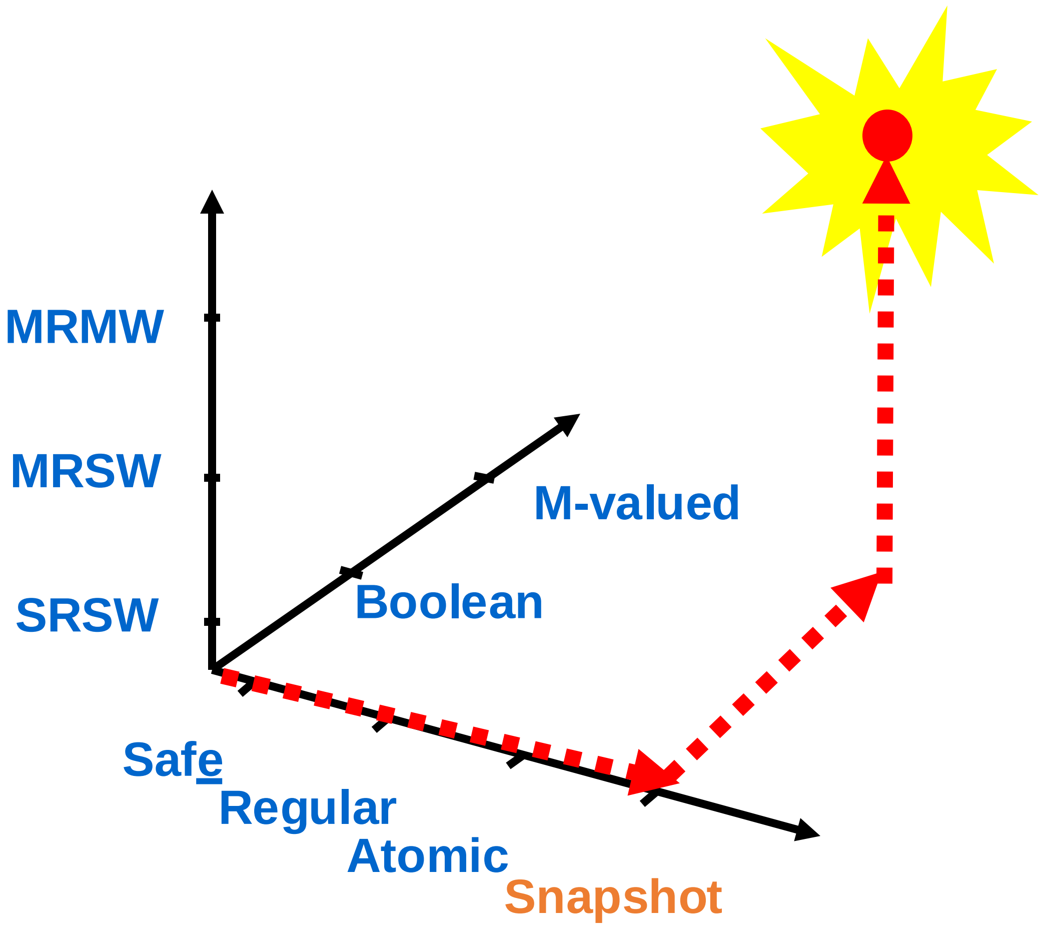
Wait-Free Implementation

- Every method call completes in finite number of steps
- Implies no mutual exclusion

From Weakest Register



All the way to a Wait-free Implementation of Atomic Snapshots



Rationale for wait-freedom

- We wanted atomic registers to implement mutual exclusion

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- We wanted atomic registers to implement mutual exclusion
- So we couldn't use mutual exclusion to implement atomic registers

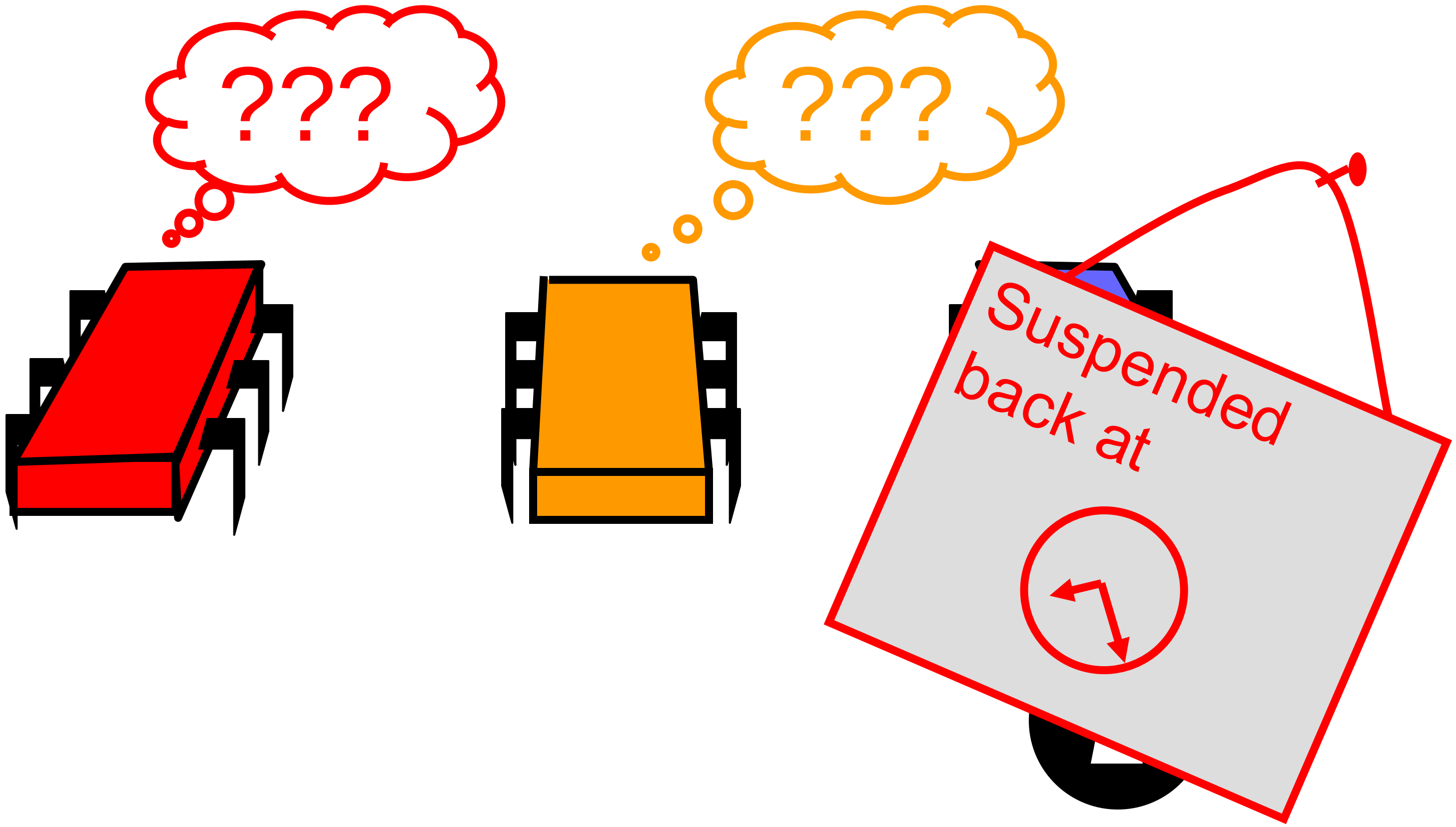
Rationale for wait-freedom

- We wanted atomic registers to implement mutual exclusion
- So we couldn't use mutual exclusion to implement atomic registers
- But wait, there's more!

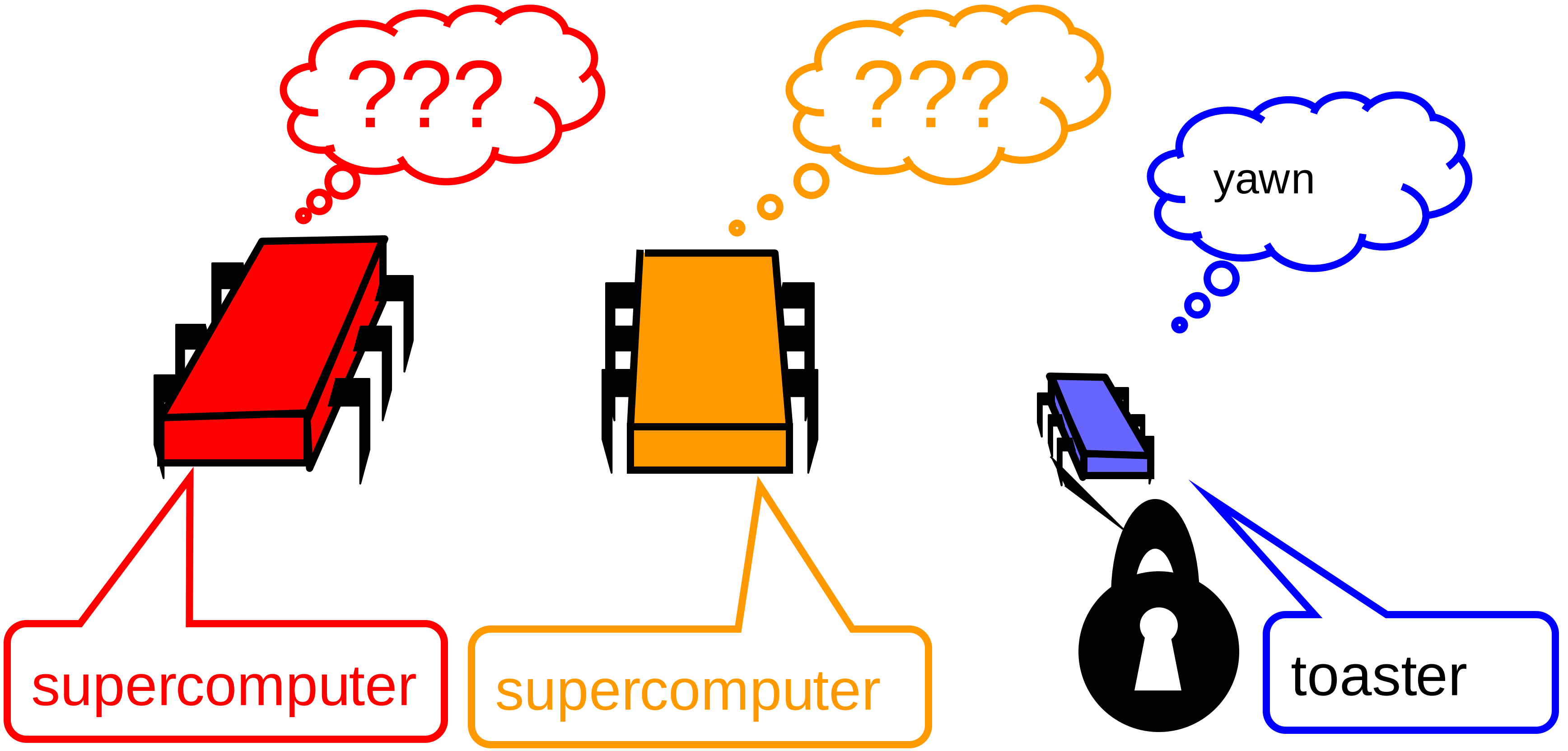
This Lecture

Why is Mutual Exclusion so wrong?

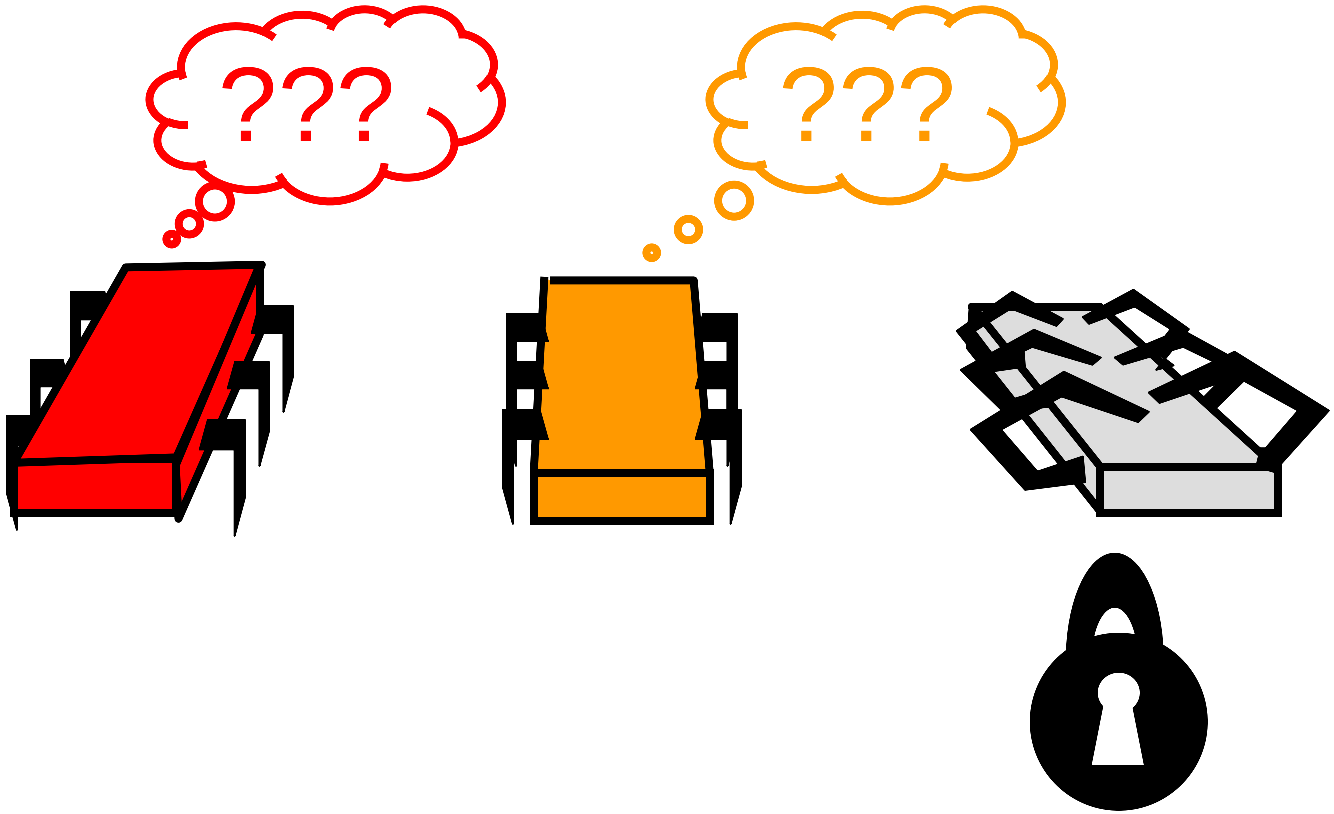
Asynchronous Interrupts



Heterogeneous Processors



Fault-tolerance



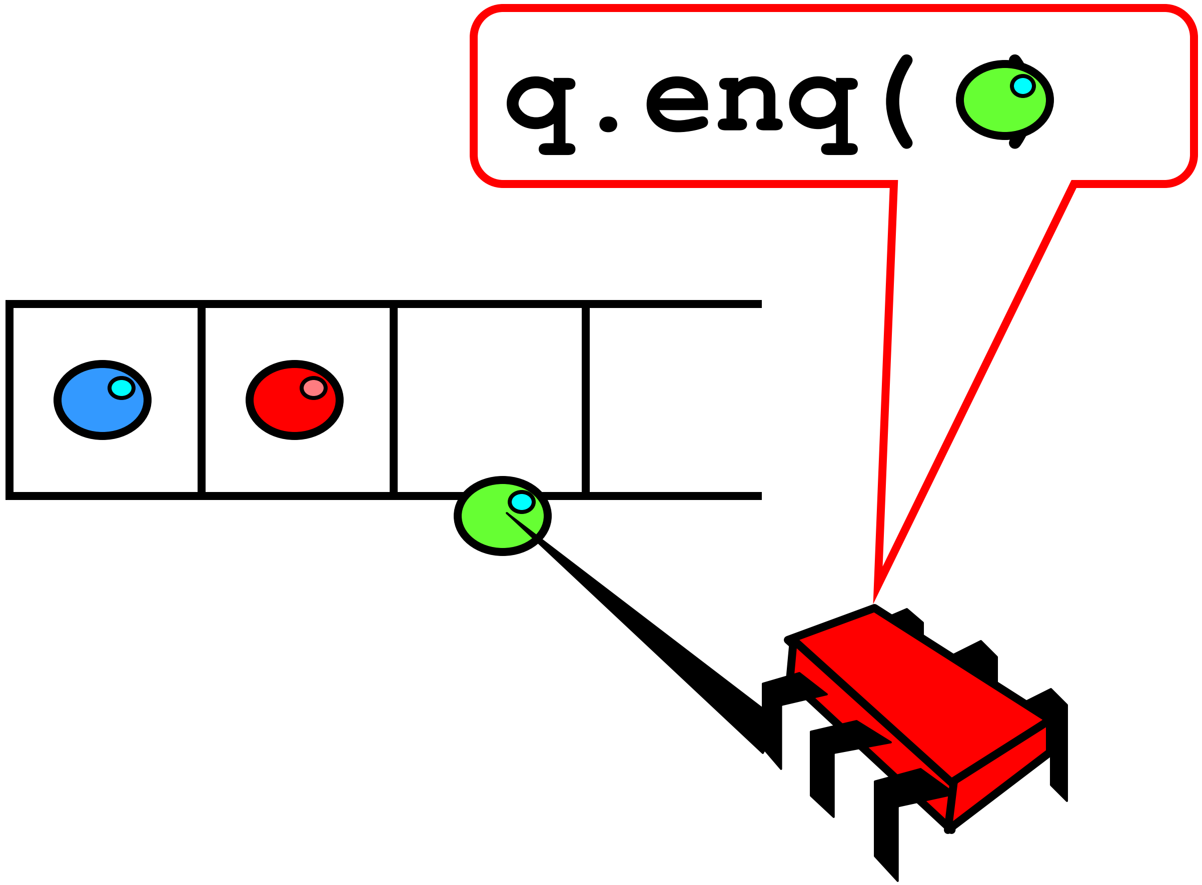
Basic Questions

- Wait-Free synchronization might be a good idea in principle

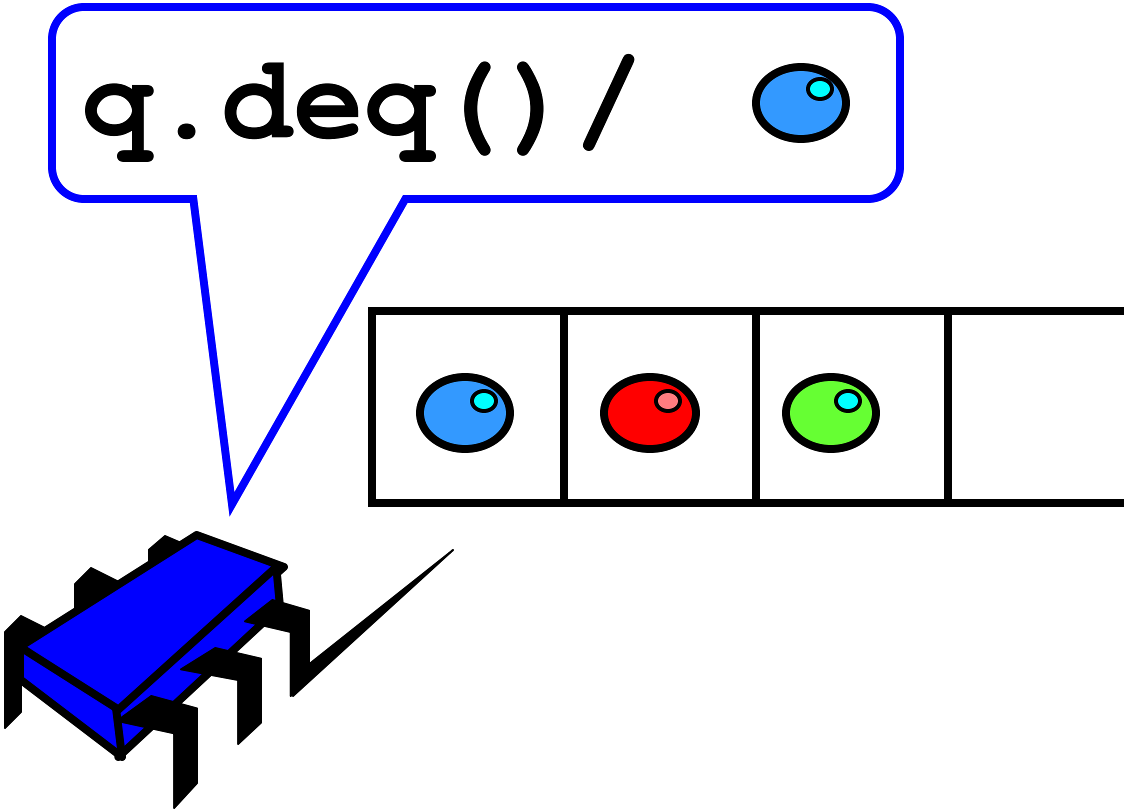
Basic Questions

- Wait-Free synchronization might be a good idea in principle
- But how do you do it …
  - Systematically?
  - Correctly?
  - Efficiently?

FIFO Queue: Enqueue Method

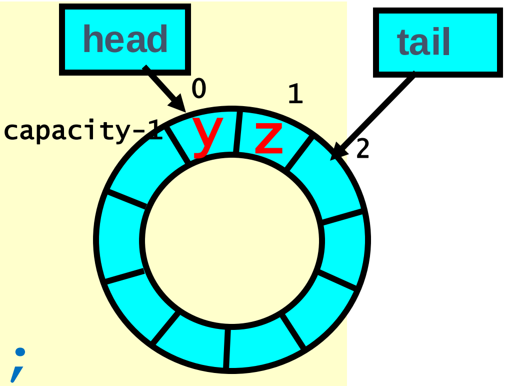


FIFO Queue: Dequeue Method

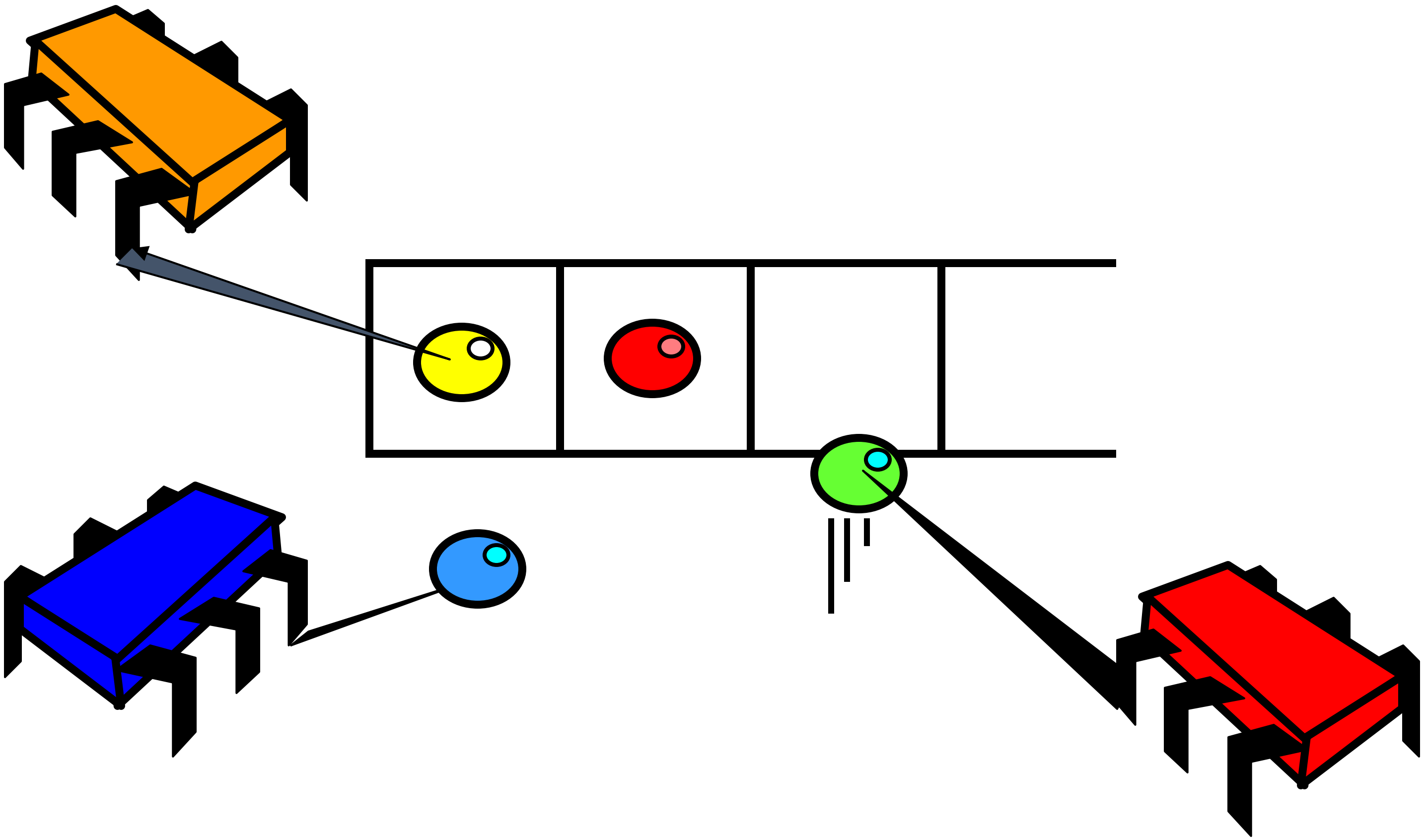


Two-Thread Wait-Free Queue

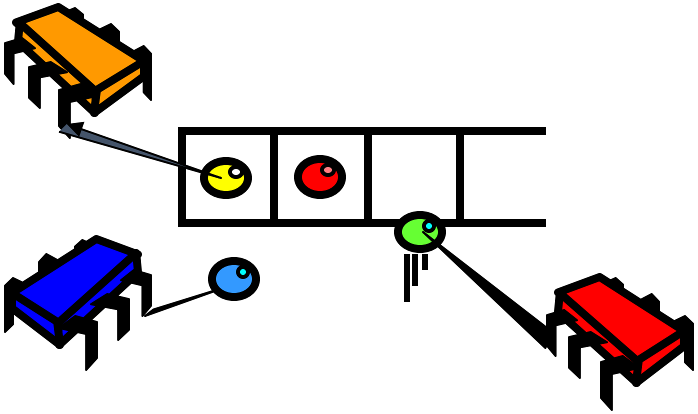
```
public class WaitFreeQueue {
 int head = 0, tail = 0;
 Item[QSIZE] items;
 public void enq(Item x) {
  while (tail-head == QSIZE) {};
  items[tail % QSIZE] = x; tail++;
  }
 public Item deq() {
  while (tail-head == 0) {}
  Item item = items[head % QSIZE];
  head++; return item;
}}
```



What About Multiple Dequeuers?



Grand Challenge



Only new aspect

- Implement a FIFO queue
  - Wait-free
  - Linearizable
  - From atomic read-write registers
  - Multiple dequeuers

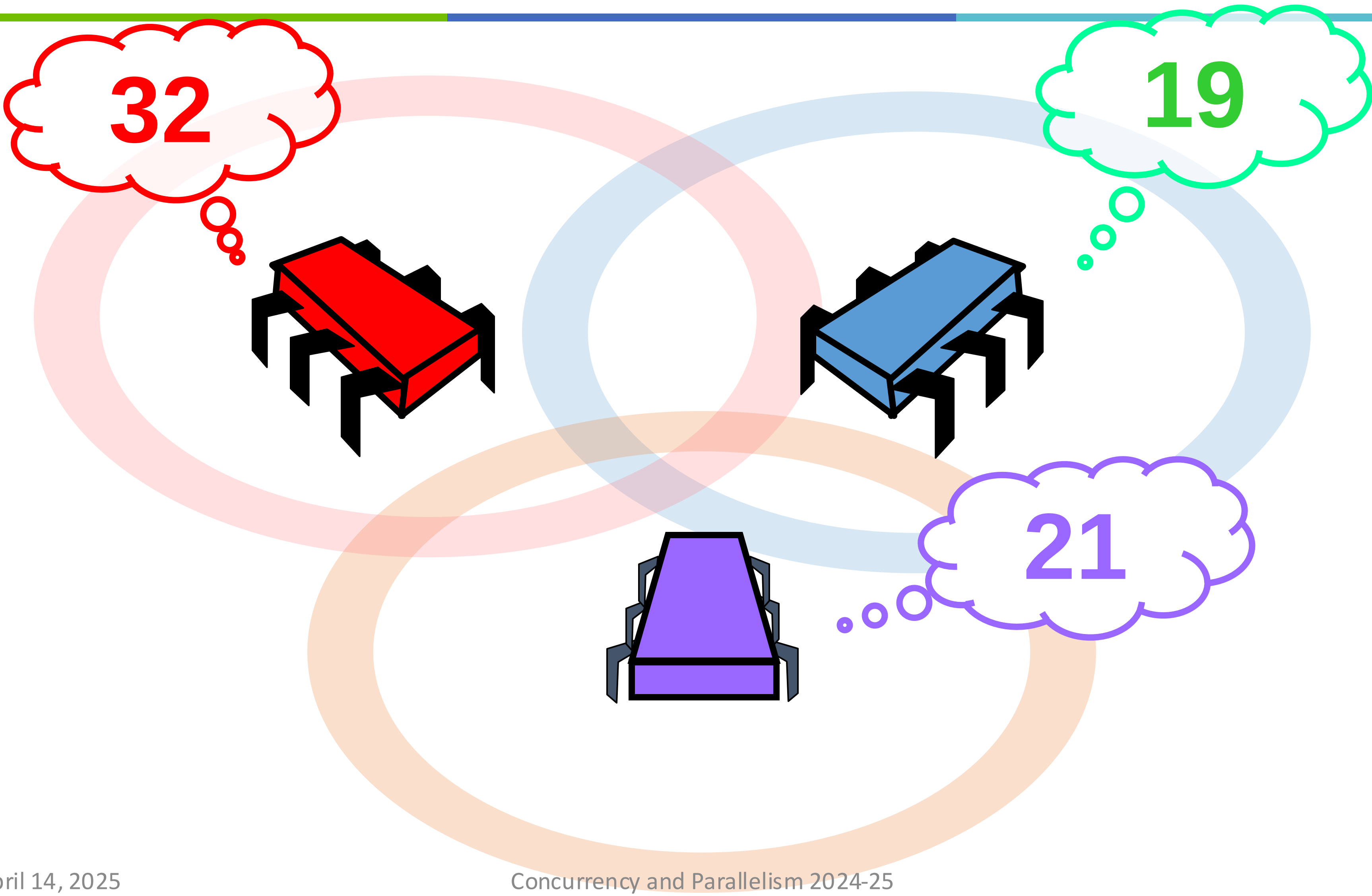
Puzzle

While you are ruminating on the grand challenge …

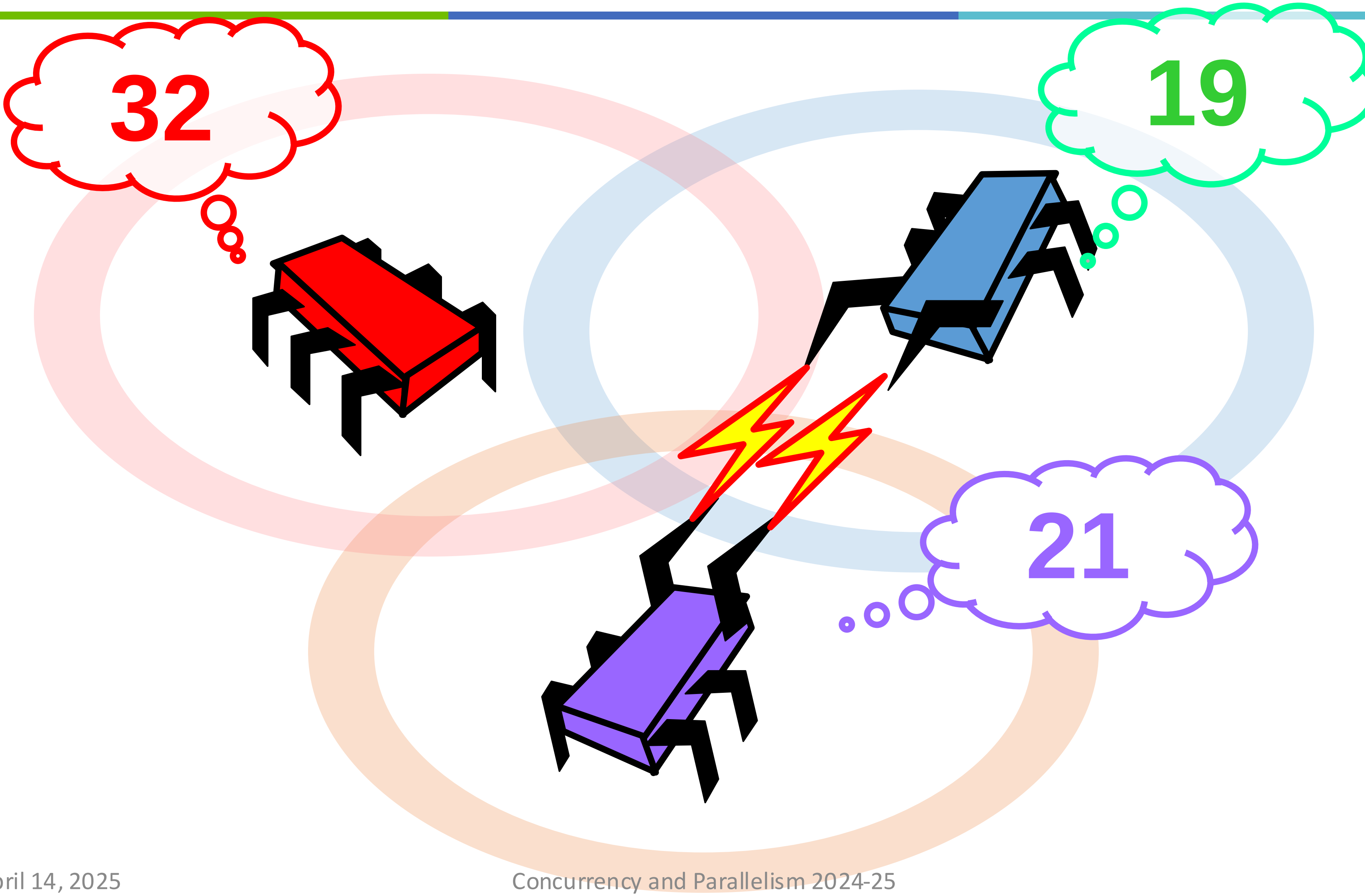
We will give you another puzzle …

Consensus!

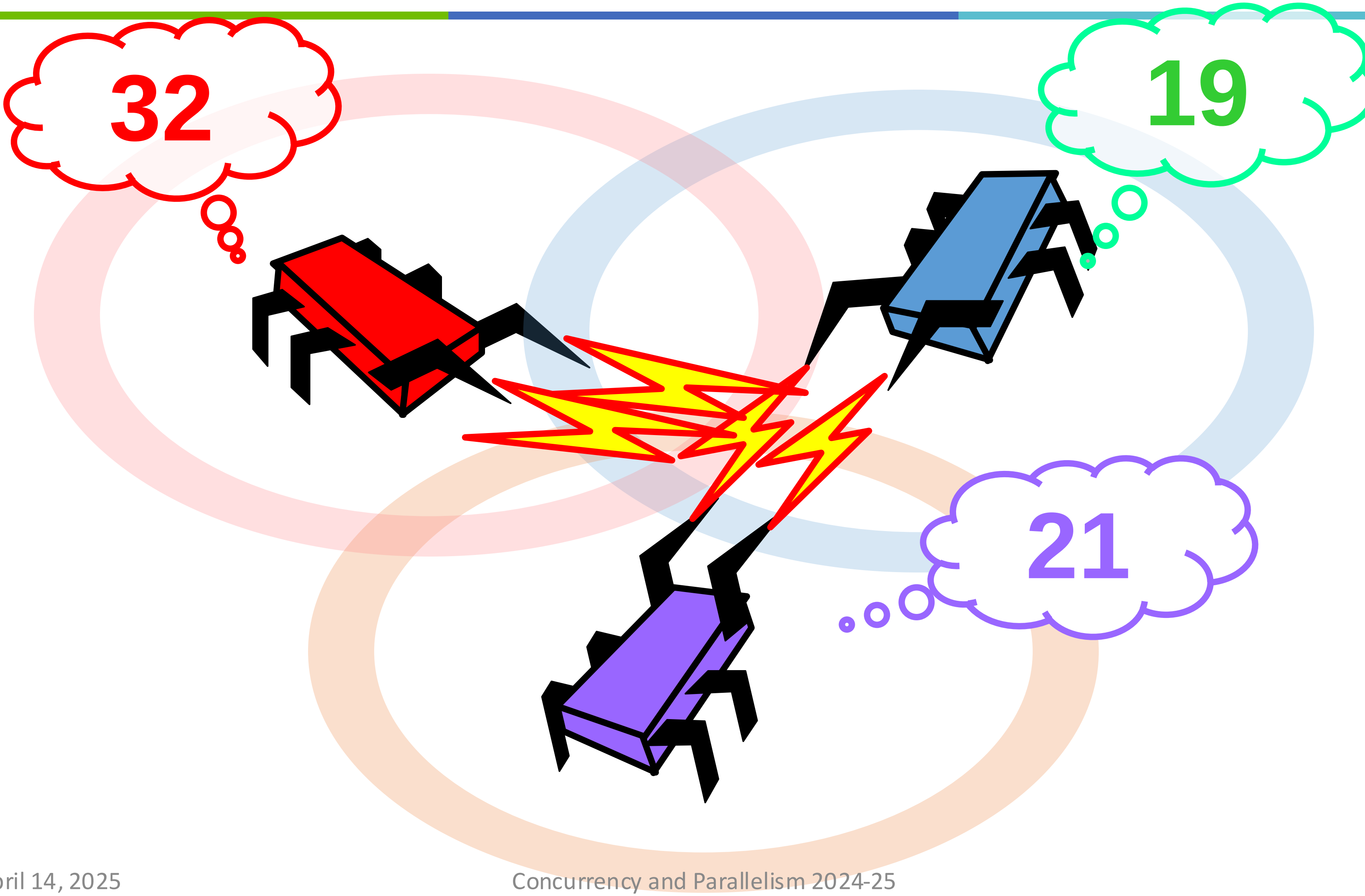
Consensus: Each Thread has a Private Input



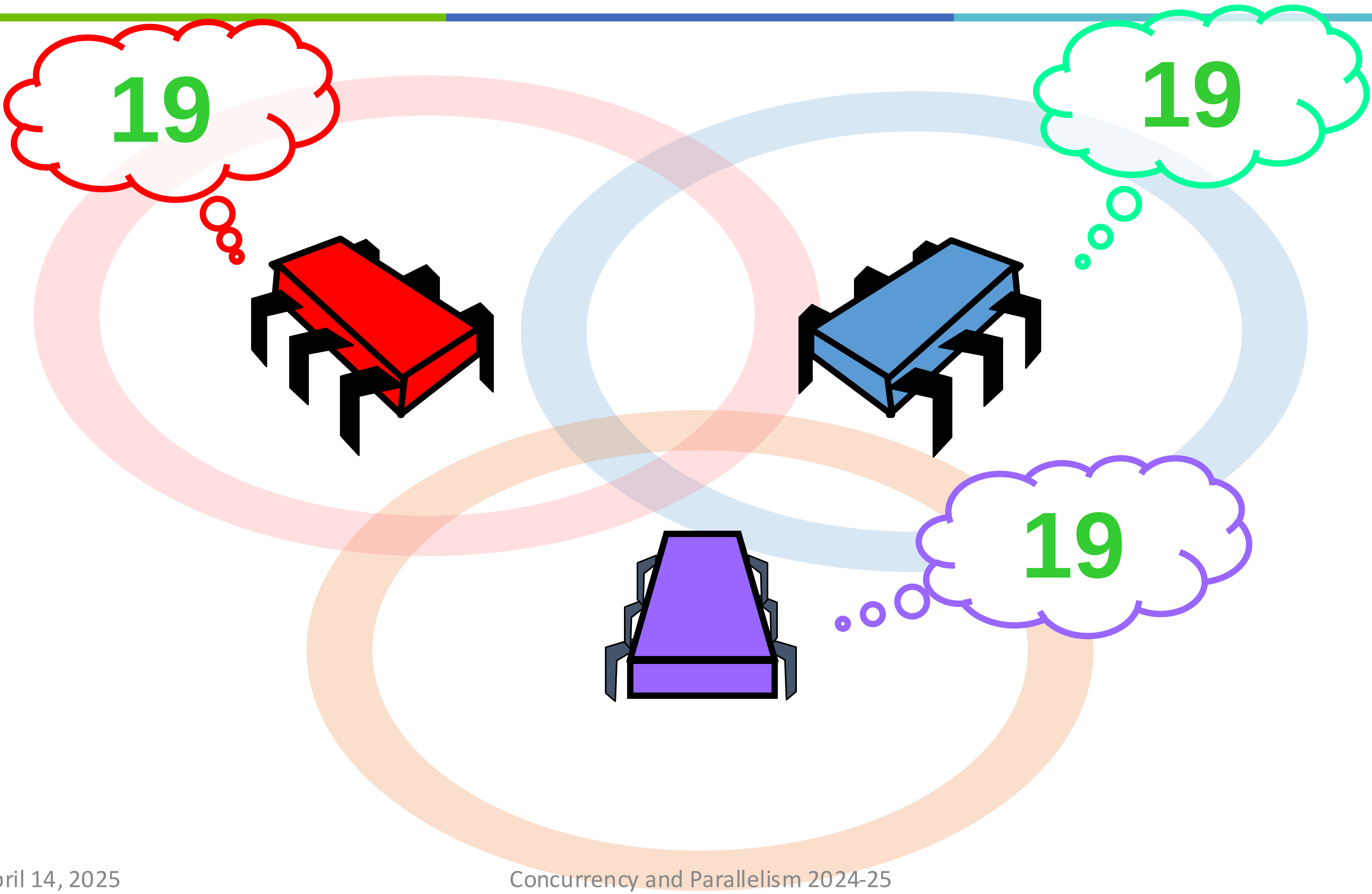
They Communicate



They Communicate



They Agree on One Thread's Input



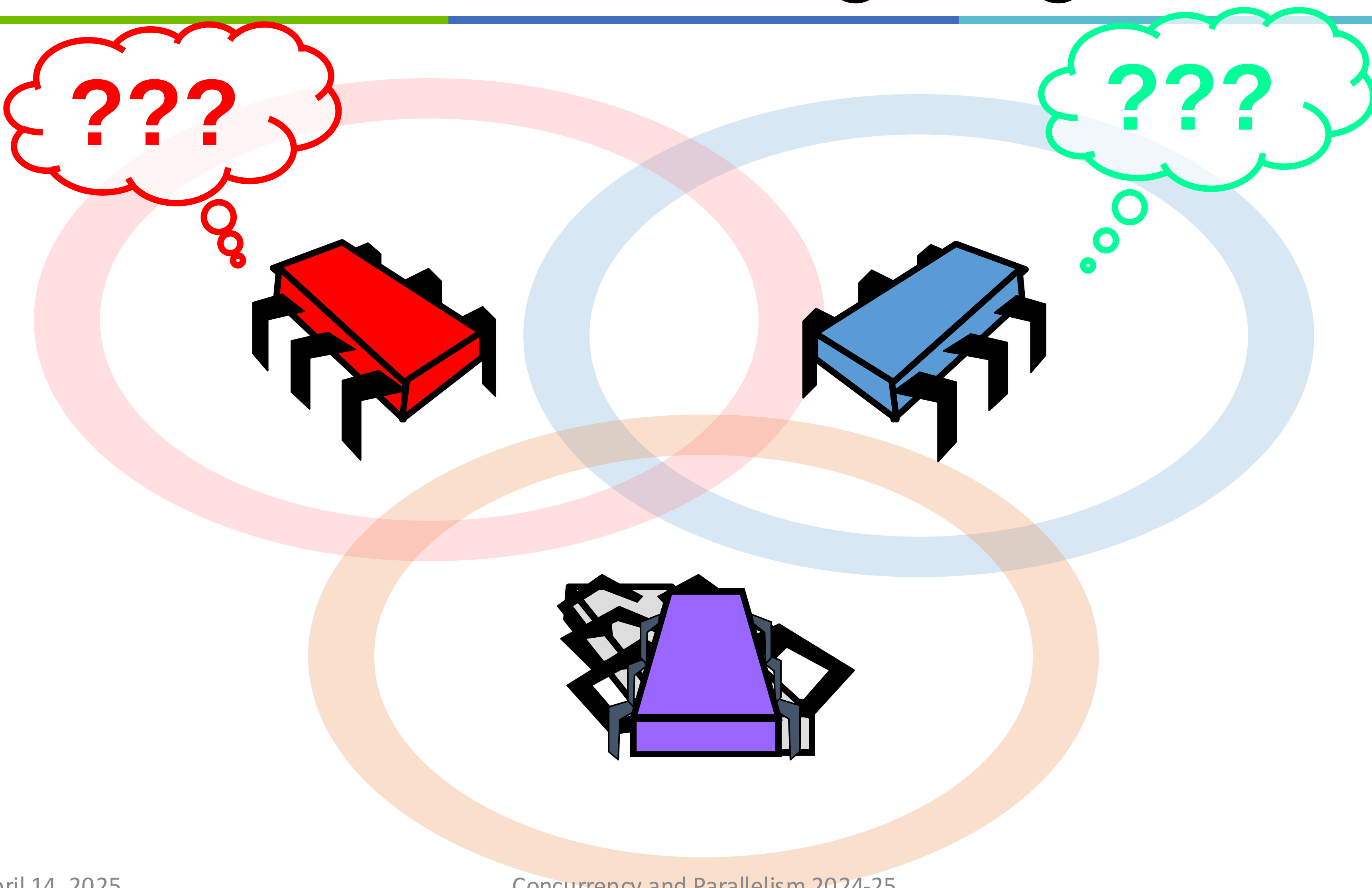
Formally: Consensus

- Consistent:
  - all threads decide the same value

Formally: Consensus

- Consistent:
  - all threads decide the same value
- Valid:
  - the common decision value is some thread's input

No Wait-Free Implementation of Consensus using Registers



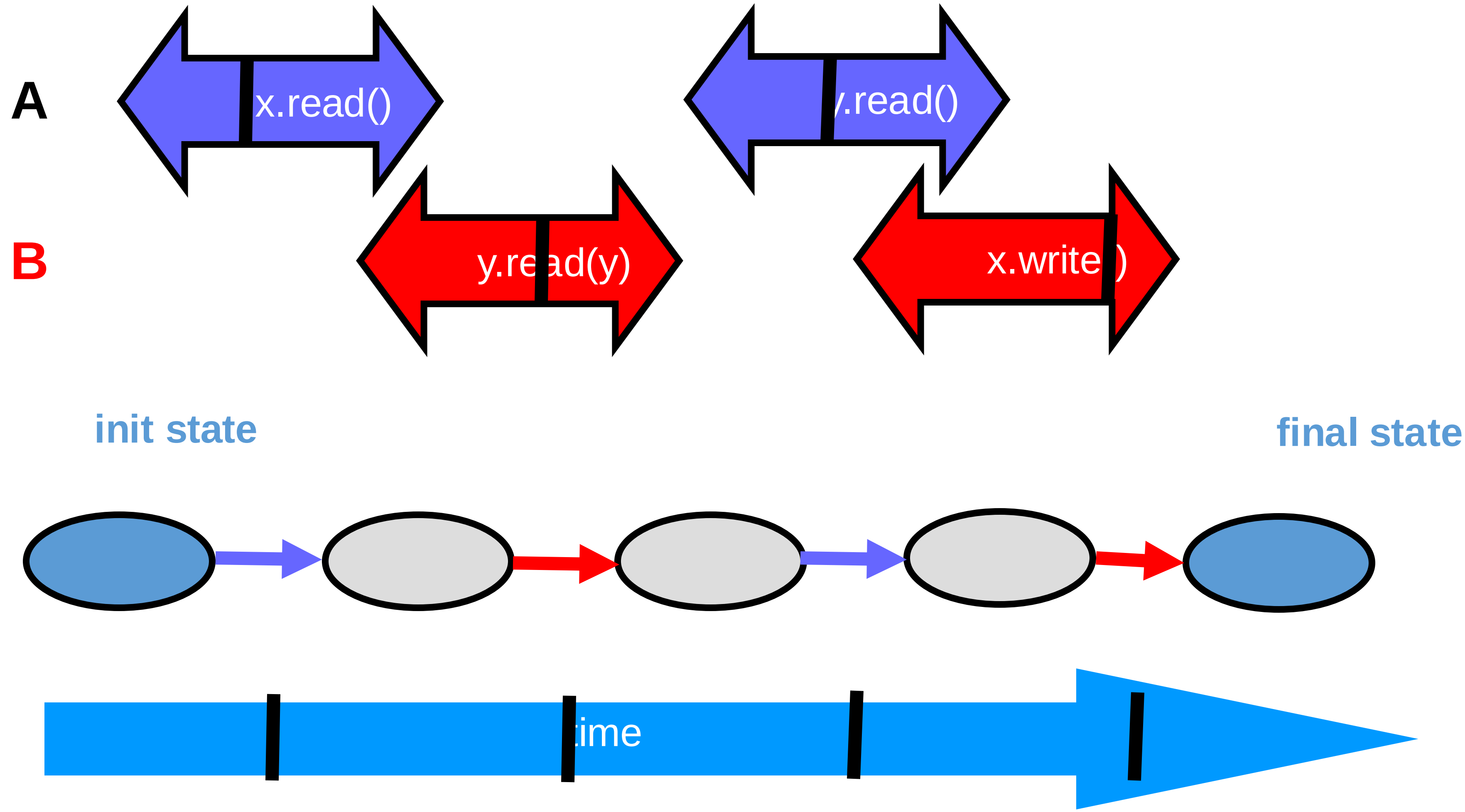
Formally

- Theorem
  - There is no wait-free implementation of *n*-thread consensus from read-write registers

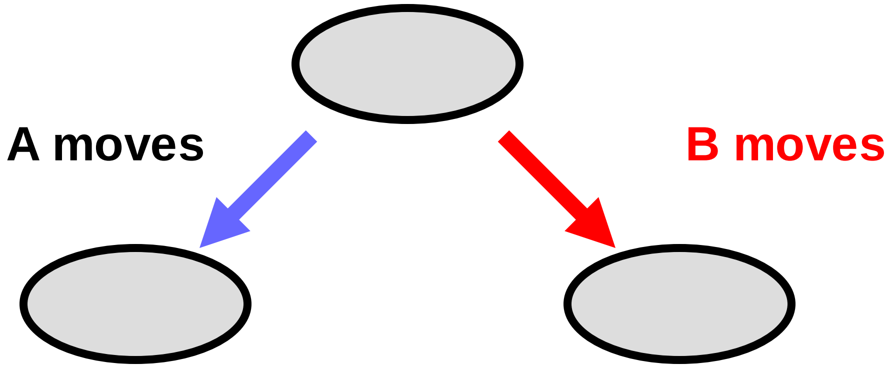
Formally

- Theorem
  - There is no wait-free implementation of *n*-thread consensus from read-write registers
- Implication
  - Asynchronous computability ≠ sequential (Turing) computability
- Proof
  - Read the book chapter 5

Protocol Histories as State Transitions

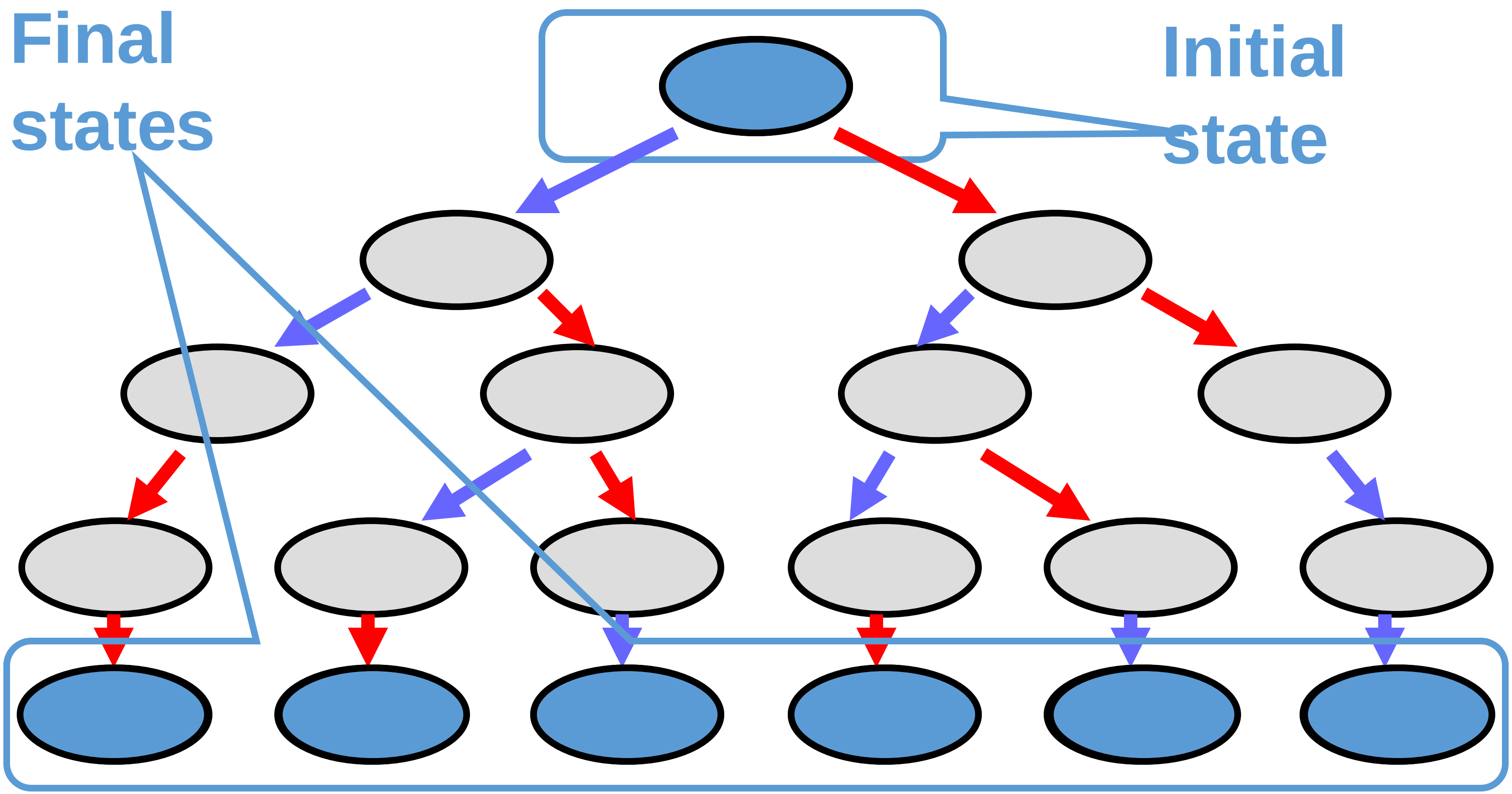


Wait-Free Computation

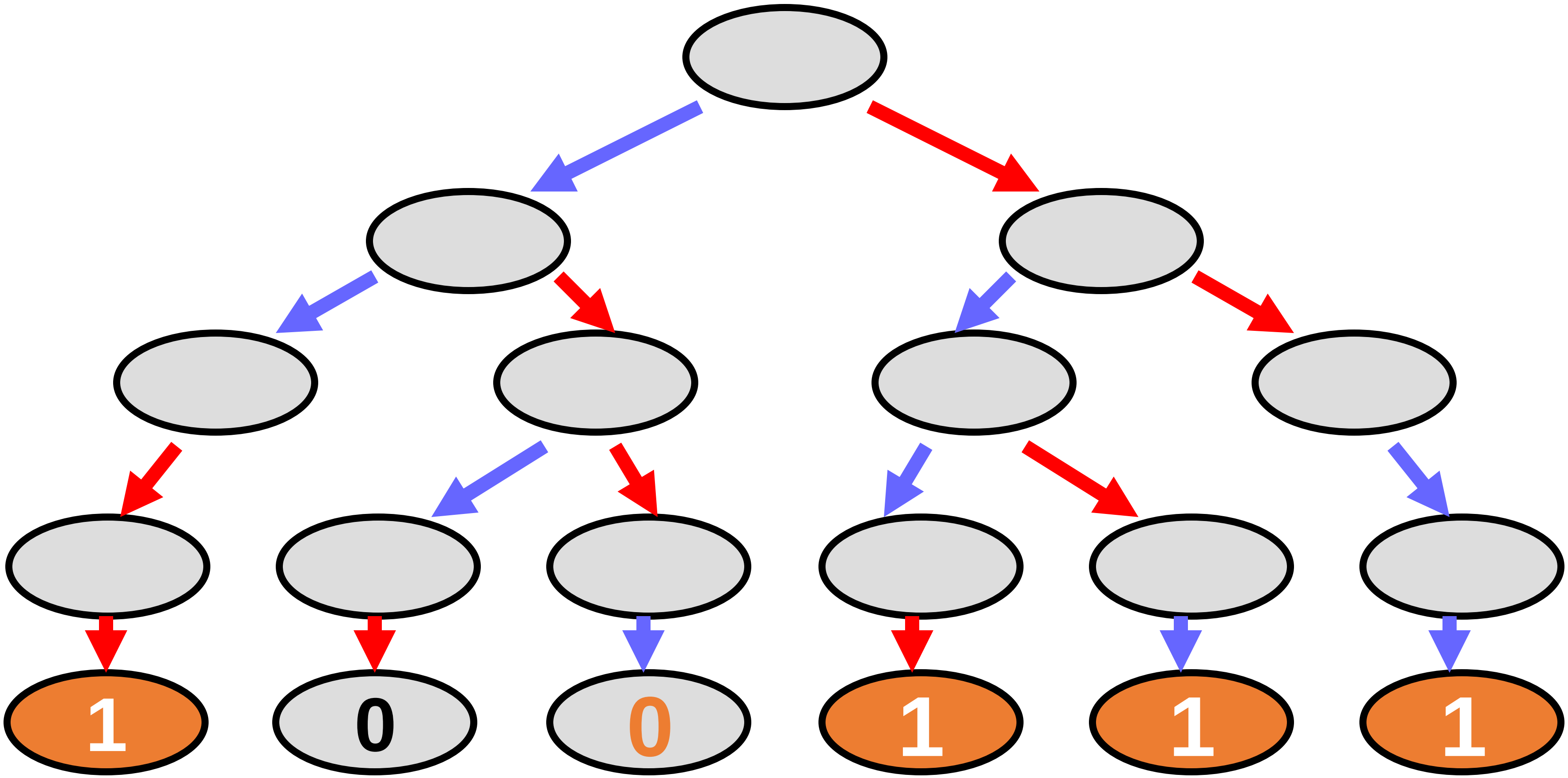


- Either A or B "moves"
- Moving means
  - Register read
  - Register write

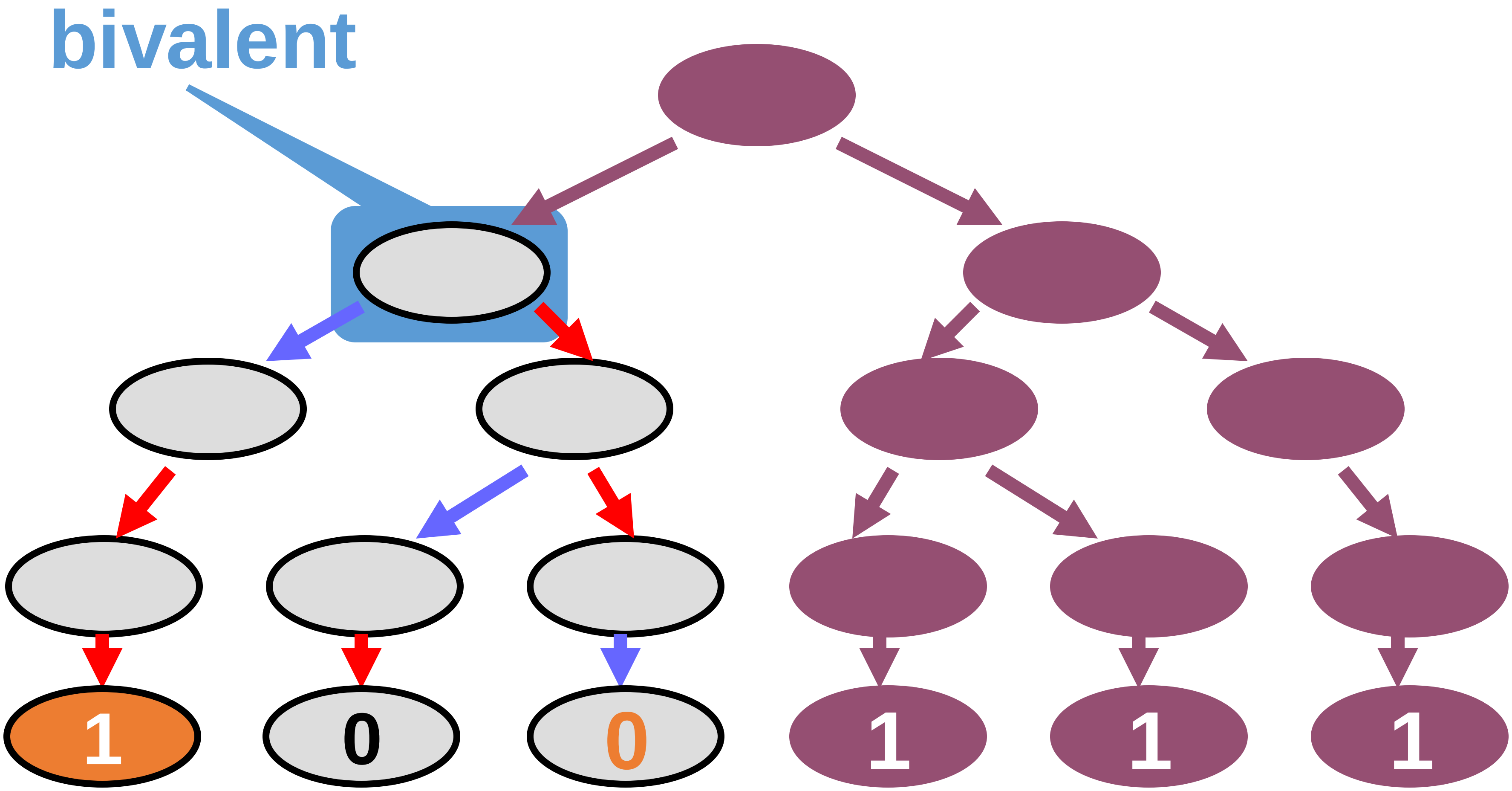
The Two-Move Tree



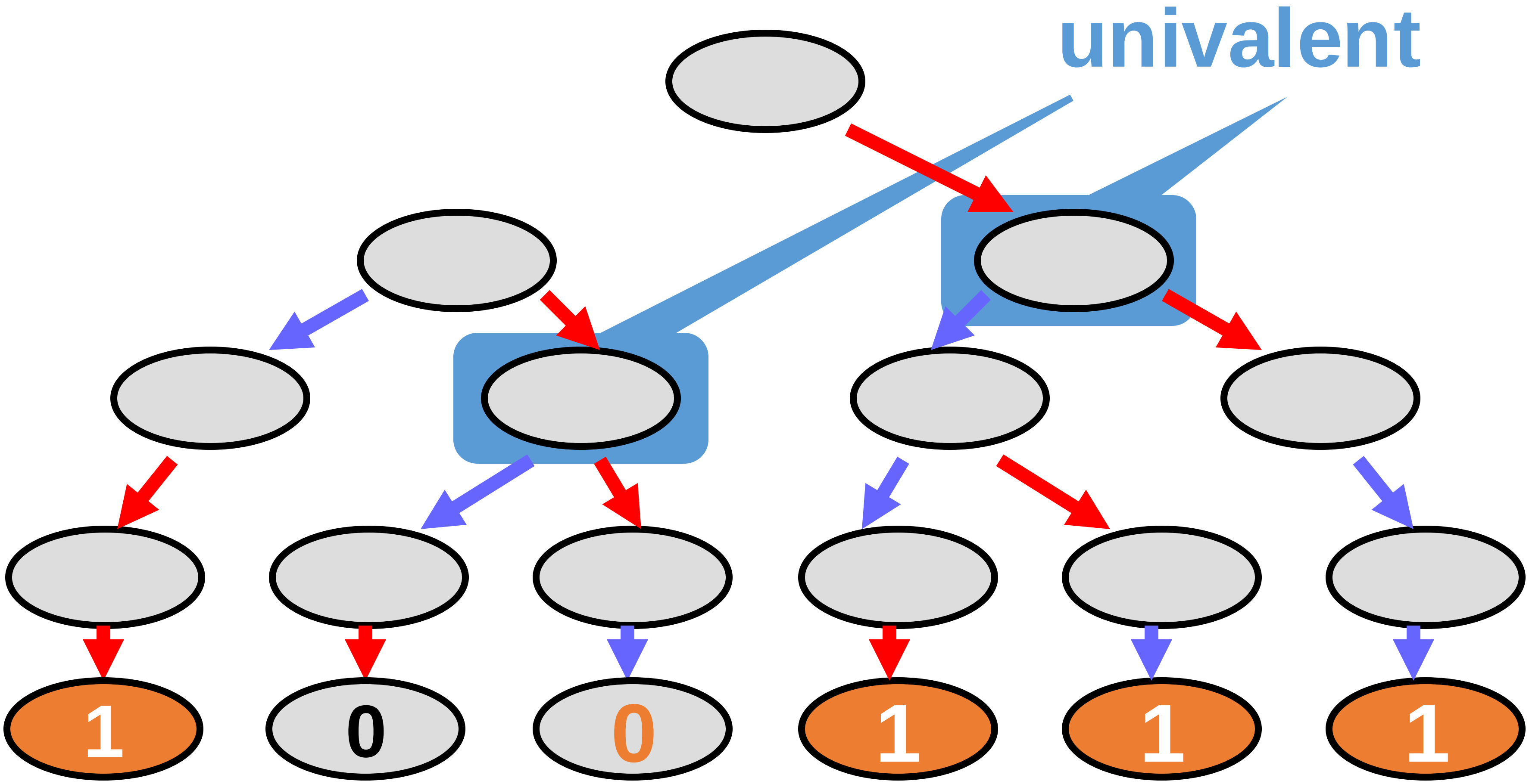
Decision Values



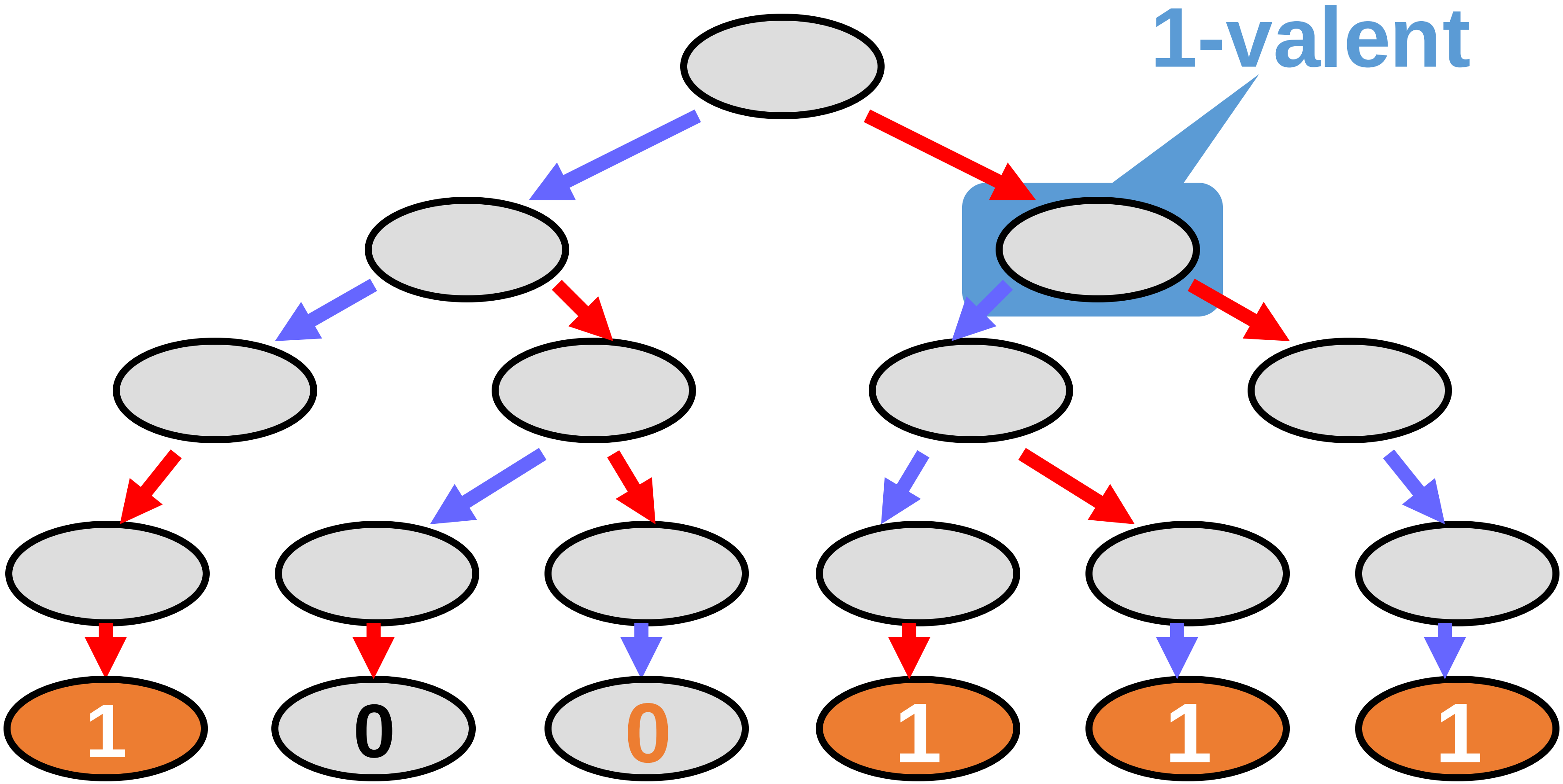
Bivalent: Both Possible



Univalent: Single Value Possible



x-valent: x Only Possible Decision



Summary

- Wait-free computation is a tree

Summary

- Wait-free computation is a tree
- Bivalent system states
  - Outcome not fixed

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- Bivalent system states
  - Outcome not fixed
- Univalent states
  - Outcome is fixed
  - May not be “known” yet

Summary

- Wait-free computation is a tree
- Bivalent system states
  - Outcome not fixed
- Univalent states
  - Outcome is fixed
  - May not be “known” yet
- 1-Valent and 0-Valent states

Claim

- Some initial state is bivalent

Claim

- Some initial state is bivalent
- Outcome depends on
  - Chance
  - Whim of the scheduler

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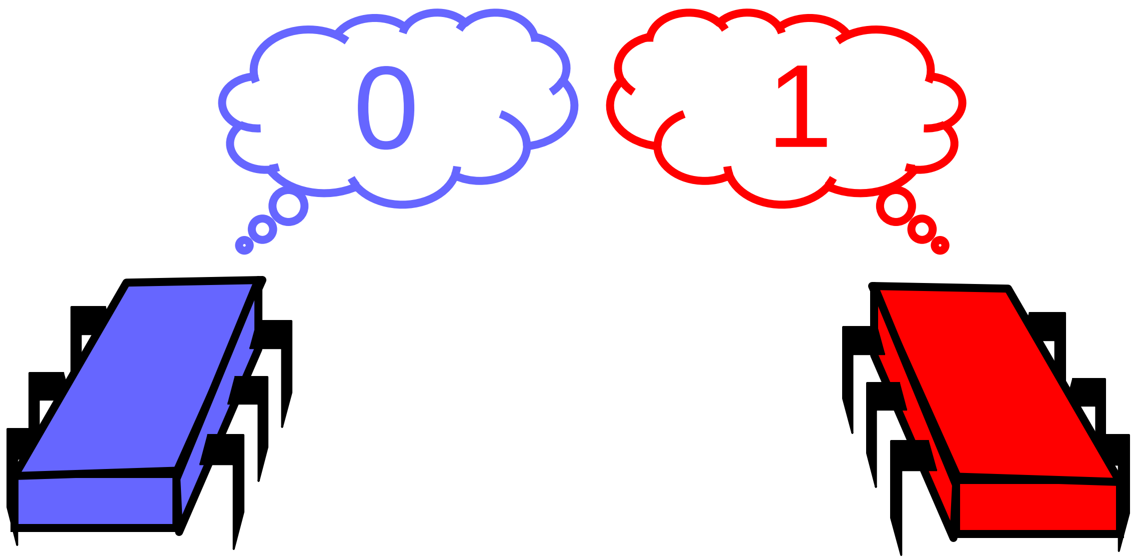
- Multicore gods procrastinate …

Claim

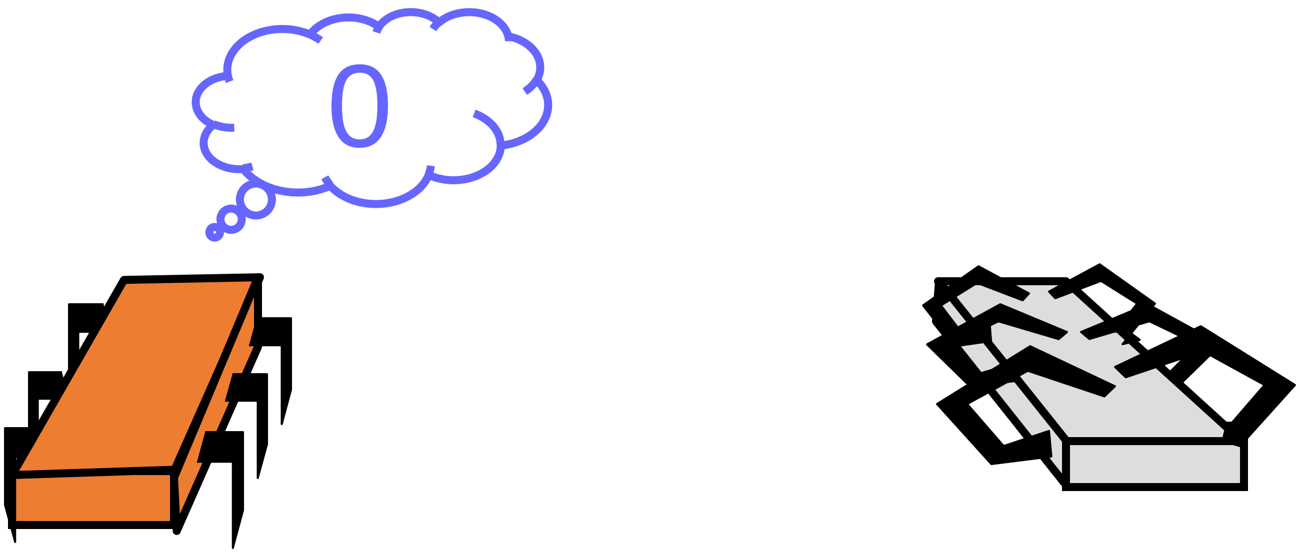
- Some initial state is bivalent
- Outcome depends on
  - Chance
  - Whim of the scheduler
- Multicore gods procrastinate …
- Let’s prove it …

What if inputs differ?



Must Decide 0

In this solo execution by A

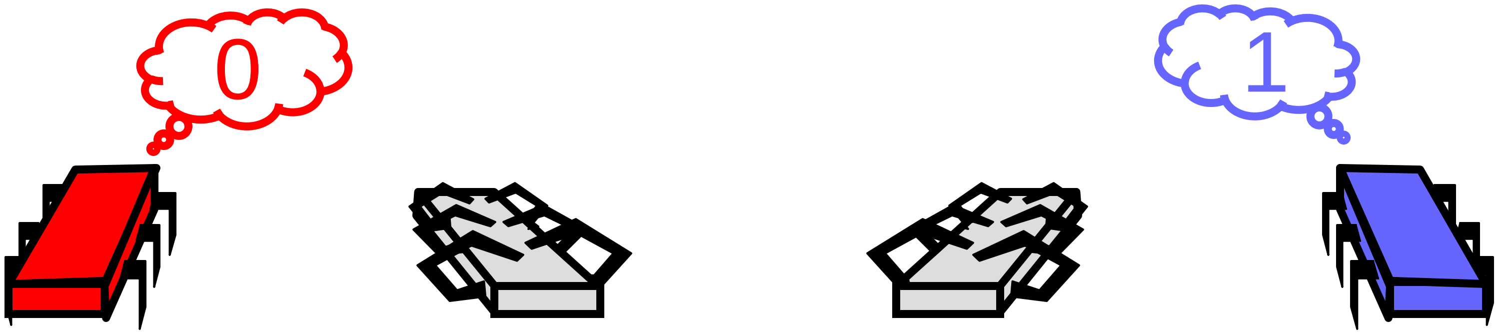


Must Decide 1

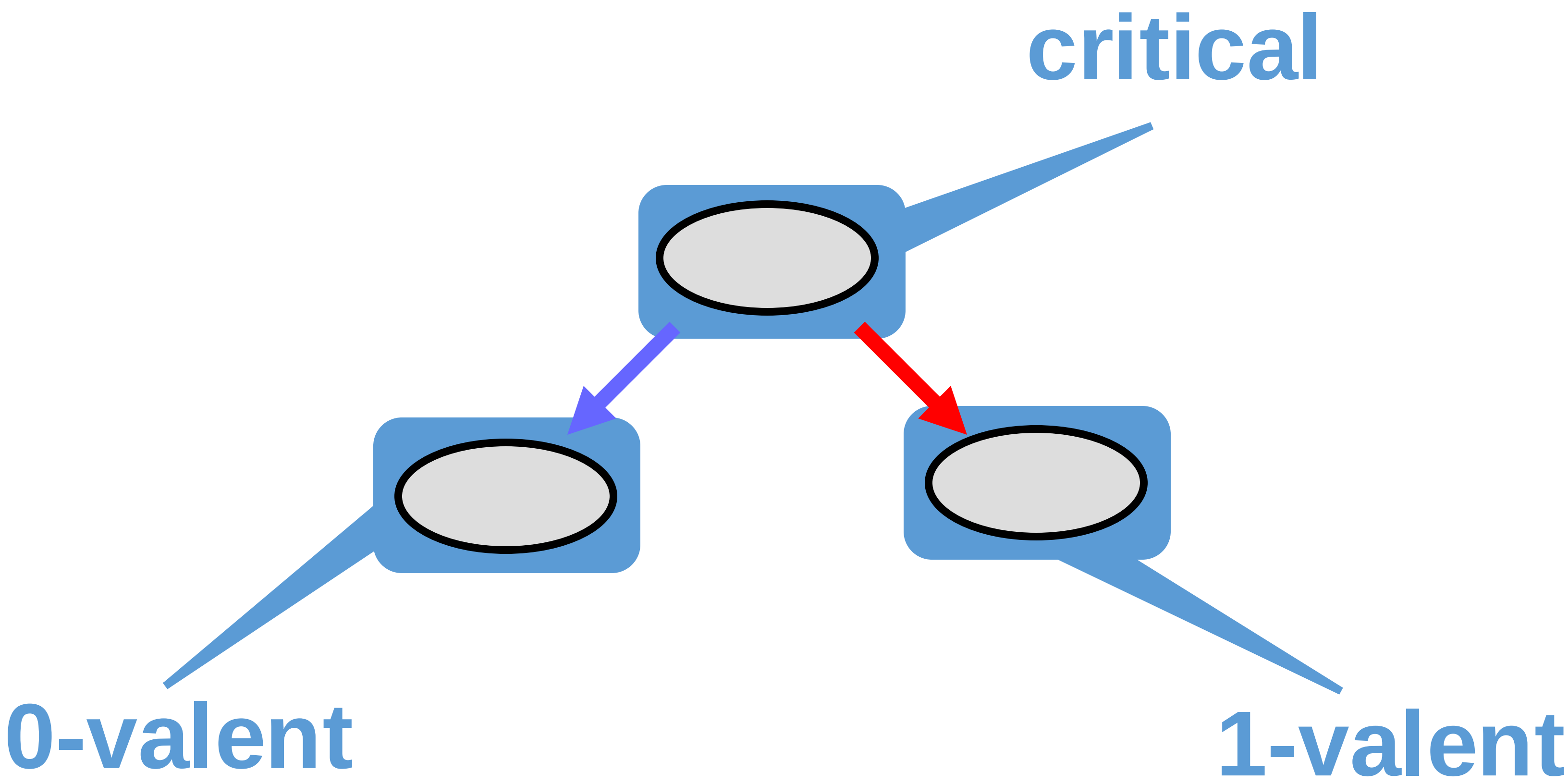
In this solo execution by B

Mixed Initial State Bivalent

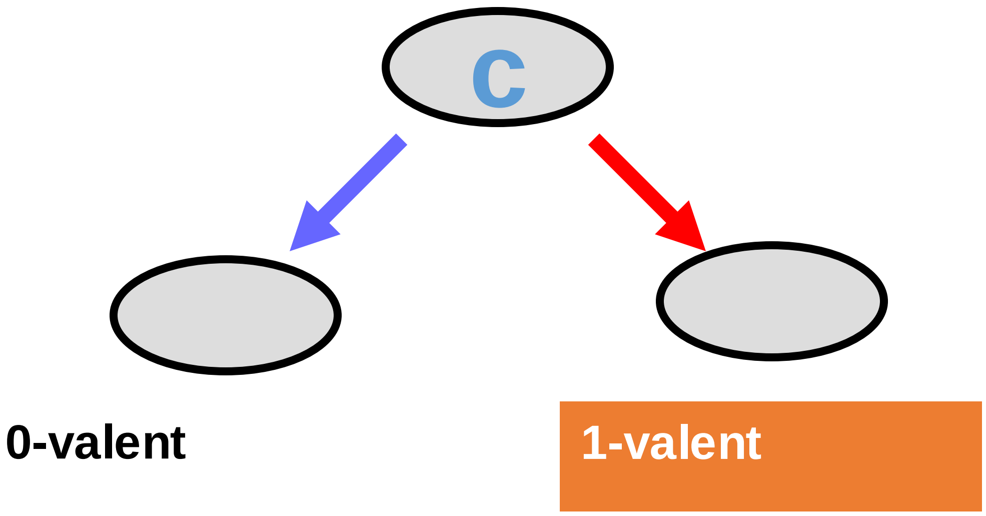
- Solo execution by A must decide 0
- Solo execution by B must decide 1



Critical States



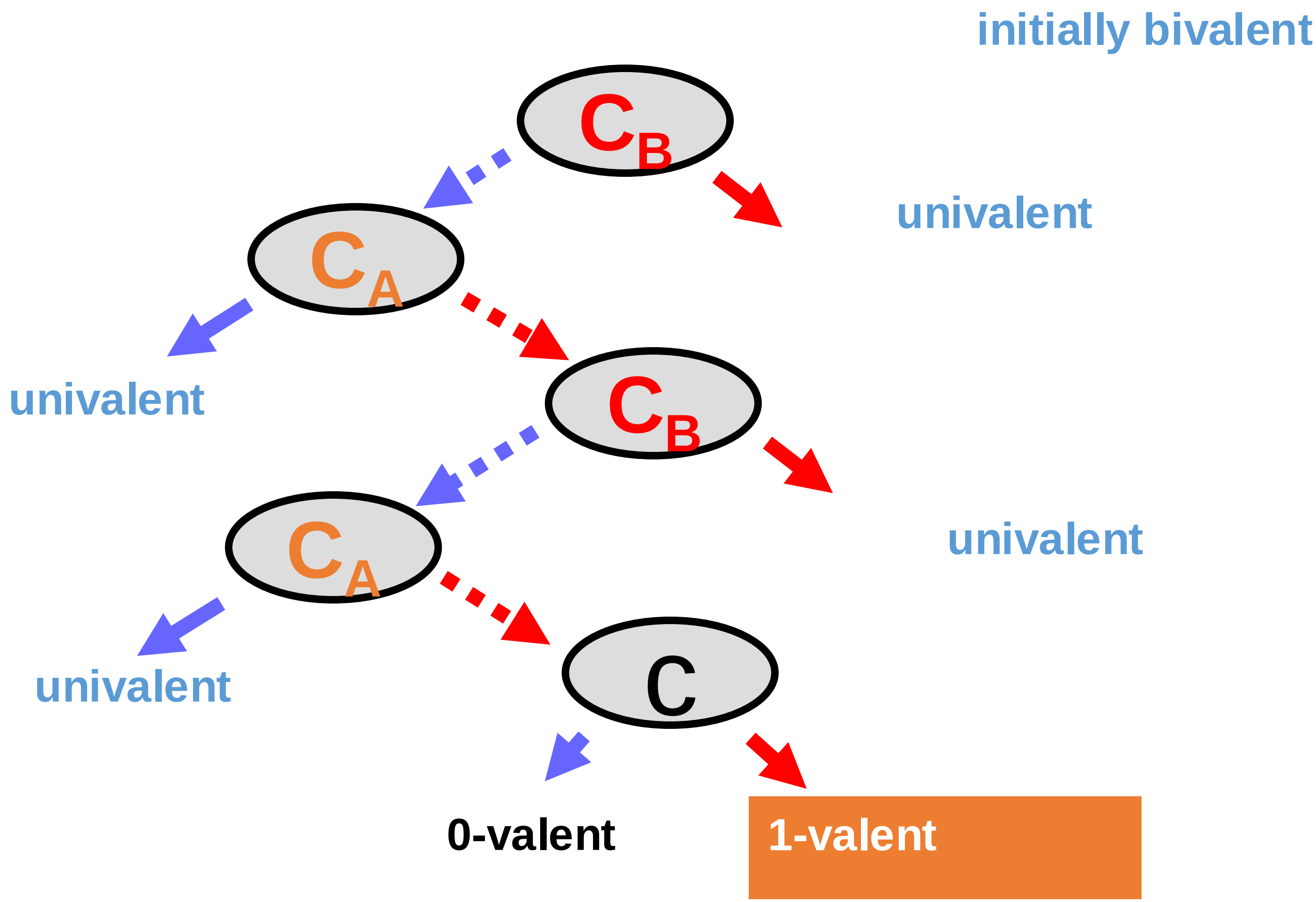
From a Critical State



If A goes first, protocol decides 0

If B goes first, protocol decides 1

Reaching Critical State



Critical States

- Starting from a bivalent initial state

Critical States

- Starting from a bivalent initial state
- The protocol must reach a critical state

Critical States

- Starting from a bivalent initial state
- The protocol must reach a critical state
  - Otherwise we could stay bivalent forever
  - And the protocol is not wait-free

Model Dependency

- So far, memory-independent!
- True for
  - Registers
  - Message-passing
  - Carrier pigeons
  - Any kind of asynchronous computation

Atomic Registers & Consensus

- Atomic registers have consensus number 1

Closing the Deal

- Start from a critical state

Closing the Deal

- Start from a critical state
- Each thread fixes outcome by
  - Reading or writing …
  - Same or different registers

Closing the Deal

- Start from a critical state
- Each thread fixes outcome by
  - Reading or writing …
  - Same or different registers
- Leading to a 0 or 1 decision …

Closing the Deal

- Start from a critical state
- Each thread fixes outcome by
  - Reading or writing …
  - Same or different registers
- Leading to a 0 or 1 decision …
- And a contradiction.

Possible Interactions

A reads x

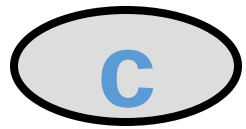
A reads y

	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	?	?	?	?
`y.read()`	?	?	?	?
`x.write()`	?	?	?	?
`y.write()`	?	?	?	?

Some Thread Reads

	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	?	?	?	?
`y.read()`	?	?	?	?
`x.write()`	?	?	?	?
`y.write()`	?	?	?	?

Some Thread Reads



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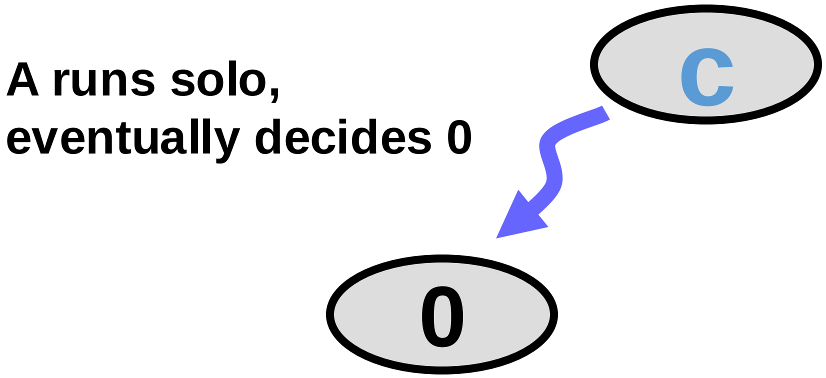

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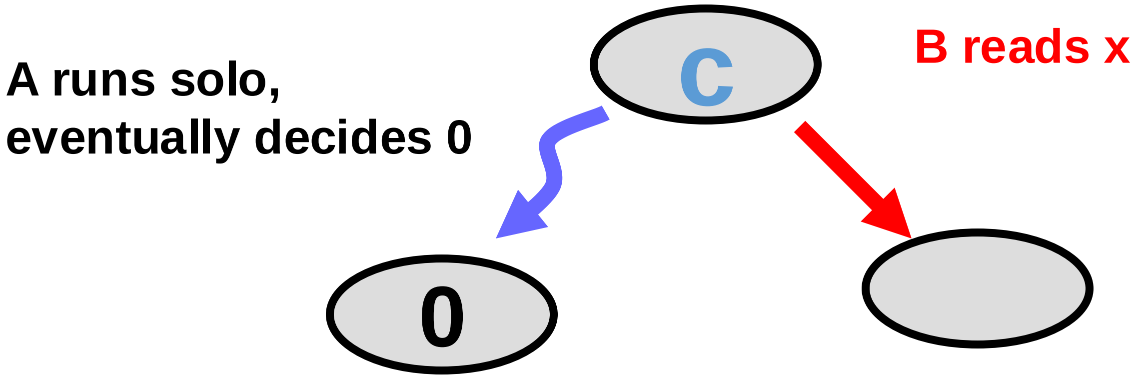

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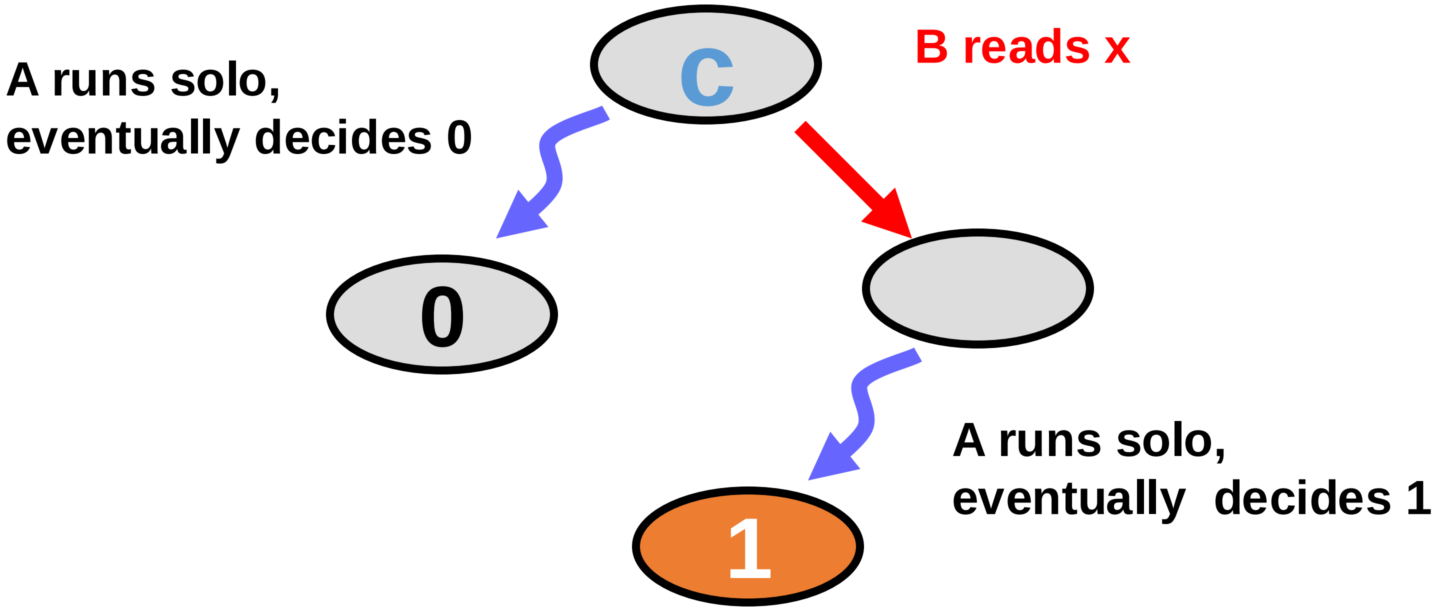

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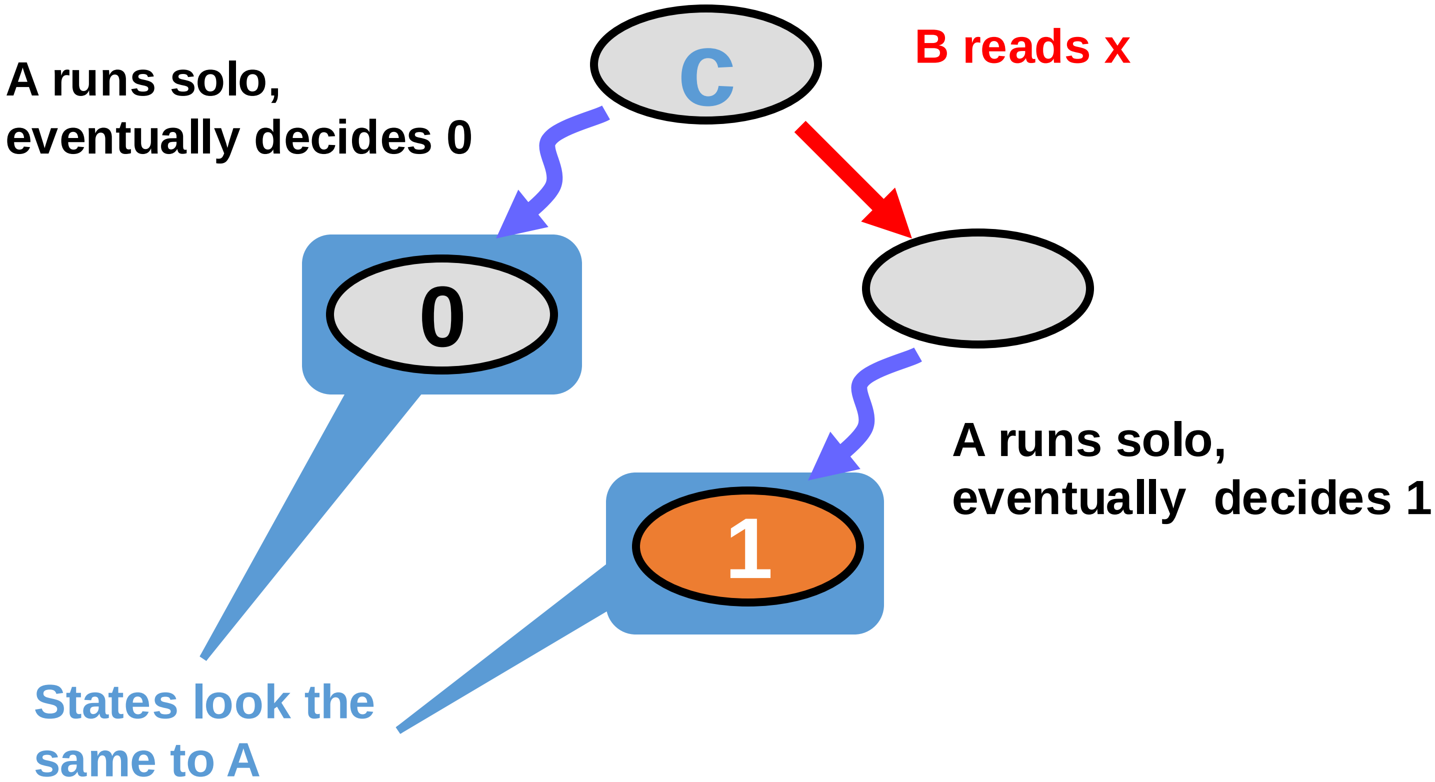

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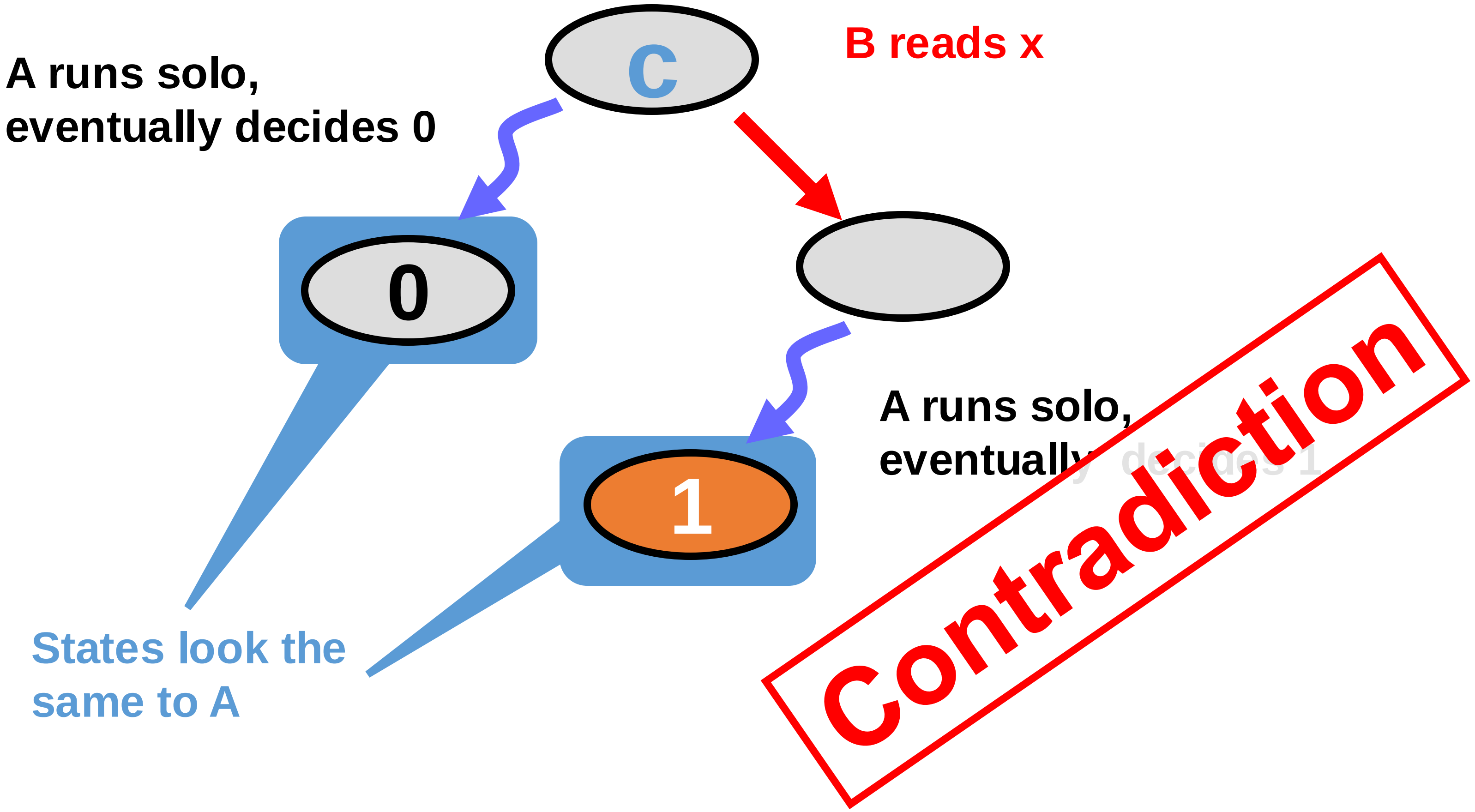


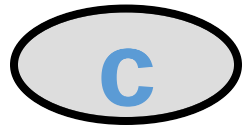
Possible Interactions

	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	no	no	no	no
`y.read()`	no	no	no	no
`x.write()`	no	no	?	?
`y.write()`	no	no	?	?

Writing Distinct Registers

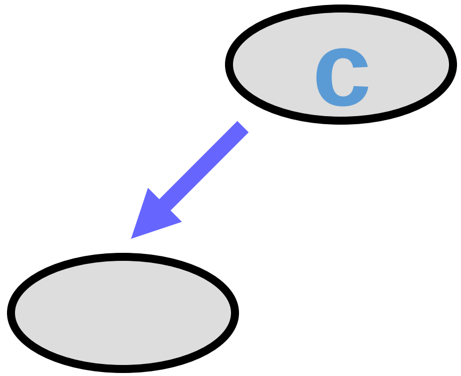
	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	no	no	no	no
`y.read()`	no	no	no	no
`x.write()`	no	no	?	?
`y.write()`	no	no	?	?

Writing Distinct Registers



Writing Distinct Registers

A writes y



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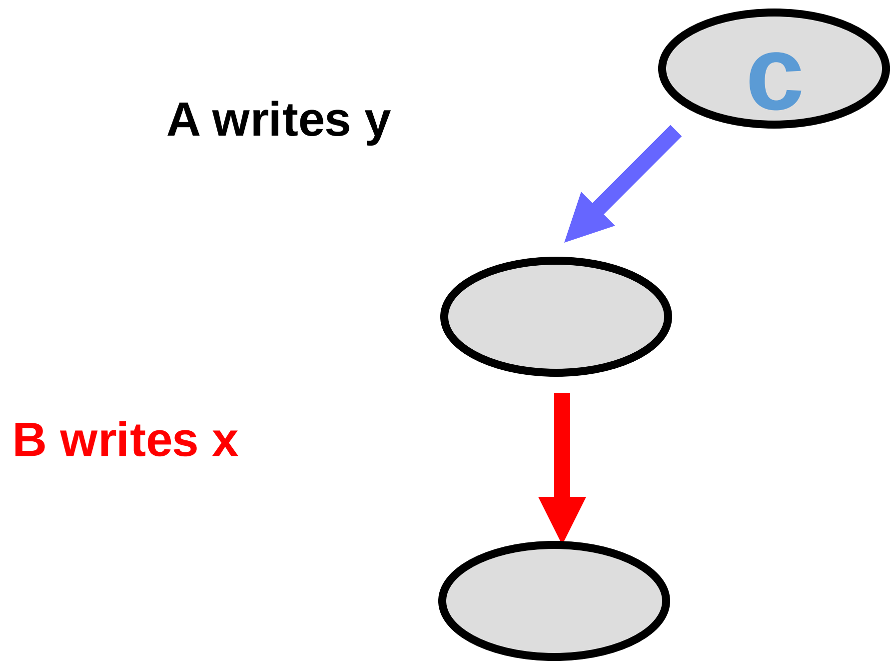

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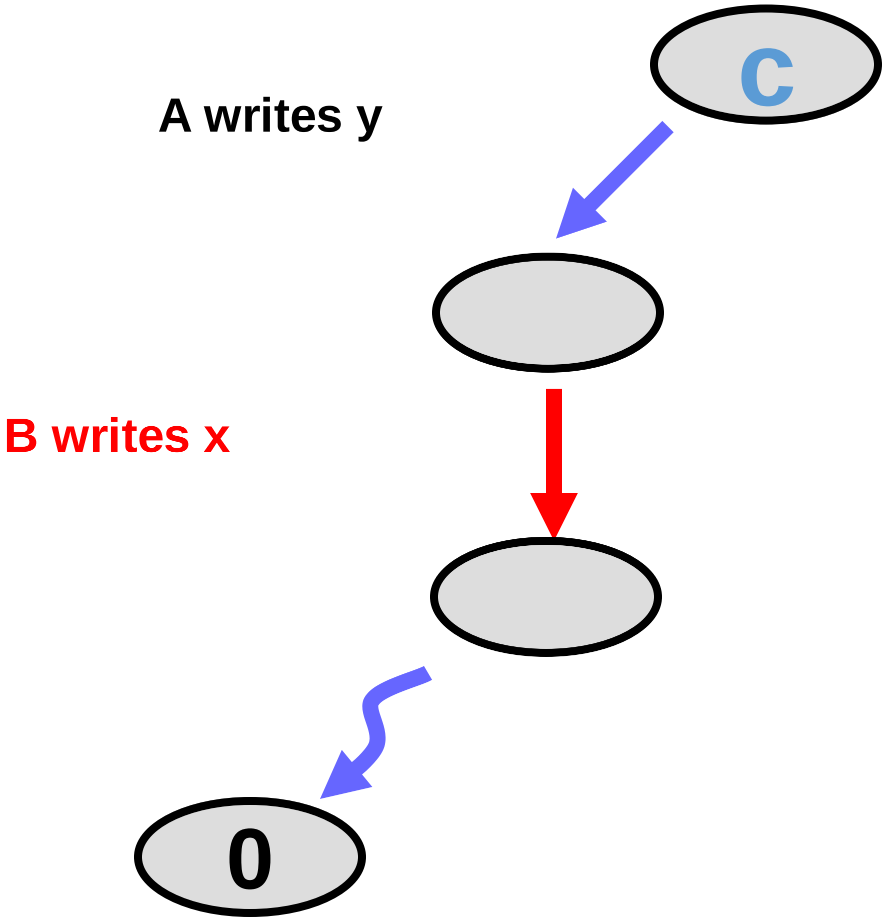

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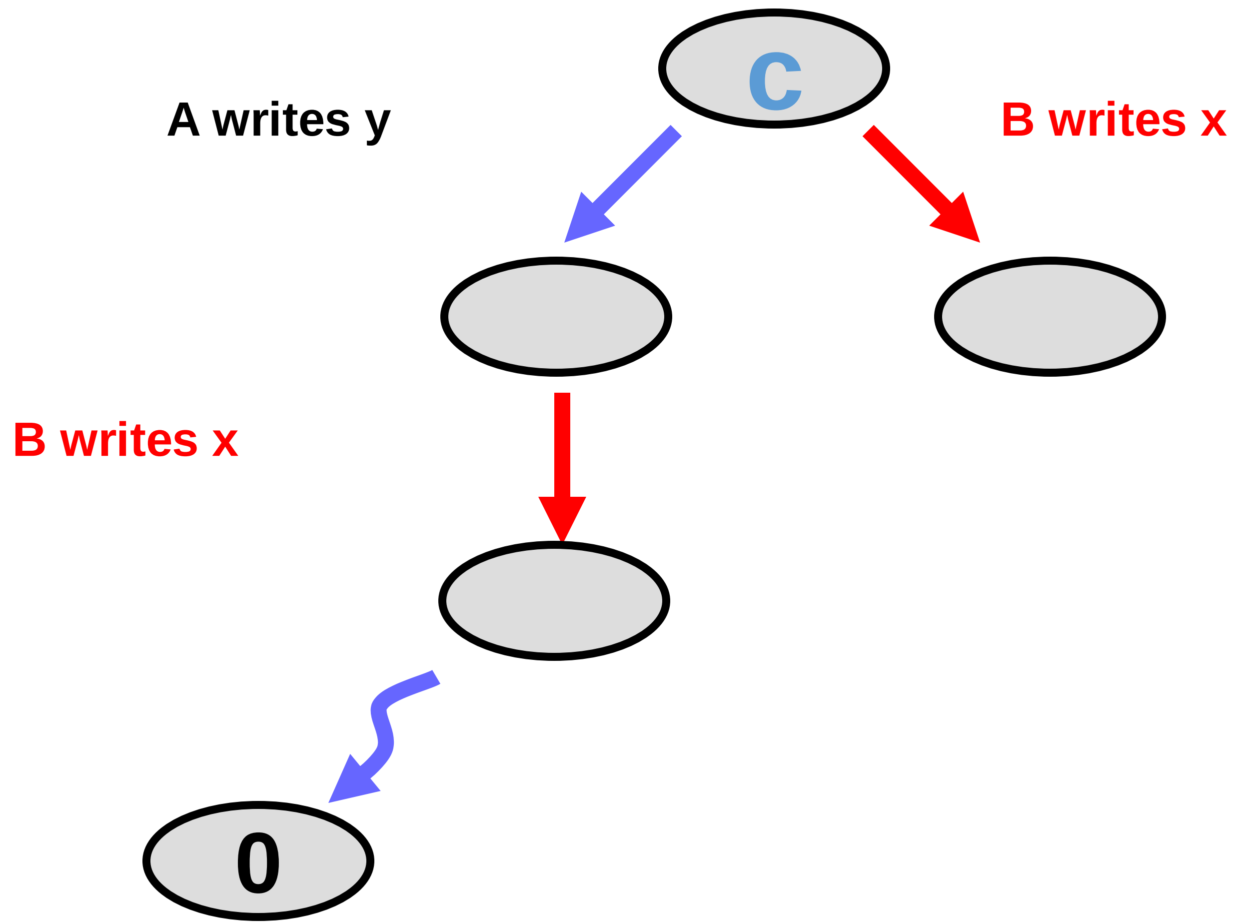

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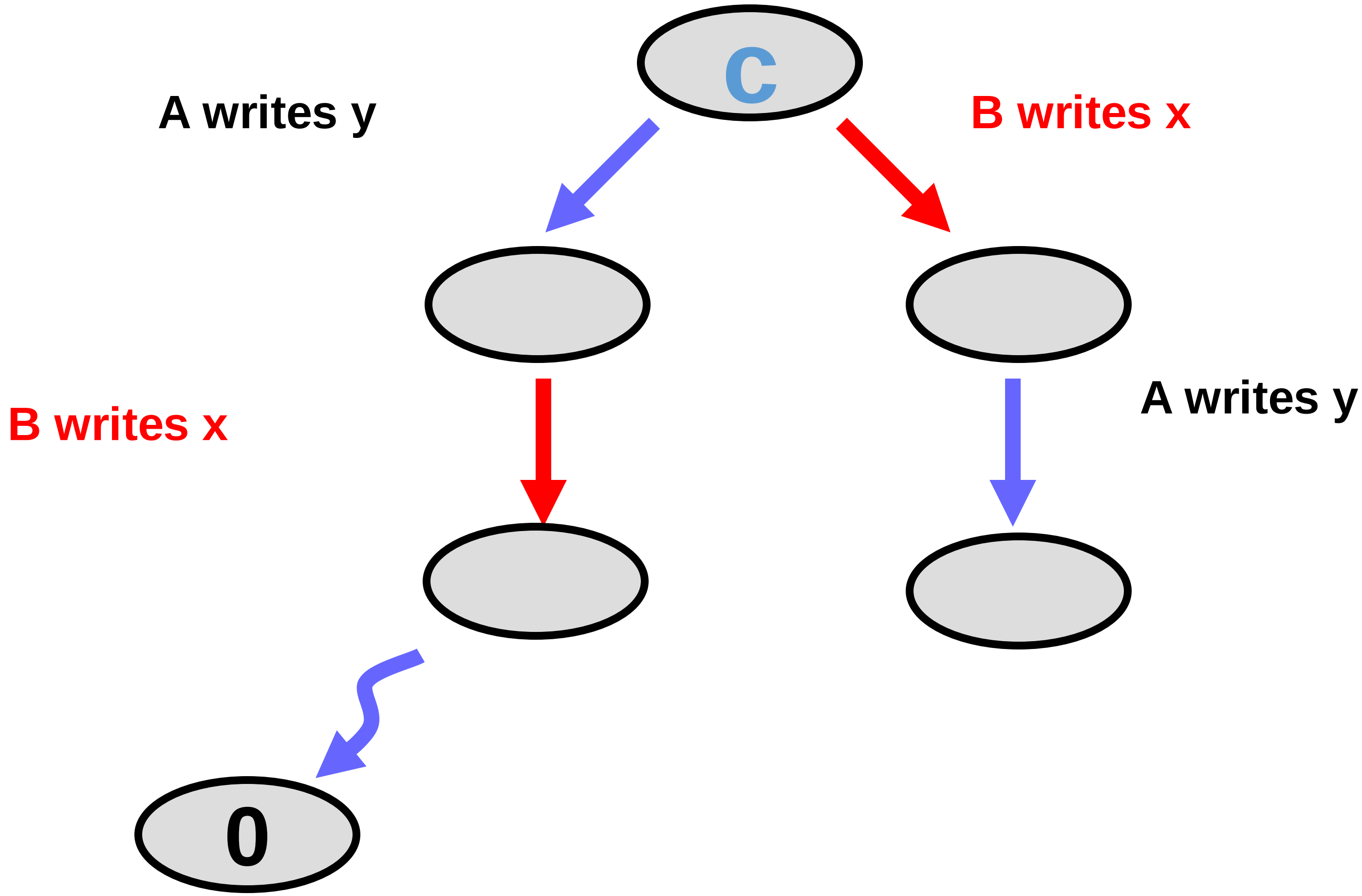

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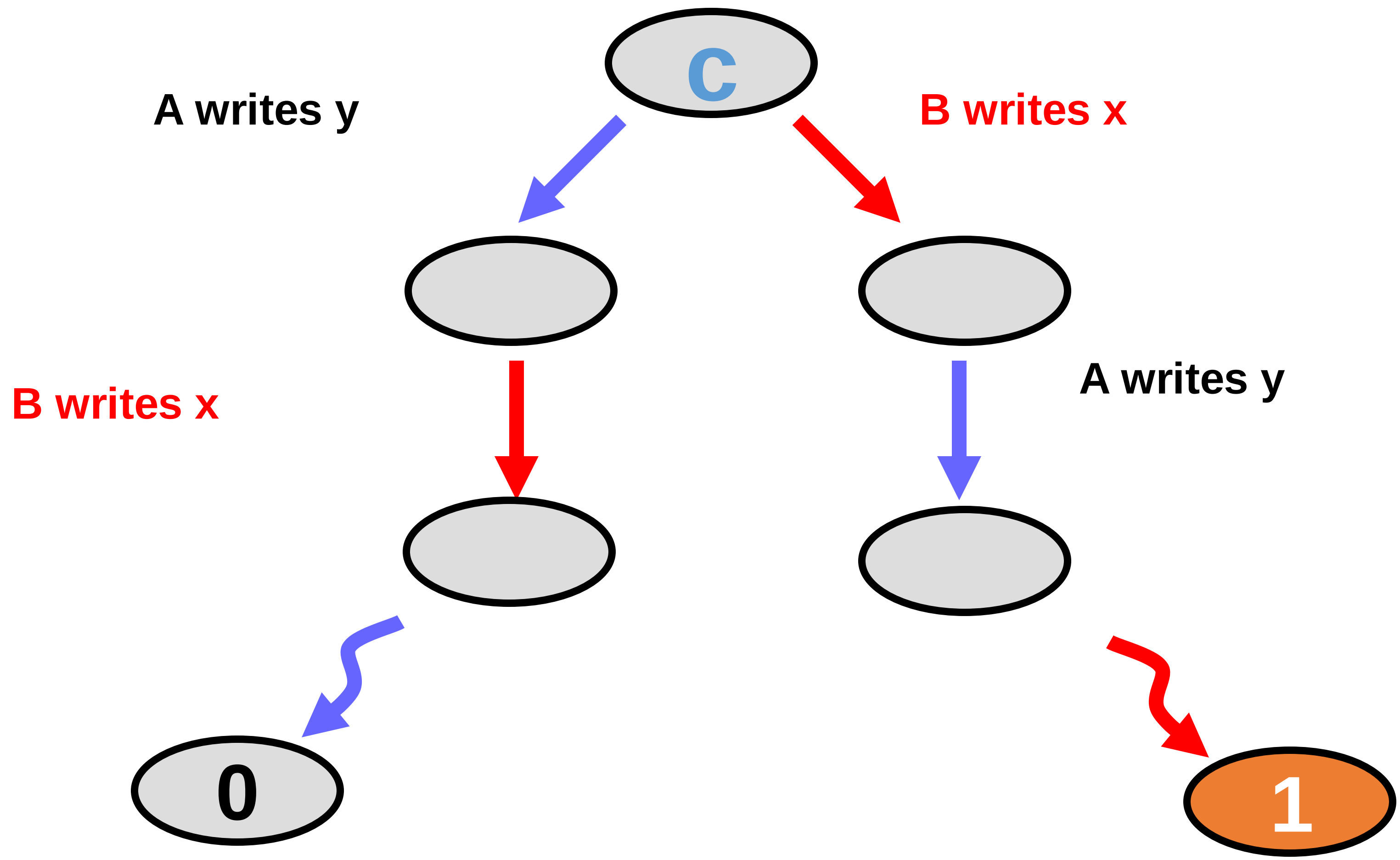

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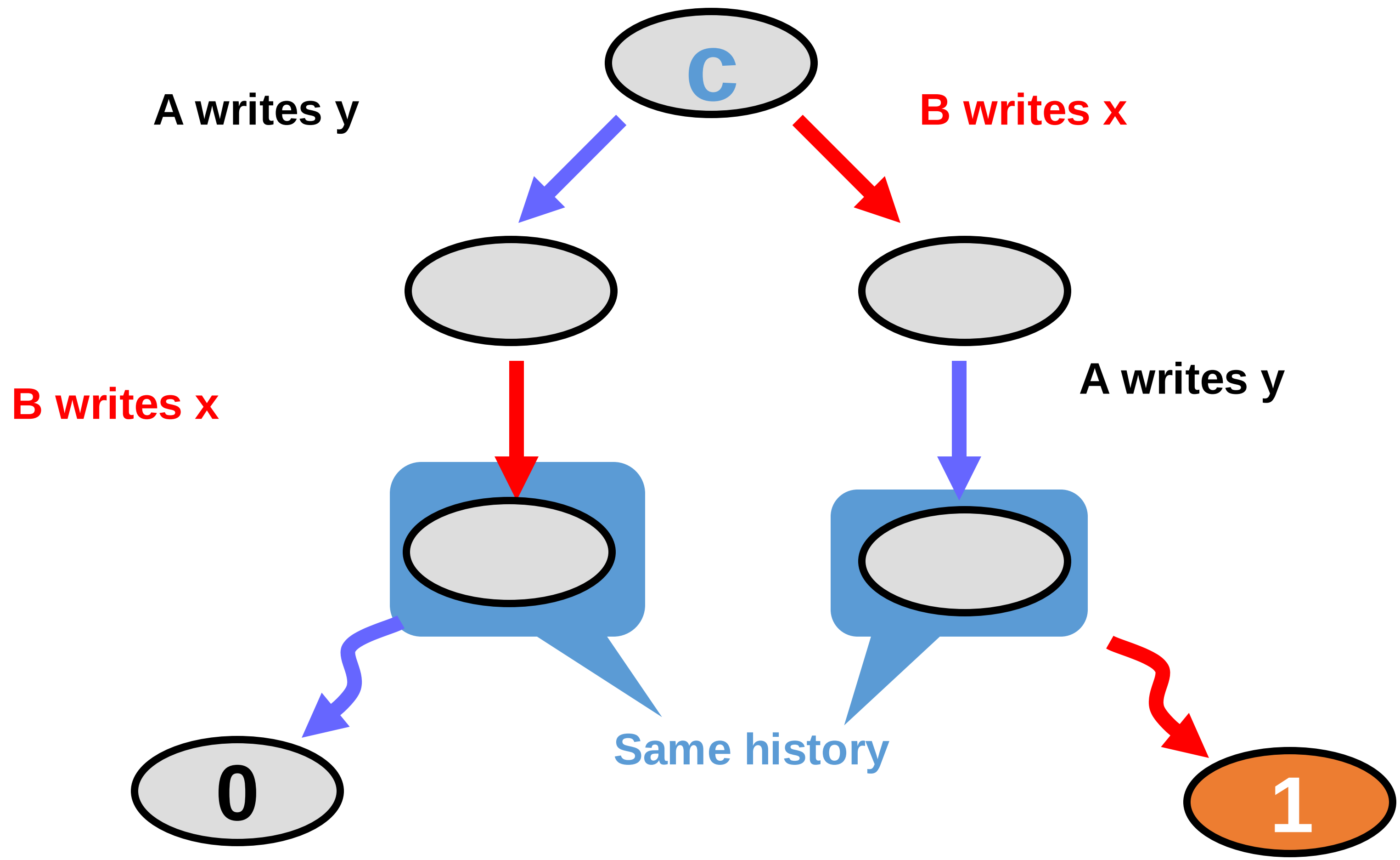

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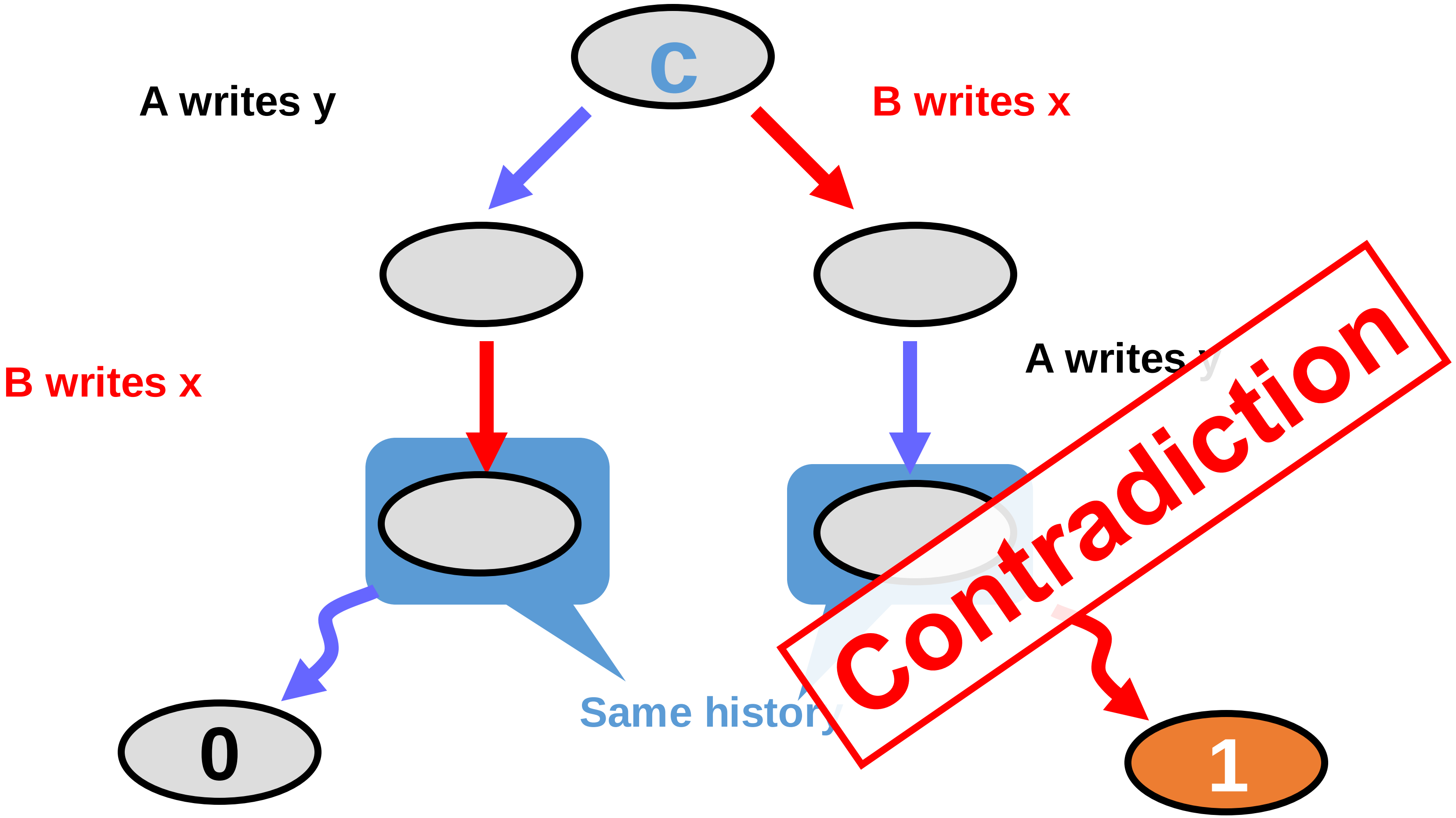


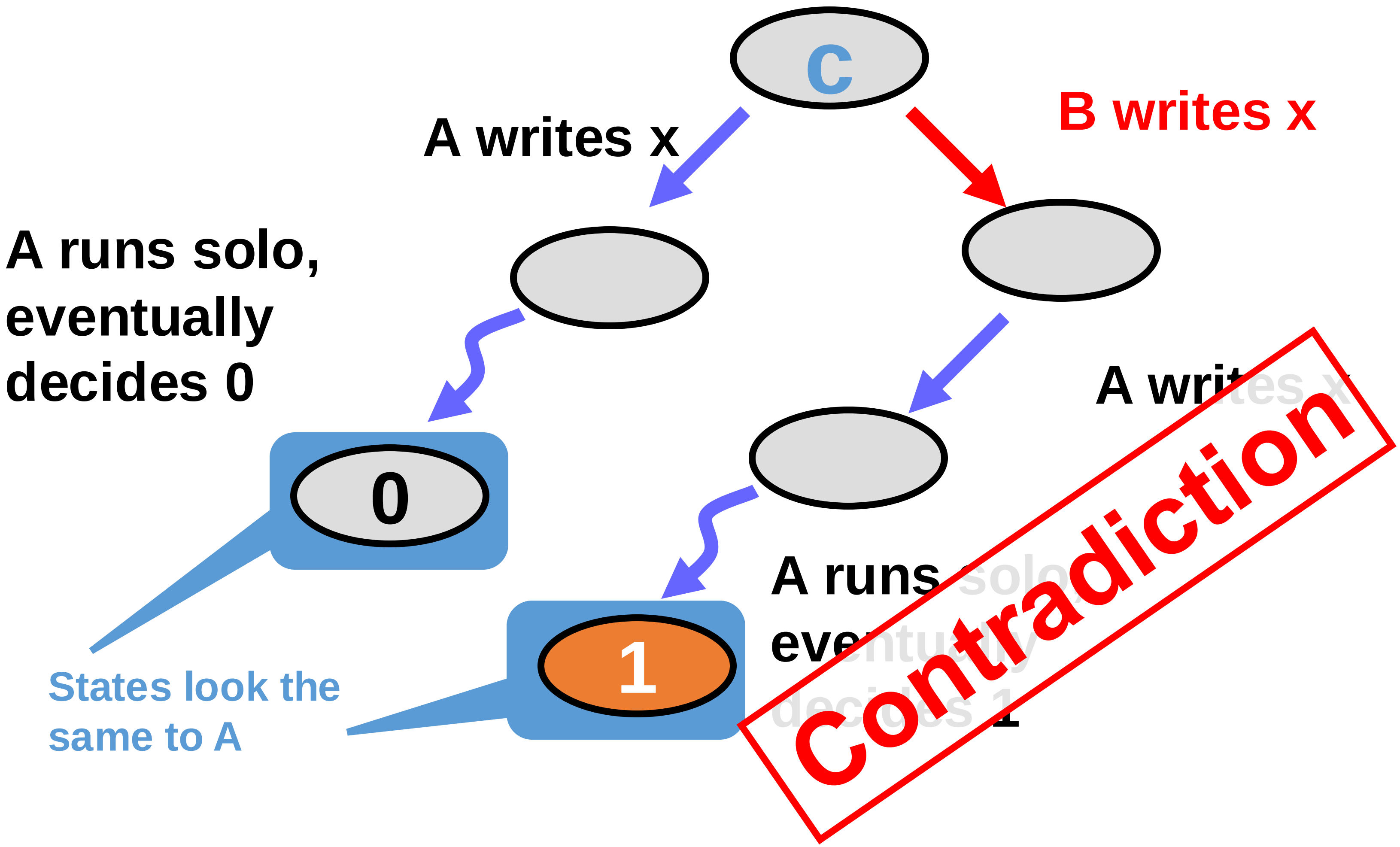
Possible Interactions

	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	no	no	no	no
`y.read()`	no	no	no	no
`x.write()`	no	no	?	no
`y.write()`	no	no	no	?

Writing Same Registers

	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	no	no	no	no
`y.read()`	no	no	no	no
`x.write()`	no	no	?	no
`y.write()`	no	no	no	?

Writing Same Registers



That's All, Folks!

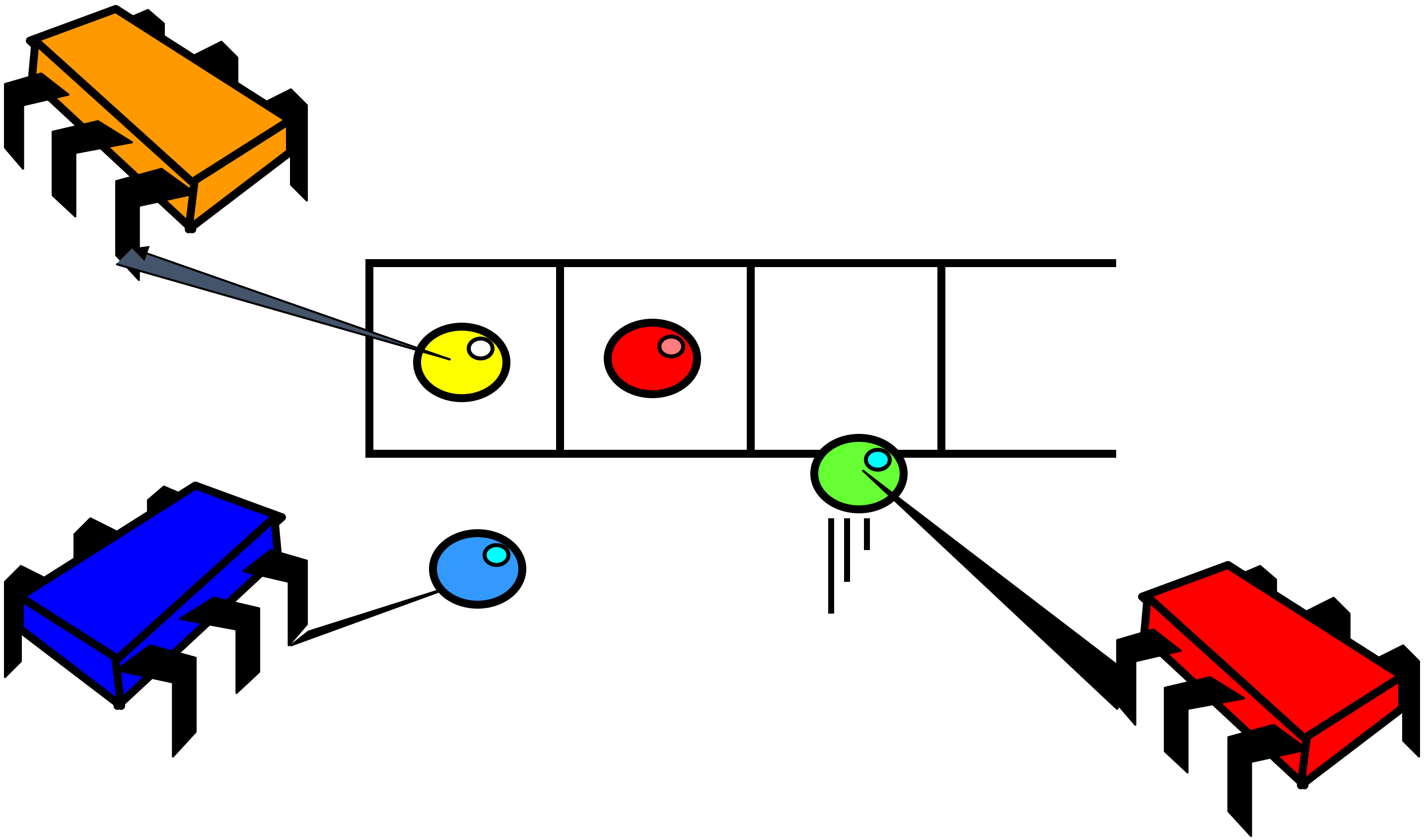
	x.read()	y.read()	x.write()	y.write()
x.read()	no	no	no	no
y.read()	no	no	no	no
x.write()	no	no	no	no
y.write()	no	no	no	no

QED

Recap: Atomic Registers Can't Do Consensus

- If protocol exists
  - It has a bivalent initial state
  - Leading to a critical state
- What's up with the critical state?
  - Case analysis for each pair of methods
  - As we showed, all lead to a contradiction

What Does Consensus have to do with Concurrent Objects?



Consensus Object

```java
public interface Consensus<T> {
 T decide(T value);
}
```

Concurrent Consensus Object

- We consider only one time objects:
  - each thread calls method only once
- Linearizable to sequential consensus object:
  - Winner's call went first

Java Jargon Watch

- Define Consensus protocol as an abstract class
- We implement some methods
- You do the rest …

Generic Consensus Protocol

```
abstract class ConsensusProtocol<T>
   implements Consensus<T> {
 protected T[] proposed = new T[N];

 protected void propose(T value) {
  proposed[ThreadID.get()] = value;
 }

 abstract public T decide(T value);
}
```

Generic Consensus Protocol

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abstract class ConsensusProtocol<T>
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 abstract public T dec
}
```

Each thread's proposed value

Generic Consensus Protocol

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```

Propose a value

Generic Consensus Protocol

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abstract class ConsensusProtocol<T>
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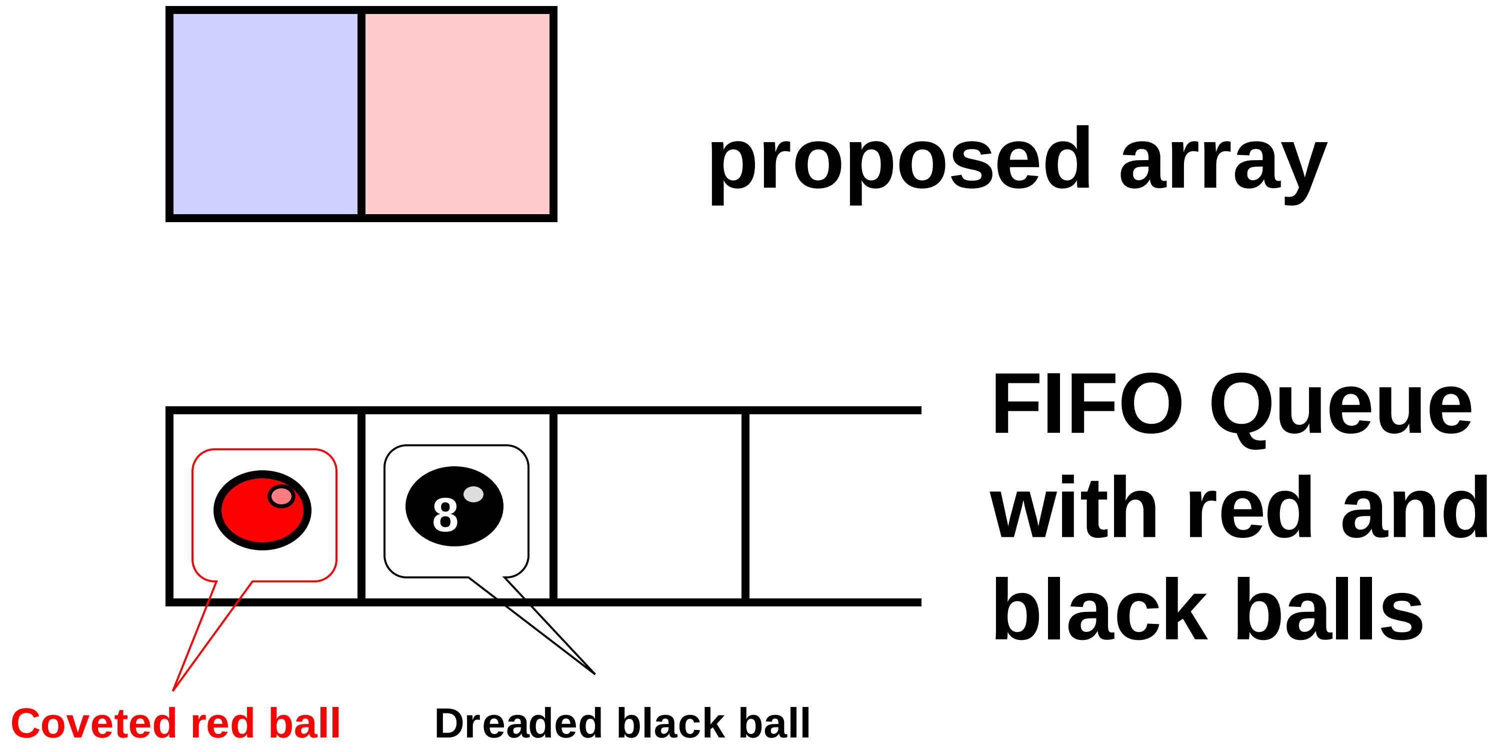
 protected void propose(T value) {
  proposed[ThreadID.get()] = value;
 }

 abstract public T decide(T value);
}
```

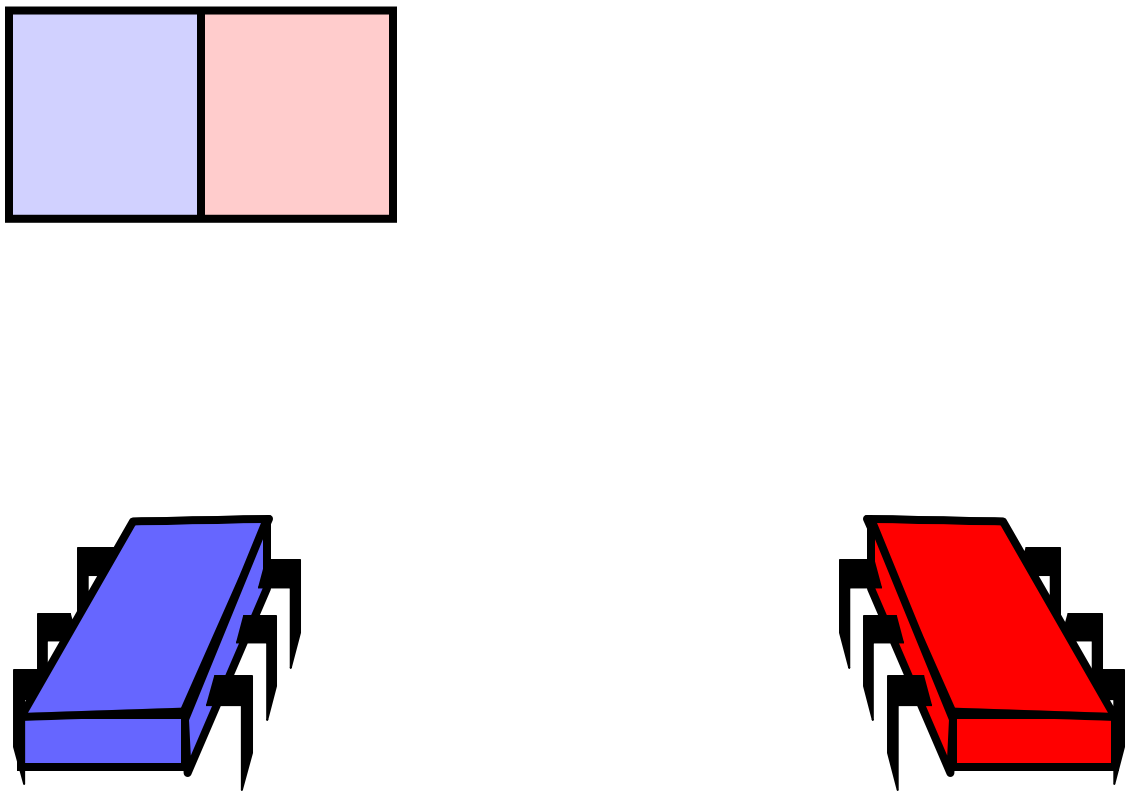
Decide a value: abstract method means subclass does the real work

Can a FIFO Queue Implement Consensus?

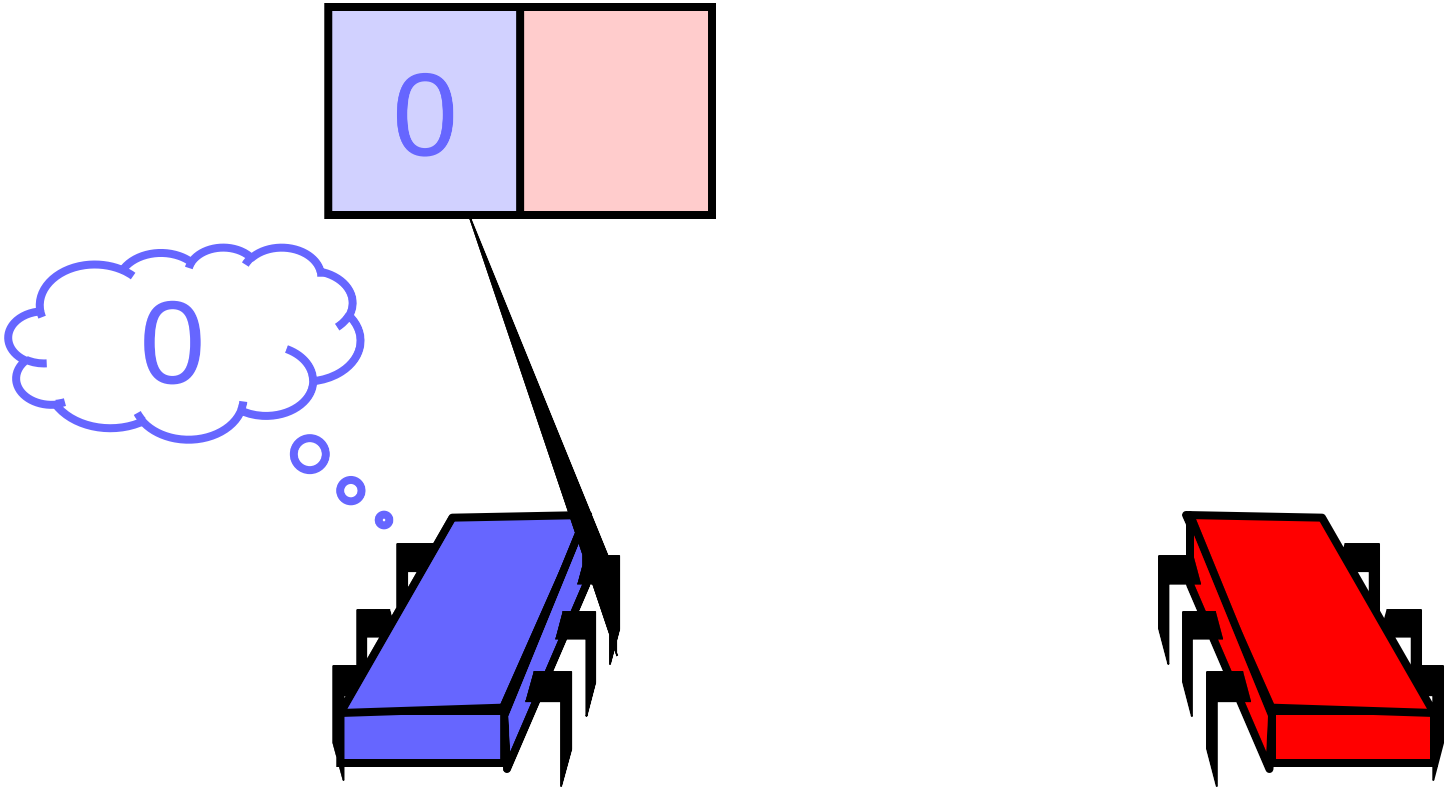
FIFO Consensus



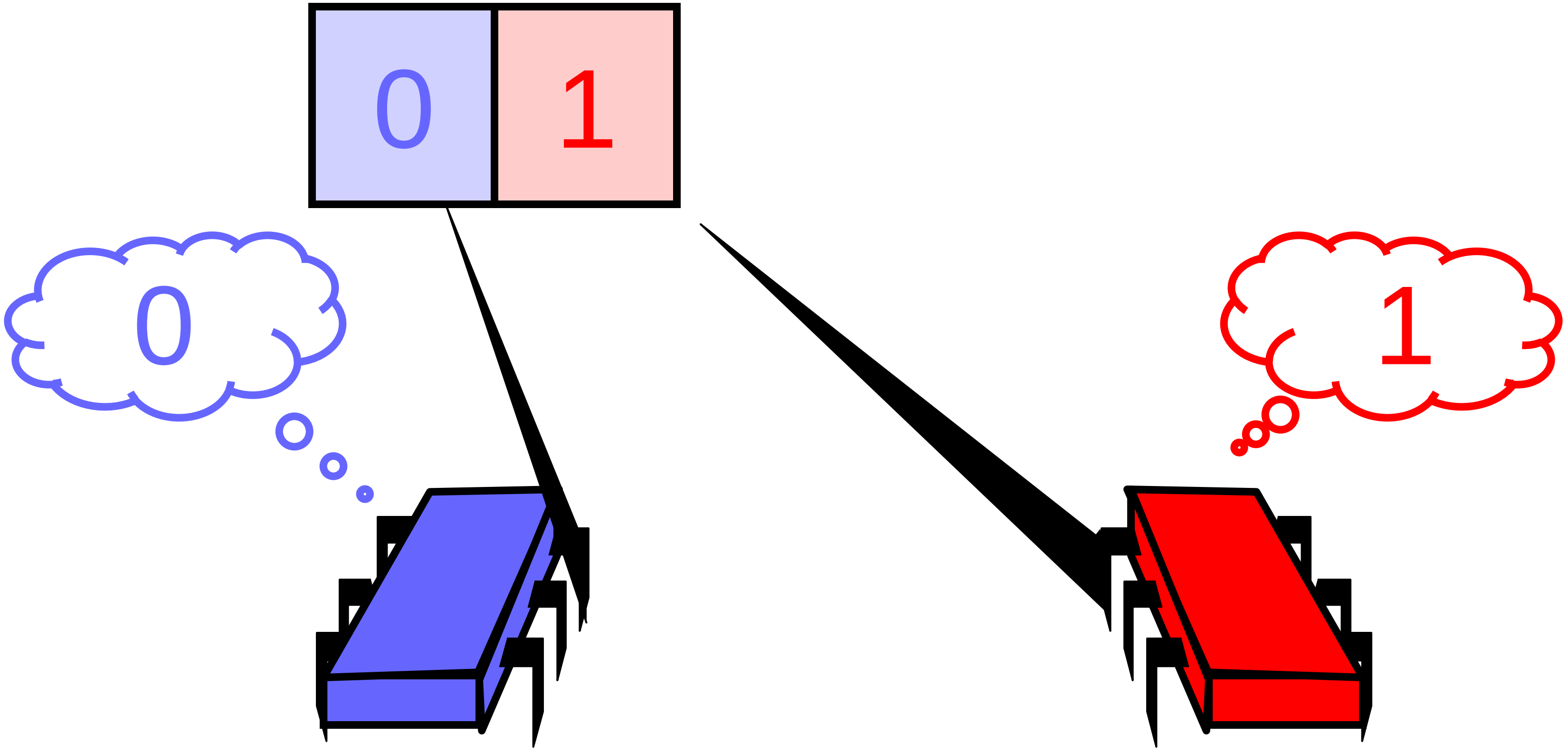
Protocol: Write Value to Array



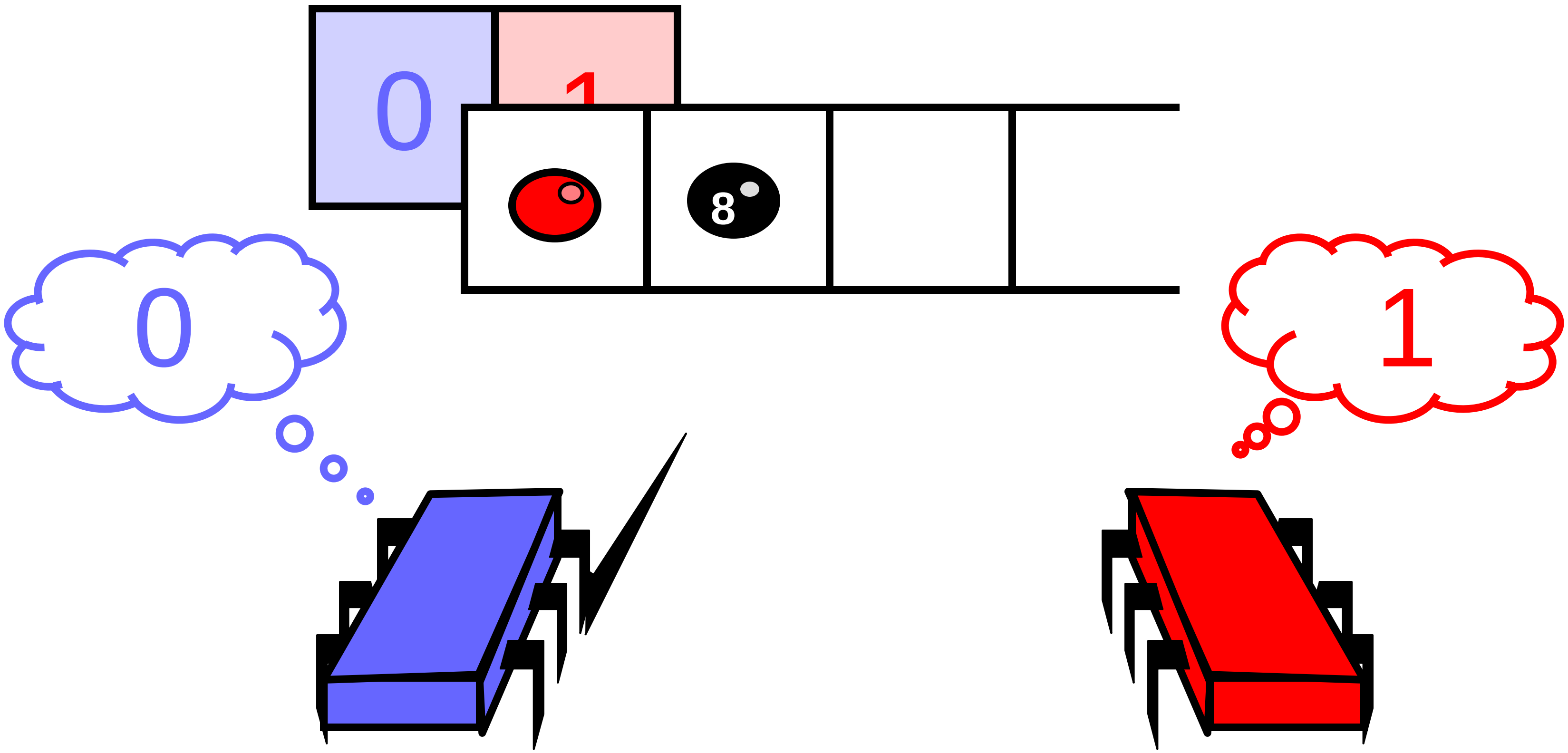
Protocol: Write Value to Array



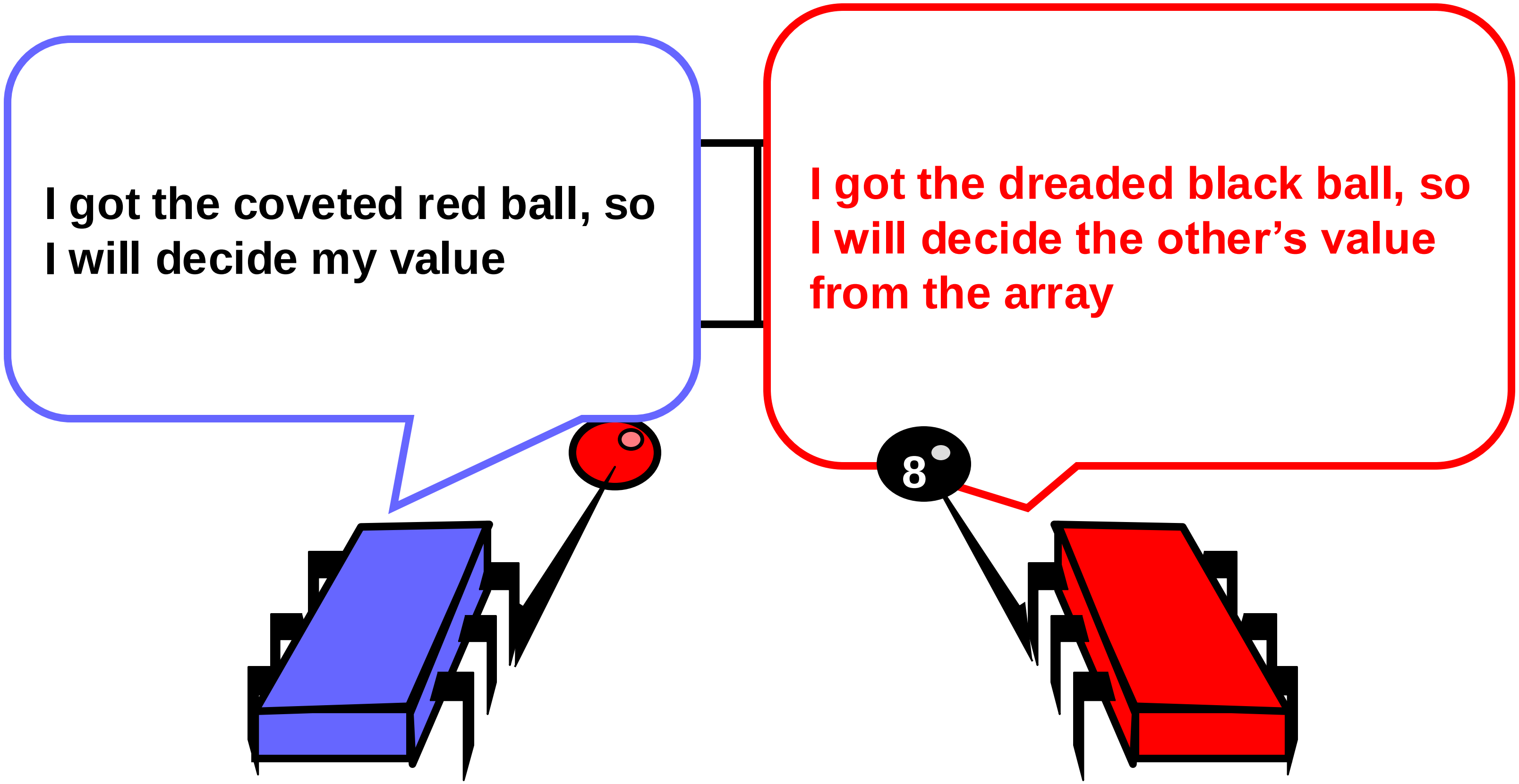
Protocol: Write Value to Array



Protocol: Take Next Item from Queue



Protocol: Take Next Item from Queue

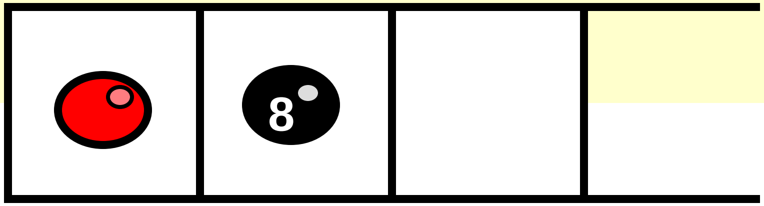


Consensus Using FIFO Queue

```
public class QueueConsensus<T>
  extends ConsensusProtocol<T> {
 private Queue queue;
 public QueueConsensus() {
  queue = new Queue();
  queue.enq(Ball.RED);
  queue.enq(Ball.BLACK);
 }
 …
}
```

Initialize Queue

```
public class QueueConsensus<T>
   extends ConsensusProtocol<T> {
 private Queue queue;
 public QueueConsensus() {
   queue = new Queue();
   queue.enq(Ball.RED);
   queue.enq(Ball.BLACK);
  }
 …
}
```



Who Won?

```
public class QueueConsensus<T>
  extends ConsensusProtocol<T> {
 private Queue queue;
 …
 public T decide(T value) {
  propose(value);
  Ball ball = queue.deq();
  if (ball == Ball.RED)
   return proposed[i];
  else
   return proposed[1-i];
 }
}
```

Who Won?

```
public class QueueConsensus<T>
  extends ConsensusProtocol<T> {
 private Queue queue;
 …
 public T decide(T value) {
  propose(value);
  Ball ball = queue.deq();
  if (ball == Ball.RED)
   return proposed[i];
  else
   return proposed[1-i];
 }
}
```

Race to dequeue first queue item

Who Won?

```
public class QueueConsensus<T>
  extends ConsensusProtocol<T> {
 private Queue queue;
 …
 public T decide(T value) {
  propose(value);
  Ball ball = queue.deq();
  if (ball == Ball.RED)
   return proposed[i];
  else
   return proposed[1-i];
 }
}
```

I win if I was first

Who Won?

```
public class QueueConsensus<T>
  extends ConsensusProtocol<T> {
 private Queue queue;
 …
 public T decide(T value) {
  propose(value);
  Ball ball = queue.deq();
  if (ball == Ball.RED)
   return proposed[i];
  else
   return proposed[1-i];
 }
}
```

Other thread wins if I was second

Why does this Work?

- If one thread gets the red ball
- Then the other gets the black ball
- Winner decides her own value
- Loser can find winner's value in array
  - Because threads write array
  - Before dequeueing from queue

Theorem

- We can solve 2-thread consensus using only
  - A two-dequeuer queue, and
  - Some atomic registers

Implications

- Given
  - A consensus protocol from queue and registers
- Assume there exists
  - A queue implementation from atomic registers
- Substitution yields:
  - A wait-free consensus protocol from at
    registers

contradiction

Corollary

- It is impossible to implement
  - a two-dequeuer wait-free FIFO queue
  - from read/write memory.

Consensus Numbers

- An object X has consensus number *n*
  - If it can be used to solve *n*-thread consensus
    - Take any number of instances of X
    - together with atomic read/write registers
    - and implement *n*-thread consensus
  - But not (*n*+1)-thread consensus

Consensus Numbers

- Theorem
  - Atomic read/write registers have consensus number 1
- Theorem
  - Multi-dequeuer FIFO queues have consensus number at least 2

Consensus Numbers Measure Synchronization Power

- Theorem
  - If you can implement X from Y
  - And X has consensus number c
  - Then Y has consensus number at least c

Synchronization Speed Limit

- Conversely
  - If X has consensus number c
  - And Y has consensus number d < c
  - Then there is no way to construct a wait-free implementation of X by Y
- This theorem will be very useful
  - Unforeseen practical implications!

Theoretical Caveat: Certain weird exceptions exist

Earlier Grand Challenge

- Snapshot means
  - Write any array element
  - Read multiple array elements atomically
- What about
  - Write multiple array elements atomically
  - Scan any array elements
- Call this problem multiple assignment

Multiple Assignment Theorem

- Atomic registers cannot implement multiple assignment
- Weird or what?
  - Single write/multi read OK
  - Multi write/multi read impossible

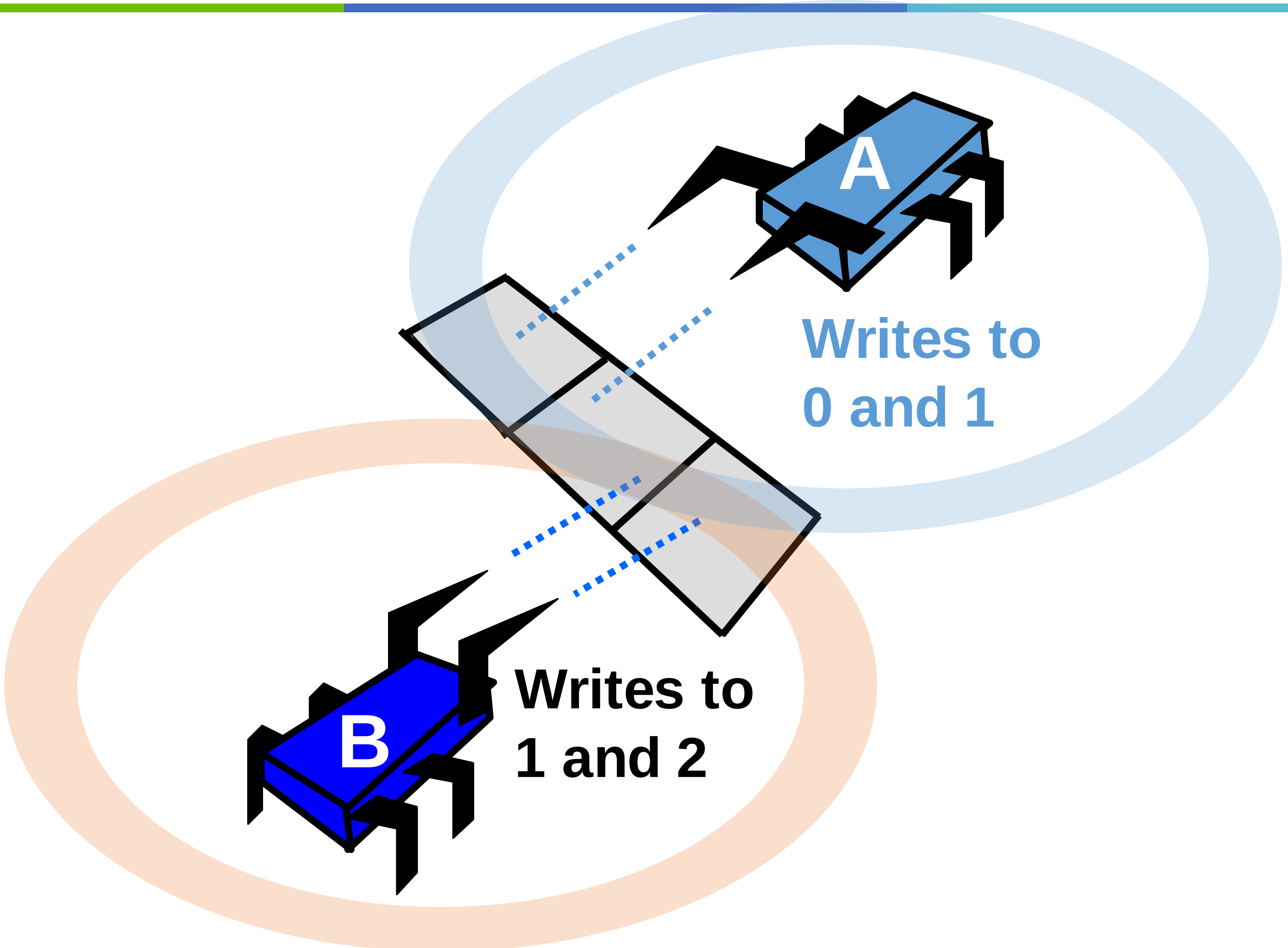
Proof Strategy

- If we can write to 2/3 array elements
  - We can solve 2-consensus
  - Impossible with atomic registers
- Therefore
  - Cannot implement multiple assignment with atomic registers

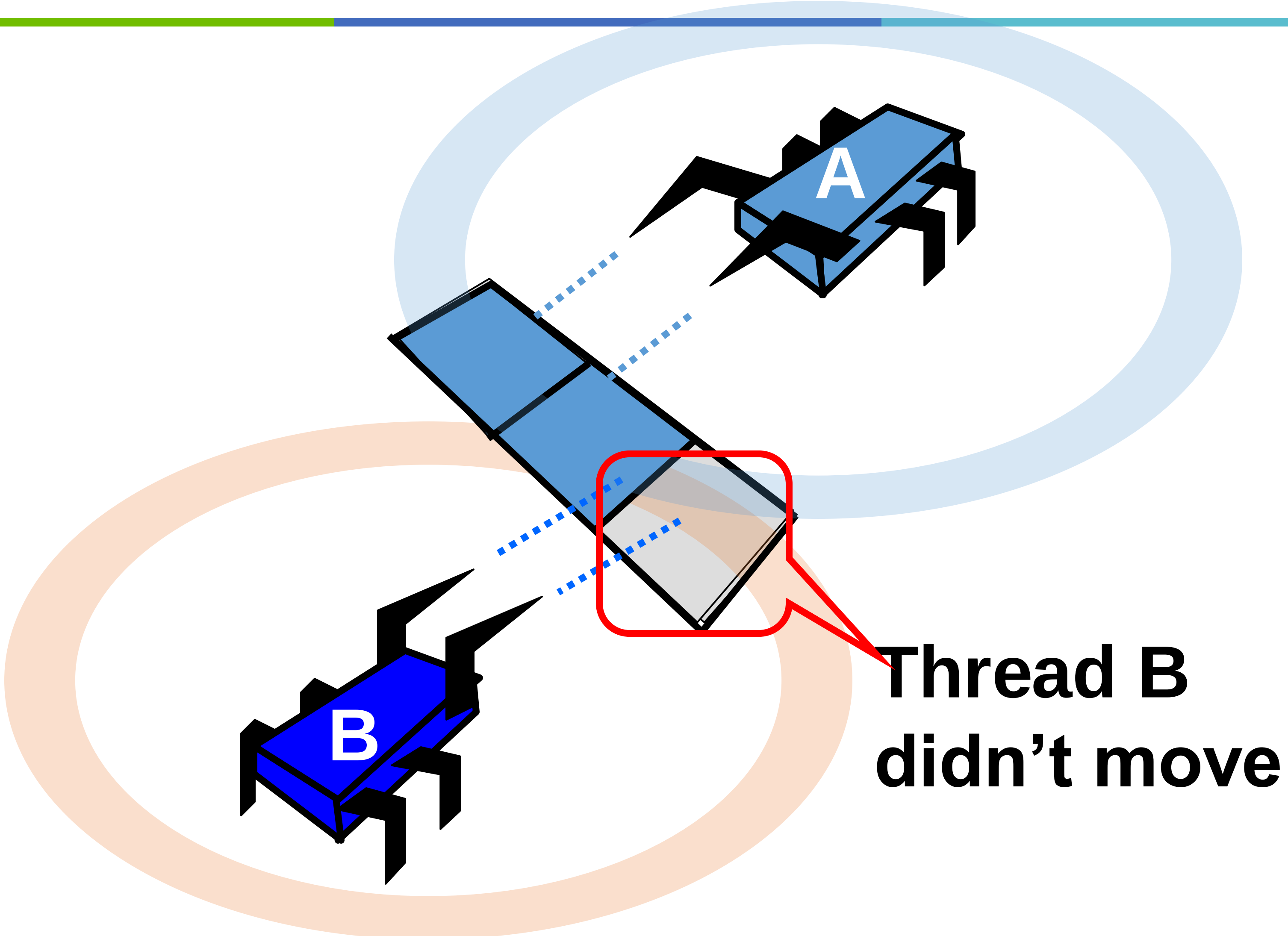
Proof Strategy

- Take a 3-element array
  - A writes atomically to slots 0 and 1
  - B writes atomically to slots 1 and 2
  - Any thread can scan any set of locations

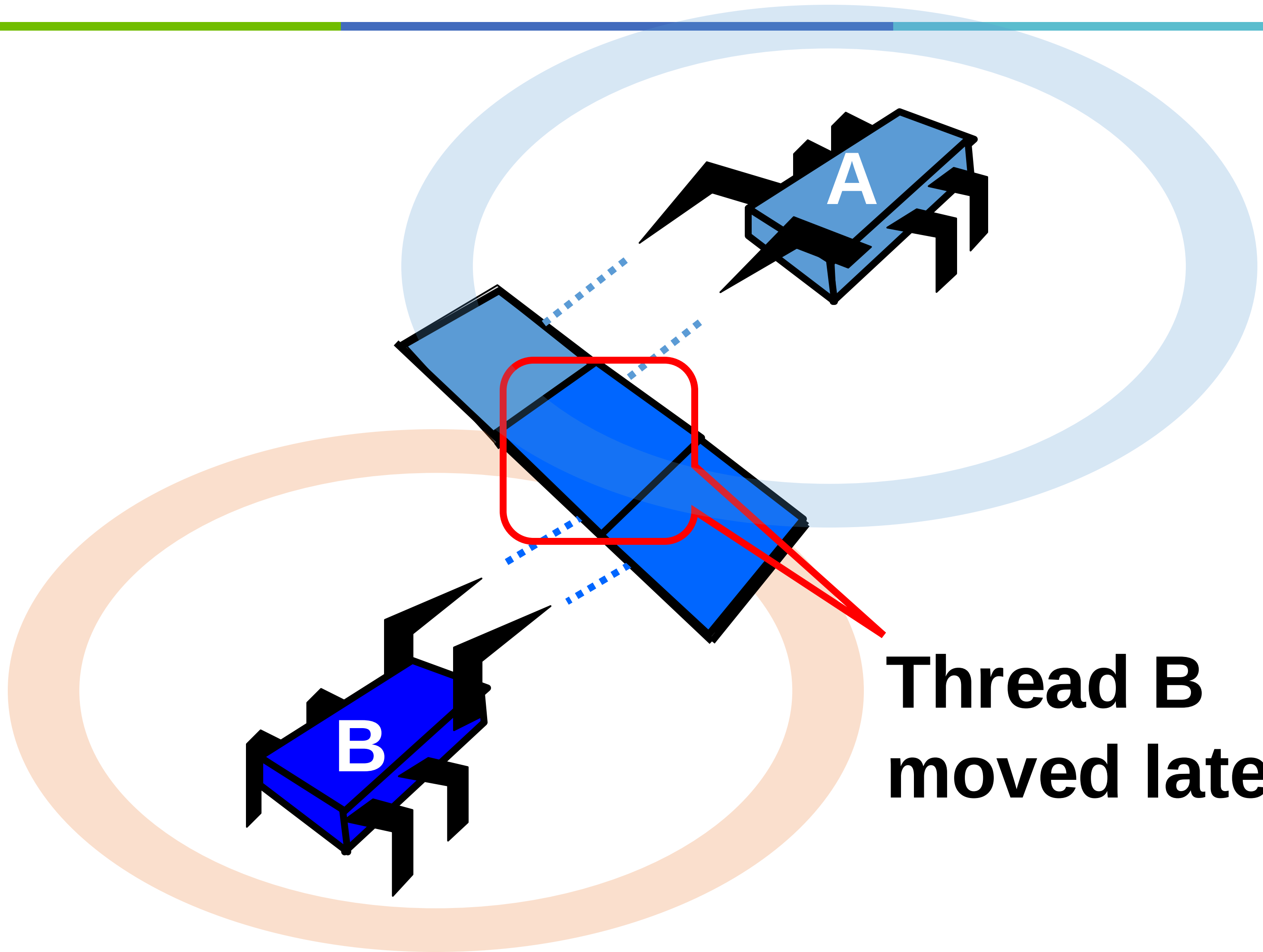
Initially



Thread A wins if

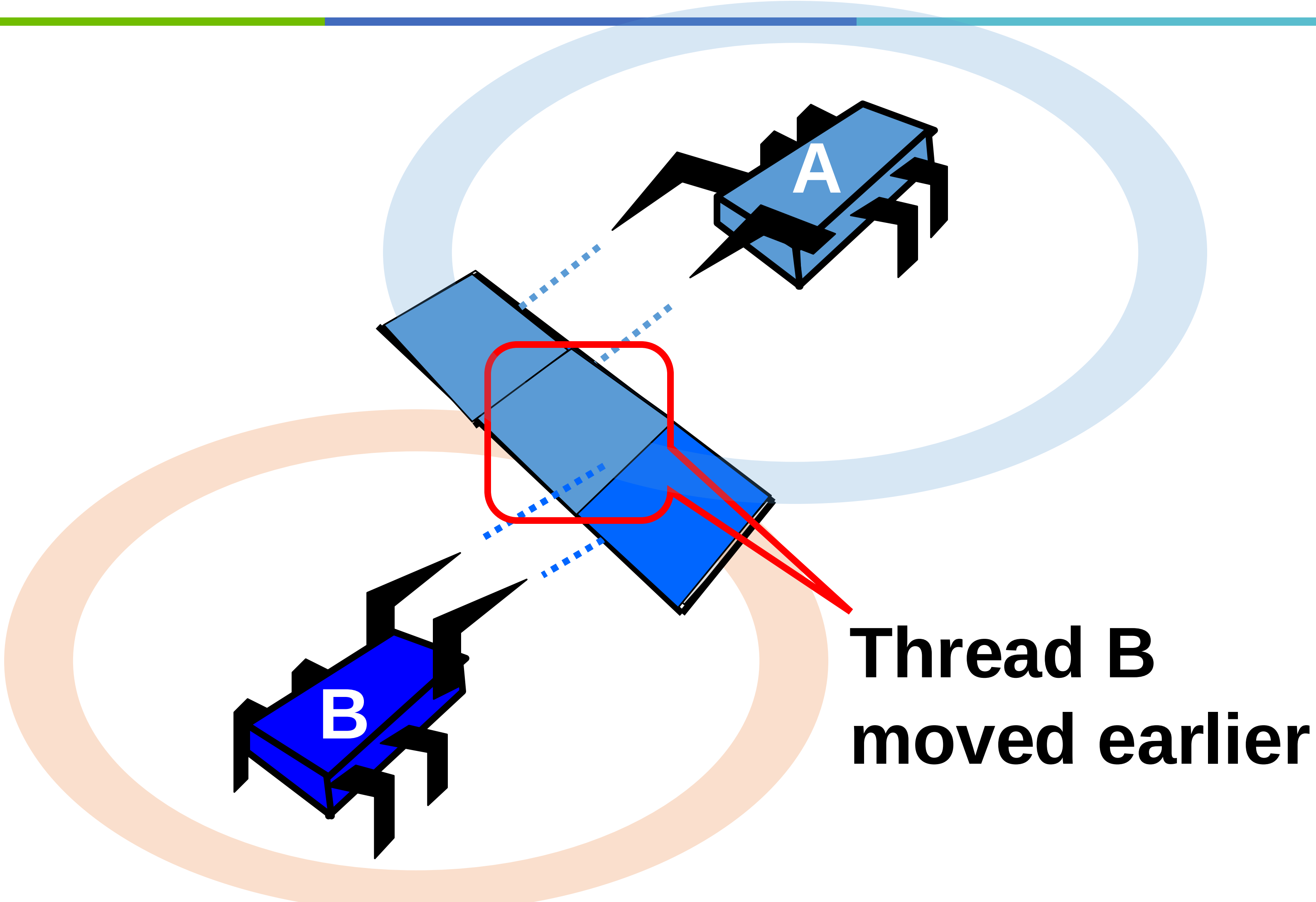


Thread A wins if



Thread B moved later

Thread A loses if



Multi-Consensus Code

```
class MultiConsensus extends …{
 Assign2 a = new Assign2(3, EMPTY);
 public T decide(T value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Multi-Consensus Code

```
class MultiConsensus extends …{
 Assign2 a = new Assign2(3, EMPTY);
 public T decide(T value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Extends ConsensusProtocol

Decide sets j=i-1 and proposes value

Multi-Consensus Code

```
class MultiConsensus extends …{
 Assign2 a = new Assign2(3, EMPTY);
 public T decide(T value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Three slots initialized to EMPTY

Multi-Consensus Code

```
class MultiConsensus extends …{
 Assign2 a = new Assign2(3, EMPTY);
 public T decide(T value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Assign ID 0 to entries 0,1
(or ID 1 to entries 1,2)

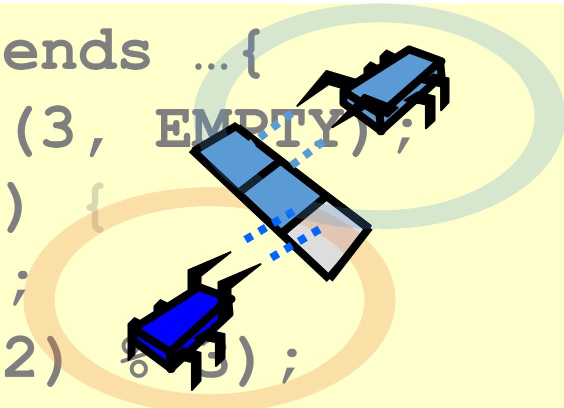
Multi-Consensus Code

```
class MultiConsensus extends …{
 Assign2 a = new Assign2(3, EMPTY);
 public T decide(T value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Read the register my thread didn't assign

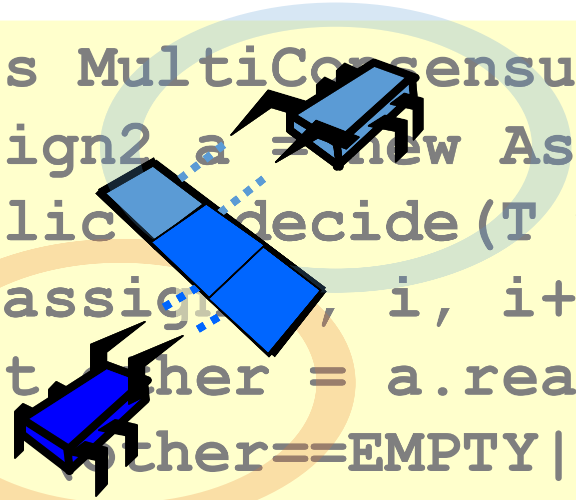
Multi-Consensus Code

```
class MultiConsensus extends …{
 Assign2 a = new Assign2(3, EMPTY);
 public T decide(T value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```



Other thread didn't move, so I win

Multi-Consensus Code



```
class MultiConsensus extends …{
 Assign2 a = new Assign2(3, EMPTY);
 public T decide(T value) {
  a.assign(value, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Other thread moved later so I win

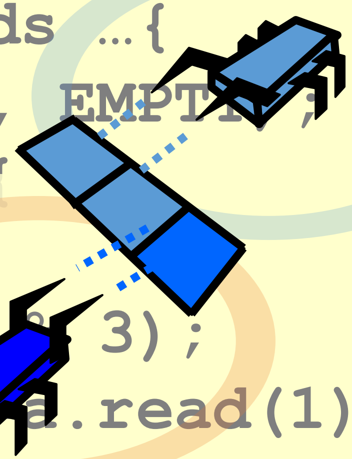
Multi-Consensus Code

```
class MultiConsensus extends …{
 Assign2 a = new Assign2(3, EMPTY);
 public T decide(T value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

OK, I win.

Multi-Consensus Code

```
class MultiConsensus extends …{
 Assign2 a = new Assign2(3, EMPTY);
 public T decide(T value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
 }}
```



Other thread moved first, so I lose

Summary

- If a thread can assign atomically to 2 out of 3 array locations
- Then we can solve 2-consensus
- Therefore
  - No wait-free multi-assignment
  - From read/write registers

Read-Modify-Write Objects

- Method call
  - Returns object's prior value **`x`**
  - Replaces **`x`** with **`mumble(x)`**

Read-Modify-Write

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
  getAndMumble() {
    int prior  = value;
    value = mumble(value);
    return prior;
  }
}
```

Read-Modify-Write

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
  getAndMumble() {
    int prior  = value;
    value = mumble(value);
    return prior;
  }
}
```

Save prior value to return later

Read-Modify-Write

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
  getAndMumble() {
    int prior  = value;
    value = mumble(value);
    return prior;
  }
}
```

Apply function to current value

Read-Modify-Write

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
  getAndMumble() {
    int prior  = value;
    value = mumble(value);
    return prior;
  }
}
```

Return prior value

RMW Everywhere!

- Most synchronization instructions
  - are RMW methods
- The rest
  - Can be trivially transformed into RMW methods

Example: Read

```
public abstract class RMWRegister {
  private int value;

  public int synchronized read() {
    int prior  = value;
    value = value;
    return prior;
  }

}
```

Example: Read

```
public abstract class RMWRegister {
  private int value;

  public int synchronized read() {
    int prior  = value;
    value = value;
    return prior;
  }
}
```

apply f(x)=x, the identity function

Example: getAndSet (swap)

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
    getAndSet(int v) {
   int prior  = value;
   value = v;
   return prior;
  }
 …
}
```

Example: getAndSet (swap)

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
   getAndSet(int v) {
  int prior  = value;
  value = v;
  return prior;
 }
 …
}
```

f(x)=v is constant

getAndIncrement

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
    getAndIncrement() {
   int prior  = value;
   value = value + 1;
   return prior;
  }
  …
}
```

getAndIncrement

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
   getAndIncrement() {
  int prior  = value;
  value = value + 1;
  return prior;
 }
 …
}
```

f(x) = x+1

getAndAdd

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
    getAndAdd(int a) {
   int prior  = value;
   value = value + a;
   return prior;
  }
 …
}
```

Example: getAndAdd

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
   getAndAdd(int a) {
  int prior  = value;
  value = value + a;
  return prior;
 }
 …
}
```

f(x) = x+a

compareAndSet

```java
public abstract class CASObject {
 private int value;
 public boolean synchronized
    compareAndSet(int expected,
                  int update) {
   if (value==expected) {
    value = update; return true;
   }
   return false;
   } … }
```

compareAndSet

```
public abstract class CASObject {
 private int value;
 public boolean synchronized
    compareAndSet(int expected,
                      int update) {
   if (value==expected) {
    value = update; return true;
   }
   return false;
   } … }
```

If value is as expected, …

compareAndSet

```
public abstract class CASObject {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  if (value==expected) {
   value = update; return true;
  }
  return false;
  } … }
```

… replace it

compareAndSet

```
public abstract class CASObject {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  if (value==expected) {
   value = update; return true;
  }
  return false;
  } … }
```

Report success

compareAndSet

```
public abstract class CASObject {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  if (value==expected) {
   value = update; return true;
  }
  return false;
  } … }
```

Otherwise report failure

Read-Modify-Write

```
public abstract class RMWRegister {
 private int value;

 public int synchronized
  getAndMumble() {
    int prior  = value;
    value = mumble(value);
    return prior;
  }
}
```

Let's characterize f(x)…

Definition

- A RMW method
  - With function mumble(x)
  - is non-trivial if there exists a value v
  - such that $v \neq \text{mumble}(v)$

Par Example

- `Identity(x) = x`
  - is trivial
- `getAndIncrement(x) = x+1`
  - is non-trivial

Theorem

- Any non-trivial RMW object has consensus number at least 2
- No wait-free implementation of RMW registers from atomic registers
- Hardware RMW instructions not just a convenience

Reminder

- Subclasses of **`consensus`** have
  - **`propose(x)`** method
    - which just stores x into **`proposed[i]`**
    - built-in method
  - **`decide(object value)`** method
    - which determines winning value
    - customized, class-specific method

Proof

```
public class RMWConsensus
      extends ConsensusProtocol {
 private RMWRegister r = v;
 public T decide(T value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

Proof

```
public class RMWConsensus
      extends ConsensusProtocol {
 private RMWRegister r = v;
 public T decide(T value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

Initialized to **v**

Proof

```
public class RMWConsensus
     extends ConsensusProtocol {
 private RMWRegister r = v;
 public T decide(T value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

Am I first?

Proof

```
public class RMWConsensus
     extends ConsensusProtocol {
 private RMWRegister r = v;
 public T decide(T value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

Yes, return my input

Proof

```
public class RMWConsensus
     extends ConsensusProtocol {
 private RMWRegister r = v;
 public T decide(T value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

No, return other's input

Proof

- We have displayed
  - A two-thread consensus protocol
  - Using any non-trivial RMW object

Interfering RMW

- Let F be a set of functions such that for all f_i and f_j, either
  - Commute: $f_i(f_j(v)) = f_j(f_i(v))$
  - Overwrite: $f_i(f_j(v)) = f_i(v)$

- Claim: Any set of RMW objects that commutes or overwrites has consensus number exactly 2

Examples

- “test-and-set” getAndSet(1) $f(v)=1$

  Overwrite $f_i(f_j(v))=f_i(v)$

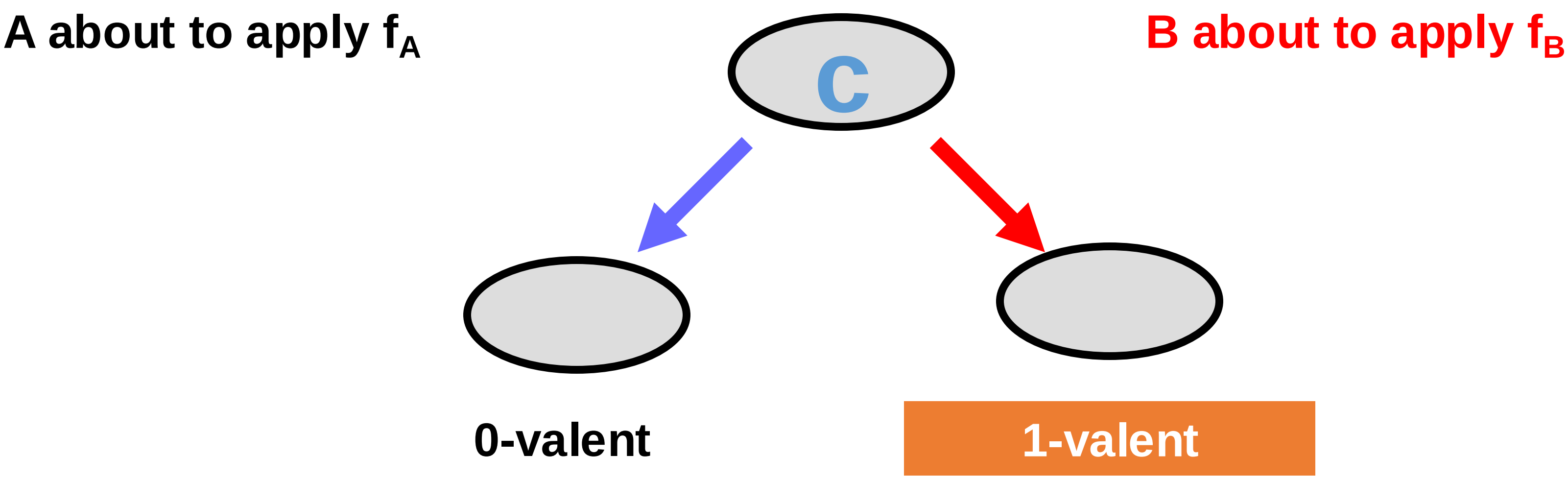
- “swap” getAndSet(x) $f(v,x)=x$

  Overwrite $f_i(f_j(v))=f_i(v)$

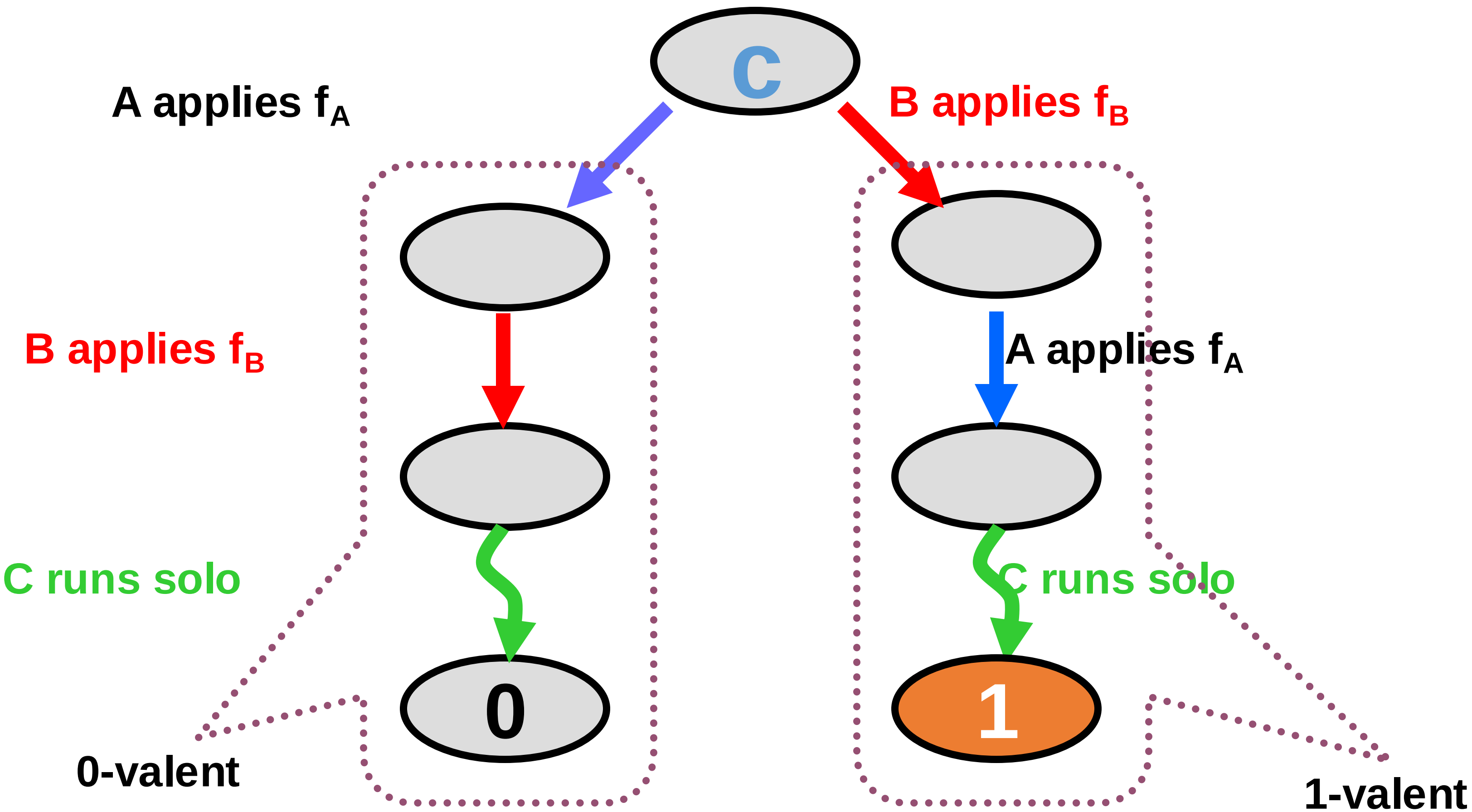
- “fetch-and-inc” getAndIncrement() $f(v)=v+1$

  Commute $f_i(f_j(v))= f_j(f_i(v))$

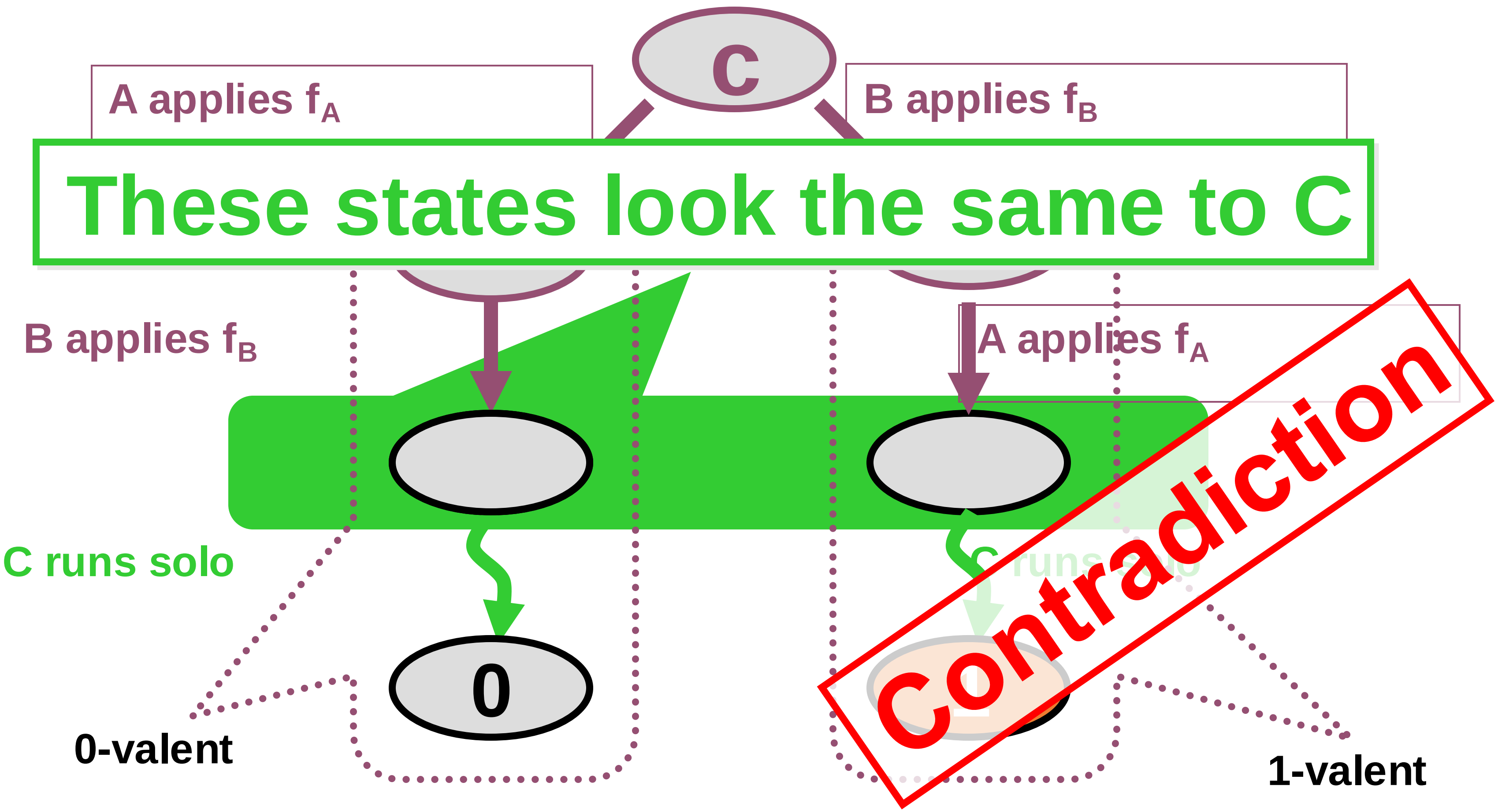
Meanwhile Back at the Critical State



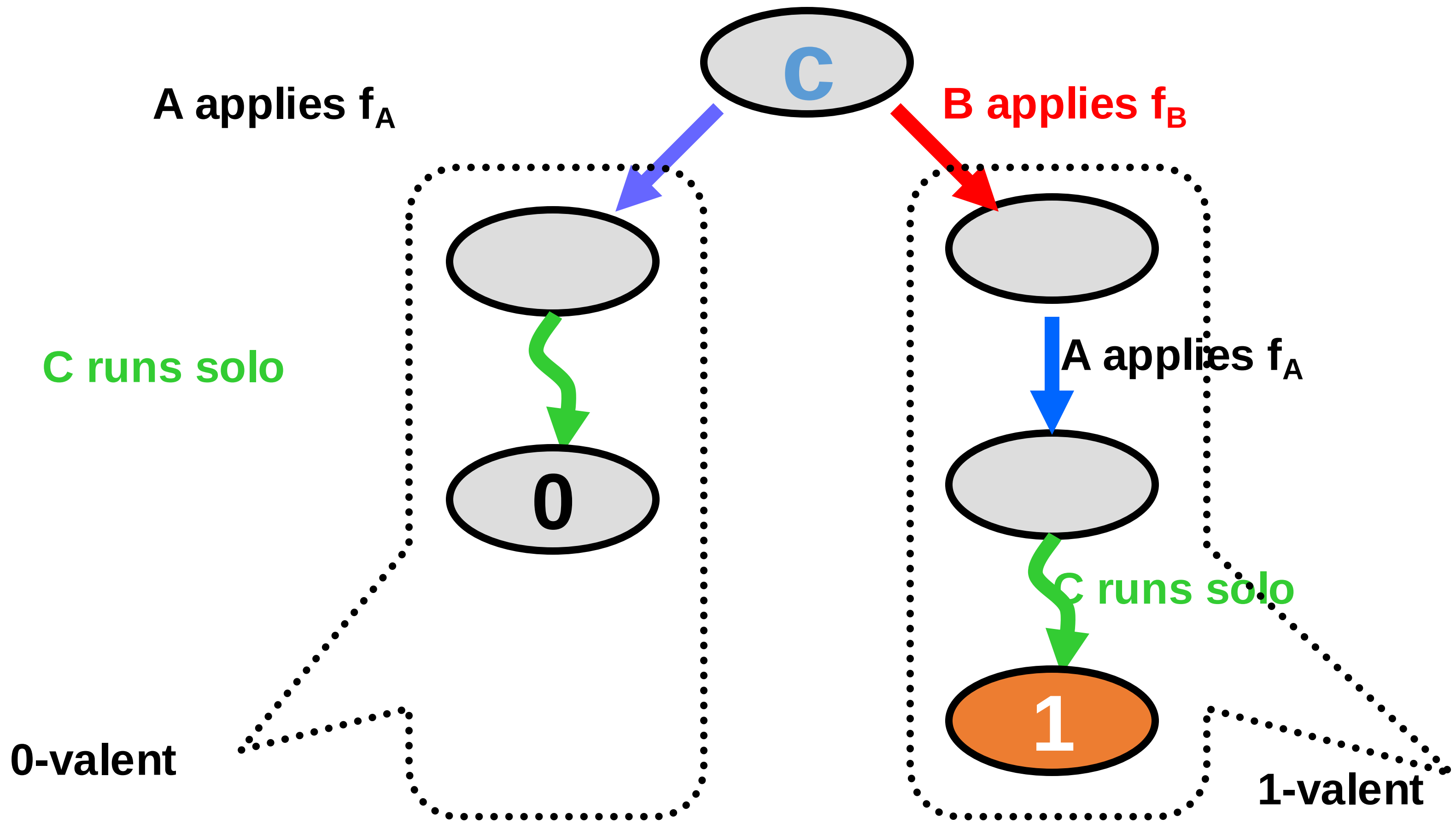
Maybe the Functions Commute



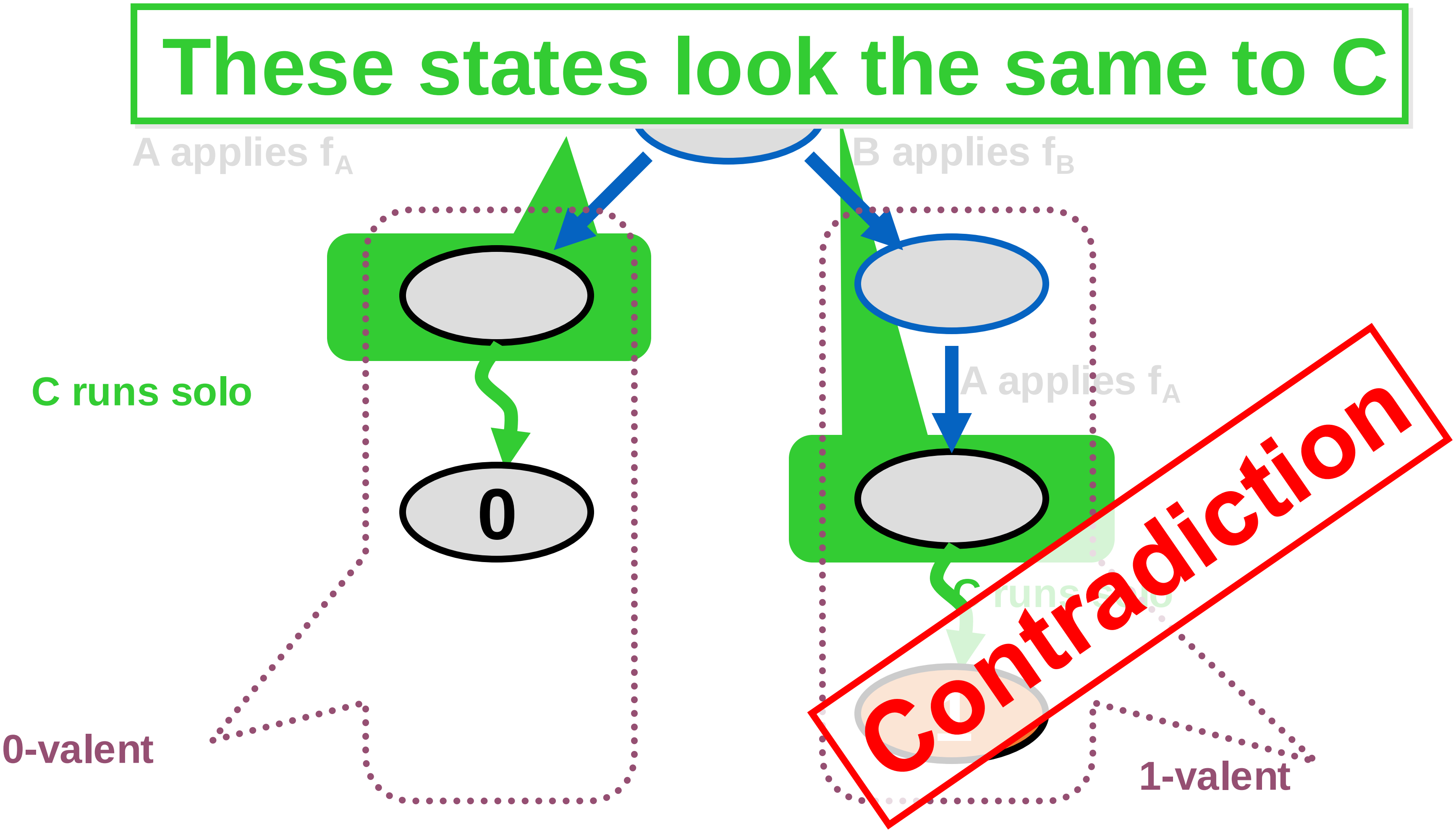
Maybe the Functions Commute



Maybe the Functions Overwrite



Maybe the Functions Overwrite



Impact

- Many early machines provided only these "weak" RMW instructions
  - Test-and-set (IBM 360)
  - Fetch-and-add (NYU Ultracomputer)
  - Swap (Original SPARCs)
- We now understand their limitations
  - But why do we want consensus anyway?

compareAndSet

```
public abstract class RMWRegister {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  int prior = value;
  if (value==expected) {
   value = update; return true;
  }
  return false;
  } … }
```

compareAndSet

```
public abstract class RMWRegister {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  int prior = value;
  if (value==expected) {
   value = update; return true;
  }
  return false;
  } … }
```

replace value if it's what we expected, …

compareAndSet Has ∞ Consensus Number

```
public class RMWConsensus
     extends ConsensusProtocol {
 private AtomicInteger r =
   new AtomicInteger(-1);
 public T decide(T value) {
  propose(value);
  r.compareAndSet(-1,i);
  return proposed[r.get()];
 }
}
```

compareAndSet Has ∞ Consensus Number

```
public class RMWConsensus
      extends ConsensusProtocol {
 private AtomicInteger r =
   new AtomicInteger(-1);
 public T decide(T value) {
  propose(value);
  r.compareAndSet(-1,i);
  return proposed[r.get()];
 }
}
```

Initialized to -1

compareAndSet Has ∞ Consensus Number

```
public class RMWConsensus
      extends ConsensusProtocol {
 private AtomicInteger r =
    new AtomicInteger(-1);
 public T decide(T value) {
  propose(value);
  r.compareAndSet(-1,i);
  return proposed[r.get()];
 }
}
```

Try to swap in my id

compareAndSet Has ∞ Consensus Number

```
public class RMWConsensus
      extends ConsensusProtocol {
 private AtomicInteger r =
    new AtomicInteger(-1);
 public T decide(T value) {
  propose(value);
  r.compareAndSet(-1,i);
  return proposed[r.get()];
 }
}
```

Decide winner's preference

The Consensus Hierarchy

1 Read/Write Registers, Snapshots…
2 getAndSet, getAndIncrement, …
. . .
∞ compareAndSet,…

Lock-Freedom

- Lock-free:
  - in an infinite execution
  - infinitely often some method call finishes
- Pragmatic approach
- Implies no mutual exclusion

Lock-Free vs. Wait-free

- Wait-Free: each method call takes a finite number of steps to finish

- Lock-free: infinitely often some method call finishes

Lock-Freedom

- Any wait-free implementation is lock-free.
- Lock-free is the same as wait-free if the execution is finite.

Lock-Free Implementations

- Lock-free consensus is as impossible as wait-free consensus
- ***All these results hold for lock-free algorithms also.***

There is More: Universality

- Consensus is universal
- From *n*-thread consensus
  - Wait-free/Lock-free
  - Linearizable
  - *n*-threaded
  - Implementation
  - Of any sequentially specified ob

Stay tuned…

The END