



The Choice of Normalization Influences Shrinkage in Regularized Regression

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The Elastic Net

Linear regression plus a combination of the ℓ_1 and ℓ_2 penalties:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^p} \left(\frac{1}{2} \|y - X\beta\|_2^2 + \underbrace{\lambda_1 \|\beta\|_1}_{\text{lasso}} + \underbrace{\frac{\lambda_2}{2} \|\beta\|_2^2}_{\text{ridge}} \right)$$

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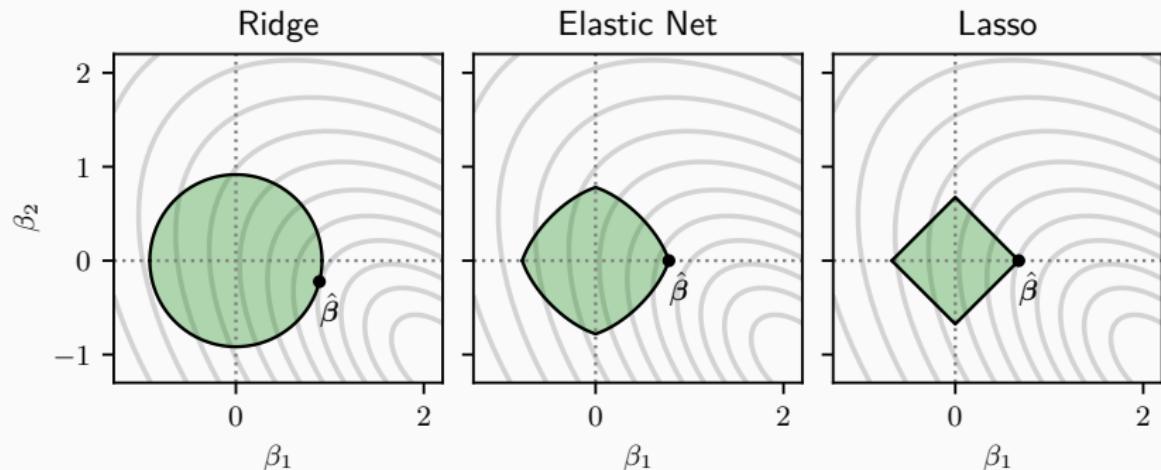


Figure 1: Elastic net is a combination of the lasso and ridge.

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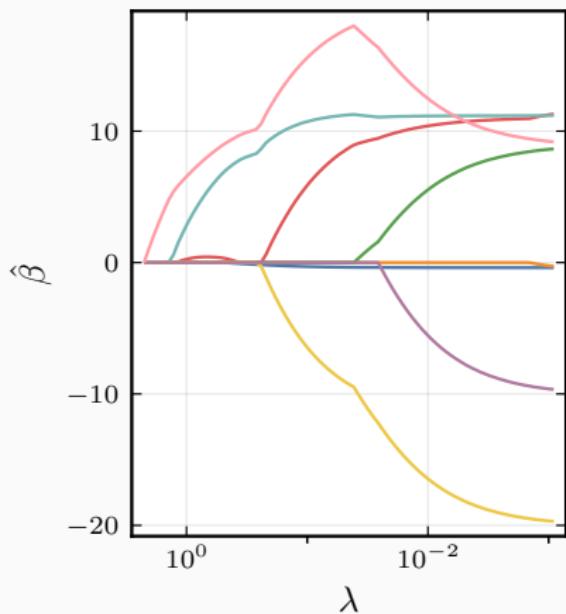


Figure 2: The elastic net path

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Large scale means **less** penalization because the size of β_j can be smaller for an equivalent effect (on y).

Normalization

Scale sensitivity can be mitigated by normalizing the features. Let $\tilde{\mathbf{X}}$ be the normalized feature matrix, with elements

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After fitting, we transform the coefficients back to their original scale via

$$\hat{\beta}_j = \frac{\hat{\beta}_j^{(n)}}{s_j} \quad \text{for } j = 1, 2, \dots, p,$$

where $\hat{\beta}_j^{(n)}$ is a coefficient from the normalized problem.

Table 1: Common ways to normalize X

Normalization	c_j	s_j
Standardization	\bar{x}_j	$\frac{1}{\sqrt{n}} \ \mathbf{x}_j - \bar{\mathbf{x}}_j\ _2$
ℓ_1 -Normalization	\bar{x}_j	$\frac{1}{\sqrt{n}} \ \mathbf{x}_j - \bar{\mathbf{x}}_j\ _1$
Max-Abs	0	$\max_i x_{ij} $
Min-Max	$\min_i(x_{ij})$	$\max_i(x_{ij}) - \min_i(x_{ij})$

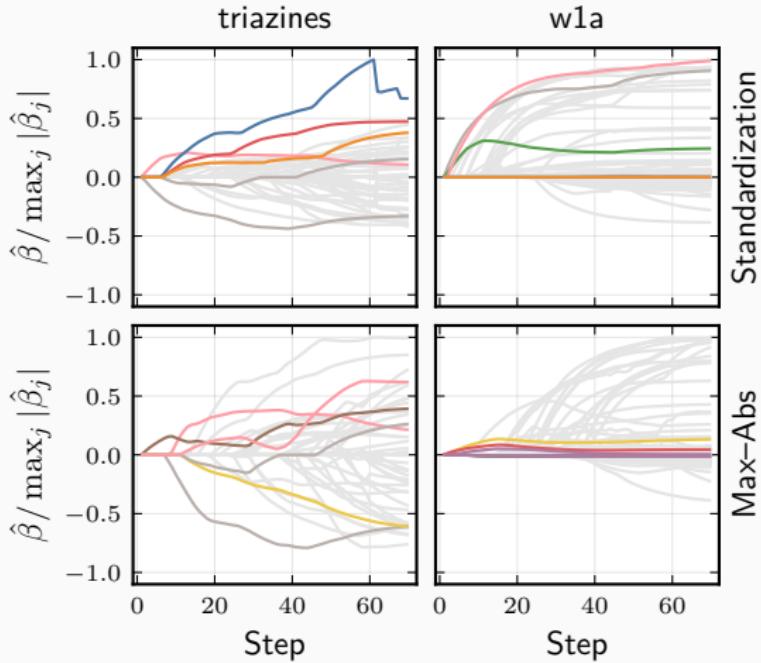


Figure 3: Normalization matters. Lasso paths under two different types of normalization (standardization and max-abs normalization). The union of the first five features selected in any of the settings are colored.

Table 2: Lasso coefficients on test sets, with λ set from 5-fold cross-validation repeated 5 times. We show the five largest coefficients in magnitude for each data set for the standardization setting and corresponding coefficients for the max-abs normalization setting.

housing		triazines		w1a	
$\hat{\beta}_{\text{std}}$	$\hat{\beta}_{\text{max-abs}}$	$\hat{\beta}_{\text{std}}$	$\hat{\beta}_{\text{max-abs}}$	$\hat{\beta}_{\text{std}}$	$\hat{\beta}_{\text{max-abs}}$
-0.63	-0.68	0.17	0.0	1.8	0.0
-1.4	-0.78	0.069	0.0	1.8	0.78
0.27	0.0	0.028	0.0	1.8	0.63
-0.99	-0.34	0.071	0.0	1.4	0.080
2.8	3.1	0.029	0.0	1.7	0.0

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When mixing binary and normal features, normalization implicitly weighs their importance.

A Little Theory

Orthogonal Features

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But if we assume that the features are orthogonal, that is

$$\tilde{\mathbf{X}}^\top \tilde{\mathbf{X}} = \text{diag}(\tilde{\mathbf{x}}_1^\top \tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_p^\top \tilde{\mathbf{x}}_p),$$

then there is:¹:

$$\hat{\beta}_j = \frac{S_{\lambda_1}(\tilde{\mathbf{x}}_j^\top \mathbf{y})}{s_j (\tilde{\mathbf{x}}_j^\top \tilde{\mathbf{x}}_j + \lambda_2)},$$

where

$$S_\lambda(z) = \text{sign}(z) \max(|z| - \lambda, 0).$$

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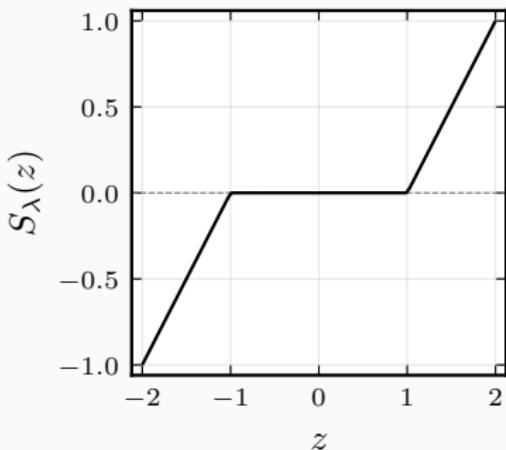


Figure 4: Soft thresholding

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Assume we have a binary feature x_j , such that $x_{ij} \in \{0, 1\}$. Let $q \in [0, 1]$ be the class balance of this feature, that is: $q = \bar{x}_j$.

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$s_j = q - q^2$ for lasso and $s_j = \sqrt{q - q^2}$ for ridge removes effect of q , which suggests the parametrization

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There is no (simple) s_j that will work for the elastic net.

Probability of Selection

Since \mathbf{X} is fixed and ε is normal, we can compute the probability of selection.

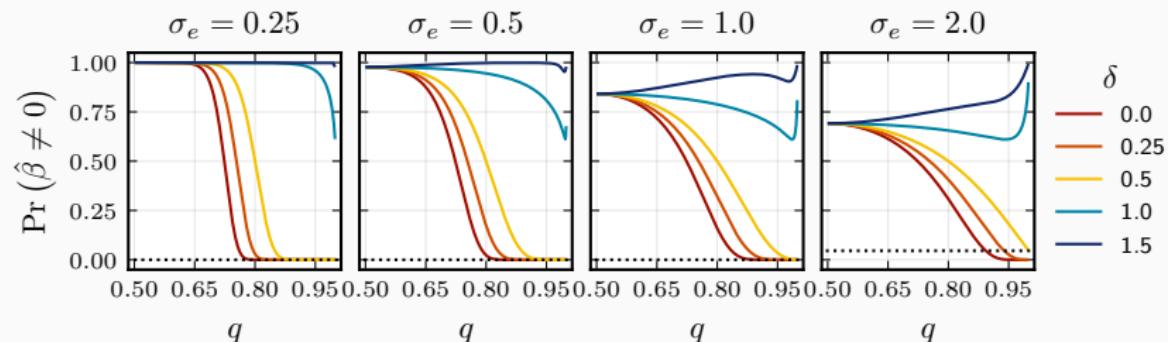


Figure 5: Probability that the elastic net selects a feature across different noise levels (σ_e), types of normalization (δ), and class balance (q). The dashed line is asymptotic behavior for $\delta = 1/2$. Scaling used is $s_j \propto (q - q^2)^\delta$.

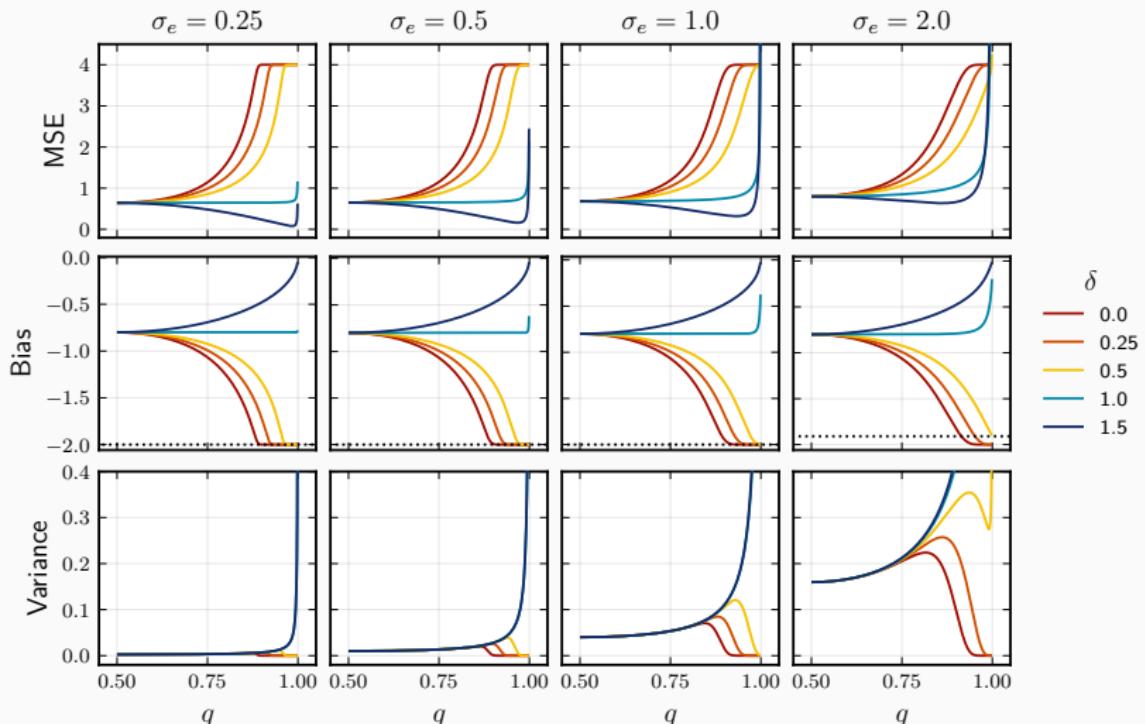


Figure 6: A bias variance tradeoff. Bias, variance, and mean-squared error for a one-dimensional lasso problem. Theoretical result for orthogonal features. Dotted line is asymptotic result or $\delta = 1/2$. Scaling used is $s_j \propto (q - q^2)^{\delta}$.

Experiments

Binary Features (Decreasing q)

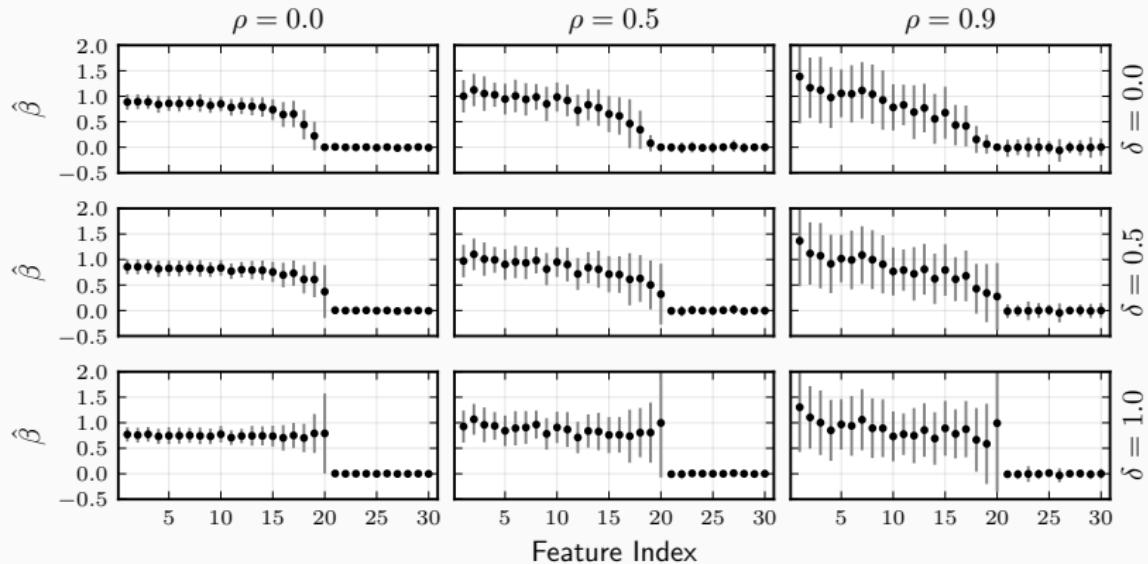


Figure 7: Lasso estimates for first 30 coefficients. First 20 features are true signals with a geometrically decreasing class balance from 0.5 to 0.99. ρ is a measure of autocorrelation.

Mixed Data

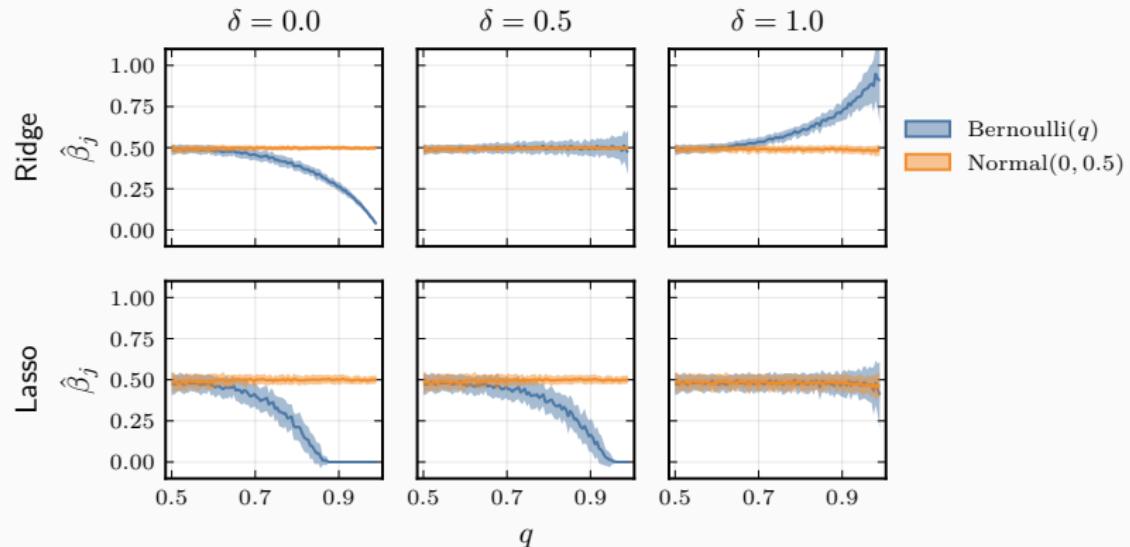


Figure 8: Comparison between lasso and ridge estimators for features generated to resemble features from various distributions.

Hyperparameter Optimization

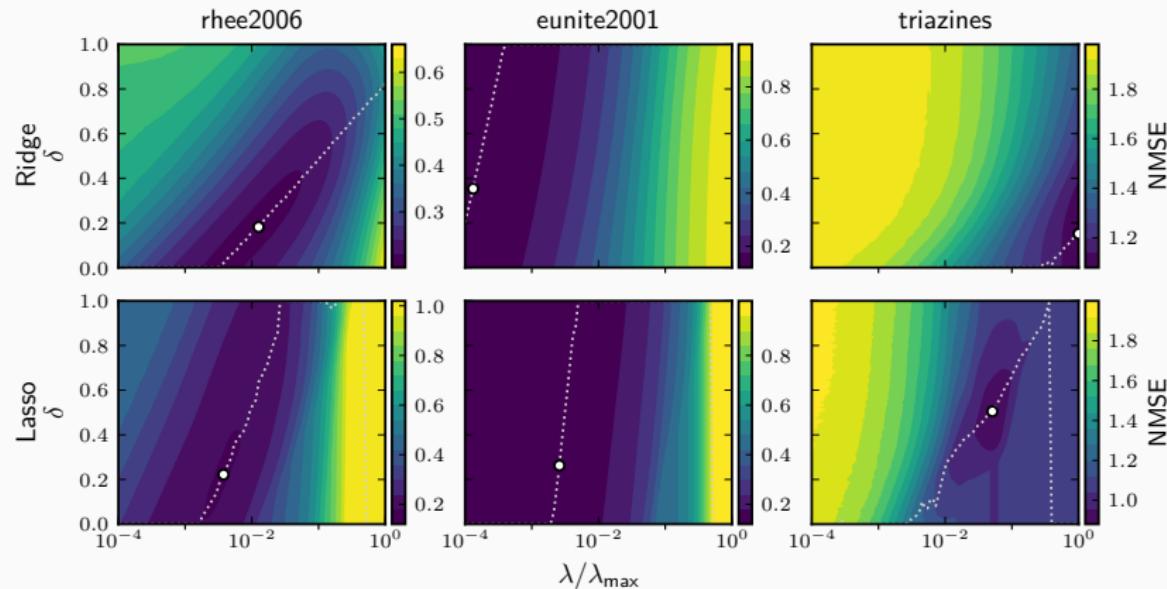


Figure 9: Contour plots of hold-out (validation set) error across a grid of δ and λ values for the lasso and ridge.

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See the paper for more details!

Thank you!