Let $\tilde{x} = X_{\mathcal{C}_k} s_{\mathcal{C}_k}$ and $\tilde{y} = X_{\bar{\mathcal{C}}_k} s_{\bar{\mathcal{C}}_k}$. The subdifferential is

$$\partial_{\tilde{\beta}} P(\beta) = (\tilde{y} - y)^T \tilde{x} + \tilde{x}^T \tilde{x} \tilde{\beta} + \partial_{\tilde{\beta}} \Big(\big(\sum_{i \in \mathcal{C}_k} \lambda_{(j)}^{-1} \big) \tilde{\beta} + \sum_{j \notin \mathcal{C}_k} \lambda_{(j)}^{-1} \beta_j \Big).$$

The gradient exists whenever $\tilde{\beta} \notin \{0, c_1, \dots c_{k-1}, c_{k+1}, \dots, c_m\}$ and the update then becomes

$$\tilde{\beta} = \frac{(y - \tilde{y})^T \tilde{x} - \operatorname{sign}(\tilde{\beta}) \sum_{j \in \mathcal{C}_k} \lambda_{(j)}^{-1}}{\tilde{x}^T \tilde{x}} = \operatorname{sign}\left((y - \tilde{y})^T \tilde{x}\right) \left(\left| \frac{(y - \tilde{y})^T \tilde{x}}{\tilde{x}^T \tilde{x}} \right| - \sum_{j \in \mathcal{C}_k} \lambda_{(j)}^{-1} \right).$$

If $\tilde{\beta} = 0$, then we have

$$0 \in (\tilde{y} - y)^T \tilde{x} + \Big[-\sum_{j = m - |\mathcal{C}(0)|}^{m} \lambda_{(j)}^{-1}, \sum_{j = m - |\mathcal{C}(0)|}^{m} \lambda_{(j)}^{-1} \Big]$$

which means that $(\tilde{y} - y)^T \tilde{x} \leq \Big| \sum_{j = m - |\mathcal{C}(0)|}^{m} \lambda_{(j)}^{-1} \Big|$, $\tilde{\beta} = 0$ satisfies the optimality condition.

If $\tilde{\beta} = c_q$, then we have

$$0 \in (\tilde{y} - y)^T \tilde{x} + \tilde{x}^T \tilde{x} c_q + \Big[\sum_{j = |\mathcal{C}_q| - |\mathcal{B}|}^{|\mathcal{C}_q|} \lambda_{(j)}^{-1}, \sum_{j=1}^{|\mathcal{B}|} \lambda_{(j)}^{-1} \Big] \implies$$

$$0 \in \frac{(\tilde{y} - y)^T \tilde{x}}{\tilde{x}^T \tilde{x}} + c_q + \frac{1}{\tilde{x}^T \tilde{x}} \Big[\quad \dots \quad \Big],$$

which means that the optimality conditions are satisfied if

$$\frac{(\tilde{y} - y)^T \tilde{x}}{\tilde{x}^T \tilde{x}} + c_q + \frac{1}{\tilde{x}^T \tilde{x}} \sum_{j = |\mathcal{C}_q| - |\mathcal{B}|}^{|\mathcal{C}_q|} \lambda_{(j)}^{-1} \leq 0 \text{ and } \frac{(\tilde{y} - y)^T \tilde{x}}{\tilde{x}^T \tilde{x}} + c_q + \frac{1}{\tilde{x}^T \tilde{x}} \sum_{j=1}^{|\mathcal{B}|} \geq 0 \implies$$

$$\frac{1}{\tilde{x}^T \tilde{x}} \sum_{j = |\mathcal{C}_q| - |\mathcal{B}|}^{|\mathcal{C}_q|} \lambda_{(j)}^{-1} \leq \frac{(y - \tilde{y})^T \tilde{x}}{\tilde{x}^T \tilde{x}} - c_q \leq \frac{1}{\tilde{x}^T \tilde{x}} \sum_{j=1}^{|\mathcal{B}|} \lambda_{(j)}^{-1}$$

If $\tilde{\beta} = -c_q$, then we have

$$0 \in (\tilde{y} - y)^T \tilde{x} - \tilde{x}^T \tilde{x} c_q + \Big[-\sum_{j=1}^{|\mathcal{B}|} \lambda_{(j)}^{-1}, -\sum_{j = |\mathcal{C}_q| - |\mathcal{B}|}^{|\mathcal{C}_q|} \lambda_{(j)}^{-1} \Big] \implies$$

$$0 \in \frac{(\tilde{y} - y)^T \tilde{x}}{\tilde{x}^T \tilde{x}} - c_q - \frac{1}{\tilde{x}^T \tilde{x}} \Big[\quad \dots \quad \Big],$$

which means that the optimality conditions are satisfied if $\frac{(\tilde{y}-y)^T\tilde{x}}{\tilde{x}^T\tilde{x}} - c_q - \frac{1}{\tilde{x}^T\tilde{x}}\sum_{j=1}^{|B|}\lambda_{(j)}^{-1} \le 0$ and $\frac{(\tilde{y}-y)^T\tilde{x}}{\tilde{x}^T\tilde{x}} - c_q - \frac{1}{\tilde{x}^T\tilde{x}}\sum_{j=1}^{|B|} > 0$

Putting everything together, we see that $|\tilde{\beta}| = c_q$ corresponds to

$$\frac{1}{\tilde{x}^T\tilde{x}}\sum_{j=|c_q|-|B|}^{|c_q|}\lambda_{(j)}^{-1} \le \left|\frac{(y-\tilde{y})^T\tilde{x}}{\tilde{x}^T\tilde{x}}\right| - c_q \le \frac{1}{\tilde{x}^T\tilde{x}}\sum_{j=1}^{|B|}\lambda_{(j)}^{-1}.$$