TD1 : Macroscopic road traffic models

1. Is the following system hyperbolic ?

$$
\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\partial_t v + \partial_x u &= 0
\end{aligned}
$$

Solution:

These equations, called Cauchy-Riemann equations, are met in the context of the study of analytic functions in complex analysis (the variables are then x, y rather than x, t).

We write the equations in the form :

$$\partial_t \left(\begin{array}{c} u \\ v \end{array} \right) + A.\partial_x \left(\begin{array}{c} u \\ v \end{array} \right) = 0$$

with A a 2×2 matrix

$$A = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$$

Characteristic polynomial :

$$|A - \lambda I| = \lambda^2 + 1$$

The roots λ are not real, the system is elliptic and thus not hyperbolic.

2. We consider the Payne-Whitham model (1971), but we ignore the driving force and viscous term. It gives the following system of equations :

$$
\begin{aligned}
\partial_t \rho + \partial_x(\rho u) &= 0 \\
\partial_t u + u \partial_x u &= -\frac{1}{\rho} p'(\rho) \partial_x \rho
\end{aligned}
$$

with $p(\rho) = \rho$.

(a) Transform the equations to put them under the form

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + A(\rho, u) \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} = 0.$$

Find the eigenvalues of A and the associated left eigenvectors.

Solution:

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + \begin{pmatrix} u & \rho \\ \frac{1}{\rho} & u \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = (u - \lambda)^2 - 1 = (u - \lambda + 1)(u - \lambda - 1)$$

Eigenvalues : $\lambda_+ = u+1$ et $\lambda_- = u-1$

Real eigenvalues, the system can be made diagonal, it is thus a hyperbolic system (even strictly hyperbolic as all eigenvalues are distinct).

Left eigenvectors $(l_1^\pm, l_2^\pm)$ must verify :

$$\begin{pmatrix} l_1^\pm & l_2^\pm \end{pmatrix} \begin{pmatrix} u & \rho \\ \frac{1}{\rho} & u \end{pmatrix} = \lambda_\pm \begin{pmatrix} l_1^\pm & l_2^\pm \end{pmatrix}$$

ie

$$\begin{aligned} l_1^+ \rho &= l_2^+ \quad \text{for } \lambda_+ = u+1, \\ l_1^- \rho &= -l_2^- \quad \text{for } \lambda_- = u-1. \end{aligned}$$

hence the left eigenvectors

$$\begin{pmatrix} l_1^+ \\ l_2^+ \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho} \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} l_1^- \\ l_2^- \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho} \\ -1 \end{pmatrix}$$

(b) Project the equations on the left eigenvectors, and find the Rieman invariants.

Solution: We have to project the equations on the left eigenvectors, which means that we have to multiply the equation by the left eigenvectors :

$$\begin{aligned} \begin{pmatrix} l_1^\pm & l_2^\pm \end{pmatrix} \partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} & + \begin{pmatrix} l_1^\pm & l_2^\pm \end{pmatrix} \begin{pmatrix} u & \rho \\ \frac{1}{\rho} & u \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} &= 0 \\ \begin{pmatrix} l_1^\pm & l_2^\pm \end{pmatrix} \partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} & + \lambda_\pm \begin{pmatrix} l_1^\pm & l_2^\pm \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} &= 0 \\ \frac{1}{\rho}\partial_t \rho \pm \partial_t u & + (u \pm 1) \left(\tfrac{1}{\rho}\partial_x \rho \pm \partial_x u \right) &= 0 \end{aligned}$$

Now, in order to find the Riemann invariants, we want to put these equations in the form

$$(\partial_t + (sth)\partial_x)\,[sth] = \ 0$$

where the first "sth" will be equal to the slope of the characteristics dX/dt, the second "sth" will be the Riemann invariant, and the r.h.s. will be here zero, but it is not always the case.

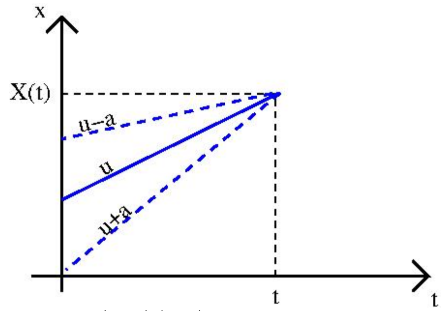
After integrating $\frac{1}{\rho}\partial_t \rho = \partial_t(\ln \rho)$ and same for ∂_x :

$$(\partial_t + (u \pm 1)\partial_x)\,[\ln \rho \pm u] = \ 0$$

So that the quantities $\ln \rho \pm u$ (called Riemann invariants) are conserved along curves [X(t),t] of slope $u \pm 1$. More precisely, these curves are such that $X'(t)$ or $\frac{dX}{dt} = u \pm 1$.

(c) These results were published in a paper entitled "Requiem for second-order fluid approximations of traffic flow" [Daganzo 1995]. Can you explain why this model fails to reproduce highway traffic ? In particular, where does the information that determines the state of a given point t,x come from ?

Solution:
The eigenvalue $u+1 > u$, so that some information always goes faster than the cars. This means that some information is coming from *behind* the car.



The state of car traffic in $(X(t), t)$ is entirely determined by what happened before between the characteristics (blue dashed) that intersect in $X(t), t$ (here we draw them as lines but it is not the case in general). The slope of the characteristics is such that $\frac{dX}{dt} = u \pm a$ where a is the sound velocity - here $a = 1$.

3. Aw-Rascle-Zhang model (2000)
Another model was proposed to solve this issue. It was entitled "Resurrection of second order models of traffic flow and numerical simulation" [Aw & Rascle 2000].

$$\begin{aligned}\partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t u + u\partial_x u &= -\frac{dp(\rho)}{dt} + \frac{1}{\tau}\left[V(\rho) - u\right]\end{aligned}$$

where $d/dt = \partial_t + u\partial_x$.
Note that the $\frac{1}{\rho}$ has been suppressed as it is now included in p.
Again, we ignore the relaxation term.

(a) Is the system hyperbolic ?

Solution:
The 2nd equation can be rewritten :

$$(\partial_t + u\partial_x)(u + p(\rho)) = \frac{1}{\tau}\left[V(\rho) - u\right] \tag{1}$$

We ignore the relaxation part.
Using that

$$(\partial_t + u\partial_x)p(\rho) = p'(\rho)(\partial_t + u\partial_x)\rho = p'(\rho)(-\rho\partial_x u) \text{ using mass conservation}$$

we get

$$
\begin{aligned}
\partial_t \rho + u\partial_x \rho + \rho \partial_x u &= 0 \\
\partial_t u + u\partial_x u - \rho p'(\rho)\partial_x u &= 0
\end{aligned}
$$

or

$$
\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + \begin{pmatrix} u & \rho \\ 0 & u - \rho p'(\rho) \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}
$$

Eigenvalues :

$$
|A - \lambda I| = (u - \lambda)(u - \rho p'(\rho) - \lambda)
$$

hence

$$
\begin{aligned}
\lambda_+ &= u \\
\lambda_- &= u - \rho p'(\rho)
\end{aligned}
$$

For $\rho > 0$, we have 2 different real eigenvalues and the system is strictly hyperbolic.

(b) Why should we have $p'(\rho) > 0$?

Solution: $\lambda_\pm$ are not only the eigenvalues of A but also the slopes of the characteristics. They correspond thus to the speed of propagation of some information along the characteristics.
We must have $p'(\rho) > 0$ to have no transport of information faster than u, ie $\lambda_\pm \leq u$ (and $\lambda_+ \neq \lambda_-$).
This ensures that the drivers respond only to informations coming from the front.

(c) Find the Rieman invariants.

Solution: • By definition, left eigenvectors $(l_1^\pm, l_2^\pm)$ must verify :

$$
\begin{pmatrix} l_1^\pm & l_2^\pm \end{pmatrix} \begin{pmatrix} u & \rho \\ 0 & u - \rho p'(\rho) \end{pmatrix} = \lambda_\pm \begin{pmatrix} l_1^\pm & l_2^\pm \end{pmatrix} \tag{3}
$$

ie for λ_+

$$
\begin{aligned}
u l_1^+ &= u l_1^+ \\
\rho l_1^+ + [u - \rho p'(\rho)]\, l_2^+ &= u l_2^+
\end{aligned}
$$

which gives

$$
\rho l_1^+ = \rho p'(\rho) l_2^+
$$

A possible choice for l^+ is thus

$$l^+ = \begin{pmatrix} p'(\rho) \\ 1 \end{pmatrix}.$$

For λ_- Eq. 3 becomes

$$\begin{aligned} ul_1^- &= [u - \rho p'(\rho)]\, l_1^- \\ \rho l_1^- + [u - \rho p'(\rho)]\, l_2^- &= [u - \rho p'(\rho)]\, l_2^- \end{aligned}$$

A possible choice for l^- is thus

$$l^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To summarize, the left eigenvectors are

$$\begin{aligned} \begin{pmatrix} l_1^+ \\ l_2^+ \end{pmatrix} &= \begin{pmatrix} p'(\rho) \\ 1 \end{pmatrix} \quad \text{for } \lambda_+ = u + 1, \\ \begin{pmatrix} l_1^- \\ l_2^- \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } \lambda_- = u - 1. \end{aligned}$$

• Projection : We multiply equations 2 by l^+ (on the left) :

$$p'(\rho)\partial_t \rho + \partial_t u + u\left(p'(\rho)\partial_x \rho + \partial_x u\right) = 0$$

or equivalently

$$\partial_t (p + u) + u\partial_x (p + u) = 0$$

$$(\partial_t + u\partial_x)(p + u) = 0$$

The Riemann invariant corresponding to $\lambda_+ = u$ is thus $p(\rho) + u$.
Now if we project on l^- :

$$\partial_t u + [u - \rho p'(\rho)]\, \partial_x u = 0$$

$$(\partial_t + [u - \rho p'(\rho)]\, \partial_x)\, u = 0$$

The Riemann invariant corresponding to $\lambda_- = u - \rho p'(\rho)$ is thus u.
Note that the characteristics are *not* straight lines in the generic case.
Let call $w = u + p(\rho)$ the first Riemann invariant. It verifies (we still ignore the relaxation term) :

$$(\partial_t + u\partial_x)w = 0 \tag{4}$$

w is thus conserved by the car along its trajectory. It can be interpreted as the desired velocity of the car in the absence of obstacle, and $p(\rho)$, which is called pressure by analogy with fluids but actually has the dimension of a velocity, is the velocity offset between the desired and real velocities of the cars.

Références

[Aw and Rascle, 2000] Aw, A. and Rascle, M. (2000). Resurrection of “second order” models of traffic flow and numerical simulation. *SIAM Journal on Applied Mathematics*, 60 :916–938.

[Daganzo, 1995] Daganzo, C. (1995). Requiem for second-order fluid approximations of traffic flow. *Transp. Res. B*, 29 :277–286.

[Zhang, 2002] Zhang, H. M. (2002). A non-equilibrium traffic model devoid of gas-like behavior. *Transportation Research Part B : Methodological*, 36 :275–290.

4. We consider a road whose characteristics correspond to a triangular fundamental diagram. The traffic obeys LWR model.

   The capacity of the road is 1600 veh/h, and there is a speed limit of 50 km/h.

   A traffic light is located in $x = 0$. It has remained green for a very long time. At $t = 0$, it becomes red, turns again green at $t = 1$ mn, then again red at $t = 2$ mn, and green at $t = 3$ mn, and so on. The period of the red+green cycle is called T.

   The incoming flow is constant and equal to 1000 veh/h.

   When cars are queuing during the red light period, the distance between the front of 2 successive cars is 10 meters.

   (a) Draw the spatio-temporal plot to show that effect of the first red light period, and indicate the various traffic phases that are produced by the traffic light variations.

   These traffic phases will be denoted by a letter and the corresponding point will be shown on the fundamental diagram. The corresponding (density,flow) coordinates will be computed.

   Solution:
   The maximum of the FD is C, corresponding to a flow = capacity = 1600 veh/h, and a critical density $\rho_C = 1600/50 = 32$ veh/km.

   A is on the free flow branch, with a flow = 1000 veh/h, and a density $\rho_A = 1000/50 = 20$ veh/km.

   B is on the congested branch, with zero flow, and $\rho_B = \rho_{max} = 1/l_{eff} = 0.1$ veh/m = 100 veh/km.

   E has density $\rho_E = 0$, flow $F_E = 0$.

   (b) Compute the speed of the various fronts.

   Solution:

$$
\begin{aligned}
\text{slope of AB} \quad v_{AB} &= \frac{F_A - 0}{\rho_A - \rho_B} = \frac{1000}{20 - 100} = -\frac{25}{2} = -12.5\text{km/h} \\
\text{slope of CB} \quad v_{CB} &= \frac{F_C - 0}{\rho_C - \rho_B} = \frac{1600}{32 - 100} = -23.5\text{km/h} \\
\text{slope of AC} \quad v_{AC} &= 50\text{km/h} \\
\text{slope of AE} \quad v_{AE} &= 50\text{km/h} \\
\text{slope of CE} \quad v_{CE} &= 50\text{km/h}
\end{aligned}
$$

(c) Is the green period between 2 red periods long enough to remove the queue? Once the periodic regime is well established, what will happen? Draw on the spatio temporal plot the various phases produced by the second red period (the scales do not have to be respected, we only want a sketch).

Solution:
The length L of the queue formed by the first red period is $L = -v_{AB}(t_1 + T/2)$ and is also $L = -v_{CB}t_1$.
By equality of these 2 expressions, we find

$$
\begin{aligned}
t_1 &= \frac{-v_{AB}}{-v_{CB} + v_{AB}} \frac{T}{2} \\
&= \frac{12.5}{23.5 - 12.5} \times 1 \text{ mn} \\
&= 1.14 \text{ mn}
\end{aligned}
$$

This is longer than the green period.
Thus the queue will be longer at each traffic light cycle.

(d) What would you do as a traffic engineer?

Solution: We could increase the green period up to a value greater than $t_1 + L/v_{AC} = t_1 + v_{CB}t_1/v_{AC} = t_1(1 + v_{CB}/v_{AC}) = 1.14 * (1 + 23.5/50) = 1.7$ mn.
We could reorient part of the flow.
We could decrase the red period.
Note that all these calculations assume infinite acceleration/deceleration capacity, which is here a strong assumption.