Random Matrix Theory and Applications

Exam

March 28th, 2023

Surname :

Name :

Master :

Write your surname & name in CAPITAL LETTERS.
Clarity and relevance of the explanations will also be evaluated for the final mark.
The answers must be written within the boxes.
No books or notes are allowed during the exam.

1. Coulomb gas approach.

(a) By assuming the effective potential $\mathcal{V}[\mathbf{x}] = \frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{i\neq j} \ln|x_i - x_j|$, discuss the physical meaning of the two terms and any analogy with a statistical mechanics framework. What is the meaning of β in the two frameworks?

(b) Go through the main steps of the Dyson technique, the so-called *Coulomb gas* formalism (no in-depth calculations needed).

(c) Can the support of the equilibrium density profile $n^*(x)$ for Gaussian (GOE, GUE, GSE) ensembles be the full real line? Explain why.

2. Real and imaginary parts.

Write the Sokhotski-Plemelj formula. Why is it introduced? What kind of information does it allow one to obtain?

3. Jpdf of the eigenvalues.

Consider the joint probability density function $\mathcal{P}_N(\mathbf{X})$ of the matrix entries, where $\mathbf{X}$ is a real Gaussian random matrix, of size $N \times N$.

$$\mathcal{P}_N(\mathbf{X}) = \mathcal{Z}_N^{-1} \exp\left[-\frac{1}{2(1-\tau^2)} \mathrm{Tr}(\mathbf{X}\mathbf{X}^{\mathbf{T}} - \tau\mathbf{X}^{\mathbf{2}})\right]$$

where the parameter τ is defined in the interval $\in [0,1]$. If $\tau = 0$, what is the mean density of eigenvalues? Discuss the case for generic τ.

4. Level spacing

Consider i.i.d. real random variables $\{X_1, ..., X_N\}$ drawn from a distribution $p(X)$, which is defined over a support Σ.

(a) What is the probability density of the spacing s between two subsequent levels?

(b) What would be the analogous level spacing distribution upon considering correlation among the levels?

(c) Deduce the probability density function $p_{N=2}(s)$ of the spacing between the two eigenvalues of a 2×2 random matrix belonging to the Gaussian Unitary Ensemble in the limit $s \to 0^+$.

5. Typical and atypical fluctuations in the Gaussian ensemble.

Consider the large deviation problem for the Wigner ensemble, $\mathbb{P}[\lambda_{\max} - 2\sqrt{\beta}\sigma > x] \equiv e^{-NC_\beta(x)}$ at finite N.

(a) Write the resulting density function for the typical and atypical fluctuations of the top eigenvalue. What is their order of magnitude respectively?

(b) Sketch the distribution(s) around $\lambda_{\max}$. What can you claim about its symmetry?

(c) By defining a generalized free energy in terms of the cumulative distribution, discuss the possibility of a phase transition, and its order, in the presence of a hard wall.

6. Cumulants

a) Give the definition of the generating function of the moments μ_k, valid at any order.

b) Re-consider the question in the Gaussian case. Represent the sixth-order moment $\tau(\mathbf{X^6})$ where $\mathbf{X}$ denotes a Gaussian matrix. Discuss the contribution of crossing pair partitions.

7. Disordered averages

a) Given the partition function $Z_N = \int \prod_{i=1}^{N} \frac{du_i}{\sqrt{2\pi/i}} \exp\{-\frac{i}{2} \sum_{i,j} u_i \, (\lambda_\epsilon \delta_{ij} - M_{ij}) \, u_j\}$, where M_{ij} denote the elements of a random symmetric matrix, write the expressions for the free-energy and the spectral density in the *annealed approximation*.

b) Write the analogous expressions in the *quenched* case. Hint: recall the Edwards-Jones formula.

c) Explain the differences between *annealed* and *quenched* averages, along with benefits and limitations.

8. Correlated random variables

The Hessian matrix – associated with a given Hamiltonian H – is defined by

$$\mathcal{M}_{ij} = \frac{1}{N}\sum_{p=1}^{P} \xi_i^p \xi_j^p \theta(-h_p) + \mu\delta_{ij}$$

where $\theta(\cdot)$ is the Heaviside function, and μ is a Lagrange multiplier resulting in a diagonal shift. The so-called *gap variables* h_p are defined by $h_p = \sum_{i=1}^{N} \frac{1}{\sqrt{N}} x_i \xi_i^p$, in terms of the vectors $\boldsymbol{\xi}^p$. Then $\boldsymbol{\xi}^p = \{\xi_1^p, , ..., \xi_N^p\}$ are i.i.d. random variables with independent $\mathcal{N}(0, 1)$ components.

(a) What ensemble is the Hessian matrix $\mathcal{M}_{ij}$ associated with?

(b) What is the spectral density in the thermodynamic limit? Which hypothesis did you make to write it? Show qualitatively $\rho(\lambda)$ in the different cases (at least two).

(c) Suppose you started with $P < N$. Can any information be deduced beforehand?

9. Symmetrization and rotational invariance

Let us define $\mathbf{M}$ a real non-symmetrix matrix whose elements are i.i.d. drawn from a Gaussian distribution with zero mean and variance $\langle M_{ij}^2 \rangle = \sigma^2$.

(a) Perform a two-line computation to show the relation between the diagonal and the off-diagonal elements of the resulting GOE matrix.

(b) If $\mathbf{R}$ is a rotation matrix such that $\mathbf{R}^T\mathbf{R} = \mathbf{1}$, show that trace of the GOE matrix obtained above times its transpose is invariant under orthogonal transformations, *i.e.* $\mathbf{H} \to \mathbf{R}^T\mathbf{H}\mathbf{R}$.