Qubits

We saw last time

Complex number
Hilbert space, vector, scalar product
orthonormal basis
Ket and bra notation

Quantum Bit

Let $S_1 = \left\{ \alpha|0\rangle + \beta|1\rangle \in \mathcal{H} \,\middle|\, |\alpha|^2 + |\beta|^2 = 1 \right\}$

We say that two elements $|\phi\rangle, |\psi\rangle \in S_1$ are **related by a global phase** if there exists an angle θ such that $|\psi\rangle = e^{i\theta}|\phi\rangle$. In this case, let us write $|\psi\rangle \equiv_{phase} |\phi\rangle$

States of quantum bits are equivalence classes under $\equiv_{phase}$.
In other words, the state of a quantum bit (qubit) Q is a subset of S_1 such that

- Elements of Q are pariwise related by a global phase:
  $|\phi\rangle, |\psi\rangle \in Q$ implies that $\exists\theta$ such that $|\psi\rangle = e^{i\theta}|\phi\rangle$
- Q is maximal:
  $|\phi\rangle \in Q$ implies that $\forall\theta$, we have $e^{i\theta}|\phi\rangle \in Q$

An element $|\psi\rangle \in Q$ is called a **representative element** of Q.

By abuse of notation, when talking about a qubit Q we refer to elements $|\psi\rangle \in Q$ instead of Q.

Canonical representation

Consider a qubit $\alpha|0\rangle + \beta|1\rangle$.

The two complex numbers α and β can be written as $\rho_a e^{i\phi_a}$ and $\rho_b e^{i\phi_b}$ with ρ_a and ρ_b non-negative and such that $\rho_a^2 + \rho_b^2 = 1$

So there exists an angle $\theta \in [0, \pi]$ such that $\rho_a = \cos\left(\frac{\theta}{2}\right)$ and $\rho_b = \sin\left(\frac{\theta}{2}\right)$

Another representative element of the same qubit is

$$e^{-i\phi_a}\left(\alpha|0\rangle + \beta|1\rangle\right)$$

which is then

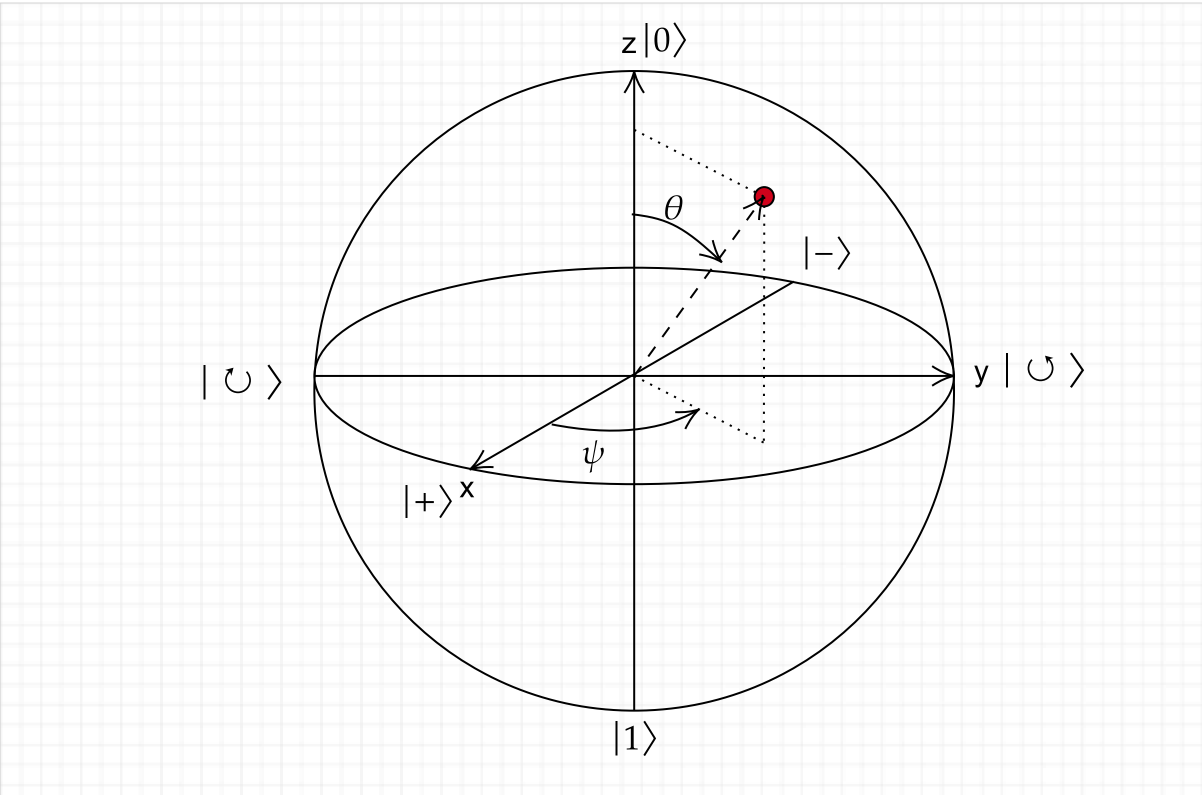
$$\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i(\phi_\beta-\phi_\alpha)}|1\rangle$$

Bloch Sphere

A qubit can therefore be parametrized by two angles $\theta \in [0,\pi]$ and $\psi \in [0,2\pi[$:

$$\cos\left(\frac{\theta}{2}\right)\cdot|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\psi}\cdot|1\rangle$$

This gives a 3-D representation of a qubit on the **Bloch sphere** :



$$\cos\left(\frac{\theta}{2}\right)\cdot|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\psi}\cdot|1\rangle$$

Where are

- $|0\rangle$: $\theta = 0$ and by convention $\psi = 0$ (but not very important)
- $|1\rangle$: $\theta = \pi$ et $\psi = 0$

- $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $\theta = \pi/2$ $\psi = 0$

- $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \qquad \theta = 3\pi/2 \quad \psi = 0$

- $|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \qquad \theta = \pi/2 \quad \psi = \pi/2$
- $|\circlearrowright\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \qquad \theta = 3\pi/2 \quad \psi = \pi/2$

$$\langle \circlearrowleft | \circlearrowright \rangle = \overline{\frac{1}{\sqrt{2}}(\langle 0| + i\langle 1|)} \cdot \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle = \frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|) \cdot \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$
$$= \frac{1}{2}(\langle 0|0\rangle - i\langle 1|0\rangle - i\langle 0|1\rangle - \langle 1|1\rangle) = 0$$

$$|| \, | \circlearrowleft \rangle || = ||\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)|| = \sqrt{\frac{1}{2} + \frac{|i|^2}{2}} = 1$$

So these are indeed **orthonormal bases**

Tensor of qubits

When given for instance 3 qubits, the state of the memory is a vector in $\mathcal{H}^{\otimes 3} \equiv \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ when $\mathcal{H}$ is the space of one single qubit.

The state of a 3-qubit system is under the form:

$$\sum_{b_1, b_2, b_3 \in \{0,1\}} \alpha_{b_1, b_2, b_3} \cdot |b_1\rangle \otimes |b_2\rangle \otimes |b_3\rangle \equiv \sum_{b_1, b_2, b_3 \in \{0,1\}} \alpha_{b_1, b_2, b_3} \cdot |b_1 b_2 b_3\rangle$$

Thus 8 basis vectors !

Tensor commutes with norm and scalar product: if $v_1, v_2 \in \mathcal{E}$ and $w_1, w_2 \in \mathcal{F}$ then
- $v_i \otimes w_j \in \mathcal{E} \otimes \mathcal{F}$
- $||v_i \otimes w_j|| = ||v_i|| \cdot ||w_j||$
- $\langle v_1 \otimes w_1 | v_2 \otimes w_2 \rangle = \langle v_1 | v_2 \rangle \langle w_1 | w_2 \rangle$

A vector under the form $v \otimes w \in \mathcal{E} \otimes \mathcal{F}$ is called **separable**, a vector $u \in \mathcal{E} \otimes \mathcal{F}$ that cannot be factorized that way is called **entangled**.

The list of vectors $|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle$ is an orthonormal basis for $\mathcal{H}^{\otimes 3}$. Recall: lexicographic order when writing line or column vector.

Also : $|...\rangle$ ≡ column vector and $\langle....|$ ≡ line vector

NOTA : There are $2^{\text{number of qubits}}$ basis vectors !

Bases of entangled states

Let us pick $\mathcal{H} \otimes \mathcal{H}$ - Canonical basis : $|00\rangle, |01\rangle, |10\rangle, |11\rangle$
It is an orthonormal basis
- For instance : $|| \, |01\rangle|| = |||0\rangle \otimes |1\rangle|| = |||0\rangle|| \cdot |||1\rangle|| = 1 \cdot 1 = 1$
- Also : $\langle 00|01\rangle = \langle 0|0\rangle\langle 0|1\rangle = 1 \cdot 0 = 0$

Example of another orthonormal basis:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \ , \ \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \ , \ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \ , \ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

(called the **Bell basis**)
Each one of such vector is **entangled**. For all scalars $\alpha, \beta, \gamma, \delta$:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

Notion of Matrix

Matrices are representations of linear maps on space states in finite dimension.
Action of a linear operator on a vector ≡ matrix multiplication with the corresponding vector.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$ So a_{ij} : i ≡ line number and j ≡ column number

If A and B are 2 matrices nxn , and if a_{ij} is the coefficient on line i and column j, then the coefficient (i, j) of the matrix $A \cdot B$ is

$$\sum_k a_{ik} b_{kj}$$

For instance :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & ... \\ ... & ... \end{pmatrix}$$

A column vector is "just" a matrix with only one column:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{pmatrix}$$

Note : If A is the matrix of a linear operation f and B the matrix of g, then $f(g(v)) = A \cdot (B \cdot v) = (A \cdot B) \cdot v$, so $f \circ g = A \cdot B$

Tensor of matrix

Pick $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

Then $A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} =$ "$\begin{pmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_{21}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{22}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{pmatrix}$"

$$= \begin{array}{c} \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \\ \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \end{array}$$

When in basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, then $A \otimes B$ applies A on the 1st qubit and B on the 2nd qubit.