Master PCS — Out of equilibrium statistical physics

TD1

1 Linear stochastic systems and causality

Let us consider a linear discrete-time Markov process for the position x of a particle

$$x = x(t+1) = ax(t) + b\eta(t) \qquad x(t+1) = \frac{1}{a} x^{ct} \tag{1}$$

with t an integer, a and b two constants and $\eta(t)$ a Gaussian random number of unit variance.

1. Write the evolution equation that relate $P(x, t+1)$ to $P(x,t)$ the probability to find the particle at x at time t.
2. Does the system satisfy detailed balance? Which function plays the role of the energy? What is the condition on a for the particle to be confined near $x = 0$?

We generalize the previous model to a vector with N components $\mathbf{x} = (x_i)_{i=1\cdots N}$

$$x_i(t+1) = a_{ij}x_j(t) + b_i\eta_i(t) \tag{2}$$

with a summation on j implied. $A = (a_{ij})$ is now a $N \times N$ matrix and $\mathbf{b} = (b_i)$ a vector. The components $\eta_i(t)$ are uncorrelated: $\langle \eta_i(t)\eta_j(t')\rangle = \delta_{ij}\delta_{tt'}$.

3. Find a condition on a_{ij} and b_i for the system to obey detailed balance in steady state.

We define the response matrix $R(\tau)$ and the correlation matrix $C(\tau)$ as

$$R_{ij}(\tau) = \frac{\langle \delta x_i(\tau)\rangle}{\delta x_j(0)}; \qquad C_{ij}(\tau) = \langle x_i(\tau)x_j(0)\rangle \tag{3}$$

for an integer τ. The brackets $\langle\cdot\rangle$ denote an average over noise realizations.

4. Reasoning by induction show that $R(\tau) = A^\tau$ and that $C(\tau) = R(\tau)C(0)$.

It is a problem in many areas of science to distinguish causation and correlation. Indeed, "correlation does not imply causation" as the saying goes. As a minimal example, let us consider the 3-variable system where x acts as a common cause of y and z

$$\begin{cases} x(t+1) &= ax(t) + b\eta_x(t) \\ y(t+1) &= ax(t) + ay(t) + b\eta_y(t) \\ z(t+1) &= ax(t) + az(t) + b\eta_z(t) \end{cases} \tag{4}$$

5. Compute directly from Eq. (4) the equal time correlation in steady state $\langle x^2\rangle$ then $\langle xy\rangle$ and $\langle yz\rangle$.
6. The response function provides a good measure of causation. Using question 4, compute the response matrix $R(\tau)$
7. Comment on the saying "correlation does not imply causation".

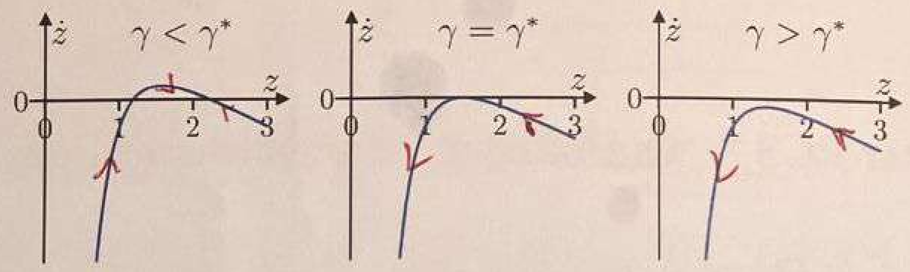

Figure 1: Sketch of $\dot z$ from Eq. (8) in the three regimes.

2 LSW coarsening

Lifshitz and Slyozov and, independently, Wagner proposed in 1961 a theory to describe the coarsening of a conserved scalar quantity in the limit when the droplets of the minority phase occupy a small fraction of the system. In this limit, the droplets can be considered independent so that their size evolve according to the Ostwald ripening equation derived in the lectures:

$$\frac{dR}{dt} = -\frac{\sigma}{2R^2}\left(1 - \frac{R}{R_c}\right). \tag{5}$$

Let $f(R,t)$ be the concentration of droplets of radius R at time t. It evolves according to the continuity equation

$$\frac{\partial f}{\partial t} + \frac{\partial J}{\partial R} = 0, \quad J = f(R,t)\frac{dR}{dt} \tag{6}$$

with $\frac{dR}{dt}$ given by Eq. (5).

Moreover, in the minority limit, the value of the field in the majority and minority phases can be assumed constant so that the volume of droplet is constant in time:

$$\int_0^\infty R^3 f(R,t)dR = \text{const.} \tag{7}$$

Eq. (5)-(6)-(7) are the starting point of the LSW theory which aims to derive the droplet distribution function $f(R,t)$ and the evolution of the characteristic droplet size $R_c(t)$.

1. We introduce the scaling ansatz $\phi(z) = R_c^4 f(R,t)$ with $z = R/R_c$. Rewrite the conservation of volume Eq. (7) in terms of z and ϕ.

2. Rewrite Eq. (6) in terms of the new variables, separating terms that depend on time and terms that depend on z. Deduce the evolution of $R_c(t)$.

3. Rewrite the spatial part of the equation derived in 2. as $g(z)\phi'(z) - h(z)\phi(z) = 0$ where g and h are functions of z that also depend on an unknown constant γ. Deduce a formal expression for ϕ as an integral of a rational function.

4. Using previous results, show that Eq. (5) can be rewritten

$$\dot z = \frac{1}{3\gamma t} g(z) \tag{8}$$

We give in Fig. 1 a sketch of $\dot z$ as a function of z in the three regimes obtained by varying the constant γ. Compute the critical value γ^*. Which choices of γ are allowed by the conservation of volume?