Elementary Notions

Complex Numbers

$\mathbb{C}$ is the field of complex numbers: $\alpha = a + b \cdot i$
with a and b reals and i the **imaginary** number : $i^2 = -1$

We denote : $\overline{\alpha} = a - b \cdot i$ called the **complex conjugate** of α.

$|\alpha| = \sqrt{a^2 + b^2}$ **absolute value** of α

$|\alpha|^2 = \alpha \cdot \overline{\alpha}$

because

$$\alpha \cdot \overline{\alpha} = (a + b \cdot i)(a - b \cdot i) = a^2 + ab \cdot i - ab \cdot i + (b \cdot i)(-b \cdot i) = a^2 - i^2 \cdot b^2 = a^2 + b^2$$

Radial representation of complex numbers

$$\alpha = |\alpha| \cdot \left(\frac{a}{|\alpha|} + \frac{b}{|\alpha|} \cdot i \right)$$

on a $\left(\frac{a}{|\alpha|} \right)^2 + \left(\frac{b}{|\alpha|} \right)^2 = 1$

So there is an angle $\theta \in [0, 2\pi[$ such that $\cos(\theta) = \frac{a}{|\alpha|}$ and $\sin(\theta) = \frac{b}{|\alpha|}$
$\alpha = |\alpha| \cdot (\cos(\theta) + \sin(\theta) \cdot i)$

Also : $\cos(\theta) + i \cdot \sin(\theta) = e^{i\theta}$

Why ?

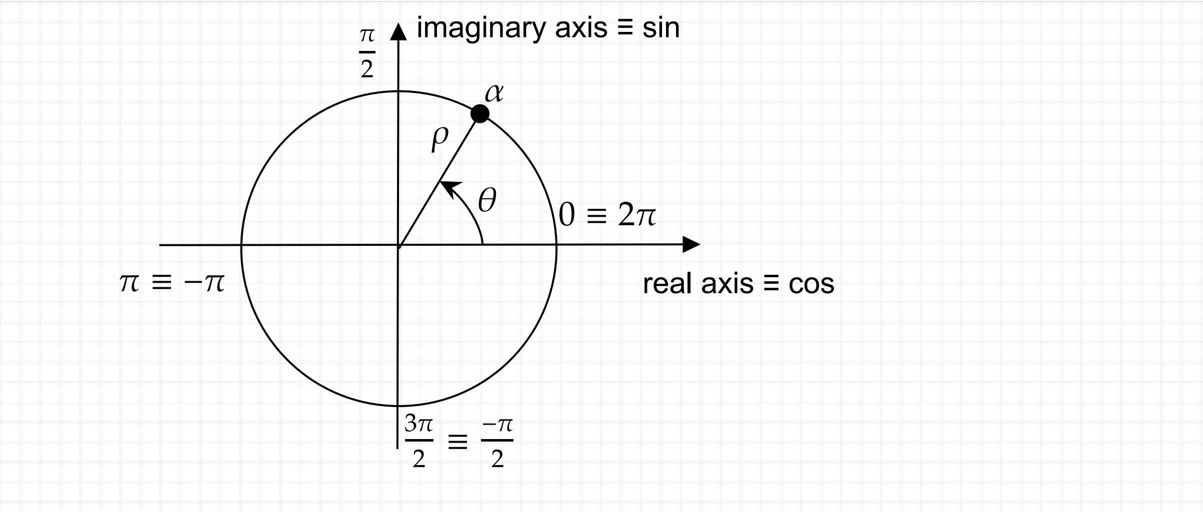
On a $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\cos(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!}$ $\sin(\theta) = \sum_{n} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}$

One can try to do

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \cdot \sum_{n} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}$$

$= \cos(\theta) + i \cdot \sin(\theta)$

Our complex number can then be written as a "canonical" form $\alpha = \rho \cdot e^{i\theta}$
with ρ positive real : the **amplitude** of α, while θ is the **phase**



$i = e^{i\frac{\pi}{2}}$
$-1 = e^{i\pi} = e^{-i\pi}$

Some equalities : $e^{a+b} = e^a e^b$ $\overline{e^a} = e^{\overline{a}}$ --- in particular, if θ is real : $\overline{e^{i\theta}} = e^{-i\theta}$

Yet another one : $e^{ab} = \left(e^a\right)^b$

Hilbert spaces

In this course, vectorial spaces have in finite dimension !
For us : we choose a **basis**, that is, a set X, for instance $\{|0\rangle, |1\rangle\}$ (for now, just notation for set elements) -- $|...\rangle$ is called a **"ket"**
One can say that $|0\rangle$ = "false" and $|1\rangle$ = "true"

From X one can build the set of linear combinations on X:

$$v = \sum_{x \in X} \alpha_x \cdot x$$

with $\alpha_x \in \mathbb{C}$
These are formal linear combinations, but they behave in the usual way:

id $w = \sum_{x \in X} \beta_x \cdot x$

then $v + w = \sum_{x \in X} (\alpha_x + \beta_x) \cdot x$

we also have 0, the empty linear combination : $0 = \sum_{x \in X} 0 \cdot x$

and scalar multiplication is distributive $\beta \cdot v = \sum_{x \in X} (\beta \alpha_x) \cdot x$

When X is $\left\{ |0\rangle, |1\rangle \right\}$, we get : $|v\rangle = \alpha \cdot |0\rangle + \beta \cdot |1\rangle$
(The "ket" notation is also used for vectors)

With the lexicographic ordering on X, : $|0\rangle < |1\rangle$ one can represent $|v\rangle$ as a column vector

$|v\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

Thus $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

A Hilbert space is a vector space with a **scalar product** and a **norm**
In two dimensions:

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \middle| \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle = \overline{\alpha_1} \cdot \alpha_2 + \overline{\beta_1} \cdot \beta_2$$

and the norm of v is $||v|| = \sqrt{\langle v|v\rangle}$
So in particular

$$||\alpha \cdot |0\rangle + \beta \cdot |1\rangle|| = \sqrt{\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \middle| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\rangle} = \sqrt{\overline{\alpha}\alpha + \overline{\beta}\beta} = \sqrt{|\alpha|^2 + |\beta|^2}$$

From the scalar product we derive a notion of **orthogonality** :
we say that $v \perp w$ when $\langle v|w\rangle = 0$
For instance, $|0\rangle \perp |1\rangle$
A basis is **orthonormal** if all of its elements are pairwise orthogonal and if they are all of norm 1,
For instance, $\left\{ |0\rangle, |1\rangle \right\}$ is an orthonormal basis.

In a Hilbert space we usually only consider orthonormal bases.

Note: $|v\rangle$ is always a column vector. Scalar product of $v = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ with $w = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ is

written

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \middle| \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle = \overline{\alpha_1} \cdot \alpha_2 + \overline{\beta_1} \cdot \beta_2 = \begin{pmatrix} \overline{\alpha_1} & \overline{\beta_1} \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

So we can say that $\langle v|w\rangle = \langle v| \cdot |w\rangle$ where $\langle v|$ is the row-vector, conjugate transpose of $|v\rangle$

We call $\langle v|$ a **"bra"**

"bra"'s are row-vectors while "ket"'s are column vectors.

Example of orthonormal basis:

$$|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$$

$$|-\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big)$$

Are they orthogonal ?

$$\langle +| \cdot |-\rangle = \frac{1}{2}\Big(\big(\langle 0| + \langle 1|\big) \cdot \big(|0\rangle - |1\rangle\big)\Big) = \frac{1}{2}(1 - 1) = 0$$

So yes...

Another example:

$$|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + i|1\rangle\big)$$

$$|\circlearrowright\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - i|1\rangle\big)$$

To conclude:

The state of a qubit is of the form $\alpha \cdot |0\rangle + \beta \cdot |1\rangle$ of **norm 1** so $|\alpha|^2 + |\beta|^2 = 1$

in general, modulo a **global phase**, $\cos(\theta/2) \cdot |0\rangle + e^{i\phi} \sin(\theta/2) \cdot |1\rangle$

with $\theta, \phi \in [0, 2\pi[$

Finally, one can work with qu-n-bits with more than 2 valeurs : $\big\{|0\rangle, |1\rangle, |2\rangle ... |n\rangle\big\}$

Tensor (Kronecker Product)

Morally, when given 2 qubits, the two corresponding particles are spacially separated.

The state of the joint system ends up being a vector in the tensor product of the two original systems.

If first qubit state space is spanned with $|0_a\rangle, |1_a\rangle$ and second qubit state space spanned with $|0_b\rangle, |1_b\rangle$
then the space of the two qubits in the tensor space is spanned with
$|0_a 0_b\rangle,\ |0_a 1_b\rangle,\ |1_a 0_b\rangle,\ |1_a 1_b\rangle$
(of dimention 4)
With a third qubit $|0_c\rangle, |1_c\rangle$
The global state spave is
$|0_c 0_a 0_b\rangle,\ |0_c 0_a 1_b\rangle,\ |0_c 1_a 0_b\rangle,\ |0_c 1_a 1_b\rangle, |1_c 0_a 0_b\rangle,\ |1_c 0_a 1_b\rangle,\ |1_c 1_a 0_b\rangle,\ |1_c 1_a 1_b\rangle$
(dimension... 8)

If I have n qubits, the memory state is of dimension 2^n : the superposition of all possible chains of n bits.

Question : how to denote this with column vector ? We need an ordering on the basis. We pick the lexicographic ordering: in the case of the 3-qbit system, we had c then a then b, so if $\mathcal{H}_a$ is the state of the a-qubit (etc), the state $\mathcal{H}_c \otimes \mathcal{H}_a \otimes \mathcal{H}_b$
has for ordered basis
$|0_c 0_a 0_b\rangle,\ |0_c 0_a 1_b\rangle,\ |0_c 1_a 0_b\rangle,\ |0_c 1_a 1_b\rangle, |1_c 0_a 0_b\rangle,\ |1_c 0_a 1_b\rangle,\ |1_c 1_a 0_b\rangle,\ |1_c 1_a 1_b\rangle$

and $\begin{pmatrix} \alpha_{000} \\ \alpha_{001} \\ \alpha_{010} \\ \alpha_{011} \\ \alpha_{100} \\ \alpha_{101} \\ \alpha_{110} \\ \alpha_{111} \end{pmatrix}$ corresponding to $\sum_{x,y,z} \alpha_{xyz} |x_c x_a x_b\rangle$

If qubits a and b are in states $|a\rangle = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ et $|b\rangle = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$
The memory state with a AND b will be
$|a\rangle \otimes |b\rangle = \left(\alpha_1 |0_a\rangle + \beta_1 |1_a\rangle\right) \otimes \left(\alpha_2 |0_b\rangle + \beta_2 |1_b\rangle\right) = (\alpha_1 \alpha_2) \cdot |0_a 0_b\rangle + \ldots$
where $|0_a 0_b\rangle \equiv |0_a\rangle \otimes |0_b\rangle$
In column vector notation :

$$|a\rangle \otimes |b\rangle = \begin{pmatrix} \alpha_1|b\rangle \\ \beta_1|b\rangle \end{pmatrix} = \begin{pmatrix} \alpha_1\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \\ \beta_1\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{pmatrix}$$

Note : $||v \otimes w|| = ||v|| \cdot ||w||$

And $\langle v_1 \otimes w_1 | v_2 \otimes w_2 \rangle = \langle v_1 | v_2 \rangle \cdot \langle w_1 | w_2 \rangle$ if the dimensions of v_1 and v_2 are the same, et and the dimensions of w_1 and w_2 are the same.

For instance : $\langle 01 | 00 \rangle = \langle 0|0\rangle\langle 1|0\rangle = 0$
(they are orthogonal)