complex systems

Academic year 2021-2022

Statistical field theory

Tutorial n°3

Ising model in one dimension: block-spin approach

One considers the Ising model in one dimension for N spins in presence of a magnetic field. The Hamiltonian is given by:

$$\beta H = -K \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i$$

1) Write the expression of the partition function $Z(N, K, h)$.

2) We shall perform the partial sum on the even-index spins in the partition function. To do that one divides the one dimensional lattice into a sublattice of odd-index spins S_i and a sublattice of even-index spins σ_i, see Fig.1.

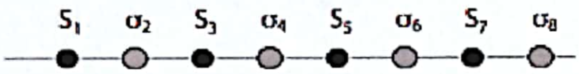

Figure 1: The two sublattices of the one dimensional Ising model.

Show that the partition function reads:

$$Z = \sum_{[S_i]} e^{-\beta H[S_i]} \equiv \sum_{[S_i]}^{N} e^{\sum_{i=1}^{N} B(S_i, S_{i+1})} = \sum_{[S_i']}^{N/2} \sum_{[\sigma_i]}^{N/2} e^{\sum_{i=1}^{N/2} B(S_i', \sigma_i) + B(\sigma_i, S_{i+1}')} \tag{1}$$

where $S_i' = S_{2i-1}$, $\sigma_i = S_{2i}$ and $B(S_i, S_{i+1}) = g + \frac{h}{2}(S_i + S_{i+1}) + K S_i S_{i+1}$ where g is a constant.

3) One assumes that:

$$e^{-\beta H'[S_i']} \equiv \prod_{i=1}^{N/2} \left[\sum_{\sigma_i = \pm 1} e^{B(S_i', \sigma_i) + B(\sigma_i, S_{i+1}')} \right] \equiv e^{\sum_{i=1}^{N/2} B'(S_i', S_{i+1}')}$$

where $B'(S_i', S_{i+1}')$ is supposed to be of the same form as $B(S_i, S_{i+1})$ but with parameters K', h' and g'.

Derive the relations between the sets of coupling constants $\{K', h', g'\}$ and $\{K, h, g\}$. It will be convenient to define the coupling constants:

$$\begin{cases} x = e^K \quad , \quad y = e^h \quad , \quad z = e^g \\ x' = e^{K'} \quad , \quad y' = e^{h'} \quad , \quad z' = e^{g'} \end{cases}$$

and show that the solution of the corresponding system is given by:

$$\begin{cases} z'^4 = z^8(x^2y + x^{-2}y^{-1})(x^{-2}y + x^2y^{-1})(y + y^{-1})^2 \\ \\ y'^2 = y^2 \, \dfrac{x^2y + x^{-2}y^{-1}}{x^{-2}y + x^2y^{-1}} \\ \\ x'^4 = \dfrac{(x^2y + x^{-2}y^{-1})(x^{-2}y + x^2y^{-1})}{(y + y^{-1})^2} \, . \end{cases}$$

Finally using the change: $x \to \tilde{x} = x^{-4}$, $y \to \tilde{y} = y^{-2}$ show that the two last equations read:

$$\begin{cases} \tilde{y}' = \dfrac{\tilde{y}(\tilde{x} + \tilde{y})}{1 + \tilde{x}\tilde{y}} \\ \\ \tilde{x}' = \dfrac{\tilde{x}(1 + \tilde{y})^2}{(\tilde{x} + \tilde{y})(1 + \tilde{x}\tilde{y})} \, . \end{cases} \tag{2}$$

Note that taking the logarithm of the solution of (2) one gets the RG flow equations for h and K:

$$\begin{cases} h' = h + \delta h(K, h) \\ \\ K' = K + \delta K(K, h) \, . \end{cases} \tag{3}$$

i) Consider the system (2) and show that the $h = 0$ subspace is closed, *i.e.* starting with a vanishing magnetic field one gets a vanishing field after one RG step.

ii) Show that there is a line of fixed points at $\tilde{x} = 1$, $\forall \, \tilde{y}$. Interpret this in terms of the original variables h and K (or even better T). Analyze the structure of the RG flow near any fixed point $\tilde{x}^* = 1$ and $\tilde{y}^* = y_0$. To do this write $x = x^* + \delta x$ and $y = y_0^* + \delta y$.

iii) Consider the fixed point $\tilde{x}^* = 0$ and $\tilde{y}^* = 0$. What are the physical properties that characterize this fixed point ? Analyze the renormalization group (RG) flow around this fixed point.

iv) Show that there is a fixed point P_c at $\tilde{x}^* = 0$ and $\tilde{y}^* = 1$. What are the physical properties that characterize this fixed point ? Analyze the structure of the RG flow near P_c.

v) Draw the RG flow diagram including all fixed points and the RG lines relating them.

vi) Write the scaling law for the correlation length ξ in terms of the scaling variables K and h. Determine the form of $\xi(K, h)$.