Master PCS — Out of equilibrium statistical physics

TD2

1 Front propagation in the ASEP

We consider the ASEP model on a 1-dimensional lattice of size L. Particles hop right and left with rates p and q respectively if the target site is empty.

1. Write the mean field equation for the evolution of the occupation $\langle n_i(t)\rangle$ of site i.
2. To take the continuum limit, we define the space variable $x = i/L$ and the density field $\rho(x,t) = \langle n_i(t)\rangle$. Write the equation of evolution for $\rho(x,t)$.
3. We first assume that the asymmetry is weak, meaning that $p - q = \lambda/L$. Rewrite the equation for $\rho(x,t)$ using a diffusive rescaling $t \to t/L^2$. Show that $u = 1 - 2\rho$ evolves according to Burgers' equation.
4. Write the equation that is obtained for $\rho(x,t)$ if the asymmetry is not weak, with $p - q = \lambda = O(1)$, using a ballsitic time rescaling.

We want to solve the previous equation (the one with strong asymmetry) using the method of characteristics. Let us define the characteristic $x_c(t)$ such that $\partial_t x_c(t) = \lambda(1 - 2\rho(x_c(t)))$.

5. Show that the density is constant along a characteristic.
6. We assume that initally the density profile is a given function $\rho_0(x)$. From each initial point x_0, there is a different characteristic with $x_c(0) = x_0$. When are two characteristics crossing? What is the condition on $\rho_0(x)$ for characteristics never to cross?
7. **Rarefaction wave.** We assume that $\rho_0(x)$ is a decreasing step function with ρ_L for $x < 0$ and ρ_R for $x > 0$ and $\rho_L > \rho_R$. How does the profile evolve qualitatively?
8. **Shock wave.** We now take $\rho_0(x)$ to be an increasing step, with $\rho_L < \rho_R$. Because the characteristics cross, the step retains the same shape but propagates at some speed c. Compute this speed by considering the mass balance at the front.

2 Propagation of a favorable allele

Let us consider a population with a fixed size of N individuals. Among them n possess a favorable allele A for a gene and $N-n$ possess the less favorable allele a of the gene. Individuals with allele A reproduce at rate α and those with allele a at rate $\beta < \alpha$. Both type of individuals die at a rate $r(n, N)$.

We first assume that the population is well mixed so that spatial considerations do not enter.

1. Write the equation of evolution for the mean number of individuals of each type. Choose the death rate r such that the total population size N remains constant in average.

2. Write the equation of evolution for the evolution of the fraction $u = \langle n \rangle / N$ of individuals with allele A. What value does u take at large time?

We now want to study how the allele A propagates spatially according to the FKPP equation in one dimension

$$\partial_t u(x,t) = D\partial_x^2 u + cu(1-u) \tag{1}$$

with constants $D > 0$ and $c > 0$. We assume that initially $u(x) = 1$ for $x < 0$ and $u(x) = 0$ for $x \geq 0$. As time passes, the front separating the regions with $u = 1$ and $u = 0$ moves to the right, with a shape that we assume to be stationary.

3. We use a mechanical analogy, the "Newton mapping", interpreting the spatial variable x as a time and u as the position of a particle moving in an external potential. Write the Newton equation. What are the two possible shapes of the propagating front depending on its speed v. Which one is physically acceptable?

4. Close to the $u = 0$ region, we can linearize the FKPP equation and look for solutions of the form $e^{-\gamma(x-vt)}$. Give the velocity of such a front as a function of γ. What is the smallest possible velocity?

The linearized FKPP equation can be solved exactly. Its solution reads

$$u_l(x,t) = \frac{e^{ct}}{\sqrt{4\pi Dt}} \int_{-\infty}^{+\infty} u_l(x_0, 0) \exp\left(-\frac{(x-x_0)^2}{4Dt}\right) dx_0 \tag{2}$$

5. Use this expression to argue that the front propagates at the minimum speed in the nonlinear problem for the initial condition given above.

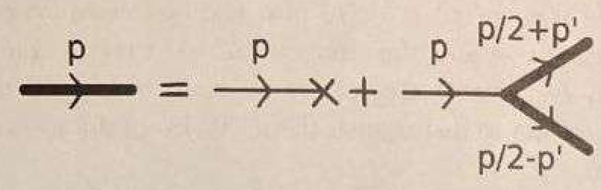

Figure 1: Diagrammatic representation of the KPZ Eq. (3)

3 Perturbative renormalization of the KPZ equation

We first want to write the KPZ equation in Fourier space for both the spatial and temporal parts. We use the convention

$$h(\omega, \mathbf{k}) = \int d^d\mathbf{r} \int dt h(\mathbf{r}, t) e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}$$

and introduce the short-hand notations $p = (\omega, \mathbf{k})$ and $\int_p = \int d^d\mathbf{k} \int d\omega \frac{1}{(2\pi)^{d+1}}$ where the integrals on $\mathbf{k}$ are cut off at large wavevectors $|\mathbf{k}| < \Lambda$. For example, for a lattice model we would take $\Lambda = 2\pi/a$ with a the lattice spacing.

1. Show that the KPZ equation can be written

$$h(p) = G_0(p)\eta(p) - \frac{\lambda}{2} G_0(p) \int_{p'} \left(\frac{\mathbf{k}}{2} + \mathbf{k}'\right) \cdot \left(\frac{\mathbf{k}}{2} - \mathbf{k}'\right) h\left(\frac{p}{2} + p'\right) h\left(\frac{p}{2} - p'\right) \quad (3)$$

where $G_0(p) = \frac{1}{\nu k^2 - i\omega}$ and $\langle \eta(p)\eta(p')\rangle = \Gamma(2\pi)^{d+1}\delta(p + p')$.

2. Assuming that λ is small, we would like to compute $h(p)$ in a perturbative expansion in λ. Show that the first-order term is divergent in $d \leq 2$.

Momentum shell RG transformation. The steps are identical to the implementation in the equilibrium case, except that we work directly with the KPZ equation rather than with the partition function:

- We first integrate out the degrees of freedom on short scale, corresponding to wavevectors $\Lambda/b < |\mathbf{k}| < \Lambda$ (with $b > 1$). At the end of this step, we obtain a KPZ equation with different coefficients λ, ν and Γ and a smaller cutoff Λ/b.
- We rescale the system to restore the original cutoff Λ and obtain the RG the flow equation which tell us how the coefficients evolve under a RG transformation.
- The fixed points of the RG flow tell us about the behavior of the system on large scale.

The coarse-graining step is more easily done using Feynman diagrams. We introduce the following rules: a thick line represents $h(p)$, a thin line $G_0(p)$, a cross $\times$ the noise $\eta(p)$, a vertex with incoming momentum p and outgoing momenta p_1 and p_2 represents the integral

$$-\frac{\lambda}{2} \int_{p_1} \int_{p_2} \mathbf{k}_1 \cdot \mathbf{k}_2 (2\pi)^{d+1} \delta(p - p_1 - p_2). \quad (4)$$

An empty circle $\circ$ denotes the contraction of two noise variables $\langle \times_p \times_{p'} \rangle = \circ_{p,p'} = \Gamma(2\pi)^{d+1}\delta(p + p')$. The representation of the KPZ Eq. (3) is shown in Fig. 1.

3. We split the field $h(p) = h^>(p) + h^<(p)$ into the fast component $h^>(p) = h(p)$ if $\Lambda/b < |\mathbf{k}| < \Lambda$ and zero otherwise and the slow one $h^<(p)$ that is non-zero for $|\mathbf{k}| < \Lambda/b$. We decompose similarly $G_0(p)$ and $\eta(p)$ and add small marks on the lines of the diagrams involving fast components to distinguish them. Write in diagrammatic form the equations for $h^>(p)$ and $h^<(p)$.

4. Write in diagrams the explicit expression of the fast variable $h^>(p)$ in terms of $h^<(p)$, G_0 and η by doing an expansion at first order in λ.

5. Replace $h^>(p)$ by the expansion in the equation for the slow part $h^<(p)$. Keep terms up to order λ^2.

6. The next step is to average over $\eta^>$, the fast part of the noise. Some diagrams correspond to terms of the KPZ equation and others to higher order (irrelevant) terms. Try to identify which diagrams contribute to the KPZ terms.

RG flow. After rescaling the system by $\mathbf{r} \to b\mathbf{r}$, $t \to b^z t$, $h \to b^\alpha h$ with $b = 1 + ds$, we obtain the RG flow equations

$$\frac{d\nu}{ds} = \left(z - 2 - \frac{(d-2)g^2}{d}\right)\nu \tag{5}$$

$$\frac{d\lambda}{ds} = (\alpha + z - 2)\,\lambda \tag{6}$$

$$\frac{d\Gamma}{ds} = (z - d - 2\alpha + g^2)\,\Gamma \tag{7}$$

with $g^2 = \Gamma\lambda^2 K_d \Lambda^{d-2}/8\nu^3$ and $K_d = S_d/(2\pi)^d$ with S_d the area of the unit sphere in d dimension.

7. Find the Edwards-Wilkinson and KPZ fixed points. What are the exponents in each class ?

8. Writing the flow equation for g^2, look at the stability of the two fixed points in $d = 1$, $d = 2$ and $d = 3$.