TD2 : stability of car/ped-following models

1. A 1D line of cars following each other drives on an infinite road. Cars are numbered so that car i is following car $i-1$.
   We assume that each car follows the OV (Optimal Velocity) model [Bando et al., 1995] :
   $$\ddot{x}_i(t) = \frac{1}{\tau}\left[V\left(\Delta x_i(t)\right) - \dot{x}_i(t)\right] \tag{1}$$
   where
   $$\Delta x_i \equiv x_{i-1} - x_i.$$
   We shall study the stability of the flow, starting from a uniform solution such that
   $$\begin{aligned} \Delta x_k^U &= d \quad \forall k \\ v_k^U &= v = V(d) \quad \forall k \end{aligned}$$
   Local stability
   We apply a small perturbation y on the follower, namely car i :
   $$x_i(t) = x_i^U + y(t)$$
   (a) • Write the corresponding relation for $\Delta x_i(t)$ on the one hand, for $\dot{x}_i(t)$ and $\ddot{x}_i(t)$ on the other hand.
   (b) • Replace in Eq.(1) in order to find the differential equation for y.
   (c) • Solve the differential equation for y.
   (d) • What can we conclude for the local linear stability of the OV model ?
   We can assume that $V'(d) > 0$.
   (e) • What can we conclude for the risk of collision ?
   (f) • We assume that $V(d)$ is such that the time headway between two successive cars in the uniform state is 2s whatever the value of the distance d. Then what would be the condition on the relaxation time τ in order to avoid collisions ?

2. The OV model of the previous section can be obtained by expanding the l.h.s. [1] of the equation defining the Newell's model of 1961 :
   $$\dot{x}_i(t+\tau) = V\left(\Delta x_i(t)\right)$$
   By contrast, the GOV (Generalized Optimal Velocity) model [Tordeux and Seyfried, 2014] that we shall study now is obtained by expanding the r.h.s. and reads
   $$\dot{x}_i(t) = V\left[\Delta x_i(t) - \tau\left[V\left(\Delta x_{i-1}(t)\right) - V\left(\Delta x_i(t)\right)\right]\right]$$
   (a) Why do we say that this model involves 2 predecessors ?
   Local stability
   We start from a uniform solution such that
   $$\begin{aligned} \Delta x_k^U &= d \quad \forall k \\ v_k^U &= v = V(d) \quad \forall k \end{aligned}$$
   We apply a small perturbation y on the follower, namely car i :
   $$x_i(t) = x_i^U + y(t)$$

1. l./r.h.s. = left/right hand side

(b) As in the first exercise, find the differential equation governing the evolution of y.

(c) Solve the equation.

(d) What can we say of the local linear stability of the system ? of the risk of collision ? We can assume that $V'(d) > 0$.

Global stability on a ring

We consider N cars driving on a ring of length L.

We have periodic boundary conditions so that

$$\Delta x_1 = x_N - x_1 + L$$

We start from a uniform solution such that

$$\begin{aligned} \Delta x_i^U &= d = \frac{L}{N} \quad \forall i = 1 \to N \\ v_i^U &= v = V(d) \quad \forall i = 1 \to N \end{aligned}$$

We apply a small perturbation y_i on each car i, for $i = 1 \to N$:

$$x_i(t) = x_i^U + y_i(t)$$

(note that y has now an index i).

(e) • Write the corresponding relations for $\Delta x_i(t)$ on the one hand, for $\dot{x}_i(t)$ on the other hand.
  • Replace in Eq.(2) in order to find the differential equation for y.

(f) • Rewrite the differential equation for y under the form

$$\dot{y}(t) = My(t)$$

where y is the vector $(y_1, \cdots, y_N)$, and M is an $N \times N$ matrix.

Preliminary calculations

Let K be a circulant $N \times N$ matrix such that $K_{1N} = 1$, $K_{ii-1} = 1$ $\forall i = 2 \to N$, and all other $K_{ij} = 0$.

(g) • Compute K^2, K^3, and infer K^N.

(h) • What are the roots in the complex plane of $\mu^N = 1$?

(i) • Express the eigenvalues of K as a function of the roots found in the previous question.

Back to our stability problem

(j) • Given that each circulant matrix can be expressed as a polynomial of matrix K, express M as a function of K.
  • What are the eigenvalues of M ?

(k) • What can we say of the sign of the real part of these eigenvalues ?

(l) • What can we conclude for the global linear stability of the GOV model on a ring ?

Références

[Bando et al., 1995] Bando, M., Hasebe, K., Nakayama, A., Shibata, A., and Sugiyama, Y. (1995). Dynamical model of traffic congestion and numerical simulation. *Phys. Rev. E*, 51 :1035.

[Tordeux and Seyfried, 2014] Tordeux, A. and Seyfried, A. (2014). Collision-free non uniform dynamics within continuous optimal velocity models. *Phys. Rev. E*, 90 :042812.