Université Paris Cité

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# M2 PCS — Statistical field theory and soft matter
# TD n°1 : Basic SFT tools

## 1 Einstein summation

1. Write the following quantities using Einstein's convention:

$$\boldsymbol{x} \cdot \boldsymbol{y}, \tag{1}$$
$$\left(\boldsymbol{a}\left(\boldsymbol{b} \cdot \boldsymbol{c}\right)\right)_i, \tag{2}$$
$$\mathsf{A} = \mathsf{BC}, \quad \text{(matrices)} \tag{3}$$
$$\mathrm{tr}(\mathsf{AB}). \tag{4}$$

2. In a space of dimension $d$, calculate $\delta_{ij}\delta_{ij}$, where $\delta_{ij}$ is the Kronecker (identity) tensor.

3. With $\partial_i = \partial/\partial x_i$, show using Einstein's summation that

$$\boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} f) = (\boldsymbol{\nabla}\alpha) \cdot (\boldsymbol{\nabla} f) + \alpha \boldsymbol{\nabla}^2 f \tag{5}$$

4. In $d = 3$, let $\epsilon_{ijk}$ be the Levi-Civita symbol. It is equal to 1 if $\{i, j, k\}$ is an even permutation of $\{1, 2, 3\}$, to $-1$ if it is an odd permutation, and to zero otherwise. Show (efficiently) that

$$\epsilon_{ijk}\,\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}. \tag{6}$$

5. Write $\boldsymbol{\nabla} \times \boldsymbol{v}$ using Einstein's summation, the Levi-Civita symbol and the basis vectors $\boldsymbol{e}_i$.

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## 2 Functional derivatives

1. Let $x_0 \in \mathbb{R}$ be a fixed point. Calculate the functional derivative $\delta f/\delta h(x)$ of

$$f[h] = h(x_0). \tag{7}$$

2. Calculate the functional derivative $\delta f/\delta h(x)$ of

$$f[h] = h'(y), \quad \forall y. \tag{8}$$

3. Calculate the functional derivative $\delta f/\delta h(x)$ of

$$f[h] = \int dx \, \frac{r}{2} \, h^2(x), \tag{9}$$

   using two different methods: (i) direct computation of $\delta f_1$, (ii) functional differentiation under the integral.

4. Consider the following functional $f[h]$ of a field $h(\boldsymbol{x})$ in a $d$-dimensional space:

$$f[h] = \int d^d x \, \frac{1}{2} a \left(\boldsymbol{\nabla} h\right)^2. \tag{10}$$

   Calculate $\delta f/\delta h(\boldsymbol{x})$, where $\boldsymbol{x}$ is a point in the bulk. By definition, the functional derivative in the bulk is calculated using functions $\delta h$ that vanish on the boundary.

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## 3 Average field and correlation function

Upon adding a formal external field to the effective Hamiltonian (it can be set to zero at the end of the calculations), the partition function and free-energy become

$$Z[h] = \int \mathcal{D}[\phi] \, e^{-\beta[\mathcal{H}[\phi] - \int d^d x \, h(\boldsymbol{x})\phi(\boldsymbol{x})]}, \qquad F[h] = -k_{\mathrm{B}} T \ln Z[h]. \tag{11}$$

We recall that the average of a quantity $Q$ is given by

$$\langle Q \rangle = \frac{1}{Z[h]} \int \mathcal{D}[\phi] \, Q \, e^{-\beta\left\{\mathcal{H}[\phi] - \int d^d y \, h(\boldsymbol{x})\phi(\boldsymbol{x})\right\}}. \tag{12}$$

1. Calculate the functional derivative $\delta Z/\delta h(\boldsymbol{x})$.

2. Show that

$$\langle \phi(\boldsymbol{x}) \rangle = - \left. \frac{\delta F}{\delta h(\boldsymbol{x})} \right|_{h=0}. \tag{13}$$

3. Show that

$$C(\boldsymbol{x}, \boldsymbol{y}) = -k_{\mathrm{B}} T \left. \frac{\delta^2 F}{\delta h(\boldsymbol{x}) \delta h(\boldsymbol{y})} \right|_{h=0}, \tag{14}$$

   where $C(\boldsymbol{x}, \boldsymbol{y}) = \langle \phi(\boldsymbol{x}) \phi(\boldsymbol{y}) \rangle - \langle \phi(\boldsymbol{x}) \rangle \langle \phi(\boldsymbol{y}) \rangle$ is the correlation function.

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## 4 Gaussian model

We consider, for a scalar field $\phi(\boldsymbol{x})$, with $\boldsymbol{x} \in \mathbb{R}^d$, the following Gaussian model:

$$\mathcal{H} = \int d^d x \left[\frac{r}{2}\phi^2 + \frac{c}{2}(\boldsymbol{\nabla}\phi)^2\right]. \tag{15}$$

In all the problem, we will neglect boundary terms whenever they appear.

1. Show that the Hamiltonian can be rewritten as
$$\mathcal{H} = \frac{1}{2}\int d^d x\, \phi(\boldsymbol{x})\mathcal{L}\phi(\boldsymbol{x}), \qquad \text{with } \mathcal{L} = r - c\boldsymbol{\nabla}^2. \tag{16}$$

2. Show that $\mathcal{L}$ is Hermitian.

3. Show that $\mathcal{H}$ can also be rewritten as
$$\mathcal{H} = \frac{1}{2}\int d^d x\, d^d y\, \phi(\boldsymbol{x})H(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{y}), \qquad \text{with } H(\boldsymbol{x},\boldsymbol{y}) = \mathcal{L}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{17}$$

4. Show that the inverse kernel $H^{-1}(\boldsymbol{x},\boldsymbol{y})$ satisfies the equation
$$\mathcal{L}_{\boldsymbol{y}} H^{-1}(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}), \tag{18}$$
where the subscript in $\mathcal{L}_{\boldsymbol{y}}$ specifies that $\mathcal{L}$ is taken for $\boldsymbol{\nabla} = \partial/\partial\boldsymbol{y}$.

5. Let $G(\boldsymbol{x})$ be the Green function of $\mathcal{L}$. Show that $H^{-1}(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{x}-\boldsymbol{y})$.

6. Show that
$$G(\boldsymbol{x}) = \int \frac{d^d q}{(2\pi)^d}\, \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{r+cq^2}. \tag{19}$$

7. Show that $G(\boldsymbol{x}) = G(|\boldsymbol{x}|)$ by working on $G(R_{ij}x_j)$ where $\mathsf{R} \in O(d)$ is any rotation.

8. Calculate the partition function and the free energy functional of an external field:
$$Z[h] = \int \mathcal{D}[\phi]\, e^{-\beta[\mathcal{H}[\phi] - \int d^d x\, h(\boldsymbol{x})\phi(\boldsymbol{x})]} \tag{20}$$
$$F[h] = -k_{\mathrm{B}}T \ln Z[h]. \tag{21}$$

9. Deduce from the relations
$$\langle\phi(\boldsymbol{x})\rangle = -\left.\frac{\delta F}{\delta h(\boldsymbol{x})}\right|_{h=0}. \tag{22}$$
$$C(\boldsymbol{x},\boldsymbol{y}) = -k_{\mathrm{B}}T \left.\frac{\delta^2 F}{\delta h(\boldsymbol{x})\delta h(\boldsymbol{y})}\right|_{h=0}, \tag{23}$$
that the average field and the correlation function of the Gaussian model are given by
$$\langle\phi(\boldsymbol{x})\rangle = 0, \tag{24}$$
$$C(\boldsymbol{x},\boldsymbol{y}) = k_{\mathrm{B}}T \int \frac{d^d q}{(2\pi)^d}\, \frac{e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{r+cq^2}. \tag{25}$$