Statistical field theory

Tutorial n°1

I - Conceptual issues

A- Properties of the free energy

We discuss here some properties of the free energy in relation to the study of phase transitions.

1) Recall the definition of a convex-up (or simply concave) function and a convex-down (or simply convex) function.

2) Starting from the general form of a partition function $Z(h, J, T)$ – but taking as example that of the Ising model in a magnetic field h – justify that the free energy $F(h, J, T)$ is a concave function of its arguments.

3) Deduce from this property that:

- F is a continuous function of its arguments.
- F is differentiable almost everywhere.
- $\dfrac{dF}{dx}$ is monotonically non-increasing

4) Prove directly the concavity of $F(T)$ by computing $\dfrac{\partial^2 F}{\partial T^2}$. Do the same for $\dfrac{\partial^2 F}{\partial h^2}$.

B- Spontaneous symmetry breaking and magnetization

We now study the properties of the magnetization. We start with a trivial lemma:

Lemma: For any function f of the $\{S_i\}$ one has: $\sum_{\{S_i\}} f(\{S_i\}) = \sum_{\{S_i\}} f(\{-S_i\})$

1) Deduce, from the lemma, that, for the partition function $Z(-h, J, T)$, one has:

$$Z(-h, J, T) = Z(h, J, T)$$

and $F(-h, J, T) = F(h, J, T)$. Where does this property come from ?

2) Deduce from the previous question that, for F analytic of h, the magnetization at vanishing magnetic field should vanish: $M(h = 0) = 0$.

3) Consider the case where $F(h) = F(0) - M|h| + \mathrm{O}(h^\sigma)$ with $\sigma > 1$ and show that, in this case, the magnetization at vanishing field does not vanish.

Clearly the previous behaviour relies on the non-commutativity of the limits $h \to 0$ and $N \to \infty$. A way to check this is to consider, at $h = 0$, and for a system with N spins, two configurations A and B related by $\mathbb{Z}_2$ symmetry. One thus has for the corresponding magnetizations $M_A = -M_B$.

4) Evaluate the ratio of the probabilities P_A and P_B to get the magnetization M_A and M_B respectively.

5) Consider successively the limits $h \to 0^+, N \to \infty$ and then $h \to 0^-, N \to \infty$.

C- Spontaneous symmetry breaking and ergodicity breaking

Statistical mechanics is based on the principle that long time average should identify with ensemble average:

$$\langle A \rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t A\{X_i(t')\}\, dt' = \int \prod_i \, dX_i \; P_{eq}(\{X_i\}) A\{X_i\}$$

where the $\{X_i(t)\}$ correspond to the dynamical degrees of freedom (position and momentum for a fluid, spins in for magnetic systems, etc) and $P_{eq}(\{X_i\})$ the probability distribution at equilibrium.

The ergodic hypothesis states that when $t \to \infty$ the " trajectory" $\{X_i(t)\}$ should be arbitrarily close to every possible configuration of the $\{X_i\}$. As a consequence, even in a ferromagnetic phase, *i.e.* at $T < T_c$, two states of magnetization M_A and $M_B = -M_A$ should be equally sampled.

1) Consider the situation where the net positive magnetization is given by M_A. According to the ergodic hypothesis, after some time, a large cluster of spins down should form leading the magnetization to reverse. Using the Arrhenius law give the expression of typical time τ of this phenomenon to occur as a function of ΔF, the free energy difference between the two different configurations.

2) Evaluate ΔF as function of the number N involved in the cluster of spins that reverses, and the dimension D. Conclude.

II - Ising model with long-range interactions

In this exercise we study the possibility of domains in the Ising model in the presence of long-range interactions. One considers a system of $N+1$ spin interacting ferromagnetically with Hamiltonian:

$$H = -\sum_{i,j} J_{ij} S_i S_j$$

with J_{ij} of the form:

$$J_{ij} \sim \frac{J}{|i-j|^{1+\sigma}}$$

with J is a positive real number and σ a real number. One assumes that the system is ordered (e.g. the spins are all up) and that there exists a domain of $p+1$ spins down indexed by $i \in [0, p]$.

1) Write the increase of energy ΔE due to the presence of this domain of spins down as a double sum running on the position of the spins down belonging to the domain and on the position of the spins up.

2) Carry out a continuum limit on the double sum obtained in question **1)** and show that ΔE is given by:

$$\Delta E = \frac{2J}{a^2\sigma(1-\sigma)}\left[(L-z)^{1-\sigma} - L^{1-\sigma} - a^{1-\sigma} + (z+a)^{1-\sigma}\right]$$

where z parametrizes the size of the domain of spins down, L is the size of the whole system, and a is the lattice space.

3) Consider a domain of size $z \gg a$ and get the simplified expression for ΔE:

$$\Delta E = \frac{2J}{a^2\sigma(1-\sigma)} z^{1-\sigma} \; .$$

4) Consider the possibility of a large domain of spins down according to the value of σ.