Modèle [Tordeux & Seyfried, PRE (2014)]

- Modèle de Pipes (1953)

$$\boxed{\dot{x}_i(t) = V(\Delta x_i(t))}$$

- Modèle de Newell (1961)

$$\boxed{\dot{x}_i(t+\tau) = V(\Delta x_i(t))}$$

- Modèle de Bando (1995)

we take the model of Newell 1961 and expand the left hand side =

$$\left.\begin{aligned} v_i(t+\tau) &\overset{\text{usv}}{\simeq} v_i(t) + \tau \frac{dv_i}{dt}(t) \\ &\underset{\uparrow_{SE}}{\simeq} V(\Delta x_i) \end{aligned}\right\} \Longrightarrow \boxed{\frac{dv_i}{dt}(t) = \frac{1}{\tau}\left[V(\Delta x_i) - v_i\right]}$$

2nd order (accelera°).

- Modèle [Tordeux & Seyfried (2014)]:

Instead of expanding the l.h.s, we expand the r.h.s in:

$$\begin{aligned} \dot{x}_i(t) &= V(\Delta x_i(t-\tau)) \\ &= V\left[\Delta x_i(t) - \tau \underbrace{\dot{\Delta x_i}(t)}_{\dot{x}_{i-1} - \dot{x}_i}\right] \end{aligned}$$

$$\boxed{\dot{x}_i(t) = V\left[\Delta x_i(t) - \tau\left[V(\Delta x_{i-1}(t)) - V(\Delta x_i(t))\right]\right]}$$

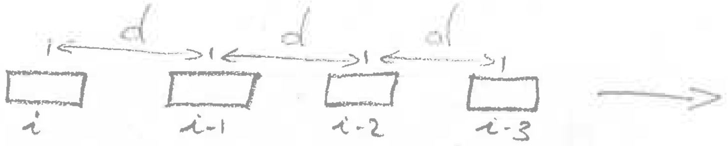
implicit in speed ; 2 predecessors ; first order (velocity and not accelera°)
sometimes called generalized optimal velocity model (GOV).

[Tordeux & Seyfried (2014)]

$$\dot{x}_i(t) = V\left[\Delta x_i(t) - \tau\left[V(\Delta x_{i-1}(t)) - V(\Delta x_i(t))\right]\right]$$

Local stability

- Uniform solu$^{\theta}$



$$\Delta x_k^u = d \quad \forall k$$

$$v_k^u = V(d) \quad \forall k$$

- Small perturbation applied on car i

$$x_i(t) = x_i^u(t) + y(t)$$

$$= x_{i-1}^u(t) - d + y(t) \quad \longrightarrow \quad \Delta x_i(t) \equiv x_{i-1}(t) - x_i(t)$$

$$= d - y(t)$$

↓ deriva$^{\theta}$

$$\dot{x}_i(t) = V(d) + \dot{y}(t)$$

We put this in the model:

$$V(d) + \dot{y}(t) = V\left[d - y(t) - \tau \underbrace{\left[V(d) - V(d - y(t))\right]}_{\text{Taylor expansion (y small)}}\right]$$

$$= V\left[d - y - \tau V'(d)\, y\right]$$

$$\underset{\substack{\uparrow \\ \text{Taylor expansion}}}{=} V(d) - y(1 + \tau V'(d))V'(d)$$

$$\iff \dot{y}(t) = -y(t)\underbrace{(1 + \tau V'(d))V'(d)}_{\equiv \alpha} \qquad \Rightarrow \quad y = y_0 e^{-\alpha t}$$

stable if $\alpha > 0$

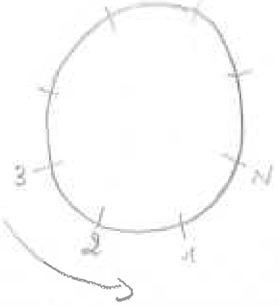
→ always stable if $V'(d) > 0$

(the solu$^{\theta s}$ $V'(d) < -\frac{1}{\tau}$ are not really physical).

Exponential convergence with no oscilla$^{\theta}$, even for $\tau > 0$ large.

→ collision free if $V'(d) > 0$.

On a ring :



Uniform sol$^{\Theta}$ U:

$d_i = d = \frac{L}{N}$; $v_i = v = V(d)$

$\Delta \overset{u}{x}_i = \overset{u}{x}_{i-1}(t) - \overset{u}{x}_i(t) = d \quad \forall i$

(up to periodic BC$_i$).

$\Delta \overset{u}{x}_1 = \overset{u}{x}_N - \overset{u}{x}_1 + L$

Small perturba$^{\Theta}$: on all cars

$$y_i(t) = x_i(t) - \overset{u}{x}_i(t) \longrightarrow \Delta x_i(t) = x_{i-1} - x_i = (y_{i-1} + x^{U}_{i-1}) - (y_i + x_i^{U}) = \Delta y_i + d$$

$$\downarrow$$

$$\dot{y}_i(t) = \dot{x}_i(t) - v = V\left[\Delta x_i(t) - \tau\left[V(\Delta x_{i-1}(t)) - V(\Delta x_i(t))\right]\right] - v$$

$$\dot{y}_i + v = V\left[\Delta y_i + d - \tau\left[V(d + \Delta y_{i-1}) - V(d + \Delta y_i)\right]\right]$$

and Δy_i small

$$\hookrightarrow \dot{y}_i + v = V\Big[d + \Delta y_i \cancel{- \tau V(d) + \tau V(d)}$$

$$- \tau \Delta y_{i-1} V'(d) + \tau \Delta y_i V'(d)\Big]$$

$$= \underbrace{V(d)}_{v} + \left[\Delta y_i - \tau \Delta y_{i-1} V'(d) + \tau \Delta y_i V'(d)\right] V'(d)$$

$$\boxed{\dot{y}_i = \underbrace{\left[1 + \tau V'(d)\right] V'(d)}_{\alpha} \Delta y_i - \underbrace{\tau \left[V'(d)\right]^2}_{\beta} \Delta y_{i-1}}$$

$\dot{y} = My$

$$M = \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \end{array} \begin{array}{ccccccc} & & i-2 & i-1 & i & & \\ & & & & & \beta & \alpha-\beta \\ -\alpha & & & & & & \\ \alpha-\beta & -\alpha & & & & & \beta \\ \beta & \ddots & \ddots & & & & \\ & & \beta & \alpha-\beta & -\alpha & & \\ & 0 & & \ddots & \ddots & \ddots & \\ & & & & \beta & \alpha-\beta & -\alpha \end{array}$$

Eigenvalues:

Preliminary results: • Let $K = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & 0 & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$ $N \times N$ matrix

$$K^2 = \begin{pmatrix} 0 & & & 1 & 0 \\ 0 & \ddots & 0 & & 1 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & 0 & 0 \end{pmatrix} \qquad K^3 = \begin{pmatrix} 0 & & & 1 & 0 & 0 \\ 0 & & & & 1 & 0 \\ 0 & \ddots & 0 & & & 1 \\ 1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ 0 & & 1 & 0 & 0 & 0 \end{pmatrix}$$

- $K^N = Id$

Eigenvalues for K : $\mu_k = \mu^k$ with $\mu = e^{\frac{2i\pi}{N}}$

- Each circulant matrix can be expressed as a polynomial of matrix K.

- Here : $M = -\alpha \, Id + (\alpha - \beta) K + \beta K^2$

Eigenvalues of M :

$$\lambda_k = -\alpha + (\alpha-\beta)\mu_k + \beta \mu_k^2$$
$$= \alpha(\mu_k - 1) + \beta \mu_k (\mu_k - 1)$$

$\lambda_0 = 0$ $(\mu_0 = 1)$ (associated to the stationary state)

$Re(\lambda_k)$? $\forall k = 1$ to $N-1$

$$Re(\lambda_k) = \alpha \underbrace{\left(\cos \frac{2k\pi}{N} - 1\right)}_{\equiv C_k} + \beta\left(\cos^2 \frac{2k\pi}{N} - \sin^2 \frac{2k\pi}{N} - \cos \frac{2k\pi}{N}\right)$$

$$= \alpha\,(C_k - 1) + \beta\left(C_k^2 - 1 + C_k^2 - C_k\right)$$

$$= \left[1 + \tau V'(d)\right] V'(d)\,(C_k - 1) - \tau\left[V'(d)\right]^2\left(2C_k^2 - \cancel{1} - C_k\right)$$

$$= V'(d)\,(C_k - 1) + \tau\left[V'(d)\right]^2\left(2C_k - 2C_k^2\right)$$

$$= V'(d)\,(1 - C_k)\left[-1 + 2\tau\,(V'(d))^2\, C_k\right]$$

$$\mathrm{Re}(\lambda_k) = V'(d)\overbrace{(1-C_k)}^{\geq 0}(-1+\tau V'(d)\,2C_k)$$

- If $V'(d) > 0$, the sign of $\mathrm{Re}(\lambda_k)$ is the same as

It should be so!

$$\text{sign}(\tau V'(d)\,2C_k - 1)$$

$$\mathrm{Re}(\lambda_k) \overset{?}{<} 0 \quad \forall k$$

$$\Leftrightarrow \tau V'(d)\,2Ck - 1 < 0 \quad \forall k$$

$$\Leftrightarrow \tau V'(d)\,Ck < \frac{1}{2} \quad \forall k$$

This is always the case if $\tau V'(d) < \frac{1}{2}$

- If $V'(d) \leq 0$, $\mathrm{Re}(\lambda_k) \overset{?}{\leq} 0 \quad \forall k$

$$\Leftrightarrow \tau V'(d)\,2Ck \geq 1 \quad \forall k$$

$$\Leftrightarrow \underbrace{\tau}_{>0}\underbrace{V'(d)}_{<0}\,Ck \geq \frac{1}{2} \quad \forall k$$

Not possible to ensure for all $k / Ck \geq 0$, ie if $N > 3$

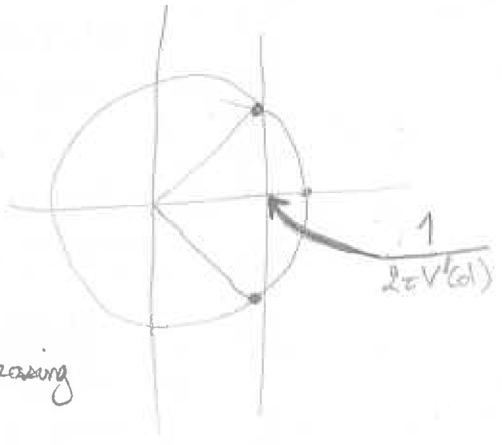
→ global stability for long lines of vehicles cannot be achieved.

Ccl: for global stability we must have

$$\boxed{\begin{array}{c} V'(d) > 0 \\ \text{and} \\ \tau V'(d) < \frac{1}{2} \end{array}}$$

If $V'(d) > 0$:

If the 2nd condition does not hold, the largest N for which we have stability is such that



$$\cos\frac{2\pi}{N} < \frac{1}{2\tau V'(d)}$$

$\downarrow$ $\cos^{-1}$ is decreasing

$$\frac{2\pi}{N} > \cos^{-1}\left(\frac{1}{2\tau V'(d)}\right).$$

$$\boxed{N < \frac{2\pi}{\cos^{-1}(1/2\tau V'(d))}}$$