Chap II

The Coulomb Gas Method: / Dyson Method.

We assume the joint probability is known:

$$P_{N\beta}(x_1,\dots x_N) = C_{N,\beta}\exp\left(\frac{-1}{2}\beta\sum_{i=1}^{N} x_i^2\right)\prod_{j<k}|x_j - x_k|^{\beta}$$

We want to find the spectral density:

$$n(x) \hat{=} \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i).$$

We know from numerical experimentation that $X \to +\infty \Rightarrow x_i \in [-\beta\sigma N, \beta\sigma N]$.

We study: $x_i = \sqrt{\beta N}\, X_i$, $x_i \in [-2\sigma, 2\sigma]$.

☐ Partition function:

$$Z_{N,\beta} = C_{N,\beta}\int\left(\prod_{j=1}^{N} dx_j\right)\exp\left\{-\beta N^2 V(x)\right\}.$$

$$C_{N,\beta} = (\beta N)^{\frac{\beta}{2}\left(N + \frac{1}{2}N(N-1)\right)}$$

$$V(x) = \frac{1}{2N}\sum_{i=1}^{N} x_i^2 - \frac{1}{2N^2}\sum_{i\neq j}\ln(|x_i - x_j|).$$

↳ We recognize the equation for repulsive coulomb gas in 1 D.

Step 1: Rewrite the $Z_{N,\beta}$ as a functional integral over $n(x)$ and $\hat{n}(x)$ which the fourier conjugate $n(x)$.

① $\int \mathcal{D}[n(x)]\ \delta\left(n(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i)\right) = 1$

② $\sum_{i=1}^{N} = N\int dx_i$ $\frac{1}{2N}\sum_{i=1}^{N} x_i^2 = \frac{1}{2}\int dx_i\, x_i^2$

$$\frac{1}{2N^2}\sum_{i\neq j}\ln(|x_i - x_j|) =$$

$$= \frac{1}{2}\int dx_i dx_j\, n(x)n(x')\ln(x - x') - \frac{1}{2N}\int dx\, n(x)\ln(\Delta x)$$

③ Replace $\delta(.)$ by an exponential representation.

$$\delta\left(n(x) - \frac{1}{N}\sum_i \delta(x-x_i)\right) = \int \mathcal{D}\hat{n}(x) \exp\Big[iN\int dx\, \hat{n}(x) n(x) - i\int dx\, \hat{n}(x) \sum_i \delta(x-x_i)$$

rq: $\hat{n}$ is the fourier conjugate of n.

$$\Rightarrow Z_{N,\beta} = \int \mathcal{D}n \int \mathcal{D}\hat{n}\, \exp\left\{-\beta N^2 V[n(x)]\right\} \times \int\left(\prod_{i=1}^{N} dx_i\right) \exp\Big\{iN\int dx\, \hat{n}(x) n(x) - i\int dx\, \hat{n}(x) \sum_{i=1}^{N} \delta(x-x_i)$$

$$\boxed{\begin{aligned} Z_{N,\beta} &= \int \mathcal{D}n \int \mathcal{D}\hat{n}\, \exp\left\{N\, S[\hat{n}|n]\right\}. \\ S[\hat{n}|n] &= i\int dx\, \hat{n}(x) n(x) + \log\left[\int dy\, e^{-i\hat{n}(y)}\right]. \\ &\text{(action of the coulomb gas)} \end{aligned}}$$

STEP 2: Compute the saddle points of S: on n^*

① $\frac{\delta S}{\delta \hat{n}} = 0 \Rightarrow \hat{n}^*(x) = i\ln(n(x)) + i\underbrace{\ln\left[\int dy\, e^{-i\hat{n}(y)}\right]}_{\text{spurious term}}$

→ We reintroduce $\hat{n}^*$ in $S[\hat{n}|n]$ and the spurious terms simplify.

$$I_{sp} = \int \mathcal{D}\hat{n}\, \exp\left(N\, S[\hat{n}^*|n]\right)$$

$$I_{sp} = \exp\left(-N\int dx\, n(x) \ln(n(x))\right)$$

$$\Sigma_{\text{shannon}} = \ln(I_{sp}) = -N\int dx\, n(x) \ln n(x)$$

• By reintroducing this result in $Z_{N,\beta}$ we optain:

$$Z_{N,\beta} = C_{N,\beta} \int \mathcal{D}[n(x)] \exp\left\{ -\beta N^2 F[n(x)] \right\}.$$

Where

$$F[n(x)] = F_0[n(x)] + \frac{1}{N} F_1[n(x)] + \frac{\beta}{2N} \ln(N) + \frac{\beta}{2N} N \ln(c) + O\left(\frac{1}{N}\right).$$

$$F_0 = \frac{1}{2}\int dx\, n(x) x^2 - \frac{1}{2}\int dx\, dx'\, n(x) n(x') \ln(|x-x'|)$$

$$F_1(x) = \int dx\, n(x) \ln(n(x))$$

• We neglect all the terms which are $O(1)$ in F (F is the free "energy per-site").

$$\rightarrow \boxed{\begin{aligned} Z_{N,\beta} &= C_{N,\beta} \int \mathcal{D}\, n(x) \int dK \exp\left[-\beta N^2 S(n(x), K)\right] \\ S(n(x), K) &= F_0(n(x)) - K \int dx (n(x) - 1) \end{aligned}}$$

lagrange multiplier to ensure normalisation.

Step 3: Saddle point approximation:

$$\frac{\delta S}{\delta n} = 0, \quad \frac{\delta S}{\delta K} = 0$$

$$\Rightarrow \begin{cases} K^* = \frac{1}{2} x^2 - \int dx'\, n(x') \ln|x' - x| \\ 0 = \int dx\, n^*(x) - 1 \end{cases}$$

• Free energy per site: $f = \frac{-1}{\beta N^2} \ln(Z_{N,\beta})$

$$= S(n^*, K^*)$$

$$f(\beta) = F_0[n^*(x), K^*] + O\left(\frac{1}{N^2}\right).$$

Rq: Here we see that the free energy is supper extensive which is typical of desordered systems.

Step 4 : Compute n^* and K^*:

① In the order:

$$F_0(n(x), K) = \int dx\; n(x)\, f(x)$$

$$f(x) = \frac{1}{2}x^2 - \int dx'\, n(x') \ln(|x-x'|) - K$$

① $n(x)$ has a compact support because otherwise $f(x) \xrightarrow[x\to\pm\infty]{} +\infty$

$$\int dx'\, n(x') \ln(|x-x'|) \xrightarrow[x\to\pm\infty]{} +\infty$$

F_0 must be finite and thus $n(x)$ must have a compact support. $\square$

② In weak sens of distribution theory, we can show that:

$$\boxed{\int dx'\, n(x') \ln(x-x') = -\mathrm{Pr}\int dx' \frac{n(x')}{x-x'}}$$

$$\Rightarrow \boxed{n^*(x) = \frac{1}{\pi\sqrt{(x-a)(x-b)}}\left[1 - x^2 + \frac{1}{2}(a+b)x + \frac{1}{8}(b-a)^2\right]}$$

Sokhotsky - Plemelj Formula
⊕ Tricomi Theorem

$$\begin{cases} K = \dfrac{a^2}{2} - \displaystyle\int_a^b dx\, n^*(x) \ln(|a-x|) \\ K = \dfrac{b^2}{2} - \displaystyle\int_a^b n^*(x) \ln(|b-x|). \end{cases}$$

$$\rightarrow \boxed{F_0[n^*] = \frac{1}{4}\int_a^b dx\, n^*(x) x^2 + \frac{a^2}{4} + \frac{1}{2}\int_a^b dx\, n^*(x) \ln(|a-x|)}$$

Wishart - Laguerre Ensemble. Chap III

I - The ensemble:

1) Definition:

def: Wishart matrice: Let ~~bet~~ be
W an $N \times N$ positive ~~demi~~-definit.
There exist H an $N \times M$ matrice with $M \leq N$
such that

$$W = H H^{\dagger} \quad \begin{cases} \beta = 1 \Rightarrow \dagger = T \\ \beta = 2 \Rightarrow \dagger = \dagger \\ \beta = 4 \Rightarrow J^{\dagger} = -J \text{ (Quantum dual)}. \end{cases}$$

2) Properties:

prop: The entry of W are correlated.

prop: $$e(W) \propto \exp\left(-\frac{1}{2} \mathrm{Tr}(W)\right)\left(\det W\right)^{\frac{\beta}{2}(M-N+1)-1}$$

$$e((\lambda_i)_i) \propto e^{-\frac{1}{2}\sum_i \lambda_i} \prod_i \left(\prod_{j \neq i} |\lambda_j - \lambda_i|^{\beta} \lambda_i^{\frac{\alpha\beta}{2}}\right)$$

$$\alpha = 1 + M - N - \frac{2}{\beta}$$

prop: This is equivalent to a coulomb gas in ~~1~~2D, confined on the positive real axis.

$$Z = \int d^2 r_1 \dots d^2 r_N \, e^{-\beta \sum_i V_{ext}(r_i)} \prod_{i<j} |r_i - r_j|^{\frac{q_i q_j}{2\pi\varepsilon_0}}$$

$$\beta V_{ext} = \frac{x}{2} - \frac{\alpha}{2} \ln x$$

prop: $e(W) \rightarrow$ is very much like Laguerre polynomials.

II - Eigenvalues:

prop: Marcenko Pastur distribution.

$$\rho_{MP}(x) = \frac{1}{2\pi x}\sqrt{(x-\xi_-)(x-\xi_+)}$$

$$\xi = \left(\frac{1}{\sqrt{c}} \pm 1\right)^2 \quad , \xi_- \xrightarrow[c \to 1]{} 0$$

Demonstration:

step 1: Write the partition function
step 2: Write $\rho(x)$ the empirical density
step 3: Saddle point approx on ρ.

Step 4: Tricomi integral equation.

prop: Largest eigenvalue statistics:

$$\lambda_{max} = \sqrt{2} + \frac{1}{\sqrt{2}} N^{-2/3} x_1$$

Tracy Widom distribution.
Typical fluctuations: cumulative function.

$$F_\beta(s) = \mathbb{P}(\chi_\beta \leq s)$$

prop:

$$F_2(s) = \det(\mathbb{1} - A_s) = 1 + \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{S_1} \det\left[A_s(x_i, x_j)\right]_{i,j=1,\dots,n} dx_1 \dots dx_n$$

def:

$$A_s(x,y) \triangleq \begin{cases} \dfrac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x-y} & \text{if } x \neq y \\ Ai'(x)^2 - x(Ai(x))^2 & \text{if } x = y. \end{cases}$$

prop: $F_2(s) = \exp\left(-\int_s^{+\infty} (x-s)\, q^2(s)\, dx\right)$

where $q''(s) = sq(s) + 2q(s)^3$
is Painlevé equation.

p: F_1 and F_4 can be defined as follow:

$$F(x) = \exp\left\{-\frac{1}{2}\int_x^\infty (y-x)\, q(y)^2\, dy\right\}.$$

$$E(x) = \exp\left\{\frac{-1}{2}\int_x^\infty q(y)\, dy\right\}.$$

$$F_1(x) = E(x)\, F(x) = \exp\left(\frac{-1}{2}\int_x^\infty q(y)\,dy\right)\left(F_2(x)\right)^{\frac{1}{2}}$$

$$F_2(x) = F(x)^2$$

$$F_4\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2}\left(E(x) + \frac{1}{E(x)}\right) F(x).$$

$$F_4\left(\frac{x}{\sqrt{2}}\right) = \cosh\left(\frac{1}{2}\int_x^\infty q(s)\, ds\right)\left(F_2(x)\right)^{\frac{1}{2}}.$$

mp: Probability density of the typical max fluctuations:

$$f_\beta(x) = F'_\beta(x) \sim \begin{cases} e^{-\frac{\beta}{24}|x|^3} & x \to -\infty \\ e^{-\frac{2\beta}{3}|x|^{3/2}} & x \to +\infty \end{cases}$$

mp: The atypical fluctuations have cumulative distribution function:

$$Q(x) = \begin{cases} \exp\left[-\beta N^2\, \Phi_-(x)\right] & x < \sqrt{2} \\ 1 - \exp\left[-\beta N\, \Phi_+(x)\right] & x > \sqrt{2}. \end{cases}$$

$$\Phi_-(x) \sim \frac{1}{6\sqrt{2}}\left(\sqrt{2} - x\right) \text{ as } x \sim \sqrt{2}$$

$$\Phi_+(x) \sim \frac{2^{7/4}}{3}\left(x - \sqrt{2}\right)^{3/2} \text{ as } x \sim \sqrt{2}.$$

After heavy computations (Use Mathematica, Sage, ...)

$$\boxed{F[a,b] = \frac{1}{9\pi^2}\Big[-9a^4 + 4a^3b + 2a^2(5b^2+48) + 4ab(b^2+16) - 256\ln(b-a) - 9b^2 + 96b^2 + 10\ln(2)\Big].}$$

We can show (I do not know how) that

$$a = \sqrt{2}$$
$$b = -\sqrt{2}.$$

$$n^*(x) = \frac{1}{\pi}\sqrt{2-x^2}$$

Layman's classification.

I- types of random matrices

1) Independent:

independent matrices: the entries are pair wise independent. = levy matrices.

mp: $p(s)ds = e^{-\rho s} \rho ds$ où s est l'écart entre deux niveaux successifs.
is "this called "poisson" distribution./law.

prop: The distribution of the eigenvalue is power law bounded.

2) Rotationaly invariant matrices.

rotationaly invariant. Let H and H' such that $H' = UHU^{-1}$
and $\rho(H') = \rho(H)$. then H and H' are rotationaly invariant.

→ orthogonal matrices: $H' = OHO^T \in \mathbb{R}$
and $OO^T = O^TO = \mathbb{1}$

→ unitary matrices: $H' = OHO^+ \in \mathbb{C}$
$OO^+ = O^+O = \mathbb{1}$.

→ simplectic matrices: $H^{-1} = -JH^TJ$
$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, $J^{-1} = -J$
↳ quaternions.

mp: Weyls lemma: A random matrix is rotationally invariant if and only if its joint probability has the same functional form before and after the rotation.

$$\rho(H) = \rho[UHU^{-1}]$$

mp: For a rotationaly invariant matrice, the distribution can be expressed only as a fct° of the

trace of the matrice
$\rightarrow \rho(H) = \rho(\mathrm{Tr}\, H, \mathrm{Tr}(H^2), \dots \mathrm{Tr}(H^N).)$

prop: ~~Then~~ a rotationnaly invariant matrix, there is correlation btw energy levels.
and
$$p(s) = C\mu(s)\exp\left(-\int^{s} ds'\,\mu(s')\right).$$
$\mu(s') = \alpha_\beta\, s$ (In general).

$\hookrightarrow$ prop $p_W = \frac{\pi}{2} s\, e^{-\frac{\pi}{4}s^2}$ (Wigner Surmise).

II) The Wigner ensemble.

def: Let be $\hat{X}$ a symetric matrice with independent identicaly distributed variables $\hat{X} = X^T$ and $E(X_i) = 0$

prop: $\tau(X) = 0$
$$\tau(X^2) = \frac{1}{N(N-1)}\left[N(N-1)\,\sigma^2_{\text{off diag}} + N\sigma_{\text{diag}}^2\right]$$
$$\tau(X^4) = \frac{1}{N}\,\mathbb{E}(\mathrm{Tr}(X^4)) = 2\sigma^4$$

prop: Eigenvalues are distributed according to Wigner semi-circle law.
$$\rho(\lambda) = \frac{\sqrt{4\sigma^2 - \lambda^2}}{2\pi\sigma^2}$$

prop: The joint probability is:
$$P_{N\beta}(x_1, \dots x_N) = C_{N,\beta}\exp\left(-\frac{1}{2}\beta\sum_{i=1}^{N} x_i^2\right)\prod_{j<k}|x_j - x_k|^\beta$$
$\beta = 1 \rightarrow$ GOE $\quad (X_i) \in SP(H)$
$\quad\;\; 2 \rightarrow$ GUE
$\quad\;\; 4 \rightarrow$ CSE $\longrightarrow$ Dyson index.

rq: The eigenvalues are correlated.

def: Weak derivative: Let u, u' two distributions.
Let be $\varphi \in \mathcal{C}^1([a,b])$, $u(a)=0$, $u(b)=0$
If
$$\int_a^b \varphi' u \, d\mu = -\int_a^b d\mu \, \varphi(x) \, u' \quad (\text{for any } \varphi)$$
Then u' is called the weak derivative of u.

def: Principal value of Cauchy:
$$\Pr_x \int dx' \frac{f(x')}{x-x'} = \lim_{\varepsilon \to 0^+} \int^{x-\varepsilon} F(x')\,dx' + \int_{x+\varepsilon} F(x')\,dx'$$

thm: Tricomi (1985)
Let be
$$g(x) = \Pr \int_a^b dx' \frac{f(x')}{x-x'}$$
$[a,b]$ a single compact support.
C an arbitrary cste
Then
$$f(x) = \frac{C - \Pr \int_a^b dt \frac{\sqrt{(t-a)(t-b)}}{x-t} g(t)}{\pi\sqrt{(x-a)(x-b)}}$$

prop: $$\Pr \int_{-\sqrt{2}}^{\sqrt{2}} dx' \frac{\sqrt{2-x'^2}}{\pi(x-x')} = x.$$

Thm. Sokhotsky - Plemelj - Formula

Let be φ an $\mathcal{L}^{2}$ function

Then ① $\lim_{\varepsilon \to 0^{+}} \int \frac{\varphi(y)}{y \pm i\varepsilon} dy = \Pr \int \frac{\varphi(y)}{y \mp i\varepsilon} dy \pm i\pi \varphi(0)$

② $\lim_{\varepsilon \to 0^{+}} \frac{1}{y \pm i\varepsilon} = \Pr\left(\frac{1}{y}\right) \pm i\pi\, \delta(y).$

corollary of the Sokhotsky Plemelj formula:

$$\rho_N(\lambda) = \frac{1}{\pi N} \lim_{\varepsilon \to 0^{+}} \operatorname{Im} \left\langle \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j - i\varepsilon} \right\rangle$$

$$\rho_N(\lambda) = \frac{1}{\pi N} \lim_{\varepsilon \to 0^{+}} \frac{\partial}{\partial \lambda} \left\langle \sum_{j=1}^{N} \ln(\lambda - \lambda_j - i\varepsilon) \right\rangle.$$

Recall: $\ln \det A = -2 \ln\left((\det(A))^{-\frac{1}{2}} \right).$

corollary:

$$\rho_N(\lambda) = \frac{-2}{\pi N} \lim_{\varepsilon \to 0^{+}} \frac{\partial}{\partial \lambda} \left\langle \ln\left(\det\left((\lambda - i\varepsilon)\mathbb{1} - \hat{H}\right)^{-\frac{1}{2}} \right) \right\rangle$$

Schur Complement Formula

Thm: Schur Complement Formula

Let be $M = \begin{array}{cc} & \\ p & \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \\ & \quad q \end{array}$ with $M \in \mathcal{M}(p+q\,;\,p+q)$

$$M = \begin{pmatrix} 1\!\!1_q & BD^{-1} \\ 0 & 1\!\!1_q \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1\!\!1_p & 0 \\ D^{-1}C & 1\!\!1_q \end{pmatrix}$$

This leads to the fact that if we write $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$ the inverse of M.

Then the "Blocks"

$$Q_{11}^{-1} = A - BD^{-1}C$$

$$Q_{22}^{-1} = D - CA^{-1}B.$$

nb: Q_{11} can be either a block of size (p, p) or a matrix element.

Normalized Trace:

def. $\tau(\hat{A}) = \frac{1}{N} \mathbb{E}\left[\mathrm{Tr}(\hat{A})\right]$

prop Let be F a polynomial of A $\left(A \in \mathrm{Sym}\, \mathbb{R}^N\right)$
Then

$$\tau\left(F(A)\right) = \frac{1}{N} \mathbb{E}\, \mathrm{Tr}\left[F(A)\right].$$

$$= \frac{1}{N} \sum_{i=1}^{N} F(\lambda_i). = \left\langle F(\lambda) \right\rangle_{Sp}.$$

$$\langle F(\lambda) \rangle_s = \int d\lambda\, w(\lambda)\, F(\lambda).$$

Diagrammatic Notations: Chap II

prop: $\mathrm{Tr}(X^k) = \int dx_1, \ldots dx_N\, \rho(x_1, \ldots x_N) \sum_{i=1}^{N} x_i^k$

$= N\left(\int dx\, \rho(x)\, x^k\right)$

For $\left| \begin{array}{l} x_i \text{ the eigenvalues of } X. \\ X \text{ in iid gaussian / GOE/GUE/GHE} \end{array} \right.$
ensemble

prop: For X a rotationaly invariant matrix with iid coeficients: (in the Wigner ensemble)

$$\lim_{N\to\infty} \frac{\langle \mathrm{Tr}(X^{2n})\rangle}{\beta^N N^{n+1}} = \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} y^{2n}\sqrt{2-y^2} = \frac{C_N}{2^n}$$

$$C_N = \frac{1}{N+1}\binom{2N}{N}.$$ Catalan numbers

prop: Wick theorem:

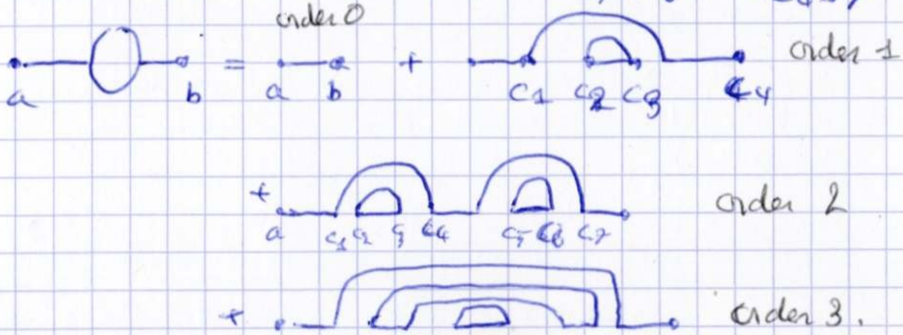
$$\tau(X^{2n}) = \frac{1}{N}\Big\{\mathbb{E}[X_{i_1 i_2} X_{i_2 i_3}] \ldots \mathbb{E}[X_{i_{2N} i_1}]$$
$$+ \text{All the permutations.}\Big\}.$$

prop: It has been show by Wilson that only the planar diagrams matter.

$$Z = z\mathbb{1}$$

$$G_N(z) = \langle (Z - \hat{X})^{-1} \rangle$$
$$= \langle [Z(1 - Z^{-1} X)]^{-1} \rangle$$
$$= Z^{-1} + \langle Z^{-1} X Z^{-1} \rangle + \dots$$

$$[G_N(z)]_{ab} = [Z^{-1}]_{ab} + \sum_{\substack{c_1, c_2 \\ c_3, c_4}} \langle [Z^{-1}]_{a c_1} X_{c_1 c_2} Z^{-1}_{c_2 c_3} X_{c_3 c_4} [Z^{-1}]_{c_4 b} \rangle$$



rq: → only the topology of the loops is important

→ each propagator $O\left(\frac{1}{N}\right)$

→ each loop $O(N)$

→ only 1PI graphs.

→ only planar graphs (Non planar are order $\left(\frac{1}{N}\right)$, Wilson)

The resolvent Method. Chap II

def: resolvent / stiltjes transform / Green Function.

$$c_N(z) = \frac{1}{N} \mathrm{Tr}\left(\frac{1}{z\mathbb{1}-H}\right) = \frac{1}{N} \sum_{x_i \in Sp(H)} \frac{1}{z-x_i}$$

prop: The resolvent is such that:

$$\left(\partial_t^2 - H\right) \tilde{c}_N(t) = \delta(t).$$

$$\tilde{c}_N(t) = \int dz\, e^{izt}\, c_N(z).$$

prop: The pôles of the resolvent are the eigenvalues of H.

prop: If H is a C° matrice / a functional kernel,

$\rightarrow \langle G_\infty^{av}(z) \rangle = \left\langle \mathrm{Tr}\left(\frac{1}{z-H}\right)\right\rangle$

$\rightarrow G_\infty(z)$ admits a continuous line of pôles on $\mathbb{R}$

$$\rightarrow \boxed{\rho(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \mathrm{Im}\, G_\infty^{av}(x - i\varepsilon).}$$

↳ spectral density of H can be expressed thanks to G_∞ and the principal value.

or equaly:

$$\rho(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \mathrm{Im}\, G_\infty^{av}(x+i\varepsilon).$$

prop: We can show in the gaussian ensemble (GOE, GUE, GSE, Wigner) that.

$$\frac{1}{2} G_N^2 - z\, G_N(z) + 1 = 1$$

$$\boxed{G_N(z) = z \pm \sqrt{z^2 - 2}}$$

Using the lema $\sqrt{a+ib} = p+iq$

and $G(x-i\varepsilon) = (x-i\varepsilon) \pm \sqrt{(x^2-\varepsilon^2-2)+i(-2\varepsilon x)}$

and $\rho(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \operatorname{Im} G(x-i\varepsilon)$.

we have: $\boxed{\rho(x) = \frac{\sqrt{2-x^2}}{\pi}}$

Methods to compute the resolvent:

① The brut force method (In this paper).

② The Cavity method.

↳ make a smart use of the shur complement formula.

③ Dyagramatic method → Schwinger Dyson.

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rep: Schwinger Dyson equations.

Self energy: $\left(\Sigma_\infty(z)\right)_{ab} = \sum_{c_1,c_2=1}^{N} \left[G_\infty(z)\right]_{c_1,c_2} \langle x_{c_1}^a x_{c_2}^b \rangle$

First Schwing-Dyson equation:

$$G_\infty(z) = \left[z - \Sigma(z)\right]^{-1}$$

Second Schwing-Dysen equations.

$$\left(\Sigma_\infty(z)\right)_{ab} = \frac{1}{N} \operatorname{Tr}\left(G_\infty(z)\right) \delta_{ab}.$$

(or) $\Sigma(z) = \tilde{G}_\infty(z) \mathbb{1}$.

$$\tilde{G}_\infty(z) \hat{=} \frac{1}{N} \operatorname{Tr}\left(G_\infty(z)\right).$$

A trivial property

prop:

$$\sqrt{a+ib} = p+iq$$

$$p = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a}$$

$$q = \frac{\mathrm{sgn}(b)}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a}$$

prop: $\displaystyle\int_a^b dx\,\frac{\sqrt{(a-x)(b-x)}}{2\pi x} = \frac{1}{4}\left[-2\sqrt{ab}+a+b\right]$

Legendre Polynomials:

def.

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

$$L_n^{(\alpha)}(x) = \frac{d^\alpha L_n(x)}{dx^\alpha}$$

$$\boxed{\int_0^{+\infty} dx \, L_n^{(\alpha)}(x) \, L_{n'}^{(\alpha)}(x) = \delta_{nn'} \frac{\Gamma(n+\alpha+1)}{n!}}$$

Formule de Ingham Siegel.

prp: Let be T an hermitian matrice, W an hermitian matrice.

$$\int dT\, e^{\frac{1}{2}\mathrm{Tr}(TW)} \det(\mu \mathbb{1}_{N,N} - T)^{-M}$$

$$\propto (\det W)^{M-N}\, e^{-\frac{1}{2}\mathrm{Tr}\, W}$$