Academic year 2021-2022

Statistical field theory

Tutorial n°4

Low-temperature expansion of the $O(N)$ model

One considers a lattice and a system of *unitary* N-components spins $\boldsymbol{S}_i = (S_{i1}, \ldots, S_{iN})^T$ at each site i of the lattice. They are supposed to be at equilibrium at temperature $T = 1/\beta$. Note that one sets $k_\mathrm{B} = 1$. The Hamiltonian of the system is assumed to have the $O(N)$ symmetry and to favor the parallel alignment of neighboring spins:

$$H = -J \sum_{\langle i,j \rangle} \boldsymbol{S}_i . \boldsymbol{S}_j - \sum_i \boldsymbol{h} . \boldsymbol{S}_i$$

with $J > 0$.

One wants to study the system at low temperatures, where the spins are almost aligned, using a continuous description. One therefore considers a field $\boldsymbol{S}(\boldsymbol{r})$, with $\boldsymbol{r} = (r_1, \ldots, r_D)^T$, such that $|\boldsymbol{S}(\boldsymbol{r})|^2 = S_\alpha(\boldsymbol{r}) S_\alpha(\boldsymbol{r}) = 1$, $\forall \boldsymbol{r}$, and one takes the simplest possible continuous Hamiltonian:

$$\mathcal{H}[\boldsymbol{S}] = \frac{c}{2} \int d^D x \, (\partial_i S_\alpha)(\partial_i S_\alpha) - h \int d^D x \, S_N,$$

where Einstein's summation is assumed and $\boldsymbol{h}$ is a uniform external field aligned along the direction of the N^{th}-component of $\boldsymbol{S}$ in the spin space, and $c > 0$.

1. Why didn't one include a generic term quadratic in $\boldsymbol{S}$ of the form $A_{\alpha\beta} S_\alpha S_\beta$?

2. Justify, on symmetry grounds, the form of the term quadratic in the gradient of $\boldsymbol{S}(\boldsymbol{r})$.

3. Show that for $h > 0$ the ground state corresponds to $\boldsymbol{S}(\boldsymbol{r}) = (0, \ldots, 1)^T$, $\forall \boldsymbol{r}$.

4. One considers small fluctuations – spin-waves – above the ground state. One writes $\boldsymbol{S} = (\boldsymbol{\pi}, \sigma)^T$, with $\boldsymbol{\pi} = (S_1, \ldots, S_{N-1})^T$ and $\sigma = S_N$. As $\boldsymbol{\pi}$ is assumed to be small one can express σ in terms of $\boldsymbol{\pi}$, *i.e.* $\sigma = \sqrt{1 - \boldsymbol{\pi}^2}$, so that it can be integrated out. Show first that

$$\delta\left(\boldsymbol{S}^2 - 1\right) = \frac{1}{2\sqrt{1 - \boldsymbol{\pi}^2}} \delta\left(\sigma - \sqrt{1 - \boldsymbol{\pi}^2}\right).$$

5. Write the partition function

$$Z[h] = \int \mathcal{D}\boldsymbol{S} \, e^{-\beta \mathcal{H}[\boldsymbol{S}]}$$

in term of fields $\boldsymbol{\pi}$ and σ.

6. Show then that the partition function can be transformed to

$$Z[h] = \int \mathcal{D}\boldsymbol{\pi} \, e^{-\beta \mathcal{G}[\boldsymbol{\pi}]} \tag{1}$$

with

$$\mathcal{G}[\boldsymbol{\pi}] = \mathcal{H}\left[(\boldsymbol{\pi}, \sqrt{1-\boldsymbol{\pi}^2})^T\right] + \frac{\rho}{2} T \int d^D x \, \log(1-\boldsymbol{\pi}^2) \, .$$

What is the relation between the constant ρ and the underlying lattice spacing a ? What is domain of variation of $\boldsymbol{\pi}$ in Eq.(1) ?

7. Write down explicitly $\mathcal{G}[\boldsymbol{\pi}] \equiv \int d^D x \, g(\boldsymbol{\pi})$, in terms of $\boldsymbol{\pi}$ and h only.

8. One will keep in g only the terms up to fourth order in $\boldsymbol{\pi}$. Show that $g = g_0 + g_1 + g_2 + \mathcal{O}(\pi^6)$ with

$$\begin{aligned} g_0(\boldsymbol{\pi}) &= \frac{c}{2}(\partial_i \pi_\alpha)(\partial_i \pi_\alpha) + \frac{h}{2}\pi_\alpha \pi_\alpha \, , \\ g_1(\boldsymbol{\pi}) &= \frac{c}{2}\pi_\alpha(\partial_i \pi_\alpha)\pi_\beta(\partial_i \pi_\beta) + \frac{h}{8}\pi_\alpha\pi_\alpha\pi_\beta\pi_\beta - T\frac{\rho}{2}\pi_\alpha\pi_\alpha \, , \\ g_2(\boldsymbol{\pi}) &= -T\frac{\rho}{4}\pi_\alpha\pi_\alpha\pi_\beta\pi_\beta \, , \end{aligned}$$

where one has discarded a constant independent of $\boldsymbol{\pi}$ and grouped the terms for reasons to be made clear in the following.

9. Show that:

$$\langle \pi_\alpha(\boldsymbol{r})\pi_\beta(\boldsymbol{r})\rangle_0 = T\delta_{\alpha\beta}\int_0^\Lambda \frac{d^D q}{(2\pi)^D}\frac{1}{h+cq^2}, \tag{2}$$

$$\langle \, \partial_j \pi_\alpha(\boldsymbol{r}) \, \pi_\beta(\boldsymbol{r})\rangle_0 = 0 \, , \tag{3}$$

$$\langle \, \partial_j \pi_\alpha(\boldsymbol{r}) \, \partial_j \pi_\beta(\boldsymbol{r}) \, \rangle_0 = T\delta_{\alpha\beta}\int_0^\Lambda \frac{d^D q}{(2\pi)^D}\frac{q^2}{h+cq^2}, \tag{4}$$

where $\langle \ldots \rangle_0$ denotes the average under the statistical weight associated with the Gaussian Hamiltonian $\mathcal{G}_0 = \int d^D x \, g_0(\boldsymbol{\pi})$ and Λ is the upper-wavevector cutoff.

Remark: One has to make the assumption that the domain of variation of π_α can be safely extended from $-\infty$ to $+\infty$. This can be justified by the fact that the minimum of the Hamiltonian $\mathcal{G}[\boldsymbol{\pi}]$ is given by $\boldsymbol{\pi} = 0$. Also one sees from Eqs.(2) and (4) that the main fluctuations satisfy $\boldsymbol{\pi} \sim \sqrt{T}$. Also one can show that field configurations such that $|\boldsymbol{\pi}| \sim 1$ give exponentially small contributions – of order $e^{-cte/T}$ – to the partition function that can be neglected within the perturbative approach. This allows to ignore the constraint $|\boldsymbol{\pi}| \leq 1$.

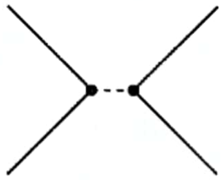
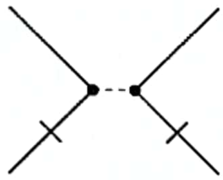
10. Show that $\langle g_{11}\rangle_0 = O(T^2)$, $\langle g_{12}\rangle_0 = O(T^2)$, $\langle g_{13}\rangle_0 = O(T^2)$, where g_{11}, g_{12} and g_{13} are the three terms entering in the expression of g_1. Show also that $\langle g_2\rangle_0 = O(T^3)$.

11. Since one is interested primarily in the low temperature behavior of the model, one will *neglect completely the term* $g_2(\boldsymbol{\pi})$ and consider $\mathcal{G}_1 = \int d^D x \, g_1(\boldsymbol{\pi})$ as a perturbation with respect to the Gaussian model with Hamiltonian $\mathcal{G}_0$. One sets up a diagrammatic representation in which π_α and $\partial_i \pi_\alpha$ are represented by the field lines below.

Draw the 3 diagrams corresponding to the three terms of $\mathcal{G}_1$, indicating as a weight their coefficients $c/2$, $h/8$ and $-T\rho/2$.

12. It is actually possible to avoid indicating α or $(\alpha, \, i)$ on the field lines if the following convention is adopted. Vertices with 4 fields, which are usually drawn as points, are splitted into two points joined by a small dotted line (figure below). The whole thing is a vertex. The convention is that on each side of a splitted vertex, the two fields have the same index,

e.g., π_α and π_α one one side, and π_β and π_β on the other side. As these indices are dummy they don't need to be specified. Likewise, on vertices with two field lines there is no need to indicate the indices. Give the expression of the diagrams below of $\mathcal{G}_1$ and indicate their weights, including h and c factors.

13. In order to perform a momentum-shell renormalization, one decomposes $\pi(r) = \pi_<(r) + \pi_>(r)$, where $\pi_<(r)$ contains the Fourier components of $\pi(r)$ in the interval $[0, \Lambda/s]$ and $\pi_>(r)$ all the remaining components in the interval $[\Lambda/s, \Lambda]$. Show that the coarse-grained effective Hamiltonian for $\pi_<$ can be expressed as

$$\mathcal{G}_<[\pi_<] = \mathcal{G}_0[\pi_<] + \langle\, \mathcal{G}_1[\pi_< + \pi_>]\,\rangle_0 + O(T^2)\,,$$

where constant terms have been discarded. What is the meaning of $\langle \ldots \rangle_0$ above ?

14. How are the Eqs.(2–4) modified if π is replaced by $\pi_>$?

15. By symmetry the coarse-grained Hamiltonian $\mathcal{G}_<$ will have the same form as $\mathcal{G}$, but with renormalized coefficients $c_< = c + \delta c$, $h_< = h + \delta h$ and $\rho_< = \rho + \delta\rho$. Write down $\mathcal{G}_<$.

16. Determine the non-vanishing diagrams arising from $\langle \int d^D x\, g_{11} \rangle_0$, with their weight, and show that they yield the contributions

$$\delta c_1 = Tc \int_{\Lambda/s}^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{1}{h + cq^2}\,,$$
$$\delta h_1 = Tc \int_{\Lambda/s}^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{q^2}{h + cq^2}$$

to δc and δh.

17. Give the non-vanishing diagrams arising from the contribution $\langle \int d^D x\, g_{12} \rangle_0$ and determine their weights.

18. Their contributions to δh are noted δh_2 and δh_3 respectively. Calculate δh_2 and δh_3.

19. The coefficient ρ can be expressed as

$$\rho = \int_0^{\Lambda} \frac{d^D q}{(2\pi)^D}.$$

What physical argument can you put forward to explain that ?

20. By directly decomposing both ρ into $\rho_< + \rho_>$ and π into $\pi_< + \pi_>$ show that $\langle \int d^D x\, g_{13} \rangle_0$ yields a natural contribution to $\mathcal{G}_<$, and produces in addition a correction to h:

$$\delta h_4 = -T \int_{\Lambda/s}^{\Lambda} \frac{d^D q}{(2\pi)^D}\,.$$

21. Calculate $\delta h_1 + \delta h_3 + \delta h_4$.

22. One now defines the renormalized fields by the relation $\pi'(r') = \zeta^{-1}\pi_<(sr')$, *i.e.* one rescales the coarse-grained field by ζ and the lengths by s. Let $\mathcal{G}'[\pi']$ be the renormalized effective Hamiltonian associated to π'. Show that its coefficients are such that

$$c_s = c\,(1+TA)\,s^{D-2}\zeta^2\,,$$
$$h_s = h\left(1+\frac{N-1}{2}TA\right)s^D\zeta^2\,,$$

where

$$A = K_D\int_{\Lambda/s}^{\Lambda} dq\,\frac{q^{D-1}}{h+cq^2}\,,$$

K_D being the area of the unit hypersphere in D dimensions divided by $(2\pi)^D$.

23. Let us now imagine that the field h acts in the direction of π_1. Explain why in the renormalization procedure it transforms into $h' = hs^d\zeta$. Deduce that in order to preserve the rotational symmetry of the model one must require

$$\zeta = \left(1+\frac{N-1}{2}TA\right)^{-1}.$$

24. One now sets $s = 1+d\ell$. Show that at zero external field ($h=0$) the renormalization flow for c is given by

$$\frac{dc}{d\ell} = (D-2)c-(N-2)TK_D\Lambda^{D-2}\,.$$

25. One assumes in the following $D>2$ and $N>2$. Determine the fixed point c^*. Determine the linearized flow equation in the vicinity of this fixed point.

26. How does c evolve at high temperature where $c<c^*$? To which phase corresponds the asymptotic fixed point reached ?

27. How does c evolve at low temperatures where $c>c^*$? To which phase corresponds the asymptotic fixed point reached ?

28. Considering now that c is taken fixed, deduce the value of T_c in this model. In the vicinity of which critical dimension D^* should one place ourselves in order for this renormalization procedure to be coherent ?

29. Deduce the Mermin-Wagner theorem.

30. Show that at the fixed point one has $h(\ell) = h\,\ell^{y_h}$ and calculate y_h. Using the general relation $\eta = 2+D-2y_h$ deduce that

$$\eta \simeq \frac{D-2}{N-2}\,.$$

in the vicinity of the critical dimension D^*.