ArTeQ Regards Croisés

INFORMATION QUANTIQUE

Introduction

· système fermé
· déterministe (états)

Théorie Quantique

1. État Quantique : vector space $\mathcal{H}$
2. Évolution états : closed system : unitary ⟷ Schrödinger multi-temps
3. Mesure : projecteurs

Cette leçon : Mesure généralisée (état pur)

↓

État quantique généralisé (matrice densité)

↓

Mesure généralisée (matrice densité)

Réduction avec "systèmes fermés"

Tout ça pour rien ?

NON → Formalisme indispensable (plus simple) lien entre

→ $\boxed{\text{système ouvert}}$ et $\boxed{\text{description probabiliste}}$

$\boxed{\text{Mesure}}$ (états purs)

Mesure projective

- $\mathcal{H}$ dim d

$\{|e_i\rangle\}_{i=1..d}$ base orthogonale de $\mathcal{H}$

$|e_i\rangle\langle e_i|$: projecteur P_i sur vect$(|e_i\rangle)$

$\sum_{i=1}^{d} P_i = \mathbb{1}$

- $\Pr(i) = |\langle e_i|\psi\rangle|^2$

$$= \langle e_i|\psi\rangle\langle\psi|e_i\rangle = \operatorname{Tr}\left[P_i|\psi\rangle\langle\psi|\right]$$

$$\left(= \sum_j \langle e_j| P_i |\psi\rangle\langle\psi| e_j\rangle = \langle e_i|\psi\rangle\langle\psi|e_i\rangle \right)$$

- État après mesure qui donne résultat "i" : $\dfrac{P_i|\psi\rangle}{\|P_i|\psi\rangle\|} = \dfrac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}$

Mesure généralisée : POVM

- Définition POVM
  $\{E_i\}$ où E_i : opérateur positif semi défini ($E_i \geq 0$)
  avec $\sum E_i = \mathbb{1}$
- $P_r(i) = Tr(E_i |\psi\rangle\langle\psi|)$

$\underline{\text{Cas particulier}}$: $\{E_i\}$ projecteurs orthogonaux $\Leftrightarrow E_i^2 = E_i \ \forall i$

Cas général : $\overset{\forall i}{\exists} F_i \quad E_i = F_i^\dagger F_i$

$$\left(F_i = W \sqrt{E_i} \right)$$

W matrice unitaire (quelconque)

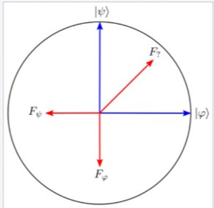
$$|\psi_{M,i}\rangle = \frac{F_i |\psi\rangle}{\sqrt{\langle\psi| F_i^\dagger F_i |\psi\rangle}}$$

Exemple : Mq on ne peut pas discriminer

$F_? \geq 0$?

An example: unambiguous quantum state discrimination [edit]

Suppose you have a quantum system with a 2-dimensional Hilbert space that you know is in either the state $|\psi\rangle$ or the state $|\varphi\rangle$, and you want to determine which one it is. If $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal, this task is easy: the set $\{|\psi\rangle\langle\psi|, |\varphi\rangle\langle\varphi|\}$ will form a PVM, and a projective measurement in this basis will determine the state with certainty. If, however, $|\psi\rangle$ and $|\varphi\rangle$ are not orthogonal, this task is *impossible*, in the sense that there is no measurement, either PVM or POVM, that will distinguish them with certainty.[2]:87 The impossibility of perfectly discriminating between non-orthogonal states is the basis for quantum information protocols such as quantum cryptography, quantum coin flipping, and quantum money.



Bloch sphere representation of states (in blue) and optimal POVM (in red) for unambiguous quantum state discrimination on the states $|\psi\rangle = |0\rangle$ and $|\varphi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Note that on the Bloch sphere orthogonal states are antiparallel.

The task of unambiguous quantum state discrimination (UQSD) is the next best thing: to never make a mistake about whether the state is $|\psi\rangle$ or $|\varphi\rangle$, at the cost of sometimes having an inconclusive result. It is possible to do this with projective measurements.[7] For example, if you measure the PVM $\{|\psi\rangle\langle\psi|, |\psi^\perp\rangle\langle\psi^\perp|\}$, where $|\psi^\perp\rangle$ is the quantum state orthogonal to $|\psi\rangle$, and obtain result $|\psi^\perp\rangle\langle\psi^\perp|$, then you know with certainty that the state was $|\varphi\rangle$. If the result was $|\psi\rangle\langle\psi|$, then it is inconclusive. The analogous reasoning holds for the PVM $\{|\varphi\rangle\langle\varphi|, |\varphi^\perp\rangle\langle\varphi^\perp|\}$, where $|\varphi^\perp\rangle$ is the state orthogonal to $|\varphi\rangle$.

This is unsatisfactory, though, as you can't detect both $|\psi\rangle$ and $|\varphi\rangle$ with a single measurement, and the probability of getting a conclusive result is smaller than with POVMs. The POVM that gives the highest probability of a conclusive outcome in this task is given by [7][8]

Mq $F_? \geq 0$ [trop dur]

$$F_\psi = \frac{1}{1+|\langle\varphi|\psi\rangle|}|\varphi^\perp\rangle\langle\varphi^\perp|$$
$$F_\varphi = \frac{1}{1+|\langle\varphi|\psi\rangle|}|\psi^\perp\rangle\langle\psi^\perp|$$
$$F_? = \mathrm{I} - F_\psi - F_\varphi.$$

A? Mq

Note that $\mathrm{tr}(|\varphi\rangle\langle\varphi|F_\psi) = \mathrm{tr}(|\psi\rangle\langle\psi|F_\varphi) = 0$, so when outcome ψ is obtained we are certain that the quantum state is $|\psi\rangle$, and when outcome φ is obtained we are certain that the quantum state is $|\varphi\rangle$.

The probability of having a conclusive outcome is given by

Exercice : Mq $\Pr(\text{conclusive}) = 1 - |\langle\varphi|\psi\rangle|$

$$1 - |\langle\varphi|\psi\rangle|,$$

$|\psi\rangle = \alpha|\varphi\rangle + \beta|\varphi^\perp\rangle$

when the quantum system is in state $|\psi\rangle$ or $|\varphi\rangle$ with the same probability. This result is known as the Ivanovic-Dieks-Peres limit, named after the authors who pioneered UQSD research.[9][10][11]

Using the above construction we can obtain a projective measurement that physically realises this POVM. The square roots of the POVM elements are given by

$$\sqrt{F_\psi} = \frac{1}{\sqrt{1+|\langle\varphi|\psi\rangle|}}|\varphi^\perp\rangle\langle\varphi^\perp|$$
$$\sqrt{F_\varphi} = \frac{1}{\sqrt{1+|\langle\varphi|\psi\rangle|}}|\psi^\perp\rangle\langle\psi^\perp|$$
$$\sqrt{F_?} = \sqrt{\frac{2|\langle\varphi|\psi\rangle|}{1+|\langle\varphi|\psi\rangle|}}|\gamma\rangle\langle\gamma|,$$

$$\mathrm{Tr}(F_\psi|\psi\rangle\langle\psi|) = \frac{|\beta|^2}{1+\alpha} = \frac{1-|\alpha|^2}{1+\alpha} = 1-|\alpha|$$

where

$$|\gamma\rangle = \frac{1}{\sqrt{2(1+|\langle\varphi|\psi\rangle|)}}(|\psi\rangle + e^{i\arg(\langle\varphi|\psi\rangle)}|\varphi\rangle).$$

Labelling the three possible states of the ancilla as $|\text{result }?\rangle$, $|\text{result }\psi\rangle$, $|\text{result }\varphi\rangle$, and initializing it on the state $|\text{result }?\rangle$, we see that the resulting unitary U_{UQSD} takes the state $|\psi\rangle$ together with the ancilla to

$$U_{\mathrm{UQSD}}(|\psi\rangle|\text{result }?\rangle) = \sqrt{1-|\langle\varphi|\psi\rangle|}|\varphi^\perp\rangle|\text{result }\psi\rangle + \sqrt{|\langle\varphi|\psi\rangle|}|\gamma\rangle|\text{result }?\rangle,$$

and similarly it takes the state $|\varphi\rangle$ together with the ancilla to

$$U_{\mathrm{UQSD}}(|\varphi\rangle|\text{result }?\rangle) = \sqrt{1-|\langle\varphi|\psi\rangle|}|\psi^\perp\rangle|\text{result }\varphi\rangle + e^{-i\arg(\langle\varphi|\psi\rangle)}\sqrt{|\langle\varphi|\psi\rangle|}|\gamma\rangle|\text{result }?\rangle.$$

A measurement on the ancilla then gives the desired results with the same probabilities as the POVM.

Etat Quantique (matrice densité)

Mesure POVM : $\{E_i\}$

Distribution probabilité d'états $\{|\psi_k\rangle, p(k)\}$

Exemples

- Bruit
- Protocole BB84
- Superposition quantique
- Allagement / accord

$$P(\text{résultat } j) = \sum_k p(k)\, P(j \,|\, |\psi_k\rangle)$$

$$= \sum_k p(k)\, \mathrm{Tr}\left(E_j\, |\psi_k\rangle\langle\psi_k|\right)$$

Définition Matrice Densité

On définit $\hat{\rho} = \sum_k p(k) |\psi_k\rangle\langle\psi_k|$

$$\Rightarrow \boxed{P(j) = Tr(\rho E_i)}$$

Propriétés de Matrice densité

1) $Tr(\rho) = 1$

Preuve : $Tr(\rho) = Tr \sum_k p(k) |\psi_k\rangle\langle\psi_k| = \sum_k p_k \overbrace{Tr(|\psi_k\rangle\langle\psi_k|)}^{1} = 1$

2) ρ Positif, semi-défini $\rho \geq 0$

$\Leftrightarrow \rho = \rho^+$ et $\langle\psi|\rho|\psi\rangle \geq 0 \quad \forall \psi \in \mathcal{H}$

Rem

$\rho = \rho^+$ ok $\qquad \langle\psi|\rho|\psi\rangle = \langle\psi| \sum_k p(k) |\psi_k\rangle\langle\psi_k|\psi\rangle$

$$= \sum_k p(k) |\langle\psi_k|\psi\rangle|^2 \geq 0$$

Exercices Matrice Densité

Écrire matrice densité pour

(a) $\begin{cases} |0\rangle & \text{proba } 1/2 \\ |+\rangle & \text{proba } 1/2 \end{cases}$ $\qquad \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$

(b) $\begin{cases} \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}, & \text{proba } p \\[2ex] \dfrac{|00\rangle - |11\rangle}{\sqrt{2}}, & \text{proba } p \end{cases}$

$$\rho = p\,\frac{|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|}{2}$$

$$+ (1-p)\,\frac{|00\rangle\langle 00| + |11\rangle\langle 11| - |00\rangle\langle 11| - |11\rangle\langle 00|}{2}$$

$$= \frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2} + \frac{(2p-1)}{2}\left(|00\rangle\langle 11| + |11\rangle\langle 00|\right)$$

$$= \left(\begin{array}{cc|cc} 1/2 & 0 & 0 & \frac{2p-1}{2} \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \frac{2p-1}{2} & 0 & 0 & 1/2 \end{array}\right)$$

si $p = 1/2$

$\rho = 1/2\,|00\rangle\langle 00| + 1/2\,|11\rangle\langle 11|$

separable

Nouveau Postulat 1 (état quantique matrice densité)

Soit $\mathcal{H}$ espace de Hilbert associé à système quantique
Alors tout système est décrit par matrice densité
ρ agissant sur $\mathcal{H}$, i.e $\underbrace{\rho \in \mathcal{L}(\mathcal{H}) \text{ tq } \begin{cases} \mathrm{Tr}(\rho)=1 \\ \rho \geq 0 \end{cases}}_{\mathcal{D}(\mathcal{H})}$

- états purs : $\rho = |\Phi\rangle\langle\Phi|$

[Rq : supprime le pb de phase globale]

- cas gal : toute matrice $\rho \in \mathcal{D}(\mathcal{H})$ définit un état quantique (éventuellement mixte)

$$\begin{cases} \rho \geq 0 \\ \mathrm{Tr}(\rho)=1 \end{cases} \xrightarrow{\text{spectral Th}} \rho = \sum_j \lambda_j |\Phi_j\rangle\langle\Phi_j| \qquad \lambda_j \geq 0$$

$\hookrightarrow \sum \lambda_j = 1$

$\Rightarrow \{\lambda_j\}$ distr proba $\Rightarrow \rho$ état quantique

Matrice densité qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow \rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

$$= \begin{pmatrix} a & \gamma \\ \gamma^* & 1-a \end{pmatrix} \qquad a \in [0,1] \qquad |\gamma| \leq \sqrt{a(1-a)}$$

cohérences

populations

Trace Partielle

Système AB

$Tr_B \hat{=}$ Trace sur B : $\mathcal{L}(H_A \otimes H_B) \longrightarrow \mathcal{L}(H_A)$

$$X \otimes Y \longmapsto Tr(Y)\, X$$

<u>Exemple</u> AB système bipartite

$\{|i\rangle_A\}$ base orth de A

$\{|k\rangle_B\}$ base orth de B

Matrice $X_{AB} = \sum\limits_{\underbrace{i,j}_{A}\ \underbrace{k,l}_{B}} x_{i,j,k,l}\ |i\rangle\langle j|_A \otimes |k\rangle\langle l|_B$

$$\Rightarrow X_A = Tr_B\left[X_{AB}\right] = \sum_{ijkl} x_{ijkl}\ |i\rangle\langle j|_A\ \underbrace{Tr_B\ |k\rangle\langle l|_B}_{\delta_{kl}}$$

$$= \sum_{ij\,k} x_{ijkk}\ |i\rangle\langle j|_A$$

Exemple 2

$$\psi_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad \rho_{AB} = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2}$$

$$= \left(\begin{array}{cc|cc} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{array}\right)$$

$$\rho_A = \mathrm{Tr}_B(\rho_{AB})$$

$$= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = I/2$$

Rq : On peut aussi écrire

$$\rho_A = \sum_i \left(\left(I_A \times \langle i|_B \right) \rho_{AB} \left(I_A \otimes |i\rangle_B \right) \right)$$

Purification

Soit ρ_A un état quelconque sur A

Purification de ρ_A est état $|\Psi_{AB}\rangle$ sur un système joint AB tq

$$\rho_A = \mathrm{Tr}_B\, |\Psi\rangle\langle\Psi|_{AB}$$

<u>Rque</u> : $\exists$ purification $\forall$ état ρ_A

Th spectral : $\rho_A = \sum \lambda_i |\psi_i\rangle\langle\psi_i|$ $\quad \lambda_i \geq 0$, $|\psi_i\rangle$ orth

Soit B isomorphe à A (même dimension)

on peut alors définir :

$$|\Phi_{AB}\rangle = \sum_i \sqrt{\lambda_i}\, |\psi_i\rangle_A |\psi_i\rangle_B$$

Exercice Matrice densité et sphère de Bloch

$$4|\gamma|^2 \leq 4\,a(1-a) = 4a-4a^2$$

$$\Rightarrow 4a^2 - 4a + 4\gamma^2 \leq 0$$

$$\rho = \begin{pmatrix} a & \gamma \\ \gamma^* & 1-a \end{pmatrix} \qquad |\gamma| \leq \sqrt{a(1-a)}$$

(a) dlq $\rho = \dfrac{I + r_x X + r_y Y + r_z Z}{2}$ avec $r_x^2 + r_y^2 + r_z^2 \leq 1$

$$\rho = \begin{pmatrix} 1/2 + (a-\frac{1}{2}) & \gamma \\ \gamma^* & \frac{1}{2} + (\frac{1}{2} - a) \end{pmatrix}$$

$$= I/2 + (a-\tfrac{1}{2})Z + \mathrm{Re}\,\gamma\, X + \mathrm{Im}\,\gamma\, Y$$

$$= I/2 + \frac{2a-1\,Z + 2\mathrm{Re}\,\gamma X + 2\mathrm{Im}\,\gamma\, Y}{2}$$

$r_x = 2\,\mathrm{Re}\,\gamma$
$r_y = 2\,\mathrm{Im}\,\gamma$
$r_z = 2a-1$

$$r_x^2 + r_y^2 + r_z^2 = (2a-1)^2 + 4\,\underbrace{\mathrm{Re}\gamma^2 + \mathrm{Im}\gamma^2} \leq 1$$

$$\underbrace{}_{4a^2-4a+1} + 4\,|\gamma|^2$$

(2) Mq états purs correspondent à
$r_x^2 + r_y^2 + r_z^2 = 1$

$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\Phi} \sin\left(\frac{\theta}{2}\right)|1\rangle$

$$|\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\left(\frac{\theta}{2}\right) & \cos(\theta/2)\sin(\theta/2)\, e^{i\Phi} \\ (*) & \sin^2\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2} + \frac{\cos(\theta)}{2}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\left.\begin{aligned} r_z &= \cos\theta \\ r_x &= \sin\theta \cos\Phi \\ r_y &= \sin\theta \sin\Phi \end{aligned}\right\} r_x^2 + r_y^2 + r_z^2 = 1$$

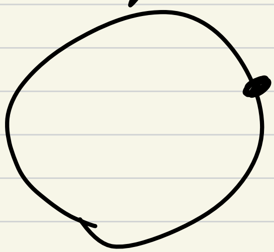
surface sphère

Etat mixte : ds la sphère

(c) Soit P_0, P_1 2 projecteurs de rang 1 orthogonaux

i.e $P_0 = |\psi\rangle\langle\psi|$ pour ψ, qulcq

Mq P_0 et P_1 correspondent à des points opposés sur sphère de Bloch

$P_0 \longleftrightarrow$  $|\psi\rangle\langle\psi|$

$\hookrightarrow r_x^0, r_y^0, r_z^0 \quad (\sum r_i^2 = 1)$

$P_1 \longleftrightarrow |\varphi\rangle\langle\varphi|$

$$P_0 P_1 = 0 \Rightarrow Tr(P_0 P_1) = 0$$

$$\Rightarrow Tr\left(\frac{I + r_x^0 X + r_y^0 Y + r_z^0 Z}{2}\right)\left(\frac{I}{2} + \cdots\right) = 0$$

$$\Rightarrow 1/4 \left(1 + r_x^0 r_x^1 + r_y^0 r_y^1 + r_z^0 r_z^1\right) = 0$$

$$\Rightarrow \vec{r}^{\,0} \cdot \vec{r}^{\,1} = -1$$

Mesure (matrice densité + POVM) [sans démonstration]

th POVM $\{E_i\}$ $\quad E_i \geq 0$

$\sum E_i = 1$

<u>mesure</u>

$$\Pr(i) = \mathrm{Tr}(\rho M_i)$$

<u>Théorème</u> : [Naimark dilation]

Soit $\{E_i\}$ POVM sur A et $\rho_A \in \mathcal{D}(A)$

$\exists$ système B et isométrie V $A \longrightarrow AB$
mesure projective $\{P_i\}$ tq

$$\mathrm{Tr}(\rho_A M_i) = \mathrm{Tr}\left[(V \rho_A V^\dagger) P_i\right]$$

État après mesure

mesure, résultat i

$$\rho_{PM,i} = \frac{\sqrt{E_i}\,\rho\,\sqrt{E_i}}{\mathrm{Tr}(\rho E_i)} \quad \ldots \quad \left(\frac{P_i \rho P_i}{\mathrm{Tr}(\rho P_i)}\right)$$