

# Random Matrix Theory and Applications

## Exam

March 28th, 2023

Surname :

Name :

Master :

**Write your surname & name in CAPITAL LETTERS.**

**Clarity and relevance of the explanations will also be evaluated for the final mark.**

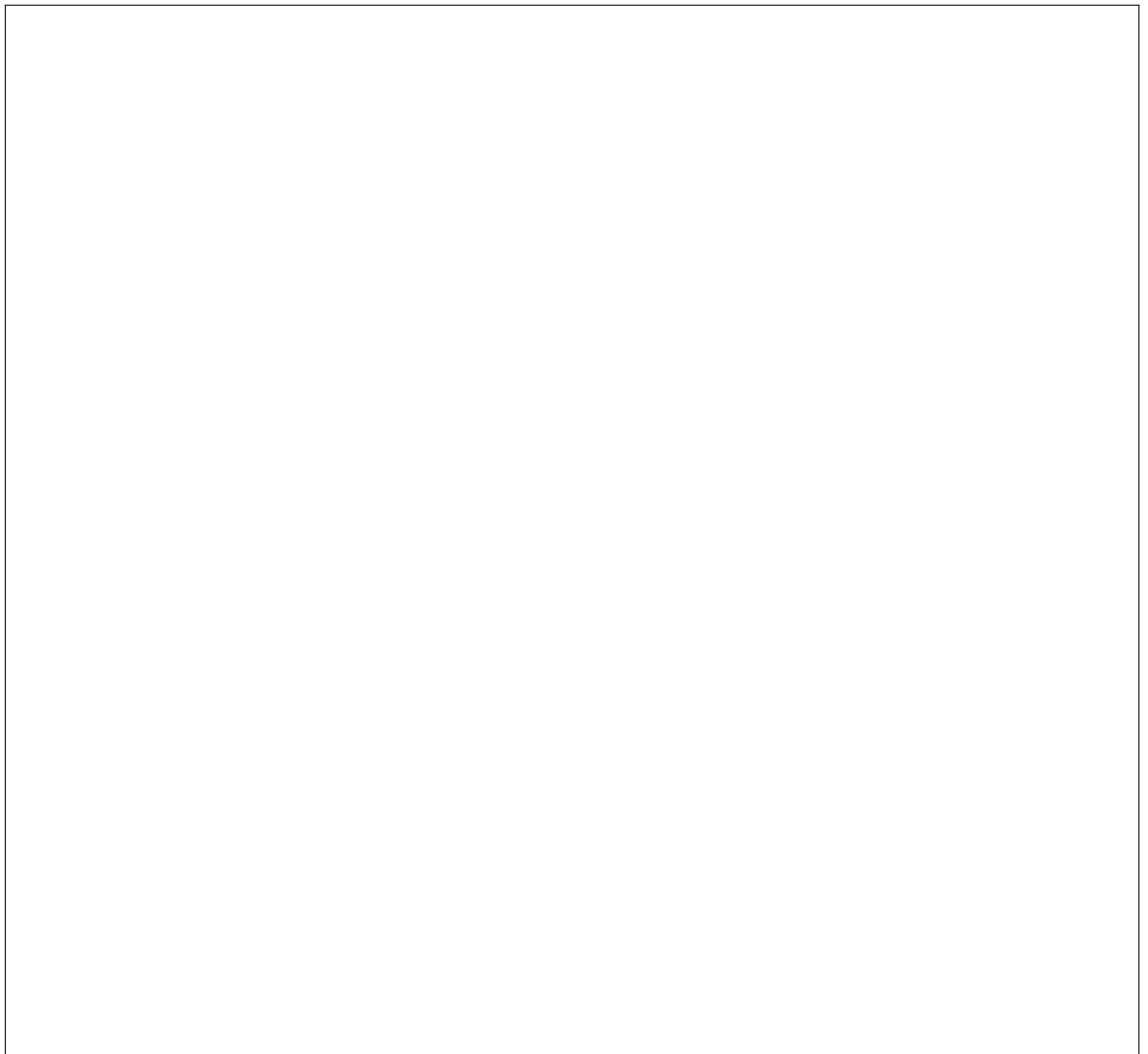
**The answers must be written within the boxes.**

**No books or notes are allowed during the exam.**

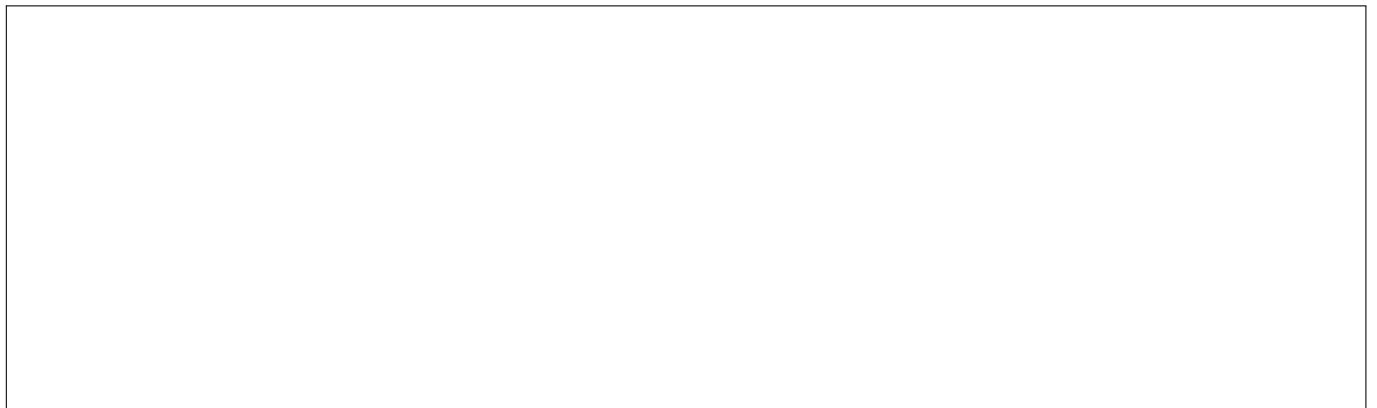
### 1. Coulomb gas approach.

(a) By assuming the effective potential  $\mathcal{V}[\mathbf{x}] = \frac{1}{2N} \sum_i x_i^2 - \frac{1}{2N^2} \sum_{i \neq j} \ln |x_i - x_j|$ , discuss the physical meaning of the two terms and any analogy with a statistical mechanics framework. What is the meaning of  $\beta$  in the two frameworks?

(b) Go through the main steps of the Dyson technique, the so-called *Coulomb gas* formalism (no in-depth calculations needed).



(c) Can the support of the equilibrium density profile  $n^*(x)$  for Gaussian (GOE, GUE, GSE) ensembles be the full real line? Explain why.



## **2. Real and imaginary parts.**

Write the Sokhotski-Plemelj formula. Why is it introduced? What kind of information does it allow one to obtain?

## **3. Jpdf of the eigenvalues.**

Consider the joint probability density function  $\mathcal{P}_N(\mathbf{X})$  of the matrix entries, where  $\mathbf{X}$  is a real Gaussian random matrix, of size  $N \times N$ .

$$\mathcal{P}_N(\mathbf{X}) = \mathcal{Z}_N^{-1} \exp \left[ -\frac{1}{2(1-\tau^2)} \text{Tr}(\mathbf{X}\mathbf{X}^T - \tau \mathbf{X}^2) \right]$$

where the parameter  $\tau$  is defined in the interval  $\in [0, 1]$ . If  $\tau = 0$ , what is the mean density of eigenvalues? Discuss the case for generic  $\tau$ .

#### 4. Level spacing

Consider i.i.d. real random variables  $\{X_1, \dots, X_N\}$  drawn from a distribution  $p(X)$ , which is defined over a support  $\Sigma$ .

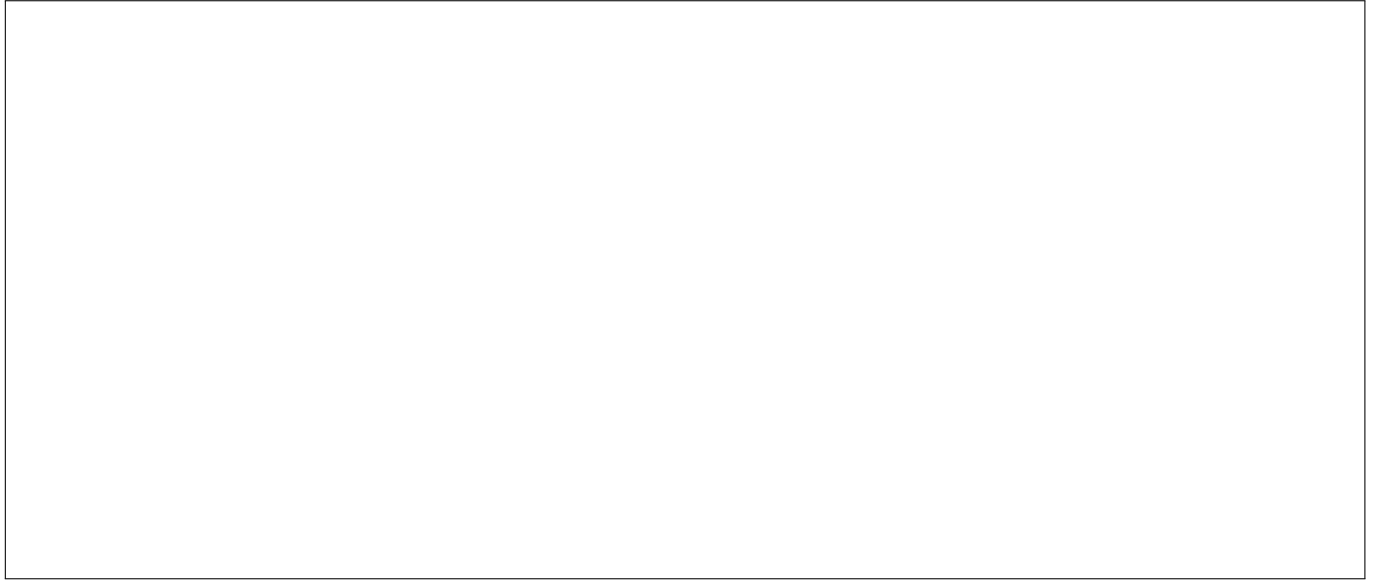
- (a) What is the probability density of the spacing  $s$  between two subsequent levels?
- (b) What would be the analogous level spacing distribution upon considering correlation among the levels?
- (c) Deduce the probability density function  $p_{N=2}(s)$  of the spacing between the two eigenvalues of a  $2 \times 2$  random matrix belonging to the Gaussian Unitary Ensemble in the limit  $s \rightarrow 0^+$ .

**5. Typical and atypical fluctuations in the Gaussian ensemble.**

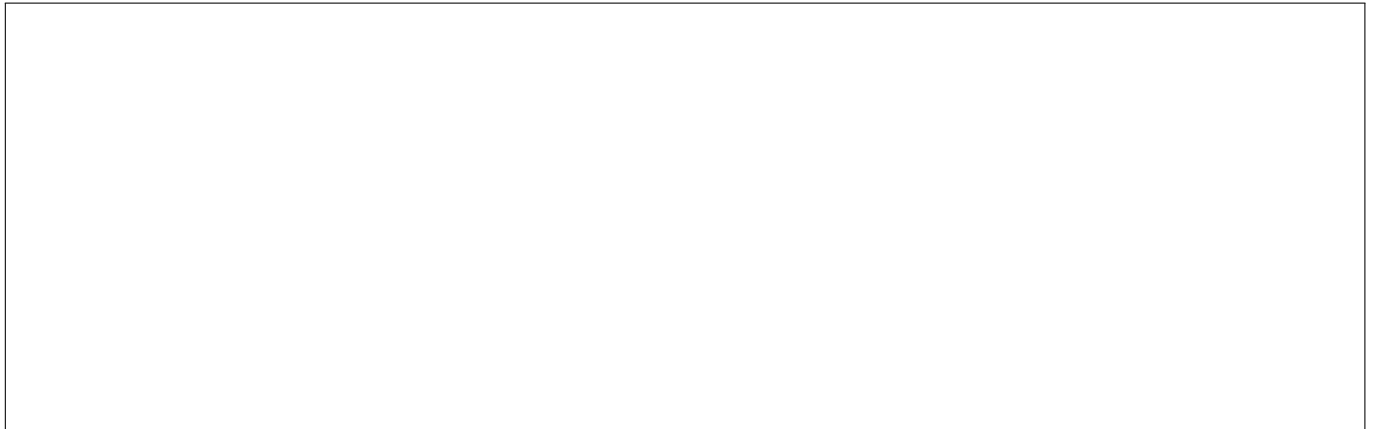
Consider the large deviation problem for the Wigner ensemble,  $\mathbb{P}[\lambda_{\max} - 2\sqrt{\beta}\sigma > x] \equiv e^{-NC_{\beta}(x)}$  at finite  $N$ .

(a) Write the resulting density function for the typical and atypical fluctuations of the top eigenvalue. What is their order of magnitude respectively?

(b) Sketch the distribution(s) around  $\lambda_{\max}$ . What can you claim about its symmetry?

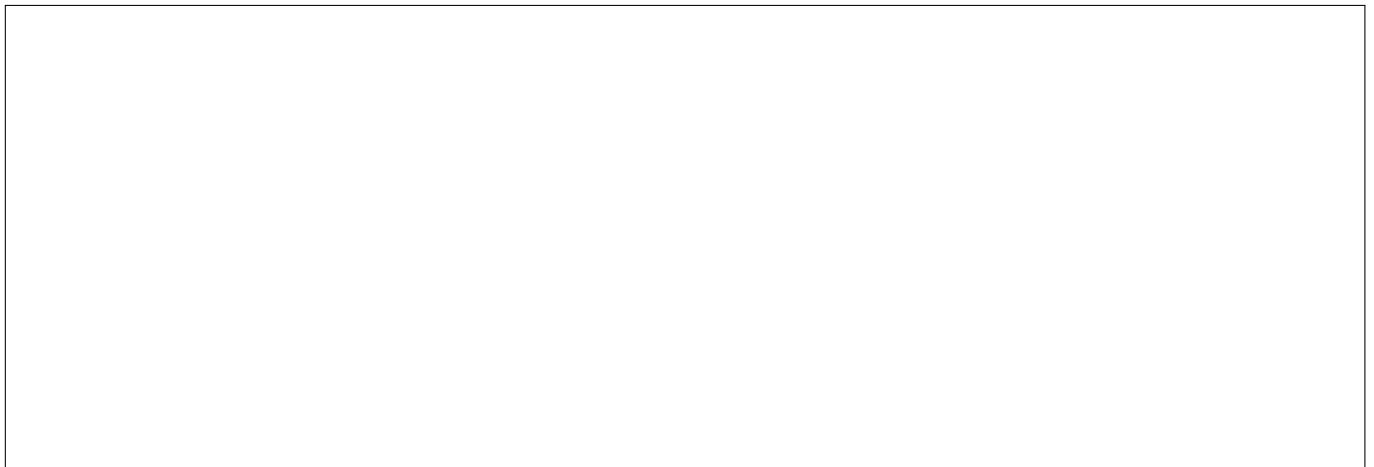


(c) By defining a generalized free energy in terms of the cumulative distribution, discuss the possibility of a phase transition, and its order, in the presence of a hard wall.



## **6. Cumulants**

a) Give the definition of the generating function of the moments  $\mu_k$ , valid at any order.



b) Re-consider the question in the Gaussian case. Represent the sixth-order moment  $\tau(\mathbf{X}^6)$  where  $\mathbf{X}$  denotes a Gaussian matrix. Discuss the contribution of crossing pair partitions.

## 7. Disordered averages

a) Given the partition function  $Z_N = \int \prod_{i=1}^N \frac{du_i}{\sqrt{2\pi/i}} \exp\{-\frac{i}{2} \sum_{i,j} u_i (\lambda_\epsilon \delta_{ij} - M_{ij}) u_j\}$ , where  $M_{ij}$  denote the elements of a random symmetric matrix, write the expressions for the free-energy and the spectral density in the *annealed approximation*.



b) Write the analogous expressions in the *quenched* case. Hint: recall the Edwards-Jones formula.

c) Explain the differences between *annealed* and *quenched* averages, along with benefits and limitations.

## 8. Correlated random variables

The Hessian matrix – associated with a given Hamiltonian  $H$  – is defined by

$$\mathcal{M}_{ij} = \frac{1}{N} \sum_{p=1}^P \xi_i^p \xi_j^p \theta(-h_p) + \mu \delta_{ij}$$

where  $\theta(\cdot)$  is the Heaviside function, and  $\mu$  is a Lagrange multiplier resulting in a diagonal shift. The so-called *gap variables*  $h_p$  are defined by  $h_p = \sum_{i=1}^N \frac{1}{\sqrt{N}} x_i \xi_i^p$ , in terms of the vectors  $\boldsymbol{\xi}^p$ . Then  $\boldsymbol{\xi}^p = \{\xi_1^p, \dots, \xi_N^p\}$  are i.i.d. random variables with independent  $\mathcal{N}(0, 1)$  components.

(a) What ensemble is the Hessian matrix  $\mathcal{M}_{ij}$  associated with?

(b) What is the spectral density in the thermodynamic limit? Which hypothesis did you make to write it? Show qualitatively  $\rho(\lambda)$  in the different cases (at least two).

(c) Suppose you started with  $P < N$ . Can any information be deduced beforehand?

### **9. Symmetrization and rotational invariance**

Let us define  $\mathbf{M}$  a real non-symmetrix matrix whose elements are i.i.d. drawn from a Gaussian distribution with zero mean and variance  $\langle M_{ij}^2 \rangle = \sigma^2$ .

(a) Perform a two-line computation to show the relation between the diagonal and the off-diagonal elements of the resulting GOE matrix.

(b) If  $\mathbf{R}$  is a rotation matrix such that  $\mathbf{R}^T \mathbf{R} = \mathbf{1}$ , show that trace of the GOE matrix obtained above times its transpose is invariant under orthogonal transformations, *i.e.*  $\mathbf{H} \rightarrow \mathbf{R}^T \mathbf{H} \mathbf{R}$ .