Preparation of the exam of Stochastic processes
List of questions on the lectures

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$\sim$ 1/3 of the exam will come from the list of questions below

Questions are in black ;
answers are in the lectures or tutorials + colored additional remarks below

Chapter 2 : Probability

Q1: Give a brief demonstration of the **central limit theorem**.

$\to$ recall the assumptions,
$\to$ use of the generating function of the cumulants $\hat{w}(k) = \ln \hat{p}(k) \underset{k\to 0}{\simeq} -\mathrm{i}\kappa_1 k - \frac{1}{2}\kappa_2 k^2$ and recover the Gaussian distribution

Q2: Discuss the concept of **large deviation** on the simple case of the binomial distribution : show that the binomial distribution presents the large deviation form $\mathcal{P}_N(n) = C_N^n\, p^n q^{N-n} \underset{N\to\infty}{\sim} \exp\left\{-N\,\Phi\left(\frac{n}{N}\right)\right\}$ and derive the large deviation function $\Phi(y)$. Recover the central limit theorem from it in this specific situation.

$\to$ understand the difference between typical and atypical fluctuations

Using the Stirling formula we have

$$\ln \mathcal{P}_N(n) \simeq \underbrace{N\ln N - n\ln n - (N-n)\ln(N-n)}_{-n\ln(n/N)-(N-n)\ln\left[(N-n)/N\right]} + n\ln p + (N-n)\ln q + \mathcal{O}(\ln N)$$

where $q = 1-p$. Therefore

$$\ln \mathcal{P}_N(n) \simeq N\left[-y\,\ln y - (1-y)\ln(1-y) + y\ln p + (1-y)\ln q\right] \tag{1}$$

i.e.

$$\Phi(y) = y\,\ln\left(\frac{y}{p}\right) + (1-y)\ln\left(\frac{1-y}{1-p}\right) \tag{2}$$

We see that $\Phi'(y) = 0$ for $y = y_* = p$. At this point $\Phi(y_*) = 0$ and $\Phi''(y_*) = 1/(pq)$.
• The quadratic behaviour of the LDF $\Phi(y) \simeq \frac{1}{2pq}(y-p)^2$ for $y \to y_* = p$ corresponds to the central limit theorem and the Gaussian distribution, $\mathcal{P}_N(S) \sim \exp\left\{\frac{1}{2Npq}(S-Np)^2\right\}$. The correspondence originates from the fact that $\mathcal{P}_N(n) = \text{Proba}\{S_N = n\}$ is the distribution of the sum of N independent Bernoulli random variables ($S_N = \sum_{i=1}^N \xi_i$ for $\xi_i = 0$ with proba q and $\xi_i = 1$ with proba p).
• For $|y - y_*| \sim \mathcal{O}(1)$ the LDF presents strong deviations to the quadratic form.

Q3: **Symmetric Lévy distribution.—** Consider the sum S_N of N i.i.d. random numbers with *symmetric* power law distribution $p(x) \underset{x\to\infty}{\sim} |x|^{-1-\mu}$ for $\mu \in]0,2[$. Using that the characteristic function presents the limiting behaviour $\hat{p}(k) \underset{k\to 0}{\simeq} 1 - c|k|^\mu$, show that the distribution of the sum S_N can be expressed in terms of the Lévy law $\mathcal{L}_{\mu,0}(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{-|k|^\mu + \mathrm{i}kx}$ and discuss the scaling of S_N with N.

$\rightarrow$ cf. derivation in the lecture notes.

Q4: Consider N correlated Gaussian random variables $x_1, \cdots, x_N$ with distribution $P(\mathbf{x}) \propto \exp\big\{-\frac{1}{2}\mathbf{x}^{\mathrm{T}} A \mathbf{x}\big\}$, where A is a positive definite real symmetric matrix.
a) Show that the correlations are $\langle x_i x_j \rangle = (A^{-1})_{ij}$.
b) Application : **discrete Ornstein-Uhlenbeck process.**– We consider random Gaussian variables $\Phi = (\cdots, \phi_n, \cdots)^{\mathrm{T}}$ with weight $P(\Phi) \propto \exp[-S(\Phi)]$ where the action is

$$S(\Phi) = \frac{1}{2} \sum_{t \in \mathbb{Z}} \left[(\phi_{t+1} - \phi_t)^2 + \mu^2 \phi_t^2 \right] \tag{3}$$

Write the action under the form $S = \frac{1}{2}\Phi^{\mathrm{T}} A \Phi$ and show that the matrix A involves the discrete Laplace operator $\Delta_{n,m} = \delta_{n,m+1} - 2\delta_{n,m} + \delta_{n,m-1}$.
c) Give the eigenvalues and the (normalised) eigenvectors of Δ on the infinite line ($n \in \mathbb{Z}$). Deduce the correlation function $\langle \phi_t \phi_{t'} \rangle$.
d) Discuss the limit $\mu \to 0$.
Hint : we give the integral $\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\sinh\lambda}{\cosh\lambda + \cos\theta}\, \mathrm{e}^{\mathrm{i}n\theta} = \mathrm{e}^{-\lambda|n|}$.

a) $\rightarrow$ use the generating function (no need to calculate an integral). Cf. lecture notes.
b) to d) $\rightarrow$ Cf. tutorial n°1

Chapter 3 : Langevin equation

Q5: **Wiener process.**– Consider the Wiener process $W(t) = \int_0^t \mathrm{d}u\, \eta(u)$ where $\eta(t)$ is a normalised Gaussian white noise such that $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = \delta(t - t')$.
a) Compute the correlator $\langle W(t)W(t') \rangle$ and deduce $\big\langle [W(t) - W(t')]^2 \big\rangle$.
b) Give the distribution of the process $P_t(W)$.

a) easy to get $\langle W(t)W(t') \rangle = \min(t, t')$ and $\big\langle [W(t) - W(t')]^2 \big\rangle = |t - t'|$.
b) The process is a sum of Gaussian variables, hence its distribution is also Gaussian. The knowledge of the two first cumulants is enough, $\langle W(t) \rangle = 0$ and $\big\langle W(t)^2 \big\rangle = t$, thus $P_t(W) = \frac{1}{\sqrt{2\pi t}} \exp[-W^2/(2t)]$.

Q6: **Langevin equation (Ornstein-Ulhenbeck process).**– consider a particle in a fluid with velocity $v(t)$ obeying the Langevin equation $\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\frac{1}{\tau}v(t) + \frac{1}{\tau}\sqrt{2D}\,\eta(t)$ where $\eta(t)$ is a normalised Gaussian white noise such that $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = \delta(t - t')$. The time τ is related to the friction coefficient $\gamma = m/\tau$, m being the mass.
a) Give the solution of the differential equation. Deduce the mean $\langle v(t) \rangle$ and the correlator $\langle v(t)v(t') \rangle_c$ for fixed $v(0) = v_0$.
b) Deduce the conditional probability $P_t(v|v_0)$.
c) Discuss the large time behaviour of $P_t(v|v_0)$ and recover the Einstein relation between the diffusion constant D, the friction coefficient $\gamma = m/\tau$ and $k_{\mathrm{B}}T$.
d) Define the overdamped regime and write the SDE for the position $x(t)$.

$\rightarrow$ cf. lecture notes.

Chapter 4 : Markov processes

Q7: Give the definition of a Markov process in few words. Give an example of Markov process and an example of non Markovian process.

Q8: Consider a Markov process $X(t)$ characterized by the probability $P_t(x)$ and the conditional probability $P_t(x|y)$.
a) Express $\langle x(t)\rangle$ in terms of these distributions (i) assuming a fixed $x(0) = x_0$, (ii) assuming a random $x(0)$.
b) Express $\langle x(t)x(t')\rangle$ in terms of the conditional probability for a fixed $x(0) = x_0$.
c) Application to the Poisson process $\mathscr{N}(t) \in \mathbb{N}$: the conditional probability is

$$P_t(n|m) = \begin{cases} \frac{(\lambda t)^{n-m}}{(n-m)!}\,\mathrm{e}^{-\lambda t} & \text{for } n \geqslant m \\ 0 & \text{for } n < m \end{cases} \tag{4}$$

Express and compute $\langle \mathscr{N}(t)\rangle$ for $\mathscr{N}(0) = 0$.
d) Express $\langle \mathscr{N}(t)\mathscr{N}(t')\rangle$ as a double sum and compute it for $\mathscr{N}(0) = 0$ when $t' < t$. Deduce the correlator $\langle \mathscr{N}(t)\mathscr{N}(t')\rangle_c = \langle \mathscr{N}(t)\mathscr{N}(t')\rangle - \langle \mathscr{N}(t)\rangle\,\langle \mathscr{N}(t')\rangle$.
Hint : the calculation involve sums of the form $\sum_{n=0}^{\infty} n\,\frac{(\lambda t)^n}{n!}$ or $\sum_{n=0}^{\infty} n(n-1)\,\frac{(\lambda t)^n}{n!}$ which are very easy to compute!

a) - c) Cf. lecture notes
d) $\langle \mathscr{N}(t)\mathscr{N}(t')\rangle = \sum_{n=m}^{\infty}\sum_{m=0}^{\infty} n\,P_{t-t'}(n|m)\,m\,P_{t'}(m|0)$. The two sums are easy to compute. Eventually, one finds $\langle \mathscr{N}(t)\mathscr{N}(t')\rangle_c = \lambda\,\min(t,t')$.

Q9: **Poisson process :** the Poisson process $\mathscr{N}(t) \in \mathbb{N}$ counts the occurences of *independent events* occuring with a constant rate λ on the interval $[0,t]$. Denote the probability $P_n(t) = \text{Proba}\{\mathscr{N}(t) = n\}$.
a) Show that $P_0(t)$ obeys the differential equation $\partial_t P_0(t) = -\lambda\,P_0(t)$. Derive a set of coupled differential equations for the probabilities $P_n(t)$.
b) Introduce the generating function $G(z;t) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} z^n\,P_n(t)$. What is the value of $G(z;0)$? Get a differential equation for $G(z;t)$ and solve it.
c) Deduce the expression for $P_n(t)$.
d) Determine the cumulants $\langle \mathscr{N}(t)^k\rangle_c$ of the Poisson process.
e) Argue that the distribution $q(\tau)$ of the time separating two successive events is related to $P_0(t)$ and give $q(\tau)$.

a) We have

$$P_n(t+\delta t) = P_n(t)\,\underbrace{(1-\lambda\delta t-\cdots)}_{\text{no event on }[t,t+\delta t]} + P_{n-1}(t)\,\underbrace{\lambda\delta t}_{\text{one event}} + \cdots \tag{5}$$

Eventually, in the limit $\delta t \to 0$ we get

$$\partial_t P_n(t) = -\lambda\,P_n(t) + \lambda\,P_{n-1}(t) \quad \text{for } n > 0 \tag{6}$$

and

$$\partial_t P_0(t) = -\lambda\,P_0(t) \text{ for } n = 0. \tag{7}$$

b) We multiply the differential equation by z^n and sum over n : $\partial_t \sum_n z^n P_n(t) = -\lambda\,\sum_n z^n P_n(t) + \lambda\,\sum_n z^n P_{n-1}(t)$, leading to

$$\partial_t G(z;t) = \lambda(z-1)\,G(z;t) \tag{8}$$

We have $G(z;0) = 1$ (normalization). The differential equation for $G(z;t)$ is straightforward to solve, we get

$$G(z;t) = \exp[\lambda t(z-1)]\,. \tag{9}$$

c) Expansion of $G(z;t)$ in powers of z gives $P_n(t) = \frac{(\lambda t)^n}{n!}\,\mathrm{e}^{-\lambda t}$
d) Direct calculation of the moments/cumulants is unpleasant... However the generating function of the moments is easy to obtain

$$\left\langle \mathrm{e}^{k\mathscr{N}(t)}\right\rangle \left(= \sum_{n=0}^{\infty}\frac{k^n}{n!}\,\langle \mathscr{N}(t)^n\rangle \right) = \sum_{n=0}^{\infty}\mathrm{e}^{kn}P_n(t) = G(\mathrm{e}^k;t) = \exp[\lambda t(\mathrm{e}^k-1)] \tag{10}$$

The generating function of the cumulants is therefore

$$\sum_{n=1}^{\infty} \frac{k^n}{n!} \left\langle \mathcal{N}(t)^n \right\rangle_c = \ln G(\mathrm{e}^k;t) = \lambda t(\mathrm{e}^k - 1) \tag{11}$$

We deduce that **all cumulants are equal**

$$\left\langle \mathcal{N}(t)^n \right\rangle_c = \lambda t \,. \tag{12}$$

e) Denote τ the time of occurence of the first event (i.e. this is the time separating two consecutive events). We have

$$P_0(t) = \text{Proba}\{\text{no event on } [0,t]\} = \text{Proba}\{\tau > t\} = \int_t^{\infty} \mathrm{d}\tau \, q(\tau) \tag{13}$$

where $q(\tau)$ is the distribution of the time. As a result, the time intervals are exponentially distributed

$$q(\tau) = \lambda \, \mathrm{e}^{-\lambda\tau} \tag{14}$$

Q10: **Compound Poisson process :** We consider the master equation

$$\frac{\partial P_t(x)}{\partial t} = \int \mathrm{d}y \, \left[W(x|y) P_t(y) - W(y|x) P_t(x) \right] \tag{15}$$

for a translation invariant kernel $W(x|y) = \lambda \, w(x-y)$; here λ is the rate of jumps and $w(\eta)$ the distribution of the jump amplitudes.

a) show that the probability is conserved.

b) Introduce the Fourier transforms $\widehat{P}_t(k) = \int \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}kx} \, P_t(x)$ and $\hat{w}(k) = \int \mathrm{d}\eta \, \mathrm{e}^{-\mathrm{i}k\eta} \, w(\eta)$. Deduce a differential equation for $\widehat{P}_t(k)$ and express the solution $P_t(x)$ under the form of an integral, for initial condition $P_0(x) = \delta(x)$.

c) Assuming that we can write $\hat{w}(k) \simeq 1 - \frac{1}{2}ck^2$ for $k \to 0$, deduce the distribution $P_t(x)$. What is the meaning of the parameter c ? How would you qualify the process in this limit ?

d) Same question for $\hat{w}(k) \simeq 1 - c|k|$ for $k \to 0$.

a) obvious

b) The master equation involves a convolution, hence, after Fourier transform we have $\partial_t \widehat{P}_t(k) = \lambda \left[\hat{w}(k) - 1 \right] \widehat{P}_t(k)$. Using $\widehat{P}_0(k) = 1$, integration is elementary. Finally we get

$$P_t(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \, \mathrm{e}^{\lambda t \, [\hat{w}(k)-1] + \mathrm{i}kx} \,. \tag{16}$$

c) for high rate (or large time), the integral selects the $k \to 0$ behaviour $\hat{w}(k) \simeq 1 - \frac{1}{2}ck^2$, where c is the variance of the jumps. Hence the integral is Gaussian

$$P_t(x) \simeq \frac{1}{\sqrt{2\pi\lambda c t}} \exp -\frac{x^2}{2\lambda c t} \tag{17}$$

i.e. we have the scaling $x \sim \sqrt{t}$. In the large time limit the random walk corresponds to the continuous Brownian motion, as a result of the central limit theorem.

d) If the jump distribution presents a symmetric power law tail $w(\eta) \sim c/\eta^2$ we have $\hat{w}(k) \simeq 1 - c|k|$ for $k \to 0$ (for example $w(\eta) = (a/\pi)(\eta^2 + a^2)^{-1}$ gives $\hat{w}(k) = \mathrm{e}^{-a|k|}$). The large time limit involves the $k \to 0$ behaviour, thus

$$P_t(x) \simeq \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \, \mathrm{e}^{-c\lambda t \, |k| + \mathrm{i}kx} = \frac{c\lambda t}{\pi \left(x^2 + (c\lambda t)^2 \right)} \tag{18}$$

corresponding to the scaling $x \sim t$ (anomalous diffusion, motion is superdiffusive). This is a Lévy flight.

Q11: **Diffusion in a ring.**– Consider a particle on the N sites of a ring. We denote $P_n(t)$ the probability to be on site $n \in \{1, \cdots, N\}$ at time $t \in \mathbb{N}$. At each time step ($\Delta t = 1$) the particle jumps from site m to site n with probability $M_{nm} \in [0, 1]$.

a) What is the name of such a stochastic process ? What is the condition on the $N \times N$ matrix M ? How do we call such a matrix ?

b) Write the master equation. Express $P_n(t+1) - P_n(t)$ in terms of $M_{nm}P_m(t) - M_{mn}P_n(t)$.

c) What is the equation for the stationary state P_n^* ? Does it always exist for a finite number N of sites ? Support your statement by a proof.

d) Recall the detailed balance condition. What is the nature of the stationary state in this case ? How do we call the stationary state if the detailed balance condition does not hold ? Give two (simple) examples corresponding to the two situations in the ring (be brief).

e) Give a (simple) example of a situation where no stationary state exists. How do we call such a process ?

a), b), c) $\rightarrow$ cf. lecture notes.

d) Detailed balance is the strong condition $M_{nm}P_m^* - M_{mn}P_n^* = 0$, corresponding to the *equilibrium state*.

If $M_{nm}P_m^* - M_{mn}P_n^* \neq 0$ but $\sum_m \left(M_{nm}P_m^* - M_{mn}P_n^*\right) = 0$ the stationary state is a NESS.

Example on the ring. Consider simple translation invariant transition probabilities between nearest-neighbour sites only, $M_{n,n-1} = p$ and $M_{n-1,n} = q = 1 - p$. On the ring, the stationary solution is obviously the uniform distribution $P_n^* = 1/N$. We have $M_{n,n-1}P_{n-1}^* - M_{n-1,n}P_n^* = (p - q)/N$. Thus :

- for $p = q$, detailed balance holds and the stationary state is an *equilibrium*.
- for $p \neq q$, detailed balance does not hold and the stationary state is a NESS. In this case there exists a non vanishing steady current around the ring, $J = (p - q)/N$.

e) Only in the limit $N \to \infty$ can P_n^* be non normalisable. Then no stationary state exists and the process is "transient". Example : the free diffusion for $p = q = 1/2$, then $P_n(t) \simeq \frac{1}{\sqrt{\pi D t}} \exp[-n^2/(4Dt)]$ for large n and large t.

Chapter 5 : SDE

Q12: **Itô calculus.**– We denote by $W(t)$ the Wiener process (a normalised BM).

a) What is $\mathrm{d}W(t)^2$?

b) Consider the SDE

$$\mathrm{d}x(t) = a(x(t))\,\mathrm{d}t + b(x(t))\,\mathrm{d}W(t) \qquad \text{(Itô)}. \tag{19}$$

What is the main assumption in Itô calculus ?

c) Recover the Itô formula for $\mathrm{d}\varphi(x(t))$ where $x(t)$ solves the Itô equation and $\varphi(x)$ a regular function.

d) **Itô SDE and FPE :** Deduce the FPE from the Itô SDE (19).

$\rightarrow$ details in lecture notes.

Q13: We recall that the Itô SDE

$$\mathrm{d}x(t) = a(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t) \qquad \text{(Itô)} \tag{20}$$

is related to the PFE $\partial_t P_t(x) = -\partial_x\big[a(x)P_t(x)\big] + \frac{1}{2}\partial_x^2\big[b(x)^2 P_t(x)\big]$, and the Stratonovich SDE

$$\mathrm{d}x(t) = \phi(x)\,\mathrm{d}t + b(x)\,\mathrm{d}W(t) \qquad \text{(Stratonovich)} \tag{21}$$

to the PFE $\partial_t P_t(x) = -\partial_x\big[\phi(x)P_t(x)\big] + \frac{1}{2}\partial_x\Big[b(x)\partial_x\big[b(x)P_t(x)\big]\Big]$.
a) What is the relation between $a(x)$ and $\phi(x)$ ensuring that the two SDE describe the same stochastic process ?
b) We consider the Stratonovich SDE

$$\mathrm{d}x(t) = \phi(x)\,\mathrm{d}t + \sqrt{2D(x)}\,\mathrm{d}W(t) \qquad \text{(Stratonovich)} \tag{22}$$

give the corresponding Itô SDE.
c) What is the drift $\phi(x)$ leading to $\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)\rangle = 0$?

a) the two FPE coincide for $\phi(x) = a(x) - \frac{1}{2}b(x)b'(x)$.
b) the corresponding Itô SDE is $\mathrm{d}x(t) = F(x)\,\mathrm{d}t + \sqrt{2D(x)}\,\mathrm{d}W(t)$ with $F(x) = \phi(x) + \frac{1}{4}\big[b(x)^2\big]'$ with $b(x) = \sqrt{2D(x)}$, i.e. $F(x) = \phi(x) + \frac{1}{2}D'(x)$,

$$\mathrm{d}x(t) = \left[\phi(x) + \frac{1}{2}D'(x)\right]\,\mathrm{d}t + \sqrt{2D(x)}\,\mathrm{d}W(t) \qquad \text{(Itô)} \tag{23}$$

c) We compute easily $\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)\rangle$ from the Itô SDE ($x(t)$ and $\mathrm{d}W(t)$ independent) : $\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)\rangle = \langle F(x(t))\rangle$. The mean velocity vanishes for the drift $\phi(x) = -\frac{1}{2}D'(x)$ in the Stratonovich SDE.

Chapter 6 : FPE

Q14: Consider the FPE

$$\frac{\partial P_t(x)}{\partial t} = -\frac{\partial}{\partial x}\left[F(x)\,P_t(x)\right] + \frac{\partial^2}{\partial x^2}\left[D(x)\,P_t(x)\right] \tag{24}$$

a) discuss the meaning of the two terms. Explain their effects (for uniform $F(x) = F$ and $D(x) = D$).
b) Give the expression of the probability current $J_t(x)$.

$\to$ cf. lecture notes.

Q15: We consider the FPE $\partial_t P_t(x) = -\partial_x\big[F(x)P_t(x)\big] + D\partial_x^2\big[P_t(x)\big]$.
a) Show that the "forward generator" $\mathscr{G}^\dagger = -\partial_x F(x) + D\partial_x^2$ can be expressed in terms of ∂_x and $\mathrm{e}^{-V(x)/D}$ where $F(x) = -V'(x)$.
b) Deduce the expression of the equilibrium state. Under what condition on $V(x)$ does the equilibrium state exists ?
c) What is the appropriate (nonunitary) transformation allowing to map the non self adjoint operator $\mathscr{G}^\dagger$ onto a self adjoint one of the form $H_+ = \mathcal{Q}^\dagger\mathcal{Q}$? Give $\mathcal{Q}$ and $\mathcal{Q}^\dagger$.

$\to \mathscr{G}^\dagger = D\partial_x\mathrm{e}^{-V(x)/D}\partial_x\mathrm{e}^{+V(x)/D}$, etc, cf. lecture notes.

Q16: **The Ornstein-Uhlenbeck process.**– Consider the SDE $\mathrm{d}x(t) = F(x)\mathrm{d}t + \sqrt{2D}\,\mathrm{d}W(t)$. We recall that the FPE $\partial_t P_t(x) = \mathscr{G}^\dagger P_t(x)$ can be mapped onto the imaginary time Schrödinger equation $-\partial_t\psi(x;t) = H_+\psi(x;t)$ through $P_t(x) = \psi(x;t)\phi_0(x)$ where $\phi_0(x) = \exp[\frac{1}{2D}\int_0^x \mathrm{d}y\,F(y)]$. The Hamiltonian has the form $H_+ = \mathcal{Q}^\dagger\mathcal{Q}$ where $\mathcal{Q} = \sqrt{D}\big(-\partial_x + \frac{1}{2D}F(x)\big)$ and $\mathcal{Q}^\dagger = \sqrt{D}\big(\partial_x + \frac{1}{2D}F(x)\big)$.
We consider the Ornstein-Uhlenbeck process such that $F(x) = -kx$.
a) Compute the commutator $[\mathcal{Q}, \mathcal{Q}^\dagger]$.
b) Give the expression of $\phi_0(x)$. Compute $\mathcal{Q}\phi_0(x)$ and deduce that $\phi_0(x)$ is an eigenstate of H_+. Give the corresponding eigenvalue λ_0. What is the solution of the FPE related to this state ?

c) Show that $\mathcal{Q}^\dagger\phi_0(x)$ is also eigenstate of H_+ and give the related eigenvalue λ_1.
d) Using the same idea, deduce the full spectrum of eigenvalues of H_+.
e) We recall that the conditional probability is expressed in terms of the spectrum of H_+ as

$$P_t(x|x_0) = \frac{\phi_0(x)}{\phi_0(x_0)} \sum_{n=0}^{\infty} \phi_n(x)\,\phi_n(x_0)\,\mathrm{e}^{-\lambda_n t} \tag{25}$$

Deduce the averaged return probability $\int \mathrm{d}x\, P_t(x|x)$ for the Ornstein-Uhlenbeck process (remember that $\phi_n(x)$ are normalised).

a) cf. correction of the test
b) Action of $\mathcal{Q} = \sqrt{D}\big(-\partial_x + \frac{1}{2D}F(x)\big)$ on $\phi_0(x) = \exp[\frac{1}{2D}\int_0^x \mathrm{d}y\, F(y)]$ is zero. Hence $\phi_0(x) = \mathrm{e}^{-kx^2/(4D)}$ has eigenvalue $\lambda_0 = 0$. This is the equilibrium state $P_{\mathrm{eq}}(x) = c\,\phi_0(x)^2 = c\,\mathrm{e}^{-kx^2/(2D)}$
c) use commutator : $\lambda_1 = \lambda_0 + k = k$.
d) By recurrence, $\big(\mathcal{Q}^\dagger\big)^n\phi_0(x)$ has an eigenvalue $\lambda_n = n\,k$ for $n \in \mathbb{N}$.
e) $\int \mathrm{d}x\, P_t(x|x) = \sum_{n=0}^{\infty} \mathrm{e}^{-nkt} = (1 - \mathrm{e}^{-kt})^{-1}$.

Q17: **Construction of the conditional probability.**— Consider the SDE $\mathrm{d}x(t) = F(x)\mathrm{d}t + \sqrt{2D}\,\mathrm{d}W(t)$. We recall that the FPE $\partial_t P_t(x|x_0) = \mathscr{G}^\dagger P_t(x|x_0)$ can be mapped onto the imaginary time Schrödinger equation $-\partial_t\psi(x;t) = H_+\psi(x;t)$ through $P_t(x|x_0) = \psi(x;t)\phi_0(x)$ where $\phi_0(x) = \exp[\frac{1}{2D}\int_0^x \mathrm{d}y\, F(y)]$. For a confining drift, the Hamiltonian H_+ has a spectrum of eigenvalues and eigenvectors $\big(\lambda_n,\,\phi_n(x)\big)$ with $\lambda_0 = 0$.
a) An initial state can be decomposed over the basis of orthonormal eigenstates $\phi_n(x)$ of H_+ as $\psi(x;0) = \sum_n c_n\phi_n(x)$. If the initial condition for the FPE is $P_0(x|x_0) = \delta(x - x_0)$, give the coefficients c_n.
b) What is the wave function $\psi(x;t)$ at time t ?
c) Deduce the corresponding solution of the FPE.
d) Check that $\int \mathrm{d}x\, P_t(x|x_0) = 1$. Analyze the $t \to \infty$ behaviour of $P_t(x|x_0)$ and discuss it.

a) Using orthonormalisation condition $\int \mathrm{d}x\, \phi_n(x)\phi_m(x) = \delta_{n,m}$, we obtain $c_n = \int \mathrm{d}x\, \psi(x;0)\,\phi_n(x)$. For $P_0(x|x_0) = \delta(x - x_0)$, we have $\psi(x;0) = P_0(x|x_0)/\phi_0(x) = \delta(x - x_0)/\phi_0(x_0)$ thus $c_n = \phi_n(x_0)/\phi_0(x_0)$.
b) Therefore $\psi(x;t) = \sum_n c_n\phi_n(x)\,\mathrm{e}^{-\lambda_n t}$
c) We get

$$P_t(x) = \psi(x;t)\phi_0(x) = \frac{\phi_0(x)}{\phi_0(x_0)} \sum_n \phi_n(x)\,\phi_n(x_0)\,\mathrm{e}^{-\lambda_n t} \equiv P_t(x|x_0) \tag{26}$$

d) $P_t(x|x_0) \simeq \phi_0(x)^2\mathrm{e}^{-\lambda_0 t} + \frac{\phi_0(x)}{\phi_0(x_0)}\phi_1(x)\phi_1(x_0)\mathrm{e}^{-\lambda_1 t} + \cdots$. The first term is the equilibrium distribution $P_{\mathrm{eq}}(x) = \phi_0(x)^2$. The correction decays as $\sim \mathrm{e}^{-\lambda_1 t}$ hence λ_1 is a relaxation rate toward equilibrium. Only the correction has the memory of the initial position x_0 :

$$P_t(x|x_0) \simeq P_{\mathrm{eq}}(x) + \frac{\phi_0(x)}{\phi_0(x_0)}\phi_1(x)\phi_1(x_0)\mathrm{e}^{-\lambda_1 t} + \cdots \tag{27}$$

Q18: **Diffusion in a ring.**— We consider the FPE $\partial_t P_t(x|x_0) = \mathscr{G}^\dagger P_t(x|x_0)$ in a ring of perimeter L (i.e. in the interval $[0, L]$ with periodic boundary conditions) for a uniform drift $F(x) = F_0$. The "forward generator" is $\mathscr{G}^\dagger = D\partial_x^2 - F_0\partial_x$.
a) Argue that the eigenfunctions of $\mathscr{G}^\dagger$ are plane waves $\Phi^{\mathrm{R}}(x) \propto \mathrm{e}^{\mathrm{i}kx}$. Argue that the boundary conditions lead to quantify $k \to k_n$ and give k_n.
b) Give the corresponding eigenvalue such that $\mathscr{G}^\dagger\Phi_n^{\mathrm{R}}(x) = -\lambda_n\,\Phi_n^{\mathrm{R}}(x)$. Give the generator $\mathscr{G}$ and deduce the left eigenfunction $\Phi_n^{\mathrm{L}}(x)$. Normalisation is chosen such that $\Phi_0^{\mathrm{R}}(x)$ is the stationary distribution and $\Phi_0^{\mathrm{L}}(x) = 1$. Check $\int_0^L \mathrm{d}x\, \Phi_n^{\mathrm{L}}(x)\Phi_m^{\mathrm{R}}(x) = \delta_{n,m}$.

c) Decompose $P_t(x|x_0)$ over the spectrum. Express it as a *real* series. Analyze the $t \to \infty$ behaviour (identify a characteristic time τ_D). Plot $P_t(x|x_0)$ for $t \gg \tau_D$ (the dominant term plus the first x-dependent correction).
d) Replace the sum by an integral in the $L \to \infty$ limit and deduce $P_t(x|x_0)$.

a) The eigenvectors of ∂_x are plane waves $\mathrm{e}^{\mathrm{i}kx}$. These are also the eigenvectors of the diffusion operator $\mathscr{G}^\dagger = D\partial_x^2 - F_0\partial_x$. For periodic boundary conditions, $\Phi^{\mathrm{R}}(x) = \Phi^{\mathrm{R}}(x+L)$, the wave vector is quantized as $k_n = 2\pi n/L$ with $n \in \mathbb{Z}$.
b) $\mathscr{G}^\dagger \Phi_n^{\mathrm{R}}(x) = -\lambda_n\, \Phi_n^{\mathrm{R}}(x)$ for

$$\lambda_n = Dk_n^2 + \mathrm{i}k_nF_0 \quad \text{for } n \in \mathbb{Z}\,. \tag{28}$$

Note that $k_n \to -k_n$ corresponds to $\lambda_n \to \lambda_n^* = \lambda_{-n}$ and $\Phi_n^{\mathrm{R}}(x) \to \Phi_n^{\mathrm{R}}(x)^*$.
The generator is $\mathscr{G} = D\partial_x^2 - F_0\partial_x$, hence, here it is related to $\mathscr{G}^\dagger$ by $F_0 \to -F_0$. Thus $\Phi_n^{\mathrm{L}}(x) \propto \Phi_n^{\mathrm{R}}(x)\big|_{k_n \to -k_n}$.
The choice of normalisation is

$$\Phi_n^{\mathrm{R}}(x) = \frac{1}{L}\mathrm{e}^{\mathrm{i}k_nx} \qquad \text{and} \qquad \Phi_n^{\mathrm{L}}(x) = \mathrm{e}^{-\mathrm{i}k_nx}\,. \tag{29}$$

We have indeed $\int_0^L \mathrm{d}x\, \Phi_n^{\mathrm{L}}(x)\Phi_m^{\mathrm{R}}(x) = \int_0^L \frac{\mathrm{d}x}{L}\, \mathrm{e}^{-\mathrm{i}k_nx+\mathrm{i}k_mx} = \delta_{n,m}$.
c) The conditional probability is

$$\begin{aligned} P_t(x|x_0) &= \sum_{n\in\mathbb{Z}} \Phi_n^{\mathrm{R}}(x)\Phi_n^{\mathrm{L}}(x_0)\,\mathrm{e}^{-\lambda_nt} = \frac{1}{L}\sum_{n\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi n(x-x_0-F_0t)/L - Dk_n^2t} \\ &= \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos(2\pi n(x-x_0-F_0t)/L)\,\mathrm{e}^{-n^2t/\tau_D} \quad \text{where } \tau_D = \frac{L^2}{(2\pi)^2D} \end{aligned} \tag{30}$$

The lowest e.v. $\lambda_0 = 0$ corresponds to the *stationary state* $P_{\mathrm{st}}(x) = \Phi_0^{\mathrm{R}}(x) = 1/L$. We identify a relaxation rate $\mathrm{Re}(\lambda_1) = 1/\tau_D$. The time $\tau_D \sim L^2/D$, known as the "*Thouless time*", is the typical time needed to explore the size of the system thanks to the diffusion.
At large time we have

$$P_t(x|x_0) \simeq \frac{1}{L} + \frac{2}{L}\cos(2\pi(x-x_0-F_0t)/L)\,\mathrm{e}^{-t/\tau_D} \tag{31}$$

which presents a (small) bump for $x \sim x_0 + F_0t$, the position of the drifted particle after time t.
d) in the $L \to \infty$ limit, we can simply replace the sum by an integral

$$P_t(x|x_0) \underset{L\to\infty}{\simeq} \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\mathrm{e}^{-(Dk^2+\mathrm{i}kF_0)t+\mathrm{i}k(x-x_0)} = \frac{1}{\sqrt{4\pi Dt}}\exp\left\{-\frac{(x-x_0-F_0t)^2}{4Dt}\right\}\,, \tag{32}$$

which describes the free diffusion on $\mathbb{R}$ with a drift.

Q19: **Boundary conditions.—** Consider the FPE $\partial_tP_t(x) = -\partial_x\big[F(x)P_t(x)\big] + D\partial_x^2\big[P_t(x)\big]$ on $\mathbb{R}_+$ for a mixed boundary condition at $x = 0$

$$\tilde{\lambda}\,P_t(0) = \partial_xP_t(x)\big|_{x=0} \qquad \forall\, t\,. \tag{33}$$

a) Give the expression of the probability current $J_t(x)$ related to $P_t(x)$
b) Show that $\partial_t\int_0^\infty \mathrm{d}x\,P_t(x)$ is expressed in term of the current. Show that for the mixed boundary condition, one finds

$$\partial_t\int_0^\infty \mathrm{d}x\,P_t(x) = -\lambda\,P_t(0) \tag{34}$$

and express λ. What is its meaning ?
c) Deduce the condition for a reflecting boundary, such that the probability is conserved. How should one choose the constant $\tilde{\lambda}$ in this case ?
d) What is an absorbing boundary ? What are λ and $\tilde{\lambda}$ in this case ?

$\to$ cf. lecture notes.

Q20: **Persistence of the free BM.**— We consider a free BM starting at $x(0) = x_0$.

a) solve the FPE $\partial_t P_t(x|x_0) = D\partial_x^2 P_t(x|x_0)$ for the conditional probability for a Dirichlet boundary condition at $x = 0$. What is the meaning of this boundary condition ?

b) Show that $S_{x_0}(t) = \int_0^\infty \mathrm{d}x\, P_t(x|x_0)$ can be expressed in terms of the error function $\mathrm{erf}(z) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{d}t\, \mathrm{e}^{-t^2}$. What does $S_{x_0}(t)$ represents ? We recall that $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z) \simeq \frac{1}{z\sqrt{\pi}}\mathrm{e}^{-z^2}$ for $z \to +\infty$. Plot neatly $S_{x_0}(t)$ as a function of t. Explain.

c) We denote by T_{x_0} the first passage time at $x = 0$. Give the relation between $S_{x_0}(t)$ and the distribution of the first passage time $\mathscr{P}_{x_0}(T)$. Give the expression of the distribution. Discuss the $T \to \infty$ behaviour.

$\to$ cf. lecture notes.

Q21: **First passage time.**— We consider the SDE $\mathrm{d}x(t) = F(x)\mathrm{d}t + \sqrt{2D}\,\mathrm{d}W(t)$. The corresponding FPE is $\partial_t P_t(x) = \mathscr{G}^\dagger P_t(x)$ where the "forward generator" is $\mathscr{G}_x^\dagger = -\partial_x F(x) + D\partial_x^2$

a) give the "generator" $\mathscr{G}_x$.

b) We consider the FPE for the conditional probability $P_t(x|x_0)$ with some reflection boundary condition at $x = a$, $\partial_{x_0} P_t(x|x_0)\big|_{x_0=a} = 0$ and some absorbing boundary condition at $x = b > a$, $P_t(x|x_0)\big|_{x_0=b} = 0$. Show that the survival probability $S_{x_0}(t) = \int_a^b \mathrm{d}x\, P_t(x|x_0)$ obeys an equation similar to the FPE. What is the initial condition $S_{x_0}(0)$?

c) Give the relation between the survival probability and the distribution of the first passage time $\mathscr{P}_{x_0}(T)$.

d) We recall that the moments $T_n(x_0) = \int_0^\infty \mathrm{d}T\, T^n\, \mathscr{P}_{x_0}(T)$ of the first passage time obey the recurrence

$$\mathscr{G}_{x_0}\, T_n(x_0) = -n\, T_{n-1}(x_0) \tag{35}$$

(with $T_0(x) = 1$). What are the boundary conditions at $x = a$ and $x = b$ for $T_n(x)$?

Show that $T_1'(x)$ obeys a first order differential equation and solve it (introduce $V(x) = -\int_0^x \mathrm{d}y\, F(y)$).

Impose the boundary condition for $T_1'(x)$ at $x = a$.

Deduce a formula for $T_1(x_0)$.

e) Consider the situation where $F(x) = -\mu$, when the reflection is at $x = 0$. Compute $T_1(x_0)$. Discuss the result : consider limiting cases (i) $\mu b/D \ll 1$, (ii) $\mu b/D \gg 1$ for $\mu > 0$, (iii) $|\mu| b/D \gg 1$ for $\mu < 0$.

a) to d) $\to$ cf. lectures.

$$T_1(x_0) = \frac{1}{D}\int_{x_0}^b \mathrm{d}x\, \mathrm{e}^{V(x)/D}\int_a^x \mathrm{d}x'\, \mathrm{e}^{-V(x')/D}\,. \tag{36}$$

e) integration is easy if $V(x) = \mu x$:

$$T_1(x_0) = \frac{D}{\mu^2}\left[\mathrm{e}^{\mu b/D} - \frac{\mu b}{D} - \mathrm{e}^{\mu x_0/D} + \frac{\mu x_0}{D}\right] \tag{37}$$

It vanishes at the absorbing boundary as it should.

(i) $\mu b/D \ll 1$: this is equivalent to send $\mu \to 0$. We find $T_1(x_0) \simeq \frac{b^2 - x_0^2}{2D}$. For $x_0 \sim 0$ we get the typical time b^2/D to diffuse over a region of size b.

(ii) $\mu b/D \gg 1$ for $\mu > 0$: we recover the Arrhenius behaviour due to the potential barrier $T_1(x_0) \simeq \frac{D}{\mu^2}\mathrm{e}^{\mu b/D} \sim \exp\left\{\frac{1}{D}[V(b) - V(0)]\right\}$

(iii) $|\mu| b/D \gg 1$ for $\mu < 0$: the time is dominated by the drift $T_1(x_0) \simeq (b - x_0)/\mu$.

Q22: We consider the free BM (the Wiener process) $x(\tau)$, defined for $0 \leqslant \tau \leqslant t$, issuing from $x(0)=0$. We denote by $\mathcal{A}[x(\tau)] = \int_0^t \mathrm{d}\tau\, x(\tau)$ the area between the Brownian curve and the real axis. We study its distribution $\mathcal{P}_t(A)$.

a) Write its characteristic function $\widetilde{\mathcal{P}}_t(p) = \left\langle \mathrm{e}^{-p\,\mathcal{A}[x(\tau)]} \,\middle|\, x(0)=0 \right\rangle$ in terms of a path integral.

b) Here, the path integral can be calculated easily, using that the integral is Gaussian : we have $\int_{x(0)=0}^{x(t)=x} \mathcal{D}x(\tau)\, \mathrm{e}^{-S[x(\tau)]} = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-S[x_{\mathrm{cl}}(\tau)]}$, where $x_{\mathrm{cl}}(\tau)$ is the solution of $\frac{\delta S}{\delta x(\tau)} = 0$ for $x_{\mathrm{cl}}(0)=0$ and $x_{\mathrm{cl}}(t)=x$. Find $x_{\mathrm{cl}}(\tau)$

c) Some calculation gives $S[x_{\mathrm{cl}}(\tau)] = -\frac{1}{6}p^2t^3 + \frac{1}{2t}(x+pt^2/2)^2$. Compute $\widetilde{\mathcal{P}}_t(p)$. Deduce $\langle \mathcal{A}[x] \rangle$ and $\left\langle \mathcal{A}[x]^2 \right\rangle$.

d) Give the distribution (it may be helpful to consider the Fourier transform $\widehat{\mathcal{P}}_t(k) = \left\langle \mathrm{e}^{-\mathrm{i}k\mathcal{A}[x]} \right\rangle = \widetilde{\mathcal{P}}_t(\mathrm{i}k)$).

a)

$$\widetilde{\mathcal{P}}_t(p) = \overbrace{\int \mathrm{d}x \int_{x(0)=0}^{x(t)=x}}^{\text{sum over paths}} \underbrace{\mathcal{D}x(\tau)\, \mathrm{e}^{-\frac{1}{2}\int_0^t \mathrm{d}\tau\, \dot{x}(\tau)^2}}_{\text{weight of a path}} \mathrm{e}^{-p\,\mathcal{A}[x(\tau)]}\,. \tag{38}$$

b) $S[x(\tau)] = \int_0^t \mathrm{d}\tau \left[\frac{1}{2}\,\dot{x}(\tau)^2 + p\,x(\tau)\right]$, thus $\frac{\delta S}{\delta x(\tau)} = -\ddot{x}(\tau)+p = 0$ has solution $x_{\mathrm{cl}}(\tau) = \frac{1}{2}p\tau^2 + v_0\tau + x_0$. Imposing the boundary conditions we get x_0 and v_0. The solution is $x_{\mathrm{cl}}(\tau) = \frac{1}{2}p\tau(\tau-t) + x\tau/t$.

c) It is easy (but a bit lengthy... not asked) to compute $S[x_{\mathrm{cl}}(\tau)] = \int_0^t \mathrm{d}\tau \left[\frac{1}{2}\,\dot{x}_{\mathrm{cl}}(\tau)^2 + p\,x_{\mathrm{cl}}(\tau)\right] = -\frac{1}{6}p^2t^3 + \frac{1}{2t}(x+pt^2/2)^2$. The path integral has a simple form, leading to

$$\widetilde{\mathcal{P}}_t(p) = \int \mathrm{d}x \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{\frac{1}{6}p^2t^3 - \frac{1}{2t}(x+pt^2/2)^2} = \mathrm{e}^{\frac{1}{6}p^2t^3} \tag{39}$$

The $p \to 0$ expansion of the characteristic function gives the moments $\widetilde{\mathcal{P}}_t(p) = 1 - p\,\langle \mathcal{A}[x] \rangle + \frac{p^2}{2}\left\langle \mathcal{A}[x]^2 \right\rangle + \cdots$, hence $\langle \mathcal{A}[x] \rangle = 0$ (obvious, by symmetry) and $\left\langle \mathcal{A}[x]^2 \right\rangle = \frac{1}{3}t^3$.

d) The distribution is obviously Gaussian. If you did not noticed this by inspection of$\widetilde{\mathcal{P}}_t(p)$, you can consider the Fourier transform

$$\widehat{\mathcal{P}}_t(k) = \widetilde{\mathcal{P}}_t(\mathrm{i}k) = \mathrm{e}^{-\frac{1}{6}k^2t^3} \tag{40}$$

which is a Gaussian with inverse Fourier transform

$$\mathcal{P}_t(A) = \sqrt{\frac{3}{2\pi t^3}}\, \mathrm{e}^{-\frac{3}{2t^3}A^2}\,. \tag{41}$$

In fact, the result could have been obtained by a simpler method : $x(\tau)$ is Gaussian thus $\mathcal{A}[x(\tau)] = \int_0^t \mathrm{d}\tau\, x(\tau)$ is a Gaussian random variable. Given the known correlation function $\langle x(\tau)x(\tau') \rangle = \min(\tau,\tau')$ we easily get $\left\langle \mathcal{A}[x]^2 \right\rangle = \frac{1}{3}t^3$.