#### Quantum teleportation is a universal computational primitive

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We present a method to create a variety of interesting gates by teleporting quantum bits through special entangled states. This allows, for instance, the construction of a quantum computer based on just single qubit operations, Bell measurements, and GHZ states. We also present straightforward constructions of a wide variety of fault-tolerant quantum gates.

Creating a quantum computer capable of realizing the theoretical promise of algorithms such as quantum factoring [1] and quantum search [2] will require both a design for a large system capable of very accurate controlled unitary evolution, and good fault-tolerant procedures to overcome inevitable residual imperfections in the physical realization of this system [3–5]. There are many suggested designs for quantum computers, but none are completely satisfactory, in the sense that none allows a large quantum computer to be built in the near future [6]; and some universal fault-tolerant protocols are known, but they can be quite complicated, and frequently require many operations to produce a specific desired transformation [7,8,3–5].

Here, we address aspects of both problems, and show how a single technique – a generalization of quantum teleportation [9] – reduces resource requirements for quantum computation and unifies known protocols for fault-tolerant quantum computation. We show, for instance, that a quantum computer can be constructed using just single quantum bit (qubit) operations, Bell-basis measurements, and Greenberger-Horne-Zeilinger states [10]. We also present straightforward constructions for a new, infinite class of fault-tolerant quantum gates. By making use of specific, pre-computable entangled states, these techniques vividly illustrate how entanglement can be a valuable resource for computation.

The heart of our discussion rests in the power of entangling measurements. Measurement, in its guise as an interface between the quantum and classical worlds, is generally considered to be an irreversible operation, destroying quantum information and replacing it with classical information. In certain carefully designed cases, however, this need not be true. For example, quantum teleportation [9] uses measurement to transfer quantum information from one place to another, and programmable quantum gates [11] can be used to probabilistically transform quantum information by an arbitrary quantum operation. Quantum error correction also allows a large set

of quantum operations, including measurement, to be reversed.

In all these applications, quantum information is preserved only in a subspace of the measured system. By selecting our initial state to lie in this preserved subspace, we can ensure, paradoxically, that the measurement tells us nothing about the quantum data. Still, the measurement can be very useful — once it has been done, the data is transformed in one of a variety of ways, indexed by the random measurement outcome. In the case of quantum teleportation or quantum error correction, this fact is used to restore the data to its initial state. Here, in contrast, we shall use quantum teleportation to transform data into a new state, corresponding to the action of some quantum gate which would otherwise be difficult or impossible to perform.

## I. UNITARY TRANSFORMS BY TELEPORTATION

We begin by showing how a controlled-NOT (CNOT) between two qubits can be deterministically accomplished using quantum teleportation. Recall how quantum teleportation works: a single qubit state  $|\alpha\rangle = a|0\rangle + b|1\rangle$ is prepared, along with an EPR state  $|\Psi\rangle = (|00\rangle +$  $|11\rangle/\sqrt{2}$ , then  $|\alpha\rangle$  and one qubit of  $|\Psi\rangle$  are measured together in the Bell basis  $|0x\rangle + (-1)^z |1\bar{x}\rangle$  (where x, z = $\{0,1\}$ , and  $\bar{x}=1-x$ ), giving a (uniformly distributed) random two-bit classical result which is xz [12]. The output qubit is then in the initial state  $|\alpha\rangle$ , but with an additional single-qubit Pauli operation X, Y, or Z [13] applied to it, with the random variable xz determining which Pauli operator it is (with 00 corresponding to the identity). We simply reverse the appropriate Pauli operator to reconstruct  $|\alpha\rangle$ , as shown in Fig. 1. Replication of this circuit allows teleportation of multiple qubits.

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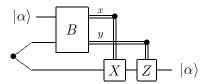


FIG. 1. Quantum circuit for teleportation. Time proceeds from left to right. < denotes the EPR state  $|\Psi\rangle$ , and the box B is a measurement in the Bell basis. The double wires carry classical bits, and the single wires, qubits.

The same basic idea can be used to teleport two qubits through a CNOT gate (a two-qubit gate which flips the "target" qubit whenever the "control" qubit is a  $|1\rangle$ ); that is, the reconstructed qubits are the original ones transformed by a CNOT gate operation. This is accomplished by the circuit shown in Fig. 2, where  $|\alpha\rangle = a|0\rangle + b|1\rangle$  and  $|\beta\rangle = c|0\rangle + d|1\rangle$  are two arbitrary single qubit states, and

$$|\chi\rangle = \frac{(|00\rangle + |11\rangle)|00\rangle + (|01\rangle + |10\rangle)|11\rangle}{\sqrt{2}}.$$
 (1)

The CNOT gate has  $|\beta\rangle$  as its control, and  $|\alpha\rangle$  as its target. This can be verified by direct computation, but it is easier to understand by realizing that  $|\chi\rangle$  can be created simply using two EPR pairs (Fig. 3). Combining this circuit with the previous one, we immediately note that the only differences with Fig. 1 are the CNOT gate appearing between the two EPR pairs, and the different classically controlled single qubit gates. For each EPR pair, the Bell basis measurement effectively introduces one of four random quantum operations (I, X, Y, Z) to the other half of the involved EPR pair, at a time which is before the CNOT gate [11].

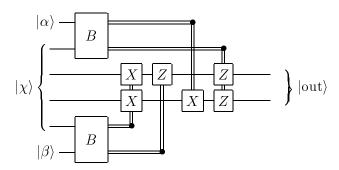


FIG. 2. Quantum circuit for teleporting two qubits through a controlled-NOT gate, giving  $|\text{out}\rangle = \text{CNOT} |\beta\rangle |\alpha\rangle$ .

However, it happens that single Pauli operations occurring before a CNOT gate are equivalent to (different) Pauli operations occurring after the CNOT gate [5]. For instance,  $\text{CNOT}(X \otimes I) = (X \otimes X) \text{CNOT}$ . This is equivalent to the statement that conjugation by CNOT preserves the Pauli group (comprised of tensor products of Pauli matrices, with overall sign  $\pm 1$ ). Thus, the quantum teleportation construction still works, but using different

controlled single-qubit operations to reconstruct the desired result.

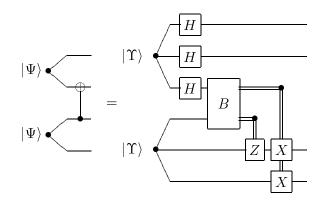


FIG. 3. Quantum circuit to create the  $|\chi\rangle$  state from two EPR pairs (left), or from two GHZ states  $|\Upsilon\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$  (right). H is the Hadamard gate.

This construction enables cnot gates to be performed between two qubits, using only classically controlled single qubit operations, prior entanglement, and Bell basis measurements. Moreover,  $|\chi\rangle$  can be created from two pairs of GHZ [10] states (Fig. 3).

### II. FAULT TOLERANT QUANTUM COMPUTATION

Fault tolerant gates come from noting that essentially the same construction works equally well for any gate U which preserves the Pauli group under conjugation; this set of gates, the Clifford group, plays an important role in the theory of quantum error-correcting codes and fault-tolerance [5,14]. To see how this is accomplished, consider an *n*-qubit state  $|\psi\rangle$ , in which each qubit is encoded using a stabilizer code, such as the 7-qubit CSS code [15,16]. 0 and 1 shall represent the corresponding encoded qubit states. Let  $|\Psi^n\rangle$  be the 2n-(encoded) qubit Bell state  $(|00\rangle + |11\rangle)^{\otimes n}$  (normalizations suppressed for clarity), rearranged so that the first n labels represent half of the EPR pairs (the upper qubits), and the last n the other half (the lower qubits). In other words,  $(I \otimes U)|\Psi^n\rangle$  (where I is the identity on n qubits) is U acting on the lower qubits of all the EPR pairs.

The goal of fault-tolerant computation is to perform gates on the logical qubits while restricting the propagation of errors among the physical qubits, which can compromise the code's ability to correct errors. The usual method for doing this is to only perform transversal gates on the code — that is, gates which interact qubits in one code block only with corresponding qubits in other code blocks. While errors may then propagate between blocks, they cannot propagate within blocks, so a single faulty gate can only cause a single error in any given block of the code.

Operators from the Pauli group (such as X, Y, and Z) can easily be performed on logical qubits which are encoded with a stabilizer code [17]. Let  $C_1$  represent the Pauli group.  $C_2$ , the Clifford group, will be the set of gates which map Pauli operators into Pauli operators under conjugation. Through an appropriate sequence of gates and measurements, any  $C_2$  operation can also be performed on any stabilizer code [17].

More difficult to perform are gates in the class defined as

$$C_3 \equiv \{ U \mid UC_1U^{\dagger} \subseteq C_2 \} \,. \tag{2}$$

 $C_3$  contains gates such as the Toffoli gate (controlled-controlled-NOT), the  $\pi/8$  gate (rotation about the Z-axis by an angle  $\pi/4$ ), and the controlled-phase gate (diag(1,1,1,i)). For instance, the  $\pi/8$  gate transforms  $X \to PX$  and  $Y \to -iPY$  (Z commutes with the gate and is thus left unchanged), where P is the  $\pi/4$  gate (diag(1,i)). Fault-tolerant constructions of these gates are known [7,8,18], but they are  $ad\ hoc$  and do not generalize easily.

However, our teleportation construction provides a straightforward way to produce any gate in  $C_3$ , as shown in Fig. 4. For  $U \in C_3$ , first construct the state

$$|\Psi_{II}^n\rangle = (I \otimes U)|\Psi^n\rangle. \tag{3}$$

Next, take the input state  $|\psi\rangle$  and do Bell basis measurements on this and the n upper qubits of  $|\Psi_U^n\rangle$ , leaving us with n qubits in the state

$$|\psi_{out}\rangle = UR_{xz}|\psi\rangle = R'_{xz}U|\psi\rangle.$$
 (4)

where  $R_{xz}$  is an operator in  $C_1$  which depends on the (random) Bell basis measurement outcomes xz, and  $R'_{xz}$  is an operator in  $C_2$ : the image of  $R_{xz}$  under conjugation by U. Since  $R'_{xz}$  is in the Clifford group, it can be performed fault-tolerantly. As long as  $|\Psi^n_U\rangle$  can be prepared fault-tolerantly, this construction allows U to be performed fault-tolerantly.

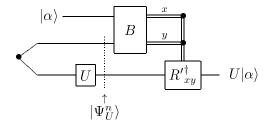


FIG. 4. Quantum circuit to perform U fault tolerantly using quantum teleportation. In general, this works for any  $U \in C_k$ , since  $R'_{xz} \in C_{k-1}$  by definition of  $C_k$ .

Of course,  $|\Psi_U^n\rangle$  must be prepared fault-tolerantly. To do this, note that the state  $|\Psi^n\rangle$  is the +1 eigenvector of the 2n operators  $X_i \otimes X_i$  and  $Z_i \otimes Z_i$  (where  $X_i$  and  $Z_i$  are

X and Z, respectively, acting on the  $i^{th}$  upper or lower qubit). Therefore,  $|\Psi^n_U\rangle$  is the +1 eigenvector of the operators  $M_i = X_i \otimes U X_i U^\dagger$  and  $N_i = Z_i \otimes U Z_i U^\dagger$ . Furthermore, the eigenvalues of these 2n operators completely determine the state, so if all these operators have eigenvalue +1, the state actually is the desired one. Therefore, to produce  $|\Psi^n_U\rangle$ , prepare n EPR pairs, which can easily be done fault-tolerantly by measuring  $X_i \otimes X_i$  and  $Z_i \otimes Z_i$  or with fault-tolerant Hadamard and CNOT gates, measure the operators  $M_i$  and  $N_i$ , and perform an appropriate Pauli operation  $Z_i \otimes I$  or  $X_i \otimes I$  as necessary to move into the +1 eigenspace of all the  $M_i$ s and  $N_i$ s.

The hard part of the preparation is measuring  $M_i$  and  $N_i$  fault-tolerantly. Since this construction is quite complicated in the case where  $M_i$  and  $N_i$  do not have transversal constructions, we defer the discussion of this point to the Appendix. Note, however, that the only point where this complex construction is necessary is in the preparation of the ancilla states  $\Psi_U$  used in the teleportation.

#### III. CONCLUSION

Our construction of quantum gates using teleportation offers tantalizing possibilities for relaxing experimental constraints on realizing quantum computers. For example, using single photons as qubits and current optical technology, one can perform nearly perfect Bell basis measurements [20], quantum teleportation [21], almost create GHZ states [22], and certainly perform single qubit operations [19]. Thus, given GHZ states, quantum computers might be constructed nearly completely from linear optical components. Similar implications can be drawn for other physical systems, particularly if entangled states can readily be prepared and stored.

The construction of a fault tolerant Toffoli gate using teleportation is a dramatic simplification of previous constructions, and generalizes through a recursive application of the construction to provide an infinite family of gates,  $C_k \equiv \{U|UC_1U^{\dagger} \subseteq C_{k-1}\}$ , all of which can be performed fault tolerantly. While the precise set of gates which form  $C_k$  is still under investigation, it is known that every  $C_k$  contains interesting gates, such as the  $\pi/2^k$  rotations, which appear in Shor's factoring algorithm [1]. The states  $|\Psi_U^n\rangle$  needed for a gate in  $C_k$ are exponentially difficult (in k) to construct, but they may be prepared offline, since  $|\Psi_U^n\rangle$  is independent of the data being acted upon. Thus,  $|\Psi_{II}^n\rangle$  are valuable generic quantum resources which might be considered a manufacturable commodity for quantum commerce! Even if  $|\Psi_{II}^n\rangle$  are not available, the construction presents a great conceptual simplification, and for small k, it can greatly reduce the number of operations needed to assemble the precise gates called for in an algorithm, clearly of benefit in efficiently performing quantum computation with

realistically imperfect gates.

# APPENDIX A: FAULT-TOLERANT PREPARATION OF $|\Psi^N_U\rangle$

A crucial step in the fault-tolerant preparation of the ancilla state  $|\Psi^n_U\rangle$  is the measurement of an operator M acting on the logical qubits in a code. The procedure for doing such measurements has been described and is straightforward; however, its adaptation to the present goal has not been clearly documented in the literature, and there are a number of potential pitfalls which we believe are useful to know about.

The basic problem can be illustrated by considering the standard non-fault-tolerant measurement method shown in Fig. 5a. We prepare a control qubit is in the state  $|0\rangle + |1\rangle$ , and perform a controlled-M gate to the block of the code containing the logical qubits we wish to measure. The M in this case must be an encoded version of M, so it acts on the data, and not the physical qubits making up the code. Then we Hadamard transform the control qubit and measure it. It is sufficient for our purposes to restrict attention to operators with eigenvalues  $\pm 1$ ; thus, the data will be collapsed on a +1 or -1 eigenstate of M when the control qubit reads 0 or 1, respectively.

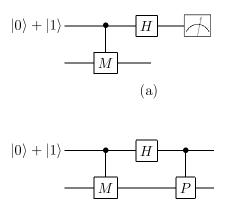


FIG. 5. a) A non-fault-tolerant procedure to measure M with eigenvalues  $\pm 1$ , b) A coherent version of this procedure.

(b)

This procedure fails to be fault-tolerant in a variety of ways. The control qubit is a single qubit; an error in it before or during the operation could be propagated to every qubit in the code block. An error on the control qubit after the measurement will tell us the wrong value of the measurement, which may cause us to act improperly later on. Furthermore, in many cases of interest to us here, the encoded version of M is itself difficult to perform, requiring a number of transversal operations and some measurements. These measurements, in turn, should be done fault-tolerantly, but since the control qubits used for those measurements are entangled with the control

qubits one level up, they cannot simply be projectively measured.

These problems can be solved using three basic ideas. First, a coherent measurement procedure can be adopted, utilizing measurement results to immediately disentangle all involved ancilla qubits. Second, by using multiple qubits prepared in a "cat" state  $|00\cdots0\rangle + |11\cdots1\rangle$  of n qubits (where each block of the code also contains n qubits), propagation of errors can be limited. Finally, since the state we are preparing,  $|\Psi_U^n\rangle$ , is known beforehand (as opposed to computing with variable data), performing the encoded version of M can be done through recursive application of the basic measurement procedure. These three steps are described in detail below.

Coherent measurement is possible in all cases of interest in this paper. In our application, the measurement, if it produces the -1 eigenstate, is followed by a (classically controlled) operation P (frequently a Pauli operation) which moves the data from a -1 eigenstate of M to a +1 eigenstate of M. Equivalently, we may instead follow the Hadamard transform on the control qubit by a controlled-P gate, as shown in Fig. 5b. Doing this leaves the control qubit disentangled with the data (which is always in the same state, a +1 eigenstate of M). Of course, since this process still depends on a single control qubit, it is not fault tolerant either, so a modification of the control scheme is needed, using cat states.

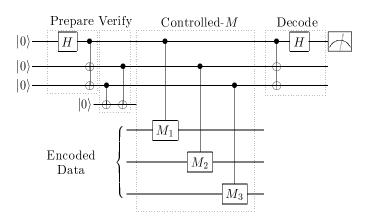


FIG. 6. Fault-tolerant measurement of a gate M with a transversal implementation.

Cat-state control is the method utilized in [7] and [23] to provide fault-tolerant measurement of Pauli operators (elements of  $C_1$ ). As shown in Fig. 6, the single control qubit is replaced with a cat state  $|00\cdots0\rangle + |11\cdots1\rangle$  of n qubits. Then given a transversal implementation of M (which, for a stabilizer code, is always available for Pauli operators), we can easily implement the controlled-M part of the measurement: the gate of M which acts on the  $k^{th}$  physical qubit of the block becomes a controlled gate, conditioned on the  $k^{th}$  qubit of the cat state. Since in the absence of errors, every qubit in the cat state is either 0 or 1, we either perform M completely or not at all.

If a single qubit of the cat state is wrong, the error can only propagate to the corresponding qubit of the code. The preparation of the cat state (which involves CNOTs between the qubits of the state) might have resulted in multiple errors, so before interacting it with the data, we should verify the cat state by comparing pairs of qubits — all should be the same.

Afterwards, we decode the cat state with a series of CNOT gates and a Hadamard; the resulting bit is again 0 or 1 depending on the eigenvalue of the state of the data. The value of this result does still depend on the bottleneck of the single qubit produced by decoding the cat state. In fact, even a single phase error on one qubit of the cat state at any time will give us the wrong value for the decoded cat state. Therefore, in order to gain sufficient confidence in the result, we repeat the procedure a number of times, and only act on the majority result. Also note that a single error in the data might cause a wrong measurement result, so we should perform error correction between measurement trials.

The case of interest here is when M is generally some element of  $C_k$  (not just  $C_1$ ), in which case recursive application of the above procedures is necessary, since implementation of M will generally consist of a series of transversal gates and measurements. For instance, to prepare the ancilla needed to teleport a  $C_3$  gate U, we must measure gates of the form  $M = P \otimes UPU^{\dagger}$ , where P is some Pauli operator. By the definition of  $C_3$ , M is in the Clifford group, and for a general stabilizer code, there will be no simple transversal implementation of M. The transversal gates in the implementation of M present no particular problem — we can condition them on the cat state just as in Fig. 6. The measurements present the difficulty. Each will require its own cat state (a sequence of them, in fact, since the measurement requires a number of trials), and we must be certain the "inner" measurement (of some operator N) does not destroy the superposition of the cat state for the "outer" measurement of M.

We proceed as follows: For each trial for the inner measurement, prepare and verify a single cat state of nqubits. Using this cat state, perform a controlled-N operation in the usual way, as per Fig. 6 (assume for the moment we have a transversal implementation of N). However, the gates making up this operation are themselves controlled by qubits from the outer cat state. For instance, if N requires NOT gates on each qubit in the code, we perform controlled-controlled-NOT gates instead, using the kth qubits of the inner and outer cat states as controls for the kth qubit in the block. Then we decode the cat state normally. If there have been no errors, the result is a single qubit in the state  $|0\rangle$  for the 0 part of the outer cat state. For the 1 part of outer cat state, the qubit is entangled with the data:  $\alpha |0\rangle |\phi_0\rangle + \beta |1\rangle |\phi_1\rangle$ (where  $|\phi_0\rangle$  and  $|\phi_1\rangle$  are eigenstates with eigenvalues  $\pm 1$ , and the data begins in the state  $\alpha |\phi_0\rangle + \beta |\phi_1\rangle$ ). Since the "data" here is actually an ancilla we are preparing, we know the values of  $\alpha$  and  $\beta$ . This fact will be important later.

We repeat the above cat state preparation, controlled-N, and decoding for each of the n cat states in a trial. Still assuming no errors, the overall state of the system at this point is

$$|00\cdots0\rangle_{\rm oc}|0\rangle_{\rm ic}|\phi\rangle_{\rm data} + (A1)$$

$$|11\cdots1\rangle_{\rm oc}\left(\alpha|0\rangle_{\rm ic}|\phi_0\rangle_{\rm data} + \beta|1\rangle_{\rm ic}|\phi_1\rangle_{\rm data}\right).$$

The subscript "oc" indicates the outer cat states, the subscript "ic" indicates the qubit produced after decoding the inner cat state for a single measurement trial, and  $|\phi\rangle = \alpha |\phi_0\rangle + \beta |\phi_1\rangle$  is the state of the data before the measurement.

An error anywhere in this procedure could potentially give us the wrong value for the decoded inner cat state, which is why we need more measurement trials. We should check, however, that a single error in the procedure will only cause a single error in the data. This is true, in fact: an error in a single qubit of the inner or outer cat states can only propagate to the corresponding qubit in the data block.

To prevent a repetition of any problem that may have occurred in the first trial, we perform an error correction operation on the code block, and reverify the outer cat state, correcting any mistakes we see in that state. Then we go through another complete trial. We continue alternating measurement trials with error correction/verification steps for a total of r trials (for some rlarge enough to give us confidence in the result). Assuming no errors, the result will look like Eq. (A1), except there will now be a total of r inner qubits, which in the absence of errors would all be the same. For each of the n qubits in the data block, we take the majority value of the r inner qubits and store the result in a new ancilla qubit. Then we perform the controlled-P "correction" step (as per Fig. 5b) based on the majority value for just the single qubit at that coordinate. Therefore, a single error in the majority calculation will only affect a single qubit in the data block.

The final step in the recursive construction is to appropriately disentangle the inner and outer qubits. Assuming no errors, when the outer cat state is 0, the inner qubits are all in the state  $|0\rangle$  as well, and the data block is in the state  $|\phi\rangle$ . When the outer cat state is 1, the data block is in the state  $|\phi_0\rangle$  (as desired), but the r+n inner qubits are in a superposition  $\alpha|00\cdots0\rangle+\beta|11\cdots1\rangle$ , so the inner qubits are still entangled with the outer cat state. Therefore, we perform sufficient CNOT operations among the inner qubits to leave one in the state  $\alpha|0\rangle+\beta|1\rangle$ , and the others all as  $|0\rangle$ . Now we use our knowledge of  $\alpha$  and  $\beta$  to rotate the single remaining inner qubit back to  $|0\rangle$ , conditioned on (any) single qubit from the outer cat state. This completely disentangles

the inner qubits from the outer cat state and the data, leaving us with the state

$$|00\cdots 0\rangle_{\rm oc}|\phi\rangle_{\rm data} + |11\cdots 1\rangle_{\rm oc}|\phi_0\rangle_{\rm data},$$
 (A2)

as desired.

As we noted before, a single error during any trial propagates to at most one qubit in the data block. A single qubit error in the outer cat state or the data block will ruin an inner measurement trial, but will not survive the subsequent verification and error correction step, so it only ruins the one set. Since we perform r inner measurement trials, a total of r/2 such errors will be required to ruin every majority calculation. Therefore, for large enough r, this will be of the same order of magnitude as other failure modes (such as having many errors in the data block itself). An error in a single majority calculation will only produce a single error in the data block. There are a number of places, however, where a single error can cause the disentanglement of the inner qubits to fail. This will effectively collapse the superposition of 0 and 1 in the outer cat state. This is annoying, but not fatal; a single qubit error directly in the outer cat state can produce the same result, which is one reason we require a number of trials for any measurement.

We have demonstrated a procedure which performs a inner measurement conditioned on an outer cat state. By stringing these together with transversal operations, we can measure any operator M in the Clifford group. This allows us to create ancillas to teleport any  $C_3$  gate. For  $C_4$  and higher gates, we will need similar, but more complicated procedures. We will need to measure  $C_3$  gates; this requires the production of an ancilla for the  $C_3$  gate, which in turn requires measurement of a  $C_2$  (Clifford group) gate, which may require measurement of Pauli group operators. Therefore, we may require 3 levels of cat states at any given time, but by simply nesting the above procedure, we can also produce the ancillas needed for  $C_4$  gates. Further nesting will allow us to build gates from  $C_5$  and higher.

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