Université Paris Cité

Academic year 2023-2024

M2 PCS — Statistical field theory and soft matter
TD n°1 : Basic SFT tools

1 Einstein summation

1. Write the following quantities using Einstein's convention:

$$\boldsymbol{x} \cdot \boldsymbol{y}, \tag{1}$$
$$\left(\boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{c})\right)_i, \tag{2}$$
$$\mathsf{A} = \mathsf{BC}, \quad \text{(matrices)} \tag{3}$$
$$\operatorname{tr}(\mathsf{AB}). \tag{4}$$

2. In a space of dimension d, calculate $\delta_{ij}\delta_{ij}$, where δ_{ij} is the Kronecker (identity) tensor.

3. With $\partial_i = \partial/\partial x_i$, show using Einstein's summation that

$$\boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} f) = (\boldsymbol{\nabla}\alpha) \cdot (\boldsymbol{\nabla} f) + \alpha \boldsymbol{\nabla}^2 f \tag{5}$$

4. In $d = 3$, let ϵ_{ijk} be the Levi-Civita symbol. It is equal to 1 if $\{i, j, k\}$ is an even permutation of $\{1, 2, 3\}$, to -1 if it is an odd permutation, and to zero otherwise. Show (efficiently) that

$$\epsilon_{ijk}\,\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}. \tag{6}$$

5. Write $\boldsymbol{\nabla} \times \boldsymbol{v}$ using Einstein's summation, the Levi-Civita symbol and the basis vectors $\boldsymbol{e}_i$.

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2 Functional derivatives

1. Let $x_0 \in \mathbb{R}$ be a fixed point. Calculate the functional derivative $\delta f/\delta h(x)$ of

$$f[h] = h(x_0). \tag{7}$$

2. Calculate the functional derivative $\delta f/\delta h(x)$ of

$$f[h] = h'(y), \quad \forall y. \tag{8}$$

3. Calculate the functional derivative $\delta f/\delta h(x)$ of

$$f[h] = \int dx \, \frac{r}{2} \, h^2(x), \tag{9}$$

using two different methods: (i) direct computation of δf_1, (ii) functional differentiation under the integral.

4. Consider the following functional $f[h]$ of a field $h(\boldsymbol{x})$ in a d-dimensional space:

$$f[h] = \int d^d x \, \frac{1}{2} a \, (\boldsymbol{\nabla} h)^2 \,. \tag{10}$$

Calculate $\delta f/\delta h(\boldsymbol{x})$, where $\boldsymbol{x}$ is a point in the bulk. By definition, the functional derivative in the bulk is calculated using functions δh that vanish on the boundary.

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3 Average field and correlation function

Upon adding a formal external field to the effective Hamiltonian (it can be set to zero at the end of the calculations), the partition function and free-energy become

$$Z[h] = \int \mathcal{D}[\phi] \, e^{-\beta[\mathcal{H}[\phi] - \int d^d x \, h(\boldsymbol{x})\phi(\boldsymbol{x})]}, \qquad F[h] = -k_{\mathrm{B}} T \ln Z[h]. \tag{11}$$

We recall that the average of a quantity Q is given by

$$\langle Q \rangle = \frac{1}{Z[h]} \int \mathcal{D}[\phi] \, Q \, e^{-\beta \left\{ \mathcal{H}[\phi] - \int d^d y \, h(\boldsymbol{x})\phi(\boldsymbol{x}) \right\}}. \tag{12}$$

1. Calculate the functional derivative $\delta Z/\delta h(\boldsymbol{x})$.

2. Show that

$$\langle \phi(\boldsymbol{x}) \rangle = - \left. \frac{\delta F}{\delta h(\boldsymbol{x})} \right|_{h=0} . \tag{13}$$

3. Show that

$$C(\boldsymbol{x}, \boldsymbol{y}) = -k_{\mathrm{B}} T \left. \frac{\delta^2 F}{\delta h(\boldsymbol{x}) \delta h(\boldsymbol{y})} \right|_{h=0} , \tag{14}$$

where $C(\boldsymbol{x}, \boldsymbol{y}) = \langle \phi(\boldsymbol{x}) \phi(\boldsymbol{y}) \rangle - \langle \phi(\boldsymbol{x}) \rangle \langle \phi(\boldsymbol{y}) \rangle$ is the correlation function.

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4 Gaussian model

We consider, for a scalar field $\phi(\boldsymbol{x})$, with $\boldsymbol{x} \in \mathbb{R}^d$, the following Gaussian model:

$$\mathcal{H} = \int d^d x \left[\frac{r}{2}\phi^2 + \frac{c}{2}(\boldsymbol{\nabla}\phi)^2\right]. \tag{15}$$

In all the problem, we will neglect boundary terms whenever they appear.

1. Show that the Hamiltonian can be rewritten as
$$\mathcal{H} = \frac{1}{2}\int d^d x\, \phi(\boldsymbol{x})\mathcal{L}\phi(\boldsymbol{x}), \qquad \text{with } \mathcal{L} = r - c\boldsymbol{\nabla}^2. \tag{16}$$
2. Show that $\mathcal{L}$ is Hermitian.
3. Show that $\mathcal{H}$ can also be rewritten as
$$\mathcal{H} = \frac{1}{2}\int d^d x\, d^d y\, \phi(\boldsymbol{x})H(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{y}), \qquad \text{with } H(\boldsymbol{x},\boldsymbol{y}) = \mathcal{L}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{17}$$
4. Show that the inverse kernel $H^{-1}(\boldsymbol{x},\boldsymbol{y})$ satisfies the equation
$$\mathcal{L}_{\boldsymbol{y}} H^{-1}(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}), \tag{18}$$
where the subscript in $\mathcal{L}_{\boldsymbol{y}}$ specifies that $\mathcal{L}$ is taken for $\boldsymbol{\nabla} = \partial/\partial\boldsymbol{y}$.
5. Let $G(\boldsymbol{x})$ be the Green function of $\mathcal{L}$. Show that $H^{-1}(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{x}-\boldsymbol{y})$.
6. Show that
$$G(\boldsymbol{x}) = \int \frac{d^d q}{(2\pi)^d}\,\frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{r + cq^2}. \tag{19}$$
7. Show that $G(\boldsymbol{x}) = G(|\boldsymbol{x}|)$ by working on $G(R_{ij}x_j)$ where $\mathsf{R} \in O(d)$ is any rotation.
8. Calculate the partition function and the free energy functional of an external field:
$$Z[h] = \int \mathcal{D}[\phi]\, e^{-\beta[\mathcal{H}[\phi] - \int d^d x\, h(\boldsymbol{x})\phi(\boldsymbol{x})]} \tag{20}$$
$$F[h] = -k_{\mathrm{B}}T \ln Z[h]. \tag{21}$$
9. Deduce from the relations
$$\langle\phi(\boldsymbol{x})\rangle = -\left.\frac{\delta F}{\delta h(\boldsymbol{x})}\right|_{h=0}. \tag{22}$$
$$C(\boldsymbol{x},\boldsymbol{y}) = -k_{\mathrm{B}}T \left.\frac{\delta^2 F}{\delta h(\boldsymbol{x})\delta h(\boldsymbol{y})}\right|_{h=0}, \tag{23}$$
that the average field and the correlation function of the Gaussian model are given by
$$\langle\phi(\boldsymbol{x})\rangle = 0, \tag{24}$$
$$C(\boldsymbol{x},\boldsymbol{y}) = k_{\mathrm{B}}T \int \frac{d^d q}{(2\pi)^d}\,\frac{e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{r + cq^2}. \tag{25}$$

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M2 PCS — Statistical field theory and soft matter
TD n°2 : Liquid crystals

1 Diagrammatic way for the nematic–isotropic Hamiltonian

The quadratic term $T_2 = a_{ijk\ell}\, q_{ij}q_{k\ell}$ in the expansion of the effective Hamiltonian in power series of the field q_{ij} is obtained by searching for the most general isotropic rank 4 tensor $a_{ijk\ell}$. Since such a tensor must be constructed only with Kronecker deltas, it is convenient to use a diagrammatic method where the sum over repeated indices is represented by a curved line joining the positions of these indices, a "contraction".

1. Find how this method works, and show that $T_2 = a\, q_{ij}q_{ij}$ is obtained from the diagram $q\,q$ and this one only.
2. Use the diagrammatic method to construct $T_3 = c\, q_{ij}q_{jk}q_{ki}$.
3. Construct likewise the two independent quartic terms $T_{41} = d_1(q_{ij}q_{ij})^2$ and $T_{42} = d_2\, q_{ij}q_{jk}q_{k\ell}q_{\ell i}$.
4. Use the diagrammatic method to show that there are three independent terms that are quadratic in the gradient $q_{ij,k}$ of the nematic field.
5. Show that they reduce to only two bulk terms $T'_{21} = L_1\, q_{ij,k}q_{ij,k}$ and $T'_{22} = L_2\, q_{ij,k}q_{kj,i}$.

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2 Fluctuations in a weakly first-order nematic

Close to a weakly first-order nematic–isotropic transition, the director exhibits large fluctuations that are clearly visible under the microscope using crossed polarizers. Neglecting the biaxial character of the nematic fluctuation[1], we assume

$$\boldsymbol{q}(\boldsymbol{r}) = s(\boldsymbol{r}) \left(\boldsymbol{\nu}\,(\boldsymbol{r}) \otimes \boldsymbol{\nu}\,(\boldsymbol{r}) - \frac{1}{3}\boldsymbol{I} \right) \qquad (\boldsymbol{\nu}^2 = 1), \tag{1}$$

[1]This is reasonable when the coarse-graining length is much larger than molecular dimensions since the phase is uniaxial

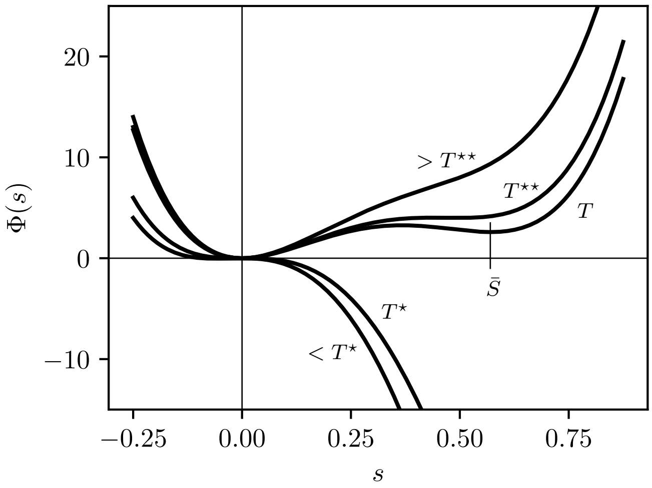

Figure 1: Hamiltonian density (4).

with, perturbatively,

$$s(\boldsymbol{r}) = \bar{S} + \sigma(\boldsymbol{r}), \tag{2}$$

$$\boldsymbol{\nu}\,(\boldsymbol{r}) = \left(\alpha, \beta, \sqrt{1-\alpha^2-\beta^2}\right)^t, \tag{3}$$

where σ, α and β are small quantities, of order ϵ. In this problem, we shall systematically neglect terms that are of order higher than ϵ^2. The expansion in (2) is made around the value $\bar{S}(T)$ that corresponds to the local nematic minimum of the Hamiltonian density $\Phi(s)$ (see Fig. 1). The full nematic Hamiltonian density has the form

$$f(\boldsymbol{q}) = \Phi(s) + \frac{1}{2}L_1 q_{ij,k}q_{ij,k}, \qquad \Phi(s) = A(T-T^\star)s^2 - Cs^3 + Ds^4, \tag{4}$$

where for the sake of simplicity we have neglected the second gradient term.

1. Assuming that σ, α and β are independent Gaussian random variables, calculate the actual scalar order parameter S and the director $\boldsymbol{n}$ of this nematic phase.

2. Show that the gradient term $T_1 = \frac{1}{2}L_1 q_{ij,k}q_{ij,k}$ takes the form

$$T_1 = \frac{1}{3}L_1\left(\nabla\sigma\right)^2 + L_1 S^2\left(\nabla\alpha\right)^2 + L_1 S^2\left(\nabla\beta\right)^2 + O(\epsilon^3) \tag{5}$$

3. Because $s = \bar{S}$ is a minimum of $\Phi(s)$, we have $\Phi(s) = \Phi_0 + \frac{1}{2}\lambda(T)\sigma^2 + O(\epsilon^3)$. Show that σ, ν_1 and ν_2 are independent random Gaussian variables

4. A simple analysis of the polynomial $\Phi(s)$ gives $\lambda(T) \sim (T^{\star\star} - T)^{1/2}$. Using your knowledge of the Gaussian model, show that the correlation length of the scalar order-parameter diverges as $(T^{\star\star} - T)^{-\nu}$ with $\nu = 1/4$.

5. Likewise, show (in 3D) that the angular fluctuations of the director are given by

$$\langle\theta^2\rangle = \frac{k_{\mathrm{B}}T\Lambda}{(2\pi)^2 L_1 S^2}, \tag{6}$$

where θ is α or β, and Λ is the upper wavevector cutoff.

6. What happens near a weakly first-order nematic–isotropic transition?

7. What happens in two dimensions? Do you recognize a theorem?

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3 Frank elasticity of the nematic phase

Deep in the nematic phase, for $T \ll T_{NI}$, the scalar-order parameter S can be considered constant. However, the director field $\boldsymbol{n}(\boldsymbol{r})$ can easily be distorted on large scales by external boundary forces[2]. This is a form of elasticity, since the nematic will relax to a uniform state if the external constraints are relaxed. F. C. Frank showed that the elastic free energy deformation has the form

$$F[\boldsymbol{n}] = \int d^3r \left[\frac{1}{2}K_1\,(\mathrm{div}\,\boldsymbol{n})^2 + \frac{1}{2}K_2\,(\boldsymbol{n}\cdot\mathrm{rot}\,\boldsymbol{n})^2 + \frac{1}{2}K_3\,(\boldsymbol{n}\times\mathrm{rot}\,\boldsymbol{n})^2\right], \tag{7}$$

where the K_i's are three independent elastic constants. The aim of this exercice is to derive this form of the free energy density f above.

1. Justify in a simple way that the three terms in the Frank elasticity correspond to deformations of the type "splay", "twist" and "bend", respectively.

For small distorsions, f can be expanded in power series of the gradient $n_{i,j} \equiv \partial n_i/\partial r_j$ of the director. Each term must be scalar (R1) and invariant under the symmetries of a uniform nematic (R2). Also, since in the ground state a nematic is uniform, the lowest-order terms must be quadratic:

$$f = A_{ijk\ell}\, n_{i,j}\, n_{k,\ell}. \tag{8}$$

The tensor $\boldsymbol{A}$ must thus have the symmetry of a uniform nematic phase (R2). Therefore it must be a linear combination of terms constructed with δ_{ij} and $n_i n_j$ only (a basis of the tensors with $D_{\infty h}$ symmetry). Thus there will be 0, 2 or 4 occurrences of n_i in $A_{ijk\ell}$.

2. Using the diagrammatic method (Kronecker contractions), show that the terms involving 0 occurrences of n_i yield only two independent bulk terms

$$f_0 = \frac{1}{2}K\, n_{i,i}\, n_{j,j} + \frac{1}{2}K'\, n_{i,j}\, n_{i,j} \tag{9}$$

[2] This is because $\boldsymbol{n}(\boldsymbol{r})$ is a "massless" field with Goldstone modes, contrary to $S(\boldsymbol{r})$, as can be seen in the previous exercice.

3. Using the diagrammatic method introduced in the first exercice, show that the terms involving 2 occurrences of n_i yields only

$$f_2 = \frac{1}{2} K'' n_j \, n_{i,j} \, n_\ell \, n_{i,\ell}. \tag{10}$$

4. Show that the contributions to $A_{ijk\ell}$ involving 4 occurrences of n_i vanish.

5. Using the relation $\epsilon_{ijk}\,\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$ calculate $(\mathrm{rot}\,\boldsymbol{n} \times \boldsymbol{n})_i$ and write f_2 using rot $\boldsymbol{n}$.

6. Show that $a^2 = (\boldsymbol{a} \cdot \boldsymbol{n})^2 + (\boldsymbol{a} \times \boldsymbol{n})^2$ for any vector $\boldsymbol{a}$.

7. Using the relation $\epsilon_{ijk}\,\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$ show that

$$(\mathrm{rot}\,\boldsymbol{n})^2 = n_{k,j} \, n_{k,j} - n_{k,j} \, n_{j,k}. \tag{11}$$

8. Deduce that the Frank form holds, with $K_1 = K + K'$, $K_2 = K'$ and $K_3 = K + K''$.

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4 Weak anchoring

A nematic liquid crystal is filled between two glass plates (Fig. 2). The upper surface is treated so as to set a "strong anchoring" of the director in the direction perpendicular to the surface. The lower surface favors the director parallel to the surface, by means of a "weak anchoring". We wish to study the elastic deformation of the nematic in the cell as a function of its thickness H. The director $\boldsymbol{n}(z)$ is assumed to lie in the (x, z) plane, such that $n_x = \cos\theta(z)$ and $n_z = \sin\theta(z)$. For the bulk elasticity, we take the Frank elasticity with $K_1 = K_2 = K_3 = K$. The upper surface sets $\theta = \pi/2$. For the weak anchoring energy density at the lower surface, we take $F_0 = \frac{1}{2} W \theta^2$, which favors $\theta = 0$.

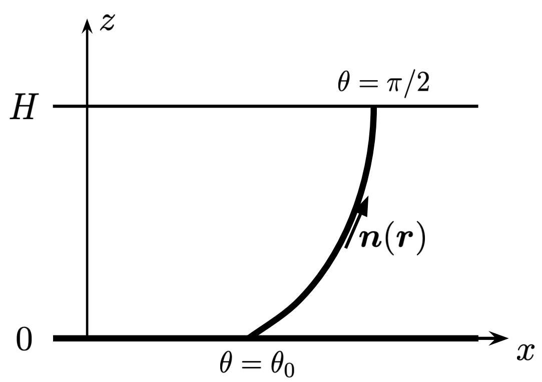

Figure 2: Nematic cell with hybrid anchoring.

1. Show that the bulk elastic energy density reduces to $\frac{1}{2} K (\partial_z \theta)^2$.

2. Show that the minimum of the free energy is obtained for

$$\partial_z^2\theta = 0 \quad (\forall z), \qquad \ell\,\partial_z\theta = \theta \quad (z=0), \tag{12}$$

3. Find the solution $\theta(z)$ and determine the boundary angle θ_0. Discuss the situations $H \ll \ell$ and $H \gg \ell$.

4. Show that the angle θ goes to 0 if one extrapolates the director profile to $z = -\ell$.

M2 PCS — Statistical field theory and soft matter
TD n°3 : Interfaces

1 Christoffel symbols (algebra homework)

The Christoffel symbols of a surface parametrized by $\boldsymbol{R}(u^1, u^2)$ are defined by the fundamental relation $\partial_i \boldsymbol{t}_j = \Gamma^k_{ij} \boldsymbol{t}_k + L_{ij} \boldsymbol{n}$, where $\boldsymbol{t}_j = \partial_j \boldsymbol{R}$, $\partial_j = \partial/\partial u^j$ and $\boldsymbol{n} \propto \boldsymbol{t}_1 \times \boldsymbol{t}_2$ is the normal to the surface. The purpose of this exercice is (i) to obtain the explicit formula:

$$\Gamma^m_{jk} = \frac{1}{2} g^{im} \left(\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk} \right), \tag{1}$$

(ii) to demonstrate the relation

$$\Gamma^j_{ji} = \partial_i \ln \sqrt{g}, \tag{2}$$

and (iii) to deduce that the covariant derivative of the metric vanishes.

1. Show that $\partial_i \boldsymbol{t}_j = \partial_j \boldsymbol{t}_i$ and deduce that $\Gamma^k_{ij} = \Gamma^k_{ji}$ (lower indices symmetry).
2. Show that $\partial_k g_{ij} = \Gamma^\ell_{ik} g_{j\ell} + \Gamma^\ell_{jk} g_{i\ell}$
3. Calculate $\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk}$ (circular permutations), and deduce (1).
4. By summing over all the values of the dummy indices i and j, show that $\Gamma^j_{j1} = \frac{1}{2} \left(g^{11} \partial_1 g_{11} + g^{22} \partial_1 g_{22} + 2 g^{12} \partial_1 g_{12} \right)$.
5. Using $g^{11} = g^{-1} g_{22}$, etc. (as g^{ij} is the inverse matrix of g_{ij}), deduce (2).
6. Using question 2 and the definition of the covariant derivative, show that

$$D_k g_{ij} = 0, \quad D_k g = 0. \tag{3}$$

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2 Laplace's law

The effective Hamiltonian of a system of two fluids in contact through an interface $\boldsymbol{R}(u^1, u^2)$ is given by the sum of a surface term and a volume pressure term:

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_V = \gamma \int d^2u \, \sqrt{g} - \int d^3r \, P(\boldsymbol{r}). \tag{4}$$

Here, γ is the surface free energy (surface tension) and $d^2u \, \sqrt{g} = dA$ is the area element along the interface and the integral. Note that the corrections due to the curvature of the interface have been neglected.

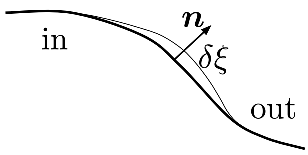

Figure 1: Normal variation of the shape of the interface.

The term $\mathcal{H}_V$ features the integral of the pressure field $P(\boldsymbol{r})$ over the entire volume of the system. It accounts for the work of the pressure forces when the interface is moved. Indeed, if the interface is shifted by $\delta\boldsymbol{R} = \boldsymbol{n}\delta\xi$, the variation of $\mathcal{H}_V$ is

$$\delta\mathcal{H}_V = -\int d^2u\sqrt{g}\,(P_{\text{in}} - P_{\text{out}})\,\delta\xi, \tag{5}$$

where P_{out} is the pressure field on the side to which the interface normal $\boldsymbol{n}$ points, and P_{in} is the pressure field on the other side. The pressure is indeed generally discontinuous at the interface. For instance, for a drop in contact with its vapor, we have $P_{\text{out}} = P_0$ in the vapor, with P_0 the constant atmospheric pressure, and $P_{\text{in}} = P_1 - \rho g z$, with P_1 a constant, ρ the mass density of the liquid and z the upward vertical coordinate

The aim of this problem is to find the equation that determines the shape of the interface that minimizes $\mathcal{H}$, i.e., Laplace's equation. It determines the "equilibrium shape" of the interface, or, more precisely, the most probable configuration of the interface.

1. For a variation $\delta\boldsymbol{R}$ of the interface, show that $\delta g = g\left(g^{22}\delta g_{22} + g^{11}\delta g_{11} + 2g^{12}\delta g_{12}\right)$.

2. Deduce that

$$\delta\sqrt{g} = \sqrt{g}\,\boldsymbol{t}^i\cdot\delta\boldsymbol{t}_i. \tag{6}$$

3. Show that $\delta\boldsymbol{t}_i = D_i\delta\boldsymbol{R}$.

4. By using the eq. (2) above, show that the covariant divergence can be expressed as

$$D_iV^i = \frac{1}{\sqrt{g}}\,\partial_i\left(\sqrt{g}\,V^i\right). \tag{7}$$

   and deduce that the surface integral $\int d^2u\sqrt{g}\,D_iV^i$ of a covariant divergence reduces to a boundary term.

5. Show then that

$$\delta\mathcal{H}_S = -\gamma\int d^2u\sqrt{g}\,K_i^i\boldsymbol{n}\cdot\delta\boldsymbol{R} + \text{boundary terms} \tag{8}$$

6. Deduce Laplace's law:

$$\gamma\,(c_1 + c_2) = P_{\text{out}} - P_{\text{in}}. \tag{9}$$

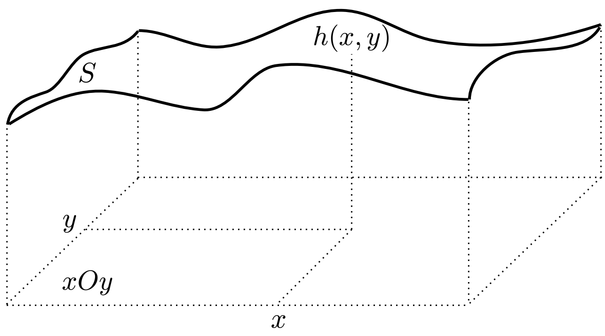

Figure 2: Monge parametrization.

3 Interface curvature in the Monge jauge

In the Monge parametrization, a surface is described by its height $z = h(x, y)$ above a reference plane (x, y) in orthonormal coordinates, as shown in Fig. 2. The corresponding shape field is therefore

$$\boldsymbol{R}(x, y) = x\boldsymbol{e}_x + y\boldsymbol{e}_y + h(x, y)\boldsymbol{e}_z. \tag{10}$$

This corresponds to a standard surface parametrization with $u^1 \equiv x$ and $u^2 \equiv y$.

1. With $h_x = \partial h/\partial x$, etc., show that the metric tensor is given by

$$g_{ij} = \begin{pmatrix} 1 + h_x^2 & h_x h_y \\ h_x h_y & 1 + h_y^2 \end{pmatrix} \tag{11}$$

2. Calculate $g = \det g_{ij}$ and the contravariant components g^{ij} of the metric tensor.

3. Deduce that the area element is given by

$$dA = \sqrt{1 + (\boldsymbol{\nabla} h)^2}\, dx\, dy. \tag{12}$$

4. Calculate the normal $\boldsymbol{n} = \boldsymbol{t}_1 \times \boldsymbol{t}_2 / |\boldsymbol{t}_1 \times \boldsymbol{t}_2|$.

5. Show that the curvature tensor is given by

$$K_{ij} = \frac{1}{\sqrt{g}} \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix}. \tag{13}$$

6. Show then that

$$K_i^j = \frac{1}{g^{3/2}} \begin{pmatrix} \left(1 + h_y^2\right) h_{xx} - h_x h_y h_{xy} & \left(1 + h_y^2\right) h_{xy} - h_x h_y h_{yy} \\ \left(1 + h_x^2\right) h_{xy} - h_x h_y h_{xx} & \left(1 + h_x^2\right) h_{yy} - h_x h_y h_{xy} \end{pmatrix}. \tag{14}$$

7. Deduce that the sum of the principal curvatures is given by the exact formula

$$c_1 + c_2 = \frac{h_{xx}\left(1 + h_y^2\right) + h_{yy}\left(1 + h_x^2\right) - 2h_{xy}h_x h_y}{\left(1 + h_x^2 + h_y^2\right)^{3/2}}. \tag{15}$$

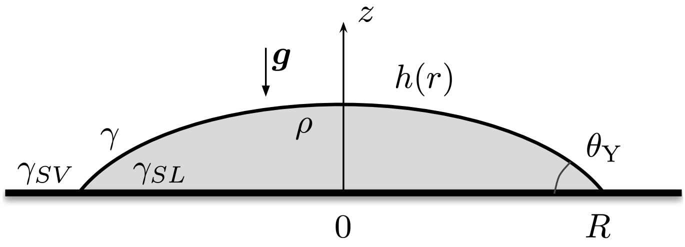

Figure 3: Drop wetting a substrate in the gravity field.

4 Wetting drop

A drop of liquid of mass density ρ is deposited on a substrate in a room on earth where gravity is g. It has the surface tension γ. The energy of the substrate–vapor interface is γ_{SV} and the energy of the substrate–liquid interface is γ_{SL}. Since we expect the drop to have rotational symmetry, we model its shape in the Monge jauge in polar coordinates by the height function

$$z = h(r), \quad r \in [0, R], \tag{16}$$

independent of the polar angle θ. Seen from above the drop takes a circular shape of radius R. We assume that we can use the Monge jauge in the limit of small deformations.

1. Work out the effective Hamiltonian $\mathcal{H}[h]$ of the drop. Show that it take the form

$$\mathcal{H}[h] = \int_0^R 2\pi r\, dr \left(\frac{1}{2}\gamma h'^2 + \frac{1}{2}\rho g h^2 \right) + (\gamma + \gamma_{SL} - \gamma_{SV})\, \pi R^2. \tag{17}$$

2. To look for the situation of minimum energy, we make the variation $h(r) \to h(r) + \delta h(r)$ and $R \to R + \delta R$, where both δh and δR are assumed to be $O(\epsilon)$. In the following, all the variations will be performed at $O(\epsilon)$. Show that

$$\delta h(R) = -h'(R)\, \delta R. \tag{18}$$

3. Let $\mathcal{H}^\star = \mathcal{H} + PV$, where V is the volume. Minimizing $\mathcal{H}$ at fixed volume is equivalent to minimizing $\mathcal{H}^\star$ without constraints. Show that

$$\delta \mathcal{H}^\star = \int_0^R 2\pi r\, dr \left[\gamma h' \delta h' + P \delta h + \rho g h \delta h \right] + 2\pi R \left[\frac{\gamma}{2} h'^2(R) + \Gamma \right] \delta R, \tag{19}$$

where $\Gamma = \gamma + \gamma_{SL} - \gamma_{SV}$.

4. Deduce the two equations that characterize the equilibrium of the drop:

$$\gamma \frac{1}{r} \left(r h' \right)' = P + \rho g h \qquad \text{(Laplace's law)}, \tag{20}$$

$$\gamma \left(1 - \frac{1}{2} h'^2 \right) = \gamma_{SV} - \gamma_{SL} \qquad \text{(Young's law)}. \tag{21}$$

5. Interpret these two equations in terms of Laplace's law, hydrostatic pressure, contact angle θ_{Y} and the balance of surface forces acting on the contact line (i.e., Young's law).

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5 The renormalization group on Gaussian interfaces

We consider an interface in d dimensions, described as $h(\boldsymbol{r})$ in the Monge jauge, with cutoff Λ in reciprocal space (i.e., $\Lambda = 2\pi/a$, where a is the small distance cutoff). It Fourier decomposition can the be written as

$$h(\boldsymbol{r}) = L^{-d} \sum_{\boldsymbol{q}\in\mathcal{E}} h_{\boldsymbol{q}}\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}}, \qquad \mathcal{E} = \{\boldsymbol{q}, q < \Lambda\}, \tag{22}$$

where L^d is the area of the projection of the interface onto the reference plane. The Hamiltonian describing the fluctuation of the interface is assumed to be Gaussian:

$$\mathcal{H}[h] = \int_{L^d} d^d r \left[\frac{\gamma}{2}\left(\boldsymbol{\nabla} h\right)^2 + \frac{\kappa}{2}\left(\boldsymbol{\nabla}^2 h\right)^2\right]. \tag{23}$$

The parameters of the system are therefore $\{\gamma, \kappa, L, \Lambda\}$. We observe the fluctuations of the interface and ask whether there is an asymptotic scale invariance (as in critical systems). In other words, do large scales behave in some way identically?

To investigate this question, we observe the interface through a renormalization group (RG) "megascope" (the opposite of a microscope). It has two parameters: ℓ (anti-zoom factor) and D (amplification factor). The image we see, $h'(\boldsymbol{r}')$, is the following transformation of the direct image:

$$h'(\boldsymbol{r}') = \ell^D\, h^<(\ell \boldsymbol{r}'), \tag{24}$$

where

$$h^<(\boldsymbol{r}) = L^{-d} \sum_{\boldsymbol{q}\in\mathcal{E}^<} h_{\boldsymbol{q}}\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}}, \qquad \mathcal{E}^< = \{\boldsymbol{q}, q < \Lambda/\ell\}. \tag{25}$$

In other words, the megascope smoothes out the image of the interface by eliminating the small deformations of wavevector larger than Λ/ℓ, then it amplifies the signal by a factor ℓ^D (a scale invariant factor), then it contracts the in-plane distances by dividing $\boldsymbol{r}$ by ℓ. The result is a smaller interface, of size $L = L/\ell$, with a smoothed, amplified shape, that fluctuates with the same cutoff Λ as the real one (thanks to the in-plane contraction). Does it look like the real one? Can one find D such that the answer is yes for $\ell \to \infty$? This is how the RG seeks scale invariance.

1. Write the Fourier expansion of $h^> = h - h^<$ in the form of Eq. (46).

2. Show that $\mathcal{H}[h] = \mathcal{H}[h^<] + \mathcal{H}[h^>]$. Would that be true for a non-Gaussian model?

3. Writing the probability density for $h^<$ as $P^<[h^<] = Z^{-1}\exp(-\beta\mathcal{H}^<[h^<])$ defines $\mathcal{H}^<[h^<]$. Show that $\mathcal{H}^<[h^<] = \mathcal{H}[h^<]$ (up to an irrelevant constant). Would this very simple result be true for a non-Gaussian model?

4. Deduce that the probability density for h' is $P'[h'] = Z^{-1}\exp(-\beta\mathcal{H}'[h'])$, with

$$\mathcal{H}'[h'] = \int_{L^d} d^d r' \left[\frac{\gamma(\ell)}{2}\left(\boldsymbol{\nabla}' h'\right)^2 + \frac{\kappa(\ell)}{2}\left(\boldsymbol{\nabla}'^2 h'\right)^2\right] \tag{26}$$

(up to an irrelevant constant), with

$$\gamma(\ell) = \gamma\,\ell^{d-2D-2}, \tag{27}$$
$$\kappa(\ell) = \kappa\,\ell^{d-2D-4}, \tag{28}$$
$$L(\ell) = L\,\ell^{-1}. \tag{29}$$

5. Assuming d is large enough, represent the renormalization flow, and deduce that the correct (interesting) setting of the megascope corresponds to $D = (d-2)/2$.

6. Deduce that κ is irrelevant at large scales when $\gamma \neq 0$.

7. Let $W(\gamma,\kappa,L)$ be the typical width of the interface (omitting Λ, which remains constant during renormalization). By comparing the real interface and the megascope image for $D = (d-2)/2$, show that

$$W(\gamma,\kappa,L) = \ell^{-D}\,W(\gamma,\kappa\ell^{-2},L\ell^{-1}). \tag{30}$$

What clever choice of ℓ will give us physical insight when $L\to\infty$? It is convenient to have $\kappa\to 0$, so we want a diverging ℓ. But we don't want $L\to 0$, otherwise the interface image would diasppear. So we choose ℓ such that $L\ell^{-1} = \bar{L}$, where $\bar{L}$ is a constant.

8. Using this choice of ℓ, show that

$$W \sim L^{1-\frac{d}{2}}. \tag{31}$$

9. Deduce that the interface is flat for $d > 2$ and rough for $d < 2$, when $\gamma \neq 0$, in agreement with the results of the course.

10. Characterize the width of an interface in the two dimensional Ising model.

11. Let $\theta(\gamma,\kappa,L)$ be the typical orientational fluctuations of the interface. Using a relation similar to (59), in the limit of small angles, show that

$$\theta \sim L^{-\frac{d}{2}}, \tag{32}$$

12. Conclude that an interface under tension is oriented in all dimensions.

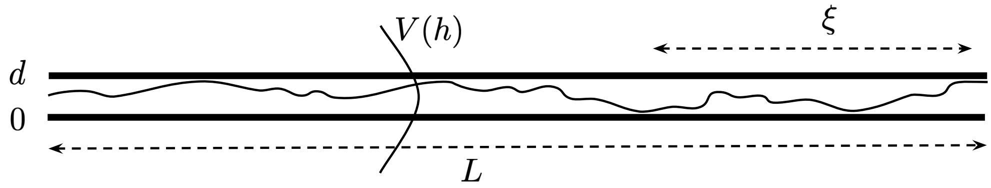

Figure 4: Fluctuating membrane exerting Helfrich forces

6 Helfrich forces

When a tensionless membrane of lateral size L is confined between two parallel plates, separated by a distance d, it exerts a pressure $P(d)$ on these plates. Indeed, a tensionless membrane is marginally rough in two dimensions, so it will always collide with the plates if it is large enough (Fig. 4). In the Monge jauge, the effective Hamiltonian of the corresponding system is given by

$$\mathcal{H} = \int d^2r \left[\frac{1}{2}\kappa(\boldsymbol{\nabla}^2 h)^2 + V(h)\right], \tag{33}$$

with

$$V(h(\boldsymbol{r})) = \begin{cases} 0 & 0 < h(\boldsymbol{r}) < d, \\ +\infty & \text{otherwise.} \end{cases} \tag{34}$$

To calculate the pressure exerted by the fluctuating membrane (Helfrich forces), we must compute the free-energy $f(d)$ per unit area of the membrane and differentiate:

$$P(d) = -\frac{\partial f}{\partial d}. \tag{35}$$

Self-consistent Gaussian approximation.—The above non-Gaussian model is very difficult to solve. A convenient strategy, which we adopt, is to approximate $V(h)$ by a quadratic potential $V(h) = \frac{1}{2}mh^2$, with $m(d)$ chosen so as to verify $\langle h^2(\boldsymbol{r})\rangle \approx d$ (as in the presence of the plates), where the symbol $\approx$ means "up to a factor of order unity". This approximation yields the correct scaling for the pressure of the Helfrich forces.

1. Using standard results on the correlation function of Gaussian models, show that

$$\langle h^2(\boldsymbol{r})\rangle \simeq \frac{k_\mathrm{B}T\xi^2}{8\kappa}, \tag{36}$$

   where $\xi = (\kappa/m)^{1/4}$. To obtain this simple result, assume that $L \to \infty$ and $\xi \gg a$, where a is the microscopic cutoff of the theory.

2. Deduce that in order to match the plate problem, we need to take

$$m \approx \frac{(k_\mathrm{B}T)^2}{\kappa d^4}. \tag{37}$$

3. Without further calculations, show that ξ is the typical membrane size below which the membrane does not collide with the plates and above which it does.

4. The pressure exerted by the membrane can be estimated using a scaling argument: $f \approx k_{\mathrm{B}}T/\xi^2$. Justify this statement and deduce that

$$P \approx \frac{(k_{\mathrm{B}}T)^2}{\kappa d^3}. \tag{38}$$

5. The partition function of the system can be expressed in the reciprocal space, with $h(\boldsymbol{r}) = L^{-2}\sum_{\boldsymbol{q}} h_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$, as

$$Z = C \int \left(\prod_{\boldsymbol{q}} dh'_{\boldsymbol{q}}\, dh''_{\boldsymbol{q}}\right) e^{-\beta L^{-2}\sum_{\boldsymbol{q}} \frac{1}{2}(\kappa q^4+m)|h_{\boldsymbol{q}}|^2}, \tag{39}$$

where $h_{\boldsymbol{q}} = h'_{\boldsymbol{q}} + ih''_{\boldsymbol{q}}$. Behind the apparences, what are the correct and precise meanings of $\sum_{\boldsymbol{q}}$ and $\prod_{\boldsymbol{q}}$? Why is there a constant C?

6. Deduce that in the thermodynamic limit and up to an irrelevant additive constant, the free energy density is given by

$$f = \frac{1}{2}k_{\mathrm{B}}T \int \frac{d^2q}{(2\pi)^2} \ln\left(\kappa q^4 + m\right). \tag{40}$$

7. Conclude again that

$$P \approx \frac{(k_{\mathrm{B}}T)^2}{\kappa d^3} \tag{41}$$

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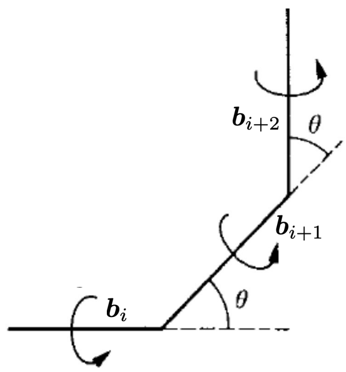
Academic year 2022-2023

M2 PCS — Statistical field theory and soft matter

TD n°4 : Polymers

1 The freely rotating chain

We consider a polymer chain with no large loop interactions (ideal), consisting of N segments $\boldsymbol{b}_i$ of length b that are forced to make a bond angle θ between them while being free to rotate in the azimuthal direction:



We recall that if x and y are two random variables, the average $\langle f(x,y)\rangle_{x,y}$ over all realisations of x and y can be computed by first averaging over y at constant x, then averaging the result over x:

$$\langle f(x,y)\rangle_{x,y} = \langle\, \langle f(x,y)\rangle_y\, \rangle_x. \tag{1}$$

1. Show that, at fixed $\boldsymbol{b}_i$, the average $\langle \boldsymbol{b}_{i+1}\rangle$ is equal to $\boldsymbol{b}_i \cos\theta$.
2. Deduce that $\langle \boldsymbol{b}_i \cdot \boldsymbol{b}_{i+1}\rangle_{i,i+1} = b^2\cos\theta$.
3. Decuce that $\langle \boldsymbol{b}_i \cdot \boldsymbol{b}_{i+2}\rangle_{i,i+1,i+2} = b^2\cos^2\theta$.
4. By generalizing, calculate $\langle \boldsymbol{b}_i \cdot \boldsymbol{b}_j\rangle$.
5. Deduce that the persistence length, i.e., the length along the chain over which correlation are negligible, is given by

$$\ell_p = \frac{b}{\ln\left(\dfrac{1}{\cos\theta}\right)}. \tag{2}$$

6. The radius of giration, i.e., the typical size of the polymer, is given by $R_g = \langle R^2 \rangle$, where $R = \sum_{i=1}^{N} \boldsymbol{b}_i$ is the end-to-end vector. Show that in the thermodynamic limit (i.e., for very long chains) it is given by

$$R_g^2 = N b^2 \frac{1+\cos\theta}{1-\cos\theta}, \tag{3}$$

7. What is the value of the exponent ν?

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2 Giration radius of a Gaussian polymer

Let us consider a Gaussian polymer in d dimension. Its shape is described by a coarse-grained field $\boldsymbol{r}(s)$, $s \in [0, N]$, with effective dimensionless Hamiltonian:

$$\mathcal{H}[\boldsymbol{r}] = \int_0^N ds \, \frac{1}{2a^2} \, \dot{\boldsymbol{r}}^2. \tag{4}$$

The giration radius $R_g = \langle (\boldsymbol{r}(N) - \boldsymbol{r}(0))^2 \rangle^{1/2}$ of the polymer (i.e., its size), can be deduced from the moments generating function:

$$G(\boldsymbol{k}) = \langle e^{i\boldsymbol{k}\cdot(\boldsymbol{r}(N)-\boldsymbol{r}(0))} \rangle = \int \mathcal{D}\boldsymbol{r} \, e^{i\boldsymbol{k}\cdot(\boldsymbol{r}(N)-\boldsymbol{r}(0))} \, e^{-\mathcal{H}[\boldsymbol{r}]}. \tag{5}$$

1. Show that

$$G(\boldsymbol{k}) = 1 - \frac{R_g^2}{2d} \, k^2 + O(k^4). \tag{6}$$

2. Compute $G(\boldsymbol{k})$ using the change of variable $\boldsymbol{r}'(s) = \boldsymbol{r}(s) - ia^2 \boldsymbol{k} \, s$.

3. Deduce that

$$R_g^2 = N a^2 d, \qquad \nu = \frac{1}{2}.$$

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3 Renormalization of a polymer under tension

A polymer in a good solvent, described by Edwards' model with a field $\boldsymbol{r}(s)$ of cutoff Λ in reciprocal space, with $s \in [0, N]$, is stretched by a pair of opposite forces of modulus f acting on both extremities of the chain. Its effective Hamiltonian is given by

$$\mathcal{H}[\boldsymbol{r}] = \int_0^N ds \, \frac{1}{2a^2} \dot{\boldsymbol{r}}^2 + \iint_0^N ds_1 \, ds_2 \, v \, \delta(\boldsymbol{r}(s_1) - \boldsymbol{r}(s_2)) \; + \mathcal{H}_{\text{ext}}, \tag{7}$$

$$\mathcal{H}_{\text{ext}} = -\beta \boldsymbol{f} \cdot (\boldsymbol{r}(N) - \boldsymbol{r}(0)), \tag{8}$$

where β is the inverse temperature. We apply renormalization group transformations:

$$\boldsymbol{r}(s) \longrightarrow \boldsymbol{r}'(s') = g^D \bar{\boldsymbol{r}}(gs'), \tag{9}$$

with coarse-graining $\boldsymbol{r}(s) = \bar{\boldsymbol{r}}(s) + \tilde{\boldsymbol{r}}(s) \longrightarrow \bar{\boldsymbol{r}}(s)$ performed by removing the Fourier modes of wavevector within Λ/g and Λ.

1. Show that $\mathcal{H}_{\text{ext}}$ corresponds to a force $\boldsymbol{f}$ acting on the last monomer and a force $-\boldsymbol{f}$ acting on the first one.

2. Explain without calculations why $\mathcal{H}_{\text{ext}}[\boldsymbol{r}] \simeq \mathcal{H}_{\text{ext}}[\bar{\boldsymbol{r}}]$. Under which condition is that correct?

3. Infer that that the parameter f is not affected by the coarse-graining, i.e., $\bar{f} = f$, while de Gennes' relations $\bar{a} = a[1 + h(u)]$ and $\bar{v} = v[1 - k(u)]$ still hold.

4. Deduce the renormalization step transformation (d is the space dimension):

$$\begin{cases} a_{n+1} = a_n\,[1 + h(u_n, g)]\, g^{\frac{1}{2}+D}, \\ v_{n+1} = v_n\,[1 - k(u_n, g)]\, g^{2+Dd}, \\ f_{n+1} = f_n\, g^{-D}. \end{cases} \tag{10}$$

5. Show, by using the π theorem, that the elongation $L = \langle |\boldsymbol{r}(N) - \boldsymbol{r}(0)| \rangle$ of the polymer can be written as

$$L = a\,\phi(u, N, \beta a f), \tag{11}$$

   where ϕ is a dimensionless function (g and d implicit).

6. By renormalizing down to a fixed $N/g = N_0$, deduce the scaling law

$$\frac{L}{R_g} = \Phi_d\left(\frac{f}{k_{\text{B}}T/R_g}\right), \tag{12}$$

   where Φ_d is a universal function[1] and $R_g \sim N^\nu$ is the giration radius.

7. Picture two identical polymers under tension chained together. What relation between L and N is expected in the regime of large forces?

8. Deduce that the elongation obeys the nontrivial power-law:

$$L \sim f^{\frac{1-\nu}{\nu}}. \tag{13}$$

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[1] Remember we showed in the course that $R_g = C a^\star N^{-D}$ with C a universal constant.

4 Polymers and the O(n) models in the limit $n \to 0$

P.-G. de Gennes, laureate of the 1991 Nobel Prize, proved that the exponent $\nu(d)$ of real polymers is equal to the analytic continuation to $n \to 0$ of the exponent $\nu(n, d)$ of the correlation length in the $O(n)$ magnets models.

Consider in d dimensions a polymer in a good solvent, described by Edwards' model with a field $\boldsymbol{r}(s)$, $s \in [0, N]$, of effective Hamiltonian

$$\mathcal{H}[\boldsymbol{r}] = \int_0^N ds\, \frac{1}{4}\dot{\boldsymbol{r}}^2 + \iint_0^N ds\, ds'\, u\, \delta(\boldsymbol{r}(s) - \boldsymbol{r}(s')). \tag{14}$$

Note that $\boldsymbol{r}(s)$ has been normalized by $a/\sqrt{2}$ so that it is now dimensionless.

1. Show that the second term $\mathcal{H}_2$ in the Hamiltonian can be rewritten as $\mathcal{H}_2 = \int d^dx\, u\rho^2$, where $\rho(\boldsymbol{r}) = \int_0^N ds\, \delta(\boldsymbol{r} - \boldsymbol{r}(s)$ is the polymer chain density.

2. The density ρ appears as a square in the Boltzmann factor $e^{-\mathcal{H}_2}$. Show that we can make ρ appear only linearly by performing a Hubbard-Stratonovich transformation through the introduction of a new field $V(\boldsymbol{x})$:

$$e^{-\mathcal{H}_2} \propto \int \mathcal{D}V\, e^{-\int d^dx \left[\frac{V^2}{4u} + iV\rho\right]}. \tag{15}$$

3. Deduce that the generating function $G(\boldsymbol{k}; N) = \langle e^{i\boldsymbol{k}\cdot(\boldsymbol{r}(N)-\boldsymbol{r}(0))}\rangle$ of the moments of the end-to-end vector can be expressed as

$$G(\boldsymbol{k}; N) \propto \int \mathcal{D}V\, e^{-\int d^dx\, \frac{V^2}{4u}} \int \mathcal{D}\boldsymbol{r}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{r}(N)-\boldsymbol{r}(0))}\, e^{-\int_0^N ds\, [\frac{1}{4}\dot{\boldsymbol{r}}^2 + iV(\boldsymbol{r}(s))]}. \tag{16}$$

Analogy with a quantum mechanics path integral.—It follows that

$$G(\boldsymbol{k}; N) = \int \mathcal{D}V\, e^{-\int d^dx\, \frac{V^2}{4u}} \int d^d\boldsymbol{r}'\, d^d\boldsymbol{r}'' e^{i\boldsymbol{k}\cdot(\boldsymbol{r}''-\boldsymbol{r}')}\, \Gamma(\boldsymbol{r}', \boldsymbol{r}''; N), \tag{17}$$

with

$$\Gamma(\boldsymbol{r} - \boldsymbol{r}'; N) = \int_{\boldsymbol{r}(0)=\boldsymbol{r}}^{\boldsymbol{r}(N)=\boldsymbol{r}'} \mathcal{D}\boldsymbol{r}\, e^{-\int_0^N ds \left[\frac{1}{4}\dot{\boldsymbol{r}}(s)^2 + iV(\boldsymbol{r}(s))\right]}, \tag{18}$$

the partition function of an *ideal* polymer with fixed extremities, interacting with an imaginary external field $iV(\boldsymbol{r})$. This quantity ressembles the path integral of a quantum particle of Hamiltonian $\hat{H} = \hat{\boldsymbol{p}}^2/(2m) + \mathcal{V}(\hat{\boldsymbol{r}})$. Indeed, the probability amplitude for detecting the particle at time T in $\boldsymbol{r}'$, knowing that it was at time 0 in $\boldsymbol{r}$, is given by

$$\psi(\boldsymbol{r}', T) = \int_{\boldsymbol{r}(0)=\boldsymbol{r}}^{\boldsymbol{r}(T)=\boldsymbol{r}'} \mathcal{D}\boldsymbol{r}\, e^{\frac{i}{\hbar}\int_0^T dt \left[\frac{1}{2}m\dot{\boldsymbol{r}}(t)^2 - \mathcal{V}(\boldsymbol{r}(t))\right]}. \tag{19}$$

4. Using this quantum analogy and the evolution operator $e^{-i\hat{H}T/\hbar}$ acting on the initial state $|\boldsymbol{r}\rangle$, show that $\Gamma(\boldsymbol{r}; N)$ is the Green function of the operator $e^{N\left(-\boldsymbol{\nabla}^2 + iV\right)}$.

5. Show (without seeking rigor), that its Laplace transform[2] $\hat{\Gamma}(\boldsymbol{r};t)$ is the Green function of $-\boldsymbol{\nabla}^2 + iV(\boldsymbol{r}) + t$.

6. Deduce that $\hat{\Gamma}(\boldsymbol{r},\boldsymbol{r}';t)$ coincides with the correlation function $\langle\phi(\boldsymbol{r})\phi(\boldsymbol{r}')\rangle$ of the Gaussian statistical field theory with Hamiltonian (normalized by $k_{\rm B}T$):
$$F[\phi] = \frac{1}{2}\int d^d r \left[t\phi^2 + iV(\boldsymbol{r})\phi^2 + (\boldsymbol{\nabla}\phi)^2\right]. \tag{20}$$

7. Deduce that the Laplace transform of $G(\boldsymbol{k};N)$ can be expressed as
$$\hat{G}(\boldsymbol{k};t) = \int \mathcal{D}V\, e^{-\int d^d x\, \frac{V^2}{4u}} \int \mathcal{D}\phi\, \phi(\boldsymbol{k})\phi(-\boldsymbol{k})\, \frac{e^{-F[\phi]}}{Z}, \tag{21}$$
with Z the partition function of ϕ. We have now matched the polymer to a field theory with two coupled fields, with a cubic term $\propto V\phi^2$.

The replica trick and the $n\to 0$ limit.—To get around a difficulty that will appear, we now replicate n times the field ϕ. We introduce a vectorial field $\vec{\phi}(\boldsymbol{x})$ with n components, $\vec{\phi} = (\phi_1,\dots,\phi_n)^t$, and $O(n)$ Hamiltonian:
$$F_n[\vec{\phi}\,] = \sum_{i=1}^{N} F[\phi_i] = \frac{1}{2}\int d^d x \left[t\vec{\phi}^{\,2} + iV(\boldsymbol{x})\vec{\phi}^{\,2} + \sum_{i=1}^{n}(\boldsymbol{\nabla}\phi_i)^2\right]. \tag{22}$$

8. Show that
$$\hat{G}(\boldsymbol{k};t) = \lim_{n\to 0}\int \mathcal{D}\vec{\phi}\, \phi_1(\boldsymbol{k})\phi_1(-\boldsymbol{k}) e^{-\int d^d x\left[\frac{1}{2}t\vec{\phi}^{\,2} + \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{\nabla}\phi_i)^2 + \frac{u}{4}\left(\vec{\phi}^{\,2}\right)^2\right]} \tag{23}$$
$$\propto \lim_{n\to 0}\langle\phi_1(\boldsymbol{k})\phi_1(-\boldsymbol{k})\rangle_4. \tag{24}$$
The correlation function $\langle\phi_1(\boldsymbol{k})\phi_1(-\boldsymbol{k})\rangle_4$ of the ϕ^4 model is therefore related, in the limit $n\to 0$, to the generating function of the moments of the size of a real polymer

9. In the $O(n)$ models, the scaling law for the correlation function close to the critical point is $\langle\phi_1(\boldsymbol{x})\phi_1(\boldsymbol{x}')\rangle_4 \sim |\boldsymbol{x}-\boldsymbol{x}'|^{-(d-2+\eta)} f(|\boldsymbol{x}-\boldsymbol{x}'|/(t-t_c)^{-\nu})$. Show that in Fourier space it corresponds to
$$\langle\phi_1(\boldsymbol{k})\phi_1(-\boldsymbol{k})\rangle_4 \sim (t-t_c)^{-\gamma}\tilde{f}(k(t-t_c)^{-\nu}), \tag{25}$$
with $\gamma = \nu(2-\eta)$.

10. Show that expanding $\hat{G}(\boldsymbol{k};t)$ in power series to second-order in $\boldsymbol{k}$ gives
$$\hat{G}(\boldsymbol{k};t) \propto a(t-t_c)^{-\gamma_0} + b(t-t_c)^{-(\gamma_0+2\nu_0)}k^2 + \dots \tag{26}$$
where $\gamma_0 = \lim_{n\to 0}\gamma(n)$ and $\nu_0 = \lim_{n\to 0}\nu(n)$.

11. Using the inverse Laplace transform, show that
$$G(\boldsymbol{k},N) \sim 1 + c\,k^2 N^{2\nu_0} + O(k^4), \qquad (N\to\infty), \tag{27}$$
and deduce that $R_g \sim N^{\nu_0}$.

[2] The Laplace transform of $f(N)$ is $\hat{f}(t) = \int_0^\infty dN\, f(N)\, e^{-Nt}$.