Lecture 4 — The Classical Wave Equation & Waves Review

Reading: Engel 4th ed., Chapter 2 (Section 2.1)

Learning Objectives

- Write and solve the classical wave equation in one dimension
- Distinguish between traveling waves and standing waves
- Express waves using complex exponential notation
- Connect the classical wave equation to the quantum-mechanical Schrödinger equation

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1. Why Study Waves?

De Broglie showed that matter has wave properties. To build a quantum theory, we need an equation that governs the behavior of "matter waves" — analogous to how the classical wave equation governs vibrating strings and electromagnetic fields.

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2. The Classical Wave Equation

A disturbance $u(x,t)$ propagating along a string satisfies:

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

where v is the wave speed.

General Solution: Traveling Waves

$u(x,t) = f(x - vt) + g(x + vt)$

- $f(x - vt)$: wave moving in $+x$ direction
- $g(x + vt)$: wave moving in $-x$ direction

Sinusoidal Traveling Wave

$u(x, t) = A \sin(kx - \omega t + \phi)$

Key parameters:

Symbol	Name	Definition
A	Amplitude	Maximum displacement
k	Wavevector	$k = 2\pi/\lambda$
ω	Angular frequency	$\omega = 2\pi\nu$
λ	Wavelength	Spatial period
ν	Frequency	$\nu = v/\lambda$
ϕ	Phase	Offset at $x = 0, t = 0$

The dispersion relation: $v = \lambda\nu = \omega/k$.

> [!NOTE] **Concept Check 4.1** If you double the frequency of a wave on a string while keeping the tension and mass density constant (so the wave speed v is constant), how does the wavelength λ change?

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3. Standing Waves

When a wave is confined (e.g., a string fixed at both ends of length L), boundary conditions restrict the allowed wavelengths:

$u_n(x, t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$

Quantization from boundary conditions:

$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots$

This is our first encounter with a recurring theme: **confinement leads to quantization**. The same principle applies to electrons confined in atoms or molecules.

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4. Complex Exponential Notation

Euler's formula connects complex exponentials to trigonometric functions:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

A traveling wave can be written compactly as:

$$u(x,t) = A\,e^{i(kx-\omega t)}$$

The physical wave is the real part: $\mathrm{Re}[u(x,t)]$.

Why Use Complex Notation?

- Derivatives are simpler: $\frac{d}{dx}e^{ikx} = ik\,e^{ikx}$
- Superposition calculations are more transparent
- Quantum mechanics inherently uses complex wavefunctions

Key Identities

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$|e^{i\theta}|^2 = e^{i\theta}\cdot e^{-i\theta} = 1$$

> [!NOTE] **Concept Check 4.2** Using complex exponential notation, write a wave that is traveling in the negative x direction with amplitude B, wavevector k, and angular frequency ω.

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5. Superposition Principle

If u_1 and u_2 are both solutions of the wave equation, then any linear combination is also a solution:

$$u(x,t) = c_1u_1(x,t) + c_2u_2(x,t)$$

This **superposition principle** holds because the wave equation is linear. The Schrödinger equation is also linear, so superposition will be a central feature of quantum mechanics.

Fourier's Theorem

Any well-behaved function on $[0, L]$ can be expanded as a sum of standing waves:

$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

This foreshadows the expansion of quantum states in terms of eigenfunctions.

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6. From Classical Waves to Quantum Waves

Classical Wave	Quantum "Wave"
$u(x,t)$ = displacement	$\Psi(x,t)$ = wavefunction
$\|u\|^2$ ∝ intensity	$\|\Psi\|^2$ = probability density
$v = \omega/k$	$E = \hbar\omega,\ p = \hbar k$
Classical wave equation	Schrödinger equation

Next lecture: we construct the Schrödinger equation by combining de Broglie's relation with the energy-momentum relationship.

Key Equations Summary

Equation	Expression
Classical wave equation	$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$
Sinusoidal wave	$u(x,t) = A\sin(kx - \omega t)$
Wavevector	$k = 2\pi/\lambda$
Angular frequency	$\omega = 2\pi\nu$
Standing wave (string)	$\lambda_n = 2L/n$
Euler's formula	$e^{i\theta} = \cos\theta + i\sin\theta$

Recent Literature Spotlight

"Ab Initio Multiple Spawning Dynamics Reveals the Photochemistry of Butadiene" *B. F. E. Curchod, T. J. Martínez*, Chemical Reviews, **2024**, 124, 3500–3529. DOI

This review showcases how the time-dependent Schrödinger equation is solved numerically to simulate ultrafast photochemical reactions. By propagating nuclear wave packets on multiple electronic surfaces, the researchers capture phenomena like conical intersections and nonadiabatic transitions — processes that determine the outcome of photochemical reactions. It illustrates how the Schrödinger equation from this lecture is the engine behind modern photochemistry simulations.

Practice Problems

1. **Standing waves.** A guitar string is 65 cm long. Calculate the wavelengths and frequencies of the first three harmonics if the wave speed on the string is 400 m/s.
2. **Complex notation.** Verify that $\Psi(x,t) = Ae^{i(kx-\omega t)}$ satisfies the classical wave equation with $v = \omega/k$.
3. **Superposition.** Two waves $u_1 = A\sin(kx - \omega t)$ and $u_2 = A\sin(kx + \omega t)$ overlap. Show that the result is a standing wave, and identify its nodes.

Next lecture: Deriving the Schrödinger Equation