AMATH 352: HOMEWORK 8

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1. INTRODUCTION AND OVERVIEW OF THE PROBLEM

This report explores the application of SVD in image processing and compression by investigating what kind of information singular vectors holds, and how the rank-r approximation works.

2. THEORETICAL BACKGROUND AND DESCRIPTION OF ALGORITHM

2.1 **Singular Value Decomposition (SVD)** is a factorization of a matrix. Specifically, given a matrix $B \in \mathbb{R}^{m\times n}$, we can decompose it as $B = U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$ is an orthogonal matrix and called left singular matrix, $\Sigma \in \mathbb{R}^{m\times n}$ is a diagonal matrix with non-negative values sorted from high to low, and $V^T \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and called right singular matrix.

2.2 **Rank-r approximation** is a minimization problem, where our primary goal is to identify the 'best' way to approximate a given matrix B with a rank-r matrix B_r. This can be applied in many areas, and in this report, we are interested in compressing image.

3. COMPUTATIONAL RESULTS

Given matrix A of size 766×713, each row of A is an image of size 31×23 of former President George W Bush and former Secretary of State Colin Powell stored as a one-dimensional array.

We first want to investigate the 'average' face of the total 766 faces/images. By computing the mean of the rows of A, we get a vector $\in R^{713}$. Plotting this vector as an image as in Figure 1, we see that it doesn't have any specific detail, but it does have the key characteristics of a face (eyes, mouth, and nose).

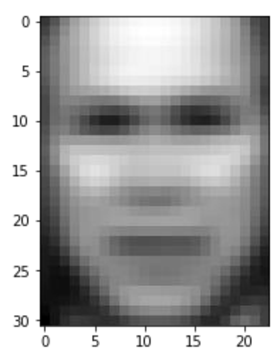

Figure 1: 'Average face', plotted with the mean of the rows of A.

Next, we construct matrix B of size 766×713 by subtracting the mean image from the rows of A and then compute the SVD of B, $B = U\Sigma \mathrm{V}^{\mathrm{T}}$. Our singular values Σ is ordered by the largest/most important to the smallest/least important, so we can see the decay rate of these values in Figure 2. It decays very quickly in about the first 50 values, then decay slowly between the ~51 and ~600 and experiences a huge drop again for the rest of the values.

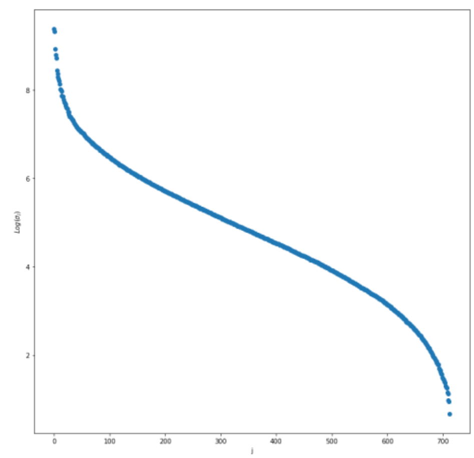

Figure 2: Plot of $\log(\sigma_j)$ vs j

Next, we investigate the right singular matrix V^T by plotting the first 5 right singular vectors (see Figure 3). These images are distorted faces without much detail. Each of these vector captures different combinations of features (dark vs light aspects). One more thing is, all these faces are on a similar scale, allowing us to compare.

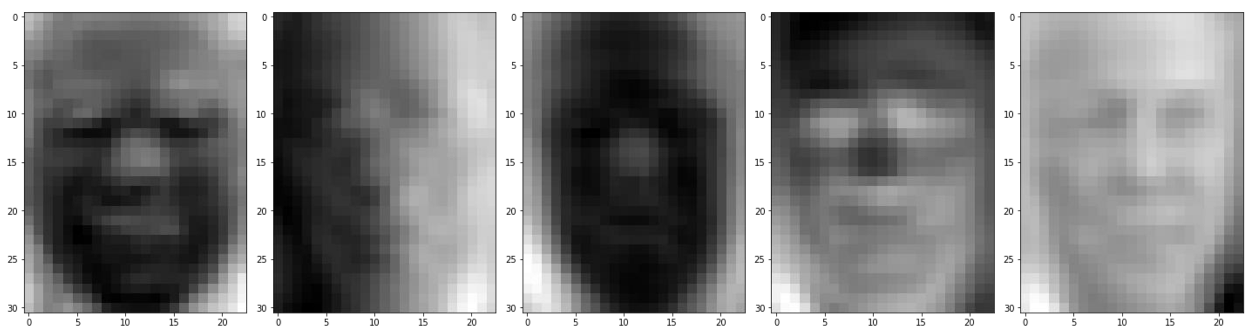

Figure 3: Plot of the first 5 right singular vectors of B

Let B_r be the rank-r approximation to B, and consider the relative error $\frac{\|B-B_r\|_F}{\|B\|_F}$. We know that the higher rank-r approximation we get, the lower error it is. We want to find the smallest r such that the error is less than or equal to 30%, 20%, 10%, and 1% (see Table 1).

Relative error (expected)	30%	20%	10%	1%
Relative error (true)	29.85%	19.95%	9.97%	0.99%
r	53	101	207	546

Table 1: Rank-r approximation with a given relative error

For the values of r found in Table 1, we want to compare the original image plotted with 1st row of matrix B and the approximated images plotted with 1st row of matrices B_r's. For B_{53}, we see that this is the matrix that have the highest relative error, and thus it can only capture the big detail like face shape and the position of the facial features. The larger r we get, the smaller the error, and thus the more detailed the image gets. So when we get to B_{546}, we can rarely spot the difference between the resolution of the original image and this approximated image. In other words, B_{546} basically captures (almost) all details of the original image.

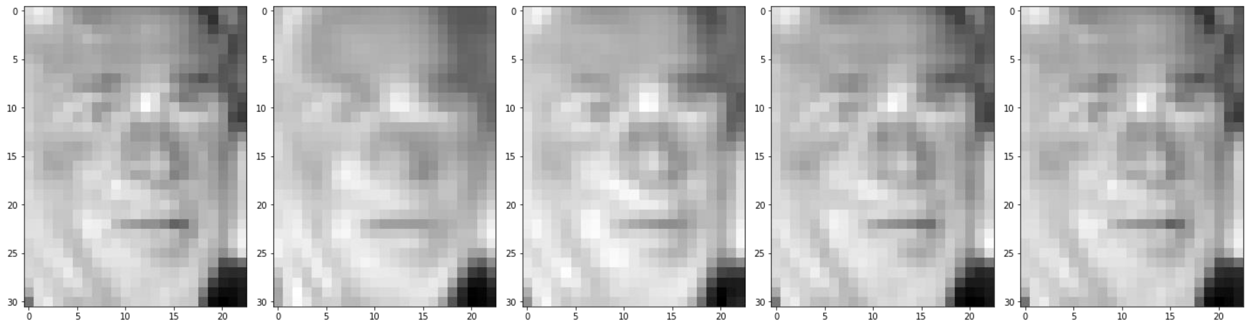

Figure 4: Original Image (first row of B) and Approximate Images (first row of $B_{53}, B_{101}, B_{207}, B_{546}$)

4. SUMMARY AND CONCLUSIONS

In this report, we learn that SVD is very powerful in image processing and compression. By decomposing matrix B, we can investigate the type of information held in the singular matrix, and specifically the right singular matrix holds the lighting aspect of the images. We also discover the quality of approximate images and how they compare to the original image. We know that the higher rank-r we get, the smaller the error, and thus the more detailed the image gets.

REFERENCES

[1] Roughgarden, T., Valiant, G., 2022: Lecture #9: The Singular Value Decomposition (SVD) and Low-Rank Matrix Approximations. CS168: The Modern Algorithmic Toolbox, 11 pp, https://web.stanford.edu/class/cs168/l/l9.pdf