AMATH 352: HOMEWORK 5

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1. INTRODUCTION AND OVERVIEW OF THE PROBLEM

This report interpolates Runge's function and investigates the numerical accuracy and stability of this problem.

2. THEORETICAL BACKGROUND AND DESCRIPTION OF ALGORITHM

2.1. **Polynomial Interpolation.** This is a method of estimating new data points based on the range of a discrete set of known data points using the polynomial of lowest possible degree that passes through the points of the dataset [2]. In other words, consider pairs of data points: $\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \dots, \begin{bmatrix} x_{N-1} \\ y_{N-1} \end{bmatrix}\right)$, we want to find a function P such that $P(x_i) = y_i$. A polynomial $P(x_i)$ of degree $\leq N-1$ that estimates the true value y_i at the point x_i has a form:

$$P(x_i) = a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_{N-1} x_i^{N-1} = \sum_{k=0}^{N-1} \alpha_k x_i^k \text{ where } a_k \in \mathbb{R}$$

We can write this at a system of linear equations $V\underline{\alpha} = \underline{y}$, where $V \in \mathbb{R}^{N\times N}$ is the Vandermonde matrix, $\underline{\alpha}, \underline{y} \in \mathbb{R}^N$:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{N-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N-1} & x_{N-1}^2 & \cdots & x_{N-1}^{N-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a^{N-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y^{N-1} \end{bmatrix}$$

2.2. **Trigonometric Interpolation.** This is a method of estimating new data points based on the range of a discrete set of known data points using the sum of sines and cosines of given periods. In other words, consider pairs of data points: $\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \dots, \begin{bmatrix} x_{N-1} \\ y_{N-1} \end{bmatrix}\right)$, we want to find a function T such that $T(x_i) = y_i$. A trigonometric interpolant $T(x_i)$ that estimates the true value y_i at the point x_i has a form:

$$T(x_i) = \sum_{k=0}^{\frac{K}{2}-1} \alpha_k \cos(k\pi x_i) + \sum_{k=\frac{K}{2}}^{K-1} \alpha_k \sin\left(\left(k - \frac{K}{2} + 1\right)\pi x_i\right) \text{ where } a_k \in \mathbb{R}$$

We can write this as a system of linear equations $A\underline{\alpha} = \underline{y}$, where $A \in \mathbb{R}^{N\times K}, \underline{\alpha} \in \mathbb{R}^K, \underline{y} \in \mathbb{R}^N$:

$$\begin{bmatrix} \cos(0) & \cos(\pi x_0) & \cdots & \cos\left(\left(\frac{K}{2}-1\right)\pi x_0\right) & \sin(\pi x_0) & \cdots & \sin\left(\left(K - \frac{K}{2}\right)\pi x_0\right) \\ \cos(0) & \cos(\pi x_1) & \cdots & \cos\left(\left(\frac{K}{2}-1\right)\pi x_1\right) & \sin(\pi x_1) & \dots & \sin\left(\left(K - \frac{K}{2}\right)\pi x_1\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \cos(0) & \cos(\pi x_{N-1}) & \cdots & \cos\left(\left(\frac{K}{2}-1\right)\pi x_{N-1}\right) & \sin(\pi x_{N-1}) & \cdots & \sin\left(\left(K - \frac{K}{2}\right)\pi x_{N-1}\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{\frac{K}{2}-1} \\ a_{\frac{K}{2}} \\ \vdots \\ a^{K-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y^{N-1} \end{bmatrix}$$

2.3. **Runge's Phenomenon.** Runge's phenomenon is a problem of oscillation at the extremes of an interval that occurs when using polynomial interpolation with polynomials of high degree over a set of uniform interpolation points [1]. In this report, we perform the analysis on the function $f(x) = \frac{1}{1+25x^2}, x \in [-1, 1]$.

3. COMPUTATIONAL RESULTS

3.1. Polynomial Interpolation

For an uniform $\underline{x} \in \mathbb{R}^N$, where each points in the interval $[-1, 1]$ space equally, we can construct a polynomial interpolation that interpolates our true function $f(x) = \frac{1}{1+25x^2},\ x \in [-1, 1]$ at the points $\underline{x}$ for different values of N. We can see in Figure 1 that when there are $N = 2^3$ interpolating points, we don't get a precise estimation to the true function in the middle and even have some oscillation towards both ends of the interval. Although we thought that the more interpolating points we have the better, in fact, it is not true for the case of uniform $\underline{x}$. The interpolation looks better in the middle part of the interval for greater N, indeed $N = 2^5$ gives us the best estimation among three interpolants. However, it also has the most aggressive oscillation the towards both ends of the interval.

Repeat the same procedure for constructing polynomial interpolation as ealier for the Chebyshev $\underline{x} \in \mathbb{R}^N$, where $x_j = \cos\left(\frac{2(j+1)-1}{2N}\pi\right),\ \ j = 0, \dots, N-1$, we could see in Figure 2 that in this case, the more interpolating points we have, the better estimation we get.

Looking at Table 1, we could see that the determinant of the Vandermonde matrix gets smaller as the interpolating points grow bigger. By theory, if the determinant is 0, the matrix is singular, so there are infinitely many solutions, and thus the solutions we got, the coefficients α_j are unexpected. The polynomial interpolation with uniform set of points actually gives us the problem while, to our surprise, the Chebyshev nodes' looks pretty good as N grows.

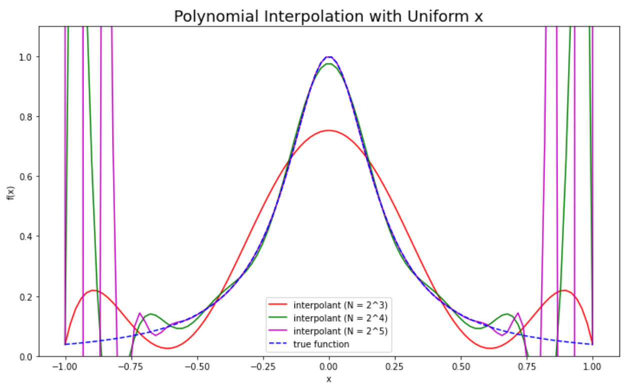

Figure 1: Polynomial Interpolation with Uniform x

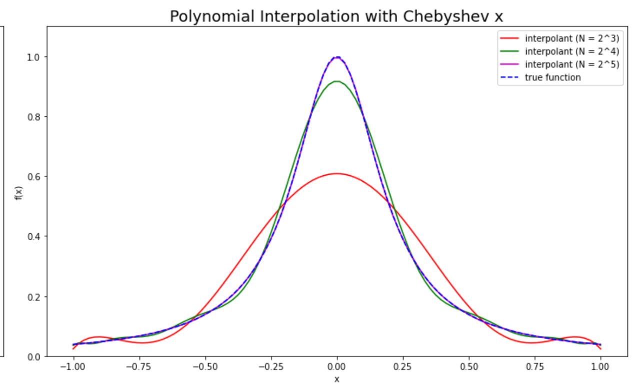

Figure 2: Polynomial Interpolation with Chebyshev x

	$N = 2^3$	$N = 2^4$	$N = 2^5$
Uniform $\underline{x}$	$\det(V) = 7.32 \times 10^{-5}$	$\det(V) = 8.89 \times 10^{-29}$	$\det(V) = 1.85 \times 10^{-141}$
Chebyshev $\underline{x}$	$\det(V) = 1.73 \times 10^{-4}$	$\det(V) = 5.85 \times 10^{-25}$	$\det(V) = 2.74 \times 10^{-121}$

Table 1: The determinant of Vandermonde matrix $V \in \mathbb{R}^{N \times N}$

3.2. Trigonometric Interpolation

For a uniform $\underline{x} \in \mathbb{R}^N$, we can construct a trigonometric interpolation that interpolates our true function $f(x)$ at the points $\underline{x}$ for different values of N. We can see in Figure 3 that when there are $N = 2^3$ interpolating points, we got quite a good estimation of the true function, but of course it could be done better. And the more interpolating points we get, the better interpolation we have. Looking at the $N = 2^4$ interpolant, we are pretty happy of how close the estimation is of the true function. Then, as we increase $N = 2^5$, it look like we get a perfect interpolation of all three.

Repeat the same procedure for constructing trigonometric interpolation as ealier for the Chebyshev $\underline{x} \in \mathbb{R}^N$ instead of uniform set of points, we could see in Figure 4 that in this case, we have a pretty bad estimation of our true function. The closest estimation we have here is with $N = 2^3$ interpolating points. Trigometric interpolation performs badly with the Chebyshev nodes.

Looking at Table 2, we could see that the determinant of the matrix $A \in \mathbb{R}^{N \times K}$ gets bigger as the interpolating points grow bigger for the uniform $\underline{x}$. In constrast, the determinant gets smaller for bigger N in the case of By theory, if the determinant is 0, the matrix is singular, so there are infinitely many solutions, and thus the Chebyshev nodes. And the results of trigonometric interpolation is consistent with our knowledge that the closer to 0 the determinant is, the more singular it is and the more likely it is to give 'bad' results.

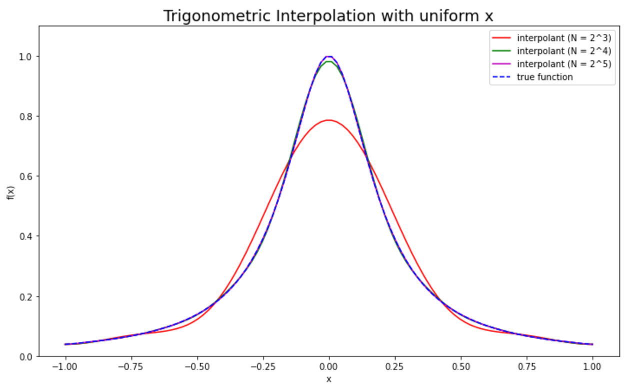

Figure 3: Trigonometric Interpolation with Uniform x

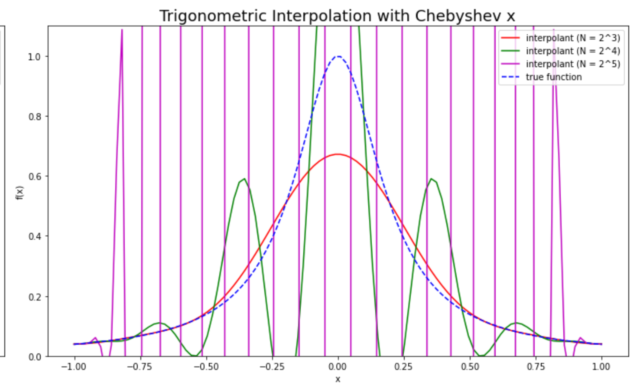

Figure 4: Trigonometric Interpolation with Chebyshev x

	$N = 2^3$	$N = 2^4$	$N = 2^5$
Uniform $\underline{x}$	$\det(A) = 1.94 \times 10^{-13}$	$\det(A) = 1.90 \times 10^{-8}$	$\det(A) = 58603.23$
Chebyshev $\underline{x}$	$\det(A) = 7.64$	$\det(A) = 0.40$	$\det(A) = 2.75 \times 10^{-14}$

Table 2: The determinant of matrix $A \in \mathbb{R}^{N \times K}$

4. SUMMARY AND CONCLUSIONS

In this report, we see the stability of a Runge's function like $f(x) = \frac{1}{1+25x^2}$. We discover that given a set of uniform interpolation points, the interpolation error increases when the degree of the polynomial is increased. It even has the same problem using the trigonometric interpolation over a set of Chebyshev nodes. One thing that is interesting to me is that the determinant of the Vandermonde matrix obtained by the Chebyshev nodes are against our knowledge as despite a very small determinant, which is close to 0, its polynomial interpolation perform a very good job on estimating the true function $f(x)$. In the case of trigonometric interpolation, the determinant of the matrix are consistent with what we know: the smaller the determinant, the worse interpolation we get.

REFERENCES

[1] Runge, C., 1901: Über empirische Funktionen und die Interpolation zwischen äquidistanten Ordinaten, *Zeitschrift für Mathematik und Physik*, 46, 224-243. Accessed 07 November 2022, https://archive.org/details/zeitschriftfrma12runggoog

[2] Tiemann, J.J., 1981: Polynomial Interpolation, *I/O News*, 1 (5), 16-19. Accessed 06 November 2022, https://archive.org/details/IoNewsVolume1Number5/mode/1up?view=theater