AMATH 352: HOMEWORK 3

JOLIE TRAN

ptran32@uw.edu

1. INTRODUCTION AND OVERVIEW OF THE PROBLEM

This report investigates the linear dependence of specific collections of vectors and studies the effectiveness of the Gram-Schmidt procedure by computing the pairwise angles between any 2 vectors.

2. THEORETICAL BACKGROUND AND DESCRIPTION OF ALGORITHM

2.1. **Inner Product and Orthogonality.** We know that two vectors $\underline{u}$ and $\underline{v}$ are orthogonal (i.e. the angle θ between them is $\frac{\pi}{2}$ rad) if their dot product is zero [1].

$$\theta = \arccos\left(\frac{\underline{u}^T\underline{v}}{\|\underline{u}\|_2\|\underline{v}\|_2}\right) = \frac{\pi}{2}rad \approx 1.57\, rad$$

2.2. **Linear Dependence and Pairwise Angle.** If the angle between any two vectors $\underline{u}, \underline{v}$ is $0\ rad$ or $\pi\ rad$, they are parallel to each other and thus they must be dependent. Any other values of angle indicate the linear independence between two vectors [3].

2.3. **Orthonormal vectors.** The collection of vectors $\underline{q}_1, \dots, \underline{q}_n$ is orthonormal if $\underline{q}_i \perp \underline{q}_j$ for $i \neq j, i, j = 1, \dots, n$ and $\left\|\underline{q}_i\right\|$=1 for $i = 1, \dots, n$ [1].

2.4. **Gram-Schmidt (GS) algorithm.** This algorithm can be used to determine if a list of n-vectors $\underline{v}_1, \dots, \underline{v}_n$ is linearly independent [1]. Classical Gram-Schmidt (CGS) can be unstable and result in the loss of orthogonalization. In CGS, we take each vector, one at a time, and make it orthogonal to all previous vectors. In modified Gram-Schmidt (MGS), we take each vector, and modify all forthcoming vectors to be orthogonal to it [2].

Classical Gram-Schmidt	Modified Gram-Schmidt
Given. n-vectors $\underline{v}_1, \dots, \underline{v}_n$ For. $i = 1, \dots, n$ 1. Orthogonalization. $\underline{q'}_i = \underline{v}_i - \sum_{j=1}^{i-1}\left(\underline{q}_j^T\underline{v}_i\right)\underline{q}_j$ 2. Test for linear dependence, if $\underline{q'}_i = 0$, quit 3. Normalization: $\underline{q}_i = \frac{1}{\|\underline{q'}_i\|_2}\underline{q'}_i$	**Given.** n-vectors $\underline{v}_1, \dots, \underline{v}_n$ For. $i = 1, \dots, n$ 1. Set $\underline{q'}_i = \underline{v}_i$ 2. Orthogonalization. $\underline{q'}_i = \underline{q'}_i - \sum_{j=1}^{i-1}\left(\underline{q}_i'^T\underline{q}_j\right)\underline{q}_j$ 3. Test for linear dependence, if $\underline{q'}_i = 0$, quit 4. Normalization: $\underline{q}_i = \frac{1}{\|\underline{q'}_i\|_2}\underline{q'}_i$

3. COMPUTATIONAL RESULTS

For a collection of N random Gaussian vectors $\underline{a}_i \in \mathbb{R}^d$ for $i \in \{1, \dots, N\}$, we can observe the linear independence of these vectors by computing the pairwise angles between the random vectors $\underline{a}_i, \underline{a}_k$ for $i \neq k$. In Figure 1, we see that the pairwise angles between the $\underline{a}_i$'s are far away from $0\ rad$, indicating that they are not parallel to each other. The distribution is bell-curved centering around $\frac{\pi}{2}rad \approx 1.57\ rad$, which suggest that majority of the vectors are orthogonal to each other and obviously all of them point in different direction. So could say that these vectors are linearly independent.

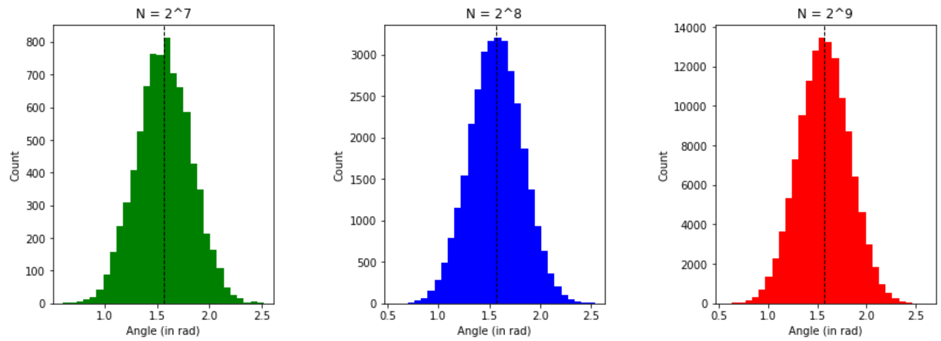

Figure 1: Pairwise Angles between $\underline{a_i}$'s, for $d = 2^4$ and $N = 2^7, 2^8, 2^9$

For a collection of vectors $\underline{b_i} \in \mathbb{R}^N$ for $i \in \{1, \dots, N\}$, where $b_{ij} = \frac{1}{i+j-1}$ we can observe the linear independence of these vectors by computing the pairwise angles between the vectors $\underline{b_i}, \underline{b_k}$, for $i \neq k$. We can see in Figure 2 that the histogram is right-skewed, and the pairwise angles are strictly smaller than $\frac{\pi}{2} rad \approx 1.57\, rad$, which imply that they are pointing in the same direction. Most of them 'look' linearly dependent because they distribute very near $0\, rad$, yet to confirm whether they are dependent, I need to use the Gram-Schmidt algorithm. In this algorithm, I check if the orthogonal vector is 0. If it's zero, I quit and declare the vectors are dependent. However, with rounding errors, I would never get a 0 in my computations unless I set a threshold for my values, i.e., if the absolute value of a number $\leq 10^{-16}$ then it is considered 0. Since the Gram-Schmitdt algorithms execute without interruption, I can say that vectors $\underline{b_i}$'s are linearly independent.

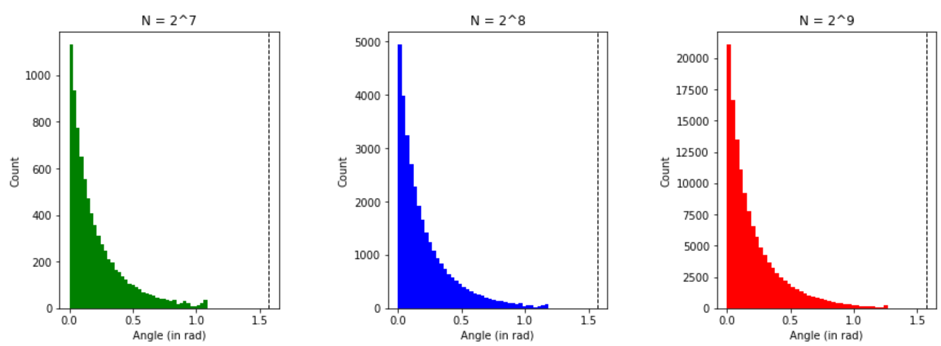

Figure 2: Pairwise Angles between $\underline{b_i}$'s, for $N = 2^7, 2^8, 2^9$

Using CGS to construct an orthonormal set of vectors $\underline{q_i} \in \mathbb{R}^N$ for $i \in \{1, \dots, N\}$ from the $\underline{b_i}$'s, we get an uneven distribution of angles and most of the values cluster near $0\, rad$ (Figure 3). This is not what I expected for an orthonormal basis. Clearly, CGS perform badly on the set of almost linearly dependent vectors $\underline{b_i}$'s.

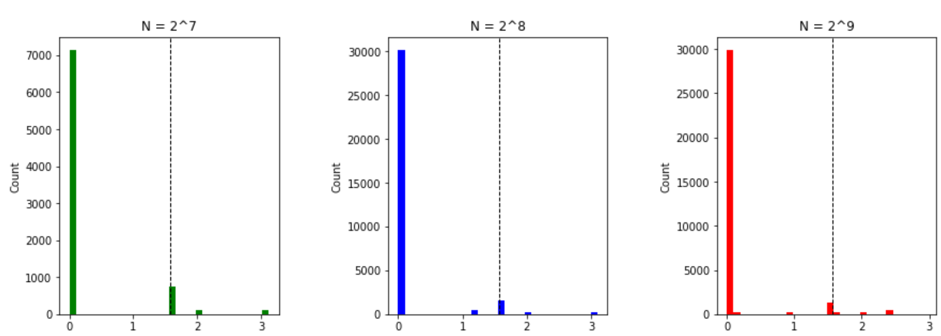

Figure 3: Pairwise Angles between $\underline{q}_i$'s, for $N = 2^7, 2^8, 2^9$ (CGS)

Using MGS to construct an orthonormal set of vectors $\underline{q}_i \in \mathbb{R}^N$ for $i \in \{1, \dots, N\}$ from the $\underline{b}_i$'s, we get the bell-curved histogram of pairwise angles between the vectors $\underline{q}_i, \underline{q}_k$ for $i \neq k$ centering around $\frac{\pi}{2} rad \approx 1.57\, rad$ (Figure 4). The distribution is very tall and thin, which suggests that most of the density around this value is very high. This is what we expect for the pairwise angle of the orthonormal basis. However, we still see some off-values due to the rounding errors within the computations. We can further see how the angles between $\underline{q}_i$'s distribute in Table 1. We can see in the table that the larger N gets, the smaller the standard deviation is.

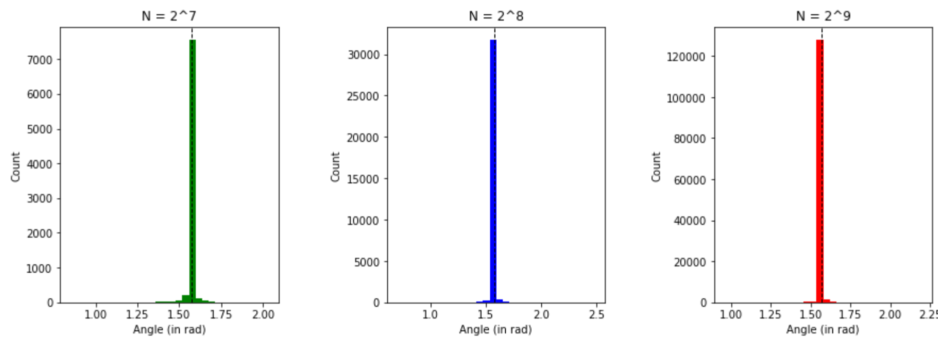

Figure 4: Pairwise Angles between $\underline{q}_i$'s, for $N = 2^7, 2^8, 2^9$ (MGS)

	$N = 2^7$	$N = 2^8$	$N = 2^9$
Mean	1.57	1.57	1.57
Standard Deviation	0.03	0.02	0.01

Table 1: Mean and Standard Deviation of Pairwise Angles between $\underline{q}_i$'s (MGS)

4. SUMMARY AND CONCLUSIONS

Given any collection of vectors, I can examine their linear independence by computing their pairwise angles and/or performing Gram-Schmidt algorithm on it. A random Gaussian set of vectors in high dimension is always linearly independent with a normal distibution centering around $\frac{\pi}{2} rad \approx 1.57\, rad$. For the set of vectors where its pairwise angles cluster near $0\, rad$, we can confirm its linearly independence with a MGS algorithm. A CGS would perform poorly on the 'almost' dependent set of vectors and result in the loss of orthogonality.

REFERENCES

[1] Boyd, S., Vandenberghe, L., 2018: *Introduction to Applied Linear Algebra – Vectors, Matrices, and Least Squares*, Cambridge University Press, 463 pp, https://web.stanford.edu/~boyd/vmls/vmls.pdf.

[2] Ford, W., 2015: Gram-Schmidt Orthonormalization, *Numerical Linear Algebra with Applications*, Academic Press, 281-197, https://doi.org/10.1016/B978-0-12-394435-1.00014-4.

[3] Sangaku, S.L., 2022: Linearly dependent and linearly independent vectors. Accessed 24 October 2022, https://www.sangakoo.com/en/unit/linearly-dependent-and-linearly-independent-vectors.