Homework 5

AMATH 352, Fall 2022

Due on Nov 7, 2022 at midnight.

Directions, Reminders and Policies

Read these instructions carefully:

- **You are required to upload a PDF report to Canvas along with a zip of your code.**
- The report should be a maximum of 3 pages long with references included. Minimum font size 10pts and margins of at least 1inch on A4 or standard letter size paper.
- Do not include your code in the report. Simply create a zip file of your main scripts and functions, without figures or data sets included, and upload the zip file to Canvas.
- Your report should be formatted as follows:
  - Title/author: Title of report, your name and email address. This is not meant to be a separate title page.
  - Sec. 1. Introduction and overview of the problem.
  - Sec. 2. Theoretical background and description of algorithms.
  - Sec. 3. Computational Results
  - Sec. 4. Summary and Conclusions
  - References
- I suggest you use LaTeX(Overleaf is a great option) to prepare your reports. A template is provided on Canvas under the Syllabus tab. You are also welcome to use Microsoft Word or any other software that properly typesets mathematical equations.
- I encourage collaborations, however, everything that is handed in (both your report and your code) should be your work.
- Your homework will be graded based on how completely you solved it as well as neatness and little things like: did you label your graphs and include figure captions. **The homework is worth 10 points. 5 points will be given for the overall layout, correctness and neatness of the report, and 5 additional points will be for specific technical things and computational results that the TAs will look for in the report itself.**

Warm up

Look up the following functions and commands in Python:

- Generating a uniform set of points in an interval: `numpy.linspace`
- Constructing Vandermonde matrices: `numpy.vander`
- Solving matrix equations and computing determinants: `numpy.linalg.solve`, `numpy.linalg.det`
- Trigonometric functions in numpy: `numpy.sin` and `numpy.cos`
- Creating high quality plots with `matplotlib.pyplot`, labeling axes and legends, setting figure sizes and axis limits.

Problem Description

Your goal in this HW is to interpolate Runge's function and investigate the numerical accuracy and stability of this problem.

Part I

1. Consider the function
$$f(x) = \frac{1}{1+25x^2}, \qquad x \in [-1,1].$$
and let $\underline{x} = (x_0, \ldots, x_{N-1}) \in \mathbb{R}^N$ be a uniform grid of N points in the interval $[-1,1]$, i.e, $x_j = \frac{2j}{N-1} - 1$. Your goal is to find a polynomial
$$P(x) = \sum_{k=0}^{N-1} \alpha_k x^k, \qquad x \in [-1,1],$$
of degree less than or equal to $N-1$, that interpolates f at the points $\underline{x}$.
2. Take $N = 2^3, 2^4, 2^5$, construct the Vandermonde matrix $V \in \mathbb{R}^{N\times N}$, and the right hand side vector $\underline{y} = (f(x_0), f(x_1), \ldots, f(x_{N-1})) \in \mathbb{R}^N$.
3. Solve the linear system
$$V\underline{\alpha} = \underline{y},$$
to find the coefficients α_j.
4. For each value of N plot your interpolating polynomial, obtained with the uniform x_j's, along with the true function f and report the determinant of V.

   Hint: use a fine uniform grid of 100 *points to plot your functions for good visualization. Make sure to generate legends if you plot multiple functions on the same figure and set your y-axis limits to* $[0, 1.1]$ *to make the functions visible. I suggest you plot all three of your interpolants along with* f *on the same figure to save space.*
5. Repeat the above steps by replacing the uniform points x_j with the Chebyshev nodes
$$x_j = \cos\left(\frac{2(j+1)-1}{2N}\pi\right), \qquad j = 0, \ldots, N-1.$$
6. For each value of N plot your new interpolating polynomial, obtained with the Chebyshev x_j's, along with the true function f and report the determinant of V.
7. Discuss your results. How does the choice of interpolating points affect the quality of the interpolating polynomial? how does it affect the determinant of V?

Part II

8. Your goal in the second part is to repeat the experiments of Part I with your interpolating polynomial replaced with a trigonometric polynomial

$$T(x) = \sum_{k=0}^{\frac{N}{2}-1} \alpha_k \cos(k\pi x) + \sum_{k=\frac{N}{2}}^{N-1} \alpha_k \sin\Big(\Big(k - \frac{N}{2} + 1\Big)\pi x\Big).$$

9. Take $\underline{x}$ to be the same set of uniform interpolation points as in Step 1 and formulate the corresponding linear system $A\underline{\alpha} = \underline{y}$ whose solution gives the coefficients α_j. Explain your derivation of A in the theory part of your report.
10. For each value of N plot your trigonometric interpolant, obtained with the uniform x_j's, along with the true function f and report the determinant of A.
11. Repeat the above experiments but this time take $\underline{x}$ to be the Chebyshev points.
12. For each value of N plot your trigonometric interpolant, obtained with the Chebyshev x_j's, along with the true function f and report the determinant of A.
13. Discuss your results. How does the choice of interpolating points affect the quality of the trigonometric interpolant? how does it affect the determinant of A? and how do your results compare with Part I?