# Hybrid implementation of observers in plant's coordinates with a finite number of approximate inversions and global convergence

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**Abstract**

We assume we are given a continuous-time observer whose dynamics are not written in the plant's coordinates and whose implementation requires the inversion of an injective immersion at each time. To avoid these costly computations, we propose methods to write the observer dynamics directly in the plant's coordinates by extending the injective immersion into a diffeomorphism, inverting its Jacobian, and using a hybrid mechanism to guarantee completeness of solutions. The obtained observers are proved to recover the same performances in terms of convergence and robustness to noise as the initial observer, and require only a finite number of approximate inversions. This methodology applies to a broad class of nonlinear observers including (low-power) high gain and Luenberger designs, and is illustrated on a Van der Pol oscillator with unknown parameters.

## 1 INTRODUCTION

Unlike for linear systems, no systematic method exists for the design of observers for nonlinear systems. However, observer design may be more or less straightforward depending on the coordinates chosen to express the plant's dynamics. That is why many observer designs consist in transforming the system, by coordinate change, into specific *normal forms*, identified for allowing a direct and easier observer construction. The high gain homogeneous designs with triangular normal forms ([20,26,15,11]), the nonlinear Luenberger design with Hurwitz normal forms ([3]), the linearizations by output injection with a linear normal form ([24] and references therein), are just examples of normal forms typically used in the literature. It follows that the dynamics of the plant and of the observer are often not expressed in the same coordinates and may even evolve in spaces of different dimensions. It is therefore necessary to invert the transformation, not only to deduce the estimate in the plant's original coordinates, but also sometimes even to define the observer dynamics as, for instance, in the high gain framework. However, although the transformation is generally well-known, its inversion can be difficult in practice. When an explicit expression of a global inverse is not available, numerical inversion usually relies on the resolution of a minimization problem with a heavy computational cost. That is why research is carried out to avoid as much as possible this inversion step.

In the case where the transformation is a diffeomorphism, one may hope to avoid this minimization by expressing the observer dynamics directly in the plant's coordinates via inversion of the Jacobian [17,28,9]. However, this observer must be treated carefully, since while the true state is known to stay in the domain of the diffeomorphism, there is no guarantee that its estimate will, in particular during transients behaviors where peaking can occur. In that case, the estimate may encounter Jacobian singularities, thus leading to non-converging non-complete solutions as pointed out in [13]. A solution proposed in [13] consists in extending the image of the diffeomorphism so that any observer trajectory evolves in it and, correspondingly, the estimate in the initial coordinates stays in the domain where the Jacobian is invertible. However, an explicit algorithm to compute this extension is not always available. Other routes consist in modifying the observer dynamics to force its state to remain in the diffeomorphism image by either adding a term in the dynamics [9] or using carefully designed complex saturations [28]. But this must be done with care since it can easily destroy the observer performances (in particular convergence): extra convexity assumptions are typically required on the diffeomorphism image to implement such methods.

In the more general case where the transformation is not a diffeomorphism, but an injective immersion, namely,

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the image space has a larger dimension than the domain, it has been proposed in [2,13] to extend the injective immersion into a diffeomorphism and implement the observer in the plant's coordinates. Although this extension is always possible, we then recover the same difficulties explained above. Other ideas have been proposed such as using Newton-like or gradient-like algorithms to inverse the transformation in [29,7], or continuation algorithms which "follow" the "optimal" inverse image in [23]. However, in those cases, the convergence is only local. Also, the design in [29,23] is restricted to high gains observers, and the transformation in [23] needs to verify a convexity assumption.

In this paper, we propose a compromise between a) the expression of the observer in the initial plant's coordinates, possibly raising issues of image completeness and requiring a difficult (maybe impossible) image extension, and b) the implementation of the observer in the observer coordinates, with a computationally demanding inversion of the transformation at each time. Assuming first that the transformation is a diffeomorphism, the idea is to implement the observer in the initial coordinates by Jacobian inversion, and to reset in an *appropriate* way the estimate in the diffeomorphism domain whenever it is about to leave it. By appropriate, we mean that the performances of the observer in terms of convergence and robustness to measurement noise must be preserved in the initial coordinates. We introduce two reset strategies satisfying this constraint: first, in Section 3, under a convexity assumption in the image space inspired by [4]; and then, in Section 4, removing this convexity assumption, and using an auxiliary practical observer (maybe of smaller dimension) implemented independently in other coordinates. Preliminary results concerning this latter approach were presented in [10]. Those resets involve the inversion of an injective map, but we show that if the measurement noise is not too large, they occur only a finite number of times (during the transient) and those inversions do not need to be exact, namely they can be achieved by solving a minimization problem on a rough grid. We show how those implementations can be used for a broad class of nonlinear observers such as the low-power high-gain observers presented in [6], specifically dealt with in Section 5. Finally, based on the extension proposed in [13], we show in Section 6 how this methodology can be used when the transformation is an injective immersion, and not a diffeomorphism. For this, we use as example the Van der Pol oscillator with unknown parameters and illustrate the efficiency of the method in simulations.

*Notations.* We denote $\mathbb{R}$ (resp. $\mathbb{N}$) the set of real numbers (resp. integers), and $\mathbb{R}_{\geq 0} = [0, +\infty)$, $\mathbb{R}_{>0} = (0, +\infty)$. An immersion $T$ on an open set $\mathcal{S}$ is a map such that $\frac{\partial T}{\partial x}(x)$ is full-rank for all $x$ in $\mathcal{S}$. For two subsets $S_1$ and $S_2$ of $\mathbb{R}^q$, we denote $d(S_1, S_2) = \min_{x_i \in S_i} |x_1 - x_2|$. We denote an inclusion $S_1 \subseteq S_2$, and a strict inclusion $S_1 \subset S_2$, the latter meaning that $d(S_1, \mathbb{R}^q \setminus S_2) > 0$. Also, $\operatorname{int} S$ stands for the interior of the set $S$, and $\partial S := \operatorname{cl}(S) \setminus \operatorname{int}(S)$ for its boundary. For a map $g$, we denote $g^k := g \circ \ldots \circ g$ $k$ times, and $g^0 = \mathrm{Id}$. For a vector $x$, $x_{i:j}$ is the vector made of the $i$th to $j$th component of $x$. For $u, v$ in $\mathbb{R}^3$, $u \times v \in \mathbb{R}^3$ denotes the cross product of $u$ and $v$.We consider hybrid dynamical systems of the form (see [22])

$$\dot{\xi} = F(\xi), \quad \xi \in C \qquad\qquad \xi^+ = G(\xi), \quad \xi \in D$$

where $F$ (resp. $G$) is the flow (resp. jump) map, and $C$ (resp. $D$) is the flow (resp. jump) set. Solutions to such systems are defined on so-called *hybrid time-domains*. A subset $E$ of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time-domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$, and it is a hybrid time domain if for any $(T, J) \in E$, $E \cap [0, T] \times \{0, \ldots, J\}$ is a compact hybrid time domain. For a solution $(t, j) \mapsto \xi(t, j)$ (see [22, Definition 2.6]), we denote $\operatorname{dom} \xi$ its domain, $\operatorname{dom}_t \xi$ (resp.$\operatorname{dom}_j \xi$) its projection on the time (resp. jump) component, and for a positive integer $j$, $t_j$ the only time defined by $(t_j, j) \in \operatorname{dom} \xi$ and $(t_j, j-1) \in \operatorname{dom} \xi$, and finally, $I_j$ the largest interval such that $I_j \times \{j\} \subseteq \operatorname{dom} \xi$. We say that $\xi$ is $t$-complete (resp. $j$-complete) if $T := \sup \operatorname{dom}_t \xi$ (resp. $J := \sup \operatorname{dom}_j \xi$) is infinite; and $\xi$ is eventually continuous if $J < +\infty$ and $T > t_J$.

## 2 PROBLEM STATEMENT

This paper deals with a nonlinear system of the form

$$\dot{x} = f(x) \quad , \quad y = h(x) + \nu \tag{1}$$

with state $x$ in $\mathbb{R}^n$ and output $y$ in $\mathbb{R}^p$ perturbed by a locally bounded disturbance $\nu$. Assume that there exist a subset $\mathcal{X}_0$ of $\mathbb{R}^n$ and a compact subset $\mathcal{X}$ of $\mathbb{R}^n$, such that for any initial condition $x_0$ in $\mathcal{X}_0$, the corresponding solution $t \mapsto x(t)$ to (1) remains in $\mathcal{X}$ for all times $t \geq 0$. We assume that we have an observer for (1), that is not necessarily expressed in the $x$-coordinates. Technically, we assume the following.

**Assumption 1 (Existence of an observer)** *There exist an open subset $\mathcal{S}$ of $\mathbb{R}^n$ and a closed subset $\mathcal{X}_s$ of $\mathcal{S}$ such that*

$$\mathcal{X} \subset \mathcal{X}_s \subset \mathcal{S} \subseteq \mathbb{R}^n \ , \tag{2}$$

*an integer $m \geq n$, an injective immersion $T : \mathcal{S} \to \mathbb{R}^m$, and continuous functions $\varphi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ and $\mathcal{T} : \mathbb{R}^m \to \mathbb{R}^n$ verifying*

$$\mathcal{T}(T(x)) = x \qquad \forall x \in \mathcal{X}_s \ , \tag{3}$$

*such that for any locally bounded disturbance $\nu$, for any solution $t \mapsto x(t)$ to (1) with $x(0) \in \mathcal{X}_0$, and for any solution $t \mapsto \hat{\xi}(t)$ to*

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, \hat{x}, y) \quad , \quad \hat{x} = \mathcal{T}(\hat{\xi}) \ , \tag{4}$$

*with $y = h(x) + \nu$, we have*

$$|\hat{\xi}(t) - T(x(t))| \leq \beta(|\hat{\xi}_0 - T(x_0)|, t) + \alpha\Big( \sup_{s \in [0,t]} |\nu(s)| \Big) \ , \tag{5}$$

*for some class-$\mathcal{K}$ function $\alpha$ and some $\mathcal{KL}$-function $\beta$.*

We observe that if the noise $\nu$ is bounded, the solutions to (4) are bounded and the asymptotic property expressed by (5) can be brought back in the $x$-coordinates if $\mathcal{T}$ is locally Lipschitz. As a matter of fact, since $x(t) \in \mathcal{X}$,

$$|\hat{x}(t) - x(t)| = |\mathcal{T}(\hat{\xi}(t)) - \mathcal{T}(T(x(t)))| ,$$

according to (3). Furthermore, since $T$ is an injective immersion, it is Lipschitz, and Lipschitz-injective on any compact subset $\mathcal{C}$ of $\mathcal{S}$, namely there exist positive scalars $L$ and $L_I$ such that for all $(x_a, x_b) \in \mathcal{C} \times \mathcal{C}$,

$$|T(x_a) - T(x_b)| \leq L\,|x_a - x_b| \tag{6a}$$

$$|x_a - x_b| \leq L_I\,|T(x_a) - T(x_b)| , \tag{6b}$$

(see [1, Lemma 3.2] for instance). According to (3), $L_I$ is actually the Lipschitz constant of $\mathcal{T}$ on $T(\mathcal{C})$ and, if $\hat{\xi}_0 = T(\hat{x}_0)$ with $\hat{x}_0$ in $\mathcal{C}$, it follows immediately that for all $t$ such that $\hat{\xi}(t) \in T(\mathcal{C})$,

$$|\hat{x}(t) - x(t)| \leq L_I\beta(L|\hat{x}_0 - x_0|, t) + L_I\alpha\Big(\sup_{s\in[0,t]} |\nu(s)|\Big) . \tag{7}$$

For instance, if $T$ is a diffeomorphism (i.e. $m = n$), since $T(x(t))$ is in the compact set $T(\mathcal{X})$, which is strictly contained in the open interior of $T(\mathcal{C})$, $\hat{\xi}$ remains in $T(\mathcal{C})$ after a certain time according to (5), unless the noise $\nu$ is too large. Therefore, (7) holds after that time. On the other hand, if $T$ is an injective immersion (i.e. $m > n$), $T(\mathcal{C})$ is only a manifold with empty interior, but (5) shows that $\hat{\xi}$ eventually gets close to $T(\mathcal{C})$. Therefore, if $\mathcal{T}$ is Lipschitz in a neighborhood of $T(\mathcal{C})$, (7) also holds after a certain time. In other words, the observer performances are preserved in the initial coordinates, modulo the injectivity constant of $T$. This constant may be large if $T$ is "poorly injective", typically if its Jacobian is poorly conditioned.

**Example 1** *Consider a strongly differentially observable system of order $m$, i.e. such that $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$T(x) = (h(x), L_f h(x), \ldots, L_f^{m-1}h(x)) \tag{8}$$

*is an injective immersion on $\mathcal{S}$ containing $\mathcal{X}$ ([21, Definition I.2.4.2]). In a high gain design ([20,25]), Assumption 1 typically holds with*

$$\varphi(\hat{\xi}, \hat{x}, y) = \Big(A\,\hat{\xi} + B\ \mathrm{sat}(L_f^m h(\hat{x})) + \mathcal{L}_1 K(y - z_1)\Big) , \tag{9}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ \vdots & & & & 1 \\ 0 & 0 & \ldots & & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} , \tag{10}$$

$$\mathcal{L}_1 = \mathrm{diag}(\ell_1, \ell_1^2, \ldots, \ell_1^m) , \tag{11}$$

*$K = (k_1, k_2, \ldots, k_m)^\top$ such that $A - KC$ Hurwitz, $\mathcal{T}$ a function verifying (3) and Lipschitz in a neighborhood of $T(\mathcal{X})$, the saturation level chosen such that*

$$\mathrm{sat}(L_f^m h(x)) = L_f^m h(x) \quad \forall x \in \mathcal{X} , \tag{12}$$

*a positive scalar $\ell_1$ chosen sufficiently large, and*

$$\beta(s, t) = \tilde{\gamma}_1 \ell_1^{m-1} e^{-\lambda\,\ell_1 t}\, s \quad , \quad \alpha(s) = \tilde{\gamma}_2\, \ell_1^{m-1}\, s \tag{13}$$

*for positive scalars $\tilde{\gamma}_i > 0$ and $\lambda > 0$.*

**Example 2** *In a Luenberger design ([3]), a backward-distinguishable system is transformed via a map $T$ into a Hurwitz normal form whose observer is simply given by*

$$\varphi(\hat{\xi}, \hat{x}, y) = M\,\hat{\xi} + N\,y$$

*with $M$ Hurwitz, and $(M, N)$ a controllable pair. If $T$ is an injective immersion, due to the linearity of the error dynamics, Assumption 1 holds with*

$$\beta(s, t) = \tilde{\gamma}_1 e^{-\lambda t}\, s \quad , \quad \alpha(s) = \tilde{\gamma}_2\, s$$

*with $\lambda$ the smallest absolute value of the eigenvalues of $M$ and positive scalars $\tilde{\gamma}_i$. Since $\varphi$ is independent from $\hat{x}$, (5) is obtained for any map $\mathcal{T}$.*

In order to use the observer given by Assumption 1, the inversion of $T$, namely the computation of the map $\mathcal{T}$, is crucial for two reasons : a) to deduce from $\hat{\xi}$ an estimate $\hat{x}$ of $x$, and b) sometimes to write the observer dynamics (4) themselves. For instance, to implement the high-gain dynamics in Example 1, we need to compute the expression of $L_f^m h(\mathcal{T}(\hat{\xi}))$. Although this can sometimes be done through elimination techniques without needing an explicit expression of $\mathcal{T}$, this step often remains problematic. We have highlighted this possible problem by making $\varphi$ explicitly depend on $\hat{x} = \mathcal{T}(\hat{\xi})$. Practically, however, an explicit analytical expression of $\mathcal{T}$ is rarely available and inversion typically relies on the resolution of a minimization problem of the type

$$\hat{x} = \operatorname*{argmin}_{x\in\mathcal{X}} |T(x) - \hat{\xi}| \tag{14}$$

at each time step, with a heavy computational cost. This motivates the interest of solutions that avoid this inversion as much as possible by implementing the observer in the $x$-coordinates.

**Example 3** *In [10], an example of a bioreactor was presented. Another is a van der Pol oscillator*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega^2 x_1 + \mu(1 - x_1^2)x_2 \end{cases} \quad , \quad y = x_1$$

*with unknown positive parameters $\omega$ and $\mu$. By adding $x_3 = \omega^2$ and $x_4 = \mu$ to the state, we get the dynamics*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_3 x_1 + x_4(1-x_1^2)x_2 \quad , \quad y = x_1 \ , \\ \dot{x}_3 = 0 \quad , \quad \dot{x}_4 = 0 \end{cases}$$

*of dimension $n = 4$. As observed in [19], the map $T$ defined in* (8) *with $m = 4$ admits singularities and we need to take $m = 5$ to obtain an injective immersion on a subset $\mathcal{S}$ inside which the solutions evolve. Its expression is straightforward to obtain but complex and omitted here. Following Example 1, Assumption 1 is satisfied with a high gain design. However, the reader can easily see that inversing globally $T$, namely computing $\mathcal{T}$, is challenging and cannot be done analytically.*

Let us assume for now that the map $T$ given by Assumption 1 is a diffeomorphism, namely $m = n$. The simplest implementation of the observer (4) in the plant's $x$-coordinates would be

$$\dot{\hat{x}} = \left(\frac{\partial T}{\partial x}(\hat{x})\right)^{-1} \varphi\Big(T(\hat{x}), \hat{x}, y\Big) \ , \tag{15}$$

as in [17,28,9] for instance. But, as mentioned in the introduction, although $x$ is known to remain in $\mathcal{S}$ where the diffeomorphism is defined and invertible, there is no guarantee that $\hat{x}$ will, so that solutions could encounter a singularity of the Jacobian. Therefore, even in this simplified case, completeness and convergence are not ensured. This problem disappears when $T$ is surjective, i.e. $T(\mathcal{S}) = \mathbb{R}^m$, since $\hat{\xi} = T(\hat{x})$ necessarily remains in $T(\mathcal{S})$, or equivalently $\hat{x}$ remains in $\mathcal{S}$. This is exploited in [11] by extending the image of $T$, namely finding a surjective diffeomorphism $T_e$ which agrees with $T$ on $\mathcal{X}$, and replacing $T$ by $T_e$ in (15). It is proved in [11] that such an extension exists if $T$ is $C^2$ and if $\mathcal{S}$ is $C^2$-diffeomorphic to $\mathbb{R}^n$. The problem is that a systematic construction of the extended diffeomorphism $T_e$ is not always available, in particular when the image set is not well-known.

Consider a compact set $\mathcal{X}'_s$ verifying

$$\mathcal{X} \subset \mathcal{X}'_s \subset \mathcal{X}_s \subset \mathcal{S} \subseteq \mathbb{R}^n \ . \tag{16}$$

To avoid the image extension, we propose to implement (15) as long as the estimate $\hat{x}$ is in the *safe set* $\mathcal{X}_s$, and reset $\hat{x}$ in the compact set $\mathcal{X}'_s$ every time it exits $\mathcal{X}_s$, to avoid singularities. It is important to notice that our definition of the strict inclusion $\subset$ implies that the minimal distance between $\mathcal{X}'_s$ and the boundary of $\mathcal{X}_s$ is non zero, so that there will always be a positive amount of flow between each resets. Observe also that the resets have to be done with care because they modify the observer trajectories and could therefore destroy its performances (convergence and robustness) given by (5). In the following, we propose two reset strategies that preserve those performances: first in Section 3 under a convexity assumption on the image of $T$, and then, removing this assumption in Section 4. Before describing those strategies, it will be useful to introduce the distances between the boundaries of $T(\mathcal{X})$, $T(\mathcal{X}'_s)$ and $T(\mathcal{X}_s)$, namely,

$$\delta := d\Big(T(\mathcal{X}), \mathbb{R}^m \setminus T(\mathcal{X}'_s)\Big) > 0 \tag{17a}$$

$$\delta' := d\Big(T(\mathcal{X}'_s), \mathbb{R}^m \setminus T(\mathcal{X}_s)\Big) > 0 \ , \tag{17b}$$

which are well-defined since $T$ is a diffeomorphism and the image spaces $T(\mathcal{X}_s)$ and $T(\mathcal{X}'_s)$ have non-empty interiors. According to (16), this means in particular that

$$d\Big(T(\mathcal{X}), \mathbb{R}^n \setminus T(\mathcal{X}_s)\Big) \geq \delta + \delta' \ . \tag{17c}$$

## 3 $T$ diffeomorphism with convex image

In this section, we draw inspiration from [4] which shows how to modify a given discrete-time observer to keep its state inside a prescribed compact convex set without disturbing its performances. We propose to combine our continuous-time observer with a similar "discrete-time strategy" in the image coordinates in order to bring $\hat{\xi} = T(\hat{x})$ back into $T(\mathcal{X}'_s)$ whenever it leaves $T(\mathcal{X}_s)$: this is indeed equivalent to bringing $\hat{x}$ back into $\mathcal{X}'_s$ when it leaves $\mathcal{X}_s$. More precisely, we still assume that $T$ is a diffeomorphism and we consider the favorable case where the image set satisfies some convexity assumption.

**Assumption 2 (Image convexity)** *There exist a $C^1$ function $c : \mathbb{R}^m \to \mathbb{R}^r$ and $\mathfrak{c}_m > 0$ such that each component $c_i : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ of $c$ is convex, namely*

$$c_i(\hat{\xi}) \leq c_i(\xi) + \frac{dc_i}{d\xi}(\hat{\xi})(\hat{\xi}-\xi) \quad \forall (\hat{\xi}, \xi) \in \mathbb{R}^m \times \mathbb{R}^m \ , \tag{18}$$

*and defining*

$$\mathcal{C}_0 := \{\xi \in \mathbb{R}^m \ : \ c(\xi) = 0\}$$

$$\mathcal{C}_m := \{\xi \in \mathbb{R}^m \ : \ |c(\xi)| \leq \mathfrak{c}_m\} \ ,$$

*we have*

$$T(\mathcal{X}) \subseteq \mathcal{C}_0 \subset \mathcal{C}_m \subset T(\mathcal{X}_s) \ . \tag{19}$$

In the case where $\mathcal{C}_0$ is compact (typically if $c$ is proper), this assumption is actually equivalent to the existence of a compact convex set $\mathcal{C}$ such that $T(\mathcal{X}) \subseteq \mathcal{C} \subset T(\mathcal{X}_s)$. Indeed, according to [16], $\mathcal{C}$ can then be outer-approximated arbitrarily close by a polytope $\mathcal{C}_0$ for which we can take $c$ of the form $c_i(\xi) = \max\{M_i\xi - b_i, 0\}^2$ for $1 \leq i \leq r$. In any case, Assumption 2 imposes that a convex set containing $T(\mathcal{X})$ can be included in $T(\mathcal{X}_s)$. This is not always verified, in particular when Jacobian singularities create "holes" in the image, or when $T$ is obtained by extending an injective immersion as will be done later in Section 6. However, this assumption is verified for the bioreator model studied in [20,13]. A graphical sketch of the sets involved in Assumption 2 is given in Figure 1.

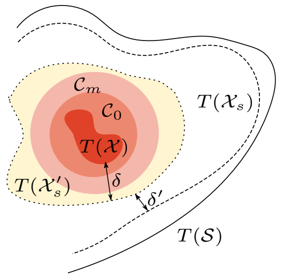


Fig. 1. Sets of interest in the $\xi$-coordinates, namely the image by the diffeomorphism $T$ of the $x$-coordinates where $\mathcal{X}$, $\mathcal{X}'_s$, $\mathcal{X}_s$ and $\mathcal{S}$ are defined verifying (16). $\mathcal{C}_0$ and $\mathcal{C}_m$ are defined in Assumption 2.

Besides Assumption 2, we assume that the stability properties of observer (4) hide a quadratic ISS Lyapunov function.

**Assumption 3 (Quadratic ISS Lyapunov function)** *There exist a positive definite matrix $P$, a class-$\mathcal{K}$ function $\alpha_0$, and a positive scalar $\lambda$ such that, defining $V : \mathbb{R}^m \to \mathbb{R}$ by*

$$V(e) = e^\top P e \ ,$$

*we have for all $x$ in $\mathcal{X}$, all $\hat{\xi}$ in $\mathbb{R}^m$, and all $\nu$ in $\mathbb{R}^p$,*

$$\frac{\partial V}{\partial e}(\hat{\xi} - T(x)) \left( \varphi(\hat{\xi}, \mathcal{T}(\hat{\xi}), h(x) + \nu) - \frac{\partial T}{\partial x}(x) f(x) \right) \leq -\lambda V(\hat{\xi} - T(x)) + \alpha_0(|\nu|) \ . \quad (20)$$

The left-hand side of (20) is nothing but the derivative of $V$ along the error trajectory $\hat{\xi} - T(x)$, with $\hat{\xi}$ and $x$ following the dynamics (4) and (1) respectively. In the following, the smallest and largest eigenvalues of $P$ are denoted $\underline{\lambda}(P)$ and $\overline{\lambda}(P)$ respectively, so that

$$\underline{\lambda}(P)|e|^2 \leq V(e) \leq \overline{\lambda}(P)|e|^2 \qquad \forall e \in \mathbb{R}^m \ . \quad (21)$$

**Remark 1** *The existence of an ISS Lyapunov function is guaranteed by Assumption 1 [31], but Assumption 3 requires that it be a quadratic function of the error $\hat{\xi} - T(x)$. This is a rather strong assumption [30], but many standard observer designs fit in this context. In the high gain design from Example 1, Assumption 3 holds with $P$ of the form $P = \mathcal{L}_1^{-1} P_0 \mathcal{L}_1^{-1}$ with $P_0$ a positive definite matrix verifying $P_0(A - KC) + (A - KC)^\top P_0 \leq -P_0$. As for the Luenberger design from Example 2, Assumption 3 holds with $P$ such that $PM + M^\top P \leq -P$. Actually, it can be shown that this method can also be applied with a time-varying $P$ as long as (21) holds uniformly in time. Therefore, Kalman-like designs such as [14] (and references therein) could also be considered. On the other hand, Assumption 3 does not hold for homogeneous/sliding mode designs [12,27] for instance.*

We now make the technical assumption that (5) is actually obtained as a consequence of (20), namely the maps $(\beta, \alpha)$ in Assumption 1 are linked to $\lambda$ and $\alpha_0$ is the following way.

**Assumption 4** *The maps $(\beta, \alpha)$ in Assumption 1 verify for all $(s,t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$,*

$$\beta(s,t) \geq \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}} e^{-\frac{\lambda}{2}t} s \quad , \quad \alpha(s) \geq \sqrt{\frac{\alpha_0(s)}{\lambda\,\underline{\lambda}(P)}} \ . \quad (22)$$

We are now ready to introduce our observer. Unlike in [4], the set $T(\mathcal{X}_s)$ where $\hat{\xi}$ should remain is not bounded. So we first start by defining a compact set $\hat{\Xi}$ where the trajectories of (4) evolve. For that, assume the measurement noise $\nu$ is bounded by $\nu_{m,0}$ and that we initialize our observer in a compact subset $\hat{\mathcal{X}}_0$ of $\mathcal{X}_s$. Consider positive scalars $v_0$ and $v_m$ such that

$$v_0 \geq \max_{(\hat{x}_0, x_0) \in \hat{\mathcal{X}}_0 \times \mathcal{X}_0} V(T(\hat{x}_0) - T(x_0)) \quad (23a)$$

$$v_m \geq \max \left\{ v_0, \frac{1}{\lambda} \alpha_0(\nu_{m,0}) \right\} , \quad (23b)$$

where $V$, $\lambda$ and $\alpha_0$ are given by Assumption 3. In other words, $v_m$ is chosen such that any function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ verifying $\frac{dv}{dt}(t) \leq -\lambda v(t) + \alpha_0(\nu_{0,m})$ for almost all $t$ and $v(0) \in [0, v_0]$, is upper-bounded by $v_m$. According to (20), and since $v_m \geq v_0$, it means that $V(\hat{\xi} - T(x))$ remains bounded by $v_m$ along the trajectories of (1)-(4) when initialized in $\mathcal{X}_0 \times T(\hat{\mathcal{X}}_0)$. Defining

$$\hat{\Xi} := \left\{ \hat{\xi} \in \mathbb{R}^m \ : \ \exists x \in \mathcal{X} \ , \ V(\hat{\xi} - T(x)) \leq v_m \right\} \supset T(\mathcal{X}) \quad (24)$$

we can infer that if the plant is initialized in $\mathcal{X}_0$, any trajectory of (4) originating in $T(\hat{\mathcal{X}}_0)$ remains in the compact set $\hat{\Xi}$. From there, inspired by [4], we define the map $g : \mathbb{R}^m \to \mathbb{R}^m$ by

$$g(\xi) = \xi - \gamma P^{-1} \frac{dc}{d\xi}(\xi)^\top c(\xi) \ , \quad (25)$$

with some positive scalar $\gamma$. As shown in Lemma 4 below, if $\gamma \leq \gamma^*$ with

$$\gamma^* = \frac{\underline{\lambda}(P)}{\pi_m^2} \quad , \quad \pi_m := \max_{\xi \in \hat{\Xi}} \left| \frac{dc}{d\xi}(\xi) \right| \ , \quad (26)$$

updating $\hat{\xi} \in \hat{\Xi}$ by $g(\hat{\xi})$ makes $V$ decrease, whatever the true value of $\xi = T(x)$ is in $T(\mathcal{X})$. Also, if $\hat{\xi}$ is outside of $\mathcal{C}_m$, $V$ decreases by $\gamma \mathfrak{c}_m^2$. Therefore, similarly to what is done in [4], when the state $\hat{x}$ of (15) reaches the boundary of the safe set $\mathcal{X}_s$, the idea is to apply repetitively $g$ to $\hat{\xi} = T(\hat{x})$ until $\hat{\xi}$ is in $\mathcal{C}_m$ (which happens after a finite number of jumps). At this point, since $\mathcal{C}_m \subset T(\mathcal{X}_s)$, $\hat{x}$ can be reset (maybe approximately) to $T^{-1}(\hat{\xi})$ which is in the interior of $\mathcal{X}_s$ and the observer can be relaunched in the $x$-coordinates with this new initial condition. The key-point is of course that the successive application of $g$ does not make $V$ increase and therefore does not alter the convergence. This process is modelled in the following hybrid system:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dot{\zeta} = \left(\dfrac{dT}{dx}(\zeta)\right)^{-1} \varphi\Big(T(\zeta), \zeta, y\Big) \\ \dot{q} = 0 \end{array}\right\} (\zeta, q) \in C \\ \\ \left.\begin{array}{l} \zeta^+ = T(\zeta) \\ q^+ = 0 \end{array}\right\} (\zeta, q) \in D_1 \\ \\ \left.\begin{pmatrix} \zeta \\ q \end{pmatrix}^+ \in G_0(\zeta, q)\right\} (\zeta, q) \in D_0 \end{array}\right. \tag{27}$$

with

$$G_0(\zeta, q) = \begin{cases} \begin{pmatrix} g(\zeta) \\ 0 \end{pmatrix}, & (\zeta, q) \in (\mathbb{R}^m \setminus \mathcal{C}_m) \times \{0\} \\ \begin{pmatrix} \mathcal{T}_\varepsilon(\zeta) \\ 1 \end{pmatrix}, & (\zeta, q) \in \mathrm{int}(\mathcal{C}_m) \times \{0\} \\ \begin{pmatrix} g(\zeta) \\ 0 \end{pmatrix} \cup \begin{pmatrix} \mathcal{T}_\varepsilon(\zeta) \\ 1 \end{pmatrix}, & (\zeta, q) \in \partial\mathcal{C}_m \times \{0\} \end{cases}$$

the flow set

$$C = \mathcal{X}_s \times \{1\} ,$$

the jump set $D = D_1 \cup D_0$ with

$$D_1 = \partial\mathcal{X}_s \times \{1\} , \quad D_0 = \mathbb{R}^m \times \{0\} ,$$

where $\mathcal{T}_\varepsilon$ is an approximation of $T^{-1}$ to be defined.

The dynamics (27) have two modes. When $q = 1$ (flow mode), $\zeta$ is in $\mathcal{X}_s$, plays the role of the estimate $\hat{x}$ for $x$, and the observer dynamics (4) are implemented in the $x$-coordinates, namely according to (15). If $\zeta$ (i.e. $\hat{x}$) reaches the boundary $\partial\mathcal{X}_s$ (and if no flow is possible into $\mathcal{X}_s$), $q$ is switched to 0, and $\zeta$ to $T(\zeta)$, namely $\zeta$ is moved to the $\xi$-coordinates: $\zeta$ is now an estimate $\hat{\xi}$ of $\xi = T(x)$. At that point, $\zeta$ jumps according to $g$ until it reaches $\mathcal{C}_m$, where we can bring $\zeta$ back to the $x$-coordinates through a (maybe approximated) inverse $\mathcal{T}_\varepsilon$ of $T$, and start again flowing. Note that according to the definition of $G_0$, when $\zeta$ in on the boundary of $\mathcal{C}_m$, we allow either to carry on applying $g$ to make the Lyapunov function decrease as much as possible, or to stop and come back to the $x$-coordinates. This choice makes the jump map outer-semicontinuous so that the hybrid basic conditions are satisfied ([22, Assumption 6.5]) and numerical robustness is ensured.

**Theorem 1** *Suppose that Assumptions 1, 2, 3 and 4 hold, with $T$ a diffeomorphism, and that*

$$\mathcal{C}_m \cap \hat{\Xi} \subseteq T(\mathcal{X}_s') . \tag{28}$$

*There exist positive scalars $\nu_m$, $\varepsilon$, $L$, and $L_I$ such that for any continuous map $\mathcal{T}_\varepsilon : \mathcal{C}_m \to \mathbb{R}^n$ verifying*

$$\mathcal{T}_\varepsilon(\mathcal{C}_m) \subset \mathcal{X}_s' , \tag{29a}$$

$$\left|\mathcal{T}_\varepsilon(\xi) - T^{-1}(\xi)\right| \leq \varepsilon \qquad \forall \xi \in \mathcal{C}_m \cap \hat{\Xi} , \tag{29b}$$

*for any $\gamma \leq \gamma^*$ defined in (26), for any $(x_0, \hat{x}_0) \in \mathcal{X}_0 \times \hat{\mathcal{X}}_0$ and for any t-complete solution $t \mapsto x(t)$ to system (1) initialized at $x_0$, any maximal solution $\phi = (\zeta, q)$ to (27) initialized at $(\hat{x}_0, 1)$ is t-complete, and for all $(t, j)$ such that $q(t, j) = 1$,*

$$|\zeta(t, j) - x(t)| \leq L_I \beta\Big(L|\hat{x}_0 - x_0|, t\Big) + L_I \alpha(\nu_m) . \tag{30}$$

*Furthermore, if $|\nu(t)| \leq \nu_m$ for all $t$, $\phi$ is eventually continuous, and $q(t, J) = 1$ for all $t \geq t_J$, with $J := \max \mathrm{dom}_j \phi < +\infty$.*

**PROOF.** See Appendix B.

The domain of the trajectories of (27) consists of a succession of flow episodes where $q = 1$, separated by a finite number of jumps where $q = 0$. If the noise is not too large, the last flow episode is infinite, i.e. the jumps stop after some time. This means that (30) is ensured at all times, except during the finite number of jump episodes. In other words, the performances of the observer given by Assumption 1 are preserved at all times in the $x$-coordinates. Actually, although it is not quantified here, the convergence is likely to be faster with the hybrid algorithm (27) than with (4). Indeed, the Lyapunov function $V$ decreases through the jumps (unless this decrease is totally compensated by the imprecise inversion of $T$), and suppresses any transient behavior that would bring the observer out of the diffeomorphism image.

In fact, every time the estimate $\zeta$ leaves the safe set $\mathcal{X}_s$, some jumps occur using the map $g$, which end by an inversion of $T$ via $\mathcal{T}_\varepsilon$. Although the definition of the dynamics (27) forces us to define $\mathcal{T}_\varepsilon$ on the possibly unbounded set $\mathcal{C}_m$, the proof enables to show that $\mathcal{T}_\varepsilon$ is actually used only on the compact set $\mathcal{C}_m \cap \hat{\Xi}$. Besides, each use of $g$ makes $V$ decrease by $\gamma \mathfrak{c}_m^2$, which gives a margin of error for the inversion of $T$. More precisely,

- either $\hat{\Xi} \subseteq \mathcal{C}_m$, and the trajectories remain in the safe set, no jump occurs, and therefore, the inverse map $\mathcal{T}_\varepsilon$ (and $\varepsilon$) can be chosen arbitrarily.
- otherwise, it is shown in Lemma 4 that $v_m \geq \gamma \mathfrak{c}_m^2$, and the inversion precision can be taken as

$$\varepsilon = \frac{\sqrt{v_m} - \sqrt{v_m - \gamma \mathfrak{c}_m^2}}{L \sqrt{\overline{\lambda}(P)}} \tag{31}$$

with $L$ defined in (6a) for $\mathcal{C} = \mathcal{X}_s'$.

This bound is actually conservative and assumes the worst case where only one jump occurs. At a given jump time, $V$ decreases by at least $k\gamma \mathfrak{c}_m^2$, with $k$ the number of successive jumps, and thus, the more jumps, the larger the decrease of $V$, and the rougher the approximation can be at this time. We conclude that, thanks to (28), the inversions can for instance be managed by a minimization on a rough grid of the compact set $\mathcal{X}_s'$, namely

$$\mathcal{T}_\varepsilon(\xi) = \operatorname*{argmin}_{x\in\mathcal{X}_s'} |T(x) - \xi| \ . \tag{32}$$

With this latter choice of $\mathcal{T}_\varepsilon$, it is not the definition of $\mathcal{T}_\varepsilon$ that is challenging, but its implementation. Hence the advantage of having only a finite number of inversions to carry out, instead of inverting the map $T$ at all times.

Finally, regarding the measurement noise, observe that $t$-completeness of solutions and performance recovery are ensured whatever the perturbation $\nu$. This is not the case of (15) where the trajectories could encounter a singularity and diverge. In fact, the role of $\nu_m$ is only to guarantee that the solutions are eventually continuous and therefore the inversions stop after some time. More precisely, consider a positive scalar $v_{\mathcal{C}}$ such that for all $\hat{\xi}$ in $\mathbb{R}^m$,

$$\exists\, x \in \mathcal{X} \ : \ V(\hat{\xi} - T(x)) \leq v_{\mathcal{C}} \quad \Longrightarrow \quad \hat{\xi} \in \operatorname{int}\mathcal{C}_m \ . \tag{33}$$

$v_{\mathcal{C}}$ is a positive threshold for $V$ below which we can deduce that $\hat{\xi} \in \operatorname{int}\mathcal{C}_m$, whatever $x$ in $\mathcal{X}$. Such a $v_{\mathcal{C}}$ exists because $T(\mathcal{X})$ is a strict subset of $\mathcal{C}_m$. It is then enough to take $\nu_m$ verifying

$$\nu_m \leq \nu_{m,0} \quad , \quad \frac{1}{\lambda}\alpha_0(\nu_m) < \max\left\{ v_{\mathcal{C}} \, , \, \underline{\lambda}(P)(\delta+\delta')^2 \right\} . \tag{34}$$

Indeed, the quantity $\frac{1}{\lambda}\alpha_0(\nu_m)$ gives an asymptotic bound for $V(\hat{\xi} - T(x))$ along the trajectories of (1)-(4) according to Assumption 3. If this bound is smaller than $v_{\mathcal{C}}$, $\hat{\xi}$ eventually remains in $\mathcal{C}_m \subset T(\mathcal{X}_s)$ ; if it is smaller than $\underline{\lambda}(P)(\delta+\delta')^2$, then $\hat{\xi} - T(x)$ is eventually smaller than $\delta + \delta'$, and with (17c), $\hat{\xi}$ is in $T(\mathcal{X}_s)$. We conclude in particular that the larger the distance between $\mathcal{X}$ and the boundary of $\mathcal{X}_s$, the larger noise we can allow.

We complete this section by observing that, inspired by [9], under Assumptions 2 and 3, and if $\mathcal{C}_m$ is compact, we could also implement

$$\dot{\hat{x}} = \left(\frac{\partial T}{\partial x}(\hat{x})\right)^{-1} \left( \varphi\Big(T(\hat{x}), \hat{x}, y\Big) - \gamma P^{-1}\frac{dc}{d\xi}(T(\hat{x}))c(T(\hat{x})) \right) . \tag{35}$$

Following [9], it can be proved that, if $\gamma$ is chosen sufficiently large, this modification makes $V$ decrease and $\mathcal{C}_m$ invariant in the $\xi$-coordinates, and thus $T^{-1}(\mathcal{C}_m)$ invariant for (35). Therefore, the trajectories never encounter singularities and are complete. Besides, no inversions are needed for this implementation. However, note that the role of the added term is to compensate for $\varphi$ when $\hat{x}$ exits $\mathcal{C}_m$, in order to force the trajectories inside it. If for numerical reasons, or because the noise is larger than expected and $\gamma$ is not taken sufficiently large, the trajectories leave the safe set, there is nothing preventing them from encountering singularities and diverging. On the contrary, the hybrid implementation (27) always has complete trajectories in the safe set. Therefore, if the noise is too large, it may trigger jumps and inversions, but at least an estimate can be recovered later if the intensity of the noise decreases. Actually, we suggest to combine both strategies: the added term in (35) makes the trajectory remain in the safe set, thus avoiding inversions, and in the unfortunate eventuality where they don't, the hybrid mechanism enables to "save" the trajectory and "have a second chance".

## 4 $T$ diffeomorphism with non-convex image

The main limitation of the previous result lies in Assumption 2. In this section, we still assume $T$ given by Assumption 1 is a diffeomorphism, i.e. $m = n$, but we consider the case where Assumption 2 does not hold. Without convexity, it is no longer clear how to bring the estimate back into the safe set without altering the Lyapunov function, and we propose here another route. Our idea is to use an additional observer whose dynamics can be run independently and give a *practical* estimation of $x$ that can be made arbitrarily precise by choosing appropriately the observer parameters.

**Assumption 5 (Existence of a practical observer)** *For any $\varepsilon > 0$, there exist $\nu_{m,\varepsilon} > 0$, an integer $m_\varepsilon$, and functions $\mathcal{T}_\varepsilon : \mathbb{R}^{m_\varepsilon} \to \mathbb{R}^n$, $\varphi_\varepsilon : \mathbb{R}^{m_\varepsilon} \times \mathbb{R}^p \to \mathbb{R}^{m_\varepsilon}$, such that*

$$\mathcal{T}_\varepsilon(\mathbb{R}^{m_\varepsilon}) \subseteq \mathcal{X}_s' \tag{36}$$

*and for any solution $t \mapsto x(t)$ to (1) initialized in $\mathcal{X}_0$, any solution to*

$$\dot{\hat{\zeta}} = \varphi_\varepsilon(\hat{\zeta}, y) \tag{37}$$

*with $y = h(x)+\nu$ and $|\nu(t)| \leq \nu_{m,\varepsilon}$, is bounded and there exists $t_\varepsilon \geq 0$ such that*

$$|\mathcal{T}_\varepsilon(\hat{\zeta}(t)) - x(t)| \leq \varepsilon \qquad \forall t \geq t_\varepsilon \ . \tag{38}$$

In many practical observer designs, there exists an injective map $T_\varepsilon : \mathcal{X}_s \to \mathbb{R}^{m_\varepsilon}$ such that $\hat{\zeta}$ estimates $T_\varepsilon(x)$ with an arbitrarily small error. In that case, Assumption 5 is satisfied by taking for $\mathcal{T}_\varepsilon$ a globally defined approximation of the left-inverse of $T_\varepsilon$ with values in $\mathcal{X}_s'$. This may involve an extension and projection, or generally, an approximate resolution of the minimization problem

$$\hat{x} = \operatorname*{argmin}_{x\in\mathcal{X}_s'} |T_\varepsilon(x) - \hat{\zeta}| \ . \tag{39}$$

For instance,

- any observer satisfying Assumption 1 and such that its dynamics are independent from the inversion (i.e. $\varphi$ explicitly expressed in function of $\hat{\xi}$ only, and not on $\hat{x}$) satisfies also Assumption 5 with $\mathcal{T}_\varepsilon$ being an approximation of $\mathcal{T}$. For instance, for any $\varepsilon$, the Luenberger design of Example 2 fits here too with

$$\varphi_\varepsilon(\hat{\zeta}, y) = M\,\hat{\zeta} + N\,y\ .$$

- in the context of dirty derivatives ([32] among many others), a high gain observer

$$\varphi_\varepsilon(\hat{\zeta}, y) = A\,\hat{\zeta} + \mathcal{L}_\varepsilon K(y - \hat{\zeta}_1) \tag{40}$$

  where $A$ is defined in (10), $\mathcal{L}_\varepsilon = \mathrm{diag}(\ell_\varepsilon, \ell_\varepsilon^2, \dots, \ell_\varepsilon^{m_\varepsilon})$, and $\ell_\varepsilon$ is chosen sufficiently large, satisfies Assumption 5 with

$$T_\varepsilon(x) = (h(x), L_f h(x), \dots, L_f^{m_\varepsilon - 1} h(x)) \tag{41}$$

  injective. Compared to (9), the nonlinearity $\mathrm{sat}(L_f^n h(\hat{x}))$ has been removed, yielding practical (instead of asymptotic) convergence, and making the observer dynamics independent from $\hat{x}$ as requested here. Also, $T_\varepsilon$ does not need to be an immersion (injectivity suffices) so it may be possible to take $m_\varepsilon < m$.
- an exact differentiator with sliding mode correction terms also fits in Assumption 5 since it provides robust finite-time convergence when the nonlinearity $\mathrm{sat}(L_f^m h(\hat{x}))$ is omitted as proved in [26].

Again, at first sight, the computation of $\mathcal{T}_\varepsilon$ may seem as difficult as the computation of $\mathcal{T}$ in Assumption 1, which we are precisely trying to avoid. But 1) since Assumption 5 only requires practical convergence, $\mathcal{T}_\varepsilon$ can be an approximation of the left-inverse, for instance through a minimization on a rough grid, whose precision depends on the required $\varepsilon$; 2) since only injectivity (and not immersion) and practical convergence are required here, the dimension $m_\varepsilon$ may be taken smaller than $m$, thus leading to a smaller grid; 3) as we will see below, computing $\mathcal{T}_\varepsilon$, i.e. possibly solving the minimization on the grid, will be necessary only a finite number of times.

The idea we pursue is the following. The $\hat{\zeta}$-dynamics can be implemented independently and provide (by computation of $\mathcal{T}_\varepsilon$) a "dirty" estimate of $x$. This estimate can be made arbitrarily precise asymptotically thanks to Assumption 5. Therefore, it can be used for a rough reinitialization of $\hat{x}$ whenever $\hat{x}$ leaves the safe set $\mathcal{X}_s$, where the Jacobian of $T$ is invertible. This leads to the following hybrid observer

$$\left\{\begin{array}{l}
\left.\begin{array}{rl}
\dot{\hat{x}} &= q\left(\frac{dT}{dx}(\hat{x})\right)^{-1}\varphi\Big(T(\hat{x}), \hat{x}, y\Big)\\
\dot{\hat{\zeta}} &= \varphi_\varepsilon(\hat{\zeta}, y+\nu)\\
\dot{q} &= 0\\
\dot{\tau} &= 1-q
\end{array}\right\}(\hat{x}, \hat{\zeta}, q, \tau) \in C\\[2ex]
\left.\begin{array}{rl}
\hat{x}^+ &= \hat{x}\\
\hat{\zeta}^+ &= \hat{\zeta}\\
q^+ &= 0\\
\tau^+ &= 0
\end{array}\right\}(\hat{x}, \hat{\zeta}, q, \tau) \in D_1\\[2ex]
\left.\begin{array}{rl}
\hat{x}^+ &= \mathcal{T}_\varepsilon(\hat{\zeta})\\
\hat{\zeta}^+ &= \hat{\zeta}\\
q^+ &= 1\\
\tau^+ &= \tau
\end{array}\right\}(\hat{x}, \hat{\zeta}, q, \tau) \in D_0
\end{array}\right. \tag{42}$$

with the flow set $C = \mathcal{X}_s \times \mathbb{R}^n \times \{0,1\} \times \mathcal{I}$, the jump set $D = D_1 \cup D_0$ with

$$D_1 = \partial\mathcal{X}_s \times \mathbb{R}^n \times \{1\} \times \mathcal{I}\ , \quad D_0 = \mathcal{X}_s \times \mathbb{R}^n \times \{0\} \times \mathcal{I}$$

where $\mathcal{I}$ is a compact subset of $\mathbb{R}_{\geq 0}$. In other words, as long as $\hat{x}$ is in the *safe set* $\mathcal{X}_s$, we run the observer given by Assumption 1 in the $x$-coordinates. If $\hat{x}$ reaches the boundary of $\mathcal{X}_s$ (and if no flow is possible within $\mathcal{X}_s$), $q$ is switched to 0 and a timer $\tau$ is initialized at 0. During the following flow phase with $q = 0$, $\dot{\hat{x}} = 0$ so $\hat{x}$ is frozen, $\hat{\zeta}$ carries on with its normal trajectory, and the timer increases. When $\tau$ reaches an element of the set $\mathcal{I}$, $\hat{x}$ is updated to $\mathcal{T}_\varepsilon(\hat{\zeta})$ in $\mathcal{X}_s'$ and $q$ is switched back to 1. The observer can then restart from this new initial condition in $\mathcal{X}_s'$. The positive distance between $\mathcal{X}_s'$ and $\partial\mathcal{X}_s$ ensures a dwell-time between a jump in $D_0$ and $D_1$, and therefore an average dwell-time. The key fact is that because an arbitrarily small error can be guaranteed on $\hat{\zeta}$ thanks to Assumption 5, and because the transient of (4) decreases with the initial error according to (5), we can ensure that at some point, $\hat{x}$ will stay in the interior of $\mathcal{X}_s$, $q$ will stay equal to 1, and $\hat{x}$ will follow the observer dynamics (4). It is important to note that this hybrid mechanism does not affect the trajectories of the practical observer $\hat{\zeta}$ which always follows its own independent dynamics $\varphi_\varepsilon$ and is always trivially reset.

The reason why we introduced a timer $\tau$ is that when $\hat{x}$ reaches $\partial\mathcal{X}_s$, for instance due to peaking, the practical estimate $\hat{\zeta}$ may also be in its transient. That is why it may be sensible to wait before using its value to reinitialize $\hat{x}$, in order to avoid unfruitful computations. This delay is determined by the set $\mathcal{I}$ which can be any compact subset of $\mathbb{R}_{\geq 0}$. In fact, it should be chosen depending on the difficulty we have in computing $\mathcal{T}_\varepsilon(\hat{\zeta})$: the longer time we wait, possibly the fewer inversions we will do, but also maybe a longer time before having a good estimation $\hat{x}$. An extreme solution is to take $\mathcal{I} = \{0\}$, namely update immediately $\hat{x}$ to $\mathcal{T}_\varepsilon(\hat{\zeta})$ after each reset of $q$ to 0 and switch $q$ immediately back to 1.

**Theorem 2** *Suppose Assumption 1 and 5 hold with $T$ a diffeomorphism. Consider a compact subset $\mathcal{I}$ of $\mathbb{R}_{>0}$. There exist positive scalars $\varepsilon$, $\nu_m$, $L$ and $L_I$ such that for any $t$-complete solution $t \mapsto x(t)$ to system (1) initialized in $\mathcal{X}_0$, any maximal solution $\phi = (\hat{x}, \hat{\zeta}, q, \tau)$ to (42) initialized in $\mathcal{X}_s' \times \mathbb{R}^n \times \{0,1\} \times [0, \max \mathcal{I}]$ is $t$-complete, and if besides $|\nu(t)| \leq \nu_m$, $\phi$ is eventually continuous, and there exists $\bar{t} \geq t_J$ such that for all $t \geq \bar{t}$,*

$$|\hat{x}(t,J) - x(t)| \leq L_I \beta\Big(L\,|\hat{x}(t_J,J) - x(t_J)|, t - t_J\Big) + L_I \alpha(\nu_m) \quad \forall t \geq \bar{t}\ , \quad (43)$$

*with $J := \max \operatorname{dom}_j \phi < +\infty$.*

**PROOF.** See Appendix C.

From (43), we conclude that this construction preserves the asymptotic properties of the main observer given by Assumption 1 (convergence and robustness to noise). The main difference with the previous design in Theorem 1, is that we only guarantee the performances starting from $t = \bar{t}$ and $|\hat{x}(t_J,J) - x(t_J)|$, instead of $t = 0$ and $|\hat{x}_0 - x_0|$. Unfortunately, $\hat{x}(t,J)$ is only known to be in the compact set $\mathcal{X}_s'$ and depends on the intermediate jumps and the values of $\hat{\zeta}$ at those jumps. Note that for $t_J$ to be uniform in the initial condition, $t_\varepsilon$ in Assumption 5 must be too.

Of course, the map $\mathcal{T}_\varepsilon$ still has to be computed, but, as before, only at a finite number of discrete times during the transient. To minimize the number of inversions, one should choose $\mathcal{X}_s$ as large as $\mathcal{S}$ allows (and therefore possibly unbounded).

Regarding the choice of the parameters, the positive scalars $\varepsilon$ and $\nu_m$ must verify the following:

$$\varepsilon < \frac{1}{L}\beta(\cdot,0)^{-1}\left(\frac{\delta+\delta'}{2}\right) \quad (44a)$$

$$\nu_m < \min\left\{\alpha^{-1}\left(\frac{\delta+\delta'}{2}\right), \nu_{m,\varepsilon}\right\}\ , \quad (44b)$$

with $(\delta, \delta')$ defined in (17a)-(17b), and $L$ defined in (6a) for $\mathcal{C} = \mathcal{X}_s'$. With this choice, we can show that the estimate $\hat{x}$ eventually remains in the compact set

$$\mathcal{C} = \Big\{\hat{x} \in \mathbb{R}^n \ :\ d(T(\hat{x}), T(\mathcal{X})) \leq \delta + \delta'\Big\} \subseteq \mathcal{X}_s\ . \quad (45)$$

The scalar $L_I$ involved in (43) is then defined by (6b) for this compact set $\mathcal{C}$.

We deduce that the inversions can be done through a rough minimization on a grid whose required precision depends on $1/L$ and the distance $\delta + \delta'$, i.e. the distance between $T(\mathcal{X})$ and the frontier of the image $T(\mathcal{X}_s)$. In fact, the larger this distance, the larger $\varepsilon$ (and thus $\nu_{m,\varepsilon}$ and thus $\nu_m$) can be. We conclude again that the set $\mathcal{X}_s$ should be chosen as large as possible. As for the choice of $\mathcal{X}_s'$, it determines $\delta'$ which gives the dwell time between successive jumps (and thus possibly the successive inversions of $\mathcal{T}_\varepsilon$). Note that the result of Theorem 2 would still hold if $\mathcal{X}$ was unbounded as long as $T$ is uniformly injective and uniformly continuous on $\mathcal{S}$, namely (6) holds globally. However, those are strong assumptions on $T$.

## 5 Application to the low-power observer

In Example 1, we have seen how the high gain observer [20] fits in the framework of this paper. However, its gain $\ell$ must be taken sufficiently large and according to (13), the maps $\beta$ and $\alpha$, bounding the decrease of its error and its robustness to noise respectively, are proportional to $\ell^{m-1}$. This leads to two major drawbacks: peaking and poor robustness to noise. Some modifications leading to the so-called *low-power* structures have been proposed in [5] to temper those drawbacks. In this section, we study how to implement them in the initial coordinates. The low-power observer proposed in [5] is of the form (4) with dimension $2m-1$, state $\hat{\xi} = (\hat{z}, \hat{\eta}) \in \mathbb{R}^m \times \mathbb{R}^{m-1}$, and dynamics $\varphi = (\varphi_z, \varphi_\eta)$ given by

$$\varphi_z((\hat{z},\hat{\eta}), \hat{x}, y) = \begin{pmatrix} \hat{\eta} \\ \operatorname{sat}(L_f^m h(\hat{x})) \end{pmatrix} + \ell K_1 e(\hat{z}, \hat{\eta}, y) \quad (46a)$$

$$\varphi_\eta((\hat{z},\hat{\eta}), \hat{x}, y) = \begin{pmatrix} \hat{\eta}_2 \\ \vdots \\ \hat{\eta}_{m-1} \\ \operatorname{sat}(L_f^m h(\hat{x})) \end{pmatrix} + \ell^2 K_2 e_{1:m-1}(\hat{z}, \hat{\eta}, y)\ , \quad (46b)$$

where the error term is

$$e(\hat{z}, \hat{\eta}, y) = (y - \hat{z}_1\ , \ \hat{\eta}_1 - \hat{z}_2\ , \ \ldots\ , \ \hat{\eta}_{m-1} - \hat{z}_m)\ ,$$

$$e_{1:m-1} = (e_1, e_2, \ldots, e_{m-1})\ ,$$

$K_1$ and $K_2$ are diagonal gain matrices chosen such that the linear part is Hurwitz, $\operatorname{sat}(L_f^m h)$ is chosen as in (12), and $\ell$ is sufficiently large. According to [5,6], the first part $\hat{z}$ of the observer state, estimates $T(x)$ with $T$ defined in (8) exactly as in a standard high gain observer. But in order to minimize the use of the noised measurement $y$ in the innovation terms and the order of $\ell$, the states $\hat{z}_2, \ldots, \hat{z}_m$ are doubled by the added variables $\hat{\eta}_1, \ldots, \hat{\eta}_{m-1}$ which are used in the correction term $e$. In other words, $\hat{\eta}_i$ doubles $\hat{z}_{i+1}$ and also estimates $T_{i+1}(x)$.

Actually, according to [6, Theorem 1, Proposition 3], (5) is satisfied using the extended map

$$\tilde{T}(x) = (T(x), T_{2:m}(x))\ .$$

If $T$ is a diffeomorphism, $\tilde{T}$ is an injective immersion, and therefore, Assumption 1 holds with the low-power observer and with $\tilde{T}$ instead of $T$. Following the method that will be explained later in Section 6, we could therefore extend $\tilde{T}$ into a diffeomorphism and apply the constructions of Section 3 and 4. However, we propose here a far easier option that avoids this extension.

In fact, the $\hat{z}$ component of $\hat{\xi}$ is sufficient to reconstruct $x$, since it estimates $T(x)$. Therefore, it is sufficient to bring the dynamics $\varphi_z$ back into the initial coordinates, and implement the dynamics $\varphi_\eta$ as additional dynamics, namely,

$$\dot{\hat{x}} = \left(\frac{\partial T}{\partial x}(\hat{x})\right)^{-1} \varphi_z\Big((T(\hat{x}),\hat{\eta}),\hat{x},y\Big)$$
$$\dot{\hat{\eta}} = \varphi_\eta\Big((T(\hat{x}),\hat{\eta}),\hat{x},y\Big) \ .$$

In other words, we propose to implement observer (27) with an extended state $\tilde{\zeta} = (\zeta,\hat{\eta})$ or observer (42) with extended state $\hat{\tilde{x}} = (\hat{x},\hat{\eta})$, with the new diffeomorphism defined on $\mathcal{S}\times\mathbb{R}^{m-1}$ by

$$\tilde{T}(x,\eta) = (T(x),\eta) \ , \tag{47}$$

and the new safe set $\mathcal{X}_s\times\mathbb{R}^{m-1}$. This is justified in the following Lemma by considering the extended system [1]

$$\begin{cases} \dot{x} \;=\; f(x) \\ \dot{\eta} \;=\; \frac{\partial T_{2-m}}{\partial x}(x) f(x) \end{cases} \quad , \quad y = h(x) \tag{48}$$

with state $\tilde{x} = (x,\eta)$, initialized in

$$\tilde{\mathcal{X}}_0 = \Big\{(x_0,\eta_0) \ : \ x_0\in\mathcal{X}_0 \ , \ \eta_0 = T_{2-m}(x_0)\Big\} \ .$$

**Lemma 1** *Assume the map $T:\mathbb{R}^n\to\mathbb{R}^m$ defined in* (8) *is an injective immersion on $\mathcal{S}$ and $f$ and $h$ are smooth. Consider a function $\mathcal{T}:\mathbb{R}^m\to\mathbb{R}^n$ verifying* (3) *and Lipschitz in a neighborhood of $T(\mathcal{X})$ as in Example 1. Then, the trajectories of* (48) *initialized in $\tilde{\mathcal{X}}_0$ remain in $\tilde{\mathcal{X}} = \mathcal{X}\times T_{2:m}(\mathcal{X})$ and Assumption 1 holds with the injective immersion $\tilde{T}$ defined in* (47) *on $\tilde{\mathcal{S}} = \mathcal{S}\times\mathbb{R}^{m-1}$, with the safe set $\tilde{\mathcal{X}}_s = \mathcal{X}_s\times\mathbb{R}^{m-1}$, with the dynamics $\tilde{\varphi}$ defined in* (46a)-(46b)*, with $\tilde{\mathcal{T}}$ defined by*

$$\tilde{\mathcal{T}}(\hat{z},\hat{\eta}) = (\mathcal{T}(\hat{z}),\hat{\eta}) \ ,$$

*and with $\alpha$ and $\beta$ of the form* (13)*. Also, there exists a positive definite matrix $P_0$ of dimension $2m-1$, depending only on $K_1$ and $K_2$ such that Assumption 3 holds with*

$$P = \mathcal{L}^{-1}P_0\mathcal{L}^{-1} \quad , \quad \mathcal{L} = \mathrm{diag}(\ell,\ell^2,\ell^2,\ell^3,\ldots,\ell^m,\ell^m) \ . \tag{49}$$

**PROOF.** The solutions to (48) are unique by smoothness, so when initialized in $\tilde{\mathcal{X}}_0$, $\eta(t) = T_{2-m}(x(t))$ for all $t$. The rest follows from [6, Theorem 1].

The advantage of having allowed $\mathcal{X}_s$ to be unbounded in Assumption 1 is that we do not need to impose any constraint on the $\hat{\eta}$ component of the observer, namely we take $\tilde{\mathcal{X}}_s = \mathcal{X}_s\times\mathbb{R}^{m-1}$. As for Assumption 2, if it is satisfied with a map $c:\mathbb{R}^m\to\mathbb{R}^{n_c}$ for the sets $T(\mathcal{X})$ and $T(\mathcal{X}_s)$, then it holds also trivially for $\tilde{T}(\tilde{\mathcal{X}})$ and $\tilde{T}(\tilde{\mathcal{X}}_s)$ with the map $\tilde{c}:\mathbb{R}^m\times\mathbb{R}^{m-1}$ defined by $\tilde{c}(\hat{z},\hat{\eta}) = c(\hat{z})$. In terms of noise, it can be shown through (30) and (43) that the filtering property is preserved in the initial coordinates, modulo the Lipschitz constant $L_I$ of $\mathcal{T}$.

It is also worth remarking that in observer (27), the jumps along the map $g$ whose role is to bring $\tilde{\zeta}$ back into $\tilde{\mathcal{C}}_m\subset\tilde{T}(\tilde{\mathcal{X}}_s)$ (or in other words, $\zeta$ back into $\mathcal{C}_m\subset T(\mathcal{X}_s)$), modify both $\zeta$ and $\hat{\eta}$, although the constraint of the safe set concerns only $\zeta$ as $\hat{\eta}$ is free in $\mathbb{R}^{m-1}$. This is because the Lyapunov function with $P$ defined in (49) depends on the whole error $(\hat{z}-T(x),\hat{\eta}-T_{2:m}(x))$, and changing $\hat{z}$ may require to change $\hat{\eta}$ in order to ensure the decrease of the Lyapunov function during jumps.

## 6 General case: $T$ injective immersion

We now see how the previous developments can be used in the more general case where the transformation $T$ in Assumption 1 is an injective immersion and not a diffeomorphism, namely $m>n$, like the van der Pol oscillator presented in Example 3.

### *6.1 From an injective immersion to a diffeomorphism*

In order to use Theorem 2 with an injective immersion, we extend $T:\mathbb{R}^n\to\mathbb{R}^m$ into a diffeomorphism $\overline{T}:\mathbb{R}^m\to\mathbb{R}^m$ by adding $m-n$ fictitious states $w$ to $x$, as done in [13] along the following explicit construction.

**Lemma 2** *Consider an open set $\mathcal{S}$ of $\mathbb{R}^n$ and an injective immersion $T:\mathcal{S}\to\mathbb{R}^m$ for some integer $m>n$. For any bounded open subset $\mathcal{O}$ of $\mathcal{S}$ and any $C^1$ function $\gamma:\mathcal{S}\to\mathbb{R}^{m\times(m-n)}$ such that*

$$\det\left(\frac{\partial T}{\partial x}(x) \ , \ \gamma(x)\right) \neq 0 \quad \forall x\in\mathrm{cl}(\mathcal{O}) \ ,$$

*there exists $\epsilon>0$ such that the map $\overline{T}$ defined on $\mathcal{S}\times\mathbb{R}^{m-n}$ by*

$$\overline{T}(x,w) = T(x) + \gamma(x)\,w \tag{50}$$

[1] The solutions to (48) verify $\eta(t) = T_{2:m}(x(t)) + k$ with $k$ a constant determined by the initial condition. $k$ is not observable from $y$, hence the particular set of initial conditions $\tilde{\mathcal{X}}_0$ for which $k=0$.

*is a diffeomorphism on $\overline{\mathcal{S}} = \mathcal{O} \times \mathbb{B}_\epsilon$, where $\mathbb{B}_\epsilon$ denotes the open ball of radius $\epsilon$ in $\mathbb{R}^{m-n}$.*

**PROOF.** See [13, Lemma 2.1].

In other words, we can build a diffeomorphism from an injective immersion by looking for $m-n$ columns that are $C^1$ in $x$ and that complete the Jacobian into an invertible matrix. Since the Jacobian of $T$ is full-rank on $\mathcal{O}$, columns $\gamma(x)$ for each $x$ are easy to find, but the difficulty is to ensure their continuity with respect to $x$. This problem is very old and related to topological questions ([18,33]). We refer the reader to [13, Section 2] for a detailed analysis of this problem and [11] for explicit examples.

**Example 4** *Continuing Example 3, we have $T : \mathbb{R}^4 \to \mathbb{R}^5$, and its Jacobian is of the form*

$$\frac{\partial T}{\partial x}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_1(x) & b_1(x) & c_1(x) & d_1(x) \\ a_2(x) & b_2(x) & c_2(x) & d_2(x) \\ a_3(x) & b_3(x) & c_3(x) & d_3(x) \end{pmatrix}$$

*with the vectors $c(x) = (c_1(x), c_2(x), c_3(x))$ and $d(x) = (d_1(x), d_2(x), d_3(x))$ independent for all $x$ in $\mathcal{S}$. To follow Lemma 2, we need to find a column vector $\gamma$ that completes it into a square matrix. Actually, when adding only one dimension, a universal completion consists in using the minors of the Jacobian, which here gives*

$$\gamma(x) = \begin{pmatrix} 0 \\ 0 \\ c(x) \times d(x) \end{pmatrix} . \tag{51}$$

*From (50), one can thus extend $T$ into a diffeomorphism $\overline{T}$ defined on $\overline{\mathcal{X}}_s = \mathcal{X}_s \times [-\epsilon, \epsilon]$.*

**Remark 2** *When an explicit construction of $\gamma$ along Example 4 or [13, Section 2] is not possible, it is useful to know that a universal completion method was proposed in [13, Section 5.2] which relies on the following simple observation. If Assumption 1 holds, it also holds with the observer dimension $m$ replaced by $m+n$, the injective immersion $T$ replaced by $\tilde{T} : \mathcal{S} \to \mathbb{R}^m \times \mathbb{R}^n$ defined by*

$$\tilde{T}(x) = (T(x), \underbrace{0, \dots, 0}_{n \text{ times}}) ,$$

*the observer dynamics $\varphi$ replaced by*

$$\tilde{\varphi}(\hat{\xi}, \hat{x}, y) = (\varphi(\hat{\xi}_1, \dots, \hat{\xi}_m, \hat{x}, y), -\hat{\xi}_{m+1}, \dots, -\hat{\xi}_{m+n})$$

*and the left-inverse $\mathcal{T}$ by $\tilde{\mathcal{T}} : \mathbb{R}^{m+n} \to \mathbb{R}^n$ defined by*

$$\tilde{\mathcal{T}}(\hat{\xi}) = \mathcal{T}(\hat{\xi}_1, \dots, \hat{\xi}_m) .$$

*In other words, we add $n$ components equal to zero to $T$ and correspondingly $n$ exponentially converging dynamics in $\varphi$. This operation may seem totally useless at first sight, but actually the benefit is that $\tilde{T}$ can easily be completed into a diffeomorphism according to Lemma 2. Indeed, its Jacobian $\frac{\partial \tilde{T}}{\partial x}(x) = \begin{pmatrix} \frac{\partial T}{\partial x}(x) \\ 0_{n\times n} \end{pmatrix}$ can always be completed by $\gamma(x) = \begin{pmatrix} -I_{m\times n} \\ \frac{\partial T}{\partial x}(x)^\top \end{pmatrix}$ according to the Schur complement.*

We conclude from this section that (maybe after increasing the observer dimension $m$), the injective immersion $T$ given by Assumption 1 can always be extended into a diffeomorphism along the construction of Lemma 2.

### *6.2 Observer implementation*

With the following lemma (proved in Appendix D), we show how the observer presented in the previous section can be used with this new diffeomorphism $\overline{T}$. For that, we define the extended system

$$\dot{\overline{x}} = \overline{f}(\overline{x}) \quad , \quad y = \overline{h}(\overline{x}) \tag{52}$$

with state $\overline{x} = (x, w) \in \mathbb{R}^n \times \mathbb{R}^{m-n}$, $\overline{f}(x, w) = (f(x), 0)$, $\overline{h}(x, w) = h(x)$, initializing set $\overline{\mathcal{X}}_0 = \mathcal{X}_0 \times \{0\}$ and trajectories staying in $\overline{\mathcal{X}} = \mathcal{X} \times \{0\}$.

**Lemma 3** *Consider an injective immersion $T : \mathcal{S} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, a diffeomorphism $\overline{T} : \overline{\mathcal{S}} \subseteq \mathbb{R}^m \to \mathbb{R}^m$, a subset $\mathcal{X}_s$ of $\mathcal{S}$, a compact subset $\mathcal{X}'_s$ of $\mathcal{X}_s$ satisfying (16), a closed subset $\mathcal{W}$ of $\mathbb{R}^{m-n}$ and a positive scalar $\epsilon$ such that*

$$\overline{\mathcal{X}}'_s := \mathcal{X}'_s \times \mathrm{cl}(\mathbb{B}_\epsilon) \quad \subset \quad \overline{\mathcal{X}}_s := \mathcal{X}_s \times \mathcal{W} \quad \subset \quad \overline{\mathcal{S}} \tag{53}$$

*and*

$$\overline{T}(x, 0) = T(x) \qquad \forall x \in \mathcal{X} . \tag{54}$$

*Consider a map $\overline{\mathcal{T}} : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ such that*

$$\overline{\mathcal{T}}(\overline{T}(x, w)) = (x, w) \qquad \forall (x, w) \in \overline{\mathcal{X}}_s , \tag{55}$$

*namely $\overline{\mathcal{T}} = \overline{T}^{-1}$ on the set $\overline{T}(\overline{\mathcal{X}}_s)$. Denote $\mathcal{T}$ the projection of $\overline{\mathcal{T}}$ on the first $n$ components. $\mathcal{T}$ is a global left inverse of $T$, and if Assumption 1 (resp. and Assumption 3) holds for system (1) with data $T$, $\mathcal{T}$, $\mathcal{X}_s$, $\mathcal{X}'_s$, and $\varphi$, then Assumption 1 (resp. and Assumption 3) also holds for the extended system (52) with data $\overline{T}$, $\overline{\mathcal{T}}$, $\overline{\mathcal{X}}_s$, $\overline{\mathcal{X}}'_s$, and $\overline{\varphi}$ defined by*

$$\overline{\varphi}(\hat{\xi}, (\hat{x}, \hat{w}), y) = \varphi(\hat{\xi}, \hat{x}, y) .$$

In other words, we can keep the same observer $\varphi$ but instead of mapping $\hat{\xi}$ to $\hat{x}$ by inverting an injective immersion $T$, we map it back to $(\hat{x}, \hat{w})$ by inverting a diffeomorphism $\overline{T}$. Since Assumption 1 is preserved, we deduce that using observer (42) with $\overline{T}$ instead of $T$ and $\hat{\overline{x}} = (\hat{x}, \hat{w})$ instead of $\hat{x}$, and applying Theorem 2 on system (52) instead of (1) gives

$$\begin{aligned} &|\hat{x}(t,J) - x(t)| + |\hat{w}(t,J)| \\ &\quad \leq \overline{L}_I \beta\Big(L\,|\hat{x}(t_J,J) - x(t_J)|, t - t_J\Big) + \overline{L}_I \alpha(\nu_m)\ , \end{aligned}$$

where $\overline{L}_I$ and $\overline{L}$ are the injectivity and Lipschitz gains of $\overline{T}$ instead of $T$. In particular, without noise, we get

$$\lim_{t\to+\infty} |\hat{x}(t,J) - x(t)| = 0 \quad , \quad \lim_{t\to+\infty} \hat{w}(t,J) = 0\ .$$

Similarly for observer (27), since Assumption 3 is also preserved, we can use $\overline{T}$ instead of $T$, but Assumption 2 has to be checked on the extended images $\overline{T}(\overline{\mathcal{X}})$ and $\overline{T}(\overline{\mathcal{X}}_s)$. Then, denoting $\zeta_x$ the first $n$ components of $\zeta$ and $\zeta_w$ the $m-n$ last, we obtain from Theorem 1

$$|\zeta_x(t,j) - x(t)| + |\zeta_w| \leq \overline{L}_I \beta\Big(\overline{L}|\hat{x}_0 - x_0|, t\Big) + L_I \alpha(\nu_m)\ .$$

Note that most of the time, the number $\epsilon$ in the definition of $\overline{\mathcal{X}}_s$ in (53) is not well-known, and besides, it may be restrictive to take it constant. Therefore, the detection of $\zeta$ or $(\hat{x}, \hat{w})$ leaving $\overline{\mathcal{X}}_s$ is unclear in the implementation of (27) or (42). A way to address this in practice, is to trigger a jump depending on the condition number of the Jacobian of $\overline{T}$. This is done in the following example.

**Example 5** *Let us apply this observer to the van der Pol oscillator with unknown parameters given in Example 3 with the diffeomorphism built in Example 4. We take for $\varphi$ a high gain observer* (9) *of dimension $m = 5$ with $\ell_1 = 2$, and for $\varphi_\varepsilon$ an exact differentiator [26] of dimension $m = 5$ with $\ell_1 = 4$ and $T_\varepsilon = \overline{T}$. To compute $\mathcal{T}_\varepsilon$ at each jump, we need an approximate left-inverse of $\overline{T}$ to have* (38)*. Exploiting the fact that the true value of $w$ is zero, and that $x_1$ and $x_2$ can be read directly from the first two components of $\overline{T}$, we take:*

$$\begin{aligned} \mathcal{T}_\varepsilon(\zeta_1, \zeta_2, \zeta_{3:5}) = \Big(&\zeta_1\ ,\ \zeta_2\ , \\ &\underset{(x_3,x_4)\in\mathcal{X}_3\times\mathcal{X}_4}{\operatorname{argmin}} |\zeta_{3:5} - T_{3:5}(\zeta_1, \zeta_2, x_3, x_4)|\ ,\ 0\Big) \end{aligned}$$

*with $\mathcal{X}_3 = [0,5]$ (resp. $\mathcal{X}_4 = [0,5]$) where the parameter $\omega^2$ (resp. $\mu$) is known to be, and the minimization problem is solved on a two-dimensional grid with precision* 0.5*. As suggested above, we trigger a jump whenever the jacobian condition number is larger than $5\times10^4$, and we wait $\tau_m = 0.5$ before reinitializing $(\hat{x}, \hat{w})$ (i.e., $\mathcal{I} = \{0.5\}$). Results of a simulation are given in Figures 2-4. It is interesting to note that the higher we need to take $\ell_1$, the more precise the practical observer must be (and therefore the grid): this can be seen in* (44a) *since $\beta(.,0)^{-1}$ decreases with $\ell_1$.*

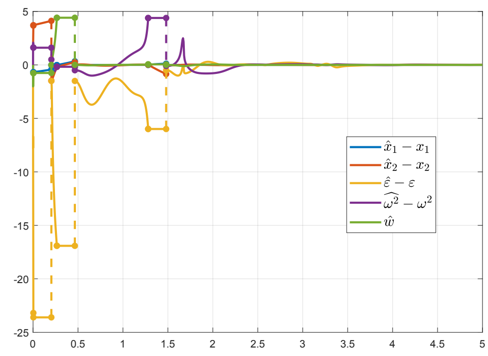


Fig. 2. Errors for the plant initialized at $x_0 = (0.8, -1, 2, 1)$, and the observer (42) initialized at $(\hat{x}_0, \hat{w}_0) = (0, 0.5, 1.5, 0.5, 0)$, with $\overline{T}$ defined by (8) with $m = 5$ and (51)-(50).

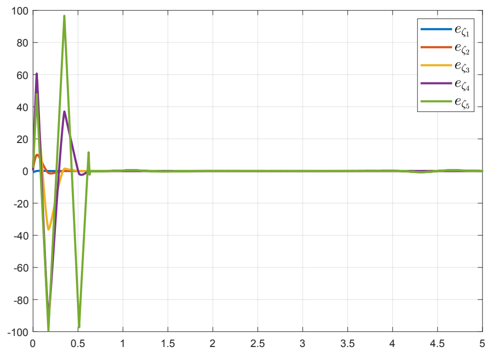


Fig. 3. Errors between $t \mapsto T(x(t))$ for a plant trajectory initialized at $x_0 = (0.8, 1.5, 2, 1)$, and the practical observer initialized at $\zeta_0 = T(\hat{x}_0)$ with $\hat{x}_0 = (0, 0.5, 1.5, 0.5)$.

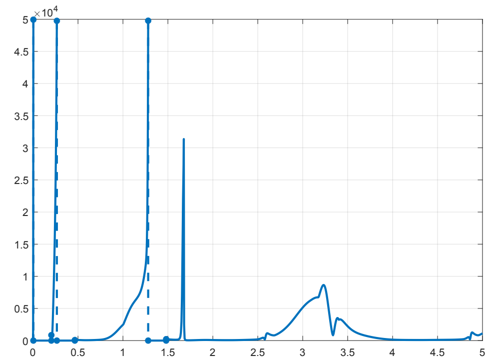

Fig. 4. Condition number of the Jacobian of $\overline{T}$.

## 7 Conclusion

We have proposed two hybrid strategies allowing to express the observer dynamics in the plant's coordinates thanks to the (always possible) extension of an injective immersion into a diffeomorphism. The first strategy is based on a convexity assumption in the image coordinates and the other on an independent practical observer. Those techniques ensure global convergence and completeness of solutions. They require only a finite number of approximate inversions of the transformation, which can be done on a grid. Note that the idea of combining an asymptotic observer with a practical one was also used in [8] to bring the estimate into the basin of attraction of a local observer. Both ideas could be combined to implement a local observer in the initial coordinates.

## A Properties of the map $g$

**Lemma 4 ([4])** *Consider a compact subset $\hat{\Xi}$ of $\mathbb{R}^m$, and a positive scalar $\gamma \leq \gamma^*$ with $\gamma^*$ defined in (26). Then, $g(\hat{\Xi}) \subseteq \hat{\Xi}$ and for all $(\xi, \hat{\xi})$ in $\mathcal{C}_0 \times \hat{\Xi}$,*

$$V\Big(g(\hat{\xi}) - \xi\Big) \leq V(\hat{\xi} - \xi) \ . \tag{A.1}$$

*If besides, $\hat{\Xi} \setminus \mathcal{C}_m \neq \emptyset$, for all $(\hat{\xi}, \xi) \in \mathrm{cl}(\hat{\Xi} \setminus \mathcal{C}_m) \times \mathcal{C}_0$,*

$$V\Big(g(\hat{\xi}) - \xi\Big) \leq V(\hat{\xi} - \xi) - \gamma \mathfrak{c}_m^2 \ , \tag{A.2}$$

*so that $v_m \geq \gamma \mathfrak{c}_m^2$. Also, denoting*[2]

$$\sigma := \left\lceil \frac{v_m - v_{\mathcal{C}}}{\gamma \mathfrak{c}_m^2} \right\rceil \ , \tag{A.3}$$

*with $v_{\mathcal{C}}$ verifying (33), for any $\hat{\xi} \in \hat{\Xi}$, there exists $k \in \{0, 1, \dots, \sigma\}$ such that $g^k(\hat{\xi}) \in \mathrm{int}\, \mathcal{C}_m$.*

**PROOF.** Take $(\xi, \hat{\xi})$ in $\mathcal{C}_0 \times \hat{\Xi}$. Using $c(\xi) = 0$, $\hat{\xi} \in \hat{\Xi}$, (18) and (26), we get,

$$\begin{aligned} V\Big(g(\hat{\xi}) - \xi\Big) &\leq V(\hat{\xi} - \xi) - 2\gamma c^\top(\hat{\xi})(c(\hat{\xi}) - c(\xi)) \\ &\qquad + \gamma^2 \overline{\lambda}(P^{-1}) \left| \frac{dc}{d\xi}(\hat{\xi}) \right|^2 \left| c(\hat{\xi}) \right|^2 \\ &\leq V(\hat{\xi} - \xi) - \gamma |c(\hat{\xi})|^2 \leq V(\hat{\xi} - \xi) \ , \end{aligned}$$

which gives (A.1)-(A.2). Now take any $\hat{\xi}$ in $\hat{\Xi}$, there exists $x$ in $\mathcal{X}$ such that $V(\hat{\xi} - T(x)) \leq v_m$. According to (18), $T(x) \in \mathcal{C}_0$, so $V(g(\hat{\xi}) - T(x)) \leq V(\hat{\xi} - T(x)) \leq v_m$ and $g(\hat{\xi}) \in \hat{\Xi}$. Therefore, $g(\hat{\Xi}) \subseteq \hat{\Xi}$. Now suppose that for all $k$ in $\{0, 1, \dots, \sigma - 1\}$, $g^k(\hat{\xi}) \notin \mathrm{int}\, \mathcal{C}_m$. Then,

$$V(g^\sigma(\hat{\xi}) - T(x)) \leq V(\hat{\xi} - T(x)) - \sigma \gamma \mathfrak{c}_m^2 \leq v_m - \sigma \gamma \mathfrak{c}_m^2 \leq v_{\mathcal{C}}$$

according to (A.3). From (33), we deduce that $g^\sigma(\hat{\xi}) \in \mathrm{int}\, \mathcal{C}_m$ and the result follows.

[2] If $v_m \leq v_{\mathcal{C}}$, necessarily $\hat{\Xi} \subseteq \mathcal{C}_m$, and $\sigma = 0$.

## B Proof of Theorem 1

Take $\hat{\Xi}$, $\nu_m$, $\varepsilon$ as defined in (24)-(34)-(31). Consider the hybrid system (27) and take solutions $t \mapsto x(t)$ and $(t, j) \mapsto \phi(t, j)$ as defined in Theorem 1. Because $q$ is a toggle state which takes value in $\{0, 1\}$, one can define subsets $\mathcal{D}_0$ and $\mathcal{D}_1$ of $\mathrm{dom}\, \phi$ such that $\mathcal{D}_1 \cup \mathcal{D}_0 = \mathrm{dom}\, \phi$,

$$q(t, j) = 0 \quad \forall (t, j) \in \mathcal{D}_0 \quad , \qquad q(t, j) = 1 \quad \forall (t, j) \in \mathcal{D}_1 \ ,$$

and $\mathcal{J}_1$ and $\mathcal{J}_0$ subsets of $\mathbb{N}$ by

$$\mathcal{J}_0 = \{ j \in \mathbb{N} \ : \ \exists t \in \mathbb{R}_{\geq 0} \ , \ (t, j) \in \mathcal{D}_0 \}$$

$$\mathcal{J}_1 = \{ j \in \mathbb{N} \ : \ \exists t \in \mathbb{R}_{\geq 0} \ , \ (t, j) \in \mathcal{D}_1 \} \ .$$

By definition of $C$ and $D$, and because $\mathcal{X}_s$ is closed, for all $(t, j)$ in $\mathcal{D}_0$, $\zeta(t, j) \in \mathcal{X}_s$, so we can define the hybrid arc $\hat{\xi}$ on $\mathrm{dom}\, \phi$ by

$$\hat{\xi}(t, j) = \begin{cases} T(\zeta(t, j)) & \text{if } (t, j) \in \mathcal{D}_0 \\ \zeta(t, j) & \text{if } (t, j) \in \mathcal{D}_1 \ . \end{cases}$$

We also define the image of the true plant's trajectory $\xi(t) = T(x(t))$. On $\mathcal{D}_0$ no flow is possible, and on $\mathcal{D}_1$, $\hat{\xi}$ follows the dynamics

$$\dot{\hat{\xi}} = \varphi\Big(\hat{\xi}, T^{-1}(\hat{\xi}), y + \nu\Big) = \varphi\Big(\hat{\xi}, \mathcal{T}(\hat{\xi}), y\Big) \ ,$$

according to (3) (verified on $\mathcal{X}_s$ by assumption). Therefore, from Assumption 3, for all $j$ in $\mathcal{J}_1$ and all $t$ in $I_j$,

$$\dot{\overbrace{V(\hat{\xi}(t, j) - \xi(t))}} \leq -\lambda V(\hat{\xi}(t, j) - \xi(t)) + \alpha_0(\nu(t)) \ . \tag{B.1}$$

Define $\overline{v}$ as

$$\overline{v}(t) = e^{-\lambda t} V(T(\hat{x}_0) - T(x_0)) + \frac{1}{\lambda} \alpha_0(\nu_m)(1 - e^{-\lambda t}) \ . \tag{B.2}$$

The following technical lemma shows that applying $T \circ \mathcal{T}_\varepsilon$ to $\hat{\xi} \in \mathcal{C}_m \cap \hat{\Xi}$ can only make $V$ increase by $\gamma \mathfrak{c}_m^2$.

**Lemma 5** *Assume $\mathcal{T}_\varepsilon : \mathcal{C}_m \to \mathcal{X}_s'$ verifies (28)-(29b). For any $\hat{\xi} \in \mathcal{C}_m \cap \hat{\Xi}$ such that there exists $\xi$ in $T(\mathcal{X})$ verifying $V(\hat{\xi} - \xi) \leq V_0 - \gamma \mathfrak{c}_m^2$ , with $V_0 \leq v_m$, we have $\hat{\xi}^+ := T(\mathcal{T}_\varepsilon(\hat{\xi})) \in \hat{\Xi}$ and $V(\hat{\xi}^+ - \xi) \leq V_0$ .*

**PROOF.** Since $\mathcal{T}_\varepsilon(\mathcal{C}_m) \subseteq \mathcal{X}_s'$ and $T^{-1}(\mathcal{C}_m \cap \hat{\Xi}) \subset \mathcal{X}_s'$ according to (28), using (6a) on $\mathcal{C} = \mathcal{X}_s'$ and (29b),

$$\left| \hat{\xi}^+ - \hat{\xi} \right| \leq L \left| \mathcal{T}_\varepsilon(\hat{\xi}) - T^{-1}(\hat{\xi}) \right| \leq \frac{\sqrt{v_m} - \sqrt{v_m - \gamma \mathfrak{c}_m^2}}{\sqrt{\overline{\lambda}(P)}} \ .$$

Therefore, denoting here $\Delta = \hat{\xi}^+ - \hat{\xi}$, we have

$$\begin{aligned} V(\hat{\xi}^+ - \xi) &= V(\hat{\xi} - \xi) + 2(\hat{\xi} - \xi)^\top P\Delta + V(\Delta) \\ &\leq V(\hat{\xi} - \xi) + 2\sqrt{V(\hat{\xi} - \xi)}\sqrt{V(\Delta)} + V(\Delta) \\ &\leq V_0 - \gamma\mathfrak{c}_m^2 + 2\sqrt{v_m - \gamma\mathfrak{c}_m^2}\sqrt{V(\Delta)} + V(\Delta) \\ &\leq V_0 \ , \end{aligned}$$

since $\sqrt{V(\Delta)} \leq \sqrt{\overline{\lambda}(P)}|\Delta| \leq \sqrt{v_m} - \sqrt{v_m - \gamma\mathfrak{c}_m^2}$. Therefore, as $\xi \in T(\mathcal{X})$ and $V_0 \leq v_m$, $\hat{\xi}^+ \in \hat{\Xi}$. □

We now study in detail the solution. First, $\hat{\xi}(0,0) = T(\hat{x}_0) \in \hat{\Xi}$ (by construction, because $v_m \geq v_0$). Since $\zeta(0,0) \in \hat{\mathcal{X}}_0 \subseteq \mathcal{X}$ is in the interior of $\mathcal{X}_s$ according to (16), and since $q(0,0) = 1$, the solution starts by flowing during a positive amount of time. Since $\nu(t) \leq \nu_{m,0}$, from Assumption 3 and the definition of $v_m$ and $\hat{\Xi}$ in (23b)-(24), $\hat{\xi}$ remains in $\hat{\Xi}$ while flowing. Also, from (B.1) and (B.2),

$$V(\hat{\xi}(t,0) - \xi(t)) \leq \overline{v}(t) \quad \forall t \in I_0 \ . \tag{B.3}$$

Since $\overline{v}$ is bounded (by $v_m$), no escape in finite time is possible during flow. Therefore, either the solution never jumps, i.e. $\mathrm{dom}_j\, \phi = \{0\}$, $\hat{\xi}$ is complete continuous, stays in $\hat{\Xi} \cap T(\mathcal{X}_s)$ and the result follows ; or the solution $\zeta$ reaches $\partial\mathcal{X}_s$ and a jump occurs, namely $1 \in \mathrm{dom}_j\, \phi$.

In the case where $\hat{\Xi} \subseteq \mathcal{C}_m \subset T(\mathcal{X}_s)$, the second case cannot happen and the conclusion follows. Therefore, in the rest of the proof, we suppose $\hat{\Xi}\backslash\mathcal{C}_m \neq \emptyset$ and we study the second item. So consider the first jump. Necessarily, $\hat{\xi}(t_1,0) \in \partial T(\mathcal{X}_s)$ and $q(t_1,0) = 1$. By definition of the jump map on $D_1$ and of the hybrid arc $\hat{\xi}$, and since $\mathcal{C}_m$ is a strict subset of $T(\mathcal{X}_s)$ from (19), $\hat{\xi}(t_1,1) = \hat{\xi}(t_1,0) \in \hat{\Xi}\backslash\mathcal{C}_m$ and $q(t_1,1) = 0$. From there, necessarily, $\hat{\xi}(t_1,2) = g(\hat{\xi}(t_1,1))$. According to Lemma 4, $\hat{\xi}(t_1,2) \in \hat{\Xi}$ and

$$V\Big(\hat{\xi}(t_1,2) - \xi(t_1)\Big) \leq V\Big(\hat{\xi}(t_1,1) - \xi(t_1)\Big) - \gamma\mathfrak{c}_m^2 \leq v_m - \gamma\mathfrak{c}_m^2 \ .$$

As long as the jumps along $g$ continue, $V$ decreases and $\hat{\xi}$ stays in $\hat{\Xi}$. From Lemma 4 and the definition of the jump map, those jumps stop after $k \leq \sigma$ jumps along $g$. At this point, $\hat{\xi}(t_1,k+1) \in \hat{\Xi} \cap \mathcal{C}_m$ and

$$\begin{aligned} V\Big(\hat{\xi}(t_1,k+1) - \xi(t_1)\Big) &\leq V\Big(\hat{\xi}(t_1,2) - \xi(t_1)\Big) \\ &\leq V\Big(\hat{\xi}(t_1,1) - \xi(t_1)\Big) - \gamma\mathfrak{c}_m^2 \ . \end{aligned}$$

At that point, we have $q(t_1,k+1) = 0$ so $\zeta(t_1,k+1) = \xi(t_1,k+1)$, $\zeta(t_1,k+2) = \mathcal{T}_\varepsilon(\zeta(t_1,k+1)) \in \mathcal{X}_s' \subset \mathcal{X}_s$ and $q(t_1,k+2) = 1$, so that

$$\hat{\xi}(t_1,k+2) = T(\mathcal{T}_\varepsilon(\hat{\xi}(t_1,k+1))) \ .$$

Since $\xi(t_1) \in T(\mathcal{X})$, we deduce from Lemma 5 that $\hat{\xi}(t_1,k+2) \in \hat{\Xi}$ and

$$V\Big(\hat{\xi}(t_1,k+2) - \xi(t_1)\Big) \leq V\Big(\hat{\xi}(t_1,1) - \xi(t_1)\Big) \ ,$$

i.e. since $\hat{\xi}(t_1,1) = \hat{\xi}(t_1,0)$,

$$V\Big(\hat{\xi}(t_1,k+2) - \xi(t_1)\Big) \leq V\Big(\hat{\xi}(t_1,0) - \xi(t_1)\Big) \ .$$

We conclude that after at most $\sigma + 2$ jumps, we are back with $\zeta$ in $\mathcal{X}_s'$ (in the interior of $\mathcal{X}_s$) and $q = 1$, i.e. $k + 2 \in \mathcal{J}_1$ with $k \leq \sigma$. Besides, from (B.3),

$$V\Big(\hat{\xi}(t_1,k+2) - \xi(t_1)\Big) \leq \overline{v}(t_1) \ .$$

Therefore, again from (B.1) and (B.2),

$$V(\hat{\xi}(t,k+2) - \xi(t)) \leq \overline{v}(t) \quad \forall t \in I_{k+2} \ .$$

Starting again the same reasoning from this new initial condition, it follows by induction that
a) the time domain is made of intervals of flow associated to $j$ in $\mathcal{J}_1$ separated by at most $\sigma + 2$ jumps. Besides, in each flow interval, $\zeta$ is initialized in the strict compact subset $\mathcal{X}_s'$ of $\mathcal{X}_s$. By absolute continuity, since no jump happens until $\zeta$ has reached $\partial\mathcal{X}_s$, the length of the flow intervals are lower-bounded by $dt$ defined by $dt := \frac{d(\mathcal{X}_s', \partial\mathcal{X}_s'')}{\varphi_m} > 0$ where $\mathcal{X}_s''$ is a compact set such that $\mathcal{X}_s' \subset \mathcal{X}_s'' \subsetneq \mathcal{X}_s$ (possible according to (16)) and

$$\varphi_m := \sup_{\hat{x} \in \mathcal{X}_s'', x \in \mathcal{X}, |\nu| \leq \nu_m} \left| \left(\frac{dT}{dx}(\hat{x})\right)^{-1} \varphi\Big(T(\hat{x}), \hat{x}, h(x) + \nu\Big) \right| \tag{B.4}$$

Therefore, each $I_j$ with $j \in \mathcal{J}_1$ has length larger than $dt$. So either $\mathcal{J}_1$ is infinite and $\mathrm{dom}_t$ is infinite, either $\mathcal{J}_1$ is finite, but then necessarily the last interval of flow is infinite because no escape in finite time is possible. Therefore, the solutions are $t$-complete.
b) $\hat{\xi}(t,j) \in \hat{\Xi}$ for all $(t,j)$ in $\mathrm{dom}\, \phi$.
c) for any $j$ in $\mathcal{J}_1$,

$$V(\hat{\xi}(t,j) - \xi(t)) \leq \overline{v}(t) \quad \forall t \in I_j \ , \tag{B.5}$$

and from Assumption 4, for all $t \in I_j$

$$|\hat{\xi}(t,j) - \xi(t)| \leq \beta(|T(\hat{x}_0) - T(x_0)|, t) + \alpha\left(\sup_{s \in [0,t]} |\nu(s)|\right) \tag{B.6}$$

Therefore, for any $j$ in $\mathcal{J}_1$ and any $t \in I_j$, $\hat{\xi}$ is in the compact set $T(\mathcal{X}_s) \cap \hat{\Xi}$. Since $\hat{\xi}(t,j) = T(\zeta(t,j))$ and $\xi(t) = T(x(t)) \in T(\mathcal{X})$, using (6b) with $\mathcal{C} = T^{-1}(T(\mathcal{X}_s) \cap \hat{\Xi})$ and (B.6) gives (30).

Finally, from the definition of $\nu_m$ in (34) and $\overline{v}$ in (B.2), there exists $\overline{t} \geq 0$ such that

$$\overline{v}(t) < \max\left\{v_{\mathcal{C}}, \underline{\lambda}(P)(\delta+\delta')^2\right\} \qquad \forall t \geq \overline{t}\ .$$

Take $j$ in $\mathcal{J}_1$ such that $I_j \cap [\overline{t}, +\infty) \neq \emptyset$. From (B.5), and since $\xi$ is in $T(\mathcal{X})$, for all $t$ in $I_j \cap [\overline{t}, +\infty)$: either $V(\hat{\xi}(t,j) - \xi(t)) < v_{\mathcal{C}}$, and with (33), $\hat{\xi}(t,j) \in \mathcal{C}_m \subset T(\mathcal{X}_s)$ ; or $V(\hat{\xi}(t,j) - \xi(t)) < \underline{\lambda}(P)(\delta+\delta')^2$, which gives $|\hat{\xi}(t,j) - \xi(t)| < \delta + \delta'$ and from (17c), $\hat{\xi}(t,j)$ is in the interior of $T(\mathcal{X}_s)$. It follows that in both cases, no jump can occurs in $I_j$, so $j = J := \sup \mathrm{dom}_j\, \phi < 0$, $[\overline{t}, +\infty) \subset I_J$ and the solution is eventually continuous.

## C Proof of Theorem 2

First, by construction $\hat{x}(t,j) \in \mathcal{X}_s$ for all $(t,j)$ in $\mathrm{dom}\,\phi$, so that we can define an hybrid arc $\hat{\xi}$ by $\hat{\xi}(t,j) = T(\hat{x}(t,j))$ on $\mathrm{dom}\,\phi$, whose continuous-time dynamics are

$$\dot{\hat{\xi}} = q\,\varphi\Big(\hat{\xi}, T^{-1}(\hat{\xi}), y\Big) = q\,\varphi\Big(\hat{\xi}, \mathcal{T}(\hat{\xi}), y\Big)\ ,$$

according to (3) verified on $\mathcal{X}_s$ by assumption. Because $q$ is again a toggle state which takes value in $\{0,1\}$, we define subsets $\mathcal{D}_0$ and $\mathcal{D}_1$ of $\mathrm{dom}\,\phi$ such that $\mathcal{D}_1 \cup \mathcal{D}_0 = \mathrm{dom}\,\phi$, and subsets and $\mathcal{J}_1$ and $\mathcal{J}_0$ subsets of $\mathbb{N}$ as in the proof of Theorem 1. Also, since a) the system satisfies the hybrid basic conditions as defined in [22, Assumption 6.5], b) $C\setminus D$ is open, c) no finite-time escape can happen during flow thanks to the ISS property and Assumption 5, d) the jump sets $D_0$ and $D_1$ are mapped through the jump map into $C \cup D$ according to [22, Proposition 6.10], the solutions are complete.

Assume a solution is $j$-complete. Since $j$ is toggled at each jump, necessarily, both $\mathcal{J}_1$ and $\mathcal{J}_0$ are infinite. Pick $j > 1$ in $\mathcal{J}_1$. Since $j > 1$, the solution has jumped at $t = t_j$ according to the jump map defined on $D_0$ so that $\hat{x}(t_j, j) \in \mathcal{X}_s'$. Consider a compact set $\mathcal{X}_s''$ such that $\mathcal{X}_s' \subset \mathcal{X}_s'' \subseteq \mathcal{X}_s$. If $T_j = \sup I_j < +\infty$, necessarily $\hat{x}(T_j, j) \in \partial\mathcal{X}_s$, and therefore, there exists $\bar{t}_j \in [t_j, T_j]$ such that $\hat{x}(t,j) \in \mathcal{X}_s''$ for $t$ in $[t_j, \bar{t}_j]$ and $\hat{x}(\bar{t}_j, j) \in \partial\mathcal{X}_s''$. We have

$$|\hat{x}(t_j, j) - \hat{x}(\bar{t}_j, j)| \geq d(\mathcal{X}_s', \partial\mathcal{X}_s'') > 0\ .$$

But for all $t$ in $[t_j, \bar{t}_j] \subset I_j$, $|\dot{\hat{x}}(t)| \leq \varphi_m$ defined in (B.4) which yields (by absolute continuity)

$$T_j - t_j \geq \bar{t}_j - t_j \geq \frac{d(\mathcal{X}_s', \partial\mathcal{X}_s'')}{\varphi_m} := dt > 0\ .$$

It follows that the time intervals associated to $j > 1$ in $\mathcal{J}_1$ have a length of at least $dt$ and since $\mathcal{J}_1$ is infinite, the solution is $t$-complete. Therefore, any solution is $t$-complete.

Now, by construction, the intervals of flow with $q = 0$ have a length that belongs to the set $\mathcal{I}$. Therefore they are bounded by $\max \mathcal{I}$. It follows that $\mathcal{J}_1$ is non empty, and, from Assumption 1, for all $j$ in $\mathcal{J}_1$ and all $t$ in $I_j$,

$$|\hat{\xi}(t,j) - \xi(t)| \leq \beta(|\hat{\xi}(t_j, j) - \xi(t_j)|, t - t_j) + \alpha(\nu_m)\ ,$$

where we have defined $\xi(t) = T(x(t))$. We need to prove that at some point, $\hat{x}(t,j)$ stays in the interior of $\mathcal{X}_s$, i.e, never reaches $D_1$, and the solution is eventually continuous. This is equivalent to showing that $\hat{\xi}(t,j)$ stays in the interior of $T(\mathcal{X}_s)$. Notice that the flow dynamics of $\hat{\zeta}$ are independent from $\hat{x}$, $q$ and $\tau$, and $\hat{\zeta}$ never changes at jumps. Therefore, from Assumption 5, there exists $t_\varepsilon$ such that for all $t \geq t_\varepsilon$ and for all $j$ such that $(t,j) \in \mathrm{dom}\,\phi$, we have $|\mathcal{T}_\varepsilon(\hat{\zeta}(t,j)) - x(t)| \leq \varepsilon$ . If for all $j$ in $\mathrm{dom}_j\,\phi$, $t_j \leq t_\varepsilon$, i.e. no jump occurs after time $t_\varepsilon$, $\hat{x}(t,j)$ stays in the interior of $\mathcal{X}_s$ after time $t_\varepsilon$ and the solution is eventually continuous. Otherwise, there exists $j \geq 1$ in $\mathcal{J}_1$ such that $t_j \geq t_\varepsilon$. Then, $\hat{x}(t_j, j) = \mathcal{T}_\varepsilon(\hat{\zeta}(t_j, j)) \in \mathcal{X}_s'$ , and from (6a), and since $t_j \geq t_\varepsilon$,

$$|\hat{\xi}(t_j, j) - \xi(t_j)| \leq L|\hat{x}(t_j, j) - x(t_j)| \leq L\,\varepsilon\ .$$

It follows that for all $t$ in $I_j$,

$$|\hat{\xi}(t,j) - \xi(t)| \leq \beta(L\varepsilon, 0) + \alpha(\nu_m) < \delta + \delta'\ ,$$

which means with (17c) that $\hat{\xi}(t,j)$ is in the interior of $T(\mathcal{X}_s)$. Therefore, $I_j$ is unbounded and the solution eventually continuous.

Now consider $J = \max \mathrm{dom}_j\,\phi \in \mathcal{J}_1$ and $\delta_{\max} = \max_{(\hat{\xi},\xi) \in T(\mathcal{X}_s') \times T(\mathcal{X})} |\hat{\xi} - \xi|$, which exists and is finite since $T(\mathcal{X}_s') \times T(\mathcal{X})$ is compact. Since $\hat{x}(t_j, j) \in \mathcal{X}_s'$, $|\hat{\xi}(t_j, j) - \xi(t_j)| \leq \delta_{\max}$, and for all $t$ in $I_J$,

$$|\hat{\xi}(t, J) - \xi(t)| \leq \beta(\delta_{\max}, t - t_J) + \alpha(\nu_m) \leq \delta + \delta'$$

if $\beta(\delta_{\max}, t - t_J) \leq \frac{\delta+\delta'}{2}$. This is achieved for all $t \geq t_0 := t_J + \beta(\delta_{\max}, \cdot)^{-1}(\frac{\delta+\delta'}{2})$ since $\beta$ is of class $\mathcal{KL}$. Therefore, for all $t \geq t_0$, $\hat{\xi}(t, J)$ is in $T(\mathcal{C})$, i.e. $\hat{x}(t, J)$ is in $\mathcal{C}$ defined in (45), and the conclusion follows.

## D Proof of Lemma 3

Let us start with Assumption 1. $\overline{\mathcal{S}}$ is open, $\overline{\mathcal{X}}_s'$ is compact and (16) holds with $\overline{\mathcal{X}}$, $\overline{\mathcal{X}}_s$, $\overline{\mathcal{X}}_s'$ and $\overline{\mathcal{S}}$. Also, according to (55), (3) holds for $\overline{T}$ and $\overline{\mathcal{T}}$ on $\overline{\mathcal{X}}_s$. Besides, given the construction of $\mathcal{T}$, the dynamics

$$\dot{\hat{\xi}} = \overline{\varphi}(\hat{\xi}, (\hat{x}, \hat{w}), y) \quad , \quad (\hat{x}, \hat{w}) = \overline{\mathcal{T}}(\hat{\xi}) \qquad \text{(D.1)}$$

are equivalent to

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, \hat{x}, y) \quad , \quad \hat{x} = \mathcal{T}(\hat{\xi})$$

whose trajectories verify (5) for any trajectory $t \mapsto x(t)$ of (1) staying in $\mathcal{X}$ according to Assumption 1. Given the definition of $\overline{\mathcal{X}}$ and using (54), we deduce that for any trajectory of (D.1), we have

$$|\hat{\xi}(t) - \overline{T}(\overline{x}(t))| \leq \beta(|\hat{\xi}_0 - \overline{T}(\overline{x}_0)|, t) + \alpha\Big(\sup_{s\in[0,t]} |\nu(s)|\Big)$$

for any trajectory $t \mapsto \overline{x}(t)$ of the extended system (52) staying in $\overline{\mathcal{X}}$. Therefore, (5) holds with $\overline{\mathcal{T}}$ and system (52), so that Assumption 1 finally holds for system (52). Now consider Assumption 3. Since
a) for all $(x, w)$ in $\overline{\mathcal{X}}$, $\overline{T}(x, w) = T(x)$
b) for all $\hat{\xi}$, and all $\overline{x} = (x, w)$, $\overline{\varphi}(\hat{\xi}, \overline{\mathcal{T}}(\hat{\xi}), \overline{h}(\overline{x}) + \nu) = \varphi(\hat{\xi}, \mathcal{T}(\hat{\xi}), h(x) + \nu)$
c) for all $\overline{x} = (x, w)$, $\frac{\partial \overline{T}}{\partial \overline{x}}(\overline{x})\overline{f}(\overline{x}) = \frac{\partial T}{\partial x}(x) f(x)$,
(20) with data $T$, $x$, $f$, $h$ and $(\varphi, \mathcal{T})$ implies (20) with $\overline{T}$, $\overline{x}$, $\overline{f}$, $\overline{h}$ and $(\overline{\varphi}, \overline{\mathcal{T}})$.

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