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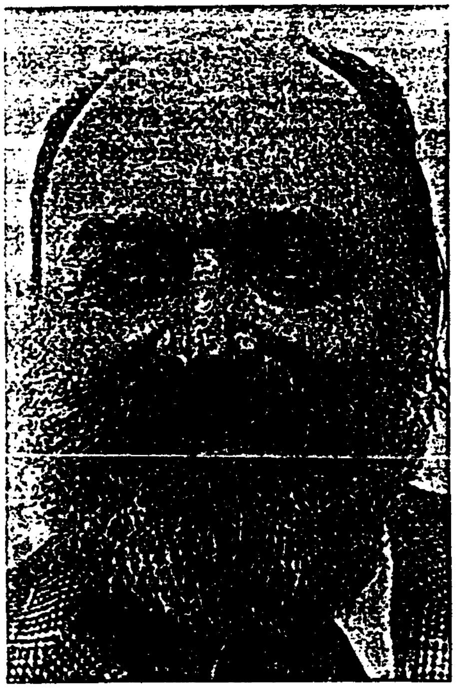
Chaos: A View of Complexity in the Physical Sciences

Leo P. Kadanoff



Leo Kadanoff is a theoretical physicist who has contributed widely to research in the properties of matter and upon the fringes of elementary particle physics, Most recently he has been involved in the understanding of the onset of chaos in simple mechanical and fluid systems.

He was born and received his early education in New York City. He did his undergraduate and graduate work at Harvard University, and after some post-doctoral work at the Niels Bohr Institute in Copenhagen he joined the staff at the University of Illinois in 1962. In 1966 and 1967 he did research on the organization of matter in "phase transitions," which led to a substantial modification of physicists' ways of looking at these changes in the state of matter. For this work he received the Buckley Prize of the American Physical Society (1977) and the Wolf Foundation Prize (1980).

He went to the University of Chicago in 1978 and became John D. MacArthur Distinguished Service Professor of Physics there in 1982.

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Introduction

Definition of chaos: complexity and order

The word *chaos* has suddenly come to be popular in the physical sciences. It is used to describe situations in which we can see a very complex behavior in space and time. Because the term is used imprecisely, it is best explained by example (see illustrations on the following two pages). Figure 1 shows a kind of atmospheric disturbance which, over the course of many years, has been observed on the surface of the planet Jupiter. This close-up shows a quite intricate pattern of atmospheric swirling or turbulence. Observers of the TV news will recognize that somewhat similar swirling patterns also exist in the Earth's atmosphere. On both planets, the turbulence takes on fantastic forms in which we can nonetheless see some underlying regularity and order. One kind of regularity is that the storm, according to what we believe, has continued to exist on the surface of Jupiter for millions of years. Another kind of regularity is that the storm contains large, rather uniform regions.

The predominant impression that one gets from weather maps on either planet is nevertheless one of considerable complexity. Chaotic patterns are characteristically quite varied in their details, but they may have quite regular general features. For example, clouds are sufficiently orderly so that one can give a meaningful classification of their general types, but each type exhibits endless variations in its detailed shapes.

Look at another example. Figure 2 shows a dried-up lake in which mud has hardened itself into a complex pattern. We can see that the pattern is almost the same in different places, but it repeats itself with apparently unpredictable variations and is hence "chaotic." Additional familiar esamples of chaotic behavior are provided by the fantastically rich patterns of snowflakes or of the frost which can appear on the inside of a window in winter. Figure 3 shows the result of the solidification of water on a cool, flat surface. New ice forms in contact with the old. Because a piece of the ice surface which sticks out can more effectively move forward, projections upon the ice surface grow into longer and longer branches. But then, if a branch has a little bump on

Figure 1. A "storm" in the atmosphere of Jupiter.

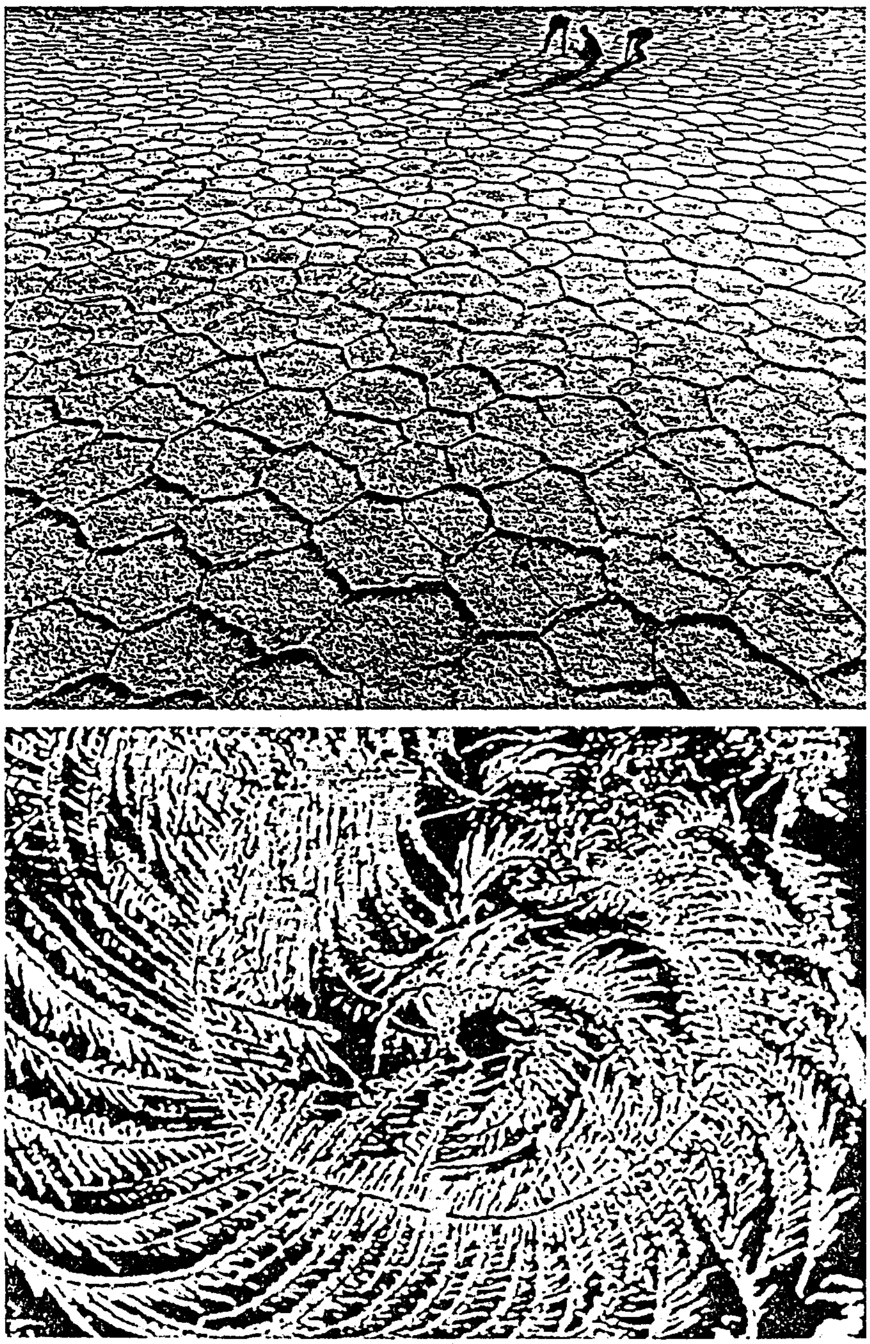
it, that bump will also tend to grow and become a branch itself. And, by the same logic, branches will grow on branches, and so on indefinitely until a beautiful treelike shape arises.

All these examples of chaos have several striking features in common. One, which I wish to emphasize now, is the outcome's sensitivity to the conditions under which the pattern is formed. A degree's change in temperature today, and next week's weather map would change totally. If you blow upon the windowpane, you can completely change the details of a branching pattern like that of Figure 3.

Chapter I: Simple laws, complex outcomes

The physical sciences are divided into many disciplinary subfields, among them meteorology, astronomy, aerodynamics, and physics. Except for physics, all these subfields try to gain a deep and solid understanding of particular areas of nature. Thus, if an astronomer looks at a galaxy and sees it to be chaotic, his or her natural reaction is likely to be a desire to understand that particular problem. Concentration on a complex behavior is natural to fields of activity like astronomy,

Physicists, on the other hand, consider themselves to be looking for the fundamental laws of nature. They seek basic principles, ideas, and mathematical formulations on which all further understanding can be



Figures 2 and 3. *"Chaotic patterns are characteristically quite varied in their details, but they may have quite regular general features."* The pattern of the dried up lake, in Figure 2 (top), and frost on a flat surface, in Figure 3, are examples.

built. To look at complexity is to some extent a new endeavor for physicists. It runs counter to the idea of physics as the science that seeks to understand nature in simplest terms. Newton gave us three simply stated laws to describe all the motions of the heavenly bodies — and many aspects of earthly motion as well. The laws of general relativity or of quantum mechanics are also simple to state. Moreover, such laws tend to result from a study of their "simplest," most elementary realizations. For Newton's gravity this realization is found in Kepler's rules for the motion of two gravitating bodies; for general relativity it is found in black holes; for quantum mechanics it lies in hydrogen atoms.

For the student of physics, or the practitioner, the science is interesting and beautiful precisely because it summarizes the complexity of the world in a few simple laws and then describes the consequences of these laws by almost equally simple examples. However, many students and even some practitioners suspect that something is lost in the process. When physics concentrates on three laws, or five, or seven, when those laws are mostly applied only to the very simplest exampies, we have lost something of the real world. We have chosen to ignore the wonderful diversity and exquisite complication that really characterize our world. This choice has led to wonderful descriptions of nature in our theories of quantum mechanics, relativity, cosmology, and so forth. But, these theories are so focused upon the simple and "basic" that they run the danger of providing a peculiar caricature of nature. Focus upon simplicity and you leave out Jupiter's storms, the diversity of galaxies, the intricacies of organic chemistry, and indeed life itself.

In recent years there has been some change in the attitude of many physicists toward complexity. Indeed, the very existence of this article reflects the change. Physicists have begun to realize that complex systems might have their own laws, and that these laws might be as simple, as fundamental, and as beautiful as any other laws of nature. Hence, more and more the attention of physicists has turned toward nature's more complex and "chaotic" manifestations, and to the attempt to construct laws for this chaos.

In some sense, this change in attention has resulted from a natural attempt to understand interesting situations such as the ones I have shown in the figures. In another sense, the concentration upon chaos has been a part of a change in our understanding of what it means for a law to be "fundamental" or "basic." Physical scientists have sometimes been tempted to take a reductionist view of nature. In this view, there are fundamental laws and everything else follows directly and immediately from them. Following this line of thought, one would construct a hierarchy of scientific problems. The "deepest" problems would be those connected with the most fundamental things, perhaps the largest issues of cosmology, or the hardest problems of mathematical logic, or maybe the physics of the very smallest observable units in the universe. To the reductionist the important problem is to understand these deepest matters and to build from them, in a step-by-step way, explanations of all other observable phenomena.

Here I wish to argue against the reductionist prejudice. It seems to me that considerable experience has been developed to show that there are levels of aggregation that represent the natural subject areas of different groups of scientists. Thus, one group may study quarks (a variety of subnuclear particle), another, atomic nuclei, another, atoms, another, molecular biology, and another, genetics. In this list, each succeeding part is made up of objects from the preceding level. Each level might be considered to be less fundamental than the one preceding it in the list. But at each level there are new and exciting valid generalizations which could not in any very natural way have been deduced from any more "basic" sciences. Starting from the "least fundamental" and going backward on the list, we can enumerate, in succession, representative and important conclusions from each of these sciences, as Mendelian inheritance, the double helix, quantum mechanics, and nuclear fission. Which is the most fundamental, the most basic? Which was derived from which? From this example, it seems rather foolish to think about a hierarchy of scientific knowledge. Rather, it would appear that grand ideas appear at any level of generalization.

With exactly this realization in mind, one might look at the rich variety of chaotic systems and wonder whether there are broad and general principles which can be derived from them. In fact, I have already mentioned one such "law": *chaotic systems show a detailed behavior which is extremely sensitive to the conditions under which they are formed*. The consequences of this sensitivity are further examined in the next section.

Practical predictability

Many of the modern concepts of chaos were formed by Henri
a nineteeth-century French astronomer and mathematician. He recognized very clearly that there was a qualitative difference between the motion of two gravitating bodies (Earth-Sun, for example) and that of three (Moon-Earth-Sun). In the former case, when we have two bodies each moving under the gravitational influence of the other, the orbits are simple and easily predictable. They are Kepler's ellipses, and these orbits are certainly not chaotic. The latter situation, the famous "three-body problem," is chaotic. Three bodies develop complex orbit structures in which the positions of the objects in the distant future are extremely sensitive to their positions now. And this sensitivity and complexity is not just theoretical nonsense. It has practical consequences. To predict the future, one needs information about the present, and the longer the forecast, the better the information required. In the

chaotic problem, the accuracy required of the input data must be very sharply improved as the forecasting period becomes longer and longer.

For astronomical systems, the data initially needed are the position and velocities of the gravitating bodies. Imagine that we are looking ahead and trying to forecast the positions of the planets, perhaps with a view to predicting the time of eclipses. Imagine further that there is a certain error in our present knowledge of planetary positions, perhaps by only a few feet. In both the nonchaotic and the chaotic cases our forecasting uncertainty will get larger as we look further forward into the future. The difference is in the type of growth. In the nonchaotic case, for each further year of forecast, the uncertainty grows by the addition of an increment proportional to the original uncertainty. In the chaotic case, for each additional year of forecast, the uncertainty grows by an increment proportional to the uncertainty in *that year's* forecast. The latter type of growth, so-called exponential growth, is akin to compound interest and is very rapid compared with the growth in the nonchaotic case, which is akin to simple interest. In the long run, the uncertainties in the "compound interest" case are far, far larger than in the corresponding case of "simple interest."

Thus our ability to forecast, for example, eclipses, is far, far worse in the chaotic case than in the more orderly example of the motion of two gravitating bodies.

This line of thought was picked up by a meteorologist from the Massachusetts Institute of Technology, Edward Lorenz. He was interested in the implications of the idea that Earth's atmosphere might exhibit a sensitivity to initial conditions similar to the one which had been thought about in the gravitational case. His work, published in 1963,* in some sense marked the beginning of our "modern era" in the study of chaotic systems. He looked at convection, that is, flows in which a heated fluid rises because it is less dense than its surroundings. He set up a simple mathematical model for convection, solved it on a computer, and showed that even in this oversimplified case the system's behavior was wonderfully rich and complex. In addition, he showed that its long-term behavior exhibited the kind of sensitivity to initial conditions that was described above for chaotic planetary systems. He made the point that if actual weather prediction were like the model he studied, it would be terribly hard to predict very far ahead.

Naturally, this practical unpredictability has very important implications for all kinds of engineering arts involving chaotic situations-not only weather prediction, but also airplane wing design, the flow of fluids through chemical plants, and many other cases. It has also inspired

*E. N. Lorenz. "Deterministic Non-Periodic Flows," *Journal of the Atmospheric Sciences* 20 (1963): 130.

some rethinking about such familiar philosophical questions as free will and determinism. For several different reasons, then, we may wish to understand this result in somewhat more detail.

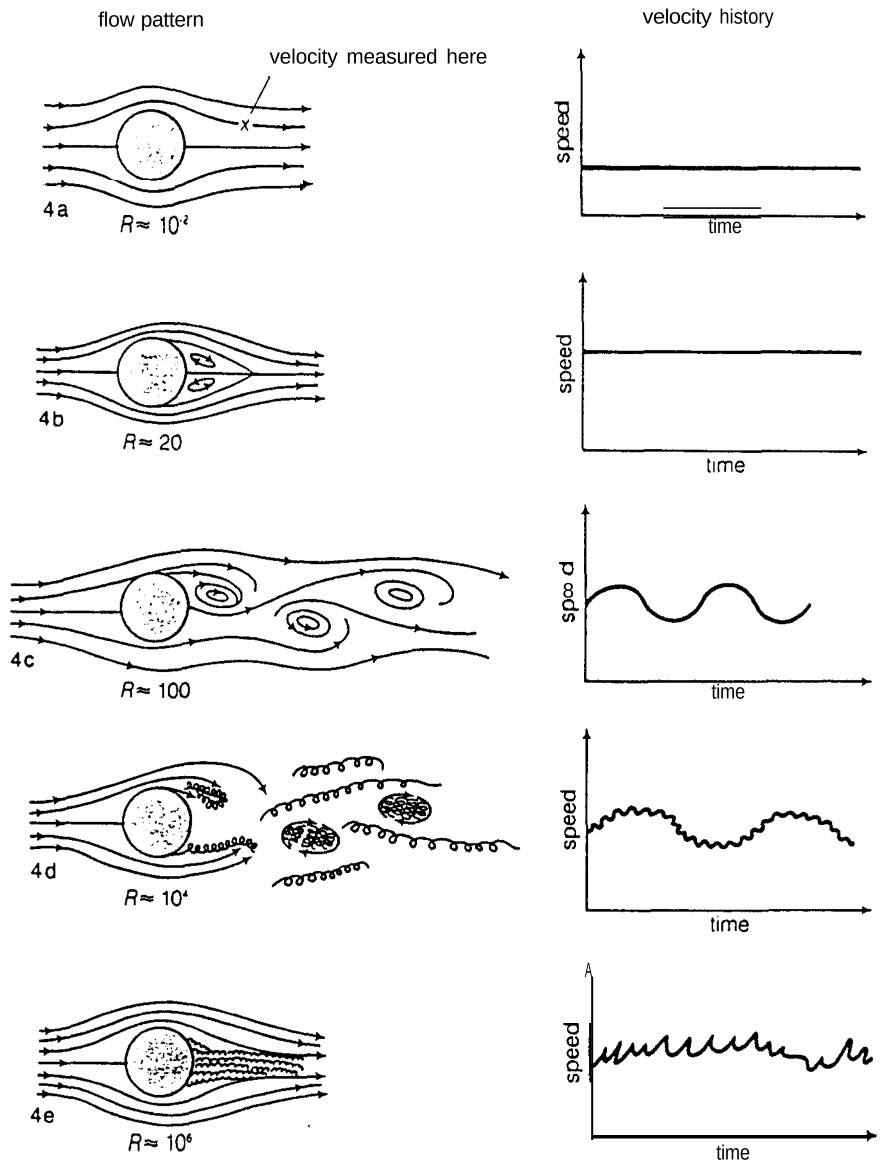
In the next chapter I shall further discuss the nature of chaos by showing how it arises. In the final chapter I shall describe how chaos reflects itself in beautiful and complicated geometrical structures and illustrate this with an esample from Lorenz's work.

Chapter II: Routes to chaos

One way of understanding the nature of chaos is to ask how it arises. One can start from a very orderly situation, gradually change the situation, and then see chaos set in little by little. Consider, for esample, the flow of water that might occur in a river as it flows past an obstacle. Imagine that we are standing on a bridge, looking down at the water as it flows past a buttress of the bridge sitting in the water. If the water is flowing slowly, it flows in smooth and unswirly paths like those in Figure 4a. The rate of flow is listed in the different parts of Figure 4 by giving the value of a "parameter," R, which is proportional to the rate or speed at which the river is flowing. (The word *parameter* is often used in the sciences to mean a numerical value which defines a natural situation; for example, the birthrate is an important parameter for determining the future quality of life on our planet.)

Successive rows in Figure 4 show the situation for successively higher values of the flow rate. As this parameter is increased, the flow gets successively more complex. This increase in complexity is depicted in two ways. The first column shows spatial patterns by depicting the flow path of typical particles in the water-or of debris on the surface-as the particles (or the debris) move around the buttress. In the second column we plot the speed of the water at a particular spot, the one marked with an x in Figure 4a. For small speeds, as in 4a, the flow pattern would be completely time-independent and totally lacking in swirls. If the river were running a bit faster, as in Figure 4b, there would be a few swirls or vortices fixed in place near the bridge, but because these are fixed, the pattern would remain time-independent. Increase the speed still more, as in Figure 4c, and the swirls come loose and start moving slowly downstream, away from the bridge. New swirls are produced near the buttress at a regular rate, and these too move downstream in a regular progression. In this case, our instrument, which measures the speed at point x, will show a repetitive time dependence in which the speed goes through a maximum as the swirl passes by.

Such periodic behavior simply repeats itself again and again as time goes by and is certainly not chaotic. But a further increase in the speed does produce chaos. Figure 4d indicates that at this higher value of R



Figures 4a–e. Flows of water past a cylinder for successively larger values of the velocity, or flow rate, defined by the parameter, R. The first column shows flow patterns, the second, time histories of the velocity at the point marked by an x in Figure 4a.

the individual swirls have begun to look a bit ragged and chaotic. The time dependence shows a basically periodic pattern, similar to that of Figure 4c, but there is a small amount of chaotic jiggling superposed upon the regular motion. Finally, if the river is moving very fast indeed, as in Figure 4e, the turbulent region moves out and fills the entire wake behind the bridge. Then the time dependence seems totally unpredictable and chaotic.

A model of chaos

Next, I would like to explain in somewhat more detail and with greater precision exactly how the chaos arose in the hydrodynamic system just described. I would like to, but I cannot. Nobody has a real understanding of chaos in any fluid dynamical context. So, instead of that, I shall turn my attention to a simpler problem, one with a very simple mathematical structure which we can encompass and understand. (A simpler problem used to illuminate a more complex one is called a "model.")

This chapter and the next are largely concerned with the description of several mathematical models of chaos. These models are sets of equations which are easier to understand and study than the realistic cases which are our actual concern. However, if deftly chosen, such a model might just capture some important feature of the real system and exhibit it in a transparent form. In fact, in the best of cases the model will capture the essential nature of the physical process under study and will leave out only insignificant details. In this best situation, the model can be used to predict the results of experiments in the real system.

Our present interest is in the onset and development of chaos in fluid mechanical systems like the one depicted in Figure 4. Our model system for understanding this onset is so simple that one might, at first glance, assume that it contains nothing of interest. But I ask the reader to suspend disbelief, at least for a time. The model is interesting and does have a connection to hydrodynamic systems.

Consider, therefore, an island with a popuiation of insects. In every year, during one month, the insects are hatched, they eat, they mate, they lay eggs, and they die. In the next year the whole process is repeated over again.

A mathematical model for this kind of process is a formula by which we can infer each year's population from the population in the previous year. We can repeat this inference again and again, and thereby generate a list of populations in the different years. We can then examine the list and see whether the result is orderly or chaotic. An orderly pattern might, for example, be one in which the population increased year by year but, after a while, started to "level out" so that, in the long run, the population approached closer and closer to some final value. A chaotic pattern could be one in which the population went up and down in an apparentlv disorderly way, like a stock market average. A given island might behave in either fashion, depending upon the formuia used to generate one year's population from the last.

Sow imagine a whole group of islands, each of which provides a different kind of environment for our insects. The different islands are distinguished by a growth-rate parameter, called r, which is a qualitative indication of how well the particular island supports the insect popuiation. We must visualize two basic processes going on. One is the natural increase in population year by year in a manner rather similar to compound interest. The other is the effect of overcrowding. If the population gets too large, destructive competition ensues and the next year's population is considerably smaller than it would otherwise have been.

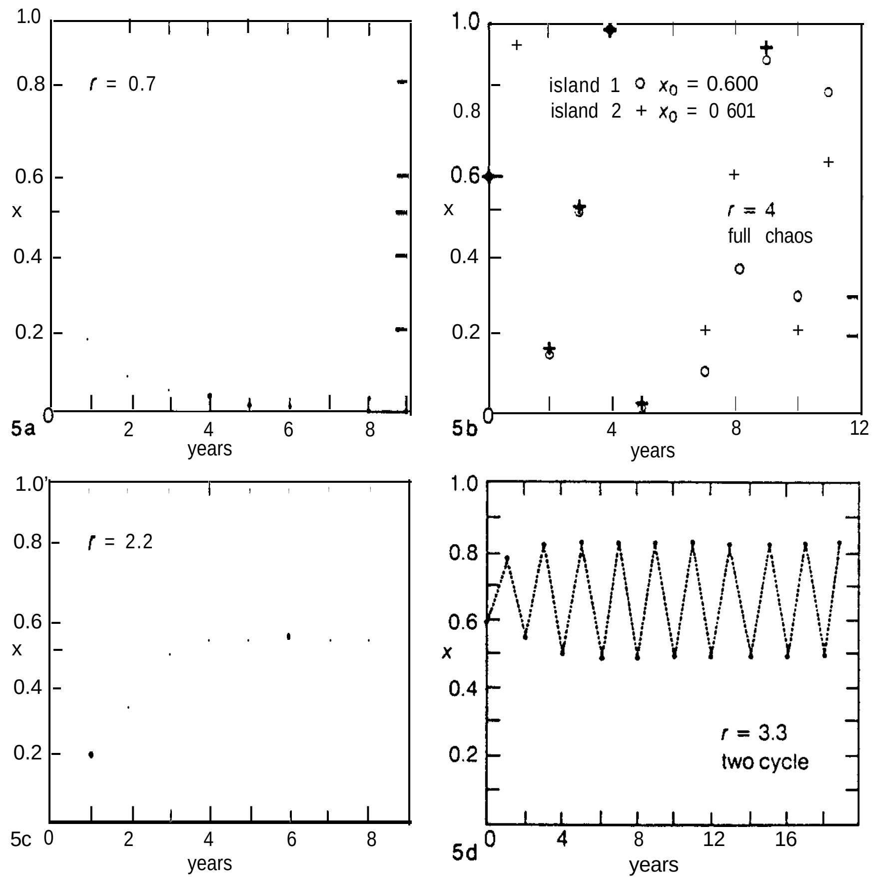
Given these two processes, there are several outcomes we may imagine. I will describe these outcomes verbally here and mathematically later on:

Case *a*. (The lowest values of r.) A poor environment provides a negative natural increase for the insects: year by year the population decreases until, finally, no insects remain. This pattern is totally orderly and not at all chaotic. The time-evolution of the population for this case is depicted in Figure 5a.

Case *b*. (Very high population increase. Large r.) In this case, a quite disorderly pattern may ensue. For example, imagine that in the first year the population is small. With a large growth factor, it could well be true that the next year's population will be very large. Then the unfavorable effects of crowding could cause a precipitous drop in the population for the third year. Over the nest few years the population could grow again, and then once again collapse due to overcrowding. We could thus have a situation in which the population increased and decreased in a disorderly and apparently chaotic fashion. This kind of behavior is depicted in Figure 5b for two separate islands.

Case c. (Intermediate values for the population increase.) Imagine a natural growth rate just large enough to sustain a slow population growth in the absence of any overcrowding effect. In this kind of island, we would see a population that increases year by year until overcrowding limits its growth. In the end, the population would settle down to a steady value in which crowding and natural growth balance each other. Such a population pattern is quite orderly. The pattern is shown in Figure 5c.

We have said, in sum, that different islands might be described by the same kind of model, but that each island would have to be distinguished by different values of the population growth parameter, r. Depending on the value of this parameter, a given island might show either orderly behavior (for low rates of population increase) or chaos (for high rates).



Figures 5a–d. Time histories of insect populations for four different values of the parameter, r. Figure 5b shows two histories with slightly different starting populations. In the course of time this difference becomes larger and larger. For the other **r-values** shown, the effect of such a small mutual difference would never become noticeable. The case shown in 5b is chaotic, the other cases depicted are not. Dotted lines in Figure 5d highlight the fact that the diagram is one oscillating insect population, not two.

Now we wish to convert the verbal argument into a mathematical one by giving a well-defined formula to determine one year's population from that of the year before.

The mathematical description of this situation will enable us to give a far more detailed account of the possible outcomes than I could give in the verbal descriptions above. There is a further advantage however. We can imagine a succession of different islands, each with a slightly larger value of the growth-rate parameter. In our minds, we can examine the history of each of these islands. Those with small values of r will be orderly; those with the largest will be chaotic. By studying the intermediate values of r we can ask ourselves, "just how does the chaos first arise?"

The model in mathematical form

One can describe an island in terms of a variable, p, which tells you the population of insects in a given year. The mathematical model is an equation that gives the population next year, p_{next}, in terms of the population this year, p. For example, the simplest such model might predict that the population will increase by 10% during each year. This process would be represented by a simple equation for next year's population, namely,

$$p_{next} = rp. \tag{1}$$

To represent the 10% per year increase in population, we will choose the growth-rate parameter, r, to be 1.1. To look for chaos, we will take the equation that predicts each year's population from the last and then use it to generate a list of populations in different years,

$$p_0, p_1, p_2, \ldots, p_j, \ldots. \tag{2}$$

Here the subscripts 0, 1, and 2 are used to describe the populations in years 0, one, and two, . . . , while p_j represents the population in the jth year. The game then is to use an equation like equation (1) to calculate each year's population in terms of the last and thereby generate, in year-by-year fashion, a list like that in expression (2). (The problem, and its solution, is exactly the same as the one for compound interest.) We then look for patterns in the list and ask whether the pattern is orderly or chaotic, and why. We especially ask whether the different islands, which are represented by different values of r, show different types of behavior. The answer is yes. There are three different categories of behavior corresponding to three qualitatively different types of environments for the insects and consisting of three different ranges of r. These are:

First case: A poor environment. Here the growth-rate parameter, r, lies in the range between zero and one. For these islands the population is smaller each year until eventually it becomes invisible. The resulting population pattern is orderly but dull.

Second case: An equally orderly and dull result will ensue in an island described by equation (1) with $r = 1$. In this balanced environment the population would simply remain unchanged year by year.

Third case: In a favorable environment, r would be greater than one. An island with this environment would have a population that increases year by year. The population would grow without limit.

This last case is unrealistic, of course. In the long run, something must limit the insect population. Thus, the simple model of equation (1) is unsatisfactory as a natural prediction. Furthermore, it shows no chaos. We must go on to develop a slightly more complex model.

The next simplest model could show that when the insect population gets large the reproductive process is inhibited and next year's population is diminished. This reduction might occur because individuals would compete for food or for nesting space. Or maybe the insects are simply shy and do not reproduce well when they are crowded. In any case, the model we need is one that reduces the population predicted in equation (1) by an amount proportional to the number of possible interactions among the different individuals in the population. Since the number of possible interactions is proportional to the population squared, we might try a model of the form:

$$p_{next} = rp - sp^2, \tag{3}$$

where s is another parameter which measures the effectiveness of the various interactions in a diminishing population.

Note that equation (3) only makes sense if the population is smaller than r/s. If the population is larger than this value, equation (3) gives the non-sense result of a population in the next year as negative. For this reason, we limit our attention to situations in which the population is a positive number but smaller than r/s.

One final step is required to convert this model into a form suitable for further study. Instead of using a variable, p, for population, we will use instead a variable x and say that $x = (s/r)p$. By this we mean that x measures the ratio of the actual population of the island to its maximum possible one. Thus, x varies between zero and one, for the population cannot be less than zero, nor greater than the maximum population the island can sustain. According to equation (3), the population ratio next year is determined from the population ratio this year as follows:

$$\begin{aligned} p_{next} &= rp\,(1 - sp^2/rp) \\ p_{next} &= rp\,[1 - (s/r)\,p] \\ (r/s)\,x_{next} &= r\,(r/s)\,x\,(1 - x). \end{aligned} \qquad [\text{since } p = (r/s)\,x]$$

We can now cancel out the common factor, r/s, and find

$$x_{next} = rx\,(1 - x). \tag{4}$$

Once again, r has the significance of a growth factor. When the population is small, i.e., x is close to zero, then $(1 - x)$ is close to one, and the population will be multiplied only by a factor of r during each year. Here then is our model. Next we can look at its consequences.

Order and chaos

For r (the growth-rate parameter) less than one, our first model, equation (1), gave a uniformly diminishing population. Since the modification that led to equations (3) and (4) will further decrease the population through decreased reproductive ability, it is reasonable to expect that this decline will also occur in our new model. To see how it works, consider, For example, the case in which $r = 0.7$. Choose some initial value of the population ratio x, for example $x_0 = 0.6$. Then it is very easy to use equation (4) to calculate the next year's population ratio to be $x_1 = rx_0(1 - x_0) = .7 \times .6 \times .4 = 0.168$. The next year's calculation gives $x_2 = 0.0978$, showing that the population has diminished. It continues to diminish, as shown in Figure 5a.

In contrast, consider an island in which the growth-rate parameter, r, has the value 4. Then, by equation (4), if the population starts out low, it will in the next year quadruple. Hence, it cannot stay low long. However, if the ratio of the actual population to the maximum ever gets close to its highest possible value, 1, then in the next year the population will become very small (from the combined obstacles to reproduction). The resulting population pattern can be seen with the data points shown as circles in Figure 5b. It goes up and down in an apparently unpredictable manner. To see this unpredictability in even more detail, compare the circle-points to the data points shown as plus signs. The only difference between the population patterns in the two cases is the starting value of the population. The circle-points represent an island in which the ratio of the initial population to the maximum is given by $x_0 = 0.600$. The plus points represent another island which has a very slightly different starting value, $x_0 = 0.601$. At the beginning and for the first few years, we cannot tell the difference between the two islands. The plus points lie on top of the circle ones. However, after a few years the difference between the two cases becomes noticeable. By the end of the twelve-year period shown in Figure 5b, there seems to be no correlation between the populations of the two islands. The diverging behavior of the population patterns, which were at first so similar, is a demonstration of the sensitivity to initial conditions that I described in the first chapter as a sign of chaos.

Now we have two situations which we can describe in words. For a growth factor, r, between zero and one the pattern is very orderly: the population simply dies down to zero. At $r = 4$, the behavior is highly chaotic; the population pattern keeps jumping around, and for most starting values of the population ratio, x, it never settles down to any orderly behavior. So this system exhibits both order and chaos. What lies in between, or how do we get from one to the other?

Period doubling and the onset of chaos

To repeat, for a growth-rate parameter, r, less than one, eventually the population dies away and x settles down to a specific value, namely, 0, by equation (4). Try to visualize what happens for r just greater than one. Imagine that for this case, too, a settling down occurs. Use the

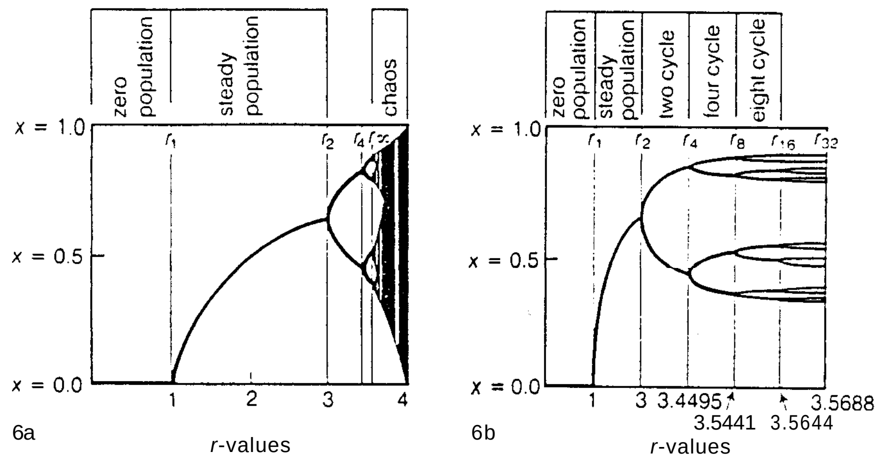


Figure 6. A summary of possible long-run behavior of insect populations for islands with different values of the growth-rate parameter, r. Two views of the same basic plot are shown. The left view has an ordinary linear scale for r in which the distance between r = 1 and r = 2 is the same as that between r = 3 and r = 4. At right, the scale of r is distorted to emphasize the region in which the higher order period doublings occur.

symbol x^* to denote the long-run value of x, i.e., the value into which the population settles after many years. Then, if the value of x this year is x^*, the value next year will also be x^*. What is the possible value of x^* itself? Look back at equation (4), substitute x^* for both x and x_{next} and we find that x^* must obey

$$x^* = rx^* (1 - x^*). \tag{5}$$

There are two possible solutions,

$$x^* = 0 \quad \text{and} \quad x^* = 1 - 1/r. \tag{6}$$

We observe that for r greater than one, the first solution is unstable in the sense given by Malthus: Even if the initial population is small, since r is greater than one, the population will grow until it is limited by overcrowding. We can see this behavior by looking at Figure 5c. This plots the population change starting from the very small value x_0 = 0.1 for the case in which r = 2.2. If we substitute this value for r in equation (6), we find x tending to 0.5454 . . . after a long period of time. From Figure 5c, we see that this value for the population ratio, x, has essentially been achieved after ten years.

In Figure 6 I will summarize the knowledge we have gained so far. This plot shows what x-values are obtained in the long run for different values of the growth-rate parameter, r. We know that at r = 4, all x-values between zero and one show up (fig. 5b), while for r between zero and one, only x^* = 0 is possible, i.e., in the long run the population will decline to nothing (fig. 5a). For r between one and three, the only possible long-term value of the insect population is the x^* given by the second part of equation (6). These three regions of behavior — $r = 4$, $0 < r < 1$, and $1 < r < 3$—correspond, respectively, to the chaotic behavior shown for the fluid in Figure 4e, and to the time-independent fluid behaviors shown in Figures 4a and 4b. In Figure 6, these three regions are marked as "chaos," "zero population," and "steady population." However, in the actual fluid, when the flow rate, R, was increased beyond the value shown in Figure 4b, then the motion became time dependent.

The analogy between the model example and the real system continues to work. As r is further increased, beyond the value 3 the steady behavior represented by equation (6) disappears. As in the fluid example of Figure 4c, a time dependence suddenly appears. One can see this time dependence by looking at Figure 5d, where the growth rate, r, is given as 3.3. Notice that after a few years the insect population settles down into a regular pattern. But the pattern is time dependent. There are two eventual population values: one high, one low. In good years the insect population is low and they reproduce avidly. The next year is a bad one in the sense that there is much overcrowding. Hence, reproduction is impaired, so that the following year's population is low. This alternation continues forever. Such a situation, in which the behavior repeats after two steps, is called a two cycle or a cycle of length two. The cycle is depicted in Figure 6 by showing two values of x for each r in this region.

But time dependence is not chaos. Chaos will not arise until the x-values cease to settle into a regular pattern. To see how that happens, imagine increasing r still further. As r increases above 3.4, a cycle of four years dominates the long-run behavior of the population. For almost all starting values of x in this region, the insect population will, after many years, fall into a pattern in which the population cycles through four different values before it repeats. This pattern is also shown in Figure 6 in the region marked four cycle. Increase r a bit more and you get an eight cycle, a tiny bit more and the period doubles yet again, until at $r = 3.59946\ldots$ an infinite number of period doublings have occurred and we reach a situation which might fairly be described as chaos. In this system, chaos first appears as a result of many successive doublings of the period of the cyclic population pattern. Hence, we call what we have just described the period-doubling route to chaos.

3.5699

The successive values of r at which cycles of length 1, 2, 4, 8, . . . first appear are denoted in Figure 6 by $r_1, r_2, r_4, r_8, \ldots$, while the r-value for which the cycle of infinite length appears is denoted by r_∞. At r_∞ the insect population never repeats itself, never settles down to a fixed value or values. We say then that we have reached the onset of chaos. Likewise, for most *higher* values of the growth-rate parameter, r, those between r_∞ and 4 (i.e., the region marked chaos in fig. 6a), the typical behavior of the insect system is one in which it does not repeat itself but shows a chaotic behavior. The crucial value of r is thus r_∞, since it is at this value that chaos first appears. We call this point on the x versus r curve a Feigenbaum point, for Mitchell J. Feigenbaum, who first elucidated its properties and thus enabled us to understand the period-doubling route to chaos.

Universality and contact with experiment

It may seem that we have lost contact with the hydrodynamic systems that served as our starting point. The last section's argument about the onset of chaos seems very specific to population problems, or at most to problems that involve a single variable, x, and a dynamics in which x is determined again and again in a step-by-step fashion. However, the work on our simple model offered a hint that the results obtained might be more generally applicable. For there are some aspects of the answers which seem quite independent of the exact form of the problem under study.

Recall that we studied a problem in which the population was determined by the equation:

$$x_{next} = rx\,(1 - x).$$

But, we could have used a slightly different equation, for example,

$$x_{next} = rx\,(1 - x^2).$$

If it were true that the answers obtained were equally valid for both types of equations, and for many others like them, we might guess that these answers would have the potential for being much more general than the particular starting point we used would suggest. They might even be applicable to real systems, in the laboratory or in nature. When some result is much more genera1 than its starting point, the situation is described by mathematicians as one of *structural stability*. Physicists describe a similar situation by saying that it exhibits *universality*.

Where can we find universality in the period-doubling route to chaos? There are two places. First, the overall structure of Figure 6, with its doublings and chaotic regions, is insensitive to the choice of the exact equation that will determine x_{next}. Changing the equation will distort the picture somewhat, as if it were drawn upon a piece of rubber and stretched, but will leave its essential features quite unchanged. Since this picture describes a variety of different mathematical problems that lead to an infinite period doubling, perhaps it also describes some real physical cases that show infinite period doubling.

In this way, we can hope to make contact between the "insect system" and real experimental systems. For example, we can set up electrical circuits that are unstable and "go chaotic" as some control parameter, roughly analogous to r, is changed. The most familiar example is an audio system, where r would describe the position of the microphone relative to the speaker. As these are brought closer together, the system may become unstable, that is, a hum may develop.

Analogous purely electrical circuits have been constructed to test the theory described above. For example, Testa, Perez, and Jeffries* performed an experiment in which an electrical circuit containing a transistor was controlled with a voltage, v_c, which played a role analogous to our parameter r. They noticed that their circuit had a natural oscillation that changed character as v_c varied. As v_c was increased, the period of the oscillation doubled, and doubled, and doubled again. By observing peak voltages, v_p, at one point in the circuit, they were able to trace out a v_p versus v_c picture that looked very much like Figure 6. Thus the electrical circuit showed a behavior very much like the one we have described. We can say, therefore, that we understand the period-doubling route to chaos in the real electrical system because we understand it in the insect model, and the two are very much the same.

Feigenbaum pointed to another way in which universality would manifest itself. As r approaches r_∞, he said, some aspects of the time pattern would remain the same even if insect behavior was different. For example, he looked at r-values for which cycles of length 1, 2, 4, 8, 16, . . . , ∞ first appeared. In Figure 6 these are denoted by r_1, r_2, r_4, r_8, r_{16}, . . . , r_∞. There is nothing universal or general about the appearance of the first few cycles. Hence there is nothing very useful to say about r_1 or r_2 or r_4. But the r-values at which very long cycles would appear turned out to be much more predictable. As the cycles get longer and longer, the spacing between successive r-values gets smaller and smaller (see the numbers at the bottom of fig. 6b). Indeed, for long cycles, the spacing forms a geometrical series in which the successive terms are divided by a constant factor called δ. It is surprising but true that *this constant has a value which is universal*, i.e., independent of insect behavior specifically. The spacing ratio, δ, takes the value $\delta = 4.8296$ . . . for *all* growth of the type indicated here, the type, that is, which follows a pattern of period doubling.

At first, other workers in the field were resistant to Feigenbaum's work, and particularly to the proposition that a number like δ could be universal. Feigenbaum derived an elaborate theory of this universality, based upon the "renormalization group" theory that Kenneth Wilson† had invented for quite another area of physics. The argument

*J. Testa, J. Perez, and C. Jeffries, "Evidence for Universal Chaotic Behavior of a Driven Non-Linear Oscillator." *Physical Review Letters* 48 (1982): 7 14.

†Kenneth C. Wilson, "Problems in Physics with Many Scales of Length," *Scientific American* 241 (August 1979): 158.

was settled by two developments: (1) a mathematical proof that in an appropriate sense the result was universal, and (2) experimental verifications that Feigenbaum's predictions about the quantitative aspects of successive period doubling held in other examples far removed from simple population growth models. In fact, δ-values are obtained for experiments involving instabilities in electrical circuits and also fluid systems. Within the limited accuracy of the experiments, Feigenbaum's predictions were fully verified.

The end result is a remarkable intellectual achievement. We can say with some truth that we understand how chaos arises in the simple model system described by equation (4). This is a satisfying achievement, and even though the system is simple, it is impressive. But we also have evidence, partially based upon theory and partially upon experiment, that exactly the same route to chaos is obtained in other much more complex systems. We believe that if we took one of these complex systems, measured the 'period doublings, and thereby found the value of δ, that number would be exactly and precisely the same number as the corresponding J-value obtained from the simple model of equation (4). Thus the pattern of the simpler system is exactly duplicated in the more complex ones, and we can see that in understanding one case, we understand many.

In the years since Feigenbaum's work, several other scenarios for the onset of chaos have been explored both experimentally and theoretically. Each of these "routes to chaos" is universal in the sense that many different systems will exhibit the same pattern. We can therefore say that we are beginning to understand how chaos arises.

In the next chapter we will look at fully developed chaos and ask how well *that* is understood.

Chapter III: The geometry of chaos

In the last chapter the onset of chaos was considered. A full understanding of chaos, beyond its onset, still eludes us. However, we have built up a few substantial ideas about its geometrical structure. The purpose of this chapter is to discuss these geometrical ideas.

Well-developed turbulence

Chaos has been defined here as a physical situation in which the basic patterns never quite repeat themselves, One vivid example of such a nonrepetitive pattern is in a flow pattern depicted by Leonardo, shown in Figure 7. Notice how within this big swirl there are smaller ones and within them smaller ones yet. This kind of flow within flow within flow is called well-developed turbulence. The nonrepetitive nature of the pattern arises precisely because it is a Chinese box in which structures

Figure 7. Turbulent flow patterns as drawn by Leonardo da Vinci. Note how the large swirls break into smaller ones, and these again break up.

appear within structures. This point was later made into a little poem by L. F. Richardson, who wrote:

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity
(in the molecular sense).*

The Soviet mathematician A. N. Kolmogorov picked up on this picture and developed a useful theory of such behavior by writing down the mathematical consequences of the idea that similar structures reappear again and again inside of one another. We do not believe that Kolmogorov's theory is entirely right, but we don't yet have a replacement for it.

The Chinese box example has to contend with two major complications. One is that any well-developed chaos is hard to understand. The other is that this particular chaos of flow within flow occurs in space.

*This poem is quoted in Benoit B. Mandelbrot. *The Fractal Geometry of Nature* (New York: W. H. Freeman and Company, 1983), p. 402.

To describe it fully, we would have to specify the velocity at every single point in the entire system. Since there are an infinite number of points, we would have to have an infinity of different numbers just to specify the chaotic situation at one time! But we have already seen a chaotic situation that could be specified by giving only one number, x, at each time. (A number like x, which can change in time, is called a "variable.") We were able to understand this example reasonably well and to see how chaos arose in it. Of course this situation was intentionally constructed to be simple. When we go out and look at the real world, we can find situations which are intermediate in complexity between the one-variable case described in the last chapter and the cases with an infinite number of variables depicted by Leonardo. These cases can often be described by specifying the time dependence of just a few variables, perhaps two, x and y, or three, x, y, and z. In the next sections, we will try to describe the kinds of behavior that could arise in these few-variable systems.

Attractors, strange and otherwise

To describe these relatively simple systems, we will direct our attention to the mathematical world in which they live. In the case of our insect island, we can fully specify its future behavior by giving one number, which describes this year's population in relation to the maximum possible one. This ratio, x, must be a number between zero and one. Indeed, for the mathematician studying our example, the relevant world is not the hypothetical island upon which the insects live, it is the mathematical world which is the set of all numbers between zero and one. That kind of world is called the "phase space" for the problem, so as to distinguish it from the physical space (the island) in which the events occur. In this example, we can depict the phase space by drawing a line segment, as in Figure 8, *case a*, and imagining that a specific value of x is depicted by putting a point upon that segment. Notice that this segment is drawn as a line that goes up and down, instead of the more conventional drawing that would go from left to right. I ignore the conventions so as to have my pictures look like the ones drawn in the previous chapter, that is, Figures 5 and 6.

Now let us return to the kind of thinking that we used in constructing these earlier figures. Consider some fixed value of the growth-rate parameter, r, say the $r = 2.2$ of Figure 5c. Then, as shown in that figure, year by year x, the ratio of the existing population to the maximum one, approaches a specific value, namely, 0.5454 . . . . For almost any value of the initial population, the long-term result is precisely the same. As the years go by, the population ratio will get closer and closer to that particular value of x. This is graphically described by drawing a point at $x = 0.5454$ within the phase space of Figure 8, as in *case c*. We then say that this point is the "attractor" for

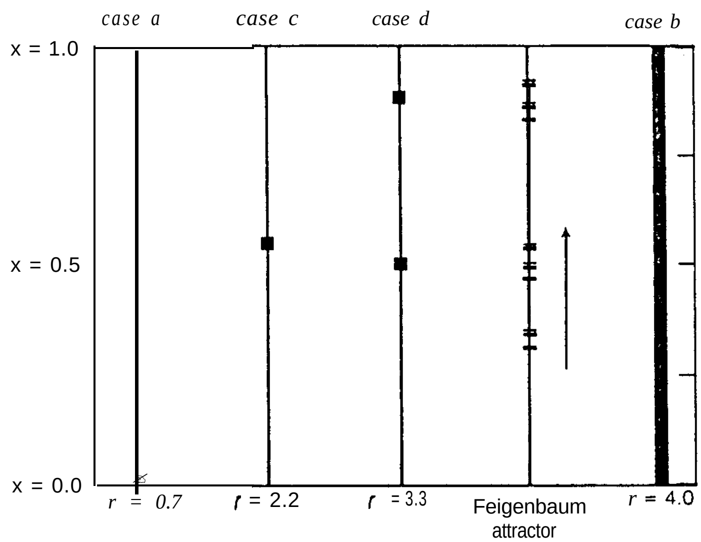


Figure 8. A description of some attractors for different *r*-values in our insect system. The vertical axis on the far left describes the "space" in which the attractor will fit. namely, the line between $x = 0.0$ and $x = 1.0$. The line labeled *case a* describes what happens when $r = 0.7$. It has a point at $x = 0.0$. showing that for this *r*-value the insect population ratio goes to zero. The filled-in regions for the other r-values similarly depict the possible population ratio values for each one of these situations.

the insect system at $r = 2.2$. By this we mean that in the year-by-year development of the insect systems, their population ratios approach or "are attracted to" this point.

For a higher growth-rate parameter, the attractor might be more complicated. For example, as we already know from Figure 5d, at $r = 3.3$ the long-term behavior of the population is a two-cycle one. Hence, the motion is attracted to a pair of points, as shown in Figure 8, *case d*. In this way, for each value of r, we can plot out the attractor, that is, the value to which the population ratio converges. In fact, Figure 6 is simply a plot that shows the attractors for all values of r.

For a still higher growth-rate parameter, the population graph exhibits chaotic behavior. The attractors in the chaotic regions of this parameter, r, are not collections of points, as in the last example, but instead *regions*. For example, in the full chaos of $r = 4.0$, as shown in Figure 8, *case b*, all values of the population ratio, x, between zero and one arise within the course of a typical pattern of time development. Hence, for this value of r, the attractor is the entire interval between zero and one. Whenever there is chaos, the attractor is an interval, or perhaps a collection of different intervals. This behavior can also be seen in the right-hand portion of Figure 6a.

In the cases described so far, the attractors are relatively simple and straightforward: a point, a few points, an interval, or a few intervals.

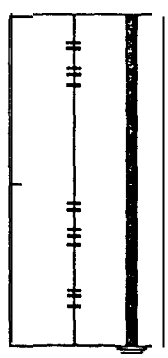


Figure 9. A portion of the Feigenbaum attractor blown up and replotted. This portion looks essentially identical to the entire attractor (see fig. 6).

However, when we follow Feigenbaum and look at the point for which a cycle of infinite length first appears, we see a much stranger and richer behavior. What we must do is draw the attractor for a two cycle, as in Figure 8, *case d*, then for a four cycle, next for an eight cycle, next sixteen. Then, in the limit, one gets the picture shown in the part of Figure 8 labeled "Feigenbaum attractor." This configuration contains an infinite number of points arranged in a rather interesting pattern. Such an attractor, in which there is structure inside of structure inside of structure, is called a "strange" attractor. The reader will notice that I have just defined strange as a technical word. In doing this, I am following the standard terminology in the field. The term *strange* attractor is due to David Ruelle of the Institut pour Haut Etude Scientifique in Paris.

Before going further, I should compare the structure within structure of Figure 7, Leonardo's drawing, with that of the Feigenbaum attractor in Figure 8. They are both "strange." However, the pictures show two different types of worlds. Leonardo draws a picture of one time in our real three-dimensional world. Figure 8, on the other hand, is drawn in phase space and is a superposition of infinitely many pictures at different times.

To see the strange character of the Feigenbaum attractor, we take the portion of it indicated by the arrow in Figure 8, blow it up, turn it over, and plot it again. The result is shown in Figure 9. Notice that, except for the values of the x-coordinates, the picture looks essentially identical to the one in Figure 8. This identity strongly suggests that there is a succession of almost identical structures nested within the Feigenbaum attractor, in the same way that' Russian dolls are nested within one another.

Such nested behavior is very common in physical systems. Look back at the solidification patterns shown in Figure 3. Notice once again how

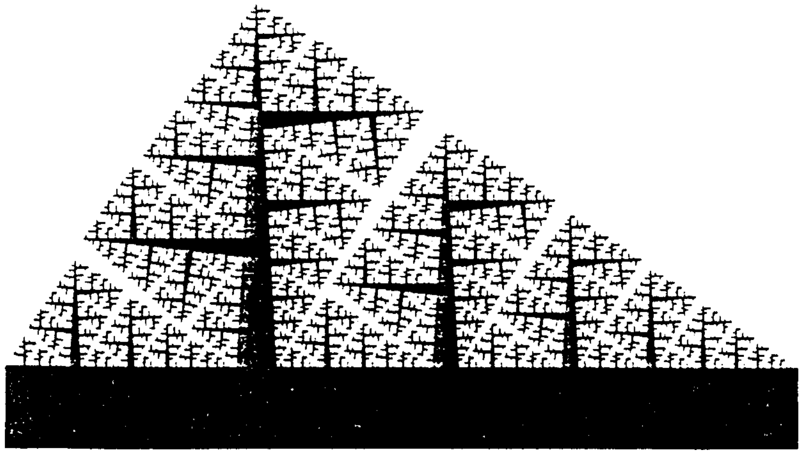


Figure 10. An example of a fractal object. Notice how the fine treelike object is represented again and again at different sizes.

the ice consists of a group of arms, upon which lie smaller arms and upon them smaller ones yet. This kind of behavior was first described by the nineteenth-century mathematician Georg Cantor, and later on by Felix Hausdorff and others. In more recent work, such nested behavior is often described by the terms *scale-invariant* or *fractal.* Figure 10 shows such a pattern. The term *scale-invariant* merely says that when you blow up the picture in Figure 9 (i.e., change its scale) and look at a portion of the result, you get much the same thing as before. Thus it is unchanged or invariant. The word *fractal* was introduced by Benoit Mandelbrot of IBM, who has discovered and publicized many examples of scale-invariant behavior. The term is intended to remind us of another property of these strange objects. They can be described by using a variant of the concept of a dimension. It is commonplace to say that a point has no dimension, a line is one-dimensional, an area two-dimensional, and a volume three-dimensional. For strange objects, we extend the meaning of *dimension* to include possibilities in which a dimension is not just an integer (1, 2, or 3) but instead any positive number (say 0.41). There is then a technical definition that enables us to calculate this fractional or "fractal" dimension from the picture of the object, as, for example, Figure 9. (This particular attractor has dimension 0.538 . . . which, as one might expect, is larger than the value for a point and smaller than the value for a line.)

Incidentally, I should note that the motion upon the Feigenbaum attractor is really rather orderly and cannot be described in any sense as chaotic. For example, imagine starting off with a population ratio of $x_0 = 0.5$, which is indeed a point lying on the attractor. We can look at the points that arise after 1, 2, 4, 8, 16, or 32, successive iterations of equation (4), using the value of the growth-rate parameter,

r, appropriate to the Feigenbaum attractor. The placement of these x-values is very orderly. The points x_1, x_2, x_4, x_8, x_{16}, . . . approach x_0 = 0.5, with alternate members of the list lying above and below x_0. Thus the Feigenbaum attractor may be "strange," but the motion on it is certainly not chaotic.

*Chaos on strange attractors**

In our insect example, chaos and strange attractors tend to occur for different values of *r*. However, in slightly more complicated systems, chaotic behavior almost always produces a strange attractor. So far, we have worked mostly with an example in which the present and future behavior of the system could be defined by giving the value of one number, x, representing a population ratio. In the nest, more complicated, example the future state of the system is defined by two numbers, called x and y. For example, these might be the populations of two different age-groups in a given year. A model system could be defined by saying how the values of x and y in the next year depended upon the values this year.

Another example of such a dependence is given by the following equations:

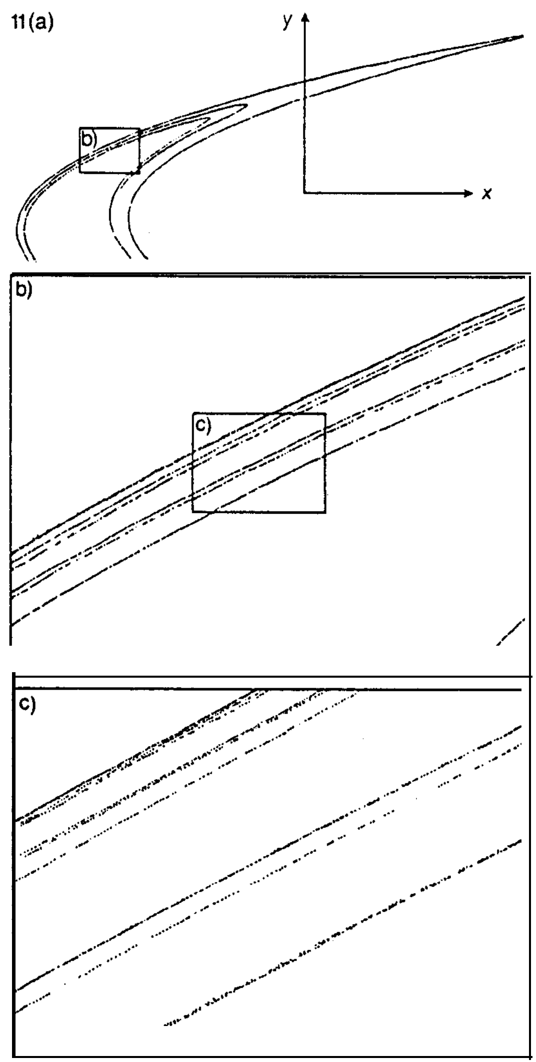
$$x_{next} = rx\,(1 - x) + y, \text{ and} \qquad (7)$$
$$y_{next} = xb.$$

We might arrive at such equations by visualizing a case in which there is once again an insect population and where x is the population in a given year. (For simplicity's sake we will here talk of x as if it were the population, rather than a population ratio, as previously.) But we could introduce one difference. We could suppose that a proportion, *b*, of the insects would live for two summers. Call the number of "old" insects y. In that case, the same analysis as before would lead us to a result like equation (7).

The point here is that, to define the situation fully, we now need to specify two numbers: the existing population, x, and the previous year's population, y. This can be rendered geometrically by drawing a common kind of graph with x and y axes and specifying some situation by a point on the graph, as shown in Figure 11. An attractor for this situation can be constructed by starting out with some initially chosen value of x and y, constructing successively the next values via equation (7), and then imagining that after some large number of steps the values of the pair (x,y) have moved in toward the attractor.

This model, or rather one equivalent to it, has been constructed by

*For a slightly more technical presentation of similar material about Henon's and Feigenbaum's work, *see* Douglas R. Hofstadter, "Metamagical Themas," *Scientific American* (November 1981): 22.



Figures 1 la-c. Figure 1 la shows the Henon attractor. The other pictures are successive blowups of this attractor, with 11 b being an expanded version of the boxed region in 11 a, and 11 c a similarly expanded version of the box in Figure 11 b.

a French astrophysicist, Michel Henon. He focuses his attention upon particular values of the parameters b and r; chooses initial values of x and y; calculates several hundred thousand successor points: throws away the first twenty thousand; and plots the rest. The result is shown in Figure 11a. This looks simply like a geometric structure containing a few parallel lines. But look more closely. Figure 11b is an expanded view of the box shown in Figure 11a. This expanded view also contains lines, but when one blows up a box within that figure, obtaining as a result 11c, one sees a familiar looking picture with a few lines within it. In this way, Henon demonstrated that the attractor from his map could be scale-invariant and fractal. And, in contrast to the previous example, we do not have to do any careful adjustment of r to find a strange attractor. On the contrary, strange attractors will pop up for many reasonable and arbitrarily chosen values of b and r.

In contrast to the motion on the Feigenbaum attractor, the motion on the Henon attractor is chaotic. In the first case, it is easy to predict the x-value that will be achieved after a large number of iterations. For example, if the large number is a high power of two, e.g., 2^{99}, the achieved x-value will be almost identical to the starting x-value. In the Henon example, if we start off on the attractor, we know that the (x,y) point will continue to lie within the attractor, but there is no similarly simple rule that permits accurate prediction of just where the points will lie after many steps. Moreover, the result is extremely sensitive to initial conditions.

Hence, in Henon's model chaos and strange attractors exist together. In fact, they are believed to have a causative relaxation: the attractors are strange exactly because they are chaotic.

Henon is an astrophysicist. He is interested in motion in the solar system and galaxies. His work on the model described above is not just the construction of a mathematical toy, unrelated to his astronomical interests. On the contrary, all kinds of astronomical systems-for example, our own solar system-can be usefully thought of as being described by a few variables that in the course of time trace out a chaotic motion on a strange attractor. The real attractors are more complicated than the one in Figure 11, but probably in many ways not essentially different.

Tracing chaos through time

Return to Figure 7, and the complicated swirls of Leonardo's picture of chaos in a fluid. From a practical point of view, it is distressing that we do not have a decent understanding of these turbulent flows. The flow of energy through real fluids like the atmosphere of the Earth, or the water cooling a nuclear reactor, or the air flowing around a body entering the Earth's atmosphere is dominated in each case by turbulent swirls. The fact is that our understanding of these swirls has hardly progressed beyond Leonardo's. Without additional understanding, we lack the tools to make predictions and reliable engineering designs in all kinds of interesting and/or technically important situations.

The meteorologist Edward Lorenz was very acutely aware of this imperfection, since it is a meteorologist's business to understand flows in the Earth's atmosphere. To describe a flow in this or any other fluid, we write equations for the rate of change of such properties as the fluid velocity, the temperature, and the pressure at each point in the fluid. As I have already mentioned, since there are an infinite number of

The Lorenz Equations

A solution of these equations is depicted in Figure 12. In this set of equations $\mathring{x}$, $\mathring{y}$, and $\mathring{z}$ stand for the rate of change of x, y, and z with respect to time:

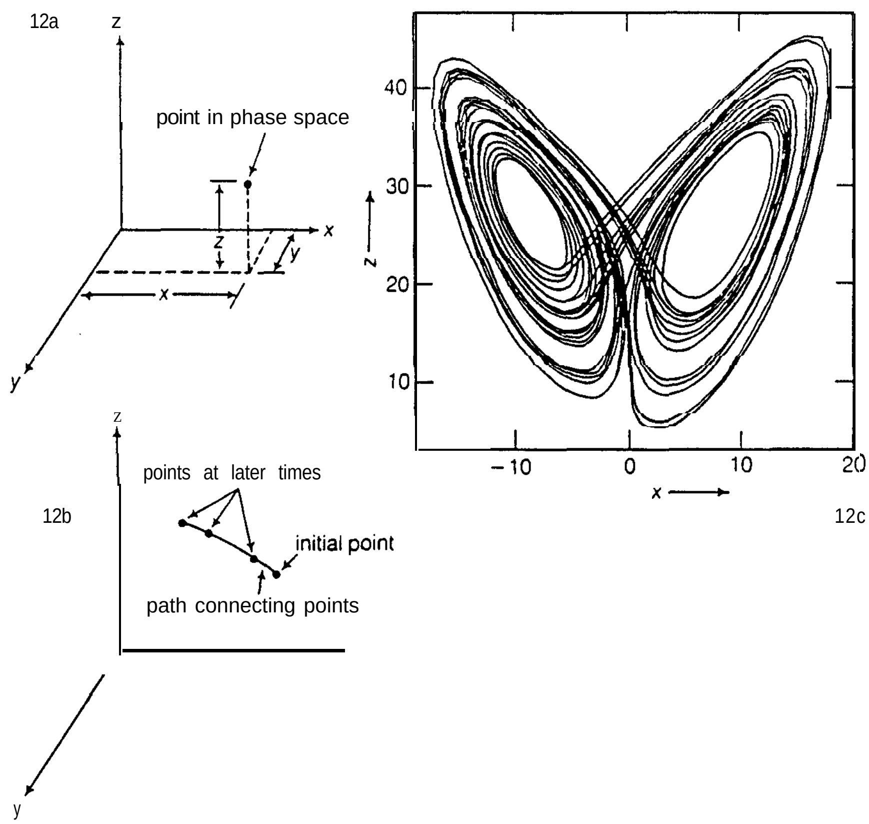
$$\mathring{x} = 10(y - x)$$
$$\mathring{y} = 28x - y - xz$$
$$\mathring{z} = 8z/3$$

"points" in every geometrical body, we must solve an infinite number of equations. Lorenz sought a ruthless simplification of the problem. Instead of describing his fluid by giving the values of an infinite number of quantities, he assumed that the fluid could be described by three, which he called, naturally enough, x, *y*, and *z*. His phase space could then be described in terms of the three coordinates. I show the exact form of his equations in the box on this page.

This detailed form is irrelevant to all the arguments that follow. The main idea, however, is not at all irrelevant. Lorenz's goal was to describe the particular kind of swirling motion called "convection." In this flow, the lower layer of a fluid heated from below rises because it is lighter (less dense) than the material above it. As the air above one portion of the Earth rises, air in another region flows downward. The net result is a complicated swirling flow. Lorenz's equations were an attempt to catch the essence of a swirling region in the very simplest fashion.

The major point about these equations-is that if you give numerical values for *x*, *y*, and *z* at a particular time, the system will determine the values of these quantities at subsequent times. Hence, we can picture the system at a given time by drawing a point on a standard x, y, z coordinate system of the kind shown in Figure 12a. The subsequent motion of the system is shown by giving the *x*, *y*, z coordinates at later times (fig. 12b) and then connecting up these points with arrows that show the direction of increasing time along the trajectory. After an initial time to settle down, the motion approaches an orbit that covers only a small portion of the x, *y*, *z* space (see fig. 12c). This orbit is, of course, a strange attractor. The path traced out by the time development of the system is an object of both impressive simplicity and imposing complexity.

First, the simplicity. Two basic kinds of motions are shown in Figure 12c. There are loops tilted leftward and loops tilted rightward. These



Figures 12a-c. Solution of the Lorenz equations. 12a shows the x, y, z space in which the equations are solved, 12b shows a fragment of the solution, while 12c is a projection upon the x, z plane of the solution over a long period of time.

come together in the region near the bottom of the diagram. As the system develops, it goes through each of these two kinds of loops in turn. To describe the sequence of events, we may list, in order, the loops that are traversed. For example, to describe the orbit in Figure 12c, which covers first a rightward loop and then two leftward ones, we may write "right-left-left." Over a very wide range of starting points, that is starting values of x, y, and z, the system will go through this loop-type behavior.

Now, the complexity: Depending upon the exact starting values of x, y, and z, the subsequent motion will be different. For one set of starting values, the motion might be

right-left-left-right-right-left-right-right-left-left-right- . . . .

Change the starting values just a little and the initial looping will change hardly at all. But the later stages may change quite considerably. Thus, with a small change in starting point we might have

right-left-left-right-right-left-right-right-left-left-left- . . . .

(For clarity, the changed values are shown in bold face.) A larger change in starting point will lead to an early change in the looping, for example,

right-left-left-right-right-left-left-le-right- . . . .

The looping structure is fully predictable in the sense that for any initial values of the x, y, z coordinates we can know and calculate the subsequent order of loops. But the structure is very sensitive to the initial conditions in that a small change in the beginning will cause a complete reshuffling of the loops at later times.

Calculating the motion in the Lorenz model is a quite nontrivial undertaking. We must carry through all the work of solving a set of differential equations. That requires a larger computer and considerable skill in its use. But the real fluids and the real world must be described with many many more variables than the three used by Lorenz. Nobody knows whether the more complicated "realistic" situations will show the same kind of complicated algebraic and geometric structure as the simplified models described here. I suspect and hope that many of the features presented here will reappear in the "real world." But now I have reached about as far as our present knowledge of the subject runs.