Fractals:

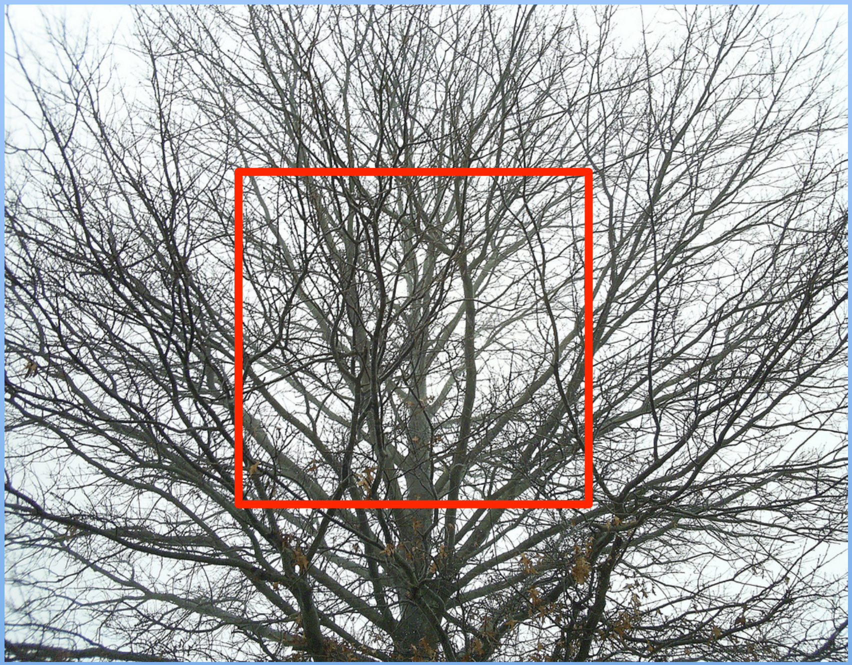
Objects with “self-similarity” at different scales

Trees are Fractal

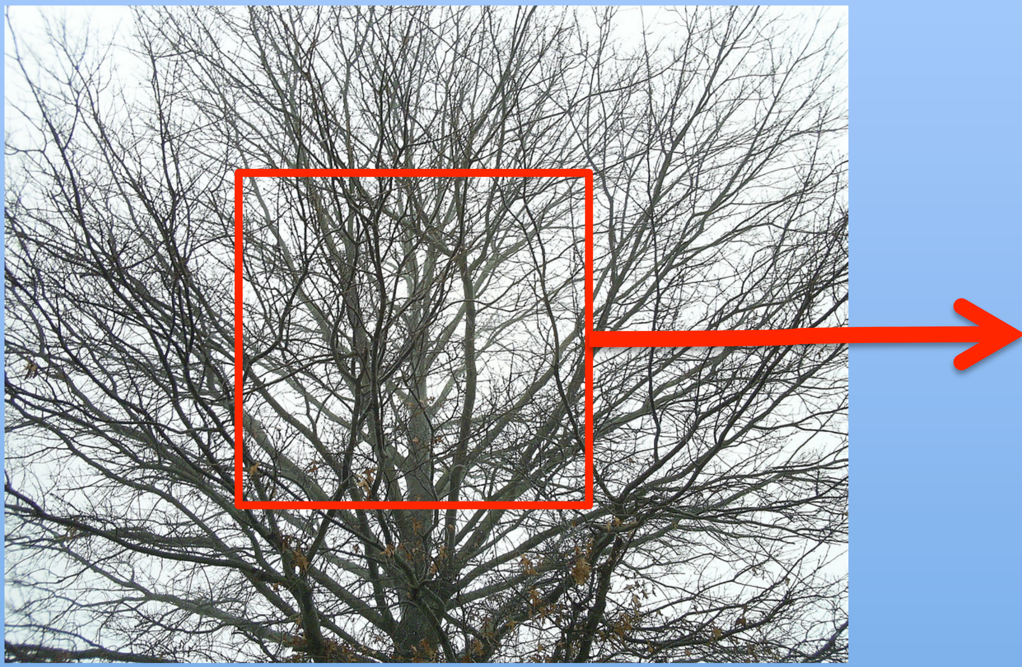
http://www.flickr.com/photos/mattmendoza/3236460615/in/photostream

Illustrating "self-similarity"

Illustrating “self-similarity”



Illustrating "self-similarity"



Illustrating “self-similarity”

Illustrating "self-similarity"

Illustrating "self-similarity"

Illustrating “self-similarity”

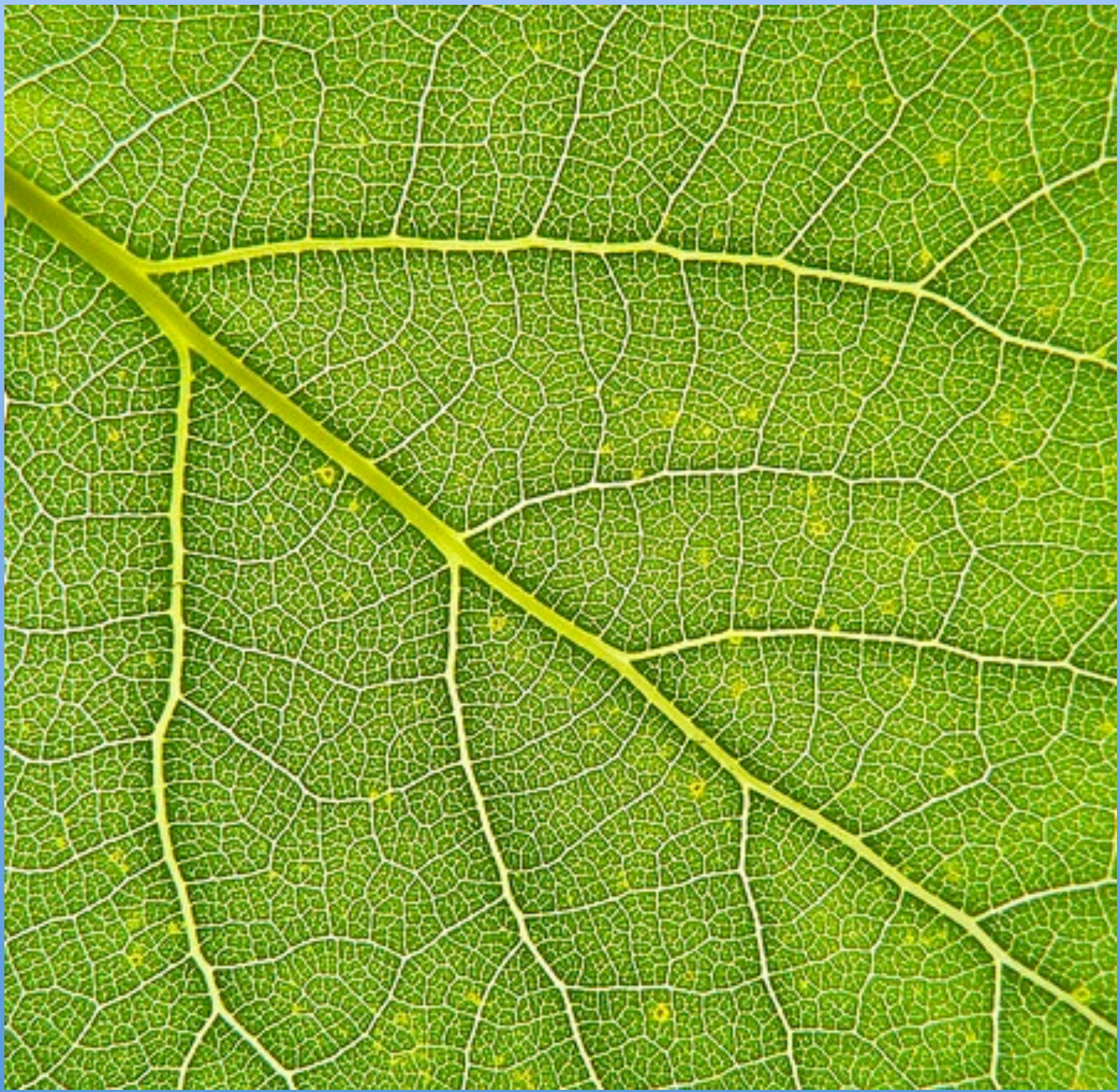
For all you mathematicians out there:

Fractal versus *Fractal-Like*

Broccoli is Fractal

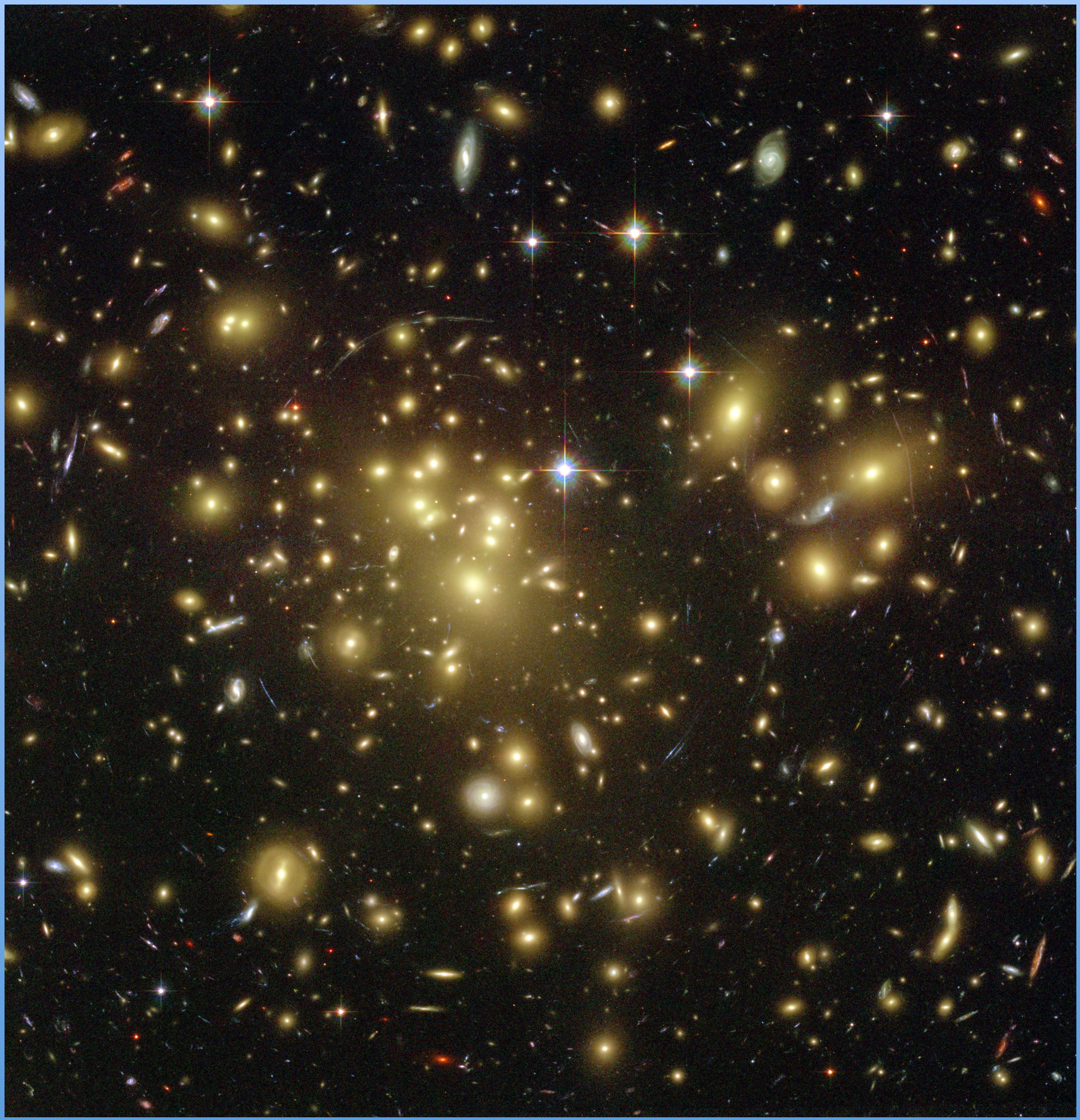
www.flickr.com/photos/xrm0/3826804608/

Leaf Veins are Fractal



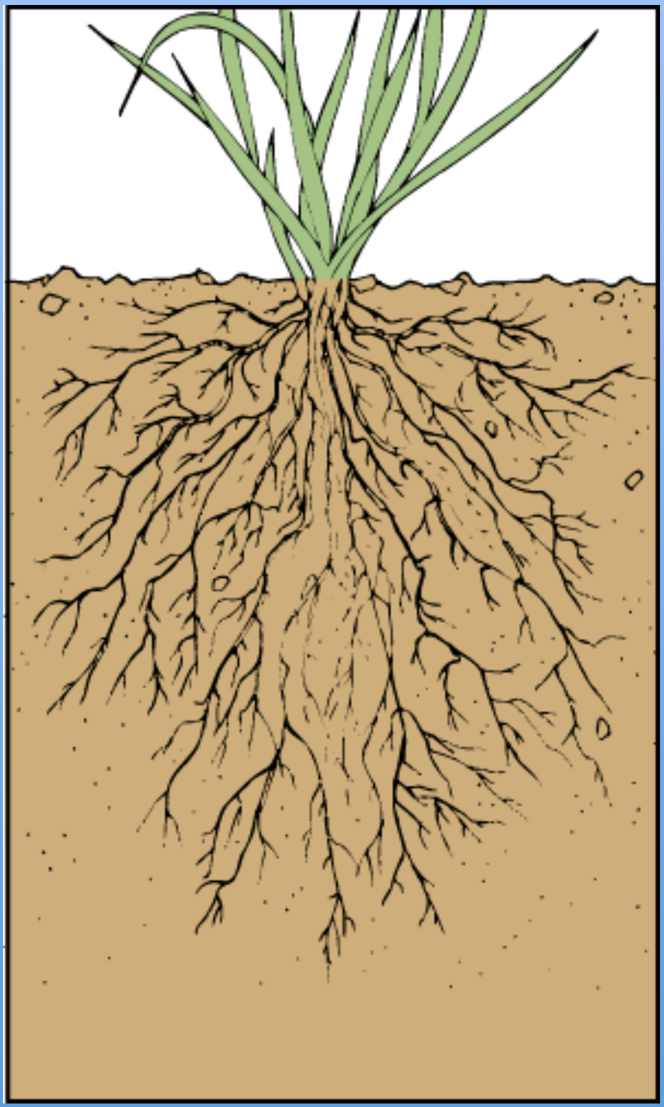
http://www.flickr.com/photos/10604632@N02/1382130717/in/set-72157602037469478/

Galaxy Clusters are Fractal



NASA Spce Telescope Science Institute

Plant Roots are Fractal

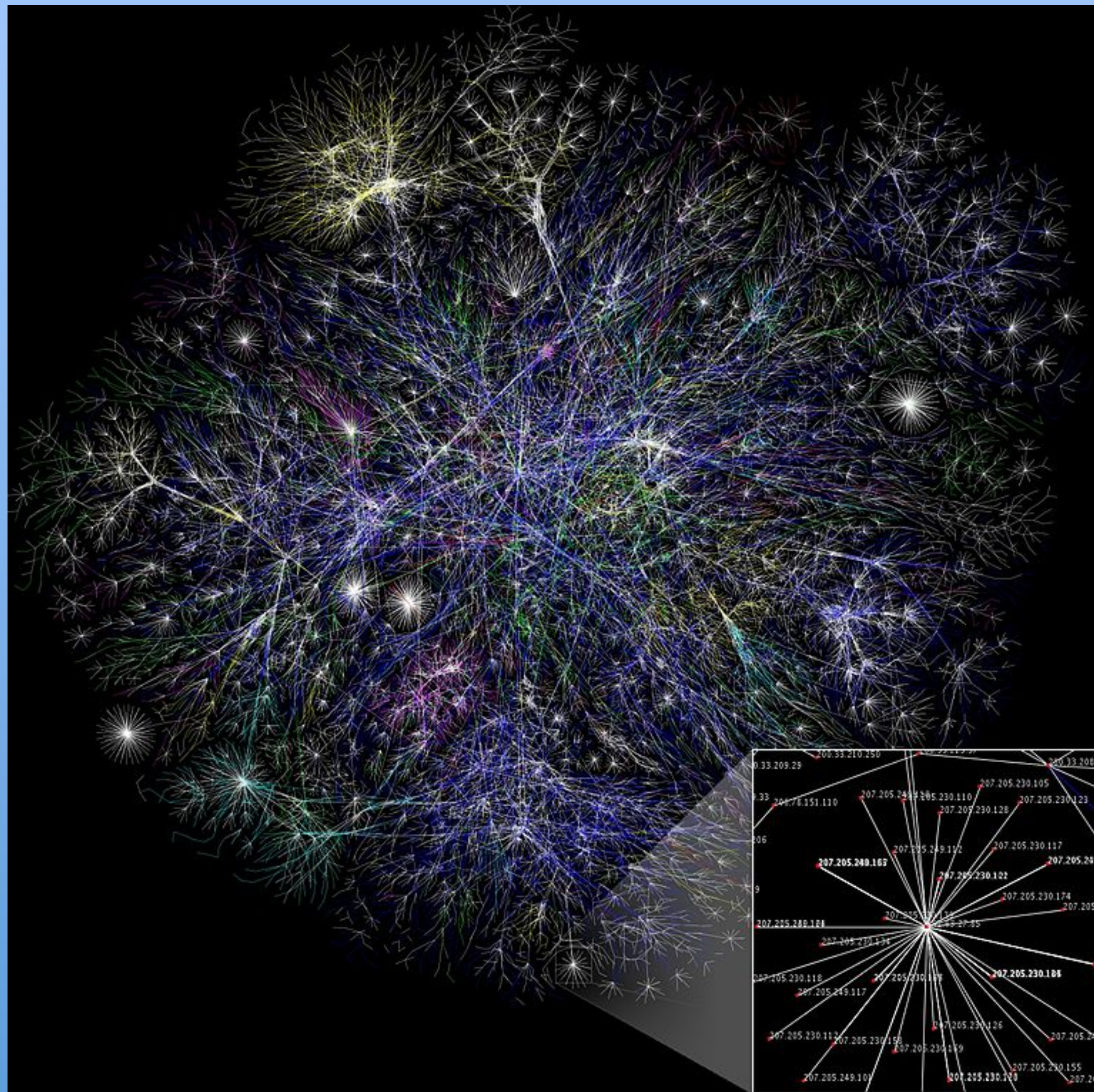


http://science.howstuffworks.com/dictionary/plant-terms/root-info.htm

Mountain Ranges are Fractal

http://geomaps.wr.usgs.gov/parks/olym/mtoly.jpg

The World-Wide-Web is Fractal



http://en.wikipedia.org/wiki/File:Internet_map_1024.jpg

A Bit of History

Benoit Mandelbrot, 1924−2010

Many mathematicians have studied the notions of self-similarity, and of “fractional dimension” and what an object with a fractional dimension would look like.

The term *fractal*, to describe such objects, was coined by the mathematician Benoit Mandelbrot, from the Latin root for “fractured”.

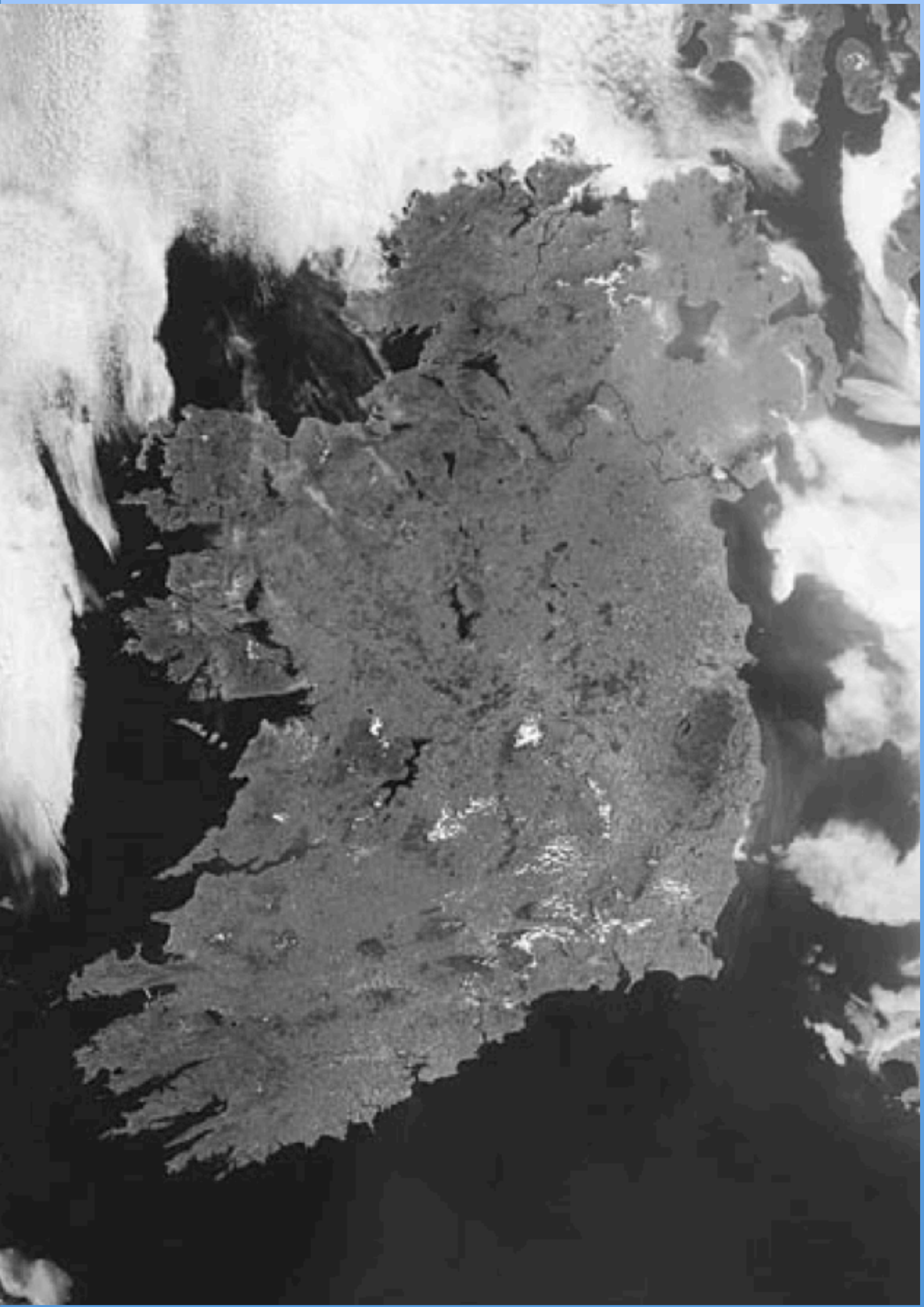
Mandelbrot’s goal was to develop a mathematical “theory of roughness” to better describe the natural world.

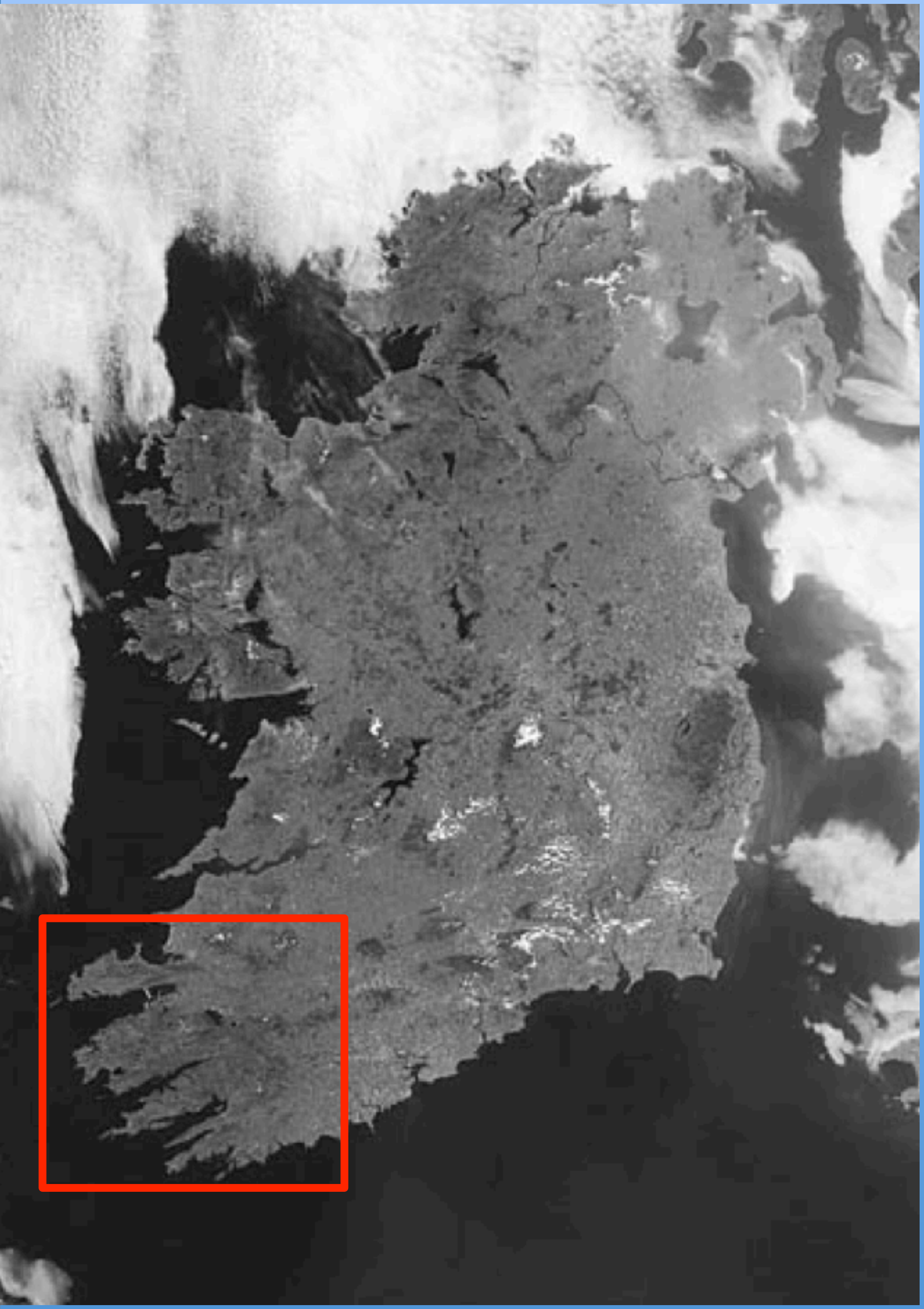
He brought together the work of different mathematicians in different fields to create the field of *Fractal Geometry.*

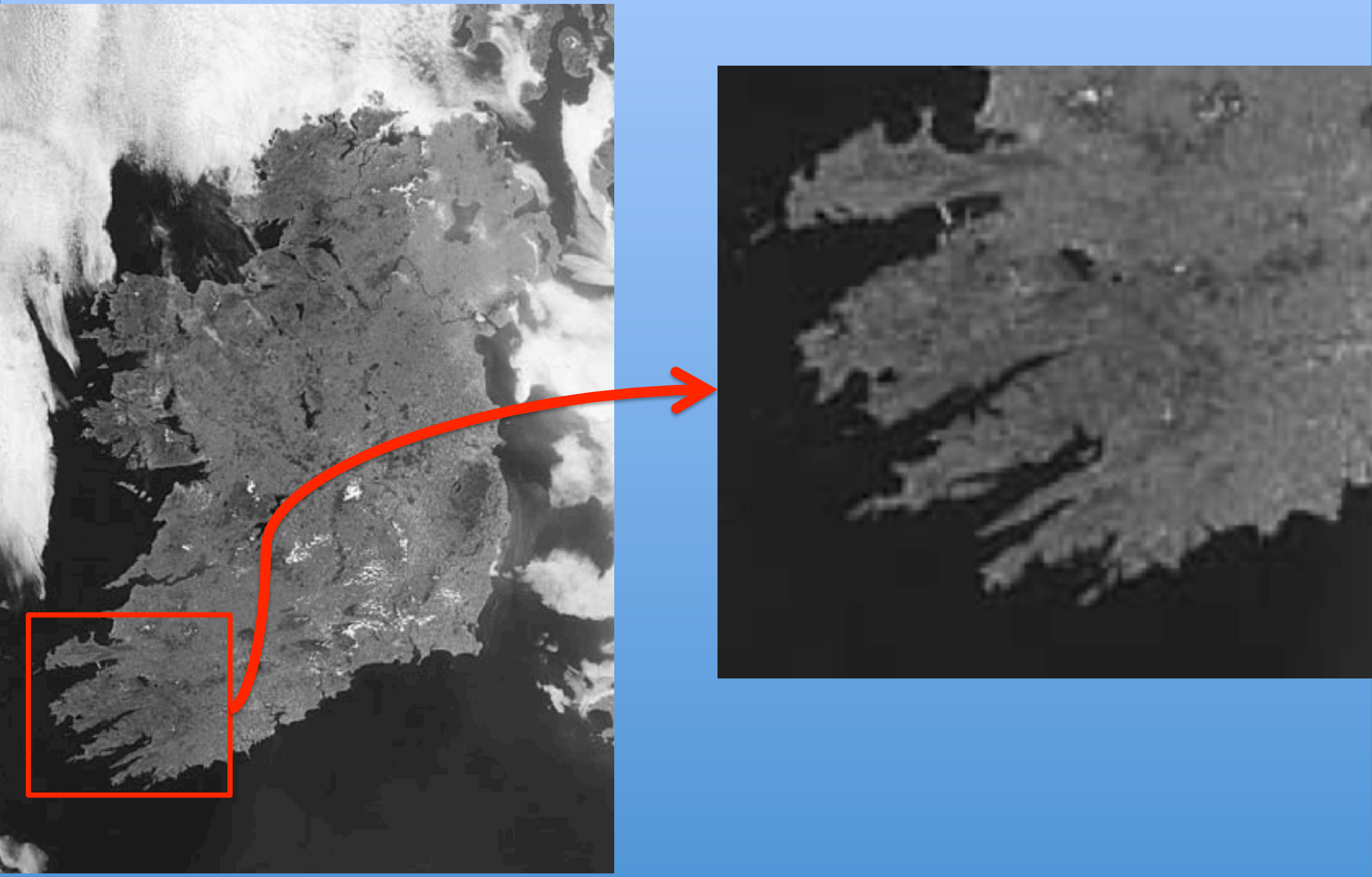
Mandelbrot's example:
Measuring the length of the coastline of Great Britain

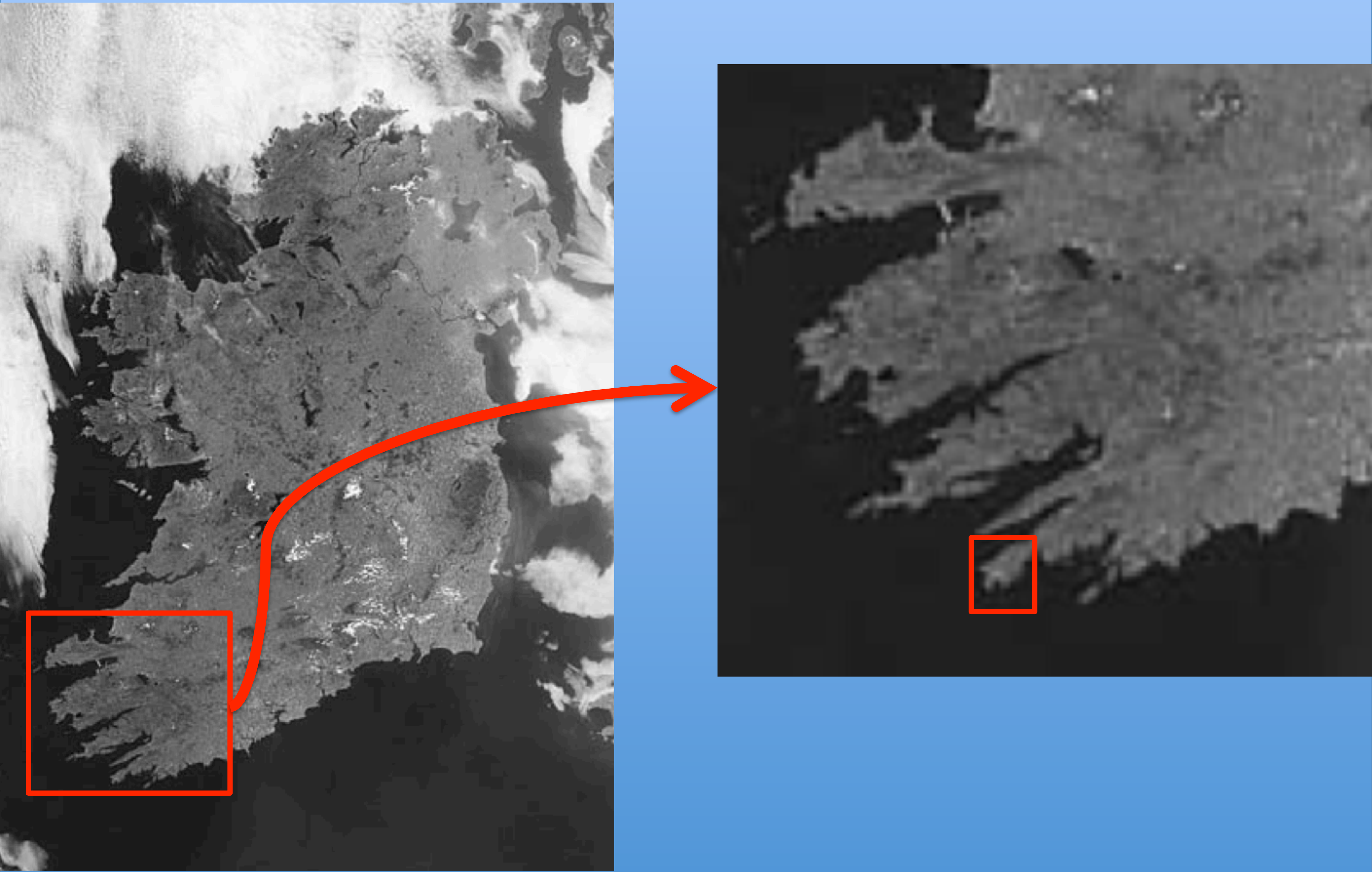
What size ruler should you use?

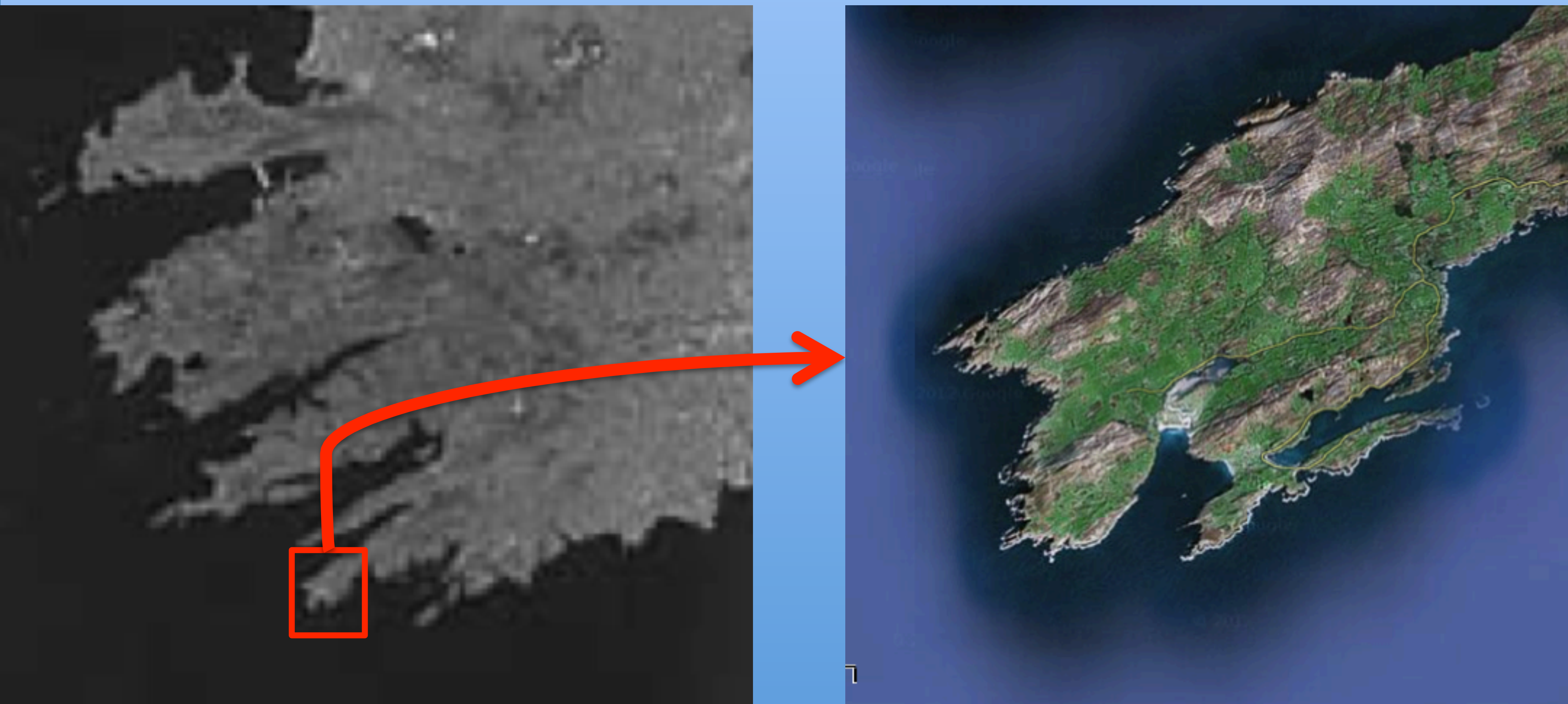
Ireland












Mizen Head
Mizen Head Lighthouse


Mizen Head
Mizen Head Lighthouse


Mizen Head
Mizen Head Lighthouse

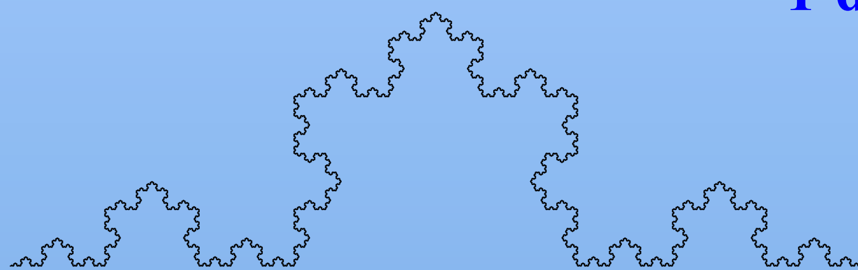
Mizen Head
Mizen Head
Lighthouse

Mandelbrot’s example:
Measuring the length of the coastline of Great Britain

What size ruler should you use?

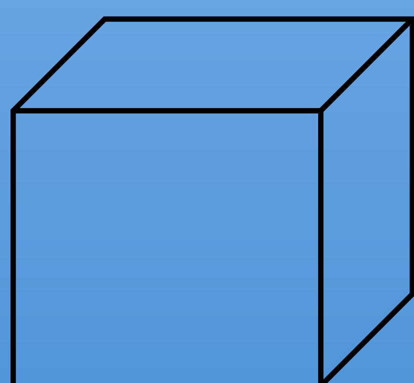
Dimension:
“Extension in a given direction”

1-dimensional



In between 1 and 2 dimensional!

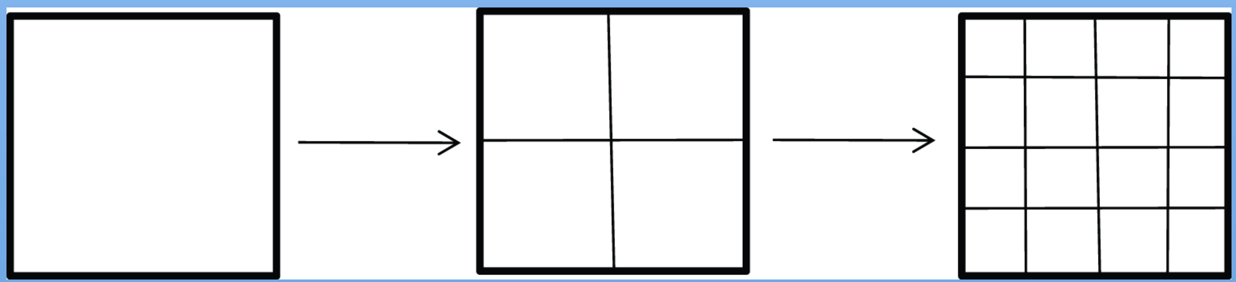
2-dimensional

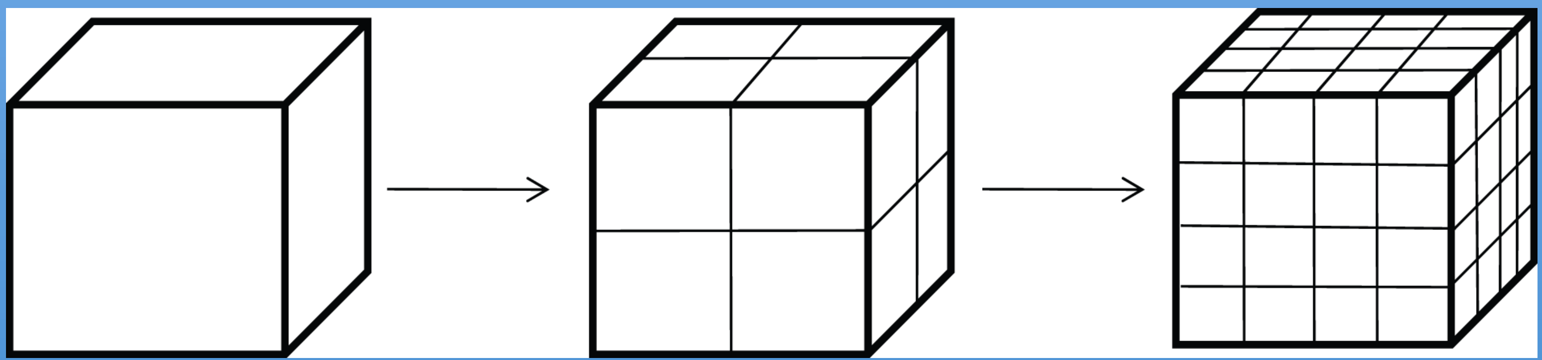


3-dimensional

Characterizing Dimension

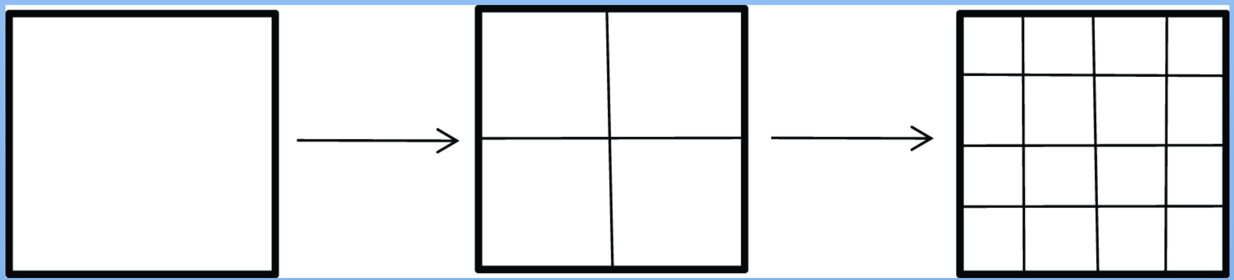
What happens when you continually bisect (cut in two equal halves) the sides of lines, squares, cubes, etc.?



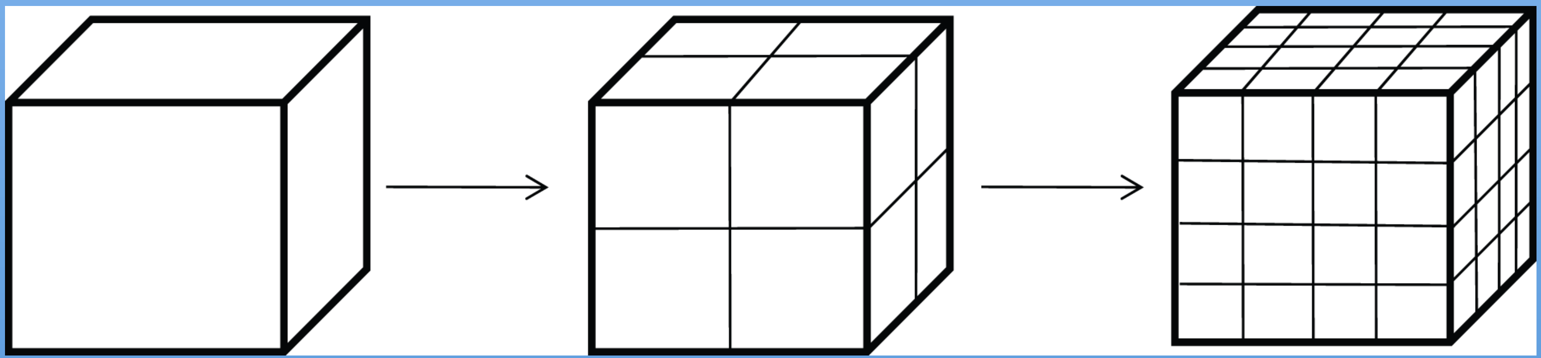


Dimension 1: Each level is made up of two 1/2-sized copies of previous level

Dimension 2: Each level is made up of four 1/4-sized copies of previous level



Dimension 3: Each level is made up of eight 1/8-sized copies of previous level

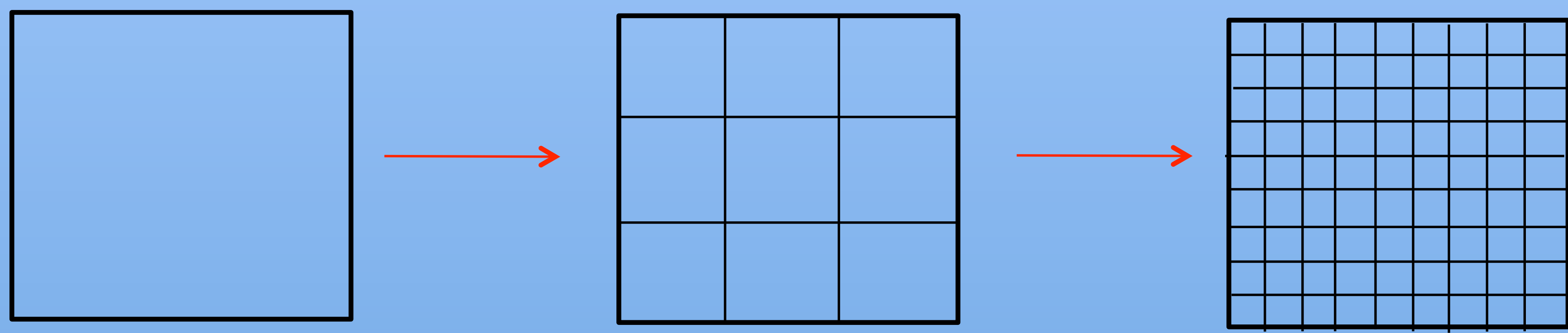


Dimension 4: Each level is made up of sixteen 1/16-sized copies of previous level

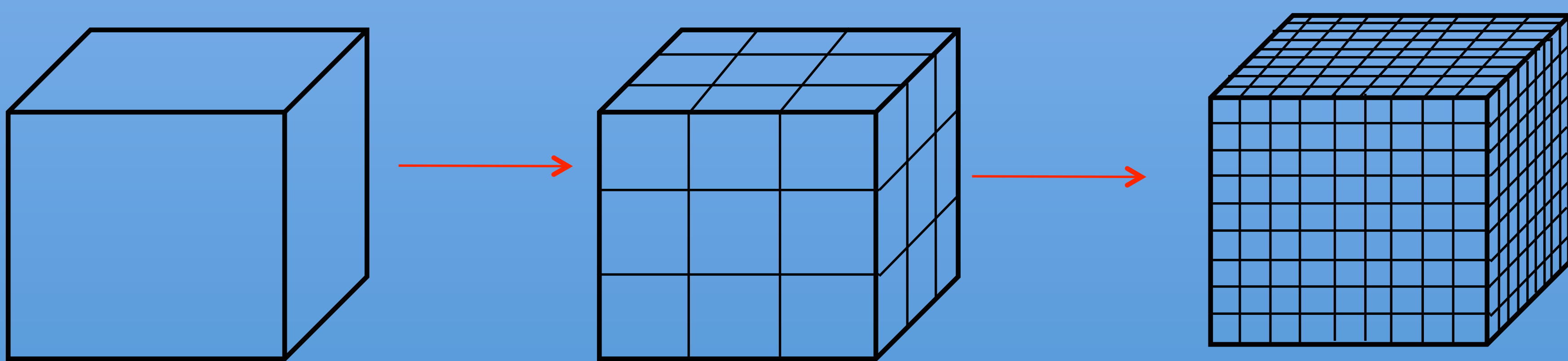
Dimension 20: Each level is made up of ??

Trisecting sides

Dimension 1: Each level is made up of three 1/3-sized copies of previous level



Dimension 2: Each level is made up of nine 1/9-sized copies of previous level



Dimension 3: Each level is made up of 27 1/27-sized copies of previous level

Trisecting sides

Dimension 1: Each level is made up of three 1/3-sized copies of previous level

Dimension 2: Each level is made up of nine 1/9-sized copies of previous level

Dimension 3: Each level is made up of 27 1/27-sized copies of previous level

~~Trisecting~~ sides

N-secting

Dimension 1: Each level is made up of ~~three 1/3-sized copies~~ of previous level

N 1/N-sized copies

Dimension 2: Each level is made up of ~~nine 1/9-sized copi~~es of previous level

N^2 1/N^2-sized copies

Dimension D: Each level is made up of N^D $1/N^D$-sized copies of previous level

Dimension 3: Each level is made up of ~~27 1/27-sized copies~~ of previous level

N^3 1/N^3-sized copies

Review of logarithms:

http://www.khanacademy.org/math/algebra/logarithms-tutorial

Definition of dimension

Create a geometric structure from a given *D*-dimensional object (e.g., line, square, cube, etc) by repeatedly dividing the length of its sides by a number *N*.

Then each level is made up of N^D copies of the previous level.

Call the number of copies *M*.

Then $M = N^D$.

We have:

$$\log M = D \log N$$

$$D = \log M / \log N$$

$\log M = D \log N$

$D = \log M / \log N$

Dimension 1: $N = 2, M = 2, D = \log 2 / \log 2 = 1$

$N = 3 , M = 3, D = \log 3 / \log 3 = 1$

Dimension 2: $N = 2, M = 4, D = \log 4 / \log 2 = 2$

$N = 3, M = 9, D = \log 9 / \log 3 = 2$

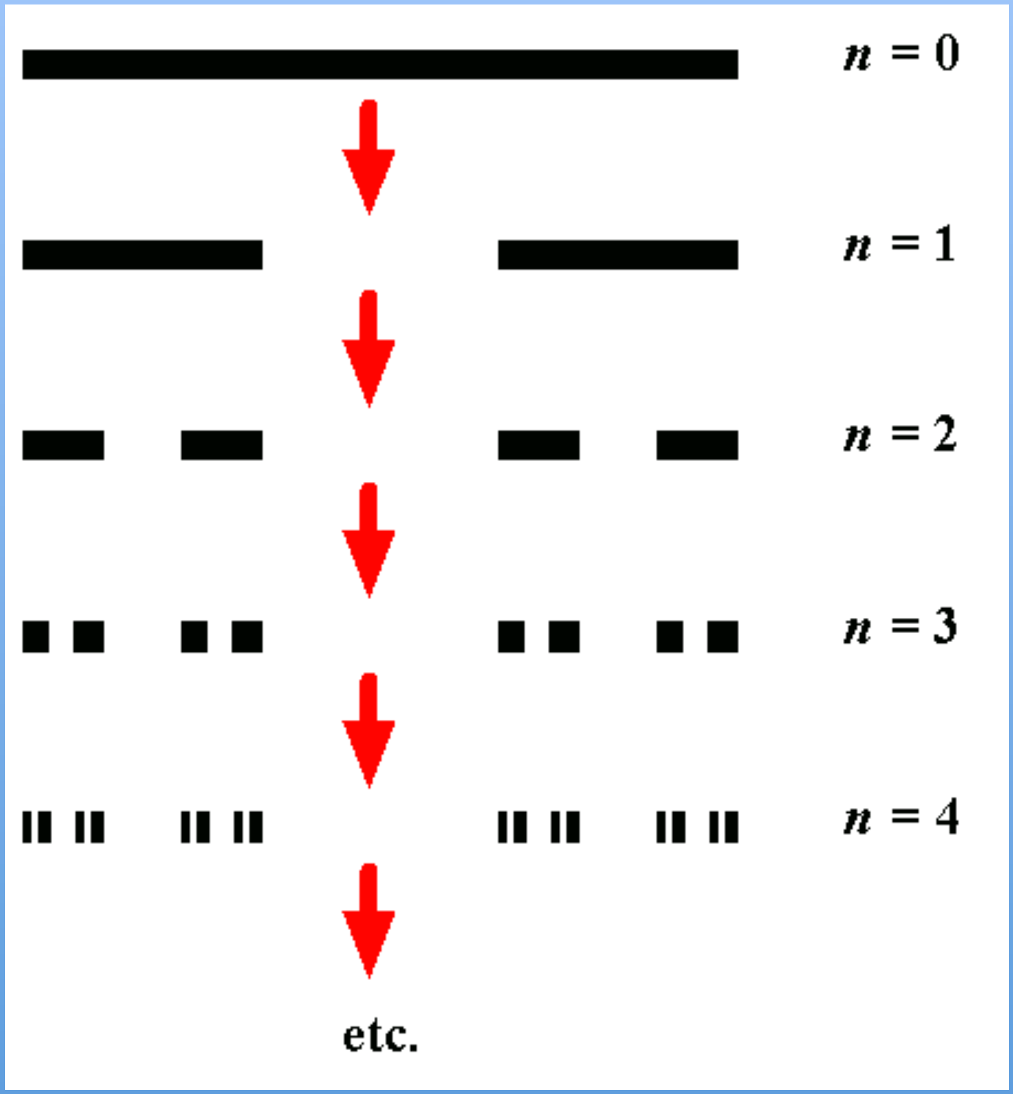
Koch curve: Here, $M = 4, N = 3$

So Fractal Dimension = $\log 4 / \log 3$

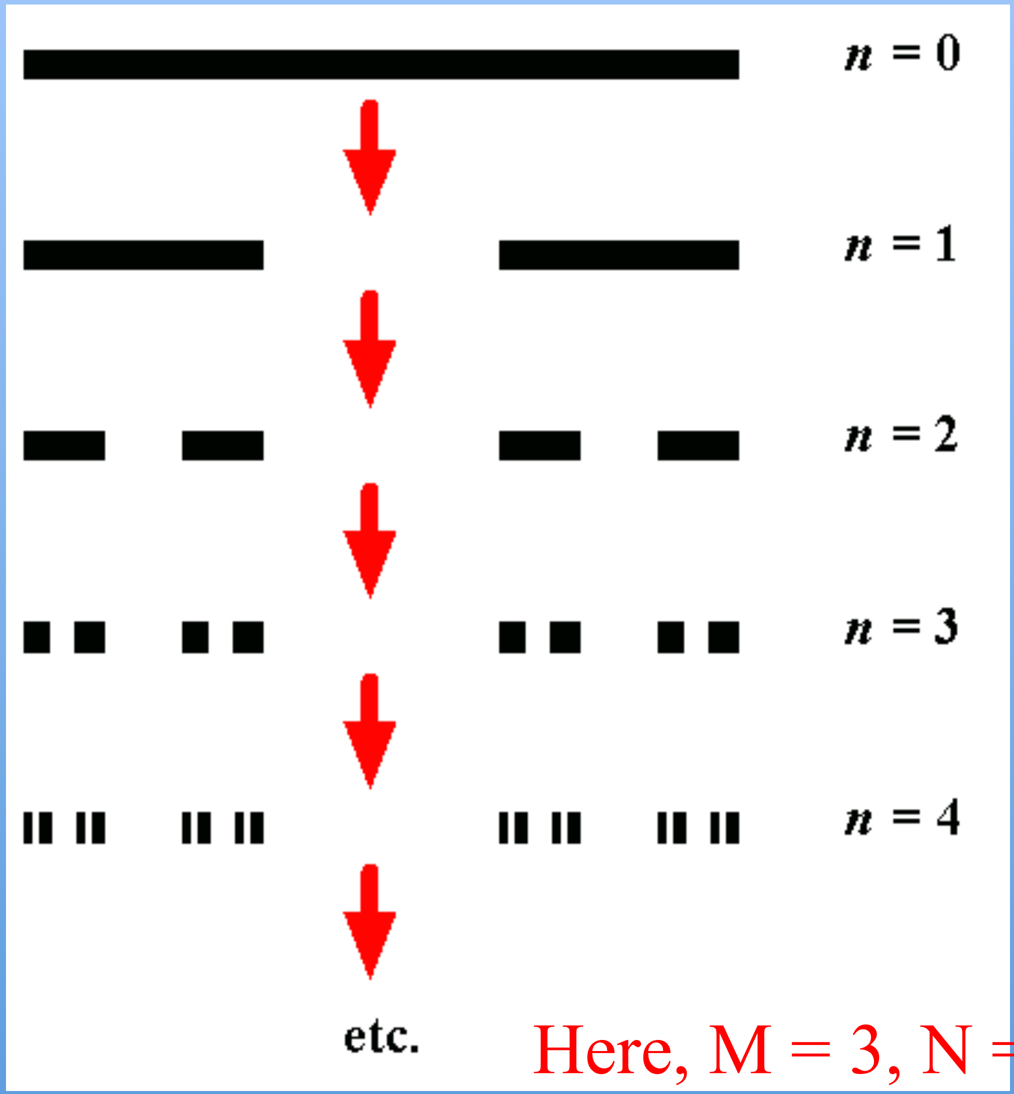
≈ 1.26

A measure of how the increase in number of copies scales with the decrease in size of the segment -- Roughly – the density of the self-similarity.

Cantor set



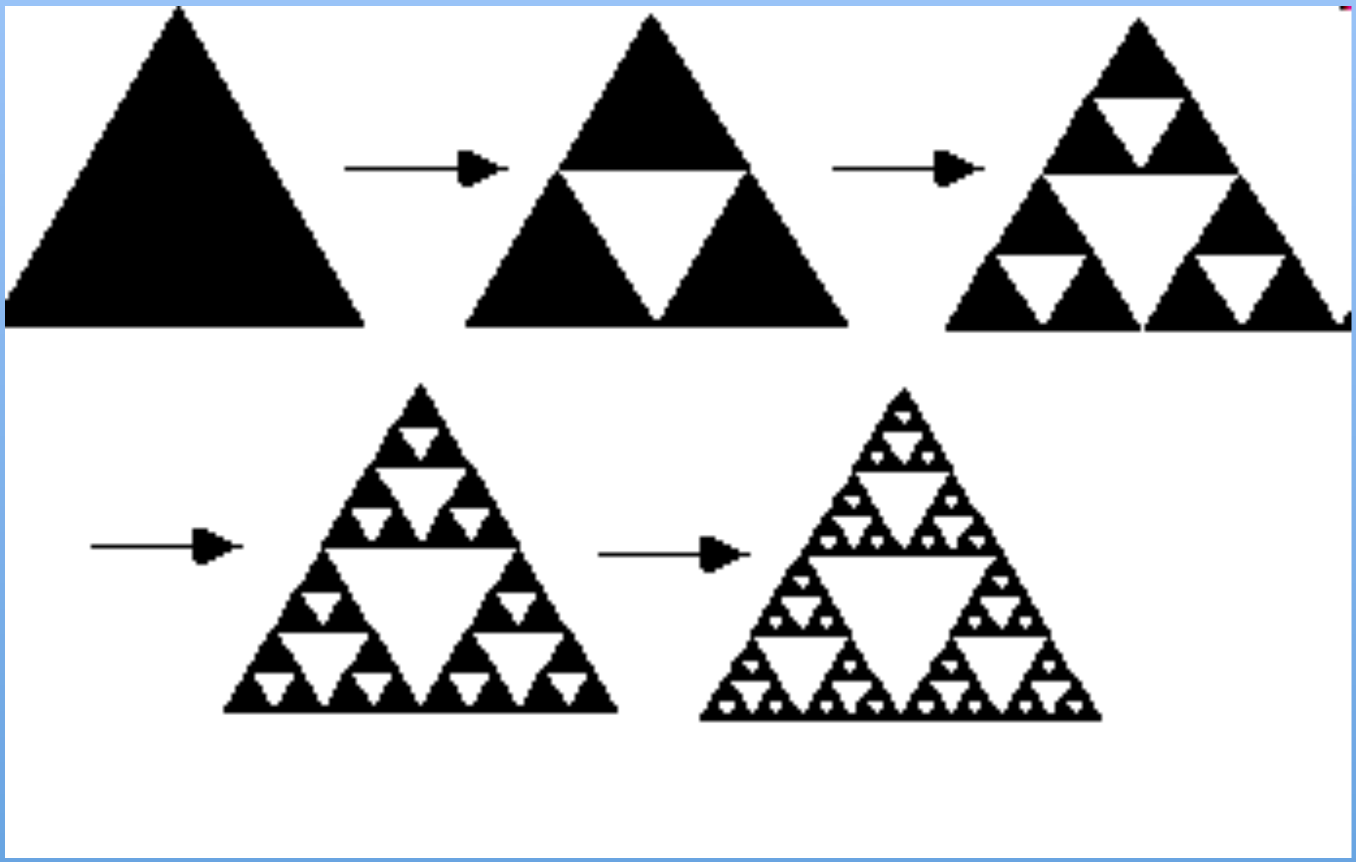
Cantor set (Homework)



Here, M = 3, N = 2

So Fractal Dimension = log 2 / log 3 ≈ .63

Sierpinski triangle



Approximate dimension of coastlines
(Shelberg, Moellering, and Lam)

Curve	Slope (β)	D (1-β)	New D
West Coast of Great Britain	-.25	1.25	1.2671
Coast of Australia	-.13	1.13	1.1490
Coast of South Africa	-.02	1.02	1.0356
Land-frontier between Spain and Portugal	-.14	1.14	1.1014

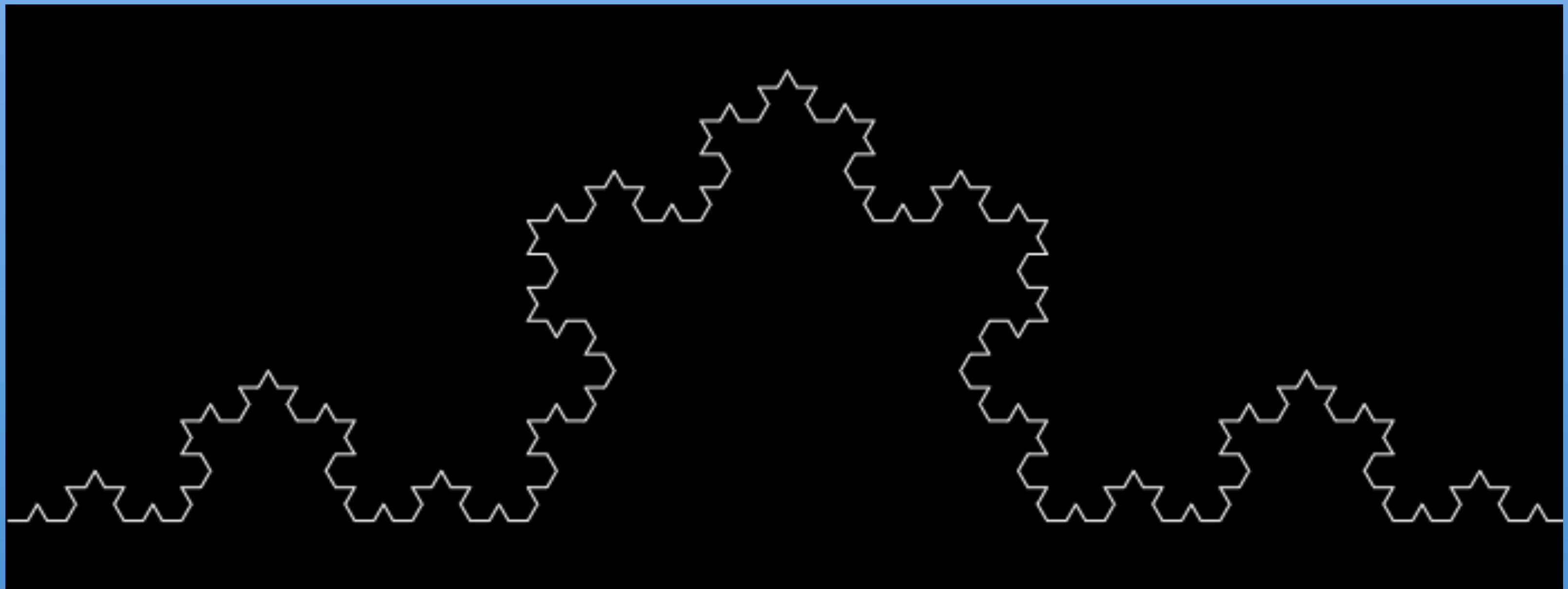
See Google Maps

Fractal Dimension

N = reduction factor from previous level = 3

M = number of copies of previous level = 4

Dimension $D = \log M / \log N = \log 4 / \log 3 \approx 1.26$



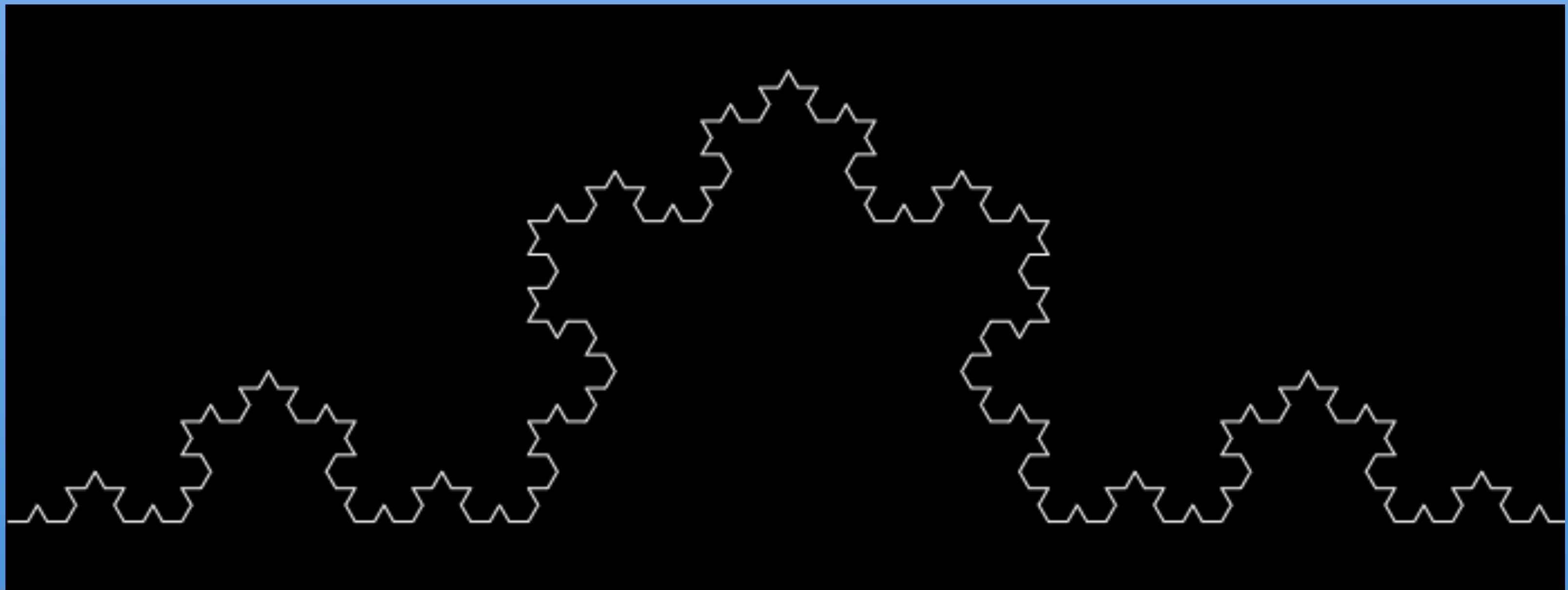
Fractal Dimension

N = reduction factor from previous level = 3

M = number of copies of previous level = 4

Dimension $D = \log M \,/\, \log N = \log 4 \,/\, \log 3 \approx 1.26$

This version of fractal dimension is called *Hausdorff Dimension*, after the German mathematician Felix Hausdorff



Sierpinski Triangle

(Waclaw Sierpinski, 1916)

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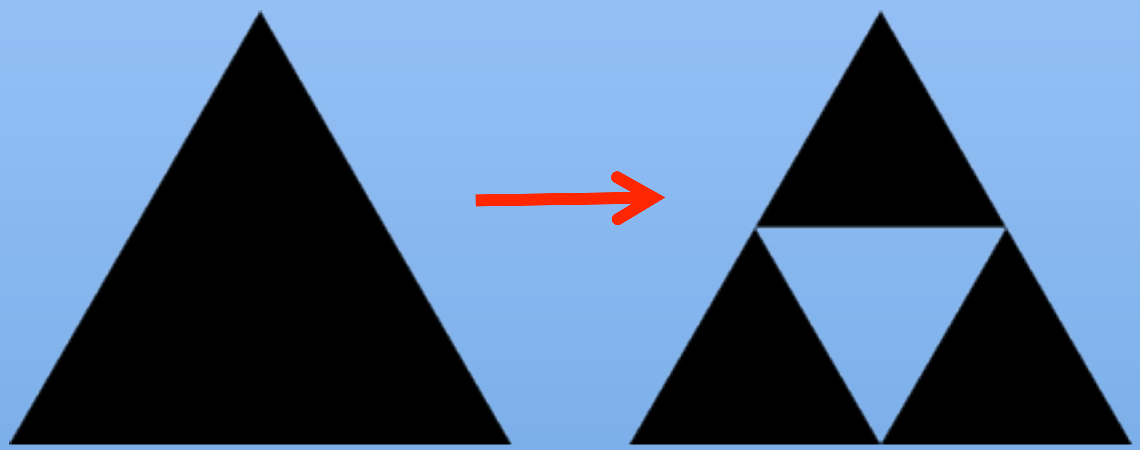
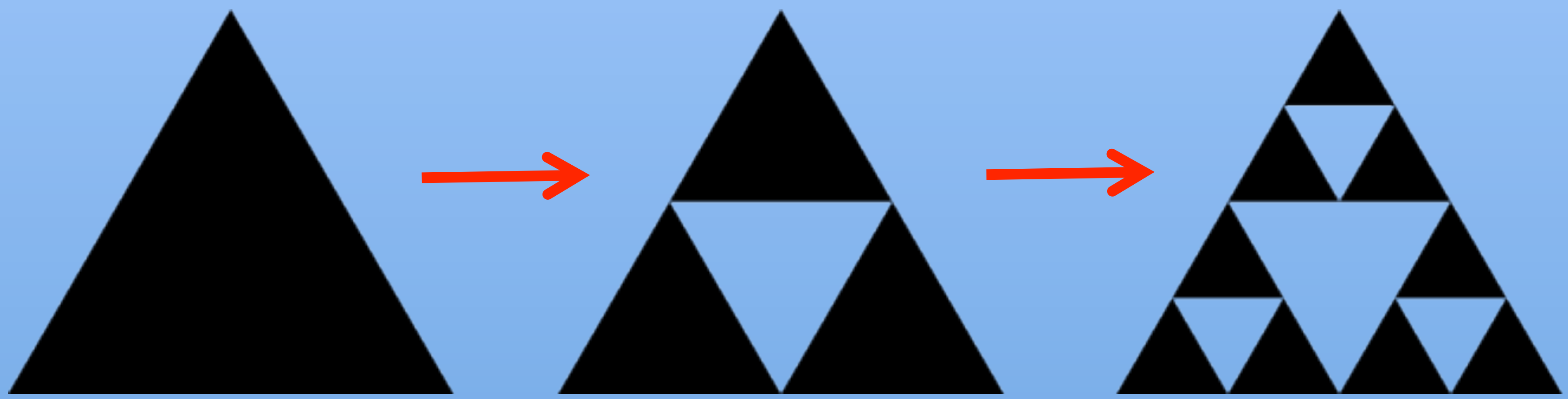


Sierpinski Triangle

(Waclaw Sierpinski, 1916)



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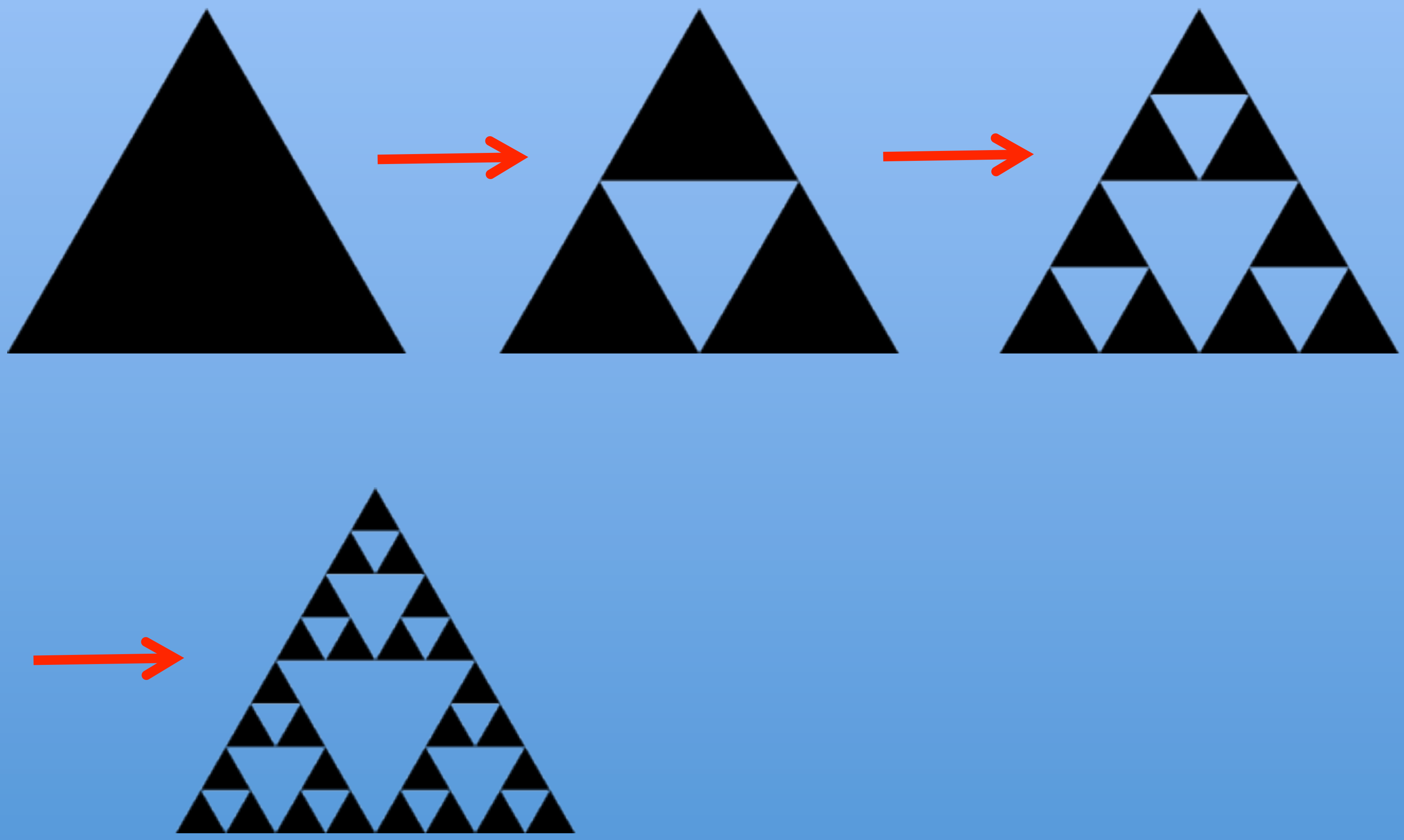
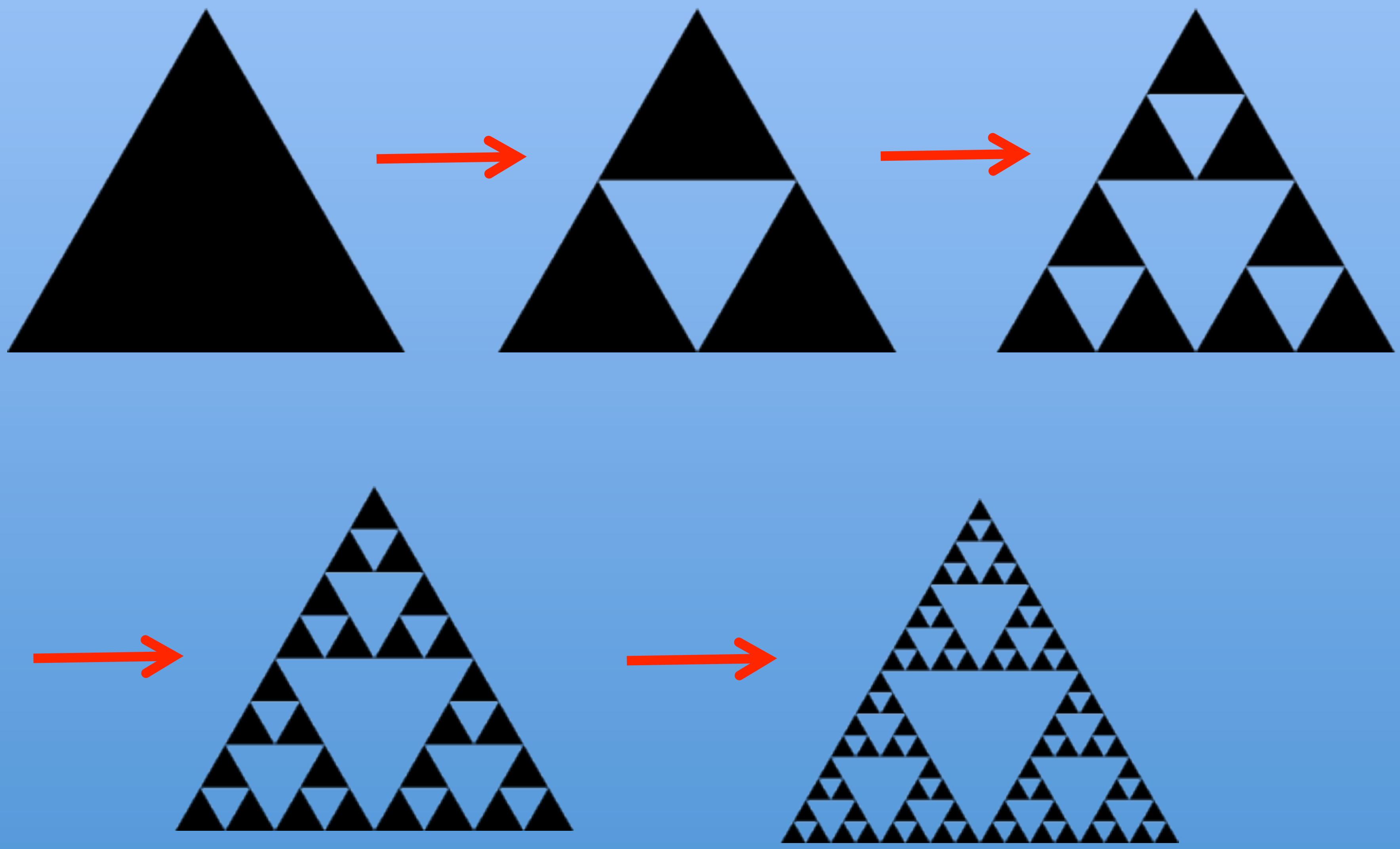
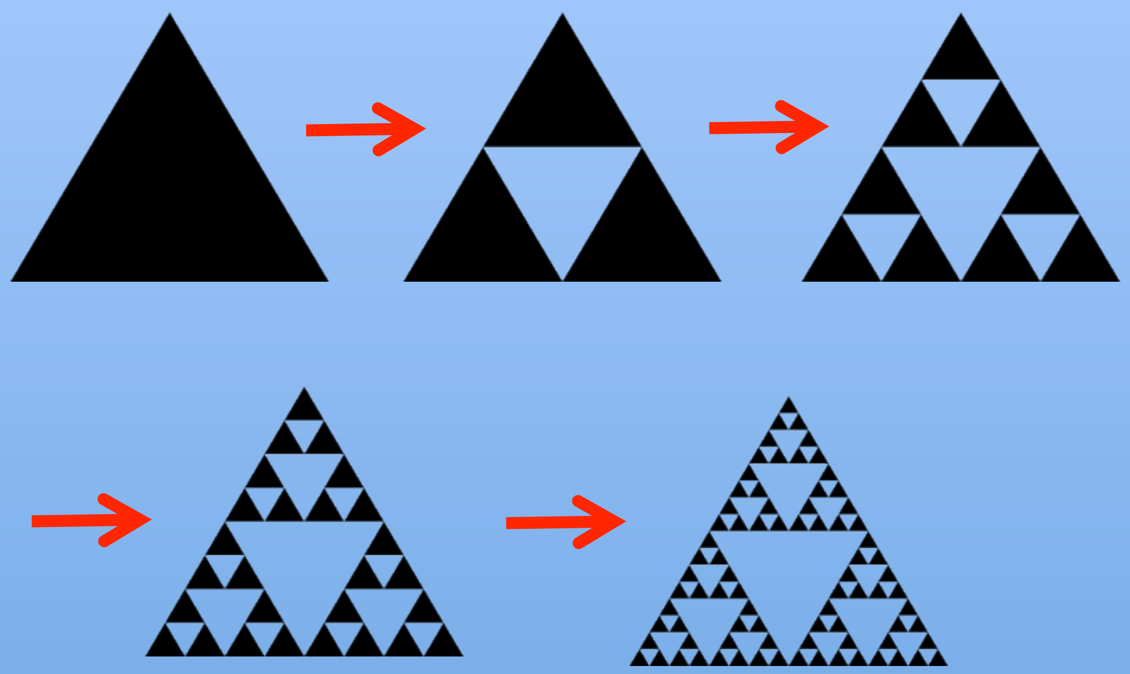
(Waclaw Sierpinski, 1916)



Sierpinski Triangle

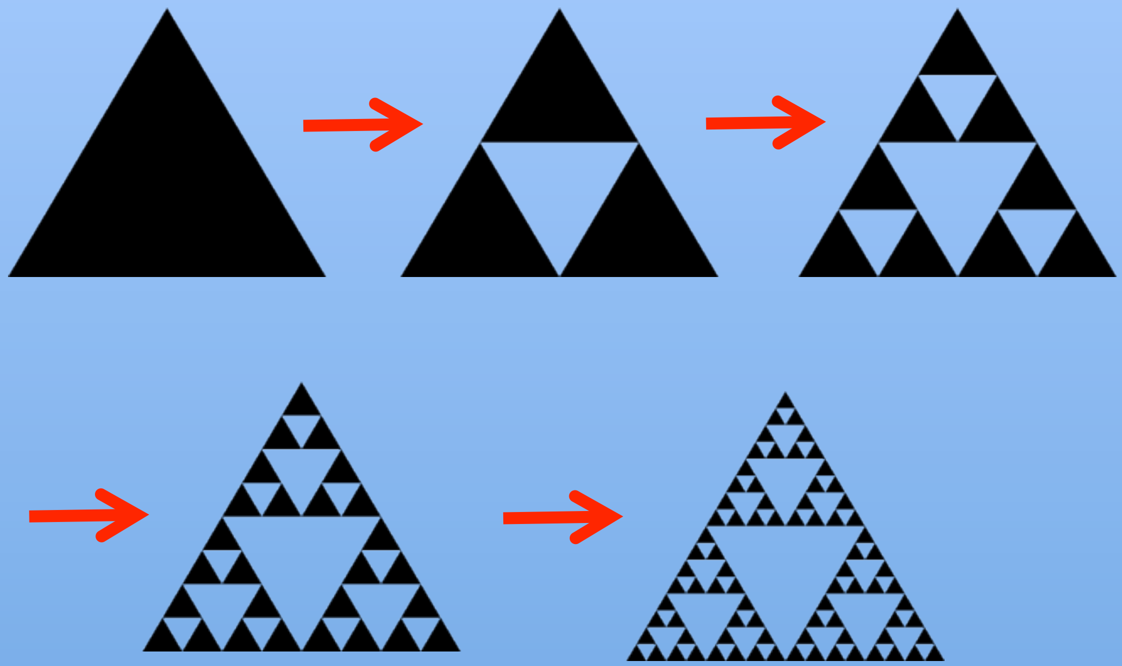
(Waclaw Sierpinski, 1916)





Fractal Dimension:

$$D = \frac{\log\ (\text{number of copies of previous level})}{\log\ (\text{reduction factor of side from previous level})}$$



Fractal Dimension:

$$D = \frac{\log\ (\text{number of copies of previous level})}{\log\ (\text{reduction factor of side from previous level})}$$

Approximate dimension of Cauliflower

Approximate dimension of Cauliflower

Fractal Structure of a White Cauliflower

Sang-Hoon Kim*

Division of Liberal Arts, Mokpo National Maritime University, Mokpo 530-729 and Institute for Condensed Matter Theory, Chonnam National University, Gwangju 500-757

(Received 17 September 2004)

The fractal structure of a white cauliflower is investigated by using the box-counting method on its cross section. The capacity dimension of the cross section is 1.88 ± 0.02, independent of the direction. From the result, we predict that the capacity dimension of a cauliflower is about 2.8. The vertical cross section of a cauliflower is modeled into a self-similar set of a rectangular tree. We discuss the condition of the fractal object in the tree and show that the vertical cross section has an angle of 67° in our model.

Approximate dimension of Cauliflower

Fractal Structure of a White Cauliflower

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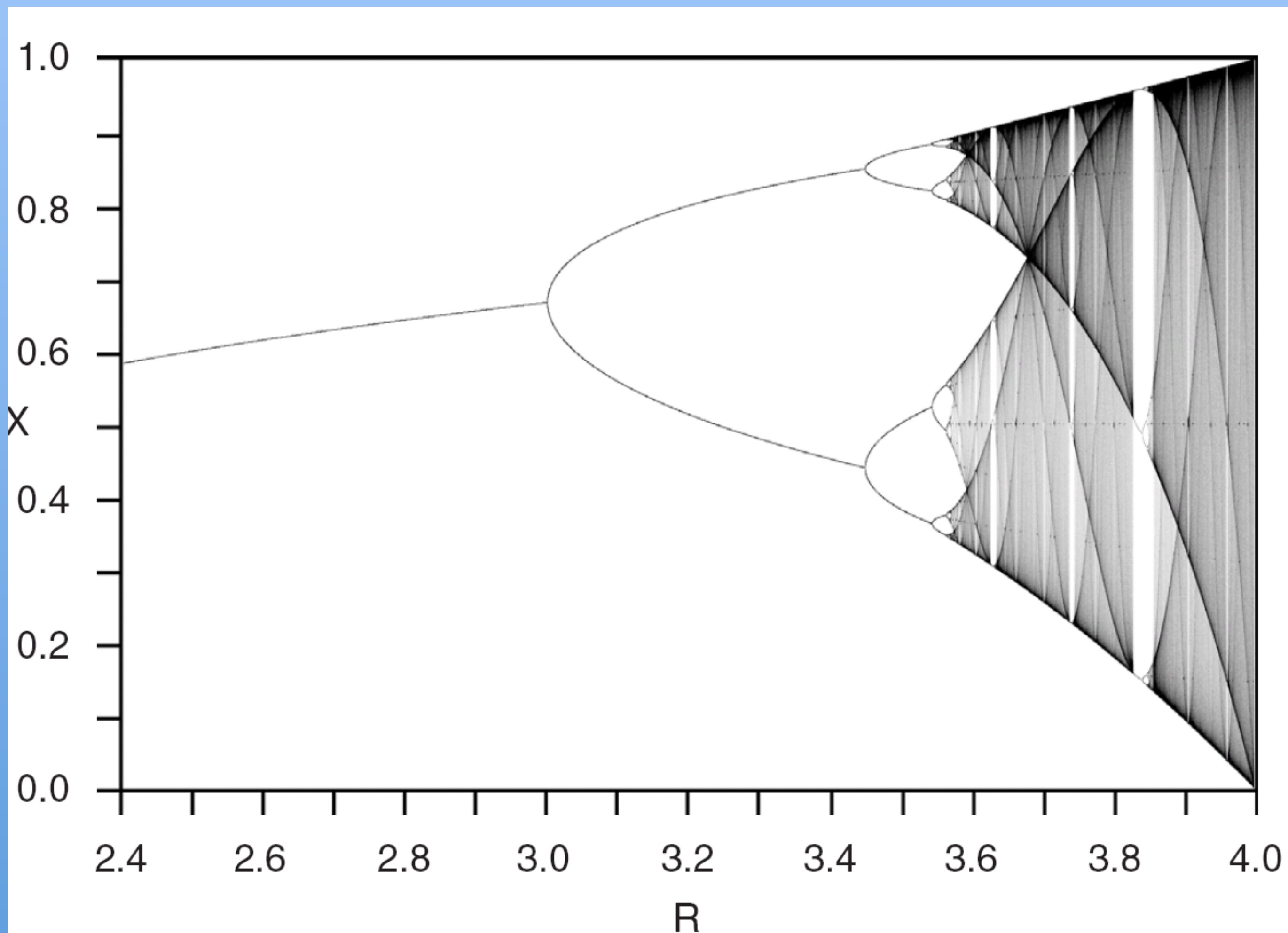
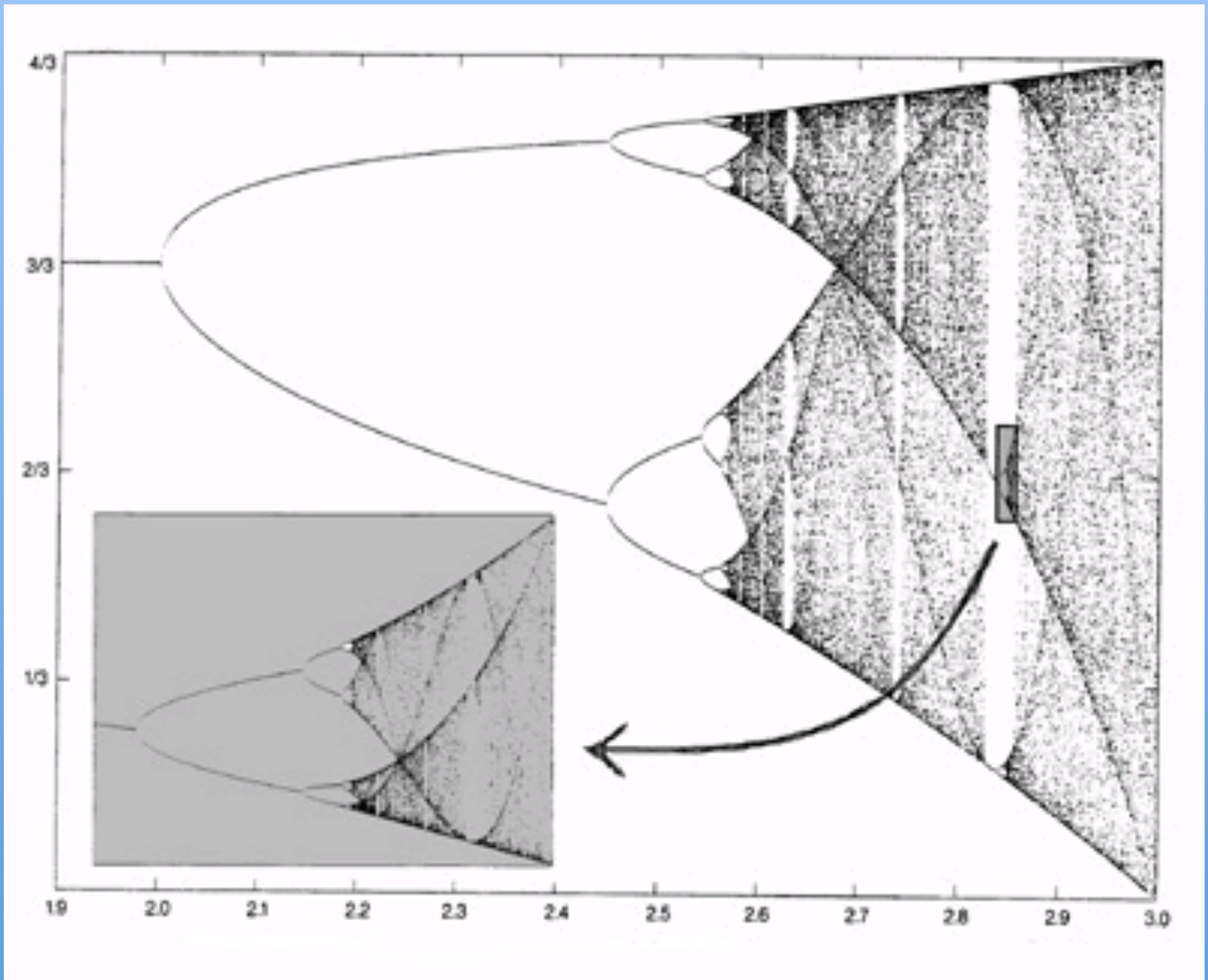
$D \approx 2.8$

Fractals in the Logistic Map Bifurcation Diagram

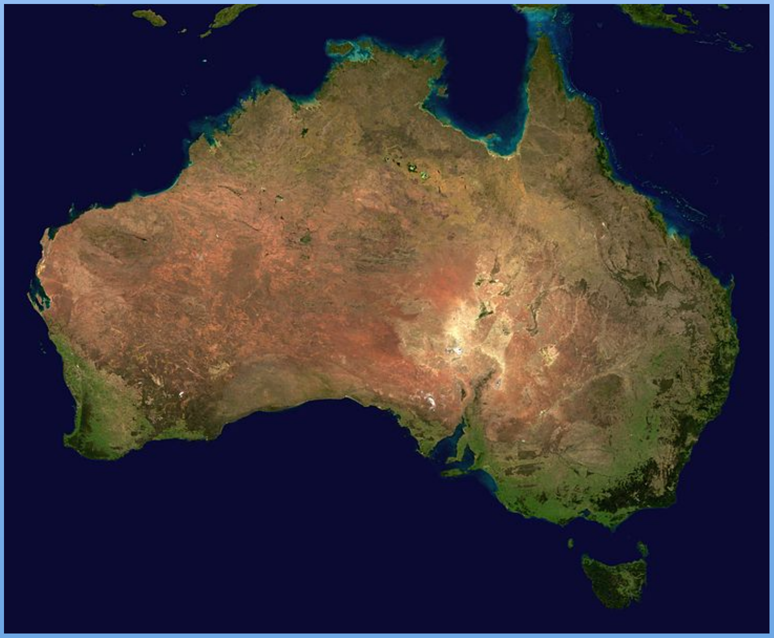




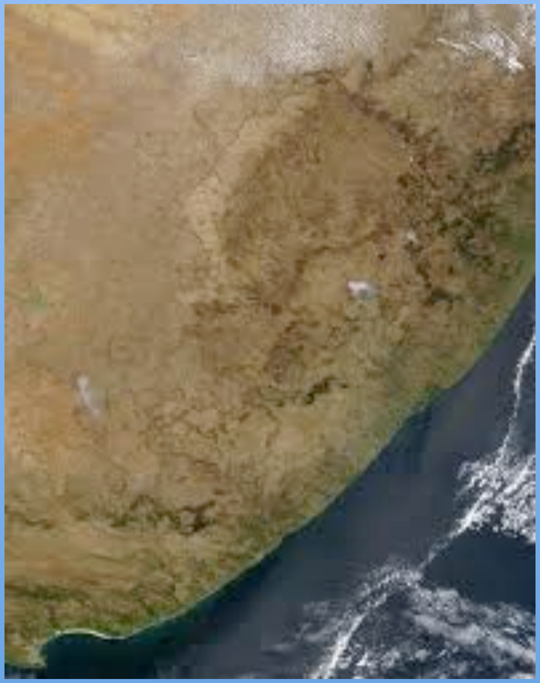
Fractal dimension ≈ 0.538

Approximate fractal dimension of coastlines
(Shelberg, Moellering, and Lam, 1982)

West Coast of Great Britain:
$D \approx 1.25$



Coast of Australia:
$D \approx 1.13$

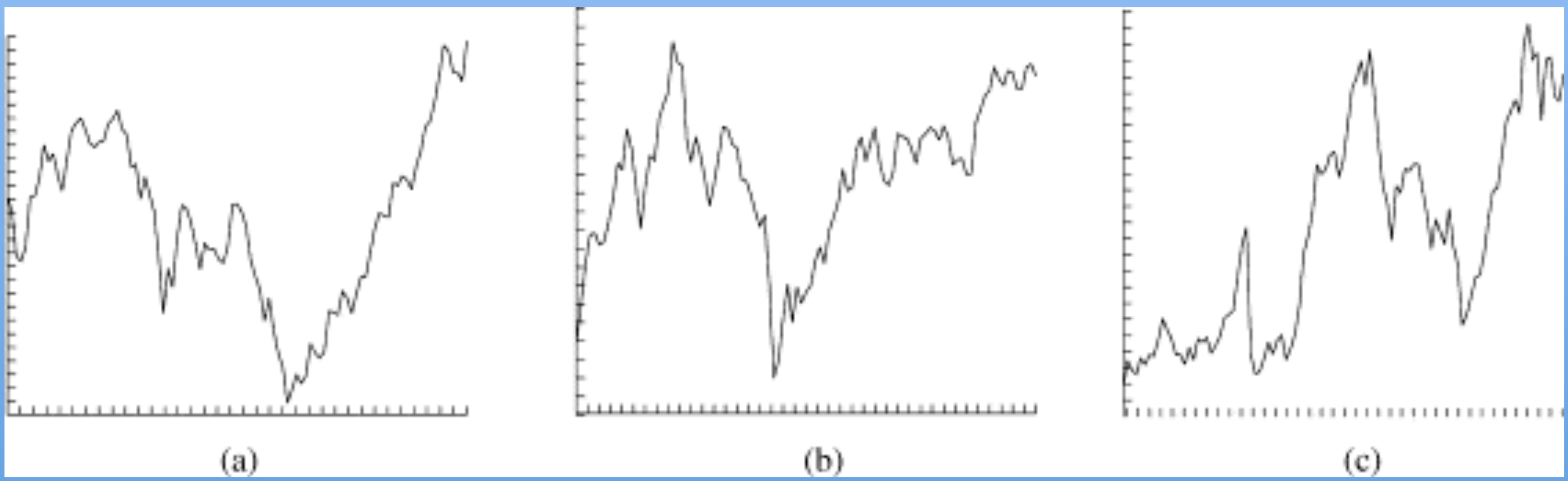


Coast of South Africa:
$D \approx 1.02$

Fractal dimension of stock prices

(J. A. Skjeltorp, Scaling in the Norwegian stock market, Physica A, 2000)

Oslo stock exchange general index



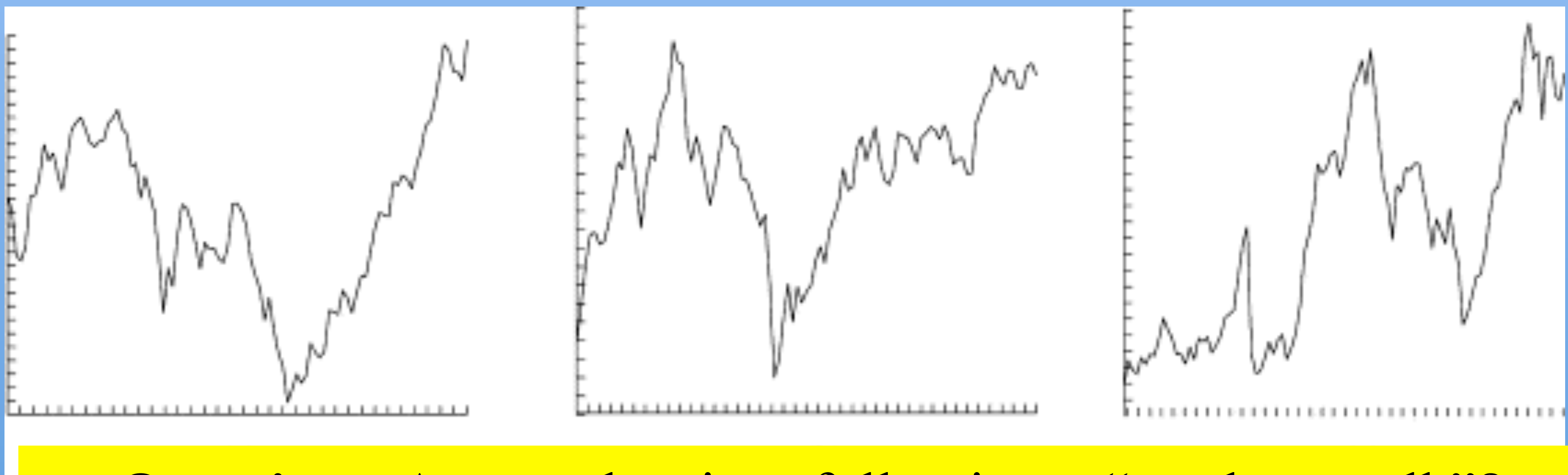
100-day daily price record	100-week weekly price record	100-month monthly price record

Self-similarity and detail at different temporal scales

Fractal dimension of stock prices

(J. A. Skjeltorp, Scaling in the Norwegian stock market, Physica A, 2000)

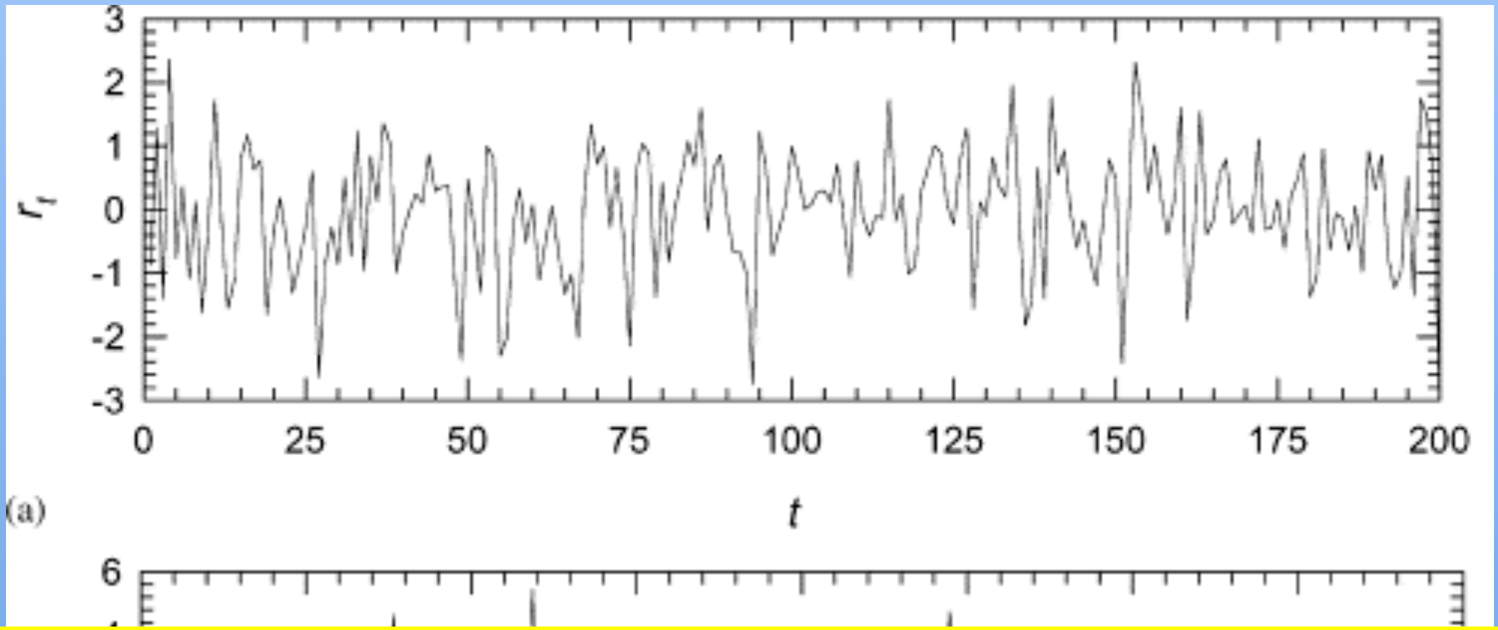
Oslo stock exchange general index



Question: Are stock prices following a "random walk"?

Project: Compare fractal dimension of stock prices with fractal dimension of "random walk"

Random Walks



Result: The answer (after some complicated mathematics) is "no": stock prices are not following a random walk.

But **note**: There are a lot of caveats in applying fractal analysis to time series such as these.

The Visual Complexity of Jackson Pollock's Drip Paintings

(Taylor, Spehar, Clifford, and Newell, 2008)

Evolution of "complexity" (fractal dimension) of Pollock's paintings: