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The Fourth Dimension of Life: Fractal Geometry and Allometric Scaling of Organisms

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Fractal-like networks effectively endow life with an additional fourth spatial dimension. This is the origin of quarter-power scaling that is so pervasive in biology. Organisms have evolved hierarchical branching networks that terminate in size-invariant units, such as capillaries, leaves, mitochondria, and oxidase molecules. Natural selection has tended to maximize both metabolic capacity, by maximizing the scaling of exchange surface areas, and internal efficiency, by minimizing the scaling of transport distances and times. These design principles are independent of detailed dynamics and explicit models and should apply to virtually all organisms.

Evolution by natural selection is one of the few universal principles in biology. It has shaped the structural and functional design of organisms in two important ways. First, it has tended to maximize metabolic capacity, because metabolism produces the energy and materials required to sustain and reproduce life; this has been achieved by increasing surface areas where resources are exchanged with the environment. Second, it has tended to maximize internal efficiency by reducing distances over which materials are transported and hence the time required for transport. A further consequence of evolution is the incredible diversity of body sizes, which range over 21 orders of magnitude, from 10^{-13} g (microbes) to 10^8 g (whales). A fundamental problem, therefore, is how exchange surfaces and transport distances change, or scale, with body size. In particular, a longstanding question has been why metabolic rate scales as the 3/4-power of body mass, M (I).

Biological scaling can be described by the allometric equation $Y = Y_0 M^b$, where Y is a variable such as metabolic rate or life span, Y_0 is a normalization constant, and b is a scaling exponent (I). Whereas Y_0 varies with the trait and type of organism, b characteristically takes on a limited number of values, all of which are simple multiples of 1/4. For

example, diameters of tree trunks and aortas scale as $M^{3/8}$ rates of cellular metabolism and heartbeat as $M^{-1/4}$, blood circulation time and life span as $M^{1/4}$, and whole-organism metabolic rate as $M^{3/4}$. The question has been why these exponents are multiples of 1/4 rather than 1/3 as expected on the basis of conventional Euclidean geometric scaling.

Recently, we presented a model which suggested that the explanation could be found in the fractal-like architecture of the hierarchical branching vascular networks that distribute resources within organisms (2). The model accurately predicts scaling exponents that have been measured for many structural and functional features of mammalian and plant vascular systems. It is not clear, however, how this model can account for the ubiquitous 3/4-power scaling of metabolic rate in diverse kinds of organisms with their wide variety of network designs, and especially in unicellular algae and protists, which

have no obvious branched anatomy. Here we present a more general model, based on the geometry rather than hydrodynamics of hierarchical networks, that does not require the existence of such explicit structures and that can account for the pervasive quarter-power scaling in biology.

We conjecture that organisms have been selected to maximize fitness by maximizing metabolic capacity, namely, the rate at which energy and material resources are taken up from the environment and allocated to some combination of survival and reproduction. This is equivalent to maximizing the scaling of whole-organism metabolic rate, B . It follows that B is limited by the geometry and scaling behavior of the total effective surface area, a , across which nutrients and energy are exchanged with the external or internal environment. Examples include the total leaf area of plants, the area of absorptive gut or capillary surface area of animals, and the total area of mitochondrial inner membranes within cells. In general, therefore, $B \propto a$. It is important to distinguish a from the relatively smooth external surface, or "skin," enclosing many organisms. We further conjecture that natural selection has acted to maximize a subject to various constraints while maintaining a compact shape. This is equivalent to minimizing the time and resistance for delivery of resources by minimizing some characteristic length or internal linear distance of the hierarchical network.

Broadly speaking, two sets of variables can be used to describe the size and shape of an organism: a conventional Euclidean set describing the external surface, A , enclosing the total volume, V ; and a "biological" set describing the internal structure, which includes the effective exchange area, a , and the

Table 1. Examples of the biological network variables l , a , and v in plant, mammalian, and unicellular systems.

Variable	Plant	Mammal	Unicellular
l	Mean path length from root to leaf, or between leaves	Mean circulation distance from heart to capillary, or between capillaries	Mean distance from cell surface to mitochondria and between mitochondria
a	Total area of leaves; area of absorptive root surface	Total area of capillaries; gut surface area	Actual cell surface area; total surface area of mitochondrial inner membranes
v	Total wood volume; total cell volume	Total blood volume; total tissue, or cell, volume	Volume of cytoplasm

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Table 2. The scaling of length, area, and volume associated with biological networks compared to the conventional Euclidean case. Allometric relations with M assume that tissue density is constant.

Variable	Conventional Euclidean	Fractal biological
Length	$L \propto A^{1/2} \propto V^{1/3} \propto M^{1/3}$	$l \propto a^{1/3} \propto v^{1/4} \propto M^{1/4}$
Area	$A \propto L^2 \propto V^{2/3} \propto M^{2/3}$	$a \propto l^3 \propto V^{3/4} \propto M^{3/4}$
Volume	$V \propto L^3 \propto M$	$v \propto l^4 \propto M$

total volume of biologically active material, v (Table 1). Although it is clearly a very difficult technical problem to calculate a , there are some general scaling properties that it must obey regardless of the detailed dynamics. Before examining these, it is instructive to consider the simpler case of how the area of skin, or external physical surface, of an organism or any Euclidean object, scales.

We first show how, and under what conditions, the classic 2/3-power Euclidean scaling law for A arises (3). In general, A is some complicated function of the various length scales, L_1, L_2, L_3, \dots , which parameterize size and shape: $A = A(L_1, L_2, L_3, \dots)$. On purely dimensional grounds this can be expressed as $A(L_1, L_2, L_3, \dots) = L_1^2 \Phi(L_2/L_1, L_3/L_1, \dots)$, where Φ is a dimensionless function of the dimensionless ratios L_2/L_1 , and so on. Suppose that we change the overall size by making a uniform scale transformation on all the lengths, $L_i \rightarrow L'_i = \Lambda L_i$ ($i = 1, 2, 3, \dots$), where Λ is some arbitrary number. This similarity transformation preserves the shape of the object as its size varies. In this case Φ clearly does not change, so A responds in the following manner:

$$A \rightarrow A' \equiv A(\Lambda L_1, \Lambda L_2, \Lambda L_3, \dots) = \Lambda^2 A(L_1, L_2, L_3, \dots) \quad (1)$$

The Euclidean volume of the object, $V = V(L_1, L_2, L_3, \dots)$, can be treated similarly; on dimensional grounds, $V = L_1^3 \Psi(L_2/L_1, L_3/L_1, \dots)$, where Ψ is a dimensionless function of the dimensionless ratios L_2/L_1 , and so on. After the scale transformation, which leaves Ψ unchanged,

$$V \rightarrow V' \equiv V(\Lambda L_1, \Lambda L_2, \Lambda L_3, \dots) = \Lambda^3 V(L_1, L_2, L_3, \dots) \quad (2)$$

From Eqs. 1 and 2, it is clear that $A'/V'^{2/3} = A/V^{2/3}$, that is, $A \propto V^{2/3}$; similarly, $L_i \propto V^{1/3}$. Notice that these are consistent with writing $V = AL$, where L is some length that is a function of the L_i and scales as $L \rightarrow L' = \Lambda L$. Assuming a size-invariant uniform density, these then give the conventional Euclidean geometric scaling results $L \propto L_i \propto M^{1/3}$ and $A \propto M^{2/3}$. These should apply, for example, to the body length and skin area of vertebrates.

The above argument ignores two basic facts of biology. First, the metabolic process relies on the hierarchical fractal-like nature of resource distribution networks. Examples include the macroscopic branching vascular

networks of plants and animals and the complicated ultrastructure within cells. We emphasize that the network can be “virtual”; it need not be a physical system of branching tubes, so long as it exhibits hierarchical pathways of material flow. Second, although organisms vary widely in size, these networks terminate at invariant units of fixed size that can be characterized by a biological length scale, l_0 . At the whole-organism level they include capillaries of mammals and leaves of plants. At the cellular and molecular levels, they include mitochondria and chloroplasts, and the metabolic rate-limiting cytochrome oxidase and RuBisCo (ribulose-1,5-bisphosphate carboxylase-oxygenase) molecules within these organelles. We now modify the above scaling argument by incorporating these two important biological features.

For a given type of organism the effective surface area is a function of the invariant length, l_0 , together with various independent length scales, l_i , that parameterize its fractal-like structure. It is important to distinguish biological length scales, l_i , which characterize the interior networks of the organism, from Euclidean ones, L_i , which characterize its exterior shape. For example, in a mammal one of the l_i is the length of the aorta, whereas one of the L_i is the overall body length; similarly, in unicellular organisms one of the l_i is the distance between mitochondria, whereas one of the L_i is the cell radius. Working as before, the effective exchange area, a , can be expressed as

$$a(l_0, l_1, l_2, \dots) = l_0^2 \phi\left(\frac{l_1}{l_0}, \frac{l_2}{l_0}, \dots\right) \quad (3)$$

where ϕ is a dimensionless function of the dimensionless ratios l_2/l_1 , and so on. Now, as the size of the organism changes, l_0 remains fixed. Consider, then, an arbitrary scale transformation on the network: $l_i \rightarrow l'_i = \lambda l_i$ ($i = 1, 2, 3, \dots$) keeping l_0 fixed. The analog of Eq. 1 reads

$$a \rightarrow a' \equiv a(l_0, \lambda l_1, \lambda l_2, \lambda l_3, \dots) = \lambda^2 l_0^2 \phi\left(\frac{l_1}{l_0}, \frac{l_2}{l_0}, \frac{l_3}{l_0}, \dots\right) \quad (4)$$

Because l_0 is fixed, the right-hand side is no longer simply proportional to λ^2 as in Eq. 1. Although we do not know the λ -dependence of ϕ , we can parameterize it as a power law reflecting the hierarchical frac-

tal-like organization:

$$\phi\left(\frac{l_1}{l_0}, \frac{l_2}{l_0}, \frac{l_3}{l_0}, \dots\right) = \lambda^{\epsilon_a} \phi\left(\frac{l_1}{l_0}, \frac{l_2}{l_0}, \frac{l_3}{l_0}, \dots\right) \quad (5)$$

where ϵ_a is an “arbitrary” exponent. In this case

$$a \rightarrow a' \equiv a(l_0, \lambda l_1, \lambda l_2, \lambda l_3, \dots) = \lambda^{2+\epsilon_a} a(l_0, l_1, l_2, l_3, \dots) \quad (6)$$

The crucial point here is that, because of the presence of l_0 , a does not scale simply as λ^2 . The assumption of a power law does not require the existence of an idealized mathematical self-similar fractal, which has no “fundamental” length scale such as l_0 . Even though the actual physical network is not a pure fractal because it has terminal units of fixed size and can be asymmetric, it is still natural to use the fractal language. We can therefore interpret the exponent in Eq. 6, $(2 + \epsilon_a) \equiv d_a$, as the fractal dimension of a (4). As such, it satisfies $0 \leq \epsilon_a \leq 1$. The lower limit, $\epsilon_a = 0$, is the conventional Euclidean case discussed above; the upper limit, $\epsilon_a = 1$, represents the “maximum fractality” of a volume-filling structure in which the effective area scales like a conventional volume.

Similarly, the biological volume, v , associated with a , can be expressed as $v(l_0, l_1, l_2, l_3, \dots) = l_0^3 \psi(l_1/l_0, l_2/l_0, l_3/l_0, \dots)$, where ψ is a dimensionless function of the dimensionless ratios l_2/l_1 , and so on. This represents the volume of protoplasm or biologically active material in the organism. It is not necessarily identical to V , because most organisms contain empty spaces enclosed by the skin; however, $v \propto V$. By analogy with ϕ , we assume that, under a scale transformation, ψ transforms as a power with an exponent ϵ_v : $\psi(l_0/l_1, l_2/l_1, l_3/l_1, \dots) = \lambda^{\epsilon_v} \psi(l_0/l_1, l_2/l_1, l_3/l_1, \dots)$. Consequently, v scales as

$$v \rightarrow v' \equiv v(l_0, \lambda l_1, \lambda l_2, \lambda l_3, \dots) = \lambda^{3+\epsilon_v} v(l_0, l_1, l_2, l_3, \dots) \quad (7)$$

with $0 \leq \epsilon_v \leq 1$. Combining Eqs. 6 and 7 straightforwardly leads to $a \propto v^{(2+\epsilon_a)/(3+\epsilon_v)}$.

Now v can always be expressed as $v = al$, where l is some length characteristic of the internal structure of the organism. We can therefore relate the scaling behaviour of v to that of a and l , with l expected to be proportional to one of the l_i . It is instructive, however, to consider the more general case and write $l = l(l_0, l_1, l_2, \dots) = l_0 \sigma(l_1/l_0, l_2/l_0, \dots)$, as was done with a and v ; σ is a dimensionless function, analogous to ϕ and ψ . This scales as $l \rightarrow l' = \lambda^{1+\epsilon_l} l$, where $d_l \equiv 1 + \epsilon_l$ is the fractal dimension of l , with $0 \leq \epsilon_l \leq 1$. Consequently, $v \rightarrow v' = \lambda^{3+\epsilon_a+\epsilon_v+\epsilon_l} v$, which, when compared to Eq. 7, gives $\epsilon_v = \epsilon_a + \epsilon_l$ (4). Assuming a uniform constant density, so that $v \propto M$, then gives

$$a \propto v^{\frac{2+\epsilon_a}{3+\epsilon_a+\epsilon_l}} \propto M^{\frac{2+\epsilon_a}{3+\epsilon_a+\epsilon_l}} \quad (8)$$

Our conjecture that organisms have evolved so as to maximize the scaling of a implies that the exponent, $b \equiv (2 + \epsilon_a)/(3 + \epsilon_a + \epsilon_l)$, must be maximized. It is straightforward to verify that this occurs when $\epsilon_a = 1$ and $\epsilon_l = 0$, thereby giving $b = 3/4$. Metabolic rate should therefore scale as $B \propto M^{3/4}$, regardless of the details of the branching architecture (5) and dynamics governing the metabolic process and distribution of resources.

This has several important consequences. First, because $a \propto M^{3/4}$, the number of invariant units in the network also scales as $M^{3/4}$. Second, the result $\epsilon_l = 0$, which gives $d_l = 1$, implies that internal distances associated with the network are not themselves fractal. This is consistent with the constraint that times for supply of resources, and therefore path lengths, should be minimized. Third, and perhaps most significant, is that $\epsilon_a = 1$, which implies that the fractal dimension of a is $d_a = 3$ rather than the canonical Euclidean value of 2. Thus, the effective surface area is "maximally fractal" and the network structure is volume-filling. It is in this sense that organisms have exploited a fourth spatial dimension (6) by evolving hierarchical fractal-like structures to maximize resource acquisition and allocation. More specifically, the area of the effective exchange surface scales as if it were a volume: $a \rightarrow a' = \lambda^3 a$, (rather than $\lambda^2 a$), whereas characteristic internal lengths associated with the fractal-like structure scale as $l \rightarrow l' = \lambda l$. Consequently, the biological volume scales as $v \rightarrow v' = \lambda^4 v$, so that in addition to $a \propto M^{3/4}$, we also have $l \propto l_i \propto M^{1/4}$.

These relationships should apply to all organisms that have been selected to maximize metabolic power under the constraint of minimizing internal transport distances and thereby having a maximally compact three-dimensional body shape (Table 2). For organisms such as roundworms and flatworms, which may be functionally one- or two-dimensional, these geometric relationships can be appropriately modified. In D dimensions, for example, our argument straightforwardly generalizes to give $a \propto B \propto M^{D/(D+1)}$ as in (2) and $l \propto M^{1/(D+1)}$ for the biological variables,

and $A \propto M^{(D-1)/D}$ and $L \propto M^{1/D}$ for the Euclidean ones. These relationships are not expected to apply to a few organisms, such as filamentous algae and fungi, that have been selected to maximize linear dimensions so as to sparsely occupy a maximal volume.

The present derivation is more general than our original model in which it was assumed that resource distribution networks were volume-filling and that energy dissipated was minimized. Incorporating dynamics led to a complete description of the physics and geometry of the networks that were shown to be fractal-like with 1/4-power scaling (2, 7). Versions of this physically explicit model show how the universal geometric derivation given here is realized in a variety of systems in different kinds of organisms. It is no accident, therefore, that many biological networks exhibit area-preserving branching, even though different anatomical designs exploit different hydrodynamic principles (2, 7). Unlike the genetic code, which has evolved only once in the history of life, fractal-like distribution networks that confer an additional effective fourth dimension have originated many times. Examples include extensive surface areas of leaves, gills, lungs, guts, kidneys, chloroplasts, and mitochondria, the whole-organism branching architectures of trees, sponges, hydrozoans, and crinoids, and the treelike networks of diverse respiratory and circulatory systems. It is not surprising, therefore, that even unicellular organisms exhibit quarter-power scaling, including the 3/4-power scaling law for metabolic rate. Although living things occupy a three-dimensional space, their internal physiology and anatomy operate as if they were four-dimensional.

Quarter-power scaling laws are perhaps as universal and as uniquely biological as the biochemical pathways of metabolism, the structure and function of the genetic code, and the process of natural selection. The vast majority of organisms exhibit scaling exponents very close to 3/4 for metabolic rate and to 1/4 for internal times and distances. These are the maximal and minimal values, respectively, for the effective surface area and linear

dimensions for a volume-filling fractal-like network. On the one hand, this is testimony to the power of natural selection, which has exploited variations on this fractal theme to produce the incredible variety of biological form and function. On the other hand, it is testimony to the severe geometric and physical constraints on metabolic processes, which have dictated that all of these organisms obey a common set of quarter-power scaling laws. Fractal geometry has literally given life an added dimension.

References and Notes

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