NYC

Topics of Quantitative Finance

HW2

Sanyueh (Michael) Yao

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Exercise 1(Replicating with futures): $R(t_k)$ **is constant for each of the subintervals** $[t_k, t_{k+1})$. **Let** $Fut(t_k, T)$ **denote the futures price at time** t_k **, k=0,1…t for tn. Let** $Fut(t_k, T)$ **denoted by S, then** $Fut(t_n, T) = S(T)$.

(i) Write a formula for $X(t_{k+1}) - X(t_k)$ **, the change in the portfolio value between tk and tk+1.**

Value of portfolio at time k+1	$X(t_{k+1}) = \Delta(t_k)\left[Fut(t_{k+1},T) - Fut(t_k,T)\right] + \left(1 + R(t_k)\left(t_{k+1} - t_k\right)\right) X(t_k)$ <Ref 1> $X(t_{k+1}) - X(t_k) = \Delta(t_k)\left[Fut(t_{k+1},T) - Fut(t_k,T)\right] + X(t_k)\cdot R(t_k)\left(t_{k+1} - t_k\right)$
<Ref 1>	It cost nothing to hold the future contract, put all the X(t_k) in the MMA $F(t_{k+1},T) - F(t_k,T)$ is the cash flow from futures position
Result	$X(t_{k+1}) - X(t_k) = \Delta(t_k)\left[F(t_{k+1},T) - F(t_k,T)\right] + X(t_k)\cdot R(t_k)\left(t_{k+1} - t_k\right)$

(ii) Continuous form for dX(t)

$dX(t)$	$= \Delta(t)dFut(t,T) + X(t)R(t)dt$

(iii) Suppose that interest rate is a constant r and that S is GBM. What must be $\Delta(t)$ **so that** $X(T) = (S(T) - K)^+$

Question Assumption	There are two securities in the market (1) Futures $Fut(t,T) = \tilde{E}\left[S(T) \mid F(t)\right] = S(t)e^{r(T-t)}$ When interest rates are constants, future price is the same as forward price all the time up to maturity theoretically. Future path converges to S(T) at maturity, but not equal to underlining before that. $dFut(t,T) = Fut(t,T)\sigma(t)d\tilde{W}_t$ (2) MMA $dS^{(0)}(t) = S^{(0)}(t)rdt$
Given that X(0)=c(0,S(0)), if dX(t)=dC(t,S(t)), we will ensure X(T)=(S(T)-K)$^+$ almost surely	$X(t) = C(t,S(t)) = \tilde{E}\left[D(T)F \mid F(t)\right]$ $d\left(D(t)X(t)\right) = d\left(D(t)C(t,S(t))\right)$ <Ref 1><Ref 2> <Ref 3> $D(t)\Delta^{(F)}(t)S(t)e^{r(T-t)}\sigma d\tilde{W}_t = D(t)C_S(t,S(t))S(t)\sigma d\tilde{W}_t$ $e^{r(T-t)}\Delta^{(F)}(t) = C_S(t,S(t)) = N(d_+)$ $\Delta^{(F)}(t) = N(d_+)e^{-r(T-t)}$ <where $\Delta^{(F)}(t) = \Delta(t)$ >
<Ref 1>	Future price is not the price for any tradable asset, so we do not discount it. The value of future contract is 0. Our first line to second line holds because the interest rate derivation and the cross variation cancels in the case of the futures. Our second line to third line holds from constant interest rate assumption and the dynamics given in the question
<Ref 2> $d\left(D(t)X(t)\right)$	$= D(t)dX(t) + X(t)dD(t) + dX(t)dD(t)$ $= D(t)\left(\Delta^{(F_0)}(t)dS^{(0)}(t) + \Delta^{(F)}(t)dFut(t,T)\right) + X(t)dD(t)$ $= D(t)\left(\Delta^{(F_0)}(t)S^{(0)}(t)rdt + \Delta^{(F)}(t)dFut(t,T)\right) + (-r)D(t)X(t)dt$ $= D(t)\left(X(t)rdt + \Delta^{(F)}(t)dFut(t,T)\right) + (-r)D(t)X(t)dt$ $= D(t)\Delta^{(F)}(t)dFut(t,T)$ <Ref 4>

	$= D(t)\Delta^{(F)}(t)Fut(t,T)\sigma d\tilde{W}_t$ $= D(t)\Delta^{(F)}(t)S(t)e^{r(T-t)}\sigma d\tilde{W}_t$
<Ref 3> $d\left(D(t)C(t,S(t))\right)$	$= D(t)C_S(t,S(t))S(t)\sigma d\tilde{W}_t$
<Ref 4> $dFut(t,T)$	$= d\left(e^{r(T-t)}S(t)\right)$ $= e^{r(T-t)}\left(-rdtS(t)+dS(t)-rdtdS(t)\right)$ $= e^{r(T-t)}\left(\sigma S(t)d\tilde{W}_t\right)$
Conclusion	We must choose $\Delta(t) = N(d_+)e^{-r(T-t)}$ so that $X(T) = (S(T)-K)^+$, where $N(d_+)$ is given in the BSM call price.

Exercise 2(Forward-futures for continuously compounding rates in the Hull-White model):

The IR in Hull-White model is $dR(t) = \left(\theta(t) - aR(t)\right)dt + \sigma d\tilde{W}(t)$.

(i) Show $For(0,T) = \frac{1}{\delta}\left[e^{-a(T)} \cdot C(T,T+\delta)R(0) + A(0,T+\delta) - A(0,T)\right]$

Continuously compounding rate For(t,T) is assumed	$e^{\delta For(t,T)} = \dfrac{B(t,T)}{B(t,T+\delta)}$ $\Leftrightarrow For(t,T) = -\frac{1}{\delta}\left[\log B(t,T+\delta) - \log B(t,T)\right]$
$For(0,T)$	$= -\frac{1}{\delta}\left[\log B(0,T+\delta) - \log B(0,T)\right]$ $= -\frac{1}{\delta}\left[\log \cdot e^{-C(0,T+\delta)R(0)-A(0,T+\delta)} - \log \cdot e^{-C(0,T)R(0)-A(0,T)}\right]$ <Ref 1> $= \frac{1}{\delta}\left[C(0,T+\delta)R(0) + A(0,T+\delta) - C(0,T)R(0) - A(0,T)\right]$ $= \frac{1}{\delta}\left[\frac{1}{a}\left(1-e^{-a(T+\delta)}\right) \cdot R(0) + A(0,T+\delta) - \frac{1}{a}\left(1-e^{-a(T)}\right)R(0) - A(0,T)\right]$ <Ref 2> $= \frac{1}{\delta}\left[R(0) \cdot \frac{1}{a}\left(e^{-a(T)}\right)\left(-e^{-a(\delta)}\right) + A(0,T+\delta) + \frac{1}{a}\left(e^{-a(T)}\right)R(0) - A(0,T)\right]$ $= \frac{1}{\delta}\left[R(0)\left(e^{-a(T)}\right) \cdot \frac{1}{a}\left(1-e^{-a(\delta)}\right) + A(0,T+\delta) - A(0,T)\right]$ $= \frac{1}{\delta}\left[R(0)e^{-a(T)} \cdot C(T,T+\delta) + A(0,T+\delta) - A(0,T)\right]$ <Ref 3>
<Ref 1>	The price at t of a zero-coupon bond that pays 1 at its maturity T is given by the affine yield formula $B(t,T) = e^{-C(t,T)R(t)-A(t,T)}, 0 \le t \le T$
<Ref 2>	Continued from <Ref 1> $C(t,T) = \frac{1}{a}\left(1-e^{-a(T-t)}\right)$ $A(t,T) = \int_t^T C(s,T)\theta(s)ds - \frac{1}{2}\sigma^2 \int_t^T C^2(s,T)ds$
<Ref 3>	$C(T,T+\delta) = \frac{1}{a}\left(1-e^{-a(T+\delta-T)}\right) = \frac{1}{a}\left(1-e^{-a(\delta)}\right)$
Conclusion	$For(0,T) = \frac{1}{\delta}\left[e^{-a(T)} \cdot C(T,T+\delta)R(0) + A(0,T+\delta) - A(0,T)\right]$

(ii) Show $Fut(0,T) = \dfrac{1}{\delta}C(T,T+\delta)\left[e^{-aT}R(0) + \int_0^T e^{-a(T-s)}\theta(s)ds\right] + \dfrac{1}{\delta}A(T,T+\delta)$

From lecture result we get	$Fut(t,T)=\tilde{E}\left[For(T,T)\mid F(t)\right],0\le t\le T$
$Fut(0,T)$	$=\tilde{E}\left[For(T,T)\mid F(0)\right]$ $=\tilde{E}\left[For(T,T)\right]$ $=-\frac{1}{\delta}\tilde{E}\left[\log B(T,T+\delta)-\log B(T,T)\right]$ $=-\frac{1}{\delta}\tilde{E}\left[\log B(T,T+\delta)\right]$ <Ref 1> $=\frac{1}{\delta}\tilde{E}\left[C(T,T+\delta)R(T)+A(T,T+\delta)\right]$ <Ref 2> $=\frac{1}{\delta}C(T,T+\delta)\tilde{E}\left[R(T)\right]+\frac{1}{\delta}A(T,T+\delta)$ <Ref 3> $=\frac{1}{\delta}C(T,T+\delta)\cdot\left[e^{-aT}R(0)+\int_0^T e^{-a(T-s)}\theta(s)ds\right]+\frac{1}{\delta}A(T,T+\delta)$ <Ref 4>
<Ref 1>	B is ZCB that pays 1 at its maturity T. So B(T,T)=1
<Ref 2>	The price at t of a zero-coupon bond that pays 1 at its maturity T is given by the affine yield formula $B(t,T)=e^{-C(t,T)R(t)-A(t,T)},0\le t\le T$ $B(T,T+\delta)=e^{-C(T,T+\delta)R(T)-A(T,T+\delta)}$
<Ref 3>	$C(T,T+\delta)=\frac{1}{a}\left(1-e^{-a(\delta)}\right)$ $A(T,T+\delta)=\int_T^{T+\delta}C(s,T+\delta)\theta(s)ds-\frac{1}{2}\sigma^2\int_T^{T+\delta}C^2(s,T+\delta)ds$ are deterministic functions
<Ref 4>	We know that $R(t)=e^{-at}R(0)+\int_0^t e^{-a(t-s)}\theta(s)ds+\sigma\int_0^t e^{-a(t-s)}d\tilde{W}(s)$ $\tilde{E}\left[R(T)\right]=e^{-aT}R(0)+\int_0^T e^{-a(T-s)}\theta(s)ds$
Conclusion	$Fut(0,T)=\frac{1}{\delta}C(T,T+\delta)\cdot\left[e^{-aT}R(0)+\int_0^T e^{-a(T-s)}\theta(s)ds\right]+\frac{1}{\delta}A(T,T+\delta)$

(iii) Show $\delta\left(For(0,T)-Fut(0,T)\right)=-\frac{1}{2}\sigma^2\int_0^T\left[C^2(s,T+\delta)-C^2(s,T)\right]ds$

$$\delta\left(For(0,T)-Fut(0,T)\right)$$

$$=A(0,T+\delta)-A(0,T)-C(T,T+\delta)\cdot\int_0^T e^{-a(T-s)}\theta(s)ds-\frac{1}{\delta}A(T,T+\delta)$$

$$=\int_0^{T+\delta}C(s,T+\delta)\theta(s)ds-\frac{1}{2}\sigma^2\int_0^{T+\delta}C^2(s,T+\delta)ds-\int_0^T C(s,T)\theta(s)ds+\frac{1}{2}\sigma^2\int_0^T C^2(s,T)ds$$

<Ref 1>

$$-C(T,T+\delta)\cdot\int_0^T e^{-a(T-s)}\theta(s)ds-\left[\int_T^{T+\delta}C(s,T+\delta)\theta(s)ds-\frac{1}{2}\sigma^2\int_T^{T+\delta}C^2(s,T+\delta)ds\right]$$

$$=\int_0^T C(s,T+\delta)\theta(s)ds-\frac{1}{2}\sigma^2\int_0^T C^2(s,T+\delta)ds-\int_0^T C(s,T)\theta(s)ds+\frac{1}{2}\sigma^2\int_0^T C^2(s,T)ds-C(T,T+\delta)\cdot\int_0^T e^{-a(T-s)}\theta(s)ds$$

$$= \int_0^T \left(C(s,T+\delta) - C(s,T)\right)\theta(s)ds - \frac{1}{2}\sigma^2 \int_0^T \left[C^2(s,T+\delta) - C^2(s,T)\right]ds - C(T,T+\delta)\cdot\int_0^T e^{-a(T-s)}\theta(s)ds$$

$$= \tfrac{1}{a}\int_0^T \left(-e^{-a(T+\delta-s)} + e^{-a(T-s)}\right)\theta(s)ds - \frac{1}{2}\sigma^2 \int_0^T \left[C^2(s,T+\delta) - C^2(s,T)\right]ds - C(T,T+\delta)\cdot\int_0^T e^{-a(T-s)}\theta(s)ds \quad \text{<Ref 2>}$$

$$= \tfrac{1}{a}\int_0^T \left(e^{-a(T-s)}\left(1 - e^{-a(\delta)}\right)\right)\theta(s)ds - \frac{1}{2}\sigma^2 \int_0^T \left[C^2(s,T+\delta) - C^2(s,T)\right]ds - C(T,T+\delta)\cdot\int_0^T e^{-a(T-s)}\theta(s)ds$$

$$= C(T,T+\delta)\int_0^T \left(e^{-a(T-s)}\right)\theta(s)ds - \frac{1}{2}\sigma^2 \int_0^T \left[C^2(s,T+\delta) - C^2(s,T)\right]ds - C(T,T+\delta)\cdot\int_0^T e^{-a(T-s)}\theta(s)ds \quad \text{<Ref 3>}$$

$$= -\frac{1}{2}\sigma^2 \int_0^T \left[C^2(s,T+\delta) - C^2(s,T)\right]ds$$

<Ref 1>	$A(0,T+\delta) = \int_0^{T+\delta} C(s,T+\delta)\theta(s)ds - \frac{1}{2}\sigma^2 \int_0^{T+\delta} C^2(s,T+\delta)ds$ $A(0,T) = \int_0^T C(s,T)\theta(s)ds - \frac{1}{2}\sigma^2 \int_0^T C^2(s,T)ds$ $A(T,T+\delta) = \int_T^{T+\delta} C(s,T+\delta)\theta(s)ds - \frac{1}{2}\sigma^2 \int_T^{T+\delta} C^2(s,T+\delta)ds$
<Ref 2>	$C(s,T) = \tfrac{1}{a}\left(1 - e^{-a(T-s)}\right);$ $C(s,T+\delta) = \tfrac{1}{a}\left(1 - e^{-a(T+\delta-s)}\right)$
<Ref 3>	$C(T,T+\delta) = \tfrac{1}{a}\left(1 - e^{-a(T+\delta-T)}\right) = \tfrac{1}{a}\left(1 - e^{-a(\delta)}\right)$
Conclusion	$\delta\left(For(0,T) - Fut(0,T)\right) = -\frac{1}{2}\sigma^2 \int_0^T \left[C^2(s,T+\delta) - C^2(s,T)\right]ds$

(iv) Show $\int_0^T \left[C^2(s,T+\delta) - C^2(s,T)\right]ds = \frac{1}{2a^3}\left(1 - e^{-a\delta}\right)^2\left(1 - e^{-2aT}\right) + \frac{1}{a^3}\left(1 - e^{-a\delta}\right)\left(1 - e^{-aT}\right)^2$

$$\int_0^T \left[C^2(s,T+\delta) - C^2(s,T)\right]ds$$

$$= \int_0^T \left[\tfrac{1}{a^2}\left(1 - e^{-a(T+\delta-s)}\right)^2 - \tfrac{1}{a^2}\left(1 - e^{-a(T-s)}\right)^2\right]ds \quad \text{<Ref 1>}$$

$$= \tfrac{1}{a^2}\int_0^T \left[\left(-2e^{-a(T+\delta-s)} + e^{-2a(T+\delta-s)}\right) - \left(-2e^{-a(T-s)} + e^{-2a(T-s)}\right)\right]ds$$

$$= \tfrac{1}{a^2}\int_0^T \left[-2e^{-a(T+\delta-s)} + e^{-2a(T+\delta-s)} + 2e^{-a(T-s)} - e^{-2a(T-s)}\right]ds$$

$$= \tfrac{1}{a^2}\left\{-2\frac{1}{a}e^{-a(T+\delta-s)} + \frac{1}{2a}e^{-2a(T+\delta-s)} + 2\frac{1}{a}e^{-a(T-s)} - \frac{1}{2a}e^{-2a(T-s)}\right\}\Bigg|_{s=0}^{s=T}$$

$$=\frac{1}{a^3}\left\{-2e^{-a(T+\delta-s)}+\frac{1}{2}e^{-2a(T+\delta-s)}+2e^{-a(T-s)}-\frac{1}{2}e^{-2a(T-s)}\right\}\Bigg|_{s=0}^{s=T}$$

$$=\frac{1}{a^3}\left\{-2e^{-a(T+\delta-T)}+\frac{1}{2}e^{-2a(T+\delta-T)}+2e^{-a(T-T)}-\frac{1}{2}e^{-2a(T-T)}\right\}-\frac{1}{a^3}\left\{-2e^{-a(T+\delta)}+\frac{1}{2}e^{-2a(T+\delta)}+2e^{-a(T)}-\frac{1}{2}e^{-2a(T)}\right\}$$

$$=\frac{1}{a^3}\left\{-2e^{-a(\delta)}+\frac{1}{2}e^{-2a(\delta)}+2e^{-a(0)}-\frac{1}{2}e^{-2a(0)}\right\}-\frac{1}{a^3}\left\{-2e^{-a(T+\delta)}+\frac{1}{2}e^{-2a(T+\delta)}+2e^{-a(T)}-\frac{1}{2}e^{-2a(T)}\right\}$$

$$=\frac{1}{a^3}\left\{-2e^{-a\delta}+\frac{e^{-2a(\delta)}}{2}+\frac{3}{2}+2e^{-a(T+\delta)}-\frac{e^{-2a(T+\delta)}}{2}-2e^{-aT}+\frac{e^{-2a(T)}}{2}\right\}$$

$$=\frac{1}{a^3}\left\{-e^{-a\delta}+\frac{e^{-2a(\delta)}}{2}+\frac{1}{2}-\frac{e^{-2a(T+\delta)}}{2}-\frac{e^{-2a(T)}}{2}+e^{-a(\delta+2T)}+\left(1-2e^{-aT}+e^{-2aT}-e^{-a\delta}+2e^{-a(\delta+T)}-e^{-a(\delta+2T)}\right)\right\}$$ <Ref 2>

$$=\frac{1}{2a^3}\left\{-2e^{-a\delta}+e^{-2a(\delta)}+1-e^{-2a(T+\delta)}-e^{-2a(T)}+2e^{-a(\delta+2T)}\right\}+\frac{1}{a^3}\left(1-e^{-aT}\right)^2\left(1-e^{-a\delta}\right)$$

$$=\frac{1}{2a^3}\left\{1-2e^{-a\delta}+e^{-2a\delta}-e^{-2aT}\left(1-2e^{-a(\delta)}+e^{-2a(\delta)}\right)\right\}+\frac{1}{a^3}\left(1-e^{-aT}\right)^2\left(1-e^{-a\delta}\right)$$ <Ref 3>

$$=\frac{1}{2a^3}\left\{\left(1-e^{-a\delta}\right)^2-e^{-2aT}\left(1-e^{-a\delta}\right)^2\right\}+\frac{1}{a^3}\left(1-e^{-aT}\right)^2\left(1-e^{-a\delta}\right)$$

$$=\frac{1}{2a^3}\left(1-e^{-a\delta}\right)^2\left(1-e^{-2aT}\right)+\frac{1}{a^3}\left(1-e^{-aT}\right)^2\left(1-e^{-a\delta}\right)$$

<Ref 1>	$C(s,T)=\frac{1}{a}\left(1-e^{-a(T-s)}\right);$ $C(s,T+\delta)=\frac{1}{a}\left(1-e^{-a(T+\delta-s)}\right)$
<Ref 2>	$\frac{1}{a^3}\left(1-e^{-aT}\right)^2\left(1-e^{-a\delta}\right)=\frac{1}{a^3}\left(1-2e^{-aT}+e^{-2aT}\right)\left(1-e^{-a\delta}\right)$ $=\frac{1}{a^3}\left(1-2e^{-aT}+e^{-2aT}-e^{-a\delta}+2e^{-a(\delta+T)}-e^{-a\delta}e^{-2aT}\right)$ $=\frac{1}{a^3}\left(1-2e^{-aT}+e^{-2aT}-e^{-a\delta}+2e^{-a(\delta+T)}-e^{-a(\delta+2T)}\right)$
<Ref 3>	$1-2e^{-a\delta}+e^{-2a\delta}=\left(1-e^{-a\delta}\right)^2$
Conclusion	$\int_0^T\left[C^2(s,T+\delta)-C^2(s,T)\right]ds=\frac{1}{2a^3}\left(1-e^{-a\delta}\right)^2\left(1-e^{-2aT}\right)+\frac{1}{a^3}\left(1-e^{-a\delta}\right)\left(1-e^{-aT}\right)^2$

- **Summary (Personal Review)**
  - **Definition for Forwards/Futures**

Forward	$For(t,T)=-\frac{1}{\delta}\left[\log B(t,T+\delta)-\log B(t,T)\right]$
	$For(0,T)=\frac{1}{\delta}\left[e^{-a(T)}\cdot C(T,T+\delta)R(0)+A(0,T+\delta)-A(0,T)\right]$
Futures	$Fut(t,T)=\tilde{E}\left[For(T,T)\mid F(t)\right],0\le t\le T$

	$Fut(0,T)=\frac{1}{\delta}C(T,T+\delta)\left[e^{-aT}R(0)+\int_0^T e^{-a(T-s)}\theta(s)ds\right]+\frac{1}{\delta}A(T,T+\delta)$

- **Relationship**

About L(t,T)	Where $L(t,T)$ is simple interest rate during [T,T+δ]: $e^{\delta For(t,T)}=\frac{B(t,T)}{B(t,T+\delta)}=1+\delta L(t,T)$
Forward Futures Spread	$\delta\left(For(0,T)-Fut(0,T)\right)$ $=A(0,T+\delta)-A(0,T)-C(T,T+\delta)\cdot\int_0^T e^{-a(T-s)}\theta(s)ds-\frac{1}{\delta}A(T,T+\delta)$ $=-\frac{1}{2}\sigma^2\int_0^T\left[C^2(s,T+\delta)-C^2(s,T)\right]ds$ $=-\frac{1}{2}\sigma^2\cdot\left\{\frac{1}{2a^3}\left(1-e^{-a\delta}\right)^2\left(1-e^{-2aT}\right)+\frac{1}{a^3}\left(1-e^{-aT}\right)^2\left(1-e^{-a\delta}\right)\right\}$
	$For(0,T)-Fut(0,T)=-\frac{\sigma^2}{4\delta a^3}\left(1-e^{-a\delta}\right)\left[\frac{1}{a}\left(1-e^{-a\delta}\right)\left(1-e^{-2aT}\right)+2a\frac{1}{a^2}\left(1-e^{-aT}\right)^2\right]$

Exercise 3(LIBOR):

(i) Show $\tilde{E}\left[D(u)B(u,v)\mid F(t)\right]=D(t)B(t,v),t\le u\le v$

$\tilde{E}\left[D(u)B(u,v)\mid F(t)\right]$	$=\tilde{E}\left[D(u)\cdot\frac{1}{D(u)}\tilde{E}\left[D(v)\mid F(u)\right]\mid F(t)\right]$	<Ref 1>
	$=\tilde{E}\left[\tilde{E}\left[D(v)\mid F(u)\right]\mid F(t)\right]$	
	$=D(t)\tilde{E}\left[\frac{1}{D(t)}\tilde{E}\left[D(v)\mid F(u)\right]\mid F(t)\right]$	
	$=D(t)\frac{1}{D(t)}\tilde{E}\left[\tilde{E}\left[D(v)\mid F(u)\right]\mid F(t)\right]$	
	$=D(t)\frac{1}{D(t)}\tilde{E}\left[D(v)\mid F(t)\right]$	<Ref 2>
	$=D(t)\cdot B(t,v)$	<Ref 3>
<Ref 1> From 5.6.1	$B(t,T)=\frac{1}{D(t)}\tilde{E}\left[D(T)\mid F(t)\right]$ $\Leftrightarrow B(u,v)=\frac{1}{D(u)}\tilde{E}\left[D(v)\mid F(u)\right]$	
<Ref 2> Iterative conditioning	$=D(t)\frac{1}{D(t)}\tilde{E}\left[D(v)\mid F(t)\right]$	
<Ref 3> From 5.6.1	$\frac{1}{D(t)}\tilde{E}\left[D(v)\mid F(t)\right]=B(t,v)$	
Conclusion	$\tilde{E}\left[D(u)B(u,v)\mid F(t)\right]=D(t)\cdot B(t,v)$	

(ii) Let $\delta > 0$ be given and let $T_j = \delta_j$. Let s be a nonnegative integer. For $j \geq s+1$, the risk-neutral price at time T_s of a payment of LIBOR $L(T_{j-1}, T_{j-1})$ at time T_j is $\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)L(T_j, T_j) \mid F(T_S)\right]$. Show that $\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)L(T_{j-1}, T_{j-1}) \mid F(T_S)\right] = \frac{1}{\delta}\left(B(T_s, T_{j-1}) - B(T_s, T_j)\right)$

$$\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)L(T_{j-1}, T_{j-1}) \mid F(T_S)\right]$$

$$= \frac{1}{D(T_S)}\tilde{E}\left[D(T_j)\cdot\frac{1}{\delta B(T_{j-1}, T_j)} - \frac{1}{\delta} \mid F(T_S)\right] \qquad \text{<Ref 1>}$$

$$= \frac{1}{\delta\cdot D(T_S)}\tilde{E}\left[\left(\frac{D(T_j)}{B(T_{j-1}, T_j)} - D(T_j)\right) \mid F(T_S)\right]$$

$$= \frac{1}{\delta\cdot D(T_S)}\left\{\tilde{E}\left[\frac{D(T_j)}{B(T_{j-1}, T_j)} \mid F(T_S)\right] - \tilde{E}\left[D(T_j) \mid F(T_S)\right]\right\}$$

$$= \frac{1}{\delta}\left\{\frac{1}{D(T_S)}\tilde{E}\left[\frac{D(T_j)}{B(T_{j-1}, T_j)} \mid F(T_S)\right] - \frac{1}{D(T_S)}\tilde{E}\left[D(T_j) \mid F(T_S)\right]\right\} \qquad \text{<Ref 2>}$$

$$= \frac{1}{\delta}\left\{\frac{1}{D(T_S)}\tilde{E}\left[\tilde{E}\left[\frac{D(T_j)}{B(T_{j-1}, T_j)} \mid F(T_{j-1})\right] \mid F(T_S)\right] - B(T_S, T_j)\right\} \qquad \text{<Ref 3>}$$

$$= \frac{1}{\delta}\left\{\frac{1}{D(T_S)}\tilde{E}\left[\frac{1}{B(T_{j-1}, T_j)}\tilde{E}\left[D(T_j) \mid F(T_{j-1})\right] \mid F(T_S)\right] - B(T_S, T_j)\right\}$$

$$= \frac{1}{\delta}\left\{\frac{1}{D(T_S)}\tilde{E}\left[\frac{1}{B(T_{j-1}, T_j)}\cdot D(T_{j-1})\cdot B(T_{j-1}, T_j) \mid F(T_S)\right] - B(T_S, T_j)\right\} \qquad \text{<Ref 4>}$$

$$= \frac{1}{\delta}\left\{\frac{1}{D(T_S)}\tilde{E}\left[D(T_{j-1}) \mid F(T_S)\right] - B(T_S, T_j)\right\}$$

$$= \frac{1}{\delta}\left(B(T_S, T_{j-1}) - B(T_S, T_j)\right) \qquad \text{<Ref 5>}$$

<Ref 1>	$L(T_{j-1}, T_{j-1}) = \frac{1}{\delta B(T_{j-1}, T_j)} - \frac{1}{\delta}$
<Ref 2> From 5.6.1	$B(t,T) = \frac{1}{D(t)}\tilde{E}\left[D(T) \mid F(t)\right] \Leftrightarrow \frac{1}{D(T_S)}\tilde{E}\left[D(T_j) \mid F(T_S)\right] = B(T_S, T_j)$
<Ref 3>	Iterative conditioning and $j \geq s+1$
<Ref 4>	From 5.6.1 $B(t,T) = \frac{1}{D(t)}\tilde{E}\left[D(T) \mid F(t)\right] \Leftrightarrow \tilde{E}\left[D(T) \mid F(t)\right] = D(t)\cdot B(t,T)$ $\Leftrightarrow \tilde{E}\left[D(T_j) \mid F(T_{j-1})\right] = D(T_{j-1})\cdot B(T_{j-1}, T_j)$ Compare result from (i) is a little bit different: $\tilde{E}\left[D(u)B(u,v) \mid F(t)\right] = D(t)\cdot B(t,v)$
<Ref 5> From 5.6.1	$B(t,T) = \frac{1}{D(t)}\tilde{E}\left[D(T) \mid F(t)\right] \Leftrightarrow \frac{1}{D(T_S)}\tilde{E}\left[D(T_{j-1}) \mid F(T_S)\right] = B(T_S, T_{j-1})$

Conclusion	$\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)L(T_{j-1},T_{j-1})\mid F(T_S)\right]=\frac{1}{\delta}\left(B(T_S,T_{j-1})-B(T_S,T_j)\right)$

❖ **Summary: Forward LIBOR (for personal review)**

Meaning	Receive a payment of L(T,T) at $T+\delta$ *Long $\frac{1}{\delta}$ T-maturity Bonds *Short $\frac{1}{\delta}$ $T+\delta$ -maturity Bonds *At T, invest Short $\frac{1}{\delta}$ receive in $T+\delta$ -maturity bonds can buy $\frac{1}{\delta B(T,T+\delta)}$ of them *At $T+\delta$, receive $\frac{1}{\delta B(T,T+\delta)}$ - $\frac{1}{\delta}$ =L(T,T)	
NO	**Formula**	**Note**
(1)	$\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)L(T_{j-1},T_{j-1})\mid F(T_S)\right]$ $=\frac{1}{\delta}\left(B(T_s,T_{j-1})-B(T_s,T_j)\right)$ $=B(T_s,T_j)L(T_s,T_{j-1})$	
(2)	$B(T_{j-1},T_j)L(T_{j-1},T_{j-1})$ $=\frac{1}{\delta}\left(1-B(T_{j-1},T_j)\right)$	$For(t,T)=\frac{S(t)}{B(t,T+\delta)}=\frac{D(t)S(t)}{D(t)B(t,T+\delta)}$ $\begin{cases} L(T_{j-1},T_{j-1})=\frac{1}{\delta B(T_{j-1},T_j)}-\frac{1}{\delta}:T\le t\le T+\delta \\ L(t,T)=\frac{B(t,T)}{\delta B(T_{j-1},T_j)}-\frac{1}{\delta}:0\le t\le T \end{cases}$ $dL(t,T)=\gamma(T)L(t,T)d\tilde{W}^{T+\delta}(t)$
(3)	$B(u,v)=\frac{1}{D(u)}\tilde{E}\left[D(v)\mid F(u)\right]$	From (4), it can derive this relationship

Exercise 4(LIBOR):

(i) Payer swap over the time period $\left[T_S,T_e\right]$ receives a payment of backset LIBOR $L(T_{j-1},T_{j-1})$ applied to a principal of 1 at each of the payment dates $T_j=\delta_j$, j=s+1,s+2....e.

Payer swap pays a fixed rate of interest K applied to a principal of 1 on each of these payment dates.

Net amount received at time T_j is $\delta\left(L(T_{j-1},T_j)-K\right)$ use exercise to show that at time Ts of a payer swap is $S_p\left(T_S,T_e;K\right)=1-B\left(T_S,T_e\right)-\delta K\sum_{j=s+1}^{e}B\left(T_s,T_j\right)$

Net amount received at time T_j	$\delta\left(L(T_{j-1},T_j)-K\right)$
$S_p\left(T_S,T_e;K\right)$	$\tilde{E}\left[\frac{1}{D(T_S)}\sum_{j=s+1}^{e}D(T_j)\delta\left(L(T_{j-1},T_{j-1})-K\right)\mid F(T_S)\right]$ $=\frac{1}{D(T_S)}\tilde{E}\left[\sum_{j=s+1}^{e}D(T_j)\delta L(T_{j-1},T_{j-1})\mid F(T_S)\right]-\frac{1}{D(T_S)}\tilde{E}\left[\sum_{j=s+1}^{e}D(T_j)\delta K\mid F(T_S)\right]$

	$=\sum_{j=s+1}^{e}\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)\delta L(T_{j-1},T_{j-1})\mid F(T_S)\right]-\sum_{j=s+1}^{e}\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)\delta K\mid F(T_S)\right]$ $=\left[1-B(T_S,T_e)\right]-\sum_{j=s+1}^{e}\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)\delta K\mid F(T_S)\right]$ <Ref 1> $=\left[1-B(T_S,T_e)\right]-\left[\delta K\sum_{j=s+1}^{e}B(T_S,T_j)\right]$ <Ref 2> $=1-B(T_S,T_e)-\delta K\sum_{j=s+1}^{e}B(T_S,T_j)$
<Ref 1>	$\because \frac{1}{D(T_S)}\tilde{E}\left[D(T_j)L(T_{j-1},T_{j-1})\mid F(T_S)\right]=\frac{1}{\delta}\left(B(T_s,T_{j-1})-B(T_s,T_j)\right)$ $\therefore \sum_{j=s+1}^{e}\delta\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)L(T_{j-1},T_{j-1})\mid F(T_S)\right]$ $=\delta\frac{1}{\delta}\left[\left(B(T_S,T_S)-B(T_S,T_{S+1})\right)+\left(B(T_S,T_{S+1})-B(T_S,T_{S+2})\right)......\left(B(T_S,T_{e-1})-B(T_S,T_e)\right)\right]$ $=B(T_S,T_S)-B(T_S,T_e)$ $=1-B(T_S,T_e)$
<Ref 2>	From 5.6.1 $\because B(t,T)=\frac{1}{D(t)}\tilde{E}\left[D(T)\mid F(t)\right]\Leftrightarrow\frac{1}{D(T_S)}\tilde{E}\left[D(T_{j-1})\mid F(T_S)\right]=B(T_S,T_{j-1})$ $\therefore \sum_{j=s+1}^{e}\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)\delta K\mid F(T_S)\right]$ $=\sum_{j=s+1}^{e}\delta K\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)\mid F(T_S)\right]$ (take out what is known) $=\delta K\sum_{j=s+1}^{e}B(T_S,T_j)$
Conclusion	$S_p\left(T_S,T_e;K\right)=1-B\left(T_S,T_e\right)-\delta K\sum_{j=s+1}^{e}B\left(T_s,T_j\right)$

(ii) A payer swaption is a zero-strike call on a payer swap. Show that at time u the option to take a long position in the payer swap in the part Ii) has value

$$SW_p\left(u,T_S,T_e;K\right)=\frac{\delta}{D(u)}\sum_{j=s+1}^{e}\tilde{E}\left[D(T_j)\left(SR(T_S,T_e)-K\right)^{+}\mid F_u\right]$$

$SW_P(u,T_S,T_e;K)$	$=\frac{1}{D(u)}\tilde{E}\left[D(T_S)\left(S_P(T_S,T_e;K)-0\right)^{+}\mid F(u)\right]$ $=\frac{1}{D(u)}\tilde{E}\left[D(T_S)\left(1-B\left(T_S,T_e\right)-\delta K\sum_{j=s+1}^{e}B\left(T_s,T_j\right)\right)^{+}\mid F(u)\right]$ <Ref 1> $=\frac{1}{D(u)}\tilde{E}\left[D(T_S)\left(SR\left(T_S,T_e\right)\cdot\delta\sum_{j=s+1}^{e}B\left(T_S,T_j\right)-\delta K\sum_{j=s+1}^{e}B\left(T_s,T_j\right)\right)^{+}\mid F(u)\right]$ <Ref 2>

	$=\frac{1}{D(u)}\tilde{E}\left[D(T_S)\delta\sum_{j=s+1}^{e}B(T_S,T_j)\left(SR(T_S,T_e)-K\right)^{+}\mid F(u)\right]$	
	$=\frac{1}{D(u)}\tilde{E}\left[D(T_S)\delta\sum_{j=s+1}^{e}\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)\mid F(T_S)\right]\left(SR(T_S,T_e)-K\right)^{+}\mid F(u)\right]$	<Ref 3>
	$=\frac{1}{D(u)}\tilde{E}\left[\delta\sum_{j=s+1}^{e}\tilde{E}\left[D(T_j)\mid F(T_S)\right]\left(SR(T_S,T_e)-K\right)^{+}\mid F(u)\right]$	
	$=\delta\frac{1}{D(u)}\sum_{j=s+1}^{e}\tilde{E}\left[\tilde{E}\left[D(T_j)\mid F(T_S)\right]\left(SR(T_S,T_e)-K\right)^{+}\mid F(u)\right]$	<Ref 4>
	$=\frac{\delta}{D(u)}\sum_{j=s+1}^{e}\tilde{E}\left[D(T_j)\left(SR(T_S,T_e)-K\right)^{+}\mid F(u)\right]$	
<Ref 1> from (i)	$S_p(T_S,T_e;K)=1-B(T_S,T_e)-\delta K\sum_{j=s+1}^{e}B(T_s,T_j)$	
<Ref 2>	**SWAP Rate at time over** $[T_s,T_e]$**:** $SR(T_S,T_e)=\frac{1-B(T_S,T_e)}{\delta\sum_{j=s+1}^{e}B(T_S,T_j)}\Leftrightarrow SR(T_S,T_e)\cdot\delta\sum_{j=s+1}^{e}B(T_S,T_j)=1-B(T_S,T_e)$	
<Ref 3> Bond price	From 5.6.1 $\because B(t,T)=\frac{1}{D(t)}\tilde{E}\left[D(T)\mid F(t)\right]\Leftrightarrow$ $\therefore\sum_{j=s+1}^{e}\frac{1}{D(T_S)}\tilde{E}\left[D(T_j)\mid F(T_S)\right]$ (take out what is known) $=\sum_{j=s+1}^{e}B(T_S,T_j)$	
<Ref 4>	$0\le u\le T_S\le T_e$	
Conclusion	$SW_P(u,T_S,T_e;K)=\frac{\delta}{D(u)}\sum_{j=s+1}^{e}\tilde{E}\left[D(T_j)\left(SR(T_S,T_e)-K\right)^{+}\mid F(u)\right]$	

(iii) A forward payer swap is an agreement to take a long payer swap position at a future date. Show that at time u the agreement to take a long position in the payer swap in part (i) has value

$$FS_p(u,T_S,T_e;K)=B(u,T_S)-B(u,T_e)-\delta K\sum_{j=s+1}^{e}B(u,T_j)$$

From RN Measure $FS_p(u,T_S,T_e;K)$	$=\frac{1}{D(u)}\tilde{E}[D(T_S)S_P(T_S,T_e;K)\mid F(u)]$	
	$=\frac{1}{D(u)}\tilde{E}\left[D(T_S)\left(1-B(T_S,T_e)-\delta K\sum_{j=s+1}^{e}B(T_s,T_j)\right)\mid F(u)\right]$	<Ref 1>
	$=\frac{1}{D(u)}\tilde{E}\left[D(T_S)-D(T_S)B(T_S,T_e)-D(T_S)\delta K\sum_{j=s+1}^{e}B(T_s,T_j)\mid F(u)\right]$	

	$= \frac{1}{D(u)}\left\{\tilde{E}\left[D(T_S)\mid F(u)\right]-\tilde{E}\left[D(T_S)B(T_S,T_e)\mid F(u)\right]-\tilde{E}\left[D(T_S)\delta K\sum_{j=s+1}^{e} B(T_s,T_j)\mid F(u)\right]\right\}$ $= \frac{1}{D(u)}\left\{D(u)B(u,T_S)-D(u)B(u,T_e)-\tilde{E}\left[D(T_S)\delta K\sum_{j=s+1}^{e} B(T_s,T_j)\mid F(u)\right]\right\}$ <Ref 2> $= \frac{1}{D(u)}\left\{D(u)B(u,T_S)-D(u)B(u,T_e)-\sum_{j=s+1}^{e} \delta K D(u)B(u,T_j)\right\}$ <Ref 3> $= B(u,T_S)-B(u,T_e)-\delta K\sum_{j=s+1}^{e} B(u,T_j)$
<Ref 1> Result from 4(i)	$S_p(T_S,T_e;K)=1-B(T_S,T_e)-\delta K\sum_{j=s+1}^{e} B(T_s,T_j)$
<Ref 2> result from Ex3(i)	$\because \tilde{E}\left[D(u)B(u,v)\mid F(t)\right]=D(t)\cdot B(t,v)$ $\therefore \tilde{E}\left[D(T_S)\mid F(u)\right]=\tilde{E}\left[D(T_S)B(T_S,T_S)\mid F(u)\right]=D(u)B(u,T_S)$ $\therefore \tilde{E}\left[D(T_S)B(T_S,T_e)\mid F(u)\right]=D(u)B(u,T_e)$
<Ref 3>	$\tilde{E}\left[D(T_S)\delta K\sum_{j=s+1}^{e} B(T_s,T_j)\mid F(u)\right]$ $=\sum_{j=s+1}^{e}\delta K\tilde{E}\left[D(T_s)B(T_s,T_j)\mid F(u)\right]$ $=\sum_{j=s+1}^{e}\delta K D(u)B(u,T_j)$
Conclusion	$FS_p(u,T_S,T_e;K)=B(u,T_S)-B(u,T_e)-\delta K\sum_{j=s+1}^{e} B(u,T_j)$

(iv) A forward payer swaption is a zero-strike call on a forward payer swap. Let 0≤t≤u≤Ts≤Te be given. Show that at time t the option to take a long position in the forward payer swap at time u, when he value of the swap is an agreement to take a long payer swap position at a future date. Show that at time u the agreement to take a long position in the payer swap in part (i) has value

$$FSW_p(t,u,T_S,T_e;K)=\frac{\delta}{D(t)}\sum_{j=s+1}^{e}\tilde{E}\left[D(T_j)\left(FSR(u,T_S,T_e)-K\right)^{+}\mid F(t)\right]$$

From RN argument $FSW_p(t,u,T_S,T_e;K)$	$=\frac{1}{D(t)}\tilde{E}\left[D(u)\left(FS_P(u,T_S,T_e-0)\right)^{+}\mid F(t)\right]$: zero-strike call $=\frac{1}{D(t)}\tilde{E}\left[D(u)\left(B(u,T_S)-B(u,T_e)-\delta K\sum_{j=s+1}^{e} B(u,T_j)\right)^{+}\mid F(t)\right]$ $=\frac{1}{D(t)}\tilde{E}\left[D(u)\left(\delta\sum_{j=s+1}^{e} B(u,T_j)\cdot FSR(u,T_S,T_e)-\delta K\sum_{j=s+1}^{e} B(u,T_j)\right)^{+}\mid F(t)\right]$ <Ref 1>

	$=\frac{1}{D(t)}\tilde{E}\left[D(u)\cdot\delta\sum_{j=s+1}^{e}B(u,T_j)\left(FSR(u,T_S,T_e)-K\right)^+\mid F(t)\right]$ $=\frac{1}{D(t)}\tilde{E}\left[D(u)\cdot\delta\sum_{j=s+1}^{e}\left(\frac{1}{D(u)}\tilde{E}\left[D(T_j)\mid F(u)\right]\right)\left(FSR(u,T_S,T_e)-K\right)^+\mid F(t)\right]$ <Ref 2> $=\frac{\delta}{D(t)}\sum_{j=s+1}^{e}\tilde{E}\left[\tilde{E}\left[D(T_j)\mid F(u)\right]\left(FSR(u,T_S,T_e)-K\right)^+\mid F(t)\right]$ $=\frac{\delta}{D(t)}\sum_{j=s+1}^{e}\tilde{E}\left[\tilde{E}\left[D(T_j)\cdot\left(FSR(u,T_S,T_e)-K\right)^+\mid F(u)\right]\mid F(t)\right]$ <Ref 3> $=\frac{\delta}{D(t)}\sum_{j=s+1}^{e}\tilde{E}\left[D(T_j)\cdot\left(FSR(u,T_S,T_e)-K\right)^+\mid F(t)\right]$
<Ref 1>	**Forward SWAP Rate at time u for period over** $[T_s,T_e]$**:** $FSR(u,T_S,T_e)=\frac{B(u,T_S)-B(u,T_e)}{\delta\sum_{j=s+1}^{e}B(u,T_j)}$ $\Leftrightarrow\delta\sum_{j=s+1}^{e}B(u,T_j)\cdot FSR(u,T_S,T_e)=B(u,T_S)-B(u,T_e)$
<Ref 2> Bond price	From 5.6.1 $\because B(t,T)=\frac{1}{D(t)}\tilde{E}\left[D(T)\mid F(t)\right]\Leftrightarrow$ $B(u,T_j)=\frac{1}{D(u)}\tilde{E}\left[D(T_j)\mid F(u)\right]$
<Ref 3>	Forward rate FSW is F(u) measurable
Conclusion	$FSW_p(t,u,T_S,T_e;K)=\frac{\delta}{D(t)}\sum_{j=s+1}^{e}\tilde{E}\left[D(T_j)\left(FSR(u,T_S,T_e)-K\right)^+\mid F(t)\right]$

❖ **Summary for personal review**

Name	**Formula**
Payer SWAP	**Meaning:** **Payer swap over the time period** $[T_S,T_e]$ **receives a payment of backset LIBOR** $L(T_{j-1},T_{j-1})$ **applied to a principal of 1 at each of the payment dates** **Payer swap pays a fixed rate of interest K applied to a principal of 1 on each of these payment dates.**
	$S_p(T_S,T_e;K)=1-B(T_S,T_e)-\delta K\sum_{j=s+1}^{e}B(T_s,T_j)$
	SWAP Rate at time over $[T_s,T_e]$**:** $SR(T_S,T_e)=\frac{1-B(T_S,T_e)}{\delta\sum_{j=s+1}^{e}B(T_S,T_j)}$
Swaption	**Meaning:** **This is a zero-strike call on a payer swap**

	$SW_P(u,T_S,T_e;K)=\dfrac{\delta}{D(u)}\sum_{j=s+1}^{e}\tilde{E}\left[D(T_j)\left(SR\left(T_S,T_e\right)-K\right)^{+}\mid F(u)\right]$
Forward Payer SWAP	**Meaning:** **This is an agreement to take a long payer swap position at a future date.** **At time u the agreement takes a long position in the payer swap in part (i) has value above**
	Value: $FS_p\left(u,T_S,T_e;K\right)=B\left(u,T_S\right)-B\left(u,T_e\right)-\delta K\sum_{j=s+1}^{e}B\left(u,T_j\right)$
	Forward SWAP Rate at time u: The value of K makes the forward payer swap have value zero at time u: This is M'gal under $\tilde{P}^{(A)}$ (TA session) $FSR\left(u,T_S,T_e\right)=\dfrac{B\left(u,T_S\right)-B\left(u,T_e\right)}{\delta\sum_{j=s+1}^{e}B\left(u,T_j\right)}$
Forward Swaption	**Meaning:** **This is a zero-strike call on a forward payer swap**
	$FSW_p\left(t,u,T_S,T_e;K\right)=\dfrac{\delta}{D(t)}\sum_{j=s+1}^{e}\tilde{E}\left[D(T_j)\left(FSR(u,T_S,T_e)-K\right)^{+}\mid F(t)\right]$