Topics of Quantitative Finance

HW1

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Exercise 1(Quanto Option): A quanto option pays off in one currency the price in another currency of an underlying assets without taking the currency conversion into account. (Quanto options are usually used in cases when investors are confident of the underlying asset's performance, but are not confident of the performance of the currency which the underlying is denominated in.)

(i) From 9.3.14, show that $S(t)=S(0)\exp\left\{\sigma_1\tilde{W}_1(t)+\left(r-\frac{1}{2}\sigma_1^2\right)t\right\}$

From Girsanov Thm and levy thm	$S(t)=S(0)\exp\left\{\sigma_1\tilde{W}_1(t)+\left(r-\frac{1}{2}\sigma_1^2\right)t\right\}$ We know that $\tilde{W}_1(t)$ would be Brownian Motion under the RN measure.
Let	$f(x,t)=S(0)\exp\left\{\sigma_1 x+\left(r-\frac{1}{2}\sigma_1^2\right)t\right\}$
Then	$f_t=\left(r-\frac{1}{2}\sigma_1^2\right)f$ $f_x=\sigma_1 f$ $f_{xx}=\sigma_1^2 f$
$df(t,\tilde{W}_1(t))$	$f_t(t,\tilde{W}_1(t))dt+f_x(t,\tilde{W}_1(t))d\tilde{W}_1(t)+\frac{1}{2}f_{xx}(t,\tilde{W}_1(t))dt$
$S(t)=f(t,\tilde{W}_1(t))$	$dS(t)=df(t,\tilde{W}_1(t))$
$dS(t)=$	$=f_t(t,\tilde{W}_1(t))dt+f_x(t,\tilde{W}_1(t))d\tilde{W}_1(t)+\frac{1}{2}f_{xx}(t,\tilde{W}_1(t))dt$ $=\left(r-\frac{1}{2}\sigma_1^2\right)S(t)dt+\sigma_1 S(t)d\tilde{W}_1(t)+\frac{1}{2}\sigma_1^2 S(t)dt$ $=rS(t)dt+\sigma_1 S(t)d\tilde{W}_1(t)$ This is what we see in 9.3.14
Conclusion	So we can say, with the assumption in 9.3.14, it would imply $S(t)=S(0)\exp\left\{\sigma_1\tilde{W}_1(t)+\left(r-\frac{1}{2}\sigma_1^2\right)t\right\}$

(ii) From 9.3.16, show that $Q(t)=Q(0)\exp\left\{\sigma_2\rho\tilde{W}_1(t)+\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)+\left(r-r^f-\frac{1}{2}\sigma_2^2\right)t\right\}$

Let	$X(t)=\sigma_2\rho\tilde{W}_1(t)+\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)+\left(r-r^f-\frac{1}{2}\sigma_2^2\right)t$ $\tilde{W}_3(t)=\rho\tilde{W}_1(t)+\sqrt{1-\rho^2}\tilde{W}_2(t)$ **(4)**
Then	$dX(t)=\sigma_2\rho d\tilde{W}_1(t)+\sigma_2\sqrt{1-\rho^2}d\tilde{W}_2(t)+\left(r-r^f-\frac{1}{2}\sigma_2^2\right)dt$ **(1)** $dX(t)dX(t)=\sigma_2^2\rho^2dt+\sigma_2^2\left(1-\rho^2\right)dt$ **(2)**
Let	$Q(t)=f(X(t))=e^x$
Then	$Q(t)=f(X(t))=f'(X(t))=f''(X(t))$ **(3)**
From (1)(2)(3)(4), we get $dQ(t)=$	$f'\left(X(t)\right)dX(t)+\frac{1}{2}f''\left(X(t)\right)dX(t)dX(t)$ $=Q(t)dX(t)+\frac{1}{2}Q(t)dt$ $=Q(t)\left(\sigma_2\rho d\tilde{W}_1(t)+\sigma_2\sqrt{1-\rho^2}d\tilde{W}_2(t)+\left(r-r^f-\frac{1}{2}\sigma_2^2\right)dt\right)+\frac{1}{2}Q(t)\left(\sigma_2^2\rho^2dt+\sigma_2^2\left(1-\rho^2\right)dt\right)$ $=Q(t)\left(\sigma_2\rho d\tilde{W}_1(t)+\sigma_2\sqrt{1-\rho^2}d\tilde{W}_2(t)+\left(r-r^f\right)dt\right)$ $=Q(t)\left(\sigma_2 d\tilde{W}_3(t)+\left(r-r^f\right)dt\right)$ This is what we see in 9.3.16

Conclusion	From 9.3.16. When the interest rate, volatility and the correlation are constant, it would imply $Q(t)=Q(0)\exp\left\{\sigma_2\rho\tilde{W}_1(t)+\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)+\left(r-r^f-\frac{1}{2}\sigma_2^2\right)t\right\}$

(iii) Show that $\frac{S(t)}{Q(t)}=\frac{S(0)}{Q(0)}\exp\left\{\sigma_4\tilde{W}_4(t)+\left(r-a-\frac{1}{2}\sigma_4^2\right)t\right\}$ **is a BM**

* $\sigma_4=\sqrt{\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2}$

** $a=r-r^f+\rho\sigma_1\sigma_2-\sigma_2^2$

*** $\tilde{W}_4(t)=\frac{\sigma_1-\sigma_2\rho}{\sigma_4}\tilde{W}_1(t)-\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_4}\tilde{W}_2(t)$

From result (i) and (ii)	$S(t)=S(0)\exp\left\{\sigma_1\tilde{W}_1(t)+\left(r-\frac{1}{2}\sigma_1^2\right)t\right\}$ $Q(t)=Q(0)\exp\left\{\sigma_2\rho\tilde{W}_1(t)+\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)+\left(r-r^f-\frac{1}{2}\sigma_2^2\right)t\right\}$	
(i)/(ii) and get the following	$=\frac{S(0)}{Q(0)}\exp\left\{\left(\sigma_1-\sigma_2\rho\right)\tilde{W}_1(t)-\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)+\left(r-\frac{1}{2}\sigma_1^2-\left(r-r^f-\frac{1}{2}\sigma_2^2\right)\right)t\right\}$ $=\frac{S(0)}{Q(0)}\exp\left\{\left(\sigma_1-\sigma_2\rho\right)\tilde{W}_1(t)-\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)+\left(-\frac{1}{2}\sigma_1^2+r^f+\frac{1}{2}\sigma_2^2\right)t\right\}$ $=\frac{S(0)}{Q(0)}\exp\left\{\sigma_4\tilde{W}_4(t)+\left(-\frac{1}{2}\sigma_1^2+r^f+\frac{1}{2}\sigma_2^2\right)t\right\}$ **<Ref 1>** $=\frac{S(0)}{Q(0)}\exp\left\{\sigma_4\tilde{W}_4(t)+\left(r-a-\frac{1}{2}\sigma_4^2\right)t\right\}$ **<Ref 2> <Ref 3>**	
<Ref1> From ***, we can get	$\tilde{W}_4(t)=\frac{\sigma_1-\sigma_2\rho}{\sigma_4}\tilde{W}_1(t)-\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_4}\tilde{W}_2(t)$ $\sigma_4\tilde{W}_4(t)=\left(\sigma_1-\sigma_2\rho\right)\tilde{W}_1(t)-\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)$	
<Ref2> From ** and * , we can get	$-\frac{1}{2}\sigma_1^2+r^f+\frac{1}{2}\sigma_2^2$ $=r-\left(r-r^f+\rho\sigma_1\sigma_2-\sigma_2^2\right)-\frac{1}{2}\left(\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2\right)$ $=r-a-\frac{1}{2}\sigma_4^2$	
<Ref3> Prove $\tilde{W}_4(t)$ is a Brownian Motion using Levy Them	(1) $\tilde{W}_4(0)=0$	(1) $\tilde{W}_4(0)=\frac{\sigma_1-\sigma_2\rho}{\sigma_4}\tilde{W}_1(0)-\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_4}\tilde{W}_2(0)=0$
	(2) $\tilde{W}_4(t)$ has continuous paths	(2) $\tilde{W}_4(t)=\frac{\sigma_1-\sigma_2\rho}{\sigma_4}\tilde{W}_1(t)-\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_4}\tilde{W}_2(t)$ has continuous paths
	(3) $\tilde{W}_4(t)$ is a Martingale	$E\left[\tilde{W}_4(t)\mid F_S\right]$ $=E\left[\frac{\sigma_1-\sigma_2\rho}{\sigma_4}\tilde{W}_1(t)-\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_4}\tilde{W}_2(t)\mid F_s\right]$ $=\frac{\sigma_1-\sigma_2\rho}{\sigma_4}E\left[\tilde{W}_1(t)\mid F_S\right]-\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_4}E\left[\tilde{W}_2(t)\mid F_S\right]$ $=\frac{\sigma_1-\sigma_2\rho}{\sigma_4}\tilde{W}_1(s)-\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_4}\tilde{W}_2(s)$ $=\tilde{W}_4(s)$
	(4) $d\tilde{W}_4(t)d\tilde{W}_4(t)=dt$	(4) $d\tilde{W}_4(t)d\tilde{W}_4(t)=\frac{(\sigma_1-\sigma_2\rho)^2}{\sigma_4^2}dt+\frac{\sigma_2^2\left(1-\rho^2\right)}{\sigma_4^2}dt$

		$= \frac{(\sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2\rho^2) + \sigma_2^2(1-\rho^2)}{\sigma_4^2} dt$ $= \frac{(\sigma_1^2 - 2\sigma_1\sigma_2\rho) + \sigma_2^2}{\sigma_4^2} dt = dt$
Conclusion	$\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)} \exp\left\{\sigma_4 \tilde{W}_4(t) + \left(r - a - \frac{1}{2}\sigma_4^2\right)t\right\}$ is a Brownian Motion	

(iv) Show that if at time t in [0,T], we have $\frac{S(t)}{Q(t)} = x$, then the price of the quanto call at this time is

$q(t,x) = xe^{-a\tau} N(d_+(\tau,x)) - e^{-r\tau} KN\left(d_-\left(\tau,x\right)\right)$

From Ch5: 5.5.8 to 5.5.12	We can replace $\frac{S(t)}{Q(t)}$ with S(t). We can the same expression $\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)} \exp\left\{\sigma_4 \tilde{W}_4(t) + \left(r - a - \frac{1}{2}\sigma_4^2 t\right)\right\}$ Derived from (iii) $S(t) = S(0) \exp\left\{\sigma \tilde{W}(t) + \left(r - a - \frac{1}{2}\sigma^2\right)t\right\}$ Shown in (5.5.8) * Replace σ by σ_4 * Replace $\tilde{W}(t)$ by $\tilde{W}_4(t)$ We can also verify it from the details below, we can see that the quanto call price is indeed: $q(t,x) = xe^{-a\tau} N(d_+(\tau,x)) - e^{-r\tau} KN\left(d_-\left(\tau,x\right)\right)$
According to RN Pricing Formula	$V(t) = \tilde{E}\left[e^{-r(T-t)} (\frac{S(T)}{Q(T)} - K)^+ \mid F(t)\right]$
C(t,x), where $\frac{S(t)}{Q(t)} = x$ and defined d $d_\pm\left(\tau,x\right)$ <Ref 2>	$= \tilde{E}\left[e^{-r(T-t)} \left(x \exp\left\{\sigma_4 \left(\tilde{W}_4(T) - \tilde{W}_4(t)\right) + \left(r - a - \frac{1}{2}\sigma_4^2\right)(T-t)\right\} - K\right)^+\right]$ $= \tilde{E}\left[e^{-r(T-t)} \left(x \exp\left\{\sigma_4 \left(\tilde{W}_4(T) - \tilde{W}_4(t)\right) + \left(r - a - \frac{1}{2}\sigma_4^2\right)(T-t)\right\} - K\right)^+\right]$ $= \tilde{E}\left[e^{-r\tau} \left(x \exp\left\{-\sigma_4 \sqrt{\tau} Y + \left(r - a - \frac{1}{2}\sigma_4^2\right)\tau\right\} - K\right)^+\right]$ <Ref1> $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau} \left(x \exp\left\{-\sigma_4 \sqrt{\tau} Y + \left(r - a - \frac{1}{2}\sigma_4^2\right)\tau\right\} - K\right) e^{-\frac{1}{2}y^2} dy$ $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} x \exp\left\{-\sigma_4 \sqrt{\tau} Y - \left(a + \frac{1}{2}\sigma_4^2\right)\tau - \frac{1}{2}y^2\right\} dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau} K e^{-\frac{1}{2}y^2} dy$ $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} xe^{-a\tau} \exp\left\{-\frac{1}{2}\left(y + \sigma\sqrt{\tau}\right)^2\right\} dy - e^{-r\tau} KN\left(d_-(\tau,x)\right)$ $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} xe^{-a\tau} \exp\left\{-\frac{1}{2}z^2\right\} dy - e^{-r\tau} KN\left(d_-(\tau,x)\right)$ <Ref3: $z = y + \sigma\sqrt{\tau}$ > $= xe^{-a\tau} N\left(d_+(\tau,x)\right) - e^{-r\tau} KN\left(d_-(\tau,x)\right)$
<Ref 1>	Let $\tau = T - t$, $Y = -\frac{\tilde{W}_4(T) - \tilde{W}_4(t)}{\sqrt{T-t}}$
<Ref 2>	Let $d_\pm\left(\tau,x\right) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r - a \pm \frac{1}{2}\sigma^2\right)T\right]$
<Ref 3> change of variable	$z = y + \sigma\sqrt{\tau}$
Quanto call price is	$q(t,x) = xe^{-a\tau} N(d_+(\tau,x)) - e^{-r\tau} KN\left(d_-\left(\tau,x\right)\right)$

Exercise 2(Exchange-rate put-call parity):

(i) What is the price in domestic currency at time zero of a contract that delivers one unit of foreign currency at time T in exchange for a payment of K unit of domestic currency.

$X_0 =$	$= \tilde{E}\left[e^{-r_d T}\left(Q(T)-K\right)\right]$ $= e^{-r_d T}\left(\tilde{E}\left[Q(T)\right]-K\right)$ $= e^{-r_d T}\left(Q(0)\exp\left\{\left(r_d - r_f\right)T\right\}-K\right)$ <Ref 1> $= Q(0)e^{-r_f T} - e^{-r_d T}K$
<Ref1> From Exercise 1 (ii)	$Q(t) = Q(0)\exp\left\{\sigma_2\rho\tilde{W}_1(t)+\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)+\left(r-r^f-\frac{1}{2}\sigma_2^2\right)t\right\}$ $\tilde{E}\left[Q\left(T\right)\right] = Q(0)\exp\left\{\left(r-r^f\right)T\right\}$
Conclusion	The price in domestic currency at time zero of a contract that delivers one unit of foreign currency at time T in exchange for a payment of K unit of domestic currency is $= Q(0)e^{-r_f T} - e^{-r_d T}K$

(ii) Show $P = e^{-r_d T}KN(-d_-) - e^{-r_f T}Q(0)N(-d_+)$

C-P =d(t)(F-K)=Xo	$Put = Call - X_0$ $Put = Call - Q(0)e^{-r_f T} + e^{-r_d T}K$ $Put = e^{-r_f T}Q(0)N\left(d_+\right) - e^{-r_d T}KN\left(d_-\right) - Q(0)e^{-r_f T} + e^{-r_d T}K$ $Put = Q(0)e^{-r_f T}\left(-1+N\left(d_+\right)\right) + e^{-r_d T}K\left(1-N\left(d_-\right)\right)$ $Put = -Q(0)e^{-r_f T}\left(1-N\left(d_+\right)\right) + e^{-r_d T}KN\left(-d_-\right)$ $Put = e^{-r_d T}KN\left(-d_-\right) - e^{-r_f T}Q(0)N\left(-d_+\right)$
Conclusion	$P = e^{-r_d T}KN(-d_-) - e^{-r_f T}Q(0)N(-d_+)$

Exercise 3(Exchange rate put-call duality):

(i) Derive the equation $\frac{N'(\pm d_+)}{N'(\pm d_-)} = e^{-\left(r_d - r_f\right)T}\frac{K}{Q(0)}$

$\frac{N'(\pm d_+)}{N'(\pm d_-)}$	$= \dfrac{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}d_+^2\right\}}{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}d_-^2\right\}}$ <Ref 1> $= \exp\left\{-\frac{1}{2}\left(d_+^2 - d_-^2\right)\right\}$ $= \exp\left\{-\frac{1}{2}\left(\frac{2}{\sigma\sqrt{T}}\left[\log\frac{Q(0)}{K}+\left(r_d - r_f\right)T\right]\right)\left(\frac{1}{\sigma\sqrt{T}}\left(\sigma^2\right)T\right)\right\}$ <Ref 2> $= \exp\left\{-\log\frac{Q(0)}{K} - \left(r_d - r_f\right)T\right\}$ $= \frac{K}{Q(0)}\exp\left\{-\left(r_d - r_f\right)T\right\}$
<Ref 1>	$\frac{N'(d_+)}{N'(d_-)} = \frac{N'(-d_+)}{N'(-d_-)} = \dfrac{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}d_+^2\right\}}{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}d_-^2\right\}}$

<Ref 2>	$d_\pm = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{Q(0)}{K} + \left(r_d - r_f \pm \frac{1}{2}\sigma^2\right)T\right]$
Conclusion:	$\frac{N'(\pm d_+)}{N'(\pm d_-)} = \frac{K}{Q(0)}\exp\left\{-\left(r_d - r_f\right)T\right\}$

(ii) Derive delta of the Call and Put

(ii.a) Delta of Call

Call on a unit of foreign currency	$C = e^{-r_f T}Q(0)N(d_+) - e^{-r_d T}KN(d_-)$ $d_\pm = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{Q(0)}{K} + \left(r_d - r_f \pm \frac{1}{2}\sigma^2\right)T\right]$
$c_x = \frac{\partial C}{\partial Q(0)}$	$= e^{-r_f T}N(d_+) + e^{-r_f T}Q(0)N'(d_+)\frac{\partial d_+}{\partial Q(0)} - e^{-r_d T}KN'(d_-)\frac{\partial d_-}{\partial Q(0)}$ $= e^{-r_f T}N(d_+) + e^{-r_f T}Q(0)\left(e^{-(r_d - r_f)T}\frac{K}{Q(0)}\cdot N'\left(d_-\right)\right)\frac{\partial d_+}{\partial Q(0)} - e^{-r_d T}KN'(d_-)\frac{\partial d_-}{\partial Q(0)}$ <Ref1> $= e^{-r_f T}N(d_+) + \left(e^{-(r_d)T}K\cdot N'\left(d_-\right)\right)\frac{\partial d_+}{\partial Q(0)} - e^{-r_d T}KN'(d_-)\frac{\partial d_-}{\partial Q(0)}$ <Ref2> $= e^{-r_f T}N(d_+)$
<Ref 1>	$N'\left(d_+\right) = e^{-(r_d - r_f)T}\frac{K}{Q(0)}\cdot N'\left(d_-\right)$
<Ref 2>	$\frac{\partial d_+}{\partial Q(0)} = \frac{\partial d_-}{\partial Q(0)}$
Conclusion	Delta of the Call (on a unit of foreign currency) is $\frac{\partial C}{\partial Q(0)} = e^{-r_f T}N(d_+)$

(ii.b) Delta of Put

Put on a unit of foreign currency	$P = e^{-r_d T}KN(-d_-) - e^{-r_f T}Q(0)N(-d_+)$ $d_\pm = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{Q(0)}{K} + \left(r_d - r_f \pm \frac{1}{2}\sigma^2\right)T\right]$
$P_x = \frac{\partial P}{\partial Q(0)}$	$= e^{-r_d T}KN'(-d_-)\frac{\partial d_-}{\partial Q(0)} - e^{-r_f T}N(-d_+) - e^{-r_f T}Q(0)N(-d_+)\frac{\partial d_+}{\partial Q(0)}$ $= -e^{-r_f T}N(-d_+) + \frac{\partial d_-}{\partial Q(0)}\left[e^{-r_d T}KN'(-d_-) - e^{-r_f T}Q(0)N(-d_+)\right]$ <Ref 1> $= -e^{-r_f T}N(-d_+) + \frac{\partial d_-}{\partial Q(0)}N'(-d_-)\left[e^{-r_d T}K - e^{-r_f T}Q(0)\frac{N'(-d_+)}{N'(-d_-)}\right]$ $= -e^{-r_f T}N(-d_+) + \frac{\partial d_-}{\partial Q(0)}N'(-d_-)\left[e^{-r_d T}K - e^{-r_f T}Q(0)\cdot e^{-(r_d - r_f)T}\frac{K}{Q(0)}\right]$ <Ref 2> $= -e^{-r_f T}N(-d_+)$
<Ref 1>	$\frac{\partial d_+}{\partial Q(0)} = \frac{\partial d_-}{\partial Q(0)}$
<Ref 2>	$\frac{N'(d_+)}{N'(d_-)} = e^{-(r_d - r_f)T}\frac{K}{Q(0)}$
Conclusion	$P_x = \frac{\partial P}{\partial Q(0)} = -e^{-r_f T}N(-d_+)$

(iii) Derive dual delta of the call and put

(iii. a) Dual Delta of Call

Call on a unit of foreign currency	$C = e^{-r_f T}Q(0)N(d_+) - e^{-r_d T}KN(d_-)$

$c_K = \frac{\partial C}{\partial K}$	$= e^{-r_f T} Q(0) N'(d_+) \frac{\partial d_+}{\partial K} - \left(e^{-r_d T} N(d_-) + e^{-r_d T} K N'(d_-) \frac{\partial d_+}{\partial K} \right)$ $= e^{-r_f T} Q(0) N'(d_+) \frac{\partial d_+}{\partial K} - e^{-r_d T} N(d_-) - e^{-r_d T} K N'(d_-) \frac{\partial d_+}{\partial K}$ $= e^{-r_f T} Q(0) \left(e^{-(r_d - r_f)T} \frac{K}{Q(0)} \cdot N'(d_-) \right) \frac{\partial d_+}{\partial K} - e^{-r_d T} N(d_-) - e^{-r_d T} K N'(d_-) \frac{\partial d_+}{\partial K}$ <Ref 1> $= -e^{-r_d T} N(d_-) + K \left(e^{-r_d T} \cdot N'(d_-) \right) \frac{\partial d_+}{\partial K} - e^{-r_d T} K N'(d_-) \frac{\partial d_+}{\partial K}$ $= -e^{-r_d T} N(d_-)$
<Ref 1>	$N'(d_+) = e^{-(r_d - r_f)T} \frac{K}{Q(0)} \cdot N'(d_-)$
Dual Delta of Call	$c_K = \frac{\partial C}{\partial K} = -e^{-r_d T} N(d_-)$

(iii.b) Dual Delta of Put

Put on a unit of foreign currency	$P = e^{-r_d T} K N(-d_-) - e^{-r_f T} Q(0) N(-d_+)$
$P_K = \frac{\partial P}{\partial K}$	$= \left(e^{-r_d T} N(-d_-) + e^{-r_d T} K N'(-d_-) \frac{\partial d_-}{\partial K} \right) - e^{-r_f T} Q(0) N(-d_+) \frac{\partial d_+}{\partial K}$ $= e^{-r_d T} N(-d_-) + \frac{\partial d_-}{\partial K} N'(-d_-) \left[e^{-r_d T} K - e^{-r_f T} Q(0) \frac{N(-d_+)}{N'(-d_-)} \right]$ <Ref 1> $= e^{-r_d T} N(-d_-) + \frac{\partial d_-}{\partial K} N'(-d_-) \left[e^{-r_d T} K - e^{-r_f T} Q(0) \cdot e^{-(r_d - r_f)T} \frac{K}{Q(0)} \right]$ <Ref 2> $= e^{-r_d T} N(-d_-) + \frac{\partial d_-}{\partial K} N'(-d_-) \left[e^{-r_d T} K - e^{-r_d T} K \right]$ $= e^{-r_d T} N(-d_-)$
<Ref 1>	$\frac{\partial d_+}{\partial K} = \frac{\partial d_-}{\partial K}$
<Ref 2>	$\frac{N'(d_+)}{N'(d_-)} = e^{-(r_d - r_f)T} \frac{K}{Q(0)}$
Dual Delta of put	$\frac{\partial P}{\partial K} = e^{-r_d T} N(-d_-)$

(iv) Find formula for P^f, the price of a put on a unit of domestic currency with a strike $\frac{1}{K}$

: $Q^f(0)$ for the current price denominated in foreign currency of a unit of a domestic currency

: $\frac{1}{K}$ the strike of a put on a unit of domestic currency

Comparison	IF C is a call on Q(t). For Dollar investor, it pays $(Q(T) - K)^+$ unit of dollar at T, or buy 1 Euro at K dollar For Euro investor, it pays $(1 - \frac{K}{Q(T)})^+ = (1 - KQ^f(T))^+ = K(\frac{1}{K} - Q^f(T))^+$ units of Euro at T. If we think it as K contracts. Each of them pays $(\frac{1}{K} - \frac{1}{Q(T)})^+ = (\frac{1}{K} - Q^f(T))^+$. Sell 1 dollar at $\frac{1}{K}$ Euro.
Recall (ii) Put on a unit of foreign currency	$P = e^{-r_d T} K N(-d_-) - e^{-r_f T} Q(0) N(-d_+)$ $d_\pm = \frac{1}{\sigma \sqrt{T}} \left[\log \frac{Q(0)}{K} + \left(r_d - r_f \pm \frac{1}{2} \sigma^2 \right) T \right]$
Put on a unit of Domestic currency	We place K for $\frac{1}{K}$. Exchange $Q^f(0)$ with $Q(0)$. Switch r_d with r_f. We updated our $d_\pm^f$ $P^f = e^{-r_f T} \cdot \frac{1}{K} N(-d_-^f) - e^{-r_d T} Q^f(0) N(-d_+^f)$

$d_{\pm}^{f}$	$=\frac{1}{\sigma\sqrt{T}}\left[\log K\cdot Q^{f}(0)+\left(r_f-r_d\pm\frac{1}{2}\sigma^2\right)T\right]$
$Q^{f}(0)$	$=\frac{1}{Q(0)}$
We can have another way to get P^{f}	$\tilde{E}^{f}\left[D^{f}(T)(\frac{1}{K}-\frac{1}{Q(T)})^{+}\right]$ $=\tilde{E}\left[\frac{D(T)Q(T)}{D^{f}(T)Q(0)}D^{f}(T)(\frac{1}{K}-\frac{1}{Q(T)})^{+}\right]$ $=\tilde{E}\left[D(T)(\frac{Q(T)}{K}-1)^{+}\right]\frac{1}{Q(0)}$
Formula for P^{f}	$P^{f}=e^{-r_fT}\cdot\frac{1}{K}N(-d_{-}^{f})-e^{-r_dT}Q^{f}(0)N(-d_{+}^{f})$

(v)Show that $C=Q(0)KP^{f}$

Verify whether $Q(0)KP^{f}$ will be definition of Call price	$=Q(0)KP^{f}$ $=Q(0)K\left(e^{-r_fT}\cdot\frac{1}{K}N(-d_{-}^{f})-e^{-r_dT}Q^{f}(0)N(-d_{+}^{f})\right)$ <Ref 1> <Ref 2> $=Q(0)\left(e^{-r_fT}N(d_{+})-e^{-r_dT}KQ^{f}(0)N(d_{-})\right)$ $=Q(0)e^{-r_fT}N(d_{+})-e^{-r_dT}KN(d_{-})$ $=C$
<Ref 1>: $-d_{-}^{f}$	$=-\frac{1}{\sigma\sqrt{T}}\left[\log K\cdot Q^{f}(0)+\left(r_f-r_d-\frac{1}{2}\sigma^2\right)T\right]$ $=-\frac{1}{\sigma\sqrt{T}}\left[\log\frac{K}{Q(0)}+\left(r_f-r_d-\frac{1}{2}\sigma^2\right)T\right]$ $=\frac{1}{\sigma\sqrt{T}}\left[\log\frac{Q(0)}{K}+\left(r_d-r_f+\frac{1}{2}\sigma^2\right)T\right]$ $=d_{+}$
<Ref 2>: $-d_{+}^{f}$	$=-\frac{1}{\sigma\sqrt{T}}\left[\log K\cdot Q^{f}(0)+\left(r_f-r_d+\frac{1}{2}\sigma^2\right)T\right]$ $=-\frac{1}{\sigma\sqrt{T}}\left[\log\frac{K}{Q(0)}+\left(r_f-r_d+\frac{1}{2}\sigma^2\right)T\right]$ $=\frac{1}{\sigma\sqrt{T}}\left[\log\frac{Q(0)}{K}+\left(r_d-r_f-\frac{1}{2}\sigma^2\right)T\right]$ $=d_{-}$
Another way (Result is still the same)	$P^{f}=\tilde{E}\left[D(T)(\frac{Q(T)}{K}-1)^{+}\right]\frac{1}{Q(0)}$ (from(v) another way) $C=\tilde{E}\left[D(T)(Q(T)-1)^{+}\right]$(From definition) $\Leftrightarrow Q(0)KP^{f}=Q(0)K\cdot\tilde{E}\left[D(T)(\frac{Q(T)}{K}-1)^{+}\right]\frac{1}{Q(0)}=\tilde{E}\left[D(T)(Q(T)-1)^{+}\right]=C$
Conclusion	The relationship for $C=Q(0)KP^{f}$ indeed exists

(vi) What is the delta of the put: $\frac{\partial P^{f}}{\partial Q^{f}(0)}$

Put on a unit of Domestic currency	$P^{f}=e^{-r_fT}\cdot\frac{1}{K}N(-d_{-}^{f})-e^{-r_dT}Q^{f}(0)N(-d_{+}^{f})$

$P_x^f = \frac{\partial P^f}{\partial Q^f(0)}$	$= e^{-r_f T} \cdot \frac{1}{K} N'(-d_-^f) \frac{\partial(-d_-^f)}{\partial Q^f(0)} - \left(e^{-r_d T} N(-d_+^f) + e^{-r_d T} Q^f(0) N'(-d_+^f) \frac{\partial(-d_+^f)}{\partial Q^f(0)} \right)$ <Ref 1> $= \frac{\partial(-d_-^f)}{\partial Q^f(0)} \left(e^{-r_f T} \cdot \frac{1}{K} N'(-d_-^f) - e^{-r_d T} Q^f(0) N'(-d_+^f) \right) - e^{-r_d T} N(-d_+^f)$ $= \frac{\partial(-d_-^f)}{\partial Q^f(0)} \cdot N'(-d_+^f) \left(\frac{N'(-d_-^f)}{N'(-d_+^f)} e^{-r_f T} \cdot \frac{1}{K} - e^{-r_d T} Q^f(0) \right) - e^{-r_d T} N(-d_+^f)$ $= \frac{\partial(-d_-^f)}{\partial Q^f(0)} \cdot N'(-d_+^f) \left(e^{-(r_d)T} \frac{1}{Q(0)} - e^{-r_d T} Q^f(0) \right) - e^{-r_d T} N(-d_+^f)$ <Ref 2> $= -e^{-r_d T} N(-d_+^f)$
<Ref 1>	$\frac{\partial d_+}{\partial Q(0)} = \frac{\partial d_-}{\partial Q(0)} \rightarrow \xrightarrow{-d_-^f = d_+, -d_+^f = d_-} \frac{\partial(-d_-^f)}{\partial Q^f(0)} = \frac{\partial(-d_+^f)}{\partial Q^f(0)}$
<Ref 2>	$\frac{N'(d_+)}{N'(d_-)} = e^{-(r_d - r_f)T} \frac{K}{Q(0)} = \frac{N'(-d_-^f)}{N'(-d_+^f)}$
Recall the result from (ii)	$P_x = \frac{\partial P}{\partial Q(0)} = -e^{-r_f T} N(-d_+)$ (We can change r_f with r_d, change $-d_+$ for $-d_+^f$, we can the same result…) $P_x^f = \frac{\partial P^f}{\partial Q^f(0)} = -e^{-r_d T} N(-d_+^f)$
Conclusion	$\frac{\partial P^f}{\partial Q^f(0)} = -e^{-r_d T} N(-d_+^f)$

(vii) Find a Formula relating $\frac{\partial P^f}{\partial Q^f(0)}$ and dual delta $\frac{\partial C}{\partial K}$

Recall (iv)	$P^f = e^{-r_f T} \cdot \frac{1}{K} N(-d_-^f) - e^{-r_d T} Q^f(0) N(-d_+^f)$
From (vi): $\frac{\partial P^f}{\partial Q^f(0)}$	$= -e^{-r_d T} N(-d_+^f)$ $= -e^{-r_d T} N(d_-)$ (from(v) we know $-d_+^f = d_-$) $= \frac{\partial C}{\partial K}$
From (iii)	$c_K = \frac{\partial C}{\partial K} = -e^{-r_d T} N(d_-)$
Conclusion	$\frac{\partial P^f}{\partial Q^f(0)} = \frac{\partial C}{\partial K}$

***Summary**

P	$= e^{-r_d T} K N(-d_-) - e^{-r_f T} Q(0) N(-d_+)$	P^f	$= e^{-r_f T} \cdot \frac{1}{K} N(-d_-^f) - e^{-r_d T} Q^f(0) N(-d_+^f)$ $= \tilde{E}\left[D(T)(\frac{Q(T)}{K} - 1)^+ \right] \frac{1}{Q(0)}$
$\frac{\partial P}{\partial Q(0)}$	$= -e^{-r_f T} N(-d_+)$	$\frac{\partial P^f}{\partial Q^f(0)}$	$= -e^{-r_d T} N(-d_+^f)$
$\frac{\partial P}{\partial K}$	$= e^{-r_d T} N(-d_-)$		
C	$= e^{-r_f T} Q(0) N(d_+) - e^{-r_d T} K N(d_-)$ $= Q(0) K P^f$ $= \tilde{E}\left[D(T)(Q(T) - 1)^+ \right]$		
$\frac{\partial C}{\partial Q(0)}$	$= e^{-r_f T} N(d_+)$		
$\frac{\partial C}{\partial K}$	$= -e^{-r_d T} N(d_-)$		

Exercise 4(Exchange rate forward delta hedge):

Long-term: Forward hedge is better than spot hedge

(i) Forward delta is the number of forward contract on foreign currency the trader should buy. Delta of the short call and long forward will be zero. Show the forward delta is $N(d_+)$

	Delta of Long Forward	Delta of short Call
Instruments	A: Price of the forward Contract (given in the question): $A = e^{-r_f T} Q(0) - e^{-r_d T} F$ Where F is the T-forward price of the foreign currency at time zero	$C = e^{-r_f T} Q(0) N(d_+) - e^{-r_d T} K N(d\_)$:Call on a unit of foreign currency
Delta	$\frac{\partial A}{\partial Q(0)} = e^{-r_f T}$ Forward Delta: Δ_F	$\frac{\partial C}{\partial Q(0)} = e^{-r_f T} N(d_+)$
	$\frac{\partial V_F}{\partial Q(0)} = \Delta_F \frac{\partial A}{\partial Q(0)} - \frac{\partial C}{\partial Q(0)} = 0$ $\Leftrightarrow \Delta_F \cdot \frac{\partial A}{\partial Q(0)} - e^{-r_f T} N(d_+) = 0$ $\Leftrightarrow \Delta_F = N(d_+)$	
Conclusion	Forward Delta $\Delta_F = N(d_+)$	

(ii) Show that V_F and V_F have the same vega and gamma

(ii.a) V_F and V_S have the same vega

	V_F	V_S
Formula	$= \Delta_F A - C$ $= \Delta_F \left(e^{-r_f T} Q(0) - e^{-r_d T} F \right) - C$	$= \Delta_S Q(0) - C$
Fixed term	$\Delta_F = N(d_+)$	$\Delta_S = e^{-r_f T} N(d_+)$
$\frac{\partial}{\partial \sigma}$	$\frac{\partial V_F}{\partial \sigma} = -\frac{\partial C}{\partial \sigma}$	$\frac{\partial V_S}{\partial \sigma} = -\frac{\partial C}{\partial \sigma}$
Conclusion	The vaga for V_F and V_S are the same	

(ii.b) V_F and V_S have the same gamma

	V_F	V_S
$\frac{\partial^2}{\partial^2 Q(0)}$	$\frac{\partial^2 V_F}{\partial^2 Q(0)} = -\frac{\partial^2 C}{\partial^2 Q(0)}$	$\frac{\partial^2 V_S}{\partial^2 Q(0)} = -\frac{\partial^2 C}{\partial^2 Q(0)}$
Conclusion	V_F and V_S have the same Gamma	

(iii) Show that the foreign rho of V_F, defined to be $\frac{\partial V_F}{\partial r^f}$, is equal o zero

V_F	$= \Delta_F \left(e^{-r_f T} Q(0) - e^{-r_d T} F \right) - C$
$\frac{\partial V_F}{\partial r^f}$	$= -T \Delta_F e^{-r_f T} Q(0) - \frac{\partial C}{\partial r_f}$ <Ref 1>

	$=-T\Delta_F e^{-r_f T}Q(0)-\left(-Te^{-r_f T}Q(0)\Delta_F\right)$ $=0$
<Ref 1> $\frac{\partial C}{\partial r_f}$	Recall that $C=e^{-r_f T}Q(0)N(d_+)-e^{-r_d T}KN(d_-)$ $\frac{\partial C}{\partial r_f}=\left[(-T)e^{-r_f T}Q(0)N(d_+)+e^{-r_f T}Q(0)N'(d_+)\frac{\partial d_+}{\partial r_f}\right]-e^{-r_d T}KN'(d_-)\frac{\partial d_-}{\partial r_f}$ Because $\frac{\partial d_+}{\partial r_f}=\frac{\partial d_-}{\partial r_f}$, $N(d_+)=\Delta_F$, $\frac{N'(d_+)}{N'(d_-)}=e^{-(r_d-r_f)T}\frac{K}{Q(0)}$ Therefore $\frac{\partial C}{\partial r_f}=-Te^{-r_f T}Q(0)\Delta_F+e^{-r_f T}Q(0)N'(d_+)\frac{\partial d_+}{\partial r_f}-e^{-r_d T}KN'(d_-)\frac{\partial d_-}{\partial r_f}$ $=-Te^{-r_f T}Q(0)\Delta_F+N'(d_-)\left[\frac{N'(d_+)}{N'(d_-)}e^{-r_f T}Q(0)\frac{\partial d_+}{\partial r_f}-e^{-r_d T}K\frac{\partial d_-}{\partial r_f}\right]$ $=-Te^{-r_f T}Q(0)\Delta_F+N'(d_-)\left[e^{-(r_d-r_f)T}\frac{K}{Q(0)}\cdot e^{-r_f T}Q(0)\frac{\partial d_+}{\partial r_f}-e^{-r_d T}K\frac{\partial d_-}{\partial r_f}\right]$ $=-Te^{-r_f T}Q(0)\Delta_F+N'(d_-)\left[e^{-(r_d)T}K\cdot\frac{\partial d_+}{\partial r_f}-e^{-r_d T}K\frac{\partial d_-}{\partial r_f}\right]$ $=-Te^{-r_f T}Q(0)\Delta_F$
Conclusion	$\frac{\partial V_F}{\partial r^f}=0$

(iv) Compute the domestic rho of V_F, defined to be $\frac{\partial V_F}{\partial r^d}$. Explain why it is generally considerably smaller in magnitude than $\frac{\partial V_S}{\partial r^d}$

(iv.a) rho for V_F

	V_F
Def	$=\Delta_F A-C$ $=\Delta_F\left(e^{-r_f T}Q(0)-e^{-r_d T}F\right)-C$
Where	$\Delta_F=N(d_+)$ $C=e^{-r_f T}Q(0)N(d_+)-e^{-r_d T}KN(d_-)$
$\dfrac{\partial V_F}{\partial r^d}$	$=\Delta_F\left(-(-T)e^{-r_d T}F\right)-\dfrac{\partial C}{\partial r_d}$, we remain Δ_F fixed <Ref 1> $=T\Delta_F e^{-r_d T}F-Te^{-r_d T}KN(d_-)$ $=Te^{-r_d T}\left(\Delta_F F-KN(d_-)\right)$ $=Te^{-r_d T}N(d_+)F-Te^{-r_d T}KN(d_-)$ (1)-(2)
<Ref 1> $\dfrac{\partial C}{\partial r_d}$	$=e^{-r_f T}Q(0)N'(d_+)\frac{\partial d_+}{\partial r_d}-\left(\left(-T\right)e^{-r_d T}KN(d_-)+e^{-r_d T}KN'(d_-)\frac{\partial d_-}{\partial r_d}\right)$ $=\frac{\partial d_-}{\partial r_d}\cdot N'(d_-)\left(e^{-r_f T}Q(0)\frac{N'(d_+)}{N'(d_-)}-e^{-r_d T}K\right)+Te^{-r_d T}KN(d_-)$ Because $\frac{\partial d_+}{\partial r_d}=\frac{\partial d_-}{\partial r_d}$, $N(d_+)=\Delta_F$, $\frac{N'(d_+)}{N'(d_-)}=e^{-(r_d-r_f)T}\frac{K}{Q(0)}$ $=\frac{\partial d_-}{\partial r_d}\cdot N'(d_-)\left(e^{-r_f T}Q(0)e^{-(r_d-r_f)T}\frac{K}{Q(0)}-e^{-r_d T}K\right)+Te^{-r_d T}KN(d_-)$

	$=\frac{\partial d_-}{\partial r_d}\cdot N'(d_-)\left(e^{-(r_d)T}K-e^{-r_dT}K\right)+Te^{-r_dT}KN(d_-)$ $=Te^{-r_dT}KN(d_-)$

(iv.b) rho for V_S

	V_S
Def	$=\Delta_S Q(0)-C$
Where	$\Delta_S=e^{-r_fT}N(d_+)$
$\frac{\partial}{\partial r_d}$	$\frac{\partial V_S}{\partial r_d}=-\frac{\partial C}{\partial r_d}$ we remain Δ_S fixed , so we shouldn't take partial derivative $=-Te^{-r_dT}KN(d_-)$

Compare $\frac{\partial V_F}{\partial r^d}$ vs $\frac{\partial V_S}{\partial r^d}$	The Magnitude of $\frac{\partial V_F}{\partial r^d}$ is generally smaller than $\frac{\partial V_S}{\partial r^d}$. Reason: we find that $\frac{\partial V_S}{\partial r^d}$ is negative. The extra positive term $\frac{\partial V_F}{\partial r^d}=Te^{-r_dT}N(d_+)F\ -Te^{-r_dT}KN(d_-)$ will offset the magnitude of itself. In practice, F tends to be close to K and $N(d_+)\,\&\,N(d_-)$ are generally of the same magnitude. So we can conclude that The Magnitude of $\frac{\partial V_F}{\partial r^d}$ is generally smaller than $\frac{\partial V_S}{\partial r^d}$ From $\frac{\partial V_F}{\partial r^f}=0$ and result above, VF is less sensitive to interest rate than spot hedge. For long-dated option, this indeed makes the forward hedge more effective. However, spot market are normally more liquid than forward market (bid-ask spread are smaller.) A common practice is to put on a spot hedge at the beginning of the trade and then replace it by a forward hedge if the option has high interest rate sensitivity. This can be done with swap.