Advanced Modeling

HW3

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Exercise 1:

1.1: Let N be a Poisson process with intensity λ>0 with respect to $F_t = \sigma(N_s)_{0\le s\le t}$ and let B be a BM, both defined on the same probability space. Define the filtrations. $G_t = \sigma(N_S, B_S)_{0\le s\le t}$, $H_t = \sigma(N_1, N_S)_{0\le s\le t}$.

What is N's intensity with respect to these two filtrations.

- **With respect to filtration: $G_t = \sigma(N_S, B_S)_{0\le s\le t}$**

From L5, page 17, we get	$M_t = N_t - \lambda t$, whose filtration will be Poisson $F_t = \sigma(N_u)_{0\le u\le t}$
Hence, we can get	λ is $N's$ F-Intensity
Because (1)N and B are independent (2) $G_t = \sigma(N_S, B_S)_{0\le s\le t}$	λ is $N's$ G-Intensity
Conclusion	λ is $N's$ G-Intensity

- **With respect to filtration: $H_t = \sigma(N_1, N_S)_{0\le s\le t}$**

Refer L6, page 9 , we use the same reasoning for Brownian Bridge and claim	$E[M_t \mid H_S] = M_S + \frac{M_1 - M_S}{1-S}(t-s)$ $E[N_t \mid H_S] = N_S + \frac{N_1 - N_S}{1-S}(t-s)$, N is not a M'g under H
We mimic the logic from L6 page 10, we define	$\tilde{N}_t = N_t - \int_0^t \frac{N_1 - N_u}{1-u} du$, $\tilde{N}_S = N_S - \int_0^S \frac{N_1 - N_u}{1-u} du$,
Take conditional expectation	$E\left[\tilde{N}_t - \tilde{N}_s \mid H_S\right] = E\left[N_t - N_s - \int_S^t \frac{N_1 - N_u}{1-u} du \mid H_S\right]$ $= \frac{N_t - N_s}{1-S}(t-s) - \int_S^t \frac{N_1 - E[N_u \mid H_S]}{1-u} du$ $= \frac{N_t - N_s}{1-S}(t-s) - \int_S^t \frac{N_1 - N_S - \frac{N_1 - N_S}{1-S}(u-s)}{1-u} du$ $= \frac{N_t - N_s}{1-S}(t-s) - \int_S^t \frac{(1-s)(N_1 - N_S) - (N_1 - N_s)(u-s)}{1-u} du$ $= \frac{N_t - N_s}{1-S}(t-s) - \int_S^t \frac{(1-u)(N_1 - N_S)}{1-u(1-s)} du = 0$ $= \frac{N_t - N_s}{1-S}(t-s) - \int_S^t \frac{(N_1 - N_S)}{(1-s)} du = 0$
Conclusion	N's H- intensity is given by $\frac{N_1 - N_s}{1-s}$

1.2: Assume that $N_t^{(i)}, i=1,2,....I$ is a family of independent Poisson process with constant intensities $\lambda^{(i)}$ generating the filtration Ft. Define the stopping time $\tau = \inf_t \left\{ \sum_{i=1}^{I} N_t^{(i)} = 1 \right\}$. What is the intensity of the {0,1}-valued process $1_{\{\tau \le t\}}$. Make sure your intensity vanish after τ

From the hint in BB, we approach the following $P(\tau > T \mid F_t)$	$= P\left(N_T^{(1)} = 0, \ldots\ldots, N_T^{(I)} = 0 \mid F_t\right)$ $= \prod_{i=1}^{I} P\left(N_T^{(i)} = 0 \mid F_t\right)$ by means of independence $= \prod_{i=1}^{I} \left(\exp(-\lambda^{(i)} \cdot T) \mid F_t\right) \cdot 1_{N_t^{(i)}=0}$ $= \prod_{i=1}^{I} \exp(-\lambda^{(i)} \cdot (T-t)) \cdot 1_{N_t^{(i)}=0}$ $= \exp\left(-(T-t) * \sum_{i=1}^{I} \lambda^{(i)}\right) \cdot 1_{N_t=0}$
We can find intensity like Shreve's book p478	$\lambda_t = -\frac{\partial}{\partial T} P(\tau > T \mid F_t)\vert_{T=t}$, where intensity vanishes after time τ $= \sum_{i=1}^{I} \lambda^{(i)} 1_{N_t=0}$
Conclusion	The intensity of the {0,1}-valued process $1_{\{\tau \le t\}}$ would be $\sum_{i=1}^{I} \lambda^{(i)}$

1.3: Under what measure. is N also has a Poisson process (constant intensity).

	Comment
$\frac{dQ^A}{dP} = \exp\left(\int_0^t \psi_s dB_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right)$	Under Q^A, the intensity is λ. Therefore, N is still Poisson
$\frac{dQ^B}{dP} = \exp\left(-\lambda \int_0^t (\phi_s - 1) ds\right) \prod_{n=1}^{N_t} \phi_{\tau_n}$	Under Q^B, the intensity becomes ϕ. Therefore N is NOT Poisson
$\frac{dQ^C}{dP} = \exp\left(t(\lambda - 1) - N_t \log(\lambda)\right)$	Under Q^C, the intensity becomes 1. Therefore N is Poisson
$\frac{dQ^D}{dP} = \frac{dQ^A}{dP}\frac{dQ^B}{dP}$	Under Q^D, due to the presence of Q^B. Therefore N is NOT Poisson
$\frac{dQ^E}{dP} = \frac{dQ^A}{dP}\frac{dQ^C}{dP}$	Under Q^E, the intensity becomes 1. Therefore N is Poisson

1.4: Let N be a Poisson Process and Let B be a BM. Let τ denote the first jump time for N. Find the density of the random variable $X \triangleq B_\tau$ (Hint: compute X's characteristic function and then use the inversion with prob 4.1 in HW1)

Characteristic Function of B_τ	$u \in \mathbb{R}, \varphi_{B_\tau}(u) = E\left[e^{iuB_\tau}\right]$
Define $f(t, B_t)$=Mt	$M_t = e^{iuB_t + \frac{1}{2}u^2 t}$

Verify that $f(t,B_t)$ is Martingale (apply Ito's lemma). Temporary replace $X \triangleq B_t$ here	$df(t,X_t)=f_t dt+f_x dX_t+\frac{1}{2}f_{xx}d\langle X\rangle_t$ $dM_t=\frac{\partial M_t}{\partial t}+\frac{\partial M_t}{\partial B_t}+\frac{1}{2}\left(\frac{\partial M_t}{\partial B_t}\right)^2$
	$f_t=\frac{1}{2}u^2\cdot e^{iuB_t+\frac{1}{2}u^2t}=\frac{1}{2}u^2\cdot f$ $f_x=iu\cdot e^{iuB_t+\frac{1}{2}u^2t}=iu\cdot f$ $f_{xx}=(iu)^2\cdot e^{iuB_t+\frac{1}{2}u^2t}=(iu)^2\cdot f$
	$df(t,X_t)=f\cdot\left(\frac{1}{2}u^2dt+iudX_t+\frac{1}{2}(iu)^2dt\right)$ $df(t,X_t)=f\cdot\left(iudX_t\right)$ $=e^{iuB_t+\frac{1}{2}u^2t}\cdot\left(iudB_t\right)\rightarrow Martingale$
From Doob's optional sampling Them (L5, page 3)	If Mt is Martingale and τ is any stopping time. Then $M_{t\wedge\tau}$ is also a Martingale $E[M_{t\wedge\tau}]=M_0=1$
Take limitation	$1=\lim_{t\to\infty}E[M_{t\wedge\tau}]$ $=\lim_{t\to\infty}E\left[e^{iuB_{t\wedge\tau}+\frac{1}{2}u^2(t\wedge\tau)}\right]$ $=E\left[\lim_{t\to\infty}e^{iuB_{t\wedge\tau}+\frac{1}{2}u^2(t\wedge\tau)}\right]$ use dominated convergence Thm
Two possible scenario of $\lim_{t\to\infty}e^{iuB_{t\wedge\tau}+\frac{1}{2}u^2(t\wedge\tau)}$	(1) when $\tau<\infty$: $\lim_{t\to\infty}M_{t\wedge\tau}=e^{iuB_\tau+\frac{1}{2}u^2\tau}$ (2) when $\tau=\infty$: $\lim_{t\to\infty}M_{t\wedge\tau}\le e^{iuB_t+\frac{1}{2}u^2t}\xrightarrow{t\to\infty}0$
Combine result of two scenario	$\lim_{t\to\infty}M_{t\wedge\tau}=e^{iuB_\tau+\frac{1}{2}u^2\tau}\cdot 1_{\{\tau<\infty\}}+0\cdot 1_{\{\tau=\infty\}}=e^{iuB_\tau+\frac{1}{2}u^2\tau}\cdot 1_{\{\tau<\infty\}}$
Take expectation	$1=E\left[e^{iuB_\tau+\frac{1}{2}u^2\tau}\cdot 1_{\{\tau<\infty\}}\right]$, we want the first jump, so $1=P\left(\tau<\infty\right)$ $1=E\left[e^{iuB_\tau+\frac{1}{2}u^2\tau}\right]$
Time $e^{-\frac{1}{2}u^2\tau}$ on both sides (deterministic function)	$e^{-\frac{1}{2}u^2\tau}=E\left[e^{iuB_\tau}\right]$ We know the characteristic function for standard normal is $\varphi(u)=e^{-\frac{1}{2}u^2\sigma^2}=E\left[e^{iux}\right]$
We also know that $B_\tau\sim N(0,\sqrt{\tau})$	$f(x)=\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{x^2}{2\tau}}$

1.5: Let N be a Poisson Process and Let B be a BM. Let $F_t=\sigma\left(N_u,B_u\right)_{0\le u\le t}$, Define the process $W_t\triangleq\int_0^t(-1)^{N_u}dB_u, t\ge 0$. Are B and N independent? Are W and N independent?

- **Bt and Nt are independent**

(From Shreve's book: thm 11.2.4) If N is Poisson with intensity λ	$N_t-\lambda t$ is a Martingale

Let $u_1 = \log(\sigma+1)$, then $e^{u_1} - 1 = \sigma$	We want both terms below to be martingale and Let them Xt $N_t \cdot u_1 - \lambda\left(e^{u_1}-1\right)t + B_t \cdot u_2 - \frac{1}{2}u_2^2 t = X_t$
Goal: Show e^{X_t} is a Martingale. Use Ito's lemma and get	$e^{X_t} = 1 + \int_0^t e^{X_u} dX_u^2 - \frac{1}{2}\int_0^t e^{X_u} d\left[X\right]_u^2 + \sum_{n=1}^{N_t}(e^{X_{T_n}} - e^{X_{T_n-}})$ (1)
$\Delta N_t = 1$: we will get contribution from $1 \cdot u_1$, so	$X_t = X_{t-} + u_1$ Take Exponential and get $e^{X_t} = e^{X_{t-}} e^{u_1}$ (2)
Continuous component of Xt	$X_t^C = -\lambda\left(e^{u_1}-1\right)t + B_t \cdot u_2 - \frac{1}{2}u_2^2 t$ $dX_t^C = u_2 dB_t - \frac{1}{2}u_2^2 dt - \lambda\left(e^{u_1}-1\right)dt$ (3)
We can rewrite (2) and combine (3), we get $e^{X_t} =$	$= 1 + \int_0^t e^{X_S}\left\{u_2 dB_S - \frac{1}{2}u_2^2 ds - \lambda\left(e^{u_1}-1\right)ds\right\} + \frac{1}{2}\int_0^t e^{X_S} u_2^2 ds$ $+ \sum_{n=1}^{N_t} e^{X_{T_n-}}(e^{u_1}-1)$
	$= 1 + \int_0^t e^{X_S} u_2 dB_S + \frac{1}{2}\int_0^t e^{X_{S-}}(e^{u_1}-1)(dN_S - \lambda ds)$. This is Martingale
Right now we know e^{X_t} is Martingale. Then	$1 = E\left[e^{X_t}\right]$ $1 = E\left[e^{u_1 N_t - \lambda\left(e^{u_1}-1\right)t + u_2 B_t - \frac{1}{2}u_2^2 t}\right]$
Move out the deterministic part	$e^{\lambda t\left(e^{u_1}-1\right)} e^{\frac{1}{2}u_2^2 t} = E\left[e^{u_1 N_t + u_2 B_t}\right]$
Conclusion	From this Generating Function Method, (we have exactly product of generating function). We can conclude Nt and Bt are independent.

- **Wt and Nt are independent**

$W_t \triangleq \int_0^t (-1)^{N_u} dB_u, t \geq 0$	$\langle W \rangle_t \triangleq \int_0^t \left[(-1)^{N_u}\right]^2 du$ $= \int_0^t 1 du$ $= t$
Conclusion	That means that Wt is a Brownian Motion And we just proved that If W is BM and Nt is Poisson Process. They will be independent.

Exercise 2 (Jump Diffusion Models):

2.1: Express $\phi_{Q_t}(u) = E^Q[e^{iuQ_t}], u \in \mathbb{R}, t \in [0, \infty]$

$\phi_{Q_t}(u)$	$= E^Q[e^{iuQ_t}]$ $= E^Q\left[e\left\{iu\sum_{i=1}^{N_t} Y^{(i)}\right\}\right]$

	$= E^Q\left[E^Q\left[e\left\{iu\sum_{i=1}^{N_t}Y^{(i)}\right\} \mid N_t\right]\right]$ $= E^Q\left[E^Q\left[e\left\{iu\sum_{i=1}^{N_t}Y^{(i)}\right\}\right]\Big|_{x=N_t}\right]$ $= E^Q\left[\prod_{i=1}^{x} E^Q\left[e^{iuY^{(i)}}\right]\Big|_{x=N_t}\right]$ $= E^Q\left[\varphi_Y(u)^x \mid_{x=N_t}\right]$ $= E^Q\left[\varphi_Y(u)^{N_t}\right]$ $= \sum_{x=0}^{\infty} \varphi_Y(u)^x \cdot e^{-\lambda t}\frac{\lambda t^x}{x!}$ $= e^{-\lambda t} \cdot e^{\lambda t \varphi_Y(u)}$ $= e^{-\lambda t}e^{\lambda t \varphi_Y(u)}$ $= e^{\lambda t\left(\varphi_Y(u)-1\right)}$
Conclusion	$\phi_{Q_t}(u) = e^{\lambda t\left(\varphi_Y^{(u)}-1\right)}$

2.2:Find a predictable process λ^M such that $M_t = Q_t - \int_0^t \lambda_u^M du$ is a Q Martingale

From text book thm 11.3.1: To get Martingale from compound Poisson Process: First: Get the mean of compound process	$E[Q_t] = \sum_{k=0}^{\infty} E\left[\sum_{i=1}^{k}Y^{(i)} \mid N_t = k\right]P\{N_t = k\}$
We know that $\beta = E^Q\left[Y^{(i)}\right]$	$= \sum_{k=0}^{\infty} \beta k\frac{(\lambda t)^k}{k!}e^{-\lambda t}$ $= \beta\lambda t e^{-\lambda t}\sum_{k=0}^{\infty} \frac{(\lambda t)^{k-1}}{k!}$ $= \beta\lambda t$
Thm 11.3.1	$Q_t - \beta\lambda t$ is a Martingale
Verify	$E[Q_t - \beta\lambda t \mid F_S] = E[Q_t - Q_S \mid F_S] + Q_S - \beta\lambda t$ $= \beta\lambda(t-s) + Q_S - \beta\lambda t$ $= Q_S - \beta\lambda s$ so this is indeed a Martingale
Conclusion	The whole Compensator is $\beta\lambda t$, its corresponding integrand with integration from 0 to t should be $\lambda_u^M = \beta\lambda$

2.3: The stock price dynamics are defined by $dS_t = -S_{t-}\beta\lambda dt + S_{t-}(\sigma_t dB_t + dQ_t)$. Where σ_t is strictly positive adapted process and B is BM under Q. Explain why S is a Q martingale and explain what happens with S at the i'th jump time for N (what happens when N jumps from i-1 to i?

- **Why S is a Q Martingale.**

Reasons	(1) Since B is Brownian Motion under Q , thus it is a Martingale
	(2) $\int_0^t -S_{t-}\beta\lambda dt + S_{t-}(dQ_t)$ $S_t = \int_0^t S_{u-}(dQ_u - \beta\lambda du)$ is Martingale
Conclusion	S is a Q Martingale

- **What happens to S at ith jump time for N**

At time $\tau^{(i)}$	Given the dynamics (0,1), when N jumps form i-1 to I ($S_{\tau^{(i)}\_}$ to $S_{\tau^{(i)}}$,). $S_{\tau^{(i)}}$ can be obtained by scaling $S_{\tau^{(i)}\_}$ with $Y^{(i)}$. We will get $S_{\tau^{(i)}} = S_{\tau^{(i)}\_}(1+Y^{(i)})$
Limitation about $Y^{(i)}$	To ensure S remains strictly positive, we need $Y^{(i)}$ to be supported on (-1, ∞). That is the reason why the problem let $(Y^{(i)})_{i=1}^{\infty}$ be a family iid with common density f_Y on (-1,∞) $Y^{(i)}$ is the relative jump for the ith jump

2.4: Solve for S in (0,1):

(a) Find explicit expression for St

From 2.2	$M_t = Q_t - \int_0^t \lambda_u^Q du = \sum_{i=1}^{N_t} Y^{(i)} - \beta\lambda t$ (1)
M's continuous part	$M_t^C = -\beta\lambda t$ (2)
Recall that	$dS_t = -S_{t-}\beta\lambda dt + S_{t-}(\sigma_t dB_t + dQ_t)$ (3)
Combine (1)(2)(3)	$dS_t = S_{t-}(\sigma_t dB_t + dQ_t - \beta\lambda dt)$ $dS_t = S_{t-}(\sigma_t dB_t + dM_t)$
Conclusion	$dS_t = S_{t-}(\sigma_t dB_t + dM_t)$

(b)Find dynamics for the R.P. $X_t = \log(\frac{S_t}{S_0})$**. Find a,b, and** $\left(d^{(i)}\right)_{i=1}^{\infty}$ **s.t.** $X_t = \int_0^t a_u du + \int_0^t b_u dB_u + \sum_{i=1}^{N_t} d^{(i)}$

We can let	$X_t = X_0 + I_t + R_t + J_t$
From 2.4.(a) we can get	$dS_t = S_{t-}(\sigma_t dB_t + dM_t)$ $S_t = S_0 \exp\left(\int_0^t \sigma_u dB_u + \int_0^t dM_u\right)$ **(1)**
By 2.3	$J_t = \sum_{i=1}^{N_t} \log(1+Y^{(i)})$ **(2)**
M_t	$= Q_t - \int_0^t \lambda_u^Q du$ $= \sum_{i=1}^{N_t} Y^{(i)} - \beta\lambda t$, where we find its continuous part is $-\beta\lambda t$ (3)

From (1)(2)(3) So we can get	$X_t = \int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du - \beta\lambda t + \sum_{i=1}^{N_t}\log(1+Y^{(i)})$ $X_t = -\frac{1}{2}\int_0^t \sigma_u^2 du - \beta\lambda t + \int_0^t \sigma_u dB_u + \sum_{i=1}^{N_t}\log(1+Y^{(i)})$
Conclusion	$X_t = \int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du - \beta\lambda t + \sum_{i=1}^{N_t}\log(1+Y^{(i)})$

2.5: Refer the expression $c(t) = E^Q\left[\left(S_T - K\right)^+ \mid F_t\right], t \in [0,T]$ Construct a measure $\tilde{Q}$ such

$c(0) = S_0\tilde{Q}(S_T \ge K) - KQ\left(S_T \ge K\right)$

Based definition above	$\frac{d\tilde{Q}}{dQ} = \frac{S_T}{S_0} = Z_T\ ; Z_t = \frac{S_t}{S_0}, t \in [0,T]\ ; Z_t > 0$, Zt is defined under Q.
$c(t, S_t, \sigma_t)$:	$= E^Q\left[\left(S_T - K\right)^+ \mid F_t\right]$ is Q Martingale $= E^Q\left[\left(S_T - K\right)\cdot 1_{\{S_T \ge K\}} \mid F_t\right]$ $= E^Q\left[\frac{Z_T}{Z_t} S_t \cdot 1_{\{S_T \ge K\}} \mid F_t\right] - KE^Q\left[1_{\{S_T \ge K\}} \mid F_t\right]$ $= S_t \cdot E^{\tilde{Q}}\left[1_{\{S_T \ge K\}} \mid F_t\right] - KQ\left(S_T \ge K \mid F_t\right)$ $= S_t \cdot \tilde{Q}\left(S_T \ge K \mid F_t\right) - KQ\left(S_T \ge K \mid F_t\right)$
t=0 just like lecture 4 slide 14 (we can remove the conditional expectation and get the following)	$S_0 \cdot \tilde{Q}\left(S_T \ge K\right) - KQ\left(S_T \ge K\right)$

2.6: Find the predicable process ϕ and $\tilde{\lambda}^M$ such that $\tilde{B}_t \triangleq B_t - \int_0^t \phi_u du$, $\tilde{M}_t \triangleq Q_t - \int_0^t \tilde{\lambda}_u^M du$ are both Martingales under the measure $\tilde{Q}$ defined in the previous question.

(a) What is N's intensity under $\tilde{Q}$?

(b) Is N still a Poisson process under $\tilde{Q}$?

(c) What is the density function $\tilde{f}_Y$ for Y^i ,i=1,2….under $\tilde{Q}$?

PART I: Combine thm 11.6.5 & 11.6.11 on page 498. The Radon-Nikodym derivative process Z(t) can be written as	$Z(t) = \exp\left\{\sum_{m=1}^{M}(\lambda_m - \tilde{\lambda}_m)\right\} \cdot \prod_{m=1}^{M}\left(\frac{\tilde{\lambda}\tilde{p}\left(y_m\right)}{\lambda p\left(y_m\right)}\right)^{N_m(t)}$ $= e^{\left(\lambda - \tilde{\lambda}\right)t} \cdot \prod_{i=1}^{N_t}\frac{\tilde{\lambda}\tilde{p}\left(Y^i\right)}{\lambda p\left(Y^i\right)}$

(Above) it suggests that if Y1,Y2….are not discrete but instead have a common density f(y), then we can change the measure so that Q(t) has intensity $\tilde{\lambda}$ and Y1,Y2…have a different density $\tilde{f}(y)$ by using the Radon-Nikodym derivative process	$Z(t)=e^{(\lambda-\tilde{\lambda})t}\cdot\prod_{i=1}^{N_t}\frac{\tilde{\lambda}\tilde{f}_Y(Y^i)}{\lambda f_Y(Y^i)}$
So given we can write change-of-measure Martingale as	$Z(t)=e^{(\lambda-\tilde{\lambda})t}\cdot\prod_{i=1}^{N_t}\frac{\tilde{\lambda}\tilde{f}(Y^i)}{\lambda f(Y^i)}$ e (1) If we match what we have in 2.4, we can get $\phi_t=\sigma_t$
PART II: We want to match the jump part of Z & Jump part from 2.4,	$e^{(M_t^c)}\cdot\prod_{0\le s\le t}(1+\Delta M_s)=e^{(-\beta\lambda t)}\cdot\prod_{i=1}^{N_t}(1+Y^{(i)})$ (2)
From (1) and (2) , we can conclude that	$\lambda-\tilde{\lambda}=-\beta\lambda$ (3) $\frac{\tilde{\lambda}\tilde{f}_Y(y)}{\lambda f_Y(y)}=1+y$ (4)
Combine (3)and (4), we can get	$\tilde{\lambda}=\lambda+\beta\lambda=\lambda(1+\beta)$ (5) $\tilde{f}_Y(y)=\frac{(1+y)\lambda f_Y(y)}{\tilde{\lambda}}=\frac{(1+y)f_Y(y)}{1+\beta}$ (6)
Verify (5) to see whether that is valid density	$\int_{-1}^{\infty}\tilde{f}_Y(y)dy=1$
Conclusion	
(a) What is N's intensity under $\tilde{Q}$?	From (5), we get $\lambda(1+\beta)$
(b) Is N still a Poisson process under $\tilde{Q}$?	YES, N still a Poisson process under $\tilde{Q}$. Where we get (1) $\phi_t=\sigma_t$ (2) $\tilde{\lambda}_t^M=\tilde{\beta}\tilde{\lambda}t$,where $\tilde{\beta}=E^{\tilde{Q}}\left[Y^{(1)}\right]$
(c) What is density function $\hat{f}_Y$ for Y^i ,i=1,2….under $\tilde{Q}$?	From (5), we get $\tilde{f}_Y=\frac{(1+y)f_Y(y)}{1+\beta}$

2.7:

(a) Show Q-characteristic fcn: $\phi_{X_T}(u)=\exp\left(iu\left(-\frac{1}{2}\sigma^2T-\beta\lambda T\right)-\frac{1}{2}u^2\sigma^2T+\lambda T\left(\exp(iu\alpha-\frac{1}{2}u^2\delta^2)-1\right)\right)$

(b) Show $\tilde{Q}$-characteristic fcn:

$$\tilde{\phi}_{X_T}(u)=\exp\left(iu\left(\frac{1}{2}\sigma^2T-\beta\lambda T\right)-\frac{1}{2}u^2\sigma^2T\right)\times\exp\left\{\lambda T\left(\exp\left\{i(u-i)\alpha-\frac{1}{2}\delta^2(u-i)^2\right\}-1-\beta\right)\right\}$$

(a) Show $\phi_{X_T}(u)$

We know from 2.4 that	$X_t=\int_0^t\sigma_u dB_u-\frac{1}{2}\int_0^t\sigma_u^2du-\beta\lambda t+\sum_{i=1}^{N_t}\log(1+Y^{(i)})$
We can rewrite X as	$X=X^{cont}+X^{jump}$

	Where $X_t^{cont} = \int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du$, $X_t^{jump} = -\beta\lambda t + \sum_{i=1}^{N_t} \log(1+Y^{(i)})$
By independence	$\phi_X(u) = E^Q\left[\exp\left(iuX_T^{cont}\right)\right] E^Q\left[\exp\left(iuX_T^{jump}\right)\right]$

❖ PART I: Continuous Part

$E^Q\left[\exp\left(iuX_T^{cont}\right)\right]$	$E^Q\left[\exp\left(iuX_T^{cont}\right)\right] = \exp(-\frac{1}{2}iu\sigma^2 T - \frac{1}{2}u^2\sigma^2 T)$

❖ PART II: Jump Part

$E^Q\left[\exp\left(iuX_T^{jump}\right)\right]$	$= \exp(-iu\beta\lambda T)\cdot E^Q[\exp(iu\sum_{i=1}^{N_T}(\alpha+\delta\varepsilon^{(i)}))]$ $\langle ref1 \rangle$ $= \exp(-iu\beta\lambda T)\cdot \exp(\lambda T(\phi_{\alpha+\delta\varepsilon^{(i)}}(u)-1))$ $\langle ref2 \rangle$ $= \exp(-iu\beta\lambda T)\cdot \exp(\lambda T(\exp(iu\alpha - \frac{1}{2}u^2\delta^2)-1))$
$\langle ref1 \rangle$	$\beta = E^Q[Y^{(i)}]$ $= E^Q\left[\exp(\alpha+\delta\varepsilon^{(i)})-1\right]$ $= \exp(\alpha+\frac{1}{2}\delta^2)-1$
$\langle ref2 \rangle$	$\phi_{\alpha+\delta\varepsilon}(u) = E^Q\left[\exp(iu(\alpha+\delta\varepsilon)\right]$ $= \exp\left(iu\alpha - \frac{1}{2}u^2\delta^2\right)$ $E^Q[\exp(iu\sum_{i=1}^{N_T}(\alpha+\delta\varepsilon))] = e^{\left(\lambda T\left(\phi_{\alpha+\delta\varepsilon}(u)-1\right)\right)}$ $= e^{\left(\lambda T\left(\exp\left(iu\alpha-\frac{1}{2}u^2\delta^2\right)-1\right)\right)}$

❖ Combine Part I & Part II

$\phi_{X_T}(u)$	$= E^Q\left[\exp\left(iuX_T^{cont}\right)\right] E^Q\left[\exp\left(iuX_T^{jump}\right)\right]$ $= \exp\left(-\frac{1}{2}iu\sigma^2 T - \frac{1}{2}u^2\sigma^2 T - iu\beta\lambda T + \lambda T\left(\exp\left(iu\alpha - \frac{1}{2}u^2\delta^2\right)-1\right)\right)$

(b) Show $\tilde{\phi}_{X_T}(u)$

We can still can X as	$X = X^{cont} + X^{jump}$ Where $X_t^{cont} = \sigma\tilde{B}_t + \frac{1}{2}\sigma^2 t$,
By independence	$\tilde{\phi}_X(u) = E^{\tilde{Q}}\left[\exp\left(iuX_T^{cont}\right)\right] E^{\tilde{Q}}\left[\exp\left(iuX_T^{jump}\right)\right]$

❖ PART I: Continuous Part

$E^{\tilde{Q}}\left[\exp\left(iuX_T^{cont}\right)\right]$	$= \exp(\frac{1}{2}iu\sigma^2 T - \frac{1}{2}u^2\sigma^2 T)$

❖ PART II: Jump Part

$E^{\tilde{Q}}\left[\exp\left(iuX_T^{jump}\right)\right]$	$=\exp(-iu\beta\lambda T)\cdot E^{\tilde{Q}}\left[\exp\left(iu\sum_{i=1}^{N_T}\log(Y^{(i)}+1)\right)\right]$ $\langle ref1\rangle\langle ref2\rangle$ $=\exp(-iu\beta\lambda T)\cdot\exp(\tilde{\lambda}T(\tilde{\phi}_{\alpha+\delta\varepsilon^{(i)}}(u)-1))$ $\langle ref4\rangle$ $=\exp(-iu\beta\lambda T)\cdot\exp(\lambda T(\phi_{\alpha+\delta\varepsilon^{(i)}}(u-i)-1-\beta))$ $\langle ref3\rangle$ $=\exp\left(iu(-\beta\lambda T)+\lambda T\left(\exp\left(i(u-i)\alpha-\frac{1}{2}\delta^2(u-i)^2\right)-1-\beta\right)\right)$ $\langle ref5\rangle$
$\langle ref1\rangle$	From result of 2.1, we get $\phi_{Q_t}(u)=e^{\lambda t\left(\varphi_Y^{(u)}-1\right)}$ From result of 2.6 (a), we know that N has intensity under $\tilde{Q}$ is $\lambda(1+\beta)$ From result of 2.6 (c): $\tilde{f}_Y=\dfrac{(1+y)f_Y(y)}{1+\beta}$ If Our goal is get $\tilde{\phi}_{\alpha+\delta\varepsilon^{(i)}}(u)=E^{\tilde{Q}}\left[\exp(iu(\log(Y^{(i)}-1))\right]$
$\langle ref2\rangle$: $f_Y(y)$	We actually can get $f(y)=\frac{\partial}{\partial y}Q\left(Y^{(i)}\le y\right)$ $=\frac{\partial}{\partial y}Q\left(\alpha+\delta\varepsilon^{(i)}\le\log(y+1)\right)$ $=k\left(\frac{\log(y+1)-\alpha}{\delta}\right)\frac{1}{\delta(1+y)}$, where $k(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$
$\langle ref3\rangle$: combine ref 1,2 for $\tilde{\phi}_{\alpha+\delta\varepsilon^{(i)}}(u)$	$=\int_{-1}^{\infty}\exp(iu\log(y+1))\tilde{f}(y)dy$ $=\int_{-1}^{\infty}\exp(iu\log(y+1))\dfrac{(1+y)f_Y(y)}{1+\beta}dy$ $=\dfrac{1}{1+\beta}\int_{-1}^{\infty}\exp(iu\log(y+1))(1+y)f_Y(y)dy$ $=\dfrac{1}{1+\beta}\int_{-1}^{\infty}\exp(iu\log(y+1))(1+y)\left[k\left(\frac{\log(y+1)-\alpha}{\delta}\right)\frac{1}{\delta(1+y)}\right]dy$ $=\dfrac{1}{\delta}\left(\dfrac{1}{1+\beta}\right)\int_{-1}^{\infty}\exp(iu\log(y+1))\left[k\left(\frac{\log(y+1)-\alpha}{\delta}\right)\right]dy$ $=\dfrac{1}{\delta(1+\beta)}\int_{-\infty}^{\infty}\exp(iux)k\left(\frac{x-\alpha}{\delta}\right)\exp(x)dx$, where we replay log(y+1)=x → dy=exp(x)dx $=\dfrac{1}{(1+\beta)}\phi_{\alpha+\delta\varepsilon^{(i)}}(u-i)$
$\langle ref4\rangle$	$E^{\tilde{Q}}\left[\exp\left(iu\sum_{i=1}^{N_T}\log(Y^{(i)}+1)\right)\right]$ $=\exp(\tilde{\lambda}T(\tilde{\phi}_{\alpha+\delta\varepsilon^{(i)}}(u)-1))$ $=\exp\left(\lambda(1+\beta)T\left(\dfrac{1}{(1+\beta)}\phi_{\alpha+\delta\varepsilon^{(i)}}(u-i)-1\right)\right)$

	: we replace $\tilde{\lambda} = \lambda(1+\beta)$, $\tilde{\phi}_{\alpha+\delta\varepsilon^{(i)}}(u) = \frac{1}{(1+\beta)}\phi_{\alpha+\delta\varepsilon^{(i)}}(u-i)$ $= \exp\left(\lambda T\left(\phi_{\alpha+\delta\varepsilon^{(i)}}(u-i)-1-\beta\right)\right)$
$\langle ref\,5\rangle$	$E^Q[\exp(iu\sum_{i=1}^{N_T}(\alpha+\delta\varepsilon))] = e^{\left(\lambda T\left(\phi_{\alpha+\delta\varepsilon}(u)-1\right)\right)}$ $= e^{\left(\lambda T\left(\exp\left(iu\alpha-\frac{1}{2}u^2\delta^2\right)-1\right)\right)}$

- Combine Part I & Part II

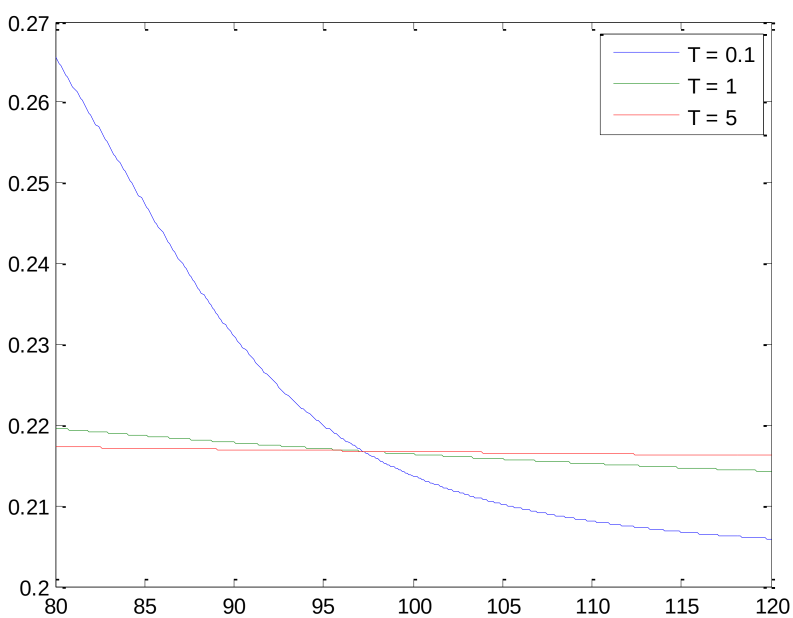
$\tilde{\phi}_{X_T}(u)$ is indeed	$= \exp\left(iu\left(\frac{1}{2}\sigma^2T-\beta\lambda T\right)-\frac{1}{2}u^2\sigma^2T\right)\times\exp\left\{\lambda T\left(\exp\left\{i(u-i)\alpha-\frac{1}{2}\delta^2(u-i)^2\right\}-1-\beta\right)\right\}$

2.8: Report ATM call price for $T \in \{0.1, 0.5, 1, 2, 5\}$ for Merton Model

	Solid	Dashed	Long-Dashed
T = 0.1	2.7578	2.6567	2.6955
T = 0.5	6.3299	5.9556	6.0877
T = 1	8.9849	8.4183	8.6154
T = 2	12.7102	11.8857	12.1707
T = 5	19.9977	18.6911	19.1416

- **Code is attached in the Appendix Section**

2.9: Create a plot with K on the horizontal axis and σ_{BS} on the vertical axis and plot the value of $C_{BS}(\sigma_{BS})$ agrees with the outcome of Merton's model



2.10: Compute X_T using Heston model with jump process under both Q and $\tilde{Q}$

Gatheral's book P66 gives the equation:

$$\phi_T(u) = \exp\left(C(u,T)\bar{v} + D(u,T)v\right) \cdot \exp\left(\psi(u)T\right)$$

Where

$$\psi(u) = -\lambda_J iu\left(e^{\alpha+\delta^2/2} - 1\right) + \lambda_J\left(e^{iu\alpha - u^2\delta^2/2} - 1\right)$$

And C, D are noted in Gatheral's book.

Alternatively, we can deduct the equations are the same as the ones we deduct below.

$$\phi_X(u) = E^Q\left[\exp\left(iuX_T^{cont}\right)\right] E^Q\left[\exp\left(iuX_T^{jump}\right)\right]$$

$$\tilde{\phi}_X(u) = E^{\tilde{Q}}\left[\exp\left(iuX_T^{cont}\right)\right] E^{\tilde{Q}}\left[\exp\left(iuX_T^{jump}\right)\right]$$

In HW2 we have

	Under Q	**Under $\tilde{Q}$:**
α	$= -\frac{u^2}{2} - \frac{iu}{2}$	$= -\frac{u^2}{2} + \frac{iu}{2}$
β	$= k\theta - \rho\sigma iu$	$= k\theta - \rho\sigma iu - \sigma\rho$
γ	$= \frac{\sigma^2}{2}$	
$C, \tilde{C}$	$C(u,\tau) = \lambda\left\{r_- \cdot \tau - \frac{2}{\sigma^2}\log\left(\frac{1-ge^{-d\tau}}{1-g}\right)\right\}$	
$D, \tilde{D}$	$D(u,\tau) = r_- \cdot \frac{1-e^{-d\tau}}{1-ge^{-d\tau}}$	
$C', \tilde{C}'$	$= k_H\theta_H \cdot D = \lambda \cdot D$	$\tilde{C}' = k_H\theta_H \cdot \tilde{D} = \lambda \cdot \tilde{D}$
$r_\pm$	$= \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma} = \frac{\beta \pm d}{2\gamma}$	
g	$= \frac{r_-}{r_+}$	
$f_{X_T}(y)$	$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iuy}\psi_H(u;0,\upsilon_0,X_0)du$	
$\psi_H, \tilde{\psi}_H$	$\psi_H(u;t,\upsilon,x)$ $= \exp\left(C\left(u,T-t\right) + D\left(u,T-t\right)\upsilon + iux\right)$	$\tilde{\psi}_H(u;t,\upsilon,x)$ $= \exp\left(\tilde{C}\left(u,T-t\right) + \tilde{D}\left(u,T-t\right)\upsilon + iux\right)$

So we know that

Continuous Part:

$$\phi^{cont}(u) = E^Q\left[\exp\left(iuX_T^{cont}\right)\middle| F_t\right] = \exp\left(C\left(u,T-t\right) + D\left(u,T-t\right)\upsilon_t + iux\right)$$

$$\tilde{\phi}^{cont}(u) = E^{\tilde{Q}}\left[\exp\left(iuX_T^{cont}\right)\middle| F_t\right] = \exp\left(\tilde{C}\left(u,T-t\right) + \tilde{D}\left(u,T-t\right)\upsilon_t + iux\right)$$

And here

X can be replaced by X_T^{cont}

From 2.7 we know the jump parts. Add them in we will have the current answers.

$E^Q\left[\exp\left(iuX_T^{jump}\right)\right]$	$=\exp(-iu\beta\lambda T)\cdot\exp(\lambda T(\exp(iu\alpha-\frac{1}{2}u^2\delta^2)-1))$
$E^{\tilde{Q}}\left[\exp\left(iuX_T^{jump}\right)\right]$	$=\exp\left(-iu\beta\lambda T+\lambda T\left(\exp\left(i(u-i)\alpha-\frac{1}{2}\delta^2(u-i)^2\right)-1-\beta\right)\right)$

Using Gatheral's notations we have

$$\phi_X(u)=E^Q\left[\exp\left(iuX_T^{cont}\right)\right]E^Q\left[\exp\left(iuX_T^{jump}\right)\right]$$

$$=\exp\left(C(u,T)\bar{v}+D(u,T)v-iu\beta\lambda T+\lambda T\left(\exp\left(iu\alpha-\frac{1}{2}u^2\delta^2\right)-1\right)\right)$$

$$\tilde{\phi}_X(u)=E^{\tilde{Q}}\left[\exp\left(iuX_T^{cont}\right)\right]E^{\tilde{Q}}\left[\exp\left(iuX_T^{jump}\right)\right]$$

$$=\exp\left(\tilde{C}(u,T)\bar{v}+\tilde{D}(u,T)v-iu\beta\lambda T+\left(\lambda T\left(\exp\left(i(u-i)\alpha-\frac{1}{2}\delta^2(u-i)^2\right)-1-\beta\right)\right)\right)$$

2.11: Transfer the numbers in the Table 5.5 in Gatheral to match in this model

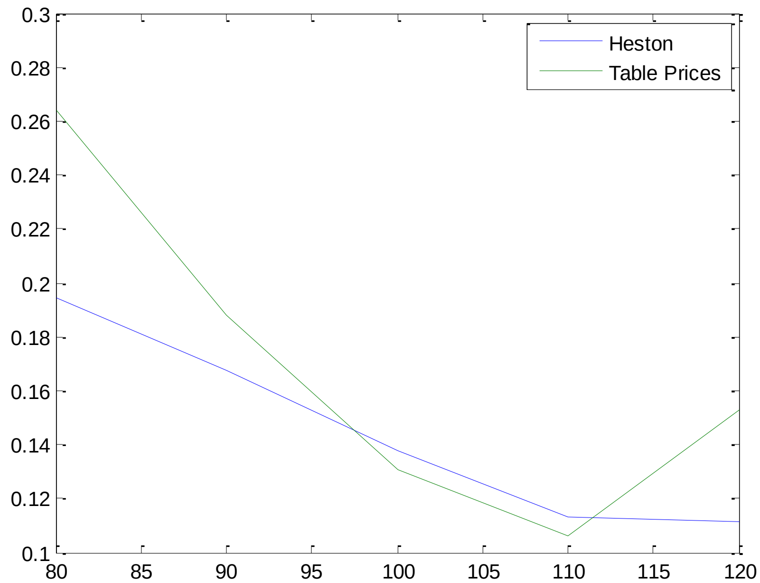
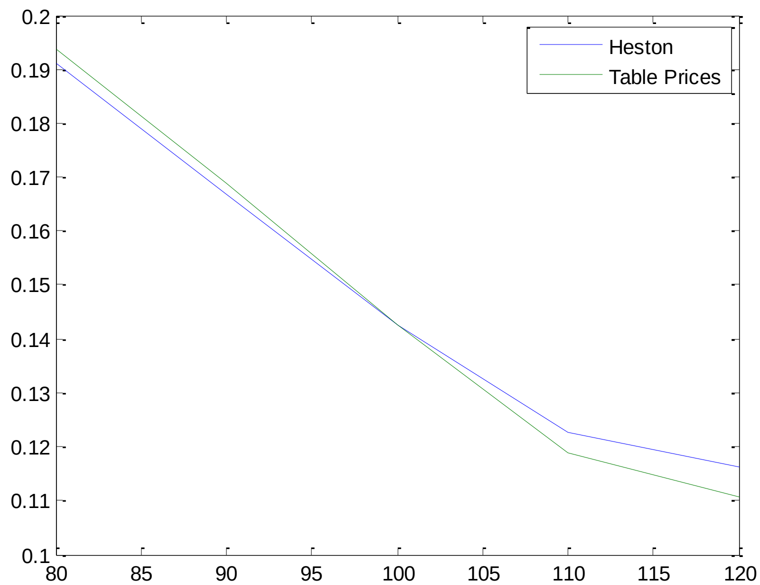
	K = 80	K = 90	K = 100	K = 110	K = 120
T = 0.1	20.0033	10.0831	1.6574	-0.0035	0.0056
T = 0.5	20.3099	11.2119	3.8555	0.412	0.022
T = 1	21.0698	12.5946	5.6859	1.4939	0.2521
T = 2	22.7142	14.9603	8.5609	3.9817	1.4647
T = 5	27.1213	20.5025	14.868	10.2927	6.7812

Exercise 3:

3.1: Report π that minimized the squared error. (The detail code are includes in the Appendix)

π	0.018824, 0.37496, 0.053682, 0.26082, -0.6691
Corresponding minimal squared error	0.10388
We also noticed that, in an alternative route of calculation, we have the following π:	$v_0=0.0174$ $\kappa=1.3253$ $\theta=0.0354$ $\beta=0.3877$ $\rho=-0.7165$
Side Note	This theoretically should give us a better result. However, because the Matlab implementation of mathematical models are slightly different. Thus it would give us a result with minor bias. Besides, because producing an alternative route of implementation can be time-consuming, we decide not to take that process.

3.2: For the optimal value of π, plot showing both implied BS volatility of the call prices in Table 1 and implied BS of the calibrate Heston Model

T=0.1	
T=1	
Comment	For T=1 Heston model results are very close to Table Prices results. For T=0.1 the differences are slightly bigger.

APPENDEX : CODE

2.8:

Note	Code

HW3Q2_8.m	`%hw3_2_8.m`
	`clear all;`
	`T=[0.1,0.5,1,2,5];`
	`sig=[0.2,0.2,0.2];`
	`lam=[0.5,1,1];`

| | ```
alpha=[-0.15,-0.07,-0.07];
delta=[0.05,0,0.05];

s0=100;
K=100;

du=1;

c=zeros(5,3);
for i=1:5
    for j=1:3
        beta = exp(alpha(j)+0.5*delta(j)*delta(j))-1;

        integ = 0;
        for u=0.5:du:100
            C = 1i*u*(-0.5*sig(j)*sig(j)*T(i)-beta*lam(j)*T(i));
            D = -0.5*u*u*sig(j)*sig(j)*T(i);
            E = lam(j)*T(i)*(exp(1i*u*alpha(j)-0.5*u*u*delta(j)*delta(j))-1);
            psi = exp(C + D + E);
            integ = integ + imag(exp(-1i*u*log(K/s0))*psi)/u*du;
        end
        Q1=1/2+1/pi*integ;

        integ = 0;
        psi = 0;
        for u=0.5:du:100
            C = 1i*u*(0.5*sig(j)*sig(j)*T(i)-beta*lam(j)*T(i));
            D = -0.5*u*u*sig(j)*sig(j)*T(i);
            E = lam(j)*T(i)*(exp(1i*(u-1i)*alpha(j)-0.5*delta(j)*delta(j)*(u-1i)*(u-1i))-1-beta);
            psi = exp(C + D) * exp(E);
            integ = integ + imag(exp(-1i*u*log(K/s0))*psi)/u*du;
        end
        Q2=1/2+1/pi*integ;
        c(i,j)=s0*Q2-K*Q1;
    end
end

c
``` |

2.9:

Note	Code
| HW3Q2_9.m | ```
%hw3_2_9c.m
clear all;
T=[0.1,1,5];
sig=[0.2,0.2,0.2];
lam=[0.5,1,1];
alpha=[-0.15,-0.07,-0.07];
delta=[0.05,0,0.05];

s0=100;
K=(80:0.1:120);

du=1;

c=zeros(1,1);
for k=1:length(K)
    for i=1:3
        for j=3:3
            beta = exp(alpha(j)+0.5*delta(j)*delta(j))-1;

            integ = 0;
            for u=0.5:du:100
                C = 1i*u*(-0.5*sig(j)*sig(j)*T(i)-beta*lam(j)*T(i));
                D = -0.5*u*u*sig(j)*sig(j)*T(i);
                E = lam(j)*T(i)*(exp(1i*u*alpha(j)-0.5*u*u*delta(j)*delta(j))-1);
                psi = exp(C + D + E);
                integ = integ + imag(exp(-1i*u*log(K(k)/s0))*psi)/u*du;
            end
            Q1=1/2+1/pi*integ;

            integ = 0;
``` |

| | ```
            psi = 0;
            for u=0.5:du:100
                C = 1i*u*(0.5*sig(j)*sig(j)*T(i)-beta*lam(j)*T(i));
                D = -0.5*u*u*sig(j)*sig(j)*T(i);
                E = lam(j)*T(i)*(exp(1i*(u-1i)*alpha(j)-0.5*delta(j)*delta(j)*(u-1i)*(u-1i))-1-
beta);
                psi = exp(C + D) * exp(E);
                integ = integ + imag(exp(-1i*u*log(K(k)/s0))*psi)/u*du;
            end
            Q2=1/2+1/pi*integ;
            c(i,k)=s0*Q2-K(k)*Q1;
        end
    end
end

r=0;
ImpliedVol = zeros(3,length(K));
options = optimset('fzero');
options = optimset(options, 'TolX', 1e-8, 'Display', 'off');
for i = 1:length(K)
    for j = 1:3
        try
            v0 = fzero(@(v0) ObjFcn(v0,s0,K(i),T(j),r,c(j,i)),[0.0001 5],options);
        catch
            v0 = NaN;
        end
        ImpliedVol(j,i) = v0;
    end
end

plot(K,ImpliedVol);
``` |

2.11:

Note	Code
| HW3Q2_11.m | ```
clear all;

v=0.0158;
v_bar=0.0439;
eta=0.3038;
rho=-0.6974;
lambda=0.5394;
lambdaJ=0.1308;
delta=0.0967;
alpha=-0.1151;

s0=100;
K=[80,90,100,110,120];
T=[0.1,0.5,1,2,5];

du=1;

c=zeros(5,1);
for k=1:length(K)
    for i=1:5
            integ = 0;
            for u=0.5:du:100
                sigma = eta;
                gamma = (eta^2)/2;
                alpha2 = -0.5*u*(1i+u);
                beta = lambda - 1i*u*rho*sigma;
                d = sqrt(beta^2 - 4*alpha2*gamma);
                rMinus = (beta-d)/(2*gamma);
                rPlus = (beta+d)/(2*gamma);
                g = rMinus/rPlus;
                beta2 = exp(alpha+0.5*delta*delta)-1;

                D = rMinus * ((1-exp(-d*T(i)))/(1-g*exp(-d*T(i))));
                C = lambda*(rMinus*T(i)-(2/sigma^2)*log((1-g*exp(-d*T(i)))/(1-g)));
                E = -1i*u*beta2*lambdaJ*T(i);
                F = lambdaJ*T(i)*(exp(1i*u*alpha-0.5*u*u*delta*delta)-1);
``` |

| | ```
                    psi = exp(C*v_bar + D*v + E + F);
                    integ = integ + imag(exp(-1i*u*log(K(k)/s0))*psi)/u*du;
                end
                Q1=1/2+1/pi*integ;

                integ = 0;
                psi = 0;
                for u=0.5:du:100
                    sigma = eta;
                    gamma = (eta^2)/2;
                    alpha2 = 0.5*u*(1i-u);
                    beta = lambda - 1i*u*rho*sigma - sigma*rho;
                    d = sqrt(beta^2 - 4*alpha2*gamma);
                    rMinus = (beta-d)/(2*gamma);
                    rPlus = (beta+d)/(2*gamma);
                    g = rMinus/rPlus;

                    D = rMinus * ((1-exp(-d*T(i)))/(1-g*exp(-d*T(i))));
                    C = lambda*(rMinus*T(i)-(2/sigma^2)*log((1-g*exp(-d*T(i)))/(1-g)));
                    E = 1i*u*(-beta2*lambdaJ*T(i));
                    F = lambdaJ*T(i)*(exp(1i*(u-1i)*alpha-0.5*delta*delta*(u-1i)*(u-1i))-1-beta2);

                    psi = exp(C*v_bar + D*v + E + F);
                    integ = integ + imag(exp(-1i*u*log(K(k)/s0))*psi)/u*du;
                end
                Q2=1/2+1/pi*integ;
                c(i,k)=s0*Q2-K(k)*Q1;
        end
end

c
``` |

3.1:

Note	Code
| HW3Q3a.m | ```
clear;

%first index is T
%second index is K

obsPrice = zeros(5,5);
obsPrice(1,1) = 20.0087;
obsPrice(1,2) = 10.0863;
obsPrice(1,3) = 1.6517;
obsPrice(1,4) = 0.0024;
obsPrice(1,5) = 0.0001;
obsPrice(2,1) = 20.3092;
obsPrice(2,2) = 11.2117;
obsPrice(2,3) = 3.8561;
obsPrice(2,4) = 0.4113;
obsPrice(2,5) = 0.0223;
obsPrice(3,1) = 21.0696;
obsPrice(3,2) = 12.5945;
obsPrice(3,3) = 5.6858;
obsPrice(3,4) = 1.4939;
obsPrice(3,5) = 0.2518;
obsPrice(4,1) = 22.7139;
obsPrice(4,2) = 14.9601;
obsPrice(4,3) = 8.5607;
obsPrice(4,4) = 3.9815;
obsPrice(4,5) = 1.4644;
obsPrice(5,1) = 27.1208;
obsPrice(5,2) = 20.5021;
obsPrice(5,3) = 14.8677;
obsPrice(5,4) = 10.2924;
obsPrice(5,5) = 6.7808;

s0 = 100;
K=[80,90,100,110,120];
T=[0.1,0.5,1,2,5];
%v0, kappa,theta,beta,rho
``` |

	`piVector=[0.018824,0.37496,0.053682,0.26082,-0.6691];`
	
	`r = 0;`
	`options = optimset('MaxFunEvals',10000);`
	`piVector = fminsearch(@(piVector) objFun(piVector,r,T,s0,K,obsPrice), piVector, options);`
	`v0 = piVector(1);`
	`kappa = piVector(2);`
	`theta = piVector(3);`
	`beta = piVector(4);`
	`rho = piVector(5);`
	`disp([num2str(v0) ',' num2str(kappa) ',' num2str(theta) ',' num2str(beta)  ',' num2str(rho)]);`

HestonPrice.m	`%HestonPrice.m`
	
	`function call = HestonPrice(kappa,theta,sig,rho,v0,r,T,s0,K)`
	`call = s0*HestonP(kappa,theta,sig,rho,v0,r,T,s0,K,1) - K*exp(-`
	`r*T)*HestonP(kappa,theta,sig,rho,v0,r,T,s0,K,2);`
	
	`function retP = HestonP(kappa,theta,sig,rho,v0,r,T,s0,K,type)`
	`retP = 1/2 + 1/pi*quad(@HestonPIntg,0,100,[],[],kappa,theta,sig,rho,v0,r,T,s0,K,type);`
	
	`function retI = HestonPIntg(phi,kappa,theta,sig,rho,v0,r,T,s0,K,type)`
	`retI = real(exp(-1i*phi*log(K)).*Hestf(phi,kappa,theta,sig,rho,v0,r,T,s0,type)./(1i*phi));`
	
	`function retf = Hestf(phi,kappa,theta,sig,rho,v0,r,T,s0,type)`
	`if type == 1`
	`    u = 0.5;`
	`    b = kappa - rho*sig;`
	`else`
	`    u = -0.5;`
	`    b = kappa;`
	`end`
	`x = log(s0);`
	`a = kappa * theta;`
	`d = sqrt((rho*sig*phi.*1i-b).^2 - sig^2*(2*u*phi.*1i-phi.^2));`
	`g = (b-rho*sig*phi*1i+d) ./ (b-rho*sig*phi*1i-d);`
	`C = r*phi.*1i*T + (a/sig^2).*((b-rho*sig*phi*1i+d)*T - 2*log((1-g.*exp(d*T))./(1-g)));`
	`D = (b-rho*sig*phi*1i+d)./sig^2 .* ((1-exp(d*T))./(1-g.*exp(d*T)));`
	`retf = exp(C + D*v0 + 1i*phi*x);`

objFun.m	`function delta = objFun(piVector,r,T,s0,K,c)`
	`v0 = piVector(1);`
	`kappa = piVector(2);`
	`theta = piVector(3);`
	`beta = piVector(4);`
	`rho = piVector(5);`
	`SquaredError = 0;`
	`PriceDiffSum = 0;`
	`myC = c;`
	`for i=1:5`
	`    for j=1:5`
	`        myC(i,j) = HestonPrice(T(i),s0,K(j),v0,theta,kappa,beta,rho,r);`
	`        PriceDiff = c(i,j) - myC(i,j);`
	`        if myC(i,j)<0`
	`            SquaredError = SquaredError + 1000;`
	`        end`
	`        if myC(i,j)<(s0-K(j))`
	`            SquaredError = SquaredError + 1000;`
	`        end`
	`        SquaredError = SquaredError + PriceDiff*PriceDiff;`
	`        PriceDiffSum = PriceDiffSum+PriceDiff;`
	`    end`
	`end`
	
	`if abs(rho) >= 1`
	`    SquaredError = SquaredError + 1000;`
	`end`
	`if beta < 0.00001`
	`    SquaredError = SquaredError + 1000;`
	`end`
	`if v0 < 0.01`
	`    SquaredError = SquaredError + 1000;`

<table>
<tr><td></td><td>

```
end
if theta <= 0.000001
    SquaredError = SquaredError + 1000;
end
if kappa <= 0.00001
    SquaredError = SquaredError + 1000;
end
%myC
disp([num2str(SquaredError) ': ' num2str(PriceDiffSum) ',' num2str(v0) ',' num2str(kappa) ','
num2str(theta) ',' num2str(beta)  ',' num2str(rho)]);
delta = SquaredError;
```

</td></tr>
<tr><td>objFun2.m</td><td>

```
%objFcn2.m

function delta = objFun2(piVector,r,T,s0,K,c)
v0 = piVector(1);
kappa = piVector(2);
theta = piVector(3);
beta = piVector(4);
rho = piVector(5);
SquaredError = 0;
PriceDiffSum = 0;
myC = c;
for i=1:5
    for j=1:5
        myC(i,j) = HestonPrice(kappa,theta,beta,rho,v0,r,T(i),s0,K(j));
        PriceDiff = c(i,j) - myC(i,j);
        if myC(i,j)<=0.0001
            SquaredError = SquaredError + 1000;
        end
        if myC(i,j)<=(s0-K(j)+0.0001)
            SquaredError = SquaredError + 1000;
        end
        %if (i==1 && j==1) || (i==1 && j==5) || ...
         %       (i==3 && j==3) || ...
         %       (i==5 && j ==1) || (i==5 && j == 5)
            SquaredError = SquaredError + PriceDiff*PriceDiff;
        %end
        PriceDiffSum = PriceDiffSum+PriceDiff;
    end
end
if myC(1,1) <20.001
     SquaredError = SquaredError + 1000;
end
c = HestonPrice(kappa,theta,beta,rho,v0,r,0.1,s0,118);
if c <= 0
     SquaredError = SquaredError + 1000;
end
if abs(rho) >= 1
    SquaredError = SquaredError + 1000;
end
if beta < 0.00001
    SquaredError = SquaredError + 1000;
end
if v0 < 0.001
    SquaredError = SquaredError + 1000;
end
if theta <= 0.000001
    SquaredError = SquaredError + 1000;
end
if kappa <= 0.00001
    SquaredError = SquaredError + 1000;
end
myC
disp([num2str(SquaredError) ': ' num2str(PriceDiffSum) ',' num2str(v0) ',' num2str(kappa) ','
num2str(theta) ',' num2str(beta)  ',' num2str(rho)]);
%disp(['squared error ' num2str(SquaredError)]);
delta = SquaredError;
```

</td></tr>
<tr><td>ObjFcn.m</td><td>

```
function delta = ObjFcn(volatility, s0, K, T, r, CallPrice)

BSprice = BSPrice(s0, K, T, r, volatility);
```

</td></tr>
</table>

	`delta = CallPrice - BSprice;`

BSPrice.m	`function BlackScholesPrice = BSPrice(s0,K,T,r,sigma)`
	
	`F=s0.*exp(r.*T);`
	`d1=log(F./K)./(sigma.*sqrt(T))+sigma.*sqrt(T)/2;`
	`d2=log(F./K)./(sigma.*sqrt(T))-sigma.*sqrt(T)/2;`
	`BlackScholesPrice = exp(-r.*T).*(F.*normcdf(d1)-K.*normcdf(d2));`

3.2:

Note	Code
hw3_3b.m	(code below)

```
clear all;
TVector=[0.1,1];
k=1; %choose between T=0.1 and 1

s0=100;
KVector=[80,90,100,110,120];

%v0, kappa,theta,beta,rho
piVector=[0.018824,0.37496,0.053682,0.26082,-0.6691];
%piVector=[0.0174,1.3253,0.0354,0.3877,-0.7165];

kappa = piVector(2);
theta = piVector(3);
beta = piVector(4);
rho = piVector(5);
v0 = piVector(1);

r=0;
ImpliedVol = zeros(length(TVector),length(KVector));
c = zeros(length(TVector),length(KVector));
options = optimset('fzero');
options = optimset(options, 'TolX', 1e-8, 'Display', 'off');
for i = 1:length(KVector)
    for j = 1:2
        v0 = piVector(1);
        T = TVector(k);
        K = KVector(i);
        if j==1
            c(j,i) = HestonPrice(T,s0,K,v0,theta,kappa,beta,rho,r);
        else
            c(2,:) = [20.0087 10.0863 1.6517 0.0024 0.0001];
            %c(2,:) = [21.0696  12.5945 5.6858  1.4939  0.2518];
        end
        try
              v0 = fzero(@(v0) ObjFcn(v0,s0,KVector(i),TVector(k),r,c(j,i)),[0.0001 5],options);
        catch
              v0 = NaN;
        end
        ImpliedVol(j,i) = v0;
    end
end

plot(KVector,ImpliedVol);
h = legend('Heston','Table Prices');
ImpliedVol;
```