Credit Derivatives

HW2

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Exercise 1:

(EX1.1)Solve a(.) and b(.) from ZCB : $B(t,T)=\exp\left(-a(T-t)-b(T-t)r_t\right), t\in\left[0,T\right]$ **under**

$dr_t = k^r(\theta^t - r_t)dt + \beta^r dB_t$

Process	We can either get a discounted B(t,T) as a Martingale. Then we get the drift as 0. Or we can direct get the risk-neutral of B(t,T) and get drift as r.
	For $dr_t = k^r(\theta^t - r_t)dt + \beta^r dB_t$, we can replace $\alpha(t,r_t)=k^r(\theta^t - r_t)$ and get $dr_t = \alpha(t,r_t)dt + \beta^r dB_t$
Apply Ito's lemma for $f(t,r_t)$	$dB(t,T)=df(t,r)= f_t dt + f_r dr_t + \frac{1}{2} f_{rr} d\left\langle r\right\rangle_t$ <Ref 1> $=\left\{\left(a'+b'r_t\right)\cdot f\right\}dt+\left\{(-b)\cdot f\right\}\left(k^r(\theta^t-r_t)dt+\beta^r dB_t\right)+\frac{1}{2}\left\{b^2\cdot f\right\}\left(\beta^{2r}dt\right)$ <Ref 2> $= f\left\{\left(a'+b'r_t\right)dt - b\left(\alpha(\bullet)dt+\beta^r d\tilde{B}_t\right)+\frac{1}{2}b^2\left(\left(\beta^r\right)^2 dt\right)\right\}$ $=B(t,T)\left\{\left(a'+b'r_t-\alpha(\cdot)b+\frac{1}{2}b^2\beta^{2r}\right)dt - b\cdot\beta^r d\tilde{B}_t\right\}$
Drift Term =r	$r_t=\left\{\left(a'+b'r_t\right)-b\cdot k^r(\theta^t-r_t)+\frac{1}{2}b^2\left(\beta^{2r}\right)\right\}$ $r_t=\left(a'+b'r_t\right)-bk^r(\theta^t-r_t)+\frac{1}{2}b^2\left(\beta^{2r}\right)$
PDE replace r_t with r	$r=\left(a'+b'r\right)-bk^r(\theta^t-r)+\frac{1}{2}b^2\left(\beta^{2r}\right)\ \forall a(.),b(.)$ with terminal value $f(t,r)=1\ \forall r,t$
Get two ODEs:	Collect r_t : $0=-1+b'(T-t)-b(T-t)k^r$, $b(T,T)=0$ Collect other term: $0=a'-bk^r\theta^t+\frac{1}{2}b^2\beta^{2r}$, $a(T,T)=0$
Solution for a(*) & b(*). Replace T-t=s	$b(s)=\frac{1}{k^r}\left(1-b'(s)\right)$ we can guess $b'(s)=e^{-Sk}$ and verify it. So $b(T-t)=\frac{1}{k^r}\left(1-e^{-k(T-t)}\right)$ we can get $a(T-t)=\frac{\beta^r}{4k^r}b^2(T-t)+\left(\theta-\frac{\beta^2}{2k^2}\right)\left(\left(T-t\right)-b(t,T)\right)$
<Ref 1 >	$dr_t = k^r(\theta^t - r_t)dt + \beta^r dB_t$ $d\left\langle r\right\rangle_t = \left(\beta^r\right)^2 dt$
<Ref 2 >	$f_t=\left(a'(T-t)+b'(T-t)r_t\right)\cdot f$ $f_r=\left(-b(T-t)\right)\cdot f$ $f_{rr}=b^2(T-t)\cdot f$
Conclusion	$a(T-t)=\frac{\beta^{r2}}{4k^r}b^2(T-t)+\left(\theta-\frac{\beta^{r2}}{2k^2}\right)\left(\left(T-t\right)-b(t,T)\right)$ $b(T-t)=\frac{1}{k^r}\left(1-e^{-k^r(T-t)}\right)$

(EX1.2) (Closed form for defaultable ZCB) Given state process $dX_t = k^X(\theta^X - X_t)dt + \beta^X dW_t$ **. pre-default intensity is** $\lambda_t = \omega r_t + X_t$**. And get** $\bar{B}(t,T), t\in\left[0,T\right]$

Assumption	1. $\left\{G_t\right\}$: information of default-free market, such as short rate r 2. $N_t^{(1)}$: std Poisson. $N^{(1)}\perp G$ under Q

	3. Default time τ = first Jump time of N_t 4. Total Filtration $F_t = G \vee F^N$

	Formula	Note
$\bar{B}(t,T)$ **Derived Process**	$= E^Q\left[\exp\left(-\int_t^T r(X_u)du\right)1_{\{T<\tau\}} \mid F_t\right]$ $= E^Q\left[E^Q\left[\exp\left(-\int_t^T r(X_u)du\right)1_{\{T<\tau\}} \mid F_t \vee G_T\right] \mid F_t\right]$ <Ref 1> $= E^Q\left[\exp\left(-\int_t^T r(X_u)du\right)E^Q\left[1_{\{t<\tau\}} \mid F_t \vee G_T\right] \mid F_t\right]$ <Ref 2> $= E^Q\left[\exp\left(-\int_t^T r(X_u)du\right)1_{\{t<\tau\}}\exp\left(-\int_t^T \lambda(X_u)du\right) \mid F_t\right]$ <Ref 3> $= E^Q\left[\exp\left(-\int_t^T \{r(X_u)+\lambda(X_u)\}du\right)1_{\{t<\tau\}} \mid F_t\right]$ $= 1_{\{t<\tau\}}E^Q\left[\exp\left(-\int_t^T \{r(X_u)+\lambda(X_u)\}du\right) \mid F_t\right]$ $= 1_{\{t<\tau\}}E^Q\left[\exp\left(-\int_t^T \{r(X_u)+\lambda(X_u)\}du\right) \mid G_t\right]$ <Ref 4>	<Ref 1> Iterated <Ref 2> $\exp\left(-\int_t^T r(X_u)du\right)$ is filtration with Ft <Ref 3> Independence lemma: $Q(T<\tau \mid F_t \vee G_T)$ $= Q(Y_T > U \mid F_t \vee G_T)$ $= 1_{\{t<\tau\}}\frac{Y_T}{Y_t}$ $= 1_{\{t<\tau\}}\exp\left(-\int_t^T \lambda(X_u)du\right)$ <Ref 4> Intuitively, Ft doesn't provide extra useful information for r and λ. So we can change it to Gt
$\lambda_t = \omega r_t + X_t$	$= 1_{\{t<\tau\}}E^Q\left[\exp\left(-\int_t^T \{r_u+\lambda_u+\omega r_u+X_u\}du\right) \mid G_t\right]$ $= 1_{\{t<\tau\}}E^Q\left[\exp\left(-\int_t^T \{(1+\omega)r_u+X_u\}du\right) \mid G_t\right]$ $= 1_{\{t<\tau\}}E^Q\left[\exp\left(-\int_t^T (1+\omega)r_u du\right) \mid G_t\right]E^Q\left[\exp\left(-\int_t^T (X_u)du\right) \mid G_t\right]$ $= 1_{\{t<\tau\}}\cdot\exp\left(-m(T-t)-n(T-t)(1+\omega)r_t\right)\cdot\exp\left(-c(T-t)-d(T-t)x_t\right)$	<Ref 5> r and x are driven by independent BW. So we can separate them. This result from Conditional Independence on jump note page 43 <Ref 6> m(.), n(.) <Ref 7> c(.), d(.)
<Ref 6>: Scaled Vasicek (1+w) for r	Let $r_t' = (1+\omega)r_t$ into $dr_t = k^r(\theta^t - r_t)dt + \beta^r dB_t$. So $dr_t = (1+\omega)dr_t' = k^r((1+\omega)\theta^t - (1+\omega)r_t)dt + (1+\omega)\beta^r dB_t$ So $\theta^{r'} = (1+\omega)\theta^r$, $\beta^{r'} = (1+\omega)\beta^r$, $k^{r'} = k^r$ $m(T-t) = \frac{(1+\omega)\beta^{r2}}{4k^r}n^2(T-t) + \left((1+\omega)\theta^r - \frac{\beta^{r2}}{2k^2}\right)\left((T-t) - m(t,T)\right)$ $n(T-t) = \frac{1}{k^r}\left(1-e^{-k^r(T-t)}\right)$	

<Ref 7>: c(.) , and d(.) for x	$c(T-t)=\frac{\beta^{X2}}{4k^X}b^2(T-t)+\left(\theta-\frac{(\beta^X)^2}{2(k^X)^2}\right)\left((T-t)-b(t,T)\right)$ $d(T-t)=\frac{1}{k^X}\left(1-e^{-k^X(T-t)}\right)$	
Conclusion	If we have no default time at time t-the Q-dynamics. $1_{\tau>t}\exp\left(-m(T-t)-n(T-t)(1+\omega)r_t-c(T-t)-d(T-t)x_t\right)$ m(.),n(.),c(.),d(.) are detailed above	

(EX1.3) (Countdown process Simulation) Simulate $Y_t=\exp\left(-\int_0^t \lambda_u du\right)$

(a) Construct X & r: $dX_t=k^X(\theta^X-X_t)dt+\beta^X dW_t$

(b)Insert into $\lambda(X)$, $\lambda:R^d\to[0,\infty)$: $\lambda_t=wr_t+X_t$

(c)Construct the countdown process: $Y_t=\exp\left(-\int_0^t \lambda_u du\right)$

	Note
Code	(see code below)

```
% for HW2:Exercise 1.3
clear all;

%define underlying parameters
r0=0.05;Theta_r=0.05;Kappa_r=0.25; Beta_r=0.05;
Theta_X=0.1; X0=0.1;Kappa_X=0.25; Beta_X=0.05;
w=1;

%Maturity
T=10;

%Time Steps
TimeSteps=50*T;
dt=T/TimeSteps;
MC_Loops=10000;

%Initialization
r=zeros(MC_Loops,TimeSteps);
X=zeros(MC_Loops,TimeSteps);
def=zeros(MC_Loops,TimeSteps);
Y=zeros(MC_Loops,TimeSteps+1);
Zr=zeros(MC_Loops,TimeSteps+1);
ZX=zeros(MC_Loops,TimeSteps+1);
TimeSeries=zeros(1,TimeSteps);

MC=zeros(1,TimeSteps);
Ana=zeros(1,TimeSteps);

%Normal Random Variable
Zr=randn(MC_Loops,TimeSteps+1);
ZX=randn(MC_Loops,TimeSteps+1);

r(:,1)=r0+Kappa_r*(Theta_r-r0)*dt+Beta_r*sqrt(dt)*Zr(:,1);
X(:,1)=X0+Kappa_X*(Theta_X-X0)*dt+Beta_X*sqrt(dt)*ZX(:,1);

%Evoluation of r and X
for i=1:TimeSteps
    r(:,i+1)=r(:,i)+Kappa_r*(Theta_r-r(:,i))*dt+Beta_r*sqrt(dt)*Zr(:,i+1);
    X(:,i+1)=X(:,i)+Kappa_X*(Theta_X-X(:,i))*dt+Beta_X*sqrt(dt)*ZX(:,i+1);
end
```

| | ```
u=rand(MC_Loops,1);
Y=exp(-dt*(w*cumsum(r,2)+cumsum(X,2)));

for i=1: TimeSteps
    def(:,i)=(Y(:,i)<=u);
end

tau=dt*(TimeSteps-sum(def,2)+1);

for i=1: TimeSteps
    time =i*dt;
    TimeSeries(1,i)=time;

    %Simulation Result
    MC(:,i)=sum(tau<=time)/MC_Loops;

    %Analytical Result
    exact_r=vas_exact(time,r0,Theta_r, Kappa_r, Beta_r);
    exact_X=vas_exact(time,X0,Theta_X, Kappa_X, Beta_X);

       %Scaled(1+w) Vasicek Distriubtion
       rs0=(1+w)*r0;
       Kappa_rs=Kappa_r;
       Theta_rs=(1+w)*Theta_r;
       Beta_rs=(1+w)*Beta_r;

       exact_rs=vas_exact(time,rs0,Theta_rs, Kappa_rs, Beta_rs);

    zcb_bar=exact_rs*exact_X;

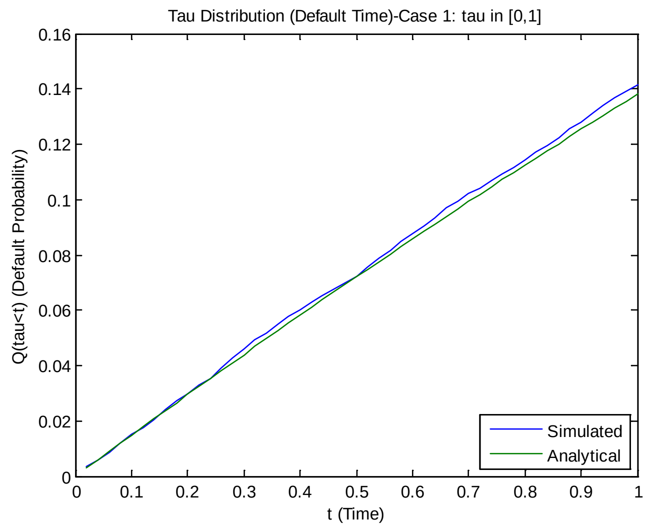
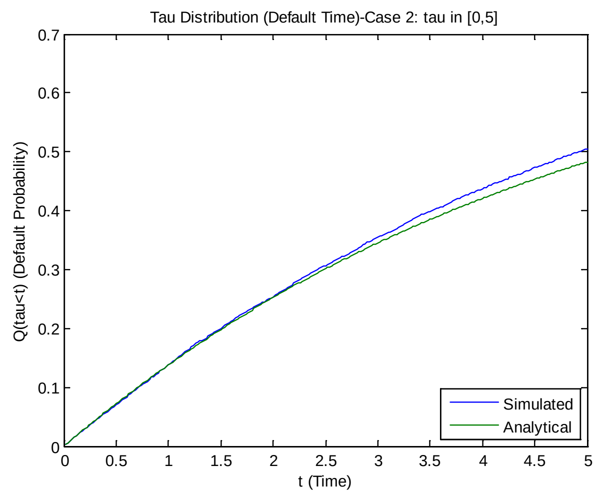
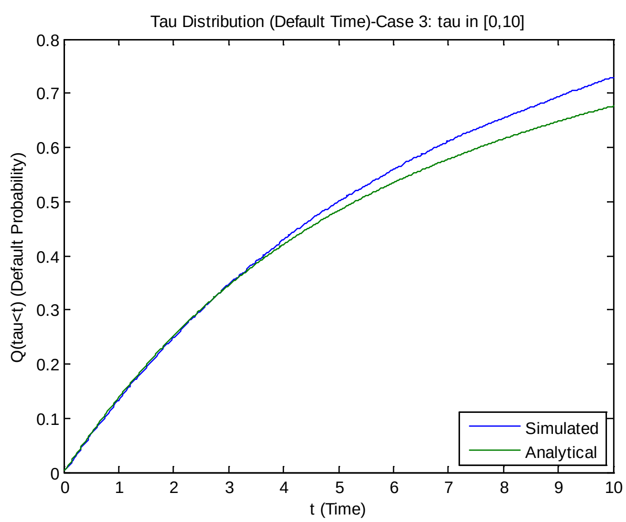
    Ana(:,i)=1-zcb_bar/exact_r;

end

%Result for Q1.3
plot(TimeSeries,MC,TimeSeries,Ana);
xlabel('t (Time)');
ylabel('Q(tau<t) (Default Probability)');
title('Tau Distribution (Default Time)-Case 3: tau in [0,10]');
legend('Simulated','Analytical','Location','SouthEast');
``` |
| **Function** | ```
function y=vas_exact(T,rt,Theta, Kappa, Beta)

b=1/Kappa*(1-exp(-Kappa*T));
a=(Beta*Beta)/(4*Kappa)*b*b+(Theta-Beta*Beta/(2*Kappa*Kappa))*(T-b);

y=exp(-a-b*rt);
``` |

<table>
<tr><td>Tau in [0,1]</td><td>
</td></tr>
<tr><td>Tau in [0,5]</td><td>
</td></tr>
<tr><td>Tau in [0,10]</td><td>
</td></tr>
</table>

(EX1.4) Based on previous result, get $E^Q\left[\tau \cdot 1_{\tau\in[0,1]}\right], E^Q\left[\tau \cdot 1_{\tau\in[0,5]}\right], E^Q\left[\tau \cdot 1_{\tau\in[0,10]}\right]$

	Note

Code	`%Result for Q1.4`
	`isDefault=(tau<=T);`
	`Expected_Tau=mean(tau.*isDefault);`
	`disp('Expected_Tau- Case 1: tau in [0,1]')`
	`disp(Expected_Tau)`

	Numerical Result
$E^Q\left[\tau \cdot 1_{\tau\in[0,1]}\right]$	Expected_Tau- Case 1: tau in [0,1] 0.0693
$E^Q\left[\tau \cdot 1_{\tau\in[0,5]}\right]$	Expected_Tau- Case 2: tau in [0,5] 1.0873
$E^Q\left[\tau \cdot 1_{\tau\in[0,10]}\right]$	Expected_Tau- Case 3: tau in [0,10] 2.7548

(EX1.5) Estimate $\bar{B}(0,T)$ for $T \in \{\frac{1}{12},\frac{1}{4},\frac{1}{2},1,2,5,10\}$ using MC-Simulation and compare these simulated results with the analytical equivalent result from Q2

	Note
Code	(see code below)

```
%Result for Q1.5
clear all;

%define underlying parameters
r0=0.05;Theta_r=0.05;Kappa_r=0.25; Beta_r=0.05;
Theta_X=0.1; X0=0.1;Kappa_X=0.25; Beta_X=0.05;
w=1;

%Vectorization
Ana_ZCB_bar=zeros(1,7);
MC_ZCB_bar=zeros(1,7);

% Time Frame
TT=[1/12,1/4,1/2,1,2,5,10];

%Scaled(1+w) Vasicek Distriubtion
rs0=(1+w)*r0;
Kappa_rs=Kappa_r;
Theta_rs=(1+w)*Theta_r;
Beta_rs=(1+w)*Beta_r;

for j=1:7

    T=TT(1,j);

    %Analytical Result
     exact_X=vas_exact(T,X0,Theta_X, Kappa_X, Beta_X);
     exact_rs=vas_exact(T,rs0,Theta_rs, Kappa_rs, Beta_rs);

    Ana_ZCB_bar(1,j)=exact_X*exact_rs;

    %MC Result
    MC_Loops=100;
    TimeSteps=50*T;
    dt=T/TimeSteps;

    ZX=randn(MC_Loops, TimeSteps+1);
    Zr=randn(MC_Loops, TimeSteps+1);

    r(:,1)=r0+Kappa_r*(Theta_r-r0)*dt+Beta_r*sqrt(dt)*Zr(:,1);
    X(:,1)=X0+Kappa_X*(Theta_X-X0)*dt+Beta_X*sqrt(dt)*ZX(:,1);

    %Evoluation of r and X
    for i=1:TimeSteps
        r(:,i+1)=r(:,i)+Kappa_r*(Theta_r-r(:,i))*dt+Beta_r*sqrt(dt)*Zr(:,i+1);
        X(:,i+1)=X(:,i)+Kappa_X*(Theta_X-X(:,i))*dt+Beta_X*sqrt(dt)*ZX(:,i+1);
```

```
    end

    ZCB_bar=exp(-dt*((1+w)*sum(r,2)+sum(X,2)));

    MC_ZCB_bar(1,j)=mean(ZCB_bar);
    MC_ZCB_bar(2,j)=sqrt(var(ZCB_bar)/MC_Loops);
end
disp('Q1.5:');
disp(['Expiries      :',num2str(TT)]);
disp(['Analytical    :',num2str(Ana_ZCB_bar)]);
disp(['Simulted      :',num2str(MC_ZCB_bar(1,:))]);
disp(['Std Error:    :',num2str(MC_ZCB_bar(2,:))]);
```

```
Q1.5:
Expiries     :0.0833333       0.25          0.5           1             2             5            10
Analytical   :0.98347         0.95126       0.90505       0.82015       0.67817       0.4118       0.21531
Simulted     :0.98021         0.94918       0.90171       0.81658       0.67572       0.41334      0.21445
Std Error:   :2.2689e-005     8.4456e-005   0.00021324    0.00050122    0.0010736     0.0020845    0.0025389
```

maturity	Analytical solution	Simulated result
1/12	0.98347	0.98021
1/4	0.95126	0.94918
1/2	0.90505	0.90171
1	0.82015	0.81658
2	0.67817	0.67572
5	0.4118	0.41334
10	0.21531	0.21445

(EX1.6) Consider $max\left\{\bar{B}\left(T,S\right)-K,0\right\}$ **.** $T\in\{\frac{1}{12},\frac{1}{4},\frac{1}{2},1,2,5,10\}$ **, and S=T+0.5. Strike=** $\bar{B}(0,T+0.5)$ **price knock out call options using MC-simulation.**

(a) Simulate the state process on [0,T]

(b) Use the analytic expression for $K=\bar{B}\left(T,S\right)$

	Note
Process	1) Draw a uniform U 2) Grow an independent path for r & X using MC, calculate Y until either default or maturity 3) If not default, at maturity T, use the analytical form for defaultable bond to calculate B_bar(T, T+0.5) with the simulated values of r and X at T 4) discount the payoff max(B_bar(T,T+0.5) - K, 0) with simulated value of r at time T back to time 0 Run the simulation n times to estimate the knock out option price.

```
%Result for Q1.6
clear all;

%define underlying parameters
r0=0.05;Theta_r=0.05;Kappa_r=0.25; Beta_r=0.05;
Theta_X=0.1; X0=0.1;Kappa_X=0.25; Beta_X=0.05;
w=1;

%Vectorization
Ana_ZCB_bar=zeros(1,7);
MC_KO_bar=zeros(1,7);

% Time Frame
TT=[1/12,1/4,1/2,1,2,5,10];

%Scaled(1+w) Vasicek Distriubtion
rs0=(1+w)*r0;
Kappa_rs=Kappa_r;
Theta_rs=(1+w)*Theta_r;
Beta_rs=(1+w)*Beta_r;

for index=1:7
```

```
    T=TT(1,index);
    S=T+0.5;

    %Analytical Result
     X_ko=vas_exact(0,S,X0,Theta_X, Kappa_X, Beta_X);
     rs_ko=vas_exact(0,S,rs0,Theta_rs, Kappa_rs, Beta_rs);

    K=X_ko*rs_ko;

    %MC Result
    MC_Loops=1000;
    TimeSteps=round(50*T);
    dt=T/TimeSteps;

    Zr=randn(MC_Loops, TimeSteps+1);
    ZX=randn(MC_Loops, TimeSteps+1);

    r(:,1)=r0+Kappa_r*(Theta_r-r0)*dt+Beta_r*sqrt(dt)*Zr(:,1);
    X(:,1)=X0+Kappa_X*(Theta_X-X0)*dt+Beta_X*sqrt(dt)*ZX(:,1);

    %Evoluation of r and X
    for i=1:TimeSteps
        r(:,i+1)=r(:,i)+Kappa_r*(Theta_r-r(:,i))*dt+Beta_r*sqrt(dt)*Zr(:,i+1);
        X(:,i+1)=X(:,i)+Kappa_X*(Theta_X-X(:,i))*dt+Beta_X*sqrt(dt)*ZX(:,i+1);
    end

    u=rand(MC_Loops,1);
    def=zeros(MC_Loops,TimeSteps);
    Y=exp(-dt*(w*cumsum(r,2)+cumsum(X,2)));

    for step=1: TimeSteps
        def(:,index)=(Y(:,step)<=u);
    end

    tau=dt*(TimeSteps-sum(def,2)+1);
    NotDef=tau>T;

    pv=exp(-dt*sum(r,2));

    for jj=1:MC_Loops
        xx=X(jj,TimeSteps);
        rr=r(jj,TimeSteps);
        rss=(1+w)*rr;

        zcb_x_TS=vas_exact(T,S,xx,Theta_rs, Kappa_rs, Beta_rs);
        zcb_rss_TS=vas_exact(T,S,rss,Theta_rs, Kappa_rs, Beta_rs);
        zcb_bar_TS(jj,1)=zcb_x_TS*zcb_rss_TS;
    end

    option_pv_payoff=NotDef.*pv.*max(0,zcb_bar_TS-K);
    MC_ko_bar(1,index)=mean(option_pv_payoff);
    MC_ko_bar(2,index)=sqrt(var(option_pv_payoff)/MC_Loops);
end

disp('Q1.6:');
disp(['Matruity     :',num2str(TT)]);
disp(['Simulted     :',num2str(MC_ko_bar(1,:))]);
disp(['Std Error:   :',num2str(MC_ko_bar(2,:))]);
```

```
Q1.6:
Maturity     :0.0833333       0.25        0.5         1          2          5          10
Simulted     :0.014945        0.042191    0.077578    0.1339     0.19814    0.21088    0.14674
Std Error:   :0.00036922      0.00073896  0.0011505   0.0021359  0.0040657  0.0077602  0.0091606
```

maturity	Strike	Simulated price
1/12	0.88970	0.014945
1/4	0.85486	0.042191
1/2	0.8122	0.077578
1	0.7564	0.1339

2	0.60835	0.19814
5	0.3927	0.21088
10	0.21329	0.14674

(EX1.7) Define the (forward survival) probability measure $\bar{Q}^T$ as for T>0. $\bar{Z}_t^T = \frac{\bar{B}(t,T)}{\bar{B}(0,T)} \exp\left(-\int_0^t r_u du\right)$**,**

$\bar{Z}_T^T = \frac{d\bar{Q}^T}{dQ}$**, Compute the Dynamics of (r,X) under** $\bar{Q}^T$

(a) Dynamics of (r,X) or $\bar{B}$ **under** $\bar{Q}^T$

	Formula	**Note**
From	$1_{\{t<\tau\}} \cdot \exp\left(-m(T-t) - n(T-t) r_t\right) \cdot \exp\left(-c(T-t) - d(T-t) x_t\right)$ Where $m(T-t) = \frac{(1+\omega)\beta^{r2}}{4k^r} n^2(T-t) + \left((1+\omega)\theta^r - \frac{\beta^{r2}}{2k^2}\right)\left((T-t) - m(t,T)\right)$ $n(T-t) = \frac{1}{k^r}\left(1 - e^{-k^r(T-t)}\right)$ $c(T-t) = \frac{\beta^{X2}}{4k^X} b^2(T-t) + \left(\theta - \frac{(\beta^X)^2}{2(k^X)^2}\right)\left((T-t) - b(t,T)\right)$ $d(T-t) = \frac{1}{k^X}\left(1 - e^{-k^X(T-t)}\right)$	We can get the dynamic of (r,X) from here. This is log Normal
$\bar{B}(t,T)$	$1_{\{t<\tau\}} \cdot \exp\left(-m(T-t) - n(T-t)(1+\omega) r_t\right) \cdot \exp\left(-c(T-t) - d(T-t) x_t\right)$ $= (1 - N_t) \cdot \exp\left(-m(T-t) - n(T-t) r_t\right) \cdot \exp\left(-c(T-t) - d(T-t) x_t\right)$ $= (1 - N_t) \cdot \exp\left(a + b r_t + h X_t\right)$	Let $a = -(m + c)$ $b = -n(T-t)(1+\omega)$ $h = -d(T-t)$
$d\bar{B}(t,T)$	$= \bar{B}(t_-,T)\left(a' dt + b' r_t dt + b dr_t + h' X_t dt + h dX_t\right) - \exp\left(a + b r_t + h X_t\right) dN_t$ $= \bar{B}(t_-,T)\left(f(.)dt + b\beta^r dB_t + h\beta^X dW_t\right) - g(.)$	
Define	$d\bar{Z}_t^T = \bar{Z}_t^T \left\{b\beta^r dB_t^Q + h\beta^X dW_t^Q - g(.)\right\}$ $= -\bar{Z}_t^T \left\{n(T-t)(1+\omega)\beta^r dB_t^Q + d(T-t)\beta^X dW_t^Q - g(.)\right\}$	
	$dB_t^{\bar{Q}(T)} = dB_t^Q + n(T-t)(1+\omega)\beta^r dt$ $dW_t^{\bar{Q}(T)} = dW_t^Q + d(T-t)\beta^X dt$	

$d\bar{B}(t,T)$	$= 1_{\{\tau>t\}} dM_t^{(T)} + M_t^{(T)} d1 + 0 + \bar{B}(t_-,T)(-dN_t)$ $= \bar{B}(t_-,T)\left[(r_t + \lambda_t) dt - n(.)\sigma_{(t,T)}^B dB_t - d(.)\sigma_{(t,T)}^W dW_t - dN_t\right]$	
$d\bar{B}(t,T)$	$= \bar{B}(t_-,T)\left[r dt - n(T-t)\beta^r d\tilde{B}_t - d(T-t)\beta^X d\tilde{W}_t - d\left(N_t - \int_0^t \lambda_u du\right)\right]$ Where $n(T-t) = \frac{1}{k^r}\left(1 - e^{-k^r(T-t)}\right)$, $d(T-t) = \frac{1}{k^X}\left(1 - e^{-k^X(T-t)}\right)$	Under Measure Q

(b) Are Q and $\bar{Q}^T$ equivalent Measures?

	Formula	Note
Given	$\bar{Z}_t^T = \frac{\bar{B}(t,T)}{\bar{B}(0,T)} \exp\left(-\int_0^t r_u du\right)$	
$d\bar{Z}_t^T$	$= d\,\frac{\bar{B}(t,T)}{\bar{B}(0,T)S_t^{(0)}}$ $= \frac{\bar{B}(t-,T)}{\bar{B}(0,T)S_t^{(0)}} \left\{-\bar{b}\sigma_t^X dW_t^Q - \bar{d}\sigma_t^r dB_t^Q - dN_t + \lambda(X_t)\right\}$	
	$= \bar{Z}_{t-}^{(T)} \left\{-\bar{b}\sigma_t^X dW_t^Q - \bar{d}\sigma_t^r dB_t^Q - dN_t + \lambda(X_t)\right\}$	Under Measure Q^T
	$dW_t^{\bar{Q}^{(T)}} = dW_t^Q + \bar{b}(T-t)\sigma_t^X dt = dW_t^Q + \bar{b}(T-t)\beta^X dt$ $dB_t^{\bar{Q}^{(T)}} = dB_t^Q + \bar{d}(T-t)\sigma_t^r dt = dB_t^Q + \bar{d}(T-t)\beta^r dt$	Under Measure Q^T

Reason 1	$\lambda_t^{\bar{Q}^{(T)}} = \lambda_t^Q(1+\varphi_t) = 0$	$\because$ the sign of $-dN_t$ is negative $\therefore \varphi_t = -1$
Result 1	There is no default. There is no different between default and pre-default.	

Reason 2	$\bar{Q}^{(T)}(\tau \le T) = E^{\bar{Q}^{(T)}}\left[1_{T\ge\tau}\right] = E^Q\left[\bar{Z}_T^{(T)} \cdot 1_{T\ge\tau}\right]$ $= E^Q\left[\frac{D(T)}{\bar{B}(0,T)} \cdot 1_{T<\tau} \cdot 1_{T\ge\tau}\right] = \bar{Q}^{(T)}\left(\bar{Z}_t^{(T)} = 0\right) = 0$	Under Measure Q^T, there is no default before T
Reason 3	$Q(\tau < T) > 0$	
Reason 4	$Q^{(T)}(T<\tau) = E^{Q^{(T)}}\left[1_{T<\tau}\right] = E^Q\left[Z_t^{(T)} 1_{T<\tau}\right]$ $= \frac{1}{B(0,T)} E^Q\left[\exp\left(-\int_0^T r(X_u)du\right) 1_{T<\tau}\right]$ $= \frac{\bar{B}(0,T)}{B(0,T)} < 1$	Default actually happen will happen Under Measure Q
Result 2 ~4	Default actually happen will happen under Measure Q	

Conclusion	**Q and $\bar{Q}^T$ are NOT equivalent Measures. We don't have Q<< $\bar{Q}^T$**

(EX1.8)

(a) Derive $\bar{B}(0,S)\bar{Q}^S\left(\bar{B}(T,S) \ge K\right) - K\bar{B}(0,T)\bar{Q}^T\left(\bar{B}(T,S) \ge K\right)$

- **Method 1**

	Formula	Note
closed form	$E^Q\left[\exp\left(-\int_0^T r(X_u)du\right) \cdot 1_{\{T<\tau\}}\left(\bar{B}(T,S) - K\right)^+\right]$ $= E^Q\left[D(T) \cdot 1_{\{T<\tau\}}\left(M_T^{(S)} - K\right)^+\right]$ $= E^Q\left[D(T)\left(M_T^{(S)} - K\right)^+ \cdot E^Q\left[1_{\{T<\tau\}} \mid G_T\right]\right]$ $= E^Q\left[D(T)\left(M_T^{(S)} - K\right)^+ \cdot D^\lambda(T)\right]$	$\bar{B}(T,S) = 1_{\{T<\tau\}} \cdot M_T^{(S)}$ $M_T^{(S)} = E^Q\left[\exp\left(-\int_T^S (\lambda_u + r_u)du\right) \mid G_T\right]$ $D^\lambda(T) = \exp\left(-\int_0^T (\lambda_u)du\right)$

<table>
<tr><td></td><td>$= E^{Q}\left[D^{R}(T)\left(M_T^{(S)}-K\right)^{+}\right]$
$= E^{Q^{R}}\left[\left(M_T^{(S)}-K\right)^{+}\right]$
$= E^{Q^{R}}\left[\left(M(0,T,S)-K\right)^{+}\right]$
$= M_0^{(T)}\left[M(0,T,S)N(d_+)-KN(d_-)\right]$</td><td>$D^{R}(T)=\exp\left(-\int_0^T(\lambda_u+r_u)du\right)$
<Ref 1>
<Ref 2>
<Ref 3></td></tr>
<tr><td><Ref 1></td><td colspan="2">$M_T^{(S)}=E^{Q}\left[\exp\left(-\int_T^S(\lambda_u+r_u)du\right)\mid G_T\right]$ here can be treated as ZCB with short rate λ_u+r_u
(1) $M(T,T,S)=M_T^{(S)}$ at time T, spot and forward value are the same.
(2)We don't distribution of $M_T^{(S)}$ but we know the $M(t,T,S)=\frac{M_t^{(S)}}{M_t^{(T)}}$ is log-Normal in forward measure.
Log-Normal distribution can help us to link to Black Formula
(3) $dM_t^{(S)}=M_t^{(S)}\left((\lambda_u+r_u)dt+\sigma^{W}(t,s)dW_t+\sigma^{B}(t,s)dB_t\right)$
$=M_t^{(S)}\left(rdt+\sigma^{W'}(t,s)dW_t^{Q}+\sigma^{B'}(t,s)dB_t^{Q}\right)$
$M_T^{(S)}=E^{Q}\left[\exp\left(-\int_T^S(\lambda_u)du\right)\mid G_T\right]+E^{Q}\left[\exp\left(-\int_T^S(r_u)du\right)\mid G_T\right]$ use product rule to find $\sigma^{W'}(t,s)$
$\sigma^{B'}(t,s)$</td></tr>
<tr><td><Ref 2></td><td colspan="2">Check dynamic of $M(t,T,S)$. If its vol is det, then we can apply Black Formula for it.
Spot dynamics:
$dM_t^{(T)}=M_t^{(T)}\left(R_t dt-b^{r}(t,T)\beta^{r}dB_t-b^{X}(t,T)\beta^{X}dW_t\right)$
Forward dynamics:
$dM(t,T,S)=M(t,T,S)\begin{pmatrix}\left(b^{r}(t,T)-b^{r}(t,S)\right)\beta^{r}dB_t\\+\left(b^{X}(t,T)-b^{X}(t,S)\right)\beta^{X}dW_t\end{pmatrix}$
The vol is indeed deterministic, so we can apply Black formula</td></tr>
<tr><td><Ref 3></td><td colspan="2">$N(d_\pm)=\dfrac{\ln\frac{M(0,T,S)}{k}+\frac{1}{2}\Sigma}{\sqrt{\Sigma}}$
$\Sigma=\int_0^T\left[\left(b^{r}(t,T)-b^{r}(t,S)\right)\beta^{r}\right]^2dt+\int_0^T\left[\left(b^{X}(t,T)-b^{X}(t,S)\right)\beta^{r}\right]^2dt$</td></tr>
<tr><td>Conclusion</td><td colspan="2">$\bar{B}(0,S)\bar{Q}^{S}\left(\bar{B}(T,S)\geq K\right)-K\bar{B}(0,T)\bar{Q}^{T}\left(\bar{B}(T,S)\geq K\right)$
$=M_0^{(T)}\left[M(0,T,S)N(d_+)-KN(d_-)\right]$
$N(d_\pm)=\dfrac{\ln\frac{M(0,T,S)}{k}+\frac{1}{2}\Sigma}{\sqrt{\Sigma}}$
$\Sigma=\int_0^T\left[\left(b^{r}(t,T)-b^{r}(t,S)\right)\beta^{r}\right]^2dt+\int_0^T\left[\left(b^{X}(t,T)-b^{X}(t,S)\right)\beta^{r}\right]^2dt$</td></tr>
</table>

❖ **Method 2**

	Formula	**Note**

Method 2	$E^Q\left[\exp\left(-\int_0^T r(X_u)du\right)\cdot 1_{\{T<\tau\}}\left(\bar{B}(T,S)-K\right)^+\right]$ $=\bar{B}(0,T)E^Q\left[\frac{\exp\left(-\int_0^T r(X_u)du\right)}{\bar{B}(0,T)}\cdot 1_{\{T<\tau\}}\left(\bar{B}(T,S)-K\right)^+\right]$ $=\bar{B}(0,T)E^{\bar{Q}^T}\left[\left(\bar{B}(T,S)-K\right)^+\right]$ $=\bar{B}(0,T)E^{\bar{Q}^T}\left[\bar{B}(T,S)\cdot 1_{\bar{B}(T,S)\geq K}\right]-K\cdot\bar{B}(0,T)E^{\bar{Q}^T}\left[1_{\bar{B}(T,S)\geq K}\right]$ $=\bar{B}(0,S)\cdot\bar{Q}^S\left(\bar{B}(T,S)\geq K\right)-K\cdot\bar{B}(0,T)E^{\bar{Q}^T}\left[1_{\bar{B}(T,S)\geq K}\right]$ $=\bar{B}(0,S)\cdot\bar{Q}^S\left(\bar{B}(T,S)\geq K\right)-K\cdot B(0,T)\bar{Q}^T\left(\bar{B}(T,S)\geq K\right)$	$1_{T<\tau}$: knock out assumption <Ref 1>
<Ref 1>	$\bar{B}(0,T)E^{\bar{Q}^T}\left[\bar{B}(T,S)\cdot 1_{\bar{B}(T,S)\geq K}\right]$ $=\bar{B}(0,T)E^{\bar{Q}}\left[\frac{\bar{B}(T,S)}{\bar{B}(0,T)}\cdot\exp\left(-\int_0^T r_u du\right)1_{\bar{B}(T,S)\geq K}\right]$ $=E^{\bar{Q}}\left[E^{\bar{Q}}\left[\bar{B}(S,S)\cdot\exp\left(-\int_0^S r_u du\right)\mid F_T\right]\cdot 1_{\bar{B}(T,S)\geq K}\right]$ $=E^{\bar{Q}}\left[\bar{B}(S,S)\cdot\exp\left(-\int_0^S r_u du\right)\cdot 1_{\bar{B}(T,S)\geq K}\right]$ $=\bar{B}(0,S)\cdot E^{\bar{Q}}\left[\frac{\bar{B}(S,S)\cdot\exp\left(-\int_0^S r_u du\right)}{\bar{B}(0,S)}\cdot 1_{\bar{B}(T,S)\geq K}\right]$ $=\bar{B}(0,S)\cdot\bar{Q}^S\left(\bar{B}(T,S)\geq K\right)$	
	Because rt and Xt are normal distribution with $\sigma^X(t,s)$ is deterministic, so $\bar{B}(T,S)$ is log-normal distribution with complicated parameters depending on $\bar{a}$ and $\bar{b}$. We can use Black Formula to get the closed form solution	

Exercise 2 (Forward CDS and Option on CDS):

Type	90 Strike	110 Strike
Payer	78.9 bps	52.3 bps
Receiver	46 bps	85.2 bps

The options "knock-out" upon a credit even by ABC. Assume the forward duration of a CDS that start one year from now and matures five years from now is 3.526. Using these instruments, how could you create a risky zero coupon bond that matures one year from now? Assume no arbitrage, what would be yield on such a bond?

	Formula	Note
CDS option price: CALL	$C=E^Q\left[D(1)\cdot 1_{\{\tau>1\}}\left(\bar{S}_{(1)}-K\right)^+V_{(1)}^{ann}\right]$	

Use Put Call Parity	$C-P=E^Q\left[D(1)\cdot 1_{\{\tau>1\}}\left(\bar{S}_{(1)}-K\right)V_{(1)}^{ann}\right]$ $=E^Q\left[D(1)\cdot 1_{\{\tau>1\}}\cdot\bar{S}_{(1)}\cdot V_{(1)}^{ann}\right]-K\cdot E^Q\left[D(1)\cdot 1_{\{\tau>1\}}KV_{(1)}^{ann}\right]$ $=E^Q\left[D(1)\cdot 1_{\{\tau>1\}}\cdot\bar{S}_{(1)}\cdot V_{(1)}^{ann}\right]-K\cdot V_{(1)}^{ann}\cdot E^Q\left[D(1)\cdot 1_{\{\tau>1\}}\right]$	$*V_{(1)}^{ann}$ *l is known at time 0
	$78.9-46=X-90*3.526\cdot\bar{B}(0,1)$............(1) $52.3-85.2=X-110*3.526\cdot\bar{B}(0,1)$.........(2)	
From (2)-(1)	$3.29=3.526\cdot\bar{B}(0,1)\Leftrightarrow\bar{B}(0,1)=0.933069$.......(3)	
Yield	$\bar{B}(0,1)\cdot(1+Yield)=1$ $\Leftrightarrow Yield=0.0717$	
Conclusion	$Yield=0.0717$	

Exercise 3:

Homer picked 10 firms and enter Part I: Sell default protection on the 10 firms through CDS contracts, one for each firm in the basket. Each CDS contract carries a spread of 170 bps. Part II: Buy protection on the first default in the basket (~the equity tranche). i.e the contract provides shelter for the first default among the 10 firms. This contract trades at the spread of 1500 bps.

Homer is trying to convince Marge that this is an excellent deal:
If no firm default, he pockets 200 bps throughout the live of the strategy.
If a firm defaults, Homer will just unwind the remaining 9 CDS contract of Part I in which case Homer pockets 200 bps until the time of the first default. Should Marge be skeptical?

Comment	Yes. Marge should be skeptical. (1)We are not sure how the correlation of those 10 firms. If they are highly correlated, after the first default, there might be consecutive default afterwards. (2) When we unwind CDS contracts by entering opposite contracts. i.e. go to the market and sell this protection at the prevailing spread to those 10 firms. When we do that, there is no guarantee that the spread you are paying for those 10 firms is the same as before. Therefore, those 200 bps are not guarantee (3) The unwinding of the position will increase the counter-party risk into portfolio.

Exercise 4:

Given $G_t=\sigma\left(B_s,W_s\right)_{s\in[0,t]}$ **and** $F_t=G_t\vee\sigma\left(N_u\right)_{u\in[0,t]}$**, where** $N_t=1_{\tau\le t}$ **CIR for spot rate r:**

$dr_t=k^r\left(\theta^r-r_t\right)dt+\beta^r\sqrt{r_t}dB_t$ **and state process X:** $dX_t=k^X\left(\theta^X-X_t\right)dt+\beta^X\sqrt{X_t}dW_t$**. In the Cox-setting, we model the firm's pre-default intensity as** $\lambda_t=\omega r_t+(1-\omega)X_t$ **for some weight** $\omega\in[0,1]$

(EX4.1) Provided a closed form zero coupon bond $\bar{B}(t,T)$ **for this setting.**

(a) Part I: Solve a(.) and b(.) from ZCB with CIR : $B(t,T)=\exp\left(-a(T-t)-b(T-t)r_t\right),t\in[0,T]$

Solution for a(*) & b(*).	$a(T-t)=-\frac{2k^r\theta}{\beta^{r2}}\left(\log(2h)+\frac{1}{2}(k+h)s-\log\left[(k+h)\left(e^{hs}-1\right)+2h\right]\right)$

	$b(T-t)=\frac{2\left(e^{hs}-1\right)}{2h+\left(k^r+h\right)\left(e^{hs}-1\right)}$
Where	$h\triangleq\sqrt{k^2+2\beta^2}$ $S\triangleq T-t$

(b) Part II: get $\overline{B}(t,T)$

	Formula
$\overline{B}(t,T)$ **Derived Process**	$=E^Q\left[\exp\left(-\int_t^T r\left(X_u\right)du\right)1_{\{T<\tau\}}\mid F_t\right]$ $=1_{\{t<\tau\}}E^Q\left[\exp\left(-\int_t^T\left\{r\left(X_u\right)+\lambda\left(X_u\right)\right\}du\right)\mid G_t\right]$
$\lambda_t=\omega r_t+\left(1-\omega\right)X_t$	$=1_{\{t<\tau\}}E^Q\left[\exp\left(-\int_t^T\left\{r_u+\omega r_u+\left(1-\omega\right)X_u\right\}du\right)\mid G_t\right]$ $=1_{\{t<\tau\}}E^Q\left[\exp\left(-\int_t^T\left\{\left(1+\omega\right)r_t+\left(1-\omega\right)X_t\right\}du\right)\mid G_t\right]$ $=1_{\{t<\tau\}}E^Q\left[\exp\left(-\int_t^T\left(\left(1+\omega\right)r_u\right)du\right)\mid G_t\right]E^Q\left[\exp\left(-\int_t^T\left(1-\omega\right)X_u du\right)\mid G_t\right]$ $=1_{\{t<\tau\}}\cdot\exp\left(-a(T-t)-b(T-t)r_t\right)\cdot\exp\left(-c(T-t)-d(T-t)x_t\right)$ <Ref 1><Ref 2>
<Ref 1> Scaled (1+w) CIR for r	Let $r_t^{'}=(1+\omega)r_t$ into $dr_t=k^r\left(\theta^r-r_t\right)dt+\beta^r\sqrt{r_t}dB_t$. So $dr_t^{'}=\left(1+\omega\right)dr_t\ =k^r((1+\omega)\theta^t-\left(1+\omega\right)r_t)dt+\left(1+\omega\right)\sqrt{r_t}\beta^r dB_t$ $dr_t^{'}=k^r(\theta^{r'}-r_t^{'})dt+\sqrt{r_t^{'}}\beta^{r'}dB_t$ So $\theta^{r'}=(1+\omega)\theta^r$, $\beta^{r'}=\sqrt{1+\omega}\beta^r$, $k^{r'}=k^r$ $a(T-t)=-\frac{2k^r\theta}{\beta^{r'2}}\left(\log\left(2h\right)+\frac{1}{2}(k+h)s-\log\left[\left(k+h\right)\left(e^{hs}-1\right)+2h\right]\right)$ $b(T-t)=\frac{2\left(e^{hs}-1\right)}{2h+\left(k^r+h\right)\left(e^{hs}-1\right)}$ $h\triangleq\sqrt{k^2+2\beta^2}$ $S\triangleq T-t$
<Ref 2> Scaled (1-w)CIR for X	Let $X_t^{'}=(1-\omega)X_t$ into $dX_t=k^x\left(\theta^x-X_t\right)dt+\beta^X\sqrt{X_t}dW_t$. So $dX_t^{'}=\left(1-\omega\right)dX_t\ =k^X((1-\omega)\theta^t-\left(1-\omega\right)r_t)dt+\left(1-\omega\right)\sqrt{X_t}\beta^X dW_t$ $dX_t^{'}=k^X(\theta^{X'}-X_t^{'})dt+\sqrt{X_t^{'}}\beta^{X'}dW_t$ So $\theta^{r'}=(1-\omega)\theta^r$, $\beta^{r'}=\sqrt{1-\omega}\beta^r$, $k^{r'}=k^r$ $c(T-t)=-\frac{2k^X\theta}{\beta^{X'2}}\left(\log\left(2h\right)+\frac{1}{2}(k+h)s-\log\left[\left(k+h\right)\left(e^{hs}-1\right)+2h\right]\right)$ $d(T-t)=\frac{2\left(e^{hs}-1\right)}{2h+\left(k^X+h\right)\left(e^{hs}-1\right)}$

	$h \triangleq \sqrt{k^2+2\beta^2}$ $S \triangleq T-t$
Conclusion	$= \exp\left(-a(T-t)-b(T-t)r_t\right)\cdot\exp\left(-c(T-t)-d(T-t)X_t\right)$ a,b,c,d are detailed above

(EX4.2) Show $T \le T_N$, $V_T^{\text{Prot}} = (1-\pi)E^Q\left[\exp\left(-\int_T^\tau r_\upsilon d\upsilon\right)\cdot 1_{\tau\in[T,T_N]} \mid F_T\right]$

$= 1_{\tau>T}(1-\pi)E^Q\left[\int_T^{T_N}\lambda_U \exp\left(-\int_T^U (r_\upsilon+\lambda_\upsilon)d\upsilon\right)dU \mid G_T\right]$.

	Formula	Note
V_T^{Prot} **Definition**	$(1-\pi)E^Q\left[\frac{D(\tau)}{D(T)}1_{\{\tau<T_N\}} \mid F_T\right]\cdot 1_{\{\tau>T\}}$ $=(1-\pi)\cdot 1_{\{\tau>T\}}E^Q\left[\frac{D(\tau)}{D(T)}1_{\tau\in\{T,T_N\}} \mid F_T\right]$ $=(1-\pi)\cdot 1_{\{\tau>T\}}E^Q\left[\exp\left(-\int_T^\tau r_\upsilon d\upsilon\right)1_{\tau\in\{T,T_N\}} \mid F_T\right]$ $=(1-\pi)\cdot 1_{\tau>T}\cdot E^Q\left[\int_T^{T_N}\exp\left(-\int_T^U (r_\upsilon+\lambda_\upsilon)d\upsilon\right)\cdot\lambda(X_U)du \mid G_T\right]$	<Ref 1>
<Ref 1>	$P(t,G_T)=P(t,G_t)=-\frac{\partial}{\partial t}Q(\tau>t \mid G_t)=-\frac{\partial}{\partial t}\exp\left(-\int_0^t \lambda(X_u)du\right)$ $=\lambda(X_t)\exp\left(-\int_0^t\lambda(X_v)dv\right)$	Conditional density of τ
Conclusion	$V_T^{\text{Prot}}=(1-\pi)1_{\tau>T}E^Q\left[\int_T^{T_N}\lambda_U\exp\left(-\int_t^T(\lambda_\upsilon+r_\upsilon)d\upsilon\right)du \mid G_T\right]$	

(EX4.3) Explain how to use the result of question 1 to compute for U≥T:

$E^Q\left[\lambda_U\exp\left(-\int_T^U(r_\upsilon+\lambda_\upsilon)d\upsilon\right) \mid G_T\right]$. Hint: compute a derivative of the expression you found in question 1. Subsequently, use the ODEs for the CIR to simplify your expression

	Formula
$\lambda_t=\omega r_t+(1-\omega)X_t$ <Ref 1> r and x are driven by independent BW. So we can separate them. This result from Conditional Independence on jump note page	$E^Q\left[\lambda_U\exp\left(-\int_T^U(r_\upsilon+\lambda_\upsilon)d\upsilon\right) \mid G_T\right]$ $=E^Q\left[(\omega r_U+(1-\omega)X_U)\exp\left(-\int_T^U(r_\upsilon+\omega r_\upsilon+(1-\omega)X_\upsilon)d\upsilon\right) \mid G_T\right]$ $=E^Q\left[(\omega r_U+(1-\omega)X_U)\exp\left(-\int_T^U((1+\omega)r_\upsilon+(1-\omega)X_\upsilon)d\upsilon\right) \mid G_T\right]$

<Ref 2> <Ref 3>	$= E^Q\left[(\omega r_U)\exp\left(-\int_T^U((1+\omega)r_\upsilon+(1-\omega)X_\upsilon)d\upsilon\right)\mid G_T\right]$ $+E^Q\left[((1-\omega)X_U)\exp\left(-\int_T^U(r_\upsilon+\omega r_\upsilon+(1-\omega)X_\upsilon)d\upsilon\right)\mid G_T\right]$ $= E^Q\left[(\omega r_U)\exp\left(-\int_T^U((1+\omega)r_\upsilon)d\upsilon\right)\mid G_T\right]\cdot E^Q\left[\exp\left(-\int_T^U((1-\omega)X_\upsilon)d\upsilon\right)\mid G_T\right]$ $+E^Q\left[\exp\left(-\int_T^U((1+\omega)r_\upsilon)d\upsilon\right)\mid G_T\right]\cdot E^Q\left[((1-\omega)X_U)\exp\left(-\int_T^U((1-\omega)X_\upsilon)d\upsilon\right)\mid G_T\right]$ $=1_{\{t<\tau\}}\cdot\left[\omega(a'(T-t)+b'(T-t))\exp(-a(T-t)-b(T-t)r_t)\right]\cdot\exp(-c(T-t)-d(T-t)x_t)$ $+1_{\{t<\tau\}}\cdot\exp(-a(T-t)-b(T-t)r_t)\cdot\left[(c'(T-t)+d'(T-t))\exp(-c(T-t)-d(T-t)x_t)\right]$
<Ref 2> PART I: Take Derivative of the expression in Q1:	$E^Q\left[(\omega r_U)\exp\left(-\int_T^U((1+\omega)r_\upsilon)d\upsilon\right)\mid G_T\right]\cdot E^Q\left[\exp\left(-\int_T^U((1-\omega)X_\upsilon)d\upsilon\right)\mid G_T\right]$ $=1_{\{t<\tau\}}\cdot\left[\omega\frac{\partial}{\partial t}\exp(-a(T-t)-b(T-t)r_t)\right]\cdot\exp(-c(T-t)-d(T-t)x_t)$ $=\frac{\omega}{1+\omega}\left[(a'(T-t)+b'(T-t)r_t)\exp(-a(T-t)-b(T-t)r_t)\right]\cdot\exp(-c(T-t)-d(T-t)x_t)$
<Ref 3> PART II: Take Derivative of the expression in Q1:	$E^Q\left[\exp\left(-\int_T^U((1+\omega)r_\upsilon)d\upsilon\right)\mid G_T\right]\cdot E^Q\left[((1-\omega)X_U)\exp\left(-\int_T^U((1-\omega)X_\upsilon)d\upsilon\right)\mid G_T\right]$ $=1_{\{t<\tau\}}\cdot\exp(-a(T-t)-b(T-t)r_t)\cdot\left[(1+\omega)\frac{\partial}{\partial t}\exp(-c(T-t)-d(T-t)x_t)\right]$ $=\exp(-a(T-t)-b(T-t)r_t)\cdot\left[(c'(T-t)+d'(T-t)x_t)\exp(-c(T-t)-d(T-t)x_t)\right]$
Conclusion	$=\frac{\omega}{1+\omega}\left[(a'(T-t)+b'(T-t)r_t)\exp(-a(T-t)-b(T-t)r_t)\right]\cdot\exp(-c(T-t)-d(T-t)x_t)$ $+\exp(-a(T-t)-b(T-t)r_t)\cdot\left[(c'(T-t)+d'(T-t)x_t)\exp(-c(T-t)-d(T-t)x_t)\right]$ $a(T-t)=-\frac{2k^r\theta}{\beta^{r'2}}\left(\log(2h)+\frac{1}{2}(k+h)s-\log\left[(k+h)(e^{hs}-1)+2h\right]\right)$ $b(T-t)=\frac{2(e^{hs}-1)}{2h+(k^r+h)(e^{hs}-1)}$ $c(T-t)=-\frac{2k^X\theta}{\beta^{X'2}}\left(\log(2h)+\frac{1}{2}(k+h)s-\log\left[(k+h)(e^{hs}-1)+2h\right]\right)$ $d(T-t)=\frac{2(e^{hs}-1)}{2h+(k^X+h)(e^{hs}-1)}$ $a'(T-t)=-\frac{2k^r\theta}{\beta^{r'2}}\left(-\frac{1}{2}(k+h)+\frac{(k+h)h(e^{hs})}{(k+h)(e^{hs}-1)+2h}\right)$ $b'(T-t)=\frac{2(e^{hs}-1)(k+h)he^{hs}}{\left[2h+(k^r+h)(e^{hs}-1)\right]^2}-\frac{2he^{hs}}{2h+(k+h)(e^{hs}-1)}$ $c'(T-t)=-\frac{2k^X\theta}{\beta^{X'2}}\left(-\frac{1}{2}(k+h)s+\frac{(k+h)h(e^{hs})}{(k+h)(e^{hs}-1)+2h}\right)$

	$d'(T-t)=\frac{2\left(e^{hs}-1\right)(k+h)he^{hs}}{\left[2h+\left(k^{r}+h\right)\left(e^{hs}-1\right)\right]^{2}}-\frac{2he^{hs}}{2h+(k+h)\left(e^{hs}-1\right)}$ $h\triangleq\sqrt{k^2+2\beta^2}\;\; S\triangleq T-t$
By changing the order of integration in Question 2, question 3 provides an explicit expression for V_T^{Prot} up to a dU-integral which we can compute numerically	$V_T^{\text{Prot}}=(1-\pi)1_{\tau>T}E^Q\left[\int_T^{T_N}\lambda_U\exp\left(-\int_t^T(\lambda_\upsilon+r_\upsilon)d\upsilon\right)du\,\vert\, G_T\right]$ $=(1-\pi)1_{\tau>T}\int_T^{T_N}E^Q\left[\lambda_U\exp\left(-\int_t^T(\lambda_\upsilon+r_\upsilon)d\upsilon\right)\vert\, G_T\right]dU$

Forward starting CDS (knock out type) that matures two years from today and has payment dates T=3,4,5. Given recovery of 0.4, $\overline{s}_2$, the F2-measurable random variable $V_2^{ann}\overline{s}_2=V_2^{prot}$.

(EX4.4) Report the fair forward starting spread such that $E^Q\left[\exp\left(-\int_0^2 r_u du\right)\left(\overline{s}_2-\overline{s}^*\right)\right]=0$.

	Formula
Process	(1) The algorithm from 1.3 will always be the same. (2) Find the constant $\overline{s}^*$ and expression for $\overline{s}_2$: from V_2^{ann} and V_2^{prot} from the previous questions. (3) Simulate r and X on [0,2] (just like in problem 1). E.g., to find V^prot_2, where we need r_2 and X_2 and then numerically compute a Riemann integral over [2,5]. This will give you one realization of $\overline{s}_2$ (4) The simulated discount factor is simulated using the same paths for r as you used to get $\overline{s}_2$ (5) There is no payoff at time 2 if there default before time 2.
About constant $\overline{s}^*$	It can be shown as $\overline{s}^*=\frac{E^Q\left[\exp(-\int_0^2 r_u du)V_2^{\text{Prot}}\right]}{E^Q\left[\exp(-\int_0^2 r_u du)V_2^{Ann}\right]}$ Simulate independent paths for r and X in [0,2]. At T=2, look at the formula for s* in this question, all random variables inside the expectation are filtration F(2) measurable. Remove the expectation because it's F(2) measurable, then we can conclude s* = $\overline{s}_2$
About $\overline{s}_2$:	$\overline{s}_2$ is a random variable with a very complicated distribution.
Code	see below

```
%Result for Q4.4
clear all;

%define underlying parameters
r0=0.05;Theta_r=0.05;Kappa_r=0.25; Beta_r=0.15;
Theta_X=0.1; X0=0.1;Kappa_X=0.25; Beta_X=0.15;
w=0.5;

%Maturity
T = 2;

%Time Settings
MC_Loops = 10000;
TimeSteps = 50 * T;
dt = T / TimeSteps;

%Initiatlization
r     =zeros(MC_Loops, TimeSteps);
X     =zeros(MC_Loops, TimeSteps);
lambda=zeros(MC_Loops, TimeSteps);

%Independent Ranadom Variables
Zr = randn(MC_Loops, TimeSteps);
ZX = randn(MC_Loops, TimeSteps+1);
```

<table>
<tr><td></td><td>

```
r(:,1) = r0 + Kappa_r *(Theta_r - r0) *dt + Beta_r *sqrt(r0 * dt) * Zr(:,1);
X(:,1) = X0 + Kappa_X *(Theta_X - X0) *dt + Beta_X *sqrt(X0 * dt) * ZX(:,1);

% State Evaluation of r and X
for i = 1:TimeSteps
    r(:,i+1) =r(:,i) +Kappa_r *(Theta_r - r(:,i)) *dt +Beta_r *sqrt(r(:,i)*dt).* Zr(:,i);
    X(:,i+1) =X(:,i) +Kappa_X *(Theta_X - X(:,i)) *dt +Beta_X *sqrt(X(:,i)*dt).* ZX(:,i+1);
end

pv=exp(-dt*sum(r,2));

lambda=w*r+(1-w)*X;

V2ann=real(Vann(w,Kappa_r, Theta_r, Beta_r, r(:, TimeSteps),Kappa_X, Theta_X, Beta_X,
X(:,TimeSteps)));

V2prot=real(Vprot(w,Kappa_r, Theta_r, Beta_r, r(:, TimeSteps),Kappa_X, Theta_X, Beta_X,
X(:,TimeSteps),3));

SS=mean(pv.*V2prot)/mean(pv.*V2ann)

disp('Q4.4:');
disp(['S_Star     :',num2str(SS)]);
```

</td></tr>
<tr><td>Function</td><td>

```
function x = Vann(w, k1, theta1, beta1, r0, k2, theta2, beta2, x0)
        b1=CIRBond(k1, (1+w)*theta1, sqrt(1+w)*beta1, (1+w)*r0, k2, (1-w)*theta2, sqrt(1-
w)*beta2, (1-w)*x0, 1);
        b2=CIRBond(k1, (1+w)*theta1, sqrt(1+w)*beta1, (1+w)*r0, k2, (1-w)*theta2, sqrt(1-
w)*beta2, (1-w)*x0, 2);
        b3=CIRBond(k1, (1+w)*theta1, sqrt(1+w)*beta1, (1+w)*r0, k2, (1-w)*theta2, sqrt(1-
w)*beta2, (1-w)*x0, 3);
        x=b1+b2+b3;
```

</td></tr>
<tr><td>

$a'(T-t)$

$b'(T-t)$

$c'(T-t)$

$d'(T-t)$

</td><td>

```
function x = Vprot(w, k1, theta1, beta1, r0, k2, theta2, beta2, x0, dt)

x1 = CIRBond(k1, (1+w)*theta1, sqrt(1+w)*beta1, (1+w)*r0, k2, (1-w)*theta2, sqrt(1-
w)*beta2, (1-w)*x0, dt);
```

$\exp\left(-c(T-t)-d(T-t)x_t\right)$

```
theta1=(1+w)*theta1;
beta1=sqrt(1+w)*beta1;

s=dt;
h=sqrt(k1^2+2*beta1^2);

at=-2*k1*theta1/beta1^2*(-0.5*(k1+h)+(k1+h)*h/((k1+h)*(exp(h*dt)-1)+2*h));
bt=2*(exp(h*s)-1)*(k1+h)*h*exp(h*s)/(2*h+(k1+h)*(exp(h*s)-1))^2-
2*h*exp(h*dt)/(2*h+(k1+h)*(exp(h*dt)-1));

x2 = -w/(1+w).*x1.*(at+bt*(1+w).*r0);
```

$\frac{\omega}{1+\omega}\left[\left(a'(T-t)+b'(T-t)r_t\right)\exp\left(-a(T-t)-b(T-t)r_t\right)\right]\cdot\exp\left(-c(T-t)-d(T-t)x_t\right)$

```
theta1=(1-w)*theta2;
beta1=sqrt(1-w)*beta2;

h=sqrt(k2^2+2*beta1^2);

at=-2*k2*theta1/beta1^2*(-0.5*(k2+h)+(k2+h)*h/((k2+h)*(exp(h*dt)-1)+2*h));

bt=2*(exp(h*s)-1)*(k2+h)*h*exp(h*s)/(2*h+(k2+h)*(exp(h*s)-1))^2-
2*h*exp(h*s)/(2*h+(k2+h)*(exp(h*s)-1));


x4=x1.*(at+bt*(1-w).*x0);
```

$\exp\left(-a(T-t)-b(T-t)r_t\right)\cdot\left[\left(c'(T-t)+d'(T-t)x_t\right)\exp\left(-c(T-t)-d(T-t)x_t\right)\right]$

</td></tr>
</table>

	`x=0.6*(x2+x4);` $=\frac{\omega}{1+\omega}\left[\left(a'(T-t)+b'(T-t)r_t\right)\exp\left(-a(T-t)-b(T-t)r_t\right)\right]\cdot\exp\left(-c(T-t)-d(T-t)x_t\right)$ $+\exp\left(-a(T-t)-b(T-t)r_t\right)\cdot\left[\left(c'(T-t)+d'(T-t)x_t\right)\exp\left(-c(T-t)-d(T-t)x_t\right)\right]$

$a(T-t)$	`function x =cira(k, theta, beta, dt)`
	`s=dt;`
	`h=sqrt(k^2+2*beta^2);`
	`x=-2*k*theta/beta^2*(log(2*h)+0.5*(k+h)*s-log((k+h)*(exp(h*s)-1)+2*h))`
	$a(T-t)=-\frac{2k^r\theta}{\beta^{r2}}\left(\log(2h)+\frac{1}{2}(k+h)s-\log\left[(k+h)\left(e^{hs}-1\right)+2h\right]\right)$ $h\triangleq\sqrt{k^2+2\beta^2}$ $S\triangleq T-t$

$b(T-t)$	`function x =cirb(k, theta, beta, dt)`
	`s=dt;`
	`h=sqrt(k^2+2*beta^2);`
	`x=2*(exp(h*s)-1)/(2*h+(k+h)*(exp(h*s)-1));`
	$b(T-t)=\frac{2\left(e^{hs}-1\right)}{2h+\left(k^r+h\right)\left(e^{hs}-1\right)}$ $h\triangleq\sqrt{k^2+2\beta^2}$ $S\triangleq T-t$

$\overline{B}(t,T)$ Based on CIR Model	`function x = CIRBond(k1, theta1, beta1, r0, k2, theta2, beta2, x0, dt)`
	`x = exp(-a(k1, theta1, beta1, dt)-b(k1, theta1, beta1, dt)*r0).*exp(-a(k2, theta2, beta2, dt)-b(k2, theta2, beta2, dt)*x0);`
	$=\exp\left(-a(T-t)-b(T-t)r_t\right)\cdot\exp\left(-c(T-t)-d(T-t)X_t\right)$

Result	`Q4.4:`
	`V2_Prot :0.1079`
	`V2_ann :2.3548`
	`S_Star :0.0454`

(EX4.5) Plot where horizontal axis has varying parameters and vertical axis is the fair forward starting spread $\overline{s}^*$

	W	β^X	β^r
Set A	**0.5**	**0.15**	$\beta^r\in[0.05,0.5]$
Set B	**0.5**	$\beta^x\in[0.05,0.5]$	**0.15**
Set C	$\omega\in[0.1,0.9]$	**0.15**	**0.15**

	Formula
Case 1: Varying Beta_r	see code below

```
BETA=[0.05:0.05:0.5];
r0=0.05; Theta_r=0.05; X0=0.1; Theta_X=0.1; Kappa_r=0.25; Kappa_X=0.25; Beta_r=0.15;
Beta_X=0.15; w=0.5;
FFS=zeros(1, 10);

for i=1:10,
   betar=BETA(i);

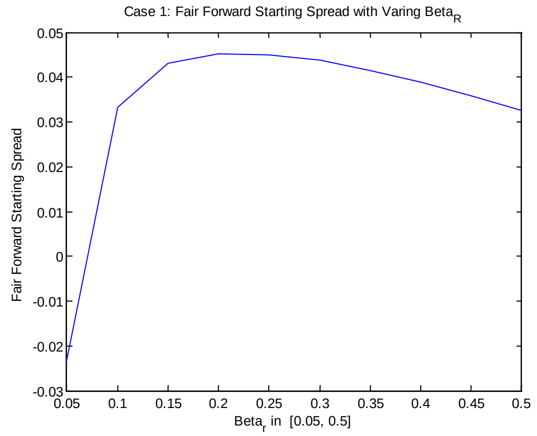
   V2Prot= real(Vprot(w, Kappa_r, Theta_r, Beta_r, r(:,TimeSteps), Kappa_X, Theta_X,
Beta_X, X(:,TimeSteps), 3));

   V2Ann=real( Vann(w, Kappa_r, Theta_r, Betar, r(:,TimeSteps), Kappa_X, Theta_X, Beta_X,
X(:,TimeSteps)) );

   FFS(i)=mean(pv.*V2Prot)/mean(pv.*V2Ann)
end;

plot(BETA, FFS);

xlabel('beta [0.05, 0.5]');
ylabel('Fair Forward Starting Spread ');
title('Case 1: with different Beta number');
```



Case 2: Varying Beta_X

```
BETA=[0.05:0.05:0.5];

FFS=zeros(1, 10);

r0=0.05; Theta_r=0.05; X0=0.1; Theta_X=0.1; Kappa_r=0.25; Kappa_X=0.25; Beta_r=0.15;
Beta_X=0.15; w=0.5;

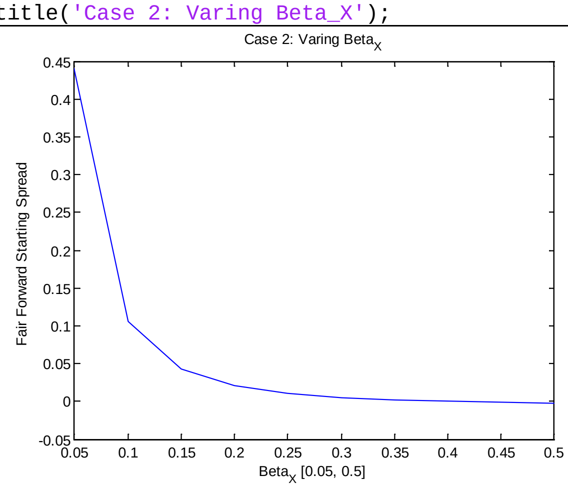
for i=1:10,
   Beta_X=BETA(i);
   V2prot= real(Vprot(w, Kappa_r, Theta_r, Beta_r, r(:,TimeSteps), Kappa_X, Theta_X,
Beta_X, X(:,TimeSteps), 3));

   V2ann=real( Vann(w, kr, Theta_r, Beta_r, r(:,TimeSteps), Kappa_X, Theta_X, Beta_X,
X(:,TimeSteps)) );

   FFS(i)=mean(pv.*V2prot)/mean(pv.*V2Ann)
end;

plot(BETA, FFS);

xlabel('Beta_X [0.05, 0.5]');
ylabel('Fair Forward Starting Spread ');
title('Case 2: Varing Beta_X');
```



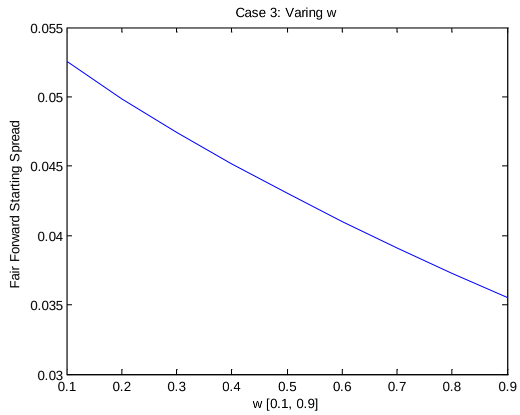
CASE 3: Varying W

```
%CASE 3: Varying w
W=[0.1:0.1:0.9];
r0=0.05; Theta_r=0.05; X0=0.1; ThetaX=0.1; Kappa_r=0.25; Kappa_X=0.25; Beta_r=0.15;
Beta_X=0.15; w=0.5;
FFS=zeros(1, 9);

for i=1:9,
   w=W(i);
   V2prot= real(Vprot(w, kr, thetar, betar, Rt(:,N), kX, thetaX, betaX, Xt(:,N), 3));

   V2ann=real( Vann(w, kr, thetar, betar, Rt(:,N), kX, thetaX, betaX, Xt(:,N)) );
```

	FFS(i)=mean(pv.*V2prot)/mean(pv.*V2ann)
	end;
	
	plot(W, FFS);
	
	xlabel('w [0.1, 0.9]');
	ylabel('Fair Forward Starting Spread ');
	title('Case 3: Varing w');

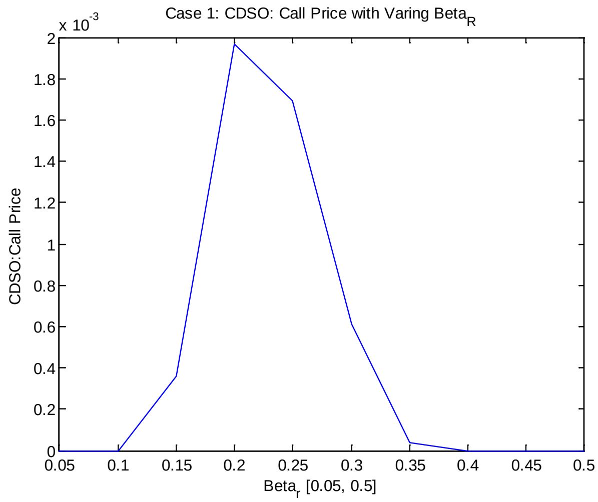
Consider a European call option of the knock-out type on this forward starting CDS, i.e. a payer swaption. The strike spread K* is always taken as the fair forward spread corresponding to the base case parameters from question 4, i.e., K* is always the value you found in question 4 in the following questions. This contract pays out a time 2 $\left(\overline{s}_2 - K^*\right)^+ V_2^{ann}$ ***, which indeed is zero if there has been a default before time 2.***

(EX4.6) Report the initial option price, the time zero option price. Plot the option value on the vertical axis against the three sets of variantion from question 5

	Formula
Given the result from before	Kstar=0.0454
Option with Base Parameters	V2Prot=real(Vprot(w, kr, Thetar, Betar, r(:,TimeSteps), kX, ThetaX, BetaX, X(:,TimeSteps), 3)); V2Ann=real(Vann(w, kr, Thetar, Betar, r(:,TimeSteps), kX, ThetaX, BetaX, X(:,TimeSteps))); Base_call = mean(Discount.*max(V2Prot./V2Ann-Kstar, 0).*V2Ann)
	Base_call = 0.0109 + 0.0000i

	Formula

Case 1: Call Price with Varying Beta_r	Kstar=0.0454; %constant from Q4.
	call = zeros(n, 1);
	r0=0.05; Thetar=0.05; X0=0.1; Theta_X=0.1;
	Kappa_r=0.25; Kappa_X=0.25; Beta_r=0.15; Beta_X=0.15; w=0.5;
	
	BETA=[0.05:0.05:0.5];
	call=zeros(1, 10);
	
	for i = 1:10,
	betar=BETA(i);
	V2prot=real(Vprot(w, Kappa_r, Theta_r, Beta_r, r(:,TimeSteps), Kappa_X, Theta_X, Beta_X, X(:,TimeSteps), 3));
	V2ann=real(Vann(w, Kappa_r, Theta_r, Beta_r, r(:,TimeSteps), Kappa_X, Theta_X, Beta_X, X(:,TimeSteps)));
	
	call(i) = mean(pv.*max(V2prot./V2ann-Kstar, 0).*V2Ann);
	end;
	
	plot(BETA, call);
	xlabel('Beta_r [0.05, 0.5]');
	ylabel('CDSO:Call Price');
	title('Case 1: CDSO: Call Price with Varing Beta_R');

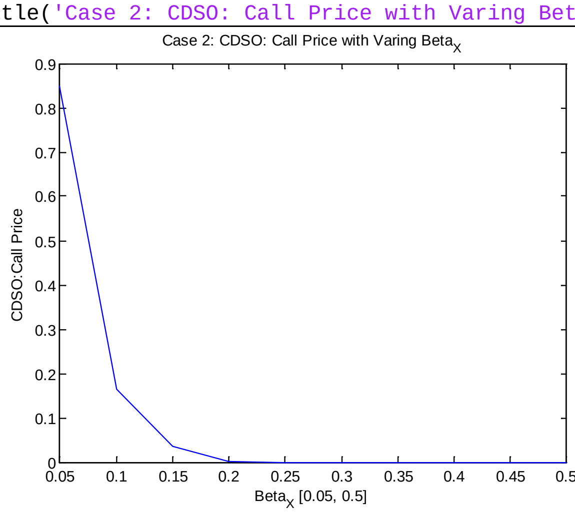


Case 2: Call Price with Varying Beta_X

```
%CASE 2: CDSO Call with Varying Beta_X
r0=0.05; Theta_r=0.05; X0=0.1; Theta_X=0.1;
Kappa_r=0.25; Kappa_X=0.25; Beta_r=0.15; Beta_X=0.15; w=0.5;
BETA=[0.05:0.05:0.5];
call=zeros(1, 10);

for i = 1:10,
    betaX=BETA(i);
    V2prot=real(Vprot(w, Kappa_r, Theta_r, Beta_r, r(:,TimeSteps), Kappa_X, Theta_X,
Beta_X, X(:,TimeSteps), 3));
    V2ann=real(Vann(w, Kappa_r, Theta_r, Beta_r, r(:,TimSteps), Kappa_X, Theta_X, BetaX,
X(:,TimeSteps)) );

    call(i) = mean( pv.*max(V2prot./V2ann-Kstar, 0).*V2Ann );
end;
plot(BETA, call);
xlabel('Beta_X [0.05, 0.5]');
ylabel('CDSO:Call Price');
title('Case 2: CDSO: Call Price with Varing Beta_X');
```



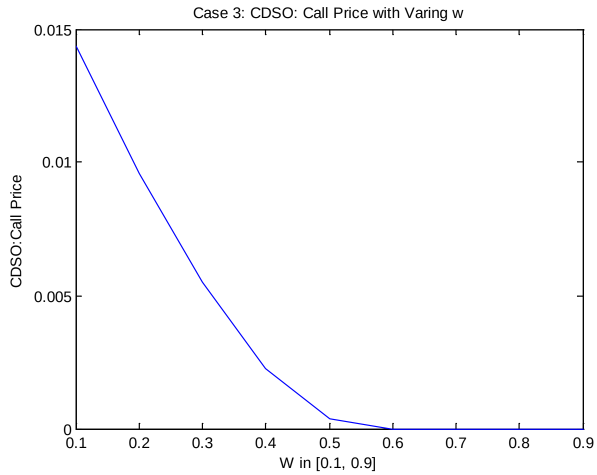
Case 3: Call Price with Varying w

```
%CASE 3: CDSO Call with Varying w
W=[0.1:0.1:0.9];
call=zeros(1, 9);
r0=0.05; Theta_r=0.05; X0=0.1; Theta_X=0.1;
Kappa_r=0.25; Kappa_X=0.25; Beta_r=0.15; Beta_X=0.15; w=0.5;

for i = 1:9,
    w=W(i);
    V2Prot=real(Vprot(w, Kappa_r, Theta_r, Betar, r(:,TimeSteps), Kappa_X, Theta_X, Beta_X,
X(:,TimeSteps), 3));
    V2Ann=real(Vann(w, Kappa_r, Theta_r, Beta_r, r(:,TimeSteps), Kappa_X, Theta_X, Beta_X,
X(:,TimeSteps)) );
```

```
    call(i) = mean( Discount.*max(V2Prot./V2Ann-Kstar, 0).*V2Ann );
end;

plot(W, call);
xlabel('W in [0.05, 0.5]');
ylabel('CDSO:Call Price');
title('Case 3: CDSO: Call Price with Varing w');
```



We assume that 2 year forward spread $\overline{s}_t^2$ satisfies the relation ($\overline{s}_t^2$ is not squared, but the value of the fair spread at time t, for a forward starting CDS at time 2), where $d\overline{s}_t^2 = \overline{s}_t^2 \sigma^{\overline{s}^2} dW_t^{Q^{ann}}$, where $W_t^{Q^{ann}}$ is a Brownian motion under the swap-measure Q^{ann}, i.e., the measure that uses Vann as numeraire (defaultable). On can prove that is always a Q^{ann} martingale, and here we specialized to the geometric Brownian motion that $\sigma^{\overline{s}^2}$ is a constant.

(EX4.7) Show analytically that we have for K>0 the relation $E^Q\left[\exp\left(-\int_0^2 r_u du\right)\left(\overline{s}_2 - K\right)^+ V_2^{ann}\right] = V_0^{ann} E^{Q^{ann}}\left[\left(\overline{s}_2 - K\right)^+\right]$ and explain how Black's formula can be used to compute the right-hand-side (recall that $\overline{s}_2$ is the fair CDS spread at time2)

	Formula
	$d\overline{S}_t = \overline{S}_t^2 \sigma^{\overline{S}^2} dW_t^{Q^{ann}}$ $E^Q\left[\exp\left(-\int_0^2 r_u du\right)\left(\overline{S}_2 - k\right)^+ V_2^{ann}\right]$ $= E^Q\left[V_0^{ann} \exp\left(-\int_0^2 r_u du\right)\left(\overline{S}_2 - k\right)^+ Z_t^{ann}\right]$ $= E^{Q^{ann}}\left[V_0^{ann}\left(\overline{S}_2 - k\right)^+\right]$ $= V_0^{ann} E^{Q^{ann}}\left[\left(\overline{S}_2 - k\right)^+\right]$
	We have $\overline{S}_2$ as the fair CDS spread at t=2, $\overline{S}_t^2$ as the 2 year forward starting spread value $\overline{S}_2 = \overline{S}_2^2$.

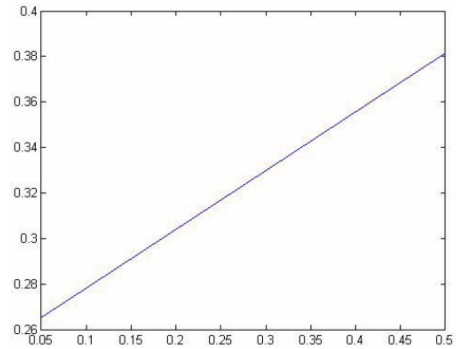
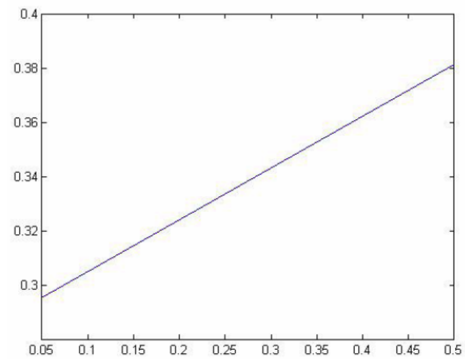
	Therefore RHS: $=V_0^{ann}E^{Q^{ann}}\left[\left(\bar{S}_2^2-k\right)^+\right]$

	Time 0	Time t	Time T
CDSO Definition	It is similar to interest rate Swaption, where the payer has the right to enter a swap(pay fix and receive floating) with the fixed rate equals the strike at maturity(of the option). For CDS option, the payer has the right to enter a CDS (pay spread receive protection) with the same spread as strike at maturity of the option, given no default happen before maturity, otherwise it pays nothing.		
Call Option on S_T Under Q Measure	$E^Q\left[\exp\left(-\int_0^T r(X_u)du\right)(S_T-K)^+V_T^{ann}\right]>0$ We are dealing non-negative RV, so it > 0		$(S_T-K)^+V_T^{ann}$ $=(S_T-K)^+\sum_{n:T_n>T}\bar{B}(T,T_n)$
Use SWAP Measure		$Z_t^{ann}:=\frac{\exp\left(-\int_0^t r(X_u)du\right)V_t^{ann}}{V_0^{ann}}$	$\frac{dQ^{ann}}{dQ}:=Z_T^{ann}$
S_T Under Q^{ann}	$E^Q\left[Z_T^{(ann)}(S_T-K)^+\right]V_0^{ann}$ $=E^{Q^{ann}}\left[(S_T-K)^+\right]V_0^{ann}$		
	(1) If $(S_t)_{t\in[0,T]}$ is lognormal under Q^{ann}, we can use Black's formula (2) Claim $(S_t)_{t\in[0,T]}$ is a Q^{ann} martingale		
Proof of Claim (2)	If (2) is true, $Z_t^{ann}S_t$ is a Q martingale $=\frac{V_t^{ann}S_t}{V_0^{ann}}\exp\left(-\int_0^t r_u du\right)=\frac{(1-\pi)V_t^{prot}}{V_0^{ann}}\exp\left(-\int_0^t r_u du\right)=\frac{(1-\pi)}{V_0^{ann}}\exp\left(-\int_0^t r_u du\right)E^Q\left[\exp\left(-\int_t^\tau r(X_u)du\right)\cdot 1_{\tau\in[T,T_N]}\mid F_t\right]$ $=\frac{(1-\pi)}{V_0^{ann}}E^Q\left[\exp\left(-\int_0^\tau r(X_u)du\right)\cdot 1_{\tau\in[T,T_N]}\mid F_t\right]$ $dS_t=\sigma_t^S dM_t^Q$		

(EX4.8) Report the implied volatility $\sigma^{\bar{S}^2}$ such that the log-normal model's output agrees with the output of the two factor model for the base case parameters. Subsequential, plot the implied vol on the vertical axies against the variation from question 5. For the plotting paurt, you may be able to find all implied vol values.

	Formula
Option with Base Parameters	`Base_call =` `0.0109 + 0.0000i`
Implied Vol from Based Call	`Base_implied vol =` `0.2319`
	$\sigma^{\bar{S}^2}=0.3414$. Option Price is 0.0351

	Formula

Case A: Implied Vol with Varying Beta_r	
Case B: Implied Vol with Varying Beta_X	
Case C: Implied Vol with Varying W