NYC

Topics of Quantitative Finance

HW3

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Exercise 1/Problem 2(Premium Adjusted Delta):

(i) Show that the premium adjusted spot delta can be written as $\Delta_S - \frac{\upsilon}{S}$

Definition	The delta is used to buy or sell spot in the corresponding amount in order to hedge the option up to first order. For consistency the premium needs to be incorporated into the delta hedge, since a premium in foreign currency will already hedge part of the option's delta risk.
Raw Delta adjusted with domestic premium	Based on the setting here, $\upsilon_S = \frac{\partial}{\partial S}\left(\frac{\upsilon}{S}\right)$ would be the value in domestic currency (DC) of an option with 1 Foreign currency (FC) notional. If Spot at S, the raw delta v denotes the number of FC to buy for the delta hedge Therefore, $S \cdot \upsilon_S = S \cdot \frac{\partial}{\partial S}\left(\frac{\upsilon}{S}\right)$ is the number USD to sell. This fits the hint in the question.
Raw Delta adjusted with Foreign premium	Right now we already have $\left(\frac{\upsilon}{S}\right) FC$, and the number of FC to buy has to be reduced by this amount. Therefore, we need to buy $\upsilon_S - \frac{\upsilon}{S} = \frac{\partial}{\partial S}\left(\frac{\upsilon}{S}\right) - \frac{\upsilon}{S}$ of FC for delta hedge or equivalent.
Conclusion	The premium adjusted spot delta can be written as $\Delta_S - \frac{\upsilon}{S}$

(ii) We define the premium adjusted forward delta as $\Delta_{F_{pa}} \triangleq \frac{S \cdot \frac{\partial}{\partial S}\left(\frac{\upsilon}{S}\right)}{\frac{\partial f}{\partial S}}$, which is the number of forward contract to buy when the premium is paid in foreign currency. Using this definition, show that the premium adjusted is given by $\phi \frac{K}{F} N(\phi d_-)$

Payment At delivery Date T	We are holding the forward price F fixed. The forward contract pays $S_T - F$ at the delivery date T.
Price At t=0 under domestic risk neutral	The value of this at t=0 is the expected value under the domestic risk neutral measure of $e^{-r_d \cdot T}\left(S_T - F\right)$
Forward Contract at t=0	Because $e^{-(r_d - r_f) \cdot t}\left(S_t\right)$ is a martingale under the domestic risk neutral measure, we write $e^{-r_d \cdot T}\left(S_T\right)$ as $e^{-r_f \cdot T} \cdot e^{-(r_d - r_f) \cdot T}\left(S_T\right)$ and take the domestic risk-neutral expectation to get $e^{-r_f \cdot T}\left(S_0\right)$. Therefore, the value of the forward contract at time zero is $f(S) = S \cdot e^{-r_f \cdot T} - F \cdot e^{-r_d \cdot T}$ where S is the initial spot price $\left(S = S_0\right)$. The derivative of this with respect to S is $\frac{\partial f(S)}{\partial S} = e^{-r_f \cdot T}$ (1)
$S \cdot \frac{\partial}{\partial S}\left(\frac{\upsilon}{S}\right)$	From (i) $S \cdot \frac{\partial}{\partial S}\left(\frac{\upsilon}{S}\right)$ can be written as $\Delta_S - \frac{\upsilon}{S}$ $\phi e^{-r_f \tau} N(\phi d_+) - \frac{1}{S} \phi e^{-r_d \tau}\left[FN\left(\phi d_+\right) - KN\left(\phi d_-\right)\right]$

	$= \phi e^{-r_f \tau} N(\phi d_+) - \phi e^{-r_d \tau}\left[\frac{Se^{(r_d - r_f)\tau}}{S} N(\phi d_+) - \frac{K}{S} N(\phi d_-)\right]$ $= \phi e^{-r_d \tau} \frac{K}{S} N(\phi d_-)$ $= \phi e^{-r_d \tau} \frac{K}{S \cdot e^{(r_d - r_f)\tau}} N(\phi d_-) e^{(r_d - r_f)\tau}$ $= \phi e^{r_f \tau} \frac{K}{F} N(\phi d_-)$ (2)
Combine (1)(2)	$\Delta_{F_{pa}} \triangleq \frac{S \cdot \frac{\partial}{\partial S}\left(\frac{\upsilon}{S}\right)}{\frac{\partial f}{\partial S}} = \frac{\phi e^{-r_f \tau} \frac{K}{F} N(\phi d_-)}{e^{-r_f \cdot T}} = \phi \frac{K}{F} N(\phi d_-)$

(iii) Show that the strike $\breve{K}$ that generates a premium adjusted forward delta position of zero of a straddle (call and put with strike $\breve{K}$) is given by $\breve{K} = F \cdot e^{-\frac{1}{2}\sigma^2 T}$. Such a straddle is used to hedge vega without affecting the delta position.

Method 1	Hint from the question itself
	$\frac{K}{F} N(d_-) - \frac{K}{F} N(-d_-) = 0$ $N(d_-) = N(-d_-)$ $N(d_-) = N(-d_-) = 1 - N(d_-)$ $N(d_-) = 0.5$
	$\ln \frac{F}{\breve{K}} = \frac{1}{2}\sigma^2 T$ $\breve{K} = F \cdot e^{-\frac{1}{2}\sigma^2 T}$
Method 2	Hint from TA
Find K*	Take derivative of $\Delta_{F_{pa}} = \phi \frac{K}{F} N(\phi d_-) = \frac{K}{F} N\left(\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$
$f_\sigma{}'(K) = 0$	$f_\sigma{}'(K) = \frac{1}{F} N\left(\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) + \frac{K}{F} N'\left(\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\left(-\frac{1}{K\sigma\sqrt{T}}\right)$ $= \frac{1}{F} N\left(\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - \left(\frac{1}{F} \cdot \frac{1}{\sigma\sqrt{T}}\right) N'\left(\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$ $= \frac{1}{F}\left[\frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{\left(\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} - \mu\right)^2}{2\sigma^2}\right)\right]$ $+ \left(-\frac{1}{F} \cdot \frac{1}{\sigma\sqrt{\tau}}\right)\left(-\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\left\{\frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{\left(\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} - \mu\right)^2}{2\sigma^2}\right)\right\}$

	$= \left(\frac{1}{F} + \frac{1}{F} \cdot \frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma^2 T} \right) \left\{ \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(- \frac{\left(\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} - \mu \right)^2}{2\sigma^2} \right) \right\} = 0$
To let $f_\sigma '(K) = 0$	$\Leftrightarrow \frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma^2 T} = 0$ $\Leftrightarrow \ln \frac{F}{K} - \frac{1}{2}\sigma^2 T = 0$ $\Leftrightarrow \ln \frac{F}{K} = \frac{1}{2}\sigma^2 T$ $\Leftrightarrow \frac{F}{K} = \exp(\frac{1}{2}\sigma^2 T)$ $\Leftrightarrow \frac{F}{\breve{K}} = \exp(\frac{1}{2}\sigma^2 T)$ $\Leftrightarrow F \exp(-\frac{1}{2}\sigma^2 T) = \breve{K}$
Conclusion	$\breve{K} = F \cdot e^{-\frac{1}{2}\sigma^2 T}$

(iv)Show that in the case of spot deltas without premium adjustment the relationship of strike and delta is monotone and show how to retrieve the strike from a given spot delta.

- **SPOT Delta without premium adjustment**

Monotone issue	ϕdt is monotone on K, therefore $N(\phi dt)$ is monotone on K. So Δ_S is a monotone function of K for both call and put
Stop Delta	$\Delta_S = \phi e^{-r_f \tau} N(\phi d_+)$ $d_+ = \frac{\ln \frac{F}{K} + \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}}$
	$\Delta_S = \phi e^{-r_f \tau} N(\phi d_+)$ $\phi e^{r_f \tau} \Delta_S = N(\phi d_+)$ $N^{-1}\left(\phi e^{r_f \tau} \Delta_S\right) = \frac{1}{\phi} \frac{\ln \frac{F}{K} + \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}}$ $N^{-1}\left(\phi e^{r_f \tau} \Delta_S\right) = -\frac{1}{\phi} \frac{\ln \frac{K}{F} - \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}}$ $-\phi N^{-1}\left(\phi e^{r_f \tau} \Delta_S\right) = \frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}}$ $-\phi N^{-1}\left(\phi e^{r_f \tau} \Delta_S\right) \cdot \sigma\sqrt{\tau} = \ln \frac{K}{F} - \frac{1}{2}\sigma^2 \tau$ $-\phi N^{-1}\left(\phi e^{r_f \tau} \Delta_S\right) \cdot \sigma\sqrt{\tau} + \frac{1}{2}\sigma^2 \tau = \ln \frac{K}{F}$ $e^{\left(-\phi N^{-1}\left(\phi e^{r_f \tau} \Delta_S\right) \cdot \sigma\sqrt{\tau} + \frac{1}{2}\sigma^2 \tau\right)} = \frac{K}{F}$ $K = F e^{\left(-\phi N^{-1}\left(\phi e^{r_f \tau} \Delta_S\right) \cdot \sigma\sqrt{\tau} + \frac{1}{2}\sigma^2 \tau\right)}$
Conclusion	SPOT Delta without premium adjustment is $K = F e^{\left(-\phi N^{-1}\left(\phi e^{r_f \tau} \Delta_S\right) \cdot \sigma\sqrt{\tau} + \frac{1}{2}\sigma^2 \tau\right)}$

- **Forward Delta without premium adjustment (For personal practice)**

Monotone issue	Δ_F is a monotone function of K for both call and put
Forward Delta	$\Delta_F = \phi N(\phi d_+)$ $d_+ = \frac{\ln\frac{F}{K}+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$
	$\Delta_f = \phi N(\phi d_+)$ $N^{-1}(\phi\Delta_F) = N(\phi d_+)$ $N^{-1}(\phi\Delta_F) = \frac{1}{\phi} d_+$ $N^{-1}(\phi\Delta_F) = \frac{1}{\phi}\frac{\ln\frac{F}{K}+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$ $N^{-1}(\phi\Delta_F) = -\frac{1}{\phi}\frac{\ln\frac{K}{F}-\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$ $-\phi N^{-1}(\phi\Delta_F) = \frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$ $-\phi N^{-1}(\phi\Delta_F)\cdot\sigma\sqrt{\tau} = \ln\frac{K}{F}-\frac{1}{2}\sigma^2\tau$ $-\phi N^{-1}(\phi\Delta_F)\cdot\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau = \ln\frac{K}{F}$ $e^{\left(-\phi N^{-1}(\phi\Delta_F)\cdot\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau\right)} = \frac{K}{F}$ $K = Fe^{\left(-\phi N^{-1}(\phi\Delta_F)\cdot\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau\right)}$
Conclusion	$K = Fe^{\left(-\phi N^{-1}(\phi\Delta_F)\cdot\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau\right)}$

(v)Show that in the case of premium adjusted forward deltas, the relationship of strike to delta for a call option is no longer guaranteed to be monotone. ON particular, show that there is another strike that generated the same delta as $\breve{K}$ if $\sigma > \sqrt{\frac{2}{\pi T}}$.

Monotone issue	If $\phi = -1$, we are fine. If $\phi = 1$, there will be a problem (Because N(d-) will move oppositely while K increases)
Goal	We want s to find K*, which is the top of the $f_\sigma(K^*)$. If K* $K \neq K^*$, $\exists K' \neq K$, such that $f_\sigma(K) = f_\sigma(K')$
Process 1.Find K* ($\phi = 1$)	$f_\sigma'(K) = \frac{1}{F}N\left(\frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)+\frac{K}{F}N'\left(\frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\left(-\frac{1}{K\sigma\sqrt{T}}\right)$ $= \frac{1}{F}N\left(\frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)-\left(\frac{1}{F}\cdot\frac{1}{\sigma\sqrt{T}}\right)N'\left(\frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$

	$=\frac{1}{F}\left[\frac{1}{\sqrt{2\pi}\cdot\sigma}\exp\left(-\frac{\left(\frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}-\mu\right)^2}{2\sigma^2}\right)\right]$ $+\left(-\frac{1}{F}\cdot\frac{1}{\sigma\sqrt{\tau}}\right)\left(-\frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)\left\{\frac{1}{\sqrt{2\pi}\cdot\sigma}\exp\left(-\frac{\left(\frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}-\mu\right)^2}{2\sigma^2}\right)\right\}$ $=\left(\frac{1}{F}+\frac{1}{F}\cdot\frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2T}{\sigma^2T}\right)\left\{\frac{1}{\sqrt{2\pi}\cdot\sigma}\exp\left(-\frac{\left(\frac{\ln\frac{F}{K}-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}-\mu\right)^2}{2\sigma^2}\right)\right\}$
From (iii) , we Plug $\breve{K}=F\cdot e^{-\frac{1}{2}\sigma^2T}$	$f_\sigma'(K)=\left(\frac{1}{F}+\frac{1}{F}\cdot\frac{\ln\frac{F}{F\cdot e^{-\frac{1}{2}\sigma^2T}}-\frac{1}{2}\sigma^2T}{\sigma^2T}\right)\left\{\frac{1}{\sqrt{2\pi}\cdot\sigma}\exp\left(-\frac{\left(\frac{\ln\frac{F}{F\cdot e^{-\frac{1}{2}\sigma^2T}}-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}-\mu\right)^2}{2\sigma^2}\right)\right\}$ $=\frac{1}{2F}-\frac{1}{F\sqrt{2\pi T}\cdot\sigma}$
When we plug $\sigma=\sqrt{\frac{2}{\pi T}}$	We got $f_\sigma'(K)=0$ $\breve{K}=K^*$
Discussion	When $\sigma<\sqrt{\frac{2}{\pi T}}$ We will get negative slope, the second term is bigger than the first term. Based on the same reasoning, when $\sigma>\sqrt{\frac{2}{\pi T}}$, we will get positive slope. As we discuss about the monotone issue before. This result makes $f_\sigma'(K)$ not a monotone anymore.
Result	When $\sigma>\sqrt{\frac{2}{\pi T}}$, we will have more than one strike.

❖ **Summary (Personal Review)**

	Non-Premium Adjusted		Premium Adjusted	
	SPOT	**FORWARD**	**SPOT**	**FORWARD**
Delta	$\phi e^{-r_f\tau}N(\phi d_+)$ $d_+=\frac{\ln\frac{F}{K}+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$	$\phi N(\phi d_+)$	$=\phi e^{r_f\tau}\frac{K}{F}N(\phi d_-)$ $=\phi e^{-r_d\tau}\frac{K}{S}N(\phi d_-)$	$=\frac{\phi e^{-r_f\tau}\frac{K}{F}N(\phi d_-)}{e^{-r_f\cdot T}}$ $=\phi\frac{K}{F}N(\phi d_-)$

			$\Delta_{S_{pa}} = \Delta_S - \frac{\upsilon}{S}$	$\Delta_{F_{pa}} \triangleq \frac{S\cdot\frac{\partial}{\partial S}\left(\frac{\upsilon}{S}\right)}{\frac{\partial f}{\partial S}}$
K	$K = Fe^{\left(-\phi N^{-1}\left(\phi e^{r_f\tau}\Delta_S\right)\cdot\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau\right)}$	$K = Fe^{\left(-\phi N^{-1}(\phi\Delta_F)\cdot\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau\right)}$		

Exercise 2/Case Study (Risk Managing an Exotic Option):

(i) By homogeneity argument that the deltas are given by the following $\frac{\partial\upsilon}{\partial S_1(t)}$, $\frac{\partial\upsilon}{\partial S_2(t)}$, $\frac{\partial\upsilon}{\partial K}$

$a\upsilon\left(t,S_1(t),S_2(t),K\right)$	$=a\phi\begin{bmatrix} S_1(t)e^{-q_1\tau}\mathcal{N}_2\left(\phi d_1,\eta d_3;\phi\eta\rho_1\right)+S_2(t)e^{-q_2\tau}\mathcal{N}_2(\phi d_2,\eta d_4;\phi\eta\rho_2) \\ -Ke^{-r\tau}\left(\frac{1-\phi\eta}{2}+\phi\mathcal{N}_2\left(\eta\left(d_1-\sigma_1\sqrt{\tau}\right)\phi,\eta\left(d_2-\sigma_2\sqrt{\tau}\right)\phi;\rho\right)\right)\end{bmatrix}$ $=\phi\begin{bmatrix} aS_1(t)e^{-q_1\tau}\mathcal{N}_2\left(\phi d_1,\eta d_3;\phi\eta\rho_1\right)+aS_2(t)e^{-q_2\tau}\mathcal{N}_2(\phi d_2,\eta d_4;\phi\eta\rho_2) \\ -aKe^{-r\tau}\left(\frac{1-\phi\eta}{2}+\phi\mathcal{N}_2\left(\eta\left(d_1-\sigma_1\sqrt{\tau}\right)\phi,\eta\left(d_2-\sigma_2\sqrt{\tau}\right)\phi;\rho\right)\right)\end{bmatrix}$ $=\upsilon\left(t,aS_1(t),aS_2(t),aK\right)$ (1)
We follow the example in the class	$a\upsilon\left(t,S_1(t),S_2(t),K\right)=\upsilon\left(t,aS_1(t),aS_2(t),aK\right)\Big\vert_{\frac{\partial}{\partial a}\vert a=1}$ We found that for a strike-defined value function, we have similar result for all a>0 We differentiae with respect to a for a=1 $\upsilon\left(S_1(t),S_2(t),K\right)$ $=\upsilon_{S_1}\left(S_1(t),S_2(t),K\right)\cdot S_1+\upsilon_{S_2}\left(S_1(t),S_2(t),K\right)\cdot S_2+\upsilon_k\left(S_1(t),S_2(t),K\right)\cdot K$ $=\Delta_{S_1}\cdot S_1+\Delta_{S_2}\cdot S_2+\Delta_k\cdot K$ (1) This is model independent. We don't need model to derive this.
Compare (1) and (2)	$a\upsilon\left(t,S_1(t),S_2(t),K\right)$ $=\phi\begin{bmatrix} S_1(t)e^{-q_1\tau}\mathcal{N}_2\left(\phi d_1,\eta d_3;\phi\eta\rho_1\right)+S_2(t)e^{-q_2\tau}\mathcal{N}_2(\phi d_2,\eta d_4;\phi\eta\rho_2) \\ -Ke^{-r\tau}\left(\frac{1-\phi\eta}{2}+\phi\mathcal{N}_2\left(\eta\left(d_1-\sigma_1\sqrt{\tau}\right)\phi,\eta\left(d_2-\sigma_2\sqrt{\tau}\right)\phi;\rho\right)\right)\end{bmatrix}$ (2)
Therefore we will get	$\frac{\partial\upsilon}{\partial S_1(t)}=\Delta_{S_1}=\phi e^{-q_1\tau}\mathcal{N}_2\left(\phi d_1,\eta d_3;\phi\eta\rho_1\right)$ $\frac{\partial\upsilon}{\partial S_2(t)}=\Delta_{S_2}=\phi e^{-q_2\tau}\mathcal{N}_2\left(\phi d_2,\eta d_4;\phi\eta\rho_2\right)$ $\frac{\partial\upsilon}{\partial K}=\Delta_K=-\phi Ke^{-r\tau}\left(\frac{1-\phi\eta}{2}+\phi\mathcal{N}_2\left(\eta\left(d_1-\sigma_1\sqrt{\tau}\right)\phi,\eta\left(d_2-\sigma_2\sqrt{\tau}\right)\phi;\rho\right)\right)$

(ii)

(ii.a) Show gamma: $\frac{\partial^2\upsilon}{\partial\left(S_1(t)\right)^2}$

Recall	$\frac{\partial \upsilon}{\partial S_1(t)} = \phi e^{-q_1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\rho_1)$
$\frac{\partial^2\upsilon}{\partial\left(S_1(t)\right)^2}$	$\frac{\partial \frac{\partial \upsilon}{\partial S_1(t)}}{\partial S_1(t)} = \phi e^{-q_1\tau}\left(\frac{\partial\mathcal{N}_2}{\partial d_1}\cdot\frac{\partial d_1}{\partial S_1} + \frac{\partial\mathcal{N}_2}{\partial d_3}\cdot\frac{\partial d_3}{\partial S_1} + \frac{\partial\mathcal{N}_2}{\partial\rho_1}\cdot\frac{\partial\rho_1}{\partial S_1}\right)$ $= \phi e^{-q_1\tau}\left(\frac{\partial\mathcal{N}_2}{\partial d_1}\cdot\frac{\partial d_1}{\partial S_1} + \frac{\partial\mathcal{N}_2}{\partial d_3}\cdot\frac{\partial d_3}{\partial S_1}\right)$ <Ref a.1><Ref a.2> $= \phi e^{-q_1\tau}\left(\mathcal{N}\left(\frac{\eta d_3 - \phi\eta\rho_1\phi d_1}{\sqrt{1-(\phi\eta\rho_1)^2}}\right)\cdot n(\phi d_1)\cdot\frac{\partial d_1}{\partial S_1} + \mathcal{N}\left(\frac{\phi d_1 - \phi\eta\rho_1\eta d_3}{\sqrt{1-(\phi\eta\rho_1)^2}}\right)\cdot n(\eta d_3)\cdot\frac{\partial d_3}{\partial S_1}\right)$ <Ref a.3> $= \phi e^{-q_1\tau}\left(\phi n(d_1)\cdot\mathcal{N}\left(\eta\frac{d_3-\rho_1 d_1}{\sqrt{1-\rho_1^2}}\right)\cdot\frac{1}{S_1\sigma_1\sqrt{\tau}} + \eta n(d_3)\cdot\mathcal{N}\left(\phi\frac{d_1-\rho_1 d_3}{\sqrt{1-\rho_1^2}}\right)\cdot\left(-\frac{1}{S_1\sigma\ \sqrt{\tau}}\right)\right)$ $= \frac{\phi e^{-q_1\tau}}{S_1\sqrt{\tau}}\left(\frac{\phi}{\sigma_1}n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right) - \frac{\eta}{\sigma}\cdot n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\right)$
<Ref a.1>	In general, the density function f is a function of x,y, sigma_x, sigma_y, mu_x, mu_y and rho. However, in our hw, N2(x,y, rho) refers to the cdf of two-dimensional jointly standard normal random variables, i.e. $\mu_X = \mu_Y = 0$, and $\sigma_X = \sigma_Y = 1$. Hence f is function of x, y and rho and $\mathcal{N}_2(x,y,\rho) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(m,n,\rho)dmdn$ $\frac{\partial\mathcal{N}_2}{\partial x} = \int_{-\infty}^{y} n_2(x,b,\rho)db = \int_{-\infty}^{y}\frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\cdot\frac{\left(x^2+b^2-2\rho xb\right)}{1-\rho^2}\right)db$ $= \frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho^2}}\left(-2\cdot\frac{1-\rho^2}{\left(x^2+b^2-2\rho xb\right)}\right)\exp\left(-\frac{1}{2}\cdot\frac{\left(x^2+b^2-2\rho xb\right)}{1-\rho^2}\right)\Bigg\vert_{-\infty}^{y}$ $= -\frac{1}{\pi}\left(\frac{\sqrt{1-\rho^2}}{\left(x^2+b^2-2\rho xb\right)}\right)\exp\left(-\frac{1}{2}\cdot\frac{\left(x^2+b^2-2\rho xb\right)}{1-\rho^2}\right)\Bigg\vert_{-\infty}^{y}$ $= -\frac{1}{\pi}\left(\frac{\sqrt{1-\rho^2}}{\left(x^2+y^2-2\rho xy\right)}\right)\exp\left(-\frac{1}{2}\cdot\frac{\left(x^2+y^2-2\rho xy\right)}{1-\rho^2}\right)$ $+\frac{1}{\pi}\left(\frac{\sqrt{1-\rho^2}}{\left(x^2+b^2-2\rho x(-\infty)\right)}\right)\exp\left(-\frac{1}{2}\cdot\frac{\left(x^2+(-\infty)^2-2\rho x(-\infty)\right)}{1-\rho^2}\right)$ $= -\frac{1}{\pi}\left(\frac{\sqrt{1-\rho^2}}{\left(x^2+y^2-2\rho xy\right)}\right)\exp\left(-\frac{1}{2}\cdot\frac{\left(x^2+y^2-2\rho xy\right)}{1-\rho^2}\right)$ $= -\frac{1}{\pi}\left(\frac{\sqrt{1-\rho^2}}{\left(x^2+y^2-2\rho xy\right)}\right)\frac{\sqrt{1-\rho^2}}{1}2\pi\cdot\frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\cdot\frac{\left(x^2+y^2-2\rho xy\right)}{1-\rho^2}\right)$

	$= -2\left(\frac{1-\rho^2}{\left(x^2+y^2-2\rho xy\right)}\right)\cdot n(x)$ <Ref a.1.i> $=\mathcal{N}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)\cdot n(x)$
<Ref a.1.i>	$\mathcal{N}(\frac{y-\rho x}{\sqrt{1-\rho^2}}) = \int_{-\infty}^{x}\int_{-\infty}^{y}\frac{n-\rho m}{\sqrt{1-\rho^2}}dmdn$ $=\int_{-\infty}^{x}\left(\frac{1}{\sqrt{1-\rho^2}}\right)\frac{1}{2}\left(n^2-\rho m\cdot n\right)\Big\vert_{n=-\infty}^{n=y} dm$ $=\int_{-\infty}^{x}\left(\frac{1}{\sqrt{1-\rho^2}}\right)\frac{1}{2}\left(y^2-\rho m\cdot y\right)\Big\vert_{n=-\infty}^{n=y} dm$ $= -2\left(\frac{1-\rho^2}{\left(x^2+y^2-2\rho xy\right)}\right)$
<Ref a.2>	We use both the formula for d(N_2(x,y;rho))/dx that your wrote in your question and the analogous formula for d(N_2(x,y;rho))/dy. $\frac{\partial\mathcal{N}_2}{\partial x}=\mathcal{N}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)\cdot n(x)$: by symmetry we can get $\frac{\partial\mathcal{N}_2}{\partial y}=\mathcal{N}\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)\cdot n(y)$ Plug $x=\phi d_1, y=\eta d_3, \rho=\eta\phi\rho_1$ $\Rightarrow\frac{\partial\mathcal{N}_2}{\partial d_1}=\mathcal{N}\left(\frac{\eta d_3-\phi\eta\rho_1\phi d_1}{\sqrt{1-(\phi\eta\rho_1)^2}}\right)\cdot n(\phi d_1)=\mathcal{N}\left(\frac{\eta d_3-\eta\rho_1 d_1}{\sqrt{1-\rho_1^2}}\right)\cdot\phi n(d_1)$ <Ref a.2.i> $=\mathcal{N}\left(\eta\frac{d_3-\rho_1 d_1}{\sqrt{1-\rho_1^2}}\right)\cdot\phi n(d_1)=\phi n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)$ <Ref a.2.ii> $\Rightarrow\frac{\partial\mathcal{N}_2}{\partial d_3}=\eta n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)$
<Ref a.2.i>	Note that $\phi^2=1,\eta^2=1,$
<Ref a.2.ii>	$\sqrt{1-\rho_1^2}=\sqrt{1-\frac{\rho^2\sigma_2^2-2\rho\sigma_2\sigma_1+\sigma_1^2}{\sigma^2}}=\sqrt{\frac{\sigma^2-\rho^2\sigma_2^2+2\rho\sigma_2\sigma_1-\sigma_1^2}{\sigma^2}}$ $=\sqrt{\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2-\rho^2\sigma_2^2+2\rho\sigma_2\sigma_1-\sigma_1^2}{\sigma^2}}=\sqrt{\frac{\sigma_2^2-\rho^2\sigma_2^2}{\sigma^2}}=\frac{\sigma_2}{\sigma}\sqrt{1-\rho^2}$
<Ref a.3>	$\frac{\partial d_1}{\partial S_1}=\frac{1}{S_1\sigma_1\sqrt{\tau}}, \frac{\partial d_3}{\partial S_1}=-\frac{1}{S_1\sigma\sqrt{\tau}}$
Conclusion	$\frac{\partial^2\upsilon}{\partial\left(S_1(t)\right)^2}=\frac{\phi e^{-q_1\tau}}{S_1\sqrt{\tau}}\left(\frac{\phi}{\sigma_1}n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)-\frac{\eta}{\sigma}\cdot n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\right)$

(ii.b) Show gamma: $\frac{\partial^2 \upsilon}{\partial \left(S_2(t)\right)^2}$

Recall	$\frac{\partial \upsilon}{\partial S_2(t)} = \phi e^{-q_2\tau}\mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\rho_2)$
$\frac{\partial^2 \upsilon}{\partial \left(S_2(t)\right)^2}$	$\frac{\partial \frac{\partial\upsilon}{\partial S_2(t)}}{\partial S_2(t)} = \phi e^{-q_2\tau}\left(\frac{\partial\mathcal{N}_2}{\partial d_2}\cdot\frac{\partial d_2}{\partial S_2}+\frac{\partial\mathcal{N}_2}{\partial d_4}\cdot\frac{\partial d_4}{\partial S_2}+\frac{\partial\mathcal{N}_2}{\partial\rho_1}\cdot\frac{\partial\rho_1}{\partial S_2}\right)$ $=\phi e^{-q_2\tau}\left(\frac{\partial\mathcal{N}_2}{\partial d_2}\cdot\frac{\partial d_2}{\partial S_2}+\frac{\partial\mathcal{N}_2}{\partial d_4}\cdot\frac{\partial d_4}{\partial S_2}\right)$ <Ref b.1> $=\phi e^{-q_2\tau}\left(\phi n(d_2)\cdot\mathcal{N}\left(\eta\sigma\frac{d_4-\rho_2 d_2}{\sigma_1\sqrt{1-\rho^2}}\right)\cdot\frac{\partial d_2}{\partial S_2}+\eta n(d_4)\cdot\mathcal{N}\left(\phi\sigma\frac{d_2-\rho_2 d_4}{\sigma_1\sqrt{1-\rho^2}}\right)\cdot\frac{\partial d_4}{\partial S_2}\right)$ <Ref b.2> $=\phi e^{-q_2\tau}\left(\phi n(d_2)\cdot\mathcal{N}\left(\eta\sigma\frac{d_4-\rho_2 d_2}{\sigma_1\sqrt{1-\rho^2}}\right)\cdot\frac{1}{S_2\sigma_2\sqrt{\tau}}+\eta n(d_4)\cdot\mathcal{N}\left(\phi\sigma\frac{d_2-\rho_2 d_4}{\sigma_1\sqrt{1-\rho^2}}\right)\cdot\left(-\frac{1}{S_2\sigma\sqrt{\tau}}\right)\right)$ $=\frac{\phi e^{-q_2\tau}}{S_2\sqrt{\tau}}\left(\frac{\phi}{\sigma_2}n(d_2)\cdot\mathcal{N}\left(\eta\sigma\frac{d_4-\rho_2 d_2}{\sigma_1\sqrt{1-\rho^2}}\right)-\frac{\eta}{\sigma}n(d_4)\cdot\mathcal{N}\left(\phi\sigma\frac{d_2-\rho_2 d_4}{\sigma_1\sqrt{1-\rho^2}}\right)\right)$
<Ref b.1>	$\frac{\partial\mathcal{N}_2}{\partial x}=\mathcal{N}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)\cdot n(x)$: by symmetry we can get $\frac{\partial\mathcal{N}_2}{\partial y}=\mathcal{N}\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)\cdot n(y)$ Plug $x=\phi d_2, y=\eta d_4, \rho=\eta\phi\rho_2$ $\Rightarrow \frac{\partial\mathcal{N}_2}{\partial d_2}=\mathcal{N}\left(\frac{\eta d_4-\phi\eta\rho_2\phi d_2}{\sqrt{1-(\phi\eta\rho_2)^2}}\right)\cdot n(\phi d_2)=\phi n(d_2)\mathcal{N}\left(\frac{\eta d_4-\eta\rho_2 d_2}{\sqrt{1-(\rho_2)^2}}\right)$ $=\phi n(d_2)\cdot\mathcal{N}\left(\eta\frac{d_4-\rho_2 d_2}{\sqrt{1-\rho_2^2}}\right)=\phi n(d_2)\cdot\mathcal{N}\left(\eta\sigma\frac{d_4-\rho_2 d_2}{\sigma_1\sqrt{1-\rho^2}}\right)$ <Ref b.2.i> $\Rightarrow\frac{\partial\mathcal{N}_2}{\partial d_4}=\eta n(d_4)\cdot\mathcal{N}\left(\phi\sigma\frac{d_2-\rho_2 d_4}{\sigma_1\sqrt{1-\rho^2}}\right)$
<Ref b.2.i>	$\sqrt{1-\rho_2^2}=\sqrt{1-\frac{\rho^2\sigma_1^2-2\rho\sigma_2\sigma_1+\sigma_2^2}{\sigma^2}}=\sqrt{\frac{\sigma^2-\rho^2\sigma_1^2+2\rho\sigma_2\sigma_1-\sigma_2^2}{\sigma^2}}$ $=\sqrt{\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2-\rho^2\sigma_1^2+2\rho\sigma_2\sigma_1-\sigma_2^2}{\sigma^2}}=\sqrt{\frac{\sigma_1^2-\rho^2\sigma_1^2}{\sigma^2}}=\frac{\sigma_1}{\sigma}\sqrt{1-\rho^2}$
<Ref b.2>	$\frac{\partial d_2}{\partial S_2}=\frac{1}{S_2\sigma_2\sqrt{\tau}}$, $\frac{\partial d_4}{\partial S_2}=-\frac{1}{S_2\sigma\sqrt{\tau}}$

Conclusion	$\frac{\partial^2 \upsilon}{\partial\left(S_2(t)\right)^2} = \frac{\phi e^{-q_2\tau}}{S_2\sqrt{\tau}}\left(\frac{\phi}{\sigma_2} n(d_2)\cdot\mathcal{N}\left(\eta\sigma\frac{d_4-\rho_2 d_2}{\sigma_1\sqrt{1-\rho^2}}\right) - \frac{\eta}{\sigma} n(d_4)\cdot\mathcal{N}\left(\phi\sigma\frac{d_2-\rho_2 d_4}{\sigma_1\sqrt{1-\rho^2}}\right)\right)$

(ii.c) Show gamma: $\frac{\partial^2 \upsilon}{\partial\left(S_1(t)\right)\partial\left(S_2(t)\right)}$

Recall	$\frac{\partial \upsilon}{\partial S_1(t)} = \phi e^{-q_1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\rho_1)$ $\frac{\partial \upsilon}{\partial S_2(t)} = \phi e^{-q_2\tau}\mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\rho_2)$
$\frac{\partial^2 \upsilon}{\partial\left(S_1(t)\right)\partial\left(S_2(t)\right)}$	$\frac{\partial \frac{\partial \upsilon}{\partial S_1(t)}}{\partial S_2(t)} = \phi e^{-q_1\tau}\left(\frac{\partial \mathcal{N}_2}{\partial d_1}\cdot\frac{\partial d_1}{\partial S_2} + \frac{\partial \mathcal{N}_2}{\partial d_3}\cdot\frac{\partial d_3}{\partial S_2} + \frac{\partial \mathcal{N}_2}{\partial \rho_1}\cdot\frac{\partial \rho_1}{\partial S_2}\right)$ $= \phi e^{-q_1\tau}\left(\frac{\partial \mathcal{N}_2}{\partial d_1}\cdot\frac{\partial d_1}{\partial S_2} + \frac{\partial \mathcal{N}_2}{\partial d_3}\cdot\frac{\partial d_3}{\partial S_2}\right)$ <Ref c. 1> $= \phi e^{-q_1\tau}\left(n(\eta d_3)\cdot\mathcal{N}\left(\phi\frac{d_1-\rho_1 d_3}{\sqrt{1-\rho_1^2}}\right)\cdot\left(\frac{1}{S_2(t)\sigma\sqrt{\tau}}\right)\right)$ $= \frac{\phi e^{-q_1\tau}}{S_2\sigma\sqrt{\tau}}\left(\eta n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\right)$ $= \frac{\phi e^{-q_1\tau}\eta}{S_2(t)\sigma\sqrt{\tau}}\cdot n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)$
<Ref c.1>	$\frac{\partial d_1}{\partial S_2} = 0$, $\frac{\partial d_3}{\partial S_2} = -\frac{1}{S_2\sigma\sqrt{\tau}}$
Conclusion	$\frac{\partial^2 \upsilon}{\partial\left(S_1(t)\right)\partial\left(S_2(t)\right)} = \frac{\phi e^{-q_1\tau}\eta}{S_2(t)\sigma\sqrt{\tau}}\cdot n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)$

(3.iii) Derive the sensitivity with respect to correlation, the vegas and the rhos

(3.iii.a) Show vega-Gamma relationship

Plackett's identity	$\frac{\partial}{\partial p_{ij}}\phi_p\left(x-\mu,\Sigma\right) = \frac{\partial^2}{\partial x_i \partial y_j}\phi_p\left(x-\mu,\Sigma\right)$ $\phi_p\left(x-\mu,\Sigma\right)$ is the density of a p-dimensional multivariate normal distribution with mean vector μ and covariance matrix $\Sigma = (\rho_{ij})$
(31) can be derived based on Plackett's identity	$\frac{\partial \upsilon}{\partial \rho} = \frac{\partial}{\partial p_{ij}}\phi_p\left(x-\mu,\Sigma\right) = \frac{\partial^2}{\partial x_i \partial y_j}\phi_p\left(x-\mu,\Sigma\right) = \sigma_1\sigma_2\tau S_1(t)S_2(t)\frac{\partial^2 \upsilon}{\partial S_1(t)\partial S_2(t)}$

From TA session, we know the following relationship $\sigma_1 \frac{\partial \upsilon}{\partial \sigma_1}$	$= \rho\sigma_1\sigma_2\tau S_1(t)S_2(t)\frac{\partial^2 \upsilon}{\partial S_1(t)\partial S_2(t)} + \sigma_1^2 S_1^2 \tau \frac{\partial^2 \upsilon}{\partial S_1(t)^2}$ <Ref 1><Ref 2> $= \rho\sigma_1\sigma_2\tau S_1(t)S_2(t)\cdot\frac{\phi e^{-q_1\tau}\eta}{S_2(t)\sigma\sqrt{\tau}}\cdot n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)$ $+\sigma_1^2 S_1^2\tau\cdot\frac{\phi e^{-q_1\tau}}{S_1\sqrt{\tau}}\left(\frac{\phi}{\sigma_1}n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)-\frac{\eta}{\sigma}\cdot n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\right)$
$\frac{\partial \upsilon}{\partial \sigma_1}$	$= \rho\sigma_2\tau S_1(t)S_2(t)\cdot\frac{\phi e^{-q_1\tau}\eta}{S_2(t)\sigma\sqrt{\tau}}\cdot n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)$ $+\sigma_1 S_1^2\tau\cdot\frac{\phi e^{-q_1\tau}}{S_1\sqrt{\tau}}\left(\frac{\phi}{\sigma_1}n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)-\frac{\eta}{\sigma}\cdot n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\right)$ $= \rho\sigma_2\sqrt{\tau}S_1\cdot\frac{\phi e^{-q_1\tau}\eta}{\sigma}\cdot n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)$ $+\sigma_1 S_1\sqrt{\tau}\phi e^{-q_1\tau}\left(\frac{\phi}{\sigma_1}n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)-\frac{\eta}{\sigma}\cdot n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\right)$ $= \sqrt{\tau}S_1\phi e^{-q_1\tau}\left[\rho\sigma_2\cdot\frac{\eta}{\sigma}\cdot n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\right]$ $+S_1\sqrt{\tau}\phi e^{-q_1\tau}\left[(\sigma_1)\cdot\left(\frac{\phi}{\sigma_1}n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)-\frac{\eta}{\sigma}\cdot n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\right)\right]$ $= \sqrt{\tau}S_1\phi e^{-q_1\tau}\begin{bmatrix}\eta\left(\frac{\rho\sigma_2-\sigma_1\tau}{\sigma}\right)\cdot n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\\ +(\sigma_1)\cdot\left(\frac{\phi}{\sigma_1}n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right)\end{bmatrix}$ $= \sqrt{\tau}S_1 e^{-q_1\tau}\left[\eta\left(\frac{\rho\sigma_2-\sigma_1}{\sigma}\right)\cdot\phi n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)+\phi^2 n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right]$ $= \sqrt{\tau}S_1 e^{-q_1\tau}\left[\eta\rho_1\cdot\phi n(d_3)\cdot\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)+n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right]$
<Ref 1>	$\frac{\partial^2\upsilon}{\partial\left(S_1(t)\right)^2}=\frac{\phi e^{-q_1\tau}}{S_1\sqrt{\tau}}\left(\frac{\phi}{\sigma_1}n(d_1)\cdot\mathcal{N}\left(\eta\sigma\frac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)-\frac{\eta}{\sigma}\cdot n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)\right)$

<Ref 2>	$\dfrac{\partial^2 \upsilon}{\partial S_1(t)\partial S_2(t)} = \dfrac{\phi e^{-q_1\tau}\eta}{S_2(t)\sigma\sqrt{\tau}} \cdot n(d_3)\cdot \mathcal{N}\left(\phi\sigma \dfrac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right)$
Conclusion	$\dfrac{\partial \upsilon}{\partial \sigma_1} = \sqrt{\tau}S_1 e^{-q_1\tau}\left[\eta\rho_1 \cdot \phi n(d_3)\cdot \mathcal{N}\left(\phi\sigma \dfrac{d_1-\rho_1 d_3}{\sigma_2\sqrt{1-\rho^2}}\right) + n(d_1)\cdot \mathcal{N}\left(\eta\sigma \dfrac{d_3-\rho_1 d_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right]$

(3.iii.b): From instruction from BB. We plug in past proved result

Similar Approach get be reached (because of symmetric relationship)	$\sigma_2 \dfrac{\partial \upsilon}{\partial \sigma_2} = \rho\sigma_1\sigma_2\tau S_1(t)S_2(t)\dfrac{\partial^2 \upsilon}{\partial S_1(t)\partial S_2(t)} + \sigma_2^2 S_2^2 \tau^2 \dfrac{\partial^2 \upsilon}{\partial S_2{}^2}$
From the previous proof result	We already know $\dfrac{\partial \upsilon}{\partial \rho}$ and $\dfrac{\partial^2 \upsilon}{\partial \left(S_2(t)\right)^2}$
Combine them we will get	$\dfrac{\partial \upsilon}{\partial \sigma_2} = \sqrt{\tau}S_2 e^{-q_2\tau}\left[\eta\rho_2 \cdot \phi n(d_4)\cdot \mathcal{N}\left(\phi\sigma \dfrac{d_2-\rho_2 d_4}{\sigma_1\sqrt{1-\rho^2}}\right) + n(d_2)\cdot \mathcal{N}\left(\eta\sigma \dfrac{d_4-\rho_2 d_2}{\sigma_1\sqrt{1-\rho^2}}\right)\right]$

(3.iii.c) Show Delta-Rho relationship

$\dfrac{\partial \upsilon}{\partial q_1} =$	$-S_1(t)\tau \dfrac{\partial \upsilon}{\partial S_1(t)} = -S_1(t)\tau \cdot \phi e^{-q_1\tau}\mathcal{N}_2\left(\phi d_1, \eta d_3; \phi\eta\rho_1\right)$
$\dfrac{\partial \upsilon}{\partial q_2} =$	$-S_2(t)\tau \dfrac{\partial \upsilon}{\partial S_2(t)} = -S_2(t)\tau \cdot \phi e^{-q_2\tau}\mathcal{N}_2\left(\phi d_2, \eta d_4; \phi\eta\rho_2\right)$
$\dfrac{\partial \upsilon}{\partial r} =$	$= -K\tau\dfrac{\partial \upsilon}{\partial K} = K^2\tau \cdot \phi e^{-r\tau}\left(\frac{1-\phi\eta}{2} + \phi\mathcal{N}_2\left(\eta\left(d_1-\sigma_1\sqrt{\tau}\right)\phi, \eta\left(d_2-\sigma_2\sqrt{\tau}\right)\phi; \rho\right)\right)$

(iv) Put on the maximum of two assets. A investment bank sell this product as part of structure retail issue for a size of 100 million USD. The retail client is given portfolio insurance for the worst performing asset at a comfortable price. After a financial crisis scenario, the spots have come down, the volatilities have risen, the correlation has moved towards 100%, the domestic rates have come down.
(iv.a)Determine the Black-Schoes theoretical value of both.
(iv.b)By looking at the Greeks state whether the issuing bank is longing or short: delta, gamma, vega, rho, correlation risk.
(iv.c)Redo the calculation assuming that back had established a delta hedge for both assets as inception and this position was kept for the first month.
*** MATLAB CODE is ALSO available in the ZIP file. (P4.m)**

	Code

Main	<pre>%Hw3.Q2.PART(A)-Before Hedging
	clear all;
	%---Parameters---
	t=0; T=3;
	s1=100; s2=100;K=98;
	q1=0.07;q2=0.1;r=0.05;
	sigma1=0.3;sigma2=0.32;
	rho=-0.2;</pre>

```
ini_wealth=100000000;

[price1 delta1 delta2 delta3] = put1(t, s1, s2, K, T, q1, q2, r, sigma1, sigma2, rho);
[gamma1 gamma2 gamma3] = put2(t, s1, s2, K, T, q1, q2, r, sigma1, sigma2, rho);

vega1=(rho*(sigma1*sigma2*(T-t)*s1*s2*gamma3)+sigma1^2*(T-t)*s1^2*gamma1)/sigma1;
vega2=(rho*(sigma1*sigma2*(T-t)*s1*s2*gamma3)+sigma2^2*(T-t)*s2^2*gamma2)/sigma2;
rho1=-s1*(T-t)*delta1;
rho2=-s2*(T-t)*delta2;
rho3=-K*(T-t)*delta3;
corr1=sigma1*sigma2*(T-t)*s1*s2*gamma3;

Hprice1=price1-s1*(-delta1)-s2*(-delta2);
Ncontract=floor(ini_wealth/price1);
NcontractH=abs(floor(ini_wealth/Hprice1));
CashH=NcontractH*Hprice1;

%after 6 months
t=0.5;T=3;
s1=70;s2=70;K=98;
q1=0.07;q2=0.1;r=0.02;
sigma1=0.8;sigma2=0.85;
rho=0.7;

[price2 delta21 delta22 delta23] = put1(t, s1, s2, K, T, q1, q2, r, sigma1, sigma2,
rho);
[gamma21 gamma22 gamma23] = put2(t, s1, s2, K, T, q1, q2, r, sigma1, sigma2, rho);

vega21=(rho*(sigma1*sigma2*(T-t)*s1*s2*gamma23)+sigma1^2*(T-t)*s1^2*gamma21)/sigma1;
vega22=(rho*(sigma1*sigma2*(T-t)*s1*s2*gamma23)+sigma2^2*(T-t)*s2^2*gamma22)/sigma2;
rho21=-s1*(T-t)*delta21;
rho22=-s2*(T-t)*delta22;
rho23=-K*(T-t)*delta23;
corr2=sigma1*sigma2*(T-t)*s1*s2*gamma23;

Hprice2=price2-s1*(-delta1)-s2*(-delta2);

PL=Ncontract*(price1-price2);
PLH=-NcontractH*Hprice2-CashH+ini_wealth;

%Before Hedge
disp('(Before Hedge)    At Inception        Six Months Later');
disp(['Initial Price:   ' num2str(price1) '                '   num2str(price2)]);
disp('                        ');
disp('Greek Summary from seller perspective');
disp(['Delta_S1:        ' num2str(-delta1) '              '   num2str(-delta21)]);
disp(['Delta_S2:        ' num2str(-delta2) '              '   num2str(-delta22)]);
disp(['Delta_K:          ' num2str(-delta3) '            '   num2str(-delta23)]);
disp('                        ');
disp(['Gamma_S1S1:      ' num2str(-gamma1) '              '   num2str(-gamma21)]);
disp(['Gamma_S2S2:      ' num2str(-gamma2) '              '   num2str(-gamma22)]);
disp(['Gamma_S1S2:      ' num2str(-gamma3) '              '   num2str(-gamma23)]);
disp('                        ');
disp(['Vega_sig1:       ' num2str(-vega1)  '               '   num2str(-vega21)]);
disp(['Vega_sig2:       ' num2str(-vega2)  '               '   num2str(-vega22)]);
disp('                        ');
disp(['Rho_q1:          ' num2str(-rho1)   '                '   num2str(-rho21)]);
disp(['Rho_q2:          ' num2str(-rho2)   '                '   num2str(-rho22)]);
disp(['Rho_r:            ' num2str(-rho3)   '            '   num2str(-rho23)]);
disp('                        ');
disp(['Correlation:     ' num2str(-corr1)   '                '   num2str(-corr2)]);
disp('                        ');
disp(['Contract sold (NonHedge):' num2str(Ncontract)]);
disp('                        ');
disp(['P&L after 6 mths:                    ' num2str(PL)]);

disp('                        ');
disp('(After Hedge)    At Inception        Six Months Later');
disp(['Initial Price:   ' num2str(Hprice1) '                '   num2str(Hprice2)]);
disp('                        ');
disp(['Contract sold:   ' num2str(NcontractH)]);
disp('                        ');
```

	`disp(['P&L after 6 mths:                    ' num2str(PLH)]);`

Function 1	<pre>function [price delta1 delta2 delta3] = put1(t, s1, s2, K, T, q1, q2, r, sigma1, sigma2, rho)
	
	tau=T-t;
	sigma=sqrt(sigma1^2+sigma2^2-2*rho*sigma1*sigma2);
	
	rho1=(rho*sigma2-sigma1)/sigma;
	rho2=(rho*sigma1-sigma2)/sigma;
	
	d1=(log(s1/K)+(r-q1+0.5*sigma1^2)*tau)/(sigma1*sqrt(tau));
	d2=(log(s2/K)+(r-q2+0.5*sigma2^2)*tau)/(sigma2*sqrt(tau));
	d3=(log(s2/s1)+(q1-q2-0.5*sigma^2)*tau)/(sigma*sqrt(tau));
	d4=(log(s1/s2)+(q2-q1-0.5*sigma^2)*tau)/(sigma*sqrt(tau));
	
	u=[0 0];
	
	price = s1*exp(-q1*tau)*mvncdf([-d1 -d3], u, [1 rho1;rho1 1])+s2*exp(-q2*tau)*mvncdf([-d2 -d4], u, [1 rho2;rho2 1])- K*exp(-r*tau)*mvncdf([-(d1-sigma1*sqrt(tau)) -(d2-sigma2*sqrt(tau))], u, [1 rho;rho 1]);
	price = -1*price;
	
	delta1=-1*exp(-q1*tau)*mvncdf([-d1 -d3], u, [1 rho1;rho1 1]);
	delta2=-1*exp(-q2*tau)*mvncdf([-d2 -d4], u, [1 rho2;rho2 1]);
	delta3=exp(-r*tau)*mvncdf([-(d1-sigma1*sqrt(tau)) -(d2-sigma2*sqrt(tau))], u, [1 rho;rho 1]);</pre>

Function 2	<pre>function [gamma1 gamma2 gamma3] = put2(t, s1, s2, K, T, q1, q2, r, sigma1, sigma2, rho)
	
	tau=T-t;
	sigma=sqrt(sigma1^2+sigma2^2-2*rho*sigma1*sigma2);
	
	rho1=(rho*sigma2-sigma1)/sigma;
	rho2=(rho*sigma1-sigma2)/sigma;
	
	d1=(log(s1/K)+(r-q1+0.5*sigma1^2)*tau)/(sigma1*sqrt(tau));
	d2=(log(s2/K)+(r-q2+0.5*sigma2^2)*tau)/(sigma2*sqrt(tau));
	d3=(log(s2/s1)+(q1-q2-0.5*sigma^2)*tau)/(sigma*sqrt(tau));
	d4=(log(s1/s2)+(q2-q1-0.5*sigma^2)*tau)/(sigma*sqrt(tau));
	
	u=[0 0];
	
	price = s1*exp(-q1*tau)*mvncdf([-d1 -d3], u, [1 rho1;rho1 1])+s2*exp(-q2*tau)*mvncdf([-d2 -d4], u, [1 rho2;rho2 1])- K*exp(-r*tau)*mvncdf([-(d1-sigma1*sqrt(tau)) -(d2-sigma2*sqrt(tau))], u, [1 rho;rho 1]);
	price = -1*price;
	
	gamma1=-1*exp(-q1*tau)/(s1*sqrt(tau))*(((-1/sigma1)*normpdf(d1)*normcdf(-1*sigma*(d3-d1*rho1)/sigma2/sqrt(1-rho^2))+(1/sigma)*normpdf(d3)*normcdf(-1*sigma*(d1-d3*rho1)/sigma2/sqrt(1-rho^2))));
	
	gamma2=-1*exp(-q2*tau)/(s2*sqrt(tau))*(((-1/sigma2)*normpdf(d2)*normcdf(-1*sigma*(d4-d2*rho2)/sigma1/sqrt(1-rho^2))+(1/sigma)*normpdf(d4)*normcdf(-1*sigma*(d2-d4*rho2)/sigma1/sqrt(1-rho^2))));
	
	gamma3=exp(-q1*tau)/(s2*sigma*sqrt(tau))*normpdf(d3)*normcdf(-1*sigma*(d1-d3*rho1)/sigma2/sqrt(1-rho^2));</pre>

Printed Result	<pre>(Before Hedge) At Inception Six Months Later
	Initial Price: 8.1027 47.6602
	
	Greek Summary from seller perspective
	Delta_S1: 0.14685 0.20203
	Delta_S2: 0.12072 0.15186
	Delta_K: -0.35571 -0.73911
	
	Gamma_S1S1: -0.0022931 -0.00069417
	Gamma_S2S2: -0.0014554 -0.00016448
	Gamma_S1S2: -0.0021289 -0.0018577
	
	Vega_sig1: -16.5508 -20.3432
	Vega_sig2: -10.1401 -14.4566</pre>

	<pre>Rho_q1: -44.0537 -35.355
	Rho_q2: -36.2156 -26.5759
	Rho_r: 104.5775 181.0816
	
	Correlation: -6.1312 -15.4747
	
	Contract sold (NonHedge):12341533
	
	P&L after 6 mths: -488200474.9718
	
	(Hedge: CASE1) At Inception Six Months Later
	Initial Price: -18.6537 28.9307
	Contract sold: 5360863
	P&L after 6 mths: 44906273.0989
	
	(Hedge: CASE2) At Inception Six Months Later
	Contract sold: 12341533
	S1 delta hedge amt: -1812301.5767
	S2 delta hedge amt: -1489852.6709
	P&L after 6 mths: -389135847.5427</pre>

- **VALUE of OPTION (Without Hedge)**

	At inception	Six Months Later
Price	8.102721	47.660244
Contract Sold	12,341,533	
P&L		-488,200,474.97

- **GREEK of OPTION: the sign of Greeks are reversed because the bank is on the selling side**

	At inception	Six Months Later	Long/Short position?
Delta_S1	0.146846	0.202029	Long
Delta_S2	0.120719	0.151863	Long
Delta_K	-0.355706	-0.739109	Short
Gamma_S1S1	-0.002293	-0.000694	Short
Gamma_S2S2	-0.001455	-0.000164	Short
Gamma_S1S2	-0.002129	-0.001858	Short
Vega_sig1	-16.550796	-20.343232	Short
Vega_sig2	-10.140111	-14.456577	Short
Rho_q1	-44.053723	-35.355036	Short
Rho_q2	-36.215582	-26.575942	Short
Rho_r	104.577468	181.081588	Long
Correlation	-6.131224	-15.474734	Short

- **VALUE of OPTION (With delta hedge):**
  - Some quick comment about P&L after six months: After six month, actually we only occur real stock position, while there is a **theoretical** P&L due change in premium. It is only theoretical, and we will not incur any real P&L on the option before expiration when potentially we assigned onto our short option position.
  - Hedge Method is the same for both Case 1 and Case 2: Bank has a short position on the option, bank need to short s1 and s2 to hedge. To hedge the short position of a put option, the bank needs to short 0.1468 share of S1 and 0.1207 share of S2, so the bank has the position = (short 1 option, short 0.1468 S1, short 0.1207 s2)
  - Assumption: We suppose there is no partial option can be sold. Therefore we that the integer option amount into consideration (Trivial cash might be left)

- CASE 1: We factor our hedge cost into our quoted price. This fact impact the contracts the bank can sell

	At inception	Six Months Later
Price	-18.653714	28.93074
Contract Sold	5,360,863	
P&L		44,906,273

- CASE 2: We hedge based on the original contract that the bank sell. Hedge S1 and S2 based on the contract.

	At inception	Six Months Later
Price	8.102721	
Contract Sold	12,341,533	
P&L		-389,135,847.54

(v)State which sources of risk can be hedged with other financial products and how.

Risk	How
Delta Risk	We can trade option and stock to hedge this risk. It can be easily hedged by holding underlying assets .
Gamma Risk	We can trade other option with positive Gamma on the same underlying to offset the negative Gamma
Vega Risk	A variance swap can be a static hedge for the vega risk of a structure if that structure P&L is linear in realized variance (which is not the same thing as constant vega). Hedge by trading other products on the volatility of the two underlying
Rho Risk	Rho of r will exposed us to the change of interest rate. We might use interest swap to hedge the risk.
Correlation Risk	We can hedge this risk by trading volatility. Or by observing the cross gamma and correlation relationship, we can find other products which has the opposite gross gamma characteristic to hedge the correlation exposure.

(vi)Assume the basis spread margins of -100 basis points for the first assets and -150 basis points for the second assets.
(vi.a) Compute the 3-year forward prices for both assets with and without basis spread. The forward prices with the spreads are the ones that we can trade.
(vi.b) Compute the rate qi implied by these forward prices keeping r fixed.
(vi.c) Compute the value of the maximum put option with these rates at inception and compare with the value computed without basis spread (The difference of these values will show the loss caused by not paying attention to the basis spread risk)

Note
1. Compare Fi(t,T) using $S_i(t)e^{(r-q_i)\tau}$ **2. Plug the forward price from (**) into (*) and back out qi** **3. Use place qi with implied** $\tilde{q}_i$ **from step 2and calculation the put option using (17)** **4. Compared step 3 with (iv)**
In practice, forward is using forward point **Get Domestic Discount factor . X, SWd from Market and use** $S_i(t)e^{(r-\tilde{q}_i)\tau} * \left(1 + x\frac{e^{rT}-1}{SWd * e^{rT}}\right)$ **to backout Foreign Discount Factor**

❖ **The Matlab Code is also attached in the ZIP file. (P6.m)**

<table>
<tr><th>Note</th><th>Code</th></tr>
<tr><td>

ZCSd: Domestic Zero Coupon SWAP Rate

$$(1+ZCSd)^T = \frac{1}{DF_d(T)} = e^{rT}$$

$$\Rightarrow ZCSd = e^r - 1$$

ZCSf: Foreign Zero Coupon SWAP Rate

$$(1+ZCSf)^T = \frac{1}{DF_f(T)} = e^{qT}$$

$$\Rightarrow ZCSf = e^q - 1$$

SWd: Domestic Interest Rate SWAP ratio

$$\frac{1-DF(T)}{\sum_{i=1}^{n} DF(t_i)\delta_i} = \frac{1-e^{-rT}}{\sum_{i=1}^{2} e^{-rt_i}\delta_i}$$

*** Without Spread**:

$$F_i(t,T) = S_i(t) * \frac{(1+ZCS_d)^T}{(1+ZCS_f)^T}$$

$$= S_i(t)e^{(r-q_i)\tau}$$

**** With Spread**:

$$\tilde{F}_i(t,T)$$

$$= S_i(t) * \frac{(1+ZCS_d)^T}{(1+ZCS_f)^T} * \left(1 + x\frac{(1+ZCSd)^T - 1}{SWd(1+ZCSd)^T}\right)$$

$$= S_i(t)e^{(r-q_i)\tau} * \left(1 + x\frac{e^{rT}-1}{SWd * e^{rT}}\right)$$

$$e^{(r-q_i)\tau} * \left(1 + x\frac{e^{rT}-1}{SWd * e^{rT}}\right) = e^{(r-q_i)\tau}$$

</td><td>

```
clear all;

%---Price 1 from (iv)
t=0; T=3;
s1=100; s2=100;K=98;
q1=0.07;q2=0.1;r=0.05;
sigma1=0.3;sigma2=0.32;
rho=-0.2;

[price1 delta1 delta2 delta3] = put1(t, s1, s2, K, T, q1, q2, r, sigma1,
sigma2, rho);

%---Parameters---
t=0;T=3;tau=T-t;
s1=100;s2=100;
q1=0.07;q2=0.1;r =0.05;
sigma1=0.3;sigma2=0.32;
K = 98;

n=12;deltaT=0.25;   % Delta=0.25

x1= -0.01;     % Spread
x2 = -0.015;

%---Formula---
ZCSd=exp(r)-1;
ZCSf1=exp(q1)-1;
ZCSf2=exp(q2)-1;
SWd=(1-exp(-r*T))/(deltaT*sum(exp(-r.*[0.25:0.25:3])));
F1=s1*exp((r-q1)*T);
F2=s2*exp((r-q2)*T);

F1_tilda=s1*exp((r-q1)*T)*(1+x1*(exp(r*T)-1)/(SWd*exp(r*T)))
F2_tilda=s1*exp((r-q2)*T)*(1+x2*(exp(r*T)-1)/(SWd*exp(r*T)))

q1_tilda=r-log(F1_tilda/s1)/T;
q2_tilda=r-log(F2_tilda/s2)/T;

[price2 delta1 delta2] = put1(0, 100, 100, 98, 3, q1_tilda, q2_tilda, 0.05,
0.3, 0.32, -0.2);

diff=price2-price1;

disp('                                   S1           S2');
disp(['Forward Price without Spread: ' num2str(F1) '         '
num2str(F2)]);
disp('                        ');
disp(['Forward Price with Spread:     ' num2str(F1_tilda) '         '
num2str(F2_tilda)]);
disp('                        ');
disp('Implied q1 and q2 ');
disp(['q1: ' num2str(q1_tilda)]);
disp(['q2: ' num2str(q2_tilda)]);
disp('                        ');
disp('Use implied q1_tilda and q2_tilda too get option at T ');
disp(['New price is :       ' num2str(price2)]);
disp(['Original price :     ' num2str(price1)]);
disp(['The difference is : ' num2str(diff)]);
```

</td></tr>
<tr><td>Print Result</td><td>

S1 S2
Forward Price without Spread: 94.1765 86.0708

Forward Price with Spread: 91.5692 82.4965

Implied q1 and q2

</td></tr>
</table>

	q1: 0.079358 q2: 0.11414 Use implied q1_tilda and q2_tilda too get option at T New price is : 9.0576 Original price : 8.1027 The difference is : 0.95483

- **Answer Summary:**
  - **Forward Price without /with spread adjustment**

	S1	**S2**
Forward Price without Spread	94.1765	86.0708
Forward Price with Spread adjustment	91.5692	82.4965

  - **Backed out q1 and q2**

	$\tilde{q}_1$	$\tilde{q}_2$
Result	0.079358	0.11414

  - **Price with adjustment**

(1) New option price using implied foreign IR	9.0576
(2) Compared to original price	8.1027
(3) Difference	0.95483