2.2 The Limit of a Sequence

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1. (a) What happens if we reverse the order of the quantifiers in Definition 2.2.3?

   Definition: A sequence (x_n) *verconges* to x if *there exists* an $\varepsilon > 0$ such that *for all* $N \in \mathbb{N}$ is it true that $n \geq N$ implies $|x_n - x| < \varepsilon$.

   i. Give an example of a vercongent sequence.

   $x_n = \frac{1}{n}$ verconges to 1.

   Proof.
   Let $\varepsilon = 2$.
   Let $N \in \mathbb{N}$ be arbitrary.
   For every $n \geq N$, $\left|\frac{1}{n} - 1\right| = \left|1 - \frac{1}{n}\right| = 1 - \frac{1}{n} \leq 1 < 2 = \varepsilon$. ■

   ii. Is there an example of a vercongent sequence that is divergent?

   The divergent sequence $x_n = (-1)^n$ verconges to 1.

   Proof.
   Let $\varepsilon = 3$.
   Let $N \in \mathbb{N}$ be arbitrary.
   For every $n \geq N$, $|(-1)^n - 1| \leq 2 < 3 = \varepsilon$. ■

   iii. Can a sequence verconge to two different values?

   Yes. For example, $x_n = \frac{1}{n}$ verconges to 1 (already shown above) and also 2.

   Proof.
   Let $\varepsilon = 2$.
   Let $N \in \mathbb{N}$ be arbitrary.
   For every $n \geq N$, $\left|\frac{1}{n} - 2\right| = \left|2 - \frac{1}{n}\right| = 2 - \frac{1}{n} < 2 = \varepsilon$. ■

   iv. What exactly is being described in this strange definition?

   A vercongence sequence is a bounded sequence.

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2. Verify, using the definition of converge of a sequence, that the following sequences converge to the proposed limit.

   (a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

   Proof.

   Let $\varepsilon > 0$ be arbitrary.

   By the ArP, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$.

   For every $n \geq n_0$, $\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{10n+5-10n-8}{5(5n+4)}\right| = \left|\frac{-3}{5(5n+4)}\right| = \frac{3}{5} \cdot \frac{1}{5n+4}$ $< \frac{3}{5} \cdot \frac{1}{5n} = \frac{3}{25} \cdot \frac{1}{n} < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$. ■

   (b) $\lim \frac{2n^2}{n^3+3} = 0$.

   Proof.

   Let $\varepsilon > 0$ be arbitrary.

   By the ArP, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$, and so $\frac{2}{n_0} < \varepsilon$.

   For every $n \geq n_0$, $\left|\frac{2n^2}{n^3+3} - 0\right| = \left|\frac{2n^2}{n^3+3}\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} \leq \frac{2}{n_0} < \varepsilon$. ■

   (c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

   Proof.

   Let $\varepsilon > 0$ be arbitrary.

   By the ArP, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$.

   For every $n \geq n_0$, $\left|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0\right| = \left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$. ■

3. Describe what we would have to demonstrate in order to disprove each of the following statements.

   (a) At every college in the United States, there is a student who is at least seven feet tall.

   Find a college in the United States in which every student is less than seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

Find a college in the United States in which every professor gives a student a grade that is not an A and is not a B.

(c) There exists a college in the United States where every student is at least six feet tall.

Show that every college in the United States has a student that is less than six feet tall.

4. Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

$(x_n) = ((-1)^n) = (-1, 1, -1, 1, -1, 1, \dots)$ has an infinite number of ones, but diverges.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

This is impossible.

Proof.

By contradiction.

Suppose (x_n) is a sequence with an infinite number of ones that converges to a limit not equal to 1.

Let (x_{n_k}) be a subsequence of (x_n) such that n_k is the k^{th} one in (x_n).

Since $x_{n_k} = 1$ for every $k \in \mathbb{N}, \lim x_{n_k} = 1$.

Thus, (x_n) also converges to 1, but this contradicts the assumption that its limit is not equal to 1. ■

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

$(x_n) = (0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots)$

5. Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

   (a) $a_n = [[5/n]]$

   $\lim a_n = 0$

   Proof.

   Let $\varepsilon > 0$ be arbitrary.

   Let $n_0 = 6$.

   For $n \geq n_0$, $\left|\left[\left[\frac{5}{n}\right]\right] - 0\right| = \left|\left[\left[\frac{5}{n}\right]\right]\right| = \left[\left[\frac{5}{n}\right]\right] = 0 < \varepsilon$. ■

   (b) $a_n = [[(12 + 4n)/3n]]$

   $\lim a_n = 1$

   Proof.

   Let $\varepsilon > 0$ be arbitrary.

   Let $n_0 = 7$.

   For $n \geq n_0$, $\left|\left[\left[\frac{12+4n}{3n}\right]\right] - 1\right| = \left|\left[\left[\frac{12+4n}{3n} - 1\right]\right]\right| = \left|\left[\left[\frac{12+4n-3n}{3n}\right]\right]\right|$
   $= \left|\left[\left[\frac{12+n}{3n}\right]\right]\right| = \left[\left[\frac{12+n}{3n}\right]\right] = 0 < \varepsilon$. ■

   Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ε-neighborhood, the larger N may have to be."

   These examples show that after a certain n_0 value (or N value in the words of the statement), the values of the sequence are equal to the limit value, regardless of the ε-value.

6. Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

   Proof.

   Let $\varepsilon > 0$ be arbitrary.

   Suppose $a, b \in \mathbb{R}$ and $\lim a_n = a$ and $\lim a_n = b$.

Then there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1, |a_n - a| < \frac{\varepsilon}{2}$.

Then there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2, |a_n - b| < \frac{\varepsilon}{2}$.

Let $n_0 = \max\{n_1, n_2\}$.

Then for all $n \geq n_0, |a - b| = |a - a_n + a_n - b| \leq |a_n - a| + |a_n - b|$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus, $a - b = 0$, and so $a = b$. ■

7. Here are two useful definitions:

   (a) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

   (b) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

      i. Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

         A. Frequently

            Proof.
            Let $N \in \mathbb{N}$ be arbitrary.
            If N is even, then let $n = N$, and so n is even.
            If N is odd, then let $n = N + 1$, and so n is even.
            Then $a_n = (-1)^n = 1 \in \{1\}$. ■

         B. Not eventually

            Proof.
            By contradiction.
            Suppose a_n is eventually in A.
            Then there exists $N \in \mathbb{N}$ such that for every $n \geq N, a_n \in A$.
            If N is even, then let $m = N + 1$, and so m is odd.
            If N is odd, then let $m = N + 2$, and so m is odd.
            Since $m \geq N$, then it must be that $m \in A$.
            However, $a_m = (-1)^m = -1 \notin A$, which is a contradiction. ■

ii. Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

Eventually is stronger, and eventually implies frequently:

Proof.

Suppose (a_n) is a sequence that is eventually in a set $A \subseteq \mathbb{R}$.

Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0, a_n \in A$.

Let $N \in \mathbb{N}$ be arbitrary.

If $N \geq n_0$, let $n = N$, and so $n \geq N$.

If $N < n_0$, let $n = n_0$, and so $n \geq N$.

Thus, $a_n \in A$.

Therefore, (a_n) is a sequence that is frequently in a set $A \subseteq \mathbb{R}$. ∎

iii. Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

Definition 2.2.3B. A sequence (a_n) converges to a if, given any ε-neighborhood $V_\varepsilon(a)$ of a, there exists a point in the sequence after which all the terms are in $V_\varepsilon(a)$.

Alternate rephrasing. A sequence (a_n) converges to a if, given any ε-neighborhood $V_\varepsilon(a)$ of a, (a_n) is eventually in $V_\varepsilon(a)$.

iv. Suppose an infinite number of terms of a sequence (x_n) are equal to 2.

A. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$?

Not necessarily, as $x_n = (-1)^n + 1$ has terms that are oscillating between 0 and 2, and so there are an infinite number of 2s in (x_n), but there is no $N \in \mathbb{N}$ at and after which there are no 0s in (x_n).

B. Is it frequently in $(1.9, 2.1)$?

Yes, for every $N \in \mathbb{N}$, we can find an $n \geq N$ such that $a_n = 2 \in (1.9, 2.1) = A$.

8. For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbb{N}$ such that for all

$N \in \mathbb{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $\{0, 1, 0, 1, 0, 1, \ldots\}$ zero heavy?

Yes.

Proof.

Let $M = 1$.

Let $N \in \mathbb{N}$ be arbitrary.

If N is odd, then let $n = N$, and so $N \leq n \leq N + 1 = N + M$ and n is odd and $x_n = 0$.

If N is even, then let $n = N + 1$, and so $N \leq n \leq N + 1 = N + M$ and n is odd and $x_n = 0$.

Therefore, $\{0, 1, 0, 1, 0, 1, \ldots\}$ is zero heavy. ∎

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

Yes.

(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counterexample.

No. For example:

$$x_n = \begin{cases} 0 & \text{if } n \text{ is a perfect square} \\ 1 & \text{otherwise} \end{cases}$$

Proof.

Let $M \in \mathbb{N}$ be arbitrary.

Let $k \in \mathbb{N}$ be such that $k > M$.

Let $N = k^2 + 1$.

Then $k^2 + 1 \leq N + M = k^2 + 1 + M < k^2 + k + 1 < k^2 + 2k + 1 < (k+1)^2$.

Then for every n satisfying $N \leq n \leq N + M, x_n = 1$.

Therefore, (x_n) is not zero-heavy. ∎

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if . . . .

A sequence is not zero-heavy if for every $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every n satisfying $N \leq n \leq N + M, x_n \neq 0$.