Chapter 1: Real Numbers - Definitions

A ***function*** f from A to B is a subset of $A \times B$ such that for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$ (denoted $f : A \to B, f(a) = b$).

Let $f : A \to B$ and $X \subseteq A$. The ***image*** of X under f (denoted $f(X)$) is the set of all output values produced when f is applied to each element of A.
If $f : A \to B$, and $X \subseteq A$, then $f(X) = \{f(a) \in B \mid a \in X\}$.

Let $f : A \to B$ and $Y \subseteq B$. The ***preimage*** of Y under f (denoted $f^{-1}(Y)$) is the set of all elements of A that map to elements of Y.
If $f : A \to B$ and $Y \subseteq B$, then $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$.

Let $A \subseteq \mathbb{R}$.
Then $b \in \mathbb{R}$ is called an ***upper bound*** for A if $a \leq b$ for every $a \in A$.
Then $b \in \mathbb{R}$ is called a ***least upper bound*** or ***supremum*** for A
(denoted $b = \sup A$) if:

1. b is an upper bound for A, and
2. if c is an upper bound for A, then $b \leq c$.

Similarly, a ***lower bound*** and a ***greatest lower bound*** or ***infimum***
(denoted $\inf A$) are defined.

Then $a_0 \in \mathbb{R}$ is called a ***maximum*** of A if $a_0 \in A$ and $a \leq a_0$ for every $a \in A$.
Similarly, a ***minimum*** is defined.

Equality of two real numbers
Let $a, b \in \mathbb{R}$. Then $a = b$ if and only if for every $\varepsilon > 0$, $|a - b| < \varepsilon$.

Chapter 1: Real Numbers - Facts

Absolute value facts: For $a, b \in \mathbb{R}$:

1. $|ab| = |a| \cdot |b|$
2. $|a + b| \le |a| + |b|$ and $|a - b| \le |a| + |b|$ (triangle inequality)
3. $||a| - |b|| \le |a - b|$

Corollary: Generalization of the triangle inequality
For $x_1, \ldots, x_n$, $|x_1 + \cdots + x_n| \le |x_1| + \cdots + |x_n|$.

Fact: Let $A \subseteq \mathbb{R}$. If A has a least upper bound, then it is unique.

Lemma: Let a_0 be an upper bound for set A. Then $a_0 = \sup A$ if and only if for every $\varepsilon > 0$ there is $a \in A$ such that $a_0 - \varepsilon < a$.

Axiom of Completeness: Every nonempty set of real numbers that has an upper bound has a least upper bound.

Infimum Principle: If S is a nonempty subset of $\mathbb{R}$ that has a lower bound, then $\inf S$ exists.

($*$) Nested Interval Property: Let $I_n = [a_n, b_n]$ and suppose for every $n \in \mathbb{N}$, $I_{n+1} \subseteq I_n$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

($*$) Archimedean Property:

1. For every $x \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that $n > x$.
   $\forall_{x \in \mathbb{R}} \exists_{n \in \mathbb{N}}\ n > x$
2. For every $y \in \mathbb{R}^+$ there is $n \in \mathbb{N}$ such that $\frac{1}{n} < y$.
   $\forall_{y \in \mathbb{R}^+} \exists_{n \in \mathbb{N}}\ \frac{1}{n} < y$

(∗) Density of $\mathbb{Q}$ in $\mathbb{R}$:

For every two real numbers a, b such that $a < b$, there is $r \in \mathbb{Q}$ such that $a < r < b$.

Corollary: (Density of $\mathbb{Q} \setminus \mathbb{R}$ in $\mathbb{R}$)

For every two real numbers a, b such that $a < b$, there is $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < t < b$.

Fact: There is no rational number m such that $m^2 = 2$.

Theorem: There exists $\alpha \in \mathbb{R}$ such that $\alpha^2 = 2$.

Fact: $\mathbb{Q}$ is countable.

Theorem: $\mathbb{R}$ is uncountable.

Chapter 2: Sequences - Definitions

A ***sequence*** is a function whose domain is $\mathbb{N}$ ($f : \mathbb{N} \to \mathbb{R}$).

A sequence (a_n) ***converges*** to a real number a (denoted $\lim a_n = a$) if $\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{n\geq n_0}\ |a_n - a| < \varepsilon$.

If a sequence does not converge for any real number, then the sequence ***diverges***.

A sequence (a_n) is ***bounded*** if there exists $M > 0$ such that $|a_n| \leq M$ for every $n \in \mathbb{N}$.

A sequence (a_n) is ***increasing*** if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$.
A sequence (a_n) is ***decreasing*** if $a_n \geq a_{n+1}$ for every $n \in \mathbb{N}$.

A sequence (a_n) is ***bounded from above*** if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for every $n \in \mathbb{N}$.
A sequence (a_n) is ***bounded from below*** if there exists $M \in \mathbb{R}$ such that $a_n \geq M$ for every $n \in \mathbb{N}$.

A sequence is ***monotone*** if it is either increasing or decreasing.

Let (a_n) be a sequence of real numbers and let $n_1 < n_2 < \ldots$ be an increasing (but not necessarily consecutive) sequence of natural numbers. Then $(a_{n_k}) = (a_{n_1}, a_{n_2}, \ldots)$ is called a ***subsequence*** of (a_n).

A sequence (a_n) is called a <u>Cauchy sequence</u> if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0, |a_n - a_m| < \varepsilon$.

Let (b_n) be a sequence. An ***infinite series*** is the expression $\Sigma_{n=1}^{\infty} b_n = b_1 + b_2 + \cdots$.
The sequence of ***partial sums*** (s_m) is such that $s_m = b_1 + \cdots + b_m$.
We say that $\Sigma_{n=1}^{\infty} b_n$ ***converges*** to B if (s_m) converges to B.
Then we write $\Sigma_{n=1}^{\infty} b_n = B$.

Chapter 2: Sequences - Facts

Theorem: If a limit exists, then it is unique.

Theorem: Every convergent sequence is bounded.

Theorem: Algebraic Limit Theorem
Let $\lim a_n = a$ and $\lim b_n = b$. Then:

1. $\lim(c \cdot a_n) = ca$, for every $c \in \mathbb{R}$
2. $\lim(a_n + b_n) = a + b$
3. $\lim(a_n b_n) = ab$
4. $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, provided $b \neq 0$

Theorem: Order Limit Theorem
Let $\lim a_n = a$ and $\lim b_n = b$. Then:

1. If $a_n \geq 0$ for every n, then $a \geq 0$.
2. If $a_n \leq b_n$ for every n, then $a \leq b$.
3. If there exists c such that $c \leq b_n$, for all n, then $c \leq b$.
   Similarly, if there exists c such that $a_n \leq c$, then $a \leq c$.

Theorem: Monotone Convergence Theorem
If a sequence is monotone and bounded, then it converges.

($*$) Theorem: MCT for increasing sequences
Let (a_n) be an increasing sequence that is bounded from above. Then (a_n) converges.

Theorem: Subsequences of a convergent sequence converge to the same limit as the original sequence.

($*$) Theorem: Bolzano-Weierstrass Theorem
Every bounded sequence contains a convergent subsequence.

Lemma: Every sequence contains a monotone subsequence.

Theorem: Every convergent sequence is a Cauchy sequence.

Lemma: Every Cauchy sequence is bounded.

($*$) Theorem: Cauchy Criterion
A sequence converges if and only if it is a Cauchy sequence.

Chapter 4: Continuity - Definitions

Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Then the ε***-neighborhood*** is defined as:
$V_\varepsilon(x) = \{y \mid |x - y| < \varepsilon\}$.

Let $A \subseteq \mathbb{R}$. Then x is called a ***limit point*** of A if for every $\varepsilon > 0$, there exists $z \in A \cap V_\varepsilon(x)$ such that $z \neq x$.

Let A be a non-degenerate interval or an interval with one point removed. Then the ***limit points of*** A are points of A together with the endpoints (if finite) and the removed point.

Let $f : A \to \mathbb{R}$ and let c be a limit point of A. Then the ***functional limit*** $lim_{x \to c} f(x) = L$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in A$, then $|f(x) - L| < \varepsilon$.
$\lim_{x \to c} f(x) = L \equiv \forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x \in A} 0 < |x - c| < \delta \to |f(x) - L| < \varepsilon$

A function $f : A \to \mathbb{R}$ is ***continuous at a point*** $c \in A$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$ and $x \in A$, then $|f(x) - f(c)| < \varepsilon$.
$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x \in A} \, |x - c| < \delta \to |f(x) - f(c)| < \varepsilon$

$f : A \to \mathbb{R}$ is ***continuous on*** A if it is continuous at c for every $c \in A$.

A function $f : A \to \mathbb{R}$ is ***uniformly continuous*** on A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in A$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.
$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x,y \in A} \, |x - y| < \delta \to |f(x) - f(y)| < \varepsilon$

Chapter 4: Continuity - Facts

Theorem: Let $A \subseteq \mathbb{R}$. Then x is a limit point of A if and only if there exists a sequence (a_n) that is contained in A and $\lim a_n = x$ and $a_n \neq x$ (for every $n \in \mathbb{N}$).

Theorem: Let $f : A \to \mathbb{R}$ and let c be a limit point of A. The following statements are equivalent:

1. $\lim_{x \to c} f(x) = L$
2. For every sequence $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $\lim x_n = c$, we have $\lim f(x_n) = L$.

Corollary: Let $f, g : A \to \mathbb{R}$ and suppose $\lim_{x \to c} f(x) = L, \lim_{x \to c} g(x) = M$. Then:

1. $\lim_{x \to c} k \cdot f(x) = k \cdot L$ for every $k \in \mathbb{R}$
2. $\lim_{x \to c}(f(x) + g(x)) = L + M$
3. $\lim_{x \to c}(f(x)g(x)) = L \cdot M$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$

Corollary: Divergence Criterion for Functional Limits
Let $f : A \to \mathbb{R}$ and let c be a limit point of A. If there exist two sequences $(x_n), (y_n)$ in A such that $x_n \neq c, y_n \neq c, \lim x_n = \lim y_n = c$ but $\lim f(x_n) \neq \lim f(y_n)$, then $\lim_{x \to c} f(x)$ does not exist.

Theorem: Characterization of Continuity

Let $f : A \to \mathbb{R}$ and let $c \in A$. The function is continuous at c if and only if any of the following conditions is satisfied:

1. $\forall_{\varepsilon>0}\exists_{\delta>0}(|x-c|<\delta \wedge x \in A) \to |f(x)-f(c)|<\varepsilon$
2. For all $V_\varepsilon(f(c))$, there exists $V_\delta(c)$ such that $x \in V_\delta(c) \cap A$ implies $f(x) \in V_\varepsilon(f(c))$.
3. For all $(x_n) \to c$ with $x_n \in A, (f(x_n)) \to f(c)$.

   In addition, if c is a limit point of A, then the above are equivalent to:

4. $\lim_{x\to c} f(x) = f(c)$

Corollary: Criterion for Discontinuity

Let $f : A \to \mathbb{R}$ and let $c \in A$ (be a limit point of A). If there is a sequence $(x_n) \subseteq A$ such that $(x_n) \to c$ but $(f(x_n))$ does not converge to $f(c)$, then f is not continuous at c.

Theorem: Algebraic Continuity Theorem

Let $f, g : A \to \mathbb{R}$ be continuous at $c \in A$. Then:

1. $k \cdot f(x)$ is continuous at c for every $k \in \mathbb{R}$.
2. $f(x) + g(x)$ is continuous at c.
3. $f(x)g(x)$ is continuous at c.
4. $\frac{f(x)}{g(x)}$ is continuous at c provided that the quotient is defined.

Theorem: Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be such that $f(A) \subseteq B$. If f, g are continuous, then $g \circ f$ is a continuous function from A to $\mathbb{R}$.

Theorem: Bolzano-Weierstrass, version 2

Let A be a closed bounded interval. Every sequence $(x_n) \subseteq A$ contains a convergent subsequence whose limit is in A.

$A = [a, b], z = \lim x_{n_k}, a \leq z \leq b$

(*) Theorem: Extreme Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exist $\alpha, \beta \in [a, b]$ such that for all $x \in [a, b], f(\alpha) \leq f(x) \leq f(\beta)$.

Theorem: A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists $\varepsilon_0 > 0$ and two sequences $(x_n), (y_n)$ in A satisfying $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

(*) Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous on $[a, b]$.

(*) Theorem: Bolzano's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose $f(a) < 0$ and $f(b) > 0$. Then there is $c \in (a, b)$ such that $f(c) = 0$.

Theorem: Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $L \in \mathbb{R}$ be such that $\min\{f(a), f(b)\} < L < \max\{f(a), f(b)\}$. Then there is $c \in (a, b)$ such that $f(c) = L$.

Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f([a, b])$ (the range of f) is a closed interval.

Continuity of the Inverse: Let $f : [a, b] \to \mathbb{R}$ be continuous and injective. Then $f([a, b]) = [c, d]$ and $f^{-1} : [c, d] \to [a, b]$.

Theorem: With the setup as above, $f^{-1} : [c, d] \to [a, b]$ is continuous.

Chapter 5: The Derivative - Definitions

Let $g : A \to \mathbb{R}$ be a function defined on interval A and let $c \in A$. The ***derivative*** of g at c is (provided that the limit exists):

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

If g exists for all points $c \in A$, then g is ***differentiable*** on A.

Chapter 5: The Derivative - Facts

Theorem: If $g : A \to \mathbb{R}$ is differentiable at c, then g is continuous at c.

Theorem: Let $f, g : A \to \mathbb{R}$ are differentiable at $c \in A$. Then:

1. $(f + g)'(c) = f'(c) + g'(c)$
2. For $k \in \mathbb{R}, (k \cdot f)'(c) = k \cdot f'(c)$
3. $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$ (Product Rule)
4. $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$ provided $g(c) \neq 0$ (Quotient Rule)

Theorem: The Chain Rule
Let $f : A \to B, g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$. If f is differentiable at $c \in A$ and g is differentiable at $f(c) \subseteq B$, then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c)$$

Theorem: Interior Extremum Theorem
Let f be differentiable on (a, b).

1. If f attains a maximum value at some point $c \in (a, b)$, then $f'(c) = 0$.
2. If f attains a minimum value at some point $c \in (a, b)$, then $f'(c) = 0$.