Chapter 1: Real Numbers - Definitions

A ***function*** f from A to B is a subset of $A \times B$ such that for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$ (denoted $f : A \to B, f(a) = b$).

Let $f : A \to B$ and $X \subseteq A$. The ***image*** of X under f (denoted $f(X)$) is the set of all output values produced when f is applied to each element of A.
If $f : A \to B$, and $X \subseteq A$, then $f(X) = \{f(a) \in B \mid a \in X\}$.

Let $f : A \to B$ and $Y \subseteq B$. The ***preimage*** of Y under f (denoted $f^{-1}(Y)$) is the set of all elements of A that map to elements of Y.
If $f : A \to B$ and $Y \subseteq B$, then $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$.

Let $A \subseteq \mathbb{R}$.
Then $b \in \mathbb{R}$ is called an ***upper bound*** for A if $a \leq b$ for every $a \in A$.
Then $b \in \mathbb{R}$ is called a ***least upper bound*** or ***supremum*** for A
(denoted $b = \sup A$) if:

1. b is an upper bound for A, and
2. if c is an upper bound for A, then $b \leq c$.

Similarly, a ***lower bound*** and a ***greatest lower bound*** or ***infimum***
(denoted $\inf A$) are defined.

Then $a_0 \in \mathbb{R}$ is called a ***maximum*** of A if $a_0 \in A$ and $a \leq a_0$ for every $a \in A$.
Similarly, a ***minimum*** is defined.

Equality of two real numbers
Let $a, b \in \mathbb{R}$. Then $a = b$ if and only if for every $\varepsilon > 0$, $|a - b| < \varepsilon$.

Chapter 1: Real Numbers - Facts

Absolute value facts: For $a, b \in \mathbb{R}$:

1. $|ab| = |a| \cdot |b|$
2. $|a + b| \leq |a| + |b|$ and $|a - b| \leq |a| + |b|$ (triangle inequality)
3. $||a| - |b|| \leq |a - b|$

Corollary: Generalization of the triangle inequality
For $x_1, \ldots, x_n$, $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$.

Fact: Let $A \subseteq \mathbb{R}$. If A has a least upper bound, then it is unique.

Lemma: Let a_0 be an upper bound for set A. Then $a_0 = \sup A$ if and only if for every $\varepsilon > 0$ there is $a \in A$ such that $a_0 - \varepsilon < a$.

Axiom of Completeness: Every nonempty set of real numbers that has an upper bound has a least upper bound.

Infimum Principle: If S is a nonempty subset of $\mathbb{R}$ that has a lower bound, then $\inf S$ exists.

(∗) Nested Interval Property: Let $I_n = [a_n, b_n]$ and suppose for every $n \in \mathbb{N}$, $I_{n+1} \subseteq I_n$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

(∗) Archimedean Property:

1. For every $x \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that $n > x$.
   $\forall_{x \in \mathbb{R}} \exists_{n \in \mathbb{N}}\ n > x$
2. For every $y \in \mathbb{R}^+$ there is $n \in \mathbb{N}$ such that $\frac{1}{n} < y$.
   $\forall_{y \in \mathbb{R}^+} \exists_{n \in \mathbb{N}}\ \frac{1}{n} < y$

(∗) Density of $\mathbb{Q}$ in $\mathbb{R}$:

For every two real numbers a, b such that $a < b$, there is $r \in \mathbb{Q}$ such that $a < r < b$.

Corollary: (Density of $\mathbb{Q} \setminus \mathbb{R}$ in $\mathbb{R}$)

For every two real numbers a, b such that $a < b$, there is $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < t < b$.

Fact: There is no rational number m such that $m^2 = 2$.

Theorem: There exists $\alpha \in \mathbb{R}$ such that $\alpha^2 = 2$.

Fact: $\mathbb{Q}$ is countable.

Theorem: $\mathbb{R}$ is uncountable.

Chapter 2: Sequences - Definitions

A ***sequence*** is a function whose domain is $\mathbb{N}$ ($f : \mathbb{N} \to \mathbb{R}$).

A sequence (a_n) ***converges*** to a real number a (denoted $\lim a_n = a$) if
$\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{n\geq n_0}\ \ |a_n - a| < \varepsilon$.

If a sequence does not converge for any real number, then the sequence ***diverges***.

A sequence (a_n) is ***bounded*** if there exists $M > 0$ such that $|a_n| \leq M$ for every $n \in \mathbb{N}$.

A sequence (a_n) is ***increasing*** if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$.
A sequence (a_n) is ***decreasing*** if $a_n \geq a_{n+1}$ for every $n \in \mathbb{N}$.

A sequence (a_n) is ***bounded from above*** if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for every $n \in \mathbb{N}$.
A sequence (a_n) is ***bounded from below*** if there exists $M \in \mathbb{R}$ such that $a_n \geq M$ for every $n \in \mathbb{N}$.

A sequence is ***monotone*** if it is either increasing or decreasing.

Let (a_n) be a sequence of real numbers and let $n_1 < n_2 < \dots$ be an increasing (but not necessarily consecutive) sequence of natural numbers. Then
$(a_{n_k}) = (a_{n_1}, a_{n_2}, \dots)$ is called a ***subsequence*** of (a_n).

A sequence (a_n) is called a <u>Cauchy sequence</u> if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, $|a_n - a_m| < \varepsilon$.

Let (b_n) be a sequence. An ***infinite series*** is the expression $\Sigma_{n=1}^{\infty} b_n = b_1 + b_2 + \cdots$.
The sequence of ***partial sums*** (s_m) is such that $s_m = b_1 + \cdots + b_m$.
We say that $\Sigma_{n=1}^{\infty} b_n$ ***converges*** to B if (s_m) converges to B.
Then we write $\Sigma_{n=1}^{\infty} b_n = B$.

Chapter 2: Sequences - Facts

Theorem: If a limit exists, then it is unique.

Theorem: Every convergent sequence is bounded.

Theorem: Algebraic Limit Theorem
Let $\lim a_n = a$ and $\lim b_n = b$. Then:

1. $\lim(c \cdot a_n) = ca$, for every $c \in \mathbb{R}$
2. $\lim(a_n + b_n) = a + b$
3. $\lim(a_n b_n) = ab$
4. $\lim \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, provided $b \neq 0$

Theorem: Order Limit Theorem
Let $\lim a_n = a$ and $\lim b_n = b$. Then:

1. If $a_n \geq 0$ for every n, then $a \geq 0$.
2. If $a_n \leq b_n$ for every n, then $a \leq b$.
3. If there exists c such that $c \leq b_n$, for all n, then $c \leq b$.
   Similarly, if there exists c such that $a_n \leq c$, then $a \leq c$.

Theorem: Monotone Convergence Theorem
If a sequence is monotone and bounded, then it converges.

($*$) Theorem: MCT for increasing sequences
Let (a_n) be an increasing sequence that is bounded from above. Then (a_n) converges.

Theorem: Subsequences of a convergent sequence converge to the same limit as the original sequence.

($*$) Theorem: Bolzano-Weierstrass Theorem

Every bounded sequence contains a convergent subsequence.

Lemma: Every sequence contains a monotone subsequence.

Theorem: Every convergent sequence is a Cauchy sequence.

Lemma: Every Cauchy sequence is bounded.

($*$) Theorem: Cauchy Criterion

A sequence converges if and only if it is a Cauchy sequence.

Chapter 4: Continuity - Definitions

Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Then the ***ε-neighborhood*** is defined as:
$V_\varepsilon(x) = \{y \mid |x - y| < \varepsilon\}$.

Let $A \subseteq \mathbb{R}$. Then x is called a ***limit point*** of A if for every $\varepsilon > 0$, there exists $z \in A \cap V_\varepsilon(x)$ such that $z \neq x$.

Let A be a non-degenerate interval or an interval with one point removed. Then the ***limit points of*** A are points of A together with the endpoints (if finite) and the removed point.

Let $f : A \to \mathbb{R}$ and let c be a limit point of A. Then the ***functional limit*** $lim_{x \to c} f(x) = L$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in A$, then $|f(x) - L| < \varepsilon$.
$\lim_{x \to c} f(x) = L \equiv \forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x \in A} 0 < |x - c| < \delta \to |f(x) - L| < \varepsilon$

A function $f : A \to \mathbb{R}$ is ***continuous at a point*** $c \in A$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$ and $x \in A$, then $|f(x) - f(c)| < \varepsilon$.
$\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x \in A} \, |x - c| < \delta \to |f(x) - f(c)| < \varepsilon$

$f : A \to \mathbb{R}$ is ***continuous on*** A if it is continuous at c for every $c \in A$.

A function $f : A \to \mathbb{R}$ is ***uniformly continuous*** on A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in A$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.
$\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x, y \in A} \, |x - y| < \delta \to |f(x) - f(y)| < \varepsilon$

Chapter 4: Continuity - Facts

Theorem: Let $A \subseteq \mathbb{R}$. Then x is a limit point of A if and only if there exists a sequence (a_n) that is contained in A and $\lim a_n = x$ and $a_n \neq x$ (for every $n \in \mathbb{N}$).

Theorem: Let $f : A \to \mathbb{R}$ and let c be a limit point of A. The following statements are equivalent:

1. $\lim_{x \to c} f(x) = L$
2. For every sequence $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $\lim x_n = c$, we have $\lim f(x_n) = L$.

Corollary: Let $f, g : A \to \mathbb{R}$ and suppose $\lim_{x \to c} f(x) = L, \lim_{x \to c} g(x) = M$. Then:

1. $\lim_{x \to c} k \cdot f(x) = k \cdot L$ for every $k \in \mathbb{R}$
2. $\lim_{x \to c}(f(x) + g(x)) = L + M$
3. $\lim_{x \to c}(f(x)g(x)) = L \cdot M$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$

Corollary: Divergence Criterion for Functional Limits
Let $f : A \to \mathbb{R}$ and let c be a limit point of A. If there exist two sequences $(x_n), (y_n)$ in A such that $x_n \neq c, y_n \neq c, \lim x_n = \lim y_n = c$ but $\lim f(x_n) \neq \lim f(y_n)$, then $\lim_{x \to c} f(x)$ does not exist.

Theorem: Characterization of Continuity

Let $f : A \to \mathbb{R}$ and let $c \in A$. The function is continuous at c if and only if any of the following conditions is satisfied:

1. $\forall_{\varepsilon>0}\exists_{\delta>0}(|x-c| < \delta \wedge x \in A) \to |f(x) - f(c)| < \varepsilon$
2. For all $V_\varepsilon(f(c))$, there exists $V_\delta(c)$ such that $x \in V_\delta(c) \cap A$ implies $f(x) \in V_\varepsilon(f(c))$.
3. For all $(x_n) \to c$ with $x_n \in A, (f(x_n)) \to f(c)$.

   In addition, if c is a limit point of A, then the above are equivalent to:

4. $\lim_{x\to c} f(x) = f(c)$

Corollary: Criterion for Discontinuity

Let $f : A \to \mathbb{R}$ and let $c \in A$ (be a limit point of A). If there is a sequence $(x_n) \subseteq A$ such that $(x_n) \to c$ but $(f(x_n))$ does not converge to $f(c)$, then f is not continuous at c.

Theorem: Algebraic Continuity Theorem

Let $f, g : A \to \mathbb{R}$ be continuous at $c \in A$. Then:

1. $k \cdot f(x)$ is continuous at c for every $k \in \mathbb{R}$.
2. $f(x) + g(x)$ is continuous at c.
3. $f(x)g(x)$ is continuous at c.
4. $\frac{f(x)}{g(x)}$ is continuous at c provided that the quotient is defined.

Theorem: Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be such that $f(A) \subseteq B$. If f, g are continuous, then $g \circ f$ is a continuous function from A to $\mathbb{R}$.

Theorem: Bolzano-Weierstrass, version 2
Let A be a closed bounded interval. Every sequence $(x_n) \subseteq A$ contains a convergent subsequence whose limit is in A.
$A = [a, b], z = \lim x_{n_k}, a \leq z \leq b$

(*) Theorem: Extreme Value Theorem
Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exist $\alpha, \beta \in [a, b]$ such that for all $x \in [a, b], f(\alpha) \leq f(x) \leq f(\beta)$.

Theorem: A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists $\varepsilon_0 > 0$ and two sequences $(x_n), (y_n)$ in A satisfying $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

(*) Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous on $[a, b]$.

(*) Theorem: Bolzano's Theorem
Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose $f(a) < 0$ and $f(b) > 0$. Then there is $c \in (a, b)$ such that $f(c) = 0$.

Theorem: Intermediate Value Theorem
Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $L \in \mathbb{R}$ be such that $\min \{f(a), f(b)\} < L < \max \{f(a), f(b)\}$. Then there is $c \in (a, b)$ such that $f(c) = L$.

Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f([a, b])$ (the range of f) is a closed interval.

Continuity of the Inverse: Let $f : [a, b] \to \mathbb{R}$ be continuous and injective. Then $f([a, b]) = [c, d]$ and $f^{-1} : [c, d] \to [a, b]$.
Theorem: With the setup as above, $f^{-1} : [c, d] \to [a, b]$ is continuous.

Chapter 5: The Derivative - Definitions

Let $g : A \to \mathbb{R}$ be a function defined on interval A and let $c \in A$. The ***derivative*** of g at c is (provided that the limit exists):

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

If g exists for all points $c \in A$, then g is ***differentiable*** on A.

Infinite limit. $\lim_{x \to a} f(x) = \infty$ if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$.

Chapter 5: The Derivative - Facts

Theorem: If $g : A \to \mathbb{R}$ is differentiable at c, then g is continuous at c.

Theorem: Let $f, g : A \to \mathbb{R}$ are differentiable at $c \in A$. Then:

1. $(f+g)'(c) = f'(c) + g'(c)$
2. For $k \in \mathbb{R}, (k \cdot f)'(c) = k \cdot f'(c)$
3. $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$ (Product Rule)
4. $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$ provided $g(c) \neq 0$ (Quotient Rule)

Theorem: The Chain Rule

Let $f : A \to B, g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$. If f is differentiable at $c \in A$ and g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c)$$

Theorem: Interior Extremum Theorem

Let f be differentiable on (a, b).

1. If f attains a maximum value at some point $c \in (a, b)$, then $f'(c) = 0$.
2. If f attains a minimum value at some point $c \in (a, b)$, then $f'(c) = 0$.

Theorem: Darboux's Theorem

If f is differentiable on an interval $[a, b]$ and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$), then there is a point $c \in (a, b)$ such that $f'(c) = \alpha$.

Theorem: Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is a point $c \in (a, b)$ such that $f'(c) = 0$.

Theorem: Mean Value Theorem

If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there is $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary: Let A be a non-degenerate interval ($A = [a,b], a < b$).

If $g : A \to \mathbb{R}$ is differentiable and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some $k \in \mathbb{R}$.

Corollary: Let A be a non-degenerate interval.

If $f, g : A \to \mathbb{R}$ are differentiable on A and satisfy $f'(x) = g'(x)$ for every $x \in A$, then $f(x) = g(x) + k$ for some $k \in \mathbb{R}$.

Theorem: Generalized Mean Value Theorem

If f, g are continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

In particular, if $g'(x) \neq 0$ for every $x \in (a,b)$, then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Theorem: L'Hospital's Rule ($\frac{0}{0}$ case)

Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).

If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Theorem: L'Hospital's Rule ($\frac{\infty}{\infty}$ case)

Let f and g be differentiable on (a,b). Suppose $g'(x) \neq 0$ for all $x \in (a,b)$. If $\lim_{x \to a} g(x) = \infty$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Theorem: Taylor's Theorem

Let $f : [a,b] \to \mathbb{R}$. Suppose for some $n \in \mathbb{N}$, $f^{(n)}(x)$ is continuous on $[a,b]$ and $f^{(n+1)}(x)$ exists on (a,b). Then for $x, x_0 \in [a,b]$, there is c between x, x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Chapter 6: Sequences of Functions - Definitions

Pointwise convergence. For $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$.
The sequence (f_n) ***converges pointwise*** on A to a function f if for every $x \in A, (f_n(x))$ converges to $f(x)$.

$$\forall_{x \in A} \forall_{\varepsilon > 0} \exists_{n_0 \in \mathbb{N}} \forall_{n \geq n_0} |f_n(x) - f(x)| < \varepsilon$$

Uniform convergence. Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$.
Then (f_n) ***converges uniformly*** on A to a function $f : A \to \mathbb{R}$ if

$$\forall_{\varepsilon > 0} \exists_{n_0 \in \mathbb{N}} \forall_{n \geq n_0} \forall_{x \in A} |f_n(x) - f(x)| < \varepsilon.$$

(f_n) does <u>not</u> converge uniformly to f on A if

$$\exists_{\varepsilon_0 > 0} \forall_{n_0 \in \mathbb{N}} \exists_{n \geq n_0} \exists_{x \in A} |f_n(x) - f(x)| \geq \varepsilon_0.$$

Chapter 6: Sequences of Functions - Facts

Theorem: Cauchy Criterion

A sequence of functions (f_n), $f_n : A \to \mathbb{R}$ converges uniformly on A if and only if

$$\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{n,m\geq n_0}\forall_{x\in A}\,|f_n(x) - f_m(x)| < \varepsilon.$$

Theorem: Continuous Limit Theorem

Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$ that converge uniformly on A to f. If each f_n is continuous at $c \in A$, then so is f.

Theorem: Differentiable Limit Theorem

Let $f_n \to f$ pointwise on $[a, b]$ and assume each f_n is differentiable.

If (f_n') converges uniformly on $[a, b]$ to g, then f is differentiable and $f' = g$.

Chapter 7: Integration - Definitions

TODO

Chapter 7: Integration - Facts

TODO

Appendix A: Theorems to Memorize

<u>Nested Interval Property (NIP)</u>

Let $I_n = [a_n, b_n]$ and suppose for every $n \in \mathbb{N}, I_{n+1} \subseteq I_n$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

. . . .

High-Level Overview of the Proof:

1. Create a set containing the left endpoints of every interval, then use its properties to show that by the AoC, a sup exists.

2. Show that the sup is the element that all the intervals contain, making their intersection nonempty.

. . . .

Proof.

Let $A = \{a_n \mid n \in \mathbb{N}\}$.

Since $a_1 \in A, A \neq \emptyset$.

Since for every $n \in \mathbb{N}, a_n \leq b_1, b_1$ is an upper bound for A.

By the AoC, $\alpha = \sup A$ exists.

- - - - - - - - - -

Since α is an upper bound for A, for every $n \in \mathbb{N}, a_n \leq \alpha$.

Since α is the least upper bound for A and for every $n \in \mathbb{N}, b_n$ is an upper bound for $A, \alpha \leq b_n$.

Thus, for every $n \in \mathbb{N}, a_n \leq \alpha \leq b_n$, and so $\alpha \in I_n$ and $\alpha \in \bigcap_{n=1}^{\infty} I_n$.

Therefore, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. ■

Monotone Convergence Theorem (MCT) for increasing sequences

Let (a_n) be an increasing sequence that is bounded from above. Then (a_n) converges.

. . . .

High-Level Overview of the Proof:

1. Create a set containing every element of the sequence, and use its properties to show that by the AoC, it has a sup.
2. Show that the limit of the sequence is that sup (starting with the sup lemma).

. . . .

Proof.
Let $A = \{a_n \mid n \in \mathbb{N}\}$.
Since $a_1 \in A, A \neq \emptyset$.
Since (a_n) is bounded from above, A has an upper bound.
By the AoC, $\alpha = \sup A$ exists.

\- - - - - - - - - -

Let $\varepsilon > 0$ be arbitrary.
Since $\alpha = \sup A$, there exists $n_0 \in \mathbb{N}$ such that $\alpha - \varepsilon < a_{n_0}$.
For $n \geq n_0$, since $a_{n_0} \leq a_n$ and $a_n \leq \alpha, -\varepsilon < a_{n_0} - \alpha \leq a_n - \alpha \leq 0 < \varepsilon$.
Thus, $|a_n - \alpha| < \varepsilon$, and so (a_n) converges to α. ■

Bolzano-Weierstrass Theorem (BW)

Every bounded sequence has a convergent subsequence.

. . . .

High-Level Overview of the Proof:

1. First prove the lemma that every sequence contains a monotone subsequence.
   (a) Define what a peak is, then show that for the case of a sequence having infinitely many peaks, it has a decreasing subsequence made up of all the peaks.
   (b) Show that for the case of a sequence having finitely many peaks, it has an increasing subsequence made up of all the points after the last peak (use induction to show that $a_{n_k} < a_{n_{k+1}}$).
2. Use the above lemma to show that a bounded sequence has a subsequence that is also bounded, and so by the MCT, that subsequence converges.

. . . .

Lemma: Every sequence contains a monotone subsequence.

Proof.

Let (a_n) be a sequence.

Then (a_m) is called a peak if for every $l \geq m, a_l \leq a_m$.

- Case 1: There are infinitely many peaks in (a_n).

  Let n_k be the index of the k^{th} peak.

  Since $n_k < n_{k+1}, a_{n_k} \geq a_{n_{k+1}}$.

  Thus, (a_{n_k}) is decreasing.

  - - - - - - - - - -

- Case 2: There are finitely many peaks in (a_n).

  Let m be the index of the last peak ($m = 0$ if there are no peaks).

  Let $n_1 = m + 1$.

Since a_{n_1} is not a peak, there exists $n_2 > n_1$ such that $a_{n_1} < a_{n_2}$.

Suppose $a_{n_1}, \ldots, a_{n_k}$ non-peaks have been chosen.

Since a_{n_k} is not a peak, there exists $n_{k+1} > n_k$ such that $a_{n_k} < a_{n_{k+1}}$.

Thus, by induction, (a_{n_k}) is increasing.

Therefore, every sequence contains a monotone subsequence. ■

- - - - - - - - - -

<u>Proof of Bolzano-Weierstrass Theorem</u>

Proof.
Let (a_n) be a bounded sequence.
By the above lemma, (a_n) contains a monotone subsequence (a_{n_k}).
Since (a_n) is bounded, (a_{n_k}) is also bounded.
Therefore, by the MCT, a_{n_k} converges. ■

Cauchy Criterion

If a sequence is Cauchy, then it converges.

....

High-Level Overview of the Proof:

1. First prove the lemma that every Cauchy sequence is bounded.

   (a) Use the definition of the Cauchy sequence (with $\varepsilon = 1$), and focus on the $n \geq n_0$ part to show that $|a_n|$ is bounded by $1 + |a_{n_0}|$.

   (b) Show that the entire sequence is bounded by the max of all the absolute values of sequence values before a_{n_0} and the bound at and after a_{n_0} $(1 + |a_{n_0}|)$.

2. Use the lemma above and BW to show that a Cauchy sequence contains a convergent subsequence.

3. Show that the limit of that subsequence is the limit of the Cauchy sequence, meaning that the Cauchy sequence converges.

   (a) Use the definitions of convergence for the subsequence and a Cauchy sequence with $\frac{\varepsilon}{2}$ for the ε-values to setup inequalities to be used later.

   (b) Choose an index to be the further out of the subsequence and Cauchy sequence convergence points (k_0 and n_0), and transform the inequalities to use that index.

   (c) Show that the Cauchy sequence converges to the value to which the subsequence converges, using the definition of convergence with the inequalities.

....

Lemma: Every Cauchy sequence is bounded.

Proof.

Let (a_n) be a Cauchy sequence.

Then there exists $n_0 \in \mathbb{N}$ such that for every $n, m \geq n_0$, $|a_n - a_m| < 1$.

In particular, for every $n \geq n_0$, $|a_n - a_{n_0}| < 1$.

Then for every $n \geq n_0, |a_n| = |a_n - a_{n_0} + a_{n_0}| \leq |a_n - a_{n_0}| + |a_{n_0}| < 1 + |a_{n_0}|$.

- - - - - - - - - -

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{n_0-1}|, 1 + |a_{n_0}|\}$.

Then for every $n \in \mathbb{N}, |a_n| \leq M$.

Therefore, (a_n) is bounded. ■

- - - - - - - - - -

<u>Proof of the Cauchy Criterion</u>

Proof.

Let (a_n) be a Cauchy sequence.

By the lemma above, (a_n) is bounded.

By Bolzano-Weierstrass, (a_n) contains a convergent subsequence (a_{n_k}).

- - - - - - - - - -

Let $a = \lim a_{n_k}$.

Let $\varepsilon > 0$ be arbitrary.

Since $\lim a_{n_k} = a$, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0, |a_{n_k} - a| < \frac{\varepsilon}{2}$.

Since (a_n) is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that for every $n, m \geq n_0, |a_n - a_m| < \frac{\varepsilon}{2}$.

- - - - - - - - - -

Let $l = \max\{k_0, n_0\}$.

Then for $n \geq n_0, |a_{n_l} - a| < \frac{\varepsilon}{2}$ and $|a_n - a_{n_l}| < \frac{\varepsilon}{2}$.

- - - - - - - - - -

Then for $n \geq n_0, |a_n - a| = |a_n - a_{n_l} + a_{n_l} - a| \leq |a_n - a_{n_l}| + |a_{n_l} - a|$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Therefore, $\lim a_n = a$, and so (a_n) converges. ■

<u>Extreme Value Theorem</u>

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exist $\alpha, \beta \in [a, b]$ such that for all $x \in [a, b]$, $f(\alpha) \leq f(x) \leq f(\beta)$.

. . . .

High-Level Overview of the Proof:

1. First show that f is bounded on $[a, b]$.

    (a) By contradiction, we suppose f is unbounded on the interval by constructing a sequence in $[a, b]$ whose absolute valued function output values are at or above their respective index values, which is possible because this assumption allows the function values to grow without bound on the closed interval.

    (b) Use BW version 2 to show that a subsequence of that sequence converges to a value in $[a, b]$, and since f is continuous, the subsequence's function output values also converge.

    (c) This contradicts the assumption that the f was unbounded, as the absolute valued function output values of the subsequence also grow without bound under that original assumption, and thus shouldn't converge.

2. By the AoC, the set of function output values on $[a, b]$ has a sup that we'll show is the max of the function output values, and we'll find the input value of f associated with that max value.

3. Construct a sequence in $[a, b]$ using the sup lemma with the function output values of that sequence (with $\varepsilon = \frac{1}{n}$), and bound that statement from above with the sup.

4. Use BW version 2 to show that the sequence has a subsequence that converges to the input value, and converting the above statement to use the subsequence and applying the squeeze theorem shows that the function output values of the subsequence converge to the sup.

5. Show that since f is continuous, the function output values of the subsequence also converge to the function value at the input value, and so the function

values on $[a, b]$ have the sup as an upper bound, which is the same as the function value at the input value.

....

Proof.

1. We will show that f is bounded on $[a, b]$.

   By contradiction.

   Suppose for every $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. $(*)$

   - - - - - - - - - - -

   By BW version 2, (x_n) contains a convergent subsequence (x_{n_k}) such that $x = \lim x_{n_k} \in [a, b]$.

   Since f is continuous on $[a, b], \lim f(x_{n_k}) = f(x)$.

   - - - - - - - - - - -

   This contradicts $(*)$, as by $(*), |f(x_{n_k})| \geq n_k \geq k$.

   - - - - - - - - - - -

2. By the AoC, $M = \sup \{f(x) \mid x \in [a, b]\}$ exists.

   - - - - - - - - - - -

   Then for every $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $M - \frac{1}{n} < f(x_n)$.

   Thus, $M - \frac{1}{n} < f(x_n) \leq M$.

   - - - - - - - - - - -

   By BW version 2, (x_n) contains a convergent subsequence (x_{n_k}) such that $\beta = \lim x_{n_k} \in [a, b]$.

   Then $M - \frac{1}{k} \leq M - \frac{1}{n_k} \leq f(x_{n_k}) \leq M$.

   Thus, by the squeeze theorem, $\lim f(x_{n_k}) = M$.

   - - - - - - - - - - -

   At the same time, since f is continuous on $[a, b], \lim f(x_{n_k}) = f(\beta)$.

   Thus, there exists $\beta \in [a, b]$ such that $f(\beta) = M$.

Therefore, for every $x \in [a, b]$, $f(x) \leq M = f(\beta)$.

$\blacksquare$

Bolzano's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose $f(a) < 0$ and $f(b) > 0$. Then there is $c \in (a, b)$ such that $f(c) = 0$.

. . . .

High-Level Overview of the Proof:

1. Use induction to create a sequence of closed intervals, each of which uses the previous interval's midpoint to be one of the new interval's endpoints. This is chosen as such to keep each interval's left endpoint negative and its right endpoint non-negative.

2. Use the NIP to show that both the left and right endpoints of the intervals converge to a value.

3. Use the definition of continuity and the order of limits theorem to show that the function output value at the above determined value is zero.

. . . .

Proof.

Let $a_1 = a, b_1 = b$, and $I_n = [a_1, b_1]$.

Suppose a sequence of closed intervals $I_1, \ldots, I_n$ with $I_j = [a_j, b_j]$ have been constructed such that $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n, f(a_j) < 0$, and $f(b_j) > 0$.

Let $z = \frac{a_n + b_n}{2}$.

1. If $f(z) < 0$, then $a_{n+1} = z$ and $b_{n+1} = b_n$.

2. If $f(z) \geq 0$, then $a_{n+1} = a_n$ and $b_{n+1} = z$.

Let $I_{n+1} = [a_{n+1}, b_{n+1}]$.

Then I_{n+1} is a closed interval, $I_{n+1} \subseteq I_n, f(a_{n+1}) < 0$, and $f(b_{n+1}) \geq 0$.

Thus, by induction we have a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \ldots$.

- - - - - - - - - -

By the NIP, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Let $c \in \bigcap_{n=1}^{\infty} I_n$.

Then for every $n \in \mathbb{N}, a_n \leq c \leq b_n$.

Since $\lim \text{length}(I_n) = 0, \lim a_n = c = \lim b_n$.

- - - - - - - - - -

Since f is continuous on $[a, b], \lim f(a_n) = f(c) = f(b_n)$.
Since $f(a_n) < 0$ for every $n \in \mathbb{N}$, by the order of limits theorem, $f(c) \leq 0$.
Since $f(b_n) \geq 0$ for every $n \in \mathbb{N}, f(c) \geq 0$.
Therefore, $f(c) = 0$ and $c \in (a, b)$. ■

The Chain Rule

Let $f : A \to B, g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$. If f is differentiable at $c \in A$ and g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c)$$

. . . .

High-Level Overview of the Proof:

TODO

. . . .

Proof.

Let $d : B \to \mathbb{R}$ be defined as follows:

$$d(y) = \begin{cases} \frac{g(y)-g(f(c))}{y-f(c)} & \text{if } y \neq f(c) \\ g'(f(c)) & \text{if } y = f(c) \end{cases}$$

Then $\lim_{y \to f(c)} d(y) = g'(f(c))$, so d is continuous at $f(c)$.

Then we have $(*)$ $d(y)(y - f(c)) = g(y) - g(f(c))$ holds for every $y \in B$ (including $y = f(c)$).

Then for $t \in A$ with $y = f(t)$, $(*)$ becomes

$d(f(t))(f(t) - f(c)) = g(f(t)) - g(f(c))$.

Thus, if $t \neq c$, $\frac{g(f(t))-g(f(c))}{t-c} = d(f(t)) \cdot \frac{f(t)-f(c)}{t-c}$.

Since f is differentiable at c, it is continuous at c, and $d(y)$ is continuous at $f(c)$.

Thus, $(d \circ f)$ is continuous at c.

As a result, $\lim_{t \to c} d(f(t)) = d(f(c)) = g'(f(c))$.

Therefore, $(g \circ f)'(c) = \lim_{t \to c} \frac{g(f(t))-g(f(c))}{t-c} = g'(f(c))f'(c)$. ■

Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

. . . .

High-Level Overview of the Proof:

TODO

. . . .

Proof.

Since $f : [a, b] \to \mathbb{R}$ is continuous, by the EVT, the max and min of f are attained on $[a, b]$.

1. If both the max and min of f are at a and b, then max of $f = f(a) = f(b) =$ min of f.

   Then f is necessarily constant.

   Thus, $f'(c) = 0$ for every $c \in (a, b)$.

2. Otherwise, by the IET, there exists $c \in (a, b)$ such that $f'(c) = 0$.

Therefore, there exists $c \in (a, b)$ such that $f'(c) = 0$. ■

Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

. . . .

High-Level Overview of the Proof:

TODO

. . . .

Proof.

Let $d(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}(x - a) + f(a)\right)$.

Then d is continuous on $[a, b]$ and differentiable on (a, b).

In addition, $d(a) = 0 = d(b)$, and $d'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.

By Rolle's Theorem, there exists $c \in (a, b)$ such that $d'(c) = 0$, and so

$0 = d'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$.

Therefore, $f'(c) = \frac{f(b)-f(a)}{b-a}$. ■

Continuous Limit Theorem

Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$ that converge uniformly on A to f. If each f_n is continuous at $c \in A$, then so is f.

....

High-Level Overview of the Proof:

TODO

....

Proof.

Let $\varepsilon > 0$.

Since (f_n) converges uniformly on A to f, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $x \in A, |f_n(x) - f(x)| < \frac{\varepsilon}{3}$.

Since f_{n_0} is continuous at c, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f_{n_0}(x) - f_{n_0}(c)| < \frac{\varepsilon}{3}$.

Then if $|x - c| < \delta$, then

$|f(x) - f(c)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(c)| + |f_{n_0}(c) - f(c)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

Therefore, f is continuous at c. ∎

Integrable Limit Theorem

Assume $f_n \to f$ uniformly on $[a,b]$ and each f_n is integrable. Then f is integrable and $\lim \int_a^b f_n = \int_a^b f$.

. . . .

High-Level Overview of the Proof:

TODO

. . . .

Proof.

Let $\varepsilon > 0$ be arbitrary.

Since (f_n) converges uniformly to f on $[a,b]$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $x \in [a,b], |f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}$.

Then $-\frac{\varepsilon}{3(b-a)} < f(x) - f_n(x) < \frac{\varepsilon}{3(b-a)}$.

f_{n_0} is integrable on $[a,b]$, and so there exists a partition P_0 of $[a,b]$ such that $U(f_{n_0}, P_0) - L(f_{n_0}, P_0) < \frac{\varepsilon}{3}$.

We have $U(f+g, P) \leq U(f,P) + U(g,P)$ and $L(f+g,P) \geq L(f,P) + L(g,P)$.

Then $U(f, P_0) - L(f, P_0) = U(f - f_{n_0} + f_{n_0}, P_0) - L(f - f_{n_0} + f_{n_0}, P_0)$

$\leq U(f - f_{n_0}, P_0) + (U(f_{n_0}, P_0) - L(f_{n_0}, P_0)) - L(f - f_{n_0}, P_0)$.

We have $U(f - f_{n_0}, P_0) \leq \frac{\varepsilon}{3(b-a)}(b-a) = \frac{\varepsilon}{3}$ and $L(f - f_{n_0}, P_0) \geq -\frac{\varepsilon}{3(b-a)}(b-a) = -\frac{\varepsilon}{3}$.

Therefore, $U(f, P_0) - L(f, P_0) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

For $n \geq n_0$, $\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f| \leq \frac{\varepsilon}{3(b-a)}(b-a) < \varepsilon$. ■

The Fundamental Theorem of Calculus (Part 1)

If $f : [a,b] \to \mathbb{R}$ is integrable and $F : [a,b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for every $x \in [a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

. . . .

High-Level Overview of the Proof:

TODO

. . . .

Proof.

Since F is differentiable on $[a,b]$, it is also continuous on $[a,b]$.
Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$.
For every $k = 1, \ldots, n$, by the MVT, there exists $t_k \in [x_{k-1}, x_k]$ such that $F'(t_k) = \frac{F(x_k)-F(x_{k-1})}{x_k - x_{k-1}}$.
Then $f(t_k) = \frac{F(x_k)-F(x_{k-1})}{x_k - x_{k-1}}$, and so $F(x_k) - F(x_{k-1}) = f(t_k)\Delta x_k$.
Then $m_k \leq f(t_k) \leq M_k$.
Therefore, $\sum_{k=1}^n (F(x_k) - F(x_{k-1})) = \sum_{k=1}^n f(t_k)\Delta x_k \leq \sum_{k=1}^n M_k \Delta x_k = U(f,P)$ and $\sum_{k=1}^n (F(x_k) - F(x_{k-1})) \geq \sum_{k=1}^n m_k \Delta x_k = L(f,P)$.
However, $\sum_{k=1}^n (F(x_k) - F(x_{k-1})) = F(x_n) - F(x_0) = F(b) - F(a)$.
Thus, $L(f,P) \leq F(b) - F(a) \leq U(f,P)$, and so $L(f) \leq F(b) - F(a) \leq U(f)$.
Since f is integrable on $[a,b]$, $L(f) = U(f)$, and so $F(b) - F(a) = L(f) = \int_a^b f$. ■

Appendix B: Theorems no longer needed for the final

Archimedean Property (ArP)

For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

....

High-Level Overview of the Proof:

1. By contradiction, assume the negation of the statement and argue that x is an upper bound for $\mathbb{N}$, and so by the AoC, a sup exists for $\mathbb{N}$.
2. Show that there exists an element in $\mathbb{N}$ that is greater than the sup (using the sup lemma with $\varepsilon = 1$), which is a contradiction.

....

Proof.
By contradiction.
Suppose there exists $x \in \mathbb{R}$ such that for every $n \in \mathbb{N}, n \leq x$.
Then x is an upper bound for $\mathbb{N}$.
Clearly, $\mathbb{N} \neq \emptyset$.
By the AoC, $\alpha = \sup \mathbb{N}$ exists.
\- - - - - - - - - -
Then there exists $n \in \mathbb{N}$ such that $\alpha - 1 < n$.
Then $\alpha < n + 1$.
Since $n \in \mathbb{N}, n + 1 \in \mathbb{N}$.
This contradicts the fact that α is an upper bound for $\mathbb{N}$. ■

<u>Theorem on continuous functions on a closed interval being uniformly continuous</u>

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous on $[a, b]$.

. . . .

High-Level Overview of the Proof:

1. By contradiction, suppose f is not uniformly continuous on $[a, b]$, and use the theorem for a function failing to be uniformly continuous.

2. Use BW version 2 and the triangle inequality to show that a subsequence of one of the above sequences converges to the same value as a subsequence of the other.

3. Use the definition of continuity and the triangle inequality to show that the absolute value of the difference of function output values between the two subsequences converges to zero, contradicting the assumption that they should converge to a positive real number.

. . . .

Proof.
By contradiction.
Suppose there exist $\varepsilon_0 > 0$ and $(x_n), (y_n)$ contained in $[a, b]$ such that
$\lim |x_n - y_n| = 0$, but $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for every $n \in \mathbb{N}$. $(*)$
\- - - - - - - - - -
By BW version 2, there exists a subsequence (x_{n_k}) of (x_n) that converges and such that $x = \lim x_{n_k} \in [a, b]$.
We have $|y_{n_k} - x| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x| \to 0$.
Thus, $\lim y_{n_k} = x$.
\- - - - - - - - - -
Since f is continuous on $[a, b], \lim f(x_{n_k}) = f(x) = \lim f(y_{n_k})$, and so
$|f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(x)| + |f(y_{n_k}) - f(x)| \to 0$, contradicting $(*)$. ■