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Solving Laplace's equation with Physics-Informed Neural Networks

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Preface

Takk til Amalie som har støttet meg med masse kjærighet

Abstract

The `ntnuthesis` document class is a customised version of the standard `LATEX report` document class. It can be used for theses at all levels – bachelor, master and PhD – and is available in English (British and American) and Norwegian (Bokmål and Nynorsk). This document is ment to serve (i) as a description of the document class, (ii) as an example of how to use it, and (iii) as a thesis template.

Sammendrag

Dokumentklassen `ntnthesis` er en tilpasset versjon av L^AT_EX' standard `report`-klasse. Den er tilrettelagt for avhandlinger på alle nivåer – bachelor, master og PhD – og er tilgjengelig på både norsk (bokmål og nynorsk) og engelsk (britisk og amerikansk). Dette dokumentet er ment å tjene (i) som en beskrivelse av dokumentklassen, (ii) som et eksempel på bruken av den, og (iii) som en mal for avhandlingen.

Zhang et al., 2022

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ΔT temperaturendring. 4

Chapter 1

Introduction

1.1 Background and motivation

The rise of deep learning has been a game changer in the field of machine learning. Deep learning has been used to solve a wide range of problems, such as image recognition, speech recognition, and natural language processing.

However, deep learning models are often considered as black boxes, as they are difficult to interpret. This is a problem, as it is important to understand how a model makes predictions in order to trust it. In addition, deep learning models are often overconfident in their predictions, which can lead to catastrophic failures in safety-critical applications. Therefore, there is a need for methods that can provide uncertainty estimates for deep learning models.

In the field of engineering, uncertainty estimation is important for decision-making under uncertainty. For example, geotechnical structure design and analysis are affected by measurement uncertainty, statistical uncertainty, and transformation model uncertainty. These geotechnical uncertainties are not capable of being taken into account by conventional deterministic analytical methods. First of all, there are many different causes of uncertainty, including variations in soil characteristics, building techniques, and environmental factors. Therefore, there is a need for methods that can provide uncertainty estimates for geotechnical structure design and analysis.

There has been very little research on uncertainty estimation for deep learning models. One approach is to use Bayesian neural networks, which can provide uncertainty estimates by placing a distribution over the weights of a neural network. However, Bayesian neural networks are computationally expensive and difficult to train. Another approach is to use ensemble methods, which can provide uncertainty estimates by training multiple models on different subsets of the data. However, ensemble methods are also computationally expensive and difficult to train.

Physics informed neural networks (PINNs) are a class of deep learning models that can be used to solve partial differential equations (PDEs). PINNs have been used to solve a wide range of PDEs, including the heat equation, the wave equation, and the Navier-Stokes equations. However, PINNs are often overconfident in their predictions, which can lead to catastrophic failures in safety-critical applications.

Therefore, there is a need for methods that can provide uncertainty estimates for PINNs.

1.2 Research question/objective

In this thesis, we use a metamodel-based importance sampling (MAIS) with a PINN as our metamodel. The objective of this thesis is to develop a method for uncertainty estimation for PINNs using MAIS. The research question is: Can we use MAIS with a PINN as our metamodel to provide uncertainty estimates for PINNs?

1.3 Scope and limitations

The thesis is limited to the use of MAIS with a PINN as our metamodel for uncertainty estimation. We do not consider other methods for uncertainty estimation, such as Bayesian neural networks or ensemble methods. We also do not consider other types of metamodels, such as Gaussian processes or radial basis functions. Furthermore, we focus on the use of MAIS with a PINN as our metamodel for uncertainty estimation. The partial differential equations considered are SO and SO

Due to

1.4 Thesis structure

Chapter 2

Preliminaries

2.1 Uncertainty Quantification

In the domain reliability analysis, the primary goal is to estimate the probability of failure, or probability of exceeding a certain threshold. Theoretically, the probability of failure is defined as the probability that the limit state function is less than zero. The limit state function is a function of the random variables and the model parameters. The random variables are the input variables to the model, and the model parameters are the parameters of the model. The limit state function is defined as $g(\mathbf{X}, \mathbf{Z})$, where \mathbf{X} is the random variables and \mathbf{Z} is the model parameters. The probability of failure is defined as the integral of the joint probability density function of the random variables and the model parameters over the region where the limit state function is less than zero. The probability of failure is given by the following equation:

$$P_f = \int_{\Omega} f_{\mathbf{X}, \mathbf{Z}}(\mathbf{x}, \mathbf{z}) d\mathbf{x} d\mathbf{z} \quad (2.1)$$

where $\Omega = \{(\mathbf{x}, \mathbf{z}) \mid g(\mathbf{x}, \mathbf{z}) < 0\}$.

This integral is often intractable and must be approximated using numerical methods. The most common method for approximating the probability of failure is the Monte Carlo simulation. The Monte Carlo simulation is a statistical method that uses random sampling to estimate the probability of failure. The Monte Carlo simulation works by generating random samples of the random variables and the model parameters, evaluating the limit state function for each sample, and counting the number of samples where the limit state function is less than zero. The probability of failure is then estimated as the ratio of the number of samples where the limit state function is less than zero to the total number of samples. The Monte Carlo simulation is a powerful and versatile method for estimating the probability of failure, but the convergence rate of MCS in number of samples is very slow.

Based on the discussion above, it is safe to conclude that the primary bottleneck of all the reliability analysis techniques is the need for running simulation to evaluate the limit-state function. Often, the

limit-state function are in form of complex nonlinear ordinary/partial differential equations (ODE/PDE) and solving it repeatedly can make the process computationally expensive. (Chakraborty, 2020)

2.2 PINNs Historical Overview

PINNs were developed by Raissi et al., 2019 in 2019. The use areas are diverse, including but not limited to... The main work of using PINNs concerns the solving of ordinary and partial differential equations (PDEs) and inverse problems governed by PDEs. The main idea is to use the PDE as a constraint in the loss function of the neural network. This is done by enforcing the PDE at a set of collocation points. The collocation points are sampled from the domain of interest. The loss function is minimized by using gradient-based optimization algorithms, as explained in 4. By relying on a PDE as a constraint, the need for labelled data is eliminated. This is a significant advantage of PINNs compared to traditional machine learning methods.

The first work applying PINNs to reliability analysis was done by Chakraborty (2020).

The probabilistic model with metamodel-based importance sampling was proposed by and when....

Uncertainty quantifications in PINNs are at what stage?...

The current work on the application of PINN to reliability analysis is still in the preliminary stage and mainly based on the first approach, i.e., physics-informed loss function. Chakraborty (2020) firstly applied PINN to reliability analysis. In his work, the LSF is expressed in the form of linear or nonlinear PDEs, and PINN is used to solve PDEs and obtain the failure probability. Based on the work in Chakraborty (2020), Zhang and Shafieezadeh (2022) introduced an active learning approach with the dual objective of training PINN for solving PDEs and characterizing the limit state. Zhou et al. (2023) applied PINN to the dynamic reliability analysis of multi-state systems. All these works transformed the reliability analysis problems into the solution of PDEs. However, the PDEs are not always available for reliability problems especially for static reliability, and PINN proposed in Raissi et al. (2019) is not available for such reliability problems that cannot be described by PDEs. For such problems, a novel NN approach combined with physical information needs to be constructed for reliability analysis.

Bai and Song, 2023

The fact that PINNs are not always available for reliability problems that cannot be described by PDEs is a significant limitation. This is a motivation for the current work. For static reliability problems, the system behaviour is typically modeled using limit state functions like $g(X)$, where X is the random variables. This is an algebraic equation, with no derivatives.

B-PINNs and E-PINNs and P-PINNs all exist..

The combination of PINNs and metamodel-based importance sampling is novel and has not been explored before.

Chapter 3

Literature study

3.1 Physics-informed neural networks for consolidation of soils

file:///C:/Users/jolu2/AppData/Local/Microsoft/Windows/INetCache/IE/79SHIDH0/Physics-informed_neural_networ[1].pdf

For the forward problem, it is difficult to obtain analytical solutions for most of the models related to consolidation.

Researchers have revealed that PINNs possess the following advantages compared with the conventional mesh-based numerical methods in tackling the forward problem. First, PINNs is capable of solving the inverse problem with the only minor change of the code that is used in a forward problem (Liu and Wang, 2019). Secondly, neural network-based methods with mesh-free features can reduce the tedious work of mesh generation (Basir and Senocak, 2022). Thirdly, PINNs can obtain remarkably accurate solutions and reliable parameter estimations with fewer data and average-quality data, to reduce the dependence on the need for large training datasets (Zhang et al., 2021). Fourthly, PINNs can produce results at any point in the domain once it has been trained (Basir and Senocak, 2022).

Uses Tanh

The values of w_f and w_b are commonly assumed to be 1 in the case that all the loss values are of the same order of magnitude. It should be noted that the choice of these two coefficients is still an open problem needing further investigations (Wang et al., 2021).

We use the Adam combined with L-BFGS optimizers

Furthermore, we employ a commonly used Glorot normal scheme (Glorot and Bengio, 2010) as the p

To deal with the stochastic nature of the training procedure, we calculated the results as an average over 5 realizations as suggested by Kadeethum et al. (2020)

Found that more with a lot of domain points, increasing the boundary points help, but not the other way around necessarily.- overfitting?

It is clear from Figure 7a that the performance of PINNs is found to be optimal at learning rates of 10^{-2} and 10^{-3}

3.2 Probabilistic physics-informed neural network for seismic petrophysical inversion

https://library.seg.org/doi/10.1190/geo2023-0214.1?utm_source=chatgpt.com

We propose a probabilistic physics-informed neural network (P-PINN) algorithm for seismic petrophysical inversion with the goal of estimating the most likely model of petrophysical properties, the unknown hyperparameters of the rock-physics model, and the uncertainty of the model predictions.

3.3 Machine Learning-Aided Monte Carlo Simulation and Subset Simulation

https://journals.sagepub.com/doi/10.1177/03611981241248166?utm_source=chatgpt.com#body-ref-bibr14-03611981241248166

(...) geotechnical structure design and analysis are also affected by measurement uncertainty, statistical uncertainty, and transformation model uncertainty. These geotechnical uncertainties are not capable of being taken into account by conventional deterministic analytical methods. First of all, there are many different causes of uncertainty, including variations in soil characteristics, building techniques, and environmental factors (13).

3.4 Bayesian neural networks

3.4.1 Simple idea

The difference between a normal neural network and a Bayesian neural network is that the latter has a distribution of weights, rather than a single value. This allows for the model to express uncertainty in the predictions.

We have a KL term as well as the likelihood term in the loss function. The KL term is a measure of how much the posterior distribution differs from the prior distribution. The likelihood term is the negative log likelihood of the data given the weights. Posterior distribution, $P(W|D)$, is the distribution of the weights given the data, and the prior distribution, $P(W)$, is the distribution of the weights before seeing the data.

$$P(W|D) = \frac{P(D|W)P(W)}{P(D)} \quad (3.1)$$

We approximate $P(W|D)$ with a Gaussian distribution, and the KL divergence describes the difference between this distribution and the posterior distribution.

Another approach is to use Markov Chain Monte Carlo. Then we use a Monte Carlo instead of a simple distribution as replacement. Less efficient.

3.4.2 In Practice

Each weight have a distribution $W \sim \mathcal{N}(\mu, \sigma^2)$

Then compute the likelihood of

A numerical example:

$$\text{NN}(x; w, b) = w^*x + b$$

$$x_1 = 1, y_1 = 1.8 \quad x_2 = 3, y_2 = 2.7$$

We have a Gaussian prior distribution of the weights, $w = b = \mathcal{N}(0, 1^2)$

The data likelihood for a single data point is

$$p(y_i|w, b) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_i - \text{NN}(x_i; w, b))^2\right) \quad (3.2)$$

We use variational inference to approximate the posterior distribution. $q(w, b) = q(w)q(b)$. We use a Gaussian distribution for each weight.

$$q(w) = \mathcal{N}(\mu_w, \sigma_w^2) \quad (3.3)$$

$$q(b) = \mathcal{N}(\mu_b, \sigma_b^2) \quad (3.4)$$

Evidence lower bound (ELBO) is the objective function we want to maximize. It is written as: $ELBO = E[\log p(D|w, b)] + \ln(p(w, b)) - \ln(q(w, b))$, where the KL divergence is the difference between the prior and the posterior: $KL(q(w, b)||p(w, b)) = -\ln(q(w, b)) + \ln(p(w, b))$. The prior is written as $p(w, b) = p(w)p(b)$.

We first sample from the prior distribution, then we compute the likelihood of the data given the weights, and then we compute the KL divergence between the prior and the posterior:

The prior distribution is $w = b = \mathcal{N}(0, 1^2)$. Let's say we draw $w' = 0.37$ and $b' = -0.11$ from the prior distribution. Then we compute the likelihood of the data given these weights:

$$ELBO = E[\log p(D|w', b')] + \ln(p(w', b')) - \ln(q(w', b'))$$

First term is the likelihood of the data given the weights. We have two data points, so we compute the log likelihood for each data point and sum them up:

$$\log p(D|w', b') = \log p(y_1|w', b') + \log p(y_2|w', b') \quad (3.5)$$

$$= \log\left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_1 - \text{NN}(x_1; w', b'))^2\right)\right) \quad (3.6)$$

$$+ \log\left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_2 - \text{NN}(x_2; w', b'))^2\right)\right) \quad (3.7)$$

Numerically, if we assume that the observation noise is gaussian with a standard deviation of 0.2, we get:

$$\begin{aligned} \log p(D|w', b') &= \log \left(\frac{1}{\sqrt{2\pi} \cdot 0.2} \exp \left(-\frac{1}{2 \cdot 0.2^2} (1.8 - \text{NN}(1; 0.37, -0.11))^2 \right) \right) \\ &\quad (3.8) \\ &\quad + \log \left(\frac{1}{\sqrt{2\pi} \cdot 0.2} \exp \left(-\frac{1}{2 \cdot 0.2^2} (2.7 - \text{NN}(3; 0.37, -0.11))^2 \right) \right) \\ &\quad (3.9) \end{aligned}$$

When computed numerically, we get log likelihood = -64,38.

Next term is the log prior, $\ln(p(w', b')) = \ln(p(w')) + \ln(p(b')) = \ln \left(\frac{1}{2\pi} e^{-\frac{0.37^2}{2}} \right) + \ln \left(\frac{1}{2\pi} e^{-\frac{0.11^2}{2}} \right)$

This becomes -0.988 - 0.926 = -1.914.

The last term is the log of the variational distribution, $\ln(q(w', b')) = \ln(q(w')) + \ln(q(b')) = \ln \left(\frac{1}{2\pi} e^{-\frac{\mu_w^2}{2\sigma_w^2}} \right) + \ln \left(\frac{1}{2\pi} e^{-\frac{\mu_b^2}{2\sigma_b^2}} \right)$

3.5 Weight Uncertainty in Neural Networks

<https://arxiv.org/pdf/1505.05424>

Bayes by Backprop is just a type of inference method.

"In general, exact Bayesian inference on the weights of a neural network is intractable as the number of parameters is very large and the functional form of a neural network does not lend itself to exact integration."

In BNN, we write $p(y|x, w)$ as the likelihood instead of the general $p(D|W)$.

3.6 from chatGPT

An alternative to the variational inference is to use Markov Chain Monte Carlo (MCMC) methods. This is less efficient, but more accurate. VI: Faster, good for large-scale problems, but can suffer from approximation bias due to the factorized assumption. MCMC: More "accurate" in theory, but can be computationally heavy and slow to converge.

ΔT is the temperature change.

Real-World Analogy

Imagine you are trying to guess **how likely it is that a student will pass an exam**.

- You **don't know their intelligence** (w), but you have some prior belief about it.
- Given their intelligence, they have a certain probability of passing ($p(D \mid w)$).
- Since you **don't know their exact intelligence**, you average over all possibilities, weighted by how likely each intelligence level is.

$$p(\text{Pass}) = \int p(\text{Pass} \mid \text{IQ})p(\text{IQ})d(\text{IQ})$$

Figure 3.1: An analogy of $p(D) = \int p(D \mid w)p(w)dw$ explained by ChatGPT.

Chapter 4

Theoretical Framework

1. Why Monte Carlo Can Struggle With Rare Events When you draw samples from the original (or “nominal”) distribution—e.g., $N(\mu, \sigma^2)$ for each parameter—the probability of seeing “extreme” samples that cause failure (like crossing $y < -1$) may be very small. Hence, most samples do not contribute to the failure count, and the probability estimate can have a high variance (you need a huge number of samples to accurately capture rare failures).

Importance Sampling (IS) addresses this by sampling from a modified distribution that focuses on the region more likely to produce failure (i.e., “importance region”). Then each sample is re-weighted to maintain an unbiased estimate of the true probability.

Chapter 5

Methodology

Chapter 6

Results

Chapter 7

Discussion

Chapter 8

Conclusion

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