

Validated enclosure of renormalization fixed points via Chebyshev series and the DFT

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Abstract

This work develops a computational framework for proving existence, uniqueness, isolation, and stability results for real analytic fixed points of m -th order Feigenbaum-Cvitanović renormalization operators. Our approach builds on the earlier work of Lanford, Eckman, Wittwer, Koch, Burbanks, Osbaldestin, and Thurlby [38, 26, 23, 8, 9], however the main point of departure between ours and previous studies is that we discretize the domain of the renormalization operators using Chebyshev rather than Taylor series. The advantage of Chebyshev series is that they are naturally adapted to spaces of real analytic functions, in the sense that they converge on ellipses containing real intervals rather than on disks in \mathbb{C} . The main disadvantage of working with Chebyshev series in this context is that the essential operations of rescaling and composition are less straight forward for Chebyshev than for Taylor series. These difficulties are overcome using a combination of a-priori information about decay rates in the Banach space with a-posteriori estimates on Chebyshev interpolation errors for analytic functions.

Our arguments are implemented in the Julia programming language and exploit extended precision floating point interval arithmetic. In addition to proving the existence of multiple renormalization fixed points of order $m = 3, \dots, 10$, and computing validated bounds on the values of their universal constants, we also reprove the existence of the classical $m = 2$ Feigenbaum renormalization fixed point and compute its universal constants to close to 500 correct decimal digits.

1 Introduction

The present work is concerned with real analytic, unimodal functions of a single variable which solve certain functional equations of Feigenbaum-Cvitanović type. By a unimodal function we mean a mapping which takes $I = [-1, 1]$ into itself and which, (a) has a single critical point $c \in I$ and (b) is strictly increasing on $[-1, c]$ and strictly decreasing on $[c, 1]$. There is no loss of generality in taking $c = 0$, and it actually simplifies somewhat the discussion as this condition can be imposed by considering even functions.

Note that a real analytic function on I can be analytically continued to an open set in \mathbb{C} containing I , and that any such open set contains an ellipse of the following form.

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Definition 1.1. For $\rho > 1$, define the closed Bernstein ellipse $\mathcal{E}_\rho \subset \mathbb{C}$ of radius ρ by

$$\mathcal{E}_\rho = \left\{ \frac{1}{2} \left(z + z^{-1} \right), z \in \mathbb{C}, 1 \leq |z| \leq \rho \right\}.$$

The remarks above motivate the following definition.

Definition 1.2. For $\rho > 1$, let X_ρ denote the set of all $f: \mathcal{E}_\rho \rightarrow \mathbb{C}$ with the following properties.

- **Analyticity:** f is analytic on the interior of \mathcal{E}_ρ ,
- **Continuity to the boundary:** f is bounded and continuous on the closed ellipse \mathcal{E}_ρ ,
- **Reality:** $f(x) \in \mathbb{R}$ when $x \in \mathcal{E}_\rho \cap \mathbb{R}$,
- **Evenness:** $f(x) = f(-x)$ for $x \in \mathcal{E}_\rho \cap \mathbb{R}$.

Denoting by $\|\cdot\|_{\mathcal{C}_\rho^0}$ the supremum norm on \mathcal{E}_ρ , i.e.,

$$\|f\|_{\mathcal{C}_\rho^0} = \sup_{z \in \mathcal{E}_\rho} |f(z)|,$$

we have that $(X_\rho, \|\cdot\|_{\mathcal{C}_\rho^0})$ is a Banach space.

Definition 1.3. Let $\rho > 1$ and

$$f^m = \underbrace{f \circ \dots \circ f}_{m \text{ times}},$$

denote the composition of f with itself m times. We refer to

$$\frac{1}{f^m(0)} f^m(f^m(0)z) = f(z), \quad z \in \mathcal{E}_\rho, f \in X_\rho, \quad (1)$$

as the (symmetric) m -th order Feigenbaum-Cvitanović equation.

We use the notation $R_m(f)$ as shorthand for the left hand side of Equation (1), and refer to R_m as an m -th order renormalization operator (for even unimodal maps). Note that if f solves Equation (1), then

$$f(0) = \frac{1}{f^m(0)} f^m(f^m(0)0) = \frac{1}{f^m(0)} f^m(0) = 1. \quad (2)$$

Remark 1.4 (Additional terminology). We refer to a solution $f \in X_\rho$ of Equation (1) as a renormalization fixed point, and distinguish such fixed points as follows.

- **Order of the solution:** we refer to $m \geq 2$ as the order of the renormalization fixed point f , when m is the number of compositions in the definition of Equation (1).
- **Degree of the solution:** Consider $f \in X_\rho$ with $f(0) = 1$. Since f is even and analytic, it has convergent Taylor series expansion of the form

$$f(z) = 1 + a_1 z^2 + a_2 z^4 + a_3 z^6 + \dots \quad (3)$$

on some disk $|z| < R$. Let $d \geq 1$ denote the first index where $a_{2d} \neq 0$. We refer to $d \geq 1$ as the degree of f .

- **Kneading sequence:** *this is a partial invariant for unimodal maps, introduced by Thurston and Milnor [51], in the sense that unimodal maps with different kneading sequences are not topologically conjugate. The definition of the kneading sequence is somewhat technical, but it involves tracking the behavior of the orbit of the critical point $x = 0$. More precisely, one constructs a sequence of symbols $L = \text{“left”}$, $R = \text{“right”}$ and $C = \text{“center”}$ where the j -th term in the sequence records whether the j -th iterate of $x = 0$ is to the left of zero, to the right of zero, or zero itself (the “center point” of I). Such a sequence may be periodic, eventually periodic, or a-periodic.*
- **Schwarzian derivative:** *We refer to*

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2, \quad (4)$$

as the Schwarzian derivative of f on $[-1, 1]$, and say that f has negative Schwarzian derivative if $(Sf)(x) < 0$ on $[-1, 1] \setminus \{0\}$. The significance of this is that a unimodal map f with negative Schwarzian derivative has at most one attracting periodic orbit.

The family of polynomials

$$q_{d,\mu}(x) = 1 - \mu x^{2d}, \quad (5)$$

provide a standard model for even, degree d , unimodal maps. Since $a_1 \neq 0$ is generic in Equation (3), the case of $d = 1$ is of special importance. We refer to $q_\mu(x) = q_{1,\mu} = 1 - \mu x^2$ as the quadratic family, and focus in the present work on renormalization fixed points of degree $d = 1$.

Note also that, while we are interested in solutions of Equation (1) with negative Schwarzian derivative, we will check this condition a-posteriori rather than imposing it a-priori in the function space. A related remark is that we do not make much use of the notion of kneading sequence in the present work, other than as a heuristic guide for locating approximate renormalization fixed points of various orders.

Given a renormalization fixed point $f \in X_\rho$, we are also interested in the spectrum of the linearized equation at f . More precisely, let $DR_m(f)$ denote Fréchet derivative of R_m at f . We seek $\xi \in X_\rho$ and a complex number $\lambda \in \mathbb{C}$ so that

$$DR_m(f)\xi = \lambda\xi,$$

with $\|\xi\|_0 \neq 0$ and $|\lambda| > 1$. As another bit of introductory notation, we write

$$\alpha = (f)^m(0).$$

The constants λ (unstable eigenvalue) and α (or its negative reciprocal $-1/\alpha$) are referred to as the universal scalings or *Feigenbaum constants* associated with the renormalization fixed point f . The dynamical meaning of the universal constants is reviewed briefly in Section 1.2.

The following theorem illustrates the utility of the constructive methods developed in the main body of the present work. These results, and others, are discussed in greater detail in Section 7.

Theorem 1.5 (Existence of renormalization fixed points through order ten). *Let \bar{f}_m , $m = 2, \dots, 10$ denote the Chebyshev polynomials whose graphs are illustrated in Figure 1 and*

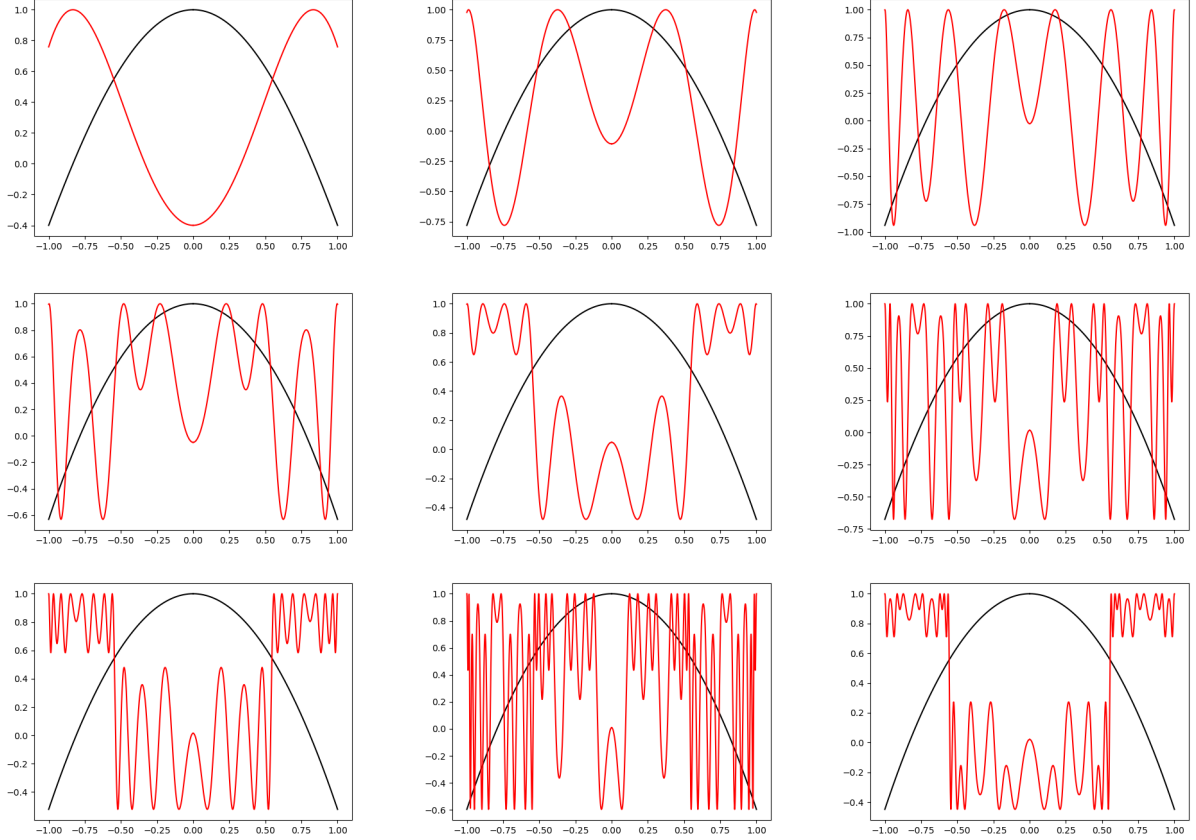


Figure 1: **Some Solutions of Equation (1) for $m = 2, \dots, 10$:** The sub-figures are arranged in order of increasing order m , with a solution for $m = 2$ in the top left and one for $m = 10$ in the bottom right. In each frame the black line depicts the graph of the approximate fixed point \bar{f}_m , described in Theorem 1.5, over the interval $[-1, 1]$. The red curve depicts the composition of f_m with itself m times (but with no rescaling, so the result is not fixed).

whose coefficients are given in the file `renor_run_final` [35]. Then there exist real numbers $0 < \epsilon_2, \dots, \epsilon_{10} \ll 1$ and functions $f_2, \dots, f_{10} \in X_2$ having that

$$\sup_{z \in \mathcal{E}_{\rho_m}} |f_m(z) - \bar{f}_m(z)| \leq \epsilon_m,$$

and that f_m is an m -th order solution of Equation (1). Each renormalization fixed point f_m is locally unique, and each ρ_m is at least 2. Interval enclosures of the associated universal constants λ_m and α_m , the degree of each polynomial approximation, and upper bounds for the ϵ_j , $2 \leq j \leq 10$, are provided in Table 1.

Note that C^0 bounds on the ellipse \mathcal{E}_{ρ_m} automatically yield error bounds on the real interval $[-1, 1]$, and that the values of the ϵ_j obtained in the computer assisted proofs are actually sharper than those reported in Table 1. See Section 7. The use of $\rho = 2$ in the proofs is convenient, but provides only a lower bound on the domain of analyticity for the renormalization fixed points f_m , $2 \leq m \leq 10$ described Theorem 1.5. In general we expect the domains to be larger.

A number of other theorems are given in Section 7, including some global non-uniqueness results. More precisely, for $m = 5, 6, 7, 8, 9$ and 10, we prove the existence of at least two (and in some cases more) distinct renormalization fixed points, together with a precise description of these fixed points and of their associated universal constants.

id	K_m	ϵ_m	α_m	λ_m
2v1	21	7×10^{-18}	$-0.39953528052313448 \frac{0}{9}$	$4.66920160910299067 \frac{0}{9}$
3v1	15	5×10^{-21}	$-0.10778950429255075546 \frac{0}{9}$	$55.247026588671997372 \frac{0}{9}$
4v1	15	7×10^{-23}	$-0.0257605318546251163452 \frac{0}{9}$	$981.5949765340714276465 \frac{0}{9}$
5v1	15	9×10^{-24}	$-0.04968100507278386813949 \frac{0}{9}$	$255.545253865903316209 \frac{0}{9}$
6v1	15	3×10^{-23}	$0.0477814797951692540990 \frac{0}{9}$	$218.411795140494963093 \frac{0}{9}$
7v1	15	1×10^{-19}	$0.0170569468962859307380 \frac{0}{9}$	$2253.7925764038328042238 \frac{0}{9}$
8v1	15	1×10^{-19}	$0.015062576067338632 \frac{0}{9}$	$2304.5578444485922703759 \frac{0}{9}$
9v1	15	1×10^{-19}	$0.008887154873072033 \frac{0}{9}$	$7918.2235631712026192866 \frac{0}{9}$
10v1	15	1×10^{-19}	$0.020848923604938857 \frac{0}{9}$	$1110.537874176532781602 \frac{0}{9}$

Table 1: **Data associated with Theorem 1.5:** the table records the rigorously verified correct digits for the universal constants associated with m -th order renormalization fixed points. In this table, we use the notation $x = 1.23\frac{4}{6}$ to mean that $x \in [1.234, 1.236]$ and note that the recorded interval enclosures are part of what is established in the computer assisted proof. Each approximate fixed point \bar{f}_m is of degree $2K_m$. The λ_m and α_m are the universal constants associated with the f_m , and the ϵ_m are the bounds on the error between the \bar{f}_m and the actual renormalization fixed points f_m .

Numerically computed kneading sequences associated with each renormalization fixed point are reported in Table 9. Note that while we do establish the existence of multiple fixed points at several orders, we do not (in this paper) prove that we have found all possible fixed points for each order m . Indeed, for degree two fixed points it is known that there exists an unique fixed point for each “allowable” kneading sequence, and that for $m \geq 4$ there are several such sequences at each order. We refer to [17] for much more complete discussion. We also remark that each of the fixed points located in the present work is of order $d = 1$. For convenience, in place of the kneading sequence, we identify the computed fixed points by an *id* of the form (m value) v (some index).

Our computations are implemented in multiple precision interval arithmetic, and to highlight the value of this we have carried out, for the classical case of $m = 2$, verified computations of the renormalization fixed point, as well as the eigenvalue and eigenfunction, using extended precision interval arithmetic and more than 600 Chebyshev modes. This results in a guaranteed interval enclosure of the $m = 2$ universal constant – also known as the Feigenbaum constant – precise to close to 500 digits.

More accurate enclosures can be obtained with longer runtimes and we refer again to Section 7 for more details.

A sketch of the proof of Theorem 1.5 is as follows. For each $m = 2, \dots, 10$, we numerically approximate a solution of Equation (1) using an iterative Newton scheme. This requires discretizing the elements of X_2 , and for this we use Chebyshev series truncated after the K_m -th mode (see [6, 61] and also Section 3.1 for precise definitions). It is also necessary to seed the Newton method with a reasonable initial guess. In the present work we start the Newton iteration from an appropriately chosen function $q_{1,\mu} \in X_\rho$ in the quadratic family (see Equation (5)). Methods for choosing good starting values of μ , depending on the desired degree $m \geq 2$ of the renormalization fixed point are well known. See for example [17].

The numerical Newton method results in a Chebyshev polynomial, which we denote by \bar{f}_m . The graphs of these polynomial are illustrated in Figure 1 for $m = 2, \dots, 10$, and there coefficients are stored in the files described in the Theorem [35]. It must be mentioned that similar numerical schemes, based on Chebyshev series approximation, were used in the works

of Mathar [48] and Molteni [53]. However these works focus on the case of $m = 2$ and do not consider validated error bounds.

After performing the numerical calculations described above, the existence of a true renormalization fixed point f_m near the approximate solution \bar{f}_m is established via a Newton-Kantorovich argument. This requires mathematically rigorous bounds on the defect associated with the approximate solution, as well as on some other condition numbers. Here we work in Banach spaces of rapidly decaying infinite sequences of Chebyshev series coefficients, endowed with weighted little ell one norms. We note however that elements of this sequence space can be interpreted as functions in X_ρ . The desired bounds are obtained by combining interval arithmetic with interpolation and asymptotic decay rate estimates, and describing them in a way that covers all $m \geq 2$ is one of the main tasks of the present work. Interval enclosures of the universal constants are obtained by applying similar arguments to the simultaneous eigenvalue/eigenfunction problem.

Remark 1.6 (Normalizations). The Feigenbaum-Cvitanovich equation can be formulated in various equivalent ways. For example, in the $m = 2$ case, the equation is often stated as

$$\frac{1}{f(1)}f(f(-f(1)x)) = f(x), \quad -1 \leq x \leq 1, \quad (6)$$

subject to the additional constraint

$$f(0) = 1.$$

The minus sign in front of the innermost $-f(1)$ term is removed by restricting to a space of even functions. Then, when representing f using a Taylor series, the $f(0) = 1$ constraint is then enforced by simply imposing that the constant term in the Taylor expansion is one, and solving for the remaining coefficients (evenness is also imposed on the level of Taylor coefficients).

When working with Chebyshev series, the $f(0) = 1$ constraint is more global, as it involves the Chebyshev coefficients of all orders (see again Section 2).

This is one reason for imposing the normalization $f(0) = 1$ directly in the functional equation, as discussed at Equation (2) above.

Remark 1.7 (Chebyshev versus Taylor series). Heuristically speaking, the reason for studying solutions of Equation (1) on Bernstein ellipses is already suggested, at least implicitly, by the theoretical work of Epstein and Lascoux in [27], and the numerical work of Nauenberg in [55]. These authors provide detailed quantitative information about the shape of the domain of analyticity of the Feigenbaum function (solutions of the $m = 2$ case of Equation (1)) and show for example that its domain has a fractal shaped boundary enclosing the set $\mathbb{R} \cup i\mathbb{R}$. In particular, there exists a disk of radius $R > 0$ large enough so that its interior is not contained in the domain. See for example Figure 1 of [55].

Then, it is only by good luck that the domain of analyticity is large enough to contain a disk at the origin of radius $r \approx 2.5 > 1$. Because of this, the fixed point has a convergent Taylor series at the origin, which is a necessary condition for the success of computer assisted methods of proof based on Taylor series. Note however that there is no guarantee that this luck will hold as either the order or the degree of the solution is increased.

Indeed, the results of Collett and Eckmann in [24] show that when $m = 2$ and the order d is increased, the corresponding solutions of Equation (1) have domains of analyticity which shrink to $\mathbb{R} \cup i\mathbb{R}$ as $d \rightarrow \infty$. That is, for d large enough, there is no single disk in the complex plane containing $[-1, 1]$ on which the desired fixed point is analytic, and it will therefore not be representable by a single convergent power series at $x = 0$. Similarly, there is no guarantee that as m varies we will always have such a disk either.

These considerations suggest the use of Bernstein ellipses, as a real analytic function on $I = [-1, 1]$ can always be analytically continued into some Bernstein ellipse, whether or not it converges on a disk containing I . After making this choice, one recalls that analytic functions on Bernstein ellipses are naturally represented using Chebyshev series. These remarks constitute a major motivation for the present work.

1.1 Computer assisted proof in Renormalization theory

The strategy of computer assisted proof sketched above is, in broad outline, the one used by Lanford in the original computer assisted existence proof of the $m = 2$ fixed point [38]. Similar arguments were later used by Eckman, Wittwer, and Koch in the complete proof of the complete Feigenbaum conjectures [38, 26, 23]. The last reference just cited discusses additional theorems in renormalization theory proven in the same style. We refer also to the works of Burbanks, Osbaldestin, and Thurlby [8, 9] for more recent computer assisted results for R_m when $m = 2, 4$. There are also computer assisted theorems for renormalization operators other than Feigenbaum-Cvitanović (that is, for situations other than period doubling in one parameter families of one dimensional unimodal maps) which use arguments similar to those in the outline sketched above. We refer for example to the works of Koch and Wittwer [41, 42], Koch [39, 40], Gaidashev and Koch [31], Arioli and Koch [1], and Gaidashev and Yampolsky [32]. We also remark that computer assisted proofs based on Newton-Kantorovich and/or contraction mapping arguments are common in a wide variety of mathematical settings including dynamical studies of ordinary, delay, and partial differential equations. The interested reader is referred to the review articles of van den Berg and Lessard [66] and of Gómez-Serrano [34] for more general discussion.

All of this being said, studying Equation (1) on \mathcal{X}_ρ and discretizing using Chebyshev series completely changes the technical character of the arguments, further justifying the need for the present work. More precisely, the operations of composition and rescaling (see again Equation (1)) are less natural for Chebyshev than for Taylor series. For example, when working with Taylor series, compositions are worked out via iterated Cauchy products – which scale poorly as m is increased. For Chebyshev series, an analogous approach leads to iterated discrete convolutions of even worse complexity.

To overcome these issues we employ techniques based on interpolation. Indeed, recalling that a Chebyshev series is “a Fourier series in disguise,” we utilize the discrete Fourier transform (DFT) to evaluate compositions and rescalings “in grid space,” where such operations are essentially “diagonal” (that is, they have algorithmic complexity which scales with N , the degree of the Chebyshev series approximation). Of course this introduces the additional cost of the DFT, but this can be made $N \log(N)$ by employing the fast Fourier transform (FFT).

The DFT techniques just described could also be adapted for computer assisted proofs with Taylor series as well. The real advantage then of using Chebyshev series is that the Bernstein ellipses, the natural domains of analyticity for Chebyshev series, are especially well adapted for studying real analytic functions on intervals. Chebyshev series are also known to have excellent uniform approximation properties on closed intervals.

For computer assisted proofs using the interpolation based DFT methods just described, bounding discretization and truncation errors requires managing the so called “aliasing error” which is, more precisely, the difference between orthogonal projection and interpolation. To obtain the necessary bounds, we start from the aliasing formula for Chebyshev series [60, Theorem 4.2], and then derive interpolation error estimates between appropriate function spaces. These developments are described in detail in Section 3. Similar ideas were used by Figueras, Haro, and Luque for computing interval enclosures of function compositions for Fourier series in their work on computer assisted KAM theory [30].

We remark that, validated methods for computing rigorous interval enclosures of functions represented by Chebyshev series were developed by Joldes and Brisbarre in [7], with extensions

by Benoit, Joldes, and Mezzarobba in [5]. Existence proofs using Chebyshev series for differential equations with polynomial nonlinearities were developed by Lessard and Reinhardt in [44], and extensions to nonpolynomial applications in celestial mechanics and mechanical engineering are developed in [64, 10] by exploiting ideas from differential algebra. Here, one computes a rigorous enclosure of a non-polynomial function by solving the polynomial differential equation it satisfies. While this idea works well for problems in mathematical physics with nonlinearities given by elementary functions, it is not clear how to extend this idea to the function compositions appearing in renormalization theory, and this motivates the interpolation based approach developed below.

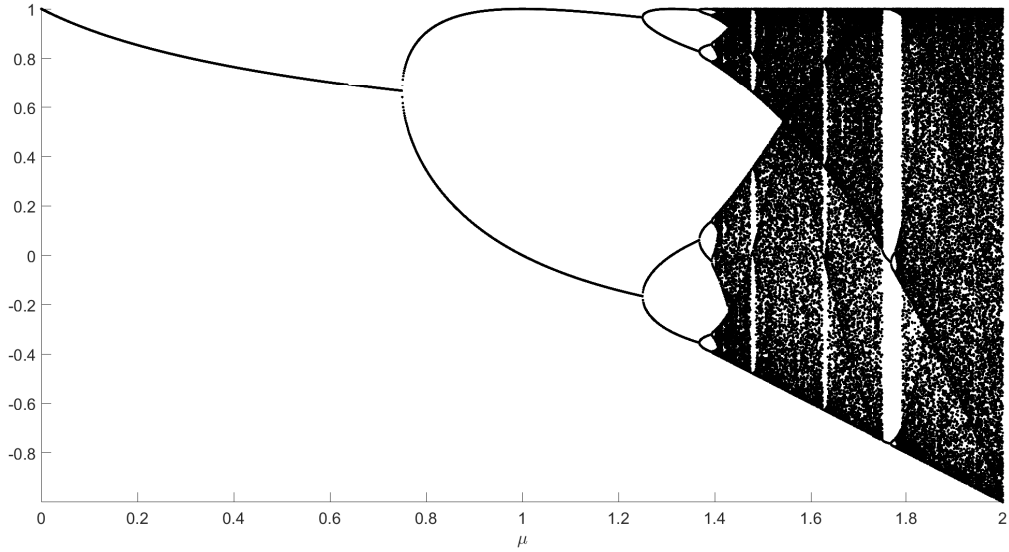


Figure 2: **Period doubling cascades:** the image illustrates several period doubling cascades for the quadratic family $q_{1,\mu}$ of Equation (5), by plotting accumulation points of orbits for different values of μ . The left side of the image depicts the cascade of stable period 2^n orbits bifurcating from the stable fixed point and accumulating to the chaotic regime. Shortly after the onset of chaos (for $\mu \approx 1.8$) one sees also the stable period three window, and its doubling bifurcations into orbits of period $3 \cdot 2^n$. Other stable period 5, and 6 windows can be seen by looking carefully after the first onset of chaos, and each undergoes its own period doubling cascade. The locations of all of these doublings are governed by the Feigenbaum constant λ_2 given in the first row of Table 1 (see Section 1.2).

1.2 Period-tupling cascades, and their renormalization fixed points

Period doubling cascades are central to study of nonlinear dynamical systems. The topic was popularized in the 1976 paper of May in Nature [49], but the story goes back much further, as illustrated for example by the 1962 paper of Myrberg on the existence of period doubling in polynomial families [54], and the 1964 results of Sharkovsky [57]. A lively discussion of more than a century of “pre-renormalization” literature on period doubling is found in the 2019 review article of Collet [11].

A typical period doubling cascade is illustrated in the top frame of Figure 2, for the quadratic family (defined in Equation (5) but recalled here for the sake of clarity)

$$q_\mu(x) = 1 - \mu x^2.$$

Figure 2 for example illustrates bifurcations behavior of q_μ as μ varies.

So for example when $\mu < 0.75$, one sees a single attracting fixed point. Near $\mu = 0.75$ an attracting period two orbit is born in a period doubling bifurcation. Further doubling bifurcations occur at $\mu \approx 1.25$ and beyond, resulting in attracting orbits of period 4, 8, 16 etcetera. After

$$\mu_\infty \approx 1.4\dots$$

orbits are attracted to a chaotic invariant set.

In the late 1970's, papers by Feigenbaum [28, 29] and Couillet and Tresser [14, 62] presented compelling numerical evidence in support of the conjecture that the period doubling bifurcations occur at parameters μ_n having

$$\mu_n \rightarrow 1.4\dots = \mu_\infty, \quad \text{as } n \rightarrow \infty,$$

and that the ratios between the lengths between successive bifurcations satisfy

$$\lim_{n \rightarrow \infty} \frac{\mu_{n+1} - \mu_n}{\mu_n - \mu_{n-1}} = \lambda_2 \approx 4.66920\dots \quad (7)$$

That is, the constant λ_2 eventually facilitates locating the next bifurcation in the sequence, once the first several are known.

The truly remarkable discovery was the “universality” of this behavior. Loosely speaking, it was observed that the numerical bifurcation diagrams associated with generic one parameter families of unimodal maps appear to satisfy Equation (7) *with the same universal constant* $\lambda_2 \approx 4.66920$, referred to as *the Feigenbaum constant*. The description of period doubling summarized above is part of what became known as *the Feigenbaum conjectures*.

The key point, from the perspective of the present work, is that the the universality properties just described are related to the renormalization fixed points described in the introduction. Indeed, it turns out that the Feigenbaum conjectures are equivalent to the statement that the $m = 2$ renormalization operator has a hyperbolic fixed point satisfying certain geometric hypotheses. More precisely, the one dimensional unstable manifold attached to the fixed point should intersect transversally the (codimension-one) set of superstable period two functions. Moreover the Feigenbaum constant is seen to be the unstable eigenvalue of the $m = 2$ renormalization operator at the fixed point. For a more detailed discussion of the equivalence between the Feigenbaum conjectures and renormalization fixed points, we refer to the work of Collet, Eckmann, Lanford in [12]. We also refer the interested reader to the paper [13] by Coppersmith for a heuristic derivation of the renormalization operator via period doubling.

The first existence proof of a locally unique fixed point for the $m = 2$ renormalization operator was given by Lanford in 1982 [38], and a complete proof of the Feigenbaum conjectures by Eckman and Wittwer appeared in 1987 [38, 26]. The results just mentioned were established with computer assistance, and we refer to the memoir of Eckman, Wittwer, and Koch [23] for a complete description of the results. Further developments are mentioned in Remark 1.9 below.

Shortly after the work of Feigenbaum-Couillet-Tresser on period doubling, it was discovered that period m -tupling cascades are similarly governed by higher order renormalization operators R_m with $m \geq 3$, and their universal constants (unstable eigenvalues).

To the best of our knowledge, the first study of this kind was the 1979 paper of Derrida, Gervois, and Pomeau [21], which considered universal properties of period tripling cascade ($m = 3$), and proposed a geometric explanation based on renormalization. A period tripling bifurcation occurs when a period k orbit bifurcates into an orbit of period $3k$, and a tripling cascade is a sequence of bifurcations into orbits of period $k3^n$ for $n \geq 1$. Further developments are found in the work of Gol'berg, Sinai, and Khanin [33], Hu and Satija [37], Delbourgo and Kenny [20], and Delbourgo, Hart, and Kenny [18, 19]. The paper [18] in particular is especially interesting

from the perspective of the present work, and many of its results – for example the values given in Table 1 of the paper just cited – are validated in the present work.

Figures 3 and 4 illustrate two different period tripling ($m = 3$) bifurcations. The first starts from a period 2 orbit and leads to orbits of period 6, 18, and beyond. The second starts from period 3 and leads to orbits of period 9, 27 and beyond. Note that for $m > 2$ the bifurcations are of fold or tangent type, rather than the pitchfork type seen in the $m = 2$ case. The cascades illustrated in Figures 3 and 4 are governed by the constant λ_3 given in Table 1. We note that similar m -tupling cascades are associated with each of the fixed points described in Theorem 1.5, but that plotting the resulting cascades is increasingly delicate.

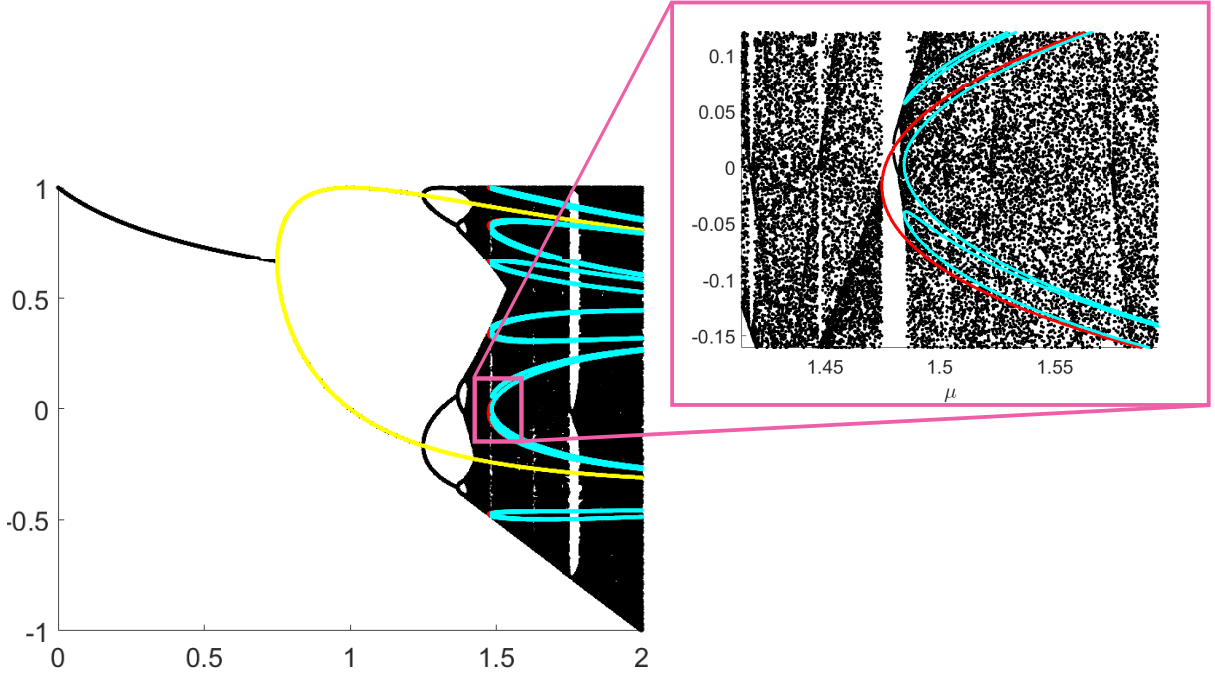


Figure 3: Tripling cascade for the stable period two orbit: the figure illustrates a period tripling cascade, starting from the stable period two orbit. Again, it is important to note that for $m > 2$, the bifurcations appearing in the cascade are of tangency or fold type and hence appear “to come out of nowhere” (the bifurcation curves are not connected as in the $m = 2$ case). The left frame illustrates the entire bifurcation diagram of the quadratic family (Equation (5) with $d = 1$) and the period two is highlighted in yellow. Just below $\mu = 1.5$ the period tripling bifurcation occurs, and a stable and unstable period 6 are born. The next bifurcation occurs almost immediately, resulting in a stable and unstable period 18 orbit. This is illustrated in the right frame of the figure where one branch of the period 6 is shown in red, followed by three branches of the period 18 in blue (the red and blue are difficult to distinguish in the frame on the right). Bifurcations to period 54 and beyond are difficult to illustrate graphically, but we note that they can be computed numerically, for instance by exploiting that they follow from the universal constant λ_3 given in the second row of Table 1. In order to better emphasize that the different orbits appearing in a period tripling cascade are not “connected”, the period 2, 6 and 18 orbits under consideration here are depicted even for values of μ for which they have become unstable, contrarily to what is typically done for such bifurcation diagram like in Figure 2 (where only stable orbits appear).

Remark 1.8 (Related existence and multiplicity results). The results of the present study are complementary to those of the 1984 paper [22] by Eckmann, Epstein, and Wittwer, where the

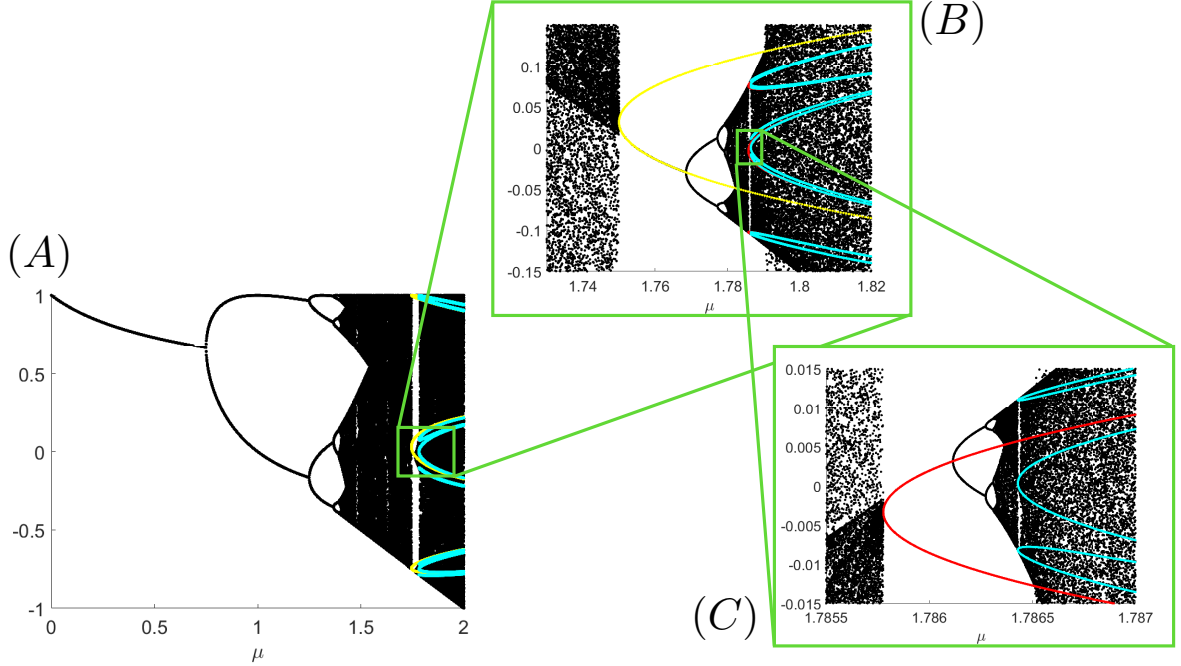


Figure 4: **Tripling cascade for the stable period three orbit:** we consider again the quadratic family (Equation (5) with $d = 1$) and focus on the period three orbit which is born in a fold or tangency bifurcation near $\mu = 1.75$. This gives rise to a stable and unstable branch of period three, shown as three yellow curves. The period three participates in a period tripling bifurcation resulting in orbits of period 9, 27, ... Frame (A) recalls the bifurcation diagram, and frame (B) zooms in on the middle component curve of the period three so that we can see the birth of the period 9 orbit near $\mu = 1.785$, depicted in red. Frame (C) zooms in further so that we can see three components of the period 27 family, depicted in blue. For period 81 and beyond it is increasingly difficult to resolve the picture graphically, but the appropriate families can be located using the universal constant λ_3 given in the second row of Table 1.

authors prove an existence theorem for fixed points of R_m , with m “large enough”. Indeed, they derive asymptotic expansions for f_m , λ_m , and α_m , which hold for large enough m . By contrast, the techniques of the present work facilitate the proof of existence results for (in principle) any non-perturbative value of m , and provide validated enclosures of the f_m , λ_m , and α_m as accurate as desired. An interesting problem for future consideration would be to determine (possibly with computer assistance) an explicit bound m_∞ such that the results of [22] hold for $m \geq m_\infty$. Mostly likely, such m_∞ would be greater than 10, but one could try to push the methods of the present work to establish the results for $11 \leq m \leq m_\infty$, and therefore obtain quantitative results for renormalization fixed points for all m .

We also mention the related perturbative work of Eckmann and Wittwer [25], where they study fixed points of the $m = 2$ renormalization operator as a function of the degree. That is, they prove the existence of renormalization fixed points of degree d near members of the standard family $q_{\mu,d}(x) = 1 - \mu x^{2d}$ in the limit as $d \rightarrow \infty$. This is a highly singular perturbation and the authors use Borel transform techniques to analyze the resulting divergent power series.

Remark 1.9 (Results from the wider literature). The discussion above is heavily skewed toward the perspective of the present work, in the sense that we have emphasized the literature on numerical simulations and computer assisted proofs. That being said, the general theory of one dimensional unimodal maps is among the most fully developed and celebrated in dynamical

systems theory. While a thorough review this literature is a task far beyond the scope of the present work, we mention briefly the existence proofs of Sullivan and McMullen [58, 59, 50], and also results on the local analysis of stable/unstable manifolds and the more global invariant renormalization horseshoe found in the work of Lyubich [45, 46]. Briefly, a renormalization horseshoe is a heteroclinic loop in function space, formed by a chain of transverse intersections between the unstable and stable manifolds of various renormalization fixed points.

The results mentioned in the preceding paragraph are proven for the $m = 2$ renormalization operator. Extensions to higher degree unimodal maps and their renormalization horseshoes are found in the work of Avila and Lyubich [3]. Of particular interest to us, they show that every fixed point of R_m , $m \geq 2$ is hyperbolic, with a single unique unstable eigenvalue and the remainder of the spectrum is stable. We refer also to the review article of Lyubich [47] for a substantive discussion of this literature complete with many additional references.

Remark 1.10 (Principle of combinatorial rescaling of exponents (PASCE)). Renormalization horseshoes are considered from a more quantitative perspective by de la Llave, Olvera, and Petrov in their paper [17]. There, the authors present compelling numerical evidence for their *Principle of Approximate Combination of Scaling Exponents*, or PACSE. Very roughly speaking, the idea of PACSE is that if p_1 and p_2 are renormalization fixed points for R_m and R_n respectively (associated with some fixed choice of finite combinatorics), having associated universal constants (unstable eigenvalues) $\lambda_1, \lambda_2 > 1$, and if the unstable manifold of p_1 intersects transversally the stable manifold of p_2 and vice versa, then in some neighborhood of the resulting heteroclinic cycle there is an infinite sequence $\{p_{k_j}\}_{j=1}^{\infty}$ of fixed points for renormalization operators R_{k_j} . Moreover, the universal constants λ_{k_j} (and resulting combinatorics) can be (approximately) computed via some explicit combinatorial formulas involving λ_1 and λ_2 . The authors provide detailed numerical evidence for the PACSE, not only for renormalization of one dimensional maps, but also for renormalization of quasiperiodic transitions in circle maps, boundaries of Segel disks, and break down of invariant circles in area-preserving twist maps. Beyond this, they argue that the PACSE should hold for many other renormalization phenomena in mathematics and mathematical physics.

Given that so many results for renormalization fixed points and renormalization horseshoes for one dimensional unimodal maps have been established using analytic arguments (Remark 1.9 above) we should stress that an advantage of constructive, computational arguments like those developed in the present work (and in many other places, as discussed above) is that they provide information about the quantitative features of fixed points and also precise bounds on the universal scalings. The tools developed in the present work form part of the toolkit needed for verifying the hypotheses, and applying the results of the PACSE theory in explicit examples. In a future work, we will combine the techniques developed below with the computer assisted methods for studying transverse connecting orbits in infinite dimensional discrete time dynamical systems developed by de la Llave and the third author in [16, 52]. This program will result in a computational framework for mathematically rigorous PACSE computations which could potentially be applied in many renormalization settings.

The remainder of the paper is organized as follows. Section 2 introduces a zero-finding problem equivalent to Equation (1), that will be used for enclosing the renormalization fixed point via a Newton-Kantorovich argument, together with the precise Banach space in which this argument will be applied. A suitable finite dimensional projection and associated a priori error estimates are presented in Section 3. The details of the Newton-Kantorovich argument are then laid out in Section 4, and the necessary estimates derived in Section 5. A similar procedure is presented in Section 6 in order to rigorously enclose the unstable eigenvalue of the obtained renormalization fixed points. Further results are then discussed in Section 7. Details about the computer-assisted parts of the proofs can be found in the appendix.

2 Setup

Our approach, being based on Newton's method, requires us to rewrite the fixed point problem for R_m as an equivalent zero finding problem. In the process we also “unwrap” one of the composition terms by introducing a scalar variable α .

So, for $h \in C([-1, 1], \mathbb{R})$ and $\alpha \in \mathbb{R}$, let

$$\tilde{R}_m(\alpha, h) := h^m(\alpha x), \quad (8)$$

and define $\phi_m: \mathbb{R} \times C([-1, 1], \mathbb{R}) \rightarrow C([-1, 1], \mathbb{R})$ by

$$\phi_m(\alpha, h)(x) = \alpha h(x) - h^m(\alpha x) = \alpha h(x) - \tilde{R}_m(\alpha, h)(x), \quad (9)$$

The equation above is one equation in two unknowns α and h . To balance the equations, we introduce a phase condition $h(0) = 1$, and define the “square” operator $\Phi_m: \mathbb{R} \times C([-1, 1], \mathbb{R}) \rightarrow \mathbb{R} \times C([-1, 1], \mathbb{R})$ by the formula

$$\Phi_m(\alpha, h) = \begin{pmatrix} h(0) - 1 \\ \phi_m(\alpha, h) \end{pmatrix}. \quad (10)$$

Our goal is to find a zero of Φ_m , in an appropriate function space. It is easy to check that, if h is zero of Φ_m then $h^m(0) = \alpha$ follows from the phase condition, so that finding a zero of Φ_m is equivalent to finding a fixed point of the m -th order Feigenbaum-Cvitanović renormalization operator defined in Equation (1).

To solve the equation, we will discretize the domain of Φ_m using Chebyshev series. To this end, choose $\rho \geq 1$ and define,

$$\ell_\rho^1 = \left\{ h = h_0 + 2 \sum_{k=1}^{\infty} h_k T_k, \quad \|h\|_{\ell_\rho^1} := |h_0| + 2 \sum_{k=1}^{\infty} |h_k| \rho^k < \infty \right\},$$

where T_k are the Chebyshev polynomials of the first kind defined recursively by

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

and

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),$$

for $k \geq 1$. By making the change of variables $x = \cos(\theta)$ one obtains that

$$T_k(\cos(\theta)) = \cos(k\theta).$$

This illustrates the usual connection between Chebyshev and Fourier cosine series.

Throughout the paper, we refer to the h_k as the Chebyshev coefficients of h . We recall that any function which is at least Lipschitz continuous on $[-1, 1]$ (which will be the case for all functions considered in this work) admits a unique Chebyshev series expansion, see for instance [60].

We also consider the subspace of even functions within ℓ_ρ^1 , namely

$$\ell_\rho^{1, \text{even}} = \left\{ h \in \ell_\rho^1, \quad h_{2k+1} = 0 \quad \forall k \geq 0 \right\}.$$

Finally, we seek zeros of Φ_m in the space

$$\mathcal{X}_\rho = \mathbb{R} \times \ell_\rho^{1, \text{even}},$$

endowed with the following norm

$$\|(\alpha, h)\|_{\mathcal{X}_\rho} = |\alpha| + \|h\|_{\ell_\rho^1}.$$

Note that $\ell_\rho^{1, \text{even}}$ can be identified with a subspace of the space X_ρ presented in the introduction (see Lemma 3.7).

3 A priori estimates

We need a finite dimensional projection on \mathcal{X}_ρ , together with some a priori projection error estimates.

Definition 3.1. For any $K \in \mathbb{N}_{\geq 0}$, we denote by $\ell_\rho^{1,K}$ the subspace of polynomial functions of degree at most K embedded in ℓ_ρ^1 , i.e.,

$$\ell_\rho^{1,K} = \left\{ h \in \ell_\rho^1, h_k = 0 \ \forall k > K \right\},$$

and similarly for $\ell_\rho^{1,K,\text{even}}$. We also introduce

$$\mathcal{X}_\rho^K = \mathbb{R} \times \ell_\rho^{1,K,\text{even}}.$$

For any $h \in \ell_\rho^1$, we denote by $\Pi^K h$ the unique polynomial of degree at most K taking the same values as h on the Chebyshev nodes x_0, x_1, \dots, x_K given by

$$x_k = \cos\left(\frac{K-k}{K}\pi\right), \quad \forall 0 \leq k \leq K.$$

Given h in ℓ_ρ^1 , we denote by \check{h}_k the coefficients (in the Chebyshev basis) of the interpolation polynomial Π^K , that is,

$$\Pi^K h = \check{h}_0 + 2 \sum_{k=1}^K \check{h}_k T_k.$$

We emphasize that the projection operator $\Pi^K : \ell_\rho^1 \rightarrow \ell_\rho^{1,K}$ is not the truncation operator, i.e. $\check{h}_k \neq h_k$ (unless $h \in \ell_\rho^{1,K}$). However, these coefficients are still related, as made explicit in the following statement (see [60, Theorem 4.2]).

Lemma 3.2. Let $\rho \geq 1$, $h \in \ell_\rho^1$, and $K \in \mathbb{N}_{\geq 1}$. Then,

$$\check{h}_0 = h_0 + 2 \sum_{l=1}^{\infty} h_{2Kl},$$

$$\check{h}_k = h_k + \sum_{l=1}^{\infty} (h_{2Kl-k} + h_{2Kl+k}) \quad k = 1, \dots, K-1,$$

and

$$\check{h}_K = h_K + \sum_{l=1}^{\infty} h_{2Kl+K}.$$

Remark 3.3. When we restrict ourselves to $\ell_\rho^{1,\text{even}}$, Π^K can be defined in terms of half the nodes only, as $x_{K-k} = -x_k$ for all $0 \leq k \leq K$, and whenever $h \in \ell_\rho^{1,\text{even}}$ we have $h(x_{K-k}) = h(x_k)$.

We will also use the notation $\Pi^\infty h := h - \Pi^K h$ for convenience, and extend Π^K to \mathcal{X}_ρ as $\Pi^K(\alpha, h) := (\alpha, \Pi^K h)$.

3.1 Useful facts about Chebyshev series and Chebyshev interpolation

We recall here a few well known facts about Chebyshev approximation which will prove useful in this work, and again refer to [60] for a wider discussion and further references.

Lemma 3.4. *Let $\rho \geq 1$, then*

$$\|T_k\|_{C_\rho^0} = \frac{1}{2} (\rho^k + \rho^{-k}).$$

Proof. This identity follows directly from the fact that

$$T_k \left(\frac{z + z^{-1}}{2} \right) = \frac{z^k + z^{-k}}{2}, \quad \forall z \in \mathbb{C} \setminus \{0\}. \quad (11)$$

In order to check that (11) holds, one can for instance notice that both sides are analytic functions of z on $\mathbb{C} \setminus \{0\}$, which coincide on the unit circle (because $T_k(\cos \theta) = \cos(k\theta)$), therefore they must coincide everywhere on $\mathbb{C} \setminus \{0\}$. \square

Lemma 3.5. *Let $\rho > \rho' > 1$. For any $h \in \ell_\rho^1$, we have that $h' \in \ell_{\rho'}^1$ and that*

$$h' = 2 \left(\left(\sum_{l=0}^{\infty} (2l+1) h_{2l+1} \right) T_0 + 2 \sum_{k=1}^{\infty} \left(\sum_{l=0}^{\infty} (k+2l+1) h_{k+2l+1} \right) T_k \right).$$

3.2 Decay of the coefficients and comparison of norms

While the space \mathcal{X}_ρ in which we will apply our fixed point argument is equipped with an ℓ_ρ^1 norm, we often prefer to discuss our final results in terms of C_ν^0 norms. Moreover, for certain steps in our arguments, the C^0 norms are easier to compute; for example when dealing with compositions. In this subsection we consider the problem of passing from one to the other. We also require bounds which relate the norm of the derivative of a function to its C^0 norm, after giving up a portion of the domain. Of course we need explicit constants throughout, and these are controlled by leveraging the decay of the Chebyshev coefficients, as described in the following lemma.

Lemma 3.6. *Let $\rho \geq 1$ and h an analytic function on \mathcal{E}_ρ . Then, for all $k \geq 0$, the Chebyshev coefficient h_k of h satisfies*

$$|h_k| \leq \frac{\|h\|_{C_\rho^0}}{\rho^k}.$$

If $\rho > 1$, then we also have an estimate on the coefficients of $\Pi^K h$, namely

$$|\check{h}_0| \leq \|h\|_{C_\rho^0} \frac{\rho^{2K} + 1}{\rho^{2K} - 1},$$

$$|\check{h}_k| \leq \frac{\|h\|_{C_\rho^0}}{\rho^k} \frac{\rho^{2K} + \rho^{2k}}{\rho^{2K} - 1}, \quad k = 1, \dots, K-1,$$

and

$$|\check{h}_K| \leq \frac{\|h\|_{C_\rho^0}}{\rho^K} \frac{\rho^{2K}}{\rho^{2K} - 1}.$$

Proof. The estimate on the coefficients h_k is nothing but [60, Theorem 8.1]. Together with Lemma 3.2 it yields the estimates on the coefficients \check{h}_k . \square

Lemma 3.7. *Let $\rho \geq 1$ and $h \in \ell_\rho^1$, then h is analytic on \mathcal{E}_ρ and*

$$\|h\|_{\mathcal{C}_\rho^0} \leq \|h\|_{\ell_\rho^1}.$$

Conversely, if $\rho < \nu$ and h is an analytic function on \mathcal{E}_ν , then

$$\|h\|_{\ell_\rho^1} \leq \frac{\nu + \rho}{\nu - \rho} \|h\|_{\mathcal{C}_\nu^0}.$$

Furthermore, if h is even, then

$$\|h\|_{\ell_\rho^1} \leq \frac{\nu^2 + \rho^2}{\nu^2 - \rho^2} \|h\|_{\mathcal{C}_\nu^0}.$$

Proof. The analyticity of h on \mathcal{E}_ρ follows from [60, Theorem 8.3], and the fact that the \mathcal{C}_ρ^0 norm is controlled by the ℓ_ρ^1 norm is straightforward:

$$\|h\|_{\mathcal{C}_\rho^0} \leq |h_0| \|T_0\|_{\mathcal{C}_\rho^0} + 2 \sum_{k=1}^{\infty} |h_k| \|T_k\|_{\mathcal{C}_\rho^0},$$

and Lemma 3.4 shows that $\|T_0\|_{\mathcal{C}_\rho^0} \leq \rho^k$. The second identity follows from Lemma 3.6. Indeed,

$$\begin{aligned} \|h\|_{\ell_\rho^1} &= |h_0| + 2 \sum_{k=1}^{\infty} |h_k| \rho^k \\ &\leq \left(1 + 2 \sum_{k=1}^{\infty} \left(\frac{\rho}{\nu} \right)^k \right) \|h\|_{\mathcal{C}_\nu^0} \\ &= \frac{\nu + \rho}{\nu - \rho} \|h\|_{\mathcal{C}_\nu^0}. \end{aligned}$$

The last estimate is obtained by using the fact that half the h_k are zero when h is even. \square

Proposition 3.8. *Let $1 \leq \rho < \nu$,*

$$\sigma_{\rho,\nu}^{\text{even}} = \sup_{n \in \mathbb{N}_{\geq 1}} \frac{2n}{\nu^{2n}} \left(\rho \frac{\rho^{2n} - 1}{\rho^2 - 1} + \rho^{-1} \frac{\rho^{-2n} - 1}{\rho^{-2} - 1} \right),$$

$$\sigma_{\rho,\nu}^{\text{odd}} = \sup_{n \in \mathbb{N}_{\geq 0}} \frac{2n+1}{\nu^{2n+1}} \left(1 + \rho^2 \frac{\rho^{2n} - 1}{\rho^2 - 1} + \rho^{-2} \frac{\rho^{-2n} - 1}{\rho^{-2} - 1} \right),$$

and

$$\sigma_{\rho,\nu} = \max \left(\sigma_{\rho,\nu}^{\text{even}}, \sigma_{\rho,\nu}^{\text{odd}} \right).$$

Then, for all h in ℓ_ν^1

$$\|h'\|_{\mathcal{C}_\rho^0} \leq \sigma_{\rho,\nu} \|h\|_{\ell_\nu^1}.$$

Furthermore, if h is even, then

$$\|h'\|_{\mathcal{C}_\rho^0} \leq \sigma_{\rho,\nu}^{\text{even}} \|h\|_{\ell_\nu^1}.$$

Proof. Starting from the formula of Lemma 3.5, taking the \mathcal{C}_ρ^0 norm and using Lemma 3.4, we get

$$\begin{aligned}
\|h'\|_{\mathcal{C}_\rho^0} &\leq 2 \left(\sum_{l=0}^{\infty} \frac{2l+1}{\nu^{2l+1}} |h_{2l+1}| \nu^{2l+1} + \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{k+2l+1}{\nu^{k+2l+1}} (\rho^k + \rho^{-k}) |h_{k+2l+1}| \nu^{k+2l+1} \right) \\
&= 2 \left(\sum_{n=0}^{\infty} |h_{2n+1}| \nu^{2n+1} \left(\frac{2n+1}{\nu^{2n+1}} + \sum_{k=1}^n \frac{2n+1}{\nu^{2n+1}} (\rho^{2k} + \rho^{-2k}) \right) \right. \\
&\quad \left. + \sum_{n=0}^{\infty} |h_{2n+2}| \nu^{2n+2} \sum_{k=0}^n \frac{2n+2}{\nu^{2n+2}} (\rho^{2k+1} + \rho^{-(2k+1)}) \right) \\
&\leq 2 \left(\sigma_{\rho,\nu}^{\text{odd}} \sum_{n=0}^{\infty} |h_{2n+1}| \nu^{2n+1} + \sigma_{\rho,\nu}^{\text{even}} \sum_{n=1}^{\infty} |h_{2n}| \nu^{2n} \right) \\
&\leq \sigma_{\rho,\nu} \|h\|_{\ell_\nu^1}.
\end{aligned}$$

When h is even, all the h_{2n+1} vanish, therefore we indeed get $\|h'\|_{\mathcal{C}_\rho^0} \leq \sigma_{\rho,\nu}^{\text{even}} \|h\|_{\ell_\nu^1}$. \square

Remark 3.9. Since $\nu > \rho$, the suprema defining $\sigma_{\rho,\nu}^{\text{even}}$ and $\sigma_{\rho,\nu}^{\text{odd}}$ are in fact maxima, and we provide in Lemma A.1 a simple procedure allowing to explicitly compute them.

3.3 Projection errors

This subsection contains crucial projection/interpolation error estimates, again with explicit constants. Because our analysis involves going back and forth between ℓ_ρ^1 and \mathcal{C}_ν^0 , we need to estimate the interpolation error in \mathcal{C}^0 in terms of the ℓ^1 norm, and vice-versa.

Lemma 3.10. Let $\rho \geq 1$ and $h \in \ell_\rho^1$. Then

$$\|\Pi^K h\|_{\ell_\rho^1} \leq \|h\|_{\ell_\rho^1}.$$

Proof. This is an immediate consequence of Lemma 3.2. \square

Remark 3.11. The above estimate is one of the reasons we conduct our fixed point argument using the ℓ_ρ^1 norm rather than the \mathcal{C}_ρ^0 norm. Indeed, in the latter norm, $\Pi^K h$ is only controlled by h times a constant behaving roughly like ρ^K , which quickly becomes detrimental when ρ is larger than 1 (in our actual computer-assisted proofs we will use $\rho = 2$).

Proposition 3.12. Let $1 \leq \rho \leq \nu$, and $h \in \ell_\nu^1$. Then

$$\|h - \Pi^K h\|_{\mathcal{C}_\rho^0} \leq \Upsilon_{\rho,\nu,K}^{0,1} \|h\|_{\ell_\nu^1},$$

where

$$\Upsilon_{\rho,\nu,K}^{0,1} = \frac{1}{2} \frac{\rho^{K-1} + \rho^{-(K-1)} + \rho^{K+1} + \rho^{-(K+1)}}{\nu^{K+1}}.$$

Furthermore, if h is even and K is even,

$$\|h - \Pi^K h\|_{\mathcal{C}_\rho^0} \leq \Upsilon_{\rho,\nu,K}^{0,1,\text{even}} \|h\|_{\ell_\nu^1},$$

where

$$\Upsilon_{\rho,\nu,K}^{0,1,\text{even}} = \frac{1}{2} \frac{\rho^{K-2} + \rho^{-(K-2)} + \rho^{K+2} + \rho^{-(K+2)}}{\nu^{K+2}}.$$

Proof. The starting point is Lemma 3.2, which allows us to write the interpolation error as

$$\begin{aligned} h - \Pi^K h &= \left(2 \sum_{l=1}^{\infty} h_{2Kl} \right) T_0 + 2 \sum_{k=1}^{K-1} \left(\sum_{l=1}^{\infty} (h_{2Kl+k} + h_{2Kl-k}) \right) T_k + 2 \left(\sum_{l=1}^{\infty} h_{2Kl+K} \right) T_K \\ &\quad + 2 \sum_{k=K+1}^{\infty} h_k T_k. \end{aligned} \quad (12)$$

Taking the \mathcal{C}_ρ^0 norm and simply using Lemma 3.4 together with the triangular inequality, we get

$$\begin{aligned} \|h - \Pi^K h\|_{\mathcal{C}_\rho^0} &\leq 2 \sum_{l=1}^{\infty} |h_{2Kl}| + \sum_{k=1}^{K-1} \sum_{l=1}^{\infty} (|h_{2Kl+k}| + |h_{2Kl-k}|) (\rho^k + \rho^{-k}) \\ &\quad + \sum_{l=1}^{\infty} |h_{2Kl+K}| (\rho^K + \rho^{-K}) + \sum_{k=K+1}^{\infty} |h_k| (\rho^k + \rho^{-k}). \end{aligned} \quad (13)$$

Reorganizing the terms slightly, and taking worst cases in k and l , we end up with

$$\begin{aligned} \|h - \Pi^K h\|_{\mathcal{C}_\rho^0} &\leq \frac{2 + \rho^{2K} + \rho^{-2K}}{\nu^{2K}} \sum_{l=1}^{\infty} |h_{2Kl}| \nu^{2Kl} \\ &\quad + \frac{\rho + \rho^{-1} + \rho^{2K+1} + \rho^{-(2K+1)}}{\nu^{2K+1}} \sum_{k=1}^{K-1} \sum_{l=1}^{\infty} |h_{2Kl+k}| \nu^{2Kl+k} \\ &\quad + \frac{\rho^{K-1} + \rho^{-(K-1)} + \rho^{K+1} + \rho^{-(K+1)}}{\nu^{K+1}} \sum_{k=1}^{K-1} \sum_{l=1}^{\infty} |h_{2Kl-k}| \nu^{2Kl-k} \\ &\quad + \frac{\rho^K + \rho^{-K} + \rho^{3K} + \rho^{3K}}{\nu^{3K}} \sum_{l=1}^{\infty} |h_{2Kl+K}| \nu^{2Kl+K} \\ &\leq \frac{1}{2} \max_{a \in \{-(K-1), 0, 1, K\}} \frac{\rho^a + \rho^{-a} + \rho^{2K+a} + \rho^{-(2K+a)}}{\nu^{2K+a}} \|h\|_{\ell_\nu^1}. \end{aligned}$$

Since $\rho \leq \nu$, the term

$$\frac{\rho^a + \rho^{-a} + \rho^{2K+a} + \rho^{-(2K+a)}}{\nu^{2K+a}} = \frac{1}{\nu^{2K}} \left(\left(\frac{\rho}{\nu} \right)^a + \left(\frac{1}{\rho\nu} \right)^a + \left(\frac{\rho}{\nu} \right)^{2K+a} + \left(\frac{1}{\rho\nu} \right)^{2K+a} \right)$$

is non-increasing with a , hence the maximum over a is reached for $a = -(K-1)$.

When h is even, only the h_k with k even remain in the above computation. If K is also even, the worst term (which is the factor in front of h_{2Kl-k} for $l = 1$ and $k = K-1$) drops out, and the next worst one (in front of h_{2Kl-k} for $l = 1$ and $k = K-2$) give the announced constant. \square

Proposition 3.13. *Let $1 \leq \rho < \nu$, and h an analytic function on \mathcal{E}_ν . Then*

$$\|h - \Pi^K h\|_{\ell_\rho^1} \leq \Upsilon_{\rho, \nu, K}^{1,0} \|h\|_{\mathcal{C}_\rho^0},$$

where

$$\Upsilon_{\rho, \nu, K}^{1,0} = \frac{2}{\nu^{2K} - 1} \left(\frac{\rho}{\nu - \rho} \left(1 - \left(\frac{\rho}{\nu} \right)^K \right) + \frac{(\nu\rho)^K - 1}{\nu\rho - 1} \right) + \frac{2\rho}{\nu - \rho} \left(\frac{\rho}{\nu} \right)^K,$$

Furthermore, if h is even and K is even,

$$\|h - \Pi^K h\|_{\ell_\rho^1} \leq \Upsilon_{\rho, \nu, K}^{1,0, \text{even}} \|h\|_{\mathcal{C}_\rho^0},$$

where

$$\Upsilon_{\rho,\nu,K}^{1,0,\text{even}} = \frac{2}{\nu^{2K}-1} \left(\frac{\rho^2}{\nu^2-\rho^2} \left(1 - \left(\frac{\rho}{\nu} \right)^K \right) + \frac{(\nu\rho)^K - 1}{(\nu\rho)^2 - 1} \right) + \frac{2\rho^2}{\nu^2-\rho^2} \left(\frac{\rho}{\nu} \right)^K.$$

Proof. The starting point is again to write the interpolation error as (12), and then to take the ℓ_ρ^1 norm instead of the \mathcal{C}_ρ^0 norm, which simply means each $\frac{\rho^k + \rho^{-k}}{2}$ should be replaced by ρ^k in (13). Next, we estimate each $|h_k|$ using the first part of Lemma 3.6 and the fact that h is analytic on \mathcal{E}_ν , which yields

$$\begin{aligned} & \|h - \Pi^K h\|_{\ell_\rho^1} \\ & \leq \left(2 \sum_{l=1}^{\infty} \frac{1}{\nu^{2Kl}} + 2 \sum_{k=1}^{K-1} \sum_{l=1}^{\infty} \left(\frac{\rho^k}{\nu^{2Kl+k}} + \frac{\rho^k}{\nu^{2Kl-k}} \right) + 2 \sum_{l=1}^{\infty} \frac{\rho^K}{\nu^{2Kl+K}} + 2 \sum_{k=K+1}^{\infty} \frac{\rho^k}{\nu^k} \right) \|h\|_{\mathcal{C}_\nu^0}, \end{aligned}$$

and obtaining the formula for $\Upsilon_{\rho,\nu,K}^{1,0}$ is just a matter of putting together the sums of all those geometric series. Also using that only the terms with even indices remain when h is even yields the second constant. \square

3.4 Practical considerations

We make here several remarks, related to the way we actually use some of the theoretical estimates presented in the previous two subsections in practice.

Remark 3.14. *In various places below, we need to compute or estimate the coefficients of $\Pi^K g$ for some function g . If g is explicit enough, we can compute each entry of the vector $(g(x_k))_{0 \leq k \leq K}$ and then get the coefficients of $\Pi^K g$ using the DFT (or more precisely, the Discrete Cosine Transform, see Appendix B). Similarly, if we are only able to get component-wise upper bounds for $(|g(x_k)|)_{0 \leq k \leq K}$, we get upper-bounds for the coefficients of $\Pi^K g$. While this is usually fine for moderately large values of k , when k becomes large such estimates may fail to capture the expected decay of the coefficients, which can be problematic when ρ is somewhat larger than 1, for instance if we need to compute or estimate $\|\Pi^K g\|_{\ell_\rho^1}$. One way to remedy this is to also compute or estimate $\|g\|_{\mathcal{C}_\nu^0}$ for some $\nu \geq \rho$, and then use the second part of Lemma 3.6 to get an estimate on the coefficients of $\Pi^K g$ with a guaranteed decay at a rate ν^{-k} . For each coefficient, we can then take the minimum between the estimate obtained via the values at the nodes, and the estimate obtained from the \mathcal{C}_ν^0 norm. This strategy is reminiscent of the one presented in [43].*

Remark 3.15. *Another recurring task will be to compute quantities like $\|g\|_{\mathcal{C}_\nu^0}$, where g is some analytic function on \mathcal{E}_ν (usually a polynomial). It is worth noticing that we in fact only need to compute the supremum of $|g|$ on the boundary of \mathcal{E}_ν , because the maximum is necessarily reached there thanks to the maximum modulus principle. An efficient way of rigorously enclosing this supremum using interval arithmetic together with the FFT, coming from [63] (see also [36]), is recalled in Appendix C.*

Remark 3.16. *Many estimates to come will be of the form $C_\nu \|g\|_{\mathcal{C}_\nu^0}$, for some function g and $\nu > 1$, where C_ν is a computable constant depending on ν . Whenever we face such a quantity, we numerically optimize the value of ν in order to make this as small as possible.*

In order to use such estimates, we need to know that g is actually analytic on some Bernstein ellipse of explicit size. The following Lemma can prove useful in determining (lower bounds on) domains of analytically.

Lemma 3.17. *Let $\nu \geq 1$, ψ analytic on \mathcal{E}_ν , and*

$$\eta = \max_{z \in \partial \mathcal{E}_\nu} |\psi(z) - 1| + |\psi(z) + 1|.$$

Then, for all $\rho \geq \frac{\eta + \sqrt{\eta^2 - 4}}{2}$, $\psi(\mathcal{E}_\nu) \subset \mathcal{E}_\rho$.

4 Fixed point reformulation

In the remainder of the paper, K denotes a positive even integer, and $(\bar{\alpha}, \bar{h})$ an element of \mathcal{X}_ρ^K , which should be thought of as an approximate zero of Φ_m (recall equation (10)), obtained numerically, satisfying $0 < \bar{\alpha} < 1$. Our goal is to prove the existence of an exact zero of Φ_m (i.e., of an exact fixed point of R_m) near $(\bar{\alpha}, \bar{h})$, and to provide an explicit and small error bound.

We define

$$J^\dagger = \Pi^K D\Phi_m(\bar{\alpha}, \bar{h})|_{\mathcal{X}_\rho^K},$$

which is a linear operator on \mathcal{X}_ρ^K .

Since \mathcal{X}_ρ^K is finite dimensional (of dimension $K/2 + 1$), we can compute numerically an approximate inverse J of J^\dagger . Finally, we define the linear operator A by

$$\begin{cases} A \Pi^K(\alpha, h) = J \Pi^K(\alpha, h) \\ A(I - \Pi^K)(\alpha, h) = \left(0, \frac{1}{\bar{\alpha}}(I - \Pi^K)h\right). \end{cases}$$

This leads to the fixed-point operator

$$T : (\alpha, h) \mapsto (\alpha, h) - A\Phi_m(\alpha, h),$$

and our goal is now to prove that T is a contraction on a small neighborhood of $(\bar{\alpha}, \bar{h})$ in \mathcal{X}_ρ . This will be accomplished thanks to a Newton-Kantorovich-type argument, which has become very common for computer-assisted proofs [2, 65, 15, 56, 67].

Theorem 4.1. *Let $r^* \in (0, +\infty]$. Assume there exist nonnegative constants Y and Z such that*

$$\|A\Phi_m(\bar{\alpha}, \bar{h})\|_{\mathcal{X}_\rho} \leq Y \tag{14a}$$

$$\sup_{(\alpha, h) \in \mathcal{B}_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)} \|I - AD\Phi_m(\alpha, h)\|_{\mathcal{X}_\rho} \leq Z, \tag{14b}$$

where $\mathcal{B}_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^)$ is the closed ball of center $(\bar{\alpha}, \bar{h})$ and radius r^* in \mathcal{X}_ρ . If*

$$Z < 1, \tag{15}$$

then, for any r satisfying

$$\frac{Y}{1 - Z} \leq r \leq r^*, \tag{16}$$

there exists a unique $(\alpha_, h_*) \in \mathcal{B}_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r)$ so that*

$$\Phi_m(\alpha_*, h_*) = \begin{pmatrix} 0 \\ \mathbf{0} \end{pmatrix}.$$

Remark 4.2. *Often in the literature, the above Z estimate is considered for an arbitrary $r < r^*$ and split into two parts:*

$$\begin{aligned} & \|I - AD\Phi_m(\bar{\alpha}, \bar{h})\|_{\mathcal{X}_\rho} \leq Z_1 \\ & \sup_{(\alpha, h) \in \mathcal{B}_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)} \|AD^2\Phi_m(\alpha, h)\|_{\mathcal{X}_\rho} \leq Z_2, \end{aligned}$$

so that

$$\sup_{(\alpha, h) \in \mathcal{B}_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r)} \|I - AD\Phi_m(\alpha, h)\|_{\mathcal{X}_\rho} \leq Z_1 + Z_2 r,$$

This allows to isolate the crucial part, namely $I - AD\Phi_m(\bar{\alpha}, \bar{h})$, and to estimate it as sharply as possible, because the derivative is now taken at a fixed and explicit point, but it then requires to also control locally the second derivative (or at least to get a Lipschitz bound on the first derivative locally). For our specific problem, where Φ_m contains a composition operator, looking at higher order derivatives means getting more and more complicated formulas, which we avoid by directly working with (14b). The supremum over $\mathcal{B}_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)$ is then handled directly by using interval arithmetic in combination with some a priori error estimates (see Section 5.2). The downside of this approach is that we get less sharp bounds, but how less sharp they are really depends on the choice of r^* . In practice, we do take r^* really small (see Section 7 for explicit values), which alleviates this drawback. The main cost of the maneuver is that we only obtain isolation results on the (now very small) neighborhood of size r_* .

5 Bounds in Theorem 4.1

In this section, we derive computable estimates satisfying assumption (14) of Theorem 4.1. We start by introducing notation for the terms appearing in the Frechet derivative of ϕ_m (recall equation (9)), and then derive separately a Y bound and a Z bound.

For any $(\tilde{\alpha}, \tilde{h})$ and (α, h) in \mathcal{X}_ρ ,

$$\partial_\alpha \phi(\tilde{\alpha}, \tilde{h})\alpha = \alpha(\tilde{h} - \tilde{f}), \quad (17)$$

where

$$\tilde{f}(x) = x \prod_{j=0}^{m-1} \tilde{h}'(\tilde{h}^j(\tilde{\alpha}x)),$$

and

$$\partial_h \phi(\tilde{\alpha}, \tilde{h})h(x) = \tilde{\alpha}h(x) - \sum_{j=0}^{m-1} \tilde{g}_j(x)h(\tilde{h}^j(\tilde{\alpha}x)), \quad (18)$$

where

$$\tilde{g}_j(x) = \prod_{l=j+1}^{m-1} \tilde{h}'(\tilde{h}^l(\tilde{\alpha}x)), \quad j = 0, \dots, m-1,$$

with the convention that the empty product is equal to 1, i.e. $\tilde{g}_{m-1} = 1$.

5.1 Y estimate in (14a)

This subsection is devoted to the estimate Y satisfying (14a). We split $A\Phi_m(\bar{\alpha}, \bar{h})$ as

$$\Pi^K(A\Phi_m(\bar{\alpha}, \bar{h})) + (I - \Pi^K)(A\Phi_m(\bar{\alpha}, \bar{h})),$$

and estimate both terms separately. The bounds for the terms are denoted by Y^K and Y^∞ respectively, and sometimes referred to as the *finite part* and the *tail part*.

For the finite part, we simply take

$$Y^K = \left\| J \Pi^K \Phi_m(\bar{\alpha}, \bar{h}) \right\|_{\mathcal{X}_\rho}.$$

That is, we compute the coefficients of $\Pi^K \Phi_m(\bar{\alpha}, \bar{h})$ using Remark 3.14, and then simply multiply the result by J and compute the ℓ_ρ^1 norm of the result.

For the tail part, we have to estimate

$$\frac{1}{|\bar{\alpha}|} \left\| (I - \Pi^K) \Phi_m(\bar{\alpha}, \bar{h}) \right\|_{\mathcal{X}_\rho} = \frac{1}{|\bar{\alpha}|} \left\| (I - \Pi^K) (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) \right\|_{\ell_\rho^1}. \quad (19)$$

Since $\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)$ is itself a polynomial, in principle one should be able to compute its coefficients exactly, and then to exactly evaluate the r.h.s. of (19). However, $\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)$ is of very large degree (K^m), therefore computing its coefficients accurately enough can be very challenging in practice. In particular, if the obtained coefficients start plateauing around machine epsilon, the resulting ℓ_ρ^1 norm could become extremely large if $\rho > 1$, which is the case here. In order to alleviate this difficulty, and estimate the r.h.s. of (19) as sharply as possible, we will split it into two parts: one which should be *close* to $(I - \Pi^K) (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot))$ but that we can compute *precisely*, and a second part we can only estimate but which should hopefully be of relatively small magnitude compared to the first part. More concretely, we consider an integer K_Y and use the triangle inequality

$$\begin{aligned} \frac{1}{|\bar{\alpha}|} \left\| (I - \Pi^K) (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) \right\|_{\ell_\rho^1} &\leq \frac{1}{|\bar{\alpha}|} \left\| \Pi^{K_Y} (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) - \Pi^K (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) \right\|_{\ell_\rho^1} \\ &\quad + \frac{1}{|\bar{\alpha}|} \left\| (I - \Pi^{K_Y}) (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) \right\|_{\ell_\rho^1}. \end{aligned}$$

In practice, K_Y should be chosen larger than K , so that $\Pi^{K_Y} (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot))$ approximates $\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)$ well, but not too large so that the coefficients of $\Pi^{K_Y} (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot))$ can still be computed accurately. The interpolation error $(I - \Pi^{K_Y})$ is then estimated using Proposition 3.13, which yields

$$\begin{aligned} \frac{1}{|\bar{\alpha}|} \left\| (I - \Pi^K) (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) \right\|_{\ell_\rho^1} &\leq \frac{1}{|\bar{\alpha}|} \left\| \Pi^{K_Y} (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) - \Pi^K (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) \right\|_{\ell_\rho^1} \\ &\quad + \frac{\Upsilon_{\rho, \nu, K_Y}^{1,0,\text{even}}}{|\bar{\alpha}|} \left\| \bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot) \right\|_{\mathcal{C}_\nu^0} := Y^\infty, \end{aligned}$$

for some $\nu > \rho$ chosen according to Remark 3.16. Note that $K_Y > K$ makes the constant $\Upsilon_{\rho, \nu, K_Y}^{1,0,\text{even}}$ smaller than the $\Upsilon_{\rho, \nu, K}^{1,0,\text{even}}$ that would have appeared if we had use Proposition 3.13 directly on (19), and therefore the second term in Y^∞ should in principle be small compared to $\frac{1}{|\bar{\alpha}|} \left\| \Pi^{K_Y} (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) - \Pi^K (\bar{\alpha}\bar{h} - \bar{h}^m(\bar{\alpha}\cdot)) \right\|_{\ell_\rho^1}$, which we can just compute.

5.2 Z estimate in (14b)

This subsection is devoted to the estimate Z satisfying (14b), which will be split in three parts:

$$\begin{aligned} \sup_{(\tilde{\alpha}, \tilde{h}) \in B_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)} \|DT(\tilde{\alpha}, \tilde{h})\|_{\mathcal{X}_\rho} &= \sup_{(\tilde{\alpha}, \tilde{h}) \in B_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)} \|I - AD\Phi_m(\tilde{\alpha}, \tilde{h})\|_{\mathcal{X}_\rho} \\ &\leq \sup_{(\tilde{\alpha}, \tilde{h}) \in B_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)} \left\| \Pi^K \left(I - AD\Phi_m(\tilde{\alpha}, \tilde{h}) \right) \Pi^K \right\|_{\mathcal{X}_\rho} \end{aligned} \quad (20)$$

$$+ \sup_{(\tilde{\alpha}, \tilde{h}) \in B_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)} \left\| \Pi^K \left(I - AD\Phi_m(\tilde{\alpha}, \tilde{h}) \right) \Pi^\infty \right\|_{\mathcal{X}_\rho} \quad (21)$$

$$+ \sup_{(\tilde{\alpha}, \tilde{h}) \in B_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)} \left\| \Pi^\infty \left(I - AD\Phi_m(\tilde{\alpha}, \tilde{h}) \right) \right\|_{\mathcal{X}_\rho}. \quad (22)$$

Remark 5.1. *In principle, one might optimize the efficiency of the whole procedure by using here a K which is different (typically smaller) than the K used for obtaining the numerical solution, but we will not do so in this work.*

5.2.1 Dealing with (20)

The bound derived in this section for (20) will be denoted by $Z^{K,K}$ when reporting numerical values or in the code.

For a given $(\tilde{\alpha}, \tilde{h}) \in B_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)$ we have that

$$\left\| \Pi^K \left(I - AD\Phi_m(\tilde{\alpha}, \tilde{h}) \right) \Pi^K \right\|_{\mathcal{X}_\rho} = \left\| I_K - J\Pi^K D\Phi_m(\tilde{\alpha}, \tilde{h})\Pi^K \right\|_{\mathcal{X}_\rho},$$

where I_K is the identity operator on \mathcal{X}_ρ^K . Therefore, we merely have to compute the norm of a finite dimensional operator, the only slight difficulty being that $(\tilde{\alpha}, \tilde{h})$ are arbitrary elements in $B_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)$. We deal with that by using interval arithmetic. In particular,

$$\begin{aligned} \tilde{\alpha} &\in \bar{\alpha} + [-r^*, r^*], \\ \forall x \in [-1, 1], \quad \tilde{h}(x) &\in \bar{h}(x) + [-r^*, r^*], \\ \forall x \in [-1, 1], \quad \tilde{h}'(x) &\in \bar{h}'(x) + \sigma_{1,\rho}^{\text{even}}[-r^*, r^*], \end{aligned}$$

the second estimate being a consequence of Lemma 3.7, and the last one following from Proposition 3.8.

5.2.2 Dealing with (21)

The bound derived in this section for (21) will be denoted by $Z^{K,\infty}$ when reporting numerical values or in the code.

For a given $(\tilde{\alpha}, \tilde{h}) \in B_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)$ we have that

$$\begin{aligned} \left\| \Pi^K \left(I - AD\Phi_m(\tilde{\alpha}, \tilde{h}) \right) \Pi^\infty \right\|_{\mathcal{X}_\rho} &= \left\| J\Pi^K D\Phi_m(\tilde{\alpha}, \tilde{h})\Pi^\infty \right\|_{\mathcal{X}_\rho} \\ &= \sup_{\|h\|_{\ell_\rho^{1,\text{even}}} \leq 1} \left\| J\Pi^K \left(D\Phi_m(\tilde{\alpha}, \tilde{h})(0, \Pi^\infty h) \right) \right\|_{\mathcal{X}_\rho}. \end{aligned}$$

Therefore, according to (18) we have to estimate, for any $h \in \ell_\rho^{1,\text{even}}$ with $\|h\|_{\ell_\rho^1} \leq 1$,

$$\begin{aligned} \Pi^K D\Phi_m(\tilde{\alpha}, \tilde{h})(0, \Pi^\infty h) &= \left(\Pi^K \left[\tilde{\alpha}(\Pi^\infty h) - \sum_{j=0}^{m-1} \tilde{g}_j(\cdot)(\Pi^\infty h)(\tilde{h}^j(\tilde{\alpha} \cdot)) \right] \right) \\ &= \left(-\sum_{j=0}^{m-1} \Pi^K \left[\tilde{g}_j(\cdot)(\Pi^\infty h)(\tilde{h}^j(\tilde{\alpha} \cdot)) \right] \right). \end{aligned}$$

First, notice that since K is even, then 0 is among the Chebyshev nodes, and thus $(\Pi^\infty h)(0) = 0$.

We then get bounds for (the absolute values of) the Chebyshev coefficients of

$$\Pi^K \left[\tilde{g}_j(\cdot) (\Pi^\infty h) (\tilde{h}^j(\tilde{\alpha}\cdot)) \right], \quad j = 0, \dots, m-1, \quad (23)$$

following Remark 3.14. We finally add these bounds back together, multiply the output by $|J|$ and take the $\|\cdot\|_{\mathcal{X}_\rho}$ norm to get a bound on (21).

Let us be more explicit about the way we bound the Chebyshev coefficients of (23). As explained in Remark 3.14, we in fact derive two different estimates and then take the minimum between the two.

First, after having checked that $\tilde{h}^j(\tilde{\alpha}x_k) \in \mathcal{E}_\beta$ for all $0 \leq k \leq K$ for some $1 \leq \beta \leq \rho$, we use Proposition 3.12 to get, for all $0 \leq k \leq K$,

$$\begin{aligned} \left| \tilde{g}_j(x_k) (\Pi^\infty h) (\tilde{h}^j(\tilde{\alpha}x_k)) \right| &\leq |\tilde{g}_j(x_k)| \|(\Pi^\infty h)\|_{\mathcal{C}_\beta^0} \\ &\leq |\tilde{g}_j(x_k)| \Upsilon_{\beta, \rho, K}^{0,1,\text{even}} \|h\|_{\ell_p^1}. \end{aligned}$$

Multiplying this estimate by $|M_K^{-1}|$ we get a bound for the Chebyshev coefficients of (23):

$$\left| \Pi^K D\Phi_m(\tilde{\alpha}, \tilde{h}) (0, \Pi^\infty h) \right| \leq \Upsilon_{\beta, \rho, K}^{0,1,\text{even}} \left(|M_K^{-1}| \sum_{j=0}^{m-1} \begin{pmatrix} 0 \\ |\tilde{g}_j(x_0)| \\ \vdots \\ |\tilde{g}_j(x_k)| \\ \vdots \\ |\tilde{g}_j(x_K)| \end{pmatrix} \right). \quad (24)$$

A second way of controlling the Chebyshev coefficients of (23) is to consider β_j and γ_j satisfying

$$1 \leq \beta_j \leq \rho \leq \gamma_j \quad \text{and} \quad \tilde{h}^j(\tilde{\alpha}\mathcal{E}_{\gamma_j}) \subset \mathcal{E}_{\beta_j}, \quad (25)$$

and then estimate

$$\begin{aligned} \left\| \tilde{g}_j(\cdot) (\Pi^\infty h) (\tilde{h}^j(\tilde{\alpha}\cdot)) \right\|_{\mathcal{C}_{\gamma_j}^0} &\leq \|\tilde{g}_j\|_{\mathcal{C}_{\gamma_j}^0} \|\Pi^\infty h\|_{\mathcal{C}_{\beta_j}^0} \\ &\leq \|\tilde{g}_j\|_{\mathcal{C}_{\gamma_j}^0} \Upsilon_{\beta_j, \rho, K}^{0,1,\text{even}} \|h\|_{\ell_p^1}. \end{aligned}$$

The second part of Lemma 3.6 then gives us a bound on the Chebyshev coefficients of

$$\Pi^K \left[\tilde{g}_j(\cdot) (\Pi^\infty h) (\tilde{h}^j(\tilde{\alpha}\cdot)) \right],$$

which decays like γ_j^{-k} . Putting all the terms together, we get a second estimate:

$$\left| \Pi^K D\Phi_m(\tilde{\alpha}, \tilde{h}) (0, \Pi^\infty h) \right| \leq \sum_{j=0}^{m-1} \Upsilon_{\beta_j, \rho, K}^{0,1,\text{even}} \|\tilde{g}_j\|_{\mathcal{C}_{\gamma_j}^0} \begin{pmatrix} 0 \\ \frac{\gamma_j^{2K+1}}{\gamma_j^{2K-1}} \\ \vdots \\ \frac{1}{\gamma_j^k} \frac{\gamma_j^{2K+\gamma_j^{2k}}}{\gamma_j^{2K-1}} \\ \vdots \\ \frac{1}{\gamma_j^K} \frac{\gamma_j^{2K}}{\gamma_j^{2K-1}} \end{pmatrix} \quad (26)$$

Remark 5.2. In practice, we optimize only over the γ_j , and take

$$\beta_j = \beta_j(\gamma_j) = \frac{1}{2} \left(\eta(\gamma_j) + \sqrt{\eta(\gamma_j)^2 - 4} \right),$$

where, according to Lemma 3.17, we take

$$\eta(\gamma_j) \geq \max_{z \in \partial \mathcal{E}_{\gamma_j}} |\tilde{h}^j(\tilde{\alpha}z) - 1| + |\tilde{h}^j(\tilde{\alpha}z) + 1|.$$

This choice ensures that $\tilde{h}^j(\tilde{\alpha} \mathcal{E}_{\gamma_j}) \subset \mathcal{E}_{\beta_j}$. However, for each j , we do this optimization independently in each mode. That is, we may in fact select different β_j and γ_j for each component k of (26).

Finally, we take the minimum component-wise between (24) and (26), multiply the result by $|J|$ and take the \mathcal{X}_ρ norm to get a bound on (21).

5.2.3 Dealing with (22)

Similarly to the previous two sections, the bound derived for (22) will be denoted by Z^∞ when reporting numerical values or in the code.

For a given $(\tilde{\alpha}, \tilde{h}) \in B_{\mathcal{X}_\rho}((\bar{\alpha}, \bar{h}), r^*)$ we have that

$$\left\| \Pi^\infty \left(I - AD\Phi_m(\tilde{\alpha}, \tilde{h}) \right) \right\|_{\mathcal{X}_\rho} = \sup_{\|(\alpha, h)\|_{\mathcal{X}_\rho} \leq 1} \left\| \Pi^\infty \left(I - AD\Phi_m(\tilde{\alpha}, \tilde{h}) \right) (\alpha, h) \right\|_{\mathcal{X}_\rho}.$$

Therefore, we have to estimate, for $\|(\alpha, h)\|_{\mathcal{X}_\rho} \leq 1$,

$$\left\| \Pi^\infty \left((\alpha, h) - AD\Phi_m(\tilde{\alpha}, \tilde{h})(\alpha, h) \right) \right\|_{\mathcal{X}_\rho} = \left\| \Pi^\infty h - \frac{1}{\tilde{\alpha}} \Pi^\infty D\phi(\tilde{\alpha}, \tilde{h})(\alpha, h) \right\|_{\ell_\rho^1}.$$

According to (17)-(18), we have

$$\begin{aligned} \Pi^\infty \left(h - \frac{1}{\tilde{\alpha}} D\phi(\tilde{\alpha}, \tilde{h})(\alpha, h) \right) &= \left(1 - \frac{\tilde{\alpha}}{\bar{\alpha}} \right) \Pi^\infty h \\ &\quad + \frac{1}{\bar{\alpha}} \sum_{j=0}^{m-1} \Pi^\infty \left(\tilde{g}_j(\cdot) h \left((\tilde{h}^j(\tilde{\alpha} \cdot)) \right) \right) \\ &\quad - \frac{\alpha}{\bar{\alpha}} \Pi^\infty (\tilde{h} - \tilde{f}), \end{aligned}$$

and we bound each term independently in the r.h.s. below

$$\begin{aligned} \left\| \Pi^\infty \left(h - \frac{1}{\tilde{\alpha}} D\phi(\tilde{\alpha}, \tilde{h})(\alpha, h) \right) \right\|_{\ell_\rho^1} &\leq \frac{r^*}{|\bar{\alpha}|} \|\Pi^\infty h\|_{\ell_\rho^1} \\ &\quad + \frac{1}{|\bar{\alpha}|} \sum_{j=0}^{m-1} \left\| \Pi^\infty \left(\tilde{g}_j(\cdot) h \left((\tilde{h}^j(\tilde{\alpha} \cdot)) \right) \right) \right\|_{\ell_\rho^1} \\ &\quad + \frac{|\alpha|}{|\bar{\alpha}|} \|\Pi^\infty (\tilde{f} - \tilde{h})\|_{\ell_\rho^1}. \end{aligned}$$

The last term is very similar to the tail part of the Y bound. Indeed, for any $\nu > \rho$ we have

$$\frac{|\alpha|}{|\bar{\alpha}|} \|\Pi^\infty (\tilde{f} - \tilde{h})\|_{\ell_\rho^1} \leq \frac{|\alpha|}{|\bar{\alpha}|} \left(\Upsilon_{\rho, \nu, K}^{1,0,\text{even}} \|\tilde{f}\|_{\mathcal{C}_\nu^0} + r^* \right).$$

In order to bound the remaining terms, we take, for $j = 0, \dots, m-1$, $\mu_j > \rho$ such that

$$\tilde{h}^j(\tilde{\alpha}\mathcal{E}_{\mu_j}) \subset \mathcal{E}_\rho$$

and then use Proposition 3.13 to estimate

$$\begin{aligned} \frac{1}{|\tilde{\alpha}|} \sum_{j=0}^{m-1} \left\| \Pi^\infty \left(\tilde{g}_j(\cdot) h \left((\tilde{h}^j(\tilde{\alpha}\cdot)) \right) \right) \right\|_{\ell_\rho^1} &\leq \frac{1}{|\tilde{\alpha}|} \sum_{j=0}^{m-1} \Upsilon_{\rho, \mu_j, K}^{1,0,\text{even}} \left\| \tilde{g}_j(\cdot) h \left((\tilde{h}^j(\tilde{\alpha}\cdot)) \right) \right\|_{\mathcal{C}_{\mu_j}^0} \\ &\leq \frac{1}{|\tilde{\alpha}|} \left(\sum_{j=0}^{m-1} \Upsilon_{\rho, \mu_j, K}^{1,0,\text{even}} \|\tilde{g}_j\|_{\mathcal{C}_{\mu_j}^0} \right) \|h\|_{\ell_\rho^1}. \end{aligned}$$

Putting everything together, we have

$$\begin{aligned} \left\| \Pi^\infty \left(h - \frac{1}{\tilde{\alpha}} D\phi(\tilde{\alpha}, \tilde{h})(\alpha, h) \right) \right\|_{\ell_\rho^1} &\leq \frac{1}{|\tilde{\alpha}|} \left(\Upsilon_{\rho, \nu, K}^{1,0,\text{even}} \|\tilde{f}\|_{\mathcal{C}_\nu^0} + r^* \right) |\alpha| \\ &\quad + \frac{1}{|\alpha|} \left(\sum_{j=0}^{m-1} \Upsilon_{\rho, \mu_j, K}^{1,0,\text{even}} \|\tilde{g}_j\|_{\mathcal{C}_{\mu_j}^0} + r^* \right) \|h\|_{\ell_\rho^1} \\ &\leq \frac{1}{|\alpha|} \max \left[\Upsilon_{\rho, \nu, K}^{1,0,\text{even}} \|\tilde{f}\|_{\mathcal{C}_\nu^0} + r^*, \right. \\ &\quad \left. \sum_{j=0}^{m-1} \Upsilon_{\rho, \mu_j, K}^{1,0,\text{even}} \|\tilde{g}_j\|_{\mathcal{C}_{\mu_j}^0} + r^* \right] (|\alpha| + \|h\|_{\ell_\rho^1}) \\ &\leq \frac{1}{|\alpha|} \left(\max \left[\Upsilon_{\rho, \nu, K}^{1,0,\text{even}} \|\tilde{f}\|_{\mathcal{C}_\nu^0}, \sum_{j=0}^{m-1} \Upsilon_{\rho, \mu_j, K}^{1,0,\text{even}} \|\tilde{g}_j\|_{\mathcal{C}_{\mu_j}^0} \right] + r^* \right), \end{aligned}$$

which upper-bounds (22).

6 Eigenvalue

In this section, we assume we have obtained a fixed point f of R_m , using the computer-assisted proof described in Section 4. Examples of such results are provided in Section 7. Our goal is now to get a rigorous enclosure of the associated universal constant λ , i.e., the unstable eigenvalue of $DR_m(f)$. To that end, we abandon the intermediate problem Φ_m that was used to obtain the fixed point, and return to the standard representation of $R_m(f)$. In this case the Jacobian $DR_m(f)$ will be fixed at the approximated fixed point throughout the analysis, and we will be able to take advantage of some of the computations already performed. We first express $R_m(f)$ in terms of $\tilde{R}_m(f^m(0), f)$ (recall equation (8)).

We have that

$$R_m(f)(x) = \frac{1}{\alpha(f)} f^m(\alpha(f)x) = \frac{1}{\alpha(f)} \tilde{R}_m(\alpha(f), f)(x) \quad \text{with} \quad \alpha(f) = f^m(0).$$

Then,

$$\begin{aligned} DR_m(f)(h)(x) &= \frac{-1}{(\alpha(f))^2} \partial_f \tilde{R}_m(\alpha(f), f)(h)(0) \times f^m(\alpha(f)x) \\ &\quad + \frac{1}{\alpha(f)} \left(\partial_f \tilde{R}_m(\alpha(f), f)(h)(x) + (f^m)'(\alpha(f)x) \times \partial_f \tilde{R}_m(\alpha(f), f)(h)(0)x \right), \end{aligned}$$

and more explicitly,

$$\begin{aligned}
DR_m(f)(h)(x) &= \frac{1}{f^m(0)} \sum_{j=0}^{m-1} \left(\prod_{l=j+1}^{m-1} f' \left(f^l(f^m(0)x) \right) \right) h \left(f^j(f^m(0)x) \right) \\
&\quad + \left(\frac{1}{f^m(0)} x \prod_{j=0}^{m-1} f' \left(f^j(f^m(0)x) \right) - \frac{1}{(f^m(0))^2} f^m(f^m(0)x) \right) \\
&\quad \times \sum_{j=0}^{m-1} \left(\prod_{l=j+1}^{m-1} f' \left(f^l(0) \right) \right) h \left(f^j(0) \right) \\
&= \frac{1}{\delta_m(0)} \left(\sum_{j=0}^{m-1} \xi_{j+1}(x) h(\delta_j(x)) + \left(x \xi_0(x) - \frac{\delta_m(x)}{\delta_m(0)} \right) \sum_{j=0}^{m-1} \xi_{j+1}(0) h(\delta_j(0)) \right),
\end{aligned}$$

where

$$\delta_j(x) = f^j(f^m(0)x), \quad \xi_j(x) = \prod_{l=j}^{m-1} f'(\delta_l(x)).$$

Let f be a fixed point of R_m . We look for $\lambda \in \mathbb{C}$ and $u \in \ell_\rho^{1,\text{even}}$ such that (λ, u) is a zero of

$$F(\lambda, u) = \begin{pmatrix} u_0 - 1 \\ DR_m(f)u - \lambda u \end{pmatrix},$$

where u_0 is the zero-th Chebyshev coefficient of u .

Remark 6.1. *Some normalization is needed to ensure that the eigenpair we try to validate is isolated, but the specific choice of enforcing $u_0 = 1$ is somewhat arbitrary.*

Similarly to what we did in Section 4, we reformulate the zero-finding problem into a fixed-point problem in order to validate a posteriori an approximate eigenpair. We have,

$$DF(\bar{\lambda}, \bar{u}) = \begin{pmatrix} 0 & E_0 \\ -\bar{u} & DR_m(f) - \bar{\lambda}I \end{pmatrix},$$

where E_0 is the map $u \mapsto u_0$. Since $DR_m(f)$ is compact, we take A as

$$\begin{cases} A \Pi^K(\lambda, u) = J \Pi^K(\lambda, u) \\ A(I - \Pi^K)(\lambda, u) = \left(0, -\frac{1}{\lambda} (I - \Pi^K)u \right), \end{cases}$$

where J is an approximate inverse of

$$J^\dagger = \Pi^K DF(\bar{\lambda}, \bar{u})|_{\mathcal{X}_\rho^K},$$

This leads to the fixed-point operator

$$T : (\lambda, u) \mapsto (\lambda, u) - A F(\lambda, u).$$

We again will use Theorem 4.1, but this time since F is merely quadratic in (λ, u) we split the Z estimate into a Z_1 and Z_2 part as explained in Remark 4.2. Specifically, we write, for an arbitrary (λ, u) in \mathcal{X}_ρ :

$$\begin{aligned}
\|DT(\lambda, u)\|_{\mathcal{X}_\rho} &\leq \|DT(\bar{\lambda}, \bar{u})\|_{\mathcal{X}_\rho} + \|DT(\lambda, u) - DT(\bar{\lambda}, \bar{u})\|_{\mathcal{X}_\rho} \\
&\leq Z_1 + Z_2 r.
\end{aligned}$$

We derive suitable estimates Y , Z_1 and Z_2 below. Many of the calculations are very similar to the ones of Section 5, and we do not repeat all the details.

6.1 Y estimate in (14a)

The finite part is simply

$$Y^K = \left\| J \Pi^K F(\bar{\lambda}, \bar{u}) \right\|_{\mathcal{X}_\rho}.$$

We estimate the coefficients of $\Pi^K F(\bar{\lambda}, \bar{u})$ using Remark 3.14, and then simply multiply the result by J and compute the ℓ_ρ^1 norm of the result.

For the tail part, we use Proposition 3.13 to estimate

$$\frac{1}{|\bar{\lambda}|} \left\| (I - \Pi^K) F(\bar{\lambda}, \bar{u}) \right\|_{\ell_\rho^1} = \frac{1}{|\bar{\lambda}|} \left\| (I - \Pi^K) (DR_m(f)\bar{u} - \bar{\lambda}\bar{u}) \right\|_{\ell_\rho^1}.$$

We then proceed as in Section 5.1, and take

$$Y^\infty = \frac{1}{|\bar{\lambda}|} \left\| (\Pi^{K_Y} - \Pi^K) (DR_m(f)\bar{u} - \bar{\lambda}\bar{u}) \right\|_{\ell_\rho^1} + \frac{\Upsilon_{\rho, \nu, K_Y}^{1,0,\text{even}}}{|\bar{\lambda}|} \|DR_m(f)\bar{u}\|_{C_\nu^0},$$

for some $K_Y > K$ and some $\nu > \rho$ chosen according to Remark 3.16.

6.2 Z_1 estimate in Remark 4.2

In this section, we estimate

$$\begin{aligned} \|DT(\bar{\lambda}, \bar{u})\|_{\mathcal{X}_\rho} &= \|I - ADF(\bar{\lambda}, \bar{u})\|_{\mathcal{X}_\rho} \\ &\leq \left\| \Pi^K (I - ADF(\bar{\lambda}, \bar{u})) \Pi^K \right\|_{\mathcal{X}_\rho} \end{aligned} \quad (27)$$

$$+ \left\| \Pi^K (I - ADF(\bar{\lambda}, \bar{u})) \Pi^\infty \right\|_{\mathcal{X}_\rho} \quad (28)$$

$$+ \left\| \Pi^\infty (I - ADF(\bar{\lambda}, \bar{u})) \right\|_{\mathcal{X}_\rho}. \quad (29)$$

6.2.1 Dealing with (27)

We have that

$$\left\| \Pi^K (I - ADF(\bar{\lambda}, \bar{u})) \Pi^K \right\|_{\mathcal{X}_\rho} = \left\| I_K - J \Pi^K DF(\bar{\lambda}, \bar{u}) \Pi^K \right\|_{\mathcal{X}_\rho},$$

where I_K is the identity operator on \mathcal{X}_ρ^K . Therefore, we merely have to compute the norm of a finite dimensional operator. This bound will be denoted by $Z_1^{K,K}$ when reporting numerical values or in the code, similar to the case of the fixed point.

6.2.2 Dealing with (28)

We have that

$$\begin{aligned} \left\| \Pi^K (I - ADF(\bar{\lambda}, \bar{u})) \Pi^\infty \right\|_{\mathcal{X}_\rho} &= \left\| J \Pi^K DF(\bar{\lambda}, \bar{u}) \Pi^\infty \right\|_{\mathcal{X}_\rho} \\ &= \sup_{\|h\|_{\ell_\rho^{1,\text{even}}} \leq 1} \left\| J \Pi^K (DF(\bar{\lambda}, \bar{u})(0, \Pi^\infty h)) \right\|_{\mathcal{X}_\rho}. \end{aligned}$$

Therefore, we have to estimate, for any $h \in \ell_\rho^{1,\text{even}}$ with $\|h\|_{\ell_\rho^1} \leq 1$,

$$\begin{aligned} \Pi^K DF(\bar{\lambda}, \bar{u})(0, \Pi^\infty h) &= \begin{pmatrix} E_0(\Pi^\infty h) \\ \Pi^K [DR_m(f)\Pi^\infty h - \bar{\lambda}\Pi^\infty h] \end{pmatrix} \\ &= \begin{pmatrix} E_0(\Pi^\infty h) \\ \Pi^K [DR_m(f)\Pi^\infty h] \end{pmatrix}. \end{aligned}$$

The first component is easy to bound. Indeed, according to Lemma 3.2 we have

$$|E_0(\Pi^\infty h)| = |h_0 - \check{h}_0| = \left| 2 \sum_{l=1}^{\infty} h_{2Kl} \right| \leq \frac{1}{\rho^{2K}} \|h\|_{\ell_\rho^1}.$$

We then get bounds for (the absolute values of) the Chebyshev coefficients of $\Pi^K [DR_m(f)\Pi^\infty h]$, following Remark 3.14. That is, we in fact derive two different estimates:

$$\left| \Pi^K DF(\bar{\lambda}, \bar{u})(0, \Pi^\infty h) \right| \leq \left(\frac{\Upsilon_{1,\rho,K}^{0,1,\text{even}}}{|\delta_m(0)|} |M_K^{-1}| \sum_{j=0}^{m-1} \left(\begin{array}{c} \rho^{-2K} \\ |\xi_{j+1}(x_0)| + \left| \left(x_0 \xi_0(x_0) - \frac{\delta_m(x_0)}{\delta_m(0)} \right) \xi_{j+1}(0) \right| \\ \vdots \\ |\xi_{j+1}(x_k)| + \left| \left(x_k \xi_0(x_k) - \frac{\delta_m(x_k)}{\delta_m(0)} \right) \xi_{j+1}(0) \right| \\ \vdots \\ |\xi_{j+1}(x_K)| + \left| \left(x_K \xi_0(x_K) - \frac{\delta_m(x_K)}{\delta_m(0)} \right) \xi_{j+1}(0) \right| \end{array} \right) \right),$$

and

$$\left| \Pi^K DF(\bar{\lambda}, \bar{u})(0, \Pi^\infty h) \right| \leq \left(\frac{1}{|\delta_m(0)|} \sum_{j=0}^{m-1} \left(\Upsilon_{\beta_j, \rho, K}^{0,1,\text{even}} \|\xi_{j+1}\|_{\mathcal{C}_{\gamma_j}^0} \left(\begin{array}{c} \frac{\gamma_j^{2K+1}}{\gamma_j^{2K-1}} \\ \vdots \\ \frac{1}{\gamma_j^k} \frac{\gamma_j^{2K} + \gamma_j^{2k}}{\gamma_j^{2K-1}} \\ \vdots \\ \frac{1}{\gamma_j^K} \frac{\gamma_j^{2K}}{\gamma_j^{2K-1}} \end{array} \right) + |\xi_{j+1}(0)| \Upsilon_{1,\rho,K}^{0,1,\text{even}} \left\| \cdot \xi_0 - \frac{\delta_m}{\delta_m(0)} \right\|_{\mathcal{C}_\gamma^0} \left(\begin{array}{c} \frac{\gamma^{2K+1}}{\gamma^{2K-1}} \\ \vdots \\ \frac{1}{\gamma^k} \frac{\gamma^{2K} + \gamma^{2k}}{\gamma^{2K-1}} \\ \vdots \\ \frac{1}{\gamma^K} \frac{\gamma^{2K}}{\gamma^{2K-1}} \end{array} \right) \right) \right)$$

take the minimum component-wise between the two, and finally multiply the result by $|J|$ and take the \mathcal{X}_ρ norm to get a bound on (28). This bound will be denoted by $Z_1^{K,\infty}$ when reporting numerical values or in the code.

6.2.3 Dealing with (29)

We have that

$$\left\| \Pi^\infty \left(I - ADF(\bar{\lambda}, \bar{u}) \right) \right\|_{\mathcal{X}_\rho} = \sup_{\|(\lambda, h)\|_{\mathcal{X}_\rho} \leq 1} \left\| \left((0, \Pi^\infty h) + A \Pi^\infty DF(\bar{\lambda}, \bar{u})(\lambda, h) \right) \right\|_{\mathcal{X}_\rho}.$$

Therefore, we have to estimate, for $\|(\lambda, h)\|_{\mathcal{X}_\rho} \leq 1$,

$$\left\| \Pi^\infty \left(h + \frac{1}{\bar{\lambda}} \left(DR_m(f)h - \bar{\lambda}h \right) \right) \right\|_{\ell_\rho^1} = \frac{1}{|\bar{\lambda}|} \left\| \Pi^\infty (DR_m(f)h) \right\|_{\ell_\rho^1}.$$

In order to bound the first terms appearing in $DR_m(f)h$, we take for $j = 0, \dots, m-1$, $\mu_j > \rho$ such that

$$\delta_j(\mathcal{E}_{\mu_j}) \subset \mathcal{E}_\rho,$$

and then use Proposition 3.13 to estimate

$$\begin{aligned} \sum_{j=0}^{m-1} \|\Pi^\infty(\xi_j h \circ \delta_j)\|_{\ell_\rho^1} &\leq \sum_{j=0}^{m-1} \Upsilon_{\rho, \mu_j, K}^{1,0,\text{even}} \|\xi_j h \circ \delta_j\|_{\mathcal{C}_{\mu_j}^0} \\ &\leq \sum_{j=0}^{m-1} \Upsilon_{\rho, \mu_j, K}^{1,0,\text{even}} \|\xi_j\|_{\mathcal{C}_{\mu_j}^0} \|h\|_{\ell_\rho^1}. \end{aligned}$$

The other terms are can be dealt with as in the Y bound, i.e.

$$\left\| \Pi^\infty \left(\cdot \xi_0 - \frac{\delta_m}{\delta_m(0)} \right) \right\|_{\ell_\rho^1} \leq \Upsilon_{\rho, \nu, K}^{1,0,\text{even}} \left\| \cdot \xi_0 - \frac{\delta_m}{\delta_m(0)} \right\|_{\mathcal{C}_\nu^0},$$

for any $\nu > \rho$. Putting everything together, we have

$$\begin{aligned} \frac{1}{|\bar{\lambda}|} \|\Pi^\infty(DR_m(f)h)\|_{\ell_\rho^1} &\leq \frac{1}{|\bar{\lambda}\delta_m(0)|} \left(\sum_{j=0}^{m-1} \Upsilon_{\rho, \mu_j, K}^{1,0,\text{even}} \|\xi_j\|_{\mathcal{C}_{\mu_j}^0} + \Upsilon_{\rho, \nu, K}^{1,0,\text{even}} \left\| \cdot \xi_0 - \frac{\delta_m}{\delta_m(0)} \right\|_{\mathcal{C}_\nu^0} \sum_{j=0}^{m-1} |\xi_{j+1}(0)| \right) \|h\|_{\ell_\rho^1} \\ &\leq \frac{1}{|\bar{\lambda}\delta_m(0)|} \left(\sum_{j=0}^{m-1} \Upsilon_{\rho, \mu_j, K}^{1,0,\text{even}} \|\xi_j\|_{\mathcal{C}_{\mu_j}^0} + \Upsilon_{\rho, \nu, K}^{1,0,\text{even}} \left\| \cdot \xi_0 - \frac{\delta_m}{\delta_m(0)} \right\|_{\mathcal{C}_\nu^0} \sum_{j=0}^{m-1} |\xi_{j+1}(0)| \right), \end{aligned}$$

which upper-bounds (29). This bound will be denoted by Z_1^∞ when reporting numerical values or in the code.

6.3 Z_2 estimate in Remark 4.2

Noticing that DF is linear, we simply have, for any (λ, u) in \mathcal{X}_ρ ,

$$\begin{aligned} \|DT(\lambda, u) - DT(\bar{\lambda}, \bar{u})\|_{\mathcal{X}_\rho} &= \|A(DF(\lambda, u) - DF(\bar{\lambda}, \bar{u}))\|_{\mathcal{X}_\rho} \\ &\leq \left\| A \begin{pmatrix} 0 & 0 \\ -(u - \bar{u}) & -(\lambda - \bar{\lambda})I \end{pmatrix} \right\|_{\mathcal{X}_\rho} \\ &\leq \|A\|_{\mathcal{X}_\rho} \|(\lambda, u) - (\bar{\lambda}, \bar{u})\|_{\mathcal{X}_\rho}, \end{aligned}$$

hence we can take $Z_2 = \|A\|_{\mathcal{X}_\rho}$.

7 Results

In this section, we first give the proof of Theorem 1.5, and then present two additional results. The first one is a 480-digits accurate proof for the classical $m = 2$ case, and the second one provides extra fixed points for m between 5 and 10.

Proof of Theorem 1.5. Fix m in $\{2, 3, \dots, 10\}$, and consider the associated map Φ_m defined in equation (10). Let $(\bar{\alpha}, \bar{h})$ be the approximate zero of Φ_m for the selected value of m , stored in the file `renor_run_final` available at [?] (note that \bar{h} is a polynomial, represented in Figure 1).

Let $\rho = 2$ and r^* be as in Table 3. Using `final_script_gen_m-Copy_final.jl` available at [?], we then evaluate the bounds Y and Z derived in Section 5, which satisfy assumption (14) of Theorem 4.1. These bounds are evaluated using interval arithmetic via the `IntervalArithmetic.jl` library [4], to account for rounding errors. We then check that $Z < 1$ and $r := \frac{Y}{1-Z} \leq r^*$. Theorem 4.1 then yields the existence of a unique zero (α_*, h_*) of Φ_m in \mathcal{X}_ρ such that $\|(\alpha_*, h_*) -$

$(\bar{\alpha}, \bar{h})\|_{\mathcal{X}_\rho} \leq r$. Since (α_*, h_*) is a zero of Φ_m , h_* is a fixed point of R_m , and $h_*^m(0) = \alpha_*$. These \bar{h} , h_* , α_* and r are the \bar{f}_m , f_m , α_m and ϵ_m of Theorem 1.5.

Next, we consider the approximate eigenpair $(\bar{\lambda}, \bar{u})$ of $DR_m(f_m)$ stored in the file `renor_run_final` available at [?]. Still using `final_script_gen_m-Copy_final.jl` available at [?], we then evaluate the bounds Y and Z (split as Z_1 and Z_2) derived in Section 6, and apply once more Theorem 4.1. Denoting by \tilde{r} the error bound obtained this time, we get the existence of a unique eigenpair (λ_m, u_m) of $DR_m(f_m)$ in \mathcal{X}_ρ such that $\|(\lambda, u_m) - (\bar{\lambda}, \bar{u})\|_{\mathcal{X}_\rho} \leq \tilde{r}$. In particular, $|\lambda_m - \bar{\lambda}| \leq \tilde{r}$, which provides the enclosure of λ_m reported in Table 1. \square

Remark 7.1. *Note that we have in fact established a slightly stronger result than the one stated in Theorem 1.5. Indeed, as seen in the above proof, the estimate $|f_m(x) - \bar{f}_m(x)| \leq \epsilon_m$ does in fact hold for all x in the Bernstein ellipse \mathcal{E}_2 thanks to Lemma 3.7.*

Remark 7.2. *Further details about the proof, such as the obtained values for each part of the bounds, are provided in Appendix D. The entire proof runs for around 0.5 minutes for $m = 2$ and 10 minutes for $m = 10$ on an Intel(R) Core(TM) i7-1065G7 CPU @ 1.30GHz 1.50GHz processor with 16GB of ram.*

Theorem 7.3. *For $m = 2$ and the fixed point f_2 of R_2 obtained in Theorem 1.5, the universal constant λ_2 and $\alpha_2 = f_2(f_2(0)) = f_2(1)$ satisfy*

$\lambda_2 \in 4.66920160910299067185320382046620161725818557747576863274565134300413433021131473713868974402394801381716598485518981513440862714202793252231244298889089085994493546323671341153248171421994745564436582379320200956105833057545861765222207038541064674949428498145339172620056875566595233987560382563722564800409510712838906118447027758542854198011134401750024285853824983357155220522360872502916788603626745272133990571316068753450834339344461037063094520191158769724322735898389037 \pm 10^{-479},$

$\alpha_2 \in -0.3995352805231344898575804686336937194335442804669527275170730449124380166088380429818445948741812667617940648468383667140945404846164364373609475570184545976789402326870225485797735028209746477510392557978775073697474932326975513734923082122088541722241308330948027391890574703944646041606699384157782298900077729901354421213971924552385259444903372376975537750905488329754433672693681140505788840461793440186571478080760841608146499827233996549139348743626575822619683926231334765 \pm 10^{-481}.$

Proof. The proof proceeds as the one of Theorem 1.5, except we use extended precision (2^{12} digits) and a polynomial approximation of higher order ($K = 680$) in order to compute a much finer approximate solution and then to evaluate the bounds, together with a much smaller r^* (equal to 10^{-470}). The entire proof can be reproduced by running `final_script_m2_long-Copy_final.jl` available at [?]. \square

Remark 7.4. *Further details regarding the different bounds can also be found in Appendix D.*

Finally, we compute several distinct fixed points for m between 5 and 10, along with their associated universal constants.

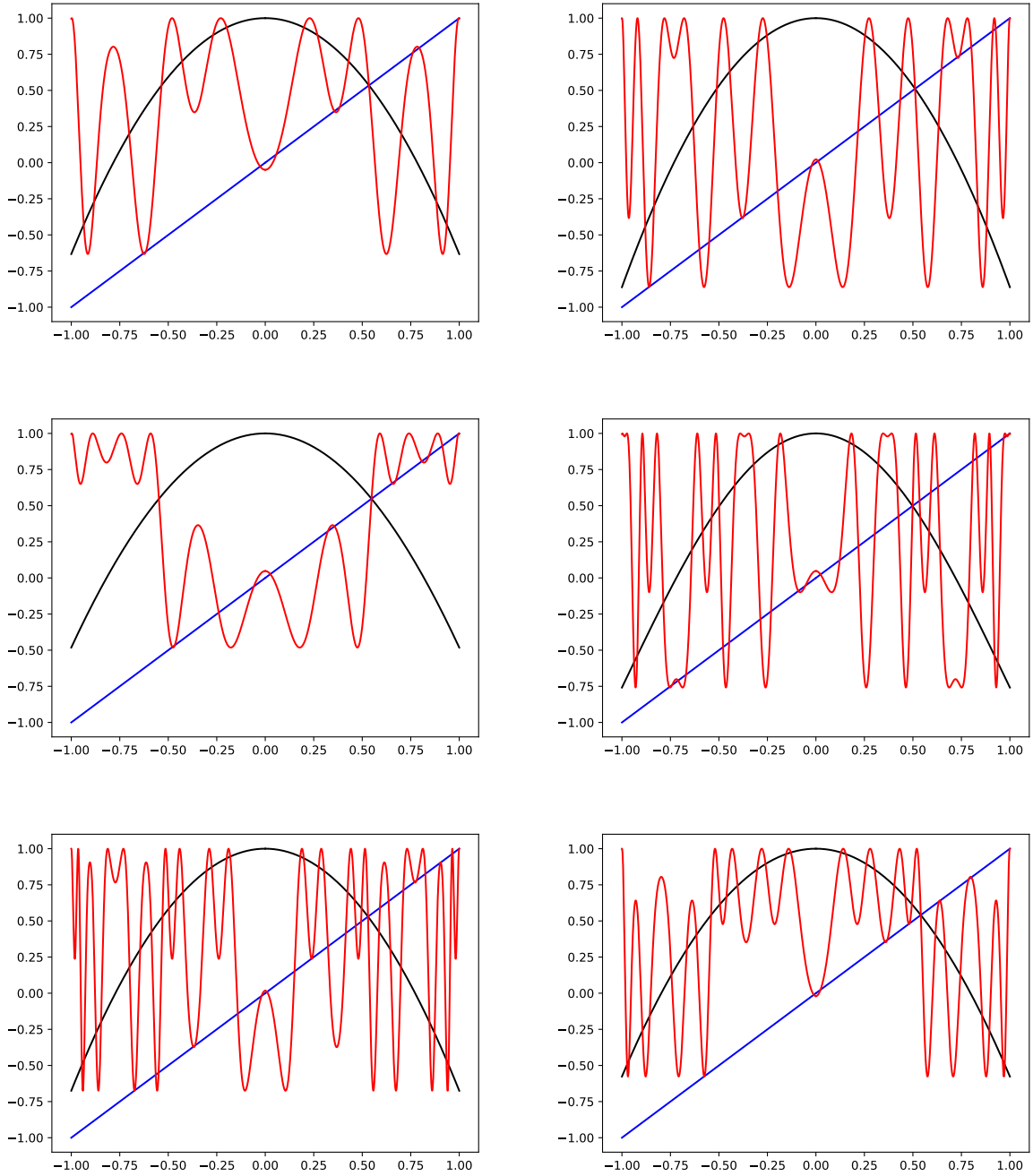


Figure 5: Distinct fixed points with $m = 5$, $m = 6$ and $m = 7$ (v1 on the left and v2 on the right). The black curves show the graph of the polynomials \bar{f}_m from Table 2 on $[-1, 1]$, and the red curve is the composition of \bar{f}_m with itself m times without rescaling.

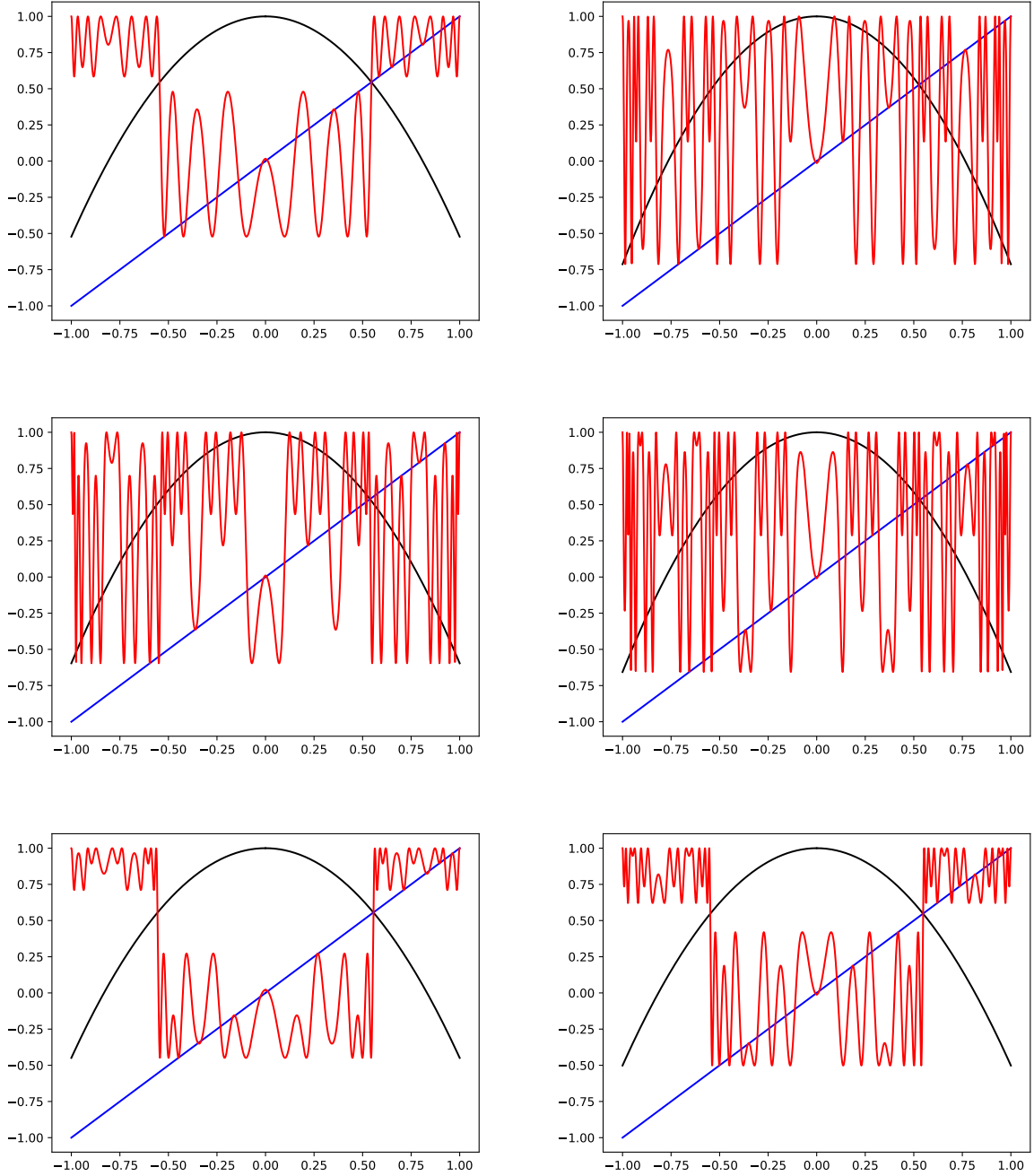


Figure 6: Distinct fixed points with $m = 8$, $m = 9$ and $m = 10$ (v1 on the left and v2 on the right). The black curves show the graph of the polynomials \bar{f}_m from Table 2 on $[-1, 1]$, and the red curve is the composition of \bar{f}_m with itself m times without rescaling.

Theorem 7.5. *For each row in Table 2, consider the corresponding polynomial \bar{f} whose precise coefficients in the Chebyshev basis can be found in the file `renor_run_final` available at [?] (many of those \bar{f} are represented in Figure 5 and Figure 6). There exists an analytic function $f \in \ell_2^1$ such that f is a fixed point of R_m , and $\|f - \bar{f}\|_{\ell_2^1} \leq \epsilon$. All of these fixed points are different from one another, and from the ones obtained in Theorem 1.5 (even when the value of m is the same). The corresponding value of $f^m(0)$ belongs to the interval α , and the unstable eigenvalue of $DR_m(f)$ belongs to the interval λ .*

id	ϵ	α	λ
5v2	10^{-18}	$0.021831959945968847 \pm 10^{-18}$	$1287.0791186707268558285 \pm 10^{-19}$
5v3	10^{-18}	$-0.006248774967523457 \pm 10^{-18}$	$16930.6456004403249060268 \pm 10^{-19}$
6v2	10^{-18}	$0.0477814797951692540 \pm 10^{-19}$	$218.411795140494963093 \pm 10^{-18}$
6v3	10^{-24}	$-0.008694742127194554107054 \pm 10^{-24}$	$8507.78078352962469461955609964 \pm 10^{-26}$
7v2	10^{-18}	$-0.020339275729028984 \pm 10^{-18}$	$1446.4412089927497250765 \pm 10^{-19}$
7v3	10^{-18}	$-0.007611915064702328 \pm 10^{-18}$	$10169.6224655537089255415 \pm 10^{-19}$
7v4	10^{-18}	$0.005220825117857598 \pm 10^{-18}$	$22840.3723154853471780350 \pm 10^{-19}$
7v5	10^{-18}	$-0.004346219158619544 \pm 10^{-18}$	$35305.7264640737380638896 \pm 10^{-19}$
8v2	10^{-18}	$-0.011348310995370730 \pm 10^{-18}$	$5829.6195636953609639965 \pm 10^{-19}$
8v3	10^{-17}	$0.01506257606733863 \pm 10^{-17}$	$2304.5578444485922703759 \pm 10^{-19}$
9v2	10^{-18}	$-0.006966968262344619 \pm 10^{-18}$	$12818.3728321957038811078 \pm 10^{-19}$
10v2	10^{-18}	$-0.010323910150325685 \pm 10^{-19}$	$4522.7721958113701625452 \pm 10^{-19}$
10v3	10^{-27}	$0.020848923604938857446598772 \pm 10^{-27}$	$1110.537874176532781602180700697 \pm 10^{-27}$

Table 2: **Data associated with Theorem 7.5:** the table records the rigorously verified enclosures for the universal constants associated with m -th order renormalization fixed points available in the file `renor_run_final` and proven to exist in Theorem 7.5.

Proof. The proof is exactly the same as the one of Theorem 4.1, except we start with different approximate solutions $(\bar{\alpha}, \bar{h})$ (and then different approximate eigenpairs $(\bar{\lambda}, \bar{u})$). Once the error bounds between the exact fixed points and the approximate ones are obtained, it is trivial to check that these fixed points are indeed all different from one another. \square

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Appendix

A Computing $\sigma_{\rho,\nu}^{\text{even}}$

We provide here a computable upper bound for the constant $\sigma_{\rho,\nu}^{\text{even}}$ introduced in Proposition 3.8, together with a criterion ensuring that this upper-bound is actually sharp. A similar procedure can be derived for $\sigma_{\rho,\nu}^{\text{odd}}$.

Lemma A.1. *Let $1 \leq \rho < \nu$, and $n_0 \in \mathbb{N}_{\geq 1}$ such that $n_0 \geq \frac{1}{2(\ln \nu - \ln \rho)}$. Then*

$$\sigma_{\rho, \nu}^{\text{even}} \leq \max \left(\max_{1 \leq n \leq n_0-1} \frac{2n}{\nu^{2n}} \left(\rho \frac{\rho^{2n} - 1}{\rho^2 - 1} + \rho^{-1} \frac{\rho^{-2n} - 1}{\rho^{-2} - 1} \right), \frac{2n_0}{\nu^{2n_0}} \left(\rho^{2n_0} \frac{\rho}{\rho^2 - 1} + \frac{\rho^{-1}}{1 - \rho^{-2}} \right) \right).$$

Moreover, if n_0 is large enough so that

$$\max_{1 \leq n \leq n_0-1} \frac{2n}{\nu^{2n}} \left(\rho \frac{\rho^{2n} - 1}{\rho^2 - 1} + \rho^{-1} \frac{\rho^{-2n} - 1}{\rho^{-2} - 1} \right) \geq \frac{2n_0}{\nu^{2n_0}} \left(\rho^{2n_0} \frac{\rho}{\rho^2 - 1} + \frac{\rho^{-1}}{1 - \rho^{-2}} \right),$$

then

$$\sigma_{\rho, \nu}^{\text{even}} = \max_{1 \leq n \leq n_0-1} \frac{2n}{\nu^{2n}} \left(\rho \frac{\rho^{2n} - 1}{\rho^2 - 1} + \rho^{-1} \frac{\rho^{-2n} - 1}{\rho^{-2} - 1} \right).$$

Proof. For all $n \in \mathbb{N}_{\geq 1}$,

$$\frac{2n}{\nu^{2n}} \left(\rho \frac{\rho^{2n} - 1}{\rho^2 - 1} + \rho^{-1} \frac{\rho^{-2n} - 1}{\rho^{-2} - 1} \right) \leq \frac{2n}{\nu^{2n}} \left(\frac{\rho}{\rho^2 - 1} \rho^{2n} + \frac{\rho^{-1}}{1 - \rho^{-2}} \right),$$

therefore

$$\sigma_{\rho, \nu}^{\text{even}} \leq \max \left(\max_{1 \leq n \leq n_0-1} \frac{2n}{\nu^{2n}} \left(\rho \frac{\rho^{2n} - 1}{\rho^2 - 1} + \rho^{-1} \frac{\rho^{-2n} - 1}{\rho^{-2} - 1} \right), \sup_{n \geq n_0} \frac{2n}{\nu^{2n}} \left(\rho^{2n} \frac{\rho}{\rho^2 - 1} + \frac{\rho^{-1}}{1 - \rho^{-2}} \right) \right),$$

and the announced estimates simply follows from the fact that the map

$$x \mapsto \frac{x}{\nu^x} \left(\frac{\rho}{\rho^2 - 1} \rho^x + \frac{\rho^{-1}}{1 - \rho^{-2}} \right)$$

is decreasing for $x \geq \frac{1}{\ln \nu - \ln \rho}$. □

B Going between Chebyshev coefficients and values at Chebyshev nodes

In practice, while we represent an even polynomial $h \in \Pi^{2K} \ell_\rho^1$ by a vector containing its coefficients (of even index) in the Chebyshev basis, i.e. $h = h_0 + 2 \sum_{k=1}^K h_{2k}$, it is sometimes convenient to work instead with the values at the Chebyshev points, for instance to efficiently compute compositions. We point out that the isomorphism between these two representations can be computed explicitly by

$$\begin{pmatrix} h(x_0) \\ h(x_1) \\ \vdots \\ h(x_K) \end{pmatrix} = M_K \begin{pmatrix} h_0 \\ h_2 \\ \vdots \\ h_{2K} \end{pmatrix},$$

where

$$M_K = \begin{pmatrix} \cos(0\theta_0) & 2\cos(2\theta_0) & \dots & 2\cos(2(K-1)\theta_0) & 2\cos(2K\theta_0) \\ \cos(0\theta_1) & 2\cos(2\theta_1) & \dots & 2\cos(2(K-1)\theta_1) & 2\cos(2K\theta_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos(0\theta_{K-1}) & 2\cos(2\theta_{K-1}) & \dots & 2\cos(2(K-1)\theta_{K-1}) & 2\cos(2K\theta_{K-1}) \\ \cos(0\theta_K) & 2\cos(2\theta_K) & \dots & 2\cos(2(K-1)\theta_K) & 2\cos(2K\theta_K) \end{pmatrix},$$

and $x_k = \cos \theta_k$ are half the Chebyshev nodes, i.e.,

$$\theta_k = \frac{K-k}{2K}\pi, \quad 0 \leq k \leq K.$$

It should be noted that this is nothing but (one version of) the Discrete Cosine Transform. Therefore, the inverse transformation can also be described explicitly and has a very similar expression

$$M_K^{-1} = \frac{1}{2K} \begin{pmatrix} \frac{1}{2}\cos(0\theta_0) & \cos(0\theta_1) & \dots & \cos(0\theta_{K-1}) & \frac{1}{2}\cos(0\theta_K) \\ \frac{1}{2}\cos(2\theta_0) & \cos(2\theta_1) & \dots & \cos(2\theta_{K-1}) & \frac{1}{2}\cos(2\theta_K) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2}\cos(2(K-1)\theta_0) & \cos(2(K-1)\theta_1) & \dots & \cos(2(K-1)\theta_{K-1}) & \frac{1}{2}\cos(2(K-1)\theta_K) \\ \frac{1}{4}\cos(2K\theta_0) & \frac{1}{2}\cos(2K\theta_1) & \dots & \frac{1}{2}\cos(2K\theta_{K-1}) & \frac{1}{4}\cos(2K\theta_K) \end{pmatrix}.$$

These transformations can also be computed efficiently via Fast Fourier Transform algorithms.

C Computing the supremum on a Bernstein Ellipse

Consider a polynomial h , written in the Chebyshev basis:

$$h(x) = h_0 + 2 \sum_{k=1}^K h_k T_k(x).$$

We describe here a strategy originating from [63] (see also [36]) allowing to efficiently compute, at least to sharply upper-bound, the \mathcal{C}_ρ^0 norm of h . For any $\rho > 1$, we have

$$\begin{aligned} \|h\|_{\mathcal{C}_\rho^0} &= \max_{z \in \mathcal{E}_\rho} |h(z)| \\ &= \max_{z \in \partial \mathcal{E}_\rho} |h(z)| \\ &= \max_{\theta \in [0, 2\pi]} \left| h \left(\frac{1}{2} \left(\rho e^{i\theta} + (\rho e^{i\theta})^{-1} \right) \right) \right| \\ &= \max_{\theta \in [0, 2\pi]} \left| \sum_{k=-K}^K h_k \rho^k e^{ik\theta} \right| \\ &= \max_{\theta \in [0, 2\pi]} |f(\theta)|, \end{aligned}$$

where

$$f(\theta) = \sum_{k=-K}^K f_k e^{ik\theta}, \quad f_k = h_k \rho^k.$$

Therefore, we simply have to compute (or at least to upper-bound) the supremum of a trigonometric polynomial on $[0, 2\pi]$.

Now, given an integer $N \geq 2K$, and considering the uniform grid

$$\theta_n = n \frac{2\pi}{N}, \quad n = 0, \dots, N,$$

we can efficiently evaluate $f(\theta_n)$ for all $n \in \{0, \dots, N\}$ using the FFT (and padding f by zeros if necessary). In order to get a rigorous enclosure of the image of f on the whole interval $[0, 2\pi]$, rather than just of the image of the grid, we notice that, for any $\theta \in [0, 2\pi]$, there exists $n \in \{0, \dots, N\}$ such that $\theta \in \theta_n + [-\delta, \delta]$, with $\delta = \frac{\pi}{N}$. Therefore, with interval arithmetic,

$$f(\theta) \in \sum_{k=-K}^K f_k e^{ik[-\delta, \delta]} e^{ik\theta_n}.$$

That is, if we consider \tilde{f} the trigonometric polynomial (having interval coefficients) given by

$$\tilde{f}(\theta) = \sum_{k=-K}^K \tilde{f}_k e^{ik\theta}, \quad \tilde{f}_k = f_k e^{ik[-\delta, \delta]},$$

and evaluate \tilde{f} on the grid $\theta_0, \dots, \theta_N$, (which can be done efficiently using the FFT), we get $N + 1$ intervals whose reunion contains $f([0, 2\pi]) = h(\partial\mathcal{E}_\rho)$. We can then easily get an upper-bound for $|h(\partial\mathcal{E}_\rho)|$, but also for any quantity of the form $|G(h(\partial\mathcal{E}_\rho))|$ if G is an analytic function which we can rigorously evaluate with intervals.

D Implementation and results details on the computer-assisted proofs

All the calculations were done using the IntervalArithmetic.jl library [4] version 0.20.8, in Julia 1.8.5. The proofs of Theorem 1.5 and Theorem 7.5 were done using an extend precision of 128, i.e., 128 digits for the mantissa of floating-point numbers. For the proof of Theorem 7.3, a precision of 4096 was used.

For each of the fixed points obtained in Theorem 1.5, we specify in Table 3 and Table 4 how many Chebyshev modes were used, what value of r^* was selected, and provide the evaluation of the bounds described in Sections 5 and 6. The same thing is done in Table 5 and Table 6 regarding Theorem 7.3, and in Table 7 and Table 8 for Theorem 7.5. For the computer codes, see [?].

Remark D.1. Notice that the fixed points *6v1* and *6v2*, *8v1* and *8v3*, *10v1* and *10v3*, have the same universal constants. This phenomena has been observed by other authors and is discussed in more detail in [17] and the references discussed therein.

id	K	r_*	Y^K	Y^∞	$Z^{K,K}$	$Z^{K,\infty}$	Z^∞	r_{\min}
2v1	21	10^{-15}	8.332e-25	4.548e-18	0.365	0.0162	0.00354	7.384e-18
3v1	15	10^{-16}	1.667e-27	5.056e-21	0.000222	4.154e-05	2.086e-14	5.058e-21
4v1	15	10^{-16}	2.384e-26	6.957e-23	0.01242	1.754e-05	2.359e-10	7.047e-23
5v1	15	10^{-16}	7.500e-27	9.391e-24	0.00626	2.373e-05	1.101e-11	9.457e-24
6v1	15	10^{-16}	8.860e-27	3.605e-23	0.012	3.1251e-05	3.768e-07	3.649e-23
7v1	15	10^{-16}	6.719e-26	8.348e-24	0.1864	7.212e-05	8.173e-06	1.035e-23
8v1	15	10^{-17}	2.723e-64	9.674e-20	0.0358	0.00213	3.819e-14	1.006e-19
9v1	15	10^{-17}	5.799e-64	9.508e-20	0.180	0.000182	1.417e-13	1.159e-19
10v1	15	10^{-17}	1.851e-64	1.461e-19	0.0408	9.194e-05	5.433e-14	1.523e-19

Table 3: **Data associated with the proof of Theorem 1.5, fixed point.** The finite dimensional projection used for the proof was Π^{2K} . We report in this table the values obtained for each part of the bounds $Y = Y^K + Y^\infty$ and $Z = Z^{K,K} + Z^{K,\infty} + Z^\infty$ described in Section 5 for the validation of the fixed point (rounded to 4 digits for readability). The value of r_{\min} is the smallest error bound provided by Theorem 4.1 for the fixed point, i.e. $\frac{Y}{1-Z}$.

id	Y^K	Y^∞	$Z_1^{K,K}$	$Z_1^{K,\infty}$	Z_1^∞	Z_2	r_{\min}
2v1	5.445e-24	2.636e-18	1.755e-24	0.0359	0.000758	6.389	2.736e-18
3v1	7.243e-27	1.701e-19	2.510e-27	0.000362	3.607e-16	5.493	1.702e-19
4v1	7.786e-26	4.890e-20	2.712e-26	0.000680	2.403e-13	5.141	4.894e-20
5v1	2.978e-26	2.052e-19	1.091e-26	0.000477	4.308e-14	5.097	2.053e-19
6v1	4.833e-26	2.885e-19	1.753e-26	0.000630	1.726e-09	5.254	2.887e-19
7v1	2.3993e-25	3.959e-20	8.183e-26	0.00423	3.627e-09	5.010	3.976e-20
8v1	1.011e-63	3.520e-20	3.436e-64	0.14091	1.629e-17	5.059	4.097e-20
9v1	2.162e-63	2.806e-20	7.370e-64	0.0204	1.775e-17	5.018	2.865e-20
10v1	1.053e-63	1.747e-19	3.993e-64	0.004392	4.849e-17	5.176	1.755e-19

Table 4: **Data associated with the proof of Theorem 1.5, eigenvalue problem.** We report in this table the values obtained for each part of the bounds $Y = Y^K + Y^\infty$, $Z_1 = Z_1^{K,K} + Z_1^{K,\infty} + Z_1^\infty$ and Z_2 described in Section 6 for the validation of the eigenpair of the fixed point (rounded to 4 digits for readability). The value of r_{\min} is the smallest error bound provided by Theorem 4.1 in that case, i.e. $\frac{1-Z_1-\sqrt{(1-Z_1)^2-4YZ_2}}{2Z_2}$.

id	K	r_*	Y^K	Y^∞	$Z^{K,K}$	$Z^{K,\infty}$	Z^∞	r_{\min}
2v1	680	10^{-470}	3.0443e-821	7.446e-482	2.0812e-59	1.511e-98	9.866e-113	7.446e-482

Table 5: **Data associated with the proof of Theorem 7.3, fixed point.** The finite dimensional projection used for the proof was Π^{2K} . We report in this table the values obtained for each part of the bounds $Y = Y^K + Y^\infty$ and $Z = Z^{K,K} + Z^{K,\infty} + Z^\infty$ described in Section 5 for the validation of the fixed point (rounded to 4 digits for readability). The value of r_{\min} is the smallest error bound provided by Theorem 4.1 for the fixed point, i.e. $\frac{Y}{1-Z}$.

id	Y^K	Y^∞	$Z_1^{K,K}$	$Z_1^{K,\infty}$	Z_1^∞	Z_2	r_{\min}
2v1	1.657e-820	3.314e-480	4.4943e-821	3.7613e-98	2.113e-113	6.389	3.314e-480

Table 6: **Data associated with the proof of Theorem 7.3, eigenvalue problem.** We report in this table the values obtained for each part of the bounds $Y = Y^K + Y^\infty$, $Z_1 = Z_1^{K,K} + Z_1^{K,\infty} + Z_1^\infty$ and Z_2 described in Section 6 for the validation of the eigenpair of the fixed point (rounded to 4 digits for readability). The value of r_{\min} is the smallest error bound provided by Theorem 4.1 in that case, i.e. $\frac{1-Z_1-\sqrt{(1-Z_1)^2-4YZ_2}}{2Z_2}$.

id	K	r_*	Y^K	Y^∞	$Z^{K,K}$	$Z^{K,\infty}$	Z^∞	r_{\min}
5v2	15	e-16	3.553e-26	1.235e-19	0.0424	1.857e-05	1.548e-13	1.289e-19
5v3	15	e-17	3.993e-25	1.232e-19	0.0762	0.000124	2.263e-13	1.333e-19
6v2	15	e-18	2.401e-23	2.625e-20	0.338	7.168e-08	2.427e-13	3.965e-20
6v3	15	e-19	6.307e-61	3.308e-26	0.6772	1.149e-07	1.163e-17	1.025e-25
7v2	15	e-16	4.6198e-26	1.045e-19	0.1229	3.526e-05	1.744e-13	1.192e-19
7v3	15	e-17	2.922e-25	1.074e-19	0.1318	0.000133	4.786e-10	1.237e-19
7v4	15	e-17	5.804e-25	1.096e-19	0.3434	0.000226	5.065e-10	1.6697e-19
7v5	15	e-17	8.879e-25	1.121e-19	0.60195	0.000317	7.644e-11	2.817e-19
8v2	15	e-16	1.381e-25	1.083e-19	0.8023	0.000107	6.2892e-11	5.477e-19
8v3	15	e-17	7.413e-25	3.509e-18	0.427	0.000235	1.9897e-06	6.121e-18
9v2	15	e-17	4.000e-25	9.714e-20	0.3173	0.000231	1.219e-08	1.424e-19
10v2	15	e-17	4.9393e-64	9.538e-20	0.184	0.000175	5.541e-10	1.169e-19
10v3	25	e-25	5.084e-58	2.9798e-28	0.00118	0.01649	9.986e-19	3.034e-28

Table 7: **Data associated with the proof of Theorem 7.5, fixed point.** Each line in this table corresponds to the same line in Table 2. The finite dimensional projection used for the proof was Π^{2K} . We report in this table the values obtained for each part of the bounds $Y = Y^K + Y^\infty$ and $Z = Z^{K,K} + Z^{K,\infty} + Z^\infty$ described in Section 5 for the validation of the fixed point (rounded to 4 digits for readability). The value of r_{\min} is the smallest error bound provided by Theorem 4.1 for the fixed point, i.e. $\frac{Y}{1-Z}$.

id	Y^K	Y^∞	$Z_1^{K,K}$	$Z_1^{K,\infty}$	Z_1^∞	Z_2	r_{\min}
5v2	1.201e-25	4.307e-20	4.305e-26	0.00086	1.167e-16	5.063	4.311e-20
5v3	1.172e-24	2.808e-20	3.955e-25	0.01984	1.327e-17	5.036	2.865e-20
6v2	1.4316e-22	5.902e-19	5.541e-23	1.397e-06	5.951e-16	6.903	5.904e-19
6v3	2.007e-60	4.3398e-27	6.9895e-61	1.321e-05	1.502e-23	5.009	4.3398e-27
7v2	1.938e-25	4.416e-20	6.864e-26	0.00174	1.174e-16	5.083	4.424e-20
7v3	9.411e-25	2.553e-20	3.177e-25	0.01738	4.706e-14	5.005	2.5981e-20
7v4	1.806e-24	2.482e-20	6.293e-25	0.04315	2.218e-14	5.007	2.594e-20
7v5	2.603e-24	2.559e-20	8.906e-25	0.0728	2.166e-15	5.012	2.7593e-20
8v2	4.878e-25	3.3291e-20	1.674e-25	0.00938	1.079e-14	5.098	3.361e-20
8v3	2.934e-24	1.991e-20	1.037e-24	0.01458	8.634e-10	7.400	2.021e-20
9v2	1.413e-24	2.616e-20	4.782e-25	0.03308	9.505e-13	5.002	2.706e-20
10v2	1.986e-63	3.608e-20	6.737e-64	0.01688	1.226e-13	5.047	3.6693e-20
10v3	2.916e-57	3.849e-29	1.142e-57	0.79051	8.992e-22	6.461	1.838e-28

Table 8: **Data associated with the proof of Theorem 7.5, eigenvalue problem.** Each line in this table corresponds to the same line in Table 2. We report in this table the values obtained for each part of the bounds $Y = Y^K + Y^\infty$, $Z_1 = Z_1^{K,K} + Z_1^{K,\infty} + Z_1^\infty$ and Z_2 described in Section 6 for the validation of the eigenpair of the fixed point (rounded to 4 digits for readability). The value of r_{\min} is the smallest error bound provided by Theorem 4.1 in that case, i.e. $\frac{1-Z_1-\sqrt{(1-Z_1)^2-4YZ_2}}{2Z_2}$.

id	Kneading seq.
2v1	$(R)^\infty$
3v1	$(R L)^\infty$
4v1	$(R L L)^\infty$
5v1	$(R L R R)^\infty$
5v2	$(R L L R)^\infty$
5v3	$(R L L L)^\infty$
6v1	$(R L R R R)^\infty$
6v2	$(R L L R L)^\infty$
6v3	$(R L L R R)^\infty$
7v1	$(R L R R L R)^\infty$
7v2	$(R L R R R R)^\infty$
7v3	$(R L L R L R)^\infty$
7v4	$(R L L R R R)^\infty$
7v5	$(R L L R R L)^\infty$
8v1	$(R L R R R R R)^\infty$
8v2	$(R L R R L R R)^\infty$
8v3	$(R L L L R L L)^\infty$
9v1	$(R L R R R R L R)^\infty$
9v2	$(R L R R L R L R)^\infty$
10v1	$(R L R R R L R L R)^\infty$
10v2	$(R L R R R R R L R)^\infty$
10v3	$(R L R R L R L R R)^\infty$

Table 9: **Kneading sequences for the fixed points:** The powers/products here are with respect to the star product for kneading sequences. See for example [14, 17] for precise definitions. The superscript indicates the infinite star product of the sequence in parenthesis. That is, if A is a kneading sequence, then $A^\infty = \lim_{n \rightarrow \infty} A^{*n}$, where $A^{*n} = A * \dots * A$ is the iterated star product of the sequence A with itself n times. Note that the kneading sequences given in the table are not periodic, due to the definition of the star operator. For example $R * R = RLR$ while $R * R * R = R * (RLR) = RLRRRLR$ etcetera. That is, this is just a convenient notation for expressing an a periodic sequences. The reported values are not validated, as they are simply computed by following the orbit of 0 for the numerically computed fixed point f_m and extracting the appropriate combinatorial sequence A from the orbit data.

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