

On the Constant Depth Implementation of Pauli Exponentials

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We decompose arbitrary weight $Z \otimes \dots \otimes Z$ exponentials into circuits of constant depth using $\mathcal{O}(n)$ ancillae and two-body XX and ZZ interactions. Consequently, a similar method works for arbitrary Pauli exponentials. Our decomposition is compatible with linear nearest neighbour architectures. We prove its correctness after introducing novel circuit rewrite rules for circuits which benefit from qubit recycling. As a novelty, the decomposition is immediately applicable for implementing fault-tolerant lattice surgery computations, expressing arbitrary stabilizer circuits using only two-body interactions, as well as reducing the depth of NISQ computations, such as VQE.

I. INTRODUCTION

The efficient compilation of Pauli exponentials has far-reaching implications in quantum computing. Whenever the exponentiation angle is $\pi/2$, Pauli strings can be used as measurement operators for stabilizing quantum systems (e.g. [1–3]). When the angle is part of the set $[\pi/2, \pi/4, \pi/8]$, the exponentiated Pauli strings can be interpreted as multi-body measurements which implement fault-tolerant, lattice surgery computations [4–6]. In the case of arbitrary rotation angles, these strings are central to VQE-algorithms (e.g. [7]) and to the form of UCCSD terms in quantum chemistry [8]. Pauli exponentials appear as phased gadgets in the ZX-calculus [9].

Herein, we present and discuss a decomposition of exponentiated Pauli strings. Our constant depth decomposition uses a linear number of ancillas (effectively doubling the number of qubits) and two-body XX and ZZ interactions. These interactions are native to Majorana computers [2], silicon spin qubits [10], as well as to fault-tolerant and error-correction circuits.

Constructions similar to ours appeared in [2, 3], but these have not been generalized to an arbitrary number of qubits or rotation angles, and were only derived and verified by applying the ZX-calculus on small scale diagrams. In contrast, we introduce novel circuit rewrite rules, derive and show the correctness of our decomposition using circuit diagrams.

Additionally, a result concerning long range entangling gates of arbitrary length has been introduced by [11]. Their adaptive circuit construction uses CNOT gates and introduces the same number of ancillae as our decomposition. Nevertheless, due to the presence of CNOT gates, their construction seems to be focusing entirely on NISQ applications as the authors indicate UCCSD as an application. In contrast, our decomposition is applicable to both NISQ and error-corrected machines. While a constant-depth implementation of CNOTs using pairwise measurements is presented in [12], it has not been generalized to arbitrary Paulis.

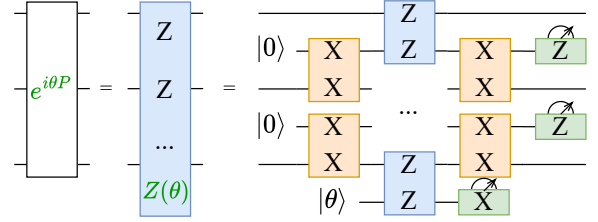


FIG. 1. An n -qubit Pauli exponential is decomposed into a constant depth sequence of XX and ZZ interactions and $\mathcal{O}(n)$ ancillae. We use the shorthand notation $e^{i\theta P} = P(\theta)$, and consider that P is formed exclusively of Z terms: $P = Z_0 \otimes \dots \otimes Z_n$. A general string can be always decomposed into single-qubit Clifford gates conjugating the Z -terms of P [9]. In this figure, we assume that θ is arbitrary, such that our construction is extending P with an additional Z term (marked green) and an ancilla initialized in $|\theta\rangle = R_z(\theta)|+\rangle$ [4].

In the following, we are describing the building blocks of our derivation in Section II, and discuss briefly the applications of the decomposition in Section III. The correctness of our method, as well as the complete proof of the decomposition from Fig 1 are in the Figs 9, 10, 8.

II. METHODS

The string P can be exponentiated to a unitary by computing $e^{i\theta P}$. We consider the measurement-based implementation of the exponentiated Pauli strings [4], such that, in practice, the exponential is implemented by a circuit that uses an additional Z term in the Pauli string (green in Fig. 1) and an ancilla rotated by θ . Consequently, $e^{i\theta P}$ is achieved by the rule **ROT** from Fig. 2.

The lhs. of Fig. 1 represents the unitary obtained by the exponentiation. The rhs. is the decomposition which includes ancillae for enabling the XX interactions, as well as the $|\theta\rangle$ ancilla initialized for performing a teleportation-based gate.

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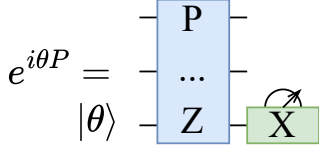


FIG. 2. The **ROT** rewrite rule implements the exponential of the Pauli P by introducing an ancilla initialized in $|\theta\rangle = Rz(\theta)|+\rangle$, extending the Pauli string with a Z -term for the ancilla, and measuring the ancilla in the X -basis.

A. Measurement-based implementation

When implementing the Pauli exponentials on fault-tolerant machines, the θ rotation is first decomposed into a sequence of rotation gates supported by the QECC. This increases the depth proportionally to the length of the gate sequence. Moreover, the depth will additionally increase by a constant factor owing to the fact that each rotation is probabilistic and might require corrections [4] (e.g. if the X -measurement results in the -1 eigenvalue, then a corrective rotation of 2θ is needed). Corrections can be tracked in a Pauli frame, when the angles correspond to Pauli gates.

Moreover, we assume that the two-body interactions are also measurement-based. The latter will introduce corrective terms (e.g.[3]), but the corrections can be tracked in a Pauli frame, as the underlying circuit will consist solely of CNOT gates and ancillae initialised or measured in a discrete set of rotated basis [13, 14].

If the exponentials are implemented on NISQ machines, the θ rotation is a single qubit gate. The corrective terms generated by the measurements of the ancillae will be applied adaptively without increasing the depth of the circuit [11].

As a conclusion, in all the following diagrams we omit the corrective terms introduced by the measurements. Corrections can be tracked through the circuit, or can be applied immediately after the exponentiated Pauli without affecting the depth of the decomposition.

B. Decomposing the Pauli exponentials

We present the rewrite rules used to obtain the generalized decomposition from Fig. 1. The result from Fig. 1 is obtained by the following algorithm:

1. *Input:* arbitrary Pauli exponential P with angle θ ;
2. Apply **ROT**, if θ is not $\pi/2$ – this extends the Pauli string by an additional Z term and the circuit will include the $|\theta\rangle$ ancilla; if the angle is $\pi/2$ the Pauli string is left unchanged and $|\theta\rangle$ is not appended;
3. Repeat the sequence of rewrite rules **PG**, **LS**, **MR**, **FUSE**, until the largest Pauli string acts on a maximum of two qubits.

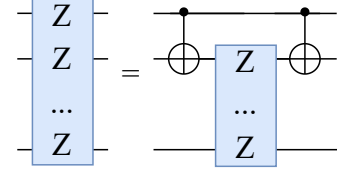


FIG. 3. The **PG** rewrite takes an n -qubit string and decomposes it into a $(n-1)$ -qubit string and two CNOTs.

We start by decomposing an n -qubit Pauli string with the **PG** rule [9] (Fig. 3). CNOTs are expressed in lattice surgery style[15] decompositions (Fig. 7). After using **PG**, we choose the first **LS** rewrite for the lhs. CNOT, and the second rewrite for the rhs. CNOT.

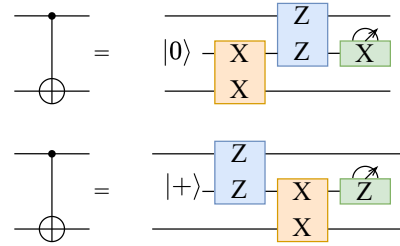


FIG. 4. The **LS** rewrite can decompose a CNOT into any of the two functionally equivalent forms. A CNOT will require an ancilla, an XX and a ZZ two-body interaction. The figure does not include the corrective terms.

We replace a sequence of measurement and reset operations in the same basis with an equivalent measurement-based implementation by using the **MR** rule (Fig. 5). This is, to the best of our knowledge, a novel quantum circuit rewrite rule. The rule is the inverse of the copy rule from the ZX-calculus [16].

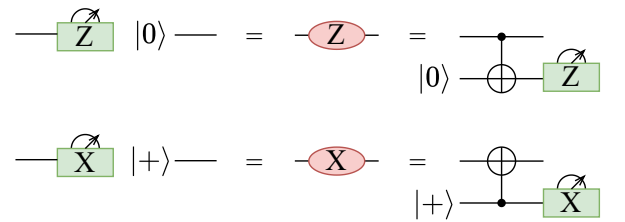


FIG. 5. The **MR** rewrite is joining a measurement and a reset in the same basis into an operation represented by the oval element (e.g. Z). Instead of explicitly measuring and resetting, we will consider the circuits on the rhs for achieving this task. The figure does not include the corrective terms.

The **MR** rewrite rule is applicable for the optimization of circuits using qubit recycling schemes[17, 18]. By merging the measurement and the reset, it is possible to commute gates across seemingly disjoint parts of the cir-

cuit. This has the potential to generate novel optimization heuristics. In particular, this approach has enabled us to derive the **FUSE** rule: we can merge two-body interactions into a single one, if a measure-and-reset acts on one of the qubits between them.

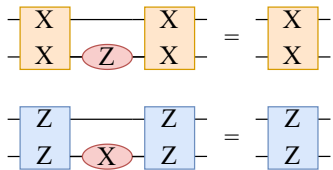


FIG. 6. The **FUSE** reduces the depth of the decomposition.

III. APPLICATIONS

Two-body interactions such as XX and ZZ are commonly used in a wide variety of quantum computations and architectures. Herein, we list some of the applications of our novel decomposition.

A. Lattice Surgery

Lattice surgery is the de facto standard way of implementing computations with QLDPC codes, such as the surface code. Therein, the logical qubits of the circuits are arranged on a 2D layout of patches (e.g. Fig. 7), and ancillary space between the patches is supporting long range multi-body measurements. The latter are expressed as generalized Pauli exponentials. The structure of the layout plays a significant role in the trade-off between a computation's speed and the required amount of physical hardware [4, 19]. Although our decomposition uses ancillae, they are readily available in most of the lattice surgery layouts considered for compiling fault-tolerant computations.

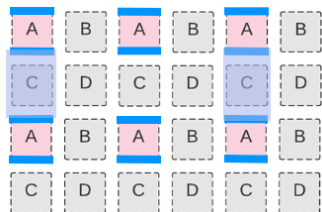


FIG. 7. Example of a lattice surgery layout, adapted from [19], which consists of pink patches for holding logical qubits, and gray patches used for allowing the logical qubits to perform XX and ZZ interactions. The blue boundaries represent the X operators of the logical qubits. The two transparent blue rectangles, covering C patches and touching on pairs of A patches, represent XX interactions. Only $1/4$ of the available patches are used for the computation.

The general compilation strategy for lattice surgery is to first decompose arbitrary algorithms into generalized Pauli exponentials [4], and then to implement these exponentials via multi-body measurements. This approach holds multiple disadvantages.

First, the resulting Pauli exponentials become very long range and cover large portions of the layout. This hinders the parallelisation of the operations [12]. Alternative compilation methods have been proposed, such as [12] (shows how to perform a constant depth CNOT using two-body measurements), and [20], which compiles to Clifford+T instead of exponentiated Paulis.

Our approach is directly applicable to the compilation method from [4] – we do not require compiling to CNOTs and then decomposing them to irregular XX and ZZ measurement patterns. The advantage of our decomposition is the very regular measurement schedule, which might enable high degrees of parallelism and massive optimizations when considering **FUSE**.

Second, long range multi-body interactions might introduce high weight correlated logical errors into the computation. Such correlated errors are difficult to decode. The advantage of using pairwise interactions is in the decoding of logical computations [21]. Nearest neighbour pairwise interactions will localize the correlated errors and will limit their weight to at most two.

B. Constant Depth Fault-Tolerant Protocols

Recently, there has been an increased interest in using only two-body interactions for implementing error-correction (e.g. [2, 3]). This is attributable to the fact that quantum computers have limited connectivity between the qubits, and very often the connectivity is not long range (e.g. [11]).

From the perspective of stabilizer QECC codes, our decomposition allows for any set of QECC stabilizers to be implemented in constant depth using pairwise XX and ZZ measurements. QECC code conversion schemes [22] can also be implemented locally using pairwise interactions. Additionally, after carefully choosing the pairwise interaction schedules, our method might lead to results similar to the ones from [23]. Our decomposition involves only two-body measurements and may find relevance to subsystem codes such as the Bacon-Shor code [24] or its Floquet version [23].

Our method is also useful to implement Clifford circuits in constant depth by using only pairwise interactions. For example, the authors of [25] show how to implement Clifford circuits fault-tolerantly in constant depth by using non-local, multi-qubit Pauli operator measurements. Our method achieves local implementation of the circuits at the cost of doubling the number of qubits.

It should be noted that the listed constructions might not be fully fault-tolerant, e.g. introduce hook errors in certain scenarios, and such analysis is left for future work.

C. NISQ computations

The recursive decomposition of Pauli exponentials (i.e. phase gadgets) using **PG** will result in a ladder of CNOTs. These ladders are frequently encountered in the circuits expressing UCCSD terms. The long range CNOT construction from [11] has already recognized that such terms can be implemented in constant depth.

Our decomposition is achieving the same constant depth, but this time using different two-body interactions. In conjunction with architectures such as [10], our decomposition can offer speed-ups to the execution of adaptive VQE [7] or other QAOA-like algorithms.

IV. CONCLUSION

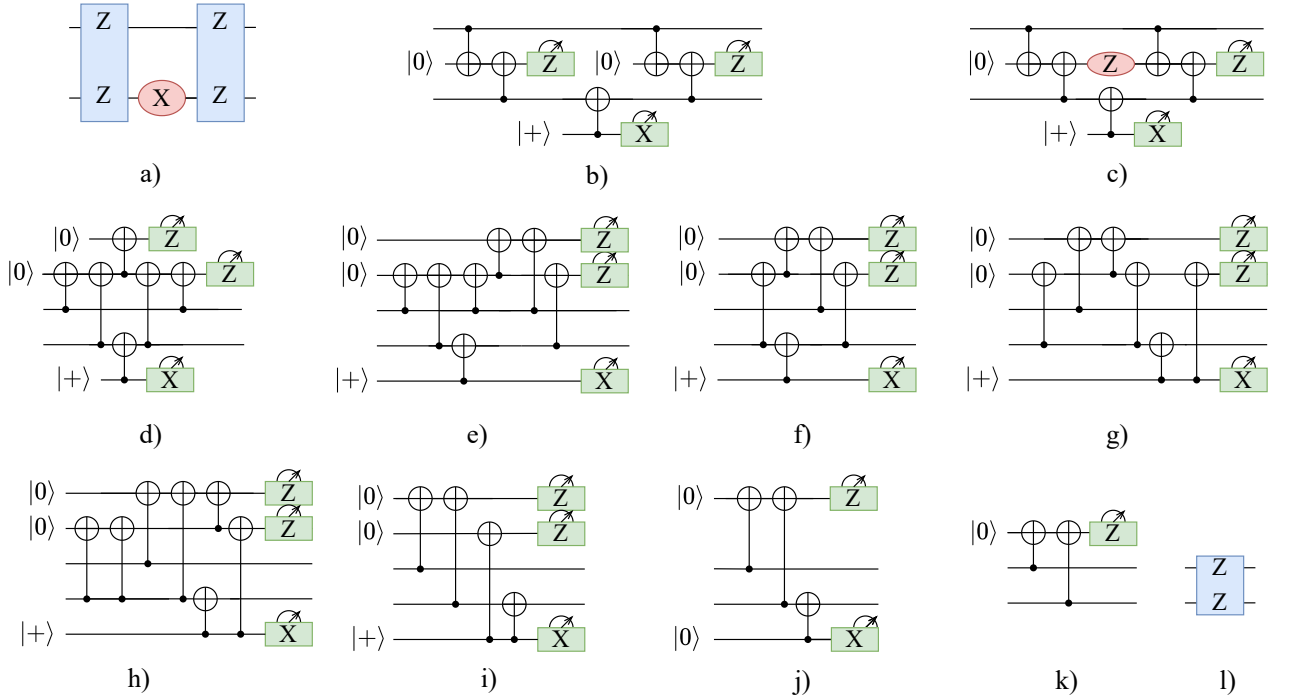
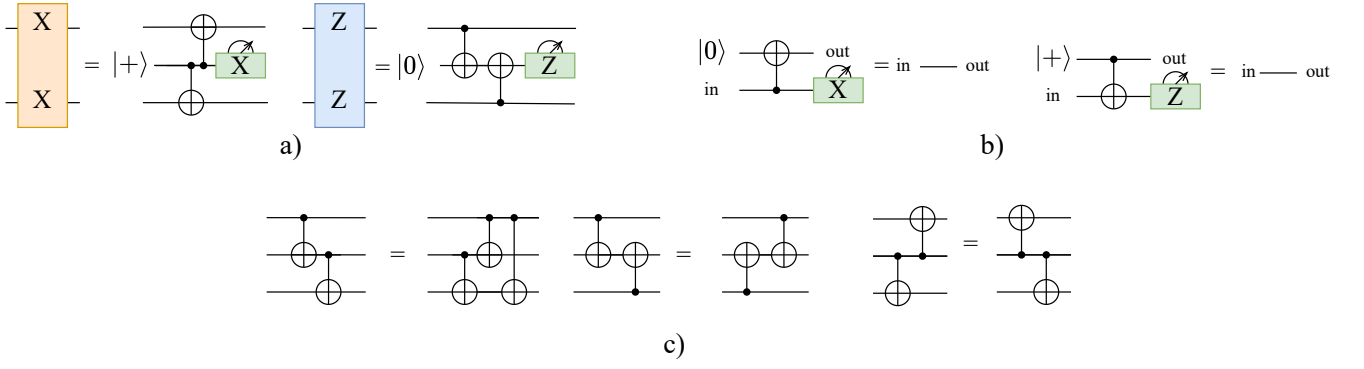
We introduced a novel decomposition of Pauli exponentials. Our method uses only local two-body interactions and is applicable to a very wide range of quantum

computing applications. Future work will focus on the fault-tolerant usage of our result.

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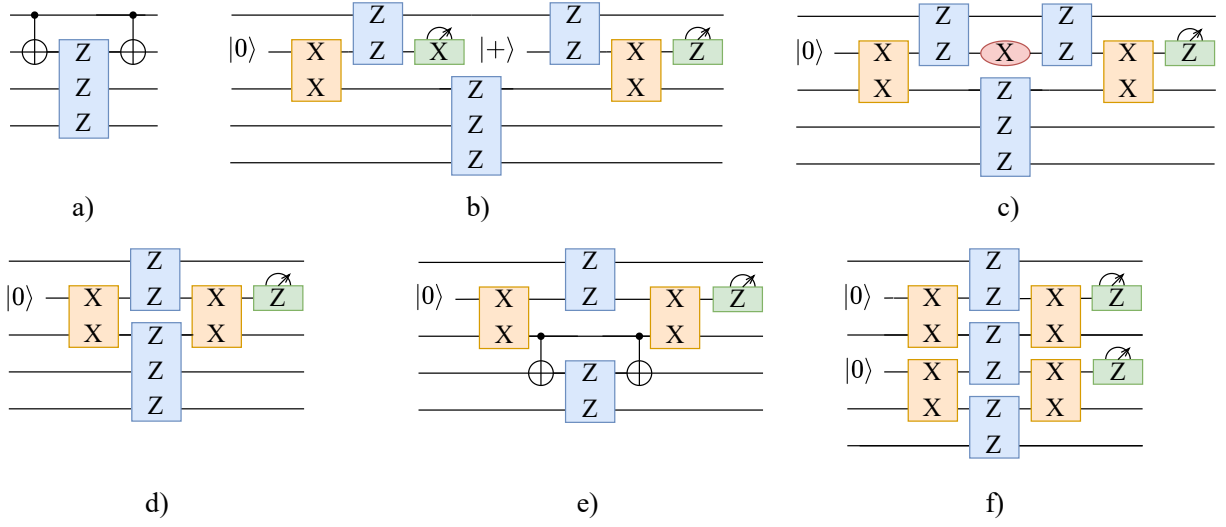


FIG. 10. Correctness of the generalized decomposition, illustrated by an example of decomposing a Pauli string of four Z terms: a) applying the **PG** rule; b) decomposing the two CNOTs using **LS**; c) the X-measurement and the initialization in $|+\rangle$ can be merged using **MR** into a red oval X element; d) the two ZZ parities surrounding the red oval are fused using **FUSE**; e) **PG** is applied again and inserts two CNOTs; f) the CNOTs are decomposed with **LS**, the measure and reset are merged with **MR** and, finally, **FUSE** is applied.