Ejercicios dinámica de Fluidos

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1 Ecuación de Navier Stokes en coordenadas cartesianas, cilíndricas y esféricas

En primer lugar definimos los factores de escala entre transformaciones de coordenadas como:

$$\left|\frac{\partial r}{\partial q_i}\right| = h_i, \qquad \hat{e}_i = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial q_i} \tag{1}$$

A partir de acá se definen los operadores diferenciales en términos de factores de escala y coordenadas ortogonales

- Gradiente

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1}\hat{q}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}\hat{q}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}\hat{q}_3$$

- Divergencia

$$\nabla\cdot\vec{F} = \frac{1}{h_1h_2h_3} = \left(\frac{\partial(F_1h_2h_3)}{\partial q_1} + \frac{\partial(h_1F_2h_3)}{\partial q_2} + \frac{\partial(h_1h_2F_3)}{\partial q_3}\right)$$

- Rotacional

$$\nabla\times\vec{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{q}_1 & h_2\hat{q}_2 & h_3\hat{q}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

Consideremos la ecuación de Navier Stokes, en primer lugar en coordenadas cartesianas

$$\rho\left[\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right] = \mu\left(u_x\frac{\partial^2 u_x}{\partial x^2} + u_y\frac{\partial^2 u_x}{\partial y^2} + u_z\frac{\partial^2 u_x}{\partial z^2}\right) - \frac{\partial P}{\partial x} \tag{2}$$

$$\rho\left[\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right] = \mu\left(u_x\frac{\partial^2 u_y}{\partial x^2} + u_y\frac{\partial^2 u_y}{\partial y^2} + u_z\frac{\partial^2 u_y}{\partial z^2}\right) - \frac{\partial P}{\partial y} \tag{3}$$

$$\rho\left[\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right] = \mu\left(u_x\frac{\partial^2 u_z}{\partial x^2} + u_y\frac{\partial^2 u_z}{\partial y^2} + u_z\frac{\partial^2 u_z}{\partial z^2}\right) - \frac{\partial P}{\partial z} \tag{4}$$

Ahora en coordenadas cilíndricas, para esto se asume que $\vec{u} = u_r\hat{r} + u_\theta\hat{\theta} + u_z\hat{z}$, por lo tanto la derivada temporal toma la forma

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(u_r\hat{r} + u_\theta\hat{\theta} + u_z\hat{z}\right) && (5)\\
&= \frac{\partial u_r}{\partial t}\hat{r} + \frac{\partial u_\theta}{\partial t}\hat{\theta} + \frac{\partial u_z}{\partial t}\hat{z} && (6)
\end{aligned}
$$

Se escribe el término $(\vec{u}\cdot\nabla)$ y $(\vec{u}\cdot\nabla)\vec{u}$

$$
\begin{aligned}
(\vec{u}\cdot\nabla) &= u_r\frac{\partial}{\partial r} + \frac{1}{r}u_\theta\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}\\
(\vec{u}\cdot\nabla)\vec{u} &= u_r\frac{\partial(u_r\hat{r})}{\partial r} + \frac{1}{r}u_\theta\frac{\partial(u_r\hat{r})}{\partial\theta} + u_z\frac{\partial(u_r\hat{r})}{\partial z}\\
&+ u_r\frac{\partial(u_\theta\hat{\theta})}{\partial r} + \frac{1}{r}u_\theta\frac{\partial(u_\theta\hat{\theta})}{\partial\theta} + u_z\frac{\partial(u_\theta\hat{\theta})}{\partial z}\\
&+ u_r\frac{\partial(u_z\hat{z})}{\partial r} + \frac{1}{r}u_\theta\frac{\partial(u_z\hat{z})}{\partial\theta} + u_z\frac{\partial(u_z\hat{z})}{\partial z}\\
&= u_r\frac{\partial u_r}{\partial r}\hat{r} + u_r^2\frac{\partial(\hat{r})}{\partial r} + \frac{u_\theta}{r}\frac{\partial(u_r)}{\partial\theta}\hat{r} + \frac{u_\theta u_r}{r}\frac{\partial(\hat{r})}{\partial\theta}u_z\frac{\partial(u_r)}{\partial z}\hat{r} + u_z u_r\frac{\partial(\hat{r})}{\partial z}\\
&+ u_r\frac{\partial u_\theta}{\partial r}\hat{\theta} + u_r u_\theta\frac{\partial(\hat{\theta})}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\theta^2}{r}\frac{\partial(\hat{\theta})}{\partial r} + u_z\frac{\partial u_\theta}{\partial z} + u_z u_\theta\frac{\partial(\hat{\theta})}{\partial z}\\
&+ u_r\frac{\partial u_z}{\partial r}\hat{z} + u_r u_z\frac{\partial(\hat{z})}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}\hat{z} + \frac{u_\theta u_z}{r}\frac{\partial(\hat{z})}{\partial\theta} + u_z\frac{\partial u_z}{\partial z}\hat{z} + u_z^2\frac{\partial(\hat{z})}{\partial z}
\end{aligned}
$$

Tomando los término correspondientes a cada componente se llegan a las tres ecuaciones en coordenadas cilndricas

$$
\begin{aligned}
&\rho\left[\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right] = -\frac{\partial P}{\partial r}\\
+\ &\mu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{u_r}{r^2}\right) && (7)\\
&\rho\left[\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + u_z\frac{\partial u_\theta}{\partial z} - \frac{u_\theta^2}{r}\right] = -\frac{\partial P}{\partial\theta}\\
+\ &\mu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right) && (8)\\
&\rho\left[\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta} + u_z\frac{\partial u_z}{\partial z}\right] = -\frac{\partial P}{\partial z}\\
+\ &\mu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) && (9)
\end{aligned}
$$

Ahora queremos escribir las ecuaciones esféricas, para esto se hace un procedimiento análogo al de las ciíndricas. Por esta razón escribimos los operadores de la derivada convectiva.

$$
\begin{aligned}
(\vec{u}\cdot\nabla) &= u_r\frac{\partial}{\partial r}+\frac{1}{r}u_\theta\frac{\partial}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\\
(\vec{u}\cdot\nabla)\vec{u} &= u_r\frac{\partial(u_r\hat{r})}{\partial r}+\frac{1}{r}u_\theta\frac{\partial(u_r\hat{r})}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial(u_r\hat{r})}{\partial\phi}\\
&+ u_r\frac{\partial(u_\theta\hat{\theta})}{\partial r}+\frac{1}{r}u_\theta\frac{\partial(u_\theta\hat{\theta})}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial(u_\theta\hat{\theta})}{\partial\phi}\\
&+ u_r\frac{\partial(u_z\hat{z})}{\partial r}+\frac{1}{r}u_\theta\frac{\partial(u_z\hat{z})}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial(u_\phi\hat{\phi})}{\partial\phi}\\
&= u_r\frac{\partial u_r}{\partial r}\hat{r}+u_r^2\frac{\partial(\hat{r})}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}\hat{r}+\frac{u_\theta u_r}{r}\frac{\partial(\hat{r})}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi}\hat{r}+\frac{u_\phi u_r}{r\sin\theta}\frac{\partial(\hat{r})}{\partial\phi}\\
&+ u_r\frac{\partial u_\theta}{\partial r}\hat{\theta}+u_ru_\theta\frac{\partial(\hat{\theta})}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}\hat{\theta}+\frac{u_\theta^2}{r}\frac{\partial(\hat{\theta})}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\hat{\theta}+\frac{u_\phi u_\theta}{r\sin\theta}\frac{\partial(\hat{\theta})}{\partial\phi}\\
&+ u_r\frac{\partial u_\phi}{\partial r}\hat{\phi}+u_ru_\phi\frac{\partial(\hat{\phi})}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}\hat{\phi}+\frac{u_\theta u_\phi}{r}\frac{\partial(\hat{\phi})}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\hat{\phi}+\frac{u_\phi^2}{r\sin\theta}\frac{\partial(\hat{\phi})}{\partial\phi}
\end{aligned}
$$

Obteniendo de esta manera las siguientes ecuaciones

$$
\begin{aligned}
&\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\theta^2+u_\phi^2}{r}\right]=-\frac{\partial P}{\partial r} \qquad (10)\\
+\quad&\mu\left(\frac{1}{r}\frac{\partial^2(ru_r)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}+\frac{\cot\theta}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}-\frac{2u_r}{r^2}-\frac{2\cot\theta}{r^2}u_\theta\right)\\
&\rho\left[\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{u_ru_\theta}{r}-\frac{u_\phi^2\cot\theta}{r}\right]=-\frac{1}{r}\frac{\partial P}{\partial\theta} \qquad (11)\\
+\quad&\mu\left(\frac{1}{r}\frac{\partial^2(ru_\theta)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi^2}+\frac{\cot\theta}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}\right)\\
&\rho\left[\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_ru_\phi}{r}+\frac{u_\theta u_\phi\cot\theta}{r}\right] \qquad (12)\\
+\quad&-\frac{1}{r\sin\theta}\frac{\partial P}{\partial\phi}+\mu\left(\frac{1}{r}\frac{\partial^2(ru_\phi)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}+\frac{\cot\theta}{r^2}\frac{\partial u_\phi}{\partial\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi^2}-\frac{u_\phi}{r^2\sin^2\theta}\right)
\end{aligned}
$$

2 Teorema de Reynolds

Este teorema es una generalización de la regla de integración de Leibniz. El teorema de Reynoldses usado para formular las reglas de conservación de la mecánica del medio continuo. En este caso, la dinámica de fluidos.

Consideremos la derivada temporal de un tensor de orden dos, sin embargo este analisis es válido para un tensor de cualquier orden,en un volumen de control arbitrario,donde este se mueve con el fluido. Se obtiene la siguiente expresión

$$
\frac{D}{Dt}\left(\int_{V(t)}T_{ij}dV\right)=\lim_{\Delta t\to 0}\left[\frac{1}{\Delta t}\left(\int_{V(t)}T_{ij}(t)dV-\int_{V(t)}T_{ij}(t)dV\right)\right] \qquad (13)
$$

Sumemos cero ...

$$
\begin{aligned}
\frac{D}{Dt}\left(\int_{V(t)} T_{ij} dV\right) &= \lim_{\Delta t \to 0}\left[\frac{1}{\Delta t}\left(\int_{V(t)} T_{ij}(t) dV - \int_{V(t)} T_{ij}(t+\Delta t) dV + \int_{V(t)} T_{ij}(t)(t+\Delta t) dV \int_{V(t)} T_{ij} dV\right)\right] \\
&= \lim_{\Delta t \to 0}\left[\int_{V(t+\Delta t)-V(t)} T_{ij}(t+\Delta t) dV\right] + \int_{V(t)} \frac{\partial T_{ij}}{\partial t} dV \qquad (14)
\end{aligned}
$$

La primera integral describe el cambio de volumen en un paso de tiempo infinitesimal , Por tanto podemos escribir

$$
\frac{D}{Dt}\left(\int_{V(t)} T_{ij} dV\right) = \lim_{\Delta t \to 0}\left[\oint_{S(t)} T_{ij}(t+\Delta t) u_k \hat{n}_k\right] + \int_{V(t)} \frac{\partial T_{ij}}{\partial t} dV \tag{15}
$$

Por el teorema de Gauus

$$
\oint_S F_{ik} n_k dS = \int_V \frac{\partial F_{ik}}{\partial x_k} \tag{16}
$$

Se llega a lo siguiente

$$
\frac{D}{Dt}\left(\int_{V(t)} T_{ij} dV\right) = \int_{V(t)} \frac{\partial}{\partial x_k}(T_{ij} u_k) dV + \int_{V(t)} \frac{\partial T_{ij}}{\partial t} dV \tag{17}
$$

Finalmente

$$
\boxed{\frac{D}{Dt}\left(\int_{V(t)} T_{ij} dV\right) = \int_{V(t)} \left[\frac{\partial}{\partial x_k}(T_{ij} u_k) + \frac{\partial T_{ij}}{\partial t}\right] dV} \tag{18}
$$

Como ejemplo se considera que la masa es constante , por lo tanto un tensor de orden cero $T_{ij} \to \rho$

$$
\frac{DM}{Dt} = 0 = \frac{D}{Dt}\left(\int_{V(t)} \rho dV\right) = \int_{V(t)} \left[\frac{\partial}{\partial x_k}(u_k \rho) + \frac{\partial \rho}{\partial t}\right] dV \tag{19}
$$

$$
\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_x}(u_k \rho) = 0} \tag{20}
$$