Demostrar las siguientes TL:

1. $\mathcal{L}\{e^{bt}u(t)\} = ?$
2. $\mathcal{L}\{\cos(\omega_0 t)u(t)\} = ?$
3. $\mathcal{L}\{\sin(\omega_0 t)u(t)\} = ?$
4. $\mathcal{L}\{e^{bt}\cos(\omega_0 t)u(t)\} = ?$
5. $\mathcal{L}\{e^{bt}\sin(\omega_0 t)u(t)\} = ?$

1. $$\mathcal{L}\{e^{bt}u(t)\} = \int_{-\infty}^{\infty} e^{bt}u(t)e^{-st}\,dt = \int_0^{\infty} e^{bt}e^{-st}\,dt = \int_0^{\infty} e^{-t(s-b)}\,dt$$

$$= \frac{1}{-(s-b)}\left[\lim_{t\to\infty} e^{-t(s-b)} - 1\right] = \frac{1}{s-b}\left[1 - \lim_{t\to\infty} e^{-t(s-b)}\right]$$

ROC: $s = \sigma + j\omega \Rightarrow \frac{1}{s-b}\left[1 - \lim_{t\to\infty} e^{-t(\sigma+j\omega-b)}\right] = \frac{1}{s-b}\left[1 - \lim_{t\to\infty} e^{-t(j\omega - \mathrm{Im}\{b\})}e^{-t(\sigma-\mathrm{Re}\{b\})}\right]$

$\sigma - \mathrm{Re}\{b\} > 0 \Rightarrow \sigma > \mathrm{Re}\{b\}$ Para que converja la transformada $= \frac{1}{s-b}$

2. $\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$ $\quad \mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$

$$\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \mathcal{L}\left\{\frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)\right\} = \frac{1}{2}\mathcal{L}\{e^{j\omega_0 t}u(t)\} + \frac{1}{2}\mathcal{L}\{e^{-j\omega_0 t}u(t)\}$$

En este caso $b = \pm j\omega_0$ y se aplica que $\mathcal{L}\{e^{\pm bt}u(t)\} = \frac{1}{s \mp b}$

$$\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \frac{1}{2}\left(\frac{1}{s-j\omega_0}\right) + \frac{1}{2}\left(\frac{1}{s+j\omega_0}\right) = \frac{1}{2}\left(\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right)$$

$$= \frac{1}{2}\left(\frac{(s+j\omega_0)+(s-j\omega_0)}{(s-j\omega_0)(s+j\omega_0)}\right) = \frac{1}{2}\left(\frac{2s}{s^2-(j\omega_0)^2}\right) = \frac{1}{2}\left(\frac{2s}{s^2+\omega_0^2}\right) = \frac{s}{s^2+\omega_0^2}$$

ROC: $\mathrm{Re}\{s\} > 0$

3. $\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$ $\quad \mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$

$$\mathcal{L}\{\sin(\omega_0 t)u(t)\} = \mathcal{L}\left\{\left(\frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}\right)u(t)\right\} = \frac{1}{2j}\mathcal{L}\{e^{j\omega_0 t}u(t)\} - \frac{1}{2j}\mathcal{L}\{e^{-j\omega_0 t}u(t)\}$$

$b = \pm j\omega_0$ y $\mathcal{L}\{e^{\pm bt}u(t)\} = \frac{1}{s \mp b}$

$\mathcal{L}\{\sin(\omega_0 t)u(t)\} = \frac{1}{2j}\left(\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right) = \frac{1}{2j}\left(\frac{(s+j\omega_0)-(s-j\omega_0)}{(s-j\omega_0)(s+j\omega_0)}\right) = \frac{1}{2j}\left(\frac{2j\omega_0}{s^2+\omega_0^2}\right)$

$= \frac{\omega_0}{s^2+\omega_0^2}$

ROC: $Re\{s\} > 0$

4. $\mathcal{L}\{e^{bt}f(t)\} = F(s-b)$; $\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \frac{s}{s^2+\omega_0^2}$

$\mathcal{L}\{e^{bt}\cos(\omega_0 t)u(t)\} = \mathcal{L}\{\cos(\omega_0 t)u(t)\}$ con $s = s-b$ por desplazamiento del exponencial

$\mathcal{L}\{e^{bt}\cos(\omega_0 t)u(t)\} = \frac{s-b}{(s-b)^2+\omega_0^2}$

ROC: $Re\{s\} > Re\{b\}$ Debido al desplazamiento del e^{bt}

5. $\mathcal{L}\{e^{bt}f(t)\} = F(s-b)$; $\mathcal{L}\{\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$

$\mathcal{L}\{e^{bt}\sin(\omega_0 t)u(t)\} = \mathcal{L}\{\sin(\omega_0 t)u(t)\}$ con $s = s-b$ por el desplazamiento dado por el e^{bt}

$\mathcal{L}\{e^{bt}\sin(\omega_0 t)u(t)\} = \frac{\omega_0^2}{(s-b)^2+\omega_0^2}$

ROC: $Re\{s\} > Re\{b\}$ Debido al desplazamiento del e^{bt}