Green's Functions: Intuition, Jump Condition, and a Simple Example

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What is a Green's function?

- Consider a linear boundary-value problem on (a, b):

$$L[u](x) = f(x), \quad \text{with boundary conditions (BCs).}$$

- The Green's function $G(x, \xi)$ is defined by

$$L_x\, G(x, \xi) = \delta(x - \xi),$$

  subject to the same BCs in x as u.

- Once G is known, the solution is the representation formula:

$$u(x) = \int_a^b G(x, \xi)\, f(\xi)\, d\xi.$$

Why does this work? (1D sketch)

- For linear L (e.g., Sturm–Liouville), integrate against $G(\cdot, \xi)$ and integrate by parts.
- Boundary terms vanish because G satisfies the same BCs as u.
- The δ picks out the value at $x = \xi$, yielding the integral representation.
- For self-adjoint L, one usually has the symmetry $G(x, \xi) = G(\xi, x)$.

Continuity and the jump condition

Let G solve $L_x G = \delta(x-\xi)$ for $L = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Then:

- $G(x,\xi)$ is continuous at $x=\xi$.
- $G'(x,\xi)$ has a jump of size 1 at $x=\xi$:

$$G'(\xi^+,\xi) - G'(\xi^-,\xi) = 1.$$

Derivation:

$$\begin{aligned} \int_{\xi-\varepsilon}^{\xi+\varepsilon} G''(x,\xi)\,dx &= \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi)\,dx = 1, \\ \Rightarrow \quad G'(\xi^+,\xi) - G'(\xi^-,\xi) &= 1. \end{aligned}$$

This means G' behaves like a Heaviside step across $x=\xi$, and G'' contains the Dirac delta.

Practical recipe in 1D (Sturm–Liouville)

For $L[y] = -(py')' + qy$ on (a, b) with homogeneous BCs:

1. Find y_1: solution of $L[y] = 0$ satisfying the left BC at a.
2. Find y_2: solution of $L[y] = 0$ satisfying the right BC at b.
3. Let $W(\xi) = p(\xi)\big(y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi)\big)$ (Wronskian times p).
4. Then

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)\, y_2(\xi)}{W(\xi)}, & x < \xi, \\ \dfrac{y_1(\xi)\, y_2(x)}{W(\xi)}, & x > \xi. \end{cases}$$

This G is continuous and enforces the jump in $p(x)G'(x,\xi)$ of size 1 at $x = \xi$.

Example: $u''(x) = f(x)$ on $(0,1)$ with $u(0) = u(1) = 0$

- Here $L = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $p \equiv 1$, $q \equiv 0$. Homogeneous Dirichlet BCs.
- Fundamental solutions of $L[y] = 0$: $y_1(x) = x$ (satisfies $y_1(0) = 0$), $y_2(x) = 1 - x$ (satisfies $y_2(1) = 0$).
- $W(\xi) = y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi) = -1$.
- Therefore

$$G(x,\xi) = \begin{cases} x(1-\xi), & x < \xi, \\ \xi(1-x), & x > \xi. \end{cases}$$

Solution formula and a check

Representation

$$u(x) = \int_0^1 G(x,\xi)\, f(\xi)\, d\xi.$$

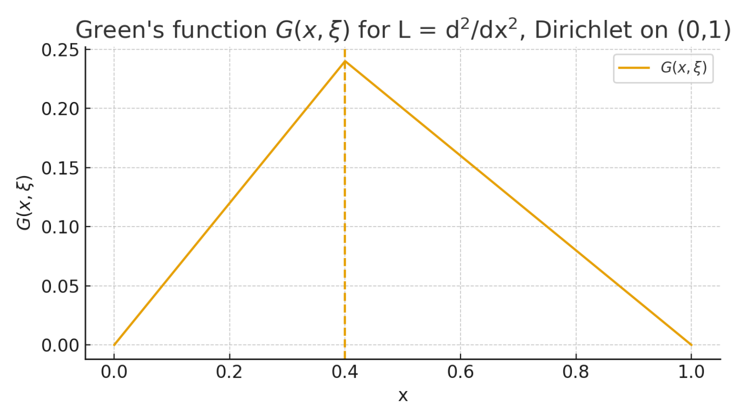
Test case $f(x) \equiv 1$

Compute

$$u(x) = \int_0^x \xi(1-x)\, d\xi + \int_x^1 x(1-\xi)\, d\xi = \frac{x(1-x)}{2}.$$

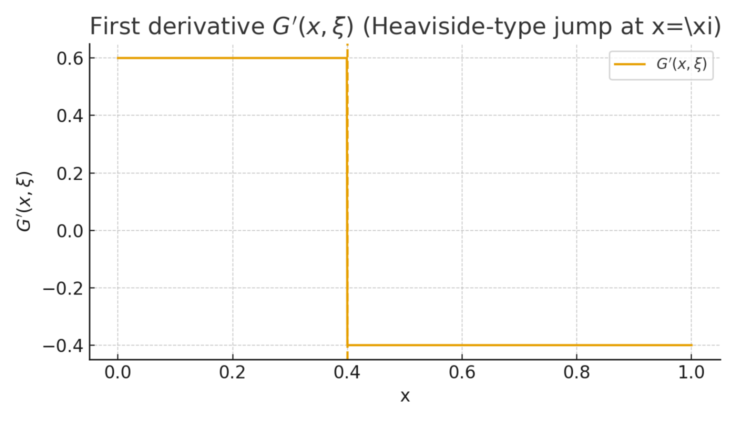
This matches the direct solution of $u'' = 1$ with $u(0) = u(1) = 0$.

Visualization: $G(x, \xi)$ for $\xi = 0.4$



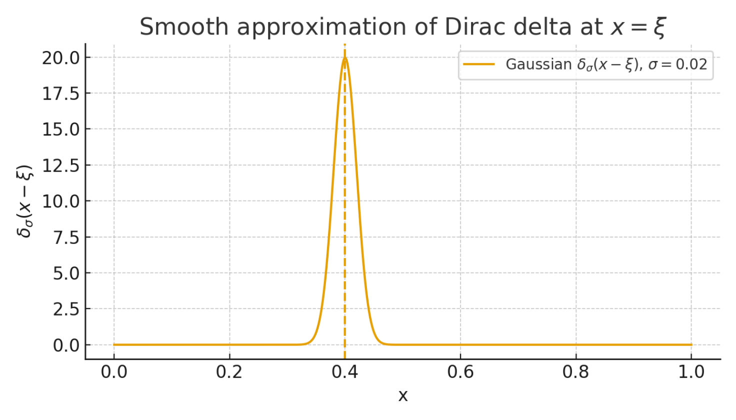
G is continuous; the slope changes at $x = \xi$.

Visualization: $G'(x, \xi)$ jump at $x = \xi$



G' behaves like a Heaviside step; the jump size is 1.

Visualization: δ as a smooth peak

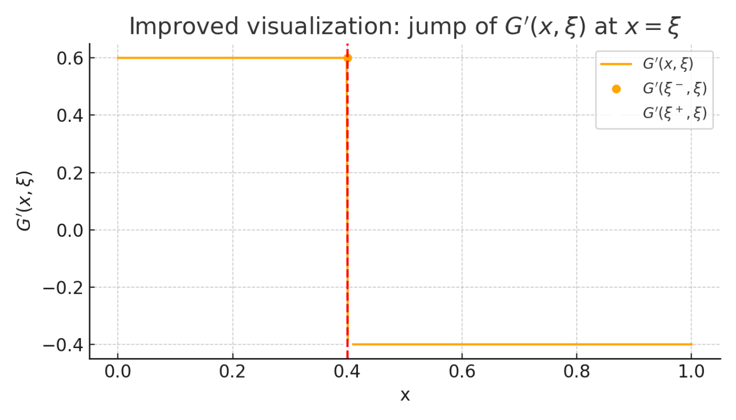


A Gaussian $\delta_\sigma(x-\xi)$ approximates the Dirac delta as $\sigma \to 0$ (area $= 1$).

Extensions and remarks

- Different BCs (Neumann/Robin) change the construction and the jump is in $p\,G'$.
- In higher dimensions, L could be Poisson/Helmholtz; G becomes a fundamental solution with BCs.
- On unbounded domains, solutions reduce to *convolutions* with the free-space Green's function.
- Spectral viewpoint: G expands in eigenfunctions $\{\phi_n\}$ with eigenvalues $\{\lambda_n\}$ as $G(x,\xi) = \sum_n \phi_n(x)\phi_n(\xi)/\lambda_n$ (when permissible).

Improved visualization: $G'(x, \xi)$ jump at $x = \xi$



- G' is piecewise constant: $1 - \xi$ for $x < \xi$, and $-\xi$ for $x > \xi$.
- At $x = \xi$, the value is not defined; we represent this with open/closed circles.
- The jump size is exactly 1: $G'(\xi^+, \xi) - G'(\xi^-, \xi) = 1$.

Example with $f(x) = \sin(\pi x)$

Integral representation

$$u(x) = \int_0^1 G(x, \xi)\, \sin(\pi \xi)\, d\xi.$$

- Using symmetry of G, this integral can be evaluated.
- The result is
$$u(x) = \frac{\sin(\pi x)}{\pi^2}.$$
- This matches the direct solution of $u'' = \sin(\pi x)$ with $u(0) = u(1) = 0$.

Other boundary conditions

- For Neumann or Robin BCs, the Green's function is still built piecewise from two fundamental solutions.
- The continuity at $x = \xi$ remains:

$$G(\xi^+, \xi) = G(\xi^-, \xi).$$

- The jump condition is modified:

$$p(\xi)\, G'(\xi^+, \xi) - p(\xi)\, G'(\xi^-, \xi) = 1,$$

where $p(x)$ is the coefficient in the Sturm–Liouville operator.
- Example: pure Neumann BCs lead to Green's functions that are not unique (constants in the kernel).