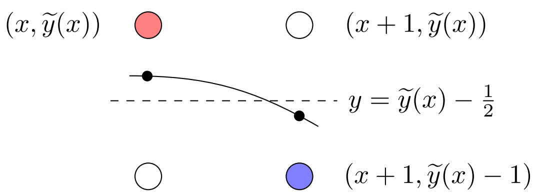
First-Order Incremental Ellipses

(Analogously to Script)

Region 1



Decision for going south-east:

$$y(x+1) < \widetilde{y}(x) - \frac{1}{2}$$

y, $\widetilde{y}$ and x all ≥ 0, therefore

$$y^2(x+1) < \widetilde{y}^2(x) - \widetilde{y}(x) + \frac{1}{4}$$

Then, to not having to compute $y^2(x+1)$, with the ellipse formula

$$y^2(x) = \frac{1}{a^2}\left(a^2b^2 - b^2x^2\right) = b^2 - \frac{b^2}{a^2}x^2$$

we get

$$\begin{aligned}
b^2 - \frac{b^2}{a^2}(x+1)^2 &< \widetilde{y}^2(x) - \widetilde{y}(x) + \frac{1}{4} \\
\underbrace{b^2 - \frac{b^2}{a^2}x^2}_{y^2(x)} - 2\frac{b^2}{a^2}x - \frac{b^2}{a^2} &< \widetilde{y}^2(x) - \widetilde{y}(x) + \frac{1}{4} \\
y^2(x) - 2\frac{b^2}{a^2}x - \frac{b^2}{a^2} &< \widetilde{y}^2(x) - \widetilde{y}(x) + \frac{1}{4}
\end{aligned}$$

And finally

$$d(x) := y^2(x) - \widetilde{y}^2(x) + \widetilde{y}(x) - \frac{b^2}{a^2}(2x+1) - \frac{1}{4} < 0 \tag{1}$$

The decision $d(x) < 0$ can be made without having to compute $y^2(x+1)$ but still requires $y^2(x)$. Now find an incremental update for $d(x)$ to bypass computation of $y^2(x)$ in every step.

East

$$x \to x+1 \quad : \quad \widetilde{y}(x+1) = \widetilde{y}(x)$$

$$\begin{aligned}
d(x+1) &= y^2(x+1) - \widetilde{y}^2(x+1) + \widetilde{y}(x+1) - \frac{b^2}{a^2}(2x+3) - \frac{1}{4} \\
&= y^2(x) - \frac{b^2}{a^2}(2x+1) - \widetilde{y}^2(x) + \widetilde{y}(x) - \frac{b^2}{a^2}(2x+3) - \frac{1}{4} \\
&= d(x) - \frac{b^2}{a^2}(2x+3) \qquad (2)
\end{aligned}$$

South-East

$$x \to x+1 \quad : \quad \widetilde{y}(x+1) = \widetilde{y}(x) - 1$$

$$\begin{aligned}
d(x+1) &= y^2(x+1) - \widetilde{y}^2(x+1) + \widetilde{y}(x+1) - \frac{b^2}{a^2}(2x+3) - \frac{1}{4} \\
&= y^2(x) - \frac{b^2}{a^2}(2x+1) - (\widetilde{y}(x)-1)^2 + \widetilde{y}(x) - 1 - \frac{b^2}{a^2}(2x+3) - \frac{1}{4} \\
&= y^2(x) - \frac{b^2}{a^2}(2x+1) - \widetilde{y}^2(x) + 2\widetilde{y}(x) - 1 + \widetilde{y}(x) - 1 - \frac{b^2}{a^2}(2x+3) - \frac{1}{4} \\
&= d(x) + 2\widetilde{y}(x) - 2 - \frac{b^2}{a^2}(2x+3) \\
&= d(x) + 2(\widetilde{y}(x) - 1) - \frac{b^2}{a^2}(2x+3) \\
&= d(x) + 2\widetilde{y}(x+1) - \frac{b^2}{a^2}(2x+3) \qquad (3)
\end{aligned}$$

Note: For an actual implementation it matters, whether the y coordinate is decremented before or after the computation of $d(x+1)$. This is reflected in the last step of the derivation. In this case, decrementing first, then computing $d(x+1)$ saves a few arithmetic operations (this is not really the case for x here).

Kick-off

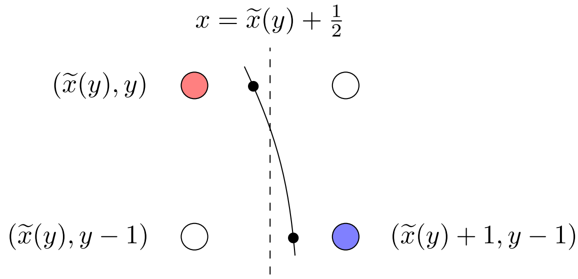
In order to start this incremental scheme, $d(x_0)$ has to be initialized correctly. For an ellipse around $(0,0)$, the first point on the ellipse is given as $(0,b)$. With Eq.(1):

$$d(x_0) = b^2 - b^2 + b - \frac{b^2}{a^2} - \frac{1}{4} = b - \frac{b^2}{a^2} - \frac{1}{4}$$

Note: As we are only interested in whether $d(x)$ is larger or smaller than 0, the $d(x)$ and their increments in Eqs.(2) and (3) can be scaled by constant factors, such as a^2 or 4, ultimately allowing integer arithmetic everywhere.

Note: Region switch at $a^2y = b^2x$. The question whether to switch as soon as $\left(x+1, y-\frac{1}{2}\right)$ falls into region 2 or as soon as the first drawn point falls into region 2 (or even other conditions; one way might be to draw bottom-up) is kind of esoteric.

Region 2



Decision for going south:

$$x(y-1) < \widetilde{x}(y) + \frac{1}{2}$$

x, $\widetilde{x}$ and y all ≥ 0, therefore

$$x^2(y-1) < \widetilde{x}^2(y) + \widetilde{x}(y) + \frac{1}{4}$$

Then, to not having to compute $x^2(y-1)$, with the ellipse formula

$$x^2(y) = \frac{1}{b^2}\left(a^2b^2 - a^2y^2\right) = a^2 - \frac{a^2}{b^2}y^2$$

we get

$$\begin{aligned} a^2 - \frac{a^2}{b^2}(y-1)^2 &< \widetilde{x}^2(y) + \widetilde{x}(y) + \frac{1}{4} \\ \underbrace{a^2 - \frac{a^2}{b^2}y^2}_{x^2(y)} + 2\frac{a^2}{b^2}y - \frac{a^2}{b^2} &< \widetilde{x}^2(y) + \widetilde{x}(y) + \frac{1}{4} \\ x^2(y) + 2\frac{a^2}{b^2}y - \frac{a^2}{b^2} &< \widetilde{x}^2(y) + \widetilde{x}(y) + \frac{1}{4} \end{aligned}$$

And finally

$$d(y) := x^2(y) - \widetilde{x}^2(y) - \widetilde{x}(y) + \frac{a^2}{b^2}(2y-1) - \frac{1}{4} < 0 \tag{4}$$

The decision $d(y) < 0$ can be made without having to compute $x^2(y-1)$ but still requires $x^2(y)$. Now find an incremental update for $d(y)$ to bypass computation of $x^2(y)$ in every step.

South-East

$$y \to y-1 \quad : \quad \widetilde{x}(y-1) = \widetilde{x}(y)+1$$

$$\begin{aligned}
d(y+1) &= x^2(y-1) - \widetilde{x}^2(y-1) - \widetilde{x}(y-1) + \frac{a^2}{b^2}(2y-3) - \frac{1}{4} \\
&= x^2(y) + \frac{a^2}{b^2}(2y-1) - (\widetilde{x}(y)+1)^2 - \widetilde{x}(y) - 1 + \frac{a^2}{b^2}(2y-3) - \frac{1}{4} \\
&= x^2(y) + \frac{a^2}{b^2}(2x+1) - \widetilde{x}^2(y) - 2\widetilde{x}(y) - 1 - \widetilde{x}(y) - 1 + \frac{a^2}{b^2}(2y-3) - \frac{1}{4} \\
&= d(y) - 2\widetilde{x}(y) - 2 + \frac{a^2}{b^2}(2y-3) \\
&= d(y) - 2(\widetilde{x}(y)+1) + \frac{a^2}{b^2}(2y-3) \\
&= d(y) - 2\widetilde{x}(y-1) + \frac{a^2}{b^2}(2y-3) \qquad (5)
\end{aligned}$$

Note: For an actual implementation it matters, whether the x coordinate is incremented before or after the computation of $d(y+1)$. This is reflected in the last step of the derivation. In this case, incrementing first, then computing $d(y{+}1)$ saves a few arithmetic operations (this is not really the case for y here).

South

$$y \to y-1 \quad : \quad \widetilde{x}(y-1) = \widetilde{x}(y)$$

$$\begin{aligned}
d(y+1) &= x^2(y-1) - \widetilde{x}^2(y-1) - \widetilde{x}(y-1) + \frac{a^2}{b^2}(2y-3) - \frac{1}{4} \\
&= x^2(y) + \frac{a^2}{b^2}(2y-1) - \widetilde{x}^2(y) - \widetilde{x}(y) + \frac{a^2}{b^2}(2y-3) - \frac{1}{4} \\
&= d(y) + \frac{a^2}{b^2}(2y-3) \qquad (6)
\end{aligned}$$

Kick-off

Let (x_e, y_e) be the last computed point from region 1. The initialization of $d(y_0)$ is then given, using Eq.(4), as:

$$\begin{aligned}
d(y_0) &= x^2(y_e) - x_e^2 - x_e + \frac{a^2}{b^2}(2y_e-1) - \frac{1}{4} \\
&= a^2 - \frac{a^2}{b^2}y_e^2 - x_e^2 - x_e + \frac{a^2}{b^2}(2y_e-1) - \frac{1}{4}
\end{aligned}$$

Note: Again we are only interested in whether $d(y)$ is larger or smaller than 0. The $d(y)$ and their increments in Eqs.(5) and (6) can be scaled by constant factors, such as b^2 or 4, ultimately allowing integer arithmetic everywhere.