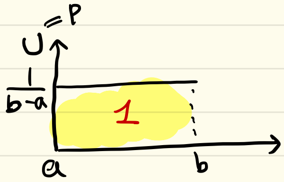
기계인공지능 HW 08 sol

Problem 1

uniform distribution



$$\text{mean} = E(x) = \int_a^b x \cdot U(x;a,b)\,dx = \int_a^b \frac{x}{b-a}\,dx$$

$$= \frac{1}{b-a} \cdot \left[\frac{x^2}{2}\right]_a^b = \frac{a+b}{2}$$

$$\text{Var} = \text{var}(x) = E(x^2) - [E(x)]^2$$

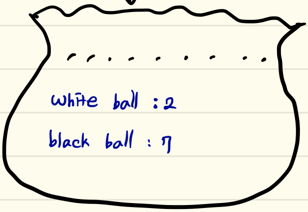
$$= \int_a^b x^2\, U(x;a,b)\,dx - \left(\frac{a+b}{2}\right)^2$$

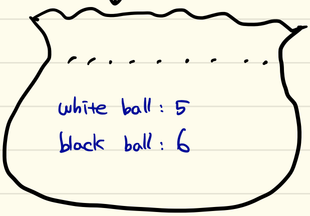
$$= \int_a^b \frac{x^2}{b-a}\,dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} = \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$$

Problem 2

Coin toss : Head / Tail





heads $\rightarrow$ white ball selected P ? (conditional prob)

Let define probability space X, Y

X : coin toss (fair)

Y : color of ball (black or white)

Then, $P(Y=w) = P(Y=w \mid X=H)P(X=H) + P(Y=w \mid X=T)P(X=T)$

$P(Y=w \mid X=H) = \frac{2}{9}$, $P(Y=w \mid X=T) = \frac{5}{11}$

$P(Y=w) = \frac{2}{9} \times \frac{1}{2} + \frac{5}{11} \times \frac{1}{2} = \frac{67}{198}$

Problem 3

Conditional probability mass function of X given that Y=1

$$\Rightarrow p(X \mid Y=1) = \begin{cases} P(X=1 \mid Y=1) \\ P(X=2 \mid Y=1) \end{cases}$$

$$P(X=1 \mid Y=1) = \frac{p(1,1)}{P(Y=1)} = \frac{p(1,1)}{p(1,1)+p(2,1)}$$

$$= \frac{0.5}{0.6} = \frac{5}{6}$$

$$P(X=2 \mid Y=1) = \frac{p(2,1)}{P(Y=1)} = \frac{0.1}{0.6} = \frac{1}{6}$$

Thus,

$$p(X \mid Y=1) = \begin{cases} P(X=1 \mid Y=1) = \frac{5}{6} \\ P(X=2 \mid Y=1) = \frac{1}{6} \end{cases}$$

problem 4

$P_A(H) = 0.7$, $P_B(H) = 0.4$

(a) $$\frac{P_B(H)\left(1-P_A(H)\right)}{P_A(H)\left(1-P_B(H)\right)+P_B(H)\left(1-P_A(H)\right)}$$

$$= \frac{0.4 \times 0.3}{0.7 \times 0.6 + 0.4 \times 0.3} = \frac{2}{9}$$

(b) $$\frac{P_B(H)\left[P_A(H) + 1 - P_A(H)\right]}{P_B(H)P_A(H) + P_B(H)\left(1-P_A(H)\right) + P_A(H)\left(1-P_B(H)\right)}$$

$$= \frac{0.4}{0.4 \times 0.7 + 0.4 \times 0.3 + 0.7 \times 0.6} = \frac{20}{41}$$

Problem 5

① $E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = \sum_x \sum_y (x+y)\, Pr(X=x,\ Y=y)$$

$$= \sum_x \sum_y x\, Pr(X=x, Y=y) + \sum_x \sum_y y\, Pr(X=x, Y=y)$$

$$= \sum_x x \sum_y Pr(X=x, Y=y) + \sum_y y \sum_x Pr(X=x, Y=y)$$

$$= \sum_x x\, Pr(X=x) + \sum_y y\, Pr(Y=y)$$

$$= E[X] + E[Y]$$

② $Var[X+Y] = Var[X] + Var[Y]$

$$Var[X+Y] = E[(X+Y)^2] - (E[X+Y])^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - (\mu_x^2 + \mu_y^2 + 2\mu_x\mu_y)$$

$$= (E[X^2] - \mu_x^2) + (E[Y^2] - \mu_y^2) + 2(\cancel{E[XY] - \mu_x\mu_y})$$ independent variable X, Y

$$= Var[X] + Var[Y] + \cancel{2cov(X,Y)}$$

problem6

$E[A] = E[B] = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} = \frac{6\cdot 7}{2}\cdot\frac{1}{6} = \frac{7}{2}$

$Cov(x, y) = E[(x-\mu_x)(y-\mu_y)]$

$x = A+B$
$y = A-B$ $\Rightarrow$ $E[A+B] = E[A] + E[B] = \frac{7}{2} + \frac{7}{2} = 7 = \mu_x$
$E[A-B] = E[A] - E[B] = \frac{7}{2} - \frac{7}{2} = 0 = \mu_y$

$Cov(x,y) = E[xy + \cancel{\mu_x\mu_y} - \cancel{x\mu_y} - y\mu_x]$

since $E[A^2] = E[B^2]$

$= E[A^2] - E[B^2] - 7E[y] = \cancel{E[A^2] - E[B^2]} - \cancel{7\mu_y} = 0$