problem 1.

(a)

Need to show

$$(T_1+T_2)(ax+by) = a(T_1+T_2)(x) + b(T_1+T_2)(y)$$

$$\begin{aligned}(T_1+T_2)(ax+by) &= T_1(ax+by) + T_2(ax+by) \\ &= aT_1(x) + aT_2(x) + bT_1(y) + bT_2(y) \\ &= a(T_1(x)+T_2(x)) + b(T_1(x)+T_2(y)) \\ &= a(T_1+T_2)x + b(T_1+T_2)y\end{aligned}$$

$\therefore$ (T_1+T_2) is a linear mapping

(b)

Need to show

$$(T_1 \cdot T_2)(ax+by) = a(T_1 \cdot T_2)(x) + b(T_1 \cdot T_2)(y)$$

$$\begin{aligned}(T_1 \cdot T_2)(ax+by) &= T_1(T_2(ax+by)) \\ &= T_1(aT_2(x)+bT_2(y)) \\ &= a(T_1 \cdot T_2)(x) + b(T_1 \cdot T_2)(y)\end{aligned}$$

$\therefore$ $T_1 \cdot T_2$ is a linear mapping problem2

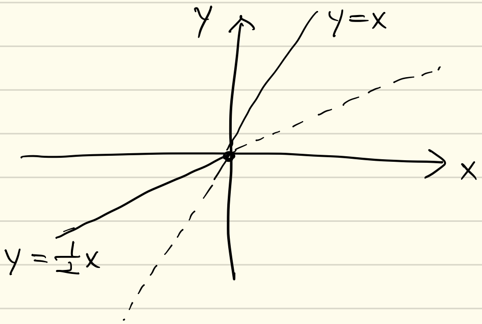
(a)

$f(1+2) = f(3) = c$

$f(1) + f(2) = 2c$

if $c \neq 0$, f is not linear mapping

(b)



$f(-2+1) = f(-1) = -\frac{1}{2}$

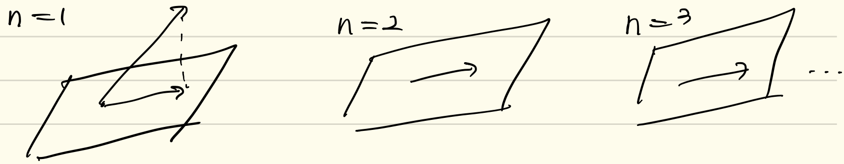
$f(-2) + f(1) = -1+1 = 0$

$\rightarrow f(-2+1) \neq f(-2) + f(1)$

$\therefore$ f is not linear mapping.

problem 3.

(a)



Once a vector projected onto a space, further projection transforms nothing.

$\rightarrow \quad P = P^n \quad (n \geq 2)$

$\therefore \quad P = P^2$

(b) Note that

$Px \perp (y - Py) \quad \forall x, y \in \mathbb{R}^n$

$\rightarrow \quad x^T P^T (y - Py) = 0$

$\rightarrow \quad x^T P^T y - x^T P^T P y = 0$

$\rightarrow \quad x^T (P^T - P^T P) y = 0$

So, $\quad P^T = P^T P$

$\rightarrow \quad P^T$ is symmetric

$\rightarrow \quad P$ is symmetric.

Problem 4

$F_{k+2} = F_{k+1} + F_k$ is the second order linear system.

$$\rightarrow \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

Eigen analysis)

$$\det(A - \lambda I) = 0 \longrightarrow \quad 1-\lambda \quad 1 \quad = 0$$

$$\rightarrow \quad \lambda^2 - \lambda - 1 = 0$$

$$\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{5}}{2}$$

x_1)

$$\begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{\quad}{2} \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = 0$$

so $\quad x_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$

x_2) Replacing $\sqrt{5} \to -\sqrt{5}$ to the previous result gives.

$$x_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

So, $$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$$= c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n x_1 + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n x_2$$

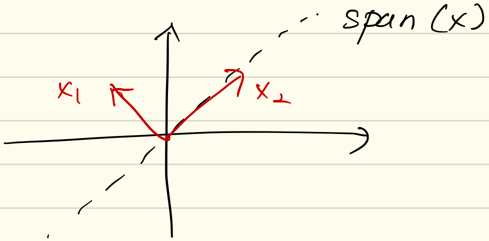
Then $$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{c_1 \left(\frac{1+\sqrt{5}}{2} \right)^{n+1}}{c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n}$$

$$= \frac{1+\sqrt{5}}{2}$$

($*$ $\lim_{n\to\infty} \left(\frac{1-\sqrt{5}}{2} \right)^n = 0$ since $\left| \frac{1-\sqrt{5}}{2} \right| < 1$)

Problem 5

(a)



We have eigen pairs (λ_1, x_1), (λ_2, x_2)

$$(\lambda_1, x_1) = \left(0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

$$(\lambda_2, x_2) = \left(1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

So
$$A = [x_1 | x_2] \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} [x_1 | x_2]^{-1}$$
$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(b)
$$A = \frac{xx^T}{\|x\|} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$