System of Linear Equations

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Table of Contents

1. System of Linear Equations

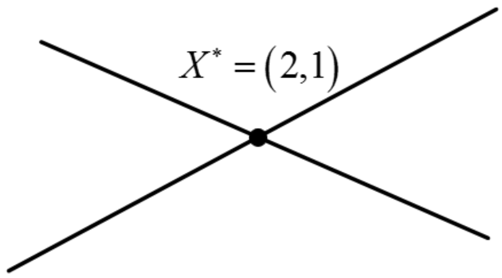
1. well-determined linear systems
2. under-determined linear systems
3. over-determined linear systems

1.1. Well-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_1^* &= 2 \\ x_2^* &= 1 \end{aligned}$$

- Geometric analysis



- Generalize

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \quad \overset{\text{Matrix form}}{\Longrightarrow} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$AX = B$$

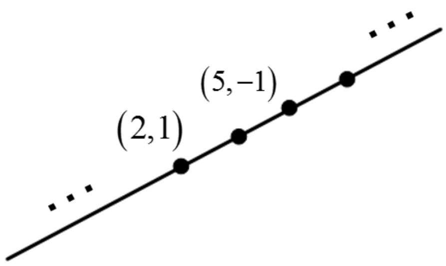
$$\therefore \quad X^* = A^{-1}B \quad \text{if } A^{-1} \text{ exists}$$

1.2. Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \quad \Longrightarrow \quad \text{Many solutions}$$

- Geometric analysis



- Generalize

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \overset{\text{Matrix form}}{\Longrightarrow} \quad \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

$$A^T X = B$$

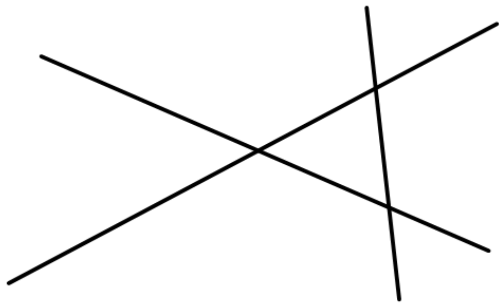
$$\therefore \quad \text{Many Solutions when } A \text{ is fat}$$

1.3. Over-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \quad \Longrightarrow \quad \text{No solutions} \\ x_1 + x_2 &= 4 \end{aligned}$$

- Geometric analysis



- Generalize

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \\ a_{31}x_1 + a_{32}x_2 &= b_3 \end{aligned} \quad \underset{\Longrightarrow}{\text{Matrix form}} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$AX = B$$

$$\therefore \quad \text{No Solutions when } A \text{ is skinny}$$

Summary of Linear Systems

$$AX = B$$

- Square: Well-determined Linear Systems

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- fat: Under-determined Linear Systems

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- skinny: Over-determined Linear Systems

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

2. Optimization Point of View

2.1. Least-Norm Solution

- For Under-Determined Linear System

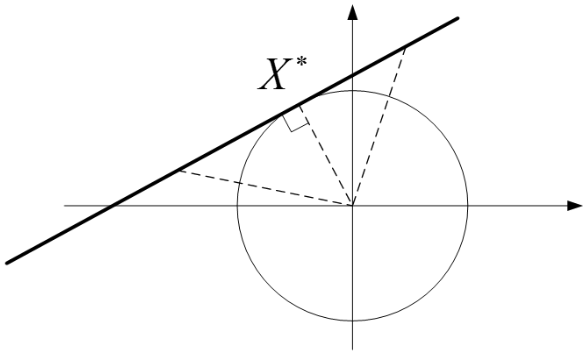
$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} `x_1 \\ x_2 \end{bmatrix} = b_1 \quad \text{or} \quad A^T X = B$$

Find the solution of $A^T X = B$ that minimize $\|X\|$ or $\|X\|^2$

i.e., optimization problem

$$\begin{aligned} \min \quad & \|X\|^2 \\ \text{s. t.} \quad & A^T X = B \end{aligned}$$

- Geometric analysis



- Select one solution among many solutions

$$X^* = \left(A^T A\right)^{-1} A^T B \quad \text{Least norm solution}$$

- Often control problem

2.2. Least-Square Solution

- For Over-Determined Linear System

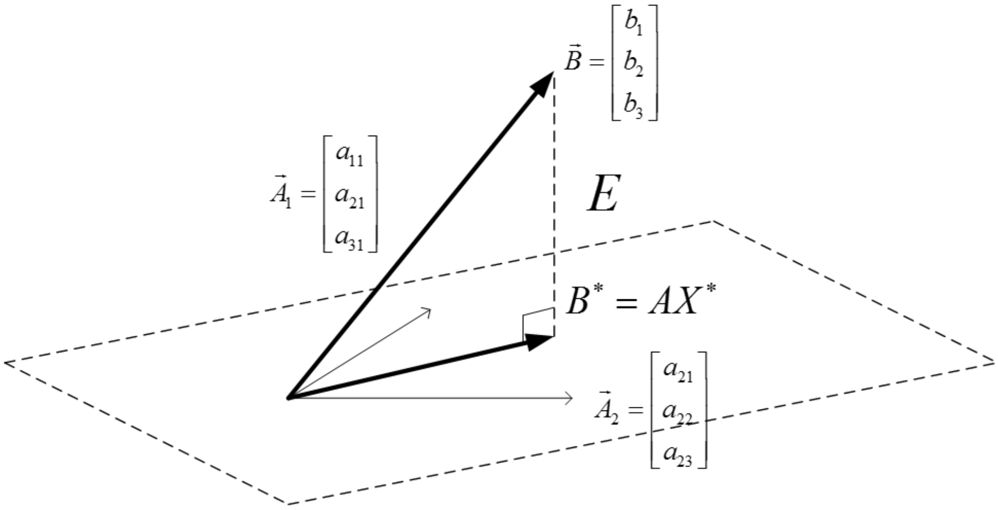
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{or} \quad AX \neq B \quad x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Find X that minimizes $\|E\|$ or $\|E\|^2$

i.e. optimization problem

$$\begin{aligned} \min_X \|E\|^2 &= \min_X \|AX - B\|^2 \\ X^* &= \left(A^T A\right)^{-1} A^T B \\ B^* = AX^* &= A\left(A^T A\right)^{-1} A^T B \end{aligned}$$

- Geometric analysis

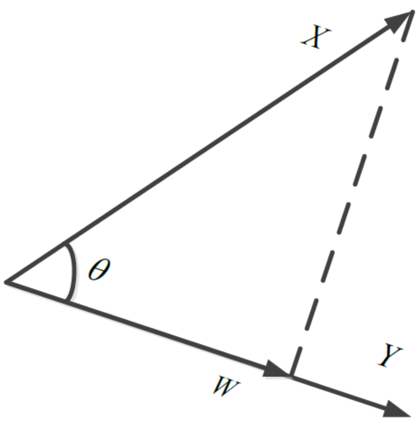


- Often estimation problem

3. Geometic Point of View: Projection

3.1. Vector Projection

- The vector projection of a vector X on (or onto) a nonzero vector Y is the orthogonal projection of X onto a straight line parallel to Y



$$
\begin{aligned}
W &= \omega \hat{Y} = \omega \frac{Y}{|Y|}, \text{ where } \omega = |W| \\
\omega &= |X| \cos\theta = |X| \frac{X \cdot Y}{|X||Y|} = \frac{X \cdot Y}{|Y|} \\
W &= \omega \hat{Y} = \frac{X \cdot Y}{|Y|} \frac{Y}{|Y|} = \frac{X \cdot Y}{|Y||Y|} Y = \frac{X \cdot Y}{Y \cdot Y} Y = \frac{\langle X, Y \rangle}{\langle Y, Y \rangle} Y
\end{aligned}
$$

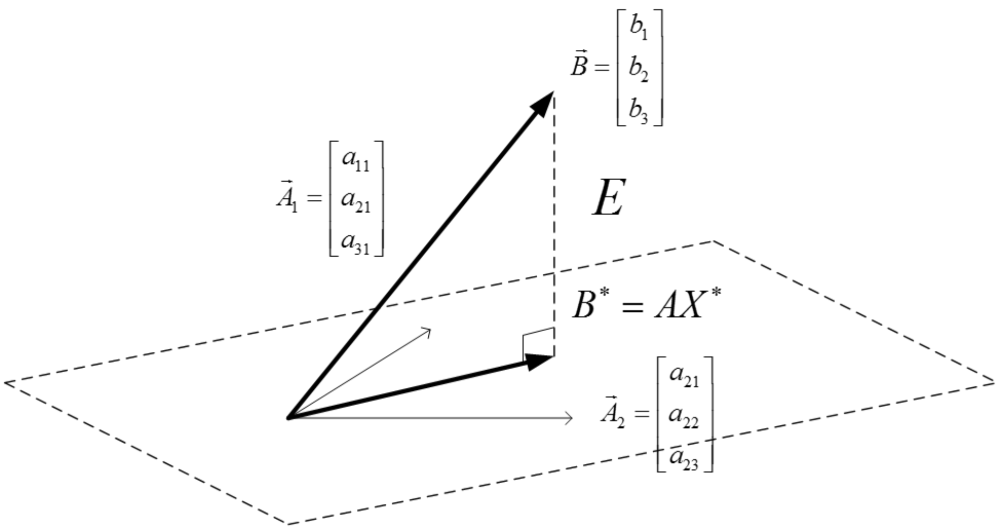
- Another way of computin ω and W

$$\begin{aligned} & Y \perp (X - W) \\ \Longrightarrow & Y^T(X - W) = Y^T\left(X - \omega\frac{Y}{|Y|}\right) = 0 \\ \Longrightarrow & \omega = \frac{Y^TX}{Y^TY}|Y| \\ \Longrightarrow & \omega = \frac{Y^TX}{Y^TY}|Y| \\ & W = \omega\frac{Y}{|Y|} = \frac{Y^TX}{Y^TY}Y = \frac{\langle X, Y\rangle}{\langle Y, Y\rangle}Y \end{aligned}$$

3.2. Orthogonal Projection onto a Subspace

- Projection of B onto a subspace U of span of A_1 and A_2
- Orthogonality

$$\begin{aligned} A \perp (AX^* - B) \\ A^T(AX^* - B) = 0 \\ A^TAX^* = A^TB \\ X^* = \left(A^TA\right)^{-1}A^TB \\ B^* = AX^* = A\left(A^TA\right)^{-1}A^TB \end{aligned}$$

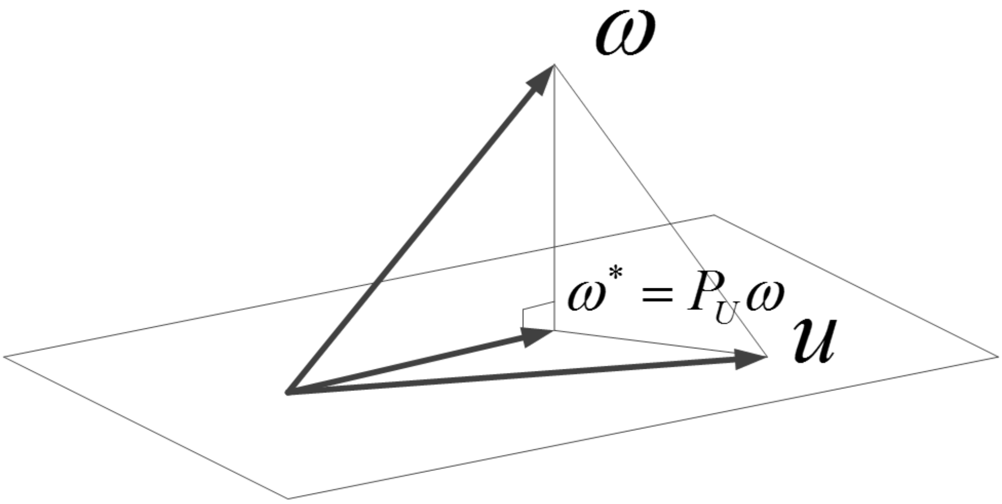


3.3. Towards Minimization Problems

- Suppose U is a subspace of W and $\omega \in W$. Then

$$\|\omega - P_U\omega\|^2 \leq \|\omega - u\|$$

for every $u \in U$. Furthermore, if $u \in U$ and the inequality above is an equality, then $u = P_u\omega$



3.4. Orthogonal Projection

- Is P a linear transformation?
- Any projection matrix P satisfies the two properties:
  - $P^2 = P$
  - P is symmetric
- It is also true that any matrix that satisfies these two properties is the projection matrix for some subspace $\mathbb{R}^n$

In [3]:

```
%%javascript
$.getScript('https://kmahelona.github.io/ipython_notebook_goodies/ipython_notebook_toc.js')
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