Regression

by Seungchul Lee
iSystems Design Lab
http://isystems.unist.ac.kr/
UNIST

Table of Contents

1. Linear Regression

Begin by considering linear regression (easy to extend to more comlex predictions later on)

$\text{Given } \begin{cases} x_i : \text{inputs} \\ y_i : \text{outputs} \end{cases}$, Find θ_1 and θ_2

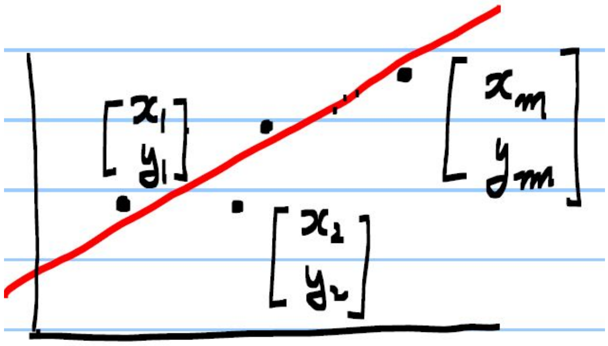
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \approx \hat{y}_i = \theta_1 x_i + \theta_2$$

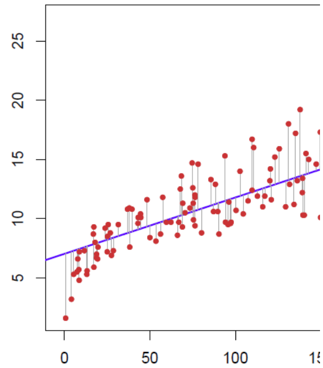
- $\hat{y}_i$: predicted output
- $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$: Model parameters

$$\hat{y}_i = f(x_i, \theta) \text{ in general}$$

- in many cases, a linear model to predict y_i used

$$\hat{y}_i = \theta_1 x_i + \theta_2 \text{ such that } \min_{\theta_1, \theta_2} \sum_{i=1}^{m} (\hat{y}_i - y_i)^2$$





1.1. Re-cast problem as a least squares

- For convenience, we define a function that maps inputs to feature vectors, ϕ

$$\begin{aligned} \hat{y}_i &= \begin{bmatrix} x_i & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \\ &= \begin{bmatrix} x_i \\ 1 \end{bmatrix}^T \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad \qquad \text{feature vector } \phi(x_i) = \begin{bmatrix} x_i \\ 1 \end{bmatrix} \\ &= \phi^T(x_i)\theta \end{aligned}$$

$$\Phi = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_m & 1 \end{bmatrix} = \begin{bmatrix} \phi^T(x_1) \\ \phi^T(x_2) \\ \vdots \\ \phi^T(x_m) \end{bmatrix} \quad \implies \quad \hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_m \end{bmatrix} = \Phi\theta$$

- optimization problem

$$\min_{\theta_1, \theta_2} \sum_{i=1}^{m} (\hat{y}_i - y_i)^2 = \min_{\theta} \|\Phi\theta - y\|_2^2 \qquad \qquad \left(\text{same as } \min_{x} \|Ax - b\|_2^2\right)$$

$$\text{solution } \theta^* = (\Phi^T\Phi)^{-1}\Phi^T y$$

Note

$$\begin{array}{ccccc} \text{input} & & \text{feature} & & \text{predicted output} \\ x_i & \rightarrow & \begin{bmatrix} x_i \\ 1 \end{bmatrix} & \rightarrow & \hat{y}_i \end{array}$$

$$\begin{array}{c} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \vec{x} & \vec{b} \end{array} \end{array} \quad \begin{array}{c} \text{over-determined or} \\ \text{projection} \end{array}$$

$$A(= \Phi) = [\vec{a}_1 \; \vec{a}_2]$$

1.2. Single Variable Linear Regression

1) use a linear algebra

- known as *least square*

$$\theta = (A^T A)^{-1} A^T y$$

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

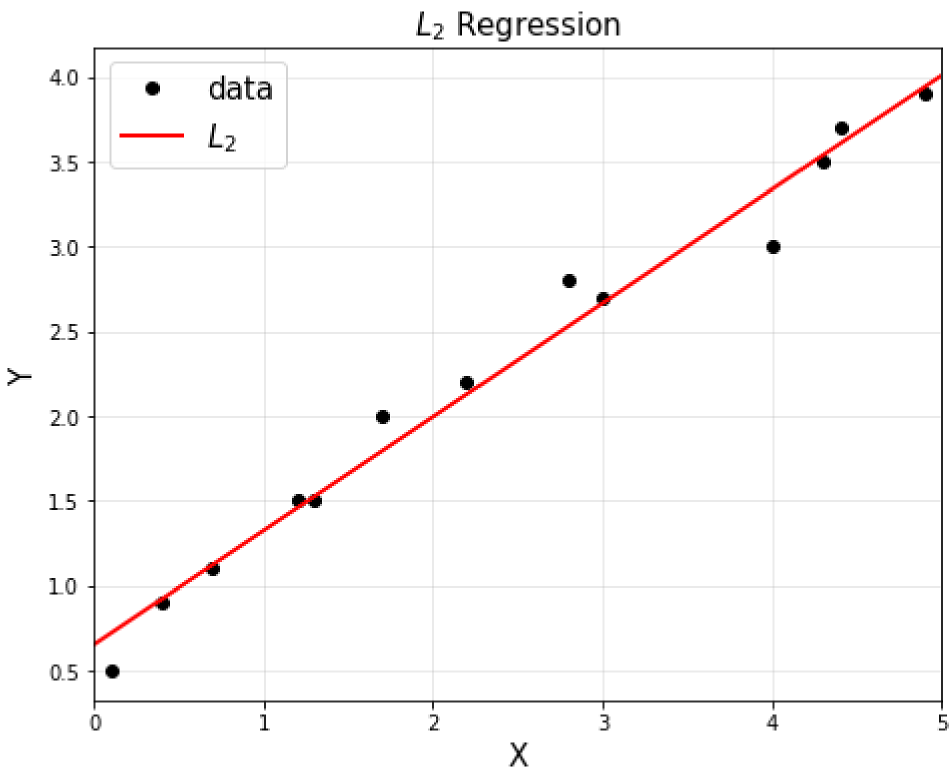
In [2]:

```
# data points in column vector [input, output]
x = np.array([0.1, 0.4, 0.7, 1.2, 1.3, 1.7, 2.2, 2.8, 3.0, 4.0, 4.3, 4.4,
4.9]).reshape(-1, 1)
y = np.array([0.5, 0.9, 1.1, 1.5, 1.5, 2.0, 2.2, 2.8, 2.7, 3.0, 3.5, 3.7,
3.9]).reshape(-1, 1)

m = y.shape[0]
A = np.hstack([x, np.ones([m, 1])])
A = np.asmatrix(A)

#theta = np.linalg.inv(A.T*A)*A.T*y
theta = (A.T*A).I*A.T*y

print('theta:\n', theta)
```

```
theta:
 [[ 0.67129519]
 [ 0.65306531]]
```

In [3]:

```
# to plot
plt.figure(figsize=(10, 6))
plt.title('$L_2$ Regression', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.plot(x, y, 'ko', label="data")

# to plot a straight line (fitted line)
xp = np.arange(0, 5, 0.01).reshape(-1, 1)
yp = theta[0,0]*xp + theta[1,0]

plt.plot(xp, yp, 'r', linewidth=2, label="$L_2$")
plt.legend(fontsize=15)
plt.axis('scaled')
plt.grid(alpha=0.3)
plt.xlim([0, 5])
plt.show()
```



2) use CVXPY optimization (least squared)

$$\min_{\theta} \ \|\hat{y} - y\|_2 = \min_{\theta} \ \|A\theta - y\|_2$$

In [4]:

```
import cvxpy as cvx

theta2 = cvx.Variable(2, 1)
obj = cvx.Minimize(cvx.norm(A*theta2-y, 2))
cvx.Problem(obj,[]).solve()

print('theta:\n', theta2.value)
```

```
theta:
 [[ 0.67129519]
 [ 0.65306531]]
```

By the way, do we have to use only L_2 norm? No.

- Let's use L_1 norm

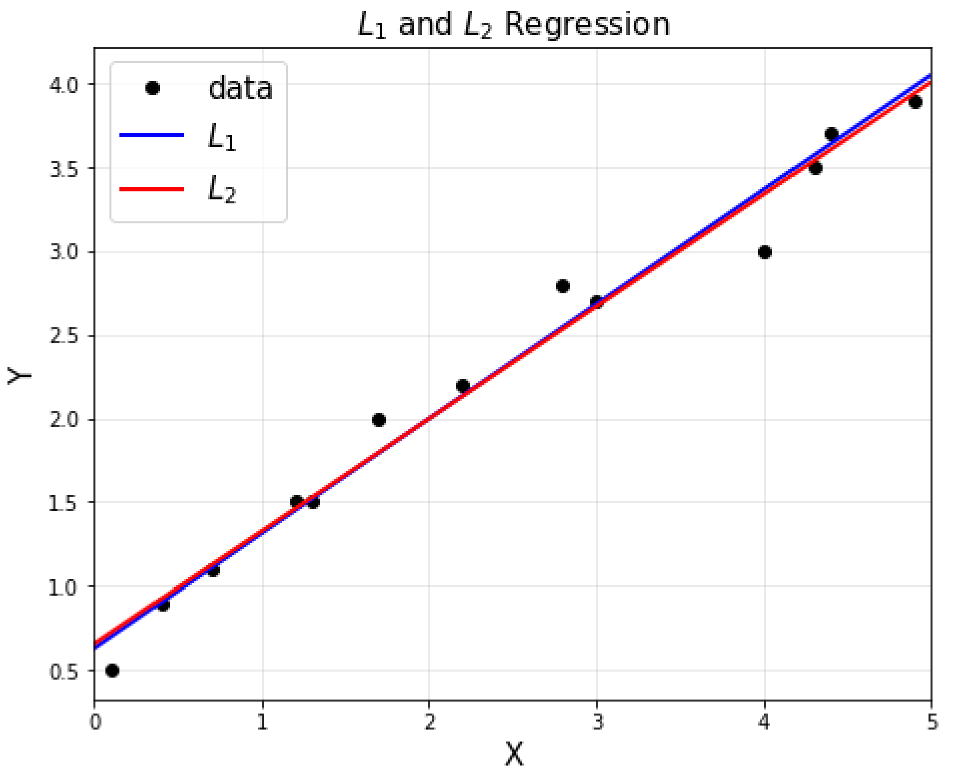
In [5]:

```
theta1 = cvx.Variable(2, 1)
obj = cvx.Minimize(cvx.norm(A*theta1-y, 1))
cvx.Problem(obj).solve()

print('theta:\n', theta1.value)
```

```
theta:
 [[ 0.68531634]
 [ 0.62587346]]
```

In [6]:

```
# to plot data
plt.figure(figsize=(10, 6))
plt.title('$L_1$ and $L_2$ Regression', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.plot(x, y, 'ko', label='data')

# to plot straight lines (fitted lines)
xp = np.arange(0, 5, 0.01).reshape(-1, 1)
yp1 = theta1.value[0,0]*xp + theta1.value[1,0]
yp2 = theta2.value[0,0]*xp + theta2.value[1,0]

plt.plot(xp, yp1, 'b', linewidth=2, label='$L_1$')
plt.plot(xp, yp2, 'r', linewidth=2, label='$L_2$')
plt.legend(fontsize=15)
plt.axis('scaled')
plt.xlim([0, 5])
plt.grid(alpha=0.3)
plt.show()
```



L_1 norm also provides a decent linear approximation. **What if outliers exist?**

- fitting with the different norms
- Discuss the result
- it is important to understand what makes them different.

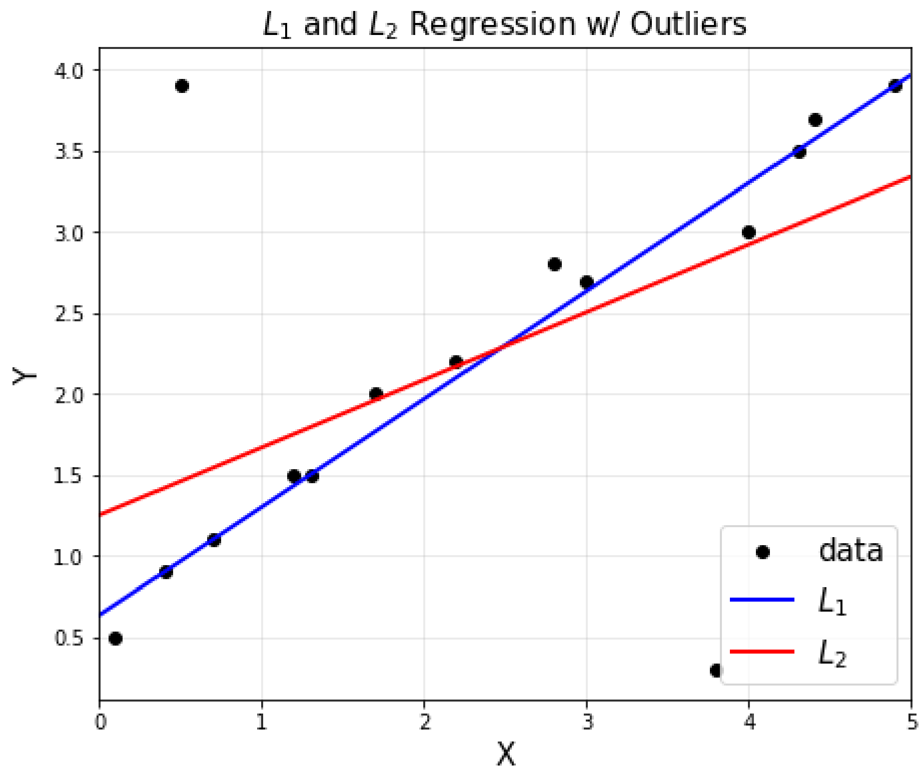
- source:
  - Week 9 of Computational Methods for Data Analysis by Coursera of Univ. of Washingon
  - Chapter 17, online book available (http://courses.washington.edu/amath582/582.pdf)

In [7]:

```
x = np.array([0.1, 0.4, 0.7, 1.2, 1.3, 1.7, 2.2, 2.8, 3.0, 4.0, 4.3, 4.4,
4.9]).reshape(-1, 1)
y = np.array([0.5, 0.9, 1.1, 1.5, 1.5, 2.0, 2.2, 2.8, 2.7, 3.0, 3.5, 3.7,
3.9]).reshape(-1, 1)

# add outliers
x = np.vstack([x, np.array([0.5, 3.8]).reshape(-1, 1)])
y = np.vstack([y, np.array([3.9, 0.3]).reshape(-1, 1)])

A = np.hstack([x, np.ones([x.shape[0], 1])])
A = np.asmatrix(A)

theta1 = cvx.Variable(2, 1)
obj1 = cvx.Minimize(cvx.norm(A*theta1-y, 1))
cvx.Problem(obj1).solve()

theta2 = cvx.Variable(2, 1)
obj2 = cvx.Minimize(cvx.norm(A*theta2-y, 2))
prob2 = cvx.Problem(obj2).solve()
```

In [8]:

```
# to plot data
plt.figure(figsize=(10, 6))
plt.plot(x, y, 'ko', label='data')
plt.title('$L_1$ and $L_2$ Regression w/ Outliers', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)

# to plot straight lines (fitted lines)
xp = np.arange(0, 5, 0.01).reshape(-1,1)
yp1 = theta1.value[0,0]*xp + theta1.value[1,0]
yp2 = theta2.value[0,0]*xp + theta2.value[1,0]

plt.plot(xp, yp1, 'b', linewidth=2, label='$L_1$')
plt.plot(xp, yp2, 'r', linewidth=2, label='$L_2$')
plt.axis('scaled')
plt.xlim([0, 5])
plt.legend(fontsize=15)
plt.grid(alpha=0.3)
plt.show()
```

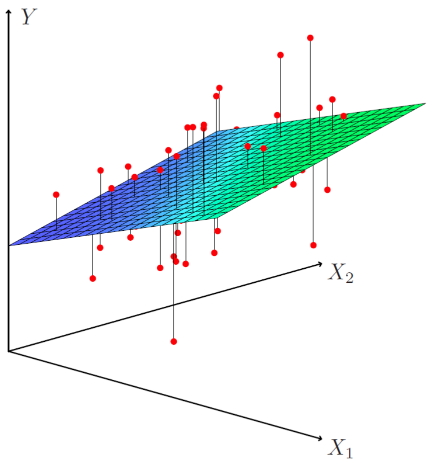


Think about what makes them different.

2.3. Multivariate Linear Regression (linear regression for multivariate data)

$$\hat{y}_i = \theta_1 x_1 + \theta_2 x_2 + \theta_3$$

$$\phi(x_i) = \begin{bmatrix} x_{1i} \\ x_{2i} \\ 1 \end{bmatrix}$$

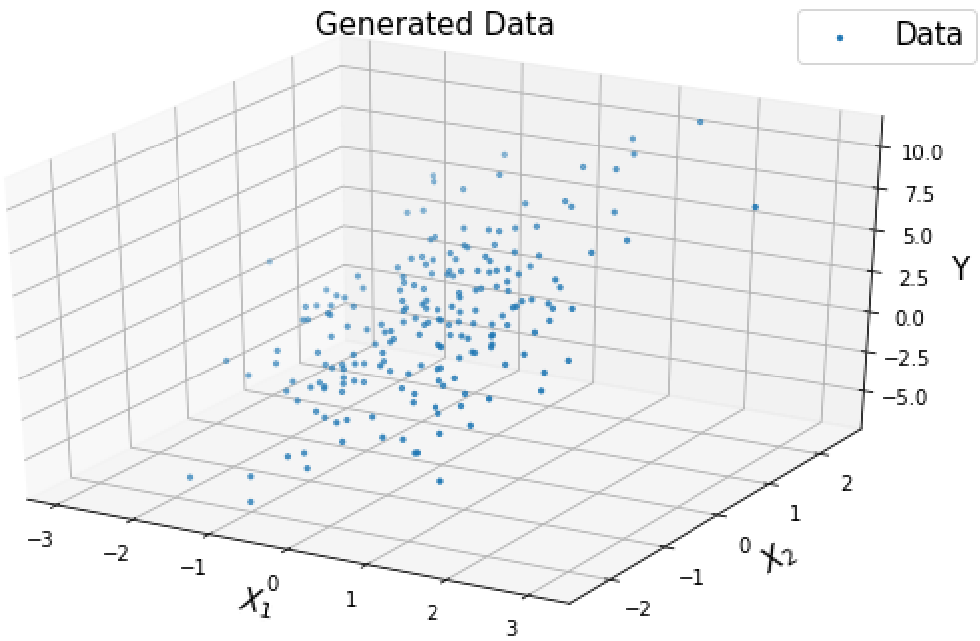


In [9]:

```
# for 3D plot
from mpl_toolkits.mplot3d import Axes3D
```

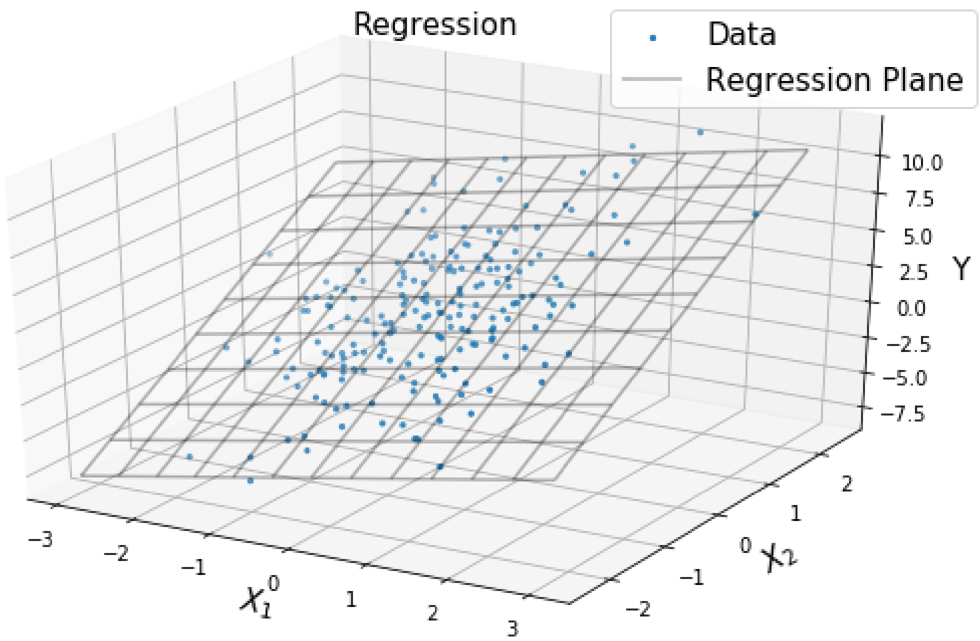
In [10]:

```
# y = theta1*x1 + theta2*x2 + theta3 + noise

n = 200
x1 = np.random.randn(n, 1)
x2 = np.random.randn(n, 1)
noise = 0.5*np.random.randn(n, 1);

y = 1*x1 + 3*x2 + 2 + noise

fig = plt.figure(figsize=(10, 6))
ax = fig.add_subplot(1, 1, 1, projection='3d')
ax.set_title('Generated Data', fontsize=15)
ax.set_xlabel('$X_1$', fontsize=15)
ax.set_ylabel('$X_2$', fontsize=15)
ax.set_zlabel('Y', fontsize=15)
ax.scatter(x1, x2, y, marker='.', label='Data')
#ax.view_init(30,30)
plt.legend(fontsize=15)
plt.show()
```

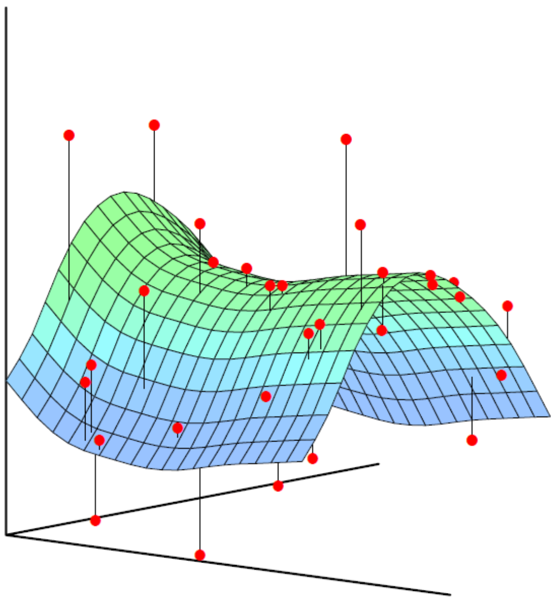


In [11]:

```
# % matplotlib qt

A = np.hstack([x1, x2, np.ones((n, 1))])
A = np.asmatrix(A)
theta = (A.T*A).I*A.T*y

X1, X2 = np.meshgrid(np.arange(np.min(x1), np.max(x1), 0.5), np.arange(np.min(x2), np.m
ax(x2), 0.5))
YP = theta[0,0]*X1 + theta[1,0]*X2 + theta[2,0]

fig = plt.figure(figsize=(10, 6))
ax = fig.add_subplot(1, 1, 1, projection='3d')
ax.set_title('Regression', fontsize=15)
ax.set_xlabel('$X_1$', fontsize=15)
ax.set_ylabel('$X_2$', fontsize=15)
ax.set_zlabel('Y', fontsize=15)
ax.scatter(x1, x2, y, marker='.', label='Data')
ax.plot_wireframe(X1, X2, YP, color='k', alpha=0.3, label='Regression Plane')
#ax.view_init(30,30)
plt.legend(fontsize=15)
plt.show()
```



3. Nonlinear Regression (Linear Regression for Non-linear Data)



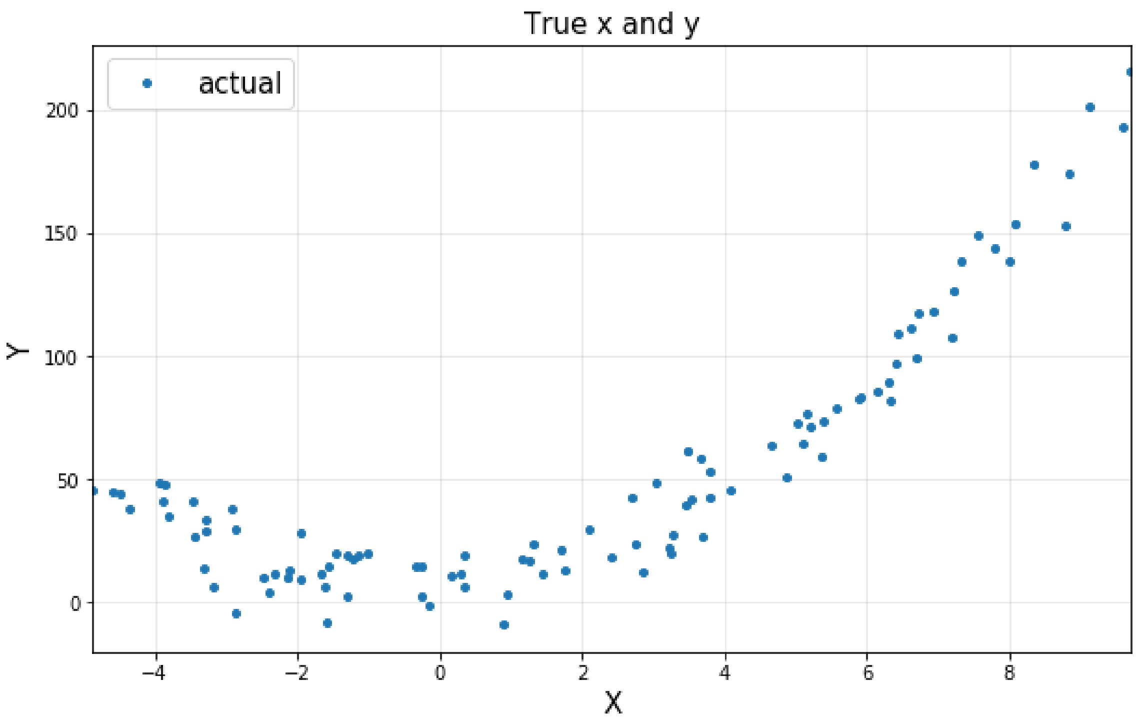
- same as linear regression, just with non-linear features
- method 1: constructing explicit feature vectors
  - polynomial features
  - Radial basis function (RBF) features
- method 2: implicit feature vectors, kernels (*optional*)
- polynomial (here, quad is used as an example)

$$y = \theta_1 + \theta_2 x + \theta_3 x^2 + \text{noise}$$

$$\phi(x_i) = A = \begin{bmatrix} 1 \\ x_i \\ x_i^2 \end{bmatrix}$$

In [12]:

```
# y = theta1 + theta2*x + theta3*x^2 + noise

n = 100
x = -5 + 15*np.random.rand(n, 1)
noise = 10*np.random.randn(n, 1)

y = 10 + 1*x + 2*x**2 + noise

plt.figure(figsize=(10, 6))
plt.title('True x and y', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.plot(x, y, 'o', markersize=4, label='actual')
plt.xlim([np.min(x), np.max(x)])
plt.grid(alpha=0.3)
plt.legend(fontsize=15)
plt.show()
```



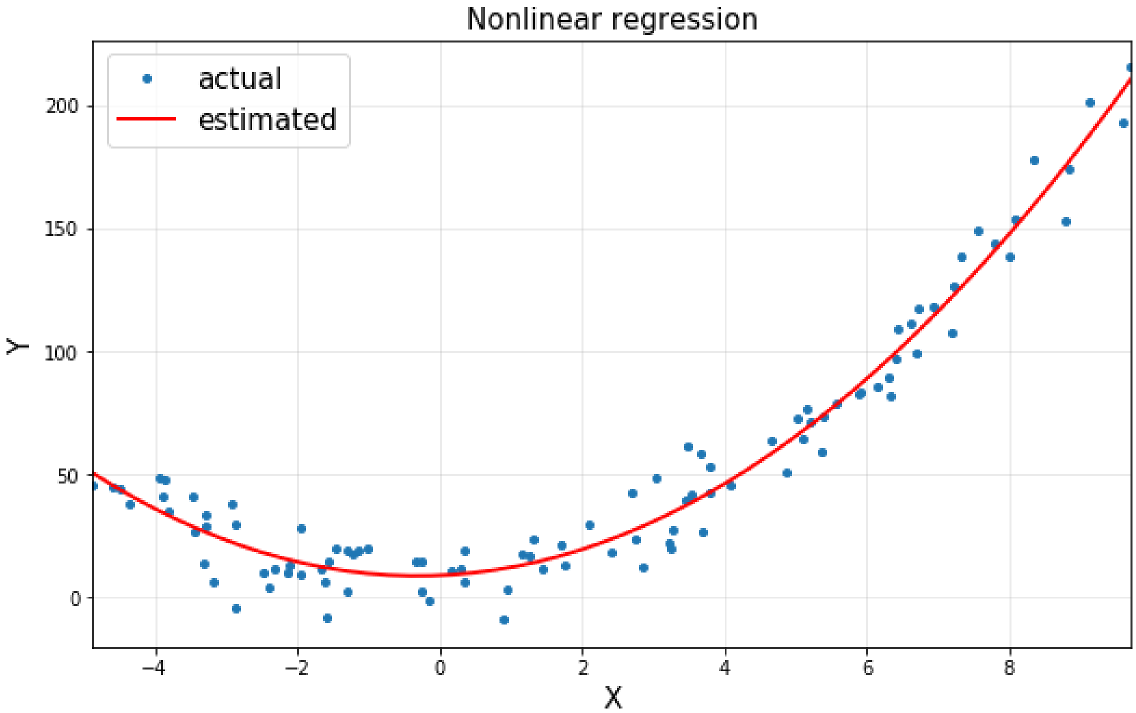
In [13]:

```
A = np.hstack([np.ones((n, 1)), x, x**2])
A = np.asmatrix(A)

theta = (A.T*A).I*A.T*y
print('theta:\n', theta)
```

```
theta:
 [[ 8.95645066]
 [ 1.2446826 ]
 [ 2.00589109]]
```

In [14]:

```
xp = np.linspace(np.min(x), np.max(x))
yp = theta[0,0] + theta[1,0]*xp + theta[2,0]*xp**2

plt.figure(figsize=(10, 6))
plt.plot(x, y, 'o', markersize=4, label='actual')
plt.plot(xp, yp, 'r', linewidth=2, label='estimated')

plt.title('Nonlinear regression', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.xlim([np.min(x), np.max(x)])
plt.grid(alpha=0.3)
plt.legend(fontsize=15)
plt.show()
```

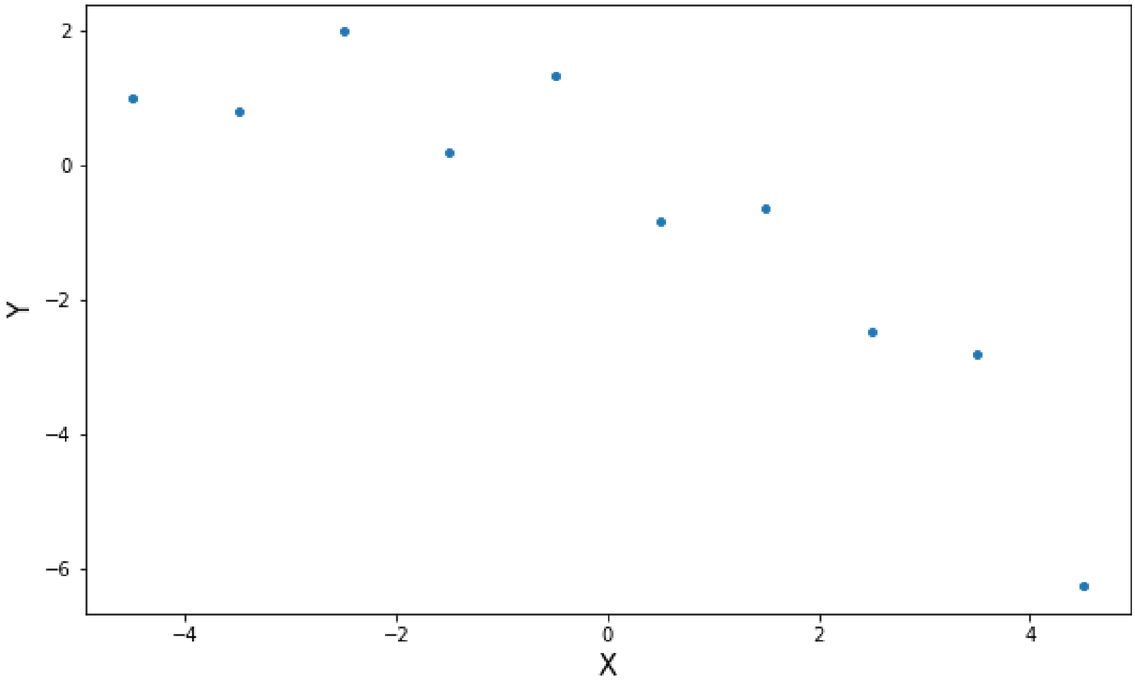


4. Overfitting

This is a very important code that you might want to fully understand or even memorize

In [15]:

```
# 10 data points
n = 10
x = np.linspace(-4.5, 4.5, 10).reshape(-1, 1)
y = np.array([0.9819, 0.7973, 1.9737, 0.1838, 1.3180, -0.8361, -0.6591, -2.4701, -2.812
2, -6.2512]).reshape(-1, 1)

plt.figure(figsize=(10, 6))
plt.plot(x, y, 'o', markersize=4, label='Data')
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.show()
```



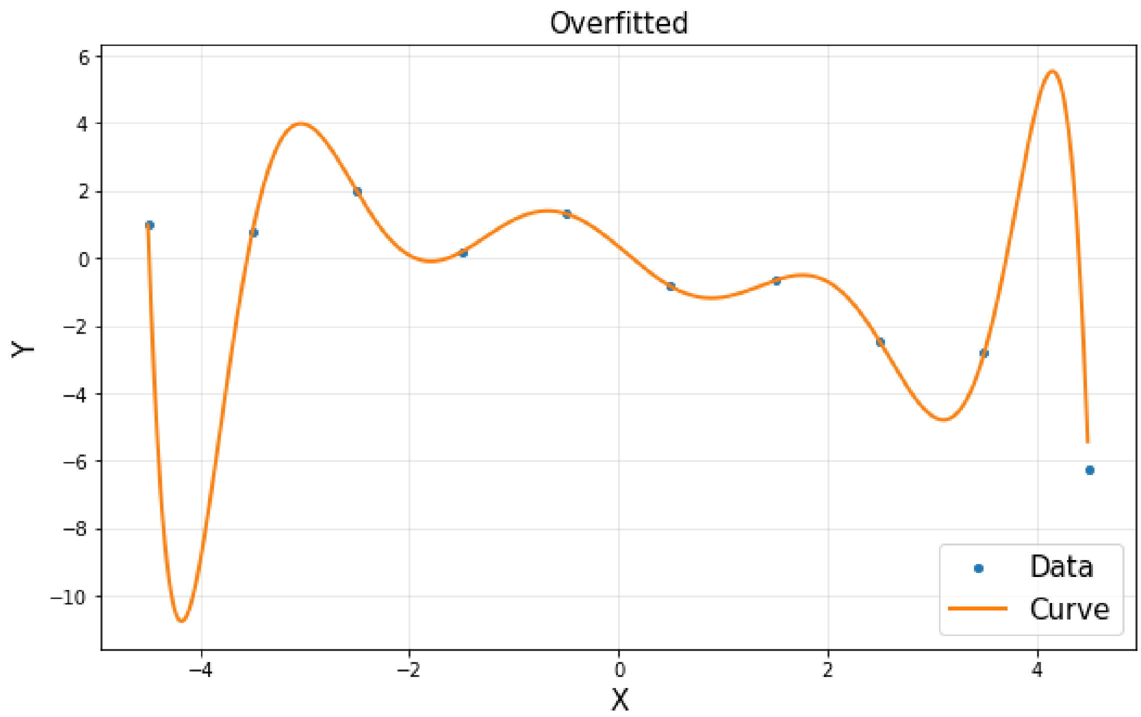
In [16]:

```
A = np.hstack([np.ones((n, 1)), x, x**2, x**3, x**4, x**5, x**6, x**7, x**8, x**9])
#A = np.hstack([x**i for i in range(10)])
A = np.asmatrix(A)
theta = (A.T*A).I*A.T*y
print(theta)
```

```
[[  3.48274701e-01]
 [ -2.58951123e+00]
 [ -4.55286474e-01]
 [  1.85022226e+00]
 [  1.06250369e-01]
 [ -4.43328786e-01]
 [ -9.25753472e-03]
 [  3.63088178e-02]
 [  2.35143849e-04]
 [ -9.24099978e-04]]
```

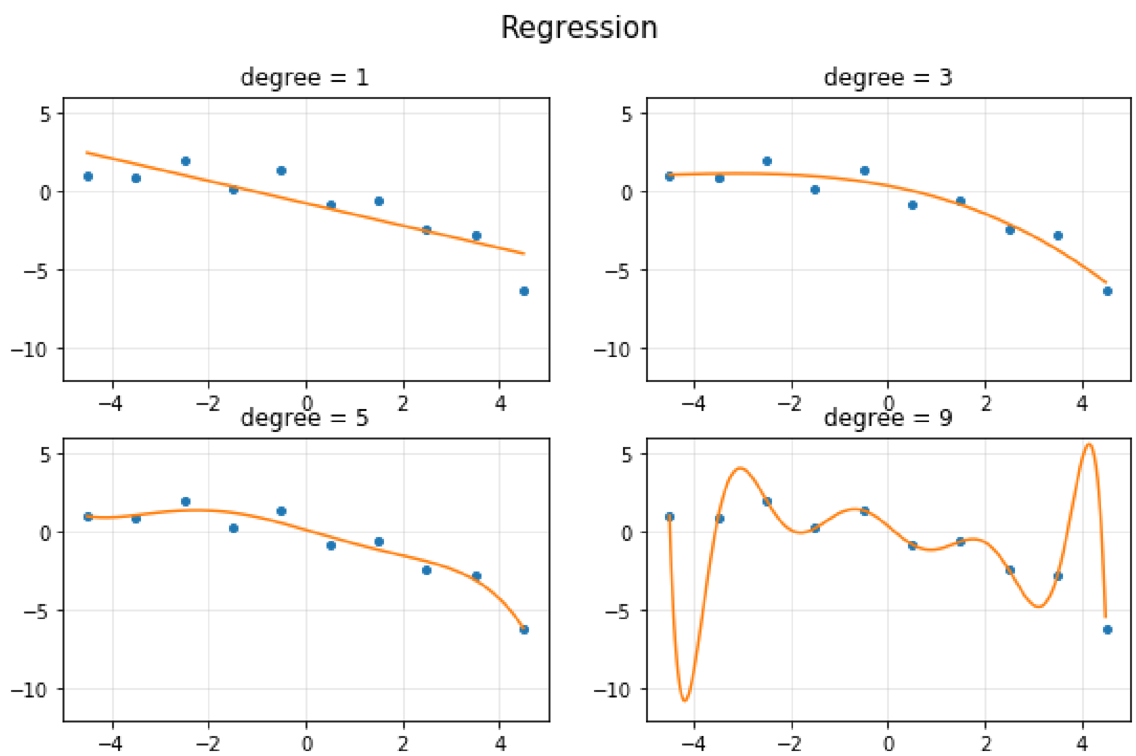
In [17]:

```
# to plot
xp = np.arange(-4.5, 4.5, 0.01).reshape(-1, 1)
yp = theta[0,0] + theta[1,0]*xp + theta[2,0]*xp**2 + theta[3,0]*xp**3 + \
     theta[4,0]*xp**4 + theta[5,0]*xp**5 + theta[6,0]*xp**6 + \
     theta[7,0]*xp**7 + theta[8,0]*xp**8 + theta[9,0]*xp**9

#polybasis = np.hstack([xp**i for i in range(10)])
#polybasis = np.asmatrix(polybasis)

#yp = polybasis*theta

plt.figure(figsize=(10, 6))
plt.plot(x, y, 'o', markersize=4, label='Data')
plt.plot(xp[:,0], yp[:,0], linewidth=2, label='Curve')
plt.title('Overfitted', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.legend(fontsize=15)
plt.grid(alpha=0.3)
plt.show()
```



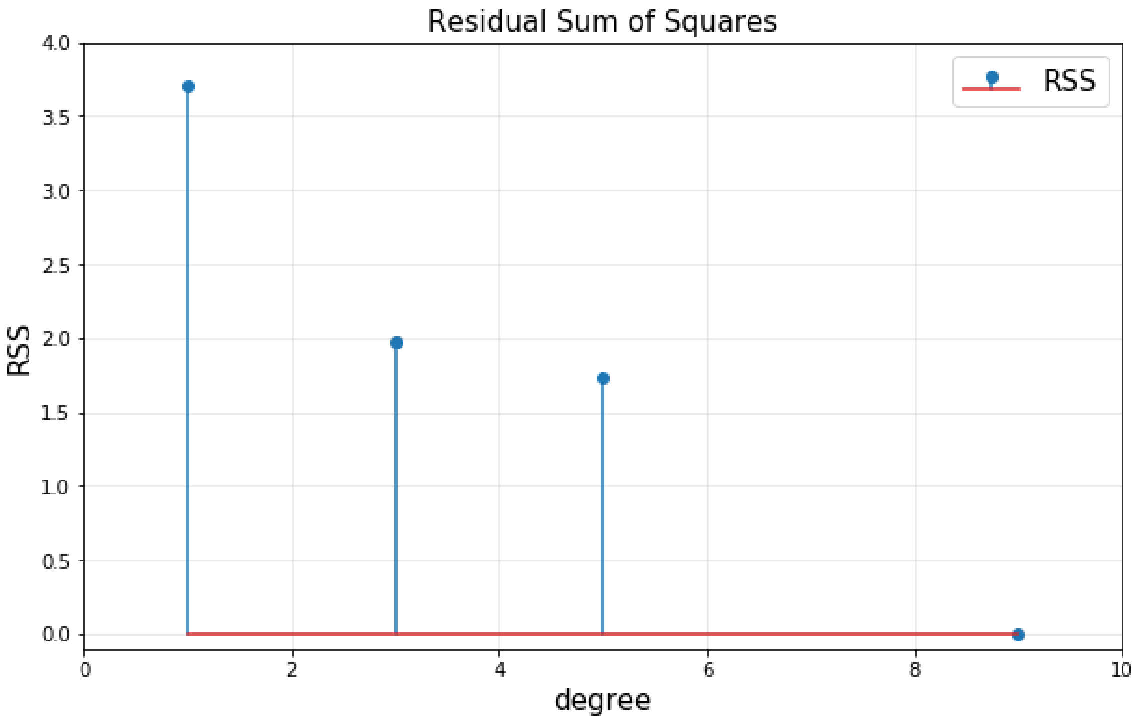
In [18]:

```
x = np.linspace(-4.5, 4.5, 10).reshape(-1, 1)
y = np.array([0.9819, 0.7973, 1.9737, 0.1838, 1.3180, -0.8361, -0.6591, -2.4701, -2.812
2, -6.2512]).reshape(-1, 1)

xp = np.arange(-4.5, 4.5, 0.01).reshape(-1, 1)

d = [1, 3, 5, 9]
RSS = []

plt.figure(figsize=(10, 6))
plt.suptitle('Regression', fontsize=15)

for k in range(4):
    A = np.hstack([x**i for i in range(d[k]+1)])
    polybasis = np.hstack([xp**i for i in range(d[k]+1)])

    A = np.asmatrix(A)
    polybasis = np.asmatrix(polybasis)

    theta = (A.T*A).I*A.T*y
    yp = polybasis*theta

    RSS.append(np.linalg.norm(y - A*theta, 2))

    plt.subplot(2, 2, k+1)
    plt.plot(x, y, 'o', markersize=4)
    plt.plot(xp, yp)
    plt.axis([-5, 5, -12, 6])
    plt.title('degree = {}'.format(d[k]))
    plt.grid(alpha=0.3)

plt.show()
```



In [19]:

```
plt.figure(figsize=(10, 6))
plt.stem(d, RSS, label='RSS')
plt.title('Residual Sum of Squares', fontsize=15)
plt.xlabel('degree', fontsize=15)
plt.ylabel('RSS', fontsize=15)
plt.axis([0, 10, -0.1, 4.0])
plt.legend(fontsize=15)
plt.grid(alpha=0.3)
plt.show()
```



5. Linear Basis Function Models

- Construct explicit feature vectors
- Consider linear combinations of fixed nonlinear functions of the input variables, of the form

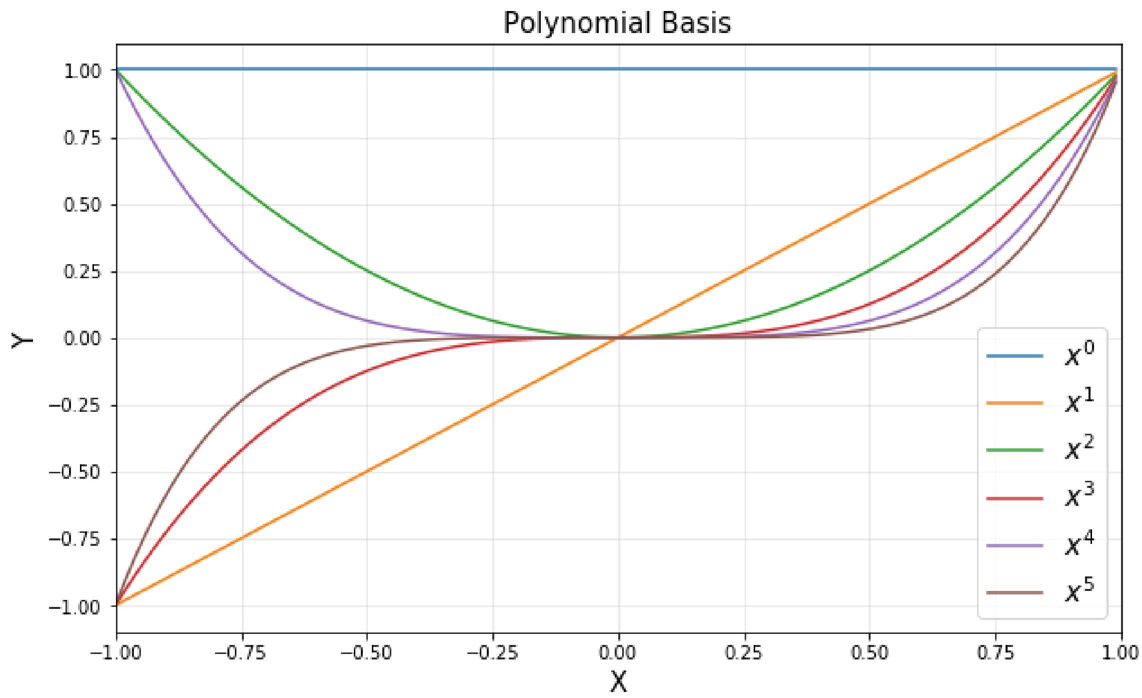
$$\hat{y} = \sum_{i=0}^{d} \theta_i \phi_i(x) = \Phi\theta$$

1) Polynomial functions

$$\phi_i(x) = x^i, \quad i = 0, \cdots, d$$

In [20]:

```
xp = np.arange(-1, 1, 0.01).reshape(-1, 1)
polybasis = np.hstack([xp**i for i in range(6)])

plt.figure(figsize=(10, 6))

for i in range(6):
    plt.plot(xp, polybasis[:,i], label='$x^{}$'.format(i))

plt.title('Polynomial Basis', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.axis([-1, 1, -1.1, 1.1])
plt.grid(alpha=0.3)
plt.legend(fontsize=15)
plt.show()
```

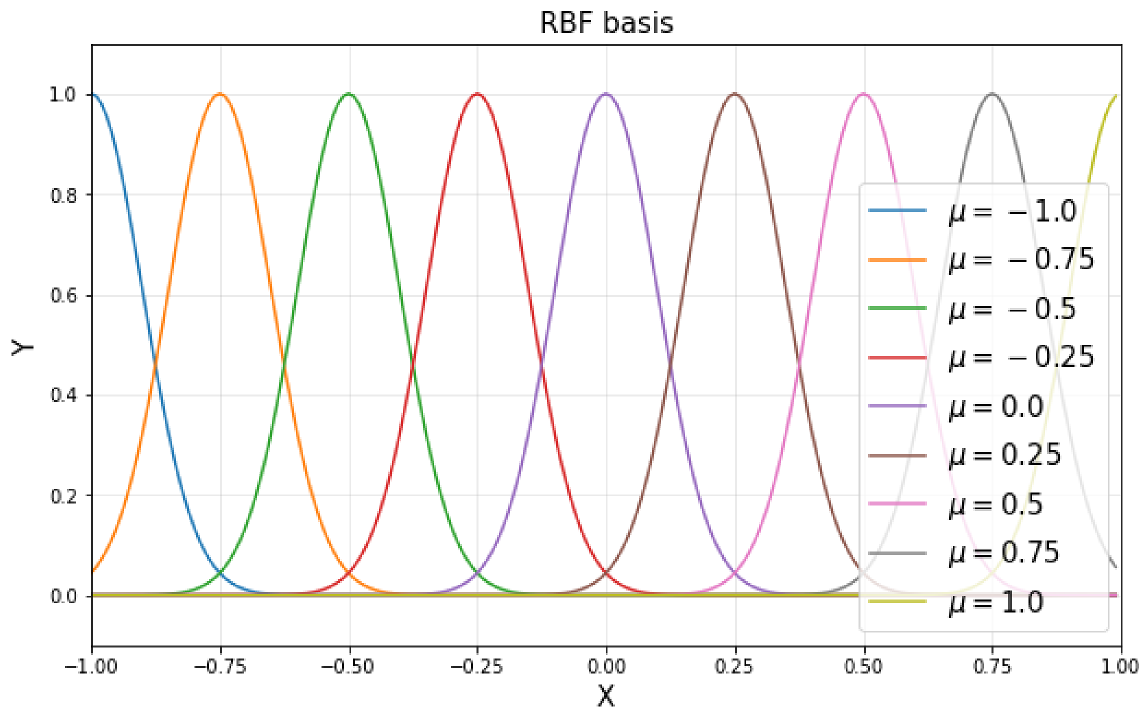


2) RBF functions with bandwidth σ and k RBF centers $\mu_i \in \mathbb{R}^n$

$$\phi_i(x) = \exp\left(-\frac{\|x-\mu_i\|^2}{2\sigma^2}\right)$$

In [21]:

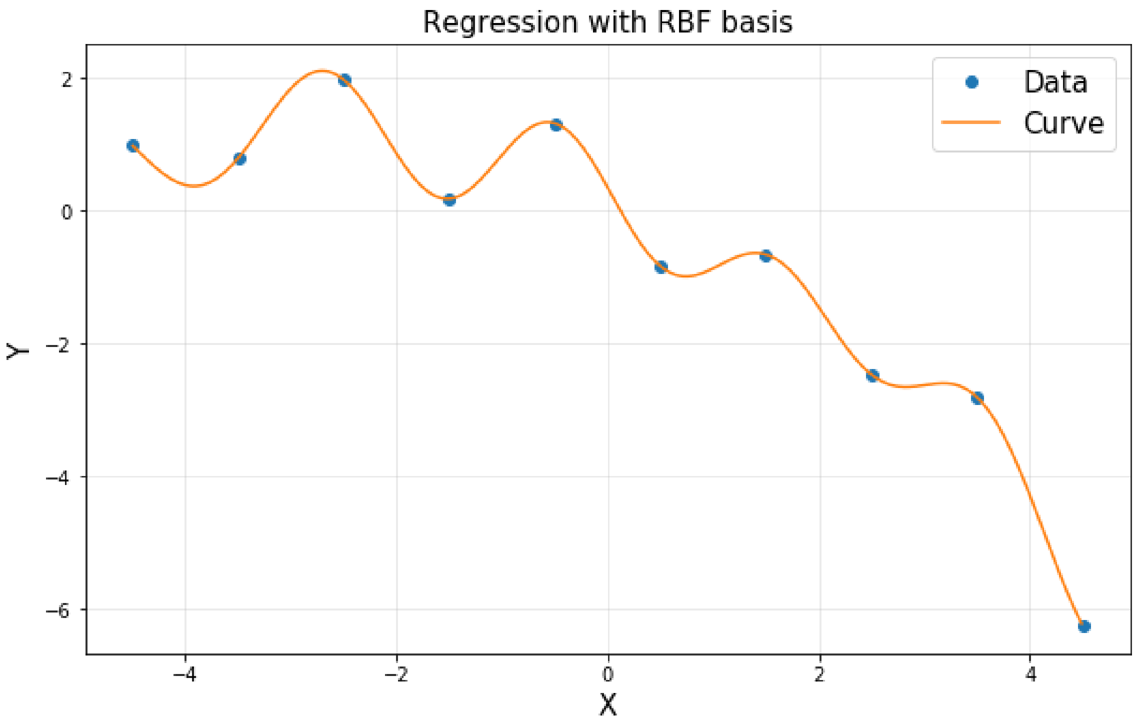
```
d = 9

u = np.linspace(-1, 1, d)
sigma = 0.1

rbfbasis = np.hstack([np.exp(-(xp-u[i])**2/(2*sigma**2)) for i in range(d)])

plt.figure(figsize=(10, 6))

for i in range(d):
    plt.plot(xp, rbfbasis[:,i], label='$\mu = {}$'.format(u[i]))

plt.title('RBF basis', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.axis([-1, 1, -0.1, 1.1])
plt.legend(loc='lower right', fontsize=15)
plt.grid(alpha=0.3)
plt.show()
```



- With many features, our prediction function becomes very expenssive
- Can lead to overfitting (low error on input data points, but high error nearby)

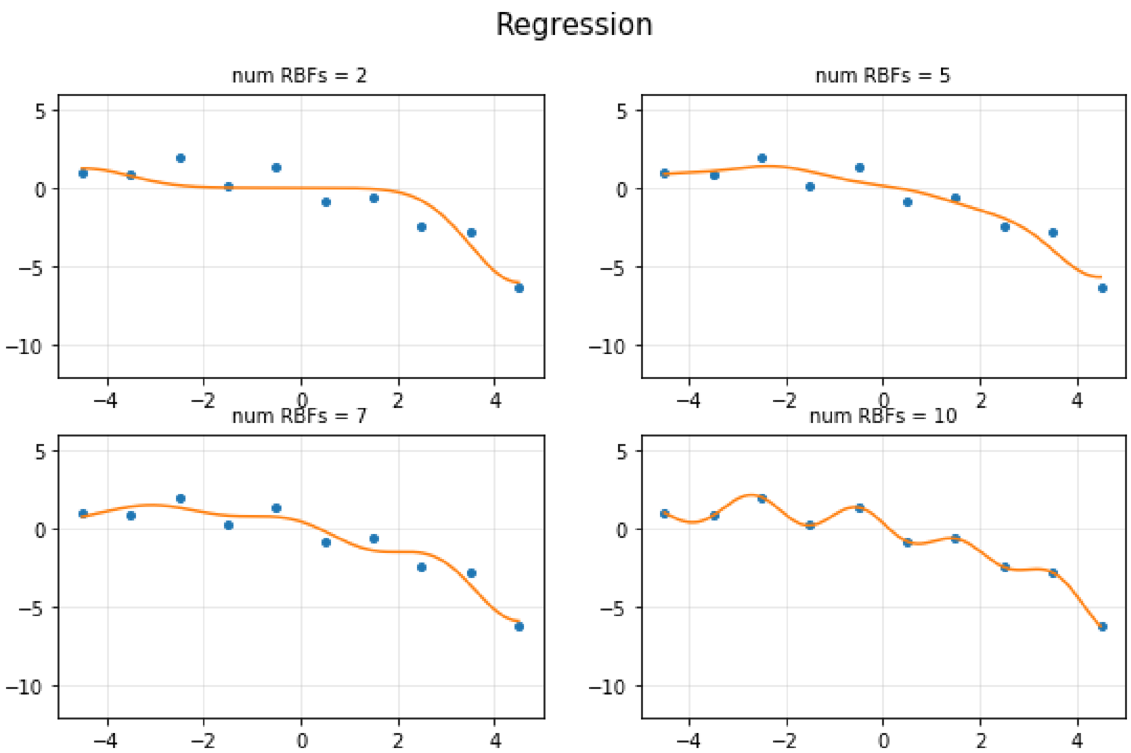
In [22]:

```
d = 10

x = np.linspace(-4.5, 4.5, d).reshape(-1, 1)
y = np.array([0.9819, 0.7973, 1.9737, 0.1838, 1.3180, -0.8361, -0.6591, -2.4701, -2.812
2, -6.2512]).reshape(-1, 1)

xp = np.arange(-4.5, 4.5, 0.01).reshape(-1, 1)

u = np.linspace(-4.5, 4.5, d)
sigma = 1

A = np.hstack([np.exp(-(x-u[i])**2/(2*sigma**2)) for i in range(10)])
rbfbasis = np.hstack([np.exp(-(xp-u[i])**2/(2*sigma**2)) for i in range(10)])
A = np.asmatrix(A)
rbfbasis = np.asmatrix(rbfbasis)

theta = (A.T*A).I*A.T*y
yp = rbfbasis*theta

plt.figure(figsize=(10, 6))
plt.plot(x, y, 'o', label='Data')
plt.plot(xp, yp, label='Curve')
plt.title('Regression with RBF basis', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.grid(alpha=0.3)
plt.legend(fontsize=15)
plt.show()
```



In [23]:

```
x = np.linspace(-4.5, 4.5, 10).reshape(-1, 1)
y = np.array([0.9819, 0.7973, 1.9737, 0.1838, 1.3180, -0.8361, -0.6591, -2.4701, -2.812
2, -6.2512]).reshape(-1, 1)

xp = np.arange(-4.5, 4.5, 0.01).reshape(-1, 1)

sigma = 1
d = [2, 5, 7, 10]

plt.figure(figsize=(10, 6))
plt.suptitle('Regression', fontsize=15)

for k in range(4):
    u = np.linspace(-4.5, 4.5, d[k])

    A = np.hstack([np.exp(-(x-u[i])**2/(2*sigma**2)) for i in range(d[k])])
    rbfbasis = np.hstack([np.exp(-(xp-u[i])**2/(2*sigma**2)) for i in range(d[k])])

    A = np.asmatrix(A)
    theta = (A.T*A).I*A.T*y
    yp = rbfbasis*theta

    plt.subplot(2, 2, k+1)
    plt.plot(x, y, 'o', markersize=4)
    plt.plot(xp, yp)
    plt.axis([-5, 5, -12, 6])
    plt.title('num RBFs = {}'.format(d[k]), fontsize=10)
    plt.grid(alpha=0.3)

plt.show()
```



6. Regularization (Shrinkage methods)

Often, overfitting associated with very large estimated parameters θ

We want to balance

- how well function fits data
- magnitude of coefficients

$$\text{Total cost} = \underbrace{\text{measure of fit}}_{RSS(\theta)} + \ \lambda \cdot \underbrace{\text{measure of magnitude of coefficients}}_{\lambda \cdot ||\theta||_2^2}$$

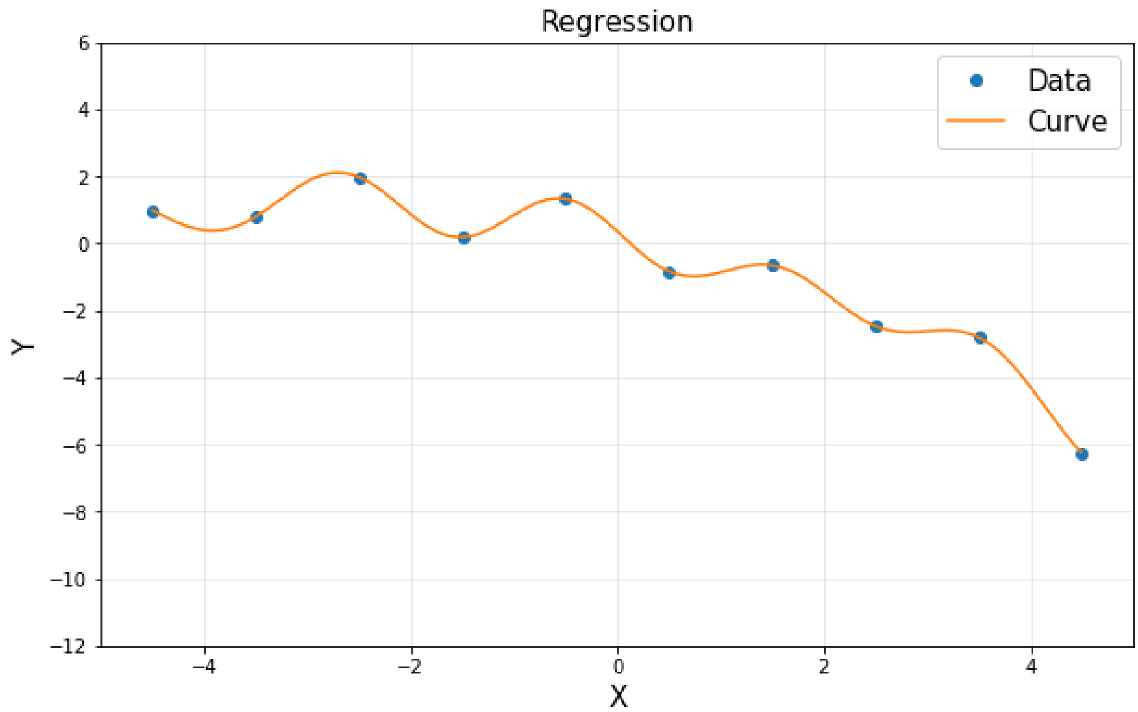
$$\implies \min \ ||\Phi\theta - y||_2^2 + \lambda||\theta||_2^2$$

where $RSS(\theta) = ||\Phi\theta - y||_2^2$, (= Rresidual Sum of Squares) and λ is a tuning parameter to be determined separately

- the second term, $\lambda \cdot ||\theta||_2^2$, called a shrinkage penalty, is small when $\theta_1, \cdots, \theta_d$ are close to zeros, and so it has the effect of shrinking the estimates of θ_j towards zero
- The tuning parameter λ serves to control the relative impact of these two terms on the regression coefficient estimates
- known as a *ridge regression*

In [24]:

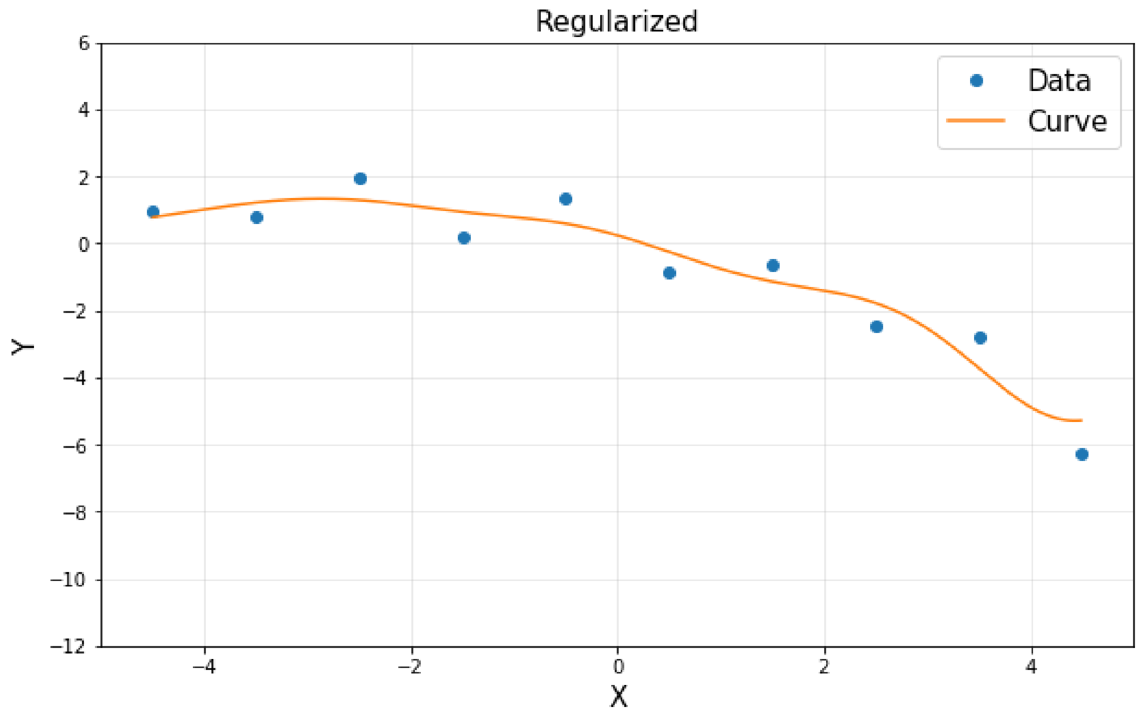
```
# CVXPY code

x = np.linspace(-4.5, 4.5, 10).reshape(-1, 1)
y = np.array([0.9819, 0.7973, 1.9737, 0.1838, 1.3180, -0.8361, -0.6591, -2.4701, -2.812
2, -6.2512]).reshape(-1, 1)

xp = np.arange(-4.5, 4.5, 0.01).reshape(-1, 1)

d = 10
u = np.linspace(-4.5, 4.5, d)

sigma = 1

A = np.hstack([np.exp(-(x-u[i])**2/(2*sigma**2)) for i in range(d)])
rbfbasis = np.hstack([np.exp(-(xp-u[i])**2/(2*sigma**2)) for i in range(d)])

theta = cvx.Variable(d, 1)
obj = cvx.Minimize(cvx.norm(A*theta-y, 2))
prob = cvx.Problem(obj).solve()

theta = theta.value

ypt = rbfbasis*theta

plt.figure(figsize=(10, 6))
plt.title('Regression', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.plot(x, y, 'o', label='Data')
plt.plot(xp, yp, label='Curve')
plt.axis([-5, 5, -12, 6])
plt.legend(fontsize=15)
plt.grid(alpha=0.3)
plt.show()

print('theta:\n', theta)
```




```
theta:
 [[  3.98406222]
 [ -7.50337699]
 [ 12.56023391]
 [-14.056845  ]
 [ 15.60808716]
 [-14.89983256]
 [ 13.30949424]
 [-13.45231001]
 [ 10.25140922]
 [-10.79133655]]
```

In [25]:

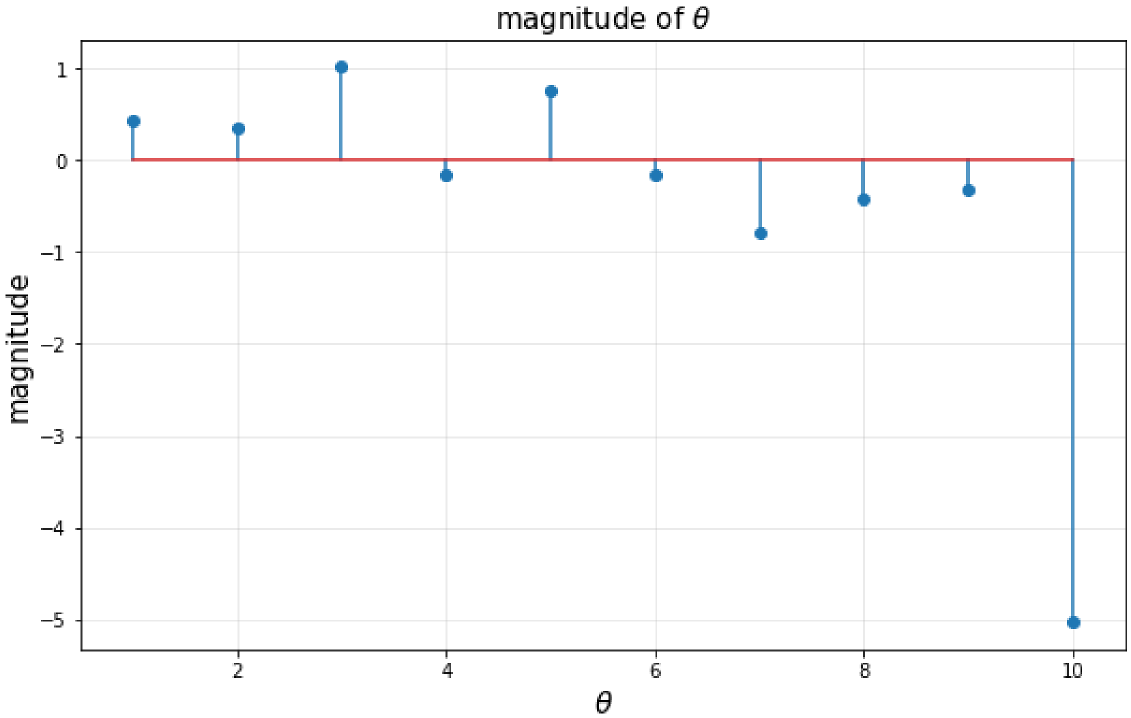
```
# ridge regression

x = np.linspace(-4.5, 4.5, 10).reshape(-1, 1)
y = np.array([0.9819, 0.7973, 1.9737, 0.1838, 1.3180, -0.8361, -0.6591, -2.4701, -2.812
2, -6.2512]).reshape(-1, 1)

xp = np.arange(-4.5, 4.5, 0.01).reshape(-1, 1)

d = 10
u = np.linspace(-4.5, 4.5, d)

sigma = 1

A = np.hstack([np.exp(-(x-u[i])**2/(2*sigma**2)) for i in range(d)])
rbfbasis = np.hstack([np.exp(-(xp-u[i])**2/(2*sigma**2)) for i in range(d)])

lamb = 0.1

theta = cvx.Variable(d, 1)
#obj = cvx.Minimize(cvx.norm(A*theta - y, 2) + lamb*cvx.norm(theta, 2))
obj = cvx.Minimize(cvx.sum_squares(A*theta - y) + lamb*cvx.sum_squares(theta))
prob = cvx.Problem(obj).solve()

theta = theta.value

yp = rbfbasis*theta

plt.figure(figsize=(10, 6))
plt.title('Regularized', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.plot(x, y, 'o', label='Data')
plt.plot(xp, yp, label='Curve')
plt.axis([-5, 5, -12, 6])
plt.legend(fontsize=15)
plt.grid(alpha=0.3)
plt.show()

print('theta:\n', theta)
```




```
theta:
 [[ 0.42397454]
 [ 0.35240809]
 [ 1.01292112]
 [-0.15712386]
 [ 0.74913783]
 [-0.16087517]
 [-0.79440461]
 [-0.41481861]
 [-0.31728828]
 [-5.02492788]]
```

In [26]:

```
# Regulization ( = ridge nonlinear regression) encourages small weights, but not exactl
y 0

plt.figure(figsize=(10, 6))
plt.title(r'magnitude of $\theta$', fontsize=15)
plt.xlabel(r'$\theta$', fontsize=15)
plt.ylabel('magnitude', fontsize=15)
plt.stem(np.linspace(1, 10, 10).reshape(-1, 1), theta)
plt.xlim([0.5, 10.5])
plt.grid(alpha=0.3)
plt.show()
```



7. Sparsity for feature selection using LASSO

- Least Squares with a penalty on the L_1-norm of the parameters
- start with full model (all possible features)
- 'Shrink' some coefficients exactly to 0
  - *i.e.*, knock out certain features
  - the l_1 penalty has the effect of forcing some of the coefficient estimates to be exactly equal to zero
- Non-zero coefficients indicate 'selected' features

Try this cost instead of ridge...

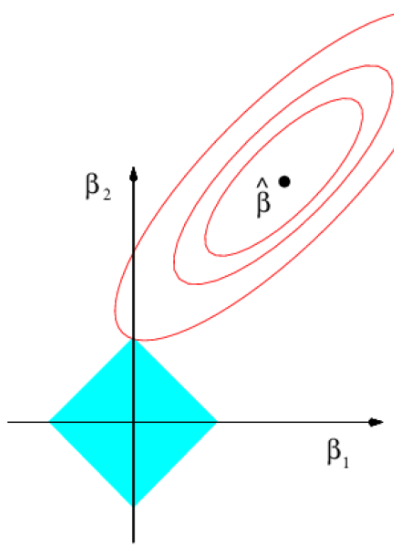
$$\text{Total cost} = \underbrace{\text{measure of fit}}_{RSS(\theta)} + \lambda \cdot \underbrace{\text{measure of magnitude of coefficients}}_{\lambda \cdot \|\theta\|_1}$$

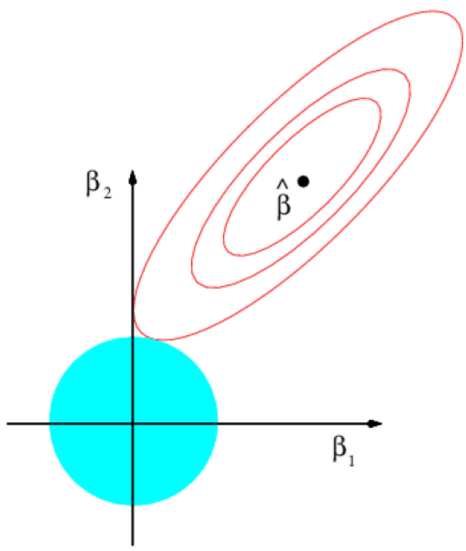
$$\Longrightarrow \min \|\Phi\theta - y\|_2^2 + \lambda\|\theta\|_1$$

- λ is a tuning parameter = balance of fit and sparsity

- Another equivalent forms of optimizations

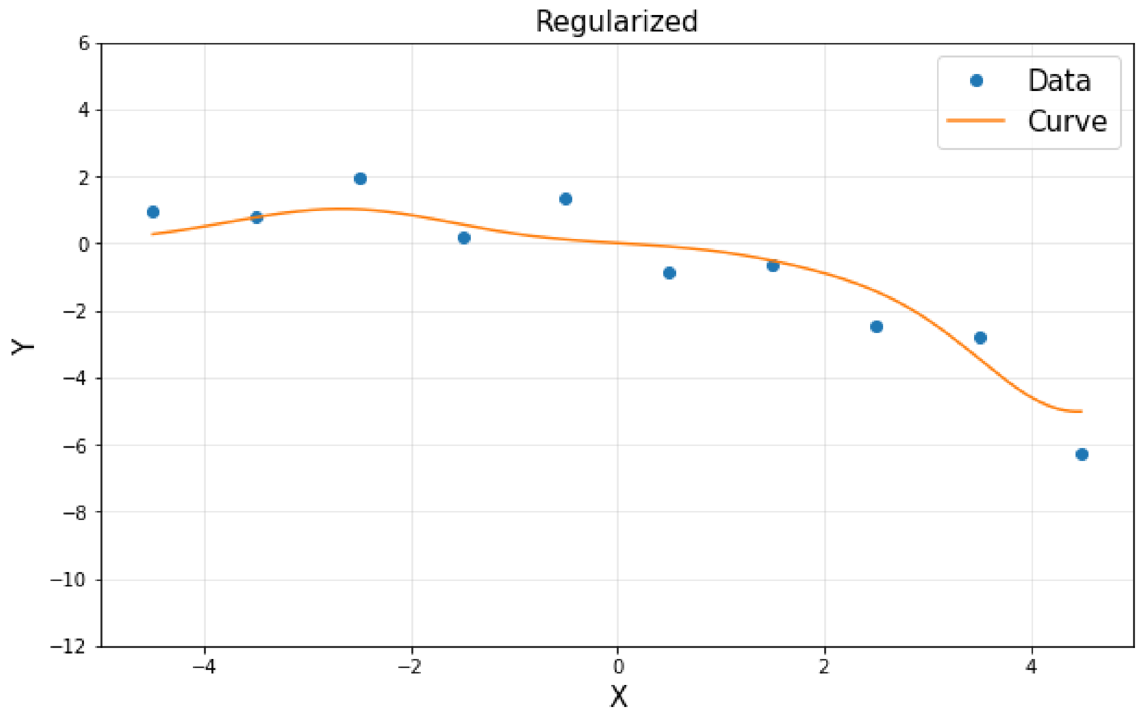
$$\begin{array}{rl} \min\limits_{\theta} & \|\Phi\theta - y\|_2^2 \\ \text{subject to} & \|\theta\|_1 \leq s \end{array} \qquad \begin{array}{ll} \min_{\theta} & \|\Phi\theta - y\|_2^2 \\ \text{subject to} & \|\theta\|_2^2 \leq s \end{array}$$





In [27]:

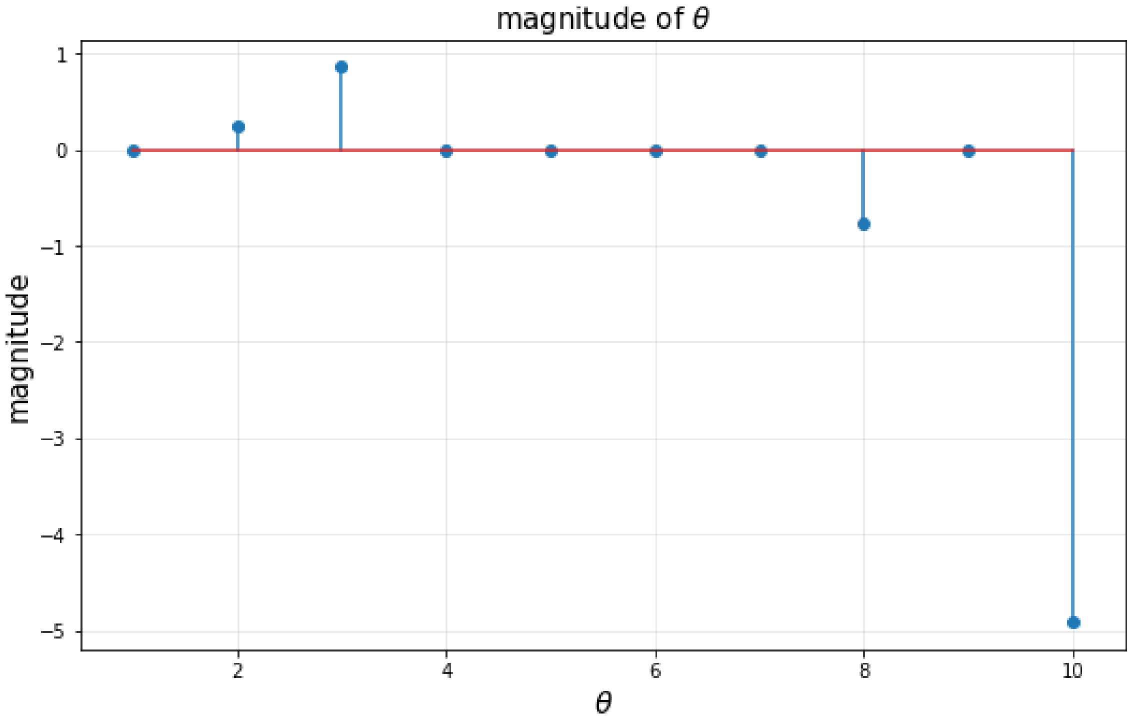
```
# LASSO regression

x = np.linspace(-4.5, 4.5, 10).reshape(-1, 1)
y = np.array([0.9819, 0.7973, 1.9737, 0.1838, 1.3180, -0.8361, -0.6591, -2.4701, -2.812
2, -6.2512]).reshape(-1, 1)

xp = np.arange(-4.5, 4.5, 0.01).reshape(-1, 1)

d = 10
u = np.linspace(-4.5, 4.5, d)

sigma = 1

A = np.hstack([np.exp(-(x-u[i])**2/(2*sigma**2)) for i in range(d)])
rbfbasis = np.hstack([np.exp(-(xp-u[i])**2/(2*sigma**2)) for i in range(d)])

lamb = 2

theta = cvx.Variable(d, 1)
obj = cvx.Minimize(cvx.sum_squares(A*theta - y) + lamb*cvx.norm(theta, 1))
prob = cvx.Problem(obj).solve()

theta = theta.value

yp = rbfbasis*theta

plt.figure(figsize=(10, 6))
plt.title('Regularized', fontsize=15)
plt.xlabel('X', fontsize=15)
plt.ylabel('Y', fontsize=15)
plt.plot(x, y, 'o', label='Data')
plt.plot(xp, yp, label='Curve')
plt.axis([-5, 5, -12, 6])
plt.legend(fontsize=15)
plt.grid(alpha=0.3)
plt.show()

print('theta:\n', theta)
```




```
theta:
 [[  1.82161814e-09]
 [  2.52775700e-01]
 [  8.60162179e-01]
 [  1.76386240e-09]
 [  6.01095568e-10]
 [ -2.09325208e-10]
 [ -3.76940233e-08]
 [ -7.64482137e-01]
 [ -1.30850933e-09]
 [ -4.90899878e+00]]
```

In [28]:

```
# Regulization ( = ridge nonlinear regression) encourages small weights, but not exactl
y 0

plt.figure(figsize=(10, 6))
plt.title(r'magnitude of $\theta$', fontsize=15)
plt.xlabel(r'$\theta$', fontsize=15)
plt.ylabel('magnitude', fontsize=15)
plt.stem(np.linspace(1, 10, 10).reshape(-1, 1), theta)
plt.xlim([0.5, 10.5])
plt.grid(alpha=0.3)
plt.show()
```



In [29]:

```
%%javascript
$.getScript('https://kmahelona.github.io/ipython_notebook_goodies/ipython_notebook_toc.
js')
```