Ellipse and Gaussian Distribution

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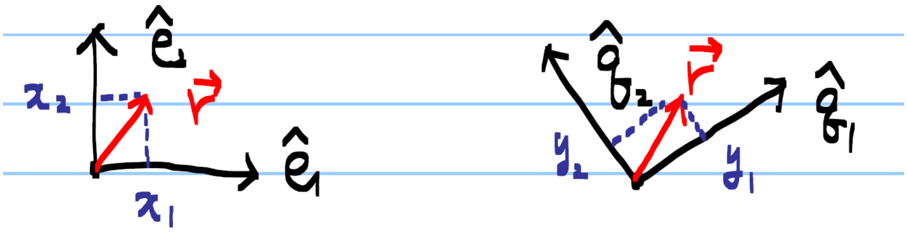
Table of Contents

1. Coordinates

- basis $\{\hat{e}_1 \;\; \hat{e}_2\}$ or basis $\{\hat{q}_1 \;\; \hat{q}_2\}$



$$\vec{r}_I = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \text{coordinate of } \vec{r} \text{ in basis } \{\hat{e}_1 \;\; \hat{e}_2\} \; (= I)$$

$$\vec{r}_Q = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : \text{coordinate of } \vec{r} \text{ in basis } \{\hat{q}_1 \;\; \hat{q}_2\} \; (= Q)$$

$$\begin{aligned} \vec{r} &= x_1\hat{e}_1 + x_2\hat{e}_2 = y_1\hat{q}_1 + y_2\hat{q}_2 \\ &= [\,\hat{e}_1 \quad \hat{e}_2\,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\,\hat{q}_1 \quad \hat{q}_2\,] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{aligned}$$

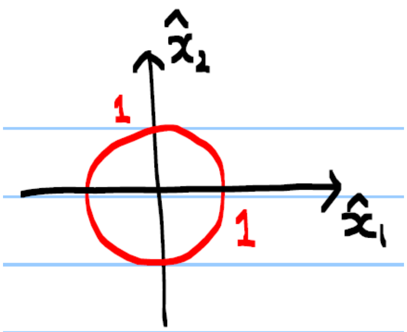
$$\begin{aligned} &\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &\Longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Q^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

- Coordinate change to basis of $\{\hat{q}_1 \hat{q}_2\}$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{coordinate in } I} \xrightarrow{Q^T} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\text{coordinate in } Q}$$

2. Equation of an Ellipse

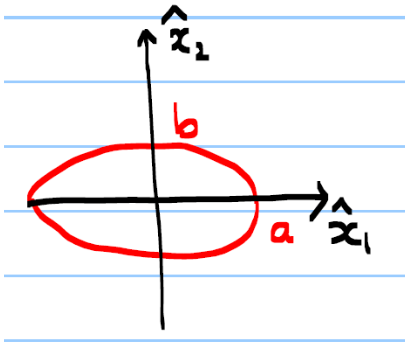
- unit circle



$$x_1^2 + x_2^2 = 1 \implies$$

$$[\,x_1 \quad x_2\,] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

- independent ellipse

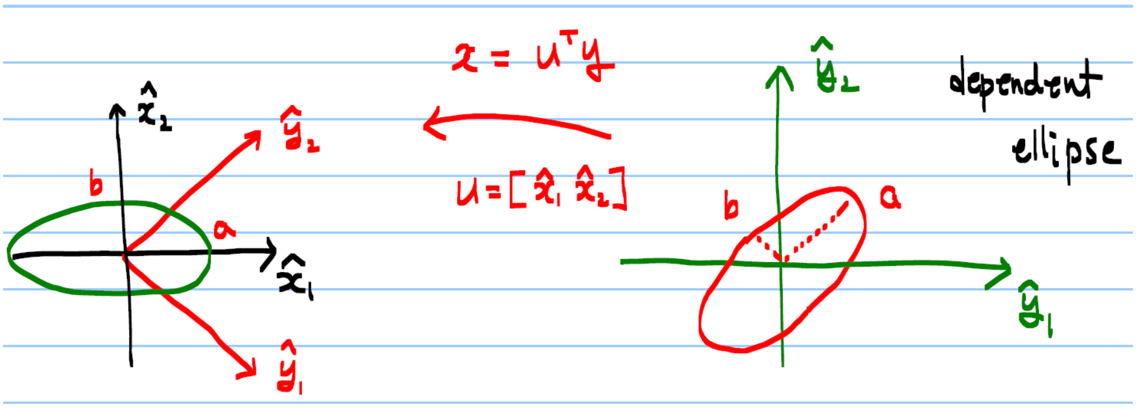


$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \implies \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

$$\implies \begin{bmatrix} x_1 & x_2 \end{bmatrix} \Sigma_x^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

$$\text{where } \Sigma_x^{-1} = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix}, \ \Sigma_x = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

- Rotated ellipse (dependent ellipse)
  - Coordinate changes



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = u^T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \begin{aligned} x &= u^T y \\ ux &= y \end{aligned}$$

- Now we know in basis $\{\hat{x}_1 \ \hat{x}_2\}$ $(= I)$

$$x^T \Sigma_x^{-1} x = 1 \quad \text{and} \quad \Sigma_x = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

- Then, we can find Σ_y such that

$$\begin{aligned} y^T \Sigma_y^{-1} y = 1 \quad &\text{and} \quad \Sigma_y = ? \\ \implies x^T \Sigma_x^{-1} x = y^T u \Sigma_x^{-1} u^T y = 1 \quad &(\Sigma_y^{-1} : \text{similar matrix to } \Sigma_x^{-1}) \end{aligned}$$

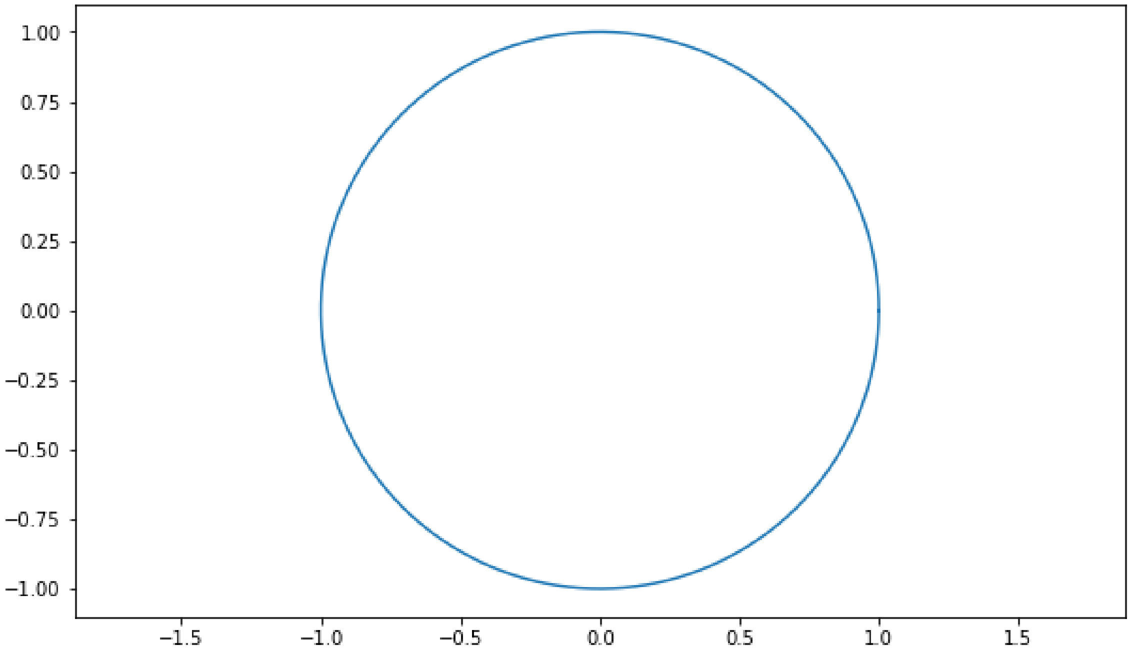
$$\begin{aligned} \therefore \ \Sigma_y^{-1} &= u \Sigma_x^{-1} u^T \quad \text{or} \\ \Sigma_y &= u \Sigma_x u^T \end{aligned}$$

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

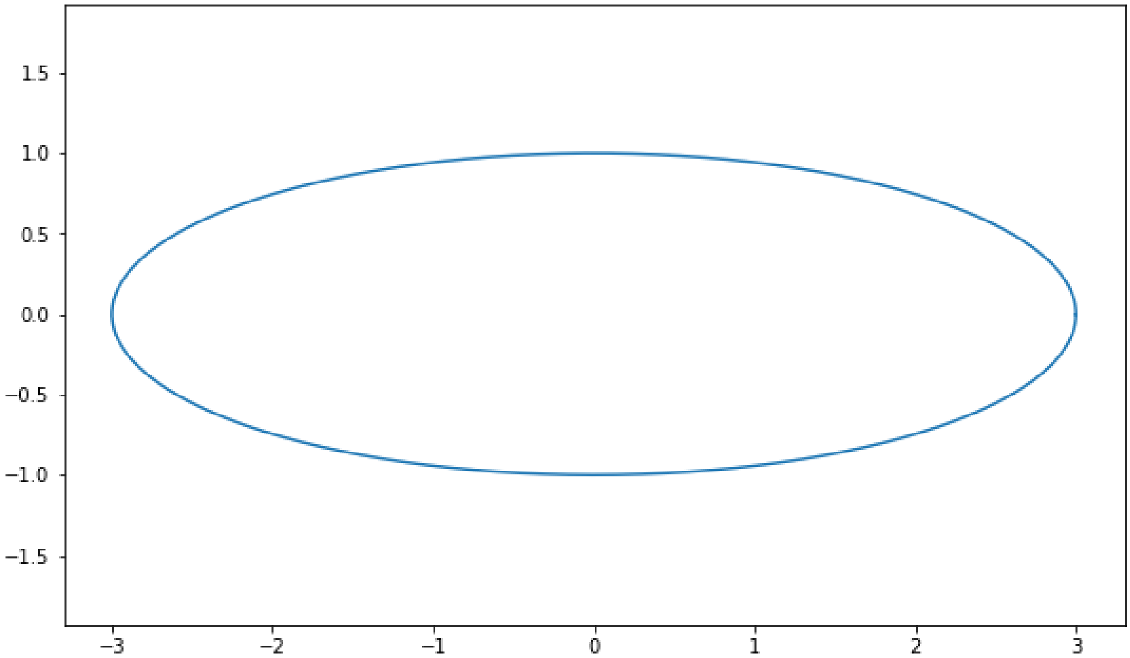
In [2]:

```
theta = np.arange(0,2*np.pi,0.01)
x1 = np.cos(theta)
x2 = np.sin(theta)

plt.figure(figsize=(10,6))
plt.plot(x1, x2)
plt.axis('equal')
plt.show()
```



In [3]:

```
x1 = 3*np.cos(theta);
x2 = np.sin(theta);

plt.figure(figsize=(10,6))
plt.plot(x1, x2)
plt.axis('equal')
plt.show()
```

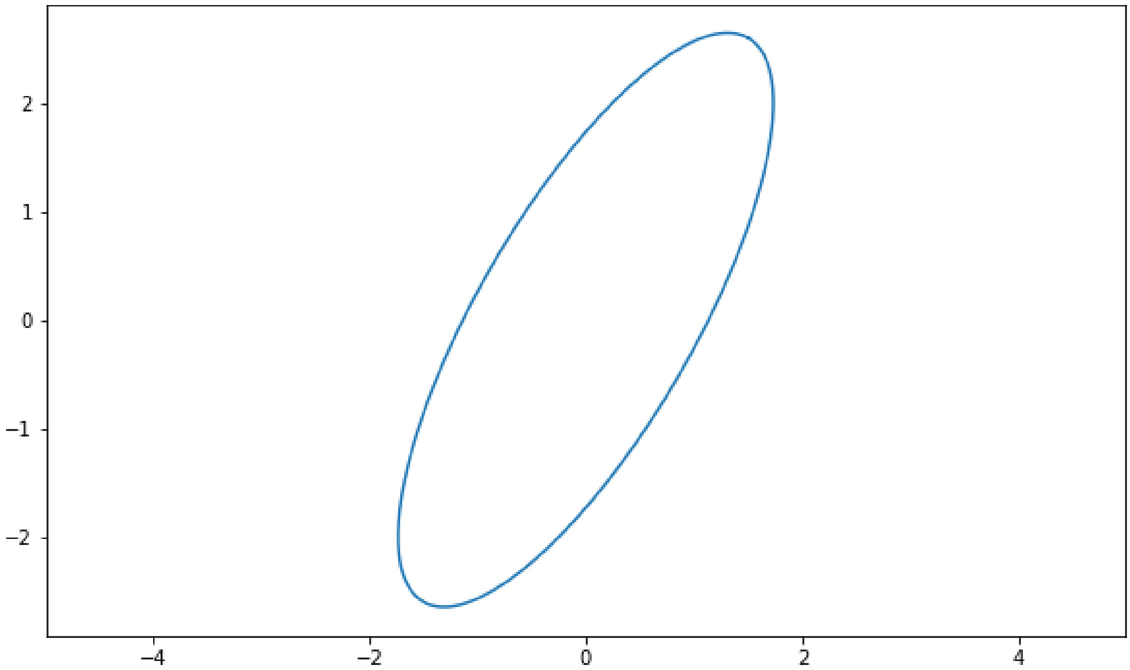


$$u = [\hat{x}_1 \ \hat{x}_2] = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

In [4]:

```
theta = np.arange(0,2*np.pi,0.01)

u = np.array([[1/2, -np.sqrt(3)/2], [np.sqrt(3)/2, 1/2]])
X = np.array([x1, x2])

u = np.asmatrix(u)
X = np.asmatrix(X)
y = u*X

print(u)
plt.figure(figsize=(10,6))
plt.plot(y[0,:].T, y[1,:].T)
plt.axis('equal')
plt.show()
```

```
[[ 0.5       -0.8660254]
 [ 0.8660254  0.5      ]]
```



In [5]:

```
Sx = np.array([[9, 0],[0, 1]])
Sx = np.asmatrix(Sx)

Sy = u*Sx*u.T

print(Sx)
print(Sy)
```

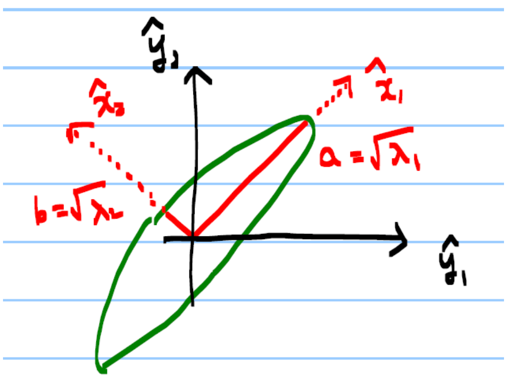
```
[[9 0]
 [0 1]]
[[ 3.          3.46410162]
 [ 3.46410162  7.        ]]
```

2.1. Question (reverse problem)

- Given Σ_y^{-1} (or Σ_y), how to find a (major axis) and b (minor axis) or **how to find the proper matrix** u
- eigenvectors of Σ

$$A = S\Lambda S^T \qquad \text{where } S = [v_1 \ v_2] \text{ eigenvector of } A, \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\text{here, } \Sigma_y = u\Sigma_x u^T = u\Lambda u^T \qquad \text{where } u = [\,\hat{x}_1 \quad \hat{x}_2\,] \text{ eigenvector of } \Sigma_y, \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} =$$



$$\text{eigen-analysis} \begin{cases} \Sigma_y \hat{x}_1 = \lambda_1 \hat{x}_1 \\ \Sigma_y \hat{x}_2 = \lambda_2 \hat{x}_2 \end{cases} \Longrightarrow \Sigma_y \underbrace{[\,\hat{x}_1 \quad \hat{x}_2\,]}_{u} = \underbrace{[\,\hat{x}_1 \quad \hat{x}_2\,]}_{u} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\Lambda}$$

$$\Sigma_y u = u\Lambda$$
$$\Sigma_y = u\Lambda u^T = u\Sigma_x u^T$$

Therefore

$$x = u^T y$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = u^T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

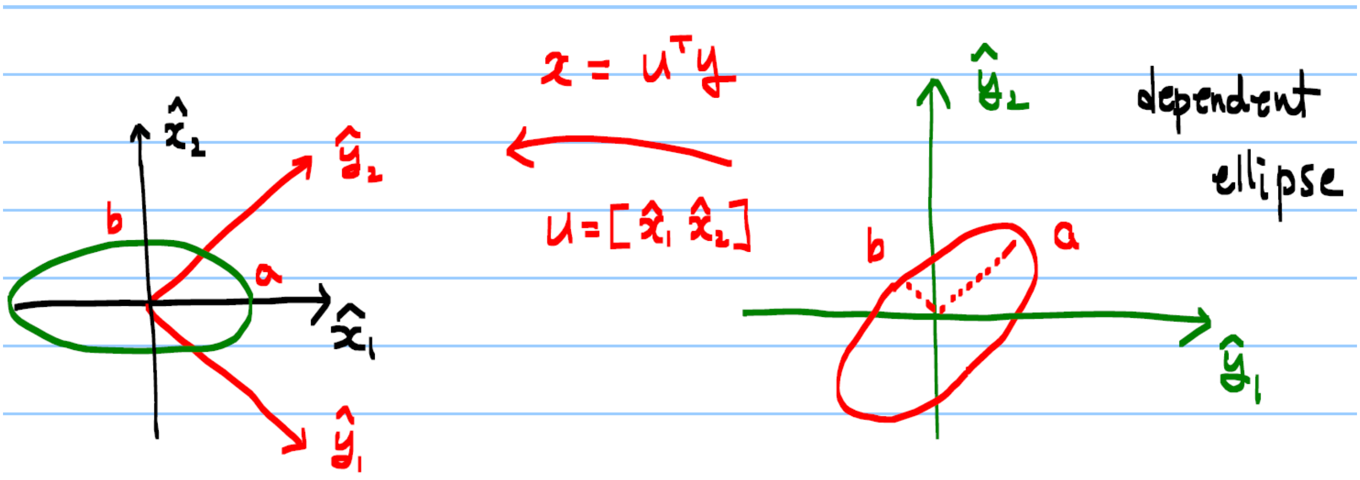
$$a = \sqrt{\lambda_1}$$
$$b = \sqrt{\lambda_2}$$
$$\text{major axis} = \hat{x}_1$$
$$\text{minor axis} = \hat{x}_2$$

In [6]:

```
D, U = np.linalg.eig(Sy)

idx = np.argsort(-D)
D = D[idx]
U = U[:,idx]

print(D)
print(np.diag(D))
print(U)
```

```
[ 9.  1.]
[[ 9.  0.]
 [ 0.  1.]]
[[-0.5        -0.8660254]
 [-0.8660254  0.5       ]]
```

2.2. Summary



$$
\begin{aligned}
x &= u^T y \\
u &= \begin{bmatrix} \hat{x}_1 & \hat{x}_2 \end{bmatrix}
\end{aligned}
$$

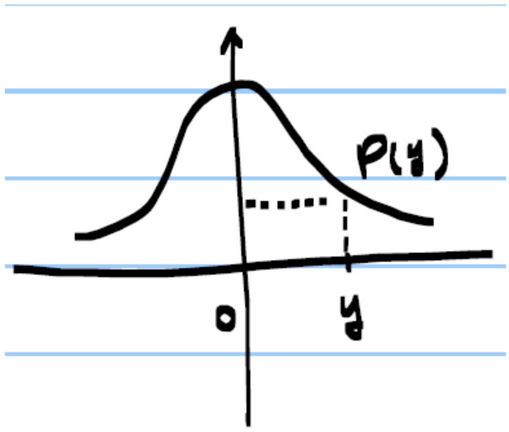
- independent ellipse in $\{\hat{x}_1, \hat{x}_2\}$
- dependent ellipse in $\{\hat{y}_1, \hat{y}_2\}$
- Decouple
  - diagonalize
  - eigen-analysis

3. Gaussian Distribution

3.1. Standard univariate normal distribution $\sim N(0, 1^2)$

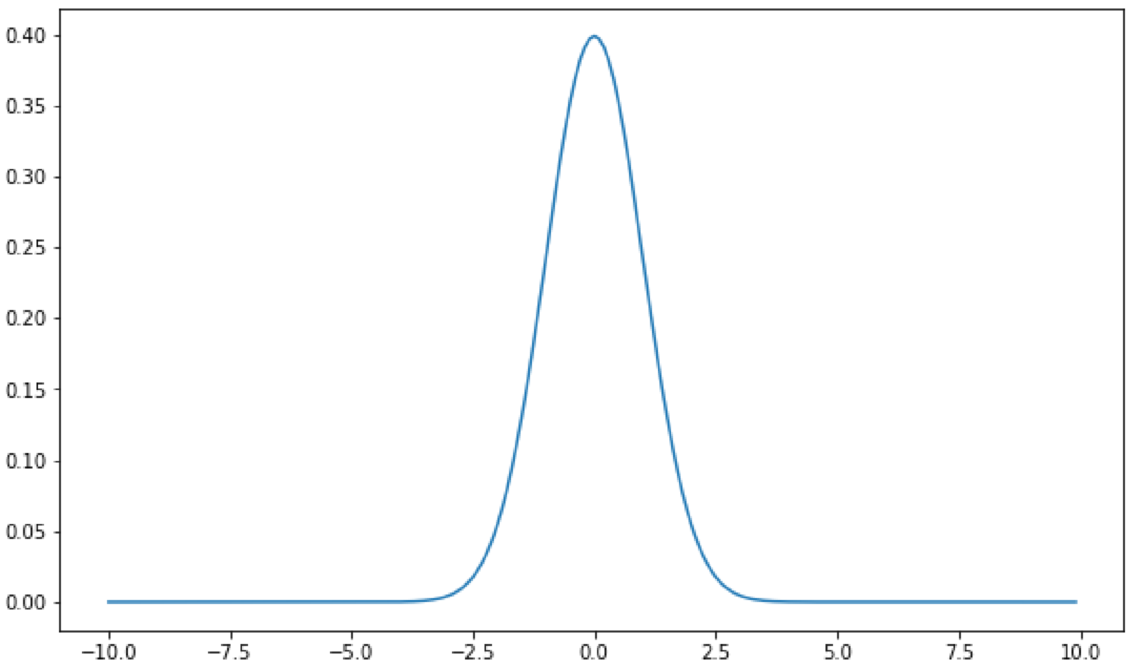
$$P_Y(Y = y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y^2\right)$$

$$\frac{1}{2} y^2 = \text{const} \implies \text{prob. contour}$$



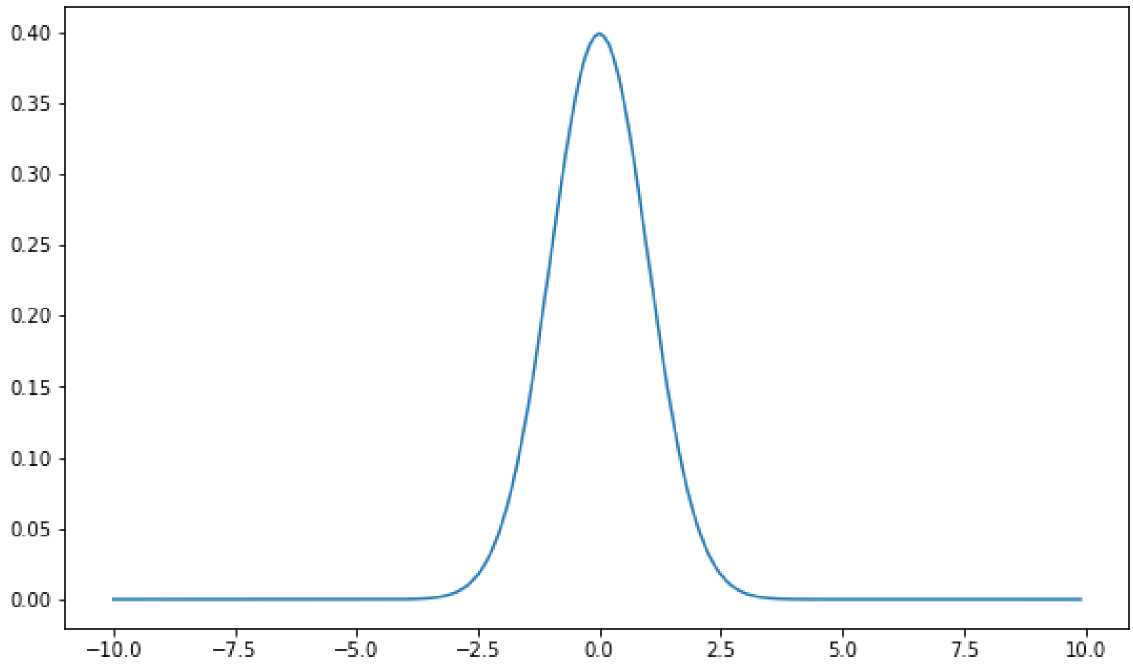
In [7]:

```
y = np.arange(-10,10,0.1)

ProbG = 1/np.sqrt(2*np.pi)*np.exp(-1/2*y**2)

plt.figure(figsize=(10,6))
plt.plot(y, ProbG)
plt.show()
```



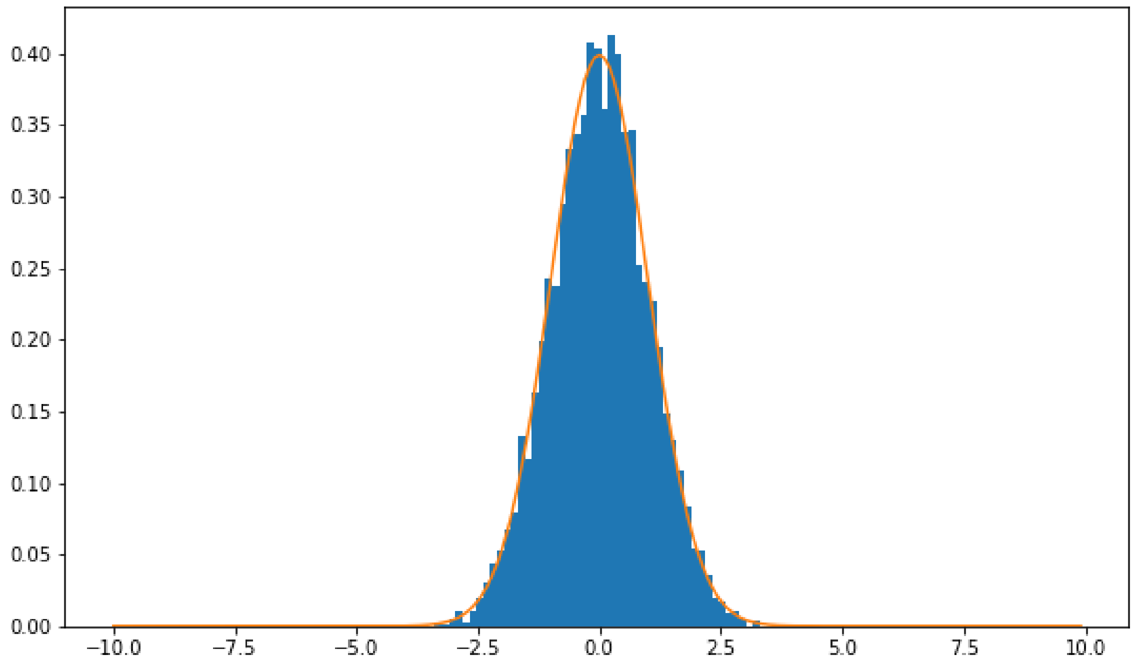
In [8]:

```
from scipy.stats import norm

ProbG2 = norm.pdf(y)

plt.figure(figsize=(10,6))
plt.plot(y, ProbG2)
plt.show()
```



In [9]:

```
x = np.random.randn(5000,1)

plt.figure(figsize=(10,6))
plt.hist(x, bins=51, normed=True)
plt.plot(y,ProbG2, label='G2')
plt.show()
```

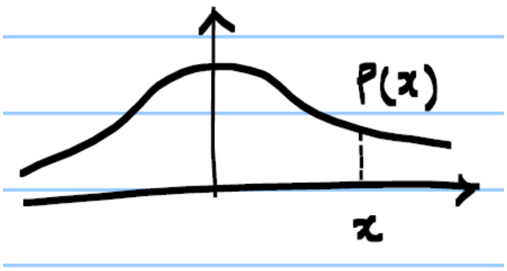


3.2. Univariate Normal distribution $\sim N\left(\mu, \sigma^2\right)$

- Gaussian or normal distribution, 1D (mean μ, variance σ^2)

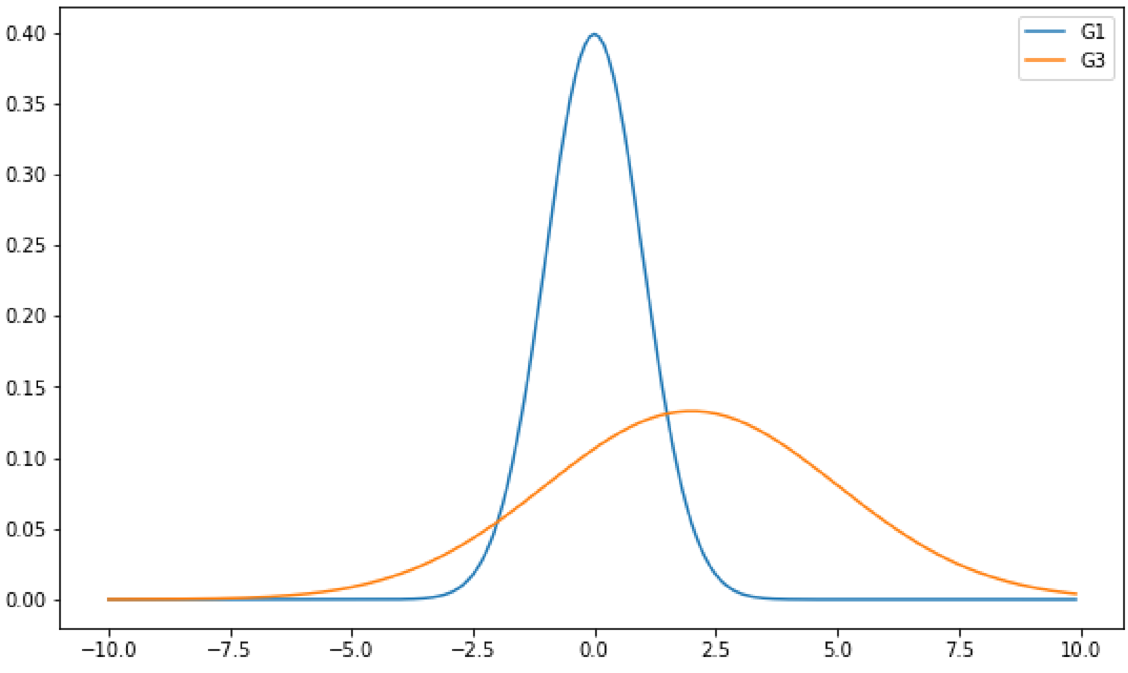
$$N(x;\ \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

$$\begin{aligned}
x &\sim N\left(\mu, \sigma^2\right) \\
&\implies P_Y\left(y\right) = P_X\left(x\right), \quad y = \frac{x-\mu}{\sigma} \\
P_X\left(X = x\right) &\sim \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \\
&= \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)
\end{aligned}$$



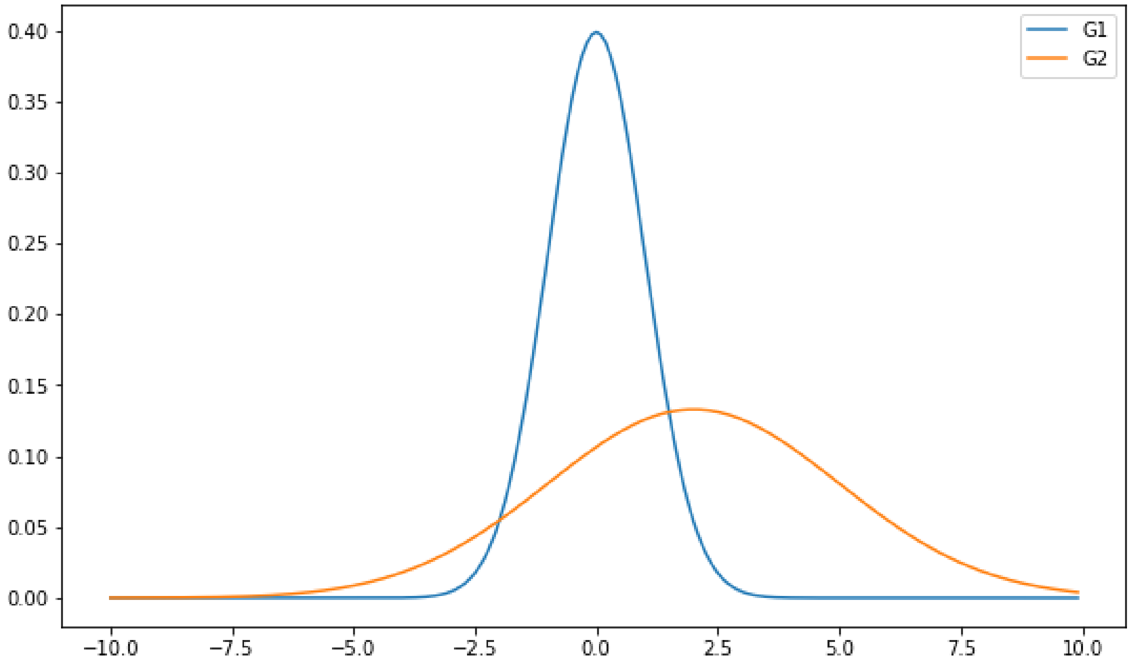
In [10]:

```
mu = 2
sigma = 3

x = np.arange(-10, 10, 0.1)

ProbG3 = 1/(np.sqrt(2*np.pi)*sigma) * np.exp(-1/2*(x-mu)**2/(sigma**2))

plt.figure(figsize=(10,6))
plt.plot(y,ProbG, label='G1')
plt.plot(x,ProbG3, label='G3')
plt.legend()
plt.show()
```



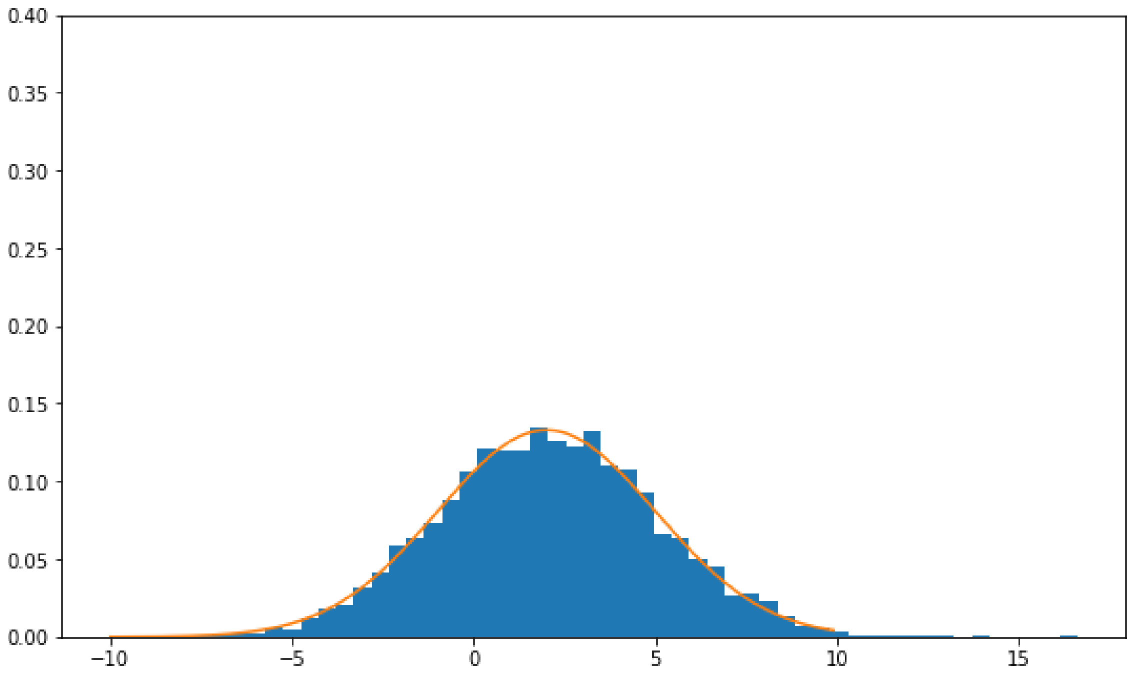
In [11]:

```
ProbG2 = norm.pdf(x, 2, 3)

plt.figure(figsize=(10,6))
plt.plot(y,ProbG, label='G1')
plt.plot(x,ProbG2, label='G2')
plt.legend()
plt.show()
```



In [12]:

```
x = mu + sigma*np.random.randn(5000,1)

plt.figure(figsize=(10,6))
plt.hist(x, bins=51, normed=True)
plt.plot(y,ProbG2, label='G2')
plt.ylim([0,0.4])
plt.show()
```



3.3. Multivariate Gaussian models

- Similar to a univariate case, but in a matrix form

$$N\big(x;\,\mu,\Sigma\big) = \frac{1}{(2\pi)^{\frac{n}{2}}\lvert\Sigma\rvert^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}\,(x-\mu)\right)$$

$$\begin{aligned} \mu &= \text{length } n \text{ column vector} \\ \Sigma &= n\times n \text{ matrix (covariance matrix)} \\ \lvert\Sigma\rvert &= \text{matrix determinant} \end{aligned}$$

- Multivariate Gaussian models and ellipse
  - Ellipse shows constant Δ^2 value...

$$\Delta^2 = (x-\mu)^T\Sigma^{-1}(x-\mu)$$

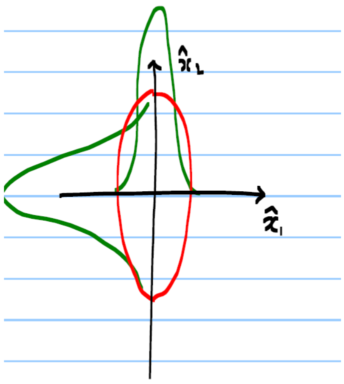
3.3.1. Two *independent* variables

$$
\begin{aligned}
P\left(X_1=x_1, X_2=x_2\right) &= P_{X_1}\left(x_1\right) P_{X_2}\left(x_2\right) \\
&\sim \exp\left(-\frac{1}{2} \frac{\left(x_1-\mu_{x_1}\right)^2}{\sigma_{x_1}^2}\right) \cdot \exp\left(-\frac{1}{2} \frac{\left(x_2-\mu_{x_2}\right)^2}{\sigma_{x_2}^2}\right) \\
&\sim \exp\left(-\frac{1}{2}\left(\frac{x_1^2}{\sigma_{x_1}^2}+\frac{x_2^2}{\sigma_{x_2}^2}\right)\right)
\end{aligned}
$$

- in a matrix form

$$
P(x_1) \cdot P(x_2) = \frac{1}{Z_1 Z_2} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)
$$

$$
\left(x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{bmatrix}\right)
$$



$$
\frac{x_1^2}{\sigma_{x_1}^2} + \frac{x_2^2}{\sigma_{x_2}^2} = c \quad \text{(ellipse)}
$$

$$
\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_{x_1}^2} & 0 \\ 0 & \frac{1}{\sigma_{x_2}^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \qquad (\sigma_{x_1} < \sigma_{x_2})
$$

- Summary in a matrix form

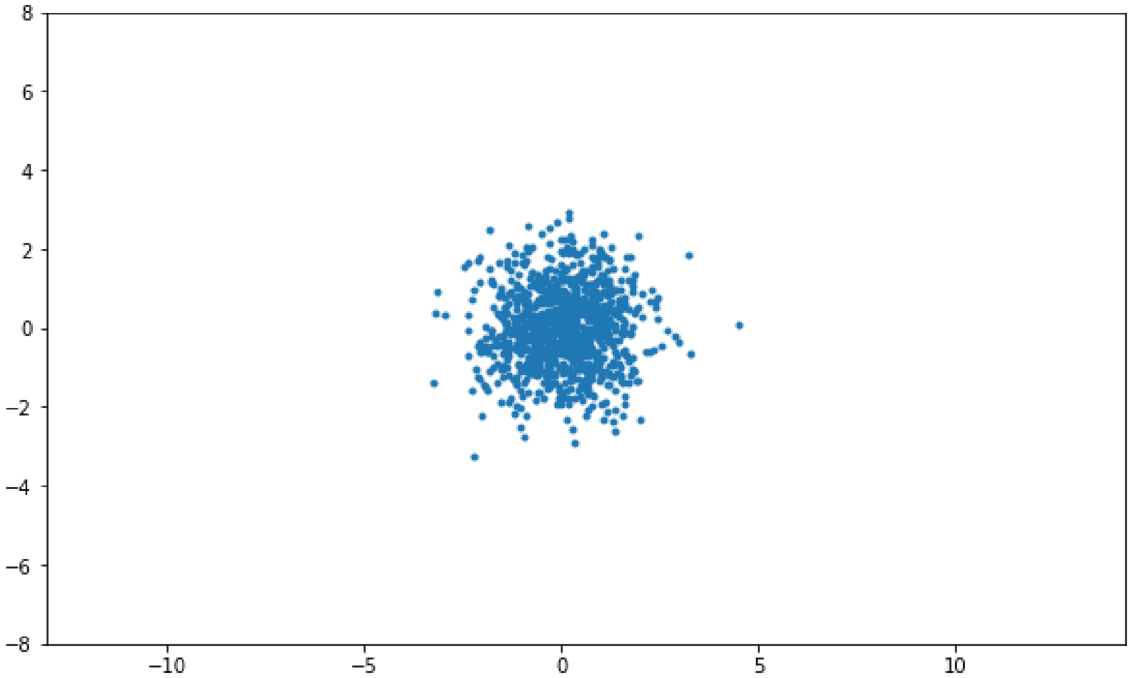
$$
N\left(0, \Sigma_x\right) \sim \exp\left(-\frac{1}{2} x^T \Sigma_x^{-1} x\right)
$$

$$
N\left(\mu_x, \Sigma_x\right) \sim \exp\left(-\frac{1}{2}\left(x-\mu_x\right)^T \Sigma_x^{-1}\left(x-\mu_x\right)\right)
$$

In [13]:

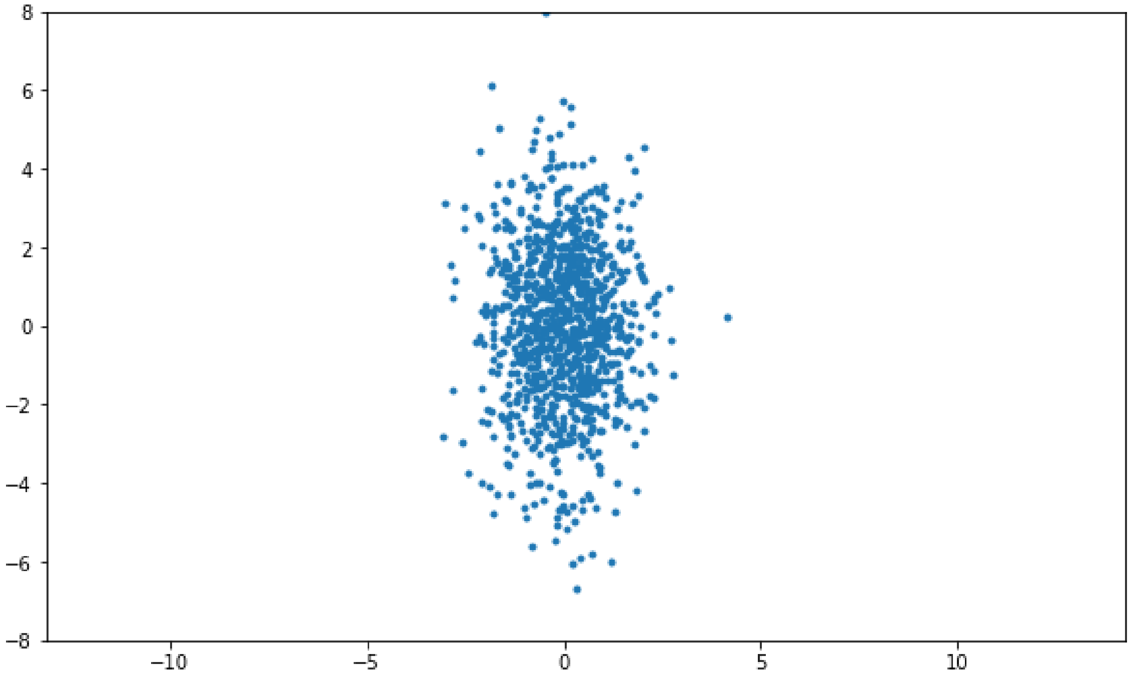
```
mu = np.array([0, 0])
sigma = np.eye(2)

m = 1000
x = np.random.multivariate_normal(mu, sigma, m)
print(x.shape)

plt.figure(figsize=(10,6))
plt.plot(x[:,0], x[:,1], '.')
plt.axis('equal')
plt.ylim([-8,8])
plt.show()
```

```
(1000, 2)
```

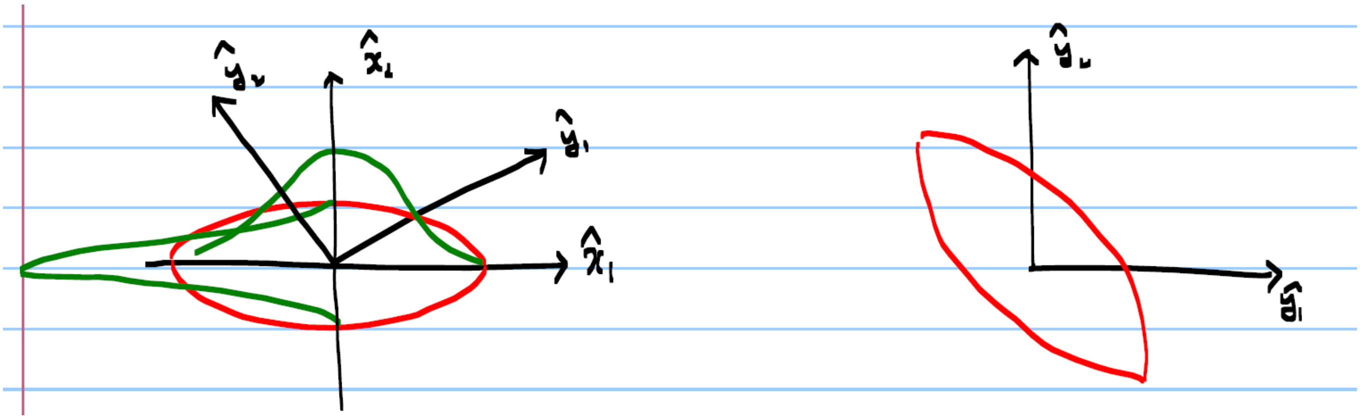


In [14]:

```
mu = np.array([0, 0])
sigma = np.array([[1, 0], [0, 4]])

m = 1000
x = np.random.multivariate_normal(mu, sigma, m)

plt.figure(figsize=(10,6))
plt.plot(x[:,0], x[:,1], '.')
plt.axis('equal')
plt.ylim([-8,8])
plt.show()
```



3.3.1. Two *dependent* variables in $\{y_1, y_2\}$



- Compute $P_Y(y)$ from $P_X(x)$

$$P_X(x) = P_Y(y) \;\; \text{where} \;\; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Relationship between y and x

$$x = [\, \hat{x}_1 \quad \hat{x}_2 \,]^T y = u^T y$$

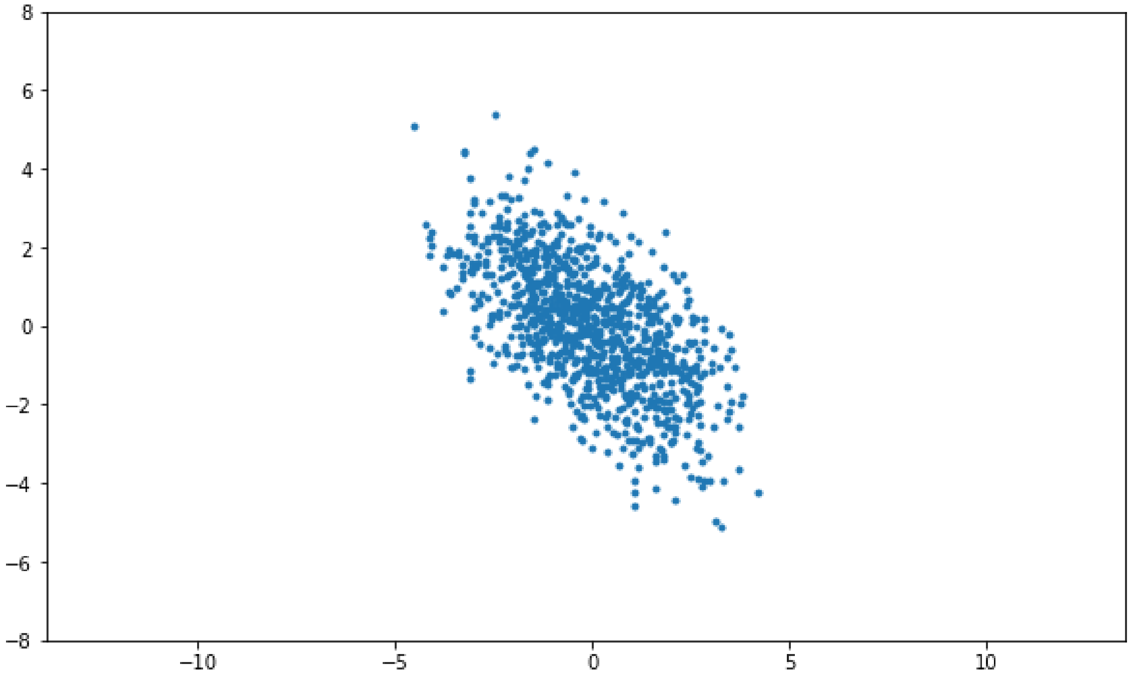
$$x^T \Sigma_x^{-1} x = y^T u \Sigma_x^{-1} u^T y = y^T \Sigma_y^{-1} y$$
$$\therefore \;\; \Sigma_y^{-1} = u \Sigma_x^{-1} u^T$$
$$\rightarrow \Sigma_y = u \Sigma_x u^T$$

- Σ_x : covariance matrix of x
- Σ_y : covariance matrix of y

- If u is an eigenvector matrix of Σ_y, then Σ_x is a diagonal matrix

In [15]:

```
mu = np.array([0, 0])
sigma = 1./2.*np.array([[5, -3], [-3, 5]])

m = 1000
x = np.random.multivariate_normal(mu, sigma, m)

plt.figure(figsize=(10,6))
plt.plot(x[:,0], x[:,1], '.')
plt.axis('equal')
plt.ylim([-8,8])
plt.show()
```



Remark

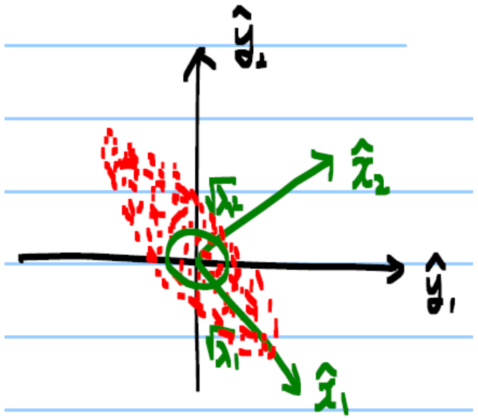
$$x \sim N(\mu_x, \Sigma_x) \text{ and } y = Ax + b \text{ affine transformation}$$

$$\implies \; y \sim N(\mu_y, \Sigma_y) = N(A\mu_x + b, A\Sigma_x A^T)$$
$$\implies \; y \; \text{ is also Gaussian with } \; \mu_y = Ax + b, \;\; \Sigma_y = A\Sigma_x A^T$$

3.4. Decouple using covariance matrix

Given data, how to find Σ_y and major (or minor) axis (assume $\mu_y = 0$)

$$\Sigma_y = \begin{bmatrix} \text{var}(y_1) & \text{cov}(y_1, y_2) \\ \text{cov}(y_2, y_1) & \text{var}(y_2) \end{bmatrix}$$



eigen-analysis

$$\Sigma_y \hat{x}_1 = \lambda_1 \hat{x}_1$$
$$\Sigma_y \hat{x}_2 = \lambda_2 \hat{x}_2$$

$$\Sigma_x^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}^2} & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}^2} \end{bmatrix}$$

$$\Sigma_x = \begin{bmatrix} \sqrt{\lambda_1}^2 & 0 \\ 0 & \sqrt{\lambda_2}^2 \end{bmatrix}$$

$$\begin{aligned} \Sigma_y \begin{bmatrix} \hat{x}_1 & \hat{x}_2 \end{bmatrix} &= \begin{bmatrix} \hat{x}_1 & \hat{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} \hat{x}_1 & \hat{x}_2 \end{bmatrix} \Sigma_x \end{aligned}$$

$$y = ux \implies u^T y = x$$
$$\begin{bmatrix} \hat{x}_1 & \hat{x}_2 \end{bmatrix} = u$$

$$\Sigma_y = u \Sigma_x u^T$$

In [16]:

```
x.shape

S = np.cov(x.T)
print ("S = \n", S)
```

```
S =
 [[ 2.59216411 -1.54924881]
 [-1.54924881  2.54567035]]
```

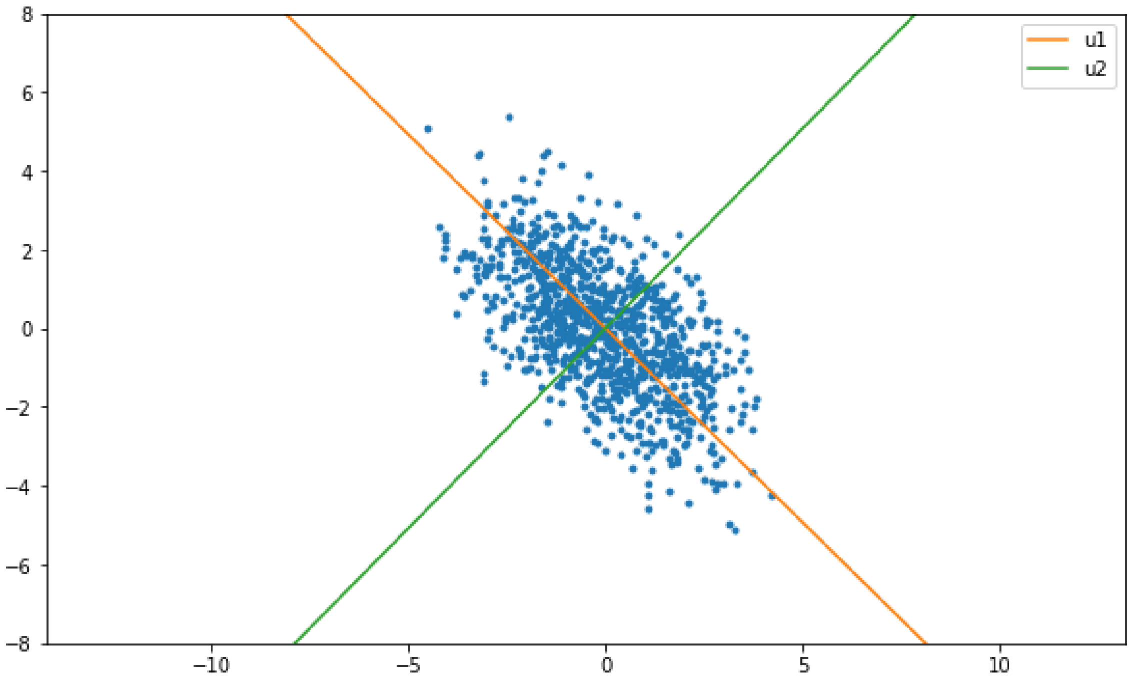
In [17]:

```
D, U = np.linalg.eig(S)

idx = np.argsort(-D)
D = D[idx]
U = U[:,idx]

print ("U = \n", U)
print ("D = \n", D)
```

```
U =
 [[ 0.7123916   0.70178217]
 [-0.70178217  0.7123916 ]]
D =
 [ 4.11834045  1.01949402]
```

In [18]:

```
xp = np.arange(-10, 10)

plt.figure(figsize=(10,6))
plt.plot(x[:,0],x[:,1],'.')
plt.plot(xp, U[1,0]/U[0,0]*xp, label='u1')
plt.plot(xp, U[1,1]/U[0,1]*xp, label='u2')
plt.axis('equal')
plt.ylim([-8, 8])
plt.legend()
plt.show()
```



In [19]:

```
%%javascript
$.getScript('https://kmahelona.github.io/ipython_notebook_goodies/ipython_notebook_toc.
js')
```