Linear Transformation and Eigen Analysis

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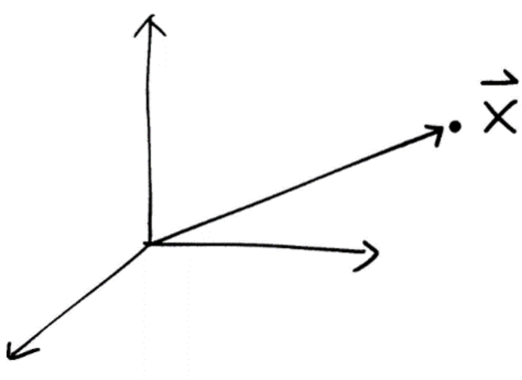
1. Matrix and Transformation

Vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Matrix and Transformation

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$\begin{aligned} \vec{y} &= M\vec{x} \\ \begin{bmatrix} \\ \\ \\ \end{bmatrix} &= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} \end{aligned}$$

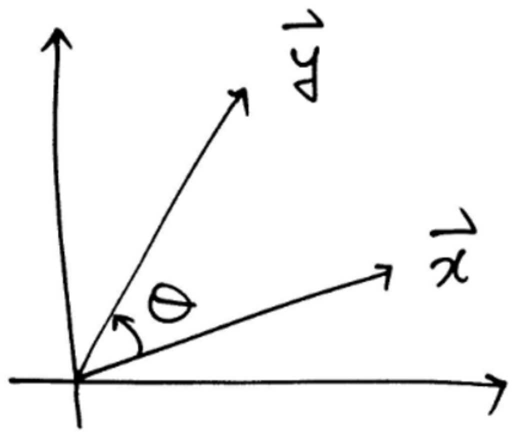
$$\begin{array}{lcl} \text{Given} & & \text{Interpret} \\ \text{Transformation} & \longrightarrow & \text{matrix} \\ \text{matrix} & \longrightarrow & \text{Transformation} \end{array}$$

$$\begin{array}{ccc} \vec{x} & \text{transformation} & \vec{y} \\ \text{input} & \Longrightarrow & \text{output} \end{array}$$

$$\text{transformation} = \text{rotate} + \text{stretch/compress}$$

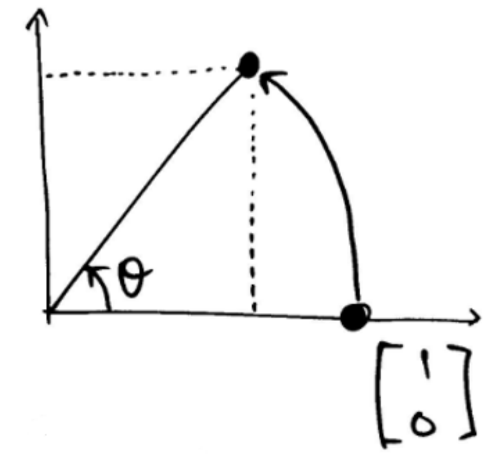
1.1. Rotation

Rotation : $R(\theta)$

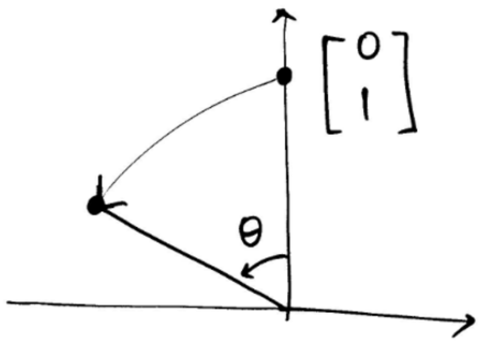


$$\vec{y} = R(\theta)\vec{x}$$

Find matrix $R(\theta)$



$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



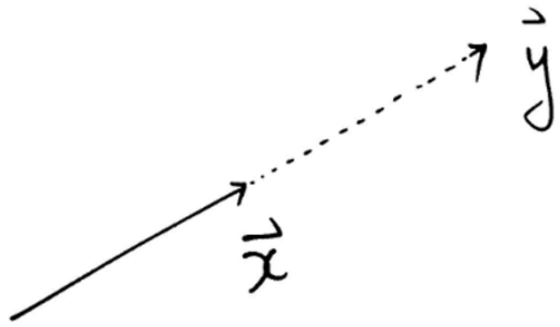
$$\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R(\theta)$$

$$\begin{aligned} M\vec{x}_1 &= \vec{y}_1 \\ M\vec{x}_2 &= \vec{y}_2 \end{aligned} \quad = \quad M\,[\,\vec{x}_1 \quad \vec{x}_2\,] \quad = \quad [\,\vec{y}_1 \quad \vec{y}_2\,]$$

1.2. Stretch/Compress

Stretch/Compress (keep the direction)



$$
\begin{aligned}
\vec{y} = \quad & k\vec{x} \\
& \uparrow \\
& \text{scalar (not matrix)} \\
\\
\vec{y} = \quad & kI\vec{x} \qquad\qquad\qquad\qquad \text{where } I = \text{ Identity martix} \\
\\
\vec{y} = \quad & \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x}
\end{aligned}
$$

Example

T: stretch a along $\hat{x}$-direction & stretch b along $\hat{y}$-direction

compute the corresponding matrix A

$$\hat{y} = A\hat{x}$$

$$
\begin{aligned}
\begin{bmatrix} ax_1 \\ bx_2 \end{bmatrix} \quad & = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow A = ? \\
& = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

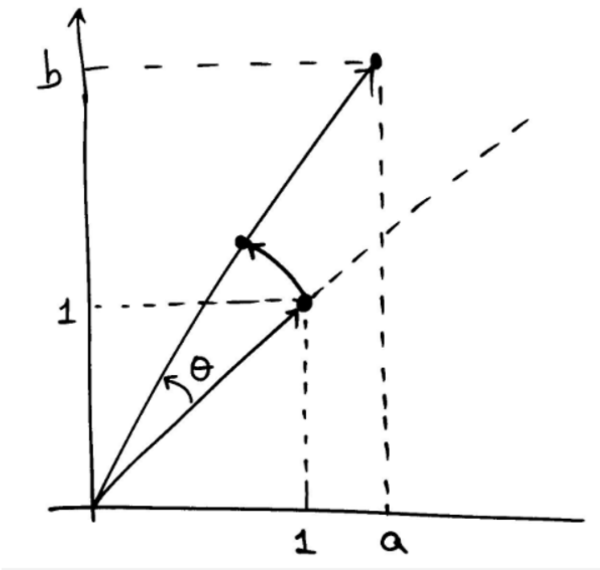
$$
\begin{aligned}
A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad & = \begin{bmatrix} a \\ 0 \end{bmatrix} \\
A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad & = \begin{bmatrix} 0 \\ b \end{bmatrix} \\
A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad & = A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}
\end{aligned}
$$

More importantly, by looking at $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, can you think of a transformation T?

Decomposition

T = rotate + stretch

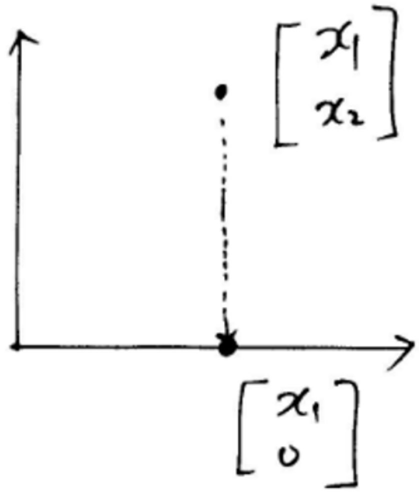
1. rotate θ, then
2. stretch

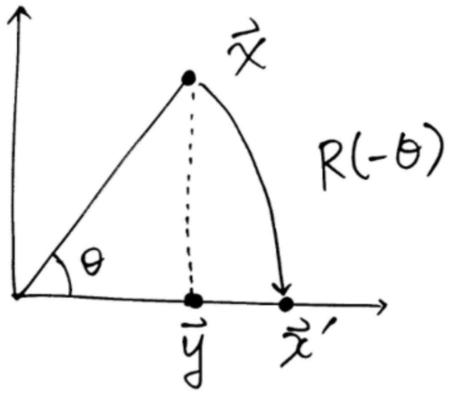


1.3. Projection

P: Projection onto $\hat{x}$ - axis

$$\underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \overset{P}{\Longrightarrow} \underset{\vec{y}}{\begin{bmatrix} x_1 \\ 0 \end{bmatrix}}$$





$$\vec{y} = P\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} P\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ P\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

1.4. Multiple Transformations

T_1 : transformation 1 : M_1

T_2 : transformation 2 : M_2

T : Do transforamtion 1, followed by transformation 2

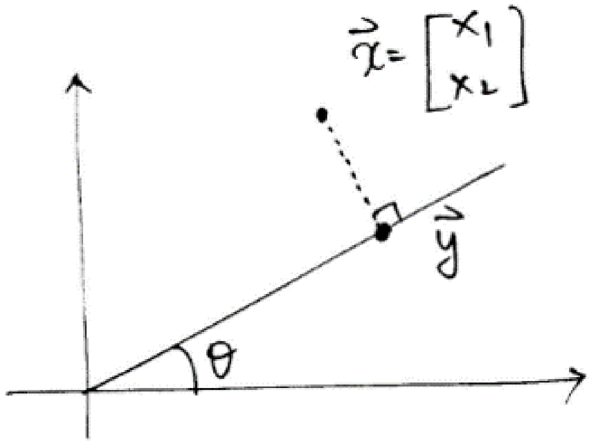
$$\therefore\ M = M_2 M_1$$

$$\vec{x} \xrightarrow{T_1} \vec{y} \xrightarrow{T_2} \vec{z}$$

$$\begin{aligned} \vec{y} &= M_1\vec{x} \\ \vec{z} &= M_2\vec{y} &&= M_2M_1\vec{x} \\ & &&= M\vec{x} \end{aligned}$$

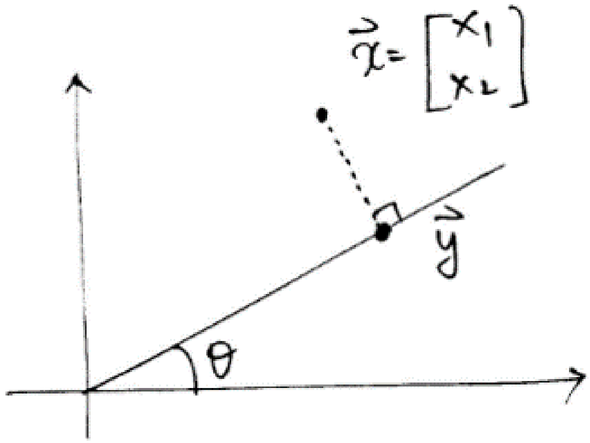
Example

P: Projection onto vector = $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



$$
\begin{aligned}
P\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{bmatrix} \\
P\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix} \\
P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}
\end{aligned}
$$

Another way to find this projection matrix

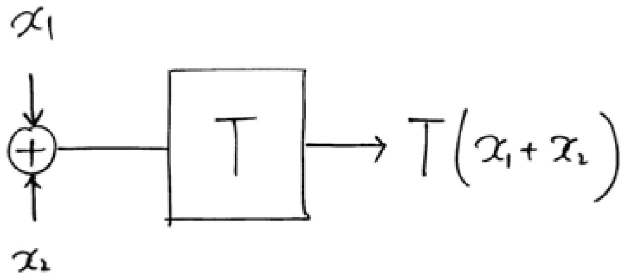


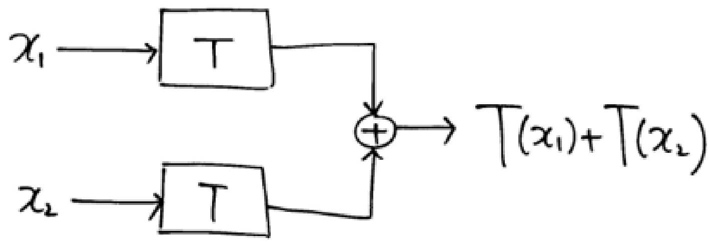
$$\begin{array}{ccccccc} & R(-\theta) & & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & & R(\theta) & \\ \vec{x} & \Longrightarrow & \vec{x}' & \Longrightarrow & \vec{x}'' & \Longrightarrow & \vec{y} \end{array}$$

$$\begin{aligned} \vec{y} &= R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(-\theta)\vec{x} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \end{aligned}$$

2. Linear Transformation

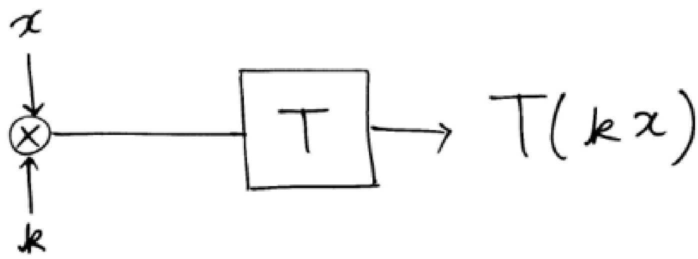
- Superposition

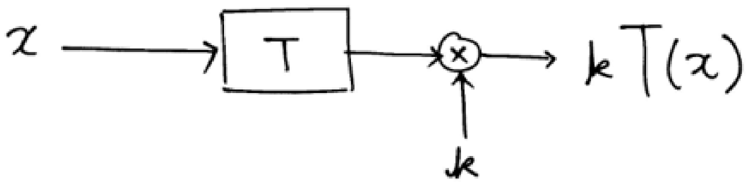




$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

- Homogeniety





$$T(kx) = kT(x)$$

Linear vs. Non-linear

$$
\begin{array}{cc}
\text{linear} & \text{non-linear} \\
f(x) = 0 & f(x) = x + c \\
f(x) = kx & f(x) = x^2 \\
f(x(t)) = \frac{dx(t)}{dt} & f(x) = \sin x \\
f(x(t)) = \int_a^b x(t)dt &
\end{array}
$$

Linear Transformation

If $\vec{v}_1$ and $\vec{v}_2$ are basis, and we know $T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = \vec{w}_2$

Then, for any $\vec{x}$

$$
\begin{aligned}
\vec{x} \quad &= a_1\vec{v}_1 + a_2\vec{v}_2 \qquad\qquad (a_1 \text{ and } a_2 \text{ unique}) \\
\\
T(\vec{x}) \quad &= T(a_1\vec{v}_1 + a_2\vec{v}_2) \\
&= a_1T(\vec{v}_1) + a_2T(\vec{v}_2) \\
&= a_1\vec{\omega}_1 + a_2\vec{\omega}_2 \\
\\
\implies \quad & \text{T: linear}
\end{aligned}
$$

3. Eigenvalue and Eigenvector

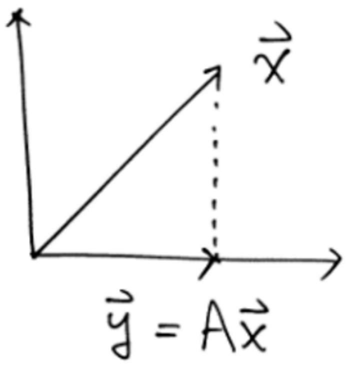
$$A\vec{v} = \lambda\vec{v}$$

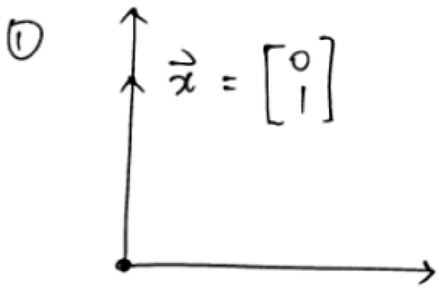
$$
\begin{array}{lcl}
\lambda & = & \begin{cases} \text{positive} \\ 0 \\ \text{negative} \end{cases} \\
\lambda & : & \text{stretched vector} \\
 & & (\text{same direction with } \vec{x}) \\
A & : & \text{transformed vector} \\
 & & (\text{generally rotate + stretch}) \\
 & & \quad A\vec{x} \text{ parallel to } \vec{x}
\end{array}
$$

Example 1

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: projection onto $\hat{x}$- axis

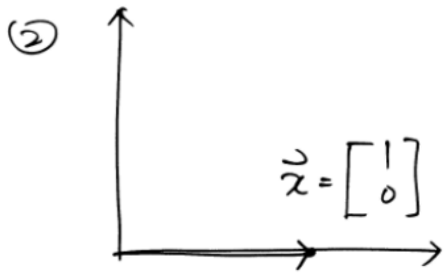
Find eigenvalues and eigenvectors.





$$\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\vec{x} = 0 \cdot \vec{x}$$

$$\lambda_1 = 0 \text{ and } \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\vec{x} = 1 \cdot \vec{x}$$

$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In [1]:

```python
import numpy as np

A = np.array([[1, 0],
              [0, 0]])
D, V = np.linalg.eig(A)
print('D :', D)
print('V :', V)
```

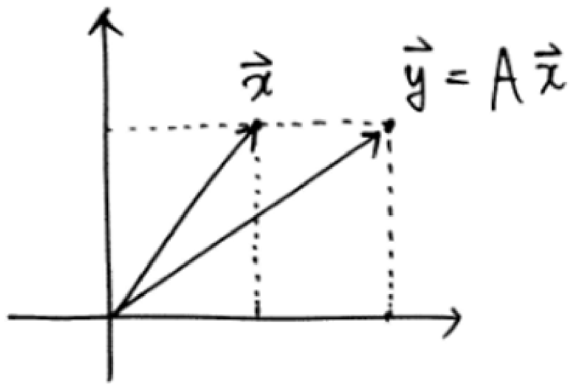
```
D : [ 1.  0.]
V : [[ 1.  0.]
 [ 0.  1.]]
```

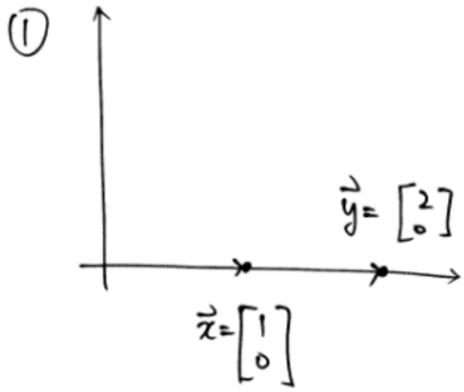
Example 2

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} : \quad \text{stretch by 2 along } \vec{x}\text{- axis}$$

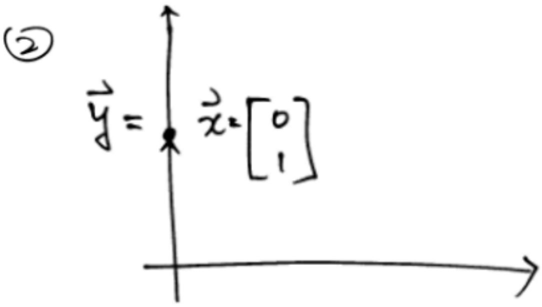
$$\text{stretch by 1 along } \vec{y}\text{- axis}$$

Find eigenvalues and eigenvectors.





$$\lambda_1 = 2 \text{ and } \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

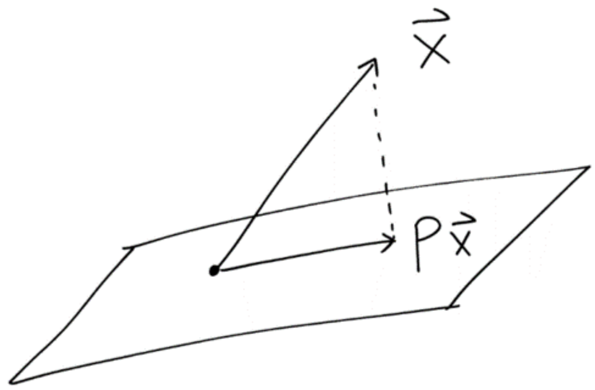
In [2]:

```
A = np.array([[2, 0],
              [0, 1]])
D, V = np.linalg.eig(A)

idx = np.argsort(-D)
D = D[idx]
V = V[idx]

print('D :', D)
print('V :', V)
```

```
D : [ 2.  1.]
V : [[ 1.  0.]
 [ 0.  1.]]
```

Example 3

Projection onto the plane. Find eigenvalues and eigenvectors.



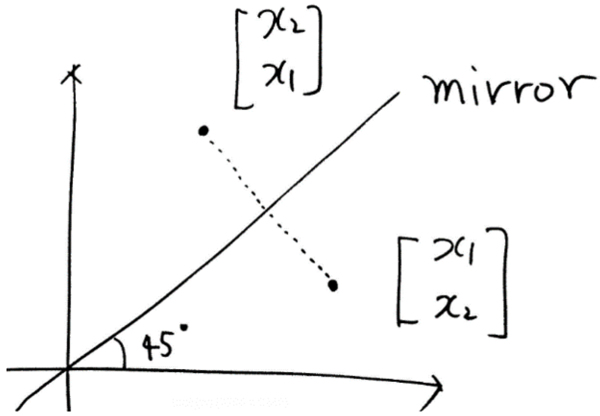
For any $\vec{x}$ in the plane, $P\vec{x} = \vec{x} \Rightarrow \lambda = 1$

For any $\vec{x}$ perpendicular to the plane, $P\vec{x} = \vec{0} \Rightarrow \lambda = 0$

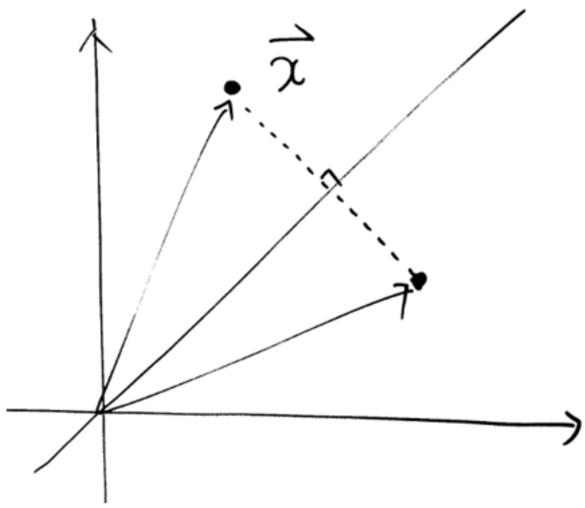
Example 4

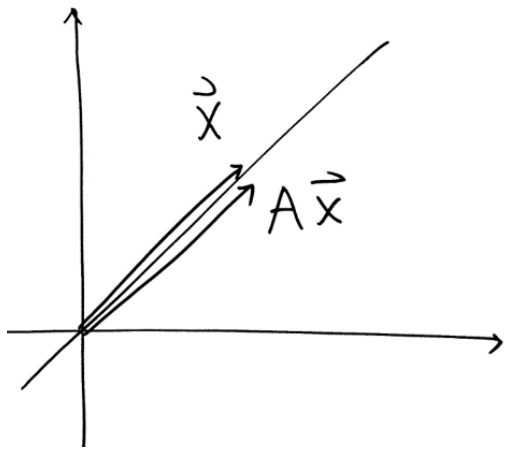
- What kind of a linear transformation?

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

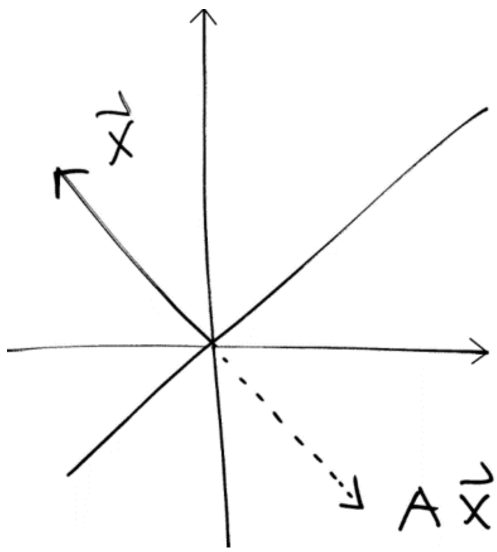


- Eigenvalues and eigenvectors?
    - can $\vec{x}$ be an eigenvector?





$$A\vec{x} = \vec{x}, \quad \lambda = 1$$



$$A\vec{x} = -\vec{x}, \quad \lambda = -1$$

- Side note : Matrix A can be seen as a multiple transformations

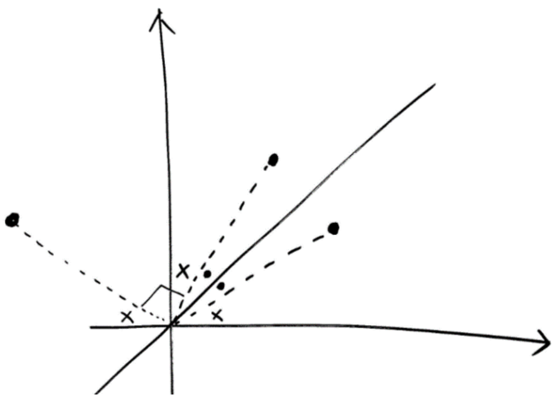
$$A = R(45)MR(-45)$$

$$\begin{aligned} R(45) \quad &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ M \quad &: \text{ mirror along } \hat{x}\text{- axis}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\ A \quad &= \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

Example 5

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = R\left(\frac{\pi}{2}\right) = R(90^\circ) = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$$



- Side note: Multiple transformations

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Eigenvalues: complex number

$$\begin{aligned} \Rightarrow \quad \det(A - \lambda I) &= 0 \\ \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} &= \lambda^2 + 1 = 0 \\ &\therefore\ \lambda = \pm i \end{aligned}$$

- What is the physical meaning?

How to Compute Eigenvalue & Eigenvector

$$\begin{gathered} A\vec{v} = \lambda\vec{v} = \lambda I\vec{v} \\ A\vec{v} - \lambda I\vec{v} = (A - \lambda I)\vec{v} = 0 \end{gathered}$$

$$\begin{aligned} \implies \quad & A - \lambda I = 0 \text{ or} \\ & \vec{v} = 0 \text{ or} \\ & (A - \lambda I)^{-1} \text{ does not exist} \\ \implies \quad & \det(A - \lambda I) = 0 \end{aligned}$$