Linear Algebra 2

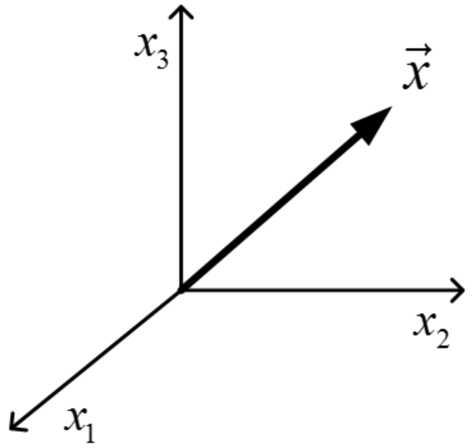
Industrial AI Lab.

Prof. Seungchul Lee

Vector

- Vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Matrix and (Linear) Transformation

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

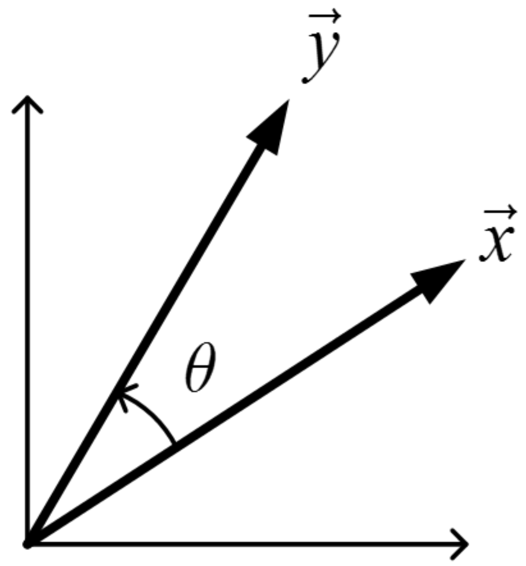
$$\begin{aligned} \vec{y} &= M\vec{x} \\ \begin{bmatrix} \\ \\ \\ \end{bmatrix} &= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix} \end{aligned}$$

Given		Interpret
linear transformation	$\longrightarrow$	matrix
matrix	$\longrightarrow$	linear transformation

$$\begin{array}{ccc} \vec{x} & \text{linear transformation} & \vec{y} \\ \text{input} & \Longrightarrow & \text{output} \end{array}$$

Rotation

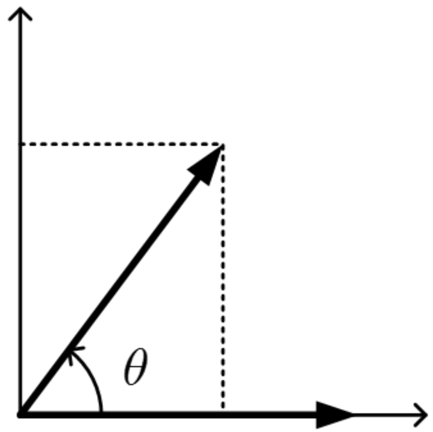
- Rotation matrix: $M = R(\theta)$
- Transformation: $\vec{y} = R(\theta)\vec{x}$



Rotation

- To find matrix $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$



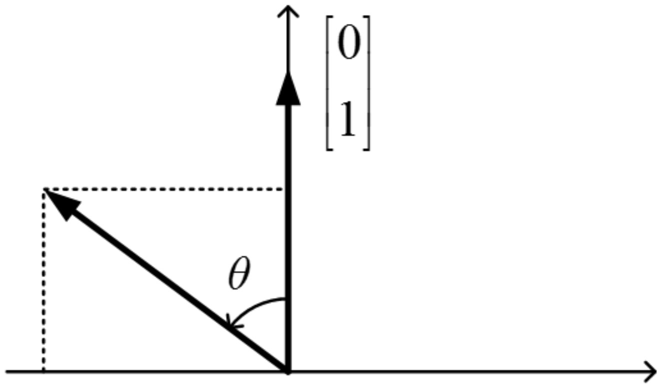
$$\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} = R(\theta)\begin{bmatrix}1\\0\end{bmatrix}$$

Rotation

- To find matrix $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$



$$\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Rotation

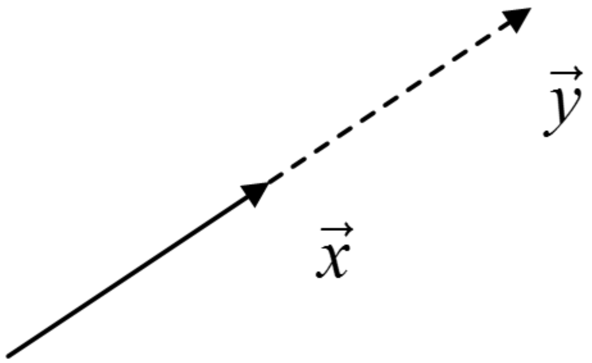
- To find matrix $M = R(\theta)$

$$\begin{aligned} M\vec{x}_1 &= \vec{y}_1 \\ M\vec{x}_2 &= \vec{y}_2 \end{aligned} \qquad = \quad M\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \quad = \quad \begin{bmatrix} \vec{y}_1 & \vec{y}_2 \end{bmatrix}$$

$$\Longrightarrow \quad \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad = \quad R(\theta)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad = \quad R(\theta)$$

Stretch/Compress

- Stretch/Compress
  - keep the direction



$$
\begin{aligned}
\vec{y} = \quad & k\vec{x} \\
& \uparrow \\
& \text{scalar (not matrix)}
\end{aligned}
$$

$$\vec{y} = \quad kI\vec{x} \qquad\qquad \text{where } I = \text{ Identity martix}$$

$$\vec{y} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x}$$

Stretch/Compress

- T: stretch by a along $\hat{x}$-direction & stretch by b along $\hat{y}$-direction
- Compute the corresponding matrix A

$$
\begin{aligned}
\begin{bmatrix} ax_1 \\ bx_2 \end{bmatrix} &= A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow A = ? \\
&= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

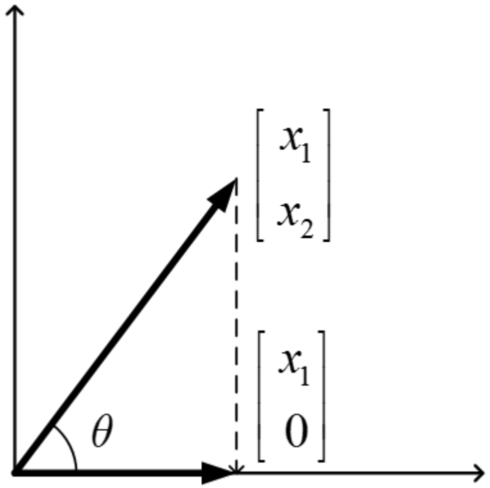
$$
\begin{aligned}
A \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} a \\ 0 \end{bmatrix} \\
A \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ b \end{bmatrix} \\
A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}
\end{aligned}
$$

- More importantly, can you think of the corresponding transformation T by looking at $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$?

Projection

- P: Projection onto $\hat{x}$ - axis

$$\begin{matrix} & P & \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \Longrightarrow & \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \\ \vec{x} & & \vec{y} \end{matrix}$$



$$\vec{y} = P\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Multiple Transformations

- T_1: transformation 1 of matrix M_1
- T_2: transformation 2 of matrix M_2
- T : Do transformation 1, followed by transformation 2

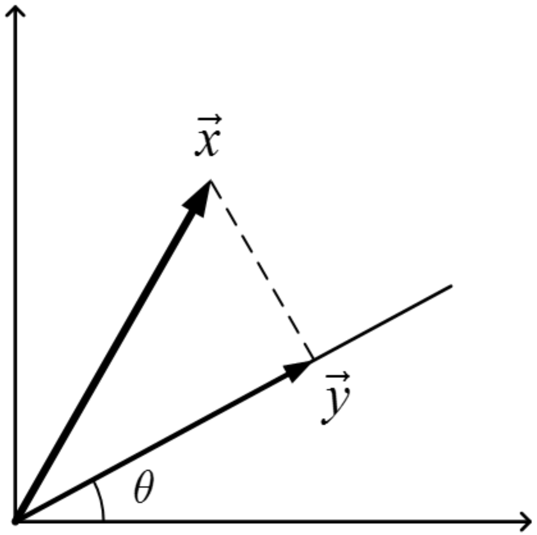
$$\vec{x} \xrightarrow{T_1} \vec{y} \xrightarrow{T_2} \vec{z}$$

$$\begin{aligned} \vec{y} &= M_1\vec{x} \\ \vec{z} &= M_2\vec{y} \quad = M_2M_1\vec{x} \\ &\qquad\qquad = M\vec{x} \end{aligned}$$

$$\therefore \; M = M_2M_1$$

POSTECH

P: Projection onto Vector = $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



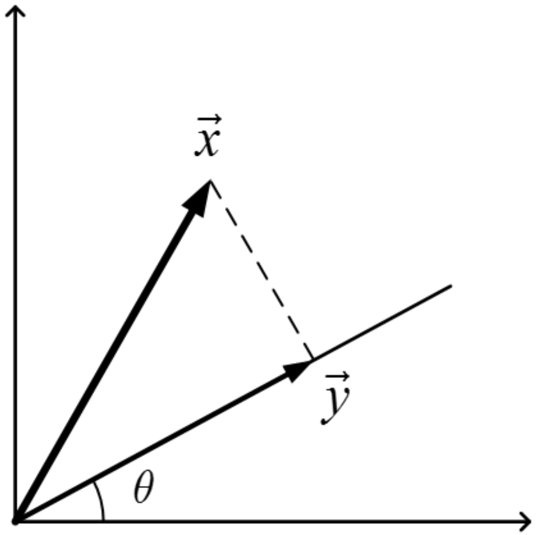
$$P\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{bmatrix}$$

$$P\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

P: Projection onto Vector = $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

- Another way to find this projection matrix



$$\vec{x} \xRightarrow{R(-\theta)} \vec{x}' \xRightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \vec{x}'' \xRightarrow{R(\theta)} \vec{y}$$

$$\begin{aligned}
\vec{y} &= R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(-\theta)\vec{x} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}
\end{aligned}$$

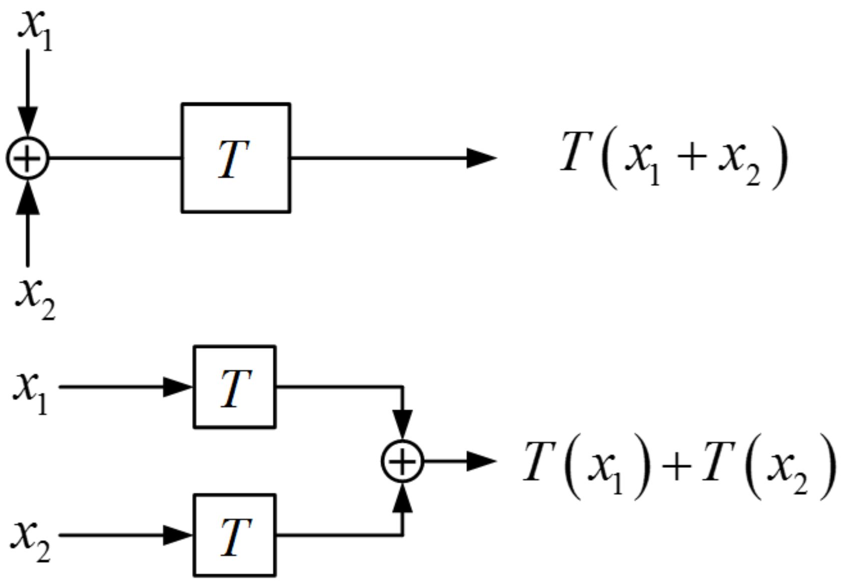
Linear Transformation

- See if the given transformation is linear
  - A linear system makes our life much easier

- Superposition
- Homogeneity

Linear Transformation

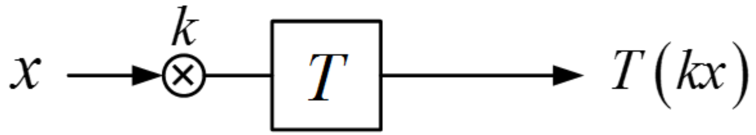
- Superposition

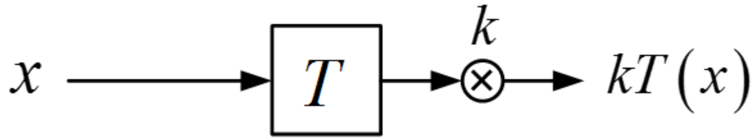


$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

Linear Transformation

- Homogeneity





$$T(kx) = kT(x)$$

Linear Transformation

- Linear vs. Non-linear

linear	non-linear
$f(x) = 0$	$f(x) = x + c$
$f(x) = kx$	$f(x) = x^2$
$f(x(t)) = \frac{dx(t)}{dt}$	$f(x) = \sin x$
$f(x(t)) = \int_a^b x(t)dt$	

Linear Transformation

- If $\vec{v}_1$ and $\vec{v}_2$ are basis, and we know $T(\vec{v}_1) = \vec{\omega}_1$ and $T(\vec{v}_2) = \vec{\omega}_2$

- Then, for any $\vec{x}$

$$\begin{aligned} \vec{x} &= a_1\vec{v}_1 + a_2\vec{v}_2 \qquad (a_1 \text{ and } a_2 \text{ unique}) \\ T(\vec{x}) &= T(a_1\vec{v}_1 + a_2\vec{v}_2) \\ &= a_1T(\vec{v}_1) + a_2T(\vec{v}_2) \\ &= a_1\vec{\omega}_1 + a_2\vec{\omega}_2 \end{aligned}$$

Eigenvalue and Eigenvector

$$A\vec{v} = \lambda\vec{v}$$

$A\vec{x}$ parallel to $\vec{x}$

$$\begin{aligned} \lambda \quad &= \begin{cases} \text{positive} \\ 0 \\ \text{negative} \end{cases} \\ \lambda\vec{v} \quad &: \quad \text{stretched vector} \\ &\qquad \text{(same direction with } \vec{x}\text{)} \\ A\vec{v} \quad &: \quad \text{transformed vector} \\ &\qquad \text{(generally rotate + stretch)} \end{aligned}$$

How to Compute Eigenvalue and Eigenvector

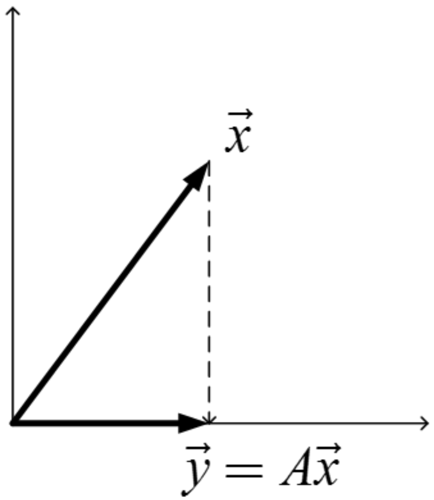
$$A\vec{v} = \lambda\vec{v} = \lambda I\vec{v}$$
$$A\vec{v} - \lambda I\vec{v} = (A - \lambda I)\vec{v} = 0$$

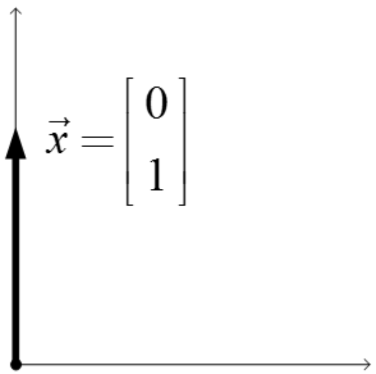
$$\implies \quad A - \lambda I = 0 \text{ or}$$
$$\vec{v} = 0 \text{ or}$$
$$(A - \lambda I)^{-1} \text{ does not exist}$$

$$\implies \quad \det(A - \lambda I) = 0$$

Eigen Analysis of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

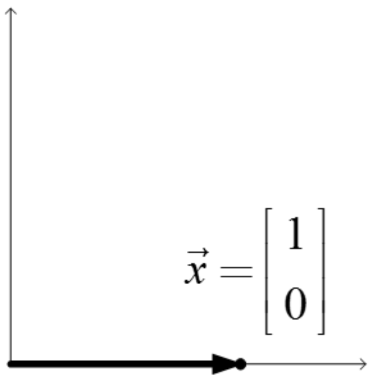
- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: projection onto $\hat{x}$ - axis
- Find eigenvalues and eigenvectors of A.





$$\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\vec{x} = 0 \cdot \vec{x}$$

$$\lambda_1 = 0 \text{ and } \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



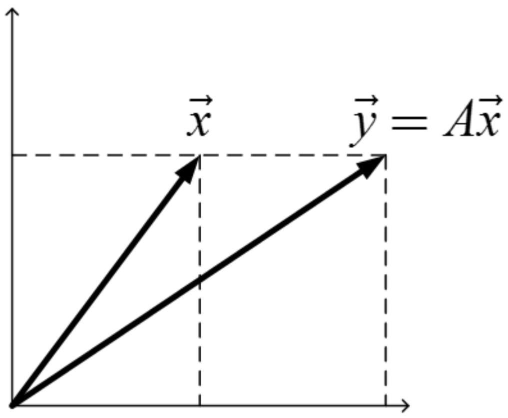
$$\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\vec{x} = 1 \cdot \vec{x}$$

$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

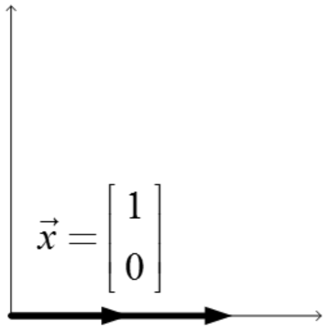
Eigen Analysis of $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

- $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$: stretch by 2 along $\vec{x}$- axis

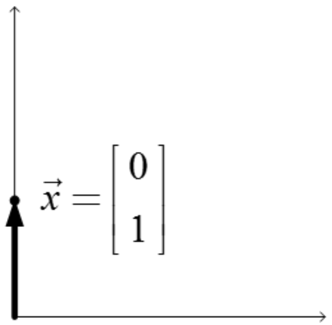
  stretch by 1 along $\vec{y}$- axis



- Find eigenvalues and eigenvectors.



$$\lambda_1 = 2 \text{ and } \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

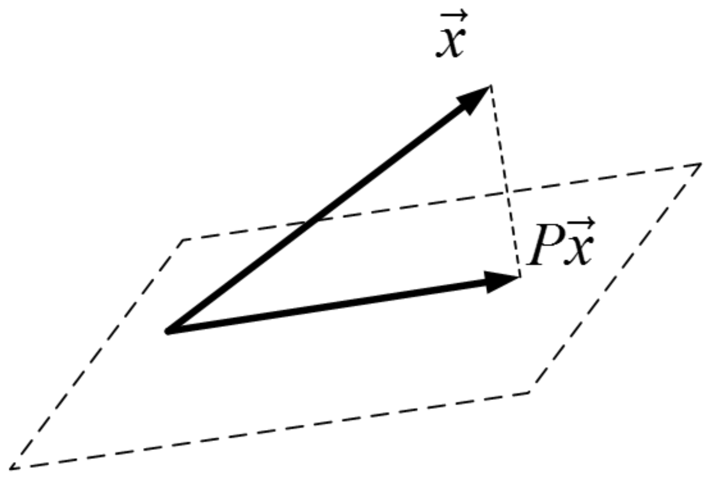
Eigen Analysis in Python

```
A = np.array([[2, 0],
              [0, 1]])
D, V = np.linalg.eig(A)

idx = np.argsort(-D)
D = D[idx]
V = V[idx]

print('D :', D)
print('V :', V)
```

```
D : [ 2.  1.]
V : [[ 1.  0.]
 [ 0.  1.]]
```

Eigen Analysis of Projection

- Projection onto the plane
- Find eigenvalues and eigenvectors

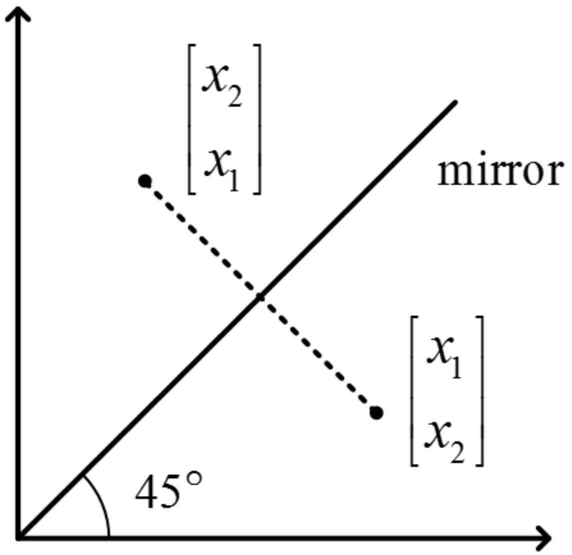


- For any $\vec{x}$ in the plane, $P\vec{x} = \vec{x} \rightarrow \lambda = 1$
- For any $\vec{x}$ perpendicular to the plane, $P\vec{x} = \vec{0} \rightarrow \lambda = 0$

Eigen Analysis of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

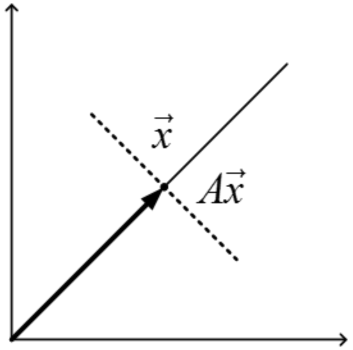
- What kind of a linear transformation?

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

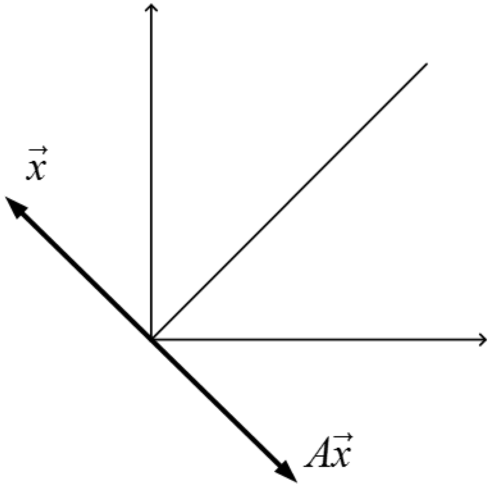


Eigen Analysis of Mirror

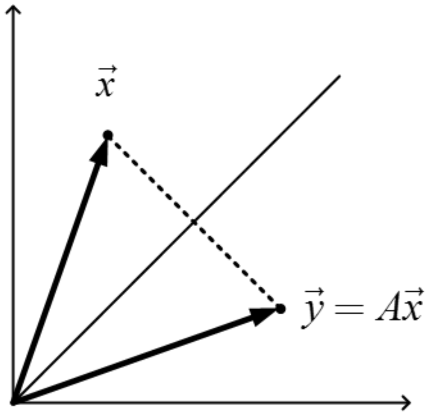
- Eigenvalues and eigenvectors?
  - can $\vec{x}$ be an eigenvector?



$$A\vec{x} = \vec{x}, \quad \lambda = 1$$



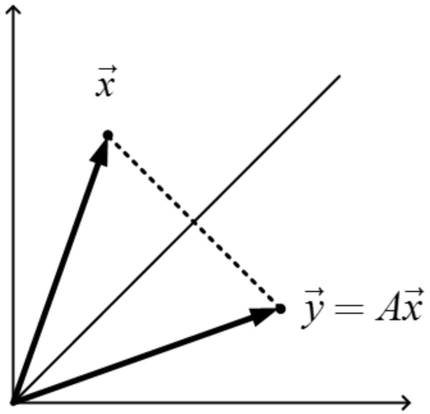
$$A\vec{x} = -\vec{x}, \quad \lambda = -1$$



Eigen Analysis of Mirror

- Side note: Matrix A can be seen as a multiple transformations

$$A = R(45)MR(-45)$$



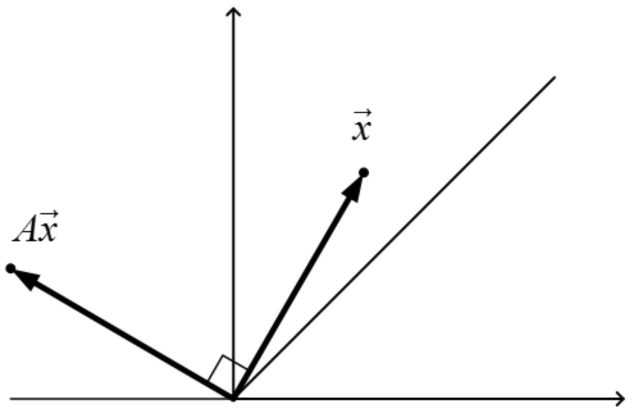
$$
\begin{aligned}
R(45) &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
M &: \text{ mirror along } \hat{x}\text{- axis, } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
A &= \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\end{aligned}
$$

Eigen Analysis of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- What kind of a linear transformation?

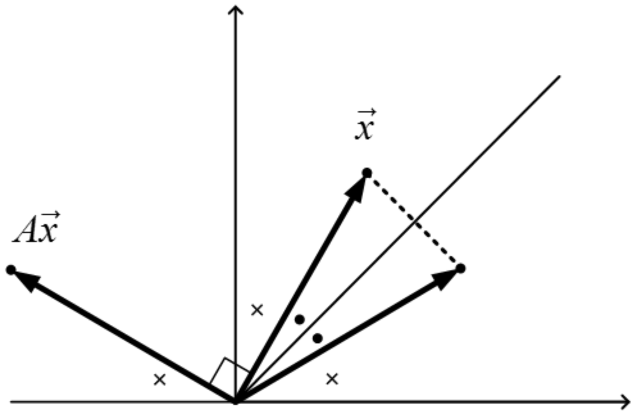
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = R\left(\frac{\pi}{2}\right) = R(90^\circ) = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$$



- Multiple transformations

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



Eigen Analysis of Rotation

- What kind of a linear transformation?

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = R\left(\frac{\pi}{2}\right) = R(90^\circ) = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$$

- Eigenvalues: complex numbers

$$\begin{aligned} \Rightarrow \quad \det(A - \lambda I) &= 0 \\ \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} &= \lambda^2 + 1 = 0 \\ \therefore \ \lambda &= \pm i \end{aligned}$$

- What is the physical meaning?

Linear Transformation and Eigenvectors

- If $\vec{v}_1$ and $\vec{v}_2$ are basis and eigenvectors, and we know $T(\vec{v}_1) = \lambda_1 \vec{v}_1$ and $T(\vec{v}_2) = \lambda_2 \vec{v}_2$

- Then, for any $\vec{x}$

$$\vec{x} = a_1 \vec{v}_1 + a_2 \vec{v}_2 \qquad (a_1 \text{ and } a_2 \text{ unique})$$

$$\begin{aligned} T(\vec{x}) &= T(a_1 \vec{v}_1 + a_2 \vec{v}_2) \\ &= a_1 T(\vec{v}_1) + a_2 T(\vec{v}_2) \\ &= a_1 \lambda_1 \vec{v}_1 + a_2 \lambda_2 \vec{v}_2 \\ &= \lambda_1 a_1 \vec{v}_1 + \lambda_2 a_2 \vec{v}_2 \end{aligned}$$

- (optional) Fourier transform
  - Sinusoids are basis and eigenvectors for functions (or signals)