Support Vector Machine

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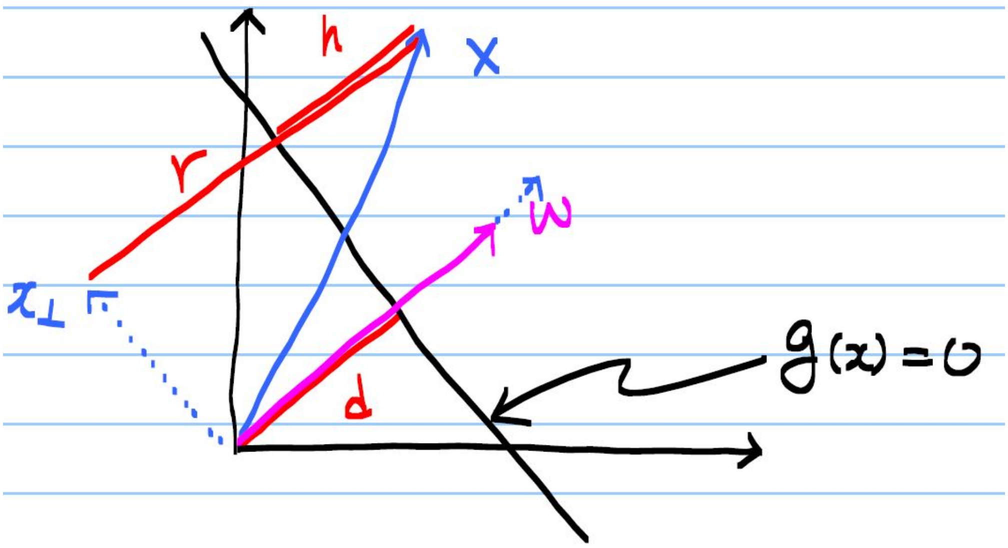
1. Classification (Linear)

- Figure out, autonomously, which category (or class) an unknown item should be categorized into
- Number of categories / classes
  - Binary: 2 different classes
  - Multiclass: more than 2 classes
- Feature
  - The measurable parts that make up the unknown item (or the information you have available to categorize)

2. Distance from a Line

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies g(x) = \omega^T x + \omega_0 = \omega_1 x_1 + \omega_2 x_2 + \omega_0$$



- If $\vec{p}$ and $\vec{q}$ are on the decision line

$$g(\vec{p}) = g(\vec{q}) = 0 \implies \omega^T \vec{p} + \omega_0 = \omega^T \vec{q} + \omega_0 = 0$$
$$\implies \omega^T (\vec{p} - \vec{q}) = 0$$

$$\therefore\ \omega : \text{normal to the line (orthogonal)}$$
$$\implies \text{tells the direction of the line}$$

- If x is on the line and $x = d\frac{\omega}{\|\omega\|}$ (where d is a normal distance from the origin to the line)

$$g(x) = \omega^T x + \omega_0 = 0$$
$$\implies \omega^T d \frac{\omega}{\|\omega\|} + \omega_0 = d\frac{\omega^T \omega}{\|\omega\|} + \omega_0 = d\|\omega\| + \omega_0 = 0$$
$$\therefore d = -\frac{\omega_0}{\|\omega\|}$$

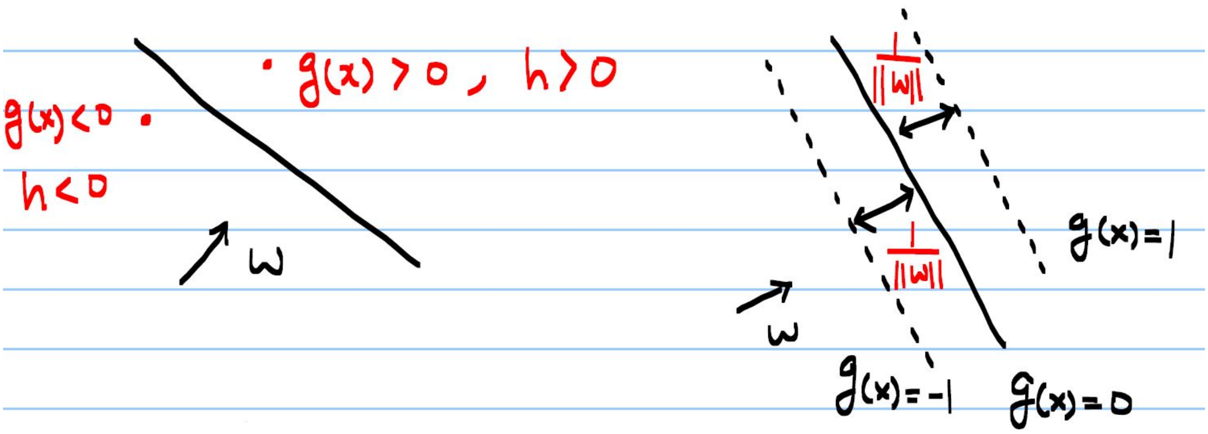
- for any vector of x

$$x = x_\perp + r\frac{\omega}{\|\omega\|}$$

$$\omega^T x = \omega^T \left(x_\perp + r\frac{\omega}{\|\omega\|}\right) = r\frac{\omega^T\omega}{\|\omega\|} = r\|\omega\|$$

$$\begin{aligned} g(x) &= \omega^T x + \omega_0 \\ &= r\|\omega\| + \omega_0 \qquad (r = d + h) \\ &= (d+h)\|\omega\| + \omega_0 \\ &= \left(-\frac{\omega_0}{\|\omega\|} + h\right)\|\omega\| + \omega_0 \\ &= h\|\omega\| \end{aligned}$$

$$\therefore\ h = \frac{g(x)}{\|\omega\|} \implies \textbf{orthogonal distance from the line}$$

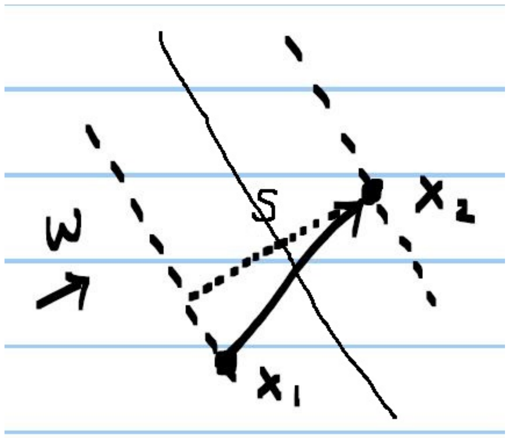


Another method to find a distance between $g(x) = -1$ and $g(x) = 1$

suppose $g(x_1) = -1, \ g(x_2) = 1$

$$\begin{aligned} \omega^T x_1 + \omega_0 &= -1 \\ \omega^T x_2 + \omega_0 &= 1 \end{aligned} \quad \Longrightarrow \quad \omega^T (x_2 - x_1) = 2$$

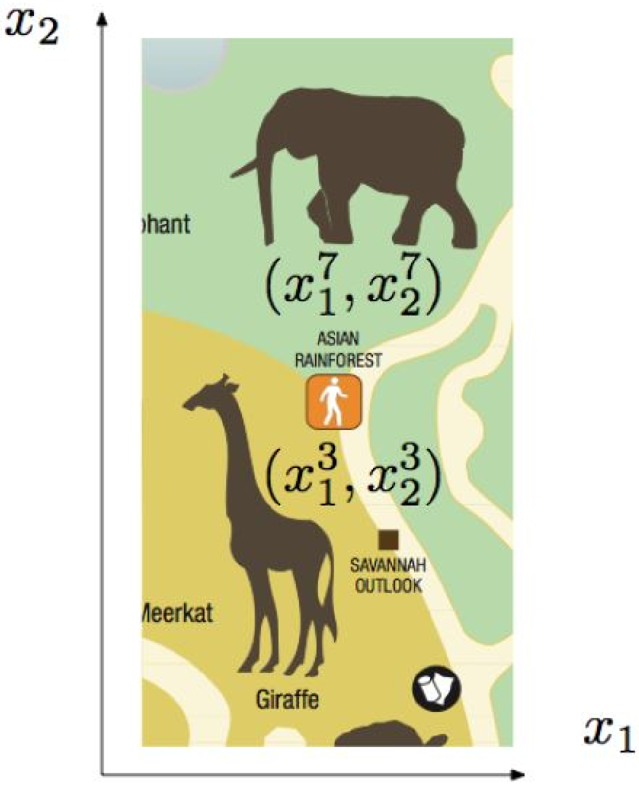
$$s = \left\langle \frac{\omega}{\|\omega\|}, x_2 - x_1 \right\rangle = \frac{1}{\|\omega\|} \omega^T (x_2 - x_1) = \frac{2}{\|\omega\|}$$



3. Illustrative Example

- Binary classification
    - C_1 and C_2
- Features
    - The coordinate of the unknown animal i in the zoo

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



- Is it possible to distinguish between C_1 and C_2 by its coordinates on a map of the zoo?
- We need to find a separating hyperplane (or a line in 2D)

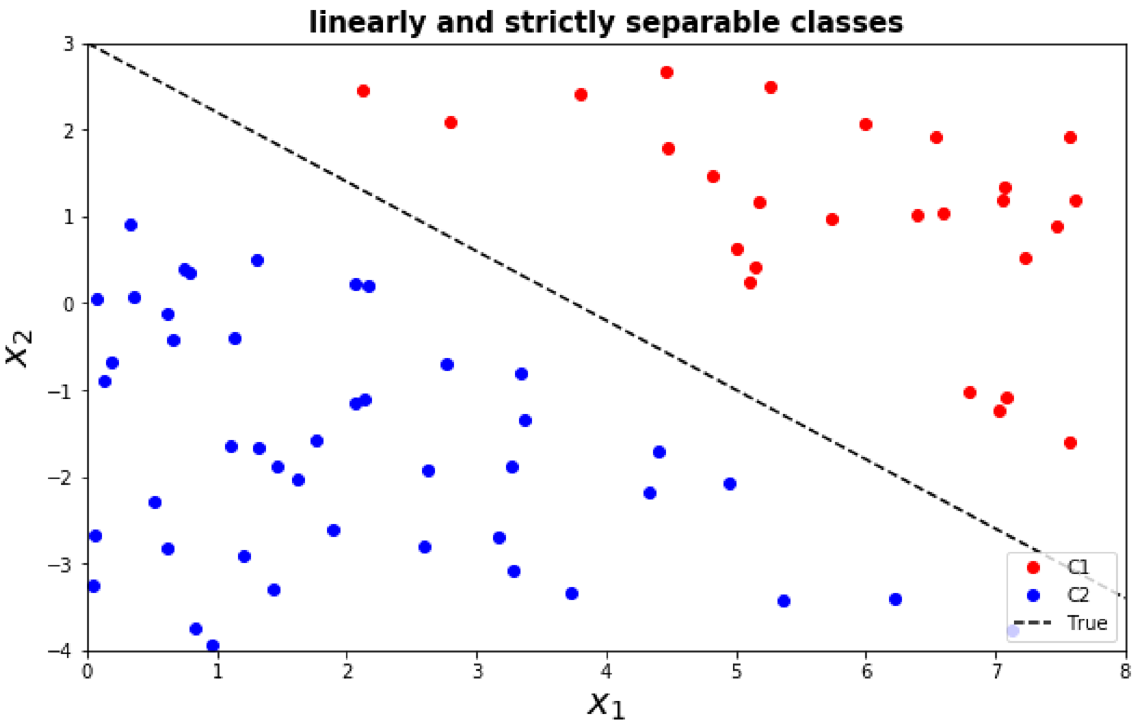
$$\begin{aligned} \omega_1 x_1 + \omega_2 x_2 + \omega_0 &= 0 \\ \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \omega_0 &= 0 \\ \omega^T x + \omega_0 &= 0 \end{aligned}$$

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

In [2]:

```
#training data gerneration
x1 = 8*np.random.rand(100, 1)
x2 = 7*np.random.rand(100, 1) - 4

g0 = 0.8*x1 + x2 - 3
g1 = g0 - 1
g2 = g0 + 1

C1 = np.where(g1 >= 0)[0]
C2 = np.where(g2 < 0)[0]

xp = np.linspace(0,8,100).reshape(-1,1)
ypt = -0.8*xp + 3

plt.figure(figsize=(10, 6))
plt.plot(x1[C1], x2[C1], 'ro', label='C1')
plt.plot(x1[C2], x2[C2], 'bo', label='C2')
plt.plot(xp, ypt, '--k', label='True')
plt.title('linearly and strictly separable classes', fontweight = 'bold', fontsize =
15)
plt.xlabel('$x_1$', fontsize = 20)
plt.ylabel('$x_2$', fontsize = 20)
plt.legend(loc = 4)
plt.xlim([0, 8])
plt.ylim([-4, 3])
plt.show()
```



- Given:
  - Hyperplane defined by ω and ω_0
  - Animals coordinates (or features) x

- Decision making:

$$\omega^T x + \omega_0 > 0 \implies x \text{ belongs to } C_1$$
$$\omega^T x + \omega_0 < 0 \implies x \text{ belongs to } C_2$$

- Find ω and ω_0 such that x given $\omega^T x + \omega_0 = 0$

  or

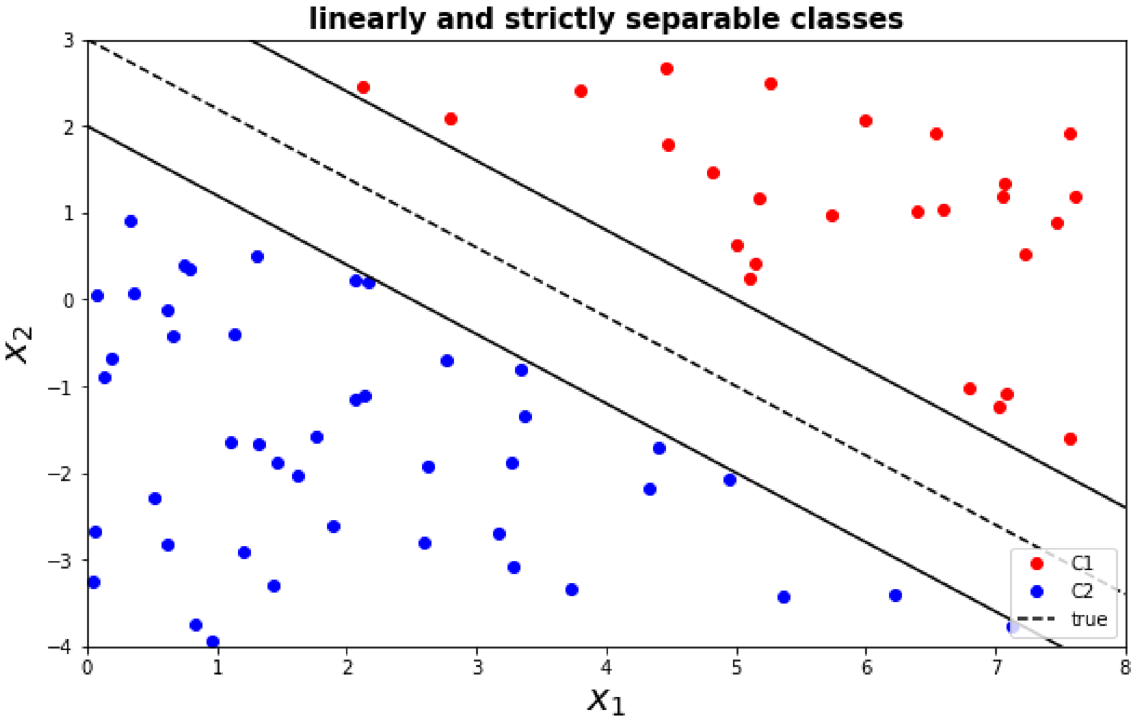
- Find ω and ω_0 such that $x \in C_1$ given $\omega^T x + \omega_0 > 1$ and $x \in C_2$ given $\omega^T x + \omega_0 < -1$

$$\begin{aligned} & \omega^T x + \omega_0 > b \\ \iff & \frac{\omega^T}{b} x + \frac{\omega_0}{b} > 1 \\ \iff & \omega'^T x + \omega'_0 > 1 \end{aligned}$$

- Same problem if strictly separable

In [3]:

```
#  see how data are generated
xp = np.linspace(0,8,100).reshape(-1,1)
ypt = -0.8*xp + 3

plt.figure(figsize=(10, 6))
plt.plot(x1[C1], x2[C1], 'ro', label='C1')
plt.plot(x1[C2], x2[C2], 'bo', label='C2')
plt.plot(xp, ypt, '--k', label='true')
plt.plot(xp, ypt-1, '-k')
plt.plot(xp, ypt+1, '-k')
plt.title('linearly and strictly separable classes', fontweight = 'bold', fontsize =
15)
plt.xlabel('$x_1$', fontsize = 20)
plt.ylabel('$x_2$', fontsize = 20)
plt.legend(loc = 4)
plt.xlim([0, 8])
plt.ylim([-4, 3])
plt.show()
```



3.1. LP Formulation 1

- $n\ (=2)$ features
- $m = N + M$ data points in training set

$$x^{(i)} = \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \end{bmatrix} \text{ with } \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \qquad \text{or} \qquad x^{(i)} = \begin{bmatrix} 1 \\ x_1^{(i)} \\ x_2^{(i)} \end{bmatrix} \text{ with } \omega = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix}$$

- N belongs to C_1 in training set
- M belongs to C_2 in training set
- ω and ω_0 are the unknown variables

$$\begin{aligned} \text{minimize} \quad & \text{something} \\ \text{subject to} \quad & \begin{cases} \omega^T x^{(1)} + \omega_0 \geq 1 \\ \omega^T x^{(2)} + \omega_0 \geq 1 \\ \quad \vdots \\ \omega^T x^{(N)} + \omega_0 \geq 1 \end{cases} \\ & \begin{cases} \omega^T x^{(N+1)} + \omega_0 \leq -1 \\ \omega^T x^{(N+2)} + \omega_0 \leq -1 \\ \quad \vdots \\ \omega^T x^{(N+M)} + \omega_0 \leq -1 \end{cases} \end{aligned} \qquad \begin{aligned} \text{minimize} \quad & \text{something} \\ \text{subject to} \quad & \begin{cases} \omega^T x^{(1)} \geq 1 \\ \omega^T x^{(2)} \geq 1 \\ \quad \vdots \\ \omega^T x^{(N)} \geq 1 \end{cases} \\ & \begin{cases} \omega^T x^{(N+1)} \leq -1 \\ \omega^T x^{(N+2)} \leq -1 \\ \quad \vdots \\ \omega^T x^{(N+M)} \leq -1 \end{cases} \end{aligned}$$

Code (CVXPY)

$$X_1 = \begin{bmatrix} \left(x^{(1)}\right)^T \\ \left(x^{(2)}\right)^T \\ \vdots \\ \left(x^{(N)}\right)^T \end{bmatrix} = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} \\ x_1^{(2)} & x_2^{(2)} \\ \vdots & \vdots \\ x_1^{(N)} & x_2^{(N)} \end{bmatrix} \qquad X_1 = \begin{bmatrix} \left(x^{(1)}\right)^T \\ \left(x^{(2)}\right)^T \\ \vdots \\ \left(x^{(N)}\right)^T \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} \\ \vdots & \vdots & \vdots \\ 1 & x_1^{(N)} & x_2^{(N)} \end{bmatrix}$$

$$X_2 = \begin{bmatrix} \left(x^{(N+1)}\right)^T \\ \left(x^{(N+2)}\right)^T \\ \vdots \\ \left(x^{(N+M)}\right)^T \end{bmatrix} = \begin{bmatrix} x_1^{(N+1)} & x_2^{(N+1)} \\ x_1^{(N+2)} & x_2^{(N+2)} \\ \vdots & \vdots \\ x_1^{(N+M)} & x_2^{(N+M)} \end{bmatrix} \qquad X_2 = \begin{bmatrix} \left(x^{(N+1)}\right)^T \\ \left(x^{(N+2)}\right)^T \\ \vdots \\ \left(x^{(N+M)}\right)^T \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(N+1)} & x_2^{(N+1)} \\ 1 & x_1^{(N+2)} & x_2^{(N+2)} \\ \vdots & \vdots & \vdots \\ 1 & x_1^{(N+M)} & x_2^{(N+M)} \end{bmatrix}$$

$$\begin{aligned} \text{minimize} \quad & \text{something} \\ \text{subject to} \quad & X_1 \omega + \omega_0 \geq 1 \\ & X_2 \omega + \omega_0 \leq -1 \end{aligned} \qquad \begin{aligned} \text{minimize} \quad & \text{something} \\ \text{subject to} \quad & X_1 \omega \geq 1 \\ & X_2 \omega \leq -1 \end{aligned}$$

Form 1

$$
\begin{aligned}
\text{minimize} \quad & \text{something} \\
\text{subject to} \quad & X_1 \omega + \omega_0 \geq 1 \\
& X_2 \omega + \omega_0 \leq -1
\end{aligned}
$$

In [4]:

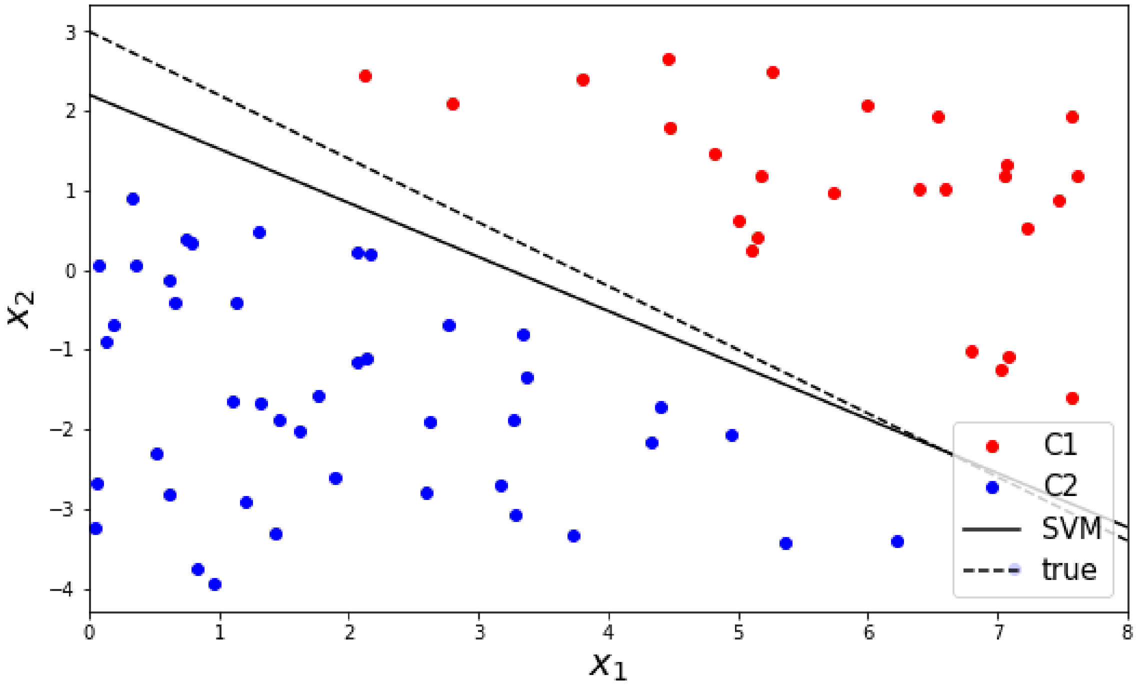
```
# CVXPY using simple classification
import cvxpy as cvx

X1 = np.hstack([x1[C1], x2[C1]])
X2 = np.hstack([x1[C2], x2[C2]])

X1 = np.asmatrix(X1)
X2 = np.asmatrix(X2)

N = X1.shape[0]
M = X2.shape[0]
```

In [5]:

```
w = cvx.Variable(2,1)
w0 = cvx.Variable(1,1)

obj = cvx.Minimize(1)
const = [X1*w + w0 >= 1, X2*w + w0 <= -1]
prob = cvx.Problem(obj, const).solve()

w = w.value
w0 = w0.value
```

In [6]:

```
xp = np.linspace(0,8,100).reshape(-1,1)
yp = - w[0,0]/w[1,0]*xp - w0/w[1,0]

plt.figure(figsize=(10, 6))
plt.plot(X1[:,0], X1[:,1], 'ro', label='C1')
plt.plot(X2[:,0], X2[:,1], 'bo', label='C2')
plt.plot(xp, yp, 'k', label='SVM')
plt.plot(xp, ypt, '--k', label='true')
plt.xlim([0,8])
plt.xlabel('$x_1$', fontsize = 20)
plt.ylabel('$x_2$', fontsize = 20)
plt.legend(loc = 4, fontsize = 15)
plt.show()
```

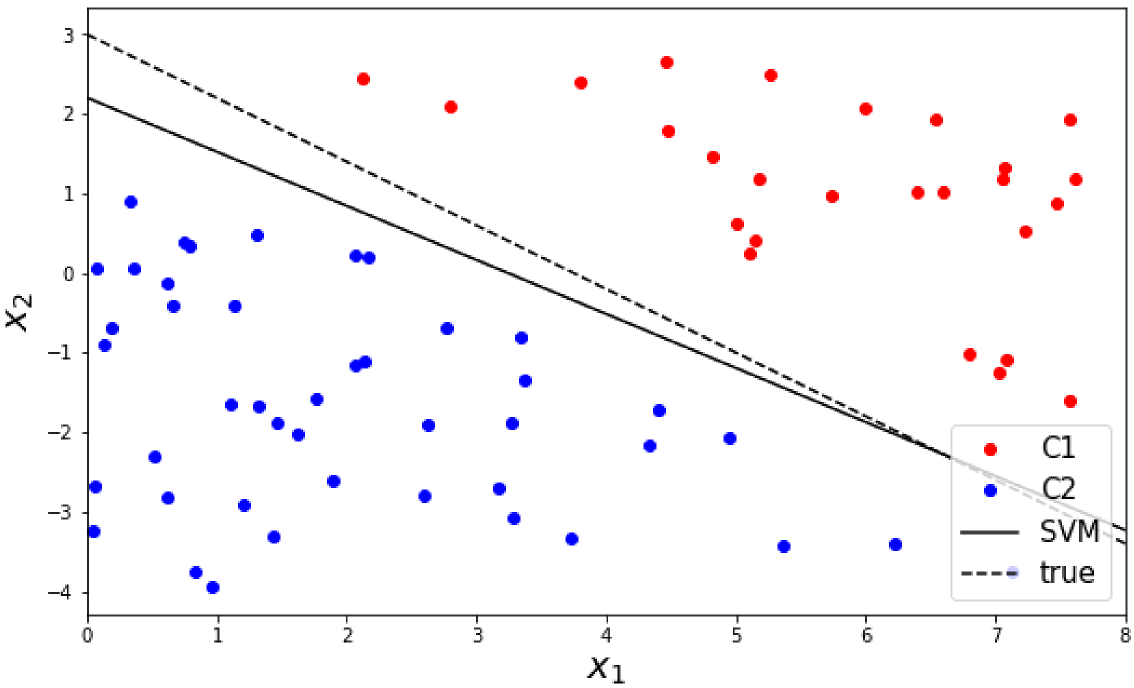


Form 2

$$
\begin{aligned}
\text{minimize} \quad & \text{something} \\
\text{subject to} \quad & X_1 \omega \geq 1 \\
& X_2 \omega \leq -1
\end{aligned}
$$

In [7]:

```
N = C1.shape[0]
M = C2.shape[0]

X1 = np.hstack([np.ones([N,1]), x1[C1], x2[C1]])
X2 = np.hstack([np.ones([M,1]), x1[C2], x2[C2]])

X1 = np.asmatrix(X1)
X2 = np.asmatrix(X2)

w = cvx.Variable(3,1)
obj = cvx.Minimize(1)
const = [X1*w >= 1, X2*w <= -1]
prob = cvx.Problem(obj, const).solve()

w = w.value

xp = np.linspace(0,8,100).reshape(-1,1)
yp = - w[1,0]/w[2,0]*xp - w[0,0]/w[2,0]

plt.figure(figsize=(10, 6))
plt.plot(X1[:,1], X1[:,2], 'ro', label='C1')
plt.plot(X2[:,1], X2[:,2], 'bo', label='C2')
plt.plot(xp, yp, 'k', label='SVM')
plt.plot(xp, ypt, '--k', label='true')
plt.xlim([0,8])
plt.xlabel('$x_1$', fontsize = 20)
plt.ylabel('$x_2$', fontsize = 20)
plt.legend(loc = 4, fontsize = 15)
plt.show()
```

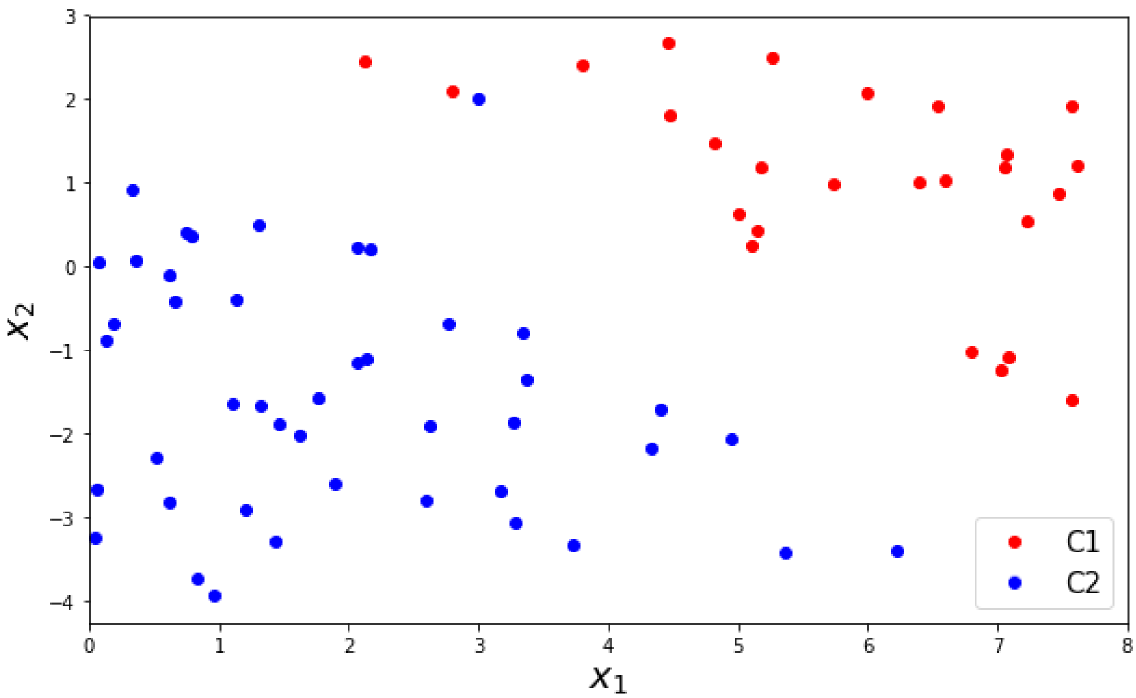


3.2. Outlier

- Note that in the real world, you may have noise, errors, or outliers that do not accurately represent the actual phenomena
- Non-separable case
- No solutions (hyperplane) exist
    - We will allow some training examples to be misclassified !
    - but we want their number to be minimized

In [8]:

```
X1 = np.hstack([np.ones([N,1]), x1[C1], x2[C1]])
X2 = np.hstack([np.ones([M,1]), x1[C2], x2[C2]])

outlier = np.array([1, 3, 2]).reshape(-1,1)
X2 = np.vstack([X2, outlier.T])

X1 = np.asmatrix(X1)
X2 = np.asmatrix(X2)

plt.figure(figsize=(10, 6))
plt.plot(X1[:,1], X1[:,2], 'ro', label='C1')
plt.plot(X2[:,1], X2[:,2], 'bo', label='C2')
plt.xlim([0,8])
plt.xlabel('$x_1$', fontsize = 20)
plt.ylabel('$x_2$', fontsize = 20)
plt.legend(loc = 4, fontsize = 15)
plt.show()
```



In [9]:

```
w = cvx.Variable(3,1)
obj = cvx.Minimize(1)
const = [X1*w >= 1, X2*w <= -1]
prob = cvx.Problem(obj, const).solve()

print(w.value)
```

```
None
```

3.3. LP Formulation 2

- $n \; (=2)$ features
- $m = N + M$ data points in a training set

$$x^i = \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \end{bmatrix}$$

- N belongs to C_1 in training set
- M belongs to C_2 in training set
- ω and ω_0 are the variables (unknown)

- For the non-separable case, we relex the above constraints
- Need slack variables u and v where all are positive

The optimization problem for the non-separable case

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{N} u_i + \sum_{i=1}^{M} v_i \\
\text{subject to} \quad & \begin{cases} \omega^T x^{(1)} + \omega_0 \geq 1 - u_1 \\ \omega^T x^{(2)} + \omega_0 \geq 1 - u_2 \\ \quad \vdots \\ \omega^T x^{(N)} + \omega_0 \geq 1 - u_N \end{cases} \\
& \begin{cases} \omega^T x^{(N+1)} + \omega_0 \leq -(1 - v_1) \\ \omega^T x^{(N+2)} + \omega_0 \leq -(1 - v_2) \\ \quad \vdots \\ \omega^T x^{(N+M)} + \omega_0 \leq -(1 - v_M) \end{cases} \\
& \begin{cases} u \geq 0 \\ v \geq 0 \end{cases}
\end{aligned}$$

- Expressed in a matrix form

$$X_1 = \begin{bmatrix} x^{(1)^T} \\ x^{(2)^T} \\ \vdots \\ x^{(N)^T} \end{bmatrix} = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} \\ x_1^{(2)} & x_2^{(2)} \\ \vdots & \vdots \\ x_1^{(N)} & x_2^{(N)} \end{bmatrix}$$

$$X_2 = \begin{bmatrix} x^{(N+1)^T} \\ x^{(N+2)^T} \\ \vdots \\ x^{(N+M)^T} \end{bmatrix} = \begin{bmatrix} x_1^{(N+1)} & x_2^{(N+1)} \\ x_1^{(N+2)} & x_2^{(N+2)} \\ \vdots & \vdots \\ x_1^{(N+M)} & x_2^{(N+M)} \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_M \end{bmatrix}$$

$$X_1 = \begin{bmatrix} \left(x^{(1)}\right)^T \\ \left(x^{(2)}\right)^T \\ \vdots \\ \left(x^{(N)}\right)^T \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} \\ \vdots & \vdots & \vdots \\ 1 & x_1^{(N)} & x_2^{(N)} \end{bmatrix}$$

$$X_2 = \begin{bmatrix} \left(x^{(N+1)}\right)^T \\ \left(x^{(N+2)}\right)^T \\ \vdots \\ \left(x^{(N+M)}\right)^T \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(N+1)} & x_2^{(N+1)} \\ 1 & x_1^{(N+2)} & x_2^{(N+2)} \\ \vdots & \vdots & \vdots \\ 1 & x_1^{(N+M)} & x_2^{(N+M)} \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}$$

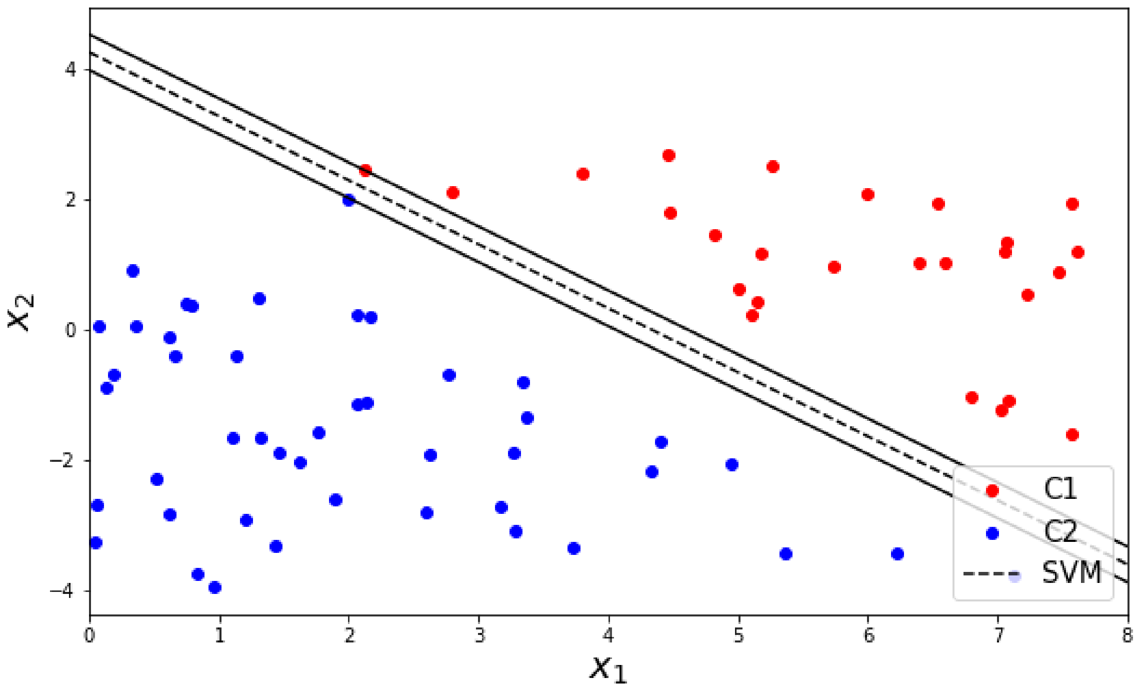
$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_M \end{bmatrix}$$

$$\begin{aligned} \text{minimize} \quad & 1^T u + 1^T v \\ \text{subject to} \quad & X_1 \omega + \omega_0 \geq 1 - u \\ & X_2 \omega + \omega_0 \leq -(1 - v) \\ & u \geq 0 \\ & v \geq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 1^T u + 1^T v \\ \text{subject to} \quad & X_1 \omega \geq 1 - u \\ & X_2 \omega \leq -(1 - v) \\ & u \geq 0 \\ & v \geq 0 \end{aligned}$$

In [10]:

```
X1 = np.hstack([np.ones([C1.shape[0],1]), x1[C1], x2[C1]])
X2 = np.hstack([np.ones([C2.shape[0],1]), x1[C2], x2[C2]])

outlier = np.array([1, 2, 2]).reshape(-1,1)
X2 = np.vstack([X2, outlier.T])

X1 = np.asmatrix(X1)
X2 = np.asmatrix(X2)

N = X1.shape[0]
M = X2.shape[0]

w = cvx.Variable(3,1)
u = cvx.Variable(N,1)
v = cvx.Variable(M,1)
obj = cvx.Minimize(np.ones((1,N))*u + np.ones((1,M))*v)
const = [X1*w >= 1-u, X2*w <= -(1-v), u >= 0, v >= 0 ]
prob = cvx.Problem(obj, const).solve()

w = w.value

xp = np.linspace(0,8,100).reshape(-1,1)
yp = - w[1,0]/w[2,0]*xp - w[0,0]/w[2,0]

plt.figure(figsize=(10, 6))
plt.plot(X1[:,1], X1[:,2], 'ro', label='C1')
plt.plot(X2[:,1], X2[:,2], 'bo', label='C2')
plt.plot(xp, yp, '--k', label='SVM')
plt.plot(xp, yp-1/w[2,0], '-k')
plt.plot(xp, yp+1/w[2,0], '-k')
plt.xlim([0,8])
plt.xlabel('$x_1$', fontsize = 20)
plt.ylabel('$x_2$', fontsize = 20)
plt.legend(loc = 4, fontsize = 15)
plt.show()
```



Further improvement

- Notice that hyperplane is not as accurately represent the division due to the outlier
- Can we do better when there are noise data or outliers?
- Yes, but we need to look beyond LP
- Idea: large margin leads to good generalization on the test data

3.4. Maximize Margin (Finally, it is Support Vector Machine)

- Distance (= margin)

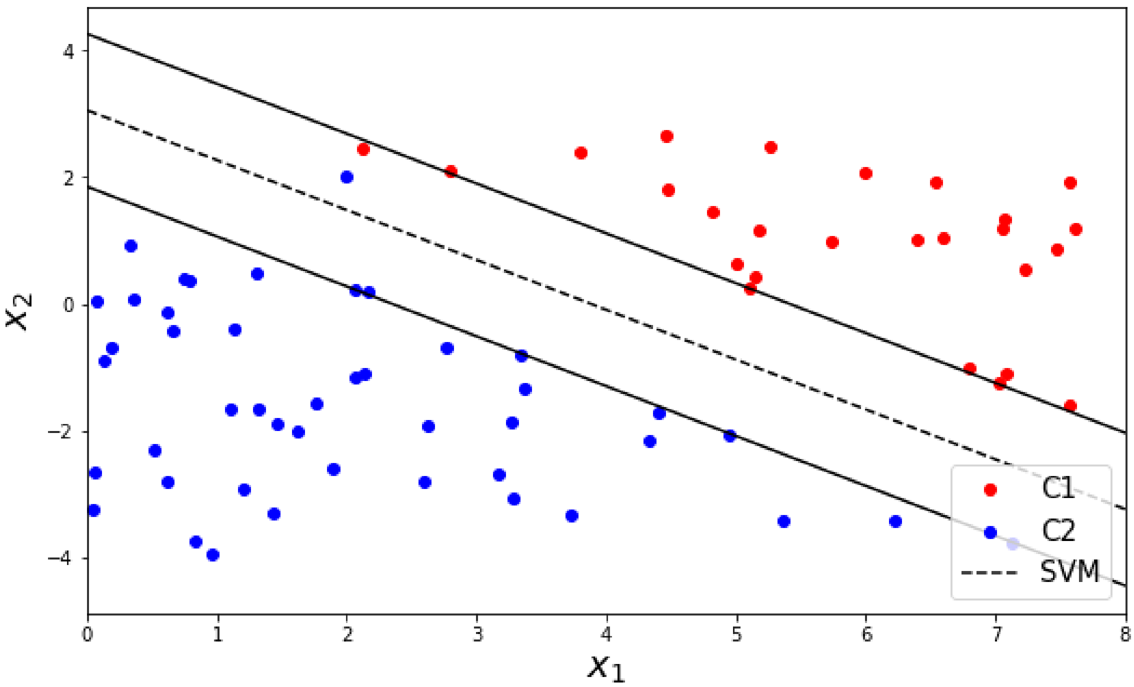
$$\text{margin} = \frac{2}{\|\omega\|_2}$$

- Minimize $\|\omega\|_2$ to maximize the margin
- Multiple objectives
- Use gamma (γ) as a weighting betwwen the followings:
  - Bigger margin given robustness to outliers
  - Hyperplane that has few (or no) errors

$$\begin{array}{ll} \text{minimize} & \|\omega\|_2 + \gamma(1^T u + 1^T v) \\ \text{subject to} & X_1 \omega + \omega_0 \geq 1 - u \\ & X_2 \omega + \omega_0 \leq -(1 - v) \\ & u \geq 0 \\ & v \geq 0 \end{array}$$

In [11]:

```
X1 = np.hstack([np.ones([C1.shape[0],1]), x1[C1], x2[C1]])
X2 = np.hstack([np.ones([C2.shape[0],1]), x1[C2], x2[C2]])

outlier = np.array([1, 2, 2]).reshape(-1,1)
X2 = np.vstack([X2, outlier.T])

X1 = np.asmatrix(X1)
X2 = np.asmatrix(X2)

N = X1.shape[0]
M = X2.shape[0]

g = 1
w = cvx.Variable(3,1)
u = cvx.Variable(N,1)
v = cvx.Variable(M,1)
obj = cvx.Minimize(cvx.norm(w,2) + g*(np.ones((1,N))*u + np.ones((1,M))*v))
const = [X1*w >= 1-u, X2*w <= -(1-v), u >= 0, v >= 0 ]
prob = cvx.Problem(obj, const).solve()

w = w.value

xp = np.linspace(0,8,100).reshape(-1,1)
yp = - w[1,0]/w[2,0]*xp - w[0,0]/w[2,0]

plt.figure(figsize=(10, 6))
plt.plot(X1[:,1], X1[:,2], 'ro', label='C1')
plt.plot(X2[:,1], X2[:,2], 'bo', label='C2')
plt.plot(xp, yp, '--k', label='SVM')
plt.plot(xp, yp-1/w[2,0], '-k')
plt.plot(xp, yp+1/w[2,0], '-k')
plt.xlim([0,8])
plt.xlabel('$x_1$', fontsize = 20)
plt.ylabel('$x_2$', fontsize = 20)
plt.legend(loc = 4, fontsize = 15)
plt.show()
```



4. Support Vector Machine

- Probably the most popular/influential classification algorithm
- A hyperplane based classifier (like the Perceptron)
- Additionally uses the maximum margin principle
  - maximize distance (margin) of closest samples from the decision line

$$\text{maximize } \{\text{minimum distance}\}$$

  - note: perceptron only utilizes a sign of distance
  - Finds the hyperplane with maximum separation margin on the training data

$$\begin{aligned}
\text{minimize} \quad & \|\omega\|_2 + \gamma(1^T u + 1^T v) \\
\text{subject to} \quad & X_1 \omega + \omega_0 \geq 1 - u \\
& X_2 \omega + \omega_0 \leq -(1 - v) \\
& u \geq 0 \\
& \nu \geq 0
\end{aligned}$$

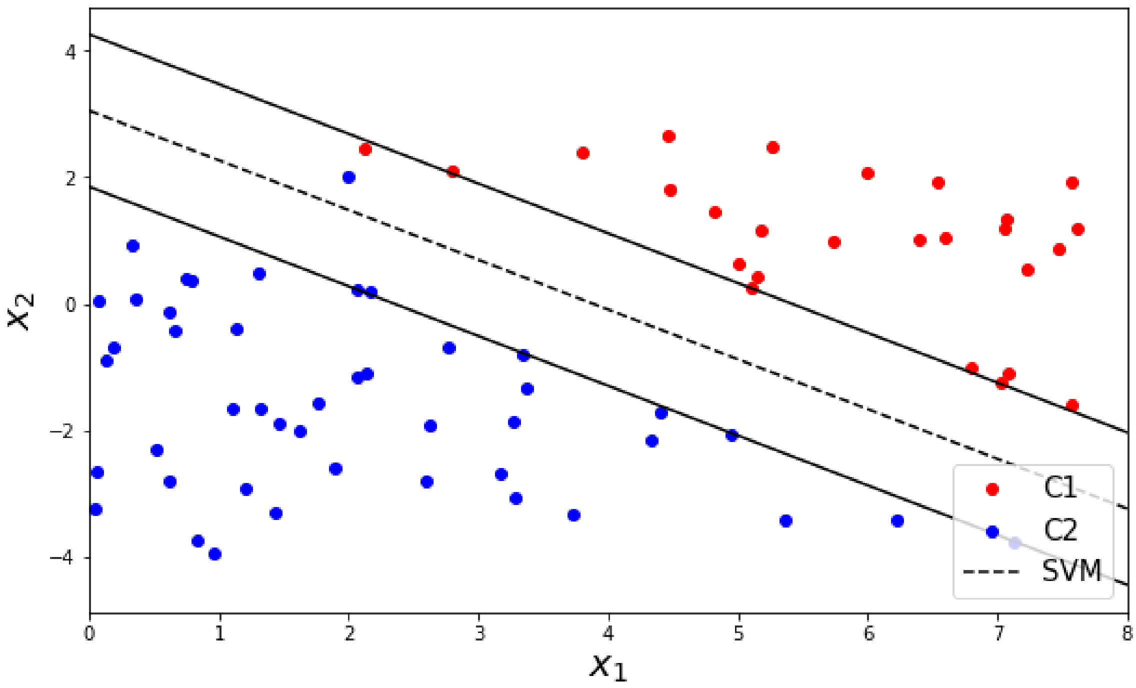
- In a more compact form

$$\begin{array}{l}
\omega^T x_n + \omega_0 \geq 1 \text{ for } \; y_n = +1 \\
\omega^T x_n + \omega_0 \leq -1 \text{ for } \; y_n = -1
\end{array} \iff y_n \left(\omega^T x_n + \omega_0\right) \geq 1$$

$$\begin{aligned}
\text{minimize} \quad & \|\omega\|_2 + \gamma(1^T \xi) \\
\text{subject to} \quad & y_n \left(\omega^T x_n + \omega_0\right) \geq 1 - \xi_n \\
& \xi \geq 0
\end{aligned}$$

In [12]:

```
# SVM in a compact form

X1 = np.hstack([np.ones([C1.shape[0],1]), x1[C1], x2[C1]])
X2 = np.hstack([np.ones([C2.shape[0],1]), x1[C2], x2[C2]])

outlier = np.array([1, 2, 2]).reshape(-1,1)
X2 = np.vstack([X2, outlier.T])

X1 = np.asmatrix(X1)
X2 = np.asmatrix(X2)

N = X1.shape[0]
M = X2.shape[0]

m = N + M
X = np.vstack([X1, X2])
y = np.vstack([np.ones([N,1]), -np.ones([M,1])])

g = 1
w = cvx.Variable(3,1)
d = cvx.Variable(m,1)
obj = cvx.Minimize(cvx.norm(w,2) + g*(np.ones([1,m])*d))
const = [cvx.mul_elemwise(y, X*w) >= 1-d, d >= 0]
prob = cvx.Problem(obj, const).solve()

w = w.value

xp = np.linspace(0,8,100).reshape(-1,1)
yp = - w[1,0]/w[2,0]*xp - w[0,0]/w[2,0]

plt.figure(figsize=(10, 6))
plt.plot(X1[:,1], X1[:,2], 'ro', label='C1')
plt.plot(X2[:,1], X2[:,2], 'bo', label='C2')
plt.plot(xp, yp, '--k', label='SVM')
plt.plot(xp, yp-1/w[2,0], '-k')
plt.plot(xp, yp+1/w[2,0], '-k')
plt.xlim([0,8])
plt.xlabel('$x_1$', fontsize = 20)
plt.ylabel('$x_2$', fontsize = 20)
plt.legend(loc = 4, fontsize = 15)
plt.show()
```



5. Nonlinear Support Vector Machine

5.1. Kernel

- Often we want to capture nonlinear patterns in the data
  - nonlinear regression: input and output relationship may not be linear
  - nonlinear classification: classes may note be separable by a linear boundary

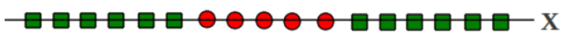
- Linear models (e.g. linear regression, linear SVM) are note just rich enough

- Kernels: make linear model work in nonlinear settings
  - by mapping data to higher dimensions where it exhibits linear patterns
  - apply the linear model in the new input feature space
  - mapping $=$ changing the feature representation

- Note: such mappings can be expensive to compute in general
  - Kernels give such mappings for (almost) free
  - in most cases, the mappings need not be even computed
  - using the Kernel trick !

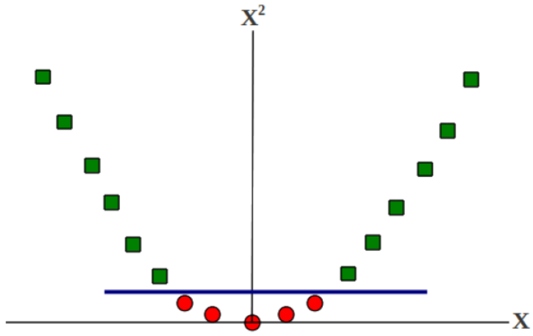
5.2. Classifying non-linear separable data

- Consider the binary classification problem
  - each example represented by a single feature x
  - No linear separator exists for this data

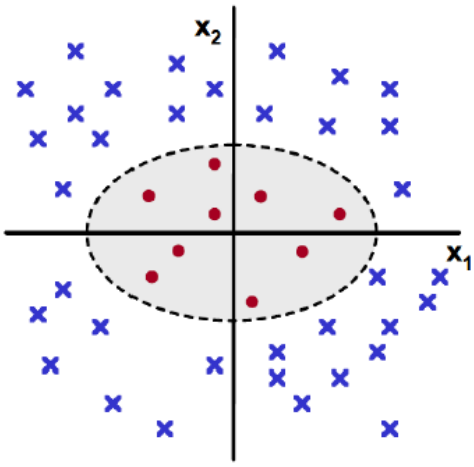


- Now map each example as $x \rightarrow \{x, x^2\}$

- Data now becomes linearly separable in the new representation

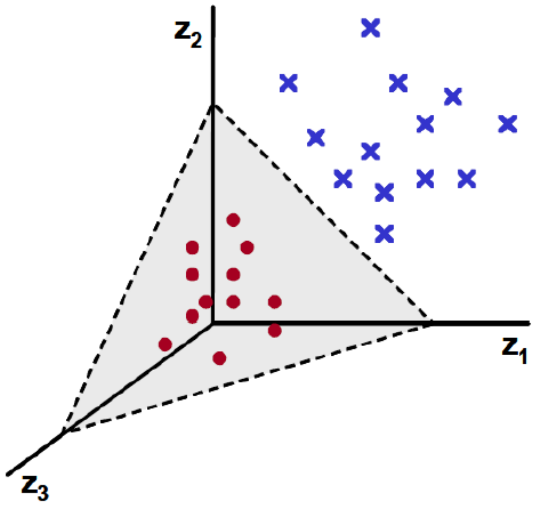


- Linear in the new representation $=$ nonlinear in the old representation
- Let's look at another example
  - Each example defined by a two features $x = \{x_1, x_2\}$
  - No linear separator exists for this data



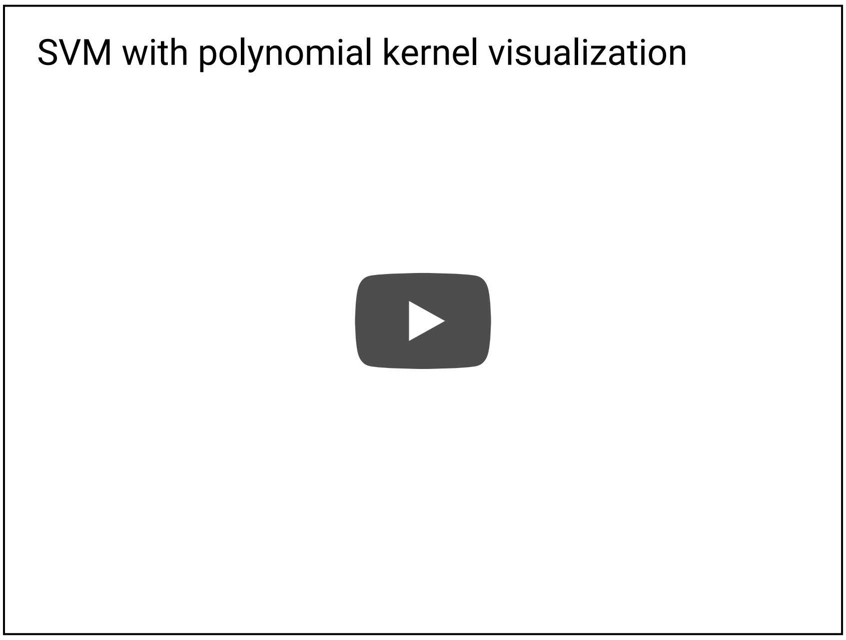
- Now map each example as $x = \{x_1, x_2\} \rightarrow z = \{x_1^2, \sqrt{2}x_1x_2, x_2^2\}$
  - Each example now has three features (derived from the old represenation)

- Data now becomes linear separable in the new representation



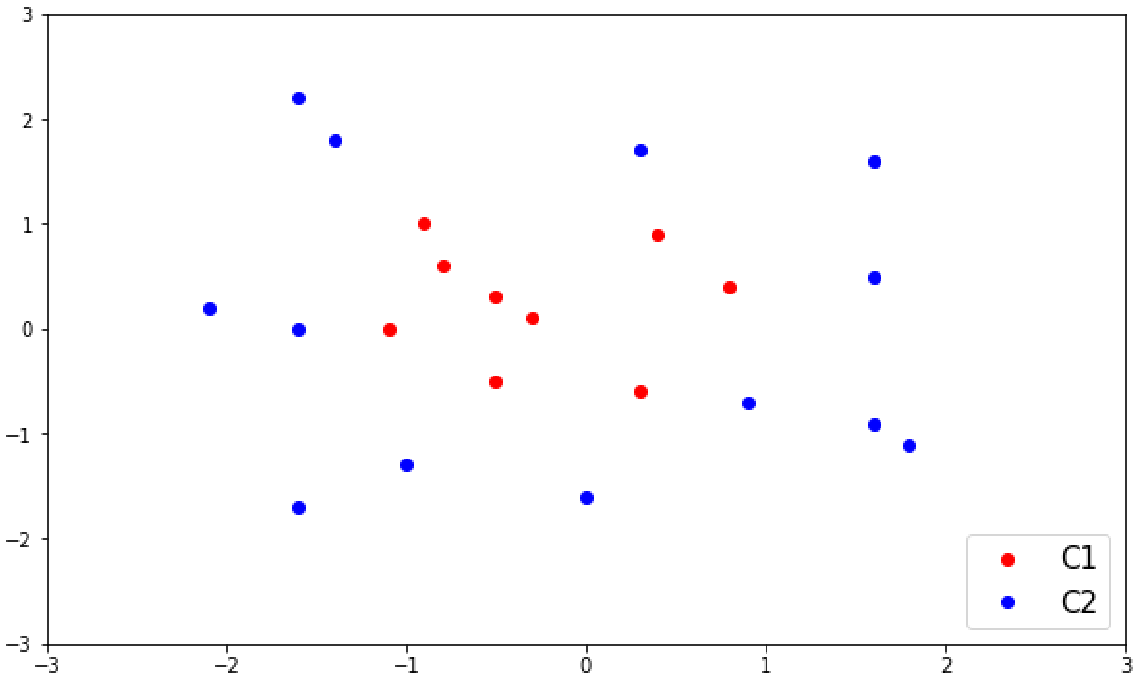
In [2]:

```
%%html
<center><iframe
width="420" height="315" src="https://www.youtube.com/embed/3liCbRZPrZA?rel=0" framebor
der="0" allowfullscreen>
</iframe></center>
```



In [14]:

```
X1 = np.array([[-1.1,  0], [-0.3,  0.1], [-0.9,  1],[0.8,  0.4],[0.4,  0.9],[0.3,-0.6],
[-0.5, 0.3],
        [-0.8,  0.6],[-0.5, -0.5]])

X2 = np.array([[-1,  -1.3], [-1.6 , 2.2],  [0.9, -0.7],[1.6,  0.5],[1.8, -1.1],[1.6,
1.6],[-1.6, -1.7],
        [-1.4,  1.8],[1.6, -0.9],[0, -1.6],[0.3, 1.7],[-1.6 , 0],[-2.1,0.2]])

X1 = np.asmatrix(X1)
X2 = np.asmatrix(X2)

plt.figure(figsize=(10, 6))
plt.plot(X1[:, 0], X1[:,1], 'ro', label='C1')
plt.plot(X2[:, 0], X2[:,1], 'bo', label='C2')
plt.axis([-3,3,-3,3])
plt.legend(loc = 4, fontsize = 15)
plt.show()
```



In [15]:

```
N = X1.shape[0]
M = X2.shape[0]

X = np.vstack([X1, X2])
y = np.vstack([np.ones([N,1]), -np.ones([M,1])])

X = np.asmatrix(X)
y = np.asmatrix(y)

m = N + M
Z = np.hstack([np.ones([m,1]), np.square(X[:,0]),
np.sqrt(2)*np.multiply(X[:,0],X[:,1]), np.square(X[:,1])])
```

In [16]:

```
g = 1
w = cvx.Variable(4, 1)
d = cvx.Variable(m, 1)

obj = cvx.Minimize(cvx.norm(w, 2) + g*np.ones([1,m])*d)
const = [cvx.mul_elemwise(y, Z*w) >= 1-d, d>=0]
prob = cvx.Problem(obj, const).solve()

w = w.value
print(w)
```

```
[[ 2.08736995]
 [-1.20600389]
 [-0.17476429]
 [-1.20600389]]
```

In [17]:

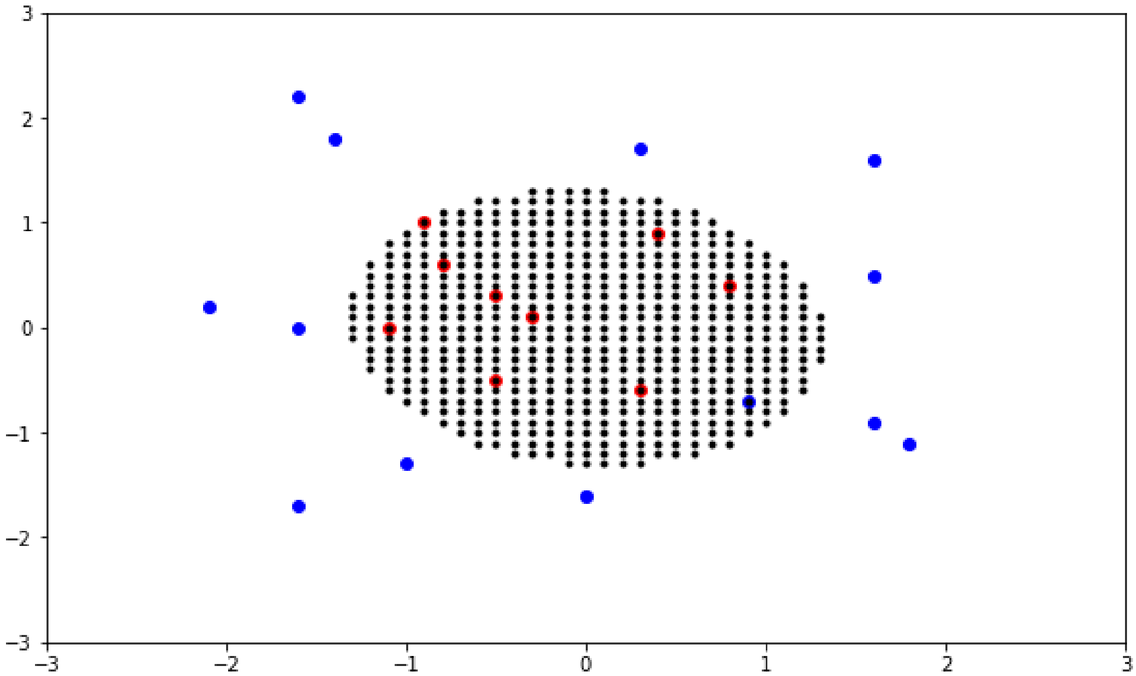
```
# to plot
[X1gr, X2gr] = np.meshgrid(np.arange(-3,3,0.1), np.arange(-3,3,0.1))

test_X = np.hstack([X1gr.reshape(-1,1), X2gr.reshape(-1,1)])
test_X = np.asmatrix(test_X)

m = test_X.shape[0]
test_Z  = np.hstack([np.ones([m,1]), np.square(test_X[:,0]), \
                     np.sqrt(2)*np.multiply(test_X[:,0],test_X[:,1]),
np.square(test_X[:,1])])
q = test_Z*w
```

In [18]:

```
B = []
for i in range(m):
    if q[i,0] > 0:
        B.append(test_X[i,:])

#B = np.array(B).reshape(-1,2)
B = np.vstack(B)
```

In [19]:

```
plt.figure(figsize=(10, 6))
plt.plot(X1[:,0], X1[:,1], 'ro')
plt.plot(X2[:,0], X2[:,1], 'bo')
plt.plot(B[:,0], B[:,1], 'k.')
plt.axis([-3, 3, -3, 3])
plt.show()
```



In [20]:

```
%%javascript
$.getScript('https://kmahelona.github.io/ipython_notebook_goodies/ipython_notebook_toc.js')
```