

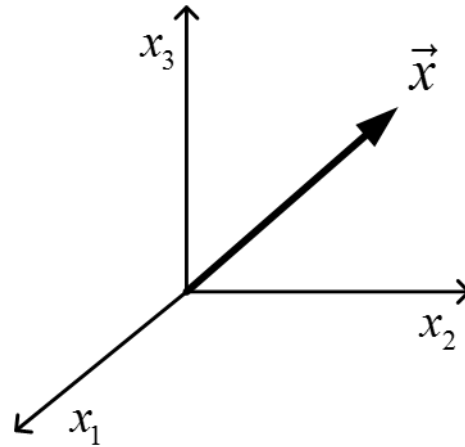
Linear Algebra 2

Industrial AI Lab.

Vector

- Vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Matrix and Transformation

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \quad \vec{y} = M\vec{x}$$

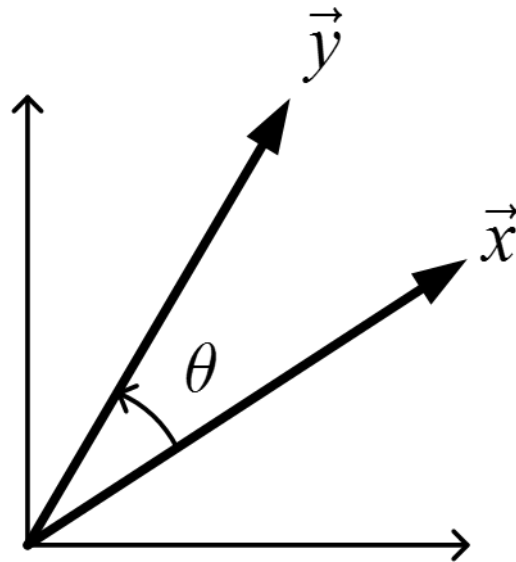
$$\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}$$

Given		Interpret
Transformation	→	matrix
matrix	→	Transformation

\vec{x} transformation \vec{y}
 input \implies output

Rotation

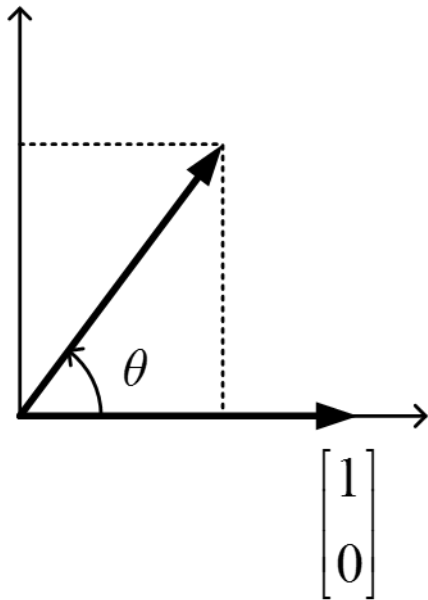
- Rotation matrix: $M = R(\theta)$
- Transformation: $\vec{y} = R(\theta)\vec{x}$



Rotation

- To find matrix $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$

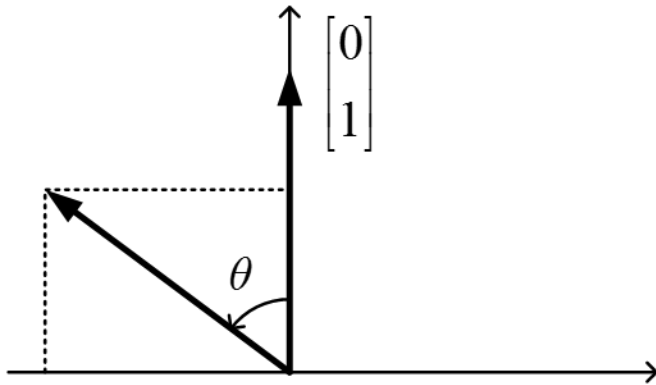


$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Rotation

- To find matrix $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$



$$\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Rotation

- To find matrix $M = R(\theta)$

$$\begin{array}{l} M\vec{x}_1 = \vec{y}_1 \\ M\vec{x}_2 = \vec{y}_2 \end{array} \quad = \quad M \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \quad = \quad \begin{bmatrix} \vec{y}_1 & \vec{y}_2 \end{bmatrix}$$

$$\implies \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R(\theta)$$

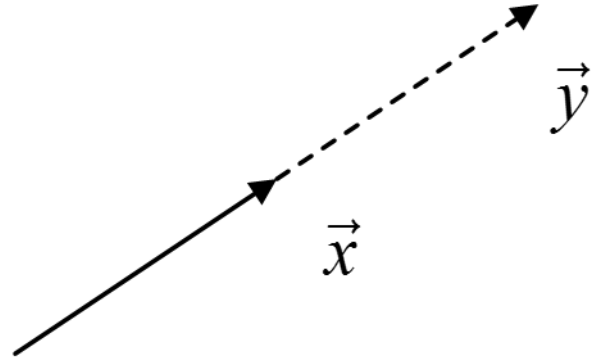
Stretch/Compress

- Stretch/Compress
 - keep the direction

$$\vec{y} = \underset{\substack{\uparrow \\ \text{scalar (not matrix)}}}{k} \vec{x}$$

$$\vec{y} = kI\vec{x} \quad \text{where } I = \text{Identity matrix}$$

$$\vec{y} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x}$$



Stretch/Compress

- T : stretch by a along \hat{x} -direction & stretch by b along \hat{y} -direction
- Compute the corresponding matrix A

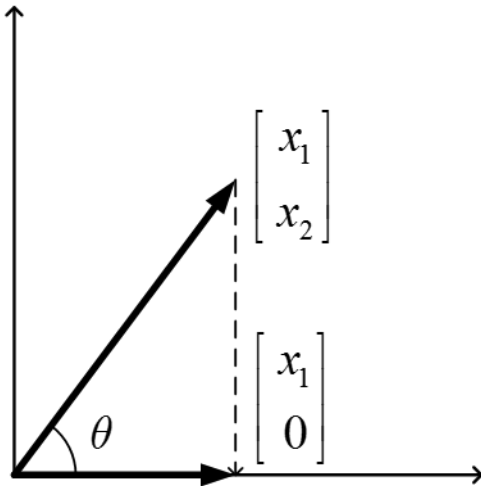
$$\begin{bmatrix} ax_1 \\ bx_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies A = ?$$
$$= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$
$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$
$$A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

- More importantly, can you think of the corresponding transformation T by looking at $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$?

Projection

- P: Projection onto \hat{x} - axis



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xRightarrow{P} \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$\vec{x} \qquad \qquad \vec{y}$

$$\vec{y} = P\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Multiple Transformations

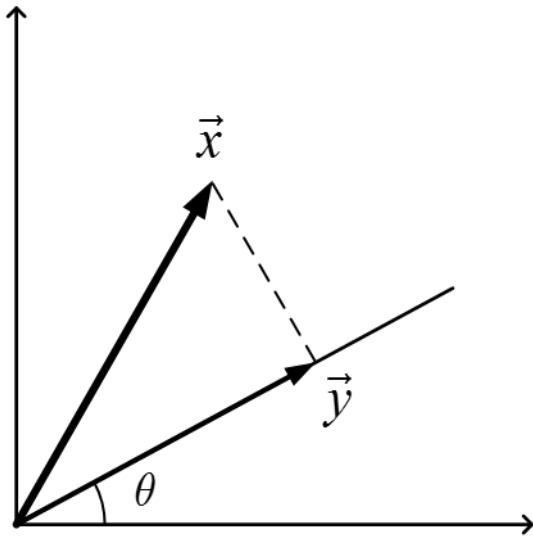
- T_1 : transformation 1 of matrix M_1
- T_2 : transformation 2 of matrix M_2
- T : Do transformation 1, followed by transformation 2

$$\vec{x} \xrightarrow{T_1} \vec{y} \xrightarrow{T_2} \vec{z}$$

$$\begin{aligned}\vec{y} &= M_1 \vec{x} \\ \vec{z} &= M_2 \vec{y} = M_2 M_1 \vec{x} \\ &= M \vec{x}\end{aligned}$$

$$\therefore M = M_2 M_1$$

P: Projection onto Vector = $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$



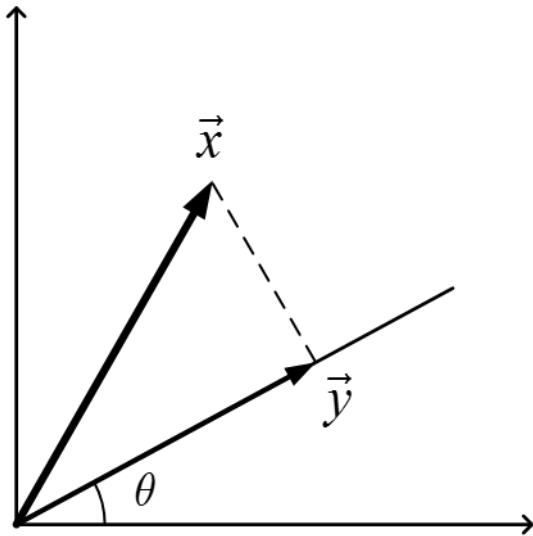
$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2 \theta \\ \cos \theta \sin \theta \end{bmatrix}$$

$$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \theta \\ \sin^2 \theta \end{bmatrix}$$

$$P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$$

P: Projection onto Vector = $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$

- Another way to find this projection matrix



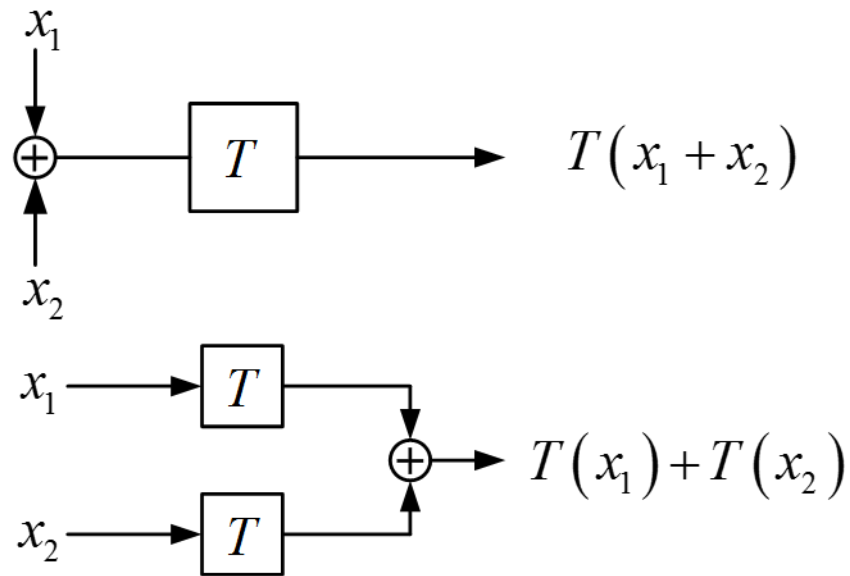
$$\begin{aligned}
 \vec{x} &\xRightarrow{R(-\theta)} \vec{x}' \xRightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \vec{x}'' \xRightarrow{R(\theta)} \vec{y} \\
 \vec{y} &= R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(-\theta) \vec{x} \\
 &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\
 &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix}
 \end{aligned}$$

Linear Transformation

- See if the given transformation is linear
 - A linear system makes our life much easier
- Superposition
- Homogeneity

Linear Transformation

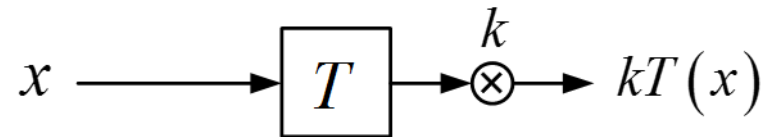
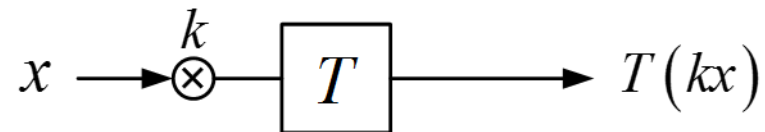
- Superposition



$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

Linear Transformation

- Homogeneity



$$T(kx) = kT(x)$$

Linear Transformation

- Linear vs. Non-linear

linear

$$f(x) = 0$$

$$f(x) = kx$$

$$f(x(t)) = \frac{dx(t)}{dt}$$

$$f(x(t)) = \int_a^b x(t)dt$$

non-linear

$$f(x) = x + c$$

$$f(x) = x^2$$

$$f(x) = \sin x$$

Linear Transformation

- If \vec{v}_1 and \vec{v}_2 are basis, and we know $T(\vec{v}_1) = \vec{\omega}_1$ and $T(\vec{v}_2) = \vec{\omega}_2$
- Then, for any \vec{x}

$$\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 \quad (a_1 \text{ and } a_2 \text{ unique})$$

$$\begin{aligned} T(\vec{x}) &= T(a_1\vec{v}_1 + a_2\vec{v}_2) \\ &= a_1T(\vec{v}_1) + a_2T(\vec{v}_2) \\ &= a_1\vec{\omega}_1 + a_2\vec{\omega}_2 \end{aligned}$$

Eigenvalue and Eigenvector

$$A\vec{v} = \lambda\vec{v}$$

$A\vec{x}$ parallel to \vec{x}

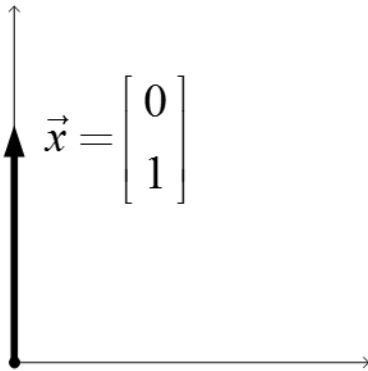
$$\lambda = \begin{cases} \text{positive} \\ 0 \\ \text{negative} \end{cases}$$

λ : stretched vector
(same direction with \vec{x})

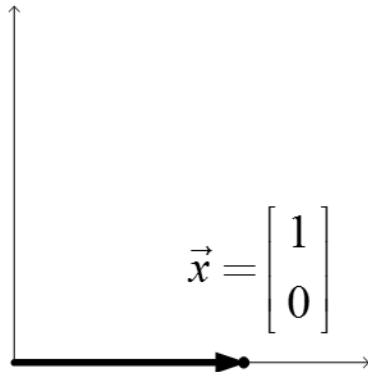
A : transformed vector
(generally rotate + stretch)

Eigen Analysis of Projection

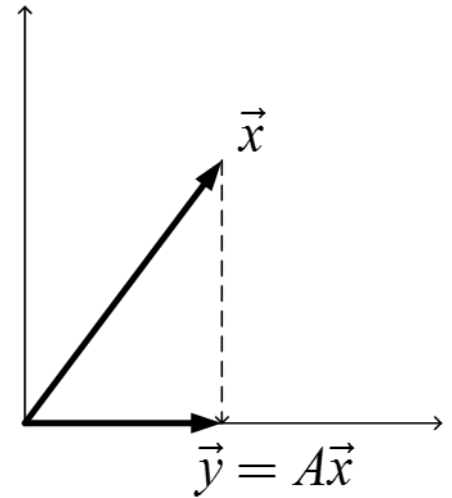
- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: projection onto \hat{x} - axis
- Find eigenvalues and eigenvectors of A .



$$\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\vec{x} = 0 \cdot \vec{x}$$
$$\lambda_1 = 0 \text{ and } \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

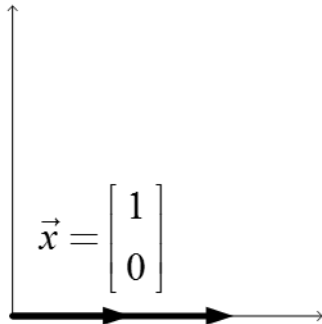


$$\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\vec{x} = 1 \cdot \vec{x}$$
$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

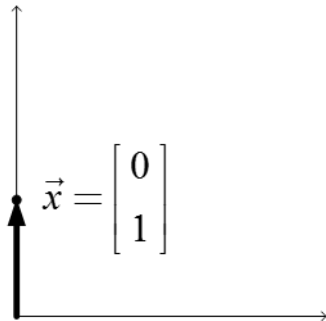


Eigen Analysis

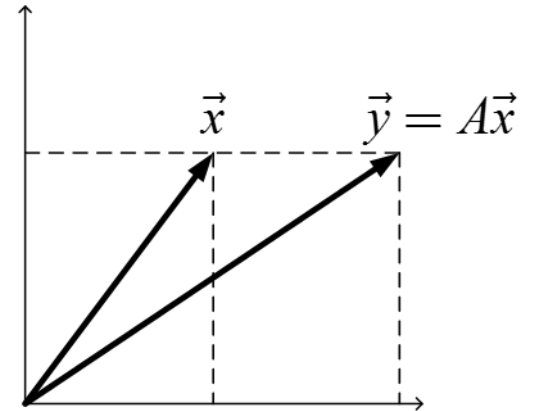
- $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$: stretch by 2 along \vec{x} - axis
stretch by 1 along \vec{y} - axis
- Find eigenvalues and eigenvectors.



$$\lambda_1 = 2 \text{ and } \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Eigen Analysis in Python

```
A = np.array([[2, 0],
              [0, 1]])
D, V = np.linalg.eig(A)

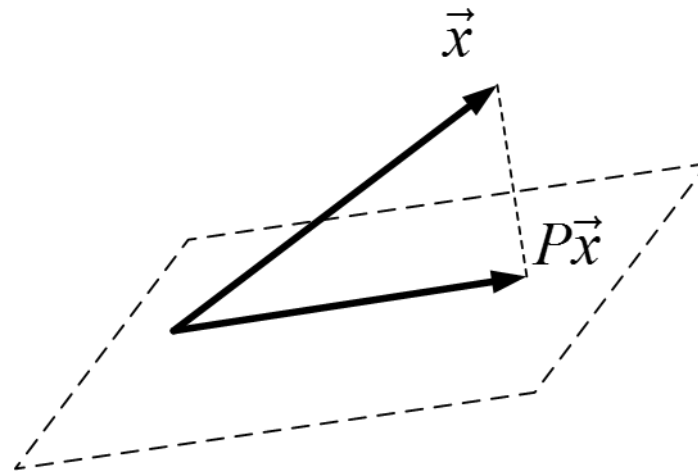
idx = np.argsort(-D)
D = D[idx]
V = V[idx]

print('D :', D)
print('V :', V)
```

```
D : [ 2.  1.]
V : [[ 1.  0.]
     [ 0.  1.]
```

Eigen Analysis of Projection

- Projection onto the plane
- Find eigenvalues and eigenvectors

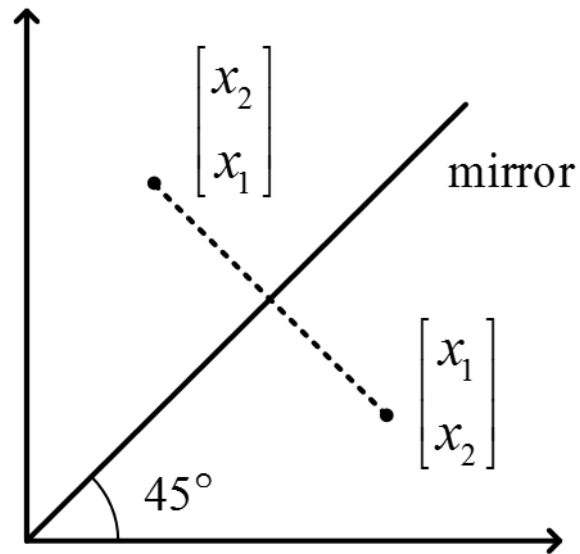


- For any \vec{x} in the plane, $P\vec{x} = \vec{x} \rightarrow \lambda = 1$
- For any \vec{x} perpendicular to the plane, $P\vec{x} = \vec{0} \rightarrow \lambda = 0$

Eigen Analysis of Mirror

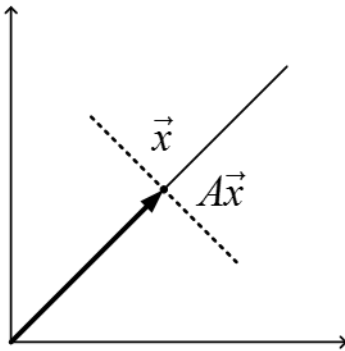
- What kind of a linear transformation?

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

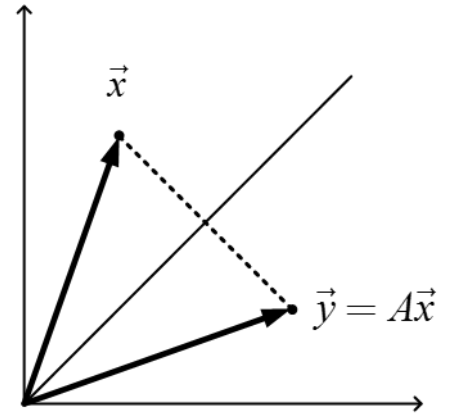


Eigen Analysis of Mirror

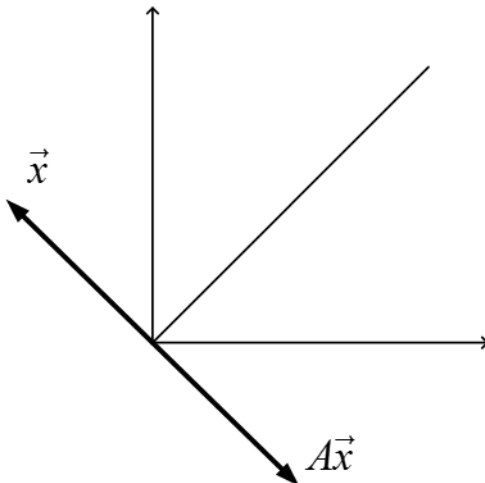
- Eigenvalues and eigenvectors?
 - can \vec{x} be an eigenvector?



$$A\vec{x} = \vec{x}, \quad \lambda = 1$$



$$A\vec{x} = -\vec{x}, \quad \lambda = -1$$



Eigen Analysis of Mirror

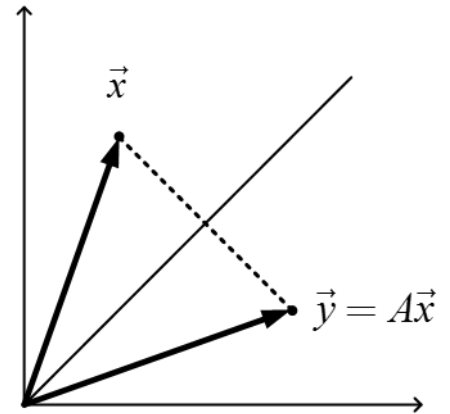
- Side note: Matrix A can be seen as a multiple transformations

$$A = R(45)MR(-45)$$

$$R(45) = \begin{bmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$M : \text{mirror along } \hat{x}\text{-axis, } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

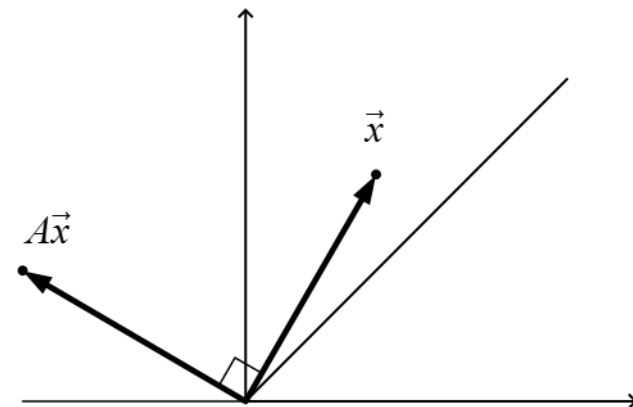


Eigen Analysis of Rotation

- What kind of a linear transformation?

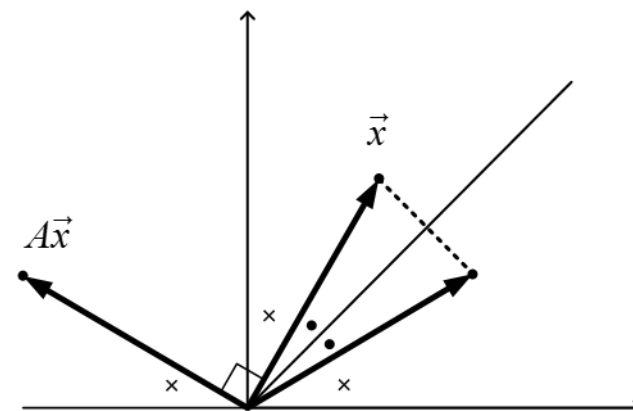
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = R\left(\frac{\pi}{2}\right) = R(90^\circ) = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix}$$



- Multiple transformations

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



Eigen Analysis of Rotation

- What kind of a linear transformation?

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = R\left(\frac{\pi}{2}\right) = R(90^\circ) = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix}$$

- Eigenvalues: complex number $\Rightarrow \det(A - \lambda I) = 0$
$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$
$$\therefore \lambda = \pm i$$
- What is the physical meaning?

Eigenvalue and Eigenvector

- How to Compute Eigenvalue and Eigenvector

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} = \lambda I\vec{v} \\ A\vec{v} - \lambda I\vec{v} &= (A - \lambda I)\vec{v} = 0 \end{aligned}$$

$$\begin{aligned} \implies A - \lambda I &= 0 \text{ or} \\ \vec{v} &= 0 \text{ or} \\ (A - \lambda I)^{-1} &\text{ does not exist} \end{aligned}$$

$$\implies \det(A - \lambda I) = 0$$