Linear Algebra 3

Industrial AI Lab.

System of Linear Equations

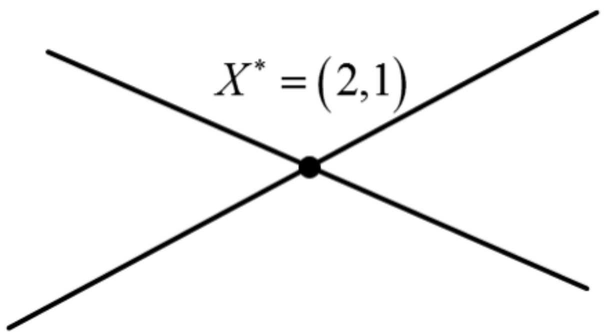
- Well-determined linear systems
- Under-determined linear systems
- Over-determined linear systems

Well-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_1^* &= 2 \\ x_2^* &= 1 \end{aligned}$$

- Geometric analysis



Well-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_1^* &= 2 \\ x_2^* &= 1 \end{aligned}$$

- Generalization

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \quad \overset{\text{Matrix form}}{\Longrightarrow} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$AX = B$$

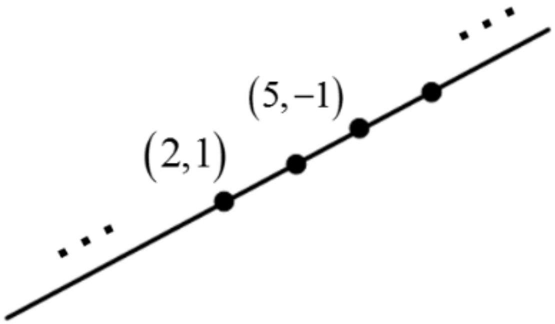
$$\therefore \quad X^* = A^{-1}B \quad \text{if } A^{-1} \text{ exists}$$

Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \quad \Longrightarrow \quad \text{Many solutions}$$

- Geometric analysis



Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \quad \Longrightarrow \quad \text{Many solutions}$$

- Generalization

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \xrightarrow{\text{Matrix form}} \quad \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

$$AX = B$$

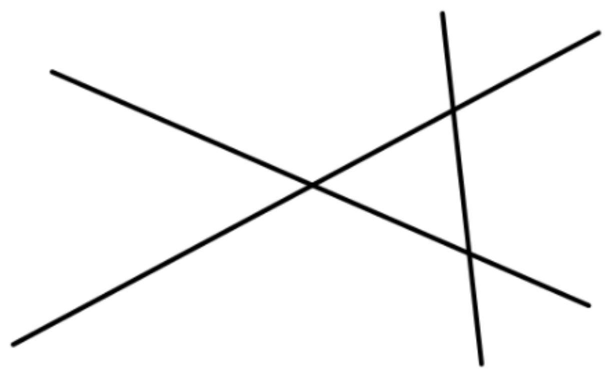
$$\therefore \text{ Many Solutions when } A \text{ is fat}$$

Over-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \\ x_1 + x_2 &= 4 \end{aligned} \implies \text{No solutions}$$

- Geometric analysis



Over-Determined Linear Systems

- System of linear equations

$$
\begin{aligned}
2x_1 + 3x_2 &= 7 \\
x_1 + 4x_2 &= 6 \quad \Longrightarrow \quad \text{No solutions} \\
x_1 + x_2 &= 4
\end{aligned}
$$

- Generalization

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2 \\
a_{31}x_1 + a_{32}x_2 &= b_3
\end{aligned}
\quad \overset{\text{Matrix form}}{\Longrightarrow} \quad
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

$$AX = B$$

$\therefore$ No Solutions when A is skinny

Summary of Linear Systems

$$AX = B$$

- Square: Well-determined

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- Fat: Under-determined

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

- Skinny: Over-determined

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Least-Norm Solution

- For Under-Determined Linear System

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1 \quad \text{or} \quad AX = B$$

- Find the solution of $AX = B$ that minimize $\|X\|$ or $\|X\|^2$

- *i.e.*, optimization problem

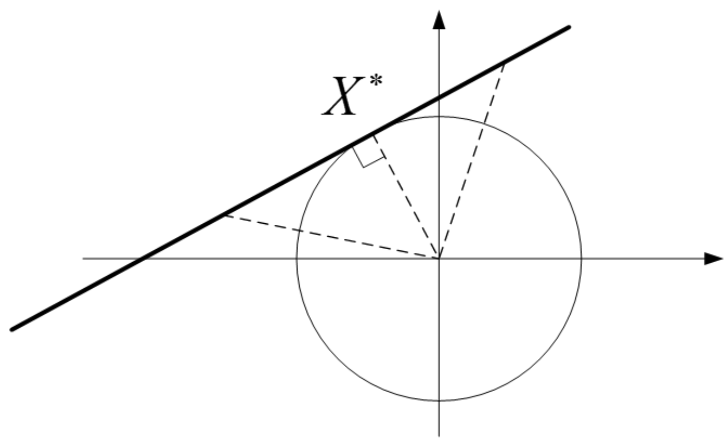
$$\begin{aligned} &\min \;\; \|X\|^2 \\ &\text{s. t. } AX = B \end{aligned}$$

Least-Norm Solution

- Optimization problem

$$\begin{aligned} &\min \ \ \|X\|^2 \\ &\text{s. t. } AX = B \end{aligned}$$

- Geometric analysis



- Select one solution among many solutions
- Often control problem

$$X^* = A^T\left(AA^T\right)^{-1}B \quad \text{Least norm solution}$$

Least-Square Solution

- For Over-Determined Linear System

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{or} \quad AX \neq B \quad x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Find X that minimizes $\|E\|$ or $\|E\|^2$

- *i.e.* optimization problem

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$

Least-Square Solution

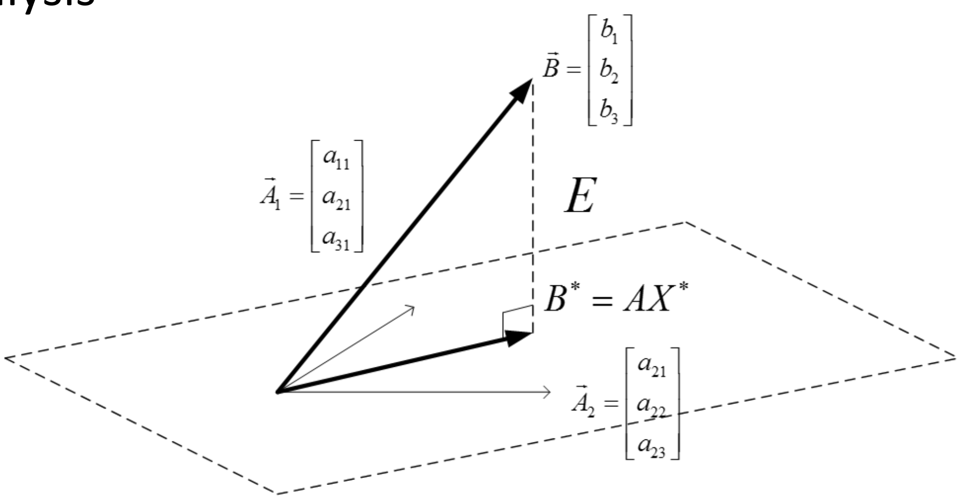
- *i.e.* optimization problem

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$

$$X^* = \left(A^T A\right)^{-1} A^T B$$

$$B^* = AX^* = A\left(A^T A\right)^{-1} A^T B$$

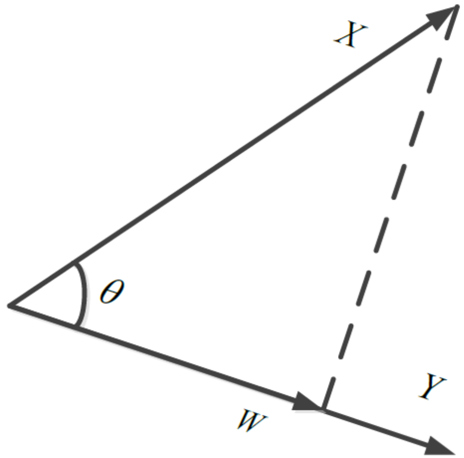
- Geometric analysis



- Often estimation problem

Vector Projection

- The vector projection of a vector X on (or onto) a nonzero vector Y is the orthogonal projection of X onto a straight line parallel to Y



$$W = \omega \hat{Y} = \omega \frac{Y}{\|Y\|}, \text{ where } \omega = \|W\|$$

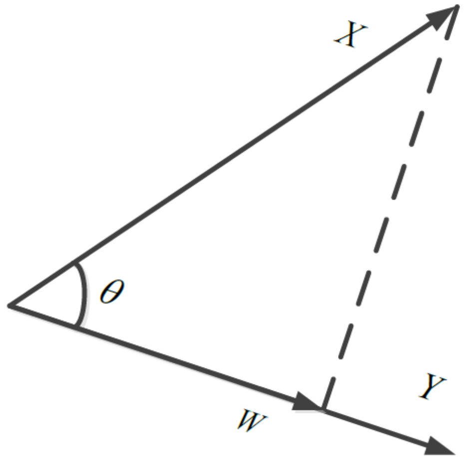
$$\omega = \|X\| \cos\theta = \|X\| \frac{X \cdot Y}{\|X\| \|Y\|} = \frac{X \cdot Y}{\|Y\|}$$

$$W = \omega \hat{Y} = \frac{X \cdot Y}{\|Y\|} \frac{Y}{\|Y\|} = \frac{X \cdot Y}{\|Y\| \|Y\|} Y = \frac{X^T Y}{Y^T Y} Y = \frac{\langle X, Y \rangle}{\langle Y, Y \rangle} Y$$

Vector Projection

- Another way of computing ω and W

$$Y \perp (X - W)$$
$$\implies Y^T(X - W) = Y^T\left(X - \omega\frac{Y}{\|Y\|}\right) = 0$$
$$\implies \omega = \frac{Y^TX}{Y^TY}\|Y\|$$
$$W = \omega\frac{Y}{\|Y\|} = \frac{Y^TX}{Y^TY}Y = \frac{\langle X, Y\rangle}{\langle Y, Y\rangle}Y$$



Orthogonal Projection onto a Subspace

- Projection of B onto a subspace U of span of A_1 and A_2
- Orthogonality

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$

$$X^* = \left(A^T A\right)^{-1} A^T B$$

$$B^* = AX^* = A\left(A^T A\right)^{-1} A^T B$$

$$\begin{aligned} A &\perp (AX^* - B) \\ A^T (AX^* - B) &= 0 \\ A^T A X^* &= A^T B \\ X^* &= \left(A^T A\right)^{-1} A^T B \\ B^* = AX^* &= A\left(A^T A\right)^{-1} A^T B \end{aligned}$$