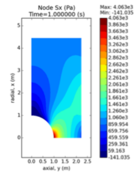
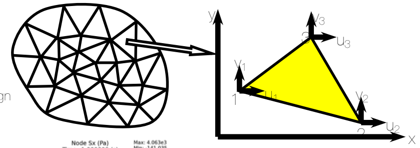
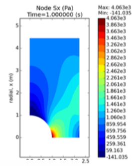
Computational Solid Mechanics MAE701

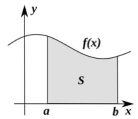
What is behind these stress fields with nice colors?



2D ELASTICITY
Based on Prof Haftka's
Finite Element Analysis and Design







$$\int_a^b f(x)dx$$

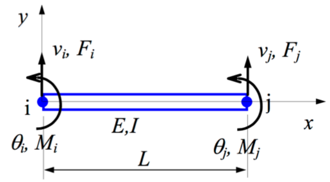
NUMERICAL INTEGRATION

Supplementary materials to read in autonomy

But before Review Beam's element?

Simple BEAM
(I=Izz)

but of course in 3D space there is also Iyy if x is the longitudinal axis



L length

I moment of inertia of the cross-sectional

E elastic modulus

$v = v(x)$ deflection (lateral displacement) of the neutral axis

$\theta = \frac{dv}{dx}$ rotation about the z-axis

$F = F(x)$ shear force

$M = M(x)$ moment about z-axis

Elementary Beam Theory:

$$\boxed{EI\frac{d^2v}{dx^2} = M(x) \qquad \sigma = -\frac{My}{I}}$$

FORMAL APPROACH

• See LMS textbook

Formal Approach

Apply the formula,

$$\mathbf{k} = \int_0^L \mathbf{B}^T EI \mathbf{B} dx$$

To derive this, we introduce the shape functions,

$$N_1(x) = 1 - 3x^2 / L^2 + 2x^3 / L^3$$
$$N_2(x) = x - 2x^2 / L + x^3 / L^2$$
$$N_3(x) = 3x^2 / L^2 - 2x^3 / L^3$$
$$N_4(x) = -x^2 / L + x^3 / L^2$$

Then, we can represent the deflection as,

$$v(x) = \mathbf{Nu}$$

$$v(x) = \mathbf{Nu}$$

$$= \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}$$

which is a cubic function. Notice that,

$$N_1 + N_3 = 1$$
$$N_2 + N_3 L + N_4 = x$$

which implies that the rigid body motion is represented by the assumed deformed shape of the beam.

FORMAL APPROACH

• See LMS textbook

Curvature of the beam is,

$$\frac{d^2 v}{dx^2} = \frac{d^2}{dx^2}\mathbf{Nu} = \mathbf{Bu}$$

where the strain-displacement matrix $\mathbf{B}$ is given by,

$$\mathbf{B} = \frac{d^2}{dx^2}\mathbf{N} = \begin{bmatrix} N_1''(x) & N_2''(x) & N_3''(x) & N_4''(x) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{6}{L^2} + \frac{12x}{L^3} & -\frac{4}{L} + \frac{6x}{L^2} & \frac{6}{L^2} - \frac{12x}{L^3} & -\frac{2}{L} + \frac{6x}{L^2} \end{bmatrix}$$

Strain energy stored in the beam element is

$$U = \frac{1}{2}\int_V \sigma^T \varepsilon dV = \frac{1}{2}\int_0^L \int_A \left(-\frac{My}{I}\right)^T \frac{1}{E}\left(-\frac{My}{I}\right) dA dx$$

$$= \frac{1}{2}\int_0^L M^T \frac{1}{EI} M dx = \frac{1}{2}\int_0^L \left(\frac{d^2 v}{dx^2}\right)^T EI \left(\frac{d^2 v}{dx^2}\right) dx$$

$$= \frac{1}{2}\int_0^L (\mathbf{Bu})^T EI (\mathbf{Bu}) dx$$

$$= \frac{1}{2}\mathbf{u}^T \left(\int_0^L \mathbf{B}^T EI \mathbf{B} dx\right)\mathbf{u}$$

We conclude that the stiffness matrix for the simple beam element is

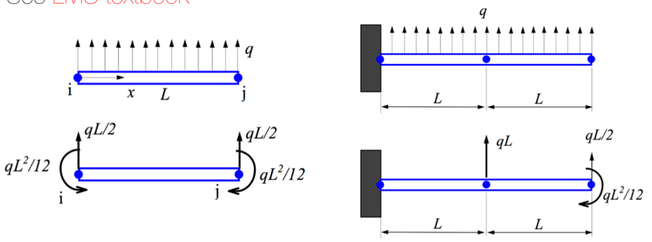
$$\mathbf{k} = \int_0^L \mathbf{B}^T EI \mathbf{B}$$

Element stiffness equation (*local node*: i, j or 1, 2):

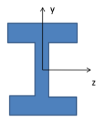
$$\frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_i & \theta_i & v_j & \theta_j \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix}$$

Equivalent Nodal Loads of Distributed Transverse Load

• See LMS textbook



Bar (Area only) vs Beam (cross section)



I_1 = inertia for bending in the xy plane = I_Z
I_2 = inertia for bending in the xz plane = I_Y

Torsion:

$$k_\phi = \frac{JG}{L}$$

$$k = \frac{AE}{L}$$

$$G = \text{Shear modulus} = \frac{E}{2(1+\upsilon)}$$

J = Torsional Factor

$J = I1 + I2$ for circular cross sections only (See tables otherwise)

Simple Beam Element (CBAR)

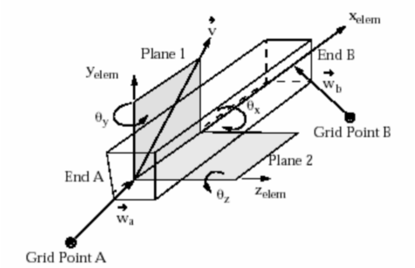
CBAR Element Characteristics

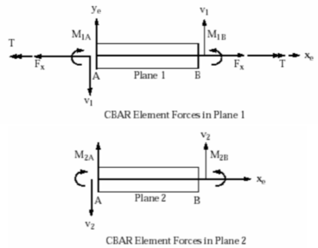
The CBAR element is a general purpose beam that supports tension and compression, torsion, bending in two perpendicular planes, and shear in two perpendicular planes. The CBAR uses two grid points, and can provide stiffness to all six DOFs of each grid point. The displacement components of the grid points are three translations and three rotations.

The characteristics and limitations of the CBAR element are summarized as follows:

- Its formulation is derived from classical beam theory (plane sections remain plane).
- It must be straight and prismatic (properties cannot vary along the length).
- The shear center and neutral axis must coincide (the CBAR element cannot model warping of open sections).
- Torsional stiffening of out-of-plane cross-sectional warping is neglected.
- It includes optional transverse shear effects (important for short beams).
- The principal axis of inertia need not coincide with the element axis.
- The neutral axis may be offset from the grid points (an internal rigid link is created). This is useful for modeling stiffened plates or gridworks.
- A pin flag capability is available to provide a moment or force release at either end of the element (this permits the modeling of linkages or mechanisms).

The element coordinate system is defined by either an orientation vector or an orientation node, both of which defines a plane *in which* bending with the second moment of area of I1 occurs. Hence, if I1 is the major principal second moment of area, the orientation node will define the minor axis of bending. The element coordinate system is then defined as shown.





Bar Element Property (PBAR)

The PBAR entry defines the properties of a CBAR element. The format of the PBAR entry is as follows:

1	2	3	4	5	6	7	8	9	10
PBAR	PID	MID	A	I1	I2	J	NSM		
	C1	C2	D1	D2	E1	E2	F1	F2	
	K1	K2	I12						

Field	Contents
PID	Property identification number. (Integer > 0)
MID	Material identification number. (Integer > 0)
A	Area of bar cross section. (Real)
I1, I2, I12	Area moments of inertia. (Real; $I1 \geq 0.0$, $I2 \geq 0.0$, $I1 \cdot I2 > I12^2$)
J	Torsional constant. (Real)
NSM	Nonstructural mass per unit length. (Real)
K1, K2	Area factor for shear. (Real)
Ci, Di, Ei, Fi	Stress recovery coefficients. (Real; Default = 0.0)

PID is the property's identification number from field 3 of the CBAR entry. MID references a MAT1 material property entry. I1 and I2 are area moments of inertia:

I1 = area moment of inertia for bending in plane 1 (same as Izz, bending about the z element axis)

I2 = area moment of inertia for bending in plane 2 (same as Iyy, bending about the y element axis)

SEE PBEAM ON QRG.pdf

J is the cross section's torsional constant (see Table 6-2). K1 and K2 depend on the shape of the cross section. K1 contributes to the shear resisting transverse force in plane 1 and K2 contributes to the shear resisting transverse force in plane 2.

Table 6-3 Area Factors for Shear

Shape of Cross Section	Value of K
Rectangular	K1 = K2 = 5/6
Solid Circular	K1 = K2 = 9/10
Thin-wall Hollow Circular	K1 = K2 = 1/2
Wide Flange Beams:	
Minor Axis	$\approx A_f / 1.2A$
Major Axis	$\approx A_w / A$

where:

A = Beam cross-sectional area
A_f = Area of flange
A_w = Area of web

The first continuation entry defines stress recovery coefficient points (Ci, Di, Ei, Fi) on the beam's cross section. These points are in the y-z plane of the element coordinate system as shown in

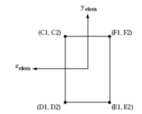
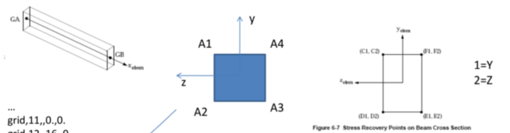


Figure 6-7 Stress Recovery Points on Beam Cross Section

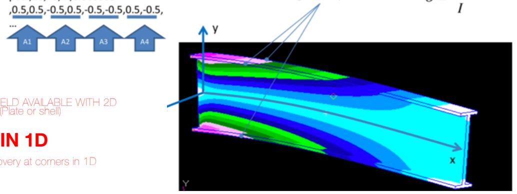
By defining stress recovery points, you are providing c in the equation $\sigma = Mc/I$, thereby allowing MSC.Nastran to calculate stresses in the beam or on its surface.

1D vs 2D




```
...
grid,11,,0.,0.
grid,12,,16.,0.
cbar,21,20,11,12,0.,1.,0.
pbar,20,40,1.,0.083333
,0.5,0.5,-0.5,0.5,-0.5,-0.5,0.5,-0.5,
...
```

Max Stress generally at corners $\sigma = \dfrac{Mc}{I}$

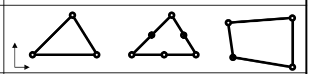
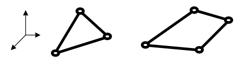
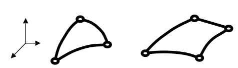


STRESS FIELD AVAILABLE WITH 2D ELEMENT (Plate or shell)

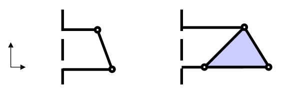
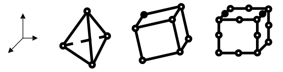
NOT IN 1D

Stress recovery at corners in 1D

2D Element types

	Name	DOF per Node	GEOMETRY
2 dimensions	2D ELASTICITY	u v	
	Plate	u v w α β	
	Shell	u v w α β	

Axi+3D

	Name	DOF per Node	GEOMETRY
Axi-symetric	Shell or Solid	u v w	
3 D	Volume	u v w	

With Me

4 courses
- 1 Matlab assignements* (#1,#2)

→publish reports on LMS.

- Slides to read in autonomy (online materials)
- Link with the Class Exercices

Today C3

1. GAUSS integration (2 exercices)
2. Plane Stress VS Plane Strain
3. 2D elements (CST VS QUAD)
4. Isoparametric element (see lecture note)
5. In plane (membranes) VS out of plane (Plates & Shells)

Today C3

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Numerical Integration

Partial evaluation of integrals over isoparametric elements

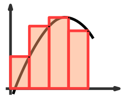
- Gauss integration   integration
- minimal number of sample points
- high level of accuracy
- higher computational efficiency

3 main methods

Convergency

Rectangle method

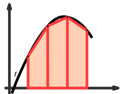
$$I_n = \frac{b-a}{n}\sum_{k=1}^{n} f(x_k)$$



$O(n^{-1})$

Trapezoidal rules (Matlab's trapz)

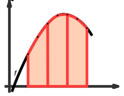
$$I_n = \frac{b-a}{2n}\sum_{k=1}^{n} \left(f(x_{k-1})+f(x_k)\right)$$



$O(n^{-2})$

Simpson's formula (Matlab's quad)

$$I_n = \frac{b-a}{6n}\sum_{k=1}^{n} \left(f(x_{k-1})+4f(x_{k-1/2})+f(x_k)\right)$$



$O(n^{-4})$

But Shape Function are polynomials function, Right?

- Stiffness matrix and distributed load calculations involve integration over the domain
- In many cases, analytical integration is very difficult
- Numerical integration based on Gauss Quadrature is commonly used in finite element programs
- Gauss Quadrature:

$$\boxed{I = \int_{-1}^{1} f(s)ds \approx \sum_{i=1}^{n} w_i f(s_i)}$$

  - Integral is evaluated using function values and weights.
  - s_i: Gauss integration points, w_i: integration weights
  - $f(s_i)$: function value at the Gauss point
  - n: number of integration points.

GAUSS QUADRATURE POINTS AND WEIGHTS

- What properties do positions and weights have?

See Lecture note

n	Integration Points (s)	Weights (w)	Exact for polynomial of degree
1	0.0	2.0	1
2	±.5773502692	1.0	3
3	±.7745966692 0.0	.5555555556 .8888888889	5
4	±.8611363116 ±.3399810436	.3478546451 .6521451549	7
5	±.9061798459 ±.5384693101 0.0	.2369268851 .4786286705 .5688888889	9

ONE INTEGRATION POINT

- Constant Function: $f(s) = 4$
  - Use one integration point $s_1 = 0$ and weight $w_1 = 2$

$$I = \int_{-1}^{1} 4ds = w_1 f(s_1) = 2 \times 4 = 8$$

  - The numerical integration is exact.
  - Linear Function: $f(s) = 2s + 1$
    - Use one integration point $s_1 = 0$ and weight $w_1 = 2$

$$I = \int_{-1}^{1} (2s+1)ds = w_1 f(s_1) = 2 \times 1 = 2$$

    - The numerical integration is exact.

→ One-point Gauss Quadrature can integrate constant and linear functions exactly.

TWO POINTS AND MORE

- Quadratic Function: $f(s) = 3s^2 + 2s + 1$
  - Let's use one-point Gauss Quadrature

$$I = \int_{-1}^{1} (3s^2 + 2s + 1)ds = 4$$

$$w_1 f(s_1) = 2 \times 1 = 2$$

  - One-point integration is not accurate for quadratic function
- Let's use two-point integration with $w_1 = w_2 = 1$ and $-s_1 = s_2 = 1/\sqrt{3}$

$$w_1 f(s_1) + w_2 f(s_2) = 1 \times f(-\tfrac{1}{\sqrt{3}}) + 1 \times f(\tfrac{1}{\sqrt{3}})$$
$$= 3 \times \tfrac{1}{3} - \tfrac{2}{\sqrt{3}} + 1 + 3 \times \tfrac{1}{3} + \tfrac{2}{\sqrt{3}} + 1 = 4$$

- Gauss Quadrature points and weights are selected such that n integration points can integrate (2n – 1)-order polynomial exactly.

3rd order in standard FEA code

- That's why FE code use third order polynomial as shape function
- Application to Stiffness Matrix Integral (note the Jacobian)

$$[\mathbf{k}^{(e)}] = h \int_{-1}^{1}\int_{-1}^{1} [\mathbf{B}]^T [\mathbf{C}][\mathbf{B}] |\mathbf{J}| ds dt$$
$$\approx h \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j [\mathbf{B}(s_i,t_j)]^T [\mathbf{C}][\mathbf{B}(s_i,t_j)] |\mathbf{J}(s_i,t_j)|$$

To apply Gauss–Legendre quadrature to the integral $\int_a^b f(x)dx$, we must first map the integration range (a, b) into the "standard" range $(-1, 1)$. We can accomplish this by the transformation

$$x = \frac{b+a}{2} + \frac{b-a}{2}\xi$$

Now $dx = d\xi(b-a)/2$, and the quadrature becomes

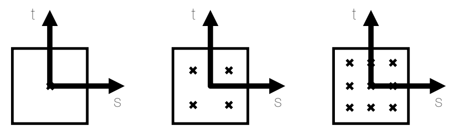
$$\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{i-1}^{n} A_i f(x_i)$$

TWO DIMENSIONAL INTEGRATION

- multiplying two one-dimensional Gauss integration formulas

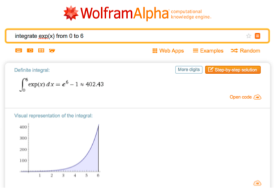
$$\boxed{I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)dsdt \approx \int_{-1}^{1}\sum_{i=1}^{m} w_i f(s_i,t)dt = \sum_{j=1}^{n}\sum_{i=1}^{m} w_i w_j f(s_i,t_j)}$$

- Total number of integration points = ***m*×*n***.



(a) 1×1 (b) 2×2 (c) 3×3

See online correction



correction on LMS

C3: Approximation & Gauss Quadrature

Exercice 1

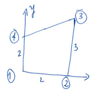
Integrate numerically $f(x) = e^x$ on $[0:6]$ using Gauss quadrature. compare with 'trapz', 'quad' Matlab's functions

$$\int_0^6 e^x\,dx = e^x\Big|_0^6 = e^6 - e^0 = e^6 - 1 \simeq 402{,}43$$

exact solution (see WOLFRAM α)

Compute the relative errors with exact solution

Exercice 2



Evaluate the integral $I = \iint_A (x^2 + y)\,dx\,dy$ over the quadrilateral shown.

Corners: $x^T = [0\ 2\ 2\ 0]$; $y^T = [0\ 0\ 3\ 2]$

Using the function gaussQuad.m, find the optimal gauss points to reach tolerance 1E-4 for abs(I_exact-I_gauss)

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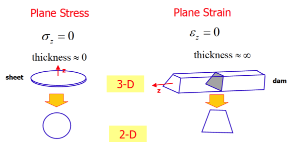
Plane solids

- All engineering problems are 3-D. It is the engineer who approximates the problem using 1-D (beam or truss) or 2-D (plane stress or strain).

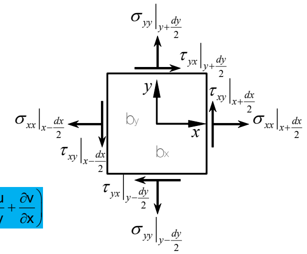


- **Plane stress**: zero stresses in the thickness direction (thin plate with in-plane forces)
- **Plane strain**: zero strains in the thickness direction (thick solid with constant thickness)
- Main variables: *u* (*x*-displacement) and *v* (*y*-displacement)

GOVERNING EQUATIONS

- Governing D.E. (equilibrium Eq1)

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$



- Strain-displacement Relation (linear)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- Stress-Strain Relation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \Leftrightarrow \{\sigma\} = [\mathbf{C}]\{\varepsilon\}$$

→ Since stress involves first-order derivative of displacements, the governing differential equation Eq1 is the second-order

GOVERNING EQUATIONS *cont.*

- Boundary Conditions
  - All differential equations must be accompanied by boundary conditions

$$\mathbf{u} = \mathbf{g}, \quad \text{on } S_g$$
$$\mathbf{s\,n} = \mathbf{T}, \quad \text{on } S_T$$

  - S_g is the essential boundary and S_T is the natural boundary
  - **g**: prescribed (specified) displacement (usually zero for linear problem)
  - **T**: prescribed (specified) surface traction force

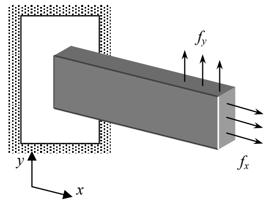
- **Objective**: to determine the displacement fields $u(x, y)$ and $v(x, y)$ that satisfy the D.E. and the B.C.

PLANE STRESS PROBLEM

- Plane Stress Problem:
  - Thickness is much smaller than the length and width dimensions
  - Thin plate or disk with applied in-plane forces
  - z-direction stresses are zero at large surfaces (side here)
  - Thus, it is safe to assume that they are also zero along the thickness

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$$

  - Non-zero stress components: σ_{xx}, σ_{yy}, τ_{xy}
  - Non-zero strain components: ε_{xx}, ε_{yy}, ε_{xy}, ε_{zz}



PLANE STRESS PROBLEM *cont.*

- Stress-strain relation for isotropic material

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \Leftrightarrow \{\sigma\} = [\mathbf{C}_\sigma]\{\varepsilon\}$$

  - Even if ε_{zz} is not zero, it is not included in the stress-strain relation because it can be calculated from the following relation:

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

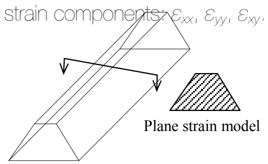
- How to derive plane stress relation?
  - Solve for $\boldsymbol{\varepsilon}_{zz}$ in terms of $\boldsymbol{\varepsilon}_{xx}$ and $\boldsymbol{\varepsilon}_{yy}$ from the relation of $\boldsymbol{\sigma}_{zz} = 0$
  - Write $\boldsymbol{\sigma}_{xx}$ and $\boldsymbol{\sigma}_{yy}$ in terms of $\boldsymbol{\varepsilon}_{xx}$ and $\boldsymbol{\varepsilon}_{yy}$

PLANE STRAIN PROBLEM (not for thin structures)

- Plane Strain Problem
  - Thickness dimension is much larger than other two dimensions.
  - Deformation in the thickness direction is constrained.
  - Strain in z-dir is zero

$$\varepsilon_{zz} = 0, \quad \varepsilon_{xz} = 0, \quad \varepsilon_{yz} = 0$$

  - Non-zero stress components: σ_{xx}, σ_{yy}, τ_{xy}, σ_{zz}.
  - Non-zero strain components: ε_{xx}, ε_{yy}, ε_{xy}.



PLANE STRAIN PROBLEM *cont.*

- Stress-strain relation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \Leftrightarrow \{\sigma\} = [\mathbf{C}_\varepsilon]\{\varepsilon\}$$

- Even if σ_{zz} is not zero, it is not included in the stress-strain relation because it can be calculated from the following relation:

$$\boxed{\sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy})}$$

Limits on Poisson's ratio

EQUIVALENCE

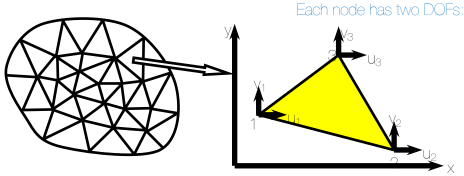
- A single program can be used to solve both the plane stress and plane strain problems by converting material properties.

From → To	E	ν
Plane strain → Plane stress	$E\left[1-\left(\frac{\nu}{1+\nu}\right)^2\right]$	$\frac{\nu}{1+\nu}$
Plane stress → Plane strain	$\frac{E}{1-\left(\frac{\nu}{1-\nu}\right)^2}$	$\frac{\nu}{1-\nu}$

CST ELEMENT

- **Constant Strain Triangular** Element
  - Decompose two-dimensional domain by a set of triangles.
  - Each triangular element is composed by **three corner nodes**.
  - Each element shares its edge and two corner nodes with an adjacent element
  - Counter-clockwise or clockwise node numbering
  - Displacements interpolation using the shape functions and nodal displacements.
  - Strain (and so Stress) are constant within this element

Each node has two DOFs: u and v



DISPLACEMENT INTERPOLATION

- Assumed form for displacements
  - $u(x,y)$ is interpolated in terms of u_1, u_2, and u_3, and $v(x,y)$ in terms of v_1, v_2, and v_3.

  →Components $u(x,y)$ and $v(x,y)$ are separately interpolated.
  interpolation function must be a three term polynomial in x and y.
  - Since we must have rigid body displacements and constant strain terms in the interpolation function, the displacement interpolation must be of the form

$$\boxed{\begin{cases} u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x,y) = \beta_1 + \beta_2 x + \beta_3 y \end{cases}}$$

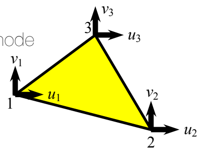
- The goal is to calculate α_i and β_i, i = 1, 2, 3, in terms of nodal displacements.

$$u(x,y) = N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3$$

CST ELEMENT *(it's more easy than quad)*

- Equations for coefficients
  - *x*-displacement: Evaluate displacement at each node

$$\begin{cases} u(x_1,y_1) \equiv u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 \\ u(x_2,y_2) \equiv u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \\ u(x_3,y_3) \equiv u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \end{cases}$$



  - In matrix notation

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

  - When is the coefficient matrix singular?

SOLUTION (Yes It's usefull Linear Algebra !)

- Explicit solution:

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

  - where

$$\begin{cases} f_1 = x_2 y_3 - x_3 y_2, & b_1 = y_2 - y_3, & c_1 = x_3 - x_2 \\ f_2 = x_3 y_1 - x_1 y_3, & b_2 = y_3 - y_1, & c_2 = x_1 - x_3 \\ f_3 = x_1 y_2 - x_2 y_1, & b_3 = y_1 - y_2, & c_3 = x_2 - x_1 \end{cases}$$

  - Area:

$$A = \frac{1}{2} \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

It gives explicitely the alpha ! Use in…INTERPOLATION FUNCTION

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\alpha_1 = \frac{1}{2A}(f_1 u_1 + f_2 u_2 + f_3 u_3)$$

$$\alpha_2 = \frac{1}{2A}(b_1 u_1 + b_2 u_2 + b_3 u_3)$$

$$\alpha_3 = \frac{1}{2A}(c_1 u_1 + c_2 u_2 + c_3 u_3)$$

- Insert to the interpolation equation

$$\begin{aligned} u(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ &= \frac{1}{2A}\left[(f_1 u_1 + f_2 u_2 + f_3 u_3) + (b_1 u_1 + b_2 u_2 + b_3 u_3)x + (c_1 u_1 + c_2 u_2 + c_3 u_3)y\right] \\ &= \boxed{\frac{1}{2A}(f_1 + b_1 x + c_1 y)} u_1 \\ &+ \boxed{\frac{1}{2A}(f_2 + b_2 x + c_2 y)} u_2 \\ &+ \boxed{\frac{1}{2A}(f_3 + b_3 x + c_3 y)} u_3 \end{aligned}$$

N1(x,y)

N2(x,y)

N3(x,y)

Shape Functions

SIMILARLY FOR V

- Displacement Interpolation
  - A similar procedure can be applied for *y*-displacement *v*(*x*, *y*).

$$u(x,y) = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$v(x,y) = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\begin{cases} N_1(x,y) = \dfrac{1}{2A}(f_1 + b_1 x + c_1 y) \\ N_2(x,y) = \dfrac{1}{2A}(f_2 + b_2 x + c_2 y) \\ N_3(x,y) = \dfrac{1}{2A}(f_3 + b_3 x + c_3 y) \end{cases}$$

Shape Functions

  - N_1, N_2, and N_3 are linear functions of *x*- and *y*-coordinates.
  - Interpolated displacement changes linearly along the each coordinate direction.

MATRIX EQUATION FOR CST ELEMENT

- Displacement Interpolation

$$\{\mathbf{u}\} \equiv \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\boxed{\{\mathbf{u}(x,y)\} = [\mathbf{N}(x,y)]\{\mathbf{q}\}}$$

- [**N**]: 2×6 matrix, {**q**}: 6×1 vector.
- For a given point (x,y) within element, calculate [N] and multiply it with {q} to evaluate displacement at the point (x,y).

STRAINS FOR CST ELEMENT

REMEMBER

$$\begin{cases} N_1(x,y) = \dfrac{1}{2A}(f_1 + b_1x + c_1y) \\ N_2(x,y) = \dfrac{1}{2A}(f_2 + b_2x + c_2y) \\ N_3(x,y) = \dfrac{1}{2A}(f_3 + b_3x + c_3y) \end{cases}$$

- Strain Interpolation
  - differentiating the displacement in x- and y-directions.
  - differentiating shape function [**N**] because {**q**} is constant.

$$\varepsilon_{xx} \equiv \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} N_i(x,y)u_i\right) = \sum_{i=1}^{3}\frac{\partial N_i}{\partial x}u_i = \sum_{i=1}^{3}\frac{b_i}{2A}u_i$$

$$\varepsilon_{yy} \equiv \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\sum_{i=1}^{3} N_i(x,y)v_i\right) = \sum_{i=1}^{3}\frac{\partial N_i}{\partial y}v_i = \sum_{i=1}^{3}\frac{c_i}{2A}v_i$$

$$\gamma_{xy} \equiv \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum_{i=1}^{3}\frac{c_i}{2A}u_i + \sum_{i=1}^{3}\frac{b_i}{2A}v_i$$

Strains are constant inside!

B-MATRIX FOR CST ELEMENT

- Strain calculation

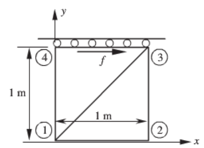
$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \equiv [\mathbf{B}]\{\mathbf{q}\}$$

- [**B**] matrix is a constant matrix and depends only on the coordinates of the three nodes of the triangular element.
- Again, the strains will be constant over a given element

Example

Given the two finite-element model in shear below with the only non-zero displacements given below
Calculate the shape function of Node 1 in the top element.

$u_3 = u_4 = 0.01m$



REMEMBER

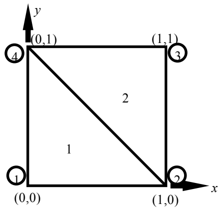
$$\begin{cases} N_1(x,y) = \dfrac{1}{2A}(f_1 + b_1x + c_1y) \\ N_2(x,y) = \dfrac{1}{2A}(f_2 + b_2x + c_2y) \\ N_3(x,y) = \dfrac{1}{2A}(f_3 + b_3x + c_3y) \end{cases}$$

$$\begin{cases} f_1 = x_2y_3 - x_3y_2, & b_1 = y_2 - y_3, & c_1 = x_3 - x_2 \\ f_2 = x_3y_1 - x_1y_3, & b_2 = y_3 - y_1, & c_2 = x_1 - x_3 \\ f_3 = x_1y_2 - x_2y_1, & b_3 = y_1 - y_2, & c_3 = x_2 - x_1 \end{cases}$$

$$A = \frac{1}{2}\det\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

EXAMPLE - Interpolation

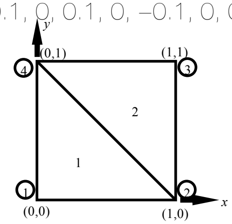
- nodal displacements
  $\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{-0.1, 0, 0.1, 0, -0.1, 0, 0.1, 0\}$
- Element 1: Nodes 1-2-4
- Given, xi,yi, ui,vi, find Ni(x,y) then COMPUTE STRAIN!



EXAMPLE - Interpolation

- nodal displacements
  $\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{-0.1, 0, 0.1, 0, -0.1, 0, 0.1, 0\}$

$x_1 = 0$	$x_2 = 1$	$x_3 = 0$
$y_1 = 0$	$y_2 = 0$	$y_3 = 1$
$f_1 = 1$	$f_2 = 0$	$f_3 = 0$
$b_1 = -1$	$b_2 = 1$	$b_3 = 0$
$c_1 = -1$	$c_2 = 0$	$c_3 = 1$



- Element 1: Nodes 1-2-4

REMEMBER

$$\begin{cases} N_1(x,y) = \frac{1}{2A}(f_1 + b_1 x + c_1 y) \\ N_2(x,y) = \frac{1}{2A}(f_2 + b_2 x + c_2 y) \\ N_3(x,y) = \frac{1}{2A}(f_3 + b_3 x + c_3 y) \end{cases}$$

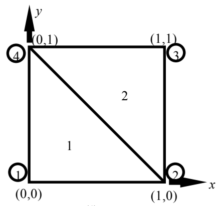
N1(x)=1*(1-1*x-1*y)
N2(x)=1*(0+1*x+0*y)
N3(x)=1*(0+0*x+1*y)

$$N_1(x,y) = 1 - x - y$$
$$N_2(x,y) = x$$
$$N_3(x,y) = y$$

EXAMPLE - Interpolation

- nodal displacements
  $\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{-0.1, 0, 0.1, 0, -0.1, 0, 0.1, 0\}$
- Element 1: Nodes 1-2-4

$x_1 = 0$	$x_2 = 1$	$x_3 = 0$
$y_1 = 0$	$y_2 = 0$	$y_3 = 1$
$f_1 = 1$	$f_2 = 0$	$f_3 = 0$
$b_1 = -1$	$b_2 = 1$	$b_3 = 0$
$c_1 = -1$	$c_2 = 0$	$c_3 = 1$



$$N_1(x,y) = 1 - x - y$$
$$N_2(x,y) = x$$
$$N_3(x,y) = y$$

Given Ni(x,y)
...Calculate strains directly

$$u^{(1)}(x,y) = \sum_{i=1}^{3} N_i(x,y)u_i = 0.1(2x + 2y - 1)$$
$$v^{(1)}(x,y) = \sum_{i=1}^{3} N_i(x,y)v_i = 0.0$$

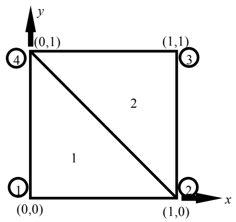
$\Rightarrow$

$$\varepsilon_{xx}^{(1)} = \frac{\partial u^{(1)}}{\partial x} = 0.2$$
$$\varepsilon_{yy}^{(1)} = \frac{\partial v^{(1)}}{\partial y} = 0.0$$
$$\gamma_{xy}^{(1)} = \frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x} = 0.2$$

EXAMPLE – Interpolation cont.

- Element 2: Nodes 2-3-4

$x_1 = 1$	$x_2 = 1$	$x_3 = 0$
$y_1 = 0$	$y_2 = 1$	$y_3 = 1$
$f_1 = 1$	$f_2 = -1$	$f_3 = 1$
$b_1 = 0$	$b_2 = 1$	$b_3 = -1$
$c_1 = -1$	$c_2 = 1$	$c_3 = 0$



$$N_1(x,y) = 1 - y$$
$$N_2(x,y) = x + y - 1$$
$$N_3(x,y) = 1 - x$$

$$u^{(2)}(x,y) = \sum_{i=1}^{3} N_i(x,y)u_i = 0.1(3 - 2x - 2y)$$
$$v^{(2)}(x,y) = \sum_{i=1}^{3} N_i(x,y)v_i = 0.0$$

$$\varepsilon_{xx}^{(2)} = \frac{\partial u^{(2)}}{\partial x} = -0.2$$
$$\varepsilon_{yy}^{(2)} = \frac{\partial v^{(2)}}{\partial y} = 0.0$$
$$\gamma_{xy}^{(2)} = \frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x} = -0.2$$

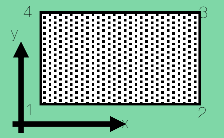
Strains are discontinuous along the element boundary !!!

RECTANGULAR ELEMENT

- Each edge is parallel to the coordinate direction (not practical)
- Lagrange interpolation for shape function calculation
- Interpolation (blue). One must note the added term xy /CST

$$\boxed{\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \\ v &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy \end{aligned}}$$

$$\begin{cases} u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 x_1 y_1 \\ u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 x_2 y_2 \\ u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 x_3 y_3 \\ u_4 = \alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 x_4 y_4 \end{cases}$$



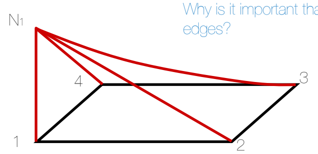
? $\Downarrow$

$$u(x,y) = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

QUAD's SHAPE FUNCTIONS

- Shape functions for rectangular elements are product of Lagrange interpolations in the two coordinate directions.
- Note that $N_1(x, y)$ is:
  - 1 at node 1 and 0 at other nodes.
  - Linear function of x along edge 1-2 and linear function of y along edge 1-4.
  - Zero along edge 2-3 and 3-4.
- Other shape functions have similar behavior.

Why is it important that it varies linearly along edges?

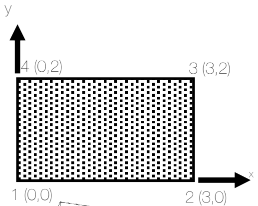


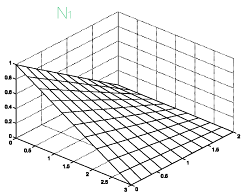
EXAMPLE

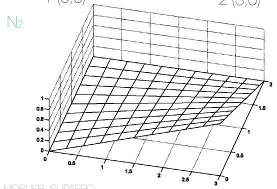
- Shape Functions Area A=6

$$N_1 = \frac{(x-3)(y-2)}{6} \qquad N_2 = \frac{-x(y-2)}{6}$$

$$N_3 = \frac{xy}{6} \qquad N_4 = \frac{-y(x-3)}{6}$$







INTERPOLATION *cont*.

- Displacement interpolation
  - Same interpolation for both u and v.

$$\{u\} \equiv \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\boxed{\{u\} = [N]_{2\times 8}\{q\}_{8\times 1}}$$

STRAIN DISPLACEMENT MATRIX

- Strain-displacement relation
  - Similarly to CST element

$$\varepsilon_{xx} \equiv \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{i=1}^{4} N_i(x,y)u_i\right) = \sum_{i=1}^{4}\frac{\partial N_i}{\partial x}u_i$$

$$\{\varepsilon\} = \frac{1}{A}\begin{bmatrix} y-y_3 & 0 & y_3-y & 0 & y-y_1 & 0 & y_1-y & 0 \\ 0 & x-x_3 & 0 & x_1-x & 0 & x-x_1 & 0 & x_3-x \\ x-x_3 & y-y_3 & x_1-x & y_3-y & x-x_1 & y-y_1 & x_3-x & y_1-y \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\equiv [\mathbf{B}]\{\mathbf{q}\}$$

BUT [**B**] is a linear function of *x* and *y*.
Strain will change linearly within the element (not completely linear in both directions)

STIFFNESS MATRIX

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

- Element stiffness matrix (from strain energy)

$$U^{(e)} = \frac{h}{2}\iint_A \{\varepsilon\}^T[\mathbf{C}]\{\varepsilon\}\,dA^{(e)} = \frac{h}{2}\{\mathbf{q}^{(e)}\}^T\iint_A [\mathbf{B}]^T_{8\times3}[\mathbf{C}]_{3\times3}[\mathbf{B}]_{3\times8}\,dA\{\mathbf{q}^{(e)}\}$$

$$\equiv \frac{1}{2}\{\mathbf{q}^{(e)}\}^T[\mathbf{k}^{(e)}]_{8\times8}\{\mathbf{q}^{(e)}\}$$

- Generally depends on aspect ratio of element NOT SIZE
- A square, plane-stress, rectangular element:

$$[\mathbf{k}^{(e)}] = \frac{Eh}{1-\nu^2}\begin{bmatrix} \frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} \\ \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} \\ -\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} \\ \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} \\ \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} \\ -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} \\ \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} \\ \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} \end{bmatrix}$$

APPLIED LOADS

- Nodal and distributed forces are the same with CST element
- Body force (constant body force **b** = $\{b_x, b_y\}^T$)

$$V^{(e)} = -h\iint_A [u \quad v]\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dA = -\{\mathbf{q}^{(e)}\}^T h\iint_A [\mathbf{N}]^T\,dA\begin{Bmatrix} b_x \\ b_y \end{Bmatrix}$$

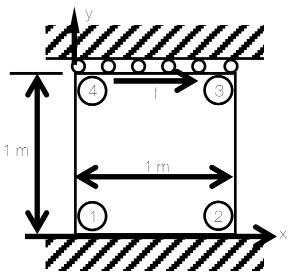
$$\equiv \{\mathbf{q}^{(e)}\}^T\{\mathbf{f}_b^{(e)}\}$$

$$\{\mathbf{f}_b^{(e)}\} = \frac{hA}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \frac{hA}{4}\begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix}$$

Equally divide the total magnitude of the body force to the four nodes

EXAMPLE: SIMPLE SHEAR

- E = 10 GPa, ν = 0.25, h=0.1m
- F = 100 kN/m^2
- $\{\mathbf{Q}_s\} = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}^T$.
- Non-zero DOFs: u_3 and u_4.



$$[\mathbf{k}^{(e)}] = \frac{Eh}{1-\nu^2}\begin{bmatrix} \frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} \\ \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} \\ -\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} \\ \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} \\ \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} & \boxed{\frac{3-\nu}{6}} & \frac{1+\nu}{8} & \boxed{-\frac{3+\nu}{12}} & \frac{-1+3\nu}{8} \\ -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} \\ \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} & \boxed{-\frac{3+\nu}{12}} & \frac{1-3\nu}{8} & \boxed{\frac{3-\nu}{6}} & -\frac{1+\nu}{8} \\ \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} \end{bmatrix}$$

EXAMPLE: SIMPLE SHEAR

- E = 10 GPa, ν = 0.25, h=0.1m
- F = 100 kN/m²
- $\{\mathbf{Q}_s\} = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}^T$.
- Non-zero DOFs: u_3 and u_4.
- Stiffness matrix

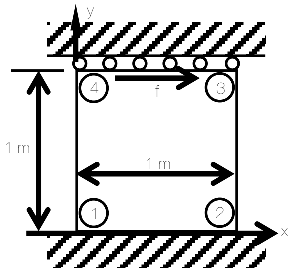
$$[\mathbf{K}] = \frac{Eh}{1-\nu^2}\begin{bmatrix} \frac{3-\nu}{6} & -\frac{3+\nu}{12} \\ -\frac{3+\nu}{12} & \frac{3-\nu}{6} \end{bmatrix}\begin{matrix} u_3 \\ u_4 \end{matrix}$$

- FEM equation (after applying BC)

$$10^8\begin{bmatrix} 4.89 & -2.89 \\ -2.89 & 4.89 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 5,000 \\ 5,000 \end{Bmatrix}$$

- Nodal displacements

$$u_3 = u_4 = 0.025\text{mm}$$



$$[\mathbf{k}^{(e)}] = \frac{Eh}{1-\nu^2}\begin{bmatrix}
\frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} \\
\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} \\
-\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} \\
\frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} \\
\frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} & \boxed{\frac{3-\nu}{6}} & \frac{1+\nu}{8} & \boxed{-\frac{3+\nu}{12}} & \frac{-1+3\nu}{8} \\
-\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} \\
\frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} & \boxed{-\frac{3+\nu}{12}} & \frac{1-3\nu}{8} & \boxed{\frac{3-\nu}{6}} & -\frac{1+\nu}{8} \\
\frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6}
\end{bmatrix}$$

EXAMPLE – SIMPLE SHEAR *cont*

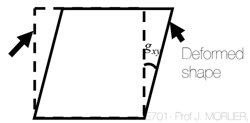
- Strain & Stress Area A=1

$$\{\varepsilon\} = \underset{3\text{x}8}{\begin{bmatrix} y-1 & 0 & 1-y & 0 & y & 0 & -y & 0 \\ 0 & x-1 & 0 & -x & 0 & x & 0 & 1-x \\ x-1 & y-1 & -x & 1-y & x & y & 1-x & -y \end{bmatrix}}\underset{8\text{x}1}{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2.5\times10^{-5} \\ 0 \\ 2.5\times10^{-5} \\ 0 \end{Bmatrix}} = \underset{3\text{x}1}{\begin{Bmatrix} 0 \\ 0 \\ 2.5\times10^{-5} \end{Bmatrix}}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \Leftrightarrow \{\sigma\} = [\mathbf{C}_\sigma]\{\varepsilon\}$$

How do you check the stresses?

E=1e10 Pa



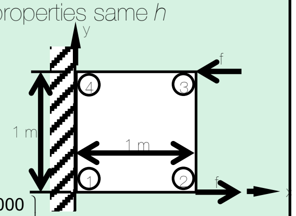
EXAMPLE – PURE BENDING

- Couple M = 100 kN.m, same material properties same *h*
- Analytical solution

$$(\sigma_{xx})_{max} = -\frac{M(-\frac{L}{2})}{I} = 6.0\text{MPa}$$

$$\sigma_{xx} = 6.0(1-2y)\,\text{MPa}$$

- FEM solution
  - Non-zero DOFs: u_2, v_2, u_3, and v_3.



$$[\mathbf{k}^{(e)}] = \frac{Eh}{1-\nu^2}\begin{bmatrix}
\frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} \\
\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} \\
-\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} \\
\frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} \\
\frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} \\
-\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} \\
\frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} \\
\frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6}
\end{bmatrix}$$

$$\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 100,000 \\ 0 \\ -100,000 \\ 0 \end{Bmatrix}$$

$$u_2 = 0.4091\,\text{mm}, \quad v_2 = 0.4091\,\text{mm}$$

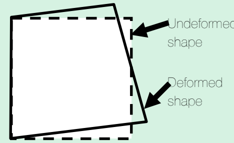
$$u_3 = -0.4091\,\text{mm}, \quad v_3 = 0.4091\,\text{mm}$$

EXAMPLE – PURE BENDING *cont*

- Strain & Stress

$$\{\varepsilon\} = \begin{bmatrix} y-1 & 0 & 1-y & 0 & y & 0 & -y & 0 \\ 0 & x-1 & 0 & -x & 0 & x & 0 & 1-x \\ x-1 & y-1 & -x & 1-y & x & y & 1-x & -y \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.41 \\ 0.41 \\ -0.41 \\ 0.41 \\ 0 \\ 0 \end{Bmatrix}\times10^{-3} = \begin{Bmatrix} 0.41\times10^{-3}(1-2y) \\ 0 \\ 0.41\times10^{-3}(1-2x) \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{10^{10}}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}\begin{Bmatrix} 0.41\times10^{-3}(1-2y) \\ 0 \\ 0.41\times10^{-3}(1-2x) \end{Bmatrix} = \begin{Bmatrix} 4.4(1-2y) \\ 1.1(1-2y) \\ 1.6(1-2x) \end{Bmatrix}\text{MPa}$$



Unable to make curvature
Trapezoidal shape -> non-zero shear stress
$(\sigma_{xx})_{max} / (\sigma_{xx})_{exact}$ = 4.364/6.0 (73%)

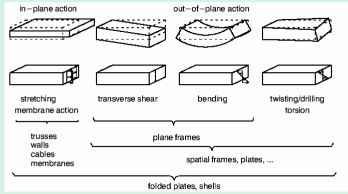
SHEAR LOCKING !

Today C3

1. GAUSS integration (2 exercices)
2. Plane Stress VS Plane Strain
3. 2D elements (CST VS QUAD)
4. Isoparametric element
5. In plane (membranes) VS out of plane (Plates & Shells)

OUT OF PLANE [in 6 slides…]

- Use (ONLY) Nastran/Samcef Plate/Shelll
- No theory (advanced course)



PLATE

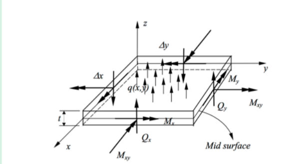
- Flat plate
- Lateral loading
- Bending behavior dominates

- Note the following similarity:

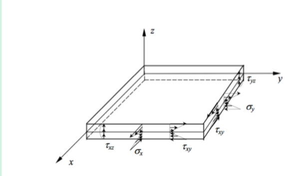
1-D straight beam model

2-D flat plate model

Forces and Moments Acting on the Plate:



Stresses:



AEROSTRUCTURES= THIN STRUCTURES

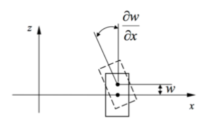
Thin Plate Theory (Kirchhoff Plate Theory)

Assumptions (similar to those in the beam theory):

A straight line along the normal to the mid surface remains straight and normal to the deflected mid surface after loading, that is, these is no transverse shear deformation:

$\gamma_{xz} = \gamma_{yz} = 0$.

Displacement:



$$w = w(x, y), \qquad (deflection)$$

$$u = -z\frac{\partial w}{\partial x},$$

$$v = -z\frac{\partial w}{\partial y}.$$

KIRCHHOFF THEORY

Governing Equation:

$$D\nabla^4 w = q(x,y),$$

where

$$\nabla^4 \equiv \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right),$$

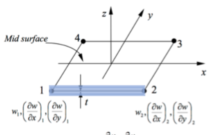
$$D = \frac{Et^3}{12(1-\nu^2)} \quad \text{(the bending rigidity of the plate)},$$

q = lateral distributed load (force/area).

Compare the 1-D equation for straight beam:

$$EI\frac{d^4 w}{dx^4} = q(x).$$

4-Node Quadrilateral Element



DOF at each node: $w, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial y}.$

On each element, the deflection *w(x,y)* is represented by

$$w(x,y) = \sum_{i=1}^{4}\left[N_i w_i + N_{xi}\left(\frac{\partial w}{\partial x}\right)_i + N_{yi}\left(\frac{\partial w}{\partial y}\right)_i\right],$$

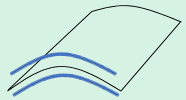
where N_i, N_{xi} and N_{yi} are shape functions. This is an incompatible element! The stiffness matrix is still of the form

$$\mathbf{k} = \int_V \mathbf{B}^T\mathbf{E}\mathbf{B}\,dV,$$

where **B** is the strain-displacement matrix, and **E** the stress-strain matrix.

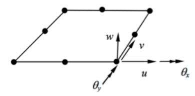
SHELL ELEMENTS

Shells – Thin structures witch span over curved surfaces.



- egg shell (the wonder of the nature);
- Containers, pipes, tanks;
- fuselage
- Roofs, buildings (the Superdome), etc.

DOF at each node:



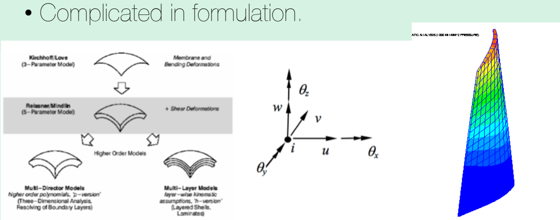
Q4 or Q8 shell element.

Forces in shells:

- Membrane forces + Bending Moments

CURVED SHELL

- Based on shell theories;
- Most general shell elements (flat shell and plate elements are subsets);
- Complicated in formulation.



Today C3

1. Plane Stress VS Plane Strain
2. 2D elements (CST VS QUAD)
3. Isoparametric element
4. In plane (membranes) VS out of plane (Plates & Shells)
5. GAUSS integration (2 exercices)

But before Assignement #4 * (deadline 23 o April)

- Play with the given 2D Beam sizing code
- Comment the given function according to the theory seen today
- Create a notebook to illustrate mesh convergency on a Quantity of Interest *(scalar)* such as Max displacement, Max VM Stress
- One bonus question… if there is a hole, what are the changes? May I change the mesh ? » → **Related to 2nd Nastran practice**