Exercice 1

Integrate numerically $f(x) = e^{x}$ on $[0 : 6]$
using Gauss quadrature. compare with 'trapz', 'quad'
Matlab's functions

$$\int_0^6 e^x \, dx = e^x \Big|_0^6 = e^6 - e^0 = e^6 - 1 \simeq 402{,}43$$

exact solution (see WOLFRAM α)

before Gauss integration, link with intrinsic coordinates

$$x = 3(1+\xi) \longrightarrow dx = 3\,d\xi$$

$$\int_0^6 e^x \, dx = 3 \int_{-1}^{+1} e^{3(1+\xi)} \, d\xi = 3 \sum_{i=1}^{nG} w_i \, e^{3(1+\xi_i)}$$

nG	
1	$3\left((2)\,e^{3(1+0)}\right) = 120{,}51$
2	$3\left((1)\,e^{3(1-0{,}577)} + (1)\,e^{3(1+0{,}577)}\right) = 351{,}244$
3	$3\left((0{,}55)\left(e^{3(1-0{,}774)} + e^{3(1+0{,}774)}\right) + (0{,}88)\,e^{3(1+0)}\right) \simeq 398{,}77$

trapz

$$1{,}5\left(\frac{e^{0}+e^{1{,}5}}{2}+\frac{e^{1{,}5}+e^{3}}{2}+\frac{e^{3}+e^{4{,}5}}{2}+\frac{e^{4{,}5}+e^{6}}{2}\right)$$

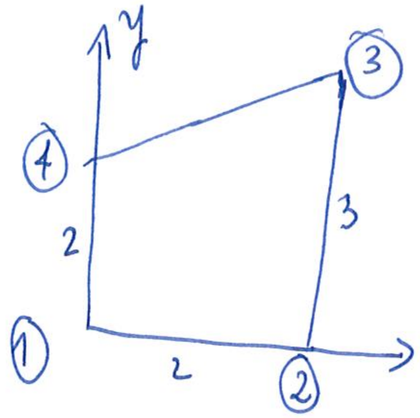
$$\simeq 475{,}198$$

quad

$$\frac{1{,}5}{3}\left(e^{0}+4e^{1{,}5}+2e^{3}+4e^{4{,}5}+e^{6}\right)$$

$$\simeq 411{,}29$$

Exercice 2



Evaluate the integral $I = \iint_A (x^2 + y)\, dx\, dy$

over the quadrilateral shown.

Corners: $x^T = [0 \quad 2 \quad 2 \quad 0]$; $y^T = [0 \quad 0 \quad 3 \quad 2]$

Mapping $x(\xi, \eta) = \sum_{k=1}^{4} N_k(\xi, \eta)\, x_k$

See end of course3 please

$$= 0 \times (\cdot)(\cdot) + \frac{(1+\xi)(1-\eta)}{4}(2) + \frac{(1+\xi)(1+\eta)}{4}(2) + 0$$

$$= 1 + \xi$$

$$y(\xi, \eta) = 0 + 0 + \frac{(1+\xi)(1+\eta)}{4}(3) + \frac{(1-\xi)(1+\eta)}{4}(2)$$

$$= \frac{(5+\xi)(1+\eta)}{4}$$

The Jacobian matrix is computed as: $J(\xi, \eta) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$

$$= \begin{bmatrix} 1 & \frac{1+\eta}{4} \\ 0 & \frac{5+\xi}{4} \end{bmatrix}$$

Thus the area scale factor is

det =

$$|J(\xi,\eta)| = \frac{5+\xi}{4}$$

$$I = \int_{-1}^{1}\int_{-1}^{1} \left[(1+\xi)^2 + \frac{(5+\xi)(1+\eta)}{4}\right] \frac{5+\xi}{4} \, d\xi \, d\eta$$

... after simplification

$\Rightarrow$ even powers of ξ & η contribute to the integral

$$I = \int_{-1}^{1}\int_{-1}^{1} \left(\frac{45}{16} + \frac{29}{16}\xi^2\right) d\xi \, d\eta = \frac{41}{3} \simeq 13{,}667$$