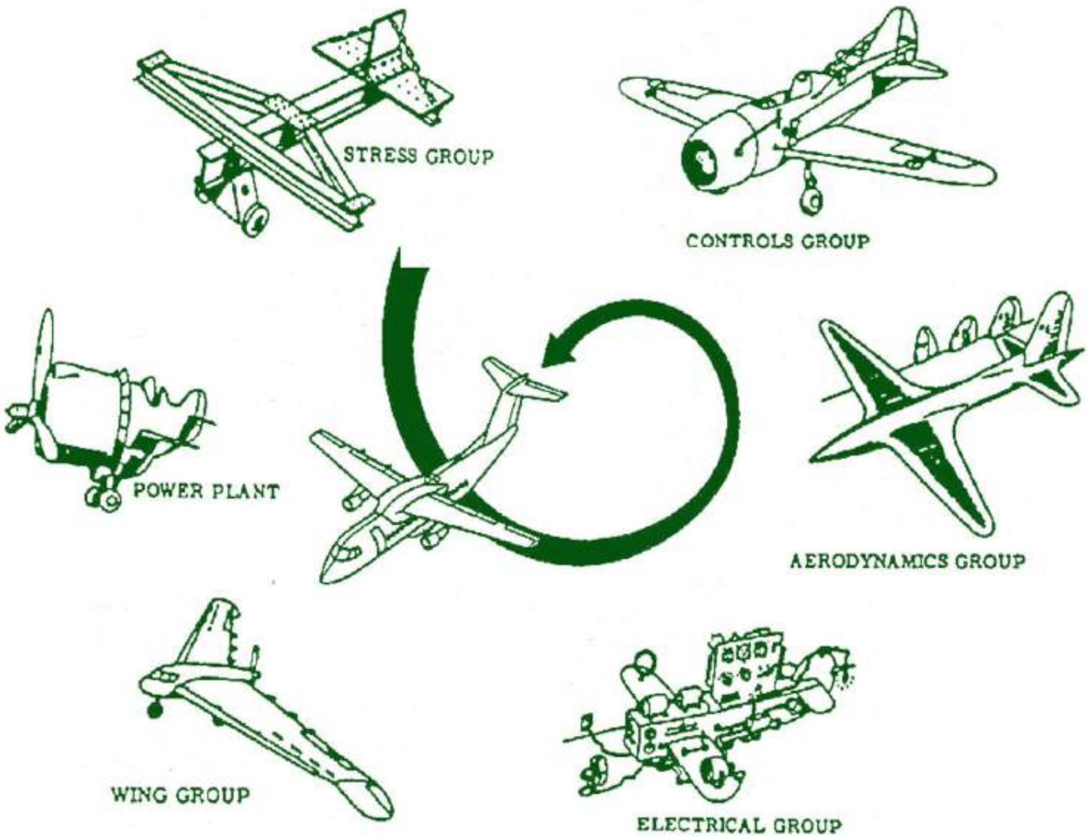
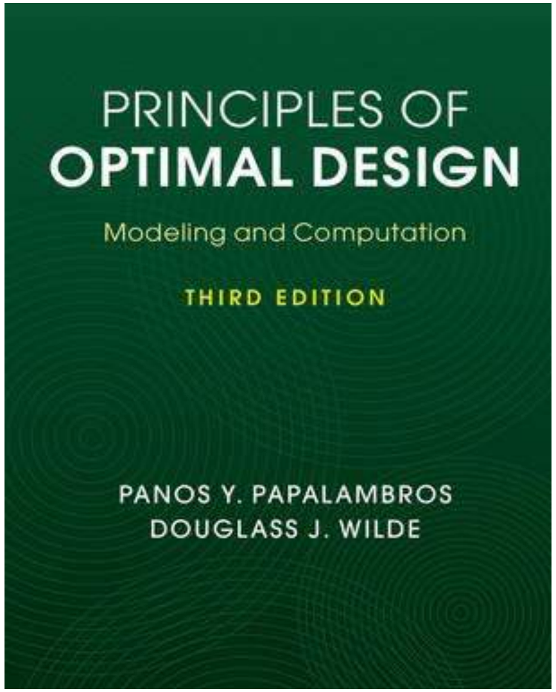
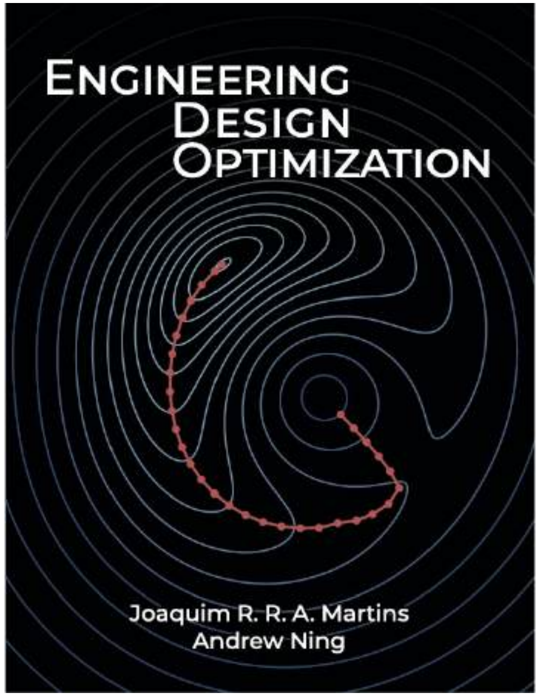


Multidisciplinary Design Optimization (MDO)

UTM Summer 2025

Good Starting Point (x0)





UTM Short course on MULTIDISCIPLINARY DESIGN OPTIMIZATION

Prof. Joseph Morlier

July 20, 2025

Abstract

This lecture note aims at giving solid mathematical foundations to an engineering design optimization course

The codes and notebooks will be available at:
https://github.com/jomorlier/mdoatutm

Contents

KEEP STUDENTS ACTIVE

HOW?

First Exercice

https://mdolab.engin.umich.edu/assets/optimizationGame/

SM305 49

Example of Optimizers

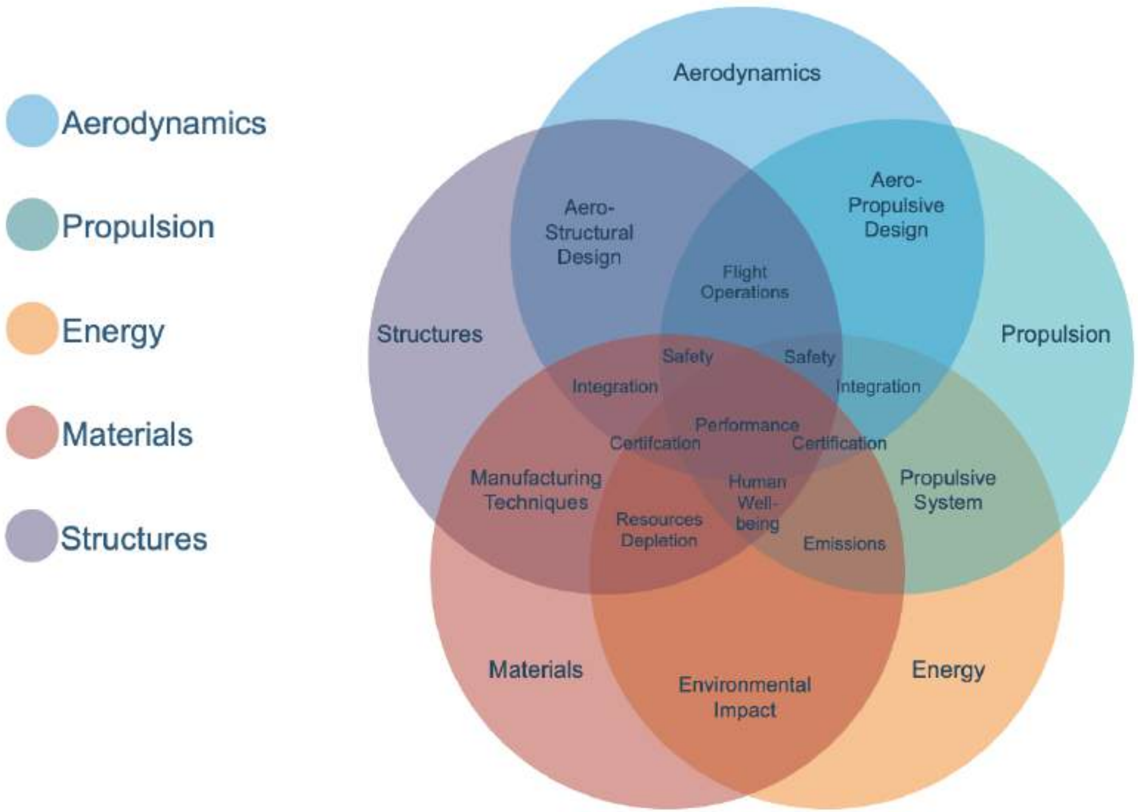
CHALLENGE #1 On Rosenbrock function minimization

https://colab.research.google.com/drive/1rGHIklpVM4Gqy9eq1Y10fIQW9umIjeEx

SM305 60

Venn Diagram

To illustrate the multidisciplinary nature of designing an aircraft and how the fundamentals disciplines are interlinked we can recur to a Venn diagram



F. Afonso, et al., "Strategies towards a more sustainable aviation: a systematic review", *Progress in Aerospace Sciences*, Vol. 137, 100878, 2023, https://doi.org/10.1016/j.paerosci.2022.100878

HOW TO

Start?

A good example

https://mdolab.engin.umich.edu/software

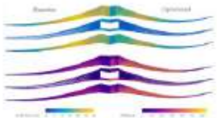
Software

The software packages listed below are all distributed under open source licenses. These are research codes, so they require a strong background in programming and some persistence to get them to work. Unfortunately we are not able to provide support except for collaborators and sponsors. However, we strive to provide as much documentation as we can and continually work towards improving the usability.

Webfoil: This is an online tool for airfoil analysis and optimization. It also includes a vast database of airfoils. [Webfoil site] [Paper]

pyOptSparse: A common Python interface to various optimization packages. pyOptSparse includes OptView, a visualization tool to explore the optimization history. [Code] [Documentation]

TACS: A general purpose structural finite-element code with adjoint derivatives that is developed by Prof. Graeme Kennedy. [Code] [Paper]

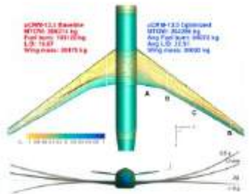


SMT: The surrogate modeling toolbox (SMT) is an open-source Python package consisting of libraries of surrogate modeling methods (e.g., radial basis functions, kriging), sampling methods, and benchmarking problems. SMT is designed to make it easy for developers to implement new surrogate models in a well-tested and well-document platform, and for users to have a library of surrogate modeling methods with which to use and compare methods. [Code] [Paper]

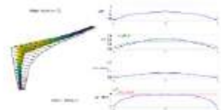
ADflow: (pronounced "A-D-flow") CFD solver that can handle structured multi-block and overset meshes. It includes an adjoint solver for computing derivatives and can be used in the MACH-Aero framework for aerodynamic shape optimization. [Code] [Documentation] [Paper]

DAFOAM: (pronounced "dahfoam") is a suite of adjoint solvers for OpenFOAM that enable the computation of derivatives for aerodynamic shape optimization. [Code] [Documentation] [Paper]

MACH-Aero: A framework for aerodynamic design optimization that couples a CFD solver (e.g. ADflow or OpenFOAM), geometry parametrization (e.g. pyGeo), mesh deformation (e.g., IDWarp), and optimizer interface (pyOptSparse). [Code] [Documentation and Tutorials]



OpenAeroStruct: A lightweight aerostructural optimization code that can optimize a wing design in minutes on a laptop. [Code] [Documentation]



OpenMDAO: A framework for coupling multiple numerical models and performing multidisciplinary analysis and optimization. OpenMDAO is developed by NASA and uses numerical techniques developed in the MDO Lab. [OpenMDAO in a nutshell] [OpenMDAO site] [Paper]

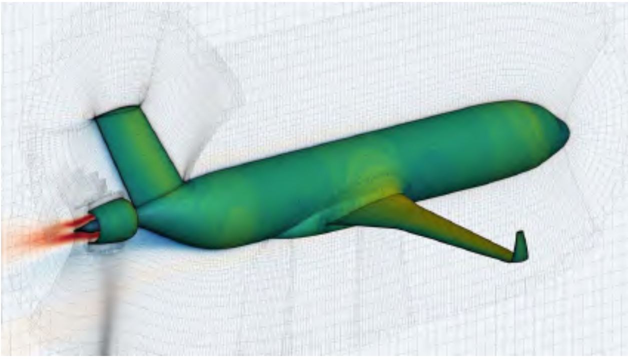
A Curated list of (mostly) opensource softwares

Tools for decarbonizing air transportation:
https://cascade.boeing.com/
https://aeromaps.eu
https://github.com/AeroMAPS/AeroMAPS
https://www.leadsresearchgroup.com/technology-dashboard
https://github.com/contrailcirrus/pycontrails?tab=readme-ov-file
https://github.com/sustainableaviation
https://github.com/Aircraft-Operations-Lab
https://psesh.github.io/aviation.html
https://github.com/leadsgroup
https://www.leadsresearchgroup.com/software
https://github.com/protontypes/open-sustainable-technology
https://www.witness4climate.org/optimizing-investments-in-energy-production-technology
https://junzis.com/open-source

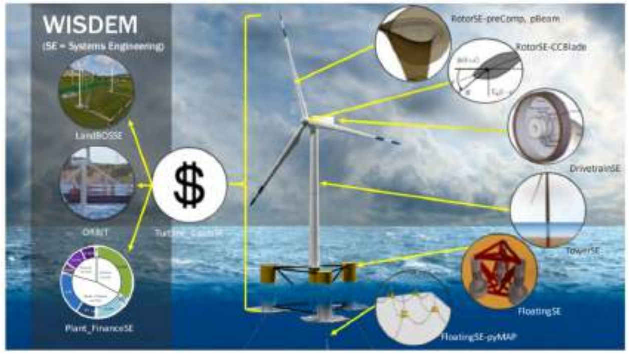
Tools for aircraft design
https://github.com/peterdsharpe/AeroSandbox/tree/master
https://openmdao.github.io/Aviary/examples/OAS_subsystem.html
https://github.com/OpenMDAO/Aviary
https://github.com/ideas-um/FAST
https://github.com/MIT-LAE/TASOPT.jl
https://github.com/mdolab/OpenAeroStruct
https://github.com/mdolab/openconcept
https://github.com/fast-aircraft-design/FAST-OAD
https://web.mit.edu/drela/Public/web/
https://lsdo.eng.ucsd.edu/software
https://github.com/mid2SUPAERO/LCA4MDAO
https://github.com/ImperialCollegeLondon/sharpy
https://www.aircraftflightmechanics.com/NotesIntroduction.html
https://github.com/MIT-LAE
https://github.com/OpenVSP/OpenVSP
https://github.com/suavecode
https://github.com/camUrban/PteraSoftware
https://github.com/cfsengineering/CEASIOMpy
https://github.com/DLR-AE/PanelAero
https://github.com/DLR-AE
https://github.com/ImperialCollegeLondon/PinhoLab-WingBox
https://commonresearchmodel.larc.nasa.gov
https://github.com/facebookarchive/FBHALE
https://github.com/mid2SUPAERO/ecoHALE

Not only Aircraft Design

OpenMDAO is a powerful tool for doing gradient-based multidisciplinary optimization



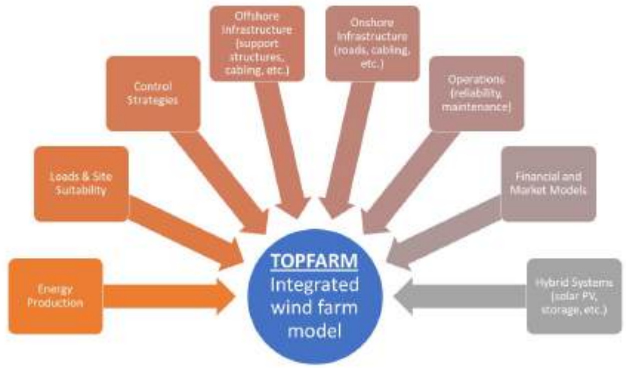
NASA designs complex multidisciplinary systems using OpenMDAO



WISDEM and WEIS are two NREL tools that use the OpenMDAO framework

https://flow.byu.edu/codes/

https://github.com/WISDEM/WISDEM

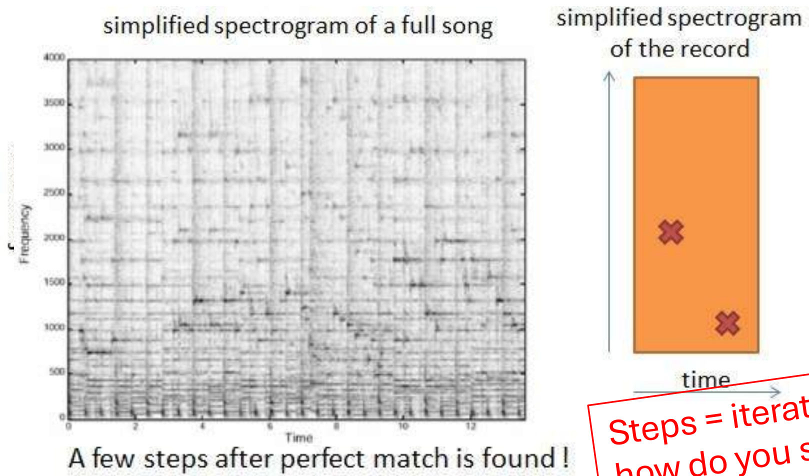


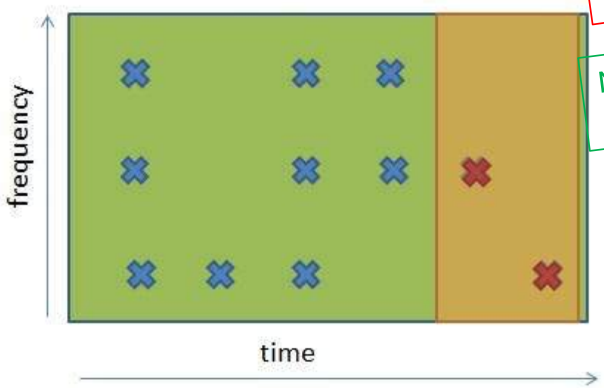
https://openmdao.org/newdocs/versions/latest/examples/betz_limit.html#optimizing-an-actuator-disc-model-to-find-betz-limit-for-wind-turbines

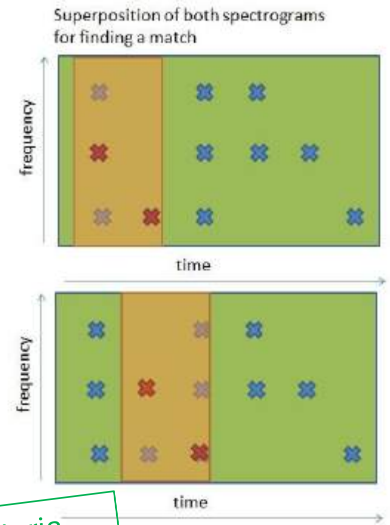
https://topfarm.pages.windenergy.dtu.dk/TopFarm2/

Optimization is everywhere

http://coding-geek.com/how-shazam-works/







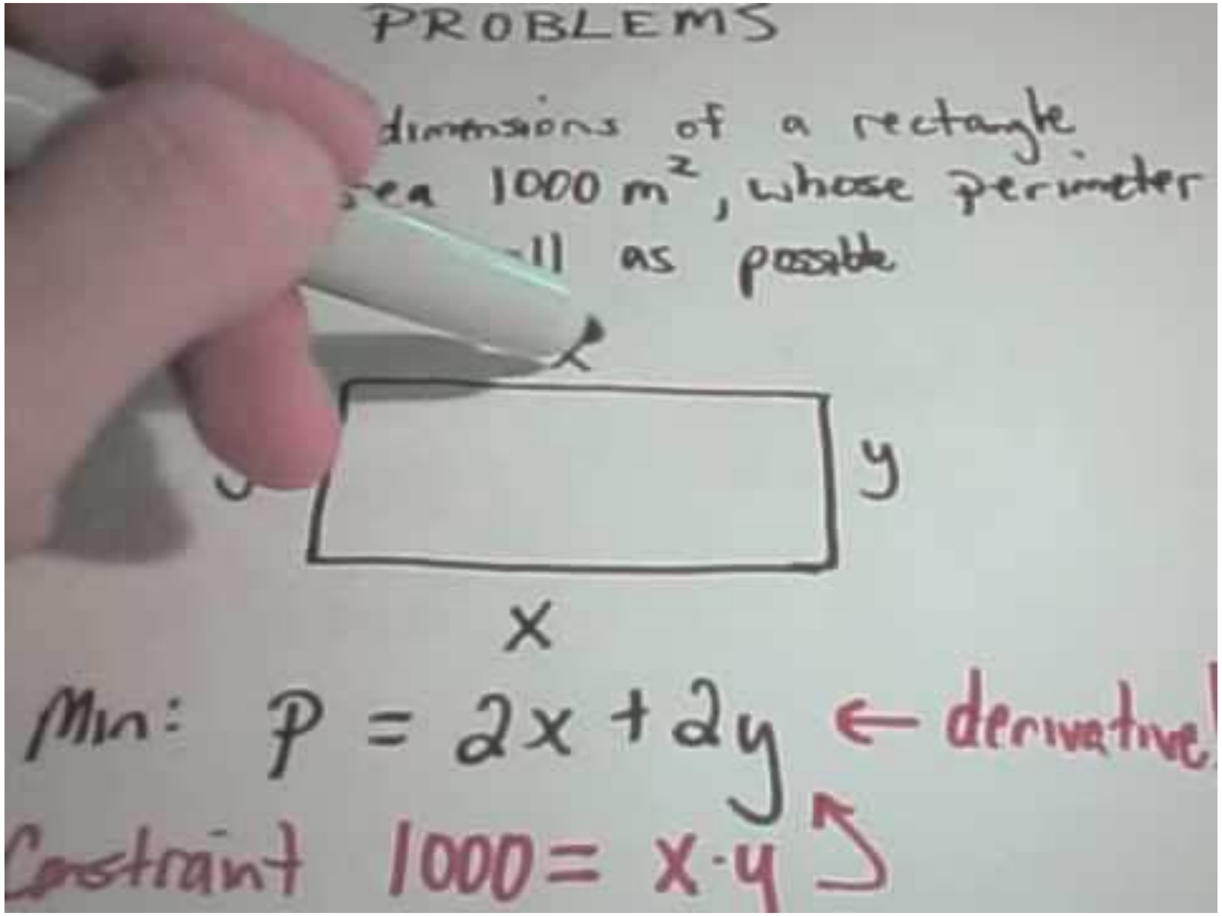
Steps = iterations,
how do you stop?

Must define a stopping criteria
according to a tolerance

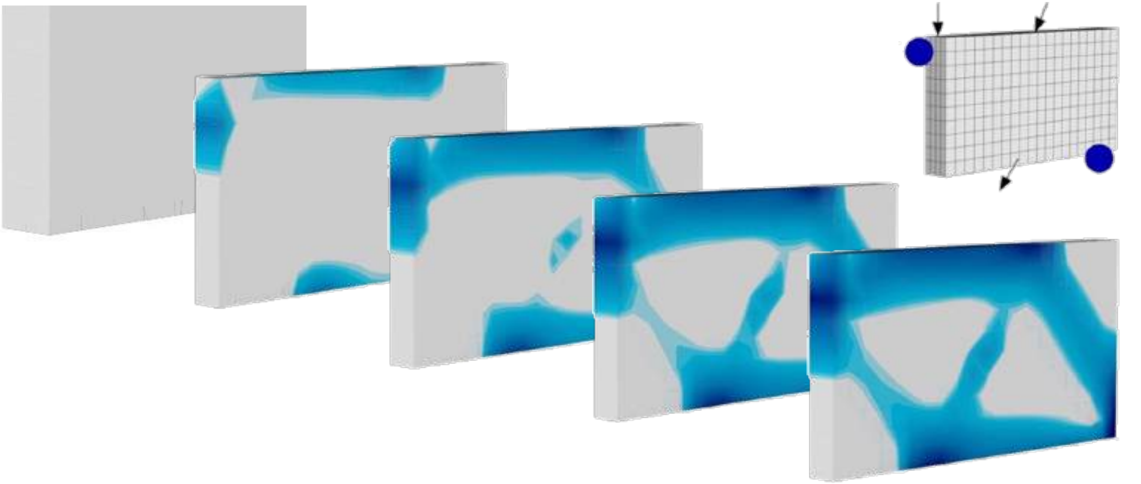
Need to match something ?

Need to define a metric
Need to minimize

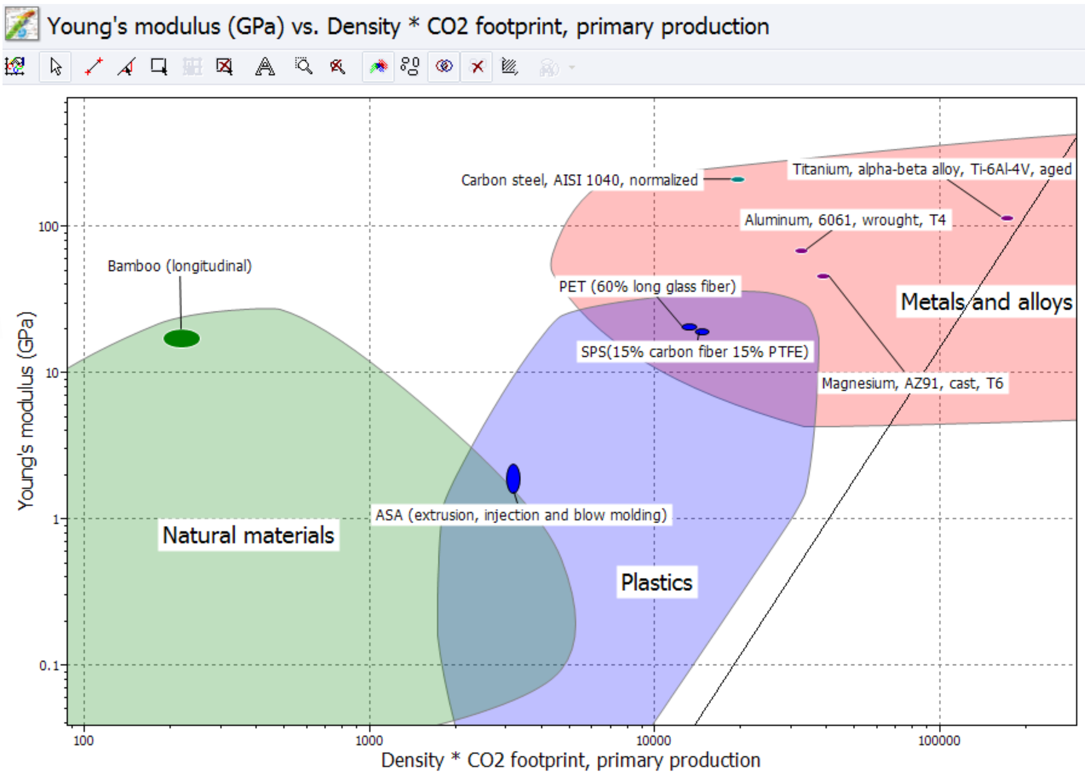
From engineering to maths (*next course)



Design of bicycle frame by optimization



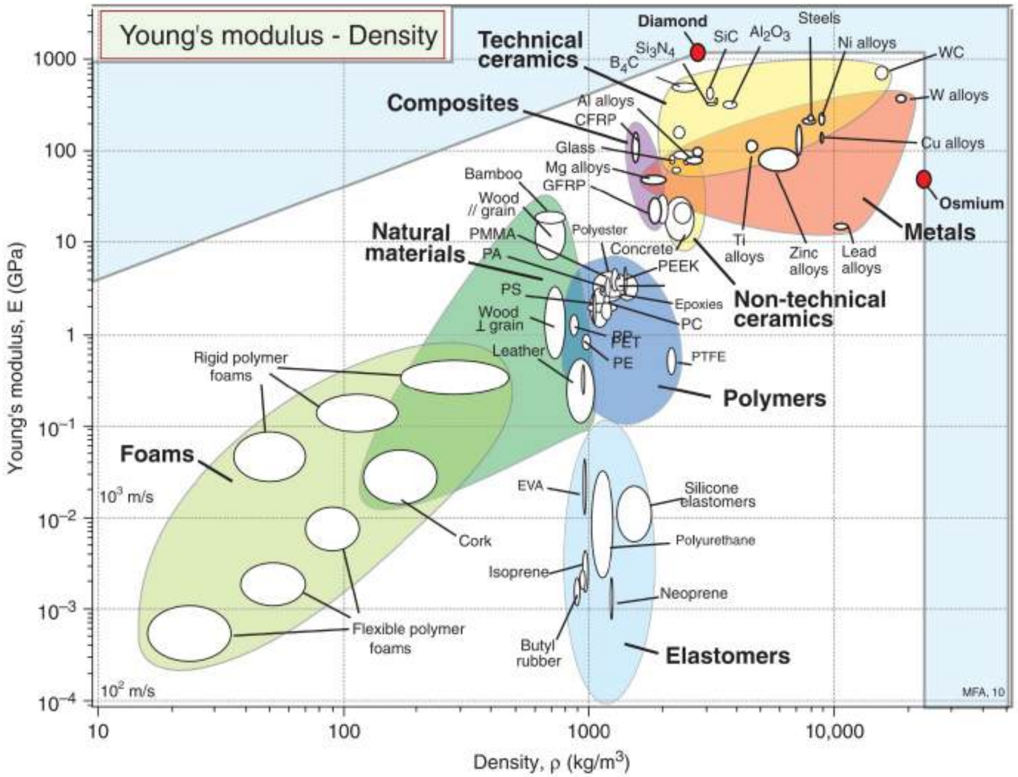
Design of bicycle frame by EcoDesign



Optimization is Everywhere

To get started, you need to select the right material. Why use steel if you can get away with aluminum?

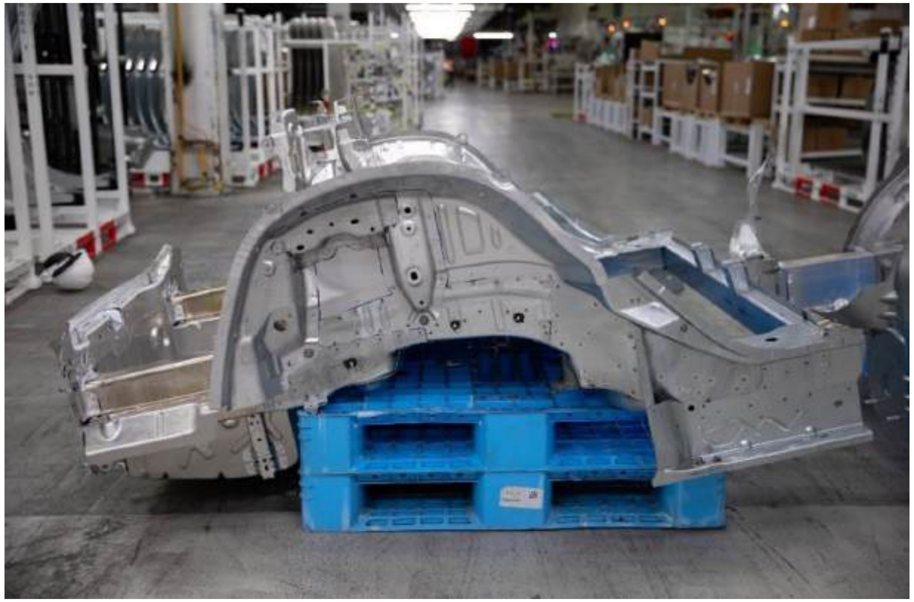
There's a weight reduction of approximately two thirds on the table, albeit with compromises: reduced stiffness, higher cost, more difficulty in welding. A great way of visualizing these tradeoffs and rankings of mechanical properties is with Ashby Charts, which essentially represent the menu of materials that an engineer can select.



Ashby Chart for evaluating stiffness-to-weight ratio of conventional materials. This essentially represents the menu of (conventional) materials that engineers have to choose from.

Optimization is Everywhere

https://www.3dprintingmedia.network/tesla-shows-massive-generatively-designed-3d-printed-part-in-model-y-underbody/

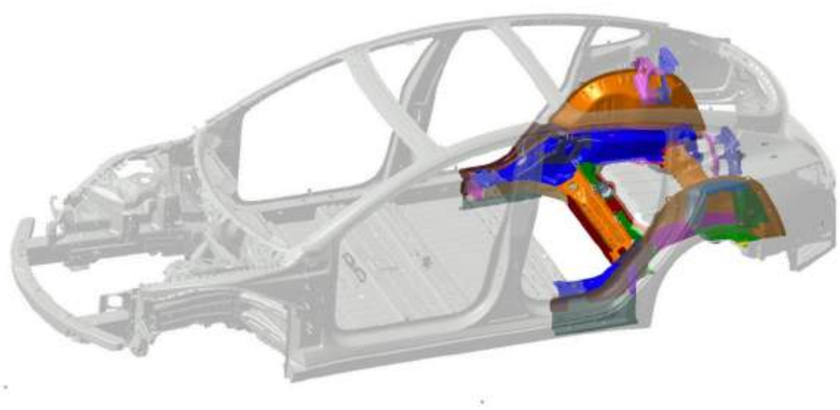


The current underbody part made of 70 different components

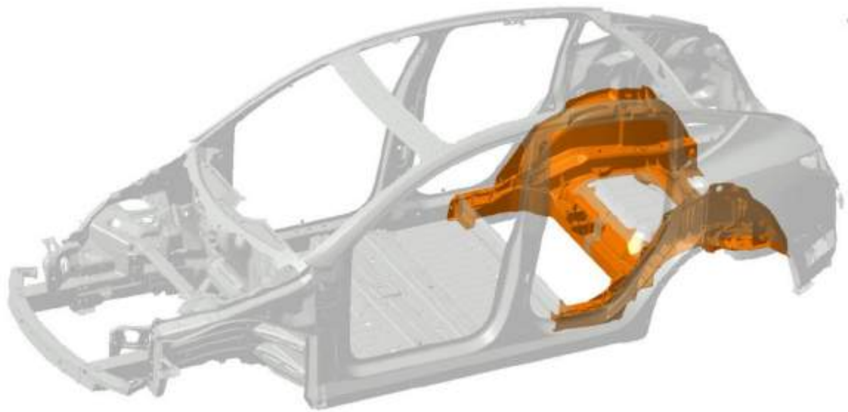
The generatively designed underbody, made of 2 and eventually 1 single piece.

Optimization is Everywhere

Think different!!!



Model 3 rear underbody
70 pieces of metal



Model Y rear underbody
2 pieces of metal (eventually a single piece)

the use of 3D printing for sand casts such as that offered by voxeljet and ExOne for to enable the reduction of subassemblies (form 70 to 1) in a custom cast can bring about a significant transition even before metal AM can be used to produce such large metal parts directly. Producing a complex cast that can reduce the number of parts to this degree needs digital casting technology

Let's take the example of a structure with 10 structural elements (10 design variables).

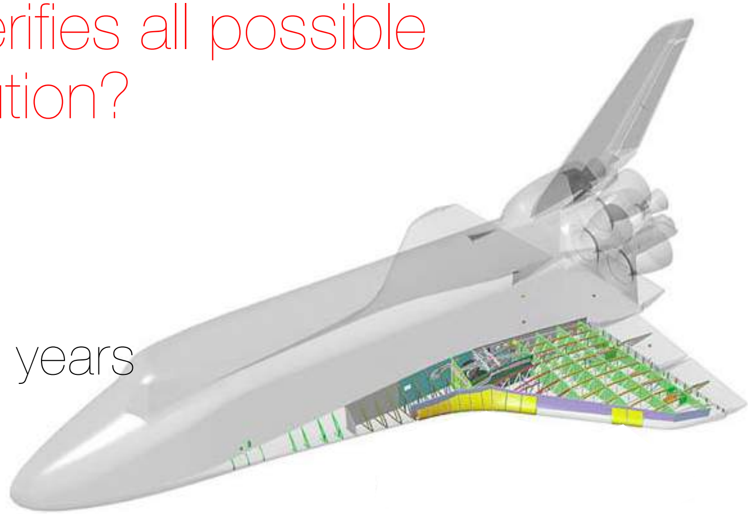
Each variable is associated with a section type (U, Z, I etc …: 10 sections available per variable).

A structural analysis "costs" only 1 second of calculation.

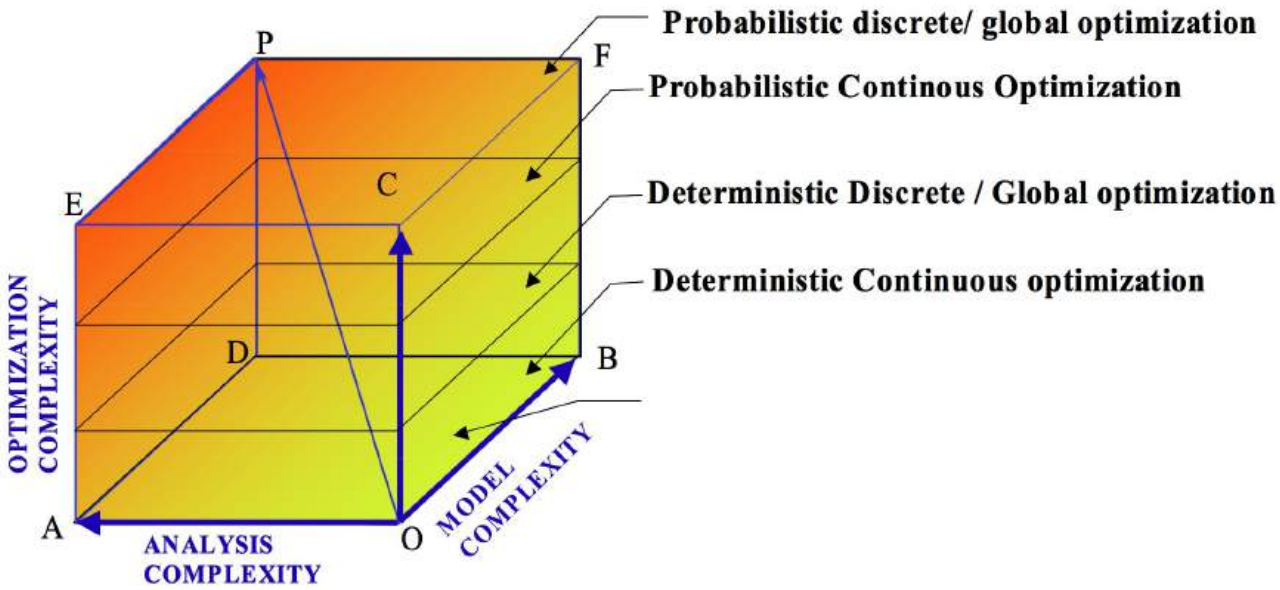
Question:

How long will the calculation last that verifies all possible combinations to ensure an optimal solution?

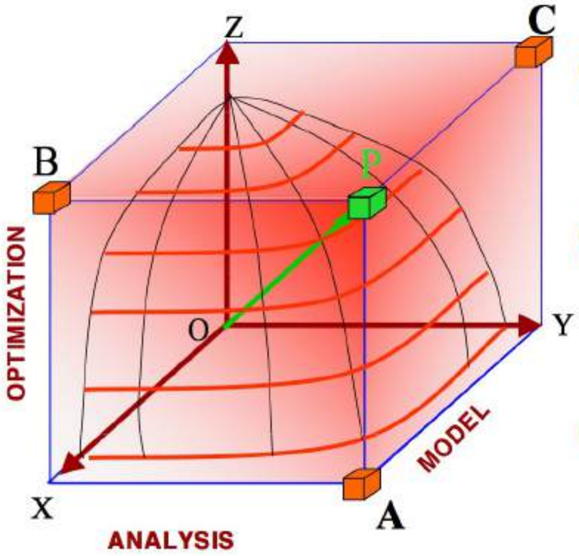
… 10^{10} seconds At least 317 years



Haftka's principle



Haftka's Examples



(A) Model and Analysis Complexity

- e.g., Complex 3-D structure requiring non-linear analysis

(B) Model and Optimization Complexity

- e.g., Probabilistic design of a complex 3-D structure

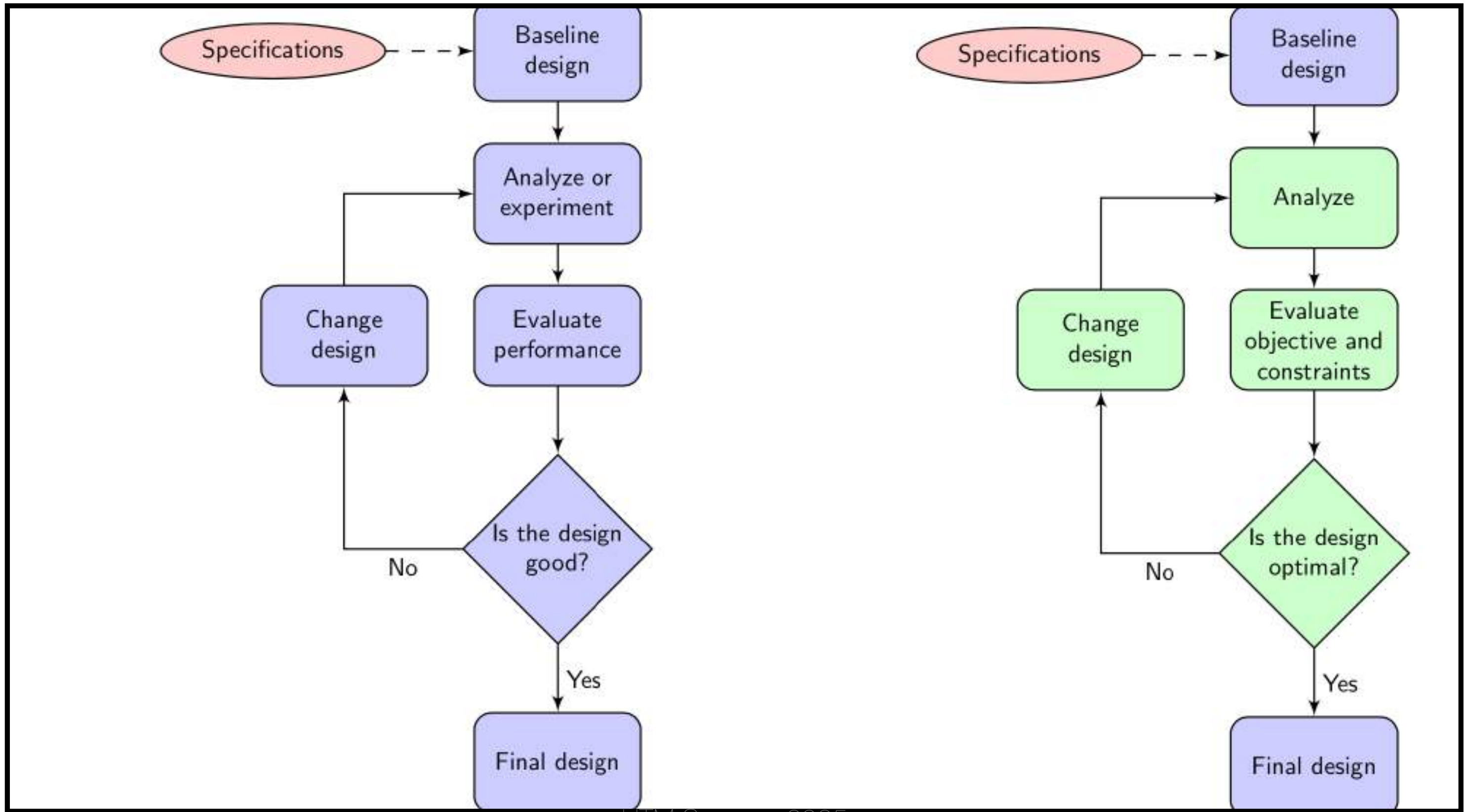
(C) Analysis and Optimization Complexity

- e.g., Non-deterministic optimization with non-linear analysis (or combined macro and micro mechanical failure models

(P) Model, Analysis and Optimization Complexity

e.g., Non-deterministic optimization of Complex 3-D structure requiring nonlinear analysis

Conventional vs Optimal Design



Definition ?

- “Making things better”
- “Generating more profit”
- “Determining the best”
- “Do more with less”

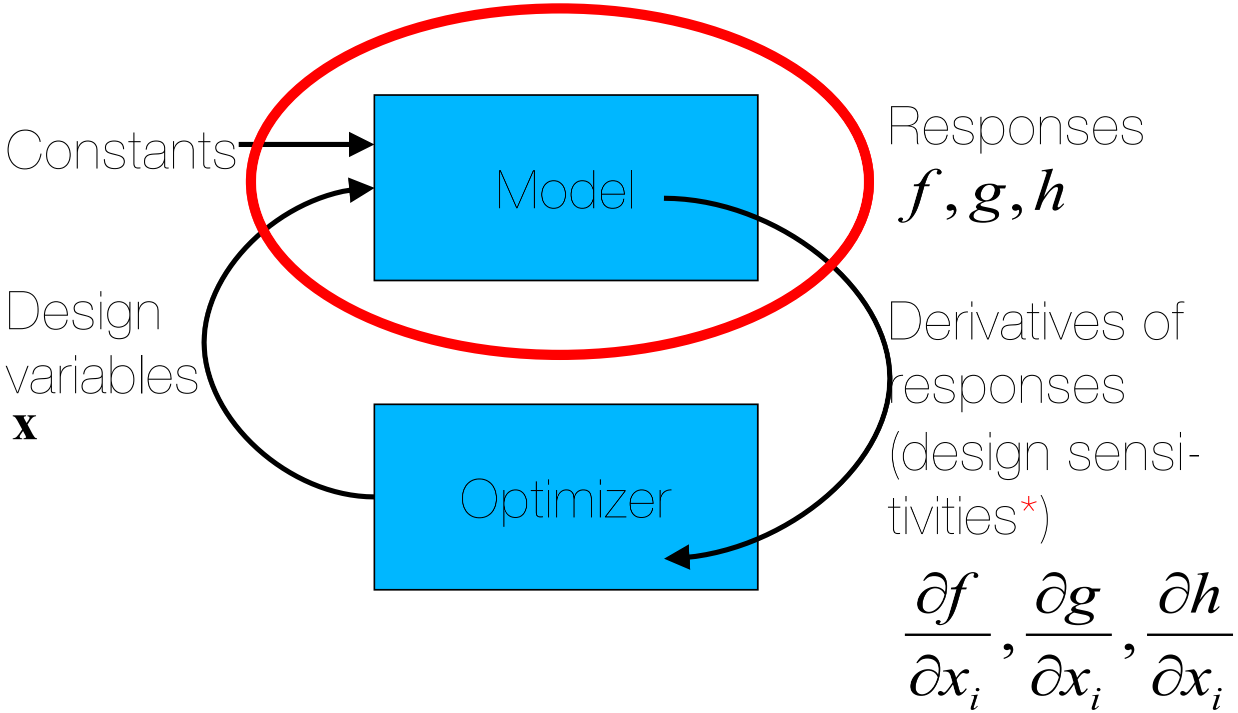
- “The determination of values for design variables which minimize (maximize) the objective, while satisfying all constraints” : Optimal design

Engineer's goals?

- The engineer must often answer a contradictory problem:
  - Airplane wing, F1 : rigidity vs mass
  - Confort in turbulence vs speed of flutter
  - Pressurized tank: use (form) / buckling load

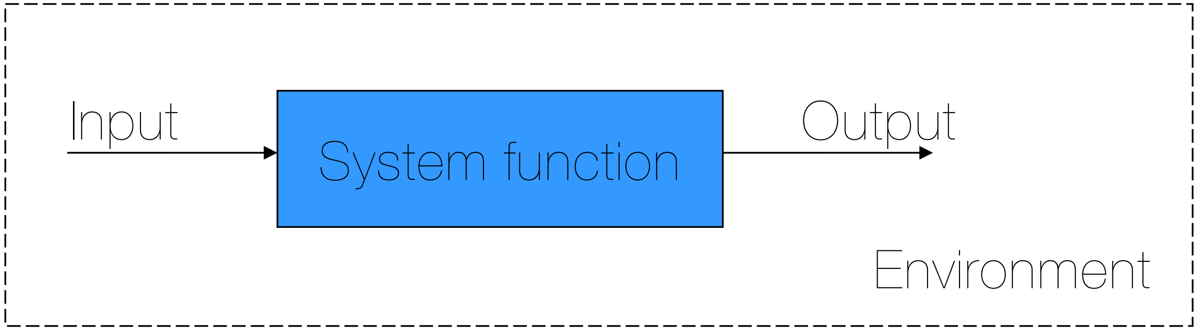
  Everything else is trivial!

How?



*see doc on LMS

System approach



- Wonder:
  - What are the inputs / outputs?
  - What is the system / environment?
  - Hierarchize the system

Optimization types

$$\min J(x) \quad x \in \mathbb{N}$$

Discrete:
Operational research, combinatory, theory of graph

$$\min J(x) \quad x \in D \subset \mathbb{R}^N$$

Finite dimension:
Optimality criteria, differentiability, convex analysis, constraints/unconstraints

$$\min J(u) \quad u \in H$$

Variational method:
Physics, mechanics, economy

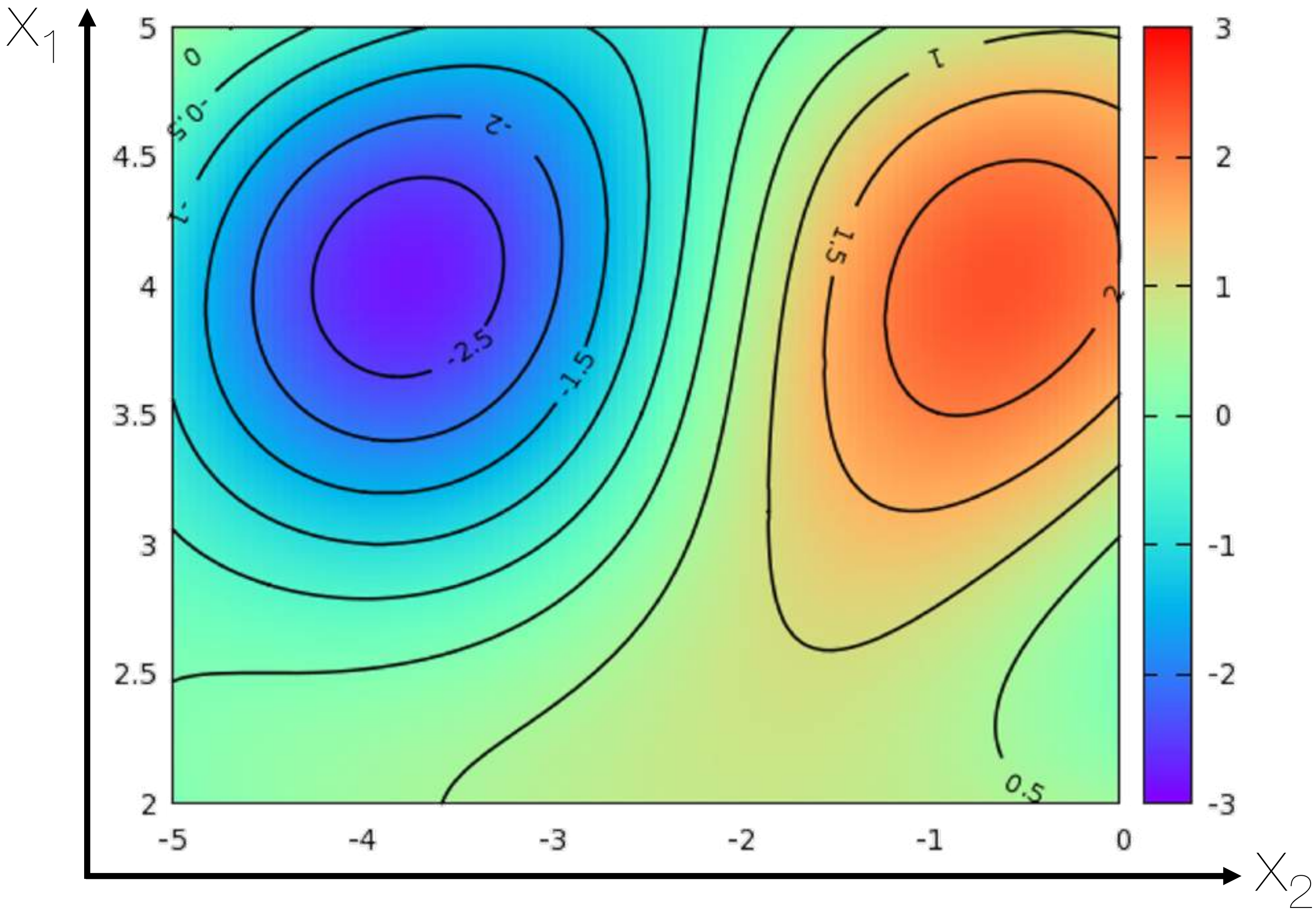
$$\min J(\Omega) \quad \Omega \subset \mathbb{R}^N$$

Shape:
Properties, optimality criteria

https://neos-guide.org/content/optimization-tree-alphabetical

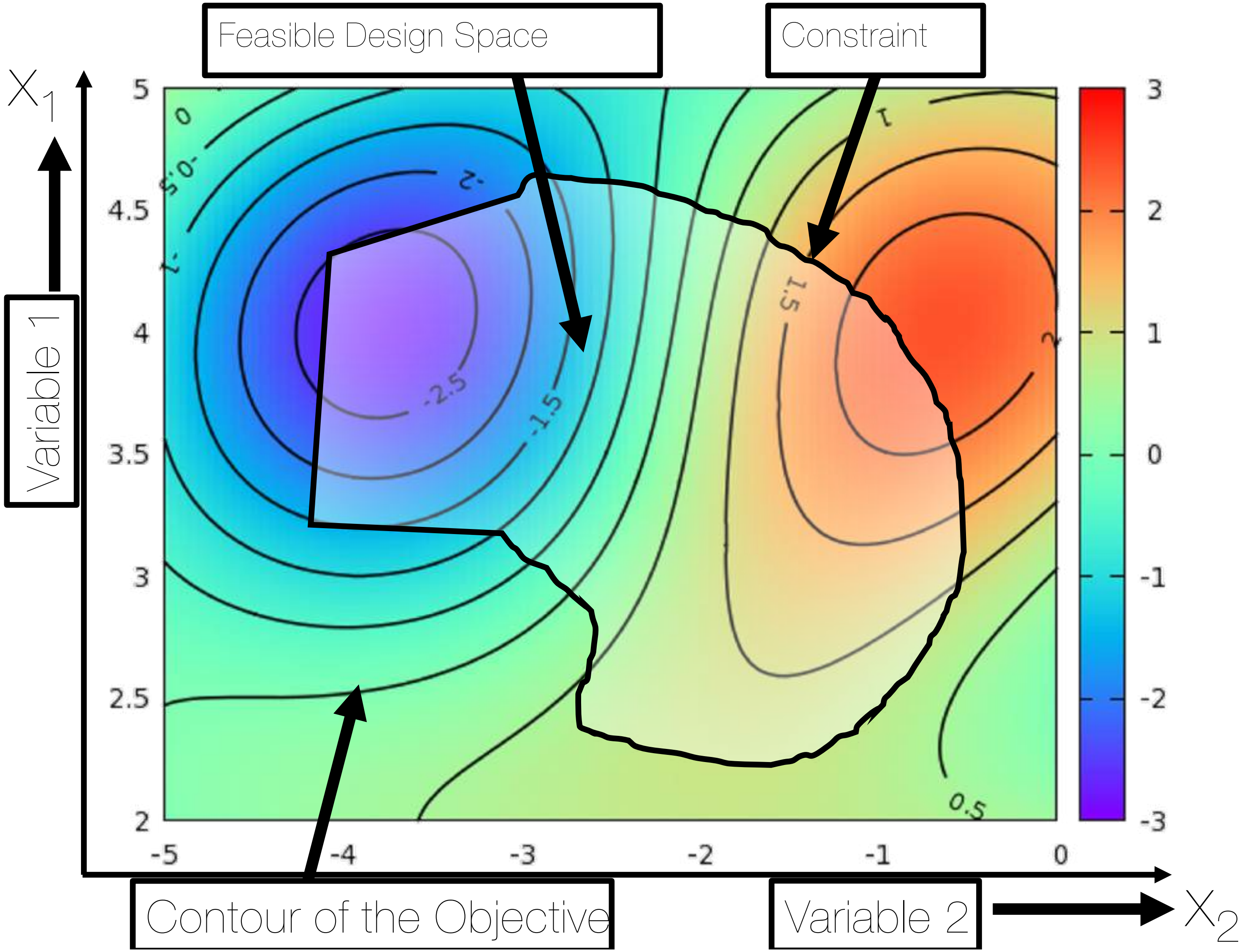
The BIG picture

Variables and the Objective



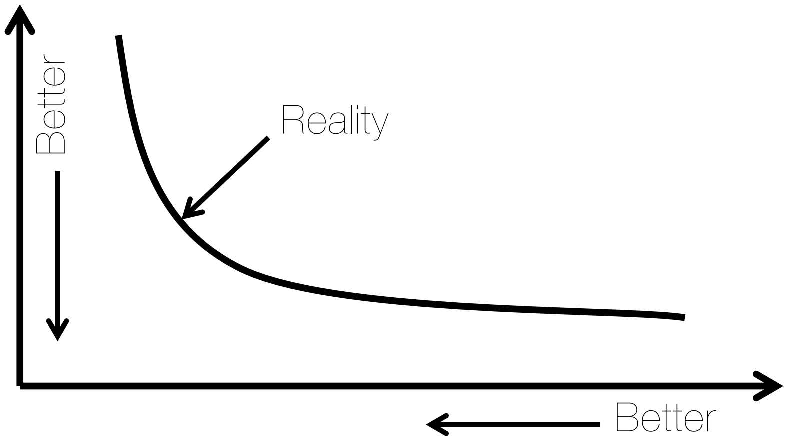
The BIG picture

Variables, Objective & Constraints

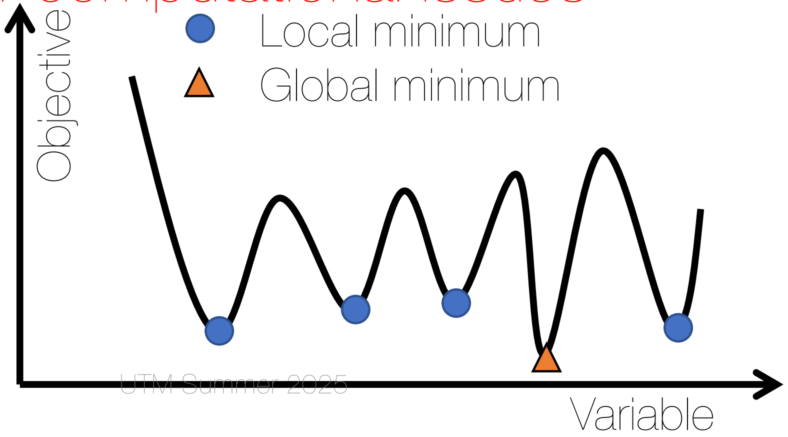


Common issues in the optimization of engineering systems

Multiple objectives

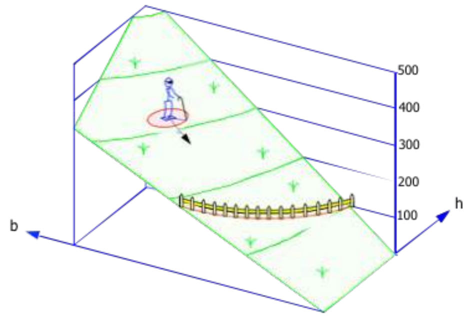


Multiple local solutions and other computational issues



Gradient based Optimization (Maximize/Minimize)

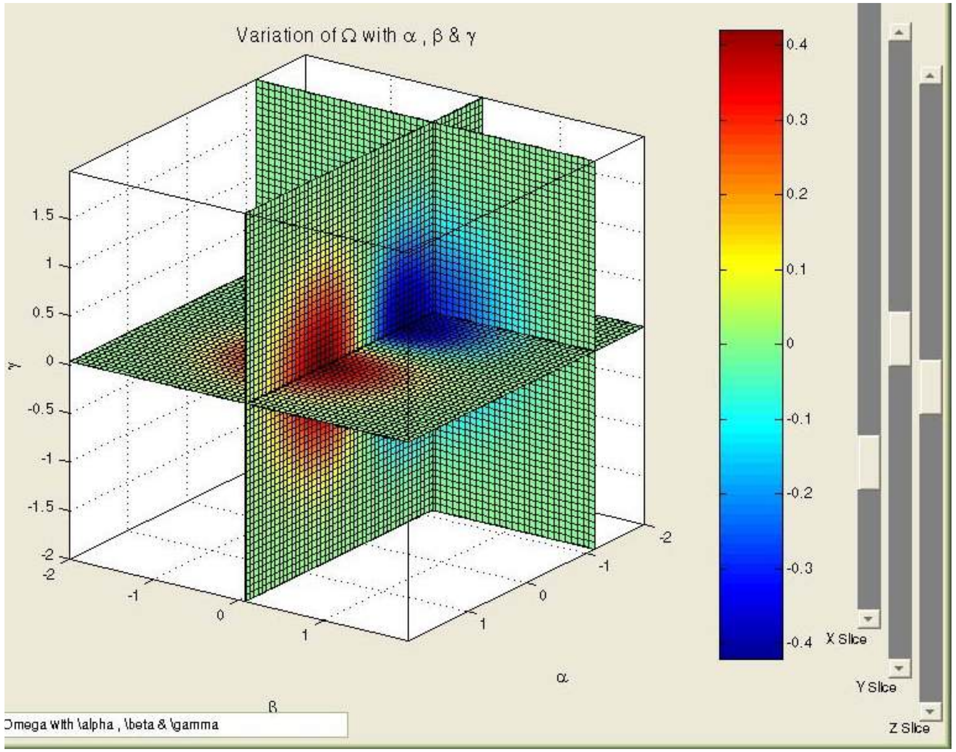
The optimum (the solution) will depend on the starting point !!! And this point of arrival may be a local minimum and not an overall minimum.



$$\min_{\tilde{x}} \quad f(\tilde{x}) \quad \Leftrightarrow \quad \max_{\tilde{x}} \quad -f(\tilde{x})$$

In N dimension?

Of course in N-D,
N Design variables
P constraints, it may be difficult to visualize…



Derivatives ?

Quasi-Newton method
Gradient descent
Gauss–Newton algorithm
Levenberg–Marquardt algorithm
Trust region
Nelder–Mead method
MATLAB, SOL200 etc…

In optimization

$$\min_{x \in \mathbf{X}} J(x)$$

Why computing derivative of criteria J ?

→Optimality criteria J'(x)=0

→To calculate an approximated solution:

→Steepest descent $X_{n+1}=X_n-\rho^* J'(X_n)$

→Newton $X_{n+1}=X_n-[D^2J(X_n)]^{-1}J'(X_n)$

Negative null form [MATLAB]

$$\begin{array}{lll} \underset{\mathbf{x}}{\text{Minimize}} & f(\mathbf{x}) & \mathbf{x} = \text{(column) vector of design variables} \\ \text{subject to} & \mathbf{h}(\mathbf{x}) = 0 & \mathbf{h} = [h_1, h_2, \ldots, h_{m_h}]^T \\ & \mathbf{g}(\mathbf{x}) \leq 0 & \mathbf{g} = [g_1, g_2, \ldots, g_{m_g}]^T \\ & \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n & \end{array}$$

Other formulations:

- positive null form ($\mathbf{g}(\mathbf{x}) \geq 0$) PYTHON
- negative unity form ($\mathbf{g}(\mathbf{x}) \leq 1$)
- positive unity form ($\mathbf{g}(\mathbf{x}) \geq 1$)

Criteria to optimize (performance)

- Some examples of criteria to minimize (or maximize)
  - Cost
  - Mass
  - Aerodynamic drag
  - Lift
  - Etc.

Design variables

The section of a beam in an assembly (lattice)
The number of ribs in a wingbox
The skin thickness of a wingbox
The orientation and sequencing of a composite
The coordinates defining a NACA profile
Etc …

In optimization of structures, we distinguish the variables of shape, geometry, and materials

Constraints

Mechanical

- Von Mises (Stresses)
- Buckling load
- Eigenfrequencies

$$g = \frac{\sigma_{\max}(\mathbf{x})}{\sigma_{allowed}} - 1 \leq 0 \qquad\qquad g = \sigma_{\max}(\mathbf{x}) - \sigma_{allowed} \leq 0$$

Scaled (Reserve Factor) vs. Unscaled

Optimization Steps

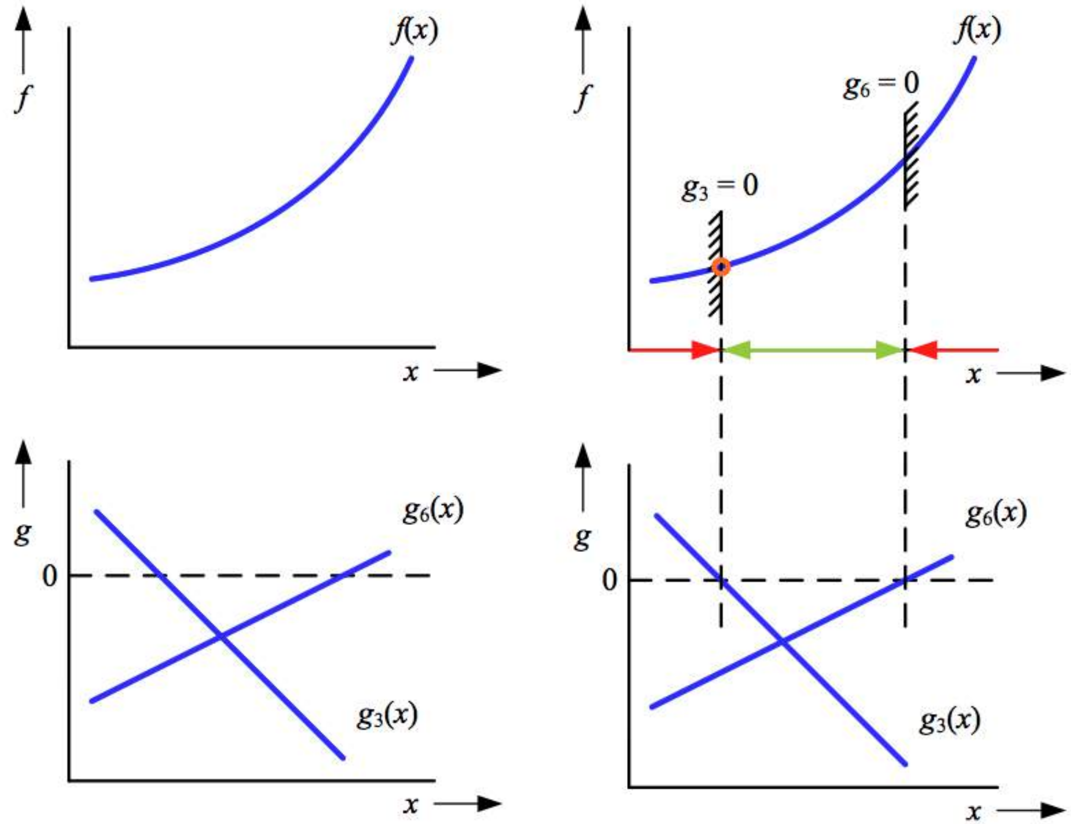
- 1 Select design variables
- 2 Select objective criterion in terms of design variables (to minimize or maximize)
- 3 Determine constraints in terms of design variables, which must be satisfied
- 4 Determine design variable values which minimize (maximize) the objective while satisfying all constraints

[Papalambros & Wilde 2000: Principles of optimal design]

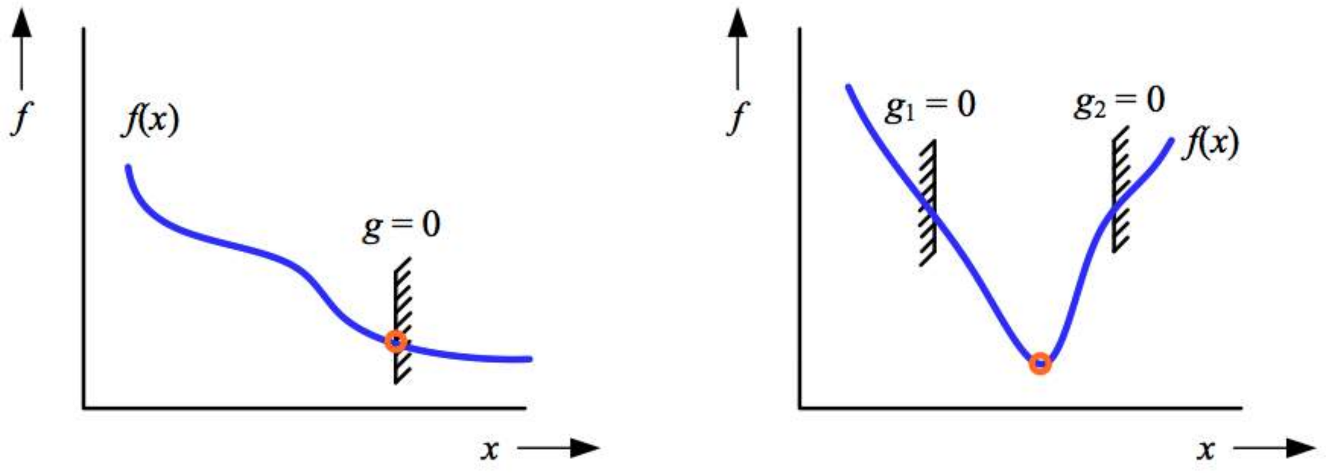
Classification

- Problems:
    - Constrained vs. unconstrained
    - Single level vs. multilevel
    - Single objective vs. multi-objective
    - Deterministic vs. stochastic
- Responses:
    - Linear vs. nonlinear
    - Convex vs. nonconvex (later!)
    - Smooth vs. nonsmooth
- Variables:
    - Continuous vs. discrete (integer)

Visualization: 1D example



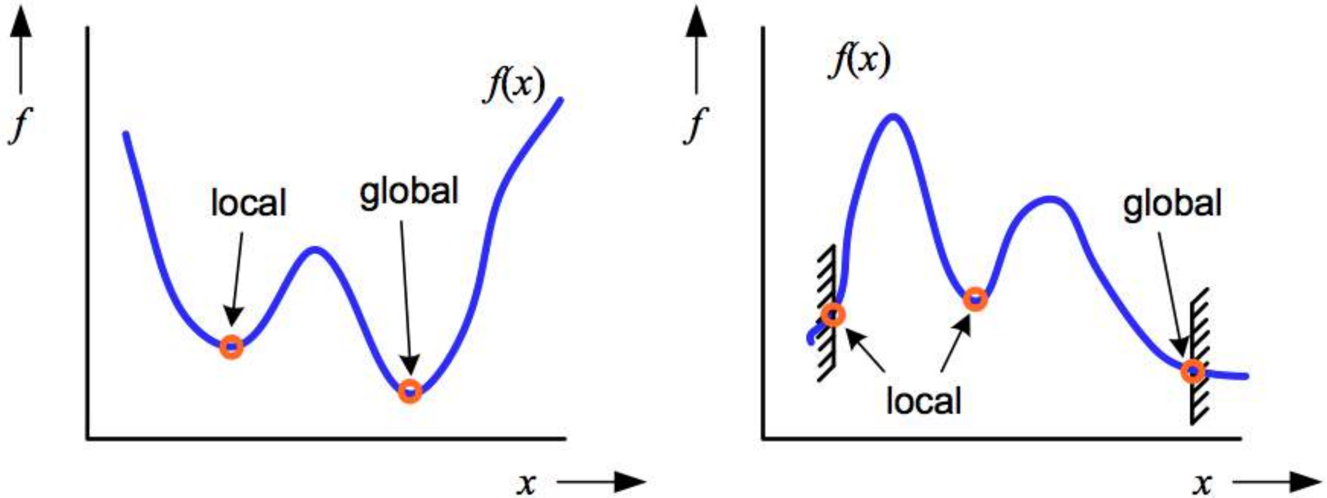
Constrained VS unconstrained optimum



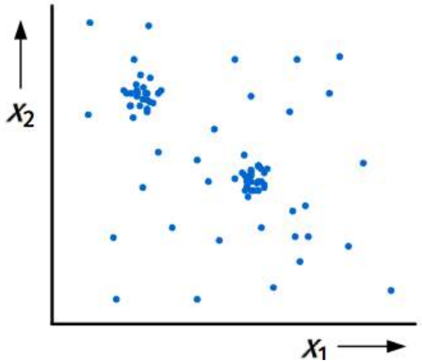
- constrained optimum
- bounded optimum

- unconstrained optimum
- interior optimum

Multimodality (multiple local minima)



Genetic Algorithms GA



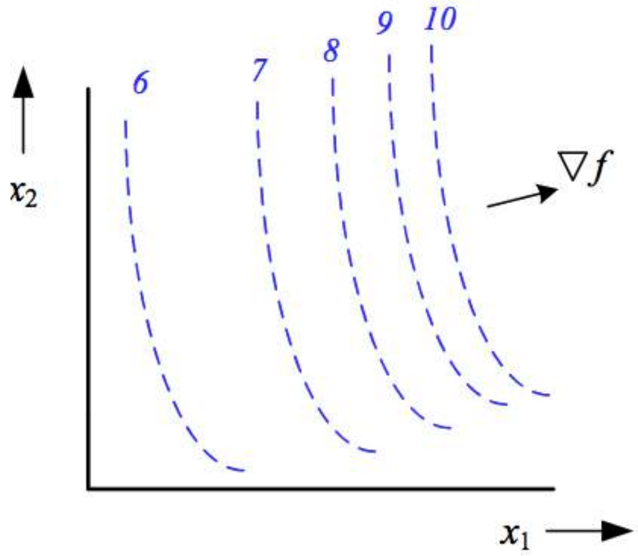
- The first "metaheuristic": 1975, John Holland publishes "Adaptation in natural and artificial systems »
- Widely used, easy to program, easy to use and robust.

1 / Problems of continuous GLOBAL optimization

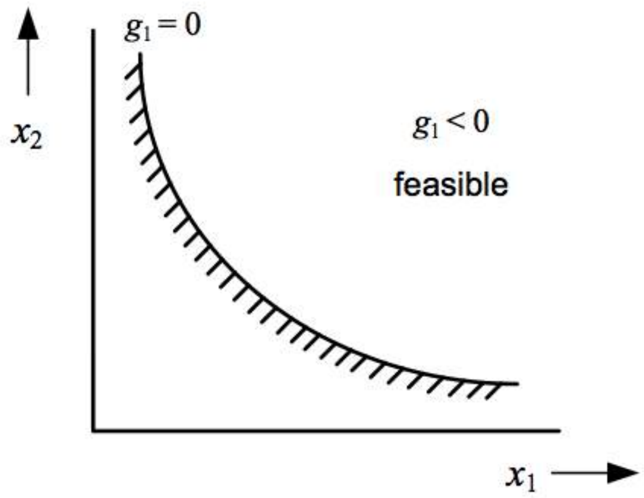
2 / Problems of combinatorial (discrete) GLOBAL optimization:

→ For these problems, either discrete values can only be used for the variables, or we must change elements of the problem to "change the values of the variables"

2D example

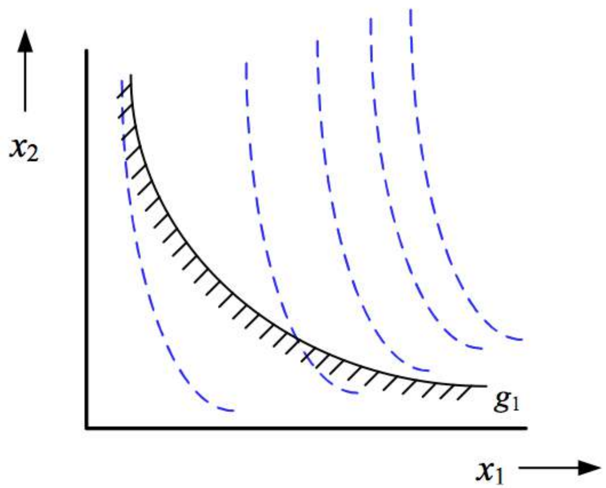


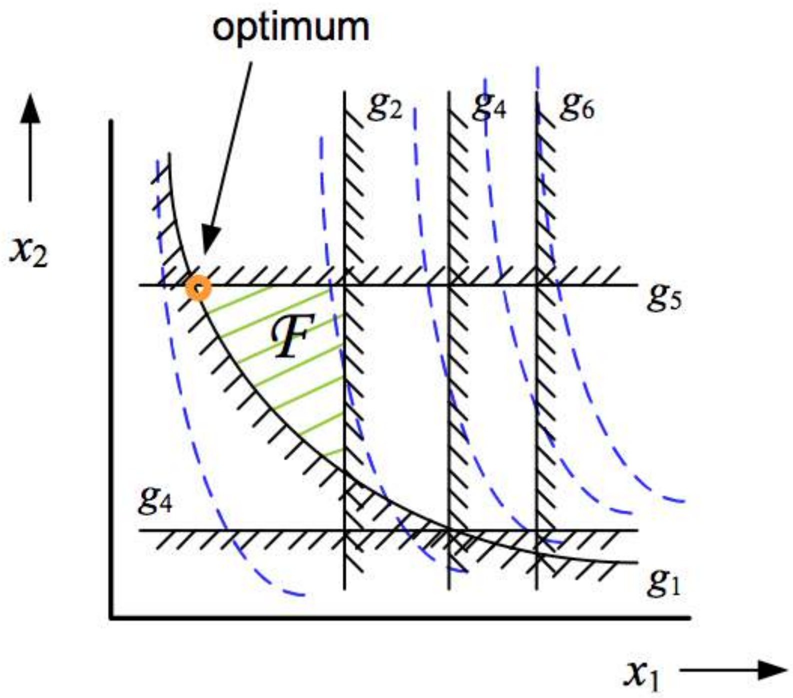
objective function



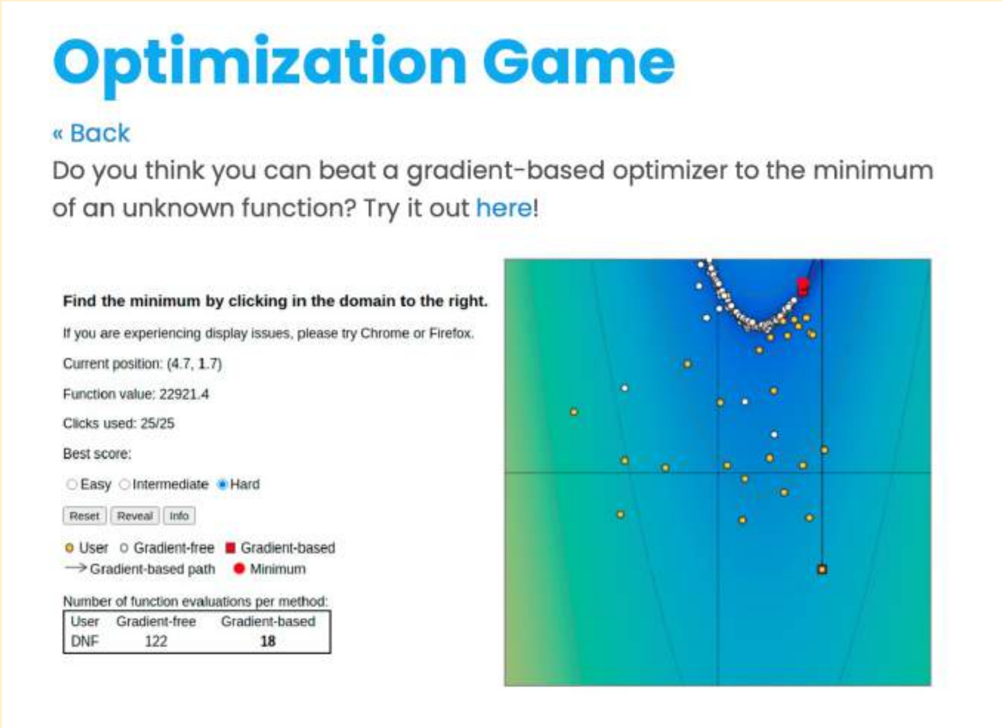
constraint g_1

2D example





First Exercice



https://mdolab.engin.umich.edu/assets/optimizationGame/

Stopping criteria

Condition on optimality

$\|\nabla L(\mathbf{x}_{k+1})\| < \varepsilon$ **(Matlab: TolFun)**

$g_j(\mathbf{x}_{k+1}) < \varepsilon$ **and** $|h_j(\mathbf{x}_{k+1})| < \varepsilon$ **(Matlab: TolCon)**

or a condition on change in x

$\|\mathbf{x}_{k+1} - \mathbf{x}_k\| < \varepsilon$ **(Matlab: TolX)**

or a condition on the number of iterations

$k \leq k_{\max}$ **(Matlab: MaxIter)**

or a combination of these, with $\varepsilon > 0$

Type	Domaine	Function
Scalar Minimization	$\min_a f(a)$ such that $a_1 < a < a_2$	`fminbnd`
Unconstrained Minimization	$\min_x f(x)$	`fminunc` `fminsearch`
Linear Programming	$\min_x f^T(x)$ such that $A.x \leq b, Aeq.x = beq, lb \leq x \leq ub$	`linprog`
Quadratic Programming	$\min_x \frac{1}{2}x^T Hx + f^T x$ such that $A.x \leq b, Aeq.x = beq, lb \leq x \leq ub$	`quadprog`
Constrained Minimization	$\min_x f(x)$ such that $c(x) \leq 0, ceq(x) = 0$ $A.x \leq b, Aeq.x = beq, lb \leq x \leq ub$	`fmincon`
Goal Attainment	$\min_{x,\gamma} \gamma$ such that $F(x) - \omega\gamma \leq goal$ $c(x) \leq 0, ceq(x) = 0$ $A.x \leq b, Aeq.x = beq, lb \leq x \leq ub$	`fgoalattain`
MiniMax	$min_x max_{\{F_i\}}\{F_i(x)\}$ such that $c(x) \leq 0, ceq(x) = 0$ $A.x \leq b, Aeq.x = beq, lb \leq x \leq ub$	`fminimax`
Semi-Infinite Minimization	$min_x f(x)$ such that $K(x,\omega) \leq 0 \forall \omega$ $c(x) \leq 0, ceq(x) = 0$ $A.x \leq b, Aeq.x = beq, lb \leq x \leq ub$	`fseminf`

Linear constraints

$$\min \quad f(x) = \left(x_1^2 + x_2^2 - 1\right)^2$$
$$\text{s.t.} \quad -1 \le x_1 \le 1, -1 \le x_2 \le 1,$$
$$x_1 + x_2 \ge 1$$
$$x_1 x_2 \ge 1/2,\ x_2 \ge x_1^2,\ x_1 \ge x_2^2$$

$$A = [-1,-1];\ b = -1;$$
$$lb = [-1;-1];\ ub = [1;1];$$
$$c(x) = \begin{bmatrix} \frac{1}{2} - x_1 x_2 \\ x_1^2 - x_2 \\ x_2^2 - x_1 \end{bmatrix}; ceq(x) = [];$$

```
% myobj.m
function  f=myobj(x)
f = (x(1)^2+x(2)^2-1)^2;

% mycon.m
function [c, ceq]=mycon(x)
c=[1/2-x(1)*x(2);
   x(1)^2-x(2);
   x(2)^2-x(1)];  % nonlinear inequalities c(x) <= 0
ceq=[ ];  % nonlinear equalities ceq(x) = 0;

% main file for fmincon
[x,fval] = fmincon(@myobj,xo,A,b,[],[],lb,ub,
                                  @mycon,options);
```

x1+x2 > 1

-x1-x2<-1

xT=[x1,x2]

A=[-1 ,-1]

b=-1

Ax=b —> 1*2*2*1=1*1

- The syntax for fmincon

  [x,fval,exitflag]=fmincon(objfun,x0,A,b,Aeq,beq,lb,ub, nonlcon,options);
  - x: optimal solution; fval: optimal value; exitflag: exit condition
  - objfun: objective function (usually written in a separate M file)
  - x0: starting point (can be infeasible)
  - A: matrix for linear inequalities; b: RHS vector for linear inequalities
  - Aeq: matrix for linear equalities; beq: RHS vector for linear equalities
  - lb: lower bounds; ub: upper bounds
  - Nonlcon: [c,ceq]=constraintfunction(x)

$$\min \quad f(x) = \left(x_1^2 + x_2^2 - a\right)^2$$
$$\text{s.t.} \quad -1 \le x_1 \le 1, -1 \le x_2 \le 1,$$
$$x_1 + x_2 \ge 1$$
$$x_1 x_2 \ge 1/2,\ x_2 \ge x_1^2,\ x_1 \ge x_2^2$$

```
% myobj.m
function  f=myobj(x, a)
f = (x(1)^2+x(2)^2-a)^2;
% main file for fmincon
a = 1;
[x,fval] = fmincon(@(x) myobj(x,a),xo,A,b,[],[],lb,ub,
                                  @mycon,options);
```

Demo Matlab

- Example1 (Local optimisation)
- Example2 (Global optimisation)
- Example3 (With/Without gradients)

- https://fr.mathworks.com/help/gads/solving-a-mixed-integer-engineering-design-problem-using-the-genetic-algorithm.html

For example, consider Rosenbrock's function

$$f(x) = 100\left(x_2 - x_1^2\right)^2 + (1 - x_1)^2,$$

which is described and plotted in Solve a Constrained Nonlinear Problem. The gradient of *f*(*x*) is

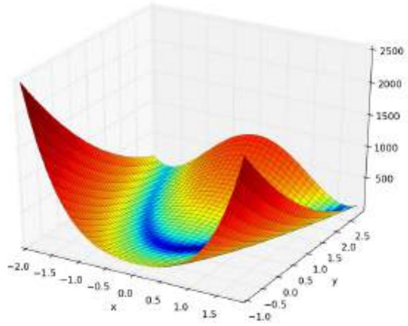
$$\nabla f(x) = \begin{bmatrix} -400\left(x_2 - x_1^2\right)x_1 - 2(1 - x_1) \\ 200\left(x_2 - x_1^2\right) \end{bmatrix},$$

and the Hessian *H*(*x*) is

$$H(x) = \begin{bmatrix} 1200x_1^2 - 400x_2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}.$$

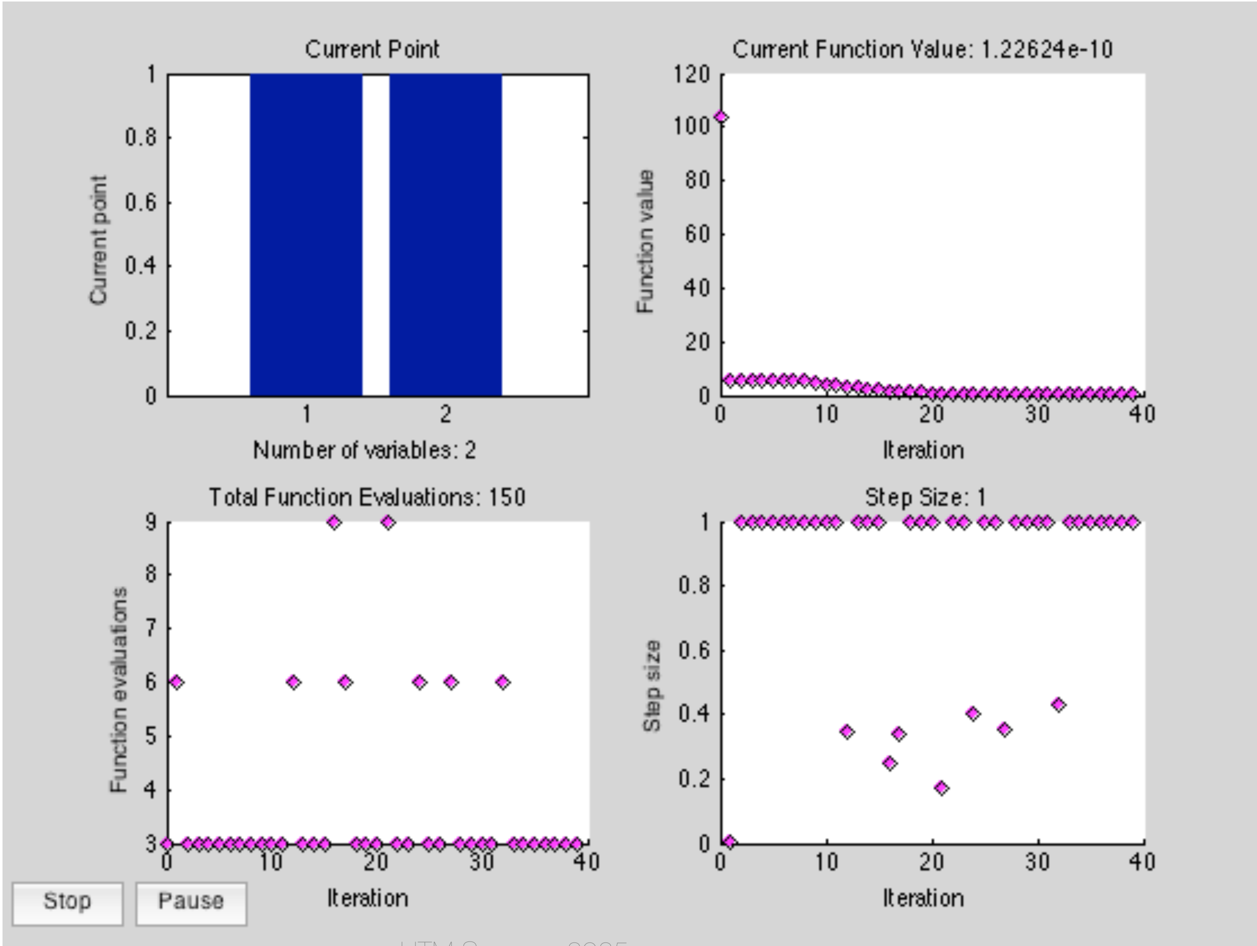
`rosenthree` is an unconditionalized function that returns the Rosenbrock function with its gradient and Hessian:

```
function [f g H] = rosenthree(x)
% Calculate objective f, gradient g, Hessian H
f = 100*(x(2) - x(1)^2)^2 + (1-x(1))^2;
g = [-400*(x(2)-x(1)^2)*x(1)-2*(1-x(1));
        200*(x(2)-x(1)^2)];
H = [1200*x(1)^2-400*x(2)+2, -400*x(1);
            -400*x(1), 200];
```

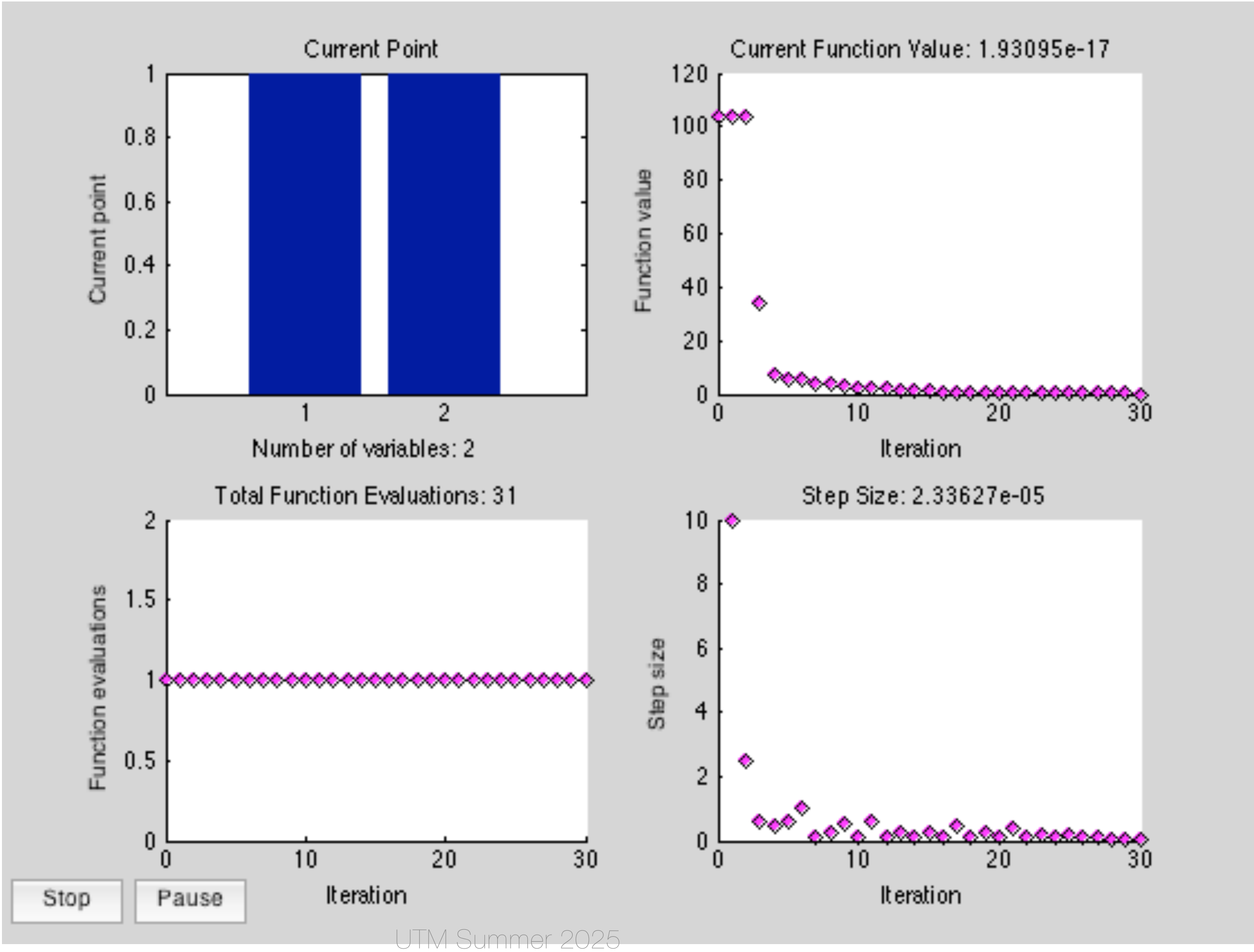



```
syms x y ;
   f=100*(y - x^2)^2 + (1-x)^2
   fx=diff(f,x)
   fy=diff(f,y)
%analytical gradients
f =(x - 1)^2 + 100*(- x^2 + y)^2
fx =2*x - 400*x*(- x^2 + y) – 2
fy =- 200*x^2 + 200*y
g=[fx;fy]
```

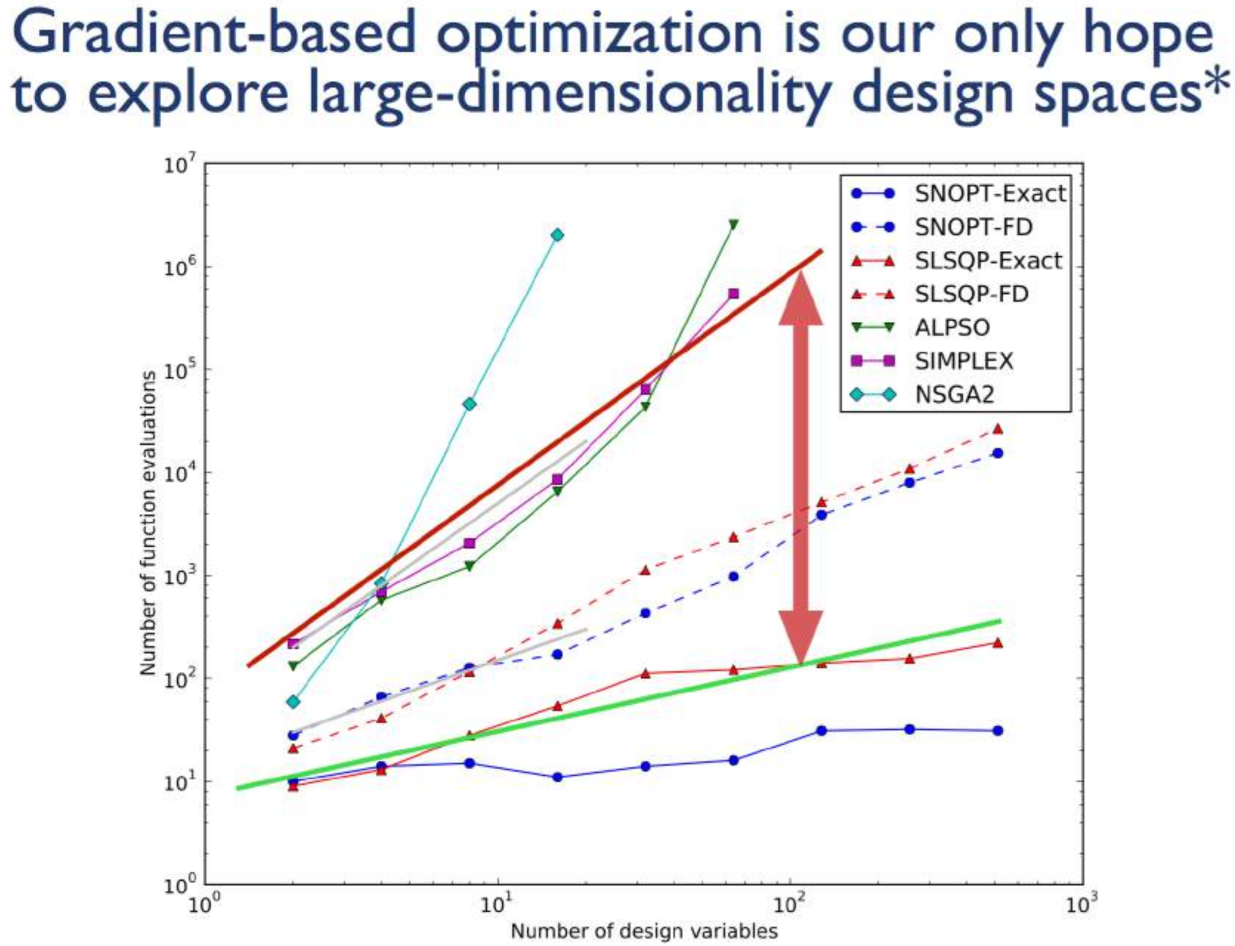
Results (w/o gradient)



Results (w/ gradient)



J. Martins, MDO course



Second exercice

1 Optimization problems in practice

Rewrite the following practical optimization problems using the following canonical form

$$\min_{x \in \Omega} f(x), \quad \Omega = \{x \in \mathbb{R}^n | h(x) = 0, g(x) \leq 0\}$$

where $h : \mathbb{R}^n \to \mathbb{R}^m$ represents m equality constraints ($h(x) = [h_i(x)]_{i=1}^m$) and $g : \mathbb{R}^n \to \mathbb{R}^p$ represents p inequality constraints ($g(x) = [g_j(x)]_{j=1}^p$).

1. Find two positive numbers whose sum is 300 and whose product is a maximum.
2. Find two positive numbers whose product is 750 and for which the sum of one and 10 times the other is a minimum.

Warning
Matlab Negative Null Form
Python Positive Null Form

Reformulation

Basic idea: convert to an unconstrained optimization problem

- Penalty function methods
- Append a penalty for violating constraints (exterior penalty methods)
- Append a penalty as you approach infeasibility (interior point methods)
- Method of Augmented Lagrange multipliers

Augmented Lagrange Method

- Adaptation of penalty method for equality constraints

$$p_{\text{Lagrange}}(\mathbf{x}) = \frac{1}{2}\rho \sum_i (h_i(\mathbf{x}))^2 - \sum_i \lambda_i h_i(\mathbf{x})$$

- λ converges towards the Lagrange multiplier

$$\boldsymbol{\lambda}^{(k+1)} = \boldsymbol{\lambda}^{(k)} - \rho \mathbf{h}(\mathbf{x})$$

https://tutorial.math.lamar.edu

See calculus III,
https://tutorial.math.lamar.edu/Problems/CalcIII/LagrangeMultipliers.aspx

Example of Optimizers

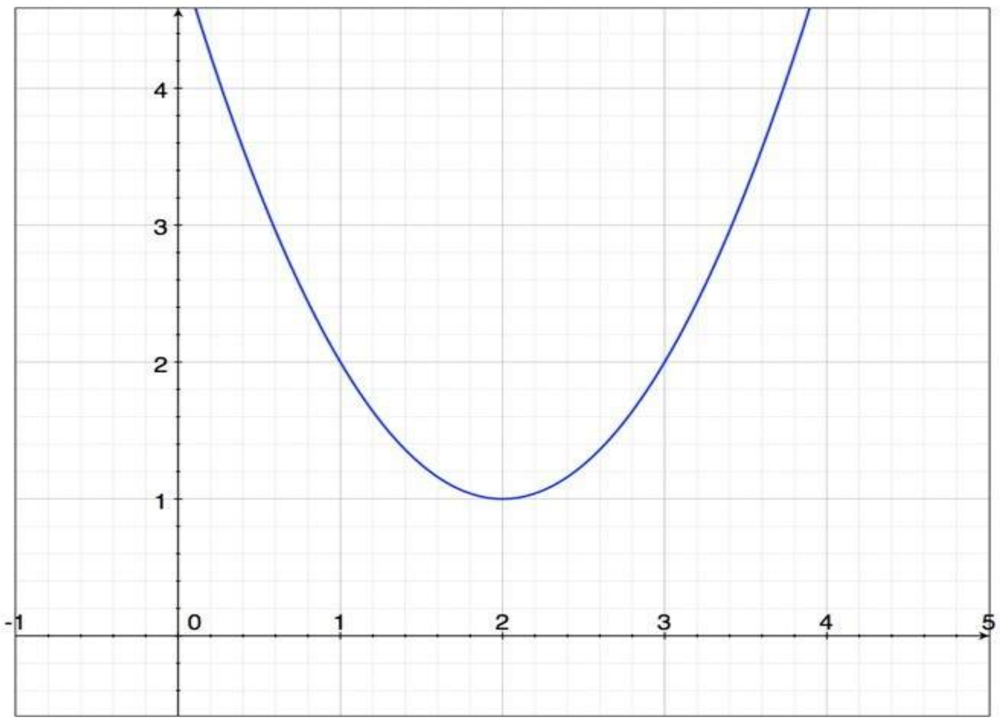
CHALLENGE #1 On Rosenbrock function minimization

https://colab.research.google.com/drive/1rGHIklpVM4Gqy9eq1Y10fIQW9umIjeEx

Example of using autograd

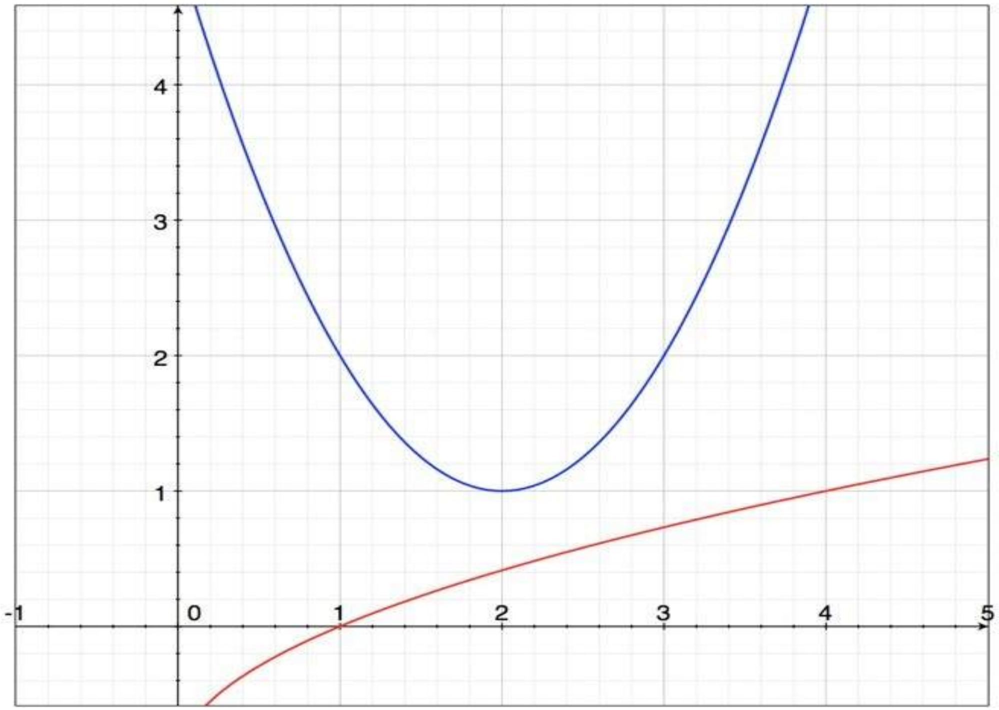
- https://colab.research.google.com/drive/1oDwKKU-WUnogMAxTg0Fx_NBuk5-8LD1x

Example



Objective: $f(x) = (x-2)^2 + 1$

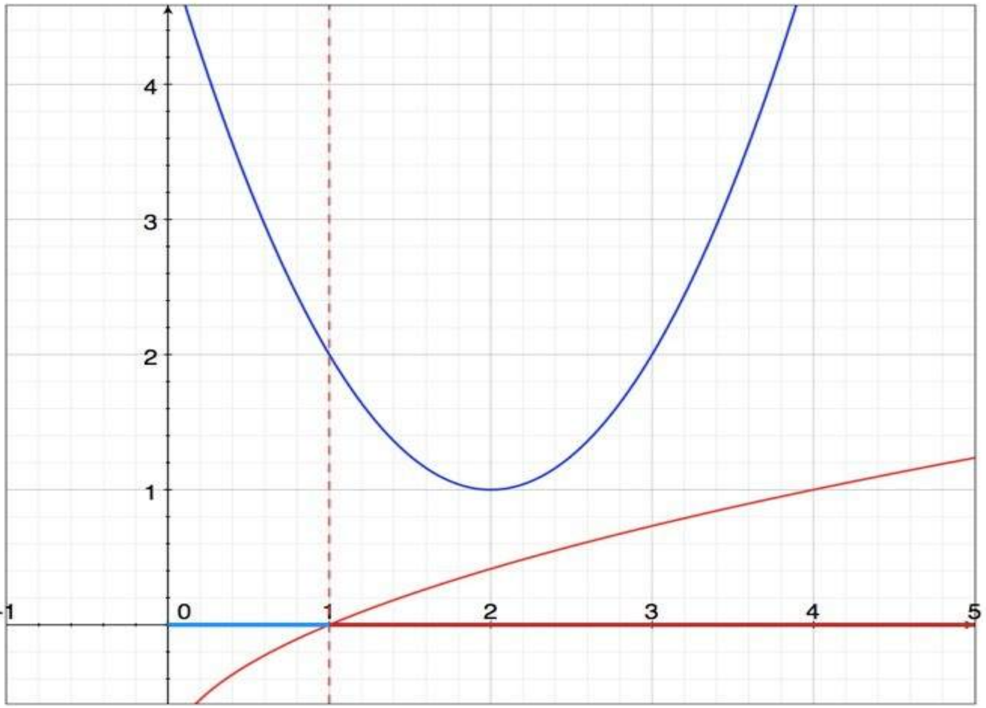
Example



Objective: $f(x) = (x - 2)^2 + 1$

Constraint: $c(x) \leq \sqrt{x} - 1$

Example



Objective: $f(x) = (x - 2)^2 + 1$

Constraint: $c(x) \leq \sqrt{x} - 1$

Bound: $x \geq 0$

Feasible Region: $0 \leq x \leq 1$

Penalty function methods

Objective Penalty function

$$\pi(x, \rho) = f(x) + \rho\phi(x)$$

Penalty parameter (non-negative)

1. Initialize penalty parameter

2. Initialize solution guess

3. Minimize penalized objective starting from guess

4. Update guess with the computed optimum

5. Go to 3., repeat

How to find the. Penalty function?

With the constraint $x - 5 \leq 0$, we need a penalty that is:

- 0 when $x - 5 \leq 0$ (the constraint is satisfied)
- positive when $x - 5$ is > 0 (the constraint is violated)

This can be done using the operation

$$P(x) = \max(0, x - 5)$$

which returns the maximum of the two values, either 0 or whatever $(x - 5)$ is.

We can make the penalty more severe by using

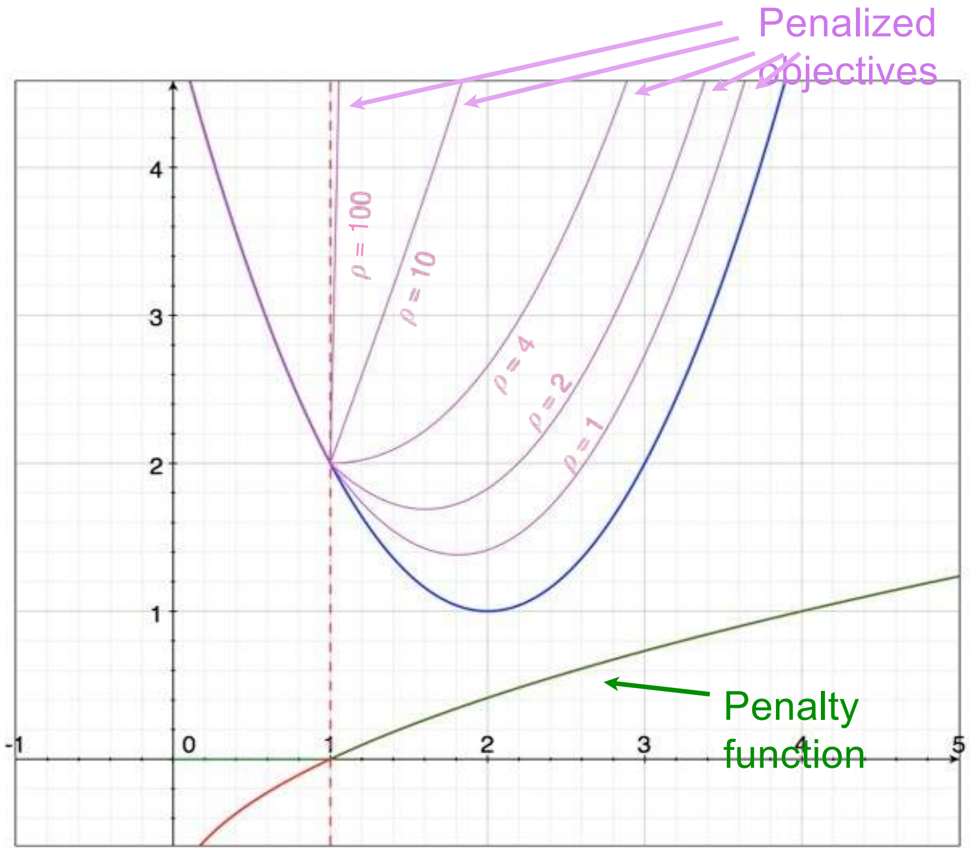
$$\boxed{P(x) = \max(0, x - 5)^2.}$$

This is known as a quadratic loss function.

Linear Exterior Penalty Function

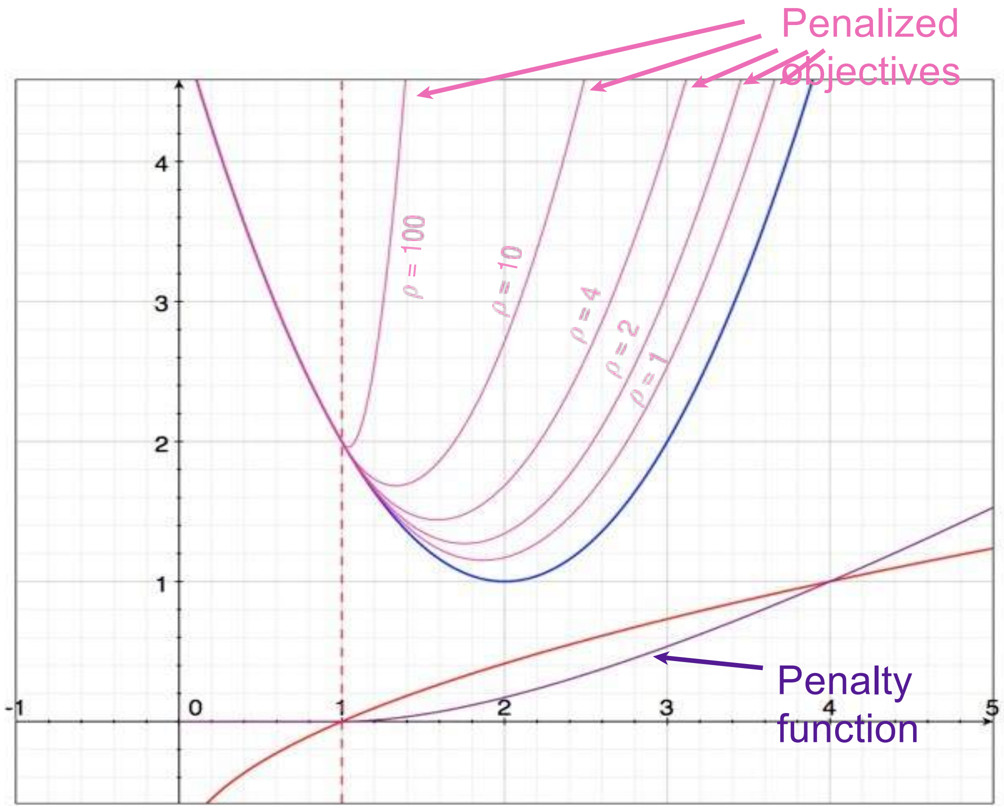
$$\phi_i(x) = \max(0, c_i(x) - u_i)$$

ui=RHS (negative null form)



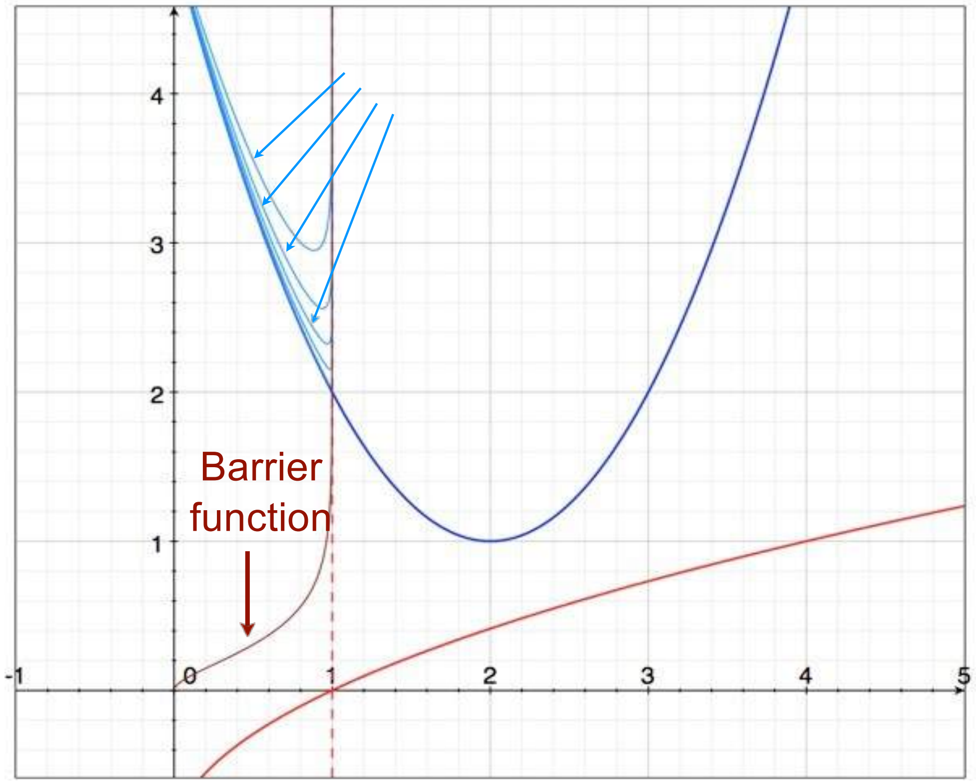
Quadratic Exterior Penalty Function

$$\phi_i(x) = \Big[\underbrace{\max(0, c_i(x) - u_i)}_{\text{Constraint violation}}\Big]^2$$



Interior-Point Methods

$$\pi(x,\mu) = f(x) - \underset{\substack{\uparrow \\ \text{Barrier parameter}}}{\mu} \overbrace{\log(u_i - c_i(x))}^{\text{Barrier function}}$$



Also called barrier methods, interior point methods ensure that each step is feasible

This allows premature termination to return a nearly optimal, feasible point

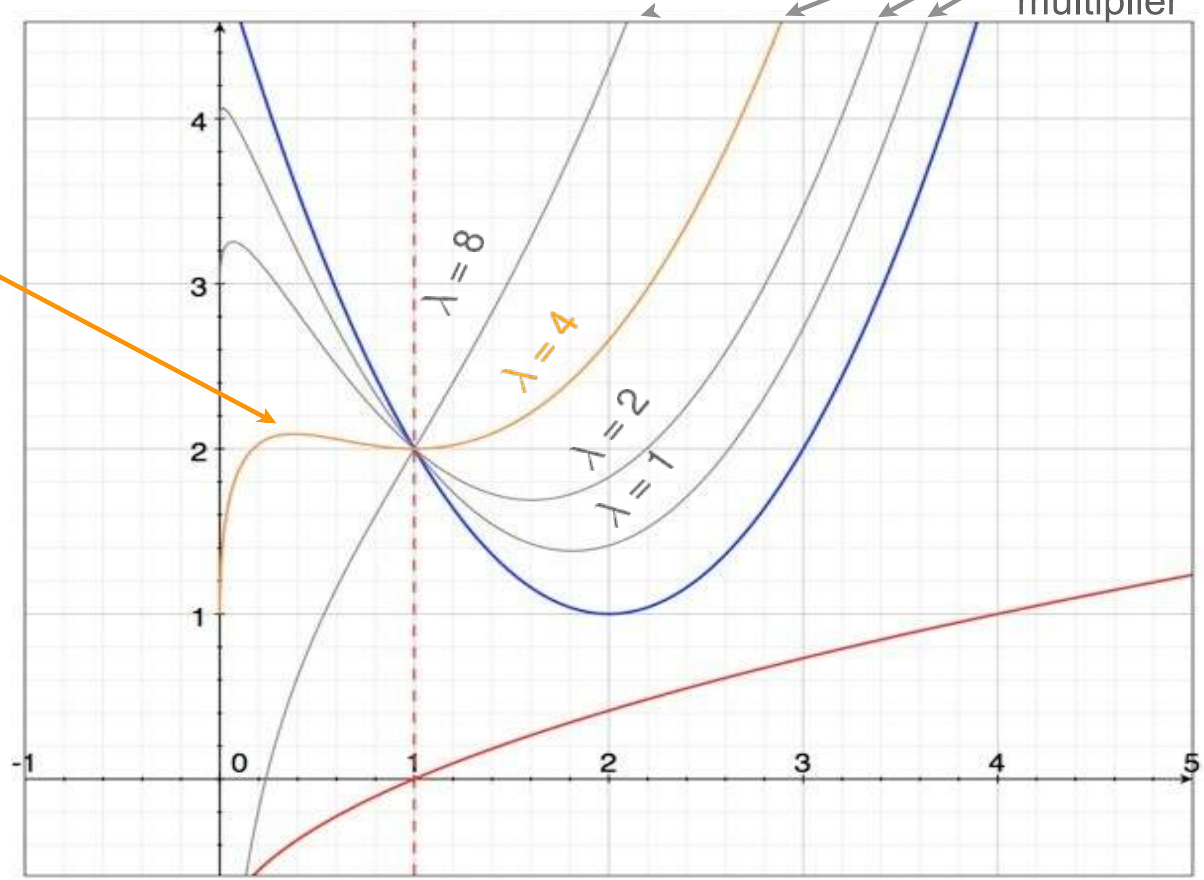
Barrier functions are implemented similar to penalties but must meet the following conditions

1. Continuous
2. Non-negative
3. Approach infinity as x approaches boundary

Lagrange Multipliers

Lagrangians for various values of the Lagrange multiplier

There is an optimal λ for which we obtain the constrained solution in x by minimizing the Lagrangian for that λ



Summary

- Constraints are requirements on the design points that a solution must satisfy
- Some constraints can be transformed or substituted into the problem to result in an unconstrained optimization problem
- Analytical methods using Lagrange multipliers yield the generalized Lagrangian and the necessary conditions for optimality under constraints
- A constrained optimization problem has a dual problem formulation that is easier to solve and whose solution is a lower bound of the solution to the original problem

Summary

- Penalty methods penalize infeasible solutions and often provide gradient information to the optimizer to guide infeasible points toward feasibility
- Interior point methods maintain feasibility but use barrier functions to avoid leaving the feasible set

Supplementary materials & lecture notes

CHALLENGE #3 Reformulate the problem

$$\min_x f(x) = (x_1 - 3)^2 + (x_2 - 3)^2$$

subject to

$$h(x) = x_1 + x_2 - 4 = 0$$

to an unconstrained optimization problem using penalization.

$$P(x, R) = f(x) + \Omega(R, g(x), h(x))$$

$$\Omega = Rh_j^2(x)$$

find $x1^*, x2^*$ for $R = [10,100,1000,10000, 10\text{^}5]$

Supplementary materials

CHALLENGE #3 Reformulate the problem

https://colab.research.google.com/drive/1rGHIklpVM4Gqy9eq1
Y10fIQW9umIjeEx

Let's start minimizing

- Read and finish the notebook called 00-intro.ipynb and 02-constrained-optimization.ipynb

MERCI

Institut Supérieur de l'Aéronautique et de l'Espace
10, avenue Edouard Belin – BP 54032
31055 Toulouse Cedex 4 – France
T +33 5 61 33 80 80
www.isae-supaero.fr