The micropolar continuum theory: a revision from the wave propagation perspective

Gary Dargush Ali Reza Hadjesfandiari
Nicolás Guarín-Zapata Juan Gomez

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1 Introduction

The classical model of continuum mechanics lacks a length scale thus it is unable to capture so-called size effects. In the early 1900s, starting with the work from the Cosserat brothers (Cosserat and Cosserat, 1909) a series of non-classical models were proposed with an intrinsic capability to account for microstructural effects while retaining a continuum approach. Broadly speaking these models could be categorized into micropolar models incorporating additional degrees of freedom and reduced or gradient models keeping additional displacement gradients in its kinematic formulation. Although most of these formulations are mathematically consistent and are also able to predict numerical results valid under particular scenarios, not all of them are fully consistent as a continuum theory. For instance in a series of recent contribution Hadjesfandiari and Dargush() and Dargush and Hadjesfandiari() presented arguments questioning the continuum mechanics consistency of non-classical models. Moreover, these authors (see Hadjesfandiari and Dargush()) formulated a valid Couple stress framework after fixing the kinematical indeterminacy in the reduced couple stress theory.

Among the non-classical models, the micropolar theory (Eringen, 1966) is very convenient since it just introduces an additional rotation independent of the macro-rotation which conserves the original structure of the displacements equations from classical theory and also eliminates compatibility issues in the corresponding numerical implementations. In this work we conduct additional scrutiny o the micropolar model but examining it from the point of view of its response under dynamic conditions and particularly from its wave propagation perspective. We start by reviewing the stress and displacement equations of motion and in general the mathematical consistency of the problem. Using vector potentials we find the existent propagation modes, derive dispersion relations and reflection coefficients. As a final verification we use Bloch analysis to obtain the band structure of a homogeneous micropolar model.

2 Fundamental relationships

2.1 Conservation of linear and angular momentum

In micropolar elasticity we have conservation of linear and angular momentum given by the following expressions

$$\sigma_{ji,i} + f_i = \rho \ddot{u}_i \tag{1a}$$
$$\sigma_{jk}\epsilon_{ijk} + \mu_{ji,j} + c_i = J\ddot{\varphi}\,. \tag{1b}$$

2.2 Kinematic relations

The (linearized) kinematic relations are given by

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\varphi_k \tag{2a}$$
$$\kappa_{ji} = \varphi_{i,j}\,. \tag{2b}$$

2.3 Constitutive equations

In the linear regime, the constitutive equations are

$$\sigma_{ji} = (\mu + \alpha)\gamma_{ji} + (\mu - \alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij} \tag{3a}$$
$$\mu_{ji} = (\gamma + \varepsilon)\kappa_{ji} + (\gamma - \varepsilon)\kappa_{ij} + \beta\kappa_{kk}\delta_{ij}\,. \tag{3b}$$

These constitutive equations can also be written as

$$\begin{aligned}\sigma_{ji} &= \mu\gamma_{(ij)} + 2\alpha\gamma_{[ij]} + \lambda\gamma_{kk}\delta_{ij}\\ \mu_{ji} &= \gamma\kappa_{(ij)} + 2\varepsilon\kappa_{[ij]} + \beta\kappa_{kk}\delta_{ij}\,,\end{aligned}$$

where we have separated the symmetric and antisymmetric parts of the

3 Derivation of the equations for displacements and rotations

We can start writing the conservation equations (1) as kinematic variables using the constitutive equations (3), to obtain:

$$(\mu+\alpha)\gamma_{ji,j} + (\mu-\alpha)\gamma_{ij,j} + \lambda\gamma_{kk,i} + f_i = \rho\ddot{u}_i$$

$$\epsilon_{ijk}(\mu+\alpha)\gamma_{jk} + \epsilon_{ijk}(\mu-\alpha)\gamma_{kj} + \cancelto{0}{\epsilon_{ijk}\delta_{jk}\lambda\gamma_{rr}} + (\gamma+\epsilon)\kappa_{ji,j} + (\gamma-\epsilon)\kappa_{ij,j} + \beta\kappa_{kk,i} + c_i = J\ddot{\varphi}_i\,.$$

Let's focus in the equations for the conservation of linear momentum first. If we replace the kinematics relations, we get

$$(\mu+\alpha)u_{i,jj} - (\mu+\alpha)\epsilon_{kji}\varphi_{k,j} + (\mu-\alpha)u_{j,ij} + (\mu-\alpha)\epsilon_{kji}\varphi_{k,j} + \lambda u_{k,ki} - \cancelto{0}{\epsilon_{krr}\lambda\varphi_{k,j}} + f_i = \rho\ddot{u}_i\,,$$

or

$$(\mu+\alpha)u_{i,jj} + (\mu-\alpha)u_{j,ij} + \lambda u_{k,kj} - 2\alpha\epsilon_{kji} + f_i = \rho\ddot{u}_i\,.$$

And, using the identity $a_{i,jj} = a_{j,ji} - \epsilon_{ijk}\epsilon_{klm}a_{m,lj}$,

$$(\mu+\alpha)u_{k,ki} - \epsilon_{ijk}\epsilon_{klm}(\mu+\alpha)u_{m,lj} + (\mu-\alpha)u_{j,ij} + \lambda u_{k,ki} - 2\alpha\epsilon_{kji}\varphi_{k,j} + f_i = \rho\ddot{u}_i\,,$$

grouping $u_{k,ki}$,

$$(\mu+\alpha+\lambda)u_{k,ki} - \epsilon_{ijk}\epsilon_{klm}(\mu+\alpha)u_{m,lj} + (\mu-\alpha)u_{j,ij} + 2\alpha\epsilon_{ijk}\varphi_{k,j} + f_i = \rho\ddot{u}_i\,,$$

but $u_{j,ij} = u_{j,ji}$, thus

$$(\lambda+2\mu)u_{k,ki} - \epsilon_{ijk}\epsilon_{klm}(\mu+\alpha)u_{m,lj} + 2\alpha\epsilon_{ijk}\varphi_{k,j} + f_i = \rho\ddot{u}_i\,.$$

Following a similar approach, we obtain

$$(\beta+2\gamma)\varphi_{k,ki} - \epsilon_{ijk}\epsilon_{klm}(\gamma+\varepsilon)u_{m,lj} + 2\alpha\epsilon_{ijk}u_{k,j} - 4\alpha\varphi + c_i = J\ddot{\varphi}_i\,,$$

for rotations.

4 Equations in index and vector notation

Authors presents the equations in slightly different ways, in this section we present the equations in two different forms.

One form is

$$(\mu + \alpha) u_{i,jj} + (\lambda + \mu - \alpha) u_{j,ji} + 2\alpha \epsilon_{ijk} \varphi_{k,j} + f_i = \rho \ddot{u}_i \tag{4a}$$

$$(\gamma + \varepsilon) \varphi_{i,jj} + (\beta + \gamma - \varepsilon) \varphi_{j,ji} + 2\alpha \epsilon_{ijk} u_{k,j} - 4\alpha \varphi_i + c_i = J \ddot{\varphi}_i \,, \tag{4b}$$

or, in vector notation

$$(\mu + \alpha) \nabla^2 \mathbf{u} + (\lambda + \mu - \alpha) \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \mathbf{u} + 2\alpha \boldsymbol{\nabla} \times \varphi + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

$$(\gamma + \varepsilon) \nabla^2 \varphi + (\beta + \gamma - \varepsilon) \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \varphi + 2\alpha \boldsymbol{\nabla} \times \mathbf{u} - 4\alpha \varphi + \mathbf{c} = J \frac{\partial^2 \varphi}{\partial t^2} \,.$$

We find the second form more appropriated for wave propagation. This one, reads

$$(\lambda + 2\mu) u_{k,ki} - \epsilon_{ijk} \epsilon_{klm} (\mu + \alpha) u_{m,lj} + 2\alpha \epsilon_{ijk} \varphi_{k,j} + f_i = \rho \ddot{u}_i, \tag{5a}$$

$$(\beta + 2\gamma) \varphi_{k,ki} - \epsilon_{ijk} \epsilon_{klm} (\gamma + \varepsilon) u_{m,lj} + 2\alpha \epsilon_{ijk} u_{k,j} - 4\alpha \varphi + c_i = J \ddot{\varphi}_i \,, \tag{5b}$$

or, in vector notation

$$(\lambda + 2\mu) \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \mathbf{u} - (\mu + \alpha) \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u} + 2\alpha \boldsymbol{\nabla} \times \varphi + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{6a}$$

$$(\beta + 2\gamma) \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \varphi - (\gamma + \varepsilon) \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \varphi + 2\alpha \boldsymbol{\nabla} \times \mathbf{u} - 4\alpha \varphi + \mathbf{c} = J \frac{\partial^2 \varphi}{\partial t^2} \,. \tag{6b}$$

5 Waves in micropolar solids

Let's write equation (6) in a slightly different way, where we regroup the material constants in a different way and neglect body forces

$$c_1^2 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \mathbf{u} - c_2^2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u} + K^2 \boldsymbol{\nabla} \times \varphi = \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{7a}$$

$$c_3^2 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \varphi - c_4^2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \varphi + Q^2 \boldsymbol{\nabla} \times \mathbf{u} - 2Q^2 \varphi = \frac{\partial^2 \varphi}{\partial t^2} \,. \tag{7b}$$

where,

$$
\begin{aligned}
c_1^2 &= \frac{\lambda + 2\mu}{\rho}, \quad c_3^2 = \frac{\beta + 2\gamma}{J}, \\
c_2^2 &= \frac{\mu + \alpha}{\rho}, \quad c_4^2 = \frac{\gamma + \varepsilon}{J}, \\
K^2 &= \frac{2\alpha}{\rho}, \quad Q^2 = \frac{2\alpha}{J},
\end{aligned}
$$

To identify types of propagating waves that can arise in the micropolar medium we expand our main variables in terms of scalar and vector potentials, as follows

$$
\begin{aligned}
\mathbf{u} &= \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\Gamma}, \\
\varphi &= \boldsymbol{\nabla}\tau + \boldsymbol{\nabla} \times \mathbf{E},
\end{aligned}
$$

with

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma} &= 0 \\
\boldsymbol{\nabla} \cdot \mathbf{E} &= 0 .
\end{aligned}
$$

If we plug these in (7a), we obtain the following, after some manipulations,

$$
c_1^2 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2} \tag{8a}
$$

$$
c_3^2 \nabla^2 \tau - 2Q^2 \tau = \frac{\partial^2 \tau}{\partial t^2} \tag{8b}
$$

$$
\begin{bmatrix} c_2^2 \nabla^2 & K^2 \boldsymbol{\nabla}\times \\ Q^2 \boldsymbol{\nabla}\times & c_4^2 \nabla^2 - 2Q^2 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Gamma} \\ \mathbf{E} \end{Bmatrix} = \frac{\partial^2}{\partial t^2} \begin{Bmatrix} \boldsymbol{\Gamma} \\ \mathbf{E} \end{Bmatrix}, \tag{8c}
$$

where we can see that the equations for the scalar potentials are uncoupled, while the ones for the vector potentials are coupled.

5.1 Dispersion relations

If we assume that the potentials are plane waves we can find the dispersion relations. Particularly, for the coupled waves, we have

$$
\begin{aligned}
\boldsymbol{\Gamma} &= \mathbf{A} \exp(i\kappa x - i\omega t) \\
\mathbf{E} &= \mathbf{B} \exp(i\kappa x - i\omega t),
\end{aligned}
$$

and plugging it into the coupled equation we end up with the following system of equations

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{A} \\ \mathbf{B} \end{Bmatrix} = \mathbf{0}\,,$$

with

$$\begin{aligned}
M_{11} &= (c_2^2\kappa^2 - \omega^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
M_{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & iK^2\kappa \\ 0 & -iK^2\kappa & 0 \end{bmatrix} \\
M_{21} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & iQ^2\kappa \\ 0 & -iQ^2\kappa & 0 \end{bmatrix} \\
M_{22} &= (2Q^2 + c_4^2\kappa^2 - \omega^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\,.
\end{aligned}$$

Since we are not interested in the null solution, we have

$$\det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = 0\,,$$

that leads to the dispersion equation

$$\omega^2 = Q^2 + \frac{c_2^2\kappa^2}{2} + \frac{c_4^2\kappa^2}{2} \mp \frac{1}{2}\sqrt{4K^2Q^2\kappa^2 + 4Q^4 - 4Q^2c_2^2\kappa^2 + 4Q^2c_4^2\kappa^2 + c_2^4\kappa^4 - 2c_2^2c_4^2\kappa^4 + c_4^4\kappa^4}\,,$$

where the minus signs corresponds to the transverse wave and the plus sign corresponds to the rotational wave.

The dispersion relations are then

$$\omega_P = c_1\kappa\,, \tag{9}$$

$$\omega_{RL} = \sqrt{2Q^2 + c_3^2\kappa^2}\,, \tag{10}$$

$$\omega_S = \sqrt{Q^2 + \frac{(c_2^2 + c_4^2)}{2}\kappa^2 - \frac{1}{2}\sqrt{4Q^4 + 4Q^2[(c_4^2 - c_2^2) + K^2]\kappa^2 + (c_4^2 - c_2^2)^2\kappa^4}}\,, \tag{11}$$

$$\omega_{RT} = \sqrt{Q^2 + \frac{(c_2^2 + c_4^2)}{2}\kappa^2 + \frac{1}{2}\sqrt{4Q^4 + 4Q^2[(c_4^2 - c_2^2) + K^2]\kappa^2 + (c_4^2 - c_2^2)^2\kappa^4}}\,, \tag{12}$$

where we can see that the only wave that is not dispersive is the P-wave, since the relationship between wavenumber and frequency is linear.

5.1.1 Wave speeds

We can write the phase speeds $v_i \equiv \omega_i/\kappa$, to obtain

$$v_P = c_1\,, \tag{13}$$

$$v_{RL} = \sqrt{\frac{2Q^2}{\kappa^2} + c_3^2}\,, \tag{14}$$

$$v_S = \sqrt{\frac{Q^2}{\kappa^2} + \frac{(c_2^2 + c_4^2)}{2} - \frac{1}{2}\sqrt{\frac{4Q^4}{\kappa^4} + \frac{4Q^2}{\kappa^2}[(c_4^2 - c_2^2) + K^2] + (c_4^2 - c_2^2)^2}}\,, \tag{15}$$

$$v_{RT} = \sqrt{\frac{Q^2}{\kappa^2} + \frac{(c_2^2 + c_4^2)}{2} + \frac{1}{2}\sqrt{\frac{4Q^4}{\kappa^4} + \frac{4Q^2}{\kappa^2}[(c_4^2 - c_2^2) + K^2] + (c_4^2 - c_2^2)^2}}\,, \tag{16}$$

if we take the limit $\kappa \to 0$, we obtain

$$\lim_{\kappa\to 0} v_S = \sqrt{c_2^2 - \frac{K^2}{2}} = \sqrt{\frac{\mu}{\rho}}\,,$$

that corresponds to the phase speed of the shear wave in classical media. And, if we take the high frequency limit ($\kappa \to \infty$), we find that

$$\begin{aligned}
\lim_{\kappa\to\infty} v_{RL} &= c_3\,,\\
\lim_{\kappa\to\infty} v_S &= c_2\,,\\
\lim_{\kappa\to\infty} v_{RT} &= c_4\,.
\end{aligned}$$

We can also compute the group speed $g_i = \partial\omega_i/\partial\kappa$, to obtain

$$g_P = c_1\,, \tag{17}$$

$$g_{RL} = \frac{c_3^2\kappa}{\omega_{RL}}\,, \tag{18}$$

$$g_S = \frac{1}{2\omega_S}\left[(c_2^2 + c_4^2)\kappa - \frac{2Q^2[(c_4^2 - c_2^2) + K^2]\kappa + (c_4^2 + c_2^2)^2\kappa^3}{\sqrt{4Q^4 + 4Q^2[(c_4^2 - c_2^2) + K^2]\kappa^2 + (c_4^2 - c_2^2)^2\kappa^4}}\right]\,, \tag{19}$$

$$g_{RT} = \frac{1}{2\omega_{RT}}\left[(c_2^2 + c_4^2)\kappa + \frac{2Q^2[(c_4^2 - c_2^2) + K^2]\kappa + (c_4^2 + c_2^2)^2\kappa^3}{\sqrt{4Q^4 + 4Q^2[(c_4^2 - c_2^2) + K^2]\kappa^2 + (c_4^2 - c_2^2)^2\kappa^4}}\right]\,, \tag{20}$$

if we take the limit $\kappa \to 0$, we obtain

$$\begin{aligned}
&\lim_{\kappa\to 0} g_{RL} = 0\,,\\
&\lim_{\kappa\to 0} g_S = \lim_{\kappa\to 0} v_S = \sqrt{c_2^2 - \frac{K^2}{2}} = \sqrt{\frac{\mu}{\rho}}\,,\\
&\lim_{\kappa\to 0} g_{RT} = 0\,.
\end{aligned}$$

And in the high frequency limit ($\kappa \to \infty$), we find that

$$\begin{aligned}
&\lim_{\kappa\to \infty} g_{RL} = \lim_{\kappa\to \infty} v_{RL} = c_3\,,\\
&\lim_{\kappa\to \infty} g_S = \lim_{\kappa\to \infty} v_S = c_2\,,\\
&\lim_{\kappa\to \infty} g_{RT} = \lim_{\kappa\to \infty} v_{RT} = c_4\,.
\end{aligned}$$

5.1.2 Dispersion relations as functions of frequency

In some applications is useful to have the dispersion relations as functions of frequency instead of wavenumber. In this case they are given by

$$\kappa_P = \frac{\omega}{c_1}\,, \tag{21}$$

$$\kappa_{RL} = \frac{\sqrt{\omega^2 - 2Q^2}}{c_3}\,, \tag{22}$$

$$\kappa_S = \sqrt{\frac{D}{2} + \frac{1}{2}\sqrt{D^2 - 4E}}\,, \tag{23}$$

$$\kappa_{RT} = \sqrt{\frac{D}{2} - \frac{1}{2}\sqrt{D^2 - 4E}}\,, \tag{24}$$

with

$$\begin{aligned}
D &= \frac{1}{2c_2^2c_4^2}\left[(c_2^2 + c_4^2)\omega^2 - 2Q^2\left(c_2^2 - \frac{K^2}{2}\right)\right]\,,\\
E &= \frac{2Q^2\omega^2}{c_2^2c_4^2} - \frac{\omega^4}{c_2^2c_4^2}\,.
\end{aligned}$$

And, equivalently, write the slowness for each wave, namely

$$s_P(\omega) \equiv \frac{1}{v_P(\omega)} = \frac{1}{c_1}\,, \tag{25}$$

$$s_{RL}(\omega) \equiv \frac{1}{v_{RL}(\omega)} = \sqrt{\frac{1}{c_3} - \frac{2Q^2}{\omega^2 c_3^2}}\,, \tag{26}$$

$$s_S(\omega) \equiv \frac{1}{v_S(\omega)} = \frac{\kappa_S}{\omega}\,, \tag{27}$$

$$s_{RT}(\omega) \equiv \frac{1}{v_{RT}(\omega)} = \frac{\kappa_R}{\omega}\,. \tag{28}$$

5.2 Reflection of a plane wave on a plane boundary

Let us consider now the problem of a plane boundary, as shown in the following schematic.

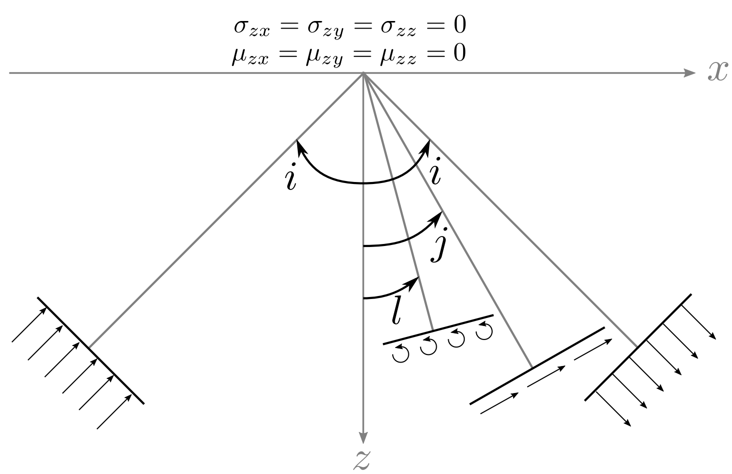

Figure 1: Schematic for analysis of reflected waves set up by a plane P-wave incident on the free surface of a micropolar half-space.

If we consider waves lying on the zx plane, the equations of motions reduce to

$$
\begin{aligned}
&c_1^2\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_z}{\partial x\partial z}\right]-c_2^2\left[\frac{\partial^2 u_z}{\partial x\partial z}-\frac{\partial^2 u_x}{\partial z^2}\right]-K^2\frac{\partial\phi_y}{\partial z}=\ddot{u}_x\,,\\
&c_1^2\left[\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_z}{\partial z^2}\right]-c_2^2\left[\frac{\partial^2 u_x}{\partial x\partial z}-\frac{\partial^2 u_z}{\partial z^2}\right]+K^2\frac{\partial\phi_y}{\partial x}=\ddot{u}_z\,,\\
&Q^2\left[\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right]+c_4^2\left[\frac{\partial^2\phi_y}{\partial x^2}+\frac{\partial^2\phi_y}{\partial z^2}\right]-2Q^2\phi_y=\ddot{\phi}_y\,.
\end{aligned}
$$

And the strain and curvature tensors are

$$
\begin{aligned}
\gamma&=\begin{bmatrix}\frac{\partial u_x}{\partial x} & 0 & \frac{\partial u_z}{\partial x}+\phi_y\\ 0 & 0 & 0\\ \frac{\partial u_x}{\partial z}-\phi_y & 0 & \frac{\partial u_z}{\partial z}\end{bmatrix}\,,\\
\kappa&=\begin{bmatrix}0 & \frac{\partial\phi_y}{\partial x} & 0\\ 0 & 0 & 0\\ 0 & \frac{\partial\phi_y}{\partial z} & 0\end{bmatrix}\,.
\end{aligned}
$$

While the force-stress and couple-stress tensor read

$$
\begin{aligned}
\sigma&=\begin{bmatrix}(\lambda+2\mu)\frac{\partial u_x}{\partial x}+\lambda\frac{\partial u_x}{\partial z} & 0 & (\mu+\alpha)\frac{\partial u_x}{\partial z}+(\mu-\alpha)\frac{\partial u_z}{\partial x}-2\alpha\phi_y\\ 0 & \lambda\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right) & 0\\ (\mu-\alpha)\frac{\partial u_x}{\partial z}+(\mu+\alpha)\frac{\partial u_z}{\partial x}+2\alpha\phi_y & 0 & \lambda\frac{\partial u_x}{\partial x}+(\lambda+2\mu)\frac{\partial u_z}{\partial z}\end{bmatrix}\,,\\
\mu&=\begin{bmatrix}0 & (\gamma-\epsilon)\frac{\partial\phi_y}{\partial x} & 0\\ (\gamma+\epsilon)\frac{\partial\phi_y}{\partial x} & 0 & (\gamma+\epsilon)\frac{\partial\phi_y}{\partial z}\\ 0 & (\gamma-\epsilon)\frac{\partial\phi_y}{\partial z} & 0\end{bmatrix}\,.
\end{aligned}
$$

For this problem the following boundary conditions should be satisfied

$$\sigma_{zx}=\sigma_{zz}=\mu_{zy}=0\,,$$

with

$$
\begin{aligned}
\sigma_{zx}&=(\mu-\alpha)\frac{\partial u_x}{\partial z}+(\mu+\alpha)\frac{\partial u_z}{\partial x}+2\alpha\phi_y\,,\\
\sigma_{zz}&=\lambda\frac{\partial u_x}{\partial x}+(\lambda+2\mu)\frac{\partial u_z}{\partial z}\,,\\
\mu_{zy}&=(\gamma-\epsilon)\frac{\partial\phi_y}{\partial z}\,.
\end{aligned}
$$

Since the problem is limited to the zx plane, all the waves can be described by 3 scalar potentials, namely:

- For P:
$$u^P = \left(\frac{\partial \psi^P}{\partial x}, 0, \frac{\partial \psi^P}{\partial z}\right) .$$
- For SV:
$$u^S = \left(-\frac{\partial \psi^S}{\partial z}, 0, \frac{\partial \psi^S}{\partial x}\right) .$$
- For RT:
$$\phi^R = \left(0, \frac{\partial \psi^R}{\partial z}, 0\right) .$$

Leading to the following (partial) traction vectors

$$\begin{aligned}
T^P &= \left(2\mu\frac{\partial^2 \psi^P}{\partial z \partial x} + 2\alpha\frac{\partial \psi^R}{\partial z}, 0, \lambda\nabla^2\psi^P + 2\mu\frac{\partial^2 \psi^P}{\partial z^2}\right) ,\\
T^S &= \left(\mu\left(\frac{\partial^2 \psi^S}{\partial x^2} - \frac{\partial^2 \psi^S}{\partial z^2}\right) + \alpha\nabla^2\psi^S + 2\alpha\frac{\partial \psi^{RT}}{\partial z}, 0, 2\mu\frac{\partial^2 \psi^S}{\partial z \partial x}\right) ,\\
M^R &= \left(0, (\gamma - \epsilon)\frac{\partial^2 \psi^R}{\partial z^2}, 0\right) .
\end{aligned}$$

The potentials can be written as

$$\begin{aligned}
\psi^P &= \psi^{P,i} + \psi^{P,r} ,\\
\psi^S &= \psi^{R,r} ,\\
\psi^R &= \psi^{R,r} ,
\end{aligned}$$

where the superscripts i and r refer to the incident and reflected waves. They can be written as plane waves

$$\begin{aligned}
\psi^{P,i} &= A\exp\left[i\omega(px - p_1 z - t)\right] ,\\
\psi^{P,r} &= B\exp\left[i\omega(px + p_1 z - t)\right] ,\\
\psi^{S,r} &= C\exp\left[i\omega(px + p_2 z - t)\right] ,\\
\psi^{R,r} &= D\exp\left[i\omega(px + p_4 z - t)\right] ,
\end{aligned}$$

with

$$p = \frac{\sin i}{v_1} = \frac{\sin j}{v_2} = \frac{\sin l}{v_4}\,,$$

the horizontal components of the slowness vectors, and,

$$p_1 = \frac{\cos i}{v_1}, \quad p_2 = \frac{\cos j}{v_2}, \quad p_4 = \frac{\cos l}{v_4},$$

the vertical components of the slowness vectors.

Taking into account that

$$\begin{aligned}
\lambda &= \rho c_1^2 - \rho c_2^2, & p_1^2 &= \frac{1}{c_1^2} - p^2, \\
\mu &= \rho c_2^2, & p_2^2 &= \frac{1}{v_2^2} - p^2, \\
2\alpha &= \rho K^2, & p_4^2 &= \frac{1}{v_4^2} - p^2,
\end{aligned}$$

we get

$$v_2^2 p p_1 (A - B) + (1 - 2v_2^2 p^2) C - \frac{K^2}{2c_2^2} C = 0\,,$$
$$(1 - 2c^2 p^2)(A + B) + 2c_2 p p_2 C = 0\,,$$

and

$$D\omega^2 p_4^2 (\gamma - \epsilon) = 0\,.$$

The last equation implies $D = 0$, meaning that an incident P-wave does not generate a reflected RT-wave.

If we solve the last two equation for the ratios B/A and C/A, we get

$$\frac{B}{A} = \frac{4c_2^4 p^2 p_1 p_2 - \left[\frac{c_2^2}{v_2^2}(1 - 2c_2^2 p^2)(1 - 2v_2^2 p^2) - \frac{K^2}{2v_2^2}(1 - 2c_2 p^2)\right]}{4c_2^4 p^2 p_1 p_2 + \left[\frac{c_2^2}{v_2^2}(1 - 2c_2^2 p^2)(1 - 2v_2^2 p^2) - \frac{K^2}{2v_2^2}(1 - 2c_2 p^2)\right]}\,,$$

$$\frac{C}{A} = \frac{4c_2^2 p p_1 (1 - 2c_2^2 p^2)}{4c_2^4 p^2 p_1 p_2 + \left[\frac{c_2^2}{v_2^2}(1 - 2c_2^2 p^2)(1 - 2v_2^2 p^2) - \frac{K^2}{2v_2^2}(1 - 2c_2 p^2)\right]}\,,$$

where the limits $v_2 \to c_2$ and $K \to 0$ clearly lead to the classical version of the coefficients (Aki and Richards, 2002). And, we can obtain the reflection coefficients for the displacements multiplying the amplitude of the potential by the slowness of each wave.

The reflection coefficients for displacements for an incident P-wave are (following the notation of (Aki and Richards, 2002))

$$\acute{P}\grave{P} = \frac{-\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2p_1p_2}{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2p_1p_2}, \tag{29}$$

$$\acute{P}\grave{S} = \frac{4\frac{c_1}{v_2}pp_1\left(\frac{1}{c_2^2} - 2p^2\right)}{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2p_1p_2}. \tag{30}$$

$$\tag{31}$$

Similarly, the reflection coefficients for displacements for an incident S-wave are

$$\acute{S}\grave{S} = \frac{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] - 4p^2p_1p_2}{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2p_1p_2}, \tag{32}$$

$$\acute{S}\grave{P} = \frac{4\frac{v_2}{c_1}pp_2\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{2pp_2K^2}{v_2^2c_2^2}}{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2p_1p_2}. \tag{33}$$

$$\tag{34}$$

where the limits $v_2 \to c_2$ and $K \to 0$ also lead to the classical version of the coefficients.

If we repeat the same process for the incident RT-wave we obtain the following coefficients

$$\acute{R}\grave{R} = -1, \tag{35}$$

$$\acute{R}\grave{P} = -\frac{3ic_1p_4K^2}{2\omega c_2^2v_4pp_1}, \tag{36}$$

$$\acute{R}\grave{S} = \frac{3iv_2p_4K^2\left(\frac{1}{c_2^2} - 2p^2\right)}{4\omega c_2^2v_4^2pp_1}, \tag{37}$$

where we highlight that the coefficients $\acute{R}\grave{P}$ and $\acute{R}\grave{S}$ have units of length and represent evanescent waves that decay with z.

In the general case we have the following scattering matrix

$$\begin{pmatrix} \acute{P}\grave{P} & \acute{S}\grave{P} & \acute{R}\grave{P} \\ \acute{P}\grave{S} & \acute{S}\grave{S} & \acute{R}\grave{S} \\ 0 & 0 & \acute{R}\grave{R} \end{pmatrix} .$$

6 Waves in Couple-Stress Theory

As presented in Hadjesfandiari and Dargush(), the equations of motion in terms of displacements for the Couple-Stress Theory (CST) are

$$c_1^2\nabla(\nabla\cdot\mathbf{u}) - c_2^2\nabla\times\nabla\times\mathbf{u} + l^2c_2^2\nabla^2\nabla\times\nabla\times\mathbf{u} = \ddot{\mathbf{u}}\,,$$

with

$$\begin{aligned} c_1^2 &= \frac{\lambda+2\mu}{\rho}\,,\\ c_2^2 &= \frac{\mu}{\rho}\,,\\ l^2 &= \frac{\eta}{\mu}\,. \end{aligned}$$

For this model the dispersion relations are much simpler, i.e.,

$$\begin{aligned} \omega_P^2 &= c_1^2\kappa^2\,,\\ \omega_S^2 &= c_2^2\kappa^2(1+\kappa^2l^2)\,, \end{aligned}$$

or, isolating κ,

$$\begin{aligned} \kappa_P^2 &= \frac{\omega^2}{c_2^2}\,,\\ \kappa_S^2 &= \frac{1}{2l^2}\left[\pm\sqrt{1+\frac{4\omega^2l^2}{c_2^2}}-1\right]\,, \end{aligned}$$

we can see that the number inside the square root is always greater than 1. We should, then, only consider the positive root. The other root presents an evanescent wave that should arise when satisfying boundary conditions.

The phase speeds are given by

$$
\begin{aligned}
v_P &= c_1\,, \\
v_S(\kappa) &= c_2\sqrt{1+\kappa^2 l^2}\,,
\end{aligned}
$$

or as a function of frequency

$$
\begin{aligned}
v_P &= c_1\,, \\
v_S(\omega) &= c_2\sqrt{1+\frac{1}{2}\left[\sqrt{1+\frac{4\omega^2 l^2}{c_2^2}}-1\right]}\,.
\end{aligned}
$$

If we take the limit $\kappa \to 0$, we obtain

$$
\lim_{\kappa\to 0} v_S = c_2\,,
$$

and, taking the limit $\kappa \to \infty$, we obtain

$$
\lim_{\kappa\to\infty} v_S \to \infty\,.
$$

Group speeds are given by

$$
\begin{aligned}
g_P &= c_1\,, \\
g_S &= c_2\frac{1+2\kappa^2 l^2}{\sqrt{1+\kappa^2 l^2}}\,,
\end{aligned}
$$

taking the same limits, we obtain

$$
\lim_{\kappa\to 0} g_S = c_2\,,
$$

and

$$
\lim_{\kappa\to\infty} g_S \to \infty\,,
$$

meaning that the speed of energy flow increases as the frequency increases.

6.1 Reflection of a plane wave on a plane boundary

We can repeat the process followed before to obtain reflection coefficients for the case of the CST. For completeness some of the steps will be repeated in this section.

The motion is constrained to the plane zx plane, what gives the following displacement vector

$$\mathbf{u} = (u_x, 0, u_z)\,,$$

the following infinitesimal strain and rotation tensor

$$\epsilon = \begin{bmatrix} \frac{\partial u_x}{\partial x} & 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & 0 & \frac{\partial u_z}{\partial z} \end{bmatrix}\,,$$

$$\omega = \begin{bmatrix} 0 & 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) & 0 & 0 \end{bmatrix}\,,$$

and the following stress tensor

$$\sigma = \begin{bmatrix} (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z} & 0 & \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \frac{\eta}{2}\nabla^2\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ 0 & 0 & 0 \\ \frac{\mu}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - \eta\nabla^2\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) & 0 & (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\frac{\partial u_x}{\partial x} \end{bmatrix}\,.$$

In this case the traction-free boundary conditions imply $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$, that is

$$\sigma_{zx} = \frac{\mu}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - \eta\nabla^2\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) = 0$$

$$\sigma_{zz} = (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\frac{\partial u_x}{\partial x} = 0\,,$$

and we know that

$$\mathbf{u}^P = \left(\frac{\partial \psi^P}{\partial x}, 0, \frac{\partial \psi^P}{\partial z}\right)\,,$$

$$\mathbf{u}^S = \left(-\frac{\partial \psi^S}{\partial z}, 0, \frac{\partial \psi^S}{\partial x}\right)\,,$$

where the superscript refer to the type of wave and ψ^i is a scalar potential.

Thus, we have the following tractions

$$T^P = \left(\mu\frac{\partial^2 \psi^P}{\partial z \partial x}, 0, \lambda\frac{\partial^2 \psi^P}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 \psi^P}{\partial z^2}\right)\,,$$

$$T^S = \left(\frac{\mu}{2}\left(\frac{\partial^2 \psi^S}{\partial x^2} - \frac{\partial^2 \psi^S}{\partial z^2}\right) + \eta\nabla^2\nabla^2\psi^S, 0, 2\mu\frac{\partial^2 \psi^S}{\partial z \partial x}\right)\,.$$

If we solve the resulting system of equations we obtain the following reflection coefficients for displacements

$$\acute{P}\grave{P} = \frac{-\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2}{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2},$$

$$\acute{P}\grave{S} = \frac{4\frac{c_1}{v_2}pp_1\left(\frac{1}{c_2^2} - 2p^2\right)}{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2},$$

$$\acute{S}\grave{S} = \frac{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) - 4p^2p_1p_2}{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2},$$

$$\acute{S}\grave{P} = \frac{4\frac{v_2}{c_1}pp_2\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right)}{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2},$$

where the limits $v_2 \to c_2$ and $K \to 0$ clearly lead to the classical version of the coefficients (Aki and Richards, 2002).

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