The micropolar continuum theory: a revision from the wave propagation perspective

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1 Introduction

The classical model of continuum mechanics based upon the fundamental postulate from Cauchy, and written in terms of force densities has been shown to produce valid engineering results provided inherent material micro-structures are neglected. This fundamental hypothesis is acceptable whenever characteristic specimen dimensions are several orders of magnitude larger than typical micro-structural features. However, if the realm of continuum mechanics is intended to be used at

lacks a length scale and thus it is unable to capture so-called size effects. In the early 1900s, starting with the work from the Cosserats (Cosserat and Cosserat, 1909) a series of non-classical models were proposed with an intrinsic capability to account for micro-structural effects while retaining a continuum approach. Broadly speaking these models could be categorized into micropolar models incorporating additional degrees of freedom and reduced or gradient models retaining additional displacement gradients in its kinematic formulation. Although most of these formulations are mathematically consistent and are also able to predict numerical results valid under particular scenarios, not all of them are fully consistent as a continuum theory and therefore their use as predictive models is questionable.

In a series of recent contributions Hadjesfandiari and Dargush() and Dargush and Hadjesfandiari() presented arguments questioning the continuum mechanics consistency of a wide variety of non-classical models. Moreover, these authors (see Hadjesfandiari and Dargush()) formulated a valid Couple stress framework after fixing the kinematical indeterminacy in the reduced couple stress theory.

Among the non-classical models, the micropolar theory (Eringen, 1966) turns out highly convenient since it just introduces an additional rotational degree of freedom independent of the macro-rotation thus preserving the original structure of the displacements equations from classical theory and also avoiding compatibility issues in the corresponding numerical implementations. In this work we conduct additional scrutiny of the micropolar model but examining it from the point of view of its response under from its wave propagation perspective. Within this context it is useful to understand a non-classical continuum model as an approach to incorporate actual micro-structural features present in the material without recurring to an explicit representation. At high frequencies these micro-structures are naturally expected to behave as micro-scatters producing wave dispersion. Clearly, one salient feature of a physically correct non-classical continuum model is the ability to capture wave dispersion.

In this report we initially review the micro-polar continuum model from Eringen (1966) in terms of conservation laws, kinematic relations, constitutive equations and displacement equations of motion. From these we identify possible propagating modes in a micro-polar medium. The resulting waves are then represented in terms of infinite plane disturbances propagating in a micro-polar half-space and producing reflections imposed by the force and moment traction-free boundary conditions. In this sense it is useful to describe the model in terms of reflection coefficients which are later used to compute the time domain response of the half-space allowing to identify the micro-scattering capabilities intrinsic in the continuous model. In the second part of the paper we conduct a similar analysis for the Skew-symmetric Couple Stress theory.

2 The Micro-polar continuum model

2.1 Conservation of linear and angular momentum

Conservation of linear and angular momentum in micropolar elasticity reads:

$$\sigma_{ji,i} + f_i = \rho \ddot{u}_i \tag{1a}$$

$$\sigma_{jk}\epsilon_{ijk} + \mu_{ji,j} + c_i = J\ddot{\varphi}\,. \tag{1b}$$

where σ_{ij} and μ_{ij} correspond respectively to Cauchy force and couple stress tensor; f_i and c_i are body forces and body couples and ρ and $J\ddot{\varphi}$ are the mass density and the rotational inertia density.

2.2 Kinematic relations

The (linearized) kinematic relations are given by

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\varphi_k \tag{2a}$$

$$\kappa_{ji} = \varphi_{i,j}\,. \tag{2b}$$

and where u_i corresponds to the displacements vector and φ_k is the micro-rotation vector.

2.3 Constitutive equations

In the linear regime, the constitutive equations are

$$\sigma_{ji} = (\mu + \alpha)\gamma_{ji} + (\mu - \alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij} \tag{3a}$$

$$\mu_{ji} = (\gamma + \varepsilon)\kappa_{ji} + (\gamma - \varepsilon)\kappa_{ij} + \beta\kappa_{kk}\delta_{ij}\,. \tag{3b}$$

where μ and λ are the known Lame parameters from classical elasticity while α , β , γ and ϵ are additional material parameters from the micro-polar model representative of additional particle interactions.

Alternatively these constitutive equations can also be written as:

$$\sigma_{ji} = \mu\gamma_{(ij)} + 2\alpha\gamma_{[ij]} + \lambda\gamma_{kk}\delta_{ij}$$
$$\mu_{ji} = \gamma\kappa_{(ij)} + 2\varepsilon\kappa_{[ij]} + \beta\kappa_{kk}\delta_{ij}\ ,$$

where we introduced parenthesis and square brackets surrounding a pair of indices to identify respectively the symmetric and skew-symmetric part of a second order tensor.

2.4 Displacements equations of motion

Using (3) and (2) in the lineal momentum and moment of momentum balance equations yields the displacement equations of motion.

$$(\lambda + 2\mu)u_{k,ki} - \epsilon_{ijk}\epsilon_{klm}(\mu + \alpha)u_{m,lj} + 2\alpha\epsilon_{ijk}\varphi_{k,j} + f_i = \rho\ddot{u}_i\ . \tag{4a}$$

$$(\beta + 2\gamma)\varphi_{k,ki} - \epsilon_{ijk}\epsilon_{klm}(\gamma + \varepsilon)u_{m,lj} + 2\alpha\epsilon_{ijk}u_{k,j} - 4\alpha\varphi + c_i = J\ddot{\varphi}_i\ , \tag{5a}$$

3 Waves in micropolar solids

3.1 Equations in terms of potentials

To identify types of propagating waves that can arise in the micropolar medium it is convenient to expand the displacement and rotation vectors in terms of scalar and vector potentials

$$\mathbf{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\Gamma}\,,$$
$$\varphi = \boldsymbol{\nabla}\tau + \boldsymbol{\nabla} \times \mathbf{E}\,,$$

subject to the conditions:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma} = 0$$
$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0\,.$$

Using the above in the displacements equations of motion (repeated here for convenience)

$$(\lambda + 2\mu)\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{u} - (\mu + \alpha)\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u} + 2\alpha\boldsymbol{\nabla} \times \varphi + \mathbf{f} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{6a}$$

$$(\beta + 2\gamma)\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \varphi - (\gamma + \varepsilon)\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \varphi + 2\alpha\boldsymbol{\nabla} \times \mathbf{u} - 4\alpha\varphi + \mathbf{c} = J\frac{\partial^2 \varphi}{\partial t^2}\,. \tag{6b}$$

yields after some manipulations:

$$c_1^2\nabla^2\phi = \frac{\partial^2\phi}{\partial t^2} \tag{7a}$$

$$c_3^2\nabla^2\tau - 2Q^2\tau = \frac{\partial^2\tau}{\partial t^2} \tag{7b}$$

$$\begin{bmatrix} c_2^2\nabla^2 & K^2\boldsymbol{\nabla}\times \\ Q^2\boldsymbol{\nabla}\times & c_4^2\nabla^2 - 2Q^2 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Gamma} \\ \mathbf{E} \end{Bmatrix} = \frac{\partial^2}{\partial t^2} \begin{Bmatrix} \boldsymbol{\Gamma} \\ \mathbf{E} \end{Bmatrix}\,. \tag{7c}$$

where we can see that the equations for the scalar potentials are uncoupled, while the ones for the vector potentials are coupled.

To arrive at (7a) we have neglected body forces and body couples and have written the displacement equations of motions in the alternative form:

$$c_1^2\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{u} - c_2^2\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u} + K^2\boldsymbol{\nabla} \times \varphi = \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{8a}$$

$$c_3^2\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \varphi - c_4^2\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \varphi + Q^2\boldsymbol{\nabla} \times \mathbf{u} - 2Q^2\varphi = \frac{\partial^2 \varphi}{\partial t^2}\,. \tag{8b}$$

where we have used the following relations:

$$
c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_3^2 = \frac{\beta + 2\gamma}{J},
$$
$$
c_2^2 = \frac{\mu + \alpha}{\rho}, \quad c_4^2 = \frac{\gamma + \varepsilon}{J},
$$
$$
K^2 = \frac{2\alpha}{\rho}, \quad Q^2 = \frac{2\alpha}{J}\,,
$$

3.2 Dispersion relations

Writing the potentials as plane waves of amplitude **A** and **B**, wave vector κ and circular frequency ω

$$
\begin{aligned}
\mathbf{\Gamma} &= \mathbf{A}\exp(i\kappa x - i\omega t)\\
\mathbf{E} &= \mathbf{B}\exp(i\kappa x - i\omega t)\,,
\end{aligned}
$$

and using these in the equations of motion yields:

$$
\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{A} \\ \mathbf{B} \end{Bmatrix} = \mathbf{0}\,,
$$

where

$$
\begin{aligned}
M_{11} &= (c_2^2\kappa^2 - \omega^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\\
M_{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & iK^2\kappa \\ 0 & -iK^2\kappa & 0 \end{bmatrix}\\
M_{21} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & iQ^2\kappa \\ 0 & -iQ^2\kappa & 0 \end{bmatrix}\\
M_{22} &= (2Q^2 + c_4^2\kappa^2 - \omega^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\,.
\end{aligned}
$$

For non-trivial solutions it is required that

$$\det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = 0\,,$$

which leads to the following frequency-wave number (or dispersion) relations

$$\omega^2 = Q^2 + \frac{c_2^2\kappa^2}{2} + \frac{c_4^2\kappa^2}{2} \mp \frac{1}{2}\sqrt{4K^2Q^2\kappa^2 + 4Q^4 - 4Q^2c_2^2\kappa^2 + 4Q^2c_4^2\kappa^2 + c_2^4\kappa^4 - 2c_2^2c_4^2\kappa^4 + c_4^4\kappa^4}\,,$$

and where the minus signs corresponds to the transverse wave while the plus sign corresponds to the rotational wave. The dispersion relations for the different phases are given by

$$\omega_P = c_1\kappa\,, \tag{9}$$

$$\omega_{RL} = \sqrt{2Q^2 + c_3^2\kappa^2}\,, \tag{10}$$

$$\omega_S = \sqrt{Q^2 + \frac{(c_2^2 + c_4^2)}{2}\kappa^2 - \frac{1}{2}\sqrt{4Q^4 + 4Q^2[(c_4^2 - c_2^2) + K^2]\kappa^2 + (c_4^2 - c_2^2)^2\kappa^4}}\,, \tag{11}$$

$$\omega_{RT} = \sqrt{Q^2 + \frac{(c_2^2 + c_4^2)}{2}\kappa^2 + \frac{1}{2}\sqrt{4Q^4 + 4Q^2[(c_4^2 - c_2^2) + K^2]\kappa^2 + (c_4^2 - c_2^2)^2\kappa^4}}\,, \tag{12}$$

Notice that the only wave that is not dispersive is the P-wave, since the relationship between wavenumber and frequency is linear.

3.2.1 Wave speeds

We can write the phase speeds $v_i \equiv \omega_i/\kappa$, to obtain

$$v_P = c_1\,, \tag{13}$$
$$v_{RL} = \sqrt{\frac{2Q^2}{\kappa^2} + c_3^2}\,, \tag{14}$$
$$v_S = \sqrt{\frac{Q^2}{\kappa^2} + \frac{(c_2^2+c_4^2)}{2} - \frac{1}{2}\sqrt{\frac{4Q^4}{\kappa^4} + \frac{4Q^2}{\kappa^2}[(c_4^2-c_2^2)+K^2] + (c_4^2-c_2^2)^2}}\,, \tag{15}$$
$$v_{RT} = \sqrt{\frac{Q^2}{\kappa^2} + \frac{(c_2^2+c_4^2)}{2} + \frac{1}{2}\sqrt{\frac{4Q^4}{\kappa^4} + \frac{4Q^2}{\kappa^2}[(c_4^2-c_2^2)+K^2] + (c_4^2-c_2^2)^2}}\,, \tag{16}$$

if we take the limit $\kappa \to 0$, we obtain

$$\lim_{\kappa\to 0} v_S = \sqrt{c_2^2 - \frac{K^2}{2}} = \sqrt{\frac{\mu}{\rho}}\,,$$

that corresponds to the phase speed of the shear wave in classical media. And, if we take the high frequency limit ($\kappa \to \infty$), we find that

$$\begin{aligned}
\lim_{\kappa\to\infty} v_{RL} &= c_3\,,\\
\lim_{\kappa\to\infty} v_S &= c_2\,,\\
\lim_{\kappa\to\infty} v_{RT} &= c_4\,.
\end{aligned}$$

We can also compute the group speed $g_i = \partial\omega_i/\partial\kappa$, to obtain

$$g_P = c_1\,, \tag{17}$$
$$g_{RL} = \frac{c_3^2\kappa}{\omega_{RL}}\,, \tag{18}$$
$$g_S = \frac{1}{2\omega_S}\left[(c_2^2+c_4^2)\kappa - \frac{2Q^2[(c_4^2-c_2^2)+K^2]\kappa + (c_4^2+c_2^2)^2\kappa^3}{\sqrt{4Q^4 + 4Q^2[(c_4^2-c_2^2)+K^2]\kappa^2 + (c_4^2-c_2^2)^2\kappa^4}}\right]\,, \tag{19}$$
$$g_{RT} = \frac{1}{2\omega_{RT}}\left[(c_2^2+c_4^2)\kappa + \frac{2Q^2[(c_4^2-c_2^2)+K^2]\kappa + (c_4^2+c_2^2)^2\kappa^3}{\sqrt{4Q^4 + 4Q^2[(c_4^2-c_2^2)+K^2]\kappa^2 + (c_4^2-c_2^2)^2\kappa^4}}\right]\,, \tag{20}$$

if we take the limit $\kappa \to 0$, we obtain

$$
\begin{aligned}
&\lim_{\kappa\to 0} g_{RL} = 0\,,\\
&\lim_{\kappa\to 0} g_S = \lim_{\kappa\to 0} v_S = \sqrt{c_2^2 - \frac{K^2}{2}} = \sqrt{\frac{\mu}{\rho}}\,,\\
&\lim_{\kappa\to 0} g_{RT} = 0\,.
\end{aligned}
$$

And in the high frequency limit ($\kappa \to \infty$), we find that

$$
\begin{aligned}
&\lim_{\kappa\to\infty} g_{RL} = \lim_{\kappa\to\infty} v_{RL} = c_3\,,\\
&\lim_{\kappa\to\infty} g_S = \lim_{\kappa\to\infty} v_S = c_2\,,\\
&\lim_{\kappa\to\infty} g_{RT} = \lim_{\kappa\to\infty} v_{RT} = c_4\,.
\end{aligned}
$$

3.2.2 Dispersion relations as functions of frequency

In some applications is useful to have the dispersion relations as functions of frequency instead of wavenumber. In this case they are given by

$$\kappa_P = \frac{\omega}{c_1}\,, \tag{21}$$

$$\kappa_{RL} = \frac{\sqrt{\omega^2 - 2Q^2}}{c_3}\,, \tag{22}$$

$$\kappa_S = \sqrt{\frac{D}{2} + \frac{1}{2}\sqrt{D^2 - 4E}}\,, \tag{23}$$

$$\kappa_{RT} = \sqrt{\frac{D}{2} - \frac{1}{2}\sqrt{D^2 - 4E}}\,, \tag{24}$$

with

$$
\begin{aligned}
D &= \frac{1}{2c_2^2 c_4^2}\left[(c_2^2 + c_4^2)\omega^2 - 2Q^2\left(c_2^2 - \frac{K^2}{2}\right)\right]\,,\\
E &= \frac{2Q^2\omega^2}{c_2^2 c_4^2} - \frac{\omega^4}{c_2^2 c_4^2}\,.
\end{aligned}
$$

And, equivalently, write the slowness for each wave, namely

$$s_P(\omega) \equiv \frac{1}{v_P(\omega)} = \frac{1}{c_1}\,, \tag{25}$$

$$s_{RL}(\omega) \equiv \frac{1}{v_{RL}(\omega)} = \sqrt{\frac{1}{c_3} - \frac{2Q^2}{\omega^2 c_3^2}}\,, \tag{26}$$

$$s_S(\omega) \equiv \frac{1}{v_S(\omega)} = \frac{\kappa_S}{\omega}\,, \tag{27}$$

$$s_{RT}(\omega) \equiv \frac{1}{v_{RT}(\omega)} = \frac{\kappa_R}{\omega}\,. \tag{28}$$

3.3 Reflection of a plane wave on a plane boundary

Let us consider now the problem of a plane boundary, as shown in the following schematic.

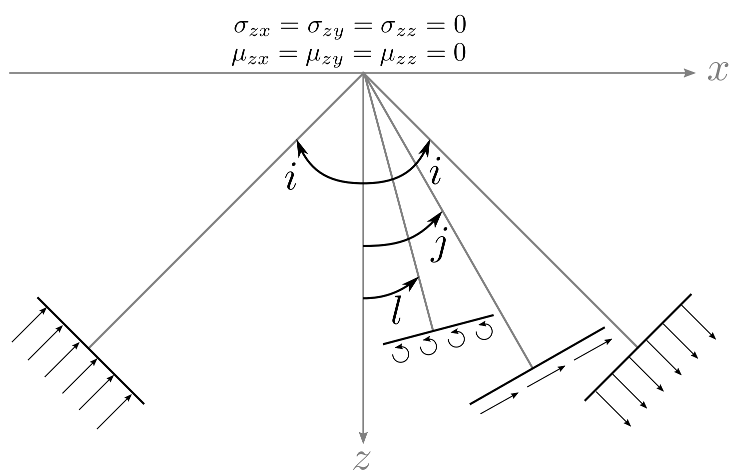

Figure 1. Schematic for analysis of reflected waves set up by a plane P-wave incident on the free surface of a micropolar half-space.

If we consider waves lying on the zx plane, the equations of motions reduce to

$$c_1^2\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_z}{\partial x\partial z}\right]-c_2^2\left[\frac{\partial^2 u_z}{\partial x\partial z}-\frac{\partial^2 u_x}{\partial z^2}\right]-K^2\frac{\partial\phi_y}{\partial z}=\ddot{u}_x\,,$$
$$c_1^2\left[\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_z}{\partial z^2}\right]-c_2^2\left[\frac{\partial^2 u_x}{\partial x\partial z}-\frac{\partial^2 u_z}{\partial z^2}\right]+K^2\frac{\partial\phi_y}{\partial x}=\ddot{u}_z\,,$$
$$Q^2\left[\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right]+c_4^2\left[\frac{\partial^2\phi_y}{\partial x^2}+\frac{\partial^2\phi_y}{\partial z^2}\right]-2Q^2\phi_y=\ddot{\phi}_y\,.$$

And the strain and curvature tensors are

$$\gamma=\begin{bmatrix}\frac{\partial u_x}{\partial x} & 0 & \frac{\partial u_z}{\partial x}+\phi_y\\ 0 & 0 & 0\\ \frac{\partial u_x}{\partial z}-\phi_y & 0 & \frac{\partial u_z}{\partial z}\end{bmatrix}\,,$$
$$\kappa=\begin{bmatrix}0 & \frac{\partial\phi_y}{\partial x} & 0\\ 0 & 0 & 0\\ 0 & \frac{\partial\phi_y}{\partial z} & 0\end{bmatrix}\,.$$

While the force-stress and couple-stress tensor read

$$\sigma=\begin{bmatrix}(\lambda+2\mu)\frac{\partial u_x}{\partial x}+\lambda\frac{\partial u_x}{\partial z} & 0 & (\mu+\alpha)\frac{\partial u_x}{\partial z}+(\mu-\alpha)\frac{\partial u_z}{\partial x}-2\alpha\phi_y\\ 0 & \lambda\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right) & 0\\ (\mu-\alpha)\frac{\partial u_x}{\partial z}+(\mu+\alpha)\frac{\partial u_z}{\partial x}+2\alpha\phi_y & 0 & \lambda\frac{\partial u_x}{\partial x}+(\lambda+2\mu)\frac{\partial u_z}{\partial z}\end{bmatrix}\,,$$
$$\mu=\begin{bmatrix}0 & (\gamma-\epsilon)\frac{\partial\phi_y}{\partial x} & 0\\ (\gamma+\epsilon)\frac{\partial\phi_y}{\partial x} & 0 & (\gamma+\epsilon)\frac{\partial\phi_y}{\partial z}\\ 0 & (\gamma-\epsilon)\frac{\partial\phi_y}{\partial z} & 0\end{bmatrix}\,.$$

For this problem the following boundary conditions should be satisfied

$$\sigma_{zx}=\sigma_{zz}=\mu_{zy}=0\,,$$

with

$$\sigma_{zx}=(\mu-\alpha)\frac{\partial u_x}{\partial z}+(\mu+\alpha)\frac{\partial u_z}{\partial x}+2\alpha\phi_y\,,$$
$$\sigma_{zz}=\lambda\frac{\partial u_x}{\partial x}+(\lambda+2\mu)\frac{\partial u_z}{\partial z}\,,$$
$$\mu_{zy}=(\gamma-\epsilon)\frac{\partial\phi_y}{\partial z}\,.$$

Since the problem is limited to the zx plane, all the waves can be described by 3 scalar potentials, namely:

- For P:
$$u^P = \left(\frac{\partial \psi^P}{\partial x}, 0, \frac{\partial \psi^P}{\partial z} \right) .$$
- For SV:
$$u^S = \left(-\frac{\partial \psi^S}{\partial z}, 0, \frac{\partial \psi^S}{\partial x} \right) .$$
- For RT:
$$\phi^R = \left(0, \frac{\partial \psi^R}{\partial z}, 0 \right) .$$

Leading to the following (partial) traction vectors

$$\begin{aligned}
T^P &= \left(2\mu \frac{\partial^2 \psi^P}{\partial z \partial x} + 2\alpha \frac{\partial \psi^R}{\partial z}, 0, \lambda \nabla^2 \psi^P + 2\mu \frac{\partial^2 \psi^P}{\partial z^2} \right) , \\
T^S &= \left(\mu \left(\frac{\partial^2 \psi^S}{\partial x^2} - \frac{\partial^2 \psi^S}{\partial z^2} \right) + \alpha \nabla^2 \psi^S + 2\alpha \frac{\partial \psi^{RT}}{\partial z}, 0, 2\mu \frac{\partial^2 \psi^S}{\partial z \partial x} \right) , \\
M^R &= \left(0, (\gamma - \epsilon) \frac{\partial^2 \psi^R}{\partial z^2}, 0 \right) .
\end{aligned}$$

The potentials can be written as

$$\begin{aligned}
\psi^P &= \psi^{P,i} + \psi^{P,r} , \\
\psi^S &= \psi^{R,r} , \\
\psi^R &= \psi^{R,r} ,
\end{aligned}$$

where the superscripts i and r refer to the incident and reflected waves. They can be written as plane waves

$$\begin{aligned}
\psi^{P,i} &= A \exp \left[i\omega (px - p_1 z - t) \right] , \\
\psi^{P,r} &= B \exp \left[i\omega (px + p_1 z - t) \right] , \\
\psi^{S,r} &= C \exp \left[i\omega (px + p_2 z - t) \right] , \\
\psi^{R,r} &= D \exp \left[i\omega (px + p_4 z - t) \right] ,
\end{aligned}$$

with

$$p = \frac{\sin i}{v_1} = \frac{\sin j}{v_2} = \frac{\sin l}{v_4}\,,$$

the horizontal components of the slowness vectors, and,

$$p_1 = \frac{\cos i}{v_1}, \quad p_2 = \frac{\cos j}{v_2}, \quad p_4 = \frac{\cos l}{v_4},$$

the vertical components of the slowness vectors.

Taking into account that

$$\begin{aligned}
\lambda &= \rho c_1^2 - \rho c_2^2, & p_1^2 &= \frac{1}{c_1^2} - p^2, \\
\mu &= \rho c_2^2, & p_2^2 &= \frac{1}{v_2^2} - p^2, \\
2\alpha &= \rho K^2, & p_4^2 &= \frac{1}{v_4^2} - p^2,
\end{aligned}$$

we get

$$v_2^2 p p_1 (A - B) + (1 - 2v_2^2 p^2) C - \frac{K^2}{2c_2^2} C = 0\,,$$
$$(1 - 2c^2 p^2)(A + B) + 2c_2 p p_2 C = 0\,,$$

and

$$D\omega^2 p_4^2 (\gamma - \epsilon) = 0\,.$$

The last equation implies $D = 0$, meaning that an incident P-wave does not generate a reflected RT-wave.

If we solve the last two equation for the ratios B/A and C/A, we get

$$\frac{B}{A} = \frac{4c_2^4 p^2 p_1 p_2 - \left[\frac{c_2^2}{v_2^2}(1 - 2c_2^2 p^2)(1 - 2v_2^2 p^2) - \frac{K^2}{2v_2^2}(1 - 2c_2 p^2)\right]}{4c_2^4 p^2 p_1 p_2 + \left[\frac{c_2^2}{v_2^2}(1 - 2c_2^2 p^2)(1 - 2v_2^2 p^2) - \frac{K^2}{2v_2^2}(1 - 2c_2 p^2)\right]}\,,$$

$$\frac{C}{A} = \frac{4c_2^2 p p_1 (1 - 2c_2^2 p^2)}{4c_2^4 p^2 p_1 p_2 + \left[\frac{c_2^2}{v_2^2}(1 - 2c_2^2 p^2)(1 - 2v_2^2 p^2) - \frac{K^2}{2v_2^2}(1 - 2c_2 p^2)\right]}\,,$$

where the limits $v_2 \to c_2$ and $K \to 0$ clearly lead to the classical version of the coefficients (Aki and Richards, 2002). And, we can obtain the reflection coefficients for the displacements multiplying the amplitude of the potential by the slowness of each wave.

The reflection coefficients for displacements for an incident P-wave are (following the notation of (Aki and Richards, 2002))

$$\acute{P}\grave{P} = \frac{-\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2 v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2 p_1 p_2}{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2 v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2 p_1 p_2}, \tag{29}$$

$$\acute{P}\grave{S} = \frac{4\frac{c_1}{v_2} p p_1 \left(\frac{1}{c_2^2} - 2p^2\right)}{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2 v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2 p_1 p_2}. \tag{30}$$

$$\tag{31}$$

Similarly, the reflection coefficients for displacements for an incident S-wave are

$$\acute{S}\grave{S} = \frac{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2 v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] - 4p^2 p_1 p_2}{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2 v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2 p_1 p_2}, \tag{32}$$

$$\acute{S}\grave{P} = \frac{4\frac{v_2}{c_1} p p_2 \left(\frac{1}{v_2^2} - 2p^2\right) - \frac{2pp_2K^2}{v_2^2 c_2^2}}{\left[\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2\right) - \frac{K^2}{2c_2^2 v_2^2}\left(\frac{1}{c_2^2} - 2p^2\right)\right] + 4p^2 p_1 p_2}. \tag{33}$$

$$\tag{34}$$

where the limits $v_2 \to c_2$ and $K \to 0$ also lead to the classical version of the coefficients.

If we repeat the same process for the incident RT-wave we obtain the following coefficients

$$\acute{R}\grave{R} = -1, \tag{35}$$

$$\acute{R}\grave{P} = -\frac{3ic_1 p_4 K^2}{2\omega c_2^2 v_4 p p_1}, \tag{36}$$

$$\acute{R}\grave{S} = \frac{3iv_2 p_4 K^2 \left(\frac{1}{c_2^2} - 2p^2\right)}{4\omega c_2^2 v_4^2 p p_1}, \tag{37}$$

where we highlight that the coefficients $\acute{R}\grave{P}$ and $\acute{R}\grave{S}$ have units of length and represent evanescent waves that decay with z.

In the general case we have the following scattering matrix

$$\begin{pmatrix} \acute{P}\grave{P} & \acute{S}\grave{P} & \acute{R}\grave{P} \\ \acute{P}\grave{S} & \acute{S}\grave{S} & \acute{R}\grave{S} \\ 0 & 0 & \acute{R}\grave{R} \end{pmatrix} .$$

4 Waves in Couple-Stress Theory

4.1 Wave speeds

As presented in Hadjesfandiari and Dargush(), the equations of motion in terms of displacements for the Couple-Stress Theory (CST) are

$$c_1^2 \nabla(\nabla \cdot \mathbf{u}) - c_2^2 \nabla \times \nabla \times \mathbf{u} + l^2 c_2^2 \nabla^2 \nabla \times \nabla \times \mathbf{u} = \ddot{\mathbf{u}} ,$$

with

$$\begin{aligned} c_1^2 &= \frac{\lambda + 2\mu}{\rho} , \\ c_2^2 &= \frac{\mu}{\rho} , \\ l^2 &= \frac{\eta}{\mu} . \end{aligned}$$

For this model the dispersion relations are much simpler, i.e.,

$$\begin{aligned} \omega_P^2 &= c_1^2 \kappa^2 , \\ \omega_S^2 &= c_2^2 \kappa^2 (1 + \kappa^2 l^2) , \end{aligned}$$

or, isolating κ,

$$\begin{aligned} \kappa_P^2 &= \frac{\omega^2}{c_2^2} , \\ \kappa_S^2 &= \frac{1}{2l^2} \left[\pm\sqrt{1 + \frac{4\omega^2 l^2}{c_2^2}} - 1 \right] , \end{aligned}$$

we can see that the number inside the square root is always greater than 1. We should, then, only consider the positive root. The other root presents an evanescent wave that should arise when satisfying boundary conditions.

The phase speeds are given by

$$\begin{aligned}
v_P &= c_1\, ,\\
v_S(\kappa) &= c_2\sqrt{1+\kappa^2 l^2}\, ,
\end{aligned}$$

or as a function of frequency

$$\begin{aligned}
v_P &= c_1\, ,\\
v_S(\omega) &= c_2\sqrt{1+\frac{1}{2}\left[\sqrt{1+\frac{4\omega^2 l^2}{c_2^2}}-1\right]}\, .
\end{aligned}$$

If we take the limit $\kappa\to 0$, we obtain

$$\lim_{\kappa\to 0} v_S = c_2\, ,$$

and, taking the limit $\kappa\to\infty$, we obtain

$$\lim_{\kappa\to\infty} v_S \to \infty\, .$$

Group speeds are given by

$$\begin{aligned}
g_P &= c_1\, ,\\
g_S &= c_2\frac{1+2\kappa^2 l^2}{\sqrt{1+\kappa^2 l^2}}\, ,
\end{aligned}$$

taking the same limits, we obtain

$$\lim_{\kappa\to 0} g_S = c_2\, ,$$

and

$$\lim_{\kappa\to\infty} g_S \to \infty\, ,$$

meaning that the speed of energy flow increases as the frequency increases.

4.2 Reflection of a plane wave on a plane boundary

We can repeat the process followed before to obtain reflection coefficients for the case of the CST. For completeness some of the steps will be repeated in this section.

The motion is constrained to the plane zx plane, what gives the following displacement vector

$$\mathbf{u} = (u_x, 0, u_z)\,,$$

the following infinitesimal strain and rotation tensor

$$\epsilon = \begin{bmatrix} \frac{\partial u_x}{\partial x} & 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & 0 & \frac{\partial u_z}{\partial z} \end{bmatrix}\,,$$

$$\omega = \begin{bmatrix} 0 & 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) & 0 & 0 \end{bmatrix}\,,$$

and the following stress tensor

$$\sigma = \begin{bmatrix} (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z} & 0 & \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \frac{\eta}{2}\nabla^2\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ 0 & 0 & 0 \\ \frac{\mu}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - \eta\nabla^2\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) & 0 & (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\frac{\partial u_x}{\partial x} \end{bmatrix}\,.$$

In this case the traction-free boundary conditions imply $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$, that is

$$\begin{aligned} \sigma_{zx} &= \frac{\mu}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - \eta\nabla^2\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) = 0 \\ \sigma_{zz} &= (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\frac{\partial u_x}{\partial x} = 0\,, \end{aligned}$$

and we know that

$$\begin{aligned} \mathbf{u}^P &= \left(\frac{\partial \psi^P}{\partial x}, 0, \frac{\partial \psi^P}{\partial z}\right)\,, \\ \mathbf{u}^S &= \left(-\frac{\partial \psi^S}{\partial z}, 0, \frac{\partial \psi^S}{\partial x}\right)\,, \end{aligned}$$

where the superscript refer to the type of wave and ψ^i is a scalar potential.

Thus, we have the following tractions

$$T^P = \left(\mu\frac{\partial^2\psi^P}{\partial z\partial x}, 0, \lambda\frac{\partial^2\psi^P}{\partial x^2} + (\lambda+2\mu)\frac{\partial^2\psi^P}{\partial z^2}\right),$$
$$T^S = \left(\frac{\mu}{2}\left(\frac{\partial^2\psi^S}{\partial x^2} - \frac{\partial^2\psi^S}{\partial z^2}\right) + \eta\nabla^2\nabla^2\psi^S, 0, 2\mu\frac{\partial^2\psi^S}{\partial z\partial x}\right).$$

If we solve the resulting system of equations we obtain the following reflection coefficients for displacements

$$\acute{P}\grave{P} = \frac{-\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2}{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2},$$
$$\acute{P}\grave{S} = \frac{4\frac{c_1}{v_2}pp_1\left(\frac{1}{c_2^2} - 2p^2\right)}{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2},$$
$$\acute{S}\grave{S} = \frac{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) - 4p^2p_1p_2}{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2},$$
$$\acute{S}\grave{P} = \frac{4\frac{v_2}{c_1}pp_2\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right)}{\left(\frac{1}{c_2^2} - 2p^2\right)\left(\frac{1}{v_2^2} - 2p^2 + \frac{2l^2\omega^2}{v_2^4}\right) + 4p^2p_1p_2},$$

where the limits $v_2 \to c_2$ and $K \to 0$ clearly lead to the classical version of the coefficients (Aki and Richards, 2002).

Figure 2. Reflection of an incident P-wave on a halfspace for different length scales. The angle of incidence is $\pi/6$ rad, the P-wave speed is $c_1 = 2$, the (classical) S-wave speed is $c_2 = 1$, and the length scales are $l = \{0.01, 0.03, 0.1\}$ (from top to bottom). It is clearly seen how the dispersion increases while the length scale parameter increases.

Appendix

Equations in index and vector notation

Authors presents the equations in slightly different ways, in this section we present the equations in two different forms.

One form is

$$(\mu+\alpha)u_{i,jj} + (\lambda+\mu-\alpha)u_{j,ji} + 2\alpha\epsilon_{ijk}\varphi_{k,j} + f_i = \rho\ddot{u}_i \tag{38a}$$

$$(\gamma+\varepsilon)\varphi_{i,jj} + (\beta+\gamma-\varepsilon)\varphi_{j,ji} + 2\alpha\epsilon_{ijk}u_{k,j} - 4\alpha\varphi_i + c_i = J\ddot{\varphi}_i\,, \tag{38b}$$

or, in vector notation

$$(\mu+\alpha)\nabla^2\mathbf{u} + (\lambda+\mu-\alpha)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u} + 2\alpha\boldsymbol{\nabla}\times\varphi + \mathbf{f} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

$$(\gamma+\varepsilon)\nabla^2\varphi + (\beta+\gamma-\varepsilon)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\varphi + 2\alpha\boldsymbol{\nabla}\times\mathbf{u} - 4\alpha\varphi + \mathbf{c} = J\frac{\partial^2\varphi}{\partial t^2}\,.$$

We find the second form more appropriated for wave propagation. This one, reads

$$(\lambda+2\mu)u_{k,ki} - \epsilon_{ijk}\epsilon_{klm}(\mu+\alpha)u_{m,lj} + 2\alpha\epsilon_{ijk}\varphi_{k,j} + f_i = \rho\ddot{u}_i, \tag{39a}$$

$$(\beta+2\gamma)\varphi_{k,ki} - \epsilon_{ijk}\epsilon_{klm}(\gamma+\varepsilon)u_{m,lj} + 2\alpha\epsilon_{ijk}u_{k,j} - 4\alpha\varphi + c_i = J\ddot{\varphi}_i\,, \tag{39b}$$

or, in vector notation

$$(\lambda+2\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u} - (\mu+\alpha)\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u} + 2\alpha\boldsymbol{\nabla}\times\varphi + \mathbf{f} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{40a}$$

$$(\beta+2\gamma)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\varphi - (\gamma+\varepsilon)\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\varphi + 2\alpha\boldsymbol{\nabla}\times\mathbf{u} - 4\alpha\varphi + \mathbf{c} = J\frac{\partial^2\varphi}{\partial t^2}\,. \tag{40b}$$

References

E. Cosserat and F. Cosserat. *Théorie des Corps Déformables.* A Hermann et Fils, 1909.

A.C Eringen. Linear theory of micropolar elasticity. *Journal of Mathematics and Mechanics*, 15:909–923, 1966.

K. Aki and P. Richards. *Quantitative Seismology.* University Science Books, second edition, 2002.